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STABILITY AND LORENTZIAN GEOMETRY FOR AN INVERSE PROBLEM OF A SEMILINEAR WAVE EQUATION

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This paper concerns an inverse boundary value problem for a semilinear wave equation on a globally hyperbolic Lorentzian manifold. We prove a Hölder stability result for recovering an unknown potential q of the nonlinear wave equation $\square_g u + qu^m = 0$, $m \geq 4$, from the Dirichlet-to-Neumann map. Our proof is based on the recent higher-order linearization method and use of Gaussian beams. We also extend earlier uniqueness results by removing the assumptions of convex boundary and that pairs of light-like geodesics can intersect only once. For this, we construct special light-like geodesics and other general constructions in Lorentzian geometry. We expect these constructions to be applicable in studies of related problems as well.

1. Introduction

We consider the stability and uniqueness of an inverse problem for the nonlinear wave equation on an $(n+1)$-dimensional, $n \geq 2$, globally hyperbolic Lorentzian manifold. As is well known, any globally hyperbolic Lorentzian manifold N is isometric to a product manifold $\mathbb{R} \times M$ equipped with the product metric

$$g = -\beta(t,x)\,dt^2 + h(t,x). \tag{1}$$

Here $\beta > 0$ is a smooth function and $h(t,\cdot)$, $t \in \mathbb{R}$, is a smooth one-parameter family of Riemannian metrics on an n-dimensional manifold M; see, e.g., [Bernal and Sánchez 2005]. Let $\Omega \subset M$ be a smooth submanifold of dimension n with smooth boundary and let us denote the lateral boundary of $[0,T] \times \Omega \subset N$ by

$$\Sigma := [0,T] \times \partial\Omega.$$

In local coordinates (x^a) the d'Alembertian wave operator $\square_g$ of g has the form

$$\square_g u = -\sum_{a,b=0}^{n} \frac{1}{\sqrt{|\det(g)|}} \frac{\partial}{\partial x^a}\Big(\sqrt{|\det(g)|}\, g^{ab} \frac{\partial u}{\partial x^b}\Big).$$

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Here we write $(g^{-1})_{ab} = (g^{ab}),\ a, b = 0, \dots, n$, as usual. We consider the nonlinear wave equation

$$\begin{cases} \Box_g u(t,x) + q(t,x)u(t,x)^m = 0 & \text{in } [0,T]\times\Omega, \\ u = f & \text{on } [0,T]\times\partial\Omega, \\ u(0,x) = \partial_t u(0,x) = 0 & \text{on } \Omega, \end{cases} \tag{2}$$

where we assume that the exponent m is an integer greater than or equal to 4. The inverse problem we study is the stability of recovery of the potential q from the Dirichlet-to-Neumann (DN) map

$$\Lambda : H_0^{s+1}(\Sigma) \to H^s(\Sigma), \quad f \mapsto \partial_\nu u_f|_\Sigma,$$

where u_f is the unique small solution of (2) and ∂_ν is the normal derivative on Σ. Here also H_0^{s+1} and H^s refer to Sobolev spaces, where $s \in \mathbb{N}$ will be specified later. See Section 1.4 for details about Sobolev spaces and Section 2 for details about the well-posedness of the forward problem. The present work is a continuation of the authors' earlier work [Lassas et al. 2022], which considered the stability of a recovery of the potential q of (2) in the Minkowski space of $\mathbb{R}^{n+1}$. We describe our main results in Section 1.1.

Studies of uniqueness and stability of the recovery of unknown parameters in inverse problems are motivated by practical applications. Let us mention some results on inverse problems for linear wave type equations. First results in this direction for the linear wave equation with vanishing initial data were obtained in [Belishev 1987; Belishev and Kurylev 1992]. The approach there is called the boundary control method and it combines both the wave propagation and controllability results [Katchalov et al. 2001]. The boundary control method allows also an effective numerical algorithm [de Hoop et al. 2018]. Recently, there have been several results on determining a Riemannian manifold from partial data boundary measurements for the linear wave equation and related equations such as the ones in [Anderson et al. 2004; Helin et al. 2018; Isozaki et al. 2017; Kian et al. 2019; Krupchyk et al. 2008; Kurylev et al. 2018b; Lassas 2018; Lassas and Oksanen 2014]. However, the boundary control method has been applicable only in the cases where the coefficients of the equation are time-independent, or when the lower-order terms are real analytic in the time variable [Eskin 2007]. In a geometric setting it has been studied if it is possible to recover a Riemannian metric g from the Dirichlet-to-Neumann map of the equation $(\partial_t^2 - \Delta_g)u = 0$ in a stable way. Earlier results for recovery of the metric are based on Tataru's unique continuation principle, which yields stability estimates of logarithmic type; see, e.g., [Bosi et al. 2022]. Later these results have been improved by using different techniques and different assumptions. For example, in [Stefanov and Uhlmann 2005] it was shown that a simple Riemannian metric g can be recovered in a Hölder stable way from the DN map. For examples of instability of inverse problems for a wide class of equations; see [Koch et al. 2021].

Concerning the unique recovery of potentials for a linear counterpart of (2) with lower-order terms we mention [Feizmohammadi et al. 2021; Stefanov 1989; Stefanov and Yang 2018]. These works make use of propagation of singularities along bicharacteristics to determine integrals of the unknown coefficients along light rays. In these results, the Dirichlet-to-Neumann or scattering operator needs to be known over all of the lateral boundary Σ.

Moving on to inverse problems for nonlinear wave equations, Kurylev, Lassas and Uhlmann [Kurylev et al. 2018a] observed that nonlinearity can be used as a beneficial tool in inverse problems for nonlinear wave equations. By exploiting the nonlinearity, some still unsolved inverse problems for linear hyperbolic equations have recently been solved for their nonlinear counterparts. The first results in [Kurylev et al. 2018a], for the scalar wave equation with a quadratic nonlinearity, already showed that local measurements of solutions of the nonlinear wave equation determine the global topology, differentiable structure and the conformal class of the metric g on a globally hyperbolic $(3+1)$-dimensional Lorentzian manifold. The results of [Kurylev et al. 2018a] use the so-called *higher-order linearization method*, which has made inverse problems for nonlinear equations more approachable. The method has given rise to many new results on inverse problems for nonlinear equations. We will explain the method later in this Introduction.

The authors of [Lassas et al. 2018] studied inverse problems for general semilinear wave equations on Lorentzian manifolds, and in [Lassas et al. 2017] they studied the analogous problem for the Einstein–Maxwell equations. The papers [Hintz et al. 2022a; 2022b] are closely related to this work. They use the higher-order linearization method to study uniqueness for the inverse problem of (2). However, these works have additional assumptions that the domain Ω of the time cylinder $[0, T] \times \Omega$ is convex and that light-like geodesics can only intersect once. These conditions are removed in the present work. Our results will in particular improve results in [Hintz et al. 2022b].

The research of inverse problems for nonlinear equations is expanding fast. By using the higher-order linearization method, inverse problems for nonlinear models have been studied for example in [Balehowsky et al. 2022; Cârstea et al. 2019; Chen et al. 2021; 2022; de Hoop et al. 2019; 2020; Feizmohammadi and Oksanen 2020; 2022; Kang and Nakamura 2002; Krupchyk and Uhlmann 2020a; 2020b; Kurylev et al. 2022; Lai et al. 2021; Lassas et al. 2021a; 2021b; Oksanen et al. 2024; Sun and Uhlmann 1997; Uhlmann and Wang 2020; Wang and Zhou 2019].

1.1. ***Main results.*** The present work is a continuation of [Lassas et al. 2022] to the setting of globally hyperbolic Lorentzian manifolds. In that work we considered a stability result for a recovery of the potential q of (2) in $\mathbb{R}^{n+1}$. We denote by (N, g) a globally hyperbolic manifold. We assume that the dimension of N is $n+1$, where $n \geq 2$. As explained earlier, we view N as the product manifold $\mathbb{R} \times M$ equipped with the product metric (1) and where M is an n-dimensional manifold. For $T > 0$, we fix a time-interval $[0, T]$. We assume that $\Omega \subset M$ is an n-dimensional submanifold of M and that Ω has a smooth nonempty boundary $\partial\Omega$.

The finite propagation speed of solutions to the wave equation and the causal structure of (N, g) cause natural limitations on the parts of $[0, T] \times \Omega$ where we can obtain information about the potential in the inverse problem. Let W be a compact set belonging to both the chronological future $I^+(\Sigma)$ and past $I^-(\Sigma)$ of the lateral boundary $\Sigma = [0, T] \times \partial\Omega$:

$$W \subset I^-(\Sigma) \cap I^+(\Sigma) \cap ([0, T] \times \Omega). \tag{3}$$

(See Section 1.2 for the definitions of $I^\pm(\Sigma)$ and other basic Lorentzian geometry concepts.) This is the domain which can be reached by sending waves from Σ so that the possible signals generated by a nonlinear interaction of the waves can also be detected on Σ. We do not assume that $[0, T] \times \partial\Omega$ is convex or that light-like geodesics of (N, g) can only intersect once.

Below we use the notation H_0^s for the closure of the space of compactly supported smooth functions, with respect to the Sobolev H^s norm. The main result of this work is the following:

Theorem 1 (stability estimate). *Suppose (N, g), $N = \mathbb{R}\times M$, is an $(n+1)$-dimensional globally hyperbolic Lorentzian manifold. Let $T > 0$ and let $\Omega \subset M$ be a submanifold with smooth nonempty boundary. Let $m \geq 4$ be an integer, $s \in \mathbb{N}$ with $s + 1 > \frac{n+1}{2}$ and $r \in \mathbb{R}$ with $r \leq s$. Let $j = 1, 2$. Assume that $q_j \in C^{s+1}(\mathbb{R}\times\Omega)$ satisfy $\|q_j\|_{C^{s+1}} \leq c$, $j = 1, 2$, for some $c > 0$. Let $\Lambda_j : H_0^{s+1}(\Sigma) \to H^r(\Sigma)$ be the corresponding Dirichlet-to-Neumann maps of the nonlinear wave equation* (2).

Let $\varepsilon_0 > 0$, $L > 0$ and $\delta \in (0, L)$ be such that

$$\|\Lambda_1(f) - \Lambda_2(f)\|_{H^r(\Sigma)} \leq \delta$$

for all $f \in H_0^{s+1}(\Sigma)$ with $\|f\|_{H^{s+1}(\Sigma)} \leq \varepsilon_0$. Then there exists a constant $C > 0$, independent of q_1, q_2 and $\delta > 0$, such that

$$\|q_1 - q_2\|_{L^\infty(W)} \leq C\delta^{\sigma(s,m)}, \tag{4}$$

where

$$\sigma(s, m) = \frac{8(m-1)}{2m(m-1)(8s - n + 13) + 2m - 1}.$$

A corollary of the theorem is a uniqueness result, which improves the main result of [Hintz et al. 2022b] by allowing nonconvex boundary and light-like geodesics to intersect more than once.

Corollary 2 (uniqueness). *Adopt the notation and assumptions of Theorem 1. Then the Dirichlet-to-Neumann map Λ uniquely determines the potential q within the set W.*

We only consider the case $m \geq 4$ in this work as the other natural cases $m = 2$ or $m = 3$ would lead to additional considerations. The reason is that our method leads to a density problem for products of $m + 1$ solutions of the wave equation. The solutions we use do not yield density in the case $m = 2$, and not even in the case $m = 3$, when light-like geodesics can intersect several times. We mention that the authors of [Hintz et al. 2022b] needed to use different types of solutions in their uniqueness proof when $m = 2$ than in the cases $m \geq 3$. We expect that both the cases $m = 2$ and $m = 3$ can be handled by a method developed in [Feizmohammadi et al. 2023] for an elliptic equation with quadratic nonlinearity transferred to the current hyperbolic setting. We consider the cases $m = 2, 3$ in a future work.

We explain next how our results are proved and how we are able to consider nonconvex boundaries and the case where light-like geodesics can intersect more than once.

1.2. *Sketch of the proof of Theorem 1.* Let us discuss the main ideas behind the proof of Theorem 1. We first discuss how to recover q uniquely from the DN map Λ associated with (2). To avoid technical details, the presentation here is slightly formal. We also only consider here the case $m = 4$ for simplicity, while the case $m > 4$ is similar.

We first recall some notation and definitions in Lorentzian geometry following the books [Beem et al. 1996; O'Neill 1983]. Let (N, g) be a Lorentzian manifold. A smooth path $\mu : (a, b) \to N$ is said to be time-like if $g(\dot{\mu}(s), \dot{\mu}(s)) < 0$ for all $s \in (a, b)$. The path μ is causal if $g(\dot{\mu}(s), \dot{\mu}(s)) \leq 0$ and $\dot{\mu}(s) \neq 0$ for all $s \in (a, b)$. For $p, q \in N$ we write $p \ll q$ if $p \neq q$ and there is a future-pointing time-like path from

p to q. Similarly, $p < q$ if $p \neq q$ and there is a future-pointing causal path from p to q, and $p \leq q$ when $p = q$ or $p < q$. The chronological future of $p \in N$ is the set $I^+(p) = \{q \in N \mid p \ll q\}$ and the causal future of p is $J^+(p) = \{q \in N \mid p \leq q\}$. The chronological past $I^-(q)$ and causal past $J^-(q)$ of $q \in N$ are defined similarly. If $A \subset N$, then we define $J^{\pm}(A) = \bigcup_{p \in A} J^{\pm}(p)$. The sets $I^{\pm}(p)$ are always open. If (N, g) is in addition globally hyperbolic, then the sets $J^{\pm}(p)$ are closed, and the sets $I^{\pm}(p)$ and $J^{\pm}(p)$ are related by $\mathrm{cl}(I^{\pm}(p)) = J^{\pm}(p)$; see [O'Neill 1983, Lemmas 14.6 and 14.22]. Finally, a geodesic from $p \in N$ with initial direction $\xi \in T_p N$ is denoted by $\gamma_{p,\xi}(t) = \exp_p(t\xi)$.

Consider $f_j \in H_0^{s+1}(\Sigma)$, $j = 1, 2, 3, 4$, with $\|f_j\|_{H^{s+1}(\Sigma)} \leq c_0$ for some constant $c_0 > 0$. Let us denote by $u_{\varepsilon_1 f_1 + \cdots + \varepsilon_4 f_4}$ the solution to (2) with boundary data $\varepsilon_1 f_1 + \cdots + \varepsilon_4 f_4$, where $\varepsilon_j > 0$ are sufficiently small parameters. We abbreviate the notation by writing $\vec{\varepsilon} = 0$ when referring to $\varepsilon_1 = \cdots = \varepsilon_4 = 0$. By taking the mixed derivative $\partial^4_{\varepsilon_1 \cdots \varepsilon_4}|_{\vec{\varepsilon}=0}$ of the solution $u_{\varepsilon_1 f_1 + \cdots + \varepsilon_4 f_4}$ to (2) with respect to the parameters $\varepsilon_1, \ldots, \varepsilon_4$, we see that the function

$$w := \frac{\partial}{\partial \varepsilon_1} \cdots \frac{\partial}{\partial \varepsilon_4}\bigg|_{\vec{\varepsilon}=0} u_{\varepsilon_1 f_1 + \cdots + \varepsilon_4 f_4}$$

solves the equation

$$\Box_g w = -16 q v_1 v_2 v_3 v_4 \quad \text{in } [0, T] \times \Omega \tag{5}$$

with vanishing Cauchy and boundary data. Here the functions v_j, $j = 1, \ldots, 4$, satisfy

$$\begin{cases} \Box_g v_j = 0 & \text{in } [0, T] \times \Omega, \\ v_j = f_j & \text{on } [0, T] \times \partial\Omega, \\ v_j|_{t=0} = \partial_t v_j|_{t=0} = 0 & \text{in } \Omega. \end{cases} \tag{6}$$

This way we have produced new linear equations from the nonlinear equation (2). If the DN map Λ is known, then the normal derivative of w is also known on Σ. This is true, because

$$\partial_\nu w = \partial^4_{\varepsilon_1 \cdots \varepsilon_4}|_{\vec{\varepsilon}=0} \Lambda(\varepsilon_1 f_1 + \cdots + \varepsilon_4 f_4).$$

Let v_0 be an auxiliary smooth function solving $\Box_g v = 0$ in $[0, T] \times \Omega$, with $v_0|_{t=T} = \partial_t v_0|_{t=T} = 0$ in Ω. The function v_0 will compensate for the fact that $\partial_\nu w$ is known only on the lateral boundary Σ, but not on $\{t = T\}$. The normal derivative $\partial_\nu w$ is known on $\{t = 0\}$ due to the initial conditions. Multiplying (5) by v_0 and integrating by parts on $[0, T] \times \Omega$, we arrive at the useful integral identity

$$\begin{aligned} \int_\Sigma v_0 \partial^4_{\varepsilon_1 \cdots \varepsilon_4}|_{\vec{\varepsilon}=0} \Lambda(\varepsilon_1 f_1 + \cdots + \varepsilon_4 f_4)\, dS &= \int_{[0,T]\times\Omega} v_0 \Box_g w \, dV_g \\ &= -16 \int_{[0,T]\times\Omega} q v_0 v_1 v_2 v_3 v_4 \, dV_g. \end{aligned} \tag{7}$$

This means that the quantity

$$\int_{[0,T]\times\Omega} q v_0 v_1 v_2 v_3 v_4 \, dV_g \tag{8}$$

is known from the knowledge of the DN map Λ. Since the functions v_j, $j = 1, \ldots, 4$, were arbitrary solutions to (6), we are able to choose suitable solutions v_j so that the products of the form $v_0 v_1 v_2 v_3 v_4$ become dense in $L^1([0, T] \times \Omega)$. This recovers the potential q uniquely. The procedure we have now explained results in new equations, and an integral identity relating the DN map and the unknown q, by differentiating solutions to the nonlinear equation (2) depending on several parameters. This procedure in general is called the *higher-order linearization method*.

The earlier work [Lassas et al. 2022] by the authors studied an analogous stability problem in the Minkowski space. There v_j were chosen to be approximate plane waves so that the product $v_1 v_2 v_3 v_4$ in the integral (8) essentially becomes a delta function of a hyperplane. Hence the integral (8) in that work became the Radon transformation of $q v_0$ in $\mathbb{R}^n$. Since the Radon transformation in $\mathbb{R}^n$ is invertible, this recovered q. In $1+1$ dimensions, the integral (8) becomes an integral of $q v_0$ against a delta distribution, in which case the recovery of pointwise values of $q v_0$ is trivial. The auxiliary function v_0 in the product $q v_0$ can be eliminated by choosing v_0 suitably.

Motivated by the above explanation, in the present work we shall consider the so-called *Gaussian beam* solutions v_j to (6). One can think of Gaussian beams as wave packets traveling on light-like geodesics. In Sections 3 and 5 we will show that by using the nonlinearity of (2) and Gaussian beams, one can produce *approximate delta distributions* from the product $v_1 v_2 v_3 v_4$ in (8). This uses the fact that Gaussian beams are solutions to the linear wave equation (6) with exponential concentration to a neighborhood of a given light-like geodesics up to a small error term. Thus, if two different geodesics intersect, then the product of the corresponding Gaussian beams concentrates near the intersection points of the geodesics. The product of four, instead of two, Gaussian beams is required to cancel oscillations of the product of the solutions. (If oscillations would not be canceled, one would expect not to be able to recover q due to the nonstationary phase.)

Let us explain how we use four Gaussian beams in (7) in more detail. Let us consider $p_0 \in W \subset I^-(\Sigma) \cap I^+(\Sigma) \cap ([0,T] \times \Omega)$. We show that there exist two different geodesics γ_1 and γ_2 that pass through p_0 and that intersect Σ in a suitable manner. We distinguish two cases depending on whether γ_1 and γ_2 intersect only once or multiple times. Let us explain first the simpler case, where the geodesics γ_1 and γ_2 intersect only at the point p_0. Let v_1 and v_2 be Gaussian beam solutions to (6) with respect to γ_1 and γ_2. Making the choice $v_3 = \bar{v}_1$ and $v_4 = \bar{v}_2$ yields $v_1 v_2 v_3 v_4 = |v_1|^2 |v_2|^2$. Evaluating this product, one finds that the product $|v_1|^2 |v_2|^2$ is an approximation of the delta distribution concentrated at p_0. Therefore, by using the integral identity (7) for this specific product $v_1 v_2 v_3 v_4$, and the knowledge of the DN map, we can recover $q v_0$ at p_0. We take v_0 to be another Gaussian beam that is nonzero at p_0. This way we have recovered q at p_0. Repeating the argument for all points of W recovers q on W.

Suppose next that γ_1 and γ_2 intersect at points $x_1 \le \cdots \le x_P$, $P \ge 2$. Using arguments similar to those above, the integral (8) reduces to an integral of $q v_0$ against a sum of approximate delta functions located at the intersection points $x_1, \dots, x_P$. That is, by using (7), we know from the DN map Λ the quantity

$$\sum_{k=1}^{P} q(x_k) v_0(x_k) \tag{9}$$

up to an error, which can be made arbitrarily small by taking a parameter associated to the Gaussian beams large enough. The task is then to decouple the information about $q v_0$ at each single point x_k from the sum above.

To decouple the information, the choice of v_0 plays a crucial role. Recall that the only requirement from v_0 was that it satisfies the wave equation $\Box_g v_0 = 0$ with Cauchy data vanishing at $t = T$. We show that there is a family $(v_0^{(k)})_{k=1}^P$ of P functions, satisfying the required conditions for v_0, with the property that the matrix

$$\mathcal{V} := \begin{pmatrix} v_0^{(1)}(x_1) & v_0^{(1)}(x_2) & \cdots & v_0^{(1)}(x_P) \\ v_0^{(2)}(x_1) & v_0^{(2)}(x_2) & \cdots & v_0^{(2)}(x_P) \\ \vdots & & \ddots & \vdots \\ v_0^{(P)}(x_1) & v_0^{(P)}(x_2) & \cdots & v_0^{(P)}(x_P) \end{pmatrix}$$

is invertible. Thus, by using (9) for each $v_0^{(k)}$ in place of v_0 separately we know the quantity

$$\mathcal{V} \begin{pmatrix} q(x_1) \\ \vdots \\ q(x_P) \end{pmatrix}$$

from the DN map Λ. Since $\mathcal{V}$ is a known invertible matrix, this uniquely recovers the values of the unknown potential q at the points $x_1, \dots, x_P$. We explain in Section 1.3 the idea of how the matrix $\mathcal{V}$ is constructed, while complete statements and proofs about the matter are in Section 5.6. The matrix $\mathcal{V}$ is called a *separation matrix*.

So far, we have sketched the proof of unique recovery of q from the DN map Λ associated with (2). We briefly discuss how to quantify the uniqueness result and thus to prove a stability estimate. To obtain a stability estimate for q in terms of Λ, instead of differentiating (2) with respect to $\varepsilon_1, \dots, \varepsilon_4$, we take the mixed finite difference $D^4_{\varepsilon_1 \cdots \varepsilon_4}$ of $u_{\varepsilon_1 f_1 + \cdots + \varepsilon_2 f_4}$ at $\vec{\varepsilon} = 0$. (Recall that $\vec{\varepsilon} = 0$ stands for $\varepsilon_1 = \cdots = \varepsilon_4 = 0$.) In this case, we obtain a slightly different version of the integral identity (7) given by

$$-16 \int_{[0,T]\times\Omega} q v_0 v_1 v_2 v_3 v_4 \, dV_g = \int_\Sigma v_0 D^4_{\varepsilon_1 \cdots \varepsilon_4}\Big|_{\vec{\varepsilon}=0} \Lambda(\varepsilon_1 f_1 + \cdots + \varepsilon_4 f_4) \, dS + \frac{1}{\varepsilon_1 \cdots \varepsilon_4} \int_{[0,T]\times\Omega} v_0 \Box_g \widetilde{\mathcal{R}} \, dV_g.$$

Here the second integral on the right is a small error term, where $\widetilde{\mathcal{R}}$ is of the size $\mathcal{O}(\langle \varepsilon_1, \dots, \varepsilon_4 \rangle^7)$ in an energy space norm. For details, see (11) and (23)–(24). Here we also denote by $\langle \varepsilon_1, \dots, \varepsilon_4 \rangle^7$ an unspecified homogeneous polynomial of order 7 in $\varepsilon_1, \dots, \varepsilon_4$. If $p_0 \in W$ is fixed, a stability result for q at p_0 follows by using Gaussian beams associated to the light-like geodesics γ_1 and γ_2 described above, optimizing with respect to the parameters $\varepsilon_1, \dots, \varepsilon_4$ and the parameters related to the Gaussian beams v_1, v_2, v_3 and v_4. The implied constant of the stability estimate at the fixed-point estimate depends on p_0. To show that the constant can in fact be taken to be independent of p_0 we must vary the geodesics γ_1 and γ_2 and the corresponding Gaussian beams smoothly. This requires some work, which is done in Section 3. In addition, we must also use different separation matrices for different points in W. These separation matrices will be constructed with respect to a suitable finite collection of solutions to $\Box_g v = 0$. The finite collection will called a *separation filter*, which is explained in the next section.

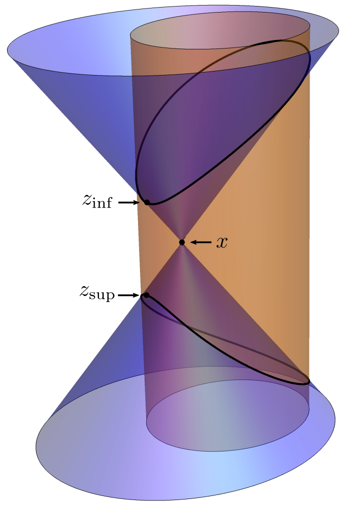

Figure 1. The lateral boundary Σ (orange cylinder) intersects the lightcone (blue cone) of a point x (apex of the cone) along the black curves. The point $z_{\sup}$ is the latest and $z_{\inf}$ the earliest point on Σ which can be reached from x by an optimal geodesics. We call these optimal geodesics boundary optimal geodesics.

1.3. ***Lorentzian geometry tools.*** To prove our main results, we make some constructions in Lorentzian geometry. The main constructions we develop are *boundary optimal geodesics* and *separation matrices*. We explain briefly what these are next. Since we expect the constructions to have applications in related inverse problems as well, and they might also be of interest in Lorentzian geometry in general, this section is written to be independent of the inverse problem we consider. We follow the terminology of the book [O'Neill 1983], and we have included the used concepts of causality in Section 1.4 for an easy access.

Boundary optimal geodesics. Let us first explain what is a boundary optimal geodesic. As before we consider the subset $[0, T]\times\Omega$ of a globally hyperbolic smooth Lorentzian manifold $\mathbb{R}\times M$, $\dim(M) = n \geq 2$, equipped with the metric (1) and where Ω is a smooth submanifold of M with boundary and of dimension n. The lateral boundary Σ refers to the set $[0, T] \times \partial\Omega$ as before. As is by now quite standard, see, e.g., [Kurylev et al. 2018a; O'Neill 1983], we say that a geodesic connecting the points $x, y \in N$, $x \leq y$, is optimal if the time separation function τ of these points vanishes, $\tau(x, y) = 0$. The time separation function is the supremum of lengths of piecewise smooth future-directed causal paths from x to y; see (49) or [O'Neill 1983] for details. An optimal geodesic is always light-like.

Let us then consider a point $x \in I^-(\Sigma)\cap([0, T]\times\Omega)$. In the inverse problem of this paper, we consider Gaussian beams that vanish on a neighborhood of $\{t = T\}$. For this, it is required to find past-directed light-like geodesics of $[0, T]\times\Omega$ from Σ to $x \in [0, T]\times\Omega$, which do not intersect the set $\{t = T\}$. In Lemma 15, we show that we may find a point $z_{\inf}$ of the lateral boundary Σ and an optimal past-directed geodesic γ from $z_{\inf}$ to x. The situation is illustrated in Figure 1. In the figure, the point $z_{\inf} \in \Sigma$, is the point which has the smallest time coordinate in the intersection of the light-like future of x (the upper cone) and Σ. The light-like geodesic γ from z_{inf} to x is not only optimal, i.e., $\tau(x, z_{\mathrm{inf}}) = 0$, but it also necessarily intersects Σ transversally even if Σ would be nonconvex. We call the geodesic γ a boundary optimal geodesic.

Note that by deforming Σ in the figure to a nonconvex manifold, it is possible to find optimal geodesics from x to points in Σ, which intersect Σ tangentially. Therefore, not all optimal geodesics are boundary optimal geodesics. Similarly, for $x \in I^+(\Sigma) \cap ([0, T] \times \Omega)$, we also prove in Lemma 15 that we may find a future-directed boundary optimal geodesic from $z_{\mathrm{sup}} \in \Sigma$ to x also presented in Figure 1.

We remark that in inverse problems related to the one studied in this paper, convexity of the lateral boundary is assumed to have light-like geodesics that intersect the boundary transversally; see, e.g., [Hintz et al. 2022b]. By using boundary optimal geodesics of this paper, the convexity assumption in that work can be dropped. We expect this to be true also in related inverse problems.

We make the notion of boundary optimal geodesics precise in the form of the following definition. Below, the time coordinate, or the time function, of N is t.

Definition 3 (boundary optimal geodesic). Let (N, g) be globally hyperbolic, $N = \mathbb{R} \times M$, $\Omega \subset M$ a manifold with boundary and $\Sigma = [0, T] \times \partial\Omega$. We call a geodesic $\gamma : [0, 1] \to [0, T] \times \Omega$ a past-directed boundary optimal geodesic to $x \in J^-(\Sigma)$ if

(1) $\gamma(0) \in \Sigma$ and $\gamma(1) = x$,

(2) the time coordinate of $\gamma(0)$ equals

$$t_{\mathrm{inf}} = \inf\{\tilde{d} \in [0, T] \mid \text{there is } \tilde{z} \in \Sigma \text{ such that } t(\tilde{z}) = \tilde{d} \text{ and } \tau(x, \tilde{z}) > 0\},$$

(3) γ is an optimal geodesic connecting the points x and $\gamma(0)$.

Similarly, we call γ a future-directed boundary optimal geodesic to $x \in J^+(\Sigma)$ if the time coordinate of $\gamma(0)$ equals instead

$$t_{\mathrm{sup}} = \sup\{\tilde{d} \in [0, T] \mid \text{there is } \tilde{z} \in \Sigma \text{ such that } t(\tilde{z}) = \tilde{d} \text{ and } \tau(\tilde{z}, x) > 0\}.$$

We refer to both past- and future-directed boundary optimal geodesics to x respectively belonging to $J^-(\Sigma)$ and $J^+(\Sigma)$ collectively as boundary optimal geodesics.

Remark 4. Boundary optimal geodesics are related to a recently introduced concept of null distance [Allen and Burtscher 2022; Sormani and Vega 2016]. A null distance turns a Lorentzian manifold admitting a suitable time function into a metric space in a conformally invariant way. In particular, a globally hyperbolic manifold N becomes a metric space with a metric $d : N \times N \to [0, \infty)$. We wish to state here the following facts, even though we do not use them.

If γ is a boundary optimal geodesic connecting $z \in \Sigma$ to x, then $|t(x) - t(z)| = d(x, z)$. Moreover, a boundary optimal geodesic minimizes the distance between x and its future causal lateral boundary $\Sigma \cap J^+(x)$ in the sense that

$$d(\Sigma \cap J^+(x), x) = d(z_{\mathrm{inf}}, x),$$

where $z_{\mathrm{inf}} \in \Sigma \cap J^+(x)$ is the starting point of a past-directed boundary optimal geodesic to x. We have similarly for the past causal lateral boundary $\Sigma \cap J^-(x)$. In this sense, boundary optimal geodesics are an analogue to Riemannian geodesics that minimize the distance to a boundary.

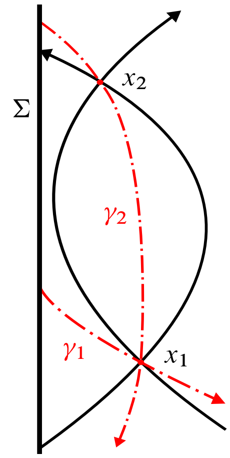

Figure 2. Past-directed light-like geodesics (red dashed lines) that separate the intersection points x_1 and x_2 of future-directed light-like geodesics (black). The geodesics in red and black intersect Σ at times $t < T$ and $t > 0$ respectively.

Separation matrices. Having explained what optimal and boundary optimal geodesics are, we are ready to present what a separation matrix is and how it is constructed.

Definition 5 (separation matrix). Let $x_1, \dots, x_P \in [0, T] \times \Omega$ and $v_1, \dots, v_P$ be solutions to $\Box_g v = 0$ in $[0, T] \times \Omega$. If the matrix

$$\begin{pmatrix} v_1(x_1) & v_2(x_1) & \cdots & v_P(x_1) \\ v_1(x_2) & v_2(x_2) & \cdots & v_P(x_2) \\ \vdots & & \ddots & \vdots \\ v_1(x_P) & v_2(x_P) & \cdots & v_P(x_P) \end{pmatrix} \tag{10}$$

is invertible, we call it a separation matrix.

In general, if $x_1, \dots, x_P \in I^-(\Sigma) \cap ([0, T] \times \partial\Omega)$ satisfy $x_1 < \cdots < x_P$ we show in Lemma 17 that there are P solutions v_k, $k = 1, \dots, P$, to the wave equation $\Box_g v = 0$ whose Cauchy data vanish on $\{t = T\}$ such that the corresponding matrix (10) is invertible and thus a separation matrix.

Let us consider here the simplest nontrivial case $P = 2$ and assume that $x_1, x_2 \in I^-(\Sigma) \cap ([0, T] \times \partial\Omega)$ satisfy $x_1 < x_2$. To construct suitable solutions v_1 and v_2 in this case, we proceed by first choosing two light-like geodesics as follows. The choice is illustrated in Figure 2, where the points x_1 and x_2 are the intersection points of the black curves. (In our inverse problem the black curves are also geodesics, but that is not important for the present discussion.) By the discussion above, we may find a boundary optimal geodesic γ_1 between x_1 and $x_{1,\mathrm{inf}} \in \Sigma$ and another boundary optimal geodesic γ_2 connecting x_2 to Σ. Next we note that if γ_1 also meets x_2, then we can perturb the initial direction of γ_1 at x_1 to have a new light-like geodesic that does not meet x_2. Indeed, if the new perturbed geodesic would still meet x_2, then it is a fact that there would be a shortcut path from x_1 to Σ which has positive length. This would contradict the condition $\tau(x_1, x_{1,\mathrm{inf}}) = 0$. We refer to the proof of Lemma 17 for the details. We also note that it is possible that γ_2 meets x_1.

By the above discussion, we have the light-like geodesic γ_1 from x_1 to Σ which does not meet x_2 and another light-like geodesic from x_2 to Σ. Corresponding to these two geodesics there are respective Gaussian beam solutions v_1 and v_2 to $\Box_g v = 0$ with vanishing Cauchy data at $\{t = T\}$. By using the properties of Gaussian beams, we know that v_1 and v_2 are concentrated to small neighborhoods of the corresponding geodesics, respectively. See Section 3 for details. Thus we have for $k, l = 1, 2$ that

$$\begin{aligned} |v_k(x_l)| &\approx 1, \quad k = l, \\ |v_k(x_l)| &\ll 1, \quad k > l, \\ |v_k(x_l)| &\le c_0, \quad k < l, \end{aligned}$$

where $c_0 > 0$ is a constant. Therefore the matrix $\mathcal{V}$ in (10) in this case is approximately a lower triangular matrix with ones on the diagonal. Thus $\mathcal{V}$ is invertible, and hence a separation matrix in our terminology. Vaguely speaking, we can separate points by solutions to the wave equation $\Box_g v = 0$. We mention that a similar condition has been used in the study of inverse problems for elliptic equations in [Guillarmou et al. 2019; Lassas et al. 2020].

Finally, we mention that when proving our stability result in this paper, we can only use finitely many separation matrices. For this, we show that there are finitely many solutions v to $\Box_g v = 0$ with vanishing Cauchy data at $\{t = T\}$ such that the separation matrices made out of these solutions can separate any fixed number of points in $I^-(\Sigma) \cap ([0, T] \times \Omega)$ that are distinct in a precise sense. In the definition below, $\bar{g}$ is an auxiliary Riemannian metric on $[0, T] \times \Omega$.

Definition 6 (separation filter). Let $K \subset [0, T] \times \Omega$ be compact and $P \in \mathbb{N}$. A finite collection $\mathcal{M} \subset C^\infty([0, T] \times \Omega)$ of solutions to $\Box_g v = 0$ is called a separation filter if the following holds: For any points $x_1, \dots, x_P \in K$ such that $x_1 < x_2 < \dots < x_P$ and $d_{\bar{g}}(x_k, x_l) > \delta$ for $x_k \neq x_l$, $k, l = 1, \dots, P$, there are $v_1, \dots, v_P \in \mathcal{M}$ such that the matrix $(v_k(x_l))_{k,l=1}^P$ in (53) is invertible (and thus a separation matrix).

In Lemma 18 we show that if $K \subset I^-(\Sigma) \cap I^+(\Sigma) \cap ([0, T] \times \Omega)$, then a separation filter exists.

1.4. ***Preliminary definitions.*** The Sobolev spaces H^s on a compact smooth manifold can be defined in several ways (up to equivalent norms). We define Sobolev spaces first on the manifold $N = \mathbb{R} \times M$ using partition of unity on charts; see, e.g., [Hörmander 1983; Roe 1988; Taylor 2011]. Sobolev spaces on the time cylinder $[0, T] \times \Omega$ are then defined by restriction:

$$H^s([0, T] \times \Omega) := \{f|_{[0,T]\times\Omega} \mid f \in H^s(\mathbb{R} \times M)\}.$$

As usual, the dual space of $H^r([0, T] \times \Omega)$, $r \ge 0$, is defined as

$$\widetilde{H}^{-r}([0, T] \times \Omega) := \{f \in H^{-r}(\mathbb{R} \times M) \mid \operatorname{supp} f \subset [0, T] \times \overline{\Omega}\}.$$

It is endowed with the norm

$$\|g\|_{\widetilde{H}^{-r}([0,T]\times\Omega)} := \sup \frac{|g(v)|}{\|v\|_{H^r([0,T]\times M)}},$$

where the supremum is over all $v \in H^r([0,T]\times M)$, $v \neq 0$, with $\operatorname{supp} v \subset [0,T]\times\overline{\Omega}$. By the Riesz representation theorem, one can always find $f_0 \in H^r(\mathbb{R}\times M)$ so that for all $v \in H^r(\mathbb{R}\times M)$

$$\|f\|_{\widetilde{H}^{-r}([0,T]\times\Omega)} = \|f_0\|_{H^r(\mathbb{R}\times M)}, \quad f(v) = \langle f_0, v\rangle.$$

Additionally, if $\operatorname{supp} v \subset [0,T]\times\overline{\Omega}$, then we have for all $v \in H^r([0,T]\times M)$ the estimate

$$|f(v)| = |\langle f_0, v\rangle| \le \|f\|_{\widetilde{H}^{-r}([0,T]\times\Omega)}\|v\|_{H^r([0,T]\times\Omega)}.$$

Sobolev spaces of the manifold Ω with boundary are defined similarly. By the notation H_0^s we mean the closure of the space of compactly supported smooth functions with respect to the Sobolev H^s norm.

Structure of the paper. This paper is organized as follows. In Section 1.1 we present our main results and explain briefly the structure of the proofs. Section 2 studies the forward problem of the nonlinear equation (2). Most of the proofs of Section 2 are included in the Appendix. Section 3 concerns the construction of Gaussian beams in Lorentzian manifolds. In Section 4 we construct the tools of Lorentzian geometry which we use in our inverse problem. This section in particular shows it is possible distinguish different points of a Lorentzian manifold by using solutions to the wave equation. The section introduces the concepts of *boundary optimal geodesics* and *separation matrices*. Finally, in Section 5 we collect the results we have obtained until that point to give a proof for our main theorem. For clarity, the proof is split into several parts.

2. Well-posedness of the forward problem

To prove existence of small solutions for the nonlinear wave equation (2), we start by recalling the corresponding results for the linear initial-boundary value problem

$$\begin{cases} \Box_g u = F & \text{in } [0,T]\times\Omega, \\ \quad u = f & \text{on } [0,T]\times\partial\Omega, \\ u|_{t=0} = u_0, \ \partial_t u|_{t=0} = u_1 & \text{in } \Omega. \end{cases}$$

Let $s \in \mathbb{N}$. Convenient spaces for solutions of the wave equation are called *energy spaces* E^s, defined as

$$E^s = \bigcap_{0\le k\le s} C^k([0,T]; H^{s-k}(\Omega)).$$

These spaces are equipped with the norm

$$\|u\|_{E^s} = \sup_{0<t<T} \sum_{0\le k\le s} \|\partial_t^k u(\cdot,t)\|_{H^{s-k}(\Omega)}. \tag{11}$$

As is the case with the Sobolev spaces $H^s(\Omega)$, the space E^s is an algebra if $s > \frac{n}{2}$ and we have the norm estimate

$$\|uv\|_{E^s} \le C_s\|u\|_{E^s}\|v\|_{E^s} \quad \text{for all } u, v \in E^s.$$

The above facts are well known, see, e.g., [Choquet-Bruhat 2009, Appendix III, Definitions 3.4(2) and 3.5], but for completeness of our presentation, we sketch a proof for them here for the case $s \in \mathbb{N}$. For this, we let $u, v \in E^s$ and show that the pointwise product uv is in E^s. Since

$$\|uv\|_{E^s} = \sup_{0<t<T} \sum_{k=0}^{s} \|\partial_t^k (uv)\|_{H^{s-k}(\Omega)},$$

it suffices to show that each term of the form

$$\sup_{0<t<T} \|\partial_t^a u \, \partial_t^b v\|_{H^{s-k}(\Omega)}$$

is finite for $a + b = k$ and for each $k = 0, \dots, s$. By using [Choquet-Bruhat 2009, Appendix III, Definition 3.4(2)] or [Behzadan and Holst 2021, Corollary 6.3 or Theorem 7.4], we see that when $s_1, s_2 \geq s \geq 0$ and $s_1 + s_2 > s + \frac{n}{2}$ the following multiplication property holds in Lipschitz domains:

$$H^{s_1}(\Omega) \times H^{s_2}(\Omega) \subset H^s(\Omega).$$

Since $u, v \in E^s$ we find $\partial_t^a u \in H^{s-a}(\Omega)$ and $\partial_t^a v \in H^{s-b}(\Omega)$ for all fixed $t \in [0, T]$ and the implied norms are uniformly bounded in t. We have $s - a,\ s - b \geq s - k \geq 0$ and $(s-a) + (s-b) > (s-k) + \frac{n}{2}$, since $s > \frac{n}{2}$ and $a + b = k$. This implies $\partial_t^a u \partial_t^b v \in H^{s-k}(\Omega)$ for all $t \in [0, T]$ with the implied norm uniformly bounded in t as required.

Remark 7. We note that $E^s \subset H^s([0, T] \times \Omega)$. Conversely, due to the standard Sobolev embedding $H^s([0, T] \times \Omega) \subset C^k([0, T] \times \Omega)$, when $s > k + \frac{n+1}{2}$, we have that $H^{s'}([0, T] \times \Omega) \subset E^s$, when $s' > s + \frac{n+1}{2}$. In particular,

$$\|u\|_{H^s([0,T]\times\Omega)} \lesssim \|u\|_{E^s} \lesssim \|u\|_{H^{s'}([0,T]\times\Omega)}. \tag{12}$$

For the wave equations we consider, we need to assume certain compatibility conditions between the boundary values and the initial data. The compatibility conditions for (2) to order 2 are given by

$$\begin{aligned} f|_{t=0} = u_0|_{\partial\Omega}, \quad \partial_t f|_{t=0} = \partial_t u|_{\{0\}\times\partial\Omega} = u_1|_{\partial\Omega}, \\ \partial_t^2 f|_{t=0} = \partial_t^2 u|_{\{0\}\times\partial\Omega} = \beta^{-1}|_{\{0\}\times\partial\Omega}(\Delta_h u_0|_{\partial\Omega} + F|_{\{0\}\times\partial\Omega}). \end{aligned} \tag{13}$$

Here the smooth function β and g are related by (1). The compatibility conditions up to general order s are obtained by setting $\partial_t^k f|_{t=0} = \partial_t^k u|_{\{0\}\times\partial\Omega}$, for $k = 0, \dots, s$, and then solving for $\partial_t^k u|_{\{0\}\times\partial\Omega}$ in terms of the initial data by using the equation $\Box_g u = F$. These conditions guarantee that at the boundary $\partial\Omega$ the initial data (u_0, u_1) is compatible with the corresponding boundary condition f. These conditions have been discussed for example in [Katchalov et al. 2001, Section 2.3.7] in the simpler case where the metric is time-independent. Especially, if $\partial_t^k f|_{t=0} = 0$ for all $k = 0, \dots, s$, or if f is supported away from the Cauchy surface $\{t = 0\}$, and $F \equiv 0$ and $u_0 \equiv u_1 \equiv 0$, then the compatibility conditions of order s hold.

Proposition 8 (existence and estimates for the linear equation [Ikawa 1968; Lasiecka et al. 1986]). *Assume that* $(\mathbb{R}\times M, g)$ *is a globally hyperbolic Lorentzian manifold as in* (1) *and* $\Omega \subset M$ *is a compact submanifold with nonempty boundary. Let* $s \in \mathbb{N}$ *be a positive integer and assume that* $F \in E^s$, $f \in H^{s+1}(\Sigma)$,

$u_0 \in H^{s+1}(\Omega)$ and $u_1 \in H^s(\Omega)$ satisfy the compatibility conditions. Then the equation

$$\begin{cases} \Box_g u = F & \text{in } [0,T]\times\Omega, \\ u = f & \text{on } \Sigma, \\ u = u_0, \ \ \partial_t u = u_1 & \text{in } \{t=0\}\times\Omega \end{cases} \tag{14}$$

has a unique solution $u \in E^{s+1}$ satisfying

$$\|u\|_{E^{s+1}} \le C(\|F\|_{E^s} + \|f\|_{H^{s+1}(\Sigma)} + \|u_0\|_{H^{s+1}(\Omega)} + \|u_1\|_{H^s(\Omega)}) \tag{15}$$

and $\partial_\nu u|_\Sigma \in H^s(\Sigma)$.

As we could not find a proof for Proposition 8 in general for globally hyperbolic Lorentzian manifolds, we have included one in the Appendix. The energy estimates of the linear problem (14) directly allow us to conclude that the nonlinear problem (2) has a unique small solution in E^{s+1}. The proof of the following lemma is similar to the one in [Lassas et al. 2022, Proof of Lemma 1, Appendix A]. We omit the proof.

Lemma 9. *Let $m \ge 2$ be an integer and $\Omega \subset M$ be a compact submanifold*, $\dim(\Omega) = \dim(M)$, *with nonempty boundary. Assume $s \in \mathbb{N}$ is such that $s+1 > \frac{n+1}{2}$. Suppose that $q \in C^{s+1}([0,T]\times\Omega)$ satisfies the a priori bound $\|q\|_{C^{s+1}} \le c$ for some $c > 0$. Then there are $\kappa > 0$ and $\rho > 0$ such that if $f \in H^{s+1}(\Sigma)$ satisfies $\|f\|_{H^{s+1}(\Sigma)} \le \kappa$, and $\partial_t^\alpha f|_{t=0} = 0$ for all $\alpha = 0,\dots,s$ on $[0,T]\times\partial\Omega$, then there is a unique solution to*

$$\begin{cases} \Box_g u + q u^m = 0 & \text{in } [0,T]\times\Omega, \\ u = f & \text{on } [0,T]\times\partial\Omega, \\ u|_{t=0} = \partial_t u|_{t=0} = 0 & \text{in } \Omega \end{cases} \tag{16}$$

in the ball

$$B_\rho(0) := \{u \in E^{s+1} \mid \|u\|_{E^{s+1}} < \rho\} \subset E^{s+1}.$$

Furthermore, the solution satisfies the estimate

$$\|u\|_{E^{s+1}} \le C_0 \|f\|_{H^{s+1}(\Sigma)},$$

where $C_0 > 0$ is a constant independent of f and q.

If the boundary data of the nonlinear equation (16) depends on small parameters, we may expand the corresponding solution u in terms of the small parameters. Indeed, let $\varepsilon_1,\dots,\varepsilon_m > 0$ be small parameters and define

$$\vec{\varepsilon} = (\varepsilon_1,\dots,\varepsilon_m).$$

Consider the boundary value in (16)

$$f(x) = \sum_{j=1}^m \varepsilon_j f_j(x),$$

where $f_j \in H^{s+1}(\Sigma)$, $j = 1,\dots,m$, satisfies the compatibility conditions to order s and $\|f\|_{H^{s+1}(\Sigma)} \le \kappa$ for some $\kappa > 0$. Let us denote in the usual multi-index notation

$$\bar{k} = (k_1,\dots,k_m),$$

where $k_j \in \{0, \dots, m\}$. Then by repeating the proof of Proposition 1 in [Lassas et al. 2022], we find that u can be expanded as

$$u = \sum_{j=1}^{m} \varepsilon_j v_j + \sum_{|\vec{k}|=m} \binom{m}{k_1, \dots, k_m} \varepsilon_1^{k_1} \cdots \varepsilon_m^{k_m} w_{\vec{k}} + \mathcal{R}. \tag{17}$$

The functions $v_j,\ j = 1, \dots, m$, satisfy

$$\begin{cases} \Box_g v_j = 0 & \text{in } [0, T] \times \Omega, \\ v_j = f_j & \text{on } [0, T] \times \partial\Omega, \\ v_j|_{t=0} = 0, \quad \partial_t v_j|_{t=0} = 0 & \text{in } \Omega \end{cases} \tag{18}$$

and the functions $w_{\vec{k}}$ satisfy

$$\begin{cases} \Box_g w_{\vec{k}} + q v_1^{k_1} \cdots v_m^{k_m} = 0 & \text{in } [0, T] \times \Omega, \\ w_{\vec{k}} = 0 & \text{on } [0, T] \times \partial\Omega, \\ w_{\vec{k}}|_{t=0} = 0, \quad \partial_t w_{\vec{k}}|_{t=0} = 0 & \text{in } \Omega. \end{cases} \tag{19}$$

The remainder $\mathcal{R}$ is bounded in the energy spaces as follows:

$$\begin{aligned} \|\mathcal{R}\|_{E^{s+2}} &\le c(s, T) \|q\|_{E^{s+1}}^2 \Big\| \sum_{j=1}^{m} \varepsilon_j f_j \Big\|_{H^{s+1}(\Sigma)}^{2m-1}, \\ \|\Box \mathcal{R}\|_{E^{s+1}} &\le C(s, T) \|q\|_{E^{s+1}}^2 \Big\| \sum_{j=1}^{m} \varepsilon_j f_j \Big\|_{H^{s+1}(\Sigma)}^{2m-1}. \end{aligned} \tag{20}$$

By using the expansion formula (17), we will next derive an integral equation which relates the potential q to the DN map Λ. In general, relating an unknown parameter/function in an inverse problem for a nonlinear equation to a formula for solutions to linear equations is called a *higher-order linearization* method. See for example [Kurylev et al. 2018a; Lassas et al. 2018; 2021b], where solutions are differentiated with respect to small parameters. However, as we are interested in stability of our inverse problem, we need accurate control on the remainder terms. For this reason, following [Lassas et al. 2022], instead of differentiating we use finite differences $D^m_{\vec{\varepsilon}}$. The mixed finite difference of u at $\vec{\varepsilon} = 0$, that is, $\varepsilon_1 = \dots = \varepsilon_m = 0$, is defined by the formula

$$D^m_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0} u_{\varepsilon_1 f_1 + \dots + \varepsilon_m f_m} = \frac{1}{\varepsilon_1 \cdots \varepsilon_m} \sum_{\sigma \in \{0,1\}^m} (-1)^{|\sigma|+m} u_{\sigma_1 \varepsilon_1 f_1 + \dots + \sigma_m \varepsilon_m f_m}, \tag{21}$$

where $u_{\varepsilon_1 f_1 + \dots + \varepsilon_m f_m}$ is the unique solution to (16) with f replaced by $\varepsilon_1 f_1 + \dots + \varepsilon_m f_m$. Then the mixed finite difference $D^m_{\vec{\varepsilon}}$ of the solution u of (16) takes the form

$$D^m_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0} u = m!\, w_{1,1,\dots,1} + D^m_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_m}|_{\vec{\varepsilon}=0} \overline{\mathcal{R}}, \tag{22}$$

where $\overline{\mathcal{R}}$ is a sum of the remainders of the solutions $u_{\sigma_1 \varepsilon_1 f_1 + \dots + \sigma_m \varepsilon_m f_m}$ in (21).

For more details about the finite differences of u, we refer the reader to [Lassas et al. 2022, Appendix C].

Let v_0 be an auxiliary function solving $\square_g v_0 = 0$ with $v_0|_{t=T} = \partial_t v_0|_{t=T} = 0$ in Ω. By multiplying the DN-map Λ associated with (2) by v_0 and integrating by parts over $[0,T]\times\Omega$, we obtain

$$\begin{aligned}\int_\Sigma v_0 D^m_{\vec\varepsilon}|_{\vec\varepsilon=0}\Lambda(\varepsilon_1 f_1+\cdots+\varepsilon_m f_m)\,dS \\ &= \int_\Sigma v_0 D^m_{\vec\varepsilon}|_{\vec\varepsilon=0}\partial_\nu u_{\varepsilon_1 f_1+\cdots+\varepsilon_m f_m}\,dS \\ &= m!\int_{[0,T]\times\Omega} v_0\square_g w_{1,1,\dots,1}\,dV_g + \frac{1}{\varepsilon_1\cdots\varepsilon_m}\int_{[0,T]\times\Omega} v_0\square_g\widetilde{\mathcal{R}}\,dV_g.\end{aligned}$$

Here we defined

$$\widetilde{\mathcal{R}} := \varepsilon_1\varepsilon_2\cdots\varepsilon_m D^m_{\vec\varepsilon}|_{\vec\varepsilon=0}\overline{\mathcal{R}} \tag{23}$$

and $\widetilde{\mathcal{R}}$ satisfies

$$\begin{aligned}\|\widetilde{\mathcal{R}}\|_{E^{s+2}} &\le c(s,T)\|q\|^2_{E^{s+1}}\sum_{\sigma\in\{0,1\}^m}\|\sigma_1\varepsilon_1 f_1+\cdots+\sigma_m\varepsilon_m f_m\|^{2m-1}_{H^{s+1}(\Sigma)}, \\ \|\square\widetilde{\mathcal{R}}\|_{E^{s+1}} &\le C(s,T)\|q\|^2_{E^{s+1}}\sum_{\sigma\in\{0,1\}^m}\|\sigma_1\varepsilon_1 f_1+\cdots+\sigma_m\varepsilon_m f_m\|^{2m-1}_{H^{s+1}(\Sigma)}.\end{aligned} \tag{24}$$

We have arrived at the following integral identity which connects the potential q with the DN-map Λ.

Integral identity.

$$\begin{aligned}-m!\int_{[0,T]\times\Omega} q v_0 v_1 v_2\cdots v_m\,dV_g \\ &= \int_\Sigma v_0 D^m_{\vec\varepsilon}|_{\vec\varepsilon=0}\Lambda(\varepsilon_1 f_1+\cdots+\varepsilon_m f_m)\,dS + \frac{1}{\varepsilon_1\varepsilon_2\cdots\varepsilon_m}\int_{[0,T]\times\Omega} v_0\square\widetilde{\mathcal{R}}\,dV_g.\end{aligned} \tag{25}$$

Our analysis of the inverse problem is based on this formula.

3. Gaussian beams

In this section we record some facts about Gaussian beams. Gaussian beams on a Lorentzian manifold (N,g), $\dim(N) = n+1 \ge 3$, are approximate solutions to the wave equation $\square_g v = 0$. If s is a geodesic parameter of a light-like geodesic $\gamma : [s_1,s_2] \to N$ and (s,y), $y = (y_1,\dots,y_n) \in \mathbb{R}^n$, are suitable Fermi coordinates (see (26) below) on a neighborhood of the graph Γ of γ, then a Gaussian beam in the coordinates (s,y) looks roughly like

$$e^{iy_1\tau - a\tau|y|^2},$$

up to a normalization. By graph of γ we mean the image set

$$\Gamma := \gamma([s_1,s_2]).$$

Here $a > 0$ and τ is a large parameter. Therefore, the qualitative behavior of a Gaussian beam is oscillation in a direction y_1 transversal to the geodesic γ and Gaussian concentration around the graph of γ.

The construction of Gaussian beams is well known; see, e.g., [Babich et al. 1985; Feizmohammadi and Oksanen 2022; Ralston 1982]. We include details about the construction since we wish to keep track of the constants that will be implicit in our stability estimate of Theorem 1. Our presentation of the construction follows closely [Feizmohammadi and Oksanen 2022, Section 4] to which we refer for omitted details. We mention here the recent work [Krupchyk et al. 2022], which constructs related Gaussian beam quasimodes in a Riemannian setting by using more sophisticated methods, which lead to better estimates.

Fermi coordinates are constructed by inverting the map

$$(s, y) \mapsto \exp_{\gamma(s)}\biggl(\sum_{k=1}^{n} y^k e_k(s)\biggr) \in N. \tag{26}$$

Here $e_k(s)$ are the parallel transportations along a light-like geodesic γ of the last n vectors of a frame $\{e_0, e_1, \dots, e_n\}$ of $T_{\gamma(0)}$ with

$$e_0 = \dot{\gamma}(0).$$

The other vectors of the frame are chosen so that, for $j, k = 2, \dots n$, it holds

$$g(e_0, e_0) = 0, \quad g(e_1, e_1) = 0, \quad g(e_0, e_1) = -2, \quad g(e_j, e_k) = \delta_{jk}. \tag{27}$$

The frame $\{e_0, e_1, \dots, e_n\}$ is called a pseudo-orthonormal frame. (Due to relation to the usual light-cone coordinates, we could also call it a lightcone frame.) Since the frame $\{e_0(s), e_1(s), \dots, e_n(s)\}$ is the parallel transportation of $\{e_0, e_1, \dots, e_n\}$ along γ, the conditions (27) hold for e_j, $j = 0, \dots, n$, replaced with $e_j(s)$ and $e_0(s) = \dot{\gamma}(s)$.

We work in the Fermi coordinates described above. In the Fermi coordinates (s, y), the geodesic γ corresponds to $(s, 0)$ and the coordinate representation $g|_\gamma = g(s, 0)$ of the metric g restricted to γ satisfies

$$g|_\gamma = -2\, ds\, dy_1 + \sum_{k=2}^{n} dy_k\, dy_k.$$

Gaussian beams are constructed by using a WKB ansatz $e^{i\tau\Theta(s,y)}a(s, y)$ to approximately solve the equation $\Box_g v = 0$ in the Fermi coordinates (s, y). We have

$$\Box_g(e^{i\tau\Theta}a) = e^{i\tau\Theta}(\tau^2 g(d\Theta, d\Theta) - 2i\tau g(d\Theta, da) + i\tau(\Box_g\Theta)a + \Box_g a). \tag{28}$$

We will choose a *phase function* Θ and an *amplitude function* a so that the right-hand side of (28) is $\mathcal{O}(\tau^{-K})$ in $H^k([0, T] \times \Omega)$ for given $k \ge 0$ and $K \in \mathbb{N}$. To do so, we first approximately solve the eikonal equation

$$g(d\Theta, d\Theta) = 0. \tag{29}$$

After finding an (approximate) solution Θ to the eikonal equation, we equate the last three terms of (28) by inserting Θ into

$$-2i\tau g(d\Theta, da) + i\tau(\Box_g\Theta)a + \Box_g a = 0.$$

By assuming an expansion of the form

$$a = a_0 + \tau^{-1}a_1 + \tau^{-2}a_2 + \cdots + \tau^{-N}a_N$$

for the amplitude a, where $N \in \mathbb{N}$ is to be chosen later, we are led by equating the powers of τ to a family of $N+1$ equations

$$-2ig(d\Theta, da_0) + i(\Box_g \Theta)a_0 = 0, \tag{30}$$

$$-2ig(d\Theta, da_j) + i(\Box_g \Theta)a_j - \Box_g a_{j-1} = 0, \tag{31}$$

$j = 1, \dots, N$. We solve these equations approximately and recursively in j starting from a_0. Equations (30) and (31) are called transport equations.

In what follows, we refer to [Feizmohammadi and Oksanen 2022] for omitted details. To solve the eikonal equation (29) approximately, one sets

$$\Theta = \sum_{j=0}^{N} \Theta_j(s, y),$$

where $\Theta_j(s, y)$ is a homogeneous polynomial of order j in $y \in \mathbb{R}^n$. We say that $g(d\Theta, d\Theta)$ vanishes to order N on Γ, or that $g(d\Theta, d\Theta) = 0$ is satisfied to order N on Γ, if

$$(\partial_y^\alpha g(d\Theta, d\Theta))(s, 0) = 0,$$

where α is any multi-index with $|\alpha| \le N$. We set

$$\Theta_0 = 0 \text{ and } \Theta_1 = y_1. \tag{32}$$

It follows that

$$g(d\Theta, d\Theta)(s, 0) = 0 \quad \text{and} \quad (\partial_{y_l} g(d\Theta, d\Theta))(s, 0) = 0,$$

where $l = 1, \dots, n$. That is, the eikonal equation (29) is satisfied to order 1 on Γ. The conditions (32) imply the invariantly written conditions

$$\Theta(\gamma(s)) = 0 \quad \text{and} \quad \nabla\Theta(\gamma(s)) = e_1(s).$$

To have that $g(d\Theta, d\Theta) = 0$ is satisfied to order 2 on Γ is more complicated. For this, one uses the quadratic ansatz

$$\Theta_2(s, y) = y \cdot H(s)y,$$

where $H(s)$ is a complex $n \times n$ matrix and "$\cdot$" refers to the usual $\mathbb{R}^n$ inner product and $y \in \mathbb{R}^n$. This ansatz leads to the Riccati equation, which is a first-order matrix-valued ODE. For our purposes, the form of the Riccati equation is not important and it suffices to say that one can find a complex solution $H(s)$ to the equation with $\mathrm{Im}(H(s)) > 0$. The conditions $\mathrm{Im}(H(s)) > 0$ and $\Theta_0 = 0$ together imply the invariantly written conditions

$$\mathrm{Im}(\nabla^2\Theta(\gamma(s))) \ge 0 \quad \text{and} \quad \mathrm{Im}(\nabla^2\Theta)(\gamma(s))|_{\dot\gamma(s)^\perp} > 0.$$

Here we use the notation $\dot\gamma(s)^\perp$ to denote the algebraic complement to $\dot\gamma(s)$ in $T_{\gamma(s)}N$. That is, $\mathbb{R}\dot\gamma(s) \oplus \dot\gamma(s)^\perp = T_{\gamma(s)}N$.

Solving the eikonal equation to order 2 is enough to understand the qualitative properties of the phase function Θ needed in our inverse problem. However, we wish to have that

$$\Box_g(e^{is\Theta(x)}a(x)) = \mathcal{O}_{H^k([0,T]\times\Omega)}(\tau^{-K}).$$

For this, we solve the eikonal equation to an order N, which depends on k and K. This can be done by solving additional ODEs, but we omit the details. After finding Θ so that $g(d\Theta, d\Theta)$ vanishes to order N on Γ, the term $\tau^2 g(d\Theta, d\Theta)$ in the expansion (28) of $\Box_g(e^{i\tau\Theta}a)$ satisfies

$$\tau^2 g(d\Theta, d\Theta) \le C_0\tau^2|y|^{N+1}. \tag{33}$$

We choose a specific N later.

Next we insert the phase function Θ that we have constructed into the transport equations (30) and (31) to find an amplitude function a. To solve the transport equations, we write

$$a_k = \chi\left(\frac{|y|}{\delta'}\right)b_k, \tag{34}$$

so that

$$a = \chi\left(\frac{|y|}{\delta'}\right)\sum_{k=0}^{N}\tau^{-k}b_k.$$

Here $\chi \in C_c^\infty(\mathbb{R})$ is a fixed cutoff function, which is identically 1 on a neighborhood of $0 \in \mathbb{R}$ and $\delta' > 0$ is chosen small enough so that $\chi(|y|/\delta')$ is compactly supported in the domain of the Fermi coordinates.

We seek the b_k, $k = 1, \dots, N$, in the form

$$b_k = \sum_{j=0}^{N} b_{k,j}(s, y), \tag{35}$$

where $b_{k,j}(s, y)$ is a complex-valued homogeneous polynomial of order j in y. We are interested in the specific form only of the leading term $b_{0,0}$. The transport equation concerning b_0 is

$$-2g(d\Theta, da_0) + (\Box_g\Theta)a_0 = 0,$$

which is satisfied to order 0 if

$$-2g(d\Theta, db_{0,0})(s, 0) + (\Box_g\Theta)b_{0,0}(s, 0) = 0.$$

Here we used that $\chi(|y|/\delta') = 1$ to order 1 at $y = 0$. We have $d\Theta(s, 0) = dy^1$ and $g^{01}(s, 0) = -1$. It is calculated in [Feizmohammadi and Oksanen 2022, Section 4.2] that $(\Box_g\Theta)(s, 0) = \frac{d}{ds}\log\det(Y(s))$, where $Y(s)$ is a one-parameter nondegenerate matrix field which solves an ODE with the initial condition $Y(0) = I_{n\times n}$, the $n \times n$ identity matrix. Thus we have that the equation for $b_{0,0}(s)$ is solved by

$$b_{0,0}(s) = \det(Y(s))^{-\frac{1}{2}}, \tag{36}$$

with

$$b_{0,0}(0) = 1. \tag{37}$$

Recall that the terms a_0, b_0 and $b_{0,j}$, $j = 1, 2, \dots, N$, are related by (34)–(35). The terms $b_{0,j}$, $j = 1, 2, \dots, N$, are constructed by solving linear ODEs so that $-2g(d\Theta, da_0) + (\Box_g\Theta)a_0 = 0$ is satisfied to order N. The higher-order transport equations (31) concerning b_k, $k \ge 1$, can be solved recursively to order N by using similar arguments. We omit the details, and only conclude that there is $C_1 > 0$ so that

$$|-2ig(d\Theta, da_0) + i(\Box_g\Theta)a_0| \le C_1|y|^{N+1},$$
$$|-2ig(d\Theta, da_k) + i(\Box_g\Theta)a_k - \Box_g a_{k-1}| \le C_1|y|^{N+1},$$

$k = 1, \dots, N$. Since $a = a_0 + \tau^{-1}a_1 + \tau^{-2}a_2 + \cdots + \tau^{-N}a_N$, we have that

$$\begin{aligned}
&-2i\tau g(d\Theta, da) + i\tau(\Box_g\Theta)a + \Box_g a \\
&= \tau\sum_{k=0}^{N}\tau^{-k}(-2ig(d\Theta, da_k) + i(\Box_g\Theta)a_k) + \sum_{k=0}^{N}\tau^{-k}\Box_g a_k \\
&= \tau\sum_{k=1}^{N}\tau^{-k}(-2ig(d\Theta, da_k) + i(\Box_g\Theta)a_k + \Box_g a_{k-1}) + \tau(-2ig(d\Theta, da_0) + i(\Box_g\Theta)a_0) + \tau^{-N}\Box_g a_N \\
&= \tau\mathcal{O}_{L^\infty}(|y|^{N+1}) + \mathcal{O}(\tau^{-N}).
\end{aligned}$$

By additionally recalling from (33) that $\tau^2 g(d\Theta, d\Theta) \le C_0\tau^2|y|^{N+1}$, we have

$$\begin{aligned}
e^{-i\tau\Theta}\Box_g(e^{i\tau\Theta}a) &= \tau^2 g(d\Theta, d\Theta) - 2i\tau g(d\Theta, da) + i\tau(\Box_g\Theta)a + \Box_g a \\
&\le C_0\tau^2|y|^{N+1} + C_1\tau|y|^{N+1} + C_2\tau^{-N}.
\end{aligned}$$

By redefining $\delta' > 0$ smaller, if necessary, we have that

$$|e^{i\tau\Theta(s,y)}| \le Ce^{-c\tau|y|^2}$$

for (s, y) in the support of a. Recall that our aim is to show that

$$\|\Box_g(e^{i\tau\Theta(s,y)}a(s, y))\|_{H^k([0,T]\times\Omega)} = \mathcal{O}(\tau^{-K}). \tag{38}$$

Taking k derivatives of $\Box_g(e^{i\tau\Theta(s,y)}a(s, y))$ gives

$$|\nabla^k\Box_g(e^{i\tau\Theta(s,y)}a(s, y))| \le C_3e^{-\tau c|y|^2}\sum_{l=0}^{k}\tau^{k-l}(\tau^2|y|^{N+1-l} + \tau|y|^{N+1-l} + \tau^{-N}). \tag{39}$$

We calculate the integral of (39) squared using polar coordinates for the y-variable and the standard formula $\int_0^\infty r^l e^{-\tau c r^2}\,\mathrm{d}r \sim \tau^{-(l+1)/2}$ for $l \ge 0$. Note that since the light-like geodesic γ of (N, g) is causal, $[0, T]\times\Omega$ compact and (N, g) globally hyperbolic, the geodesic $\gamma = \gamma(s)$ exits $[0, T]\times\Omega$ after a finite parameter time r_0. Thus the integration in the coordinate s will be over a finite interval $[0, r_0]$. The above discussion implies the estimate

$$\begin{aligned}
\|\Box_g(e^{i\tau\Theta(s,y)}a(s, y))\|^2_{H^k(M)} &\lesssim \sum_{l=0}^{k}\tau^{2(k-l)}\left(\int_0^{r_0}e^{-2\tau c r^2}r^{n-1}(\tau^4 r^{2N+2-2l} + \tau^{-2N})\,dr\right) \\
&\lesssim \sum_{l=0}^{k}\tau^{2(k-l)}(\tau^4\tau^{-\frac{n+2N+2-2l}{2}} + \tau^{\frac{-2N-n}{2}}) \\
&\lesssim \tau^{2k+4-\frac{n+2+2N}{2}} = \tau^{2k+3-\frac{n}{2}-N}
\end{aligned}$$

for τ and N large enough. (Here we have relaxed the notation and written $A \lesssim B$ if there is a constant $\widetilde{C}$ independent of τ such that $A \le \widetilde{C}B$.) If $p > 1$, we may L^p-normalize the function $e^{i\tau\Theta}a$ so that

$$\int_M |\tau^{\frac{n}{2p}} e^{i\tau\Theta} a|^p \lesssim \tau^{\frac{n}{2}} \int_0^\infty r^{n-1} e^{-\tau c r^2} \lesssim 1,$$

in which case we also have

$$\int_M |\nabla^l (\tau^{\frac{n}{2p}} e^{i\tau\Theta} a)|^2 \lesssim \tau^{\frac{n}{p}} \tau^{2l} \tau^{-\frac{n}{2}}.$$

Therefore, if we define $N = N(n, k, K, p)$ so that it satisfies

$$-2K = 2k + 3 - \frac{n}{2} - N + \frac{n}{p}, \tag{40}$$

we have (38). (If N above is not an integer, we redefine it as $\lfloor N + 1 \rfloor$.)

By collecting the details of the construction and by defining

$$v_\tau = \tau^{\frac{n}{2p}} e^{i\tau\Theta} a$$

we have:

Proposition 10 (Gaussian beams). *Let (N, g) be a globally hyperbolic Lorentzian manifold, $N = \mathbb{R} \times M$ and $\dim(N) = n + 1 \ge 3$. Let Ω be a compact submanifold of M with boundary, and $\dim(\Omega) = n$. Let $T > 0$ and let γ be a light-like geodesic of (N, g). Let $k, K, l \in \mathbb{N}$ and $p \ge 2$. There is $\tau_0 \ge 1$ and a family of functions $(v_\tau) \subset C^\infty([0, T] \times \Omega)$ such that for $\tau \ge \tau_0$*

$$\begin{aligned} \|\Box_g v_\tau\|_{H^k([0,T]\times\Omega)} &= \mathcal{O}(\tau^{-K}), \\ \|v_\tau\|_{L^p([0,T]\times\Omega)} &= \mathcal{O}(1), \\ \|v_\tau\|_{H^l([0,T]\times\Omega)} &= \mathcal{O}(\tau^{\frac{n}{2p} - \frac{n}{4} + l}) \end{aligned} \tag{41}$$

as $\tau \to \infty$. The function v_τ is called a Gaussian beam and it has the form

$$v_\tau = \tau^{\frac{n}{2p}} e^{i\tau\Theta(x)} a(x),$$

where Θ is a smooth complex function (independent of τ) on a neighborhood of $\gamma([0, L])$ satisfying

$$\begin{aligned} \Theta(\gamma(s)) = 0, \quad & \nabla\Theta(\gamma(s)) = e_1(s), \\ \operatorname{Im}(\nabla^2\Theta(\gamma(s))) \ge 0, \quad & \operatorname{Im}(\nabla^2\Theta)(\gamma(s))|_{\dot{\gamma}(s)^\perp} > 0. \end{aligned} \tag{42}$$

Here also

$$a(\gamma(s)) = a_0(\gamma(s)) + \mathcal{O}(\tau^{-1}),$$

where

$$a_0(\gamma(s)) = \det(Y(s))^{-\frac{1}{2}}$$

is nonvanishing and independent of τ. Here $Y(s)$ is a nondegenerate $n \times n$ matrix-valued function. The support of a can be taken to be in any small neighborhood U of $\gamma([0, L])$ chosen beforehand. If $s_0 \in [0, L]$, we may arrange so that $a_0(\gamma(s_0)) = 1$.

The Gaussian beams can be corrected to be exact solutions to $\Box v = 0$.

Corollary 11. *Let us adopt the assumptions and notation of Proposition 10. Assume in addition that the light-like geodesic γ does not intersect $\{t=0\}$. Assume that $l' \in \mathbb{N}$ satisfies $k > l'-1+\frac{n+1}{2}$. Then there are Gaussian beams v_τ satisfying the conditions of Proposition 10 and functions $r_\tau \in C^\infty([0,T]\times\Omega)$ such that*

$$v := v_\tau + r_\tau$$

is a solution to

$$\begin{cases} \Box_g v = 0 & \text{in } [0,T]\times\Omega, \\ v = v_\tau & \text{on } [0,T]\times\partial\Omega, \\ v|_{t=0} = \partial_t v|_{t=0} = 0 & \text{in } \Omega. \end{cases} \tag{43}$$

The functions r_τ satisfy

$$\|r_\tau\|_{H^{l'}([0,T]\times\Omega)} = \mathcal{O}(\tau^{-K}). \tag{44}$$

Proof. By assumption the graph of γ has a neighborhood U which does not intersect a neighborhood of $\{t=0\}$. Let v_τ be Gaussian beams which are supported in U and satisfy the conditions of Proposition 10. By Proposition 8, there exists a solution to

$$\begin{cases} \Box_g r_\tau = -\Box_g v_\tau & \text{in } [0,T]\times\Omega, \\ r_\tau = 0 & \text{on } [0,T]\times\partial\Omega, \\ r_\tau|_{t=0} = \partial_t r_\tau|_{t=0} = 0 & \text{in } \Omega. \end{cases}$$

Then $v = v_\tau + r_\tau$ solves (43).

By Proposition 10 we have that $\|\Box_g v_\tau\|_{H^k([0,T]\times\Omega)} = \mathcal{O}(\tau^{-K})$, where k, K can be chosen freely. By Remark 7 for $k > l'-1+\frac{n+1}{2}$ it holds that $H^k([0,T]\times\Omega) \subset E^{l'-1}$. Choosing $k > l'-1+\frac{n+1}{2}$ in Proposition 8 and using (12) shows that

$$\|r_\tau\|_{H^{l'}([0,T]\times\Omega)} \lesssim \|r_\tau\|_{E^{l'}} \lesssim \|\Box_g v_\tau\|_{E^{l'-1}} \lesssim \|\Box_g v_\tau\|_{H^k([0,T]\times\Omega)} = \mathcal{O}(\tau^{-K})$$

as claimed. ☐

Remark 12. We shall also need solutions to the wave equation

$$\begin{cases} \Box_g v = 0 & \text{in } [0,T]\times\Omega, \\ v = f & \text{on } [0,T]\times\partial\Omega, \\ v|_{t=T} = \partial_t v|_{t=T} = 0 & \text{in } \Omega, \end{cases} \tag{45}$$

where the Cauchy data of v vanishes at the top of the time cylinder. Solutions to (45) can be found as follows. Consider the isometry h given by $t \mapsto T-t$ and let $\tilde{g} = h^*g$. Let $\tilde{f} = f(T-t,x)$ and let $\tilde{v}$ be the unique solution to

$$\begin{cases} \Box_{\tilde{g}} \tilde{v} = 0 & \text{in } [0,T]\times\Omega, \\ \tilde{v} = \tilde{f} & \text{on } [0,T]\times\partial\Omega, \\ \tilde{v}|_{t=0} = \partial_t \tilde{v}|_{t=0} = 0 & \text{in } \Omega. \end{cases}$$

Because the wave operator is invariant under isometries we have

$$h^*(\Box_{\tilde{g}} \tilde{v}) = \Box_g (h^*\tilde{v}),$$

whence $v(t,x) := (h^*\tilde{v})(t,x) = \tilde{v}(T-t,x)$ solves (45).

We next vary the initial point and direction of a light-like geodesic to construct a family of Gaussian beams. The Gaussian beams will be constructed so that the implied constants of the family of Gaussian beams are uniformly bounded. This uniformity of constants is essential when proving stability estimates. We mention here a similar consideration in the Riemannian setting [Dos Santos Ferreira et al. 2020, Section 4.1].

To obtain such Gaussian beams, we start with a lemma. We define the set $\mathrm{PSO}(N)$ of pseudo-orthonormal frames as

$$\mathrm{PSO}(N) := \{(e_0, \dots, e_n) \in (TN)^{n+1} \mid g(e_0, e_0) = 0,\ g(e_1, e_1) = 0,\ g(e_0, e_1) = -2,\\ g(e_j, e_k) = \delta_{jk} \text{ for } j, k = 2, 3, \dots, n\}.$$

The lemma especially says that on a neighborhood of any point of N there is a local pseudo-orthonormal frame.

Lemma 13. *Let $z_0 \in N$ and let $V_0 \in T_{z_0}N$ be a light-like vector. The set of pseudo-orthonormal frames admits a local section $E : \mathcal{U} \to \mathrm{PSO}(N)$ such that the first component $(E(z_0))_0$ of E at z_0 is V_0. Here $\mathcal{U}$ is an open neighborhood of z_0.*

Proof. The existence of a pseudo-orthonormal frame $e = (e_0, e_1, \dots, e_n)$ of the tangent space $T_{z_0}N$ over the single point z_0 with $e_0 = V_0$ was shown in [Feizmohammadi and Oksanen 2022]. By using local coordinates (x^k) on a neighborhood $\mathcal{U} \subset M$ of z_0 let us define the mapping

$$F(x, E) : x(\mathcal{U}) \times \mathbb{R}^{(n+1)\times(n+1)} \to \mathbb{R}^{(n+1)\times(n+1)},$$

where $x(\mathcal{U}) \subset \mathbb{R}^{n+1}$, by the conditions

$$F(x, E)_{jk} = g_x(E_j, E_k) - g_{z_0}(e_j, e_k) \quad \text{if } j \ge k,$$
$$F(x, E)_{jk} = g_x(e_j, E_k) - g_{z_0}(e_j, e_k) \quad \text{if } j < k.$$

Here E_j is the j-th column vector of the $(n+1)\times(n+1)$ matrix E. Here also $g_x(E_j, E_k) = \langle E_j, g(x)E_k\rangle$ and $g_x(e_j, E_k) = \langle e_j, g(x)E_k\rangle$, where $g(x)$ is the coordinate representation matrix of g in the coordinates (x^k). The perhaps ad hoc looking conditions for $F(x, E)_{jk}$ for $j < k$ are related to the fact that local sections E of $\mathrm{PSO}(M)$ satisfying $(E(z_0))_0 = V_0$ (should they exist) are not unique without additional conditions. The conditions for $F(x, E)_{jk}$ for $j < k$ remove this ambiguity.

We apply the implicit function theorem (see, e.g., [Renardy and Rogers 2004, Theorem 10.6]) to show that there is a smooth mapping $x \mapsto E(x)$ such that $F(x, E(x)) = 0$. In this case E is a smooth section of $\mathrm{PSO}(N)$ by the conditions for $F(x, E)_{jk}$ for $j \ge k$ and by the symmetry of g. To apply the implicit function theorem, note that $F(z_0, e) = 0$ and that

$$(D_E F|_{x=z_0, E=e}(v))_{jk} = g_{z_0}(v_j, e_k) + g_{z_0}(e_j, v_k) \quad \text{if } j \ge k, \tag{46}$$
$$(D_E F|_{x=z_0, E=e}(v))_{jk} = g_{z_0}(e_j, v_k) \quad \text{if } j < k, \tag{47}$$

where $j, k = 0, 1, \dots, n$ and $v = (v_0, v_1, \dots, v_n) \in \mathbb{R}^{(n+1)\times(n+1)}$. Assume that $(D_E F|_{x=z_0, E=e}(v)) = 0$. Since g is symmetric, the condition (47) implies that $g_{z_0}(v_j, e_k) = 0$ for $j > k$. Substituting this into (46) then implies that $g_{z_0}(e_j, v_k) = 0$ for $j \geq k$. Thus we actually have that $g_{z_0}(e_j, v_k) = 0$ for all $j, k = 0, 1, \dots, n$. Since g is nondegenerate and e is a frame, it follows that each $v_k \in \mathbb{R}^{n+1}$ is the zero vector of $\mathbb{R}^{n+1}$. Thus $D_E F|_{x=z_0, E=e}$ is injective, and also surjective by dimensionality. Thus, by the implicit function theorem, and by redefining $\mathcal{U}$ smaller if necessary, there is a smooth mapping $E : \mathcal{U} \to \mathrm{PSO}(N)$. This is our desired section. □

We remark that it is likely that another proof of the above lemma can be obtained by generalizing the Gram–Schmidt procedure to the current situation. We also mention the similar construction [Dos Santos Ferreira et al. 2020, Lemma 6.1] in the Riemannian setting.

In the next result $|V_0 - \dot{\gamma}_x(s_0)|$ is defined by using local coordinates.

Corollary 14. *Let γ be a light-like geodesic of (N, g). Assume as in Proposition 10 and adopt its notation. Let s_0 be in the domain of γ and let us define $\gamma(s_0) = z_0$ and $\dot{\gamma}(t_0) = V_0$. Let also $\delta > 0$. Then there is $\tau_0 \geq 1$ and a neighborhood U of z_0 and a family of Gaussian beams*

$$v_\tau(x, \cdot)$$

solving $\Box_g v_\tau(x, \cdot) = 0$ in $[0, T] \times \Omega$ (including the correction term) parametrized by $x \in \mathcal{U}$. Here "$\cdot$" refers to points in N and $\tau \geq \tau_0$. The geodesics γ_x corresponding to the Gaussian beams $v_\tau(x, \cdot)$ satisfy $|V_0 - \dot{\gamma}_x(s_0)| \leq \delta$ and the implied constants of $v_\tau(x, \cdot)$ in Proposition 10 and Corollary 11 are uniformly bounded in x.

Proof. The proof is based on inspecting the construction of the Gaussian beams at the beginning of this section that lead to Proposition 10, and by using Corollary 11 and Lemma 13.

Let v_τ be a Gaussian beam without the error term corresponding to the geodesic γ as in Proposition 10. Note that this implies that we have chosen initial data for the certain ODEs used in the construction (such as the Riccati equation). Let us record these initial data and also define

$$v_\tau(z_0, \cdot) := v_\tau(\cdot).$$

By Lemma 13 there is a local section E of $\mathrm{PSO}(M)$ such that $(E(z_0))_0 = V_0$. We define a local vector field V by

$$V(x) = (E(x))_0.$$

By redefining the domain of E smaller, if necessary, we have that $|V(x) - \dot{\gamma}(0)| < \delta$. The section E also defines a family of Fermi coordinates by the formula (26) parametrized by x. Since E is smooth, the corresponding Fermi coordinates depend smoothly on x (say in any C^k norm in the Fréchet sense). Also the domain of the Fermi coordinates is uniformly bounded by the same reason. Let $x \in \mathcal{U}$ and let us pass to the Fermi coordinates determined by $E(x)$. We construct a Gaussian beam

$$v_\tau(x, \cdot)$$

with the following properties: (a) It corresponds to the geodesic $\gamma_{x, V(x)}$ with initial data $x \in M$ and $V(x) \in T_x M$. (b) It is constructed by exactly the same method described in the beginning of this section by using the same initial data for the corresponding ODEs that we used for v_τ. Since the coefficients of the ODEs are determined by the smooth metric g and the initial data are the same as for v_τ, the Gaussian beam $v_\tau(x,\cdot)$ differs boundedly and uniformly in x from $v_\tau(\cdot)$ (say in any $C^k(M)$ norm). In particular, the implied constants in Proposition 10 are uniform in x.

Finally, we use Corollary 11 to find correction terms for $v_\tau(x,\cdot)$ such that the implied constants in (44) are uniform in x. □

4. Separation of points

In this section (N, g) is a globally hyperbolic smooth Lorentzian manifold without boundary. The length of a piecewise smooth causal path $\alpha : [a, b] \to N$ is defined as

$$l(\alpha) := \sum_{j=0}^{k-1} \int_{a_j}^{a_{j+1}} \sqrt{-g(\dot\alpha(s), \dot\alpha(s))}\, ds, \tag{48}$$

where $a_0 < a_1 < \cdots < a_{k-1} < a_k$ are chosen such that α is smooth on each interval (a_j, a_{j+1}) for $j = 0, \ldots, k-1$. The time separation function, see, e.g., [O'Neill 1983], is denoted by $\tau : N \times N \to [0,\infty)$ and defined as

$$\tau(x, y) := \begin{cases} \sup l(\alpha), & y \in J^+(x), \\ 0, & y \notin J^+(x), \end{cases} \tag{49}$$

where the supremum is taken over all piecewise smooth future-directed causal curves $\alpha : [0, 1] \to N$ that satisfy $\alpha(0) = x$ and $\alpha(1) = y$. By [O'Neill 1983, Chapter 14, Lemma 16], we have that

$$\tau(x, z) > 0 \quad \text{if and only if} \quad x \ll z. \tag{50}$$

As before, we view N as the product manifold $\mathbb{R} \times M$ and assume that $\Omega \subset M$, $\dim(\Omega) = \dim(M)$, is a smooth compact manifold with boundary. As before, let Σ denote the lateral boundary $[0, T] \times \partial\Omega$. Let us consider $x \in I^+(\Sigma) \cap I^-(\Sigma)$. We say that $\gamma_1 : [0, 1] \to [0, T] \times \Omega$ is a future-directed *optimal geodesic* connecting Σ to x if there is

$$z_1 \in J^-(x) \cap \Sigma \quad \text{such that} \quad \gamma_1(0) = z_1,\ \gamma_1(1) = x \text{ and } \tau(z_1, x) = 0.$$

Similarly, we say that $\gamma_2 : [0, 1] \to [0, T] \times \Omega$ is a past-directed optimal geodesic connecting Σ to x if there is

$$z_2 \in J^+(x) \cap \Sigma \quad \text{such that} \quad \gamma_2(0) = z_2,\ \gamma_2(1) = x \text{ and } \tau(x, z_2) = 0.$$

We always understand optimal geodesics as their maximal extensions. Note that by definition future/past-directed optimal geodesics are always light-like. The next lemma says that such optimal geodesics always exist. We assume the notation and assumptions used earlier in this section. The situation of the lemma is illustrated in Figure 1, which can be found in Section 1.3 in the Introduction.

In the lemma we consider intersection times of geodesics and Σ. This means that if the geodesic is denoted by $\gamma : [0, 1] \to [0, T] \times \Omega$, then the first intersection time is the smallest $s_0 \in [0, 1]$ such that $\gamma(s_0) \in \Sigma$. Typically s_0 will be 0. That the intersection in the lemma is transverse means that $\dot\gamma(s_0)$ is transversal to the tangent space $T_{\gamma(s_0)}\Sigma$. We do not claim anything about possible other intersections of γ and Σ.

Lemma 15 (boundary optimal geodesics). *Let (N, g) be globally hyperbolic, $N = \mathbb{R}\times M$. If $x \in I^+(\Sigma)\cap([0, T]\times\Omega)$, there exists a future-directed optimal geodesic $\gamma : [0, 1]\to[0, T]\times\Omega$ from Σ to x and the first intersection of γ and Σ is transverse. Similarly, if $x \in I^-(\Sigma)\cap([0, T]\times\Omega)$, there exists a past-directed optimal geodesic $\gamma : [0, 1]\to[0, T]\times\Omega$ from Σ to x and the first intersection of γ and Σ is transverse.*

Proof. Existence: Let us first consider the claim about the existence of future-directed optimal geodesic. For this, let us define

$$t_{\text{sup}} = \sup\{\tilde{d} \in [0, T] \mid \text{there is } \tilde{z} \in \Sigma \text{ such that } t(\tilde{z}) = \tilde{d} \text{ and } \tau(\tilde{z}, x) > 0\}. \tag{51}$$

Here τ is defined on $N \times N$. The number t_{sup} will be the time coordinate of z_{sup} in Figure 1. By assumption $x \in I^+(\Sigma)$ and thus there is $\tilde{z} \in \Sigma$ such that $x \in I^+(\tilde{z})$ with $\tau(\tilde{z}, x) > 0$ by (50). We also have $t(\tilde{z}) \in [0, T]$. Consequently the supremum in (51) exists and $t_{\text{sup}} \in [0, T]$. Let $z_k \in \Sigma$ and $t(z_k) = t_k$ be such that $t_k \to t_{\text{sup}}$ as $k \to \infty$. Since $z_k \in \Sigma$ and Σ is compact, we may pass to a subsequence so that $z_k \to z_{\text{sup}} \in \Sigma$. We also have $t(z_{\text{sup}}) = t_{\text{sup}}$ by continuity of the time function t.

We claim that $\tau(z_{\text{sup}}, x) = 0$. We argue by contradiction and assume the opposite that $\tau(z_{\text{sup}}, x) > 0$. Then there is a timelike future-directed path $\eta : [0, 1] \to N$ connecting z_{sup} to x by (50). Since η is timelike and $I^-(x)$ is open, we may deform η slightly on a neighborhood of z_{sup} to a future-directed timelike path that connects $z' \in \Sigma$ to x so that $t(z') > t_{\text{sup}}$. Thus $x \in I^+(z')$ and we still have $\tau(z', x) > 0$ by (50). This is a contradiction to the definition of t_{sup}. We conclude that $\tau(z, x) = 0$. Since (N, g) is globally hyperbolic, there is a future-directed light-like geodesic $\gamma_1 : [0, 1] \to N$ from z_{sup} to x of length $\tau(z_{\text{sup}}, x) = 0$; see [O'Neill 1983, Chapter 14, Proposition 19].

We note that γ_1 is actually a path $[0, 1]\to[0, T]\times\Omega$. Indeed, if γ_1 meets the complement of $[0, T]\times\Omega$, then γ_1 necessarily intersects Σ at a parameter time $s_0 < 1$ before it meets z_{sup} at the parameter time 1. Since γ_1 is causal, it follows that $t(\gamma_1(s_0)) > t_{\text{sup}} = t(z_{\text{sup}})$, where $\gamma_1(s_0) \in \Sigma$. Since Σ is timelike, there is point $\hat{z} \in \Sigma$ with $t_{\text{sup}} < t(\hat{z}) < t(\gamma_1(s_0))$ and a future-directed timelike path $\hat{\eta}$ connecting $\hat{z}$ to $\gamma_1(s_0)$. Thus, a path achieved by composing the paths $\hat{\eta}$ and γ_1 has positive length by the definition (48). It follows that $\tau(\hat{z}, x) > 0$ by the definition (49). We have arrived to a contradiction with the definition of z_{sup}, since $t(\hat{z}) > t_{\text{sup}}$.

Transversality: We next show that the optimal geodesic γ constructed above intersects the lateral boundary Σ transversally. Assume that γ is parametrized so that $\gamma(0) = z_{\text{sup}}$. Let $S_{t_{\text{sup}}} = \{t_{\text{sup}}\}\times M$ be the Cauchy level surface at $t = t_{\text{sup}}$. Let $T = (T_1, \dots, T_{n-1})$ be a basis for the tangent space $T_{z_{\text{sup}}}\partial\Omega$. Then $\{T, \nu\}$, where ν is the normal vector to $\partial\Omega$ at z_{sup} in $S_{t_{\text{sup}}}$, is a basis for $T_{z_{\text{sup}}}S_{t_{\text{sup}}}$. Consequently, the tangent space $T_{z_{\text{sup}}}N$ is spanned by $\{\partial_t, T, \nu\}$, where ∂_t is the coordinate vector of $[0, T]$. Let us write $\dot{\gamma}(0) \in T_{z_{\text{sup}}}N$ in the form

$$\dot{\gamma}(0) = (\dot{\gamma}^t(0), \dot{\gamma}^T(0), \dot{\gamma}^\nu(0)).$$

Suppose now to the contrary that γ does not intersect Σ transversally. Then it follows that $\dot{\gamma}^\nu(0) = 0$. Indeed, if this is not the case, then $T_{z_{\text{sup}}}\Sigma + T_{z_{\text{sup}}}\,\text{graph}(\gamma)$ would be equal to $T_{z_{\text{sup}}}N$. Let us check whether $\dot{\gamma}(0)$ is normal to $\Sigma_{t_{\text{sup}}} := \Sigma\cap\{t = t_{\text{sup}}\}$. Since $\Sigma_{t_{\text{sup}}}$ is space-like, the normal space

$$N_{z_{\text{sup}}}\Sigma_{t_{\text{sup}}} := \{v \in T_{z_{\text{sup}}}N \mid \langle v, w\rangle_g = 0 \text{ for all } w \in T_{z_{\text{sup}}}\Sigma_{t_{\text{sup}}}\}$$

(see [O'Neill 1983, p. 98 or p. 198]) is spanned by ∂_t and ν. To see this, note that a vector $X \in T_{z_{\text{sup}}}N$, $X = a\partial_t + b \cdot T + c\nu$, is in $N_{z_{\text{sup}}}\Sigma_{t_{\text{sup}}}$ if and only if $b \in \mathbb{R}^{n-1}$ is zero. Note $\dot{\gamma}^{\nu}(0) = 0$; then if $\dot{\gamma}(0) \in N_{z_{\text{sup}}}\Sigma_{t_{\text{sup}}}$, we must have $\dot{\gamma}(0) = (\dot{\gamma}^t(0), 0, 0)$. But this is not possible, since γ is light-like. So $\dot{\gamma}(0)$ is not normal to $\Sigma_{t_{\text{sup}}}$ and by [O'Neill 1983, Chapter 10, Lemma 50] there exists a time-like curve σ from x to $\Sigma_{t_{\text{sup}}}$. By slightly deforming σ we obtain another time-like curve $\tilde{\sigma}$ connecting x to $z' \in \Sigma$ with $t(z') > t_{\text{sup}}$. This contradicts the definition of t_{sup}.

The claim about the past-directed optimal geodesic follows by defining

$$t_{\text{inf}} = \inf\{\tilde{d} \in [0, T] \mid \text{there is } \tilde{z} \in \Sigma \text{ such that } t(\tilde{z}) = \tilde{d} \text{ and } \tau(x, \tilde{z}) > 0\}$$

and by using arguments analogous to the ones above to find $z_{\text{inf}} \in \Sigma$ with $\tau(x, z_{\text{inf}}) = 0$. ☐

By using boundary optimal geodesics and related Gaussian beams we may separate points of $[0, T] \times \Omega$ by solutions to $\Box_g v = 0$. We mention here that separation of points by solutions has been beneficial in the study of inverse problems for elliptic equations [Guillarmou et al. 2019; Lassas et al. 2020].

Proposition 16 (separation of points). *Let (N, g) be globally hyperbolic, $N = \mathbb{R} \times M$. Let $x \in I^-(\Sigma) \cap ([0, T] \times \Omega)$ and $y \in N$ be such that $y \notin J^-(x)$. Denote by v_f the solution to $\Box_g v = 0$ in N with $v|_\Sigma = f$ and whose Cauchy data vanishes at $t = T$. Then there is $f \in C^\infty(\Sigma)$ such that*

$$v_f(x) \neq v_f(y).$$

If $x \in I^+(\Sigma) \cap ([0, T] \times \Omega)$ and $x \notin J^-(y)$, we have the same claim for solutions of $\Box_g v = 0$ in N with $v|_\Sigma = f$ whose Cauchy data instead vanishes at $t = 0$.

Proof. We first claim that there is a past-directed light-like geodesic from Σ that meets the point x but not y. We argue by contradiction and assume the opposite that all past-directed light-like geodesics from Σ to x meet both x and y. Since $x \in I^-(\Sigma) \cap ([0, T] \times \Omega)$, by Lemma 15 we have that there is a past-directed boundary optimal geodesic $\gamma_1 : [0, 1] \to [0, T] \times \Omega$ with $\gamma_1(0) = z \in \Sigma$ and $\gamma_1(1) = x$. The first intersection of γ_1 with Σ is transverse. If $x \notin J^-(y)$, then by the assumption $y \notin J^-(x)$ we have that x and y are not causally connected. Thus γ_1 cannot pass through y and we have found our light-like geodesic. Therefore, we may assume that $y \geq x$.

Let $\tilde{\gamma}_1 = \tilde{\gamma}_1(s)$ be a past-directed light-like geodesic with $\tilde{\gamma}_1(0) \in \Sigma$ such that $\tilde{\gamma}_1$ intersects Σ transversally at $s = 0$, and which satisfies $\tilde{\gamma}_1(\tilde{s}) = x$ for some $\tilde{s} \geq 0$. The geodesic $\tilde{\gamma}_1$ can be obtained by perturbing the tangent vector of γ_1 at $\gamma_1(1) = x$ slightly. Note that the condition of transversal intersection is an open condition. By assumption $\tilde{\gamma}_1$ meets y. In this case we have a shortcut path, which has timelike portion, obtained by traveling along $\tilde{\gamma}_1$ from x to a point y' close to y, doing a shortcut from y' to γ_1 and then by continuing along γ_1 to z; see [O'Neill 1983, Chapter 10, Proposition 46]. Since the shortcut path has timelike portion, it has positive length. Since $y \geq x$, the shortcut path is also future-directed. It follows that $\tau(x, z) > 0$. This contradicts the optimality of γ_1. We conclude that $\tilde{\gamma}_1$ is a past-directed light-like geodesic from Σ that meets x but not y.

To conclude the proof, we use Proposition 10 and choose a Gaussian beam $v_\tau = \tau^{n/4} e^{i\tau\Theta} a$ corresponding to $\tilde{\gamma}_1$ with $k > n$, $K = 1$ and $p, l = 2$. We also choose the support of the amplitude a be so small that $y \notin \operatorname{supp}(a)$ and $\operatorname{supp}(a) \cap \{t = T\} = \varnothing$. We will use the Sobolev embedding $H^{l'} \subset L^\infty$, which holds for $l' > \frac{n+1}{2}$. Since $k > n$, we have $k - \frac{n-1}{2} > \frac{n+1}{2}$, which shows that we can take l' such that $\frac{n+1}{2} < l' < k - \frac{n-1}{2}$. Applying Corollary 11 with these k and l' shows that there is $r_\tau \in C^\infty(N)$ such that

$$v_f := \tau^{-\frac{n}{4}} v = \tau^{-\frac{n}{4}} v_\tau + \tau^{-\frac{n}{4}} r_\tau$$

satisfies $\square_g v_f = 0$ and

$$\tau^{-\frac{n}{4}} v_\tau(x) = 1 \quad \text{and} \quad \tau^{-\frac{n}{4}} v_\tau(y) = 0 \quad \text{for all } \tau \geq \tau_0$$

and

$$\|\tau^{-\frac{n}{4}} r_\tau\|_{L^\infty(N)} \leq C\tau^{-\frac{n}{4}} \|r_\tau\|_{H^{l'}(N)} = \tau^{-\frac{n}{4}} \mathcal{O}(\tau^{-1}).$$

We mention for future reference that at any other point $z \in [0, T] \times \Omega$ we have

$$|v_f(z)| \leq |\tau^{-\frac{n}{4}} v_\tau(z)| + |\tau^{-\frac{n}{4}} r_\tau(z)| \leq C' + |\tau^{-\frac{n}{4}} r_\tau(z)| \leq C \tag{52}$$

for all τ large enough. Here we used the above Sobolev embedding. Taking τ large enough shows that

$$v_f(x) \neq v_f(y).$$

The claim about the case $x \in I^+(\Sigma)$ and $x \notin J^-(y)$ can be proved in similar way. □

We next consider the case where we have multiple points of $[0, T] \times \Omega$, which we wish to separate by solutions of the wave equation $\square_g v = 0$. The points will correspond to the intersection points of pairs of geodesics we use for our inverse problem. The matrix (53) below will be a *separation matrix* in the sense of Definition 5.

Lemma 17 (existence of separation matrix). *Let (N, g) be globally hyperbolic, $N = \mathbb{R} \times M$. Let $x_1, \dots, x_P \in I^-(\Sigma) \cap ([0, T] \times \Omega)$ be such that $x_1 < x_2 < \cdots < x_P$. Denote by v_f the solution of $\square_g v = 0$ in $[0, T] \times \Omega$ with $v|_\Sigma = f$ and whose Cauchy data vanishes at $t = T$. Then there are boundary values $f_k \in C^\infty(\Sigma)$ such that the matrix*

$$\begin{pmatrix} v_{f_1}(x_1) & v_{f_2}(x_1) & \cdots & v_{f_P}(x_1) \\ v_{f_1}(x_2) & v_{f_2}(x_2) & \cdots & v_{f_P}(x_2) \\ \vdots & & \ddots & \vdots \\ v_{f_1}(x_P) & v_{f_2}(x_P) & \cdots & v_{f_P}(x_P) \end{pmatrix} \tag{53}$$

is invertible.

If $x_k \in I^+(\Sigma) \cap ([0, T] \times \Omega)$, we have the similar claim for solutions of $\square_g v = 0$ in $[0, T] \times \Omega$ with $v|_\Sigma = f$ whose Cauchy data instead vanishes at $t = 0$.

Proof. The proof is an iteration of the proof Proposition 16. First we let γ_1 be a past-directed boundary optimal geodesic that connects a point $z \in \Sigma$ to the point x_1. By the shortcut argument in the proof of Proposition 16, we deduce after possibly redefining γ_1 as its small perturbation that γ_1 does not meet any of the other points x_k, $k = 2, \dots, P$. Let v_{f_1} be a Gaussian beam solution (including the correction term) as in the proof of Proposition 16, where $f_1 \in C^\infty(\Sigma)$. Then there is $\tau_1 > 0$ such that for $\tau \geq \tau_1$ we have

$$v_{f_1}(x_1) = 1 \quad \text{and} \quad v_{f_1}(x_k) = \mathcal{O}(\tau^{-1-\frac{n}{4}}), \quad k = 2, \dots, P.$$

Next, let γ_2 be a past-directed boundary optimal geodesic that connects $z \in \Sigma$ to the point x_2. By repeating the above argument we find a boundary value $f_2 \in C^\infty(\Sigma)$ and a solution v_{f_2} such that

$$v_{f_2}(x_2) = 1 \quad \text{and} \quad v_{f_2}(x_k) = \mathcal{O}(\tau^{-1-\frac{n}{4}}), \quad k = 3, \dots, P,$$

for all $\tau \geq \tau_2$. Note that we do not claim that we have much control on the value of v_{f_2} at x_1 and it might be that γ_2 meets also the point x_1. However, by (52) we know that $|v_{f_2}|$ at x_1 is bounded by C (possibly by defining τ_2 larger). This is illustrated in Figure 2, which can be found in Section 1.3 in the Introduction. By repeating the above arguments, we find other solutions v_{f_k}, $k = 3, \dots, P$, such that the matrix (53) becomes of the form

$$\mathcal{V}_\tau = \begin{pmatrix} 1 & \mathcal{O}(\tau^{-1-\frac{n}{4}}) & \mathcal{O}(\tau^{-1-\frac{n}{4}}) \\ \# & \ddots & \mathcal{O}(\tau^{-1-\frac{n}{4}}) \\ \# & \# & 1 \end{pmatrix}.$$

Here # means unspecified complex numbers bounded by some fixed constant. The determinant of this matrix tends to 1 as $\tau \to \infty$. Therefore, there is $\tau_0 \geq 1$ such that the matrix (53) is invertible for all $\tau \geq \tau_0$. □

The previous lemma shows that if we are given a set of points $x_1 < \dots < x_k$, one can find a set of Gaussian beams separating these points. However, for the proof of the stability estimate in Theorem 1, we need a finite collection of Gaussian beams that separate any sufficiently distinct $P \in \mathbb{N}$ points. The collection will be a *separation filter* in the sense of Definition 6. Existence of such a collection is the content of the next lemma.

Let $\bar{g}$ be an auxiliary Riemannian metric on $\mathbb{R} \times M$.

Lemma 18 (existence of separation filter). *Let $P \geq 1$ be an integer and let $\delta > 0$. Suppose $K \subset I^-(\Sigma) \cap I^+(\Sigma) \cap ([0, T] \times \Omega)$ is a compact set. There exists a finite collection $\mathcal{M} \subset C^\infty([0, T] \times \Omega)$ of solutions to $\Box_g v_f = 0$ with the following properties: Assume that $x_1, \dots, x_P \in K$ are any points such that $x_1 < x_2 < \dots < x_P$ and $d_{\bar{g}}(x_k, x_l) > \delta$ for $x_k \neq x_l$, $k, l = 1, \dots, P$. Then there are solutions $v_{f_1}, \dots, v_{f_P} \in \mathcal{M}$ corresponding to boundary values $f_k \in C^\infty(\Sigma)$, and which have vanishing Cauchy data at $t = T$, such that the matrix $(v_{f_k}(x_l))_{k,l=1}^P$ in (53) is invertible. Thus $\mathcal{M}$ is a separation filter.*

Proof. Case 1: If $P = 1$, then the situation is similar to the proof of Proposition 16. Applying Lemma 15 to x_1, we find a past-directed boundary optimal geodesic γ from Σ to x_1, whose first intersection with Σ is transverse. Using Corollary 11 we can then construct a Gaussian beam v (including the correction term and with vanishing Cauchy data at $\{t = T\}$) corresponding to γ such that

$$v(x_1) = 1.$$

By continuity of v, the point x_1 has a neighborhood $B(x_1)$ such that

$$|v(z)| > \tfrac{2}{3} \quad \text{for all } z \in B(x_1).$$

Doing this for all points $x \in K$ we find an open cover of K of the form

$$\bigcup_{x \in K} B(x)$$

and for each $B(x)$ the corresponding optimal geodesic and the respective Gaussian beam v. Because K is compact, there is a finite subcover

$$\bigcup_{j=1}^{R} B(x^j)$$

of K and the corresponding finite collection of Gaussian beams. Denoting this collection by $\mathcal{M}$ completes the proof for $P = 1$.

Case 2: Suppose now $P \geq 2$. To begin, consider a complex matrix of the form

$$\begin{pmatrix} d_1 & & \mathcal{O} \\ & \ddots & \\ \# & & d_P \end{pmatrix}, \tag{54}$$

where all entries # are bounded by a fixed constant $C > 0$ and the diagonal entries satisfy $|d_j| > \frac{2}{3}$, $j = 1, \ldots, P$. When the elements of the upper triangular part $\mathcal{O}$ are of the size $\varepsilon > 0$, the determinant of the matrix in (54) equals

$$d_1 \cdots d_P + O(\varepsilon).$$

This can be seen by considering the definition of the determinant in terms of minors. Thus the matrix in (54) is invertible when ε is small enough.

We construct an open cover of K as follows. Let $\widetilde{K} \subset J^+(\Sigma) \cap J^-(\Sigma) \cap ([0, T] \times \Omega)$ be an open neighborhood of K. Let us fix $x \in \widetilde{K}$ and let $B_{\delta/2}(x)$ denote a $\frac{\delta}{2}$-radius ball centered at x with respect to the metric $\bar{g}$. Let us also define

$$\mathcal{V}(x) := (J^+(x) \setminus B_{\frac{\delta}{2}}(x)) \cap ([0, T] \times \Omega).$$

Since $J^+(x)$ is closed, the set $\mathcal{V}(x)$ is compact for all $x \in \widetilde{K}$. We define the subset of $\mathcal{V}(x)^{P-1}$ of ordered points by

$$\mathcal{T}(x) := \{(x_2, \ldots, x_P) \in \mathcal{V}(x)^{P-1} : x \leq x_2 \leq \cdots \leq x_P\}.$$

Because the relation $\leq$ is closed (see, e.g., [O'Neill 1983, Section 14, Lemma 22]), the set $\mathcal{T}(x)$ is compact as a closed subset of the compact set $\mathcal{V}(x)^{P-1}$.

Let $\varepsilon > 0$ and let $X = (x_2, \ldots, x_P) \in \mathcal{T}(x)$. Recall that when constructing a Gaussian beam v, we can bound its size in absolute value by using the estimate (52). Since $x \in I^-(\Sigma) \cap ([0, T] \times \Omega)$, there is $f_X \in C^\infty(\Sigma)$ and a Gaussian beam v_X (including the correction term and with vanishing Cauchy data at $\{t = T\}$) and $\tau_X > 0$ such that there is a neighborhood $U_\varepsilon(x) \subset B_{\delta/3}(x)$ of x and neighborhoods $B(x_k)$ of x_k such that

$$\begin{aligned} |v_{f_X}| &\geq \tfrac{2}{3} && \text{on } U_\varepsilon(x), \\ |v_{f_X}| &< \varepsilon && \text{on } B(x_k), \quad k = 2, 3, \ldots, P, \\ |v_{f_X}| &\leq C && \text{on } [0, T] \times \Omega, \end{aligned} \tag{55}$$

where $C > 0$ is independent of $\varepsilon > 0$. Here we have first normalized so that $v_{f_X}(x) = 1$. Then we have chosen the τ_X large enough, so that the condition $|v_{f_X}| < \varepsilon$ holds on $B(x_k)$, and $|v_{f_X}| \leq C$ on $[0, T] \times \Omega$.

These conditions can be obtained since the correction term of a Gaussian beam can be made arbitrarily small by taking the corresponding τ large enough. Then, by the continuity of v_{f_X} and $v_{f_X}(x) = 1$, we have chosen the neighborhood $U_\varepsilon(x)$ so that $|v_{f_X}| \geq \frac{2}{3}$. Note that since here τ_X depends on ε and v_{f_X} depends on τ_X, the neighborhood $U_\varepsilon(x)$ depends on ε as indicated in the notation. See the argument in the proof of Proposition 16 for more details.

We now modify the open sets $U_\varepsilon(x)$ slightly. Let us define

$$\widetilde{U}_\varepsilon(x) := I^+(x) \cap U_\varepsilon(x).$$

We have that

$$|v_{f_X}| \geq \tfrac{2}{3} \quad \text{on } \widetilde{U}_\varepsilon(x).$$

Moreover, we have

$$x \leq z \quad \text{for all } z \in \widetilde{U}_\varepsilon(x). \tag{56}$$

We then have an open cover of $\mathcal{T}(x)$ given by

$$\bigcup_{X \in \mathcal{T}(x)} B(x_2) \times \cdots \times B(x_P).$$

Since $\mathcal{T}(x)$ is compact, we may pass to a finite open subcover

$$\bigcup_{X \in \mathcal{J}_\uparrow(x)} B(x_2) \times \cdots \times B(x_P),$$

where $\mathcal{J}_\uparrow(x)$ is a finite subset of $\mathcal{T}(x)$ and which depends on ε. Note that for each $X = (x_2, \ldots, x_P) \in \mathcal{J}_\uparrow(x)$ there are associated neighborhoods $B(x_2), \ldots, B(x_P)$ of the points $x_2, \ldots, x_P$ and an open set $\widetilde{U}_\varepsilon(x)$. This shows that to each point $x \in \widetilde{K}$ we can attach a finite collection

$$\mathcal{M}_\varepsilon(x) \subset C^\infty([0, T] \times \Omega)$$

of solutions with the following property: for any $X \in \mathcal{T}(x)$ there is some Gaussian beam solution $v_{f_X} \in \mathcal{M}_\varepsilon(x)$ corresponding to a boundary value f_X with the property (55) with $U_\varepsilon(x)$ replaced by $\widetilde{U}_\varepsilon(x)$. We repeat the above argument for all $x \in \widetilde{K}$. Note that if $x \in K$, then there is $\tilde{x} \in \widetilde{K} \cap J^-(x)$ so that $x \in \widetilde{U}_\varepsilon(\tilde{x})$. Thus, our construction yields an open cover of $K \subset [0, T] \times \Omega$ by the sets $\widetilde{U}_\varepsilon(x)$ described above. By compactness, finitely many sets $\widetilde{U}_\varepsilon(x)$ suffice to cover K. Let $x^{(j)} \in [0, T] \times \Omega$ be the corresponding points, such that

$$\bigcup_{j=1}^{R_\varepsilon} \widetilde{U}_\varepsilon(x^{(j)}) \tag{57}$$

is a finite subcover of K, where $R_\varepsilon \in \mathbb{N}$. To each of these finitely many points $x^{(j)}$ there is also attached a finite subset $\mathcal{J}_\varepsilon(x^{(j)}) \subset \mathcal{T}(x^{(j)})$, $j = 1, \ldots, R_\varepsilon$. Corresponding to this finite cover, we take as the collection of boundary values $\mathcal{M}_\varepsilon$ the set

$$\mathcal{M}_\uparrow := \bigcup_{j=1}^{R_\varepsilon} \mathcal{M}_\varepsilon(x^{(j)}).$$

Let then $x_1, x_2, \dots, x_P \in K$ with $x_1 < x_2 < \cdots < x_P$ and $d_{\bar g}(x_l, x_k) > \delta$ for $k \neq l$ with $k, l = 1, \dots, P$. Let us consider first the point $x_1 \in K$. Corresponding to x_1, there is an index $j_1 \in \{1, \dots, R_\varepsilon\}$ and a neighborhood $\widetilde U_\varepsilon(x^{(j_1)})$ of x_1, where $\widetilde U_\varepsilon(x^{(j_1)})$ belongs to the finite subcover (57) of K. The radius of $\widetilde U_\varepsilon(x^{(j_1)})$ is less than $\frac{\delta}{3}$. Note that $d_{\bar g}(x^{(j_1)}, x_k) > \frac{\delta}{2}$ for $k = 2, 3, \dots, P$. Indeed, we have that

$$d_{\bar g}(x^{(j_1)}, x_k) \geq d_{\bar g}(x_1, x_k) - d_{\bar g}(x^{(j_1)}, x_1) > \delta - \frac{\delta}{3} = \frac{2\delta}{3} > \frac{\delta}{2}. \tag{58}$$

Moreover, (56) implies $x^{(j_1)} \leq x_1$. Thus $x^{(j_1)} \leq x_2 \leq x_3 \leq \cdots \leq x_P$. Using this and (58), we obtain

$$(x_2, x_3, \dots, x_P) \in \mathcal{T}(x^{(j_1)}).$$

Consequently, using the definition of $\mathcal{J}_\varepsilon(x^{(j_1)})$, we find $X = (x_2^{(j_1)}, \dots, x_P^{(j_1)}) \in \mathcal{J}_\varepsilon(x^{(j_1)})$ with the associated neighborhoods $B(x_k^{(j_1)})$ of x_k, $k = 2, 3, \dots, P$, satisfying the following property: there is a Gaussian beam solution $v_{f_1} \in \mathcal{M}_\varepsilon$ corresponding to a boundary value f_1 such that

$$\begin{aligned} |v_{f_1}| &\geq \tfrac{2}{3} \quad \text{on } \widetilde U_\varepsilon(x^{(j_1)}), \\ |v_{f_1}| &< \varepsilon \quad \text{on } B(x_k^{(j_1)}), \quad k = 2, 3, \dots, P, \\ |v_{f_1}| &\leq C \quad \text{on } [0, T] \times \Omega. \end{aligned}$$

Let us then proceed to the point x_2. Much as above, regarding this point there is $j_2 \in \{1, \dots, R_\varepsilon\}$, $x^{(j_2)} \in [0, T] \times \Omega$ and neighborhoods $\widetilde U_\varepsilon(x^{(j_2)})$ of x_2 and neighborhoods $B(x_k^{(j_2)})$ of x_k, $k = 3, 4 \dots, x_P$, and a Gaussian beam v_{f_2}, such that

$$\begin{aligned} |v_{f_2}| &\geq \tfrac{2}{3} \quad \text{on } \widetilde U_\varepsilon(x^{(j_2)}), \\ |v_{f_2}| &< \varepsilon \quad \text{on } B(x_k^{(j_2)}), \quad k = 3, 4, \dots, P, \\ |v_{f_2}| &\leq C \quad \text{on } [0, T] \times \Omega. \end{aligned}$$

Continuing in this manner, we have indices $j_1, j_2, \dots, j_P$ and a set of Gaussian beams v_{f_k}, $k = 1, \dots, P$, such that $|v_{f_k}| \geq \frac{2}{3}$ on a neighborhood $\widetilde U_\varepsilon(x^{(j_k)})$ of x_k and $|v_{f_k}| < \varepsilon$ on a neighborhood $B(x_l^{(j_k)})$ of x_l for $l > k$ and $|v_{f_k}| < C$ on $[0, T] \times \Omega$.

The separation matrix (53) corresponding to the functions v_{f_k} and points x_k is invertible for $\varepsilon \leq \varepsilon_0$ for ε_0 small enough. We set $\mathcal{M} := \mathcal{M}_{\varepsilon_0}$. Finally, we note that the number of Gaussian beams used is

$$\#\mathcal{M} = \#\Bigg(\bigcup_{j=1}^{R_{\varepsilon_0}} \mathcal{M}_{\varepsilon_0}(x^{(j)})\Bigg) \leq \sum_{j=1}^{R_{\varepsilon_0}} \#(\mathcal{M}_{\varepsilon_0}(x^{(j)})) = \sum_{j=1}^{R_{\varepsilon_0}} \#\mathcal{J}_{\varepsilon_0}(x^{(j)}),$$

which is finite. □

Remark 19. We will apply Lemma 17 as follows. Suppose the points $x_1 < \cdots < x_P$ are the intersection points of two light-like geodesics γ_1 and γ_2. We will use Lemma 17 to show that there is a choice of P solutions $v_{f_1}, \dots, v_{f_P} \in \mathcal{M}$ which separate the points $x_1, \dots, x_P$. Moreover, these solutions have zero Cauchy data at $t = T$.

We also mention that we have a result similar to Lemma 18 for solutions that have vanishing Cauchy data at $\{t = 0\}$. The result is obtained, for example, from Lemma 18 by considering the isometry $t \mapsto T - t$ as in Remark 12.

5. Proof of the stability estimate: Theorem 1

Assume the conditions from Theorem 1, especially that

$$\|\Lambda_1(f)-\Lambda_2(f)\|_{H^r(\Sigma)} \le \delta,$$

where $r \le s+1$ and $s+1 > \frac{n+1}{2}$, and $\delta > 0$. Here Λ_1 and Λ_2 are the DN maps of the nonlinear wave equation (2) corresponding to the potentials q_1 and q_2, respectively. We show that we have explicit control on the L^∞ norm of $q_1 - q_2$ in terms of δ. The proof will be divided into several steps.

5.1. ***Step 1*****:** ***integral identity from finite differences.*** Let $j = 1, \dots, m$ and $\varepsilon_j > 0$ be small parameters. Let κ be as in Lemma 9. Assume that $f_j \in H^{s+1}(\Sigma)$ is a family of functions satisfying $\partial_t^\alpha f_j|_{t=0} = 0$ on $[0,T]\times\partial\Omega,\ \alpha = 0,\dots,s$, and that

$$\|\varepsilon_1 f_1 + \cdots + \varepsilon_m f_m\|_{H^{s+1}([0,T]\times\Omega)} \le \kappa.$$

For $l = 1,2$, we have that the boundary value problems

$$\begin{cases} \Box_g u_l + q_l u_l^m = 0 & \text{in } [0,T]\times\Omega, \\ u_l = \varepsilon_1 f_1 + \cdots + \varepsilon_m f_m & \text{on } [0,T]\times\partial\Omega, \\ u_l|_{t=0} = 0, \quad \partial_t u_l|_{t=0} = 0 & \text{in } \Omega \end{cases}$$

have unique small solutions $u_l = u_{\varepsilon_1 f_1 + \cdots + \varepsilon_m f_m}$ as described in Lemma 9. According to (17), the solutions u_l have expansions of the form

$$u_l = \varepsilon_1 v_{l,1} + \cdots + \varepsilon_m v_{l,m} + \sum_{|\vec{k}|=m} \binom{m}{k_1,\dots,k_m} \varepsilon_1^{k_1}\cdots\varepsilon_m^{k_m} w_{l,\vec{k}} + \mathcal{R}_l,$$

where $v_{l,j}$ satisfy (18) and $w_{l,\vec{k}}$ satisfy (19) with q replaced by q_l. We also used the notation $\vec{k} = (k_1,\dots,k_m)$. In particular, we know by (19) that

$$w_{l,\vec{1}} := w_{l,(1,\dots,1)}$$

satisfy

$$\begin{cases} \Box_g w_{l,\vec{1}} + q_l v_{l,1}\cdots v_{l,m} = 0 & \text{in } [0,T]\times\Omega, \\ w_{l,\vec{1}} = 0 & \text{on } [0,T]\times\partial\Omega, \\ w_{l,\vec{1}}|_{t=0} = 0, \ \partial_t w_{l,\vec{1}}|_{t=0} = 0 & \text{in } \Omega. \end{cases} \tag{59}$$

Note that since (18) for $v_{l,j}$ are independent of q_l, we have by the uniqueness of solutions that

$$v_{1,j} = v_{2,j} =: v_j, \quad j = 1,\dots,m. \tag{60}$$

Moreover, according to (20), the correction terms $\mathcal{R}_l$ for $l = 1,2$ satisfy

$$\|\mathcal{R}_l\|_{E^{s+2}} + \|\Box_g \mathcal{R}_l\|_{E^{s+1}} \le C(s,T)\|q_l\|_{E^{s+1}}^2 \|\varepsilon_1 f_1 + \cdots + \varepsilon_m f_m\|_{H^{s+1}(\Sigma)}^{2m-1}.$$

We apply the finite difference operator $D^m_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0}$ of order m, defined in (21), to u_l. By (22), we have

$$D^m_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0} u_l = m!\, w_{l,\vec{1}} + \frac{1}{\varepsilon_1\cdots\varepsilon_m}\overline{\mathcal{R}}_l,$$

where $\overline{\mathcal{R}}_l$ contains sum of the remainder terms $\mathcal{R}_l$ appearing in the finite differences. Consequently, by taking into account (59) and (60), we obtain

$$\Box_g D^m_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0} u_l = -m!\, q_l v_1 \cdots v_m + \frac{1}{\varepsilon_1 \cdots \varepsilon_m} \Box_g \widetilde{\mathcal{R}}_l,$$

where $\widetilde{\mathcal{R}}_l = \varepsilon_1 \cdots \varepsilon_m \overline{\mathcal{R}}_l$, $l = 1, 2$.

We manipulate the integral identity (25) to relate the difference of the DN maps Λ_1 and Λ_2 to the difference of the unknown potentials q_1 and q_2 in terms of v_j. For this, consider an auxiliary function v_0 which satisfies $\Box_g v_0 = 0$ in $[0, T] \times \Omega$, with $v_0|_{t=T} = \partial_t v_0|_{t=T} = 0$ in Ω. Applying (25) to the difference of the DN maps yields

$$\begin{aligned} -m! &\int_{[0,T]\times\Omega} (q_1 - q_2) v_0 v_1 \cdots v_m \, dV_g \\ &= \frac{1}{\varepsilon_1 \cdots \varepsilon_m} \int_{[0,T]\times\Omega} v_0 \Box_g (\widetilde{\mathcal{R}}_1 - \widetilde{\mathcal{R}}_2)\, dV_g + \int_\Sigma v_0 D^m_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0} (\Lambda_1 - \Lambda_2)(\varepsilon_1 f_1 + \cdots + \varepsilon_m f_m)\, dS. \end{aligned} \tag{61}$$

The finite difference $D^m_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0}$ of u_l is a sum of 2^m terms. By using (61), we calculate

$$\begin{aligned} m!\, &|\langle v_0(q_1 - q_2), v_1 \cdots v_m \rangle_{L^2([0,T]\times\Omega)}| \\ &\le |\langle v_0, D^m_{\vec{\varepsilon}=0} [(\Lambda_1 - \Lambda_2)(\varepsilon_1 f_1 + \cdots + \varepsilon_m f_m)] \rangle_{L^2(\Sigma)}| + (\varepsilon_1 \cdots \varepsilon_m)^{-1} |\langle v_0, \Box_g (\widetilde{R}_1 - \widetilde{R}_2) \rangle_{L^2([0,T]\times\Omega)}| \\ &\le 2^m (\varepsilon_1 \cdots \varepsilon_m)^{-1} |\langle v_0, (\Lambda_1 - \Lambda_2)(\varepsilon_1 f_1 + \cdots + \varepsilon_m f_m) \rangle_{L^2(\Sigma)}| \\ &\qquad\qquad + (\varepsilon_1 \cdots \varepsilon_m)^{-1} |\langle v_0, \Box_g (\widetilde{R}_1 - \widetilde{R}_2) \rangle_{L^2([0,T]\times\Omega)}| \\ &\le 2^m \delta (\varepsilon_1 \cdots \varepsilon_m)^{-1} \|v_0\|_{\widetilde{H}^{-r}(\Sigma)} + (\varepsilon_1 \cdots \varepsilon_m)^{-1} \|\Box_g (\widetilde{R}_1 - \widetilde{R}_2)\|_{H^{s+1}([0,T]\times\Omega)} \|v_0\|_{\widetilde{H}^{-(s+1)}([0,T]\times\Omega)} \\ &\le 2^m \delta (\varepsilon_1 \cdots \varepsilon_m)^{-1} \|v_0\|_{\widetilde{H}^{-r}(\Sigma)} + C_{s+1} (\varepsilon_1 \cdots \varepsilon_m)^{-1} \|\Box_g (\widetilde{R}_1 - \widetilde{R}_2)\|_{E^{s+1}} \|v_0\|_{\widetilde{H}^{-(s+1)}([0,T]\times\Omega)} \\ &\le C_{m,s+1} (\varepsilon_1 \cdots \varepsilon_m)^{-1} \big(\|v_0\|_{\widetilde{H}^{-r}(\Sigma)} + \|v_0\|_{\widetilde{H}^{-(s+1)}([0,T]\times\Omega)} \big) \\ &\qquad\qquad \times \left[2^m \delta + C(s,T)(\|q_1\|^2_{E^{s+1}} + \|q_2\|^2_{E^{s+1}}) \left(\sum_{j=1}^m \varepsilon_j \|f_j\|_{H^{s+1}(\Sigma)} \right)^{2m-1} \right] \\ &\le C\, (\varepsilon_1 \cdots \varepsilon_m)^{-1} \left[\delta + \left(\sum_{j=1}^m \varepsilon_j \|f_j\|_{H^{s+1}(\Sigma)} \right)^{2m-1} \right]. \end{aligned} \tag{62}$$

Here we used the assumption $\|\Lambda_1(f) - \Lambda_2(f)\|_{H^r(\Sigma)} \le \delta$ for $f = \varepsilon_1 f_1 + \cdots + \varepsilon_m f_m$. We also used that the norm in $H^{s+1}([0, T] \times \Omega)$ is bounded by the norm in E^{s+1} up to a multiplicative factor C_{s+1} as noticed in Remark 7. The final constant C is given by

$$C = \max\{C_{m,s+1}, C(s,T)(\|q_1\|^2_{E^{s+1}} + \|q_2\|^2_{E^{s+1}})\} \big(\|v_0\|_{\widetilde{H}^{-r}(\Sigma)} + \|v_0\|_{\widetilde{H}^{-(s+1)}([0,T]\times\Omega)} \big).$$

Here we have respectively denoted by $\widetilde{H}^{-r}(\Sigma)$ and $\widetilde{H}^{-(s+1)}([0, T] \times \Omega)$ the dual spaces of $H^r(\Sigma)$ and $H^{s+1}([0, T] \times \Omega)$.

5.2. ***Step 2*: *approximation of a delta distribution by a product of Gaussian beams.*** Recall that $(v_j)_{j=1}^m$ is a family solutions to $\Box_g v_j = 0$ as in (18). The second step of the proof of Theorem 1 is to choose the solutions v_j so that they allow us to obtain information about $q_1 - q_2$ on the left-hand side of the estimate (62). The boundary values corresponding to v_j will be denoted by f_j. We use the Gaussian beam construction of Section 3 to produce approximate delta functions from products of Gaussian beams. We shall need the following elementary results. Let $f : \mathbb{R}^d \to \mathbb{C}$ be a Lipschitz function. We define the Lipschitz seminorm of f as

$$\|f\|_{\mathrm{Lip}} := \inf\{c \geq 0 \mid |f(x) - f(y)| \leq c|x - y|\}.$$

Lemma 20. *Let $d \in \mathbb{N}$, $\tau > 0$ and b be Lipschitz. The estimate*

$$\left| b(z_0) - \left(\frac{\tau}{\pi}\right)^{\frac{d}{2}} \int_{\mathbb{R}^d} b(z) e^{-\tau|z-z_0|^2}\, dz \right| \leq c_d \|b\|_{\mathrm{Lip}} \tau^{-\frac{1}{2}}$$

holds true for all $z_0 \in \mathbb{R}^d$. In particular, the integral on the left converges uniformly to $b(z_0)$ when $\tau \to \infty$. Here $c_d := \Gamma\big(\frac{d+1}{2}\big)/\Gamma\big(\frac{d}{2}\big)$.

We omit the proof of Lemma 20 as it can be proved similarly to the following more general result:

Lemma 21. *Let $\tau > 0$, $x \in \mathbb{R}^d_+$, $d \geq 2$, and assume $x = (x_1, \ldots, x_d)$, where $x_1 \geq 0$. Let $b : \mathbb{R}^d_+ \to \mathbb{C}$ be Lipschitz. Define a map $\Phi : (-\infty, 0] \to \big[\frac{1}{2}, 1\big]$ by*

$$\Phi(s) := \frac{1}{\sqrt{\pi}} \int_s^\infty e^{-t^2}\, dt. \tag{63}$$

The estimate

$$\left| b(x) - \frac{1}{\Phi(-\sqrt{\tau} x_1)} \left(\frac{\tau}{\pi}\right)^{\frac{d}{2}} \int_{\mathbb{R}^d \cap \{z_1 \geq 0\}} b(z) e^{-\tau|z-x|^2}\, dz \right| \leq 2c_d \|b\|_{\mathrm{Lip}} \tau^{-\frac{1}{2}}$$

holds true for all $x \in \mathbb{R}^d \cap \{x_1 \geq 0\}$. In particular, the integral on the left converges uniformly to b as $\tau \to \infty$. Here $c_d = \Gamma\big(\frac{d+1}{2}\big)/\Gamma\big(\frac{d}{2}\big)$.

Proof. Let us write $x = (x_1, x')$ and assume without loss of generality that $x' = 0$. To begin, recall the identities

$$\int_{\mathbb{R}^d} e^{-|z|^2}\, dz = \pi^{\frac{d}{2}} \quad \text{and} \quad \int_{\mathbb{R}^d} |z|\, e^{-|z|^2}\, dz = c_d \pi^{\frac{d}{2}}.$$

Note also that

$$\int_{\mathbb{R}^d \cap \{z_1 \geq 0\}} e^{-\tau|z-x|^2}\, dz = \int_0^\infty \int_{\mathbb{R}^{d-1}} e^{-\tau((s-x_1)^2 + |z'|^2)}\, dz'\, ds = \left(\frac{\pi}{\tau}\right)^{\frac{d}{2}} \Phi(-\sqrt{\tau} x_1).$$

Since b is Lipschitz in $\mathbb{R}^d_+$, we have

$$|b(\tau^{-\frac{1}{2}} s + x_1, \tau^{-\frac{1}{2}} z') - b(x_1, 0)| \leq \|b\|_{\mathrm{Lip}} \tau^{-\frac{1}{2}} |(s, z')|$$

for any $x_1 \geq 0$ and$(s, z') \in \mathbb{R}^d$, $s > -\sqrt{\tau}x_1$. Thus we may calculate

$$\begin{aligned}
\left|\Phi(-\sqrt{\tau}x_1)b(x_1,0) - \left(\frac{\tau}{\pi}\right)^{\frac{d}{2}} \int_{\mathbb{R}^d \cap \{z_1 \geq 0\}} b(z)e^{-\tau|z-x|^2}\,dz\right| \\
&= \left(\frac{\tau}{\pi}\right)^{\frac{d}{2}} \int_{\mathbb{R}^d \cap \{z_1 \geq 0\}} (b(x_1,0) - b(z))e^{-\tau|z-x|^2}\,dz \\
&= \pi^{-\frac{d}{2}} \int_{-\sqrt{\tau}x_1}^{\infty} \int_{\mathbb{R}^{d-1}} (b(x_1,0) - b(\tau^{-\frac{1}{2}}s + x_1, \tau^{-\frac{1}{2}}z'))e^{-|z|^2}\,dz'\,ds \\
&\leq \|b\|_{\mathrm{Lip}}\pi^{-\frac{d}{2}} \int_{-\sqrt{\tau}x_1}^{\infty} \int_{\mathbb{R}^{d-1}} |(s,z')|e^{-|z|^2}\,dz'\,ds \\
&\leq \|b\|_{\mathrm{Lip}}\tau^{-\frac{1}{2}}\pi^{-\frac{d}{2}} \int_{\mathbb{R}^d} |z|e^{-|z|^2}\,dz = c_d\|b\|_{\mathrm{Lip}}\tau^{-\frac{1}{2}}.
\end{aligned}$$

Finally, dividing the above inequality by $\Phi(-\sqrt{\tau}x_1)$, and observing that Φ is monotone and satisfies $\Phi(0) = \frac{1}{2}$ and $\Phi(s) \to 1$ as $s \to -\infty$, we have the claim. □

We will apply Lemmas 20 and 21 with $d = n + 1$ and the function b will be a multiple of $q_1 - q_2$. Lemma 20 will be applied for recovery of points that lie in $W \setminus \Sigma$ and Lemma 21 for recovery of points on Σ.

To achieve the factor $\tau^{d/2} = \tau^{(n+1)/2}$ appearing in Lemmas 20 and 21, we use the solutions of Corollary 11 with $p = 4$ and scale them by a constant $\tau^{1/8}$. This change amounts to scaling the boundary values f_j by $\tau^{1/8}$. The estimates (41) and (44) still hold by taking k, l, K and N large enough. Moreover, when applying Lemma 21, we modify the functions of Corollary 11 by multiplying them by $\Phi(-\sqrt{\tau}x_1)$ with a suitable number $x_1 \geq 0$.

Recall that Gaussian beams concentrate on light-like geodesics. We show that at the intersection points of geodesics, the corresponding product of Gaussian beams approximates the delta function of the intersection point. Taking this approach, one can recover information about the difference of the unknown potentials q_1 and q_2 at points where the geodesics intersect. When the geodesics intersect only once, the proof is simpler and instructive. For this reason, we first analyze the case where the geodesics intersect only once and prove the general case after that.

5.3. ***Proof in the case of a single intersection point.*** Let $p_0 \in W$, where W is as in (3). In this case $p_0 \in I^+(\Sigma)$ by assumption and by Lemma 15 there is a future-directed optimal geodesic γ_1 from Σ to p_0 that does not intersect $\{t = 0\}$. By making a small perturbation of γ_1, we have another geodesic γ_2 that intersects γ_1 at p_0 and does not intersect $\{t = 0\}$. Since the geodesics are causal, they exit the compact set $[0, T] \times \Omega$ in finite parameter time. By the assumption of this simplified case, γ_1 and γ_2 intersect only at p_0. Let $\delta' > 0$ be small parameter so that the Fermi coordinates (26), associated to γ_1 and γ_2, are defined for $|y| < \delta'$.

By Proposition 10 and Corollary 11 there is $\tau_0 > 0$ such that, for $j = 1, 2$ and $\tau \geq \tau_0$, we may choose

$$v_j = \tau^{\frac{1}{8}}(v_{\tau,j} + r_j) \quad \text{and} \quad f_j = v_j|_\Sigma, \quad j = 1, 2, \tag{64}$$

so that $\Box_g(v_{\tau,j}+r_j)=0$ in $[0,T]\times\Omega$. Here the function $v_{\tau,j}$ stands for the Gaussian beam described in Section 3 corresponding to the geodesic γ_j. We also have that the correction term r_j satisfies

$$r_j|_\Sigma = 0, \quad j=1,2. \tag{65}$$

By (34) and (35) and Proposition 10 applied with $p=4$, we have for $\tau\geq\tau_0$

$$\begin{aligned} v_{\tau,j}(s,y) &= \tau^{\frac{n}{8}} e^{i\tau\Theta_j(s,y)} a^{(j)}(s,y), && \tau\geq\tau_0,\\ a^{(j)}(s,y) &= \chi\left(\frac{|y|}{\delta'}\right)\sum_{k'=0}^{N}\tau^{-k'} b_{k'}^{(j)}(s,y), && \tau\geq\tau_0,\\ b_{k'}^{(j)}(s,y) &= \sum_{k''=0}^{N} b_{k',k''}^{(j)}(s,y), \end{aligned} \tag{66}$$

where $b_{k',k''}^{(j)}(s,y)$ is a family of complex-valued homogeneous polynomials of order k'' in the variable y. We emphasize that all functions on the right-hand sides of (66) are independent of τ. Thanks to Proposition 10, see also (36) and (37), we also have

$$b_{k'}^{(j)}(0,0) = b_{0,0}^{(j)}(0,0) = 1, \quad j=1,2. \tag{67}$$

In addition, by (40), (41) and (44), we get for $j=1,2$ and $k>l+\frac{1}{2}(n-1)$

$$\begin{aligned} \|v_{\tau,j}\|_{H^l([0,T]\times\Omega)} &= O(\tau^{-\frac{n}{8}+l}), && \tau\geq\tau_0,\\ \|r_j\|_{H^l([0,T]\times\Omega)} &= O(\tau^{-K}), && \tau\geq\tau_0, \end{aligned} \tag{68}$$

if N satisfies $K=\frac{1}{2}(N+1-k)-1$. (If N defined this way is not an integer, we redefine it as $\lfloor N+1\rfloor$.) We imposed the condition $k>l+\frac{1}{2}(n-1)$ to embed the energy space E^l into $H^k([0,T]\times\Omega)$; see Remark 7. This condition is needed to control certain Sobolev norms in the following computations. Furthermore, by (41) and assuming that $l>\frac{1}{4}(n+1)$ (to embed $H^l([0,T]\times\Omega)$ into $L^4([0,T]\times\Omega)$) we get

$$\begin{aligned} \|v_{\tau,j}\|_{L^4([0,T]\times\Omega)} &= O(1), && j=1,2,\ \tau\geq\tau_0,\\ \|r_j\|_{L^4([0,T]\times\Omega)} &= O(\tau^{-K}), && j=1,2,\ \tau\geq\tau_0. \end{aligned} \tag{69}$$

Since $\Box_g$ is a linear operator, the complex conjugates of v_1 and v_2, denoted by $\bar v_1$ and $\bar v_2$, also solve $\Box_g v=0$. We set

$$v_j := \bar v_{j-2} \quad \text{and} \quad f_j := v_j|_\Sigma, \quad j=3,4.$$

Combining the trace theorem with (65) and (68) in the case $l=s+\frac{3}{2}$, we obtain an estimate for the boundary values f_j for $j=1,2,3,4$ and $\tau\geq\tau_0$, as

$$\begin{aligned} \|f_j\|_{H^{s+1}(\Sigma)} = \|v_j|_\Sigma\|_{H^{s+1}(\Sigma)} &= \tau^{\frac{1}{8}}\|(v_{\tau,j}+r_j)|_\Sigma\|_{H^{s+1}(\Sigma)}\\ &\leq \tau^{\frac{1}{8}}\|v_{\tau,j}\|_{H^{s+3/2}([0,T]\times\Omega)} \leq C\tau^{s-\frac{n}{8}+\frac{13}{8}}. \end{aligned} \tag{70}$$

For $j=5,\dots,m$, we choose Gaussian beams at fixed $\tau=\tau_0$ as

$$v_j = \tau_0^{-\frac{n+1}{8}} v_1|_{\tau=\tau_0} \quad \text{and} \quad f_j = v_j|_\Sigma, \quad j=5,\dots,m. \tag{71}$$

Let us write

$$\hat v = v_5\cdots v_m.$$

Remark 22. We remark that by making $\tau_0 > 0$ large enough, there exists $c > 0$ such that

$$|\hat{v}(s, y)| > c \tag{72}$$

in a neighborhood of $(s, y) = (0, 0)$. Indeed, by taking $l > \frac{n+1}{2}$ and combining Morrey's inequality with (68), we deduce that both $v_{\tau,1}$ and r_1 are continuous functions for $\tau \geq \tau_0$. In particular, the function v_1 is continuous according to (64). Proposition 10 ensures that $\Theta_1(0, 0) = 0$ and $b^{(1)}_{0,0}(0, 0) = 1$. Looking at (66) one has

$$a_1(0, 0) = 1 + O(\tau^{-1}), \quad \tau \geq \tau_0.$$

Hence

$$\tau^{-\frac{n+1}{8}} v_1(0, 0) = 1 + \tau^{-\frac{n}{8}} r_1(0, 0) = 1 + O(\tau^{-\frac{n}{8}}), \quad \tau \geq \tau_0,$$

where in the last equality, we have used (68) to deduce $\|r_1\|_{L^\infty([0,T]\times\Omega)} = O(1)$. Thus we have, by redefining τ_0 if necessary,

$$|\hat{v}(0, 0)| = (\tau^{-\frac{n+1}{8}} |v_1(0, 0)|)^{m-4} > \tfrac{1}{2}$$

for all $\tau \geq \tau_0$. By the continuity of $\hat{v}$, we have (72) on a neighborhood of $(0, 0)$.

With these choices, we now analyze the left-hand side of (62). We decompose the product $v_1 \cdots v_m$ as the sum of a leading term plus lower-order terms. A straightforward computation holding for $\tau \geq \tau_0$ yields

$$\begin{aligned} v_1 \cdots v_m &= |v_1|^2 |v_2|^2 \hat{v} \\ &= \tau^{\frac{1}{2}} |v_{\tau,1} + r_1|^2 |v_{\tau,2} + r_2|^2 \hat{v} \\ &= \tau^{\frac{1}{2}} (|v_{\tau,1}|^2 + v_{\tau,1}\bar{r}_1 + r_1 \bar{v}_{\tau,1} + |r_1|^2)(|v_{\tau,2}|^2 + v_{\tau,2}\bar{r}_2 + r_2 \bar{v}_{\tau,2} + |r_2|^2)\hat{v} \\ &= \tau^{\frac{1}{2}} |v_{\tau,1}|^2 |v_{\tau,2}|^2 \hat{v} + \mathcal{L}_1, \end{aligned} \tag{73}$$

where $\mathcal{L}_1$ is a sum of products of terms each containing r_1 or r_2, or their complex conjugates, as well as $\hat{v}$ as a factor. Consequently, we can choose (N, k, l, K) in (68) so that together with the Cauchy–Schwarz inequality, we obtain

$$\|\mathcal{L}_1\|_{L^1([0,T]\times\Omega)} = O(\tau^{-R}) \tag{74}$$

for some $R > 1$. Indeed, let us analyze one term of $\mathcal{L}_1$, say $\tau^{1/2} v_{\tau,1} |v_{\tau,2}|^2 \bar{r}_1 \hat{v}$. As $\hat{v}$ is continuous, it is bounded in $[0, T] \times \Omega$. Using (69), we have for $\tau \geq \tau_0$

$$\begin{aligned} \tau^{\frac{1}{2}} \|v_{\tau,1} |v_{\tau,2}|^2 \bar{r}_1 \hat{v}\|_{L^1([0,T]\times\Omega)} &\lesssim \tau^{\frac{1}{2}} \|v_{\tau,1} |v_{\tau,2}|^2 \bar{r}_1\|_{L^1([0,T]\times\Omega)} \\ &\lesssim \tau^{\frac{1}{2}} \|v_{\tau,1}\|_{L^4([0,T]\times\Omega)} \|v_{\tau,2}\|^2_{L^4([0,T]\times\Omega)} \|r_1\|_{L^4([0,T]\times\Omega)} = O(\tau^{\frac{1}{2}-K}). \end{aligned}$$

A similar analysis allows us to deduce that the $L^1([0, T] \times \Omega)$ norms of the other terms of $\mathcal{L}_1$ are $O(\tau^{1/2-K})$. Therefore

$$\|\mathcal{L}_1\|_{L^1([0,T]\times\Omega)} = O(\tau^{\frac{1}{2}-K}), \quad \tau \geq \tau_0.$$

Thus we can take $R = K - \frac{1}{2}$ in (74). Note that we can always find suitable parameters l, k, N and K satisfying $K = \frac{N+1-k}{2} - 1$, $k > l + \frac{n-1}{2}$ and $l > \frac{n+1}{2} > \frac{n+1}{4}$. One possible choice is

$$l = n+1, \quad k = 3n+1, \quad K = 2, \quad N = 3(n+1).$$

Let us now analyze the leading term in the expansion (73):

$$\tau^{\frac{1}{2}} |v_{\tau,1}|^2 |v_{\tau,2}|^2 \hat{v} = \tau^{\frac{n+1}{2}} e^{i\tau\Theta_1(x)} e^{-i\tau\overline{\Theta}_1(x)} e^{i\tau\Theta_2(x)} e^{-i\tau\overline{\Theta}_2(x)} |a^{(1)}(x)|^2 |a^{(2)}(x)|^2 \hat{v}(x).$$

For technical convenience, we consider a normal coordinate system $(x^a)_{a=0}^n$ centered at the point p_0, which is the unique intersection of the geodesics γ_1 and γ_2. At the center of the normal coordinates the metric is the identity matrix and all Christoffel symbols vanish; see, e.g., [O'Neill 1983, Section 3]. At the point p_0 both the phase functions Θ_1 and Θ_2 vanish and their gradients are real. Using the properties (42), we have the following Taylor expansion around p_0:

$$\Theta_1(x) - \overline{\Theta}_1(x) + \Theta_2(x) - \overline{\Theta}_2(x) = 2i\, x \cdot \nabla^2 \operatorname{Im}(\Theta_1 + \Theta_2)|_{x=0} x + O(|x|^3).$$

Here $\nabla^2 \operatorname{Im}(\Theta_1 + \Theta_2)$ is a positive definite matrix at p_0 (i.e., at $x = 0$ in normal coordinates) by the last two conditions of (42), because Θ_1 and Θ_2 are positive semidefinite and positive definite in directions transversal to $\dot{\gamma}_1$ and $\dot{\gamma}_2$ respectively.

Recall from (66) that the amplitude $a^{(j)}$, $j = 1, 2$, has the cut-off function χ as a factor. Therefore, we may redefine $\delta' > 0$ smaller, if necessary, so that at the intersection $U_1 \cap U_2$ of the supports

$$U_j := \operatorname{supp}(a^{(j)}) = \operatorname{supp}(v_{j,\tau})$$

we have $\operatorname{Im}(\Theta_1 + \Theta_2) > 0$. Let us write

$$\mathcal{H} := 2\nabla^2 \operatorname{Im}(\Theta_1 + \Theta_2)|_{x=0} > 0 \tag{75}$$

so that in the normal coordinates

$$\Theta_1(x) - \overline{\Theta}_1(x) + \Theta_2(x) - \overline{\Theta}_2(x) = i x \cdot \mathcal{H} x + \widehat{\Theta}(x), \tag{76}$$

where $\widehat{\Theta}(x) = O(|x|^3)$. Using the precise expressions in (66) for $a^{(j)}$, $j = 1, 2$, we see that

$$|a^{(1)}(x)|^2 \, |a^{(2)}(x)|^2 = |b_0^{(1)}(x)|^2 \, |b_0^{(2)}(x)|^2 + \tau^{-1} \mathcal{L}_2(x),$$

where

$$\|\mathcal{L}_2\|_{L^1([0,T]\times\Omega)} = O(1). \tag{77}$$

Via a calculation similar to the one done in deriving (73), we deduce in the coordinates $(x^a)_{a=0}^n$ that

$$\begin{aligned}\tau^{\frac{1}{2}} |v_{\tau,1}|^2 |v_{\tau,2}|^2 \hat{v} = \tau^{\frac{n+1}{2}} |\chi_1(x)|^2 |\chi_2(x)|^2 |b_0^{(1)}(x)|^2 |b_0^{(2)}(x)|^2 \hat{v}(x) e^{i\tau\widehat{\Theta}(x)} e^{-\tau x\cdot\mathcal{H}x} \\ + \underbrace{\tau^{-1} \tau^{\frac{n+1}{2}} |\chi_1(x)|^2 |\chi_2(x)|^2 \hat{v}(x) e^{i\tau\widehat{\Theta}(x)} e^{-\tau x\cdot\mathcal{H}x} \mathcal{L}_2(x)}_{:=\widehat{\mathcal{L}}_2(x)}.\end{aligned} \tag{78}$$

Here the functions χ_j, $j = 1, 2$, stand for the normal coordinate representations of χ_j, which in Fermi coordinates (s, y) corresponding to the geodesics γ_j take the form $\chi(|y|/\delta')$. Note that $\chi_j(0) = 1$. Recall that $\widehat{\Theta}(x) = O(|x|^3)$. By using (77), making the change of variables $x \mapsto \tau^{-1/2}x$ and using the fact $\tau\widehat{\Theta}(\tau^{-1/2}x) = \tau^{-1/2}O(|x|^3) = O(|x|^3)$ one calculates that

$$\|\widehat{\mathcal{L}}_2\|_{L^1([0,T]\times\Omega)} = O(\tau^{-1}). \tag{79}$$

(See (84) below for a similar calculation.)

For the sake of brevity, we set

$$q(x) = q_1(x) - q_2(x), \quad A(x) = |\chi_1(x)|^2|\chi_2(x)|^2|b_0^{(1)}(x)|^2|b_0^{(2)}(x)|^2\hat{v}(x). \tag{80}$$

By Proposition 10, see also (67), we have in the normal coordinates that $\Theta_j(0) = 0$ and $b_0^{(j)}(0) = 1$, $j = 1, 2$. Note also that $\widehat{\Theta}(0) = 0$. Thus one gets

$$A(0) = \hat{v}(0). \tag{81}$$

Integrating in the normal coordinates, and combining (73) and (78), we find

$$\begin{aligned}
&\int_{[0,T]\times\Omega} v_0(q_1 - q_2)v_1 \cdots v_m \, dV_g \\
&\quad = \tau^{\frac{n+1}{2}} \int_{B(p_0)} v_0(x)q(x)A(x)e^{i\tau\widehat{\Theta}(x)}e^{-\tau x\cdot\mathcal{H}x}\,dx + \int_{B(p_0)} v_0(x)q(x)(\mathcal{L}_1(x) + \widehat{\mathcal{L}}_2(x))\,dx \\
&\quad = \tau^{\frac{n+1}{2}} \int_{B(p_0)} v_0(x)q(x)A(x)e^{-\tau x\cdot\mathcal{H}x}\,dx + \int_{B(p_0)} v_0(x)q(x)(\mathcal{L}_1(x) + \widehat{\mathcal{L}}_2(x))dx \\
&\qquad\qquad + \tau^{\frac{n+1}{2}} \int_{B(p_0)} v_0(x)q(x)A(x)(e^{i\tau\widehat{\Theta}(x)} - 1)e^{-\tau x\cdot\mathcal{H}x}\,dx.
\end{aligned} \tag{82}$$

(Recall that v_0 is a function satisfying $\square_g v_0 = 0$ with $v_0|_{t=T} = \partial_t v_0|_{t=T} = 0$ in Ω.)

With slight abuse of notation, there are now two possible cases in the integral (82).

Case 1: If $U_1 \cap U_2 \cap \Sigma = \varnothing$, then $B(p_0)$ is a ball in $\mathbb{R}^{n+1}$ centered at p_0 such that $U_1 \cap U_2 \subset B(p_0)$ and we can proceed without changes.

Case 2: If $U_1 \cap U_2 \cap \Sigma \neq \varnothing$, then $B(p_0)$ is a ball in $\mathbb{R}^{n+1}_+$ centered at p_0 such that $U_1 \cap U_2 \subset B(p_0)$. In this case, we can similarly derive the identity (82) in boundary normal coordinates. As can be seen from Lemma 21, to obtain a proper normalization, we scale by the constant $1/\Phi(-\sqrt{\tau}x_1)$. This is achieved by multiplying the function v_0 by $1/\Phi(-\sqrt{\tau}x_1)$. Since $\Phi : (-\infty, 0] \to \left[\frac{1}{2}, 1\right]$, this scaling will contribute to redefining the constant of the stability estimate by a factor of at most 2. Here $\Phi(s) := \pi^{-1/2}\int_s^\infty e^{-t^2}\,\mathrm{d}t$ is as in (63) and x_1 denotes the first coordinate of p_0 in local coordinates of $\mathbb{R}^{n+1}_+$.

We now analyze each term in (82) above. Thanks to (62), we can control the term on the left-hand side of (82) in terms of $\delta, \varepsilon_1, \ldots, \varepsilon_m$ and the size of f_j. The first term after the second equality in (82) contains information about $q_1 - q_2$ and will be analyzed last. At this point, the exponential function $e^{-\tau x\cdot\mathcal{H}x}$ will play a crucial role, as it will act as an approximate delta function. This is due to the fact that $\mathcal{H}$ is a positive definite matrix, see (75). By combining (74) and (79), and using the fact that both v_0 and q are uniformly bounded, we have for $\tau \geq \tau_0$ that

$$\left|\int_{B(p_0)} v_0(x)q(x)(\mathcal{L}_1(x) + \widehat{\mathcal{L}}_2(x))\,dx\right| \lesssim \tau^{-1}. \tag{83}$$

Making the change of variables $x \mapsto \tau^{-1/2}x$, we obtain

$$\left|\tau^{\frac{n+1}{2}} \int_{B(p_0)} v_0(x)q(x)A(x)(e^{i\tau\widehat{\Theta}(x)}-1)e^{-\tau x\cdot\mathcal{H}x}\,dx\right| \\ = \left|\int_{B(p_0)} (v_0qA)(\tau^{-\frac{1}{2}}x)(e^{i\tau\widehat{\Theta}(\tau^{-1/2}x)}-1)e^{-x\cdot\mathcal{H}x}\,dx\right| \lesssim \tau^{-\frac{1}{2}}. \quad (84)$$

In the last inequality we used that $|e^{z_1}-e^{z_2}| \le |z_1-z_2|e^{\max\{|z_1|,|z_2|\}}$ for all $z_1, z_2 \in \mathbb{C}$ and $\widehat{\Theta}(x) = O(|x|^3)$ to deduce that

$$|e^{i\tau\widehat{\Theta}(\tau^{-\frac{1}{2}}x)}-1| \le \tau^{-\frac{1}{2}}|x|^3e^{\tau^{-\frac{1}{2}}|x|^3}, \quad \tau \ge \tau_0.$$

We also used that the functions v_0qA, $e^{-x\cdot\mathcal{H}x}$, $|x|^3$ and $e^{\tau^{-1/2}|x|^3}$ are uniformly bounded in $B(p_0)$.

Let us then analyze the first term after the second equality in (82). Since $\mathcal{H}$ is positive definite, there exists another positive definite matrix B so that $B^2 = \mathcal{H}$. Making the change of variables $x \mapsto Bx$, we deduce that in Case 1, where $U_1 \cap U_2 \cap \Sigma = \varnothing$, we have

$$\int_{B(p_0)} v_0(x)q(x)A(x)e^{-\tau x\cdot\mathcal{H}x}\,dx = \int_{\mathbb{R}^{n+1}} v_0(Bz)q(Bz)A(Bz)|g(z)|^{\frac{1}{2}}\,|\det B|^{-1}\,e^{-\tau|z|^2}\,dz. \quad (85)$$

For convenience, we set

$$b(z) := q(Bz)A(Bz)|g(z)|^{\frac{1}{2}}|\det B|^{-1}.$$

By using (81), we see that in normal coordinates

$$b(0) = (q_1(0)-q_2(0))\hat{v}(0)|\det\mathcal{H}|^{-\frac{1}{2}}. \quad (86)$$

The identities (82) and (85), combined with estimates (83) and (84) yield

$$\left|\left(\frac{\tau}{\pi}\right)^{\frac{n+1}{2}}\int_{\mathbb{R}^{n+1}} v_0(z)b(z)e^{-\tau|z|^2}\,dz\right| \lesssim \tau^{-\frac{1}{2}} + \left|\int_{[0,T]\times\Omega} v_0(q_1-q_2)v_1\cdots v_m\,dV_g\right|.$$

Thanks to (62), the second term on the right can be controlled in terms of $\delta, \varepsilon_1, \dots, \varepsilon_m$ and sizes of the functions f_j. Thereby, applying Lemma 20 with $z_0 = 0$ and $d = n+1$, we get

$$\begin{aligned} |b(0)| &\le \left|b(0)-\left(\frac{\tau}{\pi}\right)^{\frac{n+1}{2}}\int_{\mathbb{R}^{n+1}} v_0(z)b(z)e^{-\tau|z|^2}\,dz\right| + \left|\left(\frac{\tau}{\pi}\right)^{\frac{n+1}{2}}\int_{\mathbb{R}^{n+1}} v_0(z)b(z)e^{-\tau|z|^2}\,dz\right| \\ &\lesssim c_{n+1}\|v_0b\|_{C^1}\tau^{-\frac{1}{2}}+\tau^{-\frac{1}{2}} \\ &\qquad + \left[\delta\varepsilon_1^{-1}\cdots\varepsilon_m^{-1}+\varepsilon_1^{-1}\cdots\varepsilon_m^{-1}(\varepsilon_1\|f_1\|_{H^{s+1}(\Sigma)}+\cdots+\varepsilon_m\|f_m\|_{H^{s+1}(\Sigma)})^{2m-1}\right] \\ &\lesssim \frac{C_{\Omega,m,T,q_j,\chi}M}{\kappa_0^{2m-1}}\left[2\tau^{-\frac{1}{2}}+\frac{\kappa_0^{2m-1}\delta}{mM}\varepsilon_1^{-1}\cdots\varepsilon_m^{-1}\right. \\ &\qquad \left. +\frac{1}{m-1}\varepsilon_1^{-1}\cdots\varepsilon_m^{-1}(\varepsilon_1\|f_1\|_{H^{s+1}(\Sigma)}+\cdots+\varepsilon_m\|f_m\|_{H^{s+1}(\Sigma)})^{2m-1}\right], \end{aligned} \quad (87)$$

where v_0 can be chosen so that in normal coordinates $v_0(0) = 1$. The above holds for any $M > 0$ and $\kappa_0 > 0$. In the last step, we scaled δ by $\kappa_0^{2m-1}/(mM)$. The coefficients 2 and $1/(m-1)$ in front of $\tau^{-1/2}$ and $\varepsilon_1^{-1}\cdots\varepsilon_m^{-1}$ in (87) were included to simplify formulas later on. We will determine the constants M and κ_0 later. Their role in obtaining a stability estimate will be clarified in Lemma 23 below.

In Case 2 we arrive at the same integral (82), but the integration is only over the half-space $\mathbb{R}^{n+1}_+$ and due to scaling of v_0 the integral is scaled by a constant $1/\Phi(-\sqrt{\tau}x_1))$. All other calculations after (82) remain similar, but one needs to apply Lemma 21 instead of Lemma 20 to obtain the estimate (87). We omit the details.

5.4. *Step 3*: *optimizing the error terms.* The last step of the proof of Theorem 1 (in this simplified setting) is to choose τ and $\varepsilon_1, \dots, \varepsilon_m$ in terms of δ to have the right-hand side of (87) as small as possible. We begin by setting

$$\varepsilon_1 = \cdots = \varepsilon_m =: \varepsilon.$$

Note that by (70) and (71), we have for $\tau \geq \tau_0$ that

$$\begin{aligned} \varepsilon\|f_j\|_{H^{s+1}(\Sigma)} &\sim \varepsilon\tau^{s-\frac{n}{8}+\frac{13}{8}}, \quad j = 1, 2, 3, 4, \ \tau \geq \tau_0, \\ \varepsilon\|f_j\|_{H^{s+1}(\Sigma)} &\sim \varepsilon\tau_0^{s-\frac{n}{4}+\frac{3}{2}}, \quad j = 5, \dots, m. \end{aligned} \tag{88}$$

To guarantee the unique solvability of our nonlinear wave equation (16), we require the quantities on the right-hand sides of (88) to be bounded by κ, which was given by Lemma 9. Recall that $\tau_0 > 0$ is a fixed large parameter, which we chose at (71). The parameter was especially chosen so that the Gaussian beams v_j for $j = 5, \dots, m$ have small enough correction terms.

Lemma 23 shows how to choose the parameters τ and ε in (87) optimally given $\kappa > 0$ and $\delta \in (0, M)$. By choosing $\kappa_0 \leq \kappa$, we will see that the optimal value for τ is at least τ_0 and we also have that $\varepsilon\|f_j\|_{H^{s+1}(\Sigma)} \leq \kappa$.

Lemma 23. *Let $C, M, s > 0$ and $m \in \mathbb{N}$. Let also $\tau_0 \geq 1$, $\delta \in (0, M)$ and $\kappa \in (0, 1)$. Then there are $\varepsilon > 0$, $\tau \geq \tau_0$ and $\kappa_0 \leq \kappa$ such that*

$$\begin{aligned} f(\varepsilon, \tau) &:= 2\tau^{-\frac{1}{2}} + \frac{\kappa_0^{2m-1}\delta}{mM}\varepsilon^{-m} + \frac{1}{m-1}\varepsilon^{m-1}\,\tau^{(2m-1)(s-\frac{n}{8}+\frac{13}{8})} \\ &\leq C_{s,m,M,\kappa_0}\,\delta^{\frac{8(m-1)}{2m(m-1)(8s-n+13)+2m-1}} \end{aligned}$$

and we also have

$$\varepsilon\tau^{s-\frac{n}{8}+\frac{13}{8}} \leq C\kappa.$$

Proof. To simplify the notation, let us write $\hat{s} := (2m-1)\big(s - \frac{n}{8} + \frac{13}{8}\big)$ and $\gamma_0 = \kappa_0^{2m-1}/M$. We take $\kappa_0 \leq \kappa$ to be so that $\gamma_0 < 1$. We will redefine $\kappa_0 > 0$ smaller later if necessary. A direct computation shows that

$$\partial_\varepsilon f = -(\gamma_0\delta)\varepsilon^{-m-1} + \varepsilon^{m-1}\tau^{\hat{s}}, \quad \partial_\tau f = -\tau^{-\frac{3}{2}} + \frac{\hat{s}}{m-1}\varepsilon^{m-1}\tau^{\hat{s}-1}.$$

Making $\partial_\varepsilon f = \partial_\tau f = 0$, we obtain the critical points of f, namely

$$\begin{aligned} \tau &= ((m-1)\hat{s}^{-1})^{\frac{2(2m-1)}{2\hat{s}m+2m-1}}(\gamma_0\delta)^{-\frac{2(m-1)}{2\hat{s}m+2m-1}}, \\ \varepsilon &= ((m-1)\hat{s}^{-1})^{-\frac{2\hat{s}}{2\hat{s}m+2m-1}}(\gamma_0\delta)^{\frac{4\hat{s}m+2m-1-2\hat{s}}{(2\hat{s}m+2m-1)(2m-1)}}. \end{aligned} \tag{89}$$

(One can also verify that the Hessian of f at the critical point is positive definite, and hence the critical point is a local minimum.)

Note now that

$$\begin{aligned}\tau &= ((m-1)\hat{s}^{-1})^{\frac{2(2m-1)}{2\hat{s}m+2m-1}}(\gamma_0\delta)^{-\frac{2(m-1)}{2\hat{s}m+2m-1}} \\ &\geq ((m-1)\hat{s}^{-1})^{\frac{2(2m-1)}{2\hat{s}m+2m-1}}\kappa_0^{-\frac{2(m-1)(2m-1)}{2\hat{s}m+2m-1}},\end{aligned}$$

because by assumption $0<\delta<M$ and since $\gamma_0=\kappa_0^{2m-1}/M$. Since the constant

$$((m-1)\hat{s}^{-1})^{\frac{2(2m-1)}{2\hat{s}m+2m-1}}>0$$

and the exponent

$$-\frac{2(m-1)(2m-1)}{2\hat{s}m+2m-1}<0$$

do not depend on κ_0, we may choose κ_0 so that $\kappa_0<C\kappa$ and that τ in (90) satisfies

$$\tau=((m-1)\hat{s}^{-1})^{\frac{2(2m-1)}{2\hat{s}m+2m-1}}(\gamma_0\delta)^{-\frac{2(m-1)}{2\hat{s}m+2m-1}}\geq\tau_0.$$

With these choices, we have at the critical point of $f(\varepsilon,\tau)$ given by (89)

$$\varepsilon\tau^{s-\frac{n}{8}+\frac{13}{8}}=\varepsilon\tau^{\frac{\hat{s}}{2m-1}}=(\gamma_0\delta)^{\frac{1}{(2m-1)}}=\left(\frac{\kappa_0^{2m-1}}{M}\delta\right)^{\frac{1}{2m-1}}\leq\kappa_0<C\kappa$$

for all $0<\delta<M$. A straightforward calculation using (89) shows that $\tau^{-1/2}$, $(\gamma_0\delta)\varepsilon^{-m}$ and $\varepsilon^{m-1}\tau^{\hat{s}}$ are all bounded by $C_{s,m,M,\kappa_0}(\gamma_0\delta)^{(m-1)/(2\hat{s}m+2m-1)}$, where the constant C_{s,m,M,κ_0} is independent of ε and τ. □

Recall (86) and (87). We set $\varepsilon_1=\cdots=\varepsilon_m=:\varepsilon$ and apply Lemma 23 to obtain

$$\begin{aligned}&|v_0(p_0)||q_1(p_0)-q_2(p_0)||\hat{v}(p_0)||\det\mathcal{H}|^{-\frac{1}{2}} \\ &\qquad\lesssim\frac{C_{\Omega,T,q_j,\chi}M}{\kappa_0^{2m-1}}\left(2\tau^{-\frac{1}{2}}+\frac{\kappa_0^{2m-1}\delta}{mM}\varepsilon^{-m}+\frac{1}{m-1}\varepsilon^{m-1}\tau^{(2m-1)\left(s-\frac{n}{8}+\frac{13}{8}\right)}\right) \\ &\qquad\leq C_0\delta^{\frac{8(m-1)}{2m(m-1)(8s-n+13)+2m-1}}.\end{aligned}\tag{90}$$

Since $p_0\in I^-(\Sigma)\cap([0,T]\times\Omega)$, by Lemma 15 there exists a past-directed optimal geodesic from Σ to p_0 such that the first intersection of the geodesic and Σ is transverse. Since the intersection is transverse, the geodesic does not intersect $\{t=T\}$. Therefore, we may choose v_0 to be a Gaussian beam corresponding to the geodesic such that $v_0|_{t=T}=\partial_t v_0|_{t=T}=0$. We may assume by normalizing that $v_0(p_0)=1$. Recall also that $\hat{v}(p_0)>c>0$ and $|\det\mathcal{H}|>0$ by (72) and (75) respectively. Dividing (90) by the norm of $v_0(p_0)\hat{v}(p_0)|\det\mathcal{H}|^{-1/2}$, we have a stability estimate

$$|q_1(p_0)-q_2(p_0)|\leq C\delta^{\frac{8(m-1)}{2m(m-1)(8s-n+13)+2m-1}}\tag{91}$$

at the point p_0. We next show that the constant C can be redefined to be independent of p_0.

5.5. ***Step 4*: *uniformity of the constant C*.** So far we have obtained the estimate (91) regarding the difference of q_1 and q_2 at the single point p_0. The constant C may at this point depend on p_0. Next we argue that the constant C can be redefined to be independent of p_0. This will then yield (4) and conclude the proof of Theorem 1 in the simplified setting, where we assumed that light-like geodesics can intersect only once.

To show that C in (91) can be taken to be independent of p_0, we first construct an open cover of $W \subset I^+(\Sigma) \cap I^-(\Sigma)$ as follows. (Recall from (3) that W is a compact set which we can reach and observe from Σ.) Let $z \in W$. By Lemma 15 there are optimal light-like geodesics γ_1 and γ_2 that intersect at z and which do not intersect $\{t = 0\}$. We may reparametrize so that $\gamma_1(0) = \gamma_2(0) = z$. Let $\varepsilon = |\dot{\gamma}_1(0) - \dot{\gamma}_2(0)|$. Here and below $|\cdot|$ denotes the $\mathbb{R}^n$ norm of vectors in local coordinates.

By Corollary 14 there are open neighborhoods $\mathcal{U}_1$ and $\mathcal{U}_2$ of z and families of Gaussian beams $v_\tau(x, l, \cdot)$ (including the correction term) parametrized by $x \in \mathcal{U}_l$, $l = 1, 2$, such that all the implied constants, such as τ_0, in the construction of $v_\tau(x, l, \cdot)$ are uniform in x. Moreover, still by using Corollary 14, the geodesics $\gamma_{x,l}$ corresponding to the Gaussian beams $v_\tau(x, l, \cdot)$ satisfy $|\dot{\gamma}_l(0) - \dot{\gamma}_{x,l}(0)| \le \frac{\varepsilon}{3}$, $l = 1, 2$. Then, for $x \in \mathcal{U}_1 \cap \mathcal{U}_2$, we also have that

$$|\dot{\gamma}_{x,1}(0) - \dot{\gamma}_{x,2}(0)| \ge \frac{\varepsilon}{3} > 0. \tag{92}$$

We conclude that the geodesics $\gamma_{x,1}$ and $\gamma_{x,2}$ intersect at x and do not have the same graph. We also set

$$\hat{v}_x(\cdot) = (v_\tau(x, l, \cdot))^{m-4}|_{\tau=\tau_0, l=1}$$

for $x \in \mathcal{U}_1 \cap \mathcal{U}_2$. By redefining τ_0 larger, if necessary, we have that $|\hat{v}_x(x)| \ge d > 0$ for all $x \in \mathcal{U}_1 \cap \mathcal{U}_2$.

In deriving (91) in this Section 5, we used normal coordinates. Normal coordinates are uniquely defined by choosing an orthonormal basis at a point. By using a local orthonormal frame on a neighborhood $\mathcal{U}_3$ of z, we may find a family of normal coordinates smoothly parametrized by $x \in \mathcal{U}_3$. It follows that the contribution to C in (91) coming from the use of normal coordinates may be taken to be uniformly bounded for all $x \in \mathcal{U}_3$. All things considered, by repeating the arguments in this Section 5, we may take the constant C to be uniform for all $x \in \mathcal{U}_1 \cap \mathcal{U}_2 \cap \mathcal{U}_3$, where $\mathcal{U}_1 \cap \mathcal{U}_2 \cap \mathcal{U}_3$ is a neighborhood of z.

Recall that we aim to estimate the difference of q_1 and q_2 in the compact set $W \subset I^+(\Sigma) \cap I^-(\Sigma)$. By covering first the compact set W by the sets $\mathcal{U}_1 \cap \mathcal{U}_2 \cap \mathcal{U}_3$ as described above and then passing to a finite subcover, we have that (91) holds for all $z \in W$. Finally, we apply Lemma 18 with $P = 1$ to deduce that there is a finite family of functions $v_{z,0}$ satisfying $\Box_g v_{z,0} = 0$ in $[0, T] \times \Omega$ and $v_{z,0}|_{t=T} = \partial_t v_{z,0}|_{t=T} = 0$ and such that $|v_{z,0}(z)| \ge c > 0$. (Only finitely many of the functions $v_{z,0}$ are actually distinct.) Combining everything yields the estimate

$$|(v_{z,0}(z)\hat{v}_z(z)(q_1 - q_2))(z)|\,|\det \mathcal{H}_z|^{-\frac{1}{2}} \le C\delta^{\frac{8(m-1)}{2m(m-1)(8s-n+13)+2m-1}}, \tag{93}$$

which holds for all $z \in W$. Here the point z corresponds to the origin 0 of normal coordinates centered at z and all the quantities are expressed in these coordinates. The point z is also the point where the geodesics $\gamma_{z,1}$ and $\gamma_{z,2}$ corresponding to the Gaussian beams $v_\tau(z, 1, \cdot)$ and $v_\tau(z, 2, \cdot)$ intersect.

By Remark 22, we have that $|v_{z,0}(z)| \ge c > 0$ and hence $|\hat{v}_z(z)| \ge d > 0$ in (93). Let us estimate $|\det \mathcal{H}_z|$, where

$$\mathcal{H}_z = 2\nabla^2 \operatorname{Im}(\Theta_{z,1}(x) + \Theta_{z,2}(x))|_{x=z}.$$

Here $\Theta_{z,1}$ and $\Theta_{z,2}$ are the phase functions corresponding to the Gaussian beams $v_\tau(z, 1, \cdot)$ and $v_\tau(z, 2, \cdot)$ respectively. Here also ∇^2 is the invariant Hessian. In the normal coordinates centered at z we have that the geodesics $\gamma_{z,1}$ and $\gamma_{z,2}$ are rays emanating in from origin. Since $\gamma_{z,1}$ and $\gamma_{z,2}$ do not have the same graphs, the rays are not same and there is a positive angle (in the $\mathbb{R}^{n+1}$ metric) between the rays in the normal coordinates. Due to (92), the angle is uniformly bounded from below by a positive constant. Consequently, using also the facts that

$$\operatorname{Im}(\nabla^2\Theta_{z,l})(z) \geq 0, \quad \operatorname{Im}(\nabla^2\Theta_{z,l})(z)|_{\dot{\gamma}_{z,l}(0)^\perp} > 0$$

we conclude that there is $h > 0$ such that $|\det \mathcal{H}_z| > h$ for all $z \in W$. Dividing (93) by $|v_{z,0}(z)|$, $|\hat{v}_x(x)|$ and $|\det \mathcal{H}_z|^{-1/2}$, and redefining C larger, if necessary, concludes the proof in the special case where we assumed that light-like geodesics can intersect only once.

5.6. ***Step 5*: *multiple intersections.*** We have proven Theorem 1 in the special case, which assumed that the used light-like geodesics intersect only once. In the case of multiple intersections, we can perform a similar analysis as in the special case, but this leads to an estimate for a sum of terms regarding the difference $q_1 - q_2$ at the intersection points. To separate the contributions coming from several intersection points, we will use separation matrices and a separation filter constructed in Lemmas 17 and 18. Most of the work needed to handle the case of several intersections was already done in proving these two lemmas.

Let N be globally hyperbolic Lorentzian manifold. Let also $\bar{g}$ be an auxiliary Riemannian metric on N. The following lemma shows that given a compact set $K \subset N$ there is a bound on the number of possible intersections of pairs of causal geodesics in K. We will apply the lemma with $K = [0, T] \times \Omega$ and $N = \mathbb{R} \times M$. Let us recall some relevant facts. An open set O of N is convex if for every pair of points $p, q \in O$ with $p \neq q$ there is a unique geodesic γ of O connecting the points. Each point in N has a neighborhood that is convex [O'Neill 1983, Section 5, Proposition 7]. Let $p \in N$ and let U_p be its convex neighborhood. By [O'Neill 1983, Section 14, Exercise 10] (see also [Minguzzi 2019]), and the fact that U_p is convex, it follows that p has a neighborhood $V_p \subset U_p$ with two properties:

(i) Any causal curve starting in V_p that leaves it never returns.

(ii) Two distinct geodesic segments in V_p can intersect at most once.

We mention that in [O'Neill 1983] the sets V_p are called causality neighborhoods. It follows from conditions (i) and (ii) that any two distinct causal geodesics can intersect at most once in V_p.

Lemma 24. *Let (N, g) be a globally hyperbolic Lorentzian manifold and let $K \subset N$ be a compact set. There is $P \geq 1$ with the following property. Let γ_1 and γ_2 be two distinct causal geodesics. Then the number of intersection points of γ_1 and γ_2 is bounded by P,*

$$\#(\Gamma_1 \cap \Gamma_2) \leq P,$$

where $\Gamma_j \subset N$ are the graphs of the geodesics γ_j, $j = 1, 2$.

Proof. Let γ_1 and γ_2 be as in the statement of the lemma. Because N is globally hyperbolic, every point $p \in N$ has a neighborhood V_p satisfying the conditions (i) and (ii). Because K is compact, there exists a finite subcover

$$\bigcup_{a=1}^{P} V_{p_a} \supset K$$

formed of sets V_{p_a}. Since a pair of distinct causal geodesics can intersect at most once within each V_{p_a}, it follows that the number of intersections of γ_1 and γ_2 is bounded by P. □

Lemma 25. *Let $\bar{g}$ be an auxiliary Riemannian metric on a globally hyperbolic Lorentzian manifold (N, g) and let $K \subset N$ be compact. Then there exists $\tilde{\rho} > 0$ such that for any pair of distinct causal geodesics γ_1 and γ_2 intersecting at points $x_1, \dots, x_P$ we have*

$$d_{\bar{g}}(x_j, x_k) \geq \tilde{\rho}, \quad j \neq k,$$

where $d_{\bar{g}}(x, y)$ is the distance induced by $\bar{g}$.

Proof. Let x_j and x_k, $x_j \neq x_k$, be intersection points of γ_1 and γ_2. Let $\{V_{p_a}\}_{a=1}^P$ be a finite open cover of K consisting of sets with properties (i) and (ii). Let $\tilde{\rho} > 0$ be a Lebesgue number (see, e.g., [Munkres 1975, Lemma 27.5]) of $\{V_{p_a}\}_{a=1}^P$ with respect to the distance $d_{\bar{g}}$. It follows that the ball $B_{\bar{g}}(x_j, \tilde{\rho})$ belongs to V_{p_a} for some $a \in \{1, \dots, P\}$. Since the geodesics γ_1 and γ_2 can intersect at most once in V_{p_a}, the point x_k cannot belong to V_{p_a}. Consequently, $x_k \notin B_{\bar{g}}(x_j, \tilde{\rho})$ and thus $d_{\bar{g}}(x_j, x_k) \geq \tilde{\rho}$ as claimed. □

By Lemma 24 we know that there is $P \in \mathbb{N}$ such that light-like geodesics can intersect at most P times in $[0, T] \times \Omega$. Let also $\bar{g}$ be an auxiliary Riemannian metric on $[0, T] \times \Omega$.

Let γ_1 and γ_2 be future-directed light-like geodesics starting from Σ that intersect for the first time at z and which do not intersect $\{t = 0\}$. Let

$$z_1, \dots, z_{P_0}$$

be the intersection points of γ_1 and γ_2 arranged as $z_1 \leq z_2 \leq \cdots \leq z_{P_0}$, where $P_0 \leq P$ and

$$z = z_1.$$

As in (64), we choose

$$v_j = \tau^{\frac{1}{8}}(v_{\tau,j} + r_j), \quad j = 1, 2,$$

to be Gaussian beams associated to γ_1 and γ_2. We also choose

$$v_j = \bar{v}_{j-2}, \quad j = 3, 4, \qquad \text{and} \qquad \hat{v} = (v_1|_{\tau=\tau_0})^{m-4}$$

as before. Since the product $v_1 \cdots v_m$ is supported on neighborhoods of the intersection points, the term

$$\langle v_0(q_1 - q_2), v_1 \cdots v_m \rangle_{L^2([0,T]\times\Omega)} = \int_{[0,T]\times\Omega} v_0(q_1 - q_2) v_1 \cdots v_m \, dV_g$$

becomes a sum of terms

$$\sum_{j=1}^{P_0} \tau^{\frac{n+1}{2}} \int_{B(z_j)} v_0(x)(q_1 - q_2)(x) A(x) e^{i\tau\widehat{\Theta}(x)} e^{-\tau x \cdot \mathcal{H}_{z_j} x} \, dV_g, \tag{94}$$

where each set $B(z_j)$ is a neighborhood of z_j, $j = 1, \dots, P_0$. Here $\widehat{\Theta}(x)$ and $A(x)$ are defined similarly to (76) and (80) respectively and

$$\mathcal{H}_{z_j} = 2\nabla^2 \operatorname{Im}(\Theta_1(x) + \Theta_2(x))|_{x=z_j}, \quad j = 1, \dots, P_0$$

as before.

By Lemma 25 there is a uniform constant $\tilde{\rho} > 0$ independent of $z_1, \dots, z_{P_0}$ such that $d_{\tilde{g}}(z_i, z_j) \geq \tilde{\rho}$ for all $i \neq j$. This implies we can use Lemma 18 to find a separation filter on $[0, T] \times \Omega$. So, let $\mathcal{M} = \{v_{f_k}\}_{k \in \mathcal{K}}$ be a separation filter of $[0, T] \times \Omega$ given by Lemma 18 with the compact set W as K and P_0 as P. Here $f_k \in C^\infty(\Sigma)$ and $\mathcal{K}$ is a finite index set. According to Lemma 18, the corresponding solutions v_{f_k} to $\Box_g v = 0$ in $[0, T] \times \Omega$ can be chosen so that the associated separation matrix $(v_{f_k}(z_j))_{k,j=1}^{P_0}$ is invertible.

We note that if $B(z_j) \cap \Sigma \neq \varnothing$ in (94), then the corresponding integrals can be taken over the half-space $\mathbb{R}^{n+1}_+$ in boundary normal coordinates. As indicated by Lemma 21 we need to use the scaling factor $1/\Phi(-\sqrt{\tau} z_{j,1})$ to recover the value of $q_1 - q_2$ at z_j. This can be achieved by scaling the functions v_{f_k} of the separation matrices by $1/\Phi(-\sqrt{\tau} z_{j,1})$. This amounts to scaling the matrix element of the upper triangular parts of each of the separation matrices by $1/\Phi(-\sqrt{\tau} z_{j,1})$ if $B(z_j) \cap \Sigma \neq \varnothing$. Here $z_{j,1}$ is the first coordinate of z_j in boundary normal coordinates. Recall from (63) that $\Phi : (-\infty, 0] \to \left[\frac{1}{2}, 1\right]$. Thus by choosing a larger τ_0, if necessary, the separation matrices with scaled elements stay invertible. Much as in Step 4, it is possible to make the choices of the boundary normal coordinates so that the choices amount to redefining the constant C.

By repeating the calculation in (62) we have for each $k \in \mathcal{K}$ that

$$|\langle v_{f_k}(q_1 - q_2), v_1 \cdots v_m \rangle_{L^2([0,T]\times\Omega)}| \leq C_k \, (\varepsilon_1 \cdots \varepsilon_m)^{-1} \left[\delta + \left(\sum_{j=1}^m \varepsilon_j \, \|f_j\|_{H^{s+1}(\Sigma)} \right)^{2m-1} \right].$$

We apply (94) with v_{f_k} in place of v_0 and note that the integrals in (94) are the value of the integrand at z_k plus a term of size $O(\tau^{-1/2})$ by calculations (75)–(87) and Lemmas 20 and 21. Optimizing as in Section 5.4 in τ and $\varepsilon_1, \dots, \varepsilon_m$ yields that

$$\left| \sum_{j=1}^{P_0} v_{f_k}(z_j)(q_1(z_j) - q_2(z_j)) \hat{v}(z_j) |\det \mathcal{H}_{z_j}|^{-\frac{1}{2}} \right| \leq C \delta^{\frac{8(m-1)}{2m(m-1)(8s-n+13)+2m-1}}$$

for all $k = 1, \dots, P_0$. Let us define a matrix A and a vector $\mathcal{Q}$ as

$$A_{kj} = v_{f_k}(z_j), \quad \mathcal{Q}_j = (q_1(z_j) - q_2(z_j)) \hat{v}(z_j) |\det \mathcal{H}_{z_j}|^{-\frac{1}{2}},$$

where $j, k = 1, \dots, P_0$. Since the separation matrix $\{v_{f_k(x_j)}\}_{k,j=1}^{P_0}$ is invertible, we have that

$$|\mathcal{Q}_1| \leq \|\mathcal{Q}\| = \|A^{-1}(A\mathcal{Q})\| \leq \|A\|^{-1} \|A\mathcal{Q}\| \leq \|A\|^{-1} C \delta^{\frac{8(m-1)}{2m(m-1)(8s-n+13)+2m-1}}.$$

Recalling that $z_1 = z$, we thus have

$$\left| (q_1(z) - q_2(z)) \hat{v}(z) |\det \mathcal{H}_z|^{-\frac{1}{2}} \right| \leq C \|A\|^{-1} \delta^{\frac{8(m-1)}{2m(m-1)(8s-n+13)+2m-1}}. \tag{95}$$

In (95), $\hat{v}_z$, $\det \mathcal{H}_z$, but also $\|A\|^{-1}$ depend on the point z. We argued in Section 5.5 that $\hat{v}_z$, $|\det \mathcal{H}_z|^{-1/2}$ have norms which are uniformly bounded from below with respect to z. Since the separation filter $\mathcal{M}$ is a finite collection, we may also bound $\|A\|^{-1}$ uniformly when we consider different points in W. Using these facts and by dividing by $|\hat{v}(z) \det \mathcal{H}_z|^{-1/2}$ and redefining C shows that

$$\|q_1 - q_2\|_{L^\infty(W)} \leq C \delta^{\frac{8(m-1)}{2m(m-1)(8s-n+13)+2m-1}}.$$

This concludes the proof of Theorem 1.

Appendix: Proof of Proposition 8

Before proceeding to the proof of Proposition 8, which concerns the well-posedness of the linear wave equation (14), we need the following lemma.

Lemma 26. *Let $(\mathbb{R}\times M, g)$ be a globally hyperbolic manifold. Let also $t_0 \in \mathbb{R}$ and let $S_{t_0} = \{t = t_0\} \times M$ be the corresponding Cauchy surface. Suppose $V \subset S_{t_0}$ is a compact set in S_{t_0} and W is an open neighborhood of V in $\mathbb{R}\times M$. Then there exists $\varepsilon > 0$ such that $([t_0, t_0+\varepsilon]\times M)\cap J^+(V) \subset W$. In particular if $V \Subset U$, where U is open in S_{t_0}, there exists $\varepsilon > 0$ such that $([t_0, t_0+\varepsilon] \times M) \cap J^+(V) \subset [t_0, t_0+\varepsilon]\times U$.*

Proof. For the first claim, assume that there is no such $\varepsilon > 0$. Then there are numbers $\varepsilon_k > 0$ with $\varepsilon_k \to 0$ as $k \to \infty$ and points $p_k \in ([t_0, t_0+\varepsilon_k] \times M) \cap J^+(V)$, but $p_k \notin W$. Since W is open, any accumulation points of p_k, if existing, are not in W. As $\varepsilon_k \to 0$ there is $\varepsilon \geq \varepsilon_k$ for all sufficiently large $k \in \mathbb{N}$, say, $k \geq k_0$. It follows that $p_k \in ([t_0, t_0+\varepsilon] \times M) \cap J^+(V)$ for all $k \geq k_0$.

Because $\mathbb{R} \times M$ is foliated by the space-like Cauchy surfaces S_t, we have

$$[t_0, t_0+\varepsilon] \times M = \bigcup_{t\in[t_0,t_0+\varepsilon]} S_t.$$

Also $S_t \subset J^-(S_T)$ for all $t \leq T$, because if γ is any nonextendible future-directed causal curve with $\gamma(s) \in S_t$ for some $s \in \mathbb{R}$, then this curve intersects S_T in the future. By [Bär et al. 2007, Corollary A.5.4], the intersection $J^-(S_{t_0+\varepsilon}) \cap J^+(V)$ is compact. So $[t_0, t_0+\varepsilon] \times M$ being a closed subset of $J^-(S_{t_0+\varepsilon})$ implies that $([t_0, t_0+\varepsilon] \times M) \cap J^+(V)$ is compact and there exists a convergent subsequence $p_{k_i} \to p \in ([t_0, t_0+\varepsilon]\times M)\cap J^+(V)$. Due to the construction, as $\varepsilon_{k_i} \to 0$ we have $p_{k_i} \to p \in \{t = t_0\}\times M \cap J^+(V) = V \subset W$. Thus $p \in W$, which is a contradiction.

Suppose now that $W = (a, b) \times U$ where $t_0 \in (a, b) \subset \mathbb{R}$. Then if $\varepsilon > 0$ is so small that $([t_0, t_0+\varepsilon] \times M) \cap J^+(V) \subset (a, b) \times U$, we have $([t_0, t_0+\varepsilon] \times M) \cap J^+(V) \subset [t_0, t_0+\varepsilon] \times U$. If not, we would have some $p = (t, x) \in ([t_0, t_0+\varepsilon] \times M) \cap J^+(V)$ with $t \notin [t_0, t_0+\varepsilon]$ or $x \notin U$. Both options are invalid, so also the second claim holds. □

Proof of Proposition 8. Let us first recall results in the special case where Ω is a domain $\Omega \subset \mathbb{R}^n$. From [Lasiecka et al. 1986] we know that there exists a unique solution $v \in E^{s+1}$ to the problem

$$\begin{cases} (\partial_t^2 - \Delta_h)v = F & \text{in } [0, T] \times \Omega, \\ v = f & \text{on } [0, T] \times \partial\Omega, \\ v = u_0, \ \ \partial_t v = u_1 & \text{in } \{t = 0\} \times \Omega, \end{cases} \tag{96}$$

if $h(t, \cdot)$ is a smooth 1-parameter family of Riemannian metrics on $\mathbb{R}^n$ and if we assume that F, f, u_0 and u_1 satisfy the regularity and compatibility conditions of our proposition in $\mathbb{R}^n$. Under these assumptions, we also know from classical results such as [Ikawa 1968] that there exists a unique solution $w \in E^{s+1}$ to

$$\begin{cases} (\partial_t^2 - \Delta_h)w + Aw = G & \text{in } [0, T] \times \Omega, \\ w = 0 & \text{on } [0, T] \times \partial\Omega, \\ w = \partial_t w = 0 & \text{in } \{t = 0\} \times \Omega \end{cases} \tag{97}$$

when $A \in C^\infty([0,T]\times\Omega)$ and $G \in E^s$. By combining the mentioned results, we have that the problem

$$\begin{cases}(\partial_t^2-\Delta_h)u + Au = F & \text{in } [0,T]\times\Omega,\\ u = f & \text{on } [0,T]\times\partial\Omega,\\ u = u_0,\ \ \partial_t u = u_1 & \text{in } \{t=0\}\times\Omega\end{cases} \tag{98}$$

has a unique solution $u \in E^{s+1}$ and the regularity results of [Ikawa 1968; Lasiecka et al. 1986] also show that $\partial_\nu u \in H^s([0,T]\times\partial\Omega)$. Indeed, by solving first (96) for $v \in E^{s+1}$ and then defining $G := Av \in E^{s+1}$ for the problem (97) we find $w \in E^{s+1}$ (in fact $w \in E^{s+2}$) solving (97) and so that $u := v - w$ solves (98).

Let us then explain how these results translate to the case of a globally hyperbolic manifold $[0,T]\times M$ equipped with a Lorentzian metric $g = \beta(t,x)\,dt^2 - h(t,x)$. Here $\beta > 0$ is a smooth function and $h(t,\cdot)$ is a smooth 1-parameter family of Riemannian metrics on M. The function $\beta > 0$ is bounded from above and below by the compactness of $[0,T]\times\Omega$. Via a conformal change of variables we obtain a scaled metric $\tilde g = dt^2 - \beta^{-1}h$ for which the wave operator transforms as

$$\mathcal{P} := \beta^{\frac{3}{2}}\Box_g\beta^{-\frac{1}{2}} = \Box_{\tilde g} + V = \partial_t^2 - \Delta_{\beta^{-1}h} + V.$$

Here $V(t,x)$ is a smooth function and $\Delta_{\beta^{-1}h}$ for each $t \in [0,T]$ is the Laplace–Beltrami operator of the Riemannian metric $(\beta^{-1}h)(t,\cdot)$ on M. Then u solving (14) is equivalent to $v := \beta^{1/2}u$ solving

$$\begin{cases}\mathcal{P}v = \beta^{\frac{3}{2}}F & \text{in } [0,T]\times\Omega,\\ v = \beta^{\frac{1}{2}}f & \text{on } \Sigma,\\ v = \beta^{\frac{1}{2}}u_0,\ \ \partial_t v = \frac{1}{2}\beta^{-\frac{1}{2}}\partial_t\beta u_0 + \beta^{\frac{1}{2}}u_1 & \text{in } \{t=0\}\times\Omega.\end{cases} \tag{99}$$

From [Hörmander 1983, Theorem 24.1.1] we know that there exists a unique solution to (99). (The result of that work is not however sufficient to us.) Also, in local coordinates in Ω this equation is of the form (98). Let us define

$$R = \beta^{\frac{3}{2}}F, \quad r = \beta^{\frac{1}{2}}f, \quad r_0 = \beta^{\frac{1}{2}}u_0, \quad r_1 = \tfrac{1}{2}\beta^{-\frac{1}{2}}\partial_t\beta u_0 + \beta^{\frac{1}{2}}u_1.$$

Note that $\{t=0\}\times M$ is a space-like Cauchy surface in $\mathbb{R}\times M$. Because $\Omega \subset M$ is a compact manifold, there exists a finite atlas $\{(U_j,\varphi_j)\}_{j=1}^k$ covering Ω. Let χ_j be a partition of unity subordinate to $\{U_j\}_{j=1}^k$ and let us denote the support of χ_j as

$$V_j = \operatorname{supp}(\chi_j) \Subset U_j.$$

Let us also define

$$R_j = \chi_j R, \quad r_j = \chi_j|_\Sigma r, \quad r_{0,j} = \chi_j r_0, \quad r_{1,j} = \chi_j r_1,$$

denote the corresponding coordinate representations as

$$\widetilde{R}_j = R_j \circ \varphi_j^{-1}, \quad \tilde r_j = r \circ \varphi_j^{-1}, \quad \tilde r_{0,j} = r_0 \circ \varphi_j^{-1} \quad \tilde r_{1,j} = r_1 \circ \varphi_j^{-1},$$

and let

$$\widetilde{U}_j = \varphi_j(U_j).$$

We construct a solution to (14) by patching up local solutions following partly the proof of [Bär et al. 2007, Proposition 3.2.11]. As we will see, this is possible due to the finite speed of propagation of solutions to a wave equation. Let K_j be an open set with compact closure such that $V_j \subset K_j$ and $\overline{K}_j \subset U_j$. If $t \in \mathbb{R}$, we may use Lemma 26 to deduce that there exists $\varepsilon > 0$ so that

$$((t, t+\varepsilon) \times \Omega) \cap J^+(V_j) \subset (t, t+\varepsilon) \times K_j \subset (t, t+\varepsilon) \times U_j$$

holds. (This is similar to [Bär et al. 2007, proof of Proposition 3.2.11].) Here J^+ is defined with respect to the conformal metric $\tilde{g}$. We remark that J^+ of a set is conformally invariant. By the compactness of $[0, T]$, there is a finite set of numbers $\varepsilon_i > 0$ and $t_i \in \mathbb{R}$ so that the intervals

$$I_i := (t_i, t_i + \varepsilon_i)$$

cover $[0, T]$. We are going to find a solution to our wave equation (14) iteratively in the index i so that at each step of the iteration we have $(I_i \times \Omega) \cap J^+(V_j) \subset I_i \times U_j$, $j = 1, \dots, k$. Let us set $t_1 = 0 < t_2 < \dots < t_l$ and $t_l + \varepsilon_l = T$ and consider the set $((0, \varepsilon_1) \times \Omega) \cap J^+(V_j)$ first.

By the discussion around (98), we have that there is a unique solution $\tilde{u}_j \in E^{s+1}$ to

$$\begin{cases} \widetilde{\mathcal{P}}\tilde{u}_j = \widetilde{R}_j & \text{in } (0, \varepsilon_1) \times \widetilde{U}_j, \\ \tilde{u}_j = \tilde{r}_j & \text{on } (0, \varepsilon_1) \times \partial\widetilde{U}_j \cap \varphi_j(\partial\Omega), \\ \tilde{u}_j = 0 & \text{on } (0, \varepsilon_1) \times \partial\widetilde{U}_j \setminus \varphi_j(\partial\Omega), \\ \tilde{u}_j = \tilde{r}_{0,j}, \ \ \partial_t \tilde{u}_j = \tilde{r}_{1,j} & \text{in } \{t = 0\} \times \widetilde{U}_j \end{cases} \tag{100}$$

in each coordinate chart $\widetilde{U}_j$, $j = 1, \dots, k$, in the time interval $(0, \varepsilon_1)$. (Here and below we understand $\varphi_j(\partial\Omega) = \varnothing$ if $U_j \cap \partial\Omega = \varnothing$.) Since our (14) satisfies the compatibility conditions (13), one can verify by a direct calculation that (100) satisfies the compatibility conditions of [Ikawa 1968; Lasiecka et al. 1986] that were needed for the unique solvability of (98). In particular, at the intersection of $\{t = 0\}$ and $\partial\widetilde{U}_j \cap \varphi_j(\partial\Omega)$ the compatibility conditions follow from the assumptions of the proposition we are proving. At the intersection of $\{t = 0\}$ and a neighborhood of $\partial\widetilde{U}_j \setminus \varphi_j(\partial\Omega)$ the initial values vanish due to the cut-off functions χ_j. Thus (100) has a unique solution.

Next, let us define

$$u_j = \begin{cases} \tilde{u}_j \circ \varphi_j & \text{in } [0, \varepsilon_1] \times U_j, \\ 0 & \text{in } [0, \varepsilon_1] \times (\Omega \setminus U_j). \end{cases}$$

By the finite speed of propagation of solutions to a wave equation, see for example [Bär et al. 2007, Proposition 3.2.11], we have $\operatorname{supp}(u_j) \subset J^+(V_j)$, and by the condition $((0, \varepsilon_1) \times \Omega) \cap J^+(V_j) \subset (0, \varepsilon_1) \times K_j \subset (0, \varepsilon_1) \times U_j$, we have that

$$\tilde{u}_j = 0 \quad \text{in a neighborhood of } \partial\widetilde{U}_j \setminus \varphi_j(\partial\Omega).$$

Consequently, u_j is the smooth continuation of $\tilde{u}_j \circ \varphi_j : U_j \to \mathbb{R}$ by zero and $u_j \in E^{s+1}$. We also continue $\tilde{u}_j$ smoothly by zero to $\mathbb{R}^n$ (or to $\mathbb{R}^+_n$ if U_j is a boundary chart.)

We now patch up the functions u_j as

$$u = \sum_{j=1}^{k} u_j \in E^{s+1}$$

to have a solution to (99) in the case $T=\varepsilon_1$. Indeed, we have on $((0,\varepsilon_1)\times U_j)$ that

$$\mathcal{P}u=\sum_{j=1}^{k}(\widetilde{\mathcal{P}}\tilde{u}_j)\circ\varphi_j=\sum_{j=1}^{k}\widetilde{R}_j\circ\varphi_j=\sum_{j=1}^{k}\chi_j R=R.$$

We also have that

$$\begin{cases} u=f & \text{on } [0,\varepsilon_1]\times\partial\Omega,\\ u=r_0,\ \ \partial_t u=r_1 & \text{in } \{t=0\}\times\Omega,\end{cases}$$

which is (99) for $T=\varepsilon_1$.

We continue iteratively and extend u to a solution of (14) in increasing time steps t_i. At each iteration step, which concerns the time-interval I_i, we use as the initial values $\tilde{u}|_{t=t_i}$ and $\partial_t u|_{t=t_i}$. These are well defined since $t_i<t_{i-1}+\varepsilon_{i-1}$. In this way, we found a unique solution $u\in E^{s+1}$ to (99) in $[0,T]\times\Omega$, and consequently a unique solution to (14) in the class E^{s+1}.

Next we show that the above regularity and unique existence results of solutions for (14) can be turned into the energy estimate (15) by using the closed graph theorem. Consider the Banach space E^{s+1} and define a linear map

$$A: E^s\times H^{s+1}(\Sigma)\times H^{s+1}(\Omega)\times H^s(\Omega)\to E^{s+1}$$

by $A(F,f,u_0,u_1)=u$, where u is the unique solution to (14). To have the energy estimate (15) it is sufficient to show that A is continuous. By the closed graph theorem, this is in turn equivalent to showing that if

$$\begin{cases} (F_k,f_k,u_{0,k},u_{1,k})\to(F,f,u_0,u_1) & \text{in } E^s\times H^{s+1}(\Sigma)\times H^{s+1}(\Omega)\times H^s(\Omega),\\ A(F_k,f_k,u_{0,k},u_{1,k})\to u_\infty & \text{in } E^{s+1},\end{cases}$$

then

$$u_\infty=A(F,f,u_0,u_1).$$

Here $F_k\to\Box_g u_\infty$ in $\mathcal{D}'([0,T]\times\Omega)$, $f_k\to u_\infty|_\Sigma$ in $\mathcal{D}'(\Sigma)$, and similarly for $t=0$, $u_{0,k}\to u_\infty$ and $u_{1,k}\to\partial_t u_\infty$ in $\mathcal{D}'(\Omega)$. Due to the uniqueness of limits, we have that u_∞ solves (14). Therefore, by uniqueness of solutions to the wave equation, we have that $u=A(F,h,u_0,u_1)$. Hence A is a bounded linear map and the energy estimate follows. □

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RIGIDITY FOR VON NEUMANN ALGEBRAS OF GRAPH PRODUCT GROUPS I: STRUCTURE OF AUTOMORPHISMS

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We study various rigidity aspects of the von Neumann algebra $L(\Gamma)$, where Γ is a graph product group whose underlying graph is a certain cycle of cliques and the vertex groups are wreath-like product property (T) groups. Using an approach that combines methods from Popa's deformation/rigidity theory with new techniques pertaining to graph product algebras, we describe all symmetries of these von Neumann algebras and reduced C^*-algebras by establishing formulas in the spirit of Genevois and Martin's results on automorphisms of graph product groups.

1. Introduction

Graph product groups were introduced by E. Green [1990] in her Ph.D. thesis as natural generalizations of classical right-angled Artin and Coxeter groups. Their study has become a trendy subject over the years as they play key roles in various branches of topology and group theory. For example, over the last decade graph product groups have been intensively studied through the lens of geometric group theory resulting in many new important discoveries — [Agol 2013; Antolín and Minasyan 2015; Haglund and Wise 2008; Minasyan and Osin 2015; Wise 2009], just to enumerate a few.

In a different direction, by using techniques from measured group theory, interesting orbit equivalence rigidity results have been obtained for measure-preserving actions on probability spaces of specific classes of graph product groups, including many right-angled Artin groups [Horbez and Huang 2022; Horbez et al. 2023].

General graph product groups were considered in the analytic framework of von Neumann algebras for the first time in [Caspers and Fima 2017]. Since then several structural results such as strong solidity, absence/uniqueness of Cartan subalgebras, and classification of their tensor decompositions have been established in [Caspers 2020; Caspers and Fima 2017; Chifan and Kunnawalkam Elayavalli 2024; Chifan et al. 2018; Ding and Kunnawalkam Elayavalli 2024] for von Neumann algebras arising from these groups and their actions on probability spaces. Since general graph product groups display such a rich combinatorial structure, much remains to be done in this area, and understanding how this complexity is reflected in the von Neumann algebras remains mysterious.

This paper is the first of two which will investigate new rigidity aspects for von Neumann algebras of graph product groups through the powerful deformation/rigidity theory of Popa [2007]. This theory provides a novel conceptual framework through which a large number of impressive structural and rigidity

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results for von Neumann algebras have been discovered over the last two decades; see the surveys [Ioana 2013; 2018; Popa 2007; Vaes 2013]. These two papers will analyze new inputs in this theory from the perspective of graph product algebras. In the first paper, we completely describe the structure of all $*$-isomorphisms between von Neumann algebras arising from a large class of graph product groups; see Section 4. In the second paper [Chifan et al. 2025], we investigate superrigidity aspects of these von Neumann algebras.

1.1. ***Statements of the main results.*** To properly introduce our results, we briefly recall the construction of graph product groups. Let $\mathscr{G}=(\mathscr{V},\mathscr{E})$ be a finite *simple graph* (i.e., $\mathscr{G}$ does not admit more than one edge between any two vertices, and no edge of $\mathscr{G}$ starts and ends at the same vertex). The *graph product group* $\Gamma=\mathscr{G}\{\Gamma_v\}$ of a given family of *vertex groups* $\{\Gamma_v\}_{v\in\mathscr{V}}$ is the quotient of the free product $*_{v\in\mathscr{V}}\Gamma_v$ by the relations $[\Gamma_u,\Gamma_v]=1$ whenever u and v are connected by an edge, $(u,v)\in\mathscr{E}$. Thus, graph products can be thought of as groups that "interpolate" between the direct product $\times_{v\in\mathscr{V}}\Gamma_v$ (when $\mathscr{G}$ is complete) and the free product $*_{v\in\mathscr{V}}\Gamma_v$ (when $\mathscr{G}$ has n).

For any subgraph $\mathscr{H}=(\mathscr{U},\mathscr{F})$ of $\mathscr{G}$, we denote by $\Gamma_{\mathscr{H}}$ the subgroup generated by $\Gamma_{\mathscr{H}}=\langle\Gamma_u:u\in\mathscr{U}\rangle$, and we call it the *full subgroup* of $\mathscr{G}\{\Gamma_v\}$ corresponding to $\mathscr{H}$. A *clique* $\mathscr{C}$ of $\mathscr{G}$ is a maximal, complete subgraph of $\mathscr{G}$. The set of cliques of $\mathscr{G}$ will be denoted by $\mathrm{cliq}(\mathscr{G})$. The full subgroups $\Gamma_{\mathscr{C}}$ for $\mathscr{C}\in\mathrm{cliq}(\mathscr{G})$ are called the clique subgroups of $\mathscr{G}\{\Gamma_v\}$.

In this paper we are interested in graph product groups arising from a specific class of graphs which we introduce next. A graph $\mathscr{G}$ is called a *simple cycle of cliques* (the collection of such graphs we abbreviate CC_1) if there is an enumeration of its clique set $\mathrm{cliq}(\mathscr{G})=\{\mathscr{C}_1,\dots,\mathscr{C}_n\}$ with $n\geqslant 4$ such that the subgraphs $\mathscr{C}_{i,j}:=\mathscr{C}_i\cap\mathscr{C}_j$ satisfy

$$\mathscr{C}_{i,j}=\begin{cases}\varnothing & \text{if } \hat{i}-\hat{j}\in\mathbb{Z}_n\setminus\{\hat{1},\widehat{n-1}\},\\ \neq\varnothing & \text{if } \hat{i}-\hat{j}\in\{\hat{1},\widehat{n-1}\},\end{cases}$$
$$\mathscr{C}_i^{\mathrm{int}}:=\mathscr{C}_i\setminus(\mathscr{C}_{i-1,i}\cup\mathscr{C}_{i,i+1})\neq\varnothing \qquad \text{for all } i\in\overline{1,n},\ \text{with conventions } 0=n \text{ and } n+1=1. \tag{1-1}$$

Note this automatically implies the cardinality $|\mathscr{C}_i|\geqslant 3$ for all i. Also such an enumeration $\mathrm{cliq}(\mathscr{G})=\{\mathscr{C}_1,\dots,\mathscr{C}_n\}$ is called *a consecutive cliques enumeration.* A basic example of such a graph is any simple, length n, cycle of triangles $\mathscr{T}_n=(\mathscr{V}_n,\mathscr{E}_n)$, which essentially looks like a flower-shaped graph with n petals, shown in Figure 1. In fact any graph in CC_1 is a two-level clustered graph that is a specific retraction of $\mathscr{T}_n$; for more details the reader may consult Section 4.

The goal of this paper is to describe the structure of all $*$-isomorphisms between *graph product group von Neumann algebras* (i.e., group von Neumann algebras arising from graph product groups), where the underlying graphs belong to CC_1. To introduce our results, we first highlight a canonical family of $*$-isomorphisms between these algebras that are analogous to the graph product groups situation. Let $\mathscr{G},\mathscr{H}\in\mathrm{CC}_1$ be isomorphic graphs, and fix $\sigma:\mathscr{G}\to\mathscr{H}$ an isometry. Let $\mathrm{cliq}(\mathscr{G})=\{\mathscr{C}_1,\dots,\mathscr{C}_n\}$ be a consecutive cliques enumeration. Let $\Gamma_{\mathscr{G}}$ and $\Lambda_{\mathscr{H}}$ be graph product groups and assume that, for every $i\in\overline{1,n}$, there are $*$-isomorphisms

$$\theta_{i-1,i}:\mathcal{L}(\Gamma_{\mathscr{C}_{i-1,i}})\to\mathcal{L}(\Lambda_{\mathscr{C}_{\sigma(\mathscr{C}_{i-1,i})}}),\quad \xi_i:\mathcal{L}(\Gamma_{\mathscr{C}_i^{\mathrm{int}}})\to\mathcal{L}(\Lambda_{\sigma(\mathscr{C}_i^{\mathrm{int}})}),\quad \theta_{i,i+1}:\mathcal{L}(\Gamma_{\mathscr{C}_{i,i+1}})\to\mathcal{L}(\Lambda_{\mathscr{C}_{\sigma(\mathscr{C}_{i,i+1})}});$$

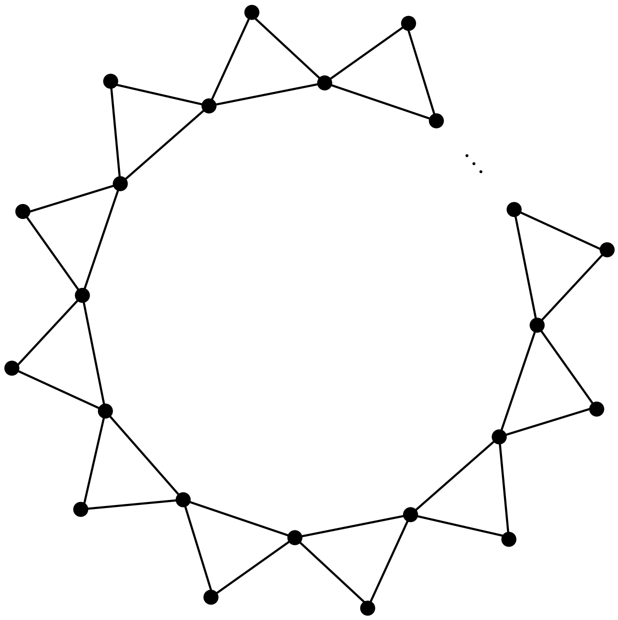

Figure 1. A simple, length n cycle of triangles, which is an example of a graph that is in CC_1.

here and in what follows we use the convention as before that $n = 0$ and $n+1 = 1$. Results in Section 7.1 show these $*$-isomorphisms induce a unique $*$-isomorphism $\phi_{\theta,\xi,\sigma} : \mathcal{L}(\Gamma_{\mathscr{G}}) \to \mathcal{L}(\Lambda_{\mathscr{H}})$ defined as

$$\phi_{\theta,\xi,\sigma}(x) = \begin{cases} \theta_{i-1,i}(x) & \text{if } x \in \mathcal{L}(\Gamma_{\mathscr{C}_{i-1,i}}), \\ \xi_i(x) & \text{if } x \in \mathcal{L}(\Gamma_{\mathscr{C}_i^{\mathrm{int}}}) \end{cases} \quad \text{for all } i \in \overline{1,n}. \tag{1-2}$$

When $\Gamma_{\mathscr{G}} = \Lambda_{\mathscr{H}}$, this construction yields a group of $*$-automorphisms of $\mathcal{L}(\Gamma_{\mathscr{G}})$, which we denote by $\mathrm{Loc}_{\mathrm{c,g}}(\mathcal{L}(\Gamma_{\mathscr{G}}))$. We also denote by $\mathrm{Loc}_{\mathrm{c}}(\mathcal{L}(\Gamma_{\mathscr{G}}))$ the subgroup of all local automorphisms satisfying $\sigma = \mathrm{Id}$. Notice that

$$\mathrm{Loc}_{\mathrm{c}}(\mathcal{L}(\Gamma_{\mathscr{G}})) \cong \bigoplus_i \mathrm{Aut}(\mathcal{L}(\Gamma_{\mathscr{C}_{i-1,i}})) \oplus \mathrm{Aut}(\mathcal{L}(\Gamma_{\mathscr{C}_i^{\mathrm{int}}})),$$

and also $\mathrm{Loc}_{\mathrm{c}}(\mathcal{L}(\Gamma_{\mathscr{G}})) \leqslant \mathrm{Loc}_{\mathrm{c,g}}(\mathcal{L}(\Gamma_{\mathscr{G}}))$ has finite index.

Next, we highlight a class of automorphisms in $\mathrm{Loc}_{\mathrm{c}}(\mathcal{L}(\Gamma_{\mathscr{G}}))$ needed to state our main results. Consider n-tuples $a = (a_{i,i+1})_i$ and $b = (b_i)_i$ of nontrivial unitaries $a_{i,i+1} \in \mathcal{L}(\Gamma_{\mathscr{C}_{i-1,i}})$ and $b_i \in \mathcal{L}(\Gamma_{\mathscr{C}_i^{\mathrm{int}}})$ for every $i \in \overline{1,n}$. If in (1-2), we let $\theta_{i,i+1} = \mathrm{ad}(a_{i,i+1})$ and $\xi_i = \mathrm{ad}(b_i)$, and then the corresponding local automorphism $\phi_{\theta,\xi,\mathrm{Id}}$ is most of the time an outer automorphism of $\mathcal{L}(\Gamma)$ and will be denoted by $\phi_{a,b}$ throughout. These automorphisms form a normal subgroup denoted by $\mathrm{Loc}_{\mathrm{c},i}(\mathcal{L}(\Gamma_{\mathscr{G}})) \lhd \mathrm{Loc}_{\mathrm{c}}(\mathcal{L}(\Gamma_{\mathscr{G}}))$; see Section 7.1 for more details.

Developing an approach which combines outgrowths of prior methods in Popa's deformation/rigidity theory [Ioana et al. 2008] with a new technique on analyzing cancellation in cyclic relations of graph von Neumann algebras (Section 5), we are able to describe all $*$-isomorphisms between these algebras solely in terms of the aforementioned local isomorphisms. This can be viewed as a von Neumann algebra counterpart of very general and deep results of Genevois and Martin [2019, Corollary C] from geometric group theory describing the structure of the automorphisms of graph product groups.

Theorem A. *Let $\mathscr{G}, \mathscr{H} \in \mathrm{CC}_1$, and let $\Gamma = \mathscr{G}\{\Gamma_v\}$ and $\Lambda = \mathscr{H}\{\Lambda_w\}$ be graph products such that*

(1) *Γ_v and Λ_w are icc property (T) groups for all $v \in \mathscr{V}$ and $w \in \mathscr{W}$,*

(2) *there is a class $\mathcal{C}$ of countable groups which satisfies the s-unique prime factorization property* (see Definition 7.6) *for which Γ_v and Λ_w belong to $\mathcal{C}$ for all $v \in \mathscr{V}$ and $w \in \mathscr{W}$.*

Let $t > 0$, and let $\Theta : \mathcal{L}(\Gamma)^t \to \mathcal{L}(\Lambda)$ be any $*$-isomorphism. Then $t = 1$ and one can find an isometry $\sigma : \mathscr{G} \to \mathscr{H}$, $*$-isomorphisms $\theta_{i-1,i} : \mathcal{L}(\Gamma_{\mathscr{C}_{i-1,i}}) \to \mathcal{L}(\Gamma_{\mathscr{C}_{\sigma(\mathscr{C}_{i-1,i})}})$ and $\xi_i : \mathcal{L}(\Gamma_{\mathscr{C}_i^{\mathrm{int}}}) \to \mathcal{L}(\Gamma_{\sigma(\mathscr{C}_i^{\mathrm{int}})})$ for all $i \in \overline{1,n}$, and a unitary $u \in \mathcal{L}(\Lambda)$ such that $\Theta = \mathrm{ad}(u) \circ \phi_{\theta,\xi,\sigma}$.

This theorem applies to fairly large classes of property (T) vertex groups, including: all fibered Rips constructions considered in [Chifan et al. 2023a; 2024], and all wreath-like product groups $\mathcal{WR}(A, B \curvearrowright I)$, where A is either abelian or icc, B is an icc subgroup of a hyperbolic group, and the action $B \curvearrowright I$ has amenable stabilizers [Chifan et al. 2023b]. The result also implies that the fundamental group [Murray and von Neumann 1936] of these graph product group II_1-factors is always trivial; this means that if Γ is a graph product group as in Theorem A, then $\{t > 0 : \mathcal{L}(\Gamma)^t \cong \mathcal{L}(\Gamma)\} = 1$. Recall that Popa [2006a] used his deformation/rigidity theory for obtaining the first examples of II_1-factors with trivial fundamental group, hence answering a longstanding open problem of Kadison; see [Ge 2003]. Subsequently, a large number of striking results on computations of fundamental groups of II_1-factors were obtained; see the introduction of [Chifan et al. 2024]. To our knowledge, Theorem A provides the first instance of computing the fundamental group for nontrivial graph product von Neumann algebras which is not a tensor product.

Specializing Theorem A to the case when the vertex groups Γ_v and Λ_w are the property (T) wreath-like product groups as in [Chifan et al. 2023c, Theorem 7.5], we obtain a fairly concrete description of all such isomorphisms between these graph product group von Neumann algebras; namely, they appear as compositions between the canonical group-like isomorphisms and the clique-inner local automorphisms of $\mathcal{L}(\Lambda)$ described above.

Theorem B. *Let $\mathscr{G}, \mathscr{H} \in \mathrm{CC}_1$, and let $\Gamma = \mathscr{G}\{\Gamma_v\}$ and $\Lambda = \mathscr{H}\{\Lambda_w\}$ be graph product groups where all vertex groups Γ_v, Λ_w are property (T) wreath-like product groups of the form $\mathcal{WR}(A, B \curvearrowright I)$, where A is abelian, B is an icc subgroup of a hyperbolic group, and $B \curvearrowright I$ has infinite orbits.*

Then, for any $t > 0$ and $*$-isomorphism $\Theta : \mathcal{L}(\Gamma)^t \to \mathcal{L}(\Lambda)$, we have $t = 1$ and one can find a character $\eta \in \mathrm{Char}(\Gamma)$, a group isomorphism $\delta \in \mathrm{Isom}(\Gamma, \Lambda)$, a $*$-automorphism $\phi_{a,b} \in \mathrm{Loc}_{\mathrm{c,i}}(\mathcal{L}(\Lambda))$, and a unitary $u \in \mathcal{L}(\Lambda)$ such that $\Theta = \mathrm{ad}(u) \circ \phi_{a,b} \circ \Psi_{\eta,\delta}$.

In the statement of Theorem B and also throughout the paper, given a character $\eta \in \mathrm{Char}(\Gamma)$ and a group isomorphism $\delta \in \mathrm{Isom}(\Gamma, \Lambda)$, we denote by $\Psi_{\eta,\delta}$ the $*$-isomorphism from $\mathcal{L}(\Gamma)$ to $\mathcal{L}(\Lambda)$ given by $\Psi_{\eta,\delta}(u_g) = \eta(g) v_{\delta(g)}$ for any $g \in \Gamma$. Here, $\{u_g : g \in \Gamma\}$ and $\{v_h : h \in \Lambda\}$ are the canonical group unitaries of $\mathcal{L}(\Gamma)$ and $\mathcal{L}(\Lambda)$, respectively.

To this end we recall that in [Chifan et al. 2023c, Corollary 2.12] it was shown that the property (T) regular wreath-like products covered by the previous theorem can be chosen to have trivial abelianization and prescribed finitely presented outer automorphism groups. Using this, Theorem A yields the following.

Corollary C. *Let* $\mathscr{G} \in \mathrm{CC}_1$, *and fix* $\mathrm{cliq}(\mathscr{G}) = \{\mathscr{C}_1, \ldots, \mathscr{C}_n\}$ *a consecutive enumeration of its cliques. Let* $\Gamma = \mathscr{G}\{\Gamma_v\}$ *be any graph product groups* (as in Theorem B). *Assume in addition that its vertex groups are pairwise nonisomorphic and have trivial abelianization and trivial outer automorphisms. Then the outer automorphisms satisfy the formula*

$$\mathrm{Out}(\mathcal{L}(\Gamma)) \cong \bigoplus_{i=1}^{n} \mathscr{U}(\mathcal{L}(\Gamma_{\mathscr{C}_{i-1,i}})) \oplus \mathscr{U}(\mathcal{L}(\Gamma_{\mathscr{C}_i^{\mathrm{int}}})).$$

By applying Corollary C to the case when the underlying graph $\mathscr{G}$ is the n-petals flower-shaped $\mathscr{F}_n = (\mathscr{V}_n, \mathscr{E}_n)$, see Figure 1, we obtain the slimmest types of outer automorphisms groups one could have in this setup. Namely, we deduce that $\mathrm{Out}(\mathcal{L}(\Gamma)) \cong \bigoplus_{v \in \mathscr{V}_n} \mathscr{U}(\mathcal{L}(\Gamma_v))$.

We conclude our introduction with Corollary D, where we describe all $*$-isomorphisms of the reduced C^*-algebras of graph product groups that we considered in Theorem B. This result can be seen as a C^*-algebraic version of [Genevois and Martin 2019, Corollary C].

Corollary D. *Let* $\mathscr{G}, \mathscr{H} \in \mathrm{CC}_1$, *and let* $\Gamma = \mathscr{G}\{\Gamma_v\}$ *and* $\Lambda = \mathscr{H}\{\Lambda_w\}$ *be graph product groups* (as in Theorem B). *Then, for any $*$-isomorphism* $\Theta : C_r^*(\Gamma) \to C_r^*(\Lambda)$, *there exist a character* $\eta \in \mathrm{Char}(\Gamma)$, *a group isomorphism* $\delta \in \mathrm{Isom}(\Gamma, \Lambda)$, *a $*$-automorphism* $\phi_{a,b} \in \mathrm{Loc}_{\mathrm{c,i}}(\mathcal{L}(\Lambda))$, *and a unitary* $u \in \mathcal{L}(\Lambda)$ *such that* $\Theta = \mathrm{ad}(u) \circ \phi_{a,b} \circ \Psi_{\eta,\delta}$.

In fact, this result is a consequence of Theorem B since the graph product groups that we consider have trivial amenable radical (see Lemma 4.3) and, consequently, their reduced C^*-algebras have unique trace [Breuillard et al. 2017].

2. Preliminaries

2.1. *Terminology.* Throughout this document all von Neumann algebras are denoted by calligraphic letters, e.g., $\mathcal{M}$, $\mathcal{N}$, $\mathcal{P}$, $\mathcal{Q}$, etc. All von Neumann algebras $\mathcal{M}$ considered in this document will be tracial, i.e., endowed with a unital, faithful, normal linear functional $\tau : \mathcal{M} \to \mathbb{C}$ satisfying $\tau(xy) = \tau(yx)$ for all $x, y \in \mathcal{M}$. This induces a norm on $\mathcal{M}$ with the formula $\|x\|_2 = \tau(x^*x)^{1/2}$ for all $x \in \mathcal{M}$. The $\|\cdot\|_2$-completion of $\mathcal{M}$ will be denoted by $L^2(\mathcal{M})$.

Given a von Neumann algebra $\mathcal{M}$, we will denote by $\mathscr{U}(\mathcal{M})$ its unitary group and by $\mathcal{Z}(\mathcal{M})$ its center. Given a unital inclusion $\mathcal{N} \subset \mathcal{M}$ of von Neumann algebras, we denote by $\mathcal{N}' \cap \mathcal{M} = \{x \in \mathcal{M} : [x, \mathcal{N}] = 0\}$ the relative commmutant of $\mathcal{N}$ inside $\mathcal{M}$, and by $\mathscr{N}_{\mathcal{M}}(\mathcal{N}) = \{u \in \mathscr{U}(\mathcal{M}) : u\mathcal{N}u^* = \mathcal{N}\}$ the normalizer of $\mathcal{N}$ inside $\mathcal{M}$. We say that the inclusion $\mathcal{N}$ is regular in $\mathcal{M}$ if $\mathscr{N}_{\mathcal{M}}(\mathcal{N})'' = \mathcal{M}$ and irreducible if $\mathcal{N}' \cap \mathcal{M} = \mathbb{C}1$.

2.2. *Graph product groups.* In this preliminary section we briefly recall the notion of graph product groups introduced by E. Green [1990] while also highlighting some of its features that are relevant to this article. Let $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ be a finite simple graph, where $\mathscr{V}$ and $\mathscr{E}$ denote its vertex and edge sets, respectively. Let $\{\Gamma_v\}_{v\in\mathscr{V}}$ be a family of groups called vertex groups. The graph product group associated with this data, denoted by $\mathscr{G}\{\Gamma_v, v\in\mathscr{V}\}$ or simply $\mathscr{G}\{\Gamma_v\}$, is the group generated by $\Gamma_v,\ v\in\mathscr{V}$, with the only relations being $[\Gamma_u,\Gamma_v]=1$ whenever $(u,v)\in\mathscr{E}$. Given any subset $\mathscr{U}\subset\mathscr{V}$, the subgroup $\Gamma_{\mathscr{U}}=\langle\Gamma_u : u\in\mathscr{U}\rangle$ of $\mathscr{G}\{\Gamma_v, v\in\mathscr{V}\}$ is called a *full subgroup*. This can be identified with the graph product $\mathscr{G}_{\mathscr{U}}\{\Gamma_u, u\in\mathscr{U}\}$ corresponding to the subgraph $\mathscr{G}_{\mathscr{U}}$ of $\mathscr{G}$, spanned by the vertices of $\mathscr{U}$. For every $v\in\mathscr{V}$, we denote by $\mathrm{lk}(v)$ the subset of vertices $w\neq v$ such that $(w,v)\in\mathscr{E}$. Similarly, for every $\mathscr{U}\subseteq\mathscr{V}$, we define $\mathrm{lk}(\mathscr{U})=\bigcap_{u\in\mathscr{U}}\mathrm{lk}(u)$. We also use the convention that $\mathrm{lk}(\varnothing)=\mathscr{V}$. Notice that $\mathscr{U}\cap\mathrm{lk}(\mathscr{U})=\varnothing$.

Graph product groups naturally admit many amalgamated free product decompositions. One such decomposition — which is essential for deriving our main results — involves full subgroup factors in [Green 1990, Lemma 3.20] as follows. For any $w\in\mathscr{V}$, we have

$$\mathscr{G}\{\Gamma_v\}=\Gamma_{\mathscr{V}\setminus\{w\}}\mathop{\ast}_{\Gamma_{\mathrm{lk}(w)}}\Gamma_{\mathrm{st}(w)},\tag{2-1}$$

where $\mathrm{st}(w)=\{w\}\cup\mathrm{lk}(w)$. Notice that $\Gamma_{\mathrm{lk}(w)}\subsetneqq\Gamma_{\mathrm{st}(w)}$, but it could be the case that $\Gamma_{\mathrm{lk}(w)}=\Gamma_{\mathscr{V}\setminus\{w\}}$ when $\mathscr{V}=\mathrm{st}(w)$. In this case the amalgam decomposition is called degenerate.

Similarly, for every subgraph $\mathscr{U}\subset\mathscr{G}$, we write $\mathrm{st}(\mathscr{U})=\mathscr{U}\cup\mathrm{lk}(\mathscr{U})$. A maximal complete subgraph $\mathscr{C}\subseteq\mathscr{G}$ is called a *clique* and the collections of all cliques of $\mathscr{G}$ will be denoted by $\mathrm{cliq}(\mathscr{G})$. Below we highlight various properties of full subgroups that will be useful in this paper. The first is [Antolín and Minasyan 2015, Lemma 3.7], the second is [Antolín and Minasyan 2015, Proposition 3.13], while the third is [Antolín and Minasyan 2015, Proposition 3.4].

Proposition 2.1 [Antolín and Minasyan 2015]. *Let $\Gamma=\mathscr{G}\{\Gamma_v\}$ be any graph product of groups with $g\in\Gamma$, and let $\mathscr{S},\mathscr{T}\subseteq\mathscr{G}$ be any subgraphs. Then the following hold*:

(1) *If $g\Gamma_{\mathscr{T}}g^{-1}\subset\Gamma_{\mathscr{S}}$, then there is $h\in\Gamma_{\mathscr{S}}$ such that $g\Gamma_{\mathscr{T}}g^{-1}=h\Gamma_{\mathscr{T}\cap\mathscr{S}}h^{-1}$. In particular, if $\mathscr{S}=\mathscr{T}$, then $g\Gamma_{\mathscr{T}}g^{-1}=\Gamma_{\mathscr{S}}$.*

(2) *The normalizer of $\Gamma_{\mathscr{T}}$ inside Γ satisfies $N_\Gamma(\Gamma_{\mathscr{T}})=\Gamma_{\mathscr{T}\cup\mathrm{link}(\mathscr{T})}$.*

(3) *There exist $\mathscr{D}\subseteq\mathscr{S}\cap\mathscr{T}$ and $h\in\Gamma_{\mathscr{T}}$ such that $g\Gamma_{\mathscr{S}}g^{-1}\cap\Gamma_{\mathscr{T}}=h\Gamma_{\mathscr{D}}h^{-1}$.*

2.3. ***Popa's intertwining-by-bimodules techniques.*** We next recall the *intertwining-by-bimodules* technique of Popa [2006b, Theorem 2.1 and Corollary 2.3], which is a powerful criterion for identifying intertwiners between arbitrary subalgebras of tracial von Neumann algebras.

Theorem 2.2 [Popa 2006b]. *Let $(\mathcal{M},\tau)$ be a tracial von Neumann algebra and $\mathcal{P}\subset p\mathcal{M}p,\ \mathcal{Q}\subset q\mathcal{M}q$ be von Neumann subalgebras. Then the following are equivalent*:

(1) *There exist projections $p_0\in\mathcal{P},\ q_0\in\mathcal{Q}$, a $*$-homomorphism $\theta: p_0\mathcal{P}p_0\to q_0\mathcal{Q}q_0$, and a nonzero partial isometry $v\in q_0\mathcal{M}p_0$ such that $\theta(x)v=vx$ for all $x\in p_0\mathcal{P}p_0$.*

(2) *There is no sequence $(u_n)_{n\geq1}\subset\mathcal{U}(\mathcal{P})$ satisfying $\|E_{\mathcal{Q}}(x^*u_ny)\|_2\to0$ for all $x,y\in p\mathcal{M}$.*

If one of these equivalent conditions holds, we write $\mathcal{P}\prec_{\mathcal{M}}\mathcal{Q}$ and say that ***a corner of $\mathcal{P}$ embeds into $\mathcal{Q}$ inside $\mathcal{M}$.*** *Moreover, if $\mathcal{P}p'\prec_{\mathcal{M}}\mathcal{Q}$ for any nonzero projection $p'\in\mathcal{P}'\cap p\mathcal{M}p$, then write $\mathcal{P}\prec^{s}_{\mathcal{M}}\mathcal{Q}$.*

Given an arbitrary graph product group, our next lemma clarifies the intertwining of subalgebras of full subgroups in the associated graph product group von Neumann algebra.

Lemma 2.3. *Let $\Gamma = \mathscr{G}\{\Gamma_v\}$ be any graph product of infinite groups, and let $\mathscr{S}, \mathscr{T} \subseteq \mathscr{G}$ be any subgraphs. If $\mathcal{L}(\Gamma_{\mathscr{S}}) \prec_{\mathcal{L}(\Gamma)} \mathcal{L}(\Gamma_{\mathscr{T}})$, then $\mathscr{S} \subset \mathscr{T}$.*

Proof. By applying [Chifan and Ioana 2018, Lemma 2.2], there is $g \in \Gamma$ such that $[\Gamma_{\mathscr{S}} : \Gamma_{\mathscr{S}} \cap g\Gamma_{\mathscr{T}} g^{-1}] < \infty$. By Proposition 2.1, one can find a subgraph $\mathscr{P} \subseteq \mathscr{S} \cap \mathscr{T}$ and $k \in \Gamma_{\mathscr{S}}$ such that $\Gamma_{\mathscr{S}} \cap g\Gamma_{\mathscr{T}} g^{-1} = k\Gamma_{\mathscr{P}} k^{-1}$. Thus $k\Gamma_{\mathscr{P}} k^{-1} < \Gamma_{\mathscr{S}}$ is a finite index subgroup. Since $k \in \Gamma_{\mathscr{S}}$, it follows that $\Gamma_{\mathscr{P}} < \Gamma_{\mathscr{S}}$ has finite index as well. Since $|\Gamma_v| = \infty$, for all $v \in \mathscr{G}$, we must have that $[\Gamma_{\mathscr{S}} : \Gamma_{\mathscr{P}}] = 1$, and hence $\Gamma_{\mathscr{S}} = \Gamma_{\mathscr{P}}$. Thus, $\mathscr{S} = \mathscr{P} \subset \mathscr{S} \cap \mathscr{T}$, and hence $\mathscr{S} \subset \mathscr{T}$. □

Remark 2.4. The proof of Lemma 2.3 shows that if $\Gamma = \mathscr{G}\{\Gamma_v\}$ is a graph product of infinite groups and $\mathscr{S}, \mathscr{T} \subseteq \mathscr{G}$ are subgraphs such that $[\Gamma_{\mathscr{S}} : \Gamma_{\mathscr{S}} \cap g\Gamma_{\mathscr{T}} g^{-1}] < \infty$ for some $g \in \Gamma$, then $\mathscr{S} \subseteq \mathscr{T}$.

2.4. ***Quasinormalizers of von Neumann algebras.*** Given an inclusion $\mathcal{P} \subset \mathcal{M}$ of tracial von Neumann algebras, we define the quasinormalizer $\mathscr{QN}_{\mathcal{M}}(\mathcal{P})$ as the subgroup of all elements $x \in \mathcal{M}$ for which there exist $x_1, \dots, x_n \in \mathcal{M}$ such that $\mathcal{P}x \subseteq \sum x_i \mathcal{P}$ and $x\mathcal{P} \subseteq \sum \mathcal{P} x_i$; see [Popa 1999, Definition 4.8].

Lemma 2.5 [Fang et al. 2011; Popa 2006b]. *Let $\mathcal{P} \subset \mathcal{M}$ be tracial von Neumann algebras. For any projection $p \in \mathcal{P}$, we have that $W^*(\mathscr{QN}_{p\mathcal{M}p}(p\mathcal{P}p)) = pW^*(\mathscr{QN}_{\mathcal{M}}(\mathcal{P}))p$.*

Given a group inclusion $H < G$, the quasinormalizer $\mathrm{QN}_G(H)$ is the group of all $g \in G$ for which there exists a finite set $F \subset G$ such that $Hg \subset FH$ and $gH \subset HF$. The following result provides a relation between the group theoretical quasinormalizer and the von Neumann algebraic one.

Lemma 2.6 [Fang et al. 2011, Corollary 5.2]. *Let $\Lambda < \Gamma$ be countable groups. Then we have that $W^*(\mathscr{QN}_{\mathcal{L}(\Gamma)}(\mathcal{L}(\Lambda))) = \mathcal{L}(\mathrm{QN}_\Gamma(\Lambda))$.*

We continue by computing the quasinormalizer of subalgebras of full subgroups in any graph product group von Neumann algebra. More generally, we show the following.

Theorem 2.7. *Let $\Gamma = \mathscr{G}\{\Gamma_v\}$ be any graph product of infinite groups, and let $\mathscr{S}, \mathscr{T} \subseteq \mathscr{G}$ be any subgraphs. Write $\mathcal{M} = \mathcal{L}(\Gamma)$, and assume there exist $x, x_1, x_2, \dots, x_n \in \mathcal{M}$ such that $\mathcal{L}(\Gamma_{\mathscr{S}})x \subseteq \sum_{k=1}^n x_k \mathcal{L}(\Gamma_{\mathscr{T}})$. Thus $\mathscr{S} \subseteq \mathscr{T}$ and $x \in \mathcal{L}(\Gamma_{\mathscr{T} \cup \mathrm{lk}(\mathscr{S})})$.*

Proof. Using the proofs of [Chifan and Ioana 2018, Lemma 2.8 and Claim 2.3], we obtain that x belongs to the $\|\cdot\|_2$-closure of the linear span of $\{u_g\}_{g\in S}$. Here, S denotes the set of all elements $g \in \Gamma$ for which $[\Gamma_{\mathscr{S}} : \Gamma_{\mathscr{S}} \cap g\Gamma_{\mathscr{T}} g^{-1}] < \infty$. By assuming that $x \neq 0$, it follows that S is nonempty. Fix $g \in S$. By using Remark 2.4, we derive that $\mathscr{S} \subseteq \mathscr{T}$, which gives the first part of the conclusion.

For proving the second part, note that by Proposition 2.1 one can find a subgraph $\mathscr{P} \subseteq \mathscr{S}$ and $k \in \Gamma_{\mathscr{S}}$ such that $\Gamma_{\mathscr{S}} \cap g\Gamma_{\mathscr{T}} g^{-1} = k\Gamma_{\mathscr{P}} k^{-1}$. Thus $k\Gamma_{\mathscr{P}} k^{-1} < \Gamma_{\mathscr{S}}$ is a finite index subgroup. Since $k \in \Gamma_{\mathscr{S}}$, this further implies that $\Gamma_{\mathscr{P}} < \Gamma_{\mathscr{S}}$ has finite index, and hence $\mathscr{P} = \mathscr{S}$. Using again that $k \in \Gamma_{\mathscr{S}}$, we get $\Gamma_{\mathscr{S}} \cap g\Gamma_{\mathscr{T}} g^{-1} = k\Gamma_{\mathscr{P}} k^{-1} = \Gamma_{\mathscr{S}}$, and thus $g^{-1}\Gamma_{\mathscr{S}} g < \Gamma_{\mathscr{T}}$. By Proposition 2.1, one can find $r \in \Gamma_{\mathscr{T}}$ such that $g^{-1}\Gamma_{\mathscr{S}} g = r\Gamma_{\mathscr{S}} r^{-1}$. This relation implies in particular that $gr \in N_\Gamma(\Gamma_{\mathscr{S}})$, and since $N_\Gamma(\Gamma_{\mathscr{S}}) = \Gamma_{\mathscr{S} \sqcup \mathrm{lk}(\mathscr{S})}$ (see Proposition 2.1), we conclude that $gr \in \Gamma_{\mathscr{S} \cup \mathrm{lk}(\mathscr{S})}$. Therefore, $g \in \Gamma_{\mathscr{S} \cup \mathrm{lk}(\mathscr{S})} \Gamma_{\mathscr{T}} \subset \Gamma_{\mathscr{T} \cup \mathrm{lk}(\mathscr{S})}$. This gives the desired conclusion. □

Corollary 2.8. *Let $\Gamma = \mathscr{G}\{\Gamma_v\}$ be any graph product of infinite groups, and let $\mathscr{C} \in \operatorname{cliq}(\mathscr{G})$ be a clique with at least two vertices. Fix a vertex $v \in \mathscr{C}$ such that $\operatorname{lk}(\mathscr{C} \setminus \{v\}) = \{v\}$. Write $\mathcal{M} = \mathcal{L}(\Gamma)$, and assume there exist $x, x_1, x_2, \dots, x_n \in \mathcal{M}$ such that $\mathcal{L}(\Gamma_{\mathscr{C}\setminus\{v\}})x \subseteq \sum_{k=1}^n x_k \mathcal{L}(\Gamma_{\mathscr{C}})$. Then $x \in \mathcal{L}(\Gamma_{\mathscr{C}})$.*

Proof. The result follows by applying Theorem 2.7 for $\mathscr{S} = \mathscr{C} \setminus \{v\}$ and $\mathscr{T} = \mathscr{C}$. □

Lemma 2.9. *Let $\Gamma = \mathscr{G}\{\Gamma_v\}$ be a graph product of groups, and let $\mathscr{C} \in \operatorname{cliq}(\mathscr{G})$ be a clique. Let $\mathcal{P} \subset p\mathcal{L}(\Gamma_{\mathscr{C}})p$ be a von Neumann subalgebra such that $\mathcal{P} \nprec_{\mathcal{L}(\Gamma_{\mathscr{C}})} L(\Gamma_{\mathscr{C}_{\hat{v}}})$ for any $v \in \mathscr{C}$. If $x \in \mathcal{L}(\Gamma)$ satisfies $x\mathcal{P} \subset \sum_{i=1}^n \mathcal{L}(\Gamma_{\mathscr{C}})x_i$ for some $x_1, \dots, x_n \in \mathcal{L}(\Gamma)$, then $xp \in \mathcal{L}(\Gamma_{\mathscr{C}})$.*

Proof. Let $g \in \Gamma \setminus \Gamma_{\mathscr{C}}$. From Proposition 2.1, there exist $h \in \Gamma_{\mathscr{C}}$ and $\mathscr{D} \subset \mathscr{C}$ such that $\Gamma_{\mathscr{C}} \cap g\Gamma_{\mathscr{C}}g^{-1} = h\Gamma_{\mathscr{D}}h^{-1}$. Note that Theorem 2.7 shows $\mathrm{QN}_\Gamma^{(1)}(\Gamma_{\mathscr{C}}) = \Gamma_{\mathscr{C}}$ and therefore $\mathscr{D} \neq \mathscr{C}$; otherwise, we would get $g \in \mathrm{QN}_\Gamma^{(1)}(\Gamma_{\mathscr{C}}) = \Gamma_{\mathscr{C}}$, a contradiction. Thus, from the assumption we deduce $\mathcal{P} \nprec_{\mathcal{L}(\Gamma_{\mathscr{C}})} L(\Gamma_{\mathscr{C}} \cap g\Gamma_{\mathscr{C}}g^{-1})$ for any $g \in \Gamma \setminus \Gamma_{\mathscr{C}}$. The conclusion now follows from [Chifan and Ioana 2018, Lemma 2.7]. □

2.5. *A result on normalizers in tensor product factors.* Our next proposition describes the normalizer of a II_1-factor $\mathcal{N}$ inside the tensor product of $\mathcal{N}$ with another II_1-factor.

Proposition 2.10. *Let $\mathcal{N}$ and $\mathcal{P}$ be II_1-factors and write $\mathcal{M} = \mathcal{N} \bar{\otimes} \mathcal{P}$. If $u \in \mathscr{U}(\mathcal{M})$ satisfies $u\mathcal{N}u^* = \mathcal{N}$, then one can find $a \in \mathscr{U}(\mathcal{N})$ and $b \in \mathscr{U}(\mathcal{P})$ such that $u = a \otimes b$.*

Proof. Let $(\xi_i)_{i\in I} \subset L^2(\mathcal{P})$ be a Pimsner–Popa basis for the inclusion $\mathcal{N} \subset \mathcal{M}$, let $u = \sum_i E_{\mathcal{N}}(u\xi_i^*) \otimes \xi_i$, and write $\eta_i = E_{\mathcal{N}}(u\xi_i^*)$. If $\theta : \mathcal{N} \to \mathcal{N}$ denotes the $*$-isomorphism $\theta = \operatorname{ad}(u)$, then we have $\theta(x)u = ux$ for all $x \in \mathcal{N}$. This combined with the above formula yields $\theta(x)\eta_i \otimes \xi_i = \theta(x)u = ux = \eta_i x \otimes \xi_i$. Hence, for all $x \in \mathcal{N}$ and all i, we have

$$\theta(x)\eta_i = \eta_i x. \tag{2-2}$$

Let $u_i \in \mathcal{N}$ be the partial isometry in the polar decomposition of η_i. Thus $\theta(x)u_i = u_i x$ for all $x \in \mathcal{N}$ and all i. In particular, we get $u_i^* u_i \in \mathcal{N}' \cap \mathcal{N} = \mathbb{C}1$, and hence $u_i \in \mathscr{U}(\mathcal{N})$ for all i. The prior relations also imply that $u_i^* x u_i = \theta(x) = u_j^* x u_j$ for all $i, j \in I$. In particular, we have $u_i u_j^* \in \mathcal{N}' \cap \mathcal{N} = \mathbb{C}1$, and thus one can find scalars $c_{i,j} \in \mathbb{T}$ such that $u_i = c_{i,j} u_j$ for all $i, j \in I$. Relation (2-2) also implies that $|\eta_i| \in \mathcal{N}' \cap L^2(\mathcal{N})$ and, since $\mathcal{N}$ is a II_1-factor, we get $|\eta_i| \in \mathbb{C}1$. In conclusion, $\eta_i \in \mathbb{C}\mathscr{U}(\mathcal{N})$ for all i, and one can find $d_{i,j} \in \mathbb{C}$ such that $\eta_i = d_{i,j}\eta_j$ for all $i, j \in I$. Fix $j \in I$ with $\eta_j \neq 0$. Using the above relations, we have $u = \sum_i \eta_i \otimes \xi_i = \sum_i d_{i,j}\eta_j \otimes \xi_i = \eta_j \otimes \left(\sum_i d_{i,j}\xi_i\right) = \eta_j \otimes b$, where we write $b = \sum_i d_{i,j}\xi_i \in L^2(\mathcal{P})$. Since $\eta_j \in \mathbb{C}\mathscr{U}(\mathcal{N})$, we get the desired conclusion. □

3. Wreath-like product groups

A new category of groups called *wreath-like product groups* were introduced in the previous work [Chifan et al. 2023b]. To briefly recall their construction, let A and B be any countable groups, and assume that $B \curvearrowright I$ is an action on a countable set. One says W is a wreath-like product of A and $B \curvearrowright I$ if it can be realized as a group extension

$$1 \to \bigoplus_{i\in I} A_i \hookrightarrow W \overset{\varepsilon}{\twoheadrightarrow} B \to 1 \tag{3-1}$$

which satisfies the following properties:

(a) $A_i \cong A$ for all $i \in I$.

(b) The action by conjugation of W on $\bigoplus_{i \in I} A_i$ permutes the direct summands according to the rule

$$wA_i w^{-1} = A_{\varepsilon(w)i} \quad \text{for all } w \in W, \ \ i \in I.$$

The class of all such wreath-like groups is denoted by $\mathcal{WR}(A, B \curvearrowright I)$. When $I = B$ and the action $B \curvearrowright I$ is by translation, this consists of so-called regular wreath-like product groups and we simply denote their class by $\mathcal{WR}(A, B)$.

Notice that every classical generalized wreath product $A \wr_I B$ belongs to $\mathcal{WR}(A, B \curvearrowright I)$. However, building examples of nonsplit wreath-like products is a far more involved problem. One way to approach this is through the use of the so-called Magnus embedding [1939]: these are quotients groups of the form $\Gamma/[\Lambda, \Lambda]$, where $\Lambda \lhd \Gamma$ is a normal subgroup. Methods of this type were used by Cohen and Lyndon to produce many such quotients in the context of one-relator groups. The following result is a particular case of [Chifan et al. 2023b, Corollary 4.6] and relies on the prior works [Dahmani et al. 2017; Osin 2007; Sun 2020].

Corollary 3.1. *Let G be an icc hyperbolic group. For every infinite order element $g \in G$, there exists $d \in \mathbb{N}$ such that, for every $k \in \mathbb{N}$ divisible by d, we have the following*:

(a) *$G/[\langle\langle g^k \rangle\rangle, \langle\langle g^k \rangle\rangle] \in \mathcal{WR}(\mathbb{Z}, G/\langle\langle g^k \rangle\rangle \curvearrowright I)$, where $\langle g^k \rangle$ is normal in $E_G(g)$, the action $G/\langle\langle g^k \rangle\rangle \curvearrowright I$ is transitive, and all the stabilizers of elements of I are isomorphic to the finite group $E_G(g)/\langle g^k \rangle$. Here, $E_G(g)$ denotes the elementary subgroup generated by g, $\langle g^k \rangle$ denotes the subgroup generated by g^k and $\langle\langle g^k \rangle\rangle$ denotes the smallest normal subgroup that contains g^k.*

(b) *$G/\langle\langle g^k \rangle\rangle$ is an icc hyperbolic group.*

Developing a new quotienting method in the context of Cohen–Lyndon triples, [Chifan et al. 2023b, Theorem 2.5] constructed many examples of property (T) regular wreath-like product groups as follows.

Theorem 3.2 [Chifan et al. 2023b]. *Let G be a hyperbolic group. For every finitely generated group A, there exists a quotient W of G such that $W \in \mathcal{WR}(A, B)$ for some hyperbolic group B.*

For further use we also recall the following result on prescribed outer automorphisms of property (T) regular wreath-like product groups established in [Chifan et al. 2023b, Theorem 6.9].

Theorem 3.3 [Chifan et al. 2023b]. *For every finitely presented group Q and every finitely generated group A_0, there exist groups A, B and a regular wreath-like product $W \in \mathcal{WR}(A, B)$ with the following properties*:

(a) *W has property (T) and has no nontrivial characters.*

(b) *A is the direct sum of $|Q|$ copies of A_0. In particular, $A = A_0$ if $Q = \{1\}$.*

(c) *B is an icc normal subgroup of a hyperbolic group H and $H/B \cong Q$. In particular, B is hyperbolic whenever Q is finite.*

(d) $\mathrm{Out}(W) \cong Q$.

Remark 3.4. Since A_0 can be any finitely generated group, it follows that if we fix the group Q there are infinitely many pairwise nonisomorphic regular wreath-like product groups $W \in \mathcal{WR}(A, B)$ which satisfy (a)–(d) in the prior theorem.

3.1. ***Unique prime factorization for von Neumann algebras of wreath-like product groups.*** In this subsection, more precisely in Theorem 3.6, we show that von Neumann algebras of certain wreath-like product groups satisfy the unique prime factorization of Ozawa and Popa [2010]. First, we point out the following structural result for commuting property (T) von Neumann subalgebras of von Neumann algebras that arise from trace-preserving actions of certain wreath-like product groups.

Lemma 3.5. *Let Γ be a wreath-like product group of the form $\mathcal{WR}(A, B \curvearrowright I)$, where A is abelian and B is an icc subgroup of a hyperbolic group. Let $\Gamma \curvearrowright \mathcal{N}$ be a trace-preserving action and write $\mathcal{M} = \mathcal{N} \rtimes \Gamma$.*

If $\mathcal{A}, \mathcal{B} \subset p\mathcal{M}p$ are commuting property (T) von Neumann subalgebras, then $\mathcal{A} \prec_{\mathcal{M}} \mathcal{N}$ or $\mathcal{B} \prec_{\mathcal{M}} \mathcal{N}$.

The proof of Lemma 3.5 follows from [Chifan et al. 2023c, Theorem 6.4], and the main ingredient of its proof is Popa and Vaes' structure theorem [2014] for normalizers in crossed products arising from actions of hyperbolic group.

Theorem 3.6. *Let Γ_i be a property (T) wreath-like product group of the form $\mathcal{WR}(A, B \curvearrowright I)$ for any $i \in \overline{1, n}$, where A is abelian and B is an icc subgroup of a hyperbolic group.*

If $\mathcal{M} := \mathcal{L}(\Gamma_1) \mathbin{\bar{\otimes}} \cdots \mathbin{\bar{\otimes}} \mathcal{L}(\Gamma_n) = \mathcal{P}_1 \mathbin{\bar{\otimes}} \mathcal{P}_2$ is a tensor product decomposition into II_1-factors, then there exist a unitary $u \in \mathcal{M}$, a decomposition $\mathcal{M} = \mathcal{P}_1^t \mathbin{\bar{\otimes}} \mathcal{P}_2^{1/t}$, and a partition $T_1 \sqcup T_2 = \overline{1, n}$ such that $\mathcal{L}\big(\times_{k \in S_j} \Gamma_k\big) = u\mathcal{P}_j^{t_j} u^*$ for any $j \in \{1, 2\}$.

Proof. To fix some notation, we have that Γ_i belongs to $\mathcal{WR}(A_i, B_i \curvearrowright I_i)$ for any $i \in \overline{1, n}$, where A_i is abelian and B_i is an icc subgroup of a hyperbolic group. Note that, since $\mathcal{P}_1$ and $\mathcal{P}_2$ have property (T), by applying Lemma 3.5, we obtain a map $\phi : \overline{1, n} \to \overline{1, 2}$ such that

$$\mathcal{P}_{\phi(i)} \prec_{\mathcal{M}} \overline{\bigotimes_{k \neq i}} \mathcal{L}(\Gamma_k) \quad \text{for any } i \in \overline{1, n}.$$

By [Drimbe et al. 2019, Lemma 2.8 (2)], there exists a partition $\overline{1, n} = S_1 \sqcup S_2$ such that $\mathcal{P}_j \prec_{\mathcal{M}} \mathcal{L}\big(\times_{k \in S_j} \Gamma_k\big)$ for any j. By passing to relative commutants, we get $\mathcal{L}\big(\times_{k \in S_j} \Gamma_k\big) \prec_{\mathcal{M}} \mathcal{P}_j$, for any j. The conclusion of the theorem follows by using standards arguments that rely on [Ozawa and Popa 2004, Proposition 12] and [Ge 1996, Theorem A]. □

Corollary 3.7. *Let $\Gamma_1, \ldots, \Gamma_n$ and $\Lambda_1, \ldots, \Lambda_m$ be property (T) wreath-like product groups of the form $\mathcal{WR}(A, B \curvearrowright I)$, where A is abelian, B is an icc subgroup of a hyperbolic group, and $B \curvearrowright I$ has infinite stabilizers.*

If there exists $t > 0$ such that $\mathcal{L}(\Gamma_1 \times \cdots \times \Gamma_n)^t = \mathcal{L}(\Lambda_1 \times \cdots \times \Lambda_m)$, then $t = 1$, $n = m$, and there is a unitary $u \in \mathcal{L}(\Gamma_1 \times \cdots \times \Gamma_n)$ such that $u\mathbb{T}(\Gamma_1 \times \cdots \times \Gamma_n)u^* = \mathbb{T}(\Lambda_1 \times \cdots \times \Lambda_m)$.

Proof. The result follows directly by combining Theorem 3.6 and [Chifan et al. 2023b, Theorem 1.3]. □

4. Graph product groups associated with cycles of cliques graphs

In this section we highlight a class of graphs considered to have good clustering properties. Specifically, a graph $\mathscr{G}$ is called a *simple cycle of cliques* (and belongs to the class CC) if there is an enumeration of its cliques set $\mathrm{cliq}(\mathscr{G}) = \{\mathscr{C}_1, \ldots, \mathscr{C}_n\}$ with $n \geqslant 4$ such that the subgraphs $\mathscr{C}_{i,j} := \mathscr{C}_i \cap \mathscr{C}_j$ satisfy the conditions

$$\mathscr{C}_{i,j} = \begin{cases} \varnothing & \text{if } \hat{i} - \hat{j} \in \mathbb{Z}_n \setminus \{\hat{1}, \widehat{n-1}\}, \\ \neq \varnothing & \text{if } \hat{i} - \hat{j} \in \{\hat{1}, \widehat{n-1}\}. \end{cases} \tag{4-1}$$

Here, the classes $\hat{i}$ and $\hat{j}$ belong to $\mathbb{Z}/n\mathbb{Z}$. We will also refer to $\mathrm{cliq}(\mathscr{G}) = \{\mathscr{C}_1, \ldots, \mathscr{C}_n\}$ satisfying the previous properties as the *the consecutive enumeration* of the cliques of $\mathscr{G}$.

For every $i \in \overline{1, n}$, we define $\mathscr{C}_i^{\mathrm{int}} := \mathscr{C}_i \setminus (\mathscr{C}_{i-1,i} \cup \mathscr{C}_{i,i+1})$, where we declare that $0 = n$ and $n + 1 = 1$. When $\mathscr{C}_i^{\mathrm{int}} \neq \varnothing$ for all $i \in \overline{1, n}$, one says that $\mathscr{G}$ belongs to the class CC_1. Most of our main results will involve graphs of this form. Throughout this article we will use all these notations consistently.

A basic example of a graph in the class CC_1 is a simple cycle of triangles called $\mathscr{T}_n$, where n is the number of cliques; see Figure 2 below for $n = 16$.

In fact every graph $\mathscr{G} \in \mathrm{CC}_1$ appears as a two-level clustered graph which is a specific retraction of $\mathscr{T}_n$ as follows. There exists a graph projection map $\Phi : \mathscr{G} \to \mathscr{T}_n$ such that, for every vertex $v \in \mathscr{T}_n$, the cluster $\Phi^{-1}(v) \subset \mathscr{G}$ is a complete subgraph of $\mathscr{G}$. In addition, whenever $v, w \in \mathscr{T}_n$ are connected in $\mathscr{T}_n$, there are edges in $\mathscr{G}$ between all vertices of the corresponding clusters $\Phi^{-1}(v)$ and $\Phi^{-1}(w)$.

We continue by recording some elementary combinatorial properties of graph product groups associated with graphs that are simple cycles of cliques. The proof of the following lemma is straightforward and we leave it to the reader.

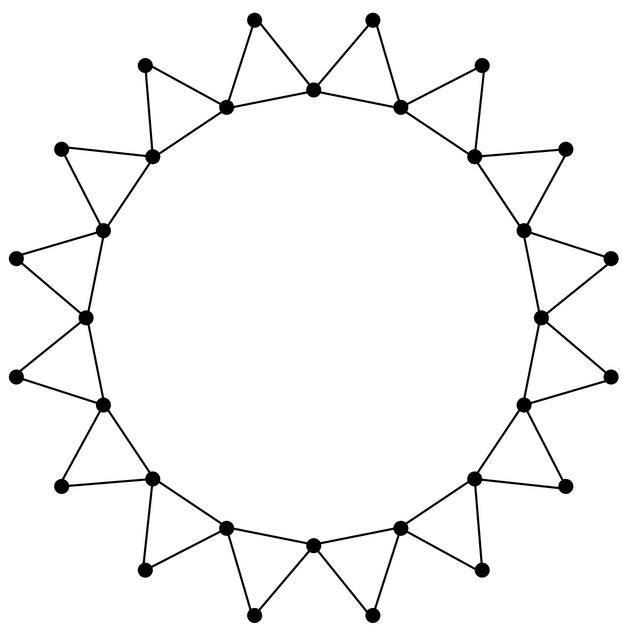

Figure 2. A cycle of 16 triangles is a simple example of a graph in CC_1.

Lemma 4.1. *Let $\mathscr{G} \in \mathrm{CC}_1$, and let $\mathscr{C}_1, \dots, \mathscr{C}_n$ be an enumeration of its consecutive cliques. Let $\{\Gamma_v, v \in \mathscr{V}\}$ be a collection of groups, and let $\Gamma_{\mathscr{G}}$ be the corresponding graph product group. We denote by $\{w_i\}_{i=1}^n$ the petal outer vertices of $\mathscr{F}_n$ and by $\{b_i\}_{i=1}^n$ the petal base vertices of $\mathscr{F}_n$.*

Then $\Gamma_{\mathscr{G}}$ can be realized as a graph product $\Gamma'_{\mathscr{F}_n}$ associated to the graph $\mathscr{F}_n$, where the vertex groups are defined by

$$\Gamma'_{w_i} = \bigoplus_{v \in \mathscr{C}_i^{\mathrm{int}}} \Gamma_v \quad \textit{and} \quad \Gamma'_{b_i} = \bigoplus_{v \in \mathscr{C}_{i-1,i}} \Gamma_v \quad \textit{for every } i \in \overline{1,n}.$$

Proposition 4.2. *Let $\mathscr{G} \in \mathrm{CC}_1$, and let $\mathscr{C}_1, \dots, \mathscr{C}_n$ be an enumeration of its consecutive cliques. Let $\{\Gamma_v, v \in \mathscr{V}\}$ be a collection of infinite groups, and let $\Gamma_{\mathscr{G}}$ be the corresponding graph product group. Then the following properties hold*:

(1) *If $g \in \Gamma_{\mathscr{C}_{i-1} \triangle \mathscr{C}_i}$ and $h \in \Gamma_{\mathscr{C}_i \triangle \mathscr{C}_{i+1}}$ satisfy $gh \in \Gamma_{\mathscr{G} \setminus \mathscr{C}_i^{\mathrm{int}}}$, then one can find $a \in \Gamma_{(\mathscr{C}_{i-1} \setminus \mathscr{C}_{i-1,i}) \cup \mathscr{C}_{i,i+1}}$, $s \in \Gamma_{\mathscr{C}_i^{\mathrm{int}}}$, and $b \in \Gamma_{(\mathscr{C}_{i+1} \setminus \mathscr{C}_{i,i+1}) \cup \mathscr{C}_{i-1,i}}$ such that $g = as$ and $h = s^{-1}b$.*

(2) *Let $g \in \Gamma_{\mathscr{C}_{i-2,i-1} \cup \mathscr{C}_{i,i+1}}$, $h \in \Gamma_{\mathscr{C}_{i-1,i} \cup \mathscr{C}_{i+1,i+2}}$, and $k \in \Gamma_{\mathscr{C}_{i,i+1} \cup \mathscr{C}_{i+2,i+3}}$ such that*

$$ghk \in \Gamma_{\mathscr{C}_{i-2,i-1} \cup \mathscr{C}_{i-1,i} \cup \mathscr{C}_{i+1,i+2} \cup \mathscr{C}_{i+2,i+3}}.$$

Then one can find $a \in \Gamma_{\mathscr{C}_{i-2,i-1}}$, $b \in \Gamma_{\mathscr{C}_{i+2,i+3}}$, and $s \in \Gamma_{\mathscr{C}_{i,i+1}}$ such that $g = as$ and $k = s^{-1}b$.

(3) *For each $i \in \overline{1,n}$, let $x_{i,i+1} \in \Gamma_{\mathscr{C}_i \cup \mathscr{C}_{i+1}}$ such that $x_{1,2}x_{2,3} \cdots x_{n-1,n}x_{n,1} = 1$. Then, for each $i \in \overline{1,n}$, one can find $a_i \in \Gamma_{\mathscr{C}_{i-1,i}}$, $b_i \in \Gamma_{\mathscr{C}_i^{\mathrm{int}}}$, and $c_i \in \Gamma_{\mathscr{C}_{i,i+1}}$ such that $x_{i,i+1} = a_i b_i c_i b_{i+1}^{-1} a_{i+2}^{-1} c_{i+1}^{-1}$. Here we use the convention that $n+1 = 1$, $n+2 = 2$, etc.*

Proof. Here Δ is the symmetric difference operation defined by $A \Delta B = (A \cup B) \setminus (A \cap B)$. We recall the *normal form* [Green 1990, Theorem 3.9], which in graph product groups plays the role that reduced words play in free product groups. If $1 \neq g \in \Gamma_{\mathscr{G}}$ is expressed as $g = g_1 \cdots g_n$, we say g is in normal form if each g_i is a nonidentity element of some vertex group (called a *syllable*) and if it is impossible, through repeated swapping of syllables (corresponding to adjacent vertices in $\mathscr{G}$), to bring together two syllables from the same vertex group. By [Green 1990, Theorem 3.9], every $1 \neq g \in \Gamma_{\mathscr{G}}$ has a normal form $g = g_1 \cdots g_n$ and it is unique up to a finite number of consecutive syllable shuffles. Moreover, given any sequence of syllables $g_1 \cdots g_n$, there is an inductive procedure for putting this sequence into normal form: if $h_1 \cdots h_r$ is the normal form of $g_1 \cdots g_m$, then the normal form of $g_1 \cdots g_{m+1}$ is either

(i) $h_1 \cdots h_r$ if $g_{m+1} = 1$,

(ii) $h_1 \cdots h_{j-1} h_{j+1} \cdots h_r$ if h_j shuffles to the end and $g_{m+1} = h_j^{-1}$,

(iii) $h_1 \cdots h_{j-1} h_{j+1} \cdots h_r (h_j g_{m+1})$ if h_j shuffles to the end, $g_{m+1} \neq h_j^{-1}$, and g_{m+1}, h_j belong to the same vertex group, or

(iv) $h_1 \cdots h_r g_{m+1}$ if g_{m+1} is in a different vertex group from that of every syllable which can be shuffled down.

Note that the normal form of an element $g \in \Gamma_{\mathscr{G}}$ has minimal syllable length with respect to all the sequences of syllables representing g.

We are now ready to prove the three assertions of the proposition. For (1), let $g = g_1 \cdots g_n$ and $h = h_1 \cdots h_m$ be the normal forms of g and h. Then gh has a normal form $gh = k_1 \cdots k_r$, determined by the procedure described in the previous paragraph. By assumption, $k_j \notin \bigcup_{v\in\mathscr{C}_i^{\text{int}}}\{\Gamma_v\}$ for all $j \in \overline{1,r}$. Now, if $g_j \notin \bigcup_{v\in\mathscr{C}_i^{\text{int}}}\{\Gamma_v\}$ for all $j \in \overline{1,n}$, then each $h_i \in \bigcup_{v\in\mathscr{C}_i^{\text{int}}}\{\Gamma_v\}$ is one of the syllables occurring in the normal form of gh. Since this cannot happen, we have $h_i \notin \bigcup_{v\in\mathscr{C}_i^{\text{int}}}\{\Gamma_v\}$ for all $i \in \overline{1,m}$, and hence we can take $a = g,\ b = h$, and s the empty word. Assume that $g_j \in \bigcup_{v\in\mathscr{C}_i^{\text{int}}}\{\Gamma_v\}$ for some $j \in \overline{1,n}$. We note that we may assume $j = n$ since, if $g_i \in \bigcup_{v\in\mathscr{C}_{i-1}\setminus\mathscr{C}_i}\{\Gamma_v\}$ for some $i \in \overline{j+1,n}$, then g_j would be a syllable in gh since it cannot be shuffled past g_i, which shows that $g_{j+1}\cdots g_n \in \bigcup_{v\in\mathscr{C}_i}\{\Gamma_v\}$. This implies that $g_n^{-1} = h_i$ for some $i \in \overline{1,m}$. Choosing the smallest such i and noting that $h_1,\ldots,h_{i-1} \in \bigcup_{v\in\mathscr{C}_i}\{\Gamma_v\}$ (since it must be possible to shuffle h_i up to g_n as in (ii) of the previous paragraph), we may assume that $h_1 = g_n^{-1}$. Continuing in this way we see that we can take $a = g_1\cdots g_{k-1},\ b = h_{n-k+2}\cdots h_m$, and $s = g_k\cdots g_n$, where $g_j \notin \bigcup_{v\in\mathscr{C}_i^{\text{int}}}\{\Gamma_v\}$ for all $j \in \overline{1,k-1}$ and $h = g_n^{-1},\ldots,g_k^{-1}h_{n-k+2},\ldots,h_m$. Notice too that none of the syllables $h_{n-k+2},\ldots,h_m$ can belong to $\bigcup_{v\in\mathscr{C}_i^{\text{int}}}\{\Gamma_v\}$ since the inverse of such a syllable cannot be any of the syllables $g_1,\ldots,g_{k-1}$. This proves (1).

For (2), let $g = g_1\cdots g_n,\ h = h_1\cdots h_m$, and $k = k_1\cdots k_r$ be normal forms. If $g_i \notin \bigcup_{v\in\mathscr{C}_{i,i+1}}\{\Gamma_v\}$ for all $i \in \overline{1,n}$, then $k_j \notin \bigcup_{v\in\mathscr{C}_{i,i+1}}\{\Gamma_v\}$ for all $j \in \overline{1,n}$ since neither h nor ghk have normal forms with syllables in $\bigcup_{v\in\mathscr{C}_{i,i+1}}\{\Gamma_v\}$ by assumption, and hence we can take $a = g,\ b = k$, and s the empty word. Otherwise we must have $g_j \in \Gamma_v$ for some $v \in \mathscr{C}_{i,i+1}$, and as in the proof of part (1) we can assume $j = n$ and $k_1 = g_n^{-1}$ (note that g_j commutes with each syllable in the normal form of h). Continuing, we see that we can take $a = g_1\cdots g_{l-1},\ b = k_{n-l+2}\cdots k_r$, and $s = g_l\cdots g_n$, where $g_j \notin \bigcup_{v\in\mathscr{C}_{i,i+1}}\{\Gamma_v\}$ for all $j \in \overline{1,l-1}$ and $k = g_n^{-1}\cdots g_l^{-1}k_{n-l+2}\cdots k_r$. This proves (2).

For (3), observe first that every $x_{i,i+1} \in \Gamma_{\mathscr{C}_i\cup\mathscr{C}_{i+1}} = \Gamma_{\mathscr{C}_{i,i+1}} \times (\Gamma_{\mathscr{C}_i\setminus\mathscr{C}_{i+1}} * \Gamma_{\mathscr{C}_{i+1}\setminus\mathscr{C}_i})$ can be written in the form $\tilde{a}_i\tilde{b}_i\tilde{c}_i\tilde{d}_i\tilde{e}_i\tilde{f}_i$, where

$$\tilde{a}_i \in \Gamma_{\mathscr{C}_{i-1,i}},\quad \tilde{b}_i \in \Gamma_{\mathscr{C}_i^{\text{int}}},\quad \tilde{c}_i \in \Gamma_{\mathscr{C}_{i,i+1}},\quad \tilde{d}_i \in \Gamma_{\mathscr{C}_{i+1}^{\text{int}}},\quad \tilde{e}_i \in \Gamma_{\mathscr{C}_{i+1,i+2}},\quad \tilde{f}_i \in \Gamma_{\mathscr{C}_i\cup\mathscr{C}_{i+1}}.$$

Moreover, we can assume that the normal form of $x_{i,i+1}$ is the sequence obtained by concatenating the normal forms of $\tilde{a}_i,\ \tilde{b}_i,\ \tilde{c}_i,\ \tilde{d}_i,\ \tilde{e}_i,\ \tilde{f}_i$, and if $\tilde{f}_i = f_1\cdots f_n$ is the normal form of $\tilde{f}_i$, then f_1 belongs to a group Γ_v, where v is vertex in $\mathscr{C}_i \setminus \mathscr{C}_{i+1}$.

We continue by showing that we can assume $\tilde{f}_i = 1$. Notice that this is the case if there is no syllable g occurring in the normal form of $x_{i,i+1}$ belonging to $\bigcup_{v\in\mathscr{C}_{i+1}\setminus\mathscr{C}_i}\{\Gamma_v\}$; indeed, in this case $\tilde{d}_i, \tilde{e}_i = 1$ and all the syllables occurring in the normal form of $\tilde{f}_i$ can be shuffled up to the normal forms of $\tilde{a}_i,\ \tilde{b}_i,\ \tilde{c}_i$. So it remains to assume that there is such a syllable g and assume by contradiction that $\tilde{f}_i \neq 1$. Notice that our hypotheses imply that f_1^{-1} is a syllable in the normal form of $x_{i-2,i-1}x_{i-1,i}$, and g^{-1} is a syllable in the normal form of $x_{i+1,i+2}x_{i+2,i+3}$. This implies that the normal form of $x_{1,2}x_{2,3}\cdots x_{i-1,i}x_{i,i+1}$ must still contain f_1 as f_1^{-1} cannot shuffle past g to cancel with f_1. Consequently, the normal form of $x_{1,2}x_{2,3}\cdots x_{n-1,n}x_{n,1}$ must still contain f_1 as f_1^{-1} cannot shuffle past g or g^{-1} to cancel with f_1. This gives a contradiction, and hence we can assume $\tilde{f}_i = 1$.

Next, we observe that $\tilde{b}_i = \tilde{d}_{i-1}^{-1}$ for each i since our hypotheses imply that all the syllables occurring in the normal form of $\tilde{b}_i^{-1}$ occur in the one for $x_{i-1,i}$, and only $\tilde{d}_{i-1}$ has normal form with syllables coming from $\bigcup_{v\in\mathscr{C}_i^{\mathrm{int}}}\{\Gamma_v\}$. To finish the proof, set $a_i=\tilde{a}_i,\ b_i=\tilde{b}_i$, and $c_i=\tilde{c}_i$ and note that, since $\tilde{e}_i\tilde{c}_{i+1}\tilde{a}_{i+2}=1$ (being the only elements in our decompositions belonging to $\Gamma_{\mathscr{C}_{i+1,i+2}}$), we have $x_{i,i+1}=\tilde{a}_i\tilde{b}_i\tilde{c}_i\tilde{d}_i\tilde{e}_i=a_ib_ic_ib_{i+1}^{-1}a_{i+2}^{-1}(a_{i+2}\tilde{e}_i)=a_ib_ic_ib_{i+1}^{-1}a_{i+2}^{-1}c_{i+1}^{-1}$. □

Lemma 4.3. *Let $\Gamma=\mathscr{G}\{\Gamma_v, v\in\mathscr{V}\}$ be a graph product of groups such that $\mathscr{G}\in \mathrm{CC}_1$. Then Γ has trivial amenable radical.*

Proof. Assume by contradiction that there exists a nontrivial amenable normal subgroup A of Γ. Since Γ is icc, we get that A is an infinite group. For any $w\in\mathscr{V}$, note that $\mathrm{st}(w)\neq\mathscr{V}$ and $\mathscr{G}\{\Gamma_v\}=\Gamma_{\mathscr{V}\setminus\{w\}}*_{\Gamma_{\mathrm{lk}(w)}}\Gamma_{\mathrm{st}(w)}$. Since A is an amenable, normal subgroup of Γ, it follows that $\mathcal{L}(A)\prec_{\mathcal{L}(\Gamma)}\mathcal{L}(\Gamma_{\mathrm{lk}(w)})$ [Vaes 2014, Theorem A]. In particular, by using [Drimbe et al. 2019, Lemma 2.4], it follows that $\mathcal{L}(A)\prec^s_{\mathcal{L}(\Gamma)}\mathcal{L}(\Gamma_{\mathscr{C}})$ for any $\mathscr{C}\in\mathrm{cliq}(\mathscr{G})$. Let $\mathscr{C},\mathscr{D}\in\mathrm{cliq}(\mathscr{G})$ such that $\mathscr{C}\cap\mathscr{D}=\varnothing$. Using [Vaes 2013, Lemma 2.7], there is $g\in\Gamma$ such that $\mathcal{L}(A)\prec^s_{\mathcal{L}(\Gamma)}\mathcal{L}(\Gamma_{\mathscr{C}}\cap g\Gamma_{\mathscr{D}}g^{-1})$. Note however that Proposition 2.1 implies that $\Gamma_{\mathscr{C}}\cap g\Gamma_{\mathscr{D}}g^{-1}=1$. This shows that $\mathcal{L}(A)\prec_{\mathcal{L}(\Gamma)}\mathbb{C}1$, and thus gives the contradiction that A is finite. □

We end this section by recording a result describing all automorphisms of graph product groups $\Gamma_{\mathscr{G}}$ associated with graphs in the class CC_1. This is a particular case of a powerful theorem in geometric group theory established recently by Genevois and Martin [2019, Corollary C].

To state the result we briefly recall a special class of automorphisms of graph product groups. For any isometry $\sigma:\mathscr{G}\to\mathscr{G}$ and any collection of group isomorphisms $\Phi=\{\phi_v:\Gamma_v\to\Gamma_{\sigma(v)}:v\in\mathscr{V}\}$, define the *local automorphism* (σ,Φ) to be the canonical automorphism of $\Gamma_{\mathscr{G}}$ induced by the maps $\bigcup_{v\in\mathscr{V}}\Gamma_v\ni g\to\phi_{\sigma(v)}(g)\in\Gamma_{\mathscr{G}}$. One can easily observe that, under composition, these form a subgroup of $\mathrm{Aut}(\Gamma_{\mathscr{G}})$ which is denoted by $\mathrm{Loc}(\Gamma_{\mathscr{G}})$. We denote by $\mathrm{Loc}_0(\Gamma_{\mathscr{G}})$ the subgroup of local automorphisms satisfying $\sigma=\mathrm{Id}$. Notice that $\mathrm{Loc}_0(\Gamma_{\mathscr{G}})$ is naturally isomorphic to $\bigoplus_{v\in\mathscr{V}}\mathrm{Aut}(\Gamma_v)$. Moreover, the inclusion $\mathrm{Loc}_0(\Gamma_{\mathscr{G}})\leqslant\mathrm{Loc}(\Gamma_{\mathscr{G}})$ has finite index.

Theorem 4.4 [Genevois and Martin 2019]. *Let $\Gamma_{\mathscr{G}}$ be a graph product associated with a graph $\mathscr{G}\in\mathrm{CC}_1$. Then its automorphism group $\mathrm{Aut}(\Gamma_{\mathscr{G}})$ is generated by the inner and the local automorphisms of $\Gamma_{\mathscr{G}}$. In fact we have $\mathrm{Aut}(\Gamma_{\mathscr{G}})=\mathrm{Inn}(\Gamma_{\mathscr{G}})\rtimes\mathrm{Loc}(\Gamma_{\mathscr{G}})$, and therefore*

$$\begin{aligned}\mathrm{Aut}(\Gamma_{\mathscr{G}})&\cong\Gamma_{\mathscr{G}}\rtimes\Bigg(\bigg(\bigoplus_{v\in\mathscr{V}}\mathrm{Aut}(\Gamma_v)\bigg)\rtimes\mathrm{Sym}(\Gamma_{\mathscr{G}})\Bigg),\\ \mathrm{Out}(\Gamma_{\mathscr{G}})&\cong\bigg(\bigoplus_{v\in\mathscr{V}}\mathrm{Aut}(\Gamma_v)\bigg)\rtimes\mathrm{Sym}(\Gamma_{\mathscr{G}}).\end{aligned}\tag{4-2}$$

Here, $\mathrm{Sym}(\Gamma_{\mathscr{G}})$ *is an explicit finite subgroup of automorphisms of* $\Gamma_{\mathscr{G}}$.

Proof. One can easily check that the graphs in CC_1 are atomic and therefore the conclusion follows immediately from [Genevois and Martin 2019, Corollary C]. □

Remark 4.5. (1) If in the hypothesis of Theorem 4.4 we assume in addition that $\{\Gamma_v\}_{v\in\mathscr{V}}$ are pairwise nonisomorphic, then we have $\mathrm{Sym}(\Gamma_{\mathscr{G}})=1$ in the automorphism group formulae (4-2). The same holds if instead we assume that any two cliques of $\mathscr{G}$ have different cardinalities and for any $\mathscr{C}\in\mathrm{cliq}(\mathscr{G})$ the set $\{\Gamma_v\}_{v\in\mathscr{C}}$ consists of pairwise nonisomorphic subgroups.

(2) One of the main goals of this paper is to establish both von Neumann algebraic and C^*-algebraic analogs of Theorem 4.4, under various assumptions on the vertex groups. For the specific statements in this direction, the reader may consult Corollaries 7.11 and 7.14 in Section 7.

5. Von Neumann algebraic cancellation in cyclic relations

In this section we establish a von Neumann algebraic analog of Proposition 4.2 (3) describing the structure of all unitaries that satisfy a similar cyclic relation (Lemma 5.1). We start by first proving the following von Neumann algebraic counterpart of Proposition 4.2 (1).

Lemma 5.1. *Let $\Lambda_1, \Lambda_2, \Sigma < \Gamma$ be groups satisfying the following properties*:

(1) $\Lambda_1 \cap \Lambda_2 = \Lambda_1 \cap \Sigma = \Lambda_2 \cap \Sigma = 1$.

(2) *For any $g_1 \in \Lambda_1 \vee \Sigma$ and $g_2 \in \Lambda_2 \vee \Sigma$ satisfying $g_1 g_2 \in \Lambda_1 \vee \Lambda_2$, one can find $a_1 \in \Lambda_1$, $a_2 \in \Lambda_2$ and $s \in \Sigma$ such that $g_1 = a_1 s$ and $g_2 = s^{-1} a_2$.*

Then, for any $y_1 \in \mathscr{U}(\mathcal{L}(\Lambda_1 \vee \Sigma))$ and $y_2 \in \mathscr{U}(\mathcal{L}(\Lambda_2 \vee \Sigma))$ satisfying $y_1 y_2 \in \mathscr{U}(\mathcal{L}(\Lambda_1 \vee \Lambda_2))$, one can find $v_1 \in \mathscr{U}(\mathcal{L}(\Lambda_1))$, $v_2 \in \mathscr{U}(\mathcal{L}(\Lambda_2))$, and $x \in \mathscr{U}(\mathcal{L}(\Sigma))$ such that $y_1 = v_1 x$ and $y_2 = x^* v_2$.

Above we used the notation that if $\Gamma_1, \Gamma_2 < \Gamma$ are groups, then we denote by $\Gamma_1 \vee \Gamma_2$ the subgroup of Γ generated by Γ_1 and Γ_2.

Proof of Lemma 5.1. For each $i = 1, 2$, consider the Fourier expansion of $y_i = \sum_{g_i \in \Lambda_i \vee \Sigma} (y_i)_{g_i} u_{g_i}$. Since $y_1 y_2 \in \mathcal{L}(\Lambda_1 \vee \Lambda_2)$ using condition (2), we have that

$$y = y_1 y_2 = \sum_{\substack{g_1 \in \Lambda_1 \vee \Sigma \\ g_2 \in \Lambda_2 \vee \Sigma \\ g_1 g_2 \in \Lambda_1 \vee \Lambda_2}} (y_1)_{g_1} (y_1)_{g_2} u_{g_1 g_2} = \sum_{\substack{a_1 \in \Lambda_1 \\ a_2 \in \Lambda_2 \\ s \in \Sigma}} (y_1)_{a_1 s} (y_2)_{s^{-1} a_2} u_{a_1 a_2}.$$

The above formula and basic approximations show that

$$1 = \sum_{s \in \Sigma} E_{\mathcal{L}(\Lambda_1)}(y_1 u_{s^{-1}}) E_{\mathcal{L}(\Lambda_2)}(u_s y_2) y^*,$$

where the right-hand side is only $\|\cdot\|_1$-summable. Using this in combination with the Cauchy–Schwarz inequality, we further get

$$\begin{aligned} 1 = \tau\Big(\sum_{s \in \Sigma} E_{\mathcal{L}(\Lambda_1)}(y_1 u_{s^{-1}}) E_{\mathcal{L}(\Lambda_2)}(u_s y_2) y^*\Big) &\leqslant \sum_{s \in \Sigma} |\tau(E_{\mathcal{L}(\Lambda_1)}(y_1 u_{s^{-1}}) E_{\mathcal{L}(\Lambda_2)}(u_s y_2) y^*)| \\ &\leqslant \sum_{s \in \Sigma} \|E_{\mathcal{L}(\Lambda_1)}(y_1 u_{s^{-1}})\|_2 \|E_{\mathcal{L}(\Lambda_2)}(u_s y_2)\|_2 \\ &\leqslant \Big(\sum_{s \in \Sigma} \|E_{\mathcal{L}(\Lambda_1)}(y_1 u_{s^{-1}})\|_2^2\Big)^{1/2} \Big(\sum_{s \in \Sigma} \|E_{\mathcal{L}(\Lambda_2)}(u_s y_2)\|_2^2\Big)^{1/2} \leqslant \|y_1\|_2 \|y_2\|_2 = 1. \end{aligned}$$

Thus, we must have equality in the Cauchy–Schwarz inequality, and hence, for every s, there is $c_s \in \mathbb{C}$ satisfying

$$E_{\mathcal{L}(\Lambda_1)}(y_1 u_{s^{-1}}) = c_s y E_{\mathcal{L}(\Lambda_2)}(y_2^* u_{s^{-1}}). \tag{5-1}$$

Taking absolute values, we get $|E_{\mathcal{L}(\Lambda_1)}(y_1u_{s^{-1}})| = |c_s||E_{\mathcal{L}(\Lambda_2)}(y_2^*u_{s^{-1}})|$, and since $\Lambda_1\cap\Lambda_2 = 1$ we conclude $|E_{\mathcal{L}(\Lambda_1)}(y_1u_{s^{-1}})| = |c_s||E_{\mathcal{L}(\Lambda_2)}(y_2^*u_{s^{-1}})| \in \mathbb{C}1$. Using the polar decomposition formula one can find $d_s, e_s \in \mathbb{C}$ and unitaries $x_s \in \mathcal{L}(\Lambda_1),\ z_s \in \mathcal{L}(\Lambda_2)$ satisfying

$$E_{\mathcal{L}(\Lambda_1)}(y_1u_{s^{-1}}) = d_sx_s \quad\text{and}\quad E_{\mathcal{L}(\Lambda_2)}(y_2^*u_{s^{-1}}) = e_sz_s.$$

Combining these with (5-1), we get $d_sx_s = c_se_syz_s$ for all $s\in\Sigma$; in particular, for every $d_s \neq 0$, we have $x_s = (e_sc_s/d_s)yz_s$. Hence, for all $s,t\in\Sigma$ with $d_s, d_t \neq 0$, we have $x_t^*x_s = (e_sc_s\overline{e_tc_t}/d_s\hat{d}_t)z_t^*z_s$. Again, as $\Lambda_1\cap\Lambda_2 = 1$, one can find $c_{s,t}, d_{s,t}\in\mathbb{C}$ such that

$$x_s = c_{s,t}x_t \quad\text{and}\quad z_s = d_{s,t}z_t. \tag{5-2}$$

Fix $t\in\Sigma$. Using the prior relations, we see that

$$y_1 = \sum_{s\in\Sigma} E_{\mathcal{L}(\Lambda_1)}(y_1u_{s^{-1}})u_s = \sum_{s\in\Sigma} d_sx_su_s = \sum_{s\in\Sigma} d_sc_{s,t}x_tu_s = x_t\Biggl(\sum_{s\in\Sigma} d_sc_{s,t}u_s\Biggr).$$

In particular, this shows there are $v_1\in\mathscr{U}(\mathcal{L}(\Lambda_1))$ and $x\in\mathscr{U}(\mathcal{L}(\Sigma))$ such that $y_1 = v_1x$. Similarly, the prior relations also imply that $y_2 = x^*v_2$ for some $v_2\in\mathscr{U}(\mathcal{L}(\Lambda_2))$. □

Theorem 5.2. *Let $\mathscr{G}$ be a graph in the class* CC_1, *and let $\mathscr{C}_1,\dots,\mathscr{C}_n$ be an enumeration of its consecutive cliques. Let Γ_v, $v\in\mathscr{V}$, be a collection of icc groups, and let $\Gamma_{\mathscr{G}}$ be the corresponding graph product group. For each $i\in\overline{1,n}$, assume $x_{i,i+1} = a_{i,i+1}b_{i,i+1}$, where $a_{i,i+1}\in\mathscr{U}(\mathcal{L}(\Gamma_{\mathscr{C}_{i,i+1}}))$ and $b_{i,i+1}\in\mathscr{U}(\mathcal{L}(\Gamma_{\mathscr{C}_i\cup\mathscr{C}_{i+1}\setminus\mathscr{C}_{i,i+1}}))$. If $x_{1,2}x_{2,3}\cdots x_{n-1,n}x_{n,1} = 1$, then for each $i\in\overline{1,n}$ one can find $a_i\in\mathscr{U}(\mathcal{L}(\Gamma_{\mathscr{C}_{i-1,i}}))$, $b_i\in\mathscr{U}(\mathcal{L}(\Gamma_{\mathscr{C}_i^{\mathrm{int}}}))$, and $c_i\in\mathscr{U}(\mathcal{L}(\Gamma_{\mathscr{C}_{i,i+1}}))$ such that $x_{i,i+1} = a_ib_ic_ib_{i+1}^*a_{i+2}^*c_{i+1}^*$. Here, we use the convention that $n+1 = 1$ and $n+2 = 2$.*

Proof. Fix an arbitrary $i\in\overline{1,n}$. Using $x_{1,2}x_{2,3}\cdots x_{n-1,n}x_{n,1} = 1$, it follows that

$$b_{i-1,i}b_{i,i+1} = a_{i-1,i}^*x_{i-2,i-1}^*\cdots x_{1,2}^*x_{n,1}^*\cdots x_{i+1,i+2}^*a_{i,i+1}^*.$$

Since $a_{i-1,i}, a_{i,i+1}\in\mathcal{L}(\Gamma_{\mathscr{G}\setminus\mathscr{C}_i^{\mathrm{int}}})$ and $x_{j,j+1}, a_{j,j+1}\in\mathcal{L}(\Gamma_{\mathscr{C}_j\cup\mathscr{C}_{j+1}})$, for any $j\in\overline{1,n}$, we get that

$$b_{i-1,i}b_{i,i+1} = a_{i-1,i}^*x_{i-2,i-1}^*\cdots x_{1,2}^*x_{n,1}^*\cdots x_{i+1,i+2}^*a_{i,i+1}^*\in\mathcal{L}(\Gamma_{\mathscr{G}\setminus\mathscr{C}_i^{\mathrm{int}}}).$$

Since $b_{i-1,i}b_{i,i+1}\in\mathcal{L}(\Gamma_{\mathscr{C}_{i-1}\cup\mathscr{C}_i\cup\mathscr{C}_{i+1}})$, we deduce that

$$b_{i-1,i}b_{i,i+1}\in\mathscr{U}(\mathcal{L}(\Gamma_{\mathscr{C}_{i-1}\cup\mathscr{C}_{i+1}})). \tag{5-3}$$

Now, fix two words $g_1\in\Gamma_{(\mathscr{C}_{i-1}\cup\mathscr{C}_i)\setminus\mathscr{C}_{i-1,i}}$ and $g_2\in\Gamma_{(\mathscr{C}_i\cup\mathscr{C}_{i+1})\setminus\mathscr{C}_{i,i+1}}$ such that $g_1g_2\in\Gamma_{\mathscr{C}_{i-1}\cup\mathscr{C}_{i+1}}$. Using Proposition 4.2 (1), there exist $a_1\in\Gamma_{(\mathscr{C}_{i-1}\setminus\mathscr{C}_i)\cup\mathscr{C}_{i,i+1}}$, $b\in\Gamma_{(\mathscr{C}_{i+1}\setminus\mathscr{C}_i)\cup\mathscr{C}_{i-1,i}}$, and $s\in\Gamma_{\mathscr{C}_i^{\mathrm{int}}}$ such that $g_1 = as$ and $g_2 = s^{-1}b$. Thus, applying Lemma 5.1 for $\Lambda_1 = \Gamma_{(\mathscr{C}_{i-1}\setminus\mathscr{C}_i)\cup\mathscr{C}_{i,i+1}}$, $\Lambda_2 = \Gamma_{(\mathscr{C}_{i+1}\setminus\mathscr{C}_i)\cup\mathscr{C}_{i-1,i}}$, and $\Sigma = \Gamma_{\mathscr{C}_i^{\mathrm{int}}}$, we derive from (5-3) that one can find unitaries $x_{i-1}\in\mathcal{L}(\Gamma_{(\mathscr{C}_{i-1}\setminus\mathscr{C}_i)\cup\mathscr{C}_{i,i+1}})$ and $z_{i-1}\in\mathcal{L}(\Gamma_{\mathscr{C}_i^{\mathrm{int}}})$ such that $b_{i-1,i} = x_{i-1}z_{i-1}$.

Lemma 5.1 also implies from (5-3) that $b_{i,i+1} = z_{i-1}^*y_i$ for some $y_i\in\mathscr{U}(\mathcal{L}(\Gamma_{(\mathscr{C}_{i+1}\setminus\mathscr{C}_i)\cup\mathscr{C}_{i-1,i}}))$. Using this with $b_{i,i+1} = x_iz_i$, we get that $b_{i,i+1} = z_{i-1}^*y_i = x_iz_i$. Hence, $x_i^*z_{i-1}^* = z_iy_i^* =: t_{i,i+1}$ and note that $t_{i,i+1} \in \mathcal{L}(\Gamma_{\mathscr{C}_{i-1,i}\cup\mathscr{C}_{i+1,i+2}})$. Thus, $x_{i,i+1} = a_{i,i+1}b_{i,i+1} = a_{i,i+1}z_{i-1}^* t_{i,i+1}^* z_i = z_{i-1}^* a_{i,i+1} t_{i,i+1}^* z_i$. In conclusion, we showed that, for any $i \in \overline{1,n}$, we have

$$x_{i,i+1} = z_{i-1}^* a_{i,i+1} t_{i,i+1} z_i. \tag{5-4}$$

Now, we note that, since $x_{1,2}x_{2,3}\cdots x_{n-1,n}x_{n,1} = 1$, we obviously have

$$a_{1,2}t_{1,2}a_{2,3}t_{2,3}\cdots a_{n-1,n}t_{n-1,n}a_{n,1}t_{n,1} = 1.$$

Again we will use this relation together with the same argument from the proof of Lemma 5.1 to show that $x_{i,i+1}$ has the form described in the conclusion of the theorem. First, observe the cyclic relation and use a similar argument as in the beginning of the proof to show that

$$\begin{aligned} w &:= t_{i-1,i}a_{i,i+1}t_{i,i+1}a_{i+1,i+2}t_{i+1,i+2}a_{i+2,i+3} \\ &= a_{i-1,i}^* \cdots t_{1,2}^* a_{1,2}^* t_{n,1}^* a_{n,1}^* \cdots t_{i+2,i+3}^* \in \mathcal{L}(\Gamma_{\mathscr{C}_{i-2,i-1}\cup\mathscr{C}_{i-1,i}\cup\mathscr{C}_{i+1,i+2}\cup\mathscr{C}_{i+2,i+3}}). \end{aligned}$$

Now, fix three words $w_1 \in \Gamma_{\mathscr{C}_{i-2,i-1}\cup\mathscr{C}_{i,i+1}}$, $w_2 \in \Gamma_{\mathscr{C}_{i-1,i}\cup\mathscr{C}_{i+1,i+2}}$, and $w_3 \in \Gamma_{\mathscr{C}_{i,i+1}\cup\mathscr{C}_{i+2,i+3}}$ satisfying $w_1w_2w_3 \in \Gamma_{\mathscr{C}_{i-2,i-1}\cup\mathscr{C}_{i-1,i}\cup\mathscr{C}_{i+1,i+2}\cup\mathscr{C}_{i+2,i+3}}$. Using Proposition 4.2 (2), we have $w_1 = as$ and $w_3 = s^{-1}b$, where $a \in \Gamma_{\mathscr{C}_{i-2,i-1}}$, $b \in \Gamma_{\mathscr{C}_{i+2,i+3}}$, and $s \in \Gamma_{\mathscr{C}_{i,i+1}}$. Since $t_{i,i+1}a_{i+1,i+2} \in \mathcal{L}(\Gamma_{\mathscr{C}_{i-1,i}\cup\mathscr{C}_{i+1,i+2}})$, we can write the Fourier expansions

$$\begin{aligned} t_{i-1,i}a_{i,i+1} &= \sum_{w_1\in\mathscr{C}_{i-2,i-1}\cup\mathscr{C}_{i,i+1}} (t_{i-1,i}a_{i,i+1})_{w_1}u_{w_1}, \\ t_{i+1,i+2}a_{i+2,i+3} &= \sum_{w_3\in\mathscr{C}_{i,i+1}\cup\mathscr{C}_{i+2,i+3}} (t_{i+1,i+2}a_{i+2,i+3})_{w_3}u_{w_3}. \end{aligned}$$

All these observations imply that

$$\begin{aligned} w &= (t_{i-1,i}a_{i,i+1})(t_{i,i+1}a_{i+1,i+2})(t_{i+1,i+2}a_{i+2,i+3}) \\ &= \sum_{s\in\Gamma_{\mathscr{C}_{i,i+1}}} E_{\mathcal{L}(\Gamma_{\mathscr{C}_{i-2,i-1}})}(t_{i-1,i}a_{i,i+1}u_{s^{-1}})t_{i,i+1}a_{i+1,i+2}E_{\mathcal{L}(\Gamma_{\mathscr{C}_{i+2,i+3}})}(u_s t_{i+1,i+2}a_{i+2,i+3}), \end{aligned}$$

where again the convergence is in $\|\cdot\|_1$. Using this and the Cauchy–Schwarz inequality, we get

$$\begin{aligned} 1 &= \sum_{s\in\Gamma_{\mathscr{C}_{i,i+1}}} |\tau(E_{\mathcal{L}(\Gamma_{\mathscr{C}_{i-2,i-1}})}(t_{i-1,i}a_{i,i+1}u_{s^{-1}})t_{i,i+1}a_{i+1,i+2}E_{\mathcal{L}(\Gamma_{\mathscr{C}_{i+2,i+3}})}(u_s t_{i+1,i+2}a_{i+2,i+3})w^*)| \\ &\leqslant \sum_{s\in\Gamma_{\mathscr{C}_{i,i+1}}} \|E_{\mathcal{L}(\Gamma_{\mathscr{C}_{i-2,i-1}})}(t_{i-1,i}a_{i,i+1}u_{s^{-1}})\|_2 \|E_{\mathcal{L}(\Gamma_{\mathscr{C}_{i+2,i+3}})}(a_{i+2,i+3}^* t_{i+1,i+2}^* u_{s^{-1}})\|_2 \\ &\leqslant \Bigg(\sum_{s\in\Gamma_{\mathscr{C}_{i,i+1}}} \|E_{\mathcal{L}(\Gamma_{\mathscr{C}_{i-2,i-1}})}(t_{i-1,i}a_{i,i+1}u_{s^{-1}})\|_2^2\Bigg)^{1/2} \Bigg(\sum_{s\in\Gamma_{\mathscr{C}_{i,i+1}}} \|E_{\mathcal{L}(\Gamma_{\mathscr{C}_{i+2,i+3}})}(a_{i+2,i+3}^* t_{i+1,i+2}^* u_{s^{-1}})\|_2^2\Bigg)^{1/2} \\ &\leqslant \|t_{i-1,i}a_{i,i+1}\|_2 \|a_{i+2,i+3}^* t_{i+1,i+2}^*\|_2 = 1. \end{aligned}$$

Therefore, one can get scalars c_s such that

$$E_{\mathcal{L}(\Gamma_{\mathscr{C}_{i-2,i-1}})}(t_{i-1,i}a_{i,i+1}u_{s^{-1}}) = c_s w E_{\mathcal{L}(\Gamma_{\mathscr{C}_{i+2,i+3}})}(a_{i+2,i+3}^* t_{i+1,i+2}^* u_{s^{-1}}) a_{i+1,i+2}^* t_{i,i+1}^*. \tag{5-5}$$

Thus, proceeding in the same fashion as in the proof of Lemma 5.1, one can find $d_s, e_s \in \mathbb{C}$, $g_s \in \mathscr{U}(\mathcal{L}(\Gamma_{\mathscr{C}_{i-2,i-1}}))$, and $h_s \in \mathscr{U}(\mathcal{L}(\Gamma_{\mathscr{C}_{i+2,i+3}}))$ such that

$$\begin{aligned} E_{\mathcal{L}(\Gamma_{\mathscr{C}_{i-2,i-1}})}(t_{i-1,i}a_{i,i+1}u_{s^{-1}}) &= d_s g_s, \\ E_{\mathcal{L}(\Gamma_{\mathscr{C}_{i+2,i+3}})}(a^*_{i+2,i+3}t^*_{i+1,i+2}u_{s^{-1}})a^*_{i+1,i+2}t^*_{i,i+1} &= e_s h_s. \end{aligned} \tag{5-6}$$

Hence, (5-5) gives that $d_s g_s = c_s e_s w h_s$ for all $s \in \Gamma_{\mathscr{C}_{i,i+1}}$, and finally employing the same arguments as in the first part one can find scalars $c_{s,t}$, $d_{s,t}$ such that $g_s = c_{s,t} g_t$ and $h_s = d_{s,t} h_t$ for all $s, t \in \Gamma_{\mathscr{C}_{i,i+1}}$. Using (5-6), we derive that

$$t_{i-1,i}a_{i,i+1} = \sum_{s\in\Gamma_{\mathscr{C}_{i,i+1}}} d_s g_s u_s = g_e \sum_{s\in\Gamma_{\mathscr{C}_{i,i+1}}} d_s c_{s,e} u_s.$$

This further implies that one can find unitaries $r_{i-1} \in \mathscr{U}(\mathcal{L}(\Gamma_{\mathscr{C}_{i-2,i-1}}))$ and $p_{i-1} \in \mathscr{U}(\mathcal{L}(\Gamma_{\mathscr{C}_{i,i+1}}))$ such that $t_{i-1,i}a_{i,i+1} = r_{i-1}p_{i-1}$, and hence $t_{i-1,i} = r_{i-1}p_{i-1}a^*_{i,i+1}$. Similarly, we get $t_{i,i+1} = r_i p_i a^*_{i+1,i+2}$, and hence, from (5-4), we deduce that

$$x_{i,i+1} = z^*_{i-1}a_{i,i+1}t_{i,i+1}z_i = z^*_{i-1}a_{i,i+1}r_i p_i a^*_{i+1,i+2}z_i = r_i z^*_{i-1}a_{i,i+1}z_i p_i a^*_{i+1,i+2}. \tag{5-7}$$

Now, one can see that, using the cyclic relation $x_{1,2}\cdots x_{n-1,n}x_{n,1} = 1$, we get that $p_i = r^*_{i+2}$. This together with (5-7) gives the desired conclusion by taking $a_i = r_i$, $b_i = z^*_{i-1}$, and $c_i = a_{i,i+1}$. □

6. Rigid subalgebras of graph product group von Neumann algebras

In this section we classify all rigid subalgebras of von Neumann algebras associated with graph product groups. This should be viewed as a counterpart of [Ioana et al. 2008, Theorem 4.3] for amalgamated free product von Neumann algebras. In fact, the latter plays an essential role in deriving our result. For convenience, we include a detailed proof of how it follows from this.

Theorem 6.1. *Let $\Gamma = \mathscr{G}\{\Gamma_v\}$ be a graph product group, let $\Gamma \curvearrowright \mathcal{P}$ be any trace-preserving action, and denote by $\mathcal{M} = \mathcal{P} \rtimes \Gamma$ the corresponding crossed product von Neumann algebra. Let $r \in \mathcal{M}$ be a projection, and let $\mathcal{Q} \subset r\mathcal{M}r$ be a property (T) von Neumann subalgebra.*

Then one can find a clique $\mathscr{C} \in \operatorname{cliq}(\mathscr{G})$ such that $\mathcal{Q} \prec_{\mathcal{M}} \mathcal{P} \rtimes \Gamma_{\mathscr{C}}$. Moreover, if $\mathcal{Q} \nprec \mathcal{P} \rtimes \Gamma_{\mathscr{C}\setminus\{c\}}$ for all $c \in \mathscr{C}$, then one can find projections $q \in \mathcal{Q}$ and $q' \in \mathcal{Q}' \cap r\mathcal{M}r$ with $qq' \neq 0$ and a unitary $u \in \mathcal{M}$ such that $uq\mathcal{Q}qq'u^* \subseteq \mathcal{P} \rtimes \Gamma_{\mathscr{C}}$. In particular, if $\mathcal{P} \rtimes \Gamma_{\mathscr{C}}$ is a factor, then one can take $q = 1$ above.

Proof. Let $\mathscr{G}_0 = (\mathscr{V}_0, \mathscr{E}_0) \subseteq \mathscr{G} = (\mathscr{V}, \mathscr{E})$ be a subgraph with $|\mathscr{V}_0|$ minimal such that $\mathcal{Q} \prec \mathcal{P} \rtimes \Gamma_{\mathscr{G}_0}$. In the remaining part we show that $\mathscr{G}_0$ is complete, which proves the conclusion.

Write $\mathcal{N} = \mathcal{P} \rtimes \Gamma_{\mathscr{G}_0}$. Since $\mathcal{Q} \prec_{\mathcal{M}} \mathcal{N}$, one can find projections $q \in \mathcal{Q}$ and $p \in \mathcal{N}$, a nonzero partial isometry $v \in p\mathcal{M}q$, and a $*$-isomorphism onto its image $\theta : q\mathcal{Q}q \to \mathcal{R} := \theta(q\mathcal{Q}q) \subseteq p\mathcal{N}p$ such that $\theta(x)v = vx$ for all $x \in q\mathcal{Q}q$. Notice that $vv^* \in \mathcal{R}' \cap p\mathcal{M}p$ and $v^*v \in (\mathcal{Q}' \cap \mathcal{M})q$. Moreover, one can assume without any loss of generality that the support projection of $E_{\mathcal{N}}(vv^*)$ equals p.

Assume by contradiction that $\mathscr{G}_0$ is not complete. Thus, one can find $v \in \mathscr{G}_0$ such that $\Gamma_{\mathscr{G}_0}$ admits a noncanonical amalgam decomposition $\Gamma_{\mathscr{G}_0} = \Gamma_{\mathscr{G}_0\setminus\{v\}} *_{\Gamma_{\mathrm{lk}(v)}} \Gamma_{\mathrm{st}(v)}$; in particular, we have $|\mathrm{st}(v)| \leqslant |\mathscr{V}_0| - 1$. Since $\mathcal{Q}$ has property (T), $\mathcal{R}$ has property (T) as well. Using [Ioana et al. 2008, Theorem 5.1], we have either (i) $\mathcal{R} \prec_{\mathcal{N}} \mathcal{P} \rtimes \Gamma_{\mathscr{G}_0\setminus\{v\}}$, or (ii) $\mathcal{R} \prec_{\mathcal{N}} \mathcal{P} \rtimes \Gamma_{\mathrm{st}(v)}$. Assume (i). Define $\mathcal{X} := \mathcal{P} \rtimes \Gamma_{\mathscr{G}_0\setminus\{v\}}$. As $\mathcal{R} \prec_{\mathcal{N}} \mathcal{X}$, one can find projections $e \in \mathcal{R}$ and $f \in \mathcal{X}$, a nonzero partial isometry $w \in f\mathcal{N}e$, and a $*$-isomorphism onto its image $\psi : e\mathcal{R}e \to \mathcal{T} := \theta(e\mathcal{R}e) \subseteq f\mathcal{X}f$ such that $\psi(x)w = wx$ for all $x \in e\mathcal{R}e$.

Next, we argue that $wv \neq 0$. Otherwise, we would have $0 = wvv^*$, and since $w \in \mathcal{N}$, we get $0 = wE_{\mathcal{N}}(vv^*)$. Therefore $0 = ws(E_{\mathcal{N}}(vv^*)) = wp = w$, which is a contradiction. Combining the previous intertwining relations, we get $\phi(\theta(x))wv = w\theta(x)v = wvx$ for all $x \in t\mathcal{Q}t$; here we write $0 \neq t = \theta^{-1}(e)$. Taking the polar decomposition of wv in the prior intertwining relation, we obtain that $\mathcal{Q} \prec_{\mathcal{M}} \mathcal{X} = \mathcal{P} \rtimes \Gamma_{\mathscr{G}_0 \setminus \{v\}}$. However, since $|\mathscr{V}_0 \setminus \{v\}| = |\mathscr{V}_0| - 1$, this contradicts the minimality of $|\mathscr{V}_0|$. In a similar manner, one can show case (ii) also leads to a contradiction.

Next, we show the moreover part. Let $\mathcal{S} = \mathcal{P} \rtimes \Gamma_{\mathscr{C}}$. From the first part of the proof one can find projections $q \in \mathcal{Q}$ and $s \in \mathcal{S}$, a nonzero partial isometry $v_0 \in s\mathcal{M}q$, and a $*$-isomorphism onto its image $\theta : q\mathcal{Q}q \to \mathcal{Y} := \theta(q\mathcal{Q}q) \subseteq s\mathcal{S}s$ such that $\theta(x)v_0 = v_0x$ for all $x \in q\mathcal{Q}q$. We note that $v_0v_0^* \in \mathcal{Y}' \cap s\mathcal{M}s$ and $v_0^*v_0 \in q\mathcal{Q}q' \cap q\mathcal{M}q$. Moreover, one can assume without loss of generality that the support projection of $E_{\mathcal{S}}(v_0v_0^*)$ equals s. Observe that we have an amalgamated free product decomposition $\mathcal{M} = (\mathcal{P} \rtimes \Gamma_{\mathscr{V} \setminus \{c\}}) *_{\mathcal{P} \rtimes \Gamma_{\mathscr{C} \setminus \{c\}}} (\mathcal{P} \rtimes \Gamma_{\mathscr{C}})$. Using the same argument as before, since $\mathcal{Q} \nprec_{\mathcal{M}} \mathcal{P} \rtimes \Gamma_{\mathscr{C} \setminus \{c\}}$, we must have that $\mathcal{Y} \nprec_{\mathcal{S}} \mathcal{P} \rtimes \Gamma_{\mathscr{C} \setminus \{c\}}$. Therefore, by [Ioana et al. 2008, Theorem 1.2.1], we have that $v_0v_0^* \in \mathcal{S}$, and hence the intertwining relation implies that $v_0q\mathcal{Q}qv_0^* = \mathcal{Y}v_0v_0^* \subseteq \mathcal{S}$. If u is a unitary extending v_0, we further see that $uq\mathcal{Q}qv_0^*v_0u^* \subseteq \mathcal{S}$. Letting $q' = v_0^*v_0$, we get the desired conclusion.

To see the last part, we note that, since $\mathcal{P} \rtimes \Gamma_{\mathscr{C}}$ is a factor, after passing to a new unitary u, one can replace q above with its central support in $\mathcal{Q}$. □

7. Symmetries of graph product group von Neumann algebras

The main result of this section is a strong rigidity result describing all $*$-isomorphisms between factors associated with a fairly large family of graph product groups arising from finite graphs in the class CC_1 (Theorem 7.10). As a by-product, we obtain concrete descriptions of all symmetries of these factors including such examples with trivial fundamental groups (Corollaries 7.11 and 7.12). However, to be able to state and prove these results, we first need to introduce some new terminology and establish a few preliminary results.

7.1. ***Local isomorphisms of graph product group von Neumann algebras.*** The isomorphism class of a von Neumann algebra associated with a graph product group tends to be fairly abundant. As in the group situation, a rich source of isomorphisms stems from both the isomorphism class of the underlying graph and the isomorphism classes of the von Neumann algebras of the vertex groups. By analogy with the group case, these are called local isomorphisms and we briefly explain their construction below.

Let $\mathscr{G}$ and $\mathscr{H}$ be simple finite graphs, and let $\Gamma\mathscr{G}$ and $\Lambda_{\mathscr{H}}$ be graph product groups, where their vertex groups are $\{\Gamma_v, v \in \mathscr{V}\}$ and $\{\Lambda_w, w \in \mathscr{W}\}$, respectively. Assume $\mathscr{G}$ and $\mathscr{H}$ are isometric and fix $\sigma : \mathscr{G} \to \mathscr{H}$ an isometry. In addition, assume that $\Phi = \{\Phi_v^\sigma, v \in \mathscr{V}\}$ is a collection of $*$-isomorphisms $\Phi_v^\sigma : \mathcal{L}(\Gamma_v) \to \mathcal{L}(\Lambda_{\sigma(v)})$ for all $v \in \mathscr{V}$. Then the following holds.

Theorem 7.1. *There exists a unique $*$-isomorphism denoted by $(\Phi, \sigma) : \mathcal{L}(\Gamma_{\mathscr{G}}) \to \mathcal{L}(\Lambda_{\mathscr{H}})$ which extends the maps $\bigcup_{v \in \mathscr{V}} \mathcal{L}(\Gamma_v) \ni x \to \Phi_v^\sigma(x) \in \mathcal{L}(\Lambda_{\mathscr{H}})$.*

Proof. We recall that a word for $\mathscr{G}$ is a finite sequence $\boldsymbol{v} = (v_1, \dots, v_n)$ of elements in $\mathscr{V}$ [Caspers and Fima 2017, Definition 1.2]. The word $\boldsymbol{v}$ is called reduced if, whenever $i < j$ and $v_{i+1}, \dots, v_{j-1} \in \mathrm{st}(v_j)$, we have $v_i \neq v_j$. Following [Caspers and Fima 2017, Section 2.3], $\mathcal{L}(\Gamma_{\mathscr{G}})$ can be presented alternatively as the graph product von Neumann algebra associated to the graph $\mathscr{G}$ and vertex von Neumann algebras $\{\mathcal{L}(\Gamma_v)\}_{v\in\mathscr{V}}$.

We continue by proving the following claim: for any reduced word $(v_1, \dots, v_n)$ in $\mathscr{G}$ and elements $a_i \in \mathcal{L}(\Gamma_{v_i})$ with $\tau(a_i) = 0$, we have $\tau(\Phi^\sigma_{v_1}(a_1)\cdots\Phi^\sigma_{v_n}(a_n)) = 0$. To show this, write $w_i = \sigma(v_i) \in \mathscr{W}$ and $b_i = \Phi^\sigma_{v_i}(a_i) \in \mathcal{L}(\Lambda_{w_i})$ for any i. Note that the word $(w_1, \dots, w_n)$ is reduced in $\mathscr{H}$ and $\tau(b_i) = 0$ for any i. By considering the Fourier series of b_i, the claim follows by proving that, whenever $h_i \in \Lambda_{w_i}$ with $h_i \neq 1$, we have $h_1 \cdots h_n \neq 1$. Since $(w_1, \dots, w_n)$ is a reduced word in $\mathscr{H}$, it is easy to see that $h_1 \cdots h_n$ is a reduced element of $\Lambda_{\mathscr{H}}$ in the sense of [Green 1990, Definition 3.5]. Applying [Green 1990, Theorem 3.9] implies that $h_1 \cdots h_n \neq 1$, hence proving the claim.

Finally, our theorem follows directly by applying [Caspers and Fima 2017, Proposition 2.22]. □

Hereafter, (Φ, σ) will be called the *local isomorphism* induced by σ and $\Phi = \{\Phi^\sigma_v, v \in \mathscr{V}\}$. Whenever $\mathscr{G} = \mathscr{H}$ and $\Gamma_v = \Lambda_v$ for all v, these are called *local automorphisms* and they form a subgroup of $\mathrm{Aut}(\mathcal{L}(\Gamma_{\mathscr{G}}))$ under composition which will be denoted by $\mathrm{Loc}_{\mathrm{v,g}}(\mathcal{L}(\Gamma_{\mathscr{G}}))$. The subgroup of local automorphisms satisfying $\sigma = \mathrm{Id}$ is denoted by $\mathrm{Loc}_{\mathrm{v}}(\mathcal{L}(\Gamma_{\mathscr{G}}))$; observe that it has finite index in $\mathrm{Loc}_{\mathrm{v}}(\mathcal{L}(\Gamma_{\mathscr{G}}))$. Moreover, we have u $\mathrm{Loc}_{\mathrm{v}}(\mathcal{L}(\Gamma_{\mathscr{G}})) = \bigoplus_{v\in\mathscr{V}} \mathrm{Aut}(\mathcal{L}(\Gamma_v))$. Next, we observe that most of the time (Φ, σ) is an outer automorphism.

Proposition 7.2. *Under the same assumptions as before, suppose in addition that $\mathscr{G}$ is a graph satisfying $\bigcap_{v\in\mathscr{V}} \mathrm{star}(v) = \varnothing$. Then (Φ, σ) is inner if and only if $\sigma = \mathrm{Id}$ and $\Phi^\sigma_v = \mathrm{Id}$ for all $v \in \mathscr{V}$.*

Proof. Let $\mathcal{M} = \mathcal{L}(\Gamma_{\mathscr{G}})$, and let $u \in \mathscr{U}(\mathcal{M})$ such that $(\Phi, \sigma) = \mathrm{ad}(u)$. Fix $v \in \mathscr{V}$. From the definitions we have $u\mathcal{L}(\Gamma_v)u^* = \mathcal{L}(\Gamma_{\sigma(v)})$. Using Theorem 2.7, we get that $v = \sigma(v)$ and $u \in \mathscr{U}(\mathcal{L}(\Gamma_{\mathrm{star}(v)}))$. As this holds for all $v \in \mathscr{V}$, we get $\sigma = \mathrm{Id}$ and also $u \in \bigcap_{v\in\mathscr{V}} \mathcal{L}(\Gamma_{\mathrm{star}(v)}) = \mathbb{C}1$. Hence $(\Phi, \sigma) = \mathrm{Id}$, and also $\Phi^\sigma_v = \mathrm{Id}$ for all $v \in \mathscr{V}$. □

When $\sigma = \mathrm{Id}$, let $\mathrm{Loc}_{\mathrm{v,i}}(\mathcal{L}(\Gamma_{\mathscr{G}}))$ be the set of all local automorphisms (Φ, σ) which satisfies the following: for any $v \in \mathscr{V}$, there exists a unitary $u_v \in \mathcal{L}(\Gamma_v)$ such that $\Phi^\sigma_v = \mathrm{ad}(u_v)$. It is easy to see that $\mathrm{Loc}_{\mathrm{v,i}}(\mathcal{L}(\Gamma_{\mathscr{G}}))$ forms a normal subgroup of $\mathrm{Loc}_{\mathrm{v}}(\mathcal{L}(\Gamma_{\mathscr{G}}))$ under composition. Thus, when there exists a $v \in \mathscr{V}$ for which Γ_v is an icc group, it follows from Proposition 7.2 that $\mathrm{Loc}_{\mathrm{v,i}}(\mathcal{L}(\Gamma_{\mathscr{G}}))$, and hence $\mathrm{Out}(\mathcal{L}(\Gamma_{\mathscr{G}}))$, is always an uncountable group. In conclusion, for this class of von Neumann algebras, in general, one cannot expect rigidity results and computations of their symmetries of the same precision level as the prior results [Popa and Vaes 2008; Vaes 2008].

Remark 7.3. It is worth mentioning that the class of local isomorphisms can be defined for all tracial graph products [Caspers and Fima 2017] (regardless if they come form groups or not) with essentially the same proofs.

Next, we highlight a family of $*$-isomorphisms between graph product von Neumann algebras that is specific to graphs in the class CC_1 and is related more to the clique algebras structure than to the vertex algebra structure as in the previous part. As before, let $\mathscr{G}, \mathscr{H} \in \mathrm{CC}_1$ be isomorphic graphs, and fix $\sigma : \mathscr{G} \to \mathscr{H}$ an isometry. Let $\mathrm{cliq}(\mathscr{G}) = \{\mathscr{C}_1, \dots, \mathscr{C}_n\}$ be a consecutive enumeration of the cliques of $\mathscr{G}$. Let $\Gamma_{\mathscr{G}}$ and $\Lambda_{\mathscr{H}}$ be graph product groups and assume, for every $i \in \overline{1, n}$, there are $*$-isomorphisms $\theta_{i-1,i} : \mathcal{L}(\Gamma_{\mathscr{C}_{i-1,i}}) \to \mathcal{L}(\Lambda_{\mathscr{C}_{\sigma(\mathscr{C}_{i-1,i})}})$ and $\xi_i : \mathcal{L}(\Gamma_{\mathscr{C}_i^{\mathrm{int}}}) \to \mathcal{L}(\Lambda_{\sigma(\mathscr{C}_i^{\mathrm{int}})})$. Here, and afterwards, we use the notation $\mathscr{C}_{0,1} = \mathscr{C}_{n,1}$. Using Lemma 4.1, we can view $\Gamma_{\mathscr{G}}$ as a graph product group $\Gamma'_{\mathscr{F}_n}$ over the graph $\mathscr{F}_n$, where the vertex groups satisfy $\Gamma'_{w_i} = \bigoplus_{v \in \mathscr{C}_i^{\mathrm{int}}} \Gamma_v$ and $\Gamma'_{b_i} = \bigoplus_{v \in \mathscr{C}_{i-1,i}} \Gamma_v$. Similarly, $\Lambda_{\mathscr{H}} = \Lambda'_{\mathscr{F}_n}$, where $\Lambda'_{\sigma(w_i)} = \bigoplus_{v \in \mathscr{C}_i^{\mathrm{int}}} \Lambda_{\sigma(v)}$ and $\Lambda'_{\sigma(b_i)} = \bigoplus_{v \in \mathscr{C}_{i-1,i}} \Lambda_{\sigma(v)}$. Therefore, using Theorem 7.1, these isomorphisms induce a unique $*$-isomorphism $\phi_{\theta,\xi,\sigma} : \mathcal{L}(\Gamma_{\mathscr{G}}) \to \mathcal{L}(\Lambda_{\mathscr{H}})$:

$$\phi_{\theta,\xi,\sigma}(x) = \begin{cases} \theta_{i-1,i}(x) & \text{if } x \in \mathcal{L}(\Gamma_{\mathscr{C}_{i-1,i}}), \\ \xi_i(x) & \text{if } x \in \mathcal{L}(\Gamma_{\mathscr{C}_i^{\mathrm{int}}}) \end{cases} \tag{7-1}$$

for all $i \in \overline{1, n}$.

When $\Gamma_{\mathscr{G}} = \Lambda_{\mathscr{H}}$, this construction yields a group of automorphisms of $\mathcal{L}(\Gamma_{\mathscr{G}})$ that will be denoted by $\mathrm{Loc}_{\mathrm{c,g}}(\mathcal{L}(\Gamma_{\mathscr{G}}))$. We also denote by $\mathrm{Loc}_{\mathrm{c}}(\mathcal{L}(\Gamma_{\mathscr{G}}))$ the subgroup of all automorphisms satisfying $\sigma = \mathrm{Id}$. Note: $\mathrm{Loc}_{\mathrm{c}}(\mathcal{L}(\Gamma_{\mathscr{G}})) \cong \bigoplus_i \mathrm{Aut}(\mathcal{L}(\Gamma_{\mathscr{C}_{i-1,i}})) \oplus \mathrm{Aut}(\mathcal{L}(\Gamma_{\mathscr{C}_i^{\mathrm{int}}}))$ and also $\mathrm{Loc}_{\mathrm{c}}(\mathcal{L}(\Gamma_{\mathscr{G}})) \leqslant \mathrm{Loc}_{\mathrm{c,g}}(\mathcal{L}(\Gamma_{\mathscr{G}}))$ has finite index.

Next, we highlight a subgroup of automorphisms in $\mathrm{Loc}_{\mathrm{c}}(\mathcal{L}(\Gamma_{\mathscr{G}}))$ that will be useful in stating our main results. Namely, consider a family of nontrivial unitaries $a_{i-1,i} \in \mathcal{L}(\Gamma_{\mathscr{C}_{i-1,i}})$ and $b_i \in \mathcal{L}(\Gamma_{\mathscr{C}_i^{\mathrm{int}}})$ for every $i \in \overline{1, n}$. If in the formula (7-1) we let $\theta_{i-1,i} = \mathrm{ad}(a_{i-1,i})$ and $\xi_i = \mathrm{ad}(b_i)$, then the corresponding automorphism $\phi_{\theta,\xi,\mathrm{Id}}$ is an (most of the times outer) automorphism of $\mathcal{L}(\Gamma)$ which we will denote by $\phi_{a,b}$ throughout this section. The set of all such automorphisms forms a normal subgroup denoted by $\mathrm{Loc}_{\mathrm{c,i}}(\mathcal{L}(\Gamma_{\mathscr{G}})) \lhd \mathrm{Loc}_{\mathrm{c}}(\mathcal{L}(\Gamma_{\mathscr{G}}))$. From the definitions we also have $\mathrm{Loc}_{\mathrm{v,i}}(\mathcal{L}(\Gamma_{\mathscr{G}})) < \mathrm{Loc}_{\mathrm{c,i}}(\mathcal{L}(\Gamma_{\mathscr{G}}))$ and $\mathrm{Loc}_{\mathrm{v}}(\mathcal{L}(\Gamma_{\mathscr{G}})) < \mathrm{Loc}_{\mathrm{c}}(\mathcal{L}(\Gamma_{\mathscr{G}}))$.

Proposition 7.4. *The automorphism $\phi_{a,b} \in \mathrm{Loc}_{\mathrm{c,i}}(\mathcal{L}(\Gamma_{\mathscr{G}}))$ is inner if and only if $a_{i,i+1} \in \mathscr{Z}(\mathcal{L}(\Gamma_{\mathscr{C}_{i-1,i}}))$ and $b_i \in \mathscr{Z}(\mathcal{L}(\Gamma_{\mathscr{C}_i^{\mathrm{int}}}))$ for all $i \in \overline{1, n}$.*

Proof. Assume that $\phi_{a,b} \in \mathrm{Loc}_{\mathrm{c,i}}(\mathcal{L}(\Gamma_{\mathscr{G}}))$ is inner, and hence, there is a unitary $u \in \mathcal{L}(\Gamma_{\mathscr{G}})$ such that $\phi_{a,b}(x)u = ux$ for any $x \in \mathcal{L}(\Gamma_{\mathscr{G}})$. Fix an arbitrary $i \in \overline{1, n}$. Then, for any $x \in \mathcal{L}(\Gamma_{\mathscr{C}_{i-1,i}})$, we have $a_{i-1,i} x a^*_{i,i-1} = uxu^*$. Together with Theorem 2.7, this yields $u^* a_{i,i-1} \in \mathcal{L}(\Gamma_{\mathscr{C}_{i-1,i}})' \cap \mathcal{L}(\Gamma_{\mathscr{G}}) \subset \mathcal{L}(\Gamma_{\mathscr{C}_{i-1} \cup \mathscr{C}_i})$. Since $a_{i,i+1} \in \mathcal{L}(\Gamma_{\mathscr{C}_{i-1,i}})$, it follows that $u \in \bigcap_{i=1}^n \mathcal{L}(\Gamma_{\mathscr{C}_{i-1} \cup \mathscr{C}_i}) = \mathbb{C}1$, and then it follows that $a_{i,i+1} \in \mathscr{Z}(\mathcal{L}(\Gamma_{\mathscr{C}_{i-1,i}}))$. Similarly, one can show that $b_i \in \mathscr{Z}(\mathcal{L}(\Gamma_{\mathscr{C}_i^{\mathrm{int}}}))$ for all $i \in \overline{1, n}$. This concludes one direction of the proof. As for the converse, note that we trivially have $\phi_{a,b} = \mathrm{Id}$. □

7.2. ***Computations of symmetries of graph product group von Neumann algebras.*** Next, we introduce a few preliminary results needed to describe the isomorphisms between von Neumann algebras arising from graph product groups with property (T) vertex groups.

Theorem 7.5. *Let $\Gamma = \mathscr{G}\{\Gamma_v\}$ and $\Lambda = \mathscr{H}\{\Lambda_w\}$ be graph product groups, and assume that Γ_v and Λ_w are icc property (T) groups for all $v \in \mathscr{V}$ and $w \in \mathscr{W}$. Let $\theta : \mathcal{L}(\Gamma) \to \mathcal{L}(\Lambda)$ be any $*$-isomorphism. Then there is a bijection $\sigma : \mathrm{cliq}(\mathscr{G}) \to \mathrm{cliq}(\mathscr{H})$ such that, for every $\mathscr{C} \in \mathrm{cliq}(\mathscr{G})$, there is a unitary $u_{\mathscr{C}} \in \mathcal{L}(\Lambda)$ such that $\theta(\mathcal{L}(\Gamma_{\mathscr{C}})) = u_{\mathscr{C}} \mathcal{L}(\Lambda_{\sigma(\mathscr{C})}) u^*_{\mathscr{C}}$.*

Proof. Fix $\mathscr{C} \in \operatorname{cliq}(\mathscr{G})$. Using the hypothesis and Corollary 2.8, it follows that $\theta(\mathcal{L}(\Gamma_{\mathscr{C}})) \subseteq \mathcal{L}(\Lambda)$ is a property (T) irreducible subfactor. By the first part of Theorem 6.1, there must exist a clique $\sigma(\mathscr{C}) \in \operatorname{cliq}(\mathscr{H})$ such that $\theta(\mathcal{L}(\Gamma_{\mathscr{C}})) \prec_{\mathcal{L}(\Lambda)} \mathcal{L}(\Gamma_{\sigma(\mathscr{C})})$. Now, we argue that, for every $c \in \sigma(\mathscr{C})$, we have $\theta(\mathcal{L}(\Gamma_{\mathscr{C}})) \nprec_{\mathcal{L}(\Lambda)} \mathcal{L}(\Gamma_{\sigma(\mathscr{C})\setminus\{c\}})$. Indeed, if we assume $\theta(\mathcal{L}(\Lambda_{\mathscr{C}})) \prec_{\mathcal{L}(\Lambda)} \mathcal{L}(\Gamma_{\sigma(\mathscr{C})\setminus\{c\}})$, then by passing to relative intertwining commutants we would get from [Vaes 2008, Lemma 3.5] that

$$\mathcal{L}(\Lambda_c) = \mathcal{L}(\Lambda_{\sigma(\mathscr{C})\setminus\{c\}})' \cap \mathcal{L}(\Lambda) \prec_{\mathcal{L}(\Lambda)} \theta(\mathcal{L}(\Gamma_{\mathscr{C}}))' \cap \mathcal{L}(\Lambda) = \theta(\mathcal{L}(\Gamma_{\mathscr{C}})' \cap \mathcal{L}(\Gamma)) = \mathbb{C}1,$$

which is a contradiction. Thus, by using that $\mathcal{L}(\Lambda_{\sigma(\mathscr{C})})$ is a factor and $\mathcal{L}(\Gamma_{\mathscr{C}}) \subset \mathcal{L}(\Gamma)$ is irreducible, it follows from the moreover part of Theorem 6.1 that there exists a unitary $u_{\mathscr{C}} \in \mathcal{L}(\Lambda)$ that satisfies $u_{\mathscr{C}}\theta(\mathcal{L}(\Gamma_{\mathscr{C}}))u_{\mathscr{C}}^* \subseteq \mathcal{L}(\Lambda_{\sigma(\mathscr{C})})$.

We now reverse the roles of Γ and Λ: in a similar manner for every $\mathscr{D} \in \operatorname{cliq}(\mathscr{H})$, one can find $\tau(\mathscr{D}) \in \operatorname{cliq}(\mathscr{G})$ and a unitary $w_{\mathscr{D}} \in \mathcal{L}(\Gamma)$ satisfying $\mathcal{L}(\Lambda_{\mathscr{D}}) \subseteq w_{\mathscr{C}}\theta(\mathcal{L}(\Gamma_{\tau(\mathscr{D})})w_{\mathscr{D}}^*$. Altogether these show that

$$u_{\mathscr{C}}\theta(\mathcal{L}(\Gamma_{\mathscr{C}}))u_{\mathscr{C}}^* \subseteq \mathcal{L}(\Lambda_{\sigma(\mathscr{C})}) \subseteq w_{\sigma(\mathscr{C})}\theta(\mathcal{L}(\Gamma_{\tau(\sigma(\mathscr{C}))})w_{\sigma(\mathscr{C})}^*.$$

In particular, Theorem 2.7 implies that $\mathscr{C} = \tau(\sigma(\mathscr{C}))$ and $u_{\mathscr{C}}^* w_{\sigma(\mathscr{C})} \in \theta(\mathcal{L}(\Gamma_{\mathscr{C}}))$. This combined with the prior containment implies that $u_{\mathscr{C}}\theta(\mathcal{L}(\Gamma_{\mathscr{C}}))u_{\mathscr{C}}^* = \mathcal{L}(\Lambda_{\sigma(\mathscr{C})})$. As $\mathscr{C} = \tau(\sigma(\mathscr{C}))$ for any clique $\mathscr{C}$ of $\mathscr{G}$, it follows in particular that σ is a bijection. □

Remarks. Theorem 7.5 still holds under the more general assumption that each vertex group Γ_v possesses an infinite property (T) normal subgroup. The proof is essentially the same and is left to the reader.

We continue by recording a notion of unique prime factorization along with some examples that will be needed in the first main result.

Definition 7.6. A family $\mathcal{C}$ of countable icc groups is said to satisfy the *s-unique prime factorization* if, whenever

$$\mathcal{M} = \mathcal{L}(\Gamma_1 \times \cdots \times \Gamma_m)^t = \mathcal{L}(\Lambda_1 \times \cdots \times \Lambda_n)$$

for some $\Gamma_1, \ldots, \Gamma_m,\ \Lambda_1, \ldots, \Lambda_n$ that belong to $\mathcal{C}$ and $t > 0$, we must have $t = 1$ and $m = n$, and there exist a unitary $u \in \mathcal{M}$ and a permutation $\tau \in \mathfrak{S}_n$ such that $u\mathcal{L}(\Gamma_i)u^* = \mathcal{L}(\Lambda_{\tau(i)})$ for all $i \in \overline{1, n}$.

There are several classes of natural examples of groups that satisfy this unique factorization condition in the literature, but for our paper only those which have property (T) will be relevant. Thus appealing to the results in [Chifan et al. 2023a; 2023b; 2024; Das 2020], we have the following.

Corollary 7.7. *Class $\mathcal{C}$ satisfies the s-unique prime factorization whenever $\mathcal{C}$ is one of the following*:

(1) *The class of property* (*T*) *fibered Rips constructions* [Chifan et al. 2023a; 2024].

(2) *The class of property* (*T*) *generalized wreath-like product groups $\mathcal{WR}(A, B \curvearrowright I)$, where A is abelian, B is an icc subgroup of a hyperbolic group, and the action $B \curvearrowright I$ has infinite stabilizers* [Chifan et al. 2023b; Chifan et al. 2023c].

Proof. Part (1) is a direct consequence of [Chifan et al. 2023a; 2024; Das 2020]. Part (2) follows from Theorem 3.6 and Corollary 3.7. □

Proposition 7.8. *Let* $\Gamma = \mathscr{G}\{\Gamma_v\}$ *and* $\Lambda = \mathscr{H}\{\Lambda_w\}$ *be graph products such that*

(1) Γ_v *and* Λ_w *are icc property* (*T*) *groups for all* $v \in \mathscr{V}$ *and* $w \in \mathscr{W}$,

(2) *there is a class* $\mathcal{C}$ *of countable groups which satisfies the s-unique prime factorization property* (see Definition 7.6) *for which* Γ_v *and* Λ_w *belong to* $\mathcal{C}$ *for all* $v \in \mathscr{V}$ *and* $w \in \mathscr{W}$.

Let $0 < t < 1$ *be a scalar and* $\Theta : \mathcal{L}(\Gamma)^t \to \mathcal{L}(\Lambda)$ *be any* $*$-*isomorphism.*

Then $t = 1$ *and there is a bijection* $\sigma : \mathrm{cliq}(\mathscr{G}) \to \mathrm{cliq}(\mathscr{H})$ *such that*, *for every* $\mathscr{C} \in \mathrm{cliq}(\mathscr{G})$, *there is a unitary* $u_{\mathscr{C}} \in \mathcal{L}(\Lambda)$ *such that* $\Theta(\mathcal{L}(\Gamma_{\mathscr{C}})) = u_{\mathscr{C}} \mathcal{L}(\Lambda_{\sigma(\mathscr{C})}) u_{\mathscr{C}}^*$.

Proof. First we observe that it suffices to show that $t = 1$, as the rest of the statement follows from Theorem 7.5.

Let $\mathscr{D}$ be a clique in $\mathscr{G}$. Since $\mathcal{L}(\Gamma_{\mathscr{D}})$ is a II_1-factor, there is a projection $p \in \mathcal{L}(\Gamma_{\mathscr{D}})$ of trace $\tau(p) = t$ with $\mathcal{L}(\Gamma)^t = p\mathcal{L}(\Gamma)p$. As $\mathcal{L}(\Gamma_{\mathscr{D}})$ has property (T) then so does $p\mathcal{L}(\Gamma_{\mathscr{D}})p$. Since $p\mathcal{L}(\Gamma_{\mathscr{D}})p \subset \Theta^{-1}(\mathcal{L}(\Lambda)) := \mathcal{N}$, then by Theorem 6.1 one can find a clique $\mathscr{F} \in \mathrm{cliq}(\mathscr{H})$ such that $p\mathcal{L}(\Gamma_{\mathscr{D}})p \prec_{\mathcal{N}} \Theta^{-1}(\mathcal{L}(\Lambda_{\mathscr{F}}))$. Now, observe that since the inclusion $p\mathcal{L}(\Gamma_{\mathcal{D}})p \subset \mathcal{N}$ is irreducible, we can proceed as in the proof of Theorem 7.5 to deduce that $p\mathcal{L}(\Gamma_{\mathscr{D}})p \nprec_{\mathcal{N}} \Theta^{-1}(\mathcal{L}(\Lambda_{\mathscr{F}\setminus\{c\}}))$ for every $c \in \mathscr{F}$. Thus, using the irreducibility condition and the moreover part of Theorem 6.1, there exists $u \in \mathscr{U}(\Theta^{-1}(\mathcal{L}(\Lambda)))$ satisfying

$$up\mathcal{L}(\Gamma_{\mathscr{D}})pu^* \subset \Theta^{-1}(\mathcal{L}(\Lambda_{\mathscr{F}})). \tag{7-2}$$

Also observe that $(up\mathcal{L}(\Gamma_{\mathscr{D}})pu^*)' \cap \Theta^{-1}(\mathcal{L}(\Lambda_{\mathscr{F}})) \subseteq up(\mathcal{L}(\Gamma_{\mathscr{D}}))' \cap \mathcal{L}(\Gamma_{\mathscr{G}})pu^* = \mathbb{C}p$. Hence (7-2) is an irreducible inclusion of II_1-factors.

Next, since $\Theta^{-1}(\mathcal{L}(\Lambda_{\mathscr{F}}))$ has property (T) and $\Theta^{-1}(\mathcal{L}(\Lambda_{\mathscr{F}})) \subset p\mathcal{L}(\Gamma)p \subset \mathcal{L}(\Gamma) := \mathcal{M}$, by Theorem 6.1 one can find $\mathscr{D}' \in \mathrm{cliq}(\mathscr{G})$ such that $\Theta^{-1}(\mathcal{L}(\Lambda_{\mathscr{F}})) \prec_{\mathcal{M}} \mathcal{L}(\Gamma_{\mathscr{D}'})$. Combining this with (7-2), we further get $p\mathcal{L}(\Gamma_{\mathscr{D}})p \prec_{\mathcal{M}} \mathcal{L}(\Gamma_{\mathscr{D}'})$, which further implies by Lemma 2.3 that $\mathscr{D} \subseteq \mathscr{D}'$, and since these are cliques we conclude that $\mathscr{D} = \mathscr{D}'$. In conclusion, the prior intertwining relation amounts to $\Theta^{-1}(\mathcal{L}(\Lambda_{\mathscr{F}})) \prec_{\mathcal{M}} \mathcal{L}(\Gamma_{\mathscr{D}})$. Since $\mathcal{L}(\Gamma_{\mathscr{D}})$ is a II_1-factor, we further obtain $\Theta^{-1}(\mathcal{L}(\Lambda_{\mathscr{F}})) \prec_{\mathcal{M}} up\mathcal{L}(\Gamma_{\mathscr{D}})pu^*$. Since $u \in p\mathcal{M}p$ is a unitary this further implies

$$\Theta^{-1}(\mathcal{L}(\Lambda_{\mathscr{F}})) \prec_{p\mathcal{M}p} up\mathcal{L}(\Gamma_{\mathscr{D}})pu^*. \tag{7-3}$$

By irreducibility, we have $\Theta^{-1}(\mathcal{L}(\Lambda_{\mathscr{F}})) \nprec_{p\mathcal{M}p} \Theta^{-1}(\mathcal{L}(\Lambda_{\mathscr{F}\setminus\{c\}})$ for all $c \in \mathscr{F}$. Thus, (7-3), (7-2), and Lemma 2.9 further imply $\Theta^{-1}(\mathcal{L}(\Lambda_{\mathscr{F}})) \prec_{\Theta^{-1}(\mathcal{L}(\Lambda_{\mathscr{F}}))} up\mathcal{L}(\Gamma_{\mathscr{D}})pu^*$. Using [Chifan and Das 2018, Lemma 2.3], this requires that the inclusion (7-2) has finite index, and consequently, we have

$$\Theta^{-1}(\mathcal{L}(\Lambda_{\mathscr{F}})) \subset \mathscr{QN}''_{p\mathcal{M}p}(up\mathcal{L}(\Gamma_{\mathscr{D}})pu^*). \tag{7-4}$$

Since $\mathscr{D}$ is a clique, Theorem 2.7 and Lemma 2.6 imply that $QN_\Gamma(\Gamma_{\mathscr{D}}) = \Gamma_{\mathscr{D}}$. Using this together with Lemmas 2.5 and 2.6, we obtain

$$\mathscr{QN}''_{p\mathcal{M}p}(up\mathcal{L}(\Gamma_{\mathscr{D}})pu^*) = up\mathscr{QN}''_{\mathcal{M}}(\mathcal{L}(\Gamma_{\mathscr{D}}))pu^* = up\mathcal{L}(QN_\Gamma(\Gamma_{\mathscr{D}}))pu^* = up\mathcal{L}(\Gamma_{\mathscr{D}})pu^*,$$

which together with (7-4) implies that $\Theta^{-1}(\mathcal{L}(\Lambda_{\mathscr{F}})) \subset up\mathcal{L}(\Gamma_{\mathscr{D}})pu^*$. Together with (7-2) it follows that $\Theta^{-1}(\mathcal{L}(\Lambda_{\mathscr{F}})) = up\mathcal{L}(\Gamma_{\mathscr{D}})pu^*$. Finally, the strong unique prime factorization property implies $p = 1$ and thus $t = 1$, as desired. □

7.3. ***Proofs of the main results.*** With all the previous preparations at hand we are ready to prove the first main result, namely Theorem A.

Theorem 7.9. *Let $\mathscr{G}$ and $\mathscr{H}$ be graphs in the class* CC_1, *and let* $\Gamma = \mathscr{G}\{\Gamma_v\}$ *and* $\Lambda = \mathscr{H}\{\Lambda_w\}$ *be graph product groups satisfying the following conditions*:

(1) Γ_v *and* Λ_w *are icc property* (T) *groups for all* $v \in \mathscr{V}$ *and* $w \in \mathscr{W}$.

(2) *There is a class* $\mathcal{C}$ *of countable groups which satisfies the s-unique prime factorization property* (see Definition 7.6) *for which* Γ_v *and* Λ_w *belong to* $\mathcal{C}$ *for all* $v \in \mathscr{V}$ *and* $w \in \mathscr{W}$.

Let $t > 0$, *and let* $\Theta : \mathcal{L}(\Gamma)^t \to \mathcal{L}(\Lambda)$ *be any* $*$*-isomorphism. Then* $t = 1$ *and one can find an isometry* $\sigma : \mathscr{G} \to \mathscr{H}$, $*$*-isomorphisms* $\theta_{i-1,i} : \mathcal{L}(\Gamma_{\mathscr{C}_{i-1,i}}) \to \mathcal{L}(\Gamma_{\mathscr{C}_{\sigma(\mathscr{C}_{i-1,i})}})$ *and* $\xi_i : \mathcal{L}(\Gamma_{\mathscr{C}_i^{\mathrm{int}}}) \to \mathcal{L}(\Gamma_{\sigma(\mathscr{C}_i^{\mathrm{int}})})$ *for all* $i \in \overline{1,n}$, *and a unitary* $u \in \mathcal{L}(\Lambda)$ *such that* $\Theta = \mathrm{ad}(u) \circ \phi_{\theta,\xi}$.

Proof. Without loss of generality, we can assume $t \leqslant 1$ and from the prior theorem we have $t = 1$. Also for simplicity we will omit Θ from the formulas. Using condition (2) in conjunction with Theorem 7.5, one can find a bijection $\sigma : \mathrm{cliq}(\mathscr{G}) \to \mathrm{cliq}(\mathscr{H})$ and unitaries $u_1, \dots, u_n \in \mathcal{M}$ such that, for any $i \in \overline{1,n}$, we have

$$u_i \mathcal{L}(\Gamma_{\mathscr{C}_i}) u_i^* = \mathcal{L}(\Lambda_{\sigma(\mathscr{C}_i)}). \tag{7-5}$$

Next, condition (2) implies that, for any $i \in \overline{1,n}$, there exist a complete subgraph $\mathscr{D}_i \subset \sigma(\mathscr{C}_i)$ and a unitary $\tilde{u}_i \in \mathcal{L}(\Lambda_{\sigma(\mathscr{C}_i)})$ such that $\tilde{u}_i \mathcal{L}(\Gamma_{\mathscr{C}_{i,i+1}}) \tilde{u}_i^* = \mathcal{L}(\Lambda_{\mathscr{D}_i})$. Note that relation (7-5) still holds if we replace u_i by $\tilde{u}_i$. Therefore, for ease of notation, we denote $\tilde{u}_i$ by u_i. Hence, $\mathcal{L}(\Gamma_{\mathscr{C}_{i,i+1}}) = u_i^* \mathcal{L}(\Lambda_{\mathscr{D}_i}) u_i = u_{i+1}^* \mathcal{L}(\Lambda_{\mathscr{D}_{i+1}}) u_{i+1}$, and therefore $\mathcal{L}(\Lambda_{\mathscr{D}_i}) u_i u_{i+1}^* = u_i u_{i+1}^* \mathcal{L}(\Lambda_{\mathscr{D}_{i+1}})$. By Theorem 2.7 this further implies $\mathscr{D}_i \subseteq \mathscr{D}_{i+1}$, and similarly we get $\mathscr{D}_i \supseteq \mathscr{D}_{i+1}$; thus, $\mathscr{D}_i = \mathscr{D}_{i+1}$. Furthermore, we see that $u_i \mathcal{L}(\Gamma_{\mathscr{C}_{i,i+1}}) u_i^* = u_{i+1} \mathcal{L}(\Gamma_{\mathscr{C}_{i,i+1}}) u_{i+1}^* = \mathcal{L}(\Lambda_{\mathscr{D}_i})$, and hence

$$u_i^* u_{i+1} \in \mathscr{N}_{\mathcal{M}}(\mathcal{L}(\Gamma_{\mathscr{C}_{i,i+1}})) = \mathcal{L}(\Gamma_{\mathscr{C}_{i,i+1} \sqcup \mathrm{lk}(\mathscr{C}_{i,i+1})}) = \mathcal{L}(\Gamma_{\mathscr{C}_i \cup \mathscr{C}_{i+1}}).$$

Moreover, using Proposition 2.10, we further have that $u_i^* u_{i+1} = a_{i,i+1} b_{i,i+1}$, where $a_{i,i+1} \in \mathscr{U}(\mathcal{L}(\mathscr{C}_{i,i+1}))$ and $b_{i,i+1} \in \mathscr{U}(\mathcal{L}(\Gamma_{(\mathscr{C}_i \cup \mathscr{C}_{i+1}) \setminus \mathscr{C}_{i,i+1}}))$. To this end observe that if we let $x_{i,i+1} := u_i^* u_{i+1}$ then we have $x_{1,2} x_{2,3} \cdots x_{n,1} = 1$. Thus, using Theorem 5.2 for each $i \in \overline{1,n}$, one can find $a_i \in \mathscr{U}(\mathcal{L}(\Gamma_{\mathscr{C}_{i-1,i}}))$, $b_i \in \mathscr{U}(\mathcal{L}(\Gamma_{\mathscr{C}_i^{\mathrm{int}}}))$, and $c_i \in \mathscr{U}(\mathcal{L}(\Gamma_{\mathscr{C}_{i,i+1}}))$ such that

$$u_i^* u_{i+1} = x_{i,i+1} = a_i b_i c_i b_{i+1}^* a_{i+2}^* c_{i+1}^*. \tag{7-6}$$

Using these relations recursively together with the commutation relations and performing the appropriate cancellations, we see that, for every $i \in \overline{2,n}$, we have

$$\begin{aligned} u_i &= u_1 (u_1^* u_2)(u_2^* u_3) \cdots (u_{i-2}^* u_{i-1})(u_{i-1}^* u_i) \\ &= u_1 (a_1 b_1 c_1 b_2^* a_3^* c_2^*)(a_2 b_2 c_2 b_3^* a_4^* c_3^*) \cdots (a_{i-1} b_{i-1} c_{i-1} b_i^* a_{i+1}^* c_i^*) \\ &\;\;\vdots \\ &= u_1 a_1 b_1 c_1 a_2 a_i^* b_i^* a_{i+1}^* c_i^*. \end{aligned} \tag{7-7}$$

Since $a_i^* b_i^* a_{i+1}^* c_i^* \in \mathscr{U}(\mathcal{L}(\Gamma_{\mathscr{C}_i}))$, we can see that by replacing each u_i by $u = u_1 a_1 b_1 c_1 a_2$ the relations in (7-5) still hold. By writing $\mathscr{F}_i = \sigma(\mathscr{C}_i)$ for all i, we observe that in particular these relations imply that $u\mathcal{L}(\Gamma_{\mathscr{C}_{i,i+1}})u^* = \mathcal{L}(\Lambda_{\mathscr{F}_{i,i+1}})$ for all i. Passing to relative commutants in each clique algebra, we also have $u\mathcal{L}(\Gamma_{\mathscr{C}_i^{\text{int}}})u^* = \mathcal{L}(\Lambda_{\mathscr{F}_i^{\text{int}}})$. We now notice that the s-unique prime factorization property of the groups implies that the map σ arises from an isometry $\sigma : \mathscr{G} \to \mathscr{H}$ still denoted by the same letter. Altogether these relations give the desired statement. □

Using the W*-superrigid property (T) wreath-like product groups recently discovered in [Chifan et al. 2023b] as vertex groups in the previous result, one obtains an even more precise description of the $*$-isomorphisms between these von Neumann algebras; hence, we provide a proof for Theorem B.

Theorem 7.10. *Let $\mathscr{G}$ and $\mathscr{H}$ be graphs in the class* CC_1, *and let $G = \mathscr{G}\{\Gamma_v\}$ and $\Lambda = \mathscr{H}\{\Lambda_w\}$ be graph product groups where all vertex groups Γ_v and Λ_w are property (T) wreath-like product groups* (as described in the second part of Corollary 7.7).

Then, for any $t > 0$ and $*$-isomorphism $\Theta : \mathcal{L}(\Gamma)^t \to \mathcal{L}(\Lambda)$, we have $t = 1$, and one can find a character $\eta \in \mathrm{Char}(\Gamma)$, *a group isomorphism* $\delta \in \mathrm{Isom}(\Gamma, \Lambda)$, *an automorphism of $\mathcal{L}(\Lambda)$ of the form $\phi_{a,b}$* (see the notation after (7-1)), *and a unitary $u \in \mathcal{L}(\Lambda)$ such that* $\Theta = \mathrm{ad}(u^*) \circ \phi_{a,b} \circ \Psi_{\eta,\delta}$.

Proof. From the prior result we have $t = 1$. From Theorem 7.9 one can find a graph isomorphism $\sigma : \mathscr{G} \to \mathscr{H}$ and a unitary $u \in \mathcal{L}(\Lambda)$ such that, for every clique $\mathscr{C}_i \in \mathrm{cliq}(\mathscr{G})$, we have $u\Theta(\mathcal{L}(\Gamma_{\mathscr{C}_i}))u^* = \mathcal{L}(\Lambda_{\sigma(\mathscr{C}_i)})$. In particular, these relations imply that $u\Theta(\mathcal{L}(\Gamma_{\mathscr{C}_{i,i+1}}))u^* = \mathcal{L}(\Lambda_{\sigma(\mathscr{C}_{i,i+1})})$ and also $u\mathcal{L}(\Gamma_{\mathscr{C}_i^{\text{int}}})u^* = \mathcal{L}(\Lambda_{\sigma(\mathscr{C}_i^{\text{int}})})$ for all $i \in \overline{1,n}$. Furthermore, using Corollary 3.7, one can find unitaries $a_{i,i+1} \in \Theta(\mathcal{L}(\Gamma_{\mathscr{C}_{i,i+1}}))$ and $b_i \in \Theta(\mathcal{L}(\Gamma_{\mathscr{C}_{i,i+1}}))$ such that

$$\mathbb{T}ua_{i,i+1}\Theta(\Gamma_{\mathscr{C}_{i,i+1}})a_{i,i+1}^*u^* = \mathbb{T}\Lambda_{\sigma(\mathscr{C}_{i,i+1})} \quad \text{and} \quad \mathbb{T}ub_i\Theta(\Gamma_{\mathscr{C}_i^{\text{int}}})b_i^*u^* = \mathbb{T}\Lambda_{\sigma(\mathscr{C}_i^{\text{int}})}.$$

Hence, there exists an automorphism of $\mathcal{L}(\Lambda)$ of the form $\phi_{a,b}$ such that, by letting $\widetilde{\Theta} = \phi_{a,b}^{-1} \circ \mathrm{ad}(u) \circ \Theta$, we have

$$\mathbb{T}\widetilde{\Theta}(\Gamma_{\mathscr{C}_{i,i+1}}) = \mathbb{T}\Lambda_{\sigma(\mathscr{C}_{i,i+1})} \quad \text{and} \quad \mathbb{T}\widetilde{\Theta}(\Gamma_{\mathscr{C}_i^{\text{int}}}) = \mathbb{T}\Lambda_{\sigma(\mathscr{C}_i^{\text{int}})}$$

for any $i \in \overline{1,n}$. The conclusion trivially follows. □

Next, we record four immediate consequences of the prior result, and hence provide proofs to the other main results of the introduction.

Corollary 7.11. *Let $\mathscr{G}$ be a graph in the class* CC_1, *and let $\Gamma = \mathscr{G}\{\Gamma_v\}$ be the graph product groups where all vertex groups Γ_v are property (T) wreath-like product groups* (as described in the second part of Corollary 7.7).

Then, for any automorphism $\Theta \in \mathrm{Aut}(\mathcal{L}(\Gamma))$, *one can find* $\eta \in \mathrm{Char}(\Gamma)$, $\delta \in \mathrm{Aut}(\Gamma)$, *an automorphism of $\mathcal{L}(\Gamma)$ of the form $\phi_{a,b}$, and a unitary $u \in \mathcal{L}(\Gamma)$ such that* $\Theta = \mathrm{ad}(u) \circ \phi_{a,b} \circ \Psi_{\eta,\delta}$.

Corollary 7.12. *Let $\mathscr{G}$ be a graph in the class* CC_1, *and let $\Gamma = \mathscr{G}\{\Gamma_v\}$ be the graph product groups where all vertex groups Γ_v are property (T) wreath-like product groups* (as described in the second part of Corollary 7.7). *Then the fundamental group* $\mathcal{F}(\mathcal{L}(\Gamma)) = \{1\}$.

In particular, combining these results with Theorem 3.3 and Remark 3.4, we obtain examples when the only outer automorphisms of von Neumann algebras of graph products are the only options discussed in relation (7-1).

Corollary 7.13. *Let $\mathscr{G} \in \mathrm{CC}_1$, and fix $\mathrm{cliq}(\mathscr{G}) = \{\mathscr{C}_1, \dots, \mathscr{C}_n\}$ a consecutive enumeration of its cliques. Let $\Gamma = \mathscr{G}\{\Gamma_v\}$ be the graph product groups where all vertex groups Γ_v are property (T) regular wreath-like product groups* (as described in the second part of Corollary 7.7) *which in addition are pairwise nonisomorphic, and have trivial abelianization and trivial outer automorphisms. Then*

$$\mathrm{Out}(\mathcal{L}(\Gamma)) \cong \bigoplus_{i=1}^{n} \mathscr{U}(\mathcal{L}(\Gamma_{\mathscr{C}_{i-1,i}})) \oplus \mathscr{U}(\mathcal{L}(\Gamma_{\mathscr{C}_i^{\mathrm{int}}})).$$

Proof. Let $\Theta \in \mathrm{Out}(\mathcal{L}(\Gamma))$. By Theorem 7.10, one can find a character $\eta \in \mathrm{Char}(\Gamma)$, a group automorphism $\delta \in \mathrm{Aut}(\Gamma)$, and an automorphism of $\mathcal{L}(\Gamma)$ of the form $\phi_{a,b}$ such that $\Theta = \phi_{a,b} \circ \Psi_{\eta,\delta}$. Note that, for any $v \in \mathscr{V}$, the restriction of η to Γ_v is a character of Γ_v and, by assumption, we get that $\eta(g) = 1$ for any $v \in \mathscr{V}$ and $g \in \Gamma_v$. Next, recall that by Theorem 4.4 we have $\mathrm{Aut}(\Gamma) \cong \Gamma \rtimes \big(\big(\bigoplus_{v\in\mathscr{V}} \mathrm{Aut}(\Gamma_v)\big) \rtimes \mathrm{Sym}(\Gamma)\big)$. Now, because the vertex groups are pairwise nonisomorphic, then $\mathrm{Sym}(\Gamma) = 1$. Moreover, since all automorphisms of the vertex groups are inner, it follows that $\Psi_{\eta,\delta}$ is essentially an automorphism of the form $\phi_{a',b'}$, where a' and b' are collections of unitaries implemented by group elements. In conclusion, we have that $\Theta = \phi_{c,d}$, where c and d are some collections of unitaries, and the formula follows. □

Corollary 7.14. *Let $\mathscr{G}$ and $\mathscr{H}$ be graphs in the class CC_1, and let $G = \mathscr{G}\{\Gamma_v\}$ and $\Lambda = \mathscr{H}\{\Lambda_w\}$ be graph product groups where all vertex groups Γ_v and Λ_w are property (T) wreath-like product groups* (as described in the second part of Corollary 7.7).

Then, for any $*$-isomorphism $\Theta : C_r^*(\Gamma) \to C_r^*(\Lambda)$, one can find a character $\eta \in \mathrm{Char}(\Gamma)$, a group isomorphism $\delta \in \mathrm{Isom}(\Gamma, \Lambda)$, an automorphism of $\mathcal{L}(\Lambda)$ of the form $\phi_{a,b}$, and a unitary $u \in \mathcal{L}(\Lambda)$ such that $\Theta = \mathrm{ad}(u^*) \circ \phi_{a,b} \circ \Psi_{\eta,\delta}$.

Proof. From Lemma 4.3, we get that Γ has trivial amenable radical, and hence, by [Breuillard et al. 2017, Theorem 1.3], it follows that $C_r^*(\Gamma)$ has unique trace. This implies that any $*$-isomorphism between $C_r^*(\Gamma)$ and $C_r^*(\Lambda)$ lifts to a $*$-isomorphism of the associated von Neumann algebras. Now the result follows from Theorem 7.10. □

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OBSERVABILITY OF THE SCHRÖDINGER EQUATION WITH SUBQUADRATIC CONFINING POTENTIAL IN THE EUCLIDEAN SPACE

ANTOINE PROUFF

We consider the Schrödinger equation in $\mathbb{R}^d$, $d \geq 1$, with a confining potential growing at most quadratically. Our main theorem characterizes open sets from which observability holds, provided they are sufficiently regular in a certain sense. The observability condition involves the Hamiltonian flow associated with the Schrödinger operator under consideration. It is obtained using semiclassical analysis techniques. It allows us to provide an accurate estimation of the optimal observation time. We illustrate this result with several examples. In the case of two-dimensional harmonic potentials, focusing on conical or rotation-invariant observation sets, we express our observability condition in terms of arithmetical properties of the characteristic frequencies of the oscillator.

1. Introduction and main results

We are concerned with the observability of the Schrödinger equation with a confining potential in the Euclidean space:

$$i\partial_t \psi = P\psi, \quad P = V(x) - \tfrac{1}{2}\Delta, \quad t \in \mathbb{R}, x \in \mathbb{R}^d, \tag{1-1}$$

where V is a real-valued potential, bounded from below. Specific assumptions shall be stated below. The general problem reads as follows: we wonder which measurable sets $\omega \subset \mathbb{R}^d$ and times $T > 0$ satisfy

$$\exists C > 0 : \forall u \in L^2(\mathbb{R}^d), \quad \|u\|^2_{L^2(\mathbb{R}^d)} \leq C \int_0^T \|e^{-itP}u\|^2_{L^2(\omega)}\, dt. \qquad \mathrm{Obs}(\omega, T)$$

When this property $\mathrm{Obs}(\omega, T)$ is true, we say that the Schrödinger equation (1-1) is observable from ω in time T, or that ω observes the Schrödinger equation. The question consists in finding conditions on

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the pair (ω, T) ensuring that one can recover a fraction of the mass of the initial data u, by observing the solution $\psi(t) = e^{-itP}u$ of (1-1) in ω during a time T. We will often call ω the observation set and T the observation time. As for the constant C in the inequality, we will refer to it as the observation cost throughout the text. When an observation set ω is fixed, the infimum of times $T > 0$ such that $\mathrm{Obs}(\omega, T)$ holds is called the optimal observation time, and is denoted by $T_\star = T_\star(\omega)$. It is clear that this so-called observability inequality holds for $\omega = \mathbb{R}^d$ in any time $T > 0$. This is because the propagator solving the Schrödinger equation e^{-itP} is an isometry on $L^2(\mathbb{R}^d)$.[1] But from the viewpoint of applications, one would like to find the smallest possible observation sets and the corresponding optimal times for which the observability inequality holds.

The observability question for Schrödinger-type equations has been extensively investigated over the past decades, mainly in compact domains of $\mathbb{R}^d$ or compact Riemannian manifolds. See the surveys of Laurent [2014] or Macià [2015] for an overview. In a compact Riemannian manifold, Lebeau [1992] showed that the so-called geometric control condition (introduced for the wave equation in [Rauch and Taylor 1974; Bardos et al. 1992]) is sufficient to get observability of the Schrödinger equation in any time $T > 0$. This means that all billiard trajectories have to enter the observation set in finite time. See for instance [Phung 2001] for later developments in Euclidean domains. However, works by Haraux [1989] and Jaffard [1990] on the torus show that this condition is not always necessary. Since then, considerable efforts have been made to find the good geometric condition characterizing the observability of the Schrödinger equation, depending on the geometrical context. This question is closely related to that of understanding the concentration or delocalization of Laplace eigenfunctions or quasimodes, which rule the propagation of states through the Schrödinger evolution; see [Burq and Zworski 2004]. The latter properties are linked to the behavior of the underlying classical dynamics, which is supposed to drive the quantum dynamics at high frequency. In the literature, mainly two different dynamical situations have been investigated. On the one hand, complete integrability, meaning existence of many conserved quantities, usually features symmetries that result in high multiplicity in the spectrum at the quantum level. This allows for possible concentration of eigenfunctions. On the other hand, chaotic systems, epitomized by the geodesic flow of negatively curved Riemannian manifolds, go along with strong instability properties. For instance, quantum ergodicity states that most[2] Laplace eigenfunctions are delocalized on manifolds with ergodic geodesic flow. Here we collect a nonexhaustive list of references illustrating this diversity of situations. On the torus, observability was investigated by several authors; see, for example, [Haraux 1989; Jaffard 1990; Burq and Zworski 2004; 2012; 2019; Bourgain et al. 2013; Macià 2010; Anantharaman and Macià 2012; 2014]. General completely integrable systems were studied by Anantharaman, Fermanian-Kammerer and Macià [Anantharaman et al. 2015]. As for the disk, the question of characterizing open sets from which observability holds was solved by Anantharaman, Léautaud and Macià [Anantharaman et al. 2016a; 2016b]. Macià and Rivière [2016; 2019] thoroughly

[1] Another consequence of this is that the condition $\mathrm{Obs}(\omega, T)$ is "open" with respect to T: if $\mathrm{Obs}(\omega, T)$ is true with cost $C > 0$, then $\mathrm{Obs}(\omega, T - \varepsilon)$ is true as soon as $\varepsilon < 1/C$. See Lemma A.3 in Appendix A for a precise statement.

[2] In fact, the situation is more complicated due to the possible existence of a sparse subsequence of eigenmodes concentrating around unstable closed classical trajectories — a phenomenon known as scarring.

described what happens on the sphere and on Zoll manifolds. In the negatively curved setting, we refer to [Anantharaman 2008; Anantharaman and Rivière 2012; Eswarathasan and Rivière 2017; Dyatlov and Jin 2018; Jin 2018; Dyatlov et al. 2022]. See also [Privat et al. 2016] in connection with quantum ergodicity.

Recently, there has been a growing interest in the question of observability for the Schrödinger equation in the Euclidean space, for which new difficulties arise due to the presence of infinity in space. Täufer [2023] dealt with the observability of the free Schrödinger equation in $\mathbb{R}^d$, showing that it is observable from any nonempty periodic open set in any positive time. It relies on the Floquet–Bloch transform and the theory of lacunary Fourier series. It was later generalized by Le Balc'h and Martin [2023] to the case of periodic measurable observation sets with a periodic L^∞ potential, in dimension 2.

Huang, Wang and Wang [Huang et al. 2022] characterized measurable sets for which the Schrödinger equation (1-1) is observable, in dimension $d = 1$ when $V(x) = |x|^{2m}$, $m \in \mathbb{N}$. They proved that, in the case where $m = 1$ (resp. $m \geq 2$), one has observability from $\omega \subset \mathbb{R}$ in some time (resp. in any time) if and only if

$$\liminf_{x\to+\infty} \frac{|\omega \cap [-x, x]|}{|[-x, x]|} > 0, \tag{1-2}$$

where $|\bullet|$ is the one-dimensional Lebesgue measure. Such a set is called "weakly thick". Simultaneously, Martin and Pravda-Starov [2021] provided a generalization of this condition in dimension d which turns out to be necessary if $d \geq 1$ and sufficient if $d = 1$ for observability to hold, in the case of the fractional harmonic Schrödinger equation, namely (1-1) with $P = (-\Delta + |x|^2)^s$, where $s \geq 1$. In the particular cases of potentials or operators discussed above, the techniques that are used, mainly relying on abstract harmonic analysis tools, provide very strong results. However, it seems that more general potentials remain out of reach, since the arguments involved require the knowledge of precise spectral estimates on eigenvalues and eigenfunctions, explicit asymptotics and symmetry properties. Moreover, regarding the case of the harmonic oscillator, the existing results focus on the properties of the sets for which observability holds, but given such a set, they do not give a hint of what would be the minimal time for which the observability inequality holds. In fact they provide an upper bound for this optimal time independent of the open set, corresponding to half a period of the classical harmonic oscillator. But it is reasonable to think that this upper bound can be improved taking into account the geometry of the observation set.

To complete the picture, let us mention the study of observability for time-dependent quadratic Hamiltonians in $\mathbb{R}^d$ by Waters [2023]. As for bounded potentials in dimension 1, a quantitative observability result was obtained by Su, Sun and Yuan [Su et al. 2025]. See also [Wei et al. 2023] on the half-line.

Motivations, assumptions and notation. This work aims to address the issues discussed above, namely:

(a) find a robust method to prove that the Schrödinger equation is observable from a given set with less restrictions on the dimension or the potential (e.g., variations of the harmonic potential like $x \cdot Ax$ where A is a real symmetric positive-definite $d \times d$ matrix, or potentials of the form $\langle x\rangle^{2m}$ with $m > 0$ a real number);

(b) provide a more accurate upper bound for the optimal observation time depending on the shape of the observation set.

Throughout this work, we make the following assumptions on the potential:

Assumption 1.1. *The potential V is C^∞ smooth and satisfies, for some $m > 0$,*

$$\exists C, r > 0 : \forall |x| \geq r, \qquad \frac{1}{C}\langle x\rangle^{2m} \leq V(x) \leq C\langle x\rangle^{2m}, \tag{1-3}$$

$$\forall \alpha \in \mathbb{N}^d, \exists C_\alpha > 0 : \forall x \in \mathbb{R}^d, \quad |\partial^\alpha V(x)| \leq C_\alpha \langle x\rangle^{2m-|\alpha|}. \tag{1-4}$$

Unless stated otherwise, we assume that the potential is subquadratic, namely $0 < m \leq 1$.

Throughout the article, we shall refer to the left-hand side inequality in (1-3) by saying that the potential is *elliptic*. In addition, the notion of *principal symbol* that we will use is made clear below.

Definition 1.2 (principal symbol). Let V_0 and V be two potentials satisfying Assumption 1.1 above with a power $m > 0$. We say that V_0 and V have the same principal symbol if

$$\forall \alpha \in \mathbb{N}^d, \exists C_\alpha > 0 : \forall x \in \mathbb{R}^d, \quad |\partial^\alpha (V - V_0)(x)| \leq C_\alpha \langle x\rangle^{2m-1-|\alpha|}.$$

This defines an equivalence relation. The equivalence class of such a potential V is called the principal symbol of V.

Classical spectral theory arguments ensure that the operator $V(x) - \frac{1}{2}\Delta$ with domain $C_c^\infty(\mathbb{R}^d)$ is essentially self-adjoint (from now on, its closure will be denoted by P) and that the evolution problem (1-1) on $L^2(\mathbb{R}^d)$ is well-posed. In fact, most of our results will depend only on the principal symbol of V, namely they will not depend on perturbations of the potential of order $\langle x\rangle^{2m-1}$.

Our strategy emphasizes the role of the underlying classical dynamics ruling the evolution of high-energy solutions to the Schrödinger equation (1-1), by means of the so-called quantum-classical correspondence principle. This motivates the introduction of the symbol of the operator P, defined by

$$p(x, \xi) := V(x) + \tfrac{1}{2}|\xi|^2, \quad (x, \xi) \in \mathbb{R}^{2d}.$$

This is a smooth function on the phase space $\mathbb{R}^{2d} \simeq \mathbb{R}^d_x \times \mathbb{R}^d_\xi$, tending to $+\infty$ as $(x, \xi) \to \infty$, since the potential is elliptic. Throughout this text, typical phase space points will be denoted by $\rho = (x, \xi)$, and we will sometimes use the notation $\pi : \mathbb{R}^{2d} \to \mathbb{R}^d$ for the projection $(x, \xi) \mapsto x$. We will often refer to p as the classical Hamiltonian, and to its quantization P as the quantum Hamiltonian. The Hamiltonian flow $(\phi^t)_{t\in\mathbb{R}}$ on $\mathbb{R}^{2d}$, which preserves p, is defined as the flow generated by the Hamilton equation:

$$\frac{d}{dt}\phi^t(\rho) = J\nabla p(\phi^t(\rho)), \quad \phi^0(\rho) = \rho. \tag{1-5}$$

It is well-defined for all times under our assumptions. Here $J = \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix}$ is the symplectic matrix. Introducing $(x^t, \xi^t) = \phi^t(\rho)$ the position and momentum components of the flow, this can be rewritten as

$$\begin{cases} \dfrac{d}{dt}x^t = \xi^t, \\ \dfrac{d}{dt}\xi^t = -\nabla V(x^t), \end{cases} \qquad (x^0, \xi^0) = \rho. \tag{1-6}$$

Hereafter, we will refer to the x-component of a trajectory of the Hamiltonian flow as a *projected trajectory*.

1.1. ***Main result.*** Let us insist on the fact that the result below applies for confining potentials having a subquadratic growth, i.e., $0 < m \leq 1$. We will explain later why we restrict ourselves to this case. Throughout the article, the open ball of radius r centered at $x \in \mathbb{R}^d$ is denoted by $B_r(x)$. Our main result reads as follows.

Theorem 1.3. *Let V_0 and V be potentials on $\mathbb{R}^d$ satisfying Assumption 1.1 with some $m \in (0, 1]$, having the same principal symbol. Set $P = V(x) - \frac{1}{2}\Delta$ and denote by e^{-itP} the propagator solving the Schrödinger equation*

$$i\partial_t \psi = P\psi.$$

Also denote by $(\phi_0^t)_{t\in\mathbb{R}}$ the Hamiltonian flow associated with the symbol $p_0(x, \xi) = V_0(x) + \frac{1}{2}|\xi|^2$. For any Borel set $\omega \subset \mathbb{R}^d$, define for any $R > 0$ the thickened set

$$\omega_R = \bigcup_{x\in\omega} B_R(x),$$

and introduce for any $T > 0$ the classical quantity[3]

$$\mathfrak{K}_{p_0}^{\infty}(\omega, T) = \liminf_{\rho\to\infty} \int_0^T \mathbf{1}_{\omega\times\mathbb{R}^d}(\phi_0^t(\rho))\, dt = \liminf_{\rho\to\infty} |\{t \in (0, T) : (\pi \circ \phi_0^t)(\rho) \in \omega\}|.$$

Fix a Borel set $\omega \subset \mathbb{R}^d$.

(i) (*sufficient condition*) *Assume there exists $T_0 > 0$ such that*

$$\mathfrak{K}_{p_0}^{\infty} := \mathfrak{K}_{p_0}^{\infty}(\omega, T_0) > 0. \tag{1-7}$$

Then there exists a constant $L = L(d, T_0, p_0, p) > 0$ such that for $R = L/\mathfrak{K}_{p_0}^{\infty}$, for any compact set $K \subset \mathbb{R}^d$ and any $T > T_0$, $\mathrm{Obs}(\omega_R \setminus K, T)$ *is true, namely*:

$$\exists C > 0 : \forall u \in L^2(\mathbb{R}^d), \quad \|u\|_{L^2(\mathbb{R}^d)}^2 \leq C \int_0^T \|e^{-itP}u\|_{L^2(\omega_R\setminus K)}^2\, dt.$$

(ii) (*necessary condition*) *Assume there exists a time $T > 0$ such that* $\mathrm{Obs}(\omega, T)$ *is true with cost $C_{\mathrm{obs}} > 0$, that is to say*,

$$\forall u \in L^2(\mathbb{R}^d), \quad \|u\|_{L^2(\mathbb{R}^d)}^2 \leq C_{\mathrm{obs}} \int_0^T \|e^{-itP}u\|_{L^2(\omega)}^2\, dt. \tag{1-8}$$

Then there is a constant $c = c(d, T, p_0, p)$ such that for any $R \geq 1$ and any compact set $K \subset \mathbb{R}^d$, one has

$$\mathfrak{K}_{p_0}^{\infty}(\omega_R \setminus K, T) \geq \frac{1}{C_{\mathrm{obs}}} - c\frac{\langle \log R\rangle^{1/2}}{R}.$$

The rest of the introduction is organized as follows: in Section 1.2, we comment on Theorem 1.3 and describe the main ideas of the proof. Then we discuss various examples of application. We begin with examples in dimension 1 in Section 1.3. In Section 1.4, we investigate the particular case of harmonic

[3] The integral makes sense when ω is Borel. Indeed, the map $(t, \rho) \mapsto \mathbf{1}_{\omega\times\mathbb{R}^d}(\phi_0^t(\rho))$ is then Lebesgue-measurable, so that the same is true for $t \mapsto \mathbf{1}_{\omega\times\mathbb{R}^d}(\phi_0^t(\rho))$ when ρ is fixed. Tonelli's theorem [Lerner 2014, Theorem 4.2.5] then shows that the map $\rho \mapsto \int_0^T \mathbf{1}_{\omega\times\mathbb{R}^d}(\phi_0^t(\rho))\, dt$ is Lebesgue-measurable.

oscillators in two dimensions. We specifically focus on conical and rotation-invariant observation sets in Sections 1.4.1 and 1.4.3 respectively. These are cases where one can prove accurate estimates on the optimal observation time — see for instance Proposition 1.5. Arithmetical properties of the characteristic frequencies of the harmonic oscillator under consideration also play a key role, as evidenced by Proposition 1.11. Then in Section 1.5, we present other consequences of Theorem 1.3 concerning observability of eigenfunctions of the Schrödinger operator P and energy decay of the damped wave equation. Lastly, we discuss the links between our work and the Kato smoothing effect in Section 1.6, and provide with further explanations regarding the natural semiclassical scaling of the problem and the criticality of quadratic potentials in Section 1.7.

1.2. *Idea of proof and comments.* The core of our work consists in establishing a suitable version of Egorov's theorem to relate the evolution through the Schrödinger flow of high-energy initial data on the quantum side, to the action of the associated Hamiltonian flow on the classical side. This is done using semiclassical analysis. To apply this theory, we approximate the indicator function of ω by a smooth and sufficiently flat cut-off function. This is how the larger set ω_R arises. Although Theorem 1.3 is not a complete characterization of sets for which observability holds, it provides an almost necessary and sufficient condition of observability, up to thickening the observation set, and it gives sharp results in many concrete situations. See the examples given in Sections 1.3, 1.4 and 1.5 below. We review remarkable features of this statement.

- The observability condition (1-7) we find is reminiscent of the geometric control condition that rules the observability or control of the wave equation in a number of geometrical contexts, especially compact Riemannian manifolds [Rauch and Taylor 1974; Bardos et al. 1988; 1992]. It reflects the importance of the quantum-classical correspondence in this problem: high-energy solutions to the Schrödinger equation, lifted to phase space, propagate along the trajectories of the Hamiltonian flow. Our constant $\mathfrak{K}_{p_0}^{\infty}(\omega, T)$ is to some extent different from the one quantifying the geometric control condition for the wave equation (see the constant $C(t)$ of Lebeau [1996] or the constant $\mathfrak{K}(T)$ of Laurent and Léautaud [2016]). Indeed, the latter constant consists in averaging some function (typically the indicator function of ω) along speed-one geodesics in a time interval $[0, T]$. In contrast, our constant $\mathfrak{K}_{p_0}^{\infty}(\omega, T)$ does the same, except that the length of trajectories tends to infinity as their initial datum ρ goes to infinity in phase space. This is consistent with the infinite speed of propagation of singularities for the Schrödinger equation.

- The necessary condition of Theorem 1.3 gives an estimate of the observation cost (from the set ω) of the form $C_{\mathrm{obs}} \geq \mathfrak{K}_{p_0}^{\infty}(\omega_R, T)^{-1} - o(1)$ as $R \to +\infty$. This is the expected lower bound while using Egorov's theorem to prove observability results; see [Laurent and Léautaud 2016] for a similar statement in the context of the wave equation. As for an upper bound, it could be that C_{obs} is much larger than the lower bound, due to localization of low-energy eigenmodes away from ω. In this respect, [Bourgain et al. 2013, Appendix A] gives a hint of how one could quantify the unique continuation argument that we use in the proof of Theorem 1.3 (see Appendix A). See also [Laurent and Léautaud 2016] for the wave equation.

- In the sufficient condition of Theorem 1.3, if one takes $K = \varnothing$, the unique continuation step (Appendix A) turns out to be unnecessary to prove observability from ω_R. Indeed, it suffices to take R large enough so as to cover a sufficiently large compact set in phase space. This allows to capture all trajectories of the Hamiltonian flow and have

$$\inf_{\rho \in \mathbb{R}^{2d}} |\{t \in (0, T) : (\pi \circ \phi_0^t)(\rho) \in \omega_R\}| > 0, \tag{1-9}$$

instead of a positive lower bound for the liminf.[4] From a unique continuation perspective, this corresponds to taking R large enough so that ω_R covers the region where low-energy modes might be localized. This indicates, through Gårding inequality, that the observation cost from the set ω_R is bounded from above by the inverse of (1-9), up to a small error that vanishes in the limit $R \to +\infty$.

- Let us insist on the fact that the Schrödinger equation (1-1) does not contain any semiclassical parameter. Instead, we artificially introduce a semiclassical parameter $R \to +\infty$, which we use to enlarge the observation set. This is natural in view of the fact that remainders in the quantum-classical correspondence are expressed in terms of derivatives of the symbol under consideration: scaling these symbols by $1/R$ thus produces remainders of the same order.

- On the technical side, the noncompactness of the Euclidean space yields new difficulties. In our problem, the use of semiclassical defect measures seems to be limited to very particular geometries of the observation set: roughly speaking, only homogeneous symbols can be paired with such measures, which would theoretically restrict the scope of the result to conical observation sets. Instead, we use (and prove) a version of Egorov's theorem to study the operator $e^{itP}\mathbf{1}_\omega e^{-itP}$. The idea of using Egorov's theorem was introduced in control theory by Dehman and Lebeau [2009] and Laurent and Léautaud [2016]. Of course, we must pay particular attention to the remainder terms, in connection with the noncompactness of the ambient space. The great advantage of this is that we can describe the evolution of a fairly large class of symbols on the phase space, which in turn allows to study observability for a variety of observation sets.

- Our result is very robust since it is valid for a fairly large class of potentials, with the noteworthy property that the statement only involves the principal symbol of the potential. Indeed, up to enlarging the parameter R, the fact that the dynamical condition (1-7) is fulfilled or not in ω_R is independent of the representative of the equivalence class of V_0 (introduced in Definition 1.2) chosen to compute $\mathfrak{K}_p^\infty(\omega_R, T)$. This is a consequence of Corollary 2.4. This was already evidenced in the context of propagation of singularities for solutions to the perturbed harmonic Schrödinger equation; see [Mao and Nakamura 2009]. The stability under subprincipal perturbation of the potential fails to be true if one considers superquadratic potentials ($m > 1$), as we can see by the examination of the trajectories of the flow. Take V_0 satisfying Assumption 1.1 for some $m > 1$, and perturb this potential with some W behaving like $\langle x \rangle^{2m-1}$. Consider the Hamiltonian flow associated with the potential $V = V_0 + W$. Then the second derivative of a trajectory of the classical flow is given by

$$\frac{d^2}{dx^2} x^t = -\nabla V_0(x^t) - \nabla W(x^t).$$

We remark that the perturbation is of order $\nabla W(x^t) \approx \langle x^t \rangle^{2(m-1)}$, which may blow up when x^t is large. When $m \leq 1$, the perturbation of the trajectory remains bounded, and can therefore be absorbed by

[4] In the proof of Theorem 1.3 (Section 3), we would be able to take $A = 0$ in (3-9) and $b_0 = 0$ in (3-10), provided R is large enough, in order to bypass the use of Appendix A.

thickening the observation set. See Section 2.1 and the proof of Theorem 1.3 at the end of Section 3 for further details.

- At the level of the Hamiltonian flow, the difference between $m \leq 1$ and $m > 1$ can also be understood by looking at the equation solved by the differential of the flow: differentiating the Hamilton equation (1-5) yields

$$\frac{d}{dt} d\phi^t(\rho) = J \operatorname{Hess} p(\phi^t(\rho))\, d\phi^t(\rho).$$

We deduce that the differential of the flow behaves as

$$|d\phi^t| \lesssim e^{t|\operatorname{Hess} p|},$$

which means that the norm of the Hessian of the Hamiltonian plays the role of a local Lyapunov exponent for the classical dynamics. Yet Hess p is uniformly bounded on phase space if and only if $m \leq 1$. Incidentally, it is likely that for $m < 1$, one can exploit the decay of Hess p at infinity in the space variable in order to get small remainders in the proof of Egorov's theorem (see Proposition 3.3) instead of taking R large. This might allow us to thicken ω by any positive ε rather than by a large parameter R. Since we are mostly interested in quadratic potentials in this work, we chose not to refine our result in this direction.

- Going through the details of the proof, it appears that one could replace assumption (1-4) on the potential by the weaker assumption[5]

$$\forall \alpha \in \mathbb{N}^d, \exists C_\alpha > 0 : \forall x \in \mathbb{R}^d, \quad |\partial^\alpha V(x)| \leq C_\alpha \langle x \rangle^{\max(0, 2m - |\alpha|)}.$$

This is consistent with the fact that there exist versions of Egorov's theorem requiring only $\partial^\alpha V(x) = O(1)$ for all $|\alpha| \geq 2$; see [Robert 1987, Theorem (IV-10)].

- It is possible that the necessary condition can be slightly improved by propagating coherent states rather than using Egorov's theorem on quantum observables. This is discussed in more detail in Section 1.4.2.

1.3. ***Examples in dimension* 1.** The one-dimensional case gives an insight of how the potential can influence the geometry of sets for which observability holds.

1.3.1. *Harmonic potential.* The one-dimensional harmonic oscillator corresponds to $V(x) = \frac{1}{2}x^2$. The Hamiltonian flow reads

$$\phi^t(x, \xi) = (x \cos t + \xi \sin t, -x \sin t + \xi \cos t), \quad (x, \xi) \in \mathbb{R}^2, \ t \in \mathbb{R}.$$

Our dynamical condition (1-7) can then be written as

$$\liminf_{(x,\xi)\to\infty} \int_0^T \mathbf{1}_\omega(x \cos t + \xi \sin t)\, dt > 0.$$

[5] In fact, in the case $m < \frac{1}{2}$, we make use of the decay of $\nabla V(x) = O(\langle x \rangle^{2m-1})$ in Proposition 2.5 (see the computation (2-16)). But in fact, $\nabla V(x) = O(1)$ already gives (2-13), which is sufficient to obtain Corollary 2.6, that is used later in the proof of Theorem 1.3.

In view of the periodicity of the flow, it is relevant to consider $T = 2\pi$. Under this additional assumption, condition (1-7) reduces to

$$\mathfrak{K}^\infty := \liminf_{A\to\infty} \int_0^{2\pi} \mathbf{1}_\omega(A\sin t)\,dt > 0, \tag{1-10}$$

where A has to be thought as (the square-root of) the energy $p(x,\xi) = \frac{1}{2}(x^2+\xi^2)$. We claim that this is equivalent to the weak thickness (1-2) condition of [Huang et al. 2022]. Suppose that $\mathfrak{K}^\infty > 0$. First, notice that

$$\int_0^{2\pi} \mathbf{1}_\omega(A\sin t)\,dt = 2\int_{-\pi/2}^{\pi/2} \mathbf{1}_\omega(A\sin t)\,dt.$$

Second, fix $c \in (0, \mathfrak{K}^\infty/2)$. Since the integrand is bounded by 1, we can slightly reduce the time interval to $\left[-\frac{\pi}{2}+\frac{c}{3}, \frac{\pi}{2}-\frac{c}{3}\right]$ so that $y = A\sin t$ defines a proper change of variables:

$$\begin{aligned}\frac{c}{3} &\le \liminf_{A\to\infty}\int_{-\pi/2}^{\pi/2} \mathbf{1}_\omega(A\sin t)\,dt - \frac{2}{3}c \le \liminf_{A\to\infty}\int_{-\pi/2+c/3}^{\pi/2-c/3} \mathbf{1}_\omega(A\sin t)\,dt \\ &\le \liminf_{A\to\infty}\int_{-\pi/2+c/3}^{\pi/2-c/3} \mathbf{1}_\omega(A\sin t)\frac{A|\cos t|}{A\frac{2}{\pi}\times\frac{c}{3}}\,dt = \frac{3\pi}{2c}\liminf_{A\to\infty}\frac{1}{A}\int_{-A\sin(\pi/2-c/3)}^{A\sin(\pi/2-c/3)} \mathbf{1}_\omega(y)\,dy.\end{aligned}$$

We used the concavity inequality

$$\cos t \ge 1 - \frac{2}{\pi}|t| \quad \text{on } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

to get the third inequality. This gives

$$\liminf_{A\to\infty} \frac{|\omega \cap [-A, A]|}{|[-A, A]|} > 0,$$

namely ω is weakly thick. Conversely, we can follow the same lines, using that the Jacobian $|\cos t|$ is less than 1, to show that any weakly thick set satisfies (1-10). Although our main theorem allows us to conclude that observability is true only on a slightly larger set, it is more precise than the previous result from [Huang et al. 2022] with respect to the optimal observation time: we can estimate this optimal time depending on the geometry of the observation set. In addition, our result is stable under subprincipal perturbation of the potential. In particular, weak thickness of ω implies observability from ω_R (for some R given by Theorem 1.3) for any potential whose principal symbol is $\frac{1}{2}x^2$ (or any positive multiple of x^2). Anticipating on the next paragraph, observe that a weakly thick set can contain arbitrarily large gaps, hence is not necessarily thick (see [Huang et al. 2022, Example 4.12]).

1.3.2. *Potentials having critical points.* An interesting phenomenon appears when the potential possesses a sequence of critical points going to infinity. To construct such a potential, we proceed as follows. We set

$$V(x) = (2+\sin(a\log\langle x\rangle))x^2, \quad x \in \mathbb{R}, \tag{1-11}$$

where a is a positive parameter to be chosen properly. See Figure 1, left, for an illustration.

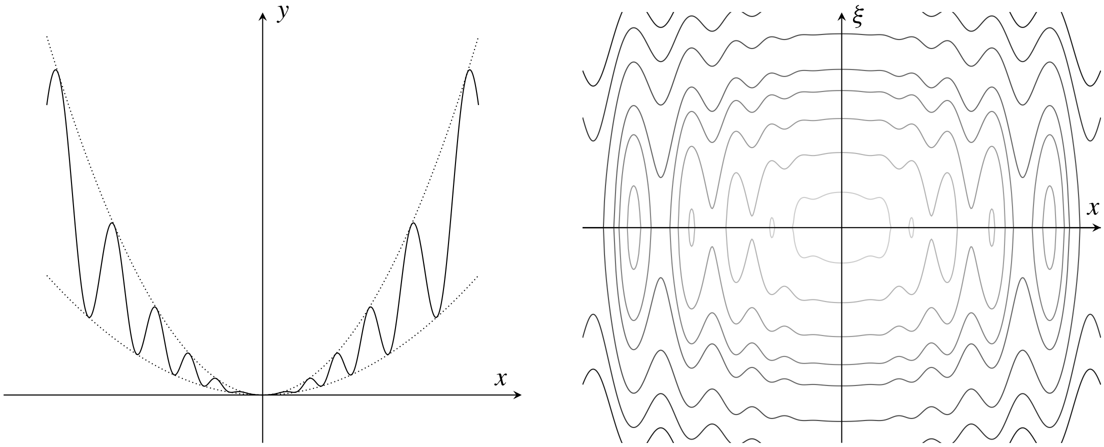


Figure 1. Case of a potential with critical points. Left: a potential V of the form (1-11). The dotted lines correspond to the potentials x^2 and $3x^2$. Right: some level sets of the Hamiltonian $p(x,\xi) = V(x) + \frac{1}{2}|\xi|^2$. The corresponding picture for the harmonic potential is just a collection of concentric ellipses.

One can check that Assumption 1.1 is fulfilled: V is subquadratic, elliptic (bounded from below by x^2) and each derivative yields a gain of $\langle x\rangle^{-1}$. Notice however that this is not a subprincipal perturbation of the harmonic potential. For any $x \in \mathbb{R}$, we have

$$\begin{aligned} V'(x) &= \frac{x}{\langle x\rangle^2}\big(2\langle x\rangle^2(2+\sin(a\log\langle x\rangle)) + ax^2\cos(a\log\langle x\rangle)\big) \\ &= \frac{x}{\langle x\rangle^2}\big((4+2\sin(a\log\langle x\rangle)) + x^2(4+2\sin(a\log\langle x\rangle)) + a\cos(a\log\langle x\rangle)\big). \end{aligned}$$

Factorizing the last two terms, we can write, for a certain angle φ_a,

$$\begin{aligned} V'(x) &= \frac{x}{\langle x\rangle^2}\big((4+2\sin(a\log\langle x\rangle)) + x^2\big(4+\sqrt{4+a^2}\sin(\varphi_a + a\log\langle x\rangle)\big)\big) \\ &= \frac{x}{\langle x\rangle^2}\left((4+2\sin(a\log\langle x\rangle)) + 4x^2\left(1+\sqrt{\frac{1}{4}+\left(\frac{a}{4}\right)^2}\sin(\varphi_a + a\log\langle x\rangle)\right)\right). \end{aligned}$$

When $\frac{1}{4}+\left(\frac{a}{4}\right)^2 > 1$, which is true if and only if $a > 2\sqrt{3}$, we can find two sequences $(x_n^+)_{n\in\mathbb{N}}$ and $(x_n^-)_{n\in\mathbb{N}}$ tending to infinity such that

$$\begin{cases} \sqrt{\frac{1}{4}+\left(\frac{a}{4}\right)^2}\sin(\varphi_a + a\log\langle x_n^+\rangle) \geq -1+\eta, \\ \sqrt{\frac{1}{4}+\left(\frac{a}{4}\right)^2}\sin(\varphi_a + a\log\langle x_n^-\rangle) \leq -1-\eta \end{cases}$$

for some sufficiently small $\eta > 0$. The intermediate value theorem then implies that there exist infinitely many points x_n^0, with $|x_n^0|$ tending to infinity, where $V'(x_n^0) = 0$. Now we observe from (1-6) that the trajectories of the Hamiltonian flow with initial data $\rho_n = (x_n^0, 0)$ are stationary, that is

$$\phi^t(\rho_n) = \rho_n \quad \forall t \in \mathbb{R}.$$

We deduce the following: assume that the Schrödinger equation (1-1) is observable from $\omega \subset \mathbb{R}$ in some time for this potential. Then the necessary condition of Theorem 1.3 tells us that there exists $R > 0$ such that, for any n large enough, $x_n^0 \in \omega_R$. We can rephrase this as

$$\exists n_0 \in \mathbb{N} : \forall n \geq n_0, \quad \omega \cap B_R(x_n^0) \neq \varnothing. \tag{1-12}$$

This is consistent with the phase portrait depicted in Figure 1, right: some energy might be trapped around small closed trajectories encircling stable critical points. Hence, in order to have observability, ω cannot be too far away from those points. In fact, one observes that (1-12) concerns all critical points, whatever the sign of $V''(x_n^0)$ is.

In conclusion, the situation of a potential of the form (1-11) is in contrast with the previous case of the harmonic potential $\frac{1}{2}x^2$ where the weak thickness condition allowed for large gaps around any sequence of points $x_n \to \infty$ satisfying $|x_{n+1}| \gg |x_n|$. Notice that ω can still have large gaps away from critical points though.

1.3.3. *Sublinear potentials.* Our last remark in the one-dimensional case concerns potentials having a sublinear growth, namely $m \in \left(0, \frac{1}{2}\right]$. In this situation, the trajectories of the Hamiltonian flow whose initial datum has purely potential energy (namely $\xi = 0$) do not escape far away from their initial location. This is because

$$\frac{d}{dt}\xi^t = -V'(x^t) = O(\langle x^t \rangle^{2m-1}),$$

which remains bounded uniformly as soon as $m \leq \frac{1}{2}$. For the same reason, $m = \frac{1}{2}$ also appears to be critical in Proposition 2.5. If observability from $\omega \subset \mathbb{R}$ holds in some time for such a potential, the necessary condition of Theorem 1.3 leads to the conclusion that ω has to intersect any interval of length $2R$, for some $R > 0$. Likewise, in higher dimension, any set from which the Schrödinger equation is observable must satisfy

$$\exists R > 0 : \forall x \in \mathbb{R}^d, \quad \omega \cap B_R(x) \neq \varnothing. \tag{1-13}$$

Therefore, sets observing the Schrödinger equation (1-1) for a sublinear potential cannot have arbitrarily large holes.[6] Although the case of bounded potentials (i.e., $m = 0$) is not in the scope of this article, let us mention that this observation is consistent with recent results on the free Schrödinger equation. See [Huang et al. 2022; Täufer 2023], as well as [Le Balc'h and Martin 2023] for the case of bounded periodic potentials in two dimensions.

1.4. ***Observability of two-dimensional harmonic oscillators.*** As an application of Theorem 1.3, we study the observability of harmonic oscillators in conical or rotation-invariant sets. Our results mainly concern the two-dimensional case. The examples presented in this subsection suggest that there is no general reformulation of our dynamical condition (1-7) in purely geometrical terms. That is to say, it seems difficult to find an equivalent condition that would not involve the Hamiltonian flow (e.g., thickness, weak thickness...). In contrast, by restricting ourselves to a certain class of potentials (harmonic oscillators at the principal level here) and a certain class of observation sets (conical or rotation-invariant), one can indeed

[6]Notice that (1-13) is much weaker that the usual thickness condition of control theory:

$$\exists R, c > 0 : \forall x \in \mathbb{R}^d, \quad |\omega \cap B_R(x)| \geq c|B_R(x)|.$$

transform the dynamical condition into a geometrical one. Along the way, we will see that observability properties are very sensitive to slight modifications of the coefficients of the harmonic oscillator under consideration. This subsection culminates in Proposition 1.11, where we show that observability of rotation-invariant sets is governed by Diophantine properties of the oscillator's coefficients.

Let us first recall basics about general harmonic oscillators. Let A be a real symmetric positive-definite $d \times d$ matrix and set $H_A = \frac{1}{2}(x \cdot Ax - \Delta)$. Up to an orthonormal change of coordinates, one can assume that A is diagonal, so that the potential can be written

$$V_A(x) = \frac{1}{2} x \cdot Ax = \frac{1}{2} \sum_{j=1}^{d} \nu_j^2 x_j^2.$$

The *characteristic frequencies* of H_A are those numbers $\nu_1, \nu_2, \dots, \nu_d$, that we will always assume to be positive. The corresponding Hamiltonian flow is explicit: denoting by $x_1(t), x_2(t), \dots, x_d(t)$ and $\xi_1(t), \xi_2(t), \dots, \xi_d(t)$ the components of ϕ^t, we can solve the Hamilton equations (1-6):

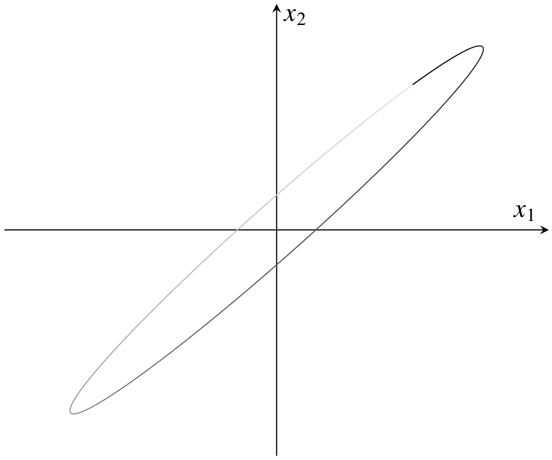
$$\begin{cases} x_j(t) = \cos(\nu_j t) x_j(0) + \dfrac{1}{\nu_j} \sin(\nu_j t) \xi_j(0), \\ \xi_j(t) = -\nu_j \sin(\nu_j t) x_j(0) + \cos(\nu_j t) \xi_j(0) \end{cases} \qquad \forall j \in \{1, 2, \dots, d\}. \tag{1-14}$$

From this expression, we see that each coordinate is periodic, so the trajectories whose initial conditions are of the form $x_j(0) = x_0 \delta_{j=j_0}$, $\xi_j(0) = \xi_0 \delta_{j=j_0}$ with $x_0, \xi_0 \in \mathbb{R}$, are periodic, with period $2\pi/\nu_{j_0}$ (unless both x_0 and ξ_0 vanish, in which case the trajectory is a point). Assuming $d = 2$, we can classify harmonic oscillators into three categories. See Figure 2 for an illustration.

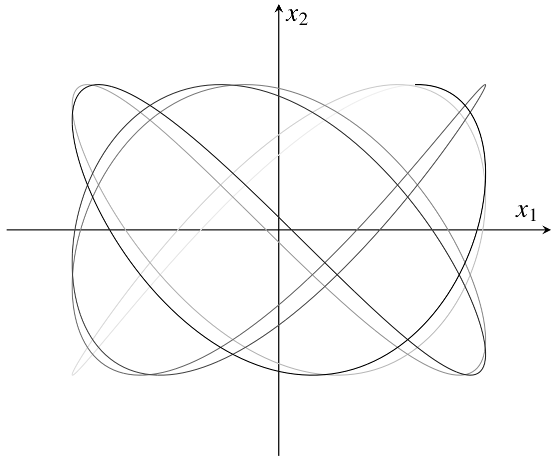
- The harmonic oscillator is *isotropic*[7] if $\nu_1 = \nu_2 = \nu$. In this situation, energy surfaces, that is, level sets of the classical Hamiltonian, are concentric spheres in phase space (up to a symplectic change of coordinates). Trajectories of the Hamiltonian flow are great circles on these spheres, so that their projection on the x-variable "physical space" are ellipses. The flow is periodic, with period $2\pi/\nu$.
- The harmonic oscillator is said to be *anisotropic rational* when ν_2/ν_1 is a rational number different from 1. Trajectories, although all closed, exhibit a more complicated behavior. Writing $\nu_2/\nu_1 = p/q$ with p and q coprime positive integers, the period of the flow is $2p\pi/\nu_2 = 2q\pi/\nu_1$. Projected trajectories are known in the physics literature as Lissajous curves [1857].
- We say a harmonic oscillator is *anisotropic irrational* when $\nu_2/\nu_1 \in \mathbb{R} \setminus \mathbb{Q}$. In that case, the Hamiltonian flow is aperiodic. Trajectories are dense in invariant tori (see (1-15) below), yielding projected trajectories that fill rectangles parallel to the eigenspaces of the matrix A.

In the multidimensional setting, the description of the flow can be achieved by examining the $\mathbb{Q}$-vector space generated by the characteristic frequencies. The dimension of the latter gives the number of periodic decoupled "suboscillators" from which we can reconstruct the dynamics of the whole oscillator. This was thoroughly explained by Arnaiz and Macià [2022a], who computed the set of quantum limits of general harmonic oscillators, and studied their behavior when bounded perturbations of the potential are added [2022b].

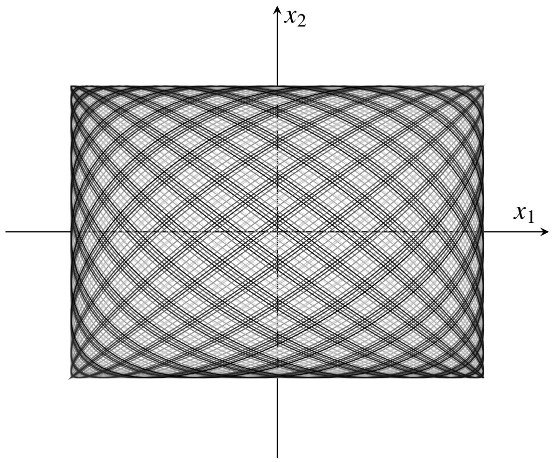
[7]For general dimension, we still call isotropic any harmonic oscillator having all its characteristic frequencies equal.



Isotropic harmonic oscillator: $\frac{\nu_2}{\nu_1} = 1$.



Rational anisotropic harmonic oscillator: $\frac{\nu_2}{\nu_1} = \frac{7}{5}$.



Irrational anisotropic harmonic oscillator: $\frac{\nu_2}{\nu_1} = \frac{\pi}{3}$.

Figure 2. Typical projected trajectories of two-dimensional harmonic oscillators. Shading indicates the course of the trajectory.

In order to understand well the classical dynamics of the harmonic oscillator, it is convenient to take advantage of the complete integrability of this dynamical system. Here, the classical Hamiltonian is the sum of the one-dimensional Hamiltonians $\frac{1}{2}(\nu_j^2 x_j^2 + \xi_j^2)$, which are conserved by the flow, as one can see from the explicit expression (1-14). This property implies that energy levels are foliated in (possibly degenerate) invariant d-dimensional tori of the form

$$\mathbb{T}_E = \left\{(x,\xi) \in \mathbb{R}^{2d} : \forall j,\ \tfrac{1}{2}(\nu_j^2 x_j^2 + \xi_j^2) = E_j\right\}, \quad E = (E_1, E_2, \dots, E_d) \in \mathbb{R}_+^d. \tag{1-15}$$

The projection of these tori on the x-variable space yields rectangles, as in Figure 2, bottom.

The goal of the following examples is to highlight the fact that observability is sensitive to the global properties of the Hamiltonian flow. We will show that isotropic and anisotropic harmonic oscillators behave differently with respect to observability, i.e., the sets that observe the Schrödinger equation are not the same. One can already anticipate that the isotropic oscillator $\nu_1 = \nu_2$ has less such sets since its classical trajectories are all ellipses, that is, they are very simple and only explore a small part of the classically allowed region. It contrasts with the anisotropic situation $\nu_1 \neq \nu_2$, where, in the rational case for instance, trajectories visit more exhaustively the classically allowed region. It makes it harder to find a set that is not reached by any of these trajectories. It is even more the case when ν_1 and ν_2 are rationally independent, since the trajectories are then dense in the invariant torus to which they belong, as we already discussed.

1.4.1. *Observability from conical sets.* We first investigate the case where the observation set ω is conical, namely it is invariant by dilations with positive scaling factor:

$$\forall x \in \mathbb{R}^d, \forall \lambda > 0, \quad (x \in \omega \iff \lambda x \in \omega). \tag{1-16}$$

We will see that exploiting the symmetries of harmonic oscillators is sometimes sufficient to obtain satisfactory results, without the need of our main theorem (see Section 1.4.2). However, Theorem 1.3 will prove useful to estimate precisely the optimal observation time in some situations.

As we already noticed, it follows from the expression of the flow (1-14) that, whatever the characteristic frequencies, the classical dynamics exhibits periodic trajectories contained in the coordinate axes. Those starting from the origin are of the form

$$x_j(t) = \frac{1}{\nu_j} \sin(\nu_j t)\xi_j(0), \quad \xi_j(t) = \cos(t)\xi_j(0)$$

for one $j \in \{1, 2, \dots, d\}$, and with all the other components being equal to zero. Thus it appears that a general necessary condition for a conical ω to observe the Schrödinger equation (1-1), working for any harmonic oscillator, is that it contains at least half of each line spanned by an eigenvector of A. Note that this works in any dimension.

Proposition 1.4. *Consider $P = V(x) - \frac{1}{2}\Delta$, where V is a potential fulfilling Assumption 1.1 and having principal symbol $V_A(x) = \frac{1}{2} x \cdot Ax$, A being a real symmetric positive-definite $d \times d$ matrix. Let $\omega \subset \mathbb{R}^d$ be a conical set and assume that it observes the Schrödinger equation in some time $T > 0$. Then $v \in \overline{\omega}$ or $-v \in \overline{\omega}$ for any eigenvector v of A.*

Now we place ourselves in dimension $d = 2$. We know from the above Proposition 1.4 that the closure of a conical set which observes the Schrödinger equation has to contain at least half of any line spanned by an eigenvector of the matrix A. Here, we exhibit a conical observation set, illustrated in Figure 3, that behaves differently according to whether the harmonic oscillator is isotropic or not.

Proposition 1.5 (conical sets and anisotropy). *Let $d = 2$ and consider a potential V fulfilling Assumption 1.1, and with principal symbol*

$$V_A(x) = \tfrac{1}{2} x \cdot Ax, \quad x \in \mathbb{R}^2,$$

where A is a real symmetric positive-definite matrix. Denote by $\nu_+ \geq \nu_- > 0$ its characteristic frequencies. Choose an orthonormal basis of eigenvectors (e_+, e_-) of A, so that $Ae_\pm = \nu_\pm^2 e_\pm$. For any $\varepsilon \in (0, \pi/2)$, define the two cones with aperture ε:

$$C_\varepsilon^\pm = \left\{x \in \mathbb{R}^2 : |x \cdot e_\mp| < \tan\left(\tfrac{1}{2}\varepsilon\right) x \cdot e_\pm\right\}. \tag{1-17}$$

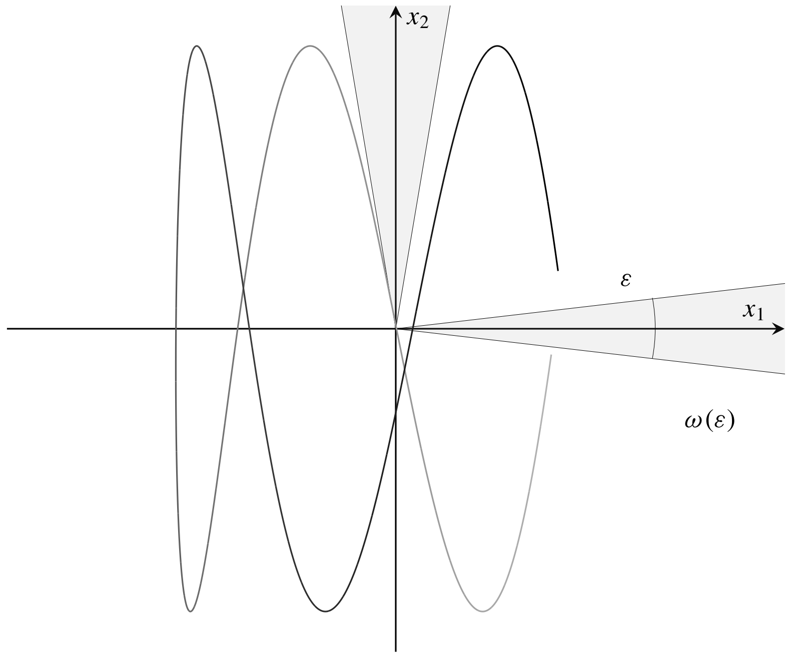


Figure 3. The above projected trajectory is responsible for the lower bound on the optimal observation time in (1-18). It is obtained with an oscillator such that $\nu_+/\nu_- = 3.9$. For $\nu_+/\nu_- = 4$, one can choose the initial datum so that the curve goes back to the upper-right quadrant, passing through the origin, without crossing the two cones. This yields a larger lower bound on the optimal time, corresponding to the jump from $\lfloor 3.9 \rfloor = 3$ to $\lfloor 4 \rfloor = 4$ in (1-18).

Then the set $\omega(\varepsilon) = C_\varepsilon^+ \cup C_\varepsilon^-$ *observes the Schrödinger equation if and only if the oscillator is anisotropic, that is,* $\nu_- < \nu_+$. *In that case, there exist constants* $C, c > 0$, *possibly depending on* ν_+, ν_-, *such that for any* $\varepsilon \in (0, \pi/2)$,

$$T_0 - C\varepsilon^2 \leq T_\star(\omega(\varepsilon)) \leq T_0 - c\varepsilon^2, \quad \text{where } T_0 = \frac{\pi}{\nu_+}\left(2 + \left\lfloor \frac{\nu_+}{\nu_-} \right\rfloor\right). \tag{1-18}$$

This result does not distinguish between rational and irrational anisotropic oscillators: one cannot guess, from the knowledge that observability from $\omega(\varepsilon)$ holds, whether the oscillator is rational or irrational.

Remark 1.6 (discontinuous behavior of $T_\star$). The time T_0, obtained formally as the limiting optimal observation time when $\varepsilon \to 0$, does not vary continuously with respect to ν_+ and ν_- because of the floor function. This is related to special symmetry properties of the Hamiltonian flow that appear when ν_+ is a multiple of ν_-, namely the projected trajectories can go from a quadrant to another one crossing the origin, and thus avoiding to cross the observation cones. See Figure 3. From the proof, especially (4-29), the constant C in the lower bound of (1-18) can be estimated by

$$C \lesssim \frac{1/\nu_+}{\min(1, \nu_+/\nu_- - 1)},$$

up to a constant independent of ε and ν_-, ν_+. (A similar but more complicated lower bound is available for the constant c from (4-14).) In particular we have

$$0 \leq T_0 - T_\star(\omega(\varepsilon)) \lesssim \frac{\varepsilon^2/\nu_+}{\min(1, \nu_+/\nu_- - 1)}.$$

Therefore, if we fix $\nu_- = 1$ and let $\nu_+ \to 2$ with $\nu_+ < 2$, the optimal observation time is of order $T_\star \approx \frac{3}{2}\pi + O(\varepsilon^2)$, while in the limit $\nu_+ = 2$, we have $T_\star \approx 2\pi + O(\varepsilon^2)$. Since the constants involved in the $O(\varepsilon^2)$ remainder are uniform in the limit $\nu_+ \to 2$, taking ε small enough gives a case where the optimal observation time depends discontinuously on ν_-, ν_+.

It is interesting to see what happens when $\nu_+, \nu_- \to \nu$, that is to say, when the operator P becomes closer to an isotropic harmonic oscillator. As mentioned earlier, we know from Proposition 1.4 that observability is not true for a set of the form $C_\varepsilon^+ \cup C_\varepsilon^-$ for an isotropic oscillator ($\varepsilon < \pi/2$ is important here). Thus it can seem surprising that the optimal observation time for such a set is bounded uniformly in ν_+, ν_- as the frequencies tend to ν. Actually, degeneracy in this limit should be seen on the observation cost, rather than on the optimal observation time. Indeed, computations suggest that the value of the dynamical constant $\mathfrak{K}_p^\infty(\omega(\varepsilon), T)$ tends to zero; see (4-13) in the proof. This would imply a blow up of the observation cost as $\nu_+, \nu_- \to \nu$, in virtue of the necessary condition part of Theorem 1.3.

1.4.2. *Refinement for the unperturbed isotropic harmonic oscillator.* Theorem 1.3 allows us to conclude whether an open set ω observes the Schrödinger equation provided this open set is in a sense "regular": the thickening process yields open sets that are sufficiently close to a cut-off function. But the quest of characterizing general measurable sets seems to be more delicate. To understand the limitation of our main theorem, we investigate the very particular case of the isotropic harmonic oscillator and conical observation sets in dimension $d \geq 1$. In this setting, we can take advantage of symmetries and exact propagation of coherent states.

For the purpose of the statement, let us introduce some notation. A conical set in $\mathbb{R}^d$ is determined by the subset $\Sigma = \omega \cap \mathbb{S}^{d-1}$ in the unit sphere. When $\Sigma \subset \mathbb{S}^{d-1}$ we denote by $\omega(\Sigma)$ the conical set defined by

$$\omega(\Sigma) = \left\{ x \in \mathbb{R}^d \setminus \{0\} : \frac{x}{|x|} \in \Sigma \right\}. \tag{1-19}$$

Moreover, for any subset $\Sigma \subset \mathbb{S}^{d-1}$, we introduce the notation

$$-\Sigma = \{\theta \in \mathbb{S}^{d-1} : -\theta \in \Sigma\}.$$

The lower density of a measurable set $\Sigma \subset \mathbb{S}^{d-1}$, denoted by Θ_Σ^-, is the function $\mathbb{S}^{d-1} \to [0, 1]$ defined by

$$\Theta_\Sigma^-(\theta) = \liminf_{r \to 0} \frac{\sigma(\Sigma \cap B_r(\theta))}{\sigma(B_r(\theta))} \quad \forall \theta \in \mathbb{S}^{d-1}, \tag{1-20}$$

where $B_r(\theta)$ is the ball of radius r centered at θ in $\mathbb{R}^d$, and σ is the uniform probability measure on the unit sphere $\mathbb{S}^{d-1}$.

We insist on the fact that the statement below is proved for exact isotropic harmonic oscillators, and not for perturbations of it.

Proposition 1.7. *Let $P = \frac{1}{2}(\nu^2|x|^2 - \Delta)$ be an isotropic oscillator with characteristic frequency $\nu > 0$. Let $\Sigma \subset \mathbb{S}^{d-1}$ be measurable, and $\omega(\Sigma)$ be the corresponding conical set. Set $\widehat{\Sigma} = \Sigma \cup -\Sigma$ the symmetrized version of Σ.*

(i) *If the Schrödinger equation is observable from $\omega(\Sigma)$ in some time, then*

$$\inf_{\mathbb{S}^{d-1}} \Theta^-_{\widehat{\Sigma}} > 0.$$

(ii) *If $\widehat{\Sigma} = \Sigma \cup -\Sigma$ has full measure, namely $\sigma(\mathbb{S}^{d-1} \setminus \widehat{\Sigma}) = 0$, or equivalently $\Theta^-_{\widehat{\Sigma}}(\theta) = 1$ for all $\theta \in \mathbb{S}^{d-1}$, then $\omega(\Sigma)$ observes the Schrödinger equation, with optimal observation time $T_\star < 2\pi/\nu$.*

Remark 1.8. The gap between the sufficient and the necessary conditions above can be thought as the difference between Σ being the complement of a Cantor set (thus having full measure) and Σ being the complement of a fat Cantor set; see [Stromberg 1981, Chapter 2, p. 80]. Regarding the estimate on the optimal observation time, the strict inequality is due to Lemma A.3.

In fact, considering the propagation of coherent state, as investigated for instance by Combescure and Robert [1997], one could conjecture that observability is characterized by the property

$$\exists R > 0 : \quad \liminf_{\rho \to \infty} \int_0^T |\omega \cap B_R(x^t(\rho))| \, dt > 0, \tag{1-21}$$

with $x^t(\rho) = (\pi \circ \phi^t)(\rho)$. This type of integral can be rewritten as

$$\int_0^T |\omega \cap B_R(x^t(\rho))| \, dt = \int_0^T \|\mathbf{1}_\omega\|_{L^1(B_R(x^t(\rho)))} \, dt.$$

The necessary condition of Theorem 1.3, namely $\mathfrak{K}^\infty_p(\omega_R, T) > 0$ for some R large enough, involves the quantity

$$\int_0^T \mathbf{1}_{\omega_R}(x^t(\rho)) \, dt = \int_0^T \|\mathbf{1}_\omega\|_{L^\infty(B_R(x^t(\rho)))} \, dt. \tag{1-22}$$

Since the L^1 norm in a ball of radius R is controlled by the L^∞ norm (times a constant of order R^d), we know that the dynamical condition (1-21) is stronger than the condition $\mathfrak{K}^\infty_p(\omega_R, T) > 0$, involving the L^∞ norm as written in (1-22). In particular, if ω is dense but Lebesgue negligible, the condition $\mathfrak{K}^\infty_p(\omega_R, T) > 0$ will be satisfied, since then $\omega_R = \mathbb{R}^d$ for any $R > 0$, whereas (1-21) will not. In this situation, Theorem 1.3 would then yield a trivial result, namely that observability holds from the whole space, although it clearly does not hold from ω itself. Thus (1-21) seems to be a good guess to free ourselves from thickening the observation set. In addition, this condition would be consistent with the generalized geometric control condition introduced by Burq and Gérard [2020] in the context of stabilization of the wave equation.

1.4.3. *Observability from spherical sets.* In this section, we investigate the observability properties of a set consisting in a union of spherical layers. In the sequel, we refer to rotation-invariant (measurable) sets as *spherical* sets. Such a set ω is completely determined by the data of a measurable set $I \subset \mathbb{R}_+$, such that

$$\omega = \omega(I) = \{x \in \mathbb{R}^d : |x| \in I\}. \tag{1-23}$$

Due to the thickening process that occurs when applying Theorem 1.3, we shall generally make further assumptions, that ensure that a set and its thickened version are somewhat equivalent.

The existence of many periodic circular orbits of the Hamiltonian flow for radial potentials implies that observability from $\omega(I)$ does not hold for such Hamiltonians if I contains large gaps. In fact, the proposition below works for slightly more general potentials.

Proposition 1.9. *Let $d \geq 2$. Suppose the Hamiltonian P is of the form $P = V(x) - \frac{1}{2}\Delta$ with a potential V satisfying Assumption 1.1 together with*

(i) $V(-x) = V(x), \forall x \in \mathbb{R}^d$;

(ii) *there exists an orthogonal change of coordinates M such that*

$$V(MS_\theta M^{-1}x) = V(x) \quad \forall x \in \mathbb{R}^d, \forall \theta \in \mathbb{R},$$

where S_θ is the rotation of angle θ acting on the first two coordinates; *in particular*, *for every $y \in \mathbb{R}^{d-2}$, the map $V_y : (x_1, x_2) \mapsto V(M(x_1, x_2, y))$ is radial*;

(iii) *the map $\widetilde{V}_0$ such that $V_{y=0}(x_1, x_2) = \widetilde{V}_0(|(x_1, x_2)|)$ is nondecreasing.*

Then for any spherical set $\omega(I)$, if observability holds from $\omega(I)$ in some time $T > 0$, one has

$$\exists r > 0 : \forall s \in \mathbb{R}_+ : \quad I \cap [s, s+r] \neq \varnothing. \tag{1-24}$$

Remark 1.10. The hypotheses are fulfilled for harmonic oscillators in d dimensions having at least two identical characteristic frequencies.

In dimension 2, Proposition 1.9 allows to conclude that spherical sets observing the Schrödinger equation for isotropic harmonic oscillators have to occupy space somewhat uniformly — they cannot contain arbitrarily large gaps. Therefore, we shall rule out isotropic harmonic oscillators from our study of observability from spherical sets. Instead, we investigate how the anisotropy of a harmonic oscillator can help to get observability from an observation set made of concentric rings. The proposition below investigates, in dimension 2, the observability from spherical sets of the form $\omega(I)$, where $I = \bigcup I_n$ is a countable union of open intervals in $\mathbb{R}_+$. We require additionally that $|I_n| \to +\infty$ (we drop this assumption if there are only finitely many I_n's). To any such set, we associate a number between 0 and 1 that quantifies the distribution of the annuli $\omega(I_n)$ at infinity:

$$\kappa_\star(I) = \min\left\{\kappa \in [0, 1] : \liminf_{r \to +\infty} \frac{1}{r}|I \cap [\kappa r, r]| = 0\right\} \in [0, 1]. \tag{1-25}$$

While investigating the observability property from such a set $\omega(I)$, we will see that it is relevant to compare the geometrical quantity $\kappa_\star(I)$ with a dynamical quantity that encodes relevant features of the underlying Hamiltonian flow. This dynamical constant is expressed in terms of a function $\Lambda : \mathbb{R}_+^\star \to [0, 1]$ defined by

$$\Lambda(\mu) = \begin{cases} \tan\left(\dfrac{\pi/2}{p+q}\right) & \text{if } \mu = \dfrac{p}{q}, \ \gcd(p, q) = 1, \ p - q \equiv 0 \pmod 2, \\ \sin\left(\dfrac{\pi/2}{p+q}\right) & \text{if } \mu = \dfrac{p}{q}, \ \gcd(p, q) = 1, \ p - q \equiv 1 \pmod 2, \\ 0 & \text{if } \mu \in \mathbb{R} \setminus \mathbb{Q}. \end{cases} \tag{1-26}$$

As far as the optimal observation time $T_\star$ is concerned, we shall use Diophantine properties of μ to approximate irrational oscillators by rational ones, for which we can control $T_\star$ by the period of the flow. This motivates the introduction of the irrationality exponent of an irrational number μ, defined by

$$\tau(\mu) = \sup\left\{ s \in \mathbb{R} : \left| \mu - \frac{p}{q} \right| < \frac{1}{q^s} \text{ for infinitely many coprime couples } (p, q) \right\}. \tag{1-27}$$

Dirichlet's approximation theorem tells us that $\tau(\mu) \in [2, +\infty]$ for any irrational number. Also keep in mind that $\tau(\mu) = 2$ is achieved for Lebesgue-almost every irrational. See the lecture notes [Durand 2015] or the books [Einsiedler and Ward 2011; Schmidt 1991] for further details.

Proposition 1.11 (spherical sets and anisotropy). *Let $d = 2$ and consider a potential V fulfilling Assumption 1.1, and with principal symbol*

$$V_A(x) = \tfrac{1}{2} x \cdot Ax, \quad x \in \mathbb{R}^2,$$

where A is a real symmetric positive-definite matrix. Denote by ν_1 and ν_2 the characteristic frequencies of A, and assume that $\nu_1 \neq \nu_2$. We fix $I = \bigcup I_n$ a union of open intervals in $\mathbb{R}_+$, assuming that $|I_n| \to +\infty$. Denote by $\omega(I)$ the corresponding open spherical set in $\mathbb{R}^2$, as defined in (1-23). *Then observability from $\omega(I)$ holds in some time T if and only if*

$$\kappa_\star(I) > \Lambda\left(\frac{\nu_2}{\nu_1}\right). \tag{1-28}$$

Moreover, the optimal observation time $T_\star$ can be estimated as follows:

- *If $\nu_2/\nu_1 \in \mathbb{Q}$, writing $\nu_2/\nu_1 = p/q$ with p, q positive coprime integers, then*

$$T_\star < \frac{\pi}{\nu_2} p = \frac{\pi}{\nu_1} q.$$

- *If $\nu_2/\nu_1 \in \mathbb{R} \setminus \mathbb{Q}$ is Diophantine, that is $\tau = \tau(\nu_2/\nu_1) < \infty$, then*

$$\forall \varepsilon > 0, \exists c_\varepsilon, C_\varepsilon > 0 : \quad c_\varepsilon \left(\frac{1}{\kappa_\star(I)} \right)^{1/(\tau - 1 + \varepsilon)} \leq T_\star \leq C_\varepsilon \left(\frac{1}{\kappa_\star(I)} \right)^{\tau - 1 + \varepsilon}. \tag{1-29}$$

  The constants c_ε and C_ε may depend on ν_1, ν_2, but not on I.

Let us review the meaning of the different quantities involved in this statement.

The number $\kappa_\star(I)$ introduced in (1-25) encodes some notion of density of the set I. For instance, $\kappa_\star(I) = 1$ means that I has positive density in any window $[\kappa r, r]$ with $\kappa < 1$ as $r \to +\infty$. In contrast, $\kappa_\star(I)$ close to zero means that the annuli are extremely sparse at infinity. This quantity is well-defined, for the map

$$\kappa \mapsto \liminf_{r \to +\infty} \frac{1}{r} \int_{\kappa r}^{r} \mathbf{1}_I(s)\, ds$$

is nonincreasing and lower semicontinuous (even Lipschitz-continuous in fact). That it is nonincreasing comes from the monotonicity of the integral and of the lower limit, whereas the continuity follows from the fact that

$$\left| \frac{1}{r} \int_{\kappa_2 r}^{r} \mathbf{1}_I(s)\, ds - \frac{1}{r} \int_{\kappa_1 r}^{r} \mathbf{1}_I(s)\, ds \right| \leq |\kappa_2 - \kappa_1|.$$

Note. Beware of the fact that $\kappa_\star(I)$ does not coincide in general with the lower density of I defined by

$$\Theta_\infty(I) = \liminf_{r\to+\infty} \frac{|I \cap [0,r]|}{|[0,r]|}.$$

In fact, the two quantities satisfy

$$\Theta_\infty(I) \le \kappa_\star(I) \quad \text{and} \quad \kappa_\star(I) = 0 \quad \Longleftrightarrow \quad \Theta_\infty(I) = 0.$$

The second assertion follows from the definition of $\kappa_\star(I)$. To check the first assertion, we write

$$\frac{|I \cap [0,r]|}{|[0,r]|} = \kappa_\star \frac{|I \cap [0,\kappa_\star r]|}{|[0,\kappa_\star r]|} + \frac{1}{r}|I \cap [\kappa_\star r, r]| \le \kappa_\star + \frac{1}{r}|I \cap [\kappa_\star r, r]|.$$

Then taking lower limits as $r \to +\infty$ and using the definition of $\kappa_\star$ yield the desired inequality. Notice that the equality $\Theta_\infty(I) = \kappa_\star(I)$ is not true in general, as one can see from the example $I = \bigcup_{n\in\mathbb{N}} \bigl(n, n+\frac{1}{2}\bigr)$, for which we have $\Theta_\infty(I) = \frac{1}{2}$ but $\kappa_\star(I) = 1$.

Given $\mu \in \mathbb{R}_+^\star$, the constant $\Lambda(\mu)$ defined in (1-26) is related to the flow of a harmonic oscillator with characteristic frequencies ν_1, ν_2 such that $\mu = \nu_2/\nu_1$. More precisely, it corresponds to the largest ratio between the minimum and the maximum of the distance to the origin of a projected trajectory. This is the content of the following lemma that we prove in Section 5.2.

Lemma 1.12. *For all $\nu_1, \nu_2 > 0$, one has*

$$\Lambda\Bigl(\frac{\nu_2}{\nu_1}\Bigr) = \sup_{\rho_0 \in \mathbb{R}^4\setminus\{0\}} \frac{\inf_{t\in\mathbb{R}} |(\pi \circ \phi^t)(\rho_0)|}{\sup_{t\in\mathbb{R}} |(\pi \circ \phi^t)(\rho_0)|}, \tag{1-30}$$

where $(\phi^t)_{t\in\mathbb{R}}$ is the Hamiltonian flow of any two-dimensional harmonic oscillator with characteristic frequencies ν_1, ν_2.

Thus we can refer to $\Lambda(\mu)$ as the optimal "radial aspect ratio" of projected trajectories. Observability from $\omega(I)$ will depend on whether the critical trajectories that attain this maximal ratio spend sufficient time in $\omega(I)$, hence the criterion $\kappa_\star(I) > \Lambda(\nu_2/\nu_2)$. See Figure 4 for an illustration of the case where such trajectories are not seen by the observation set. Notice that maximizing the ratio in (1-30) with respect to any nonzero initial data is the same as taking the upper limit as $\rho_0 \to \infty$ since the Hamiltonian flow is homogeneous. Thus $\Lambda(\mu)$ can be understood as a quantity that captures the behavior of the flow at infinity. In addition, we remark that $\Lambda(\mu) = \Lambda(1/\mu)$, which means that this value depends only on the spectrum of the matrix A, and not on the choice of a specific basis of $\mathbb{R}^2$. The maximum of Λ is reached exactly at 1, where it is equal to $\tan(\pi/4) = 1$. This is consistent with the fact that in two dimensions, isotropic harmonic oscillators are the only ones possessing circular orbits: the norm of the trajectory $|x^t(\rho_0)|$ is constant for well-chosen initial data.

The distinction between rational and irrational values of μ is natural in light of the complete integrability of the flow of harmonic oscillators. When the ratio of characteristic frequencies $\mu = \nu_2/\nu_1$ is rational, writing $\mu = p/q$ with p, q a couple of coprime integers, one can check that the Hamiltonian flow of the corresponding harmonic oscillator is periodic of period $2\pi p/\nu_2 = 2\pi q/\nu_1$. In that case, there are many orbits of the flow whose projection on the x-variable space stays away from the origin, thus producing a

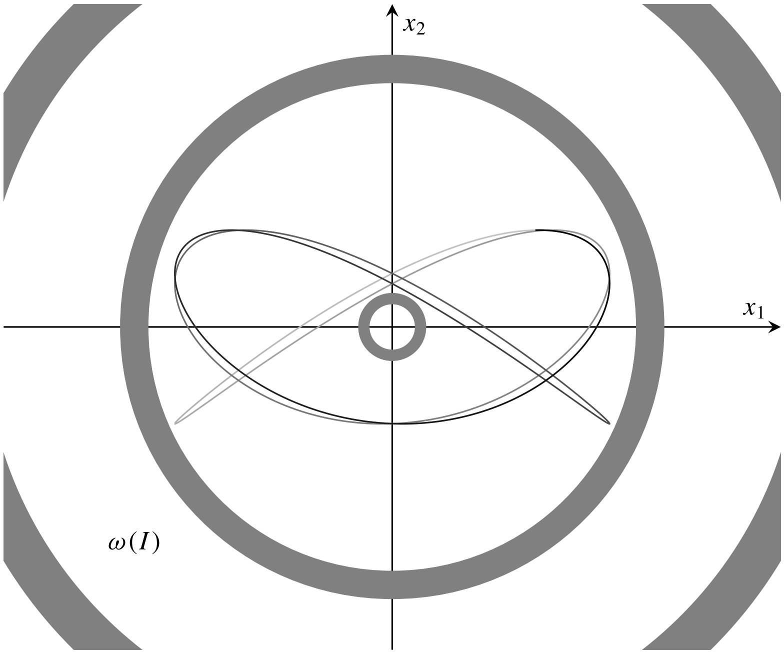


Figure 4. The above curve is a projected trajectory of a harmonic oscillator with $\nu_2/\nu_1 = \frac{4}{3}$, that does not intersect the observation set $\omega(I)$. The existence of a sequence of energy layers $\{p = E_n\}$, $E_n \to +\infty$, containing such curves would imply that observability from $\omega(I)$ fails.

positive $\Lambda(\mu)$, as one can see in Figure 2, top right. When μ is irrational, it is known that (nondegenerate) trajectories are dense in the invariant torus to which they belong. In particular, any projected trajectory can get arbitrarily close to the origin, up to waiting a long enough time, so that $\Lambda(\mu) = 0$; see Figure 2, bottom.

Lastly, let us point out that the estimate (1-29) of the optimal observation time for Diophantine irrational does not give any precise information for a given open set I, but is relevant for fixed ν_1, ν_2 in the asymptotics $\kappa_\star(I) \ll 1$.

Remark 1.13. It can look surprising that Proposition 1.11 gives an exact characterization of spherical sets for which observability holds, whereas Theorem 1.3 provides a necessary and sufficient condition up to thickening the observation set. This improvement is made possible by the extra assumption that $|I_n| \to +\infty$. This ensures that thickening the observation set by a radius R is negligible compared to the width of the annulus $\omega(I_n)$, for n large.

Remark 1.14 (non-Diophantine irrationals). When $\mu = \nu_2/\nu_1 \in \mathbb{R} \setminus \mathbb{Q}$, one can estimate $T_\star$, even if $\tau = \tau(\mu) = +\infty$, using the so-called convergents of μ. These are the rational numbers arising in the continued fraction expansion algorithm. Denote them in irreducible form by $\mu_j = p_j/q_j$. It is known that this sequence is the most efficient way to approximate an irrational number by rationals (a result known as Lagrange theorem; see [Durand 2015, Theorem 1.3] or [Einsiedler and Ward 2011; Schmidt 1991]). These convergents satisfy

$$\forall j \in \mathbb{N}, \quad \left|\mu - \frac{p_j}{q_j}\right| < \frac{1}{q_j^2}. \tag{1-31}$$

(This is why $\tau(\mu) \geq 2$ holds for any irrational.) We will show in the proof of Proposition 1.11 the following: when $\mu \in \mathbb{R} \setminus \mathbb{Q}$, there exist constants $c_1, c_2 > 0$ and $\delta_1, \delta_2 > 0$, possibly depending on ν_1, ν_2, such that

$$c_1 q_{j_1} \leq T_\star \leq c_2 q_{j_2} \tag{1-32}$$

(see (5-51) in the proof), where j_1 is the largest index for which $q_j \leq \delta_1/\kappa_\star$, and j_2 is the smallest index for which $q_j \geq \delta_2/\kappa_\star$.

The bounds (1-29) are particularly interesting when τ has the smallest possible value, that is, $\tau = 2$, which is the case of Lebesgue-almost every irrational. However, we see that the lower and upper bounds (1-29) get far apart as τ goes to infinity. This reflects the fact that the gaps between the denominators of consecutive convergents get wider at each step of the continued fraction expansion. Irrationals having an infinite irrationality exponent are known as Liouville numbers. There are many of them: the set of Liouville numbers is an instance of a Lebesgue negligible set having the cardinality of the continuum. This set is also Baire generic, as it can be written as a countable intersection of dense open sets. When ν_2/ν_1 is a Liouville number, the bounds (1-32) on the optimal observation time are very poor, owing to the lacunary behavior of the q_j's.

1.5. *Other applications.* Let us briefly discuss two other applications of Theorem 1.3.

1.5.1. *Uniform observability of eigenfunctions.* Under Assumption 1.1, the operator P is self-adjoint with compact resolvent. Thus, its spectrum consists in a collection of eigenvalues with finite multiplicity. A direct consequence of an observability inequality $\mathrm{Obs}(\omega, T)$ in a set ω is the fact that the eigenfunctions of P are uniformly observable from ω:

$$\exists c > 0 : \forall u \in L^2(\mathbb{R}^d), \quad \big(Pu = \lambda u \quad \Longrightarrow \quad \|u\|_{L^2(\omega)} \geq c\|u\|_{L^2(\mathbb{R}^d)}\big).$$

Theorem 1.3 thus furnishes a sufficient condition for this to hold. In particular, for anisotropic oscillators, Proposition 1.5 implies that uniform observability of eigenfunctions from the two cones defined in (1-17) is true. This can certainly be deduced from [Arnaiz and Macià 2022a], which characterizes quantum limits of harmonic oscillators. From Proposition 1.11, we obtain a similar uniform estimate in spherical sets satisfying the assumptions of the proposition together with the condition (1-28). This time, it is not clear that one can deduce this result as easily from the knowledge of quantum limits [Arnaiz and Macià 2022a]. See also [Dicke et al. 2023] for details about spectral inequalities for the Hermite operator, and [Martin 2022] for anisotropic Shubin operators.

1.5.2. *Energy decay of the damped wave equation.* Lastly, our study leads to stabilization results concerning the damped wave equation

$$\begin{cases} \partial_t^2 \psi + P\psi + \mathbf{1}_\omega \partial_t \psi = 0, \\ (\psi, \partial_t \psi)_{|t=0} = U_0 \in \operatorname{Dom} P^{1/2} \times L^2 \end{cases} \tag{1-33}$$

with damping in $\omega \subset \mathbb{R}^d$, provided $P \geq 0$ (assume for instance that the potential V is nonnegative). This equation comes with a natural energy

$$\mathscr{E}(U_0, t) = \tfrac{1}{2}\big(\|P^{1/2}\psi(t)\|_{L^2}^2 + \|\partial_t \psi(t)\|_{L^2}^2\big),$$

which decays over time. Let us recall that Anantharaman and Léautaud [2014, Theorem 2.3] proved that an observability inequality $\mathrm{Obs}(\omega, T)$ implies a decay at rate $t^{-1/2}$ for the damped wave equation (1-33), meaning that there exists a constant $C > 0$ such that

$$\mathscr{E}(U_0, t) \leq \frac{C}{t}\big(\|Pu_0\|_{L^2}^2 + \|P^{1/2}u_1\|_{L^2}^2\big) \quad \forall t > 0$$

for all initial data in the domain of the damped wave operator, where $U_0 = (u_0, u_1) \in \mathrm{Dom}\, P \times \mathrm{Dom}\, P^{1/2}$. Their result applies in our setting since P has compact resolvent under Assumption 1.1. Our examples thus provide concrete situations where such a decay occurs.

1.6. ***Link with the Kato smoothing effect.*** The dynamical condition (1-7) concerns only what happens at infinity in phase space. We will see that trajectories of the Hamiltonian flow escape from any compact set (in the x variable) most of the time provided the initial data has large enough energy, namely $p(\rho)$ is large enough. This is the reason why one can remove any compact set from the observation without losing observability: no energy can be trapped in a compact set. Quantitatively, we will check that, given $T > 0$, there exist a constant $C > 0$ and $E_0 > 0$ such that

$$\forall r \geq 0, \forall \rho \in \{p \geq E_0\}, \quad |\{t \in [0, T] : \phi^t(\rho) \in B_r(0) \times \mathbb{R}^d\}| = \int_0^T \mathbf{1}_{B_r(x^t)}\, dt \leq C \frac{r}{\sqrt{p(\rho)}} \tag{1-34}$$

(see Corollary 2.6). We can rephrase this by saying that compact sets are not *classically observable*. This property is related to the Kato smoothing effect as follows. Writing $(x^t, \xi^t) = \phi^t(\rho)$, for any $\varepsilon > 0$ we compute, using Fubini's theorem,

$$\int_0^T \frac{\sqrt{p(\rho)}}{\langle x^t\rangle^{1+\varepsilon}}\, dt = \int_0^T \bigg(\int_{\langle x^t\rangle}^{+\infty} (1+\varepsilon)\frac{\sqrt{p(\rho)}}{r^{2+\varepsilon}}\, dr\bigg)\, dt = \int_1^{+\infty} (1+\varepsilon)\frac{\sqrt{p(\rho)}}{r}\bigg(\int_0^T \mathbf{1}_{B_r(0)}(\langle x^t\rangle)\, dt\bigg)\frac{dr}{r^{1+\varepsilon}}.$$

From (1-34), we deduce that

$$\int_0^T \frac{\sqrt{p(\rho)}}{\langle x^t\rangle^{1+\varepsilon}}\, dt \leq C\int_1^{+\infty} (1+\varepsilon)\frac{dr}{r^{1+\varepsilon}},$$

and the latter integral is indeed convergent when $\varepsilon > 0$. This is the classical analogue to the so-called Kato smoothing effect. In our context, the latter says roughly that

$$\int_0^T \|\langle x\rangle^{-(1+\varepsilon)/2} P^{1/4} e^{-itP} u\|_{L^2(\mathbb{R}^d)}^2\, dt \leq C\|u\|_{L^2(\mathbb{R}^d)}^2.$$

See for instance [Doi 2005] for a thorough discussion on this topic. See also the survey of Robbiano [2013], as well as [Robbiano and Zuily 2008; 2009; Burq 2004] for related results. The main phenomenon responsible for this smoothing effect is the fact that P contains a Laplace–Beltrami operator associated with a nontrapping metric (here a flat metric), that is to say all geodesics escape at infinity forward and backward in time. In our case, working with a flat Laplacian enables us to compare the trajectories of the Hamiltonian flow to straight lines, at least for some time near the origin. It would be interesting to see whether our study can be adapted to operators of the form $P = V(x) - \frac{1}{2}\Delta_g$ with a nontrapping metric g on $\mathbb{R}^d$ (sufficiently flat at infinity). See [Macià and Nakamura, Lemma 3.1] for an alternative proof that nontrapping implies failure of observability from *bounded* observation sets. The argument relies on semiclassical defect measures.

1.7. ***Natural semiclassical scaling for homogeneous potentials.*** A way to comprehend what goes wrong when the potential is superquadratic is to introduce the natural semiclassical scales associated to our problem, based on an observation of [Macià and Nakamura]. Take for simplicity $p(x,\xi) = |x|^{2m} + |\xi|^2$. Following classical arguments, we recall in Appendix A that the observability inequality reduces to a high-energy observability inequality: roughly speaking, we can restrict ourselves to L^2 functions u that are microlocalized around some level set $\{p = E\}$ with $E \gg 1$. Writing

$$p(x,\xi) = E \quad \Longleftrightarrow \quad \left|\frac{x}{E^{1/2m}}\right|^{2m} + \left|\frac{\xi}{E^{1/2}}\right|^2 = 1,$$

we may introduce a small Planck parameter h such that $E = h^{-\gamma}$ for some power $\gamma > 0$. Thus we have

$$|h^{\gamma/2m}x|^{2m} + |h^{\gamma/2}\xi|^2 = 1.$$

This motivates the definition of an h-dependent Weyl quantization (see Appendix B)

$$\mathrm{Op}_h(a) := \mathrm{Op}_1(a(h^{\gamma/2m}x, h^{\gamma/2}\xi))$$

for any classical observable a on the phase space. This quantization is properly “normalized” by choosing $\gamma = 2m/(m+1)$: with this choice, the corresponding pseudodifferential calculus is expressed in powers of h, since then $h^{\gamma/2m}h^{\gamma/2} = h$. Therefore the relevant semiclassical Schrödinger operator is

$$P_h = \mathrm{Op}_h(p) = h^{\gamma}P.$$

If one wants to express the observability inequality in terms of the associated propagator, one is then lead to study

$$e^{-itP}u = e^{-ith^{1-\gamma}P_h/h}u.$$

In other words, running the Schrödinger evolution on a time interval $[0,T]$ amounts to consider a semiclassical time scale of order $h^{1-\gamma} = h^{(1-m)/(1+m)}$. It is then clear that this time blows up as $h \to 0$ when $m > 1$. Yet the analysis of the quantum-classical correspondence, for long times, is much more difficult. In particular, it restricts considerably the amount of classical observables whose evolution can be described through the usual Egorov theorem. For this reason, we will not pursue in this direction and stick to the case $m \le 1$. An interesting approach to study this would be to consider first particular potentials for which the classical flow is completely integrable, e.g., anharmonic oscillators; see [Bambusi et al. 2022]. Indeed, observability of the Schrödinger equation has been successfully investigated taking advantage of the completely integrable nature of the underlying classical dynamics in some particular geometrical contexts (e.g., in the disk [Anantharaman et al. 2016a; 2016b] which corresponds morally to $m = \infty$; see also [Anantharaman and Macià 2014] on the torus and [Anantharaman et al. 2015]).

1.8. ***Plan of the article.*** Section 2 is devoted to the study of the underlying classical dynamics: we show that the Hamiltonian flow is roughly stable under subprincipal perturbations of the potential, and that high-energy projected trajectories can cross compact sets only on a very short period of time. Then we establish an instance of quantum-classical correspondence adapted to our context in Section 3, and subsequently prove Theorem 1.3. This is the core of the article. Next, in Sections 4 and 5, we deal with the examples presented in Sections 1.4.1, 1.4.2 and 1.4.3 (observability from conical and spherical sets respectively). Finally, we recall in Appendix A a classical result, related to the notion of unique continuation, that shows that the sought observability inequality is equivalent to a similar high-energy inequality. Appendix B collects reminders about pseudodifferential operators, as well as refined estimates on the pseudodifferential calculus and the Gårding inequality needed for Section 3.

2. Study of the classical dynamics

In this section, we investigate the properties of the Hamiltonian flow $(\phi^t)_{t\in\mathbb{R}}$ associated with p. This study consists essentially in analyzing the ODE system that defines ϕ^t, namely the Hamilton equation (1-5). The dynamical condition of Theorem 1.3

$$\mathfrak{K}_p^\infty(\omega, T) = \liminf_{\rho\to\infty} \int_0^T \mathbf{1}_{\omega\times\mathbb{R}^d}(\phi^t(\rho))\,dt > 0$$

motivates the study of what can be referred to as "classical observability".

Definition 2.1 (classical observability). Let $q = q(t;\rho)$ be a Borel-measurable[8] function on $\mathbb{R}\times\mathbb{R}^{2d}$. Then we say that q is classically observable if

$$\mathfrak{K}_p^\infty(q) := \liminf_{\rho\to\infty} \int_{\mathbb{R}} q(t;\phi^t(\rho))\,dt > 0. \tag{2-1}$$

Of course, we will be specifically interested in the case where p contains a subquadratic potential and $q = \mathbf{1}_{(0,T)\times\omega\times\mathbb{R}^{2d}}$, but it is interesting to work out this problem in a more general setting in order to understand to what extent quadratic potentials are critical for the Schrödinger equation.

2.1. ***Invariance of classical observability under subprincipal perturbation.*** In this subsection, we consider a set of classical symbols on $\mathbb{R}^{2d}$ of order n_1 in x and n_2 in ξ, defined by

$$S^{n_1,n_2} = \left\{ a \in C^\infty(\mathbb{R}^{2d}) : \forall\alpha\in\mathbb{N}^{2d},\ \sup_{(x,\xi)\in\mathbb{R}^{2d}} \frac{|\partial^\alpha a(x,\xi)|}{\langle x\rangle^{n_1-|\alpha|} + \langle\xi\rangle^{n_2-|\alpha|}} < \infty \right\}.$$

A basic example is the classical Hamiltonian $p(x,\xi) = V(x) + \frac{1}{2}|\xi|^2$ that we consider: it belongs to $S^{2m,2}$. We draw the reader's attention to the fact that this is not a standard symbol class in microlocal analysis. Our aim here is simply to study symbols whose derivatives have similar decay properties as the classical Hamiltonian p. We will not make use of any notion of pseudodifferential calculus in this subsection.

It is clear that these symbol classes are nested in the following way: if $n_1 \le n_1'$ and $n_2 \le n_2'$, then $S^{n_1,n_2} \subset S^{n_1',n_2'}$ (and this inclusion is even continuous with respect to the associated Fréchet structure). Given $n_1, n_2 \in \mathbb{R}$, a real-valued symbol $a \in S^{n_1,n_2}$ is said to be elliptic in S^{n_1,n_2} if $a(x,\xi) \ge c(\langle x\rangle^{n_1} + \langle\xi\rangle^{n_2})$ provided $|(x,\xi)|$ is large enough. In addition, the binary relation

$$\forall f, g \in S^{n_1,n_2}, \quad f = g \pmod{S^{n_1-1,n_2-1}} \quad\Longleftrightarrow\quad f - g \in S^{n_1-1,n_2-1} \tag{2-2}$$

[8] Recall that Borel-measurability is slightly stronger than Lebesgue-measurability. This restriction ensures that $t \mapsto q(t;\phi^t(\rho))$ is Lebesgue-measurable. This is not a problem in our context since we will consider functions q that are continuous, or at worse, indicator functions of Borel sets.

is an equivalence relation, and the projection on the quotient space $S^{n_1,n_2}/S^{n_1-1,n_2-1}$ is called the principal symbol. Two symbols are said to have the same principal symbol if they belong to the same equivalence class through this projection. In the example of our classical Hamiltonian p, these notions of ellipticity and principal symbol are consistent with the terminology used right after Assumption 1.1 regarding the potential V.

The proposition below is essentially an application of Grönwall's lemma.

Proposition 2.2 (stability estimate). *Fix $n_1, n_2 > 0$ and let $p_1, p_2 \in S^{n_1,n_2}$ be elliptic symbols in S^{n_1,n_2}. Assume they have the same principal symbol in the sense of* (2-2). *Consider the Hamiltonian flows $(\phi_1^t)_{t\in\mathbb{R}}$ and $(\phi_2^t)_{t\in\mathbb{R}}$ associated with p_1 and p_2 respectively. Then there exists a constant $C > 0$ such that*

$$|\phi_2^t(\rho) - \phi_1^t(\rho)| \le e^{Ct\langle p_1(\rho)\rangle^{\max(0,1-2/n_+)}} \quad \forall \rho \in \mathbb{R}^{2d}, \forall t \ge 0,$$

where $n_+ = \max(n_1, n_2)$. In particular, when $n_1, n_2 \le 2$, there exists $C > 0$ such that

$$|\phi_2^t(\rho) - \phi_1^t(\rho)| \le e^{Ct} \quad \forall \rho \in \mathbb{R}^{2d}, \forall t \ge 0.$$

Remark 2.3. This result ensures that the distance between $\phi_1^t(\rho)$ and $\phi_2^t(\rho)$ is bounded provided $n_+ \le 2$, on a time interval $[0, T]$ independent of ρ. In our problem, this condition on n_+ means exactly that the potential is subquadratic.

Proof. In this proof, we write $n_+ = \max(n_1, n_2)$ and $n_- = \min(n_1, n_2)$. Set $\tilde{p} = p_2 - p_1$, which belongs to S^{n_1-1,n_2-1} by assumption. The Hamilton equation (1-5) gives

$$\begin{aligned}\left|\frac{d}{dt}(\phi_2^t(\rho) - \phi_1^t(\rho))\right| &= \left|J(\nabla p_2(\phi_2^t(\rho)) - \nabla p_1(\phi_1^t(\rho)))\right| \\ &\le \left|\nabla p_2(\phi_2^t(\rho)) - \nabla p_2(\phi_1^t(\rho))\right| + |\nabla \tilde{p}(\phi_1^t(\rho))|. \end{aligned} \tag{2-3}$$

By assumption, p_1 and p_2 are elliptic at infinity in S^{n_1,n_2}, so that for any $\rho = (x, \xi)$ large enough, one has

$$\frac{1}{C}(\langle x\rangle^{n_1} + \langle \xi\rangle^{n_2}) \le |p_j(\rho)| \le C(\langle x\rangle^{n_1} + \langle \xi\rangle^{n_2}), \quad j \in \{1, 2\}. \tag{2-4}$$

From the definition of S^{n_1-1,n_2-1}, which contains $\tilde{p}$, we have

$$|\nabla \tilde{p}(\rho)| \le C(\langle x\rangle^{n_1-2} + \langle \xi\rangle^{n_2-2}).$$

The ellipticity of p_2, that is, the left-hand side of (2-4), then yields

$$|\nabla \tilde{p}(\rho)| \le C\big(|p_1(\rho)|^{\max(0,1-2/n_1)} + |p_1(\rho)|^{\max(0,1-2/n_2)}\big) \le C'|p_1(\rho)|^{\max(0,1-2/n_+)},$$

provided $|\rho|$ is large enough. On the whole phase space we obtain

$$|\nabla \tilde{p}(\rho)| \le C + C|p_1(\rho)|^{\max(0,1-2/n_+)} \quad \forall \rho \in \mathbb{R}^{2d}. \tag{2-5}$$

Now we deal with the other term in (2-3): the mean-value inequality yields

$$\left|\nabla p_2(\phi_2^t(\rho)) - \nabla p_2(\phi_1^t(\rho))\right| \le |\phi_2^t(\rho) - \phi_1^t(\rho)| \times \sup_{s\in[0,1]} \left|\operatorname{Hess} p_2\big((1-s)\phi_1^t(\rho) + s\phi_2^t(\rho)\big)\right|. \tag{2-6}$$

Write for short $\rho_s^t = (1-s)\phi_1^t(\rho) + s\phi_2^t(\rho)$. Using that $p_2 \in S^{n_1,n_2}$, we obtain

$$|\operatorname{Hess} p_2(\rho_s^t)| \le C\big(\langle(1-s)x_1^t + sx_2^t\rangle^{n_1-2} + \langle(1-s)\xi_1^t + s\xi_2^t\rangle^{n_2-2}\big),$$

where we wrote $\phi_j^t(\rho) = (x_j^t, \xi_j^t)$, $j \in \{1, 2\}$. Then we use the classical inequality $\langle a+b\rangle \le 2(\langle a\rangle + \langle b\rangle)$ to get

$$\begin{aligned}|\mathrm{Hess}\, p_2(\rho_s^t)| &\le C\big((\langle x_1^t\rangle + \langle x_2^t\rangle)^{\max(0,n_1-2)} + (\langle \xi_1^t\rangle + \langle \xi_2^t\rangle)^{\max(0,n_2-2)}\big)\\ &\le C'\big(\langle x_1^t\rangle^{\max(0,n_1-2)} + \langle \xi_1^t\rangle^{\max(0,n_2-2)}\big) + C'\big(\langle x_2^t\rangle^{\max(0,n_1-2)} + \langle \xi_2^t\rangle^{\max(0,n_2-2)}\big).\end{aligned}$$

Next we use the ellipticity of p_1 and p_2 and the fact that they are conserved by the corresponding flows:

$$\begin{aligned}|\mathrm{Hess}\, p_2(\rho_s^t)| &\le C\big(|p_1(\phi_1^t(\rho))|^{\max(0,1-2/n_+)} + |p_2(\phi_2^t(\rho))|^{\max(0,1-2/n_+)}\big)\\ &= C\big(|p_1(\rho)|^{\max(0,1-2/n_+)} + |p_2(\rho)|^{\max(0,1-2/n_+)}\big),\end{aligned}$$

which holds for $|\rho|$ large enough. Up to adding a constant, this works for all $\rho \in \mathbb{R}^d$. Finally we use the fact that p_1 and p_2 are comparable (a consequence of ellipticity) to obtain

$$|\mathrm{Hess}\, p_2(\rho_s^t)| \le C + C|p_1(\rho)|^{\max(0,1-2/n_+)} \quad \forall \rho \in \mathbb{R}^{2d}.$$

Plugging this into (2-6), that results in

$$\big|\nabla p_2(\phi_2^t(\rho)) - \nabla p_2(\phi_1^t(\rho))\big| \le C|\phi_2^t(\rho) - \phi_1^t(\rho)| \times (1 + |p_1(\rho)|^{\max(0,1-2/n_+)})$$

for all $\rho \in \mathbb{R}^{2d}$. Putting this together with (2-5), we estimate the right-hand side of (2-3) from above as

$$\left|\frac{d}{dt}(\phi_2^t(\rho) - \phi_1^t(\rho))\right| \le C(1 + |\phi_2^t(\rho) - \phi_1^t(\rho)|) \times (1 + |p_1(\rho)|^{\max(0,1-2/n_+)}).$$

We deduce that

$$\begin{aligned}\left|\frac{d}{dt}\langle \phi_2^t(\rho) - \phi_1^t(\rho)\rangle\right| &= \left|\frac{d}{dt}(\phi_2^t(\rho) - \phi_1^t(\rho)) \cdot \frac{\phi_2^t(\rho) - \phi_1^t(\rho)}{\langle \phi_2^t(\rho) - \phi_1^t(\rho)\rangle}\right|\\ &\le C\langle \phi_2^t(\rho) - \phi_1^t(\rho)\rangle (1 + |p_1(\rho)|^{\max(0,1-2/n_+)})\end{aligned}$$

for any $\rho \in \mathbb{R}^{2d}$. We conclude by Grönwall's lemma that

$$\langle \phi_2^t(\rho) - \phi_1^t(\rho)\rangle \le e^{Ct\langle p_1(\rho)\rangle^{\max(0,1-2/n_+)}} \quad \forall \rho \in \mathbb{R}^{2d}, \forall t \ge 0,$$

which gives the sought result. □

The result below roughly states that our dynamical condition is invariant under subprincipal perturbation of the potential V, under the assumption that V is subquadratic.

Corollary 2.4. *Fix $0 < n_1, n_2 \le 2$ and let $p_1, p_2 \in S^{n_1,n_2}$ be elliptic symbols in S^{n_1,n_2}, and assume they have the same principal symbol in the sense of* (2-2). *Consider the Hamiltonian flows $(\phi_1^t)_{t\in\mathbb{R}}$ and $(\phi_2^t)_{t\in\mathbb{R}}$ associated with p_1 and p_2 respectively. For any $T > 0$, there exists a constant $C = C_T > 0$ such that the following holds: for any function $q = q(t;\rho)$, Lipschitz in ρ and such that*

$$\operatorname{supp} q \subset [-T, T] \times \mathbb{R}^{2d},$$

one has

$$\left|\int_{\mathbb{R}} q(t; \phi_2^t(\rho))\, dt - \int_{\mathbb{R}} q(t; \phi_1^t(\rho))\, dt\right| \le C\|\nabla_\rho q\|_{L^\infty(\mathbb{R}\times\mathbb{R}^{2d})} \quad \forall \rho \in \mathbb{R}^{2d}.$$

In particular,

$$|\mathfrak{K}_{p_2}^\infty(q) - \mathfrak{K}_{p_1}^\infty(q)| \le C\|\nabla_\rho q\|_{L^\infty(\mathbb{R}\times\mathbb{R}^{2d})}.$$

Proof. This is a direct application of the mean-value inequality and Proposition 2.2, observing that $n_+ = \max(n_1, n_2) \leq 2$:

$$\left| \int_{\mathbb{R}} q(t; \phi_2^t(\rho))\, dt - \int_{\mathbb{R}} q(t; \phi_1^t(\rho))\, dt \right| \\ \leq \int_{-T}^{T} \|\nabla_\rho q\|_{L^\infty(\mathbb{R}\times\mathbb{R}^{2d})} |\phi_2^t(\rho) - \phi_1^t(\rho)|\, dt \leq 2Te^{CT} \|\nabla_\rho q\|_{L^\infty(\mathbb{R}\times\mathbb{R}^{2d})}.$$

Taking lower limits in ρ yields the second claim. □

2.2. ***Quantitative estimates of classical (non)observability.*** In this subsection, we show that $\mathbf{1}_{(0,T)\times B_r(0)\times\mathbb{R}^d}$ is not classically observable in the sense of Definition 2.1 when the Hamiltonian is of the form $p(x,\xi) = V(x) + \frac{1}{2}|\xi|^2$. Actually for this class of Hamiltonians, we can prove a more precise result.

Proposition 2.5. *Let p be a symbol of the form $p(x,\xi) = V(x) + \frac{1}{2}|\xi|^2$, with V fulfilling Assumption 1.1 with an arbitrary $m > 0$.*

- *If $m \geq \frac{1}{2}$, there exists a constant $C > 0$ and $E_0 > 0$ such that for all $E \geq E_0$, one has*

$$\forall r \geq 0, \forall \rho \in \{p = E\}, \quad \left|\left\{t \in [0, E^{(1/2)(1/m-1)}] : \phi^t(\rho) \in B_r(0) \times \mathbb{R}^d\right\}\right| \leq C\frac{r}{\sqrt{E}}.$$

- *If $m < \frac{1}{2}$, then for any $\varepsilon > 0$ small enough, there exists a constant $C > 0$ and $E_0 > 0$ such that for all $E \geq E_0$, one has*

$$\forall r \geq 0, \forall \rho \in \{p = E\}, \quad \left|\left\{t \in [0, E^{(1/2)(1/m-1)-\varepsilon}] : \phi^t(\rho) \in B_r(0) \times \mathbb{R}^d\right\}\right| \leq C\frac{r}{\sqrt{E}}.$$

Corollary 2.6 (classical nonobservability). *Under the assumptions of the proposition above, one has:*

- *If $m < 1$, then for any $T \geq 0$, there exists a constant $C > 0$ and $E_0 > 0$ such that for all $E \geq E_0$, one has*

$$\forall r \geq 0, \forall \rho \in \{p = E\}, \quad \left|\left\{t \in [0, T] : \phi^t(\rho) \in B_r(0) \times \mathbb{R}^d\right\}\right| \leq C\frac{r}{\sqrt{E}}.$$

- *If $m \geq 1$, there exists a constant $C > 0$ and $E_0 > 0$ such that for all $E \geq E_0$ and for all $T \geq 0$, one has*

$$\forall r \geq 0, \forall \rho \in \{p = E\}, \quad \left|\left\{t \in [0, T] : \phi^t(\rho) \in B_r(0) \times \mathbb{R}^d\right\}\right| \leq C\frac{r(1+T)}{E^{1/(2m)}}.$$

Remark 2.7. The corollary implies in particular that when r and T are fixed, the function $\mathbf{1}_{(0,T)\times B_r(0)\times\mathbb{R}^d}$ is not classically observable in the sense of Definition 2.1.

Let us explain the meaning of the typical scales appearing in Proposition 2.5 and the subsequent corollary. When V satisfies Assumption 1.1 with an arbitrary $m > 0$, one can single out a typical time scale in the energy layer $\{p(\rho) = E\}$ of order $\tau \approx E^{(1/2)(1/m-1)}$, which corresponds roughly speaking to the "period" of the trajectories of the flow, or rather, to the time needed to go from one turning point of a projected trajectory to another. We observe that for the harmonic oscillator, one has $m = 1$; hence $\tau \approx 1$ is indeed independent of the energy layer. Following this observation, we understand the criticality of quadratic potentials in our problem: if $m > 1$, the typical time scale of evolution of the flow tends to zero as the energy goes to infinity, which means that the flow mixes the phase space more and more in the high-energy limit in a time interval of the form $[0, T]$ with $T > 0$ fixed. On the contrary, for $m < 1$, the flow gets nicer on such a time interval because $\tau \to +\infty$ as $E \to +\infty$. We also have a typical scale with respect to the space variable, which is $r \approx E^{1/(2m)}$. This is the approximate diameter of the classically allowed region $K_E = \{x \in \mathbb{R}^d : V(x) \le E\}$. This scale also appears naturally when one looks for a trajectory $t \mapsto \phi^t(\rho) = (x^t(\rho), \xi^t(\rho))$ such that $|x^t(\rho)| =$ constant (think for instance of the case of radial potentials). Differentiating $|x^t(\rho)|^2$ with respect to time, one gets $x^t(\rho) \cdot \xi^t(\rho) = 0$ for all t, and differentiating again leads to $|\xi^t(\rho)|^2 - x^t(\rho) \cdot \nabla V(x^t(\rho)) = 0$. Yet $|\nabla V(x^t(\rho))| \lesssim |x^t(\rho)|^{2m-1}$, and p is preserved by the flow. From this we can deduce that $|x^t(\rho)| \approx p(\rho)^{1/(2m)}$. So if r is larger than $p(\rho)^{1/(2m)}$, such trajectories will always stay in $B_r(0) \times \mathbb{R}^d$. Finally, if $\rho_0 = (x_0, \xi_0) \in \{p(\rho) = E\}$ is such that $|x_0| \le r$, with $r \le \varepsilon p(\rho)^{1/(2m)}$, ε being sufficiently small, the momentum of the trajectory satisfies $|\xi_0| \gtrsim \sqrt{p(\rho)}$. Therefore, we can expect that the measure of times $t \in [0, \tau]$ such that $|x^t(\rho)| \lesssim r$ will be of order $r/\sqrt{p(\rho)}$.

The proof of Proposition 2.5 relies on the lemma below.

Lemma 2.8. *Let $a, b, c > 0$. Let $I \subset \mathbb{R}$ be a measurable set such that*

$$\forall (t_1, t_2) \in I \times I, \quad a|t_2 - t_1|^2 - b|t_2 - t_1| + c \ge 0.$$

Then

$$|I \cap [0, \tau]| \le \frac{8ac}{b^2}\tau \quad \forall \tau \ge \frac{b}{2a}. \tag{2-7}$$

Remark 2.9. Observe that the left-hand side of (2-7) is always bounded by τ. Thus, the lemma is mainly relevant in the case where $ac \ll b^2$, in which case the discriminant of the polynomial $aX^2 - bX + c$ is positive.

Proof of Lemma 2.8. First assume that the discriminant of the polynomial $aX^2 - bX + c$ is positive. Denote by $z_- \le z_+$ the (real) roots of the polynomial. Then

$$\frac{b}{2a} = \frac{z_+ + z_-}{2} \le z_+ \le z_+ + z_- = \frac{b}{a} \quad \text{and} \quad z_- = \frac{z_+ z_-}{z_+} = \frac{c/a}{z_+} \le \frac{2c}{b}.$$

Since $a > 0$, we deduce that any t such that $at^2 - bt + c \ge 0$ satisfies

$$t \le z_- \le \frac{2c}{b} \quad \text{or} \quad t \ge z_+ \ge \frac{b}{2a}. \tag{2-8}$$

We deduce that

$$\left| I \cap \left[0, \frac{b}{2a}\right]\right| \le \left|\left\{t \in [0, z_+] : at^2 - bt + c \ge 0\right\}\right| \le |[0, z_-]| \le \frac{2c}{b}. \tag{2-9}$$

Now if $\tau \ge b/(2a)$, we split the interval $[0, \tau]$ as follows:

$$[0, \tau] = \bigcup_{k=1}^{n} \left[\frac{k-1}{n}\tau, \frac{k}{n}\tau\right], \quad \text{with } n = \left\lceil \frac{\tau}{b/2a} \right\rceil \ge 1.$$

On each piece, we have

$$\left| I \cap \left[\frac{k-1}{n}\tau, \frac{k}{n}\tau\right]\right| = \left|\left(I - \frac{k-1}{n}\tau\right) \cap \left[0, \frac{1}{n}\tau\right]\right| \le \left|\left(I - \frac{k-1}{n}\tau\right) \cap \left[0, \frac{b}{2a}\right]\right|,$$

where the last inequality is due to the definition of n. We can apply (2-9) with $I-(k-1)\tau/n$ instead of I, since the former set satisfies the assumptions of the lemma. Then, summing over k yields

$$|I\cap[0,\tau]|\le n\frac{2c}{b}\le\Big(\frac{\tau}{b/2a}+1\Big)\frac{2c}{b}\le\frac{8ac}{b^2}\tau,$$

which is the desired estimate. Finally if the discriminant is nonpositive, i.e., $b^2\le 4ac$, then

$$|I\cap[0,\tau]|\le\tau\le\frac{4ac}{b^2}\tau. \qquad \square$$

Proof of Proposition 2.5. Let us write for short $E=p(\rho)$, and introduce the components of the flow $(x^t,\xi^t)=\phi^t(\rho)$. Assume $E>0$. The core of the argument is to compare x^t to the straight trajectory $t\mapsto x^0+t\xi^0$, which is of course easier to handle. In order to have two distinct points of the initial trajectory to be in the ball $B_r(0)$, its distance to the straight trajectory has to be very small or very large, which is possible in a time interval which is either small or large respectively. Introduce

$$I=I_{\rho,r}=\{t\in\mathbb{R}:x^t\in B_r(0)\}.$$

This set is measurable. Moreover, for any $t_1\le t_2$, using the Hamilton equation and the Taylor formula at order 1 with integral remainder, one has

$$x^{t_2}=x^{t_1}+(t_2-t_1)\xi^{t_1}-(t_2-t_1)^2\int_0^1(1-s)\nabla V(x^{(1-s)t_1+st_2})\,ds.$$

Assume now that $t_1,t_2\in I$. Then the inverse triangle inequality leads to

$$2r\ge|t_2-t_1||\xi^{t_1}|-(t_2-t_1)^2\sup_{t\in[t_1,t_2]}|\nabla V(x^t)|. \tag{2-10}$$

At this stage we have to estimate differently the term involving ∇V, depending on whether m is greater or less than $\frac12$ (or roughly speaking on whether the potential is approximately convex of concave).

Case $m\ge\frac12$: Using that V satisfies Assumption 1.1, we have

$$|\xi^{t_1}|=\sqrt{2(E-V(x^{t_1}))}\ge\sqrt{\max(0,E-C\langle r\rangle^{2m})}$$

for some constant $C\ge1$. Moreover, one can roughly estimate the remainder using the triangle inequality:

$$\sup_{t\in[t_1,t_2]}|\nabla V(x^t)|\le C\sup_{t\in[t_1,t_2]}\langle x^t\rangle^{2m-1}.$$

Now we take advantage of the fact that V is elliptic: up to enlarging the constant C, one has

$$-C+\frac1C\langle x\rangle^{2m}\le V(x)\le V(x)+\frac12|\xi|^2\quad\forall(x,\xi)\in\mathbb{R}^{2d}.$$

Therefore if E is large enough (say larger than C), we obtain $\langle x^t\rangle^{2m-1}\le CE^{1-1/(2m)}$, with a possibly larger constant C (we use $m\ge\frac12$ here). Inequality (2-10) then becomes

$$2r\ge|t_2-t_1|\sqrt{\max(0,E-C\langle r\rangle^{2m})}-CE^{1-1/(2m)}|t_2-t_1|^2.$$

Set

$$a = CE^{1-1/(2m)}, \quad b = \sqrt{\max(0, E - C\langle r\rangle^{2m})}, \quad c = 2r \quad \text{and} \quad \tau = E^{(1/2)(1/m-1)}. \tag{2-11}$$

We have $\tau \geq b/(2a)$ since we can assume that $C \geq 1$:

$$\frac{b}{2a} = \frac{\sqrt{\max(0, E - C\langle r\rangle^{2m})}}{2C} E^{1/(2m)-1} \leq \frac{1}{2C} E^{(1/2)(1/m-1)} \leq \tau.$$

With this notation, we have that any $t_1, t_2 \in I$ satisfy

$$a|t_2 - t_1|^2 - b|t_2 - t_1| + c \geq 0.$$

Therefore, assuming first that $C\langle r\rangle^{2m} \leq E/2$, we have $b \geq \sqrt{E/2} > 0$, so that Lemma 2.8 applies. We obtain

$$|I \cap [0, E^{(1/2)(1/m-1)}]| \leq \frac{8ac}{b^2}\tau \leq \frac{8CE^{1-1/(2m)} \times 2r}{E/2} E^{(1/2)(1/m-1)} = 32C\frac{r}{\sqrt{E}}.$$

If on the contrary we have $\langle r\rangle \geq (E/2C)^{1/(2m)}$, as soon as $E \geq 2^{2m+1}C$ we have $r \geq \frac{1}{2}(E/2C)^{1/(2m)}$, and we check that

$$|I \cap [0, E^{(1/2)(1/m-1)}]| \leq E^{(1/2)(1/m-1)} = \frac{r}{\sqrt{E}} \times \frac{E^{1/(2m)}}{r} \leq \frac{r}{\sqrt{E}} \times 2^{1+1/(2m)} C^{1/(2m)}.$$

This is valid for any $r > 0$, but in fact $r = 0$ works as well since $B_0(0) = \varnothing$. In addition, this is independent of the point $\rho \in \{p = E\}$, whence the result.

Case $m < \frac{1}{2}$: In the situation where the potential is "sublinear", the inequality $\langle x^t\rangle^{2m-1} \lesssim E^{1-1/(2m)}$ is false in general since the power $2m - 1$ is nonpositive (such an inequality would require $V(x^t)$ to be controlled from below by E, which is possible near turning points of the trajectory but not in the well). Thus, a priori we can only have $|\nabla V(x^t)| \leq C$, which leads to

$$\begin{aligned} 2r &\geq |t_2 - t_1||\xi^{t_1}| - C|t_2 - t_1|^2 \geq |t_2 - t_1|\sqrt{\max(0, E - C\langle r\rangle^{2m})} - C|t_2 - t_1|^2 \\ &\geq |t_2 - t_1|\sqrt{\max(0, E - C\langle r\rangle)} - C|t_2 - t_1|^2. \end{aligned} \tag{2-12}$$

This coincides with the previous case for the critical value $m = \frac{1}{2}$: for any $t_1, t_2 \in I$, we have

$$a|t_2 - t_1|^2 - b|t_2 - t_1| + c \geq 0,$$

where a, b, c are defined in (2-11) $\left(\text{with } m = \frac{1}{2}\right)$. Then the first step of the proof tells us that there exists $C > 0$ such that for all E large enough, we have

$$\left|\left\{t \in [0, \sqrt{E}] : \phi^t(\rho) \in B_r(0) \times \mathbb{R}^d\right\}\right| \leq C\frac{r}{\sqrt{E}} \quad \forall r \geq 0, \forall \rho \in \{p = E\}. \tag{2-13}$$

We shall use this additional information to improve (2-12), and then bootstrap this procedure to reach the critical time $E^{(1/2)(1/m-1)}$. We will work this out by induction, taking (2-13) as our basis step. Consider $n \geq 0$ and suppose there exist $\gamma_n \in \left[\frac{1}{2}, \frac{1}{2}\left(\frac{1}{m} - 1\right)\right)$ and $C_n \geq 1$ such that when E is large enough, one has

$$\left|\left\{t \in [0, E^{\gamma_n}] : \phi^t(\rho) \in B_r(0) \times \mathbb{R}^d\right\}\right| \leq C_n\frac{r}{\sqrt{E}} \quad \forall r \geq 0, \forall \rho \in \{p = E\}. \tag{2-14}$$

We first deduce from the Taylor formula a bound slightly more precise than (2-10):

$$2r \geq |t_2 - t_1|\,|\xi^{t_1}| - |t_2 - t_1| \int_{t_1}^{t_2} |\nabla V(x^t)|\,dt$$
$$\geq |t_2 - t_1| \sqrt{\max(0, E - C\langle r\rangle^{2m})} - |t_2 - t_1| \int_{t_1}^{t_2} |\nabla V(x^t)|\,dt. \qquad (2\text{-}15)$$

Take $\delta \in [0, 1]$ to be chosen later. We have

$$\int_{t_1}^{t_2} |\nabla V(x^t)|\,dt \leq C \int_{t_1}^{t_2} \langle x^t\rangle^{2m-1}\,dt = C \int_0^{+\infty} \biggl(\int_{t_1}^{t_2} \mathbf{1}_{u \leq \langle x^t\rangle^{2m-1}}\,dt \biggr) du$$
$$\leq C \int_0^{+\infty} |\{t \in [t_1, t_2] : u \leq |x^t|^{2m-1}\}|\,du$$
$$\leq C \int_0^{E^{\delta(1-1/(2m))}} |t_2 - t_1|\,du + C \int_{E^{\delta(1-1/(2m))}}^{+\infty} |\{t \in [t_1, t_2] : |x^t| \leq u^{-1/(1-2m)}\}|\,du. \qquad (2\text{-}16)$$

The first inequality follows from our assumptions on V, the equality is a consequence of Fubini's theorem, then we use that $2m - 1 \leq 0$ to deduce $\langle x^s\rangle^{2m-1} \leq |x^s|^{2m-1}$, and finally we split the integral over u into two pieces. To estimate the second piece, we split the interval $[t_1, t_2]$ into $N = \lceil |t_2 - t_1| / E^{\gamma_n} \rceil$ intervals of length less than E^{γ_n}. On the k-th piece, we use the induction hypothesis (2-14), with $\rho_k = \phi^{t_1 + (k-1)|t_2 - t_1|/N}(\rho)$ instead of ρ, namely setting $\tilde{t}_k = t_1 + (k-1)|t_2 - t_1|/N$, we have

$$\bigl|\bigl\{t \in [\tilde{t}_k, \tilde{t}_{k+1}] : |x^t| \leq u^{-1/(1-2m)}\bigr\}\bigr| \leq \bigl|\bigl\{s \in [0, E^{\gamma_n}] : |x^{s+\tilde{t}_k}| \leq u^{-1/(1-2m)}\bigr\}\bigr| \leq \frac{C_n}{\sqrt{E}} u^{-1/(1-2m)}.$$

Summing over $k \in \{1, 2, \ldots, N\}$ yields

$$\bigl|\bigl\{t \in [t_1, t_2] : |x^t| \leq u^{-1/(1-2m)}\bigr\}\bigr| \leq \frac{C_n}{\sqrt{E}} u^{-1/(1-2m)} \biggl\lceil \frac{|t_2 - t_1|}{E^{\gamma_n}} \biggr\rceil,$$

provided E is large enough. Integrating over u, we obtain a bound for the second term in (2-16):

$$\int_{E^{\delta(1-1/(2m))}}^{+\infty} \bigl|\bigl\{t \in [t_1, t_2] : |x^t| \leq u^{-1/(1-2m)}\bigr\}\bigr|\,du$$
$$\leq \frac{C_n}{E^{1/2}} \biggl(\frac{|t_2 - t_1|}{E^{\gamma_n}} + 1 \biggr) \int_{E^{\delta(1-1/(2m))}}^{+\infty} u^{-1/(1-2m)}\,du$$
$$= \frac{C_n}{E^{1/2}} \biggl(\frac{|t_2 - t_1|}{E^{\gamma_n}} + 1 \biggr) \times \frac{-1}{1 - \frac{1}{1-2m}} E^{\delta(1-1/(2m))(1-1/(1-2m))}$$
$$= \biggl(\frac{1/2}{m} - 1 \biggr) \times \frac{C_n}{E^{1/2}} \biggl(\frac{|t_2 - t_1|}{E^{\gamma_n}} + 1 \biggr) E^{\delta}.$$

In the end we obtain

$$\int_{t_1}^{t_2} |\nabla V(x^t)|\,dt \leq \frac{C}{2} |t_2 - t_1| (E^{\delta(1-1/(2m))} + E^{\delta - 1/2 - \gamma_n}) + C E^{\delta - 1/2}$$

for some constant $C > 0$. By choosing $\delta = m(2\gamma_n + 1)$ (we have indeed $\delta \in [2m, 1) \subset [0, 1)$ when $\gamma_n \in \bigl[\frac{1}{2}, \frac{1}{2}\bigl(\frac{1}{m} - 1\bigr)\bigr)$), we obtain

$$\int_{t_1}^{t_2} |\nabla V(x^t)|\,dt \leq C |t_2 - t_1| E^{(2m-1)\gamma_n + m - 1/2} + C E^{\delta - 1/2}.$$

Going back to (2-15), if $t_1, t_2 \in I$, i.e., x^{t_1} and x^{t_2} lie in $B_r(0)$, we deduce

$$2r \geq |t_2 - t_1|\big(\sqrt{\max(0, E - C\langle r\rangle^{2m})} - CE^{\delta-1/2}\big) - CE^{1/2-\gamma_{n+1}}|t_2 - t_1|^2,$$

where we set $\gamma_{n+1} = (1-2m)\gamma_n + 1 - m$. Now set

$$a = CE^{1/2-\gamma_{n+1}}, \quad b = \sqrt{\max(0, E - C\langle r\rangle^{2m})} - CE^{\delta-1/2} \quad \text{and} \quad c = 2r.$$

Assuming first that $C\langle r\rangle^{2m} \leq E/2$ and recalling that $\delta < 1$, we know that for E large enough, we have $b \geq \sqrt{E/3}$. Any $t_1, t_2 \in I$ satisfy

$$a|t_2 - t_1|^2 - b|t_2 - t_1| + c \geq 0,$$

so we apply Lemma 2.8 with $\tau = E^{\gamma_{n+1}} \geq b/(2a)$ to get

$$|I \cap [0, E^{\gamma_{n+1}}]| \leq \frac{8ac}{b^2} E^{\gamma_{n+1}} \leq \frac{16CE^{1/2}}{E/3} r = \frac{48C}{\sqrt{E}} r.$$

When $C\langle r\rangle^{2m} \geq E/2$, assuming that E is large enough we have $r \geq \frac{1}{2}(E/2C)^{1/(2m)}$ and we conclude as in the previous step that

$$|\{t \in [0, E^{\gamma_{n+1}}] : x^t \in B_r(0)\}| \leq \frac{r}{\sqrt{E}} \frac{E^{\gamma_{n+1}+1/2}}{r} \leq \frac{r}{\sqrt{E}} 2^{1+1/(2m)} C^{1/(2m)} E^{\gamma_{n+1}-(1/2)(1/m-1)}.$$

Since by the induction hypothesis we have $\gamma_n \in \big[\frac{1}{2}, \frac{1}{2}\big(\frac{1}{m} - 1\big)\big)$, then γ_{n+1} belongs to the same interval because by definition, $\gamma_{n+1} \geq 1 - m \geq \frac{1}{2}$, and we have

$$\frac{\gamma - \gamma_{n+1}}{\frac{1}{2}} = (1-2m)\frac{\gamma - \gamma_n}{\frac{1}{2}}, \quad \text{where } \gamma = \frac{1}{2}\Big(\frac{1}{m} - 1\Big). \tag{2-17}$$

Therefore we see that $\gamma_{n+1} - \gamma < 0$, so as soon as E is large enough, we have

$$|\{t \in [0, E^{\gamma_{n+1}}] : x^t \in B_r(0)\}| \leq C_{n+1}\frac{r}{\sqrt{E}}$$

for any $r \geq 0$, and for some constant C_{n+1}. Thus we have constructed by induction a nondecreasing sequence $(\gamma_n)_{n\in\mathbb{N}}$ for which (2-14) holds. We deduce from (2-17) that it converges to $\gamma = \frac{1}{2}\big(\frac{1}{m} - 1\big)$, which yields the final result. □

Proof of Corollary 2.6. Firstly we treat the case where $m < 1$. For ε small enough, $E^{(1/2)(1/m-1)-\varepsilon} \to +\infty$ as $E \to +\infty$, so we can write, using Proposition 2.5,

$$\big|\big\{t \in [0, T] : \phi^t(\rho) \in B_r(0) \times \mathbb{R}^d\big\}\big| \leq \big|\big\{t \in [0, E^{(1/2)(1/m-1)-\varepsilon}] : \phi^t(\rho) \in B_r(0) \times \mathbb{R}^d\big\}\big| \leq C\frac{r}{\sqrt{E}},$$

provided E is large enough, for all $\rho \in \{p = E\}$ and all $r \geq 0$. Now in the case where $m \geq 1$, we know that $E^{(1/2)(1/m-1)}$ remains bounded as $E \to +\infty$. By Proposition 2.5 again, there is a $E_0 > 0$ such that for any $E \geq E_0$, we have

$$\big|\big\{t \in [0, E^{(1/2)(1/m-1)}] : \phi^t(\rho) \in B_r(0) \times \mathbb{R}^d\big\}\big| \leq C\frac{r}{\sqrt{E}}, \tag{2-18}$$

whenever $r \geq 0$ and $\rho \in \{p(\rho) = E\}$. Let $n = \lceil T/E^{(1/2)(1/m-1)} \rceil$. Writing $t_k = kT/n$ and $\rho_k = \phi^{t_k}(\rho)$ for any $k \in \{0, 1, \dots, n\}$, we have

$$\begin{aligned}\left|\left\{t \in [0, T] : \phi^t(\rho) \in B_r(0) \times \mathbb{R}^d\right\}\right| &\leq \sum_{k=1}^{n} \left|\left\{t \in [t_{k-1}, t_k] : \phi^t(\rho) \in B_r(0) \times \mathbb{R}^d\right\}\right| \\ &= \sum_{k=1}^{n} \left|\left\{t \in \left[0, \tfrac{1}{n}T\right] : \phi^{t+t_{k-1}}(\rho) \in B_r(0) \times \mathbb{R}^d\right\}\right| \\ &\leq \sum_{k=1}^{n} \left|\left\{t \in [0, E^{(1/2)(1/m-1)}] : \phi^t(\rho_{k-1}) \in B_r(0) \times \mathbb{R}^d\right\}\right|.\end{aligned}$$

The last inequality comes from the definition of n. Estimate (2-18) applies to each piece of this sum. We conclude that

$$\left|\left\{t \in [0, T] : \phi^t(\rho) \in B_r(0) \times \mathbb{R}^d\right\}\right| \leq nC\frac{r}{\sqrt{E}} \leq \frac{1+T}{E^{(1/2)(1/m-1)}} \times C\frac{r}{\sqrt{E}} = C\frac{r(1+T)}{E^{1/(2m)}}$$

(we can ensure that $n \leq (1+T)/E^{(1/2)(1/m-1)}$ in the second equality up to enlarging E_0 so that it is larger than 1, independently of T). □

2.3. ***Continuity of the composition by the flow in symbol classes.*** From now on we go back to a *sub-quadratic* potential, that is to say we suppose our classical Hamiltonian is of the form $p(x, \xi) = V(x) + \frac{1}{2}|\xi|^2$, with V satisfying Assumption 1.1 with $m \in (0, 1]$. In the course of our study, we will need to check that the composition of a symbol with the Hamiltonian flow is still well-behaved in a suitable symbol class, in the sense that its derivatives remain controlled properly. The following lemma is common in the context of the quantum-classical correspondence; see for instance [Bouzouina and Robert 2002, Lemma 2.2]. We reproduce a proof to obtain an estimate adapted to our context and to keep track of the dependence of constants on the parameters of the problem. We recall that a function $a \in C^\infty(\mathbb{R}^{2d})$ is said to be a symbol in the class $S(1)$ if all its derivatives are bounded. The quantities

$$|a|^{\ell}_{S(1)} = \max_{\substack{\alpha \in \mathbb{N}^{2d} \\ 0 \leq |\alpha| \leq \ell}} \sup_{\rho \in \mathbb{R}^{2d}} |\partial^\alpha a(\rho)|, \quad \ell \in \mathbb{N},$$

endow the vector space $S(1)$ with a Fréchet structure (see Appendix B for further details).

Lemma 2.10. *Let a be a symbol in $S(1)$. Then the function $a \circ \phi^t$ still belongs to $S(1)$, and stays in a bounded subset of $S(1)$ locally uniformly with respect to t. More precisely, for any fixed $T > 0$, for any nonzero multi-index $\alpha \in \mathbb{N}^{2d}$, we have the derivative estimate*

$$\|\partial^\alpha (a \circ \phi^t)\|_\infty \leq C_\alpha(T, p) \max_{1 \leq |\beta| \leq |\alpha|} \|\partial^\beta a\|_\infty,$$

uniformly in $t \in [-T, T]$. The constants $C_\alpha(T, p)$ depend only on T and on the sup-norm of derivatives of order $\{2, 3, \dots, |\alpha| + 1\}$ of p.

Proof. In all the proof, t ranges in a compact set, say $[-T, T]$ for some fixed $T > 0$.

Step 1: *Control of differentials of the Hamiltonian flow.* Differentiating the Hamilton equation (1-5) defining the flow ϕ^t, we get

$$\frac{d}{dt} d\phi^t(\rho) = J \operatorname{Hess} p(\phi^t(\rho))\, d\phi^t(\rho).$$

By assumption on the potential V (see (1-4)), we observe that the Hessian of p is bounded. Since $d\phi^0(\rho) = \mathrm{Id}$ for any $\rho \in \mathbb{R}^{2d}$, we classically deduce using Grönwall's lemma that

$$\|d\phi^t(\rho)\| \le e^{T|J \operatorname{Hess} p|_\infty} \le e^{T|\operatorname{Hess} p|_\infty} \quad \forall \rho \in \mathbb{R}^{2d}, \forall t \in [-T, T].$$

For higher-order differentials, we proceed by induction. Suppose that for some $k \ge 1$, all the differentials of order $\le k$ of ϕ^t are bounded uniformly in t on $\mathbb{R}^{2d}$, with a bound involving derivatives of order $k+1$ of p. Differentiating the Hamilton equation $k+1$ times, the Faà di Bruno formula shows that $\frac{d}{dt} d^{k+1}\phi^t(\rho)$ is a sum of terms of the form

$$J\, d^\ell(\nabla p)(\phi^t(\rho)).\big(d^{k_1}\phi^t(\rho), d^{k_2}\phi^t(\rho), \dots, d^{k_\ell}\phi^t(\rho)\big),$$

where $1 \le \ell \le k+1$ and $k_1 + k_2 + \cdots + k_\ell = k+1$. Such terms are bounded uniformly in $t \in [-T, T]$ by the induction hypothesis as soon as $\ell \ge 2$ (note that all the differentials of order ≥ 2 of p are bounded). So in fact the ODE on $d^{k+1}\phi^t(\rho)$ can be written

$$\frac{d}{dt} d^{k+1}\phi^t(\rho) = J \operatorname{Hess} p(\phi^t(\rho))\, d^{k+1}\phi^t(\rho) + R(t, \rho),$$

where $R(t, \rho)$ satisfies

$$|R(t, \rho)|_\infty \le C(T, p) \quad \forall \rho \in \mathbb{R}^{2d}, \forall t \in [-T, T],$$

where the constant $C(T, p)$ depends only on the sup-norm of derivatives of order $\{2, 3, \dots, k+2\}$ of p. We conclude by Grönwall's lemma again, together with Duhamel's formula that $d^{k+1}\phi^t(\rho)$ is bounded similarly: given that $k+1 \ge 2$, we have $d^{k+1}\phi^0(\rho) = 0$ for every $\rho \in \mathbb{R}^{2d}$, so that

$$\|d^{k+1}\phi^t(\rho)\| \le \int_0^{|t|} C(T, p) e^{|\operatorname{Hess} p|_\infty |t-s|}\, ds \le T C'(T, p).$$

This finishes the induction.

Step 2: *Estimates of derivatives of* $a \circ \phi^t$. We estimate the derivatives in x or ξ. Let $\alpha \in \mathbb{N}^{2d} \setminus \{0\}$, and denote by $(x_1^t, x_2^t, \dots, x_d^t, \xi_1^t, \xi_2^t \dots, \xi_d^t)$ the components of the flow. The chain rule together with the Faà di Bruno formula yield that $\partial^\alpha(a \circ \phi^t)$ can be expressed as a sum of terms of the form

$$(\partial_x^{\tilde\alpha} \partial_\xi^{\tilde\beta} a) \circ \phi^t \times \prod_{j_1 \in \tilde\alpha} \partial^{\alpha_{j_1}} x_{j_1} \times \prod_{j_2 \in \tilde\beta} \partial^{\beta_{j_2}} \xi_{j_2},$$

where $\tilde\alpha, \tilde\beta \in \mathbb{N}^d$ are such that $1 \le |\tilde\alpha| + |\tilde\beta| \le |\alpha|$ and $\alpha_{j_1}, \beta_{j_2} \in \mathbb{N}^{2d} \setminus \{0\}$ satisfy $\sum_{j_1} \alpha_{j_1} + \sum_{j_2} \beta_{j_2} = \alpha$. (By $j_1 \in \tilde\alpha$, $j_2 \in \tilde\beta$, we mean that $j_1, j_2 \in \{1, 2, \dots, d\}$ are indices for which $\tilde\alpha$ and $\tilde\beta$ are nonzero.) The claim follows immediately from the bounds on the derivatives of x_j^t and ξ_j^t proved in Step 1. □

3. Proof of the main theorem

We start with a lemma that will enable us to replace $\mathbf{1}_{\omega_R \setminus B_r(0)}$ in the observability inequality with a well-behaved symbol.

Lemma 3.1 (mollifying the observation set). *Let $\omega \subset \mathbb{R}^d$ and denote by ω_R the open set*

$$\omega_R = \bigcup_{x \in \omega} B_R(x), \qquad R > 0.$$

There exists a symbol $a = a_R \in S(1)$ depending only on the x variable such that

$$\mathbf{1}_{\omega_{R/2}}(x) \leq a_R(x) \leq \mathbf{1}_{\omega_R}(x) \quad \forall x \in \mathbb{R}^d.$$

In addition, it satisfies the seminorm estimates

$$\forall \ell \in \mathbb{N}, \exists C_\ell > 0 : \forall R \geq 1, \quad |a_R|^{\ell}_{S(1)} \leq C_\ell \quad \text{and} \quad |\nabla a_R|^{\ell}_{S(1)} \leq \frac{C_\ell}{R}.$$

The constants involved do not depend on ω.

Proof. Fix $\kappa \in C_c^\infty(\mathbb{R}^d)$ a mollifier with the following properties:

$$\kappa(x) \geq 0, \forall x \in \mathbb{R}^d, \quad \operatorname{supp} \kappa \subset B_1(0) \quad \text{and} \quad \int_{\mathbb{R}^d} \kappa(x)\, dx = 1.$$

For any $r > 0$, set $\kappa_r = r^{-d} \kappa(\bullet / r)$, so that $\|\kappa_r\|_{L^1(\mathbb{R}^d)} = 1$. Set, for any $R > 0$,

$$a_R(x) = (\kappa_{R/4} * \mathbf{1}_{\omega_{3R/4}})(x) \quad \forall x \in \mathbb{R}^d.$$

We check that a_R defined in this way satisfies the required properties. We first observe that, by definition, a_R is nonnegative, and that $a_R \leq 1$ by Young's inequality. Now by standard properties of convolution, the support of a_R is contained in $\omega_{3R/4} + B_{R/4}(0) \subset \omega_R$ (recall that the support of κ is a compact subset of $B_1(0)$), which proves that $a_R \leq \mathbf{1}_{\omega_R}$. On the other hand, if $x \in \omega_{R/2}$, then $\kappa_{R/4}(x - \bullet)$ is supported in $\omega_{3R/4}$, so that $a_R(x) = 1$, which proves that $a_R \geq \mathbf{1}_{\omega_{R/2}}$. Differentiating under the integral sign, we see that $\|\partial^\alpha a_R\|_\infty$ is of order $1/R^{|\alpha|}$ for any multi-index $\alpha \in \mathbb{N}^d$, which yields the desired seminorm estimates ($R \geq 1$ is important here). The constants depend only on the supremum norms of derivatives of κ, and not on ω. □

Remark 3.2. The symbol a_R can be considered as a semiclassical symbol, with Planck parameter $1/R^2$, since by construction each derivative yields a gain of $1/R$. However in view of Lemma 2.10, this property is not preserved by composition by the Hamiltonian flow, since all the derivatives of $a_R \circ \phi^t$ of order ≥ 1 behave as $1/R$. This comes from the fact that, when differentiating $a_R \circ \phi^t$ twice or more, the second, third, and higher-order derivatives can hit ϕ^t instead of a_R.

We prove a version of Egorov's theorem taking into account the above remark. Our approach is very classical; see [Bouzouina and Robert 2002] or [Zworski 2012, Chapter 11] for refinements. We refer again to Appendix B for an account on the Weyl quantization Op.

Proposition 3.3 (Egorov). *Let $a \in S(1)$. Then the symbol $a \circ \phi^t$ lies in $S(1)$ with seminorm estimates*

$$\forall T > 0, \forall \ell \in \mathbb{N}, \exists C_\ell(T, p) > 0 : \quad |a \circ \phi^t|^\ell_{S(1)} \le C_\ell(T, p)|a|^\ell_{S(1)} \quad \forall t \in [-T, T],$$

and one has

$$e^{itP}\mathrm{Op}(a)e^{-itP} = \mathrm{Op}(a \circ \phi^t) + \mathcal{R}_a(t), \tag{3-1}$$

where the remainder term $\mathcal{R}_a(t)$ is a bounded operator satisfying

$$\forall T > 0, \exists C(T, p) > 0 : \quad \|\mathcal{R}_a(t)\|_{L^2\to L^2} \le C(T, p)|\nabla a|^{k_d}_{S(1)} \quad \forall t \in [-T, T],$$

for some integer k_d depending only on the dimension.

Proof. The claim that $a \circ \phi^t \in S(1)$ and the subsequent seminorm estimates are provided by Lemma 2.10. To prove (3-1), we follow the classical method that consists in differentiating the time dependent operator

$$Q(s) = e^{-isP}\mathrm{Op}(a \circ \phi^s)e^{isP},$$

and estimating this derivative. For the sake of simplicity, let us introduce $a_s = a \circ \phi^s$. All the operators in this composition map $\mathcal{S}(\mathbb{R}^d)$ to itself continuously, so that $Q(s)u$ can be differentiated using the chain rule, for any $u \in \mathcal{S}(\mathbb{R}^d)$. From now on, we will omit to write u. Recalling that $\frac{d}{ds}a_s = \{p, a_s\}$ by definition of ϕ^s, we have

$$\frac{d}{ds}\mathrm{Op}(a_s) = \mathrm{Op}(\{p, a_s\})$$

(rigorously, one may apply the dominated convergence theorem to the pairing $\langle v, \mathrm{Op}(a_s)u\rangle_{\mathcal{S}',\mathcal{S}(\mathbb{R}^d)}$ for two Schwartz functions u and v). Therefore we get

$$\frac{d}{ds}Q(s) = -ie^{-isP}\big([P, \mathrm{Op}(a_s)] + i\mathrm{Op}(\{p, a_s\})\big)e^{isP} = -ie^{-isP}\mathrm{Op}(\mathcal{R}_3(s))e^{isP}. \tag{3-2}$$

The symbol $\mathcal{R}_3(s)$ above is nothing but the remainder of order 3 in the pseudodifferential calculus between p and a_s. Proposition B.5 provides a bound on this remainder in terms of seminorms of a_s. Recall that, in the subcritical case $m \le 1$, $\partial^\alpha p \in S(1)$ whenever $|\alpha| \ge 2$. Therefore according to Proposition B.5, for any seminorm index $\ell \in \mathbb{N}$, there exist a constant $C_\ell > 0$ as well as an integer $k \ge 0$ such that

$$|\mathcal{R}_3(s)|^\ell_{S(1)} \le C_\ell|d^3 a_s|^k_{S(1)}|d^3 p|^k_{S(1)}.$$

Then we use Lemma 2.10 to obtain

$$|\mathcal{R}_3(s)|^\ell_{S(1)} \le C_\ell(T, p)|\nabla a|^k_{S(1)}$$

for any $s \in [-T, T]$. Therefore, the Calderón–Vaillancourt theorem (Theorem B.2) tells us that the norm of $\mathrm{Op}(\mathcal{R}_3(s))$ is bounded, uniformly in $s \in [-T, T]$, by a seminorm of ∇a, and a constant depending only on T and p. Plugging this into (3-2), given that the propagator e^{isP} is an isometry, we obtain the same bound on $\frac{d}{ds}Q(s)$. Integrating this in s, we obtain from the mean-value inequality

$$\forall t \in [-T, T], \quad \|Q(t) - Q(0)\|_{L^2\to L^2} \le 2T \sup_{s\in[-T,T]} \left\|\frac{d}{ds}Q(s)\right\|_{L^2\to L^2} \le C(T, p)|\nabla a|^{k_d}_{S(1)},$$

where the integer k_d depends only on the dimension. Conjugating by the propagator, which is an isometry on L^2, yields the desired result. ☐

We are now in a position to prove our main result.

Proof of Theorem 1.3. We fix $\omega \subset \mathbb{R}^d$, a compact set $K \subset \mathbb{R}^d$, and we introduce $\tilde{\omega}(R) = (\omega \setminus K_R)_R$ for $R > 0$. One can verify that $\tilde{\omega}(R) \subset \omega_R \setminus K$. By Lemma 3.1, there exists a symbol $a_R \in S(1)$ depending on the parameter $R > 0$ such that

$$\mathbf{1}_{(\omega\setminus K_R)\times\mathbb{R}^d} \le a_R \le \mathbf{1}_{\tilde{\omega}(R)\times\mathbb{R}^d} \quad \forall R > 0, \tag{3-3}$$

and $|\nabla a_R|^{\ell}_{S(1)} \le c_{d,\ell}/R$ for any $\ell \in \mathbb{N}$, with a constant $c_{d,\ell}$ depending only on the dimension and ℓ, uniformly in $R \ge 1$. Notice that the symbol depends on ω and K but not its seminorms. On the quantum side, one can regard the functions in (3-3) as multiplication operators, and understand the inequalities in the sense of self-adjoint operators. Conjugating by the Schrödinger propagator does not change the inequalities, so that

$$e^{itP}\mathbf{1}_{\omega\setminus K_R}e^{-itP} \le e^{itP}\mathrm{Op}(a_R)e^{-itP} \le e^{itP}\mathbf{1}_{\tilde{\omega}(R)}e^{-itP} \quad \forall t \in \mathbb{R}.$$

Then we use Egorov's theorem (Proposition 3.3) and we integrate with respect to t to get

$$\int_0^{T_0} e^{itP}\mathbf{1}_{(\omega\setminus K_R)\times\mathbb{R}^d}e^{-itP}\,dt \le \int_0^{T_0}\mathrm{Op}(a_R\circ\phi^t)\,dt + \mathcal{R}_R \le \int_0^{T_0} e^{itP}\mathbf{1}_{\tilde{\omega}(R)}e^{-itP}\,dt, \tag{3-4}$$

where the remainder term $\mathcal{R}_R$ is a bounded operator with

$$\|\mathcal{R}_R\|_{L^2\to L^2} \le C|\nabla a_R|^{k_d}_{S(1)} \le \frac{C'}{R} \quad \forall R \ge 1. \tag{3-5}$$

The constant C' above depends only on p and T_0 (and of course on the dimension d), but not on ω or K. One can check that the quantization and the integral over t in the middle term of (3-4) commute.[9]

On the classical side, using the same notation as in (2-1), we introduce the quantity

$$\mathfrak{K}^{\infty}_{p_0}(a_R\mathbf{1}_{(0,T)}) = \liminf_{\rho\to\infty}\int_0^T a_R(\phi_0^t(\rho))\,dt,$$

and similarly for p instead of p_0, replacing the flow ϕ_0^t by ϕ^t. We claim that, for any $T > 0$, there exists a constant $C'' > 0$ depending only on the dimension, on T and on the Hamiltonians p_0 and p, such that, for any compact $\widetilde{K}$, and for any $R > 0$,

$$\begin{cases} \mathfrak{K}^{\infty}_{p_0}(\omega, T) \le \mathfrak{K}^{\infty}_{p}(a_R\mathbf{1}_{(0,T)}) + \dfrac{C''}{R}, \\ \mathfrak{K}^{\infty}_{p}(a_R\mathbf{1}_{(0,T)}) \le \mathfrak{K}^{\infty}_{p_0}(\omega_R\setminus\widetilde{K}, T) + \dfrac{C''}{R}. \end{cases} \tag{3-6}$$

The constant C'' does not depend on ω or K from which we built a_R neither. The proof of the first inequality in (3-6) reads as follows: Corollary 2.6 shows that the quantity in the left-hand side does not change if we remove a compact set:

$$\mathfrak{K}^{\infty}_{p_0}(\omega, T) = \mathfrak{K}^{\infty}_{p_0}(\omega\setminus K_R, T) \quad \forall R > 0. \tag{3-7}$$

Now we use that $\mathbf{1}_{(\omega\setminus K_R)\times\mathbb{R}^d} \le a_R$ to get

$$\mathfrak{K}^{\infty}_{p_0}(\omega\setminus K_R, T) \le \mathfrak{K}^{\infty}_{p_0}(a_R\mathbf{1}_{(0,T)}). \tag{3-8}$$

[9]One can see this by pairing the operator under consideration with two Schwartz functions and use the dominated convergence theorem.

Then we switch from p_0 to p, having the same principal symbol, using Corollary 2.4: the function $a_R\mathbf{1}_{(0,T)}$ is compactly supported in time and $c_{d,1}/R$-Lipschitz in the variable ρ, so that

$$\mathfrak{K}_{p_0}^{\infty}(a_R\mathbf{1}_{(0,T)}) \le \mathfrak{K}_{p}^{\infty}(a_R\mathbf{1}_{(0,T)}) + \frac{C''}{R} \quad \forall R>0.$$

Putting this together with (3-7) and (3-8) yields the first inequality in (3-6). The second inequality in (3-6) is proved using similar arguments: Corollary 2.4 leads to

$$\mathfrak{K}_{p}^{\infty}(a_R\mathbf{1}_{(0,T)}) \le \mathfrak{K}_{p_0}^{\infty}(a_R\mathbf{1}_{(0,T)}) + \frac{C''}{R} \quad \forall R>0.$$

Then we use from the construction of a_R in (3-3) that a_R is supported in $\omega_R\times\mathbb{R}^d$, and we apply Corollary 2.6 to remove a compact set $\widetilde{K}$. This leads to the sought inequality.

Sufficient condition. We wish to bound the left-hand side of (3-4) from below. The high-energy classical observability constant $\mathfrak{K}_{p_0}^{\infty} := \mathfrak{K}_{p_0}^{\infty}(\omega, T_0)$ is assumed to be positive. From the first inequality in (3-6), with T_0 in place of T, we can write

$$\exists A>0 : \forall |\rho|\ge A, \qquad \int_0^{T_0} (a_R\circ\phi^t)(\rho)\,dt \ge \frac{1}{2}\mathfrak{K}_{p_0}^{\infty} - \frac{C''}{R} = c_R. \tag{3-9}$$

Take a cut-off function $\chi\in C_c^{\infty}(\mathbb{R}^{2d})$ such that $\chi\equiv 1$ on the unit ball, and set $\chi_R=\chi(\bullet/(A+R))$. Then χ_R has compact support, equals one on the ball $B_A(0)$, and it satisfies $\|\partial^{\alpha}\chi_R\|_{\infty}=O(1/R^{|\alpha|})$, with constants independent of ω again.[10] We split the symbol in the left-hand side of (3-9) using this cut-off function: we write

$$\int_0^{T_0} a_R\circ\phi^t\,dt = b_0 + b_{\infty}, \tag{3-10}$$

where we set

$$b_0 = \chi_R \times \left(\int_0^{T_0} a_R\circ\phi^t\,dt - c_R\right) \quad\text{and}\quad b_{\infty} = (1-\chi_R)\int_0^{T_0} a_R\circ\phi^t\,dt + c_R\chi_R.$$

Using the Leibniz formula and Lemma 2.10, we can prove that $b_0\in S(1)$. Moreover, b_0 is compactly supported in $\mathbb{R}^{2d}$, so that $\mathrm{Op}(b_0)$ is a compact operator by [Zworski 2012, Theorem 4.28]. As for b_{∞}, the Leibniz formula and Lemma 2.10 lead to the following estimates on derivatives: for all $\alpha\in\mathbb{N}^{2d}$, one has

$$\|\partial^{\alpha}b_{\infty}\|_{\infty} \le C_{\alpha}\max_{\alpha_1+\alpha_2=\alpha}\|\partial^{\alpha_1}(1-\chi_R)\|_{\infty}\times\int_0^{T_0}\|\partial^{\alpha_2}(a_R\circ\phi^t)\|_{\infty}\,dt + c_R\frac{C_{\alpha}}{R^{|\alpha|}} \le C_{\alpha,T_0,p}\left(\frac{1}{R^{|\alpha|}}+\frac{1}{R}\right).$$

The last inequality comes from distinguishing the cases $\alpha_2=0$ and $\alpha_2\ne 0$. In the first case, we have $\partial^{\alpha_1}(1-\chi_R)=O(R^{-|\alpha|})$ and $|a_R\circ\phi^t|\le 1$. Otherwise, Lemma 2.10 tells us that $\partial^{\alpha_2}(a_R\circ\phi^t)$ behaves like $|\nabla a_R|_{S(1)}^{|\alpha|}=O(1/R)$, $R\ge 1$. In particular, $b_{\infty}\in S(1)$ and $|\mathrm{Hess}\, b_{\infty}|_{S(1)}^{\ell}=O(1/R)$ for any $\ell\in\mathbb{N}$, with a constant independent of ω and K. In addition, we have $b_{\infty}\ge c_R$ in view of (3-9). Therefore, the Gårding inequality (Proposition B.6) yields

$$\mathrm{Op}(b_{\infty}) \ge \left(c_R - \frac{C_1}{R}\right)\mathrm{Id}.$$

[10] The parameter A depends on R, but this will not be a problem in the sequel. The phase space region localized at distance $\le A$ from the origin will be handled by Proposition A.1.

The constant C_1 is independent of ω and K in view of the seminorm estimates of $\operatorname{Hess} b_\infty$ discussed above. Going back to (3-4), we have proved

$$\int_0^T e^{itP}\mathbf{1}_{\tilde\omega(R)}e^{-itP}\,dt \geq c_R \operatorname{Id} + \operatorname{Op}(b_0) + \mathcal{R} \quad \forall R \geq 1.$$

As we have seen in the course of the proof, $\operatorname{Op}(b_0)$ is a compact self-adjoint operator and $\|\mathcal{R}\|_{L^2\to L^2} \leq C_2/R$, with a constant C_2 depending only on the dimension, on T_0 and on the Hamiltonians p_0, p. In view of the definition of c_R in (3-9), taking $R = 4(C'' + C_2 + T_0)/\mathfrak{K}^\infty_{p_0}$, we obtain the desired observability inequality, up to a compact operator:

$$\int_0^{T_0} e^{itP}\mathbf{1}_{\tilde\omega(R)}e^{-itP}\,dt - \operatorname{Op}(b_0) \geq \tfrac{1}{4}\mathfrak{K}^\infty_{p_0}\operatorname{Id}.$$

Notice that indeed $R \geq 1$, since $\mathfrak{K}^\infty_{p_0} \leq T_0$. Proposition A.1 then applies (see Remark A.2). It yields the sought observability inequality on $\tilde\omega(R) \subset \omega_R \setminus K$, in any time $T > T_0$.

Necessary condition. Consider the symbol a_R from (3-3) with $K = \varnothing$. We fix $R \geq 1$ (not necessarily large), $\widetilde{K}$ compact, and we estimate the observation cost C_{obs} in (1-8) using the quantity $\mathfrak{K}^\infty_{p_0}(\omega_R \setminus \widetilde{K}, T)$. We will track carefully the dependence of remainders on the parameter R. Write for short

$$\langle a_R\rangle_T(\rho) = \int_0^T (a_R \circ \phi^t)(\rho)\,dt, \qquad \rho \in \mathbb{R}^{2d},$$

and pick a point $\rho_0 \in \mathbb{R}^{2d}$ such that

$$\langle a_R\rangle_T(\rho_0) \leq \inf_{\rho\in\mathbb{R}^{2d}} \langle a_R\rangle_T(\rho) + \frac{1}{R}.$$

Notice that in virtue of the second inequality of (3-6), we have

$$\langle a_R\rangle_T(\rho_0) \leq \mathfrak{K}^\infty_p(a_R\mathbf{1}_{(0,T)}) + \frac{1}{R} \leq \mathfrak{K}^\infty_{p_0}(\omega_R \setminus \widetilde{K}, T) + \frac{C''+1}{R}. \tag{3-11}$$

Differentiating under the integral sign and using Lemma 2.10, we check that $\langle a_R\rangle_T$ is Lipschitz as a function of ρ:

$$\forall \rho \in \mathbb{R}^{2d}, \quad |\nabla\langle a_R\rangle_T(\rho)| \leq T \sup_{t\in[0,T]} |\nabla(a_R\circ\phi^t)(\rho)| \leq C(T,p)\|\nabla a_R\|_\infty \leq \frac{c}{R}. \tag{3-12}$$

Consider a Gaussian wave packet centered at ρ_0, namely, writing $\rho_0 = (x_0, \xi_0)$, we define

$$w(x) = \pi^{-d/4}\exp\left(-\tfrac{1}{2}|x - x_0|^2\right)e^{i\xi_0\cdot x}, \qquad x \in \mathbb{R}^d.$$

It is properly normalized: $\|w\|_{L^2} = 1$. A classical computation (see [Folland 1989, Proposition (1.48)]) shows that the Wigner transform of w is the Gaussian in the phase space centered at ρ_0, defined by $\rho \mapsto \pi^{-d}\exp(-|\rho - \rho_0|^2)$, that is to say,

$$(w, \operatorname{Op}(\langle a_{2R}\rangle_T)w)_{L^2} = \pi^{-d}\int_{\mathbb{R}^{2d}} \langle a_{2R}\rangle_T(\rho)\exp(-|\rho-\rho_0|^2)\,d\rho = \pi^{-d}\int_{\mathbb{R}^{2d}} \langle a_{2R}\rangle_T(\rho_0+\rho)\exp(-|\rho|^2)\,d\rho.$$

Note that it is a nonnegative quantity. Taking an arbitrary $A > 0$ and splitting the integral over $\mathbb{R}^{2d}$ into two pieces, we obtain

$$\begin{aligned}(w, \mathrm{Op}(\langle a_{2R}\rangle_T)w)_{L^2} &\\ &\le \int_{B_A(0)} \big(\langle a_{2R}\rangle_T(\rho_0) + A\|\nabla\langle a_{2R}\rangle_T\|_\infty\big)\pi^{-d}e^{-|\rho|^2}\,d\rho + \int_{\mathbb{R}^{2d}\setminus B_A(0)} \|\langle a_{2R}\rangle_T\|_\infty \pi^{-d}e^{-|\rho|^2}\,d\rho \\ &\le \langle a_{2R}\rangle_T(\rho_0) + A\frac{c}{R} + T\int_{\mathbb{R}^{2d}\setminus B_A(0)} \pi^{-d}e^{-|\rho|^2}\,d\rho \\ &\le \mathfrak{K}^\infty_{p_0}(\omega_R \setminus \widetilde{K}, T) + \frac{C'' + 1 + Ac}{R} + Te^{-A^2/2}2^d.\end{aligned}$$

We used (3-12) and (3-11) to obtain the last two inequalities. We take $A = |2\log R|^{1/2}$ to obtain

$$(w, \mathrm{Op}(\langle a_{2R}\rangle_T)w)_{L^2} \le \mathfrak{K}_{p_0}(\omega_R \setminus \widetilde{K}, T) + \widetilde{C}\frac{1 + |\log R|^{1/2}}{R}$$

for some constant $\widetilde{C} > 0$ independent of R. Going back to the left-hand side of (3-4) (recall that we chose $K = \varnothing$ here) with T in place of T_0, as well as (3-5), taking the inner product with w on both sides, we deduce that

$$\int_0^T \|e^{-itP}w\|^2_{L^2(\omega)}\,dt \le \mathfrak{K}_{p_0}(\omega_R \setminus \widetilde{K}, T) + \widetilde{C}\frac{1 + |\log R|^{1/2}}{R} + \frac{C'}{R}.$$

By assumption, $\mathrm{Obs}(\omega, T)$ is true with a cost $C_{\mathrm{obs}} > 0$. Recalling that $\|w\|_{L^2} = 1$, we can bound the left-hand side from below by C_{obs}^{-1}. We arrive at

$$\mathfrak{K}_{p_0}(\omega_R \setminus \widetilde{K}, T) \ge \frac{1}{C_{\mathrm{obs}}} - \widetilde{C}\frac{1 + |\log R|^{1/2}}{R} - \frac{C'}{R},$$

which yields the sought result. □

4. Proofs of observability results from conical sets

In this section, we give proofs of the results presented in Sections 1.4.1 and 1.4.2, which concern observation sets that are conical in the sense of (1-16). Propositions 1.4, 1.5 and 1.7 are proved in Sections 4.1, 4.2 and 4.3 respectively.

4.1. ***Proof of Proposition 1.4.*** Let us prove the converse statement: assume there exists a normalized eigenvector e of A such that $e \notin \overline{\omega}$ and $-e \notin \overline{\omega}$. Let $\nu > 0$ be such that $Ae = \nu^2 e$. We claim the following.

Lemma 4.1. *There exists a constant $c > 0$ such that for any $R > 0$, one has*

$$\forall s \in \mathbb{R}, \quad (se \in \omega_R \quad \Longrightarrow \quad |s| \le cR).$$

Proof. If $s \in \mathbb{R}$ is such that $se \in \omega_R$, then there exists $y \in \omega \setminus \{0\}$ such that $|se - y| \le R$. Moreover, since e belongs to the complement of the closed set $\overline{\omega} \cup -\overline{\omega}$, there exists $\varepsilon > 0$ such that

$$\forall x \in (\omega \cup -\omega) \setminus \{0\}, \qquad \left| e - \frac{x}{|x|} \right| \ge \varepsilon.$$

We apply this to $x = \operatorname{sign}(s) y$ to obtain

$$|s| \le \frac{1}{\varepsilon}\left| se - |s|\frac{y}{|y|}\right| \le \frac{1}{\varepsilon}|se-y| + \frac{1}{\varepsilon}|y|\left|1 - \frac{|s|}{|y|}\right| \le \frac{1}{\varepsilon}|se-y| + \frac{1}{\varepsilon}|se-y| \le \frac{2R}{\varepsilon}.$$

We used the inverse triangle inequality to obtain the second to last inequality. □

Using this lemma, for any $T > 0$ and any $\eta > 0$, we can estimate the quantity

$$\int_0^T \mathbf{1}_{\omega_R\times\mathbb{R}^d}(\phi^t(0,\eta e))\,dt = \int_0^T \mathbf{1}_{\omega_R}\Big(\frac{\eta}{\nu}\sin(\nu t)e\Big)\,dt \le \int_0^{2N\pi/\nu} \mathbf{1}_{\omega_R}\Big(\frac{\eta}{\nu}\sin(\nu t)e\Big)\,dt,$$

where $N = \lceil \nu T/2\pi \rceil$. Using the periodicity of the sine and a change of variable, we deduce

$$\int_0^T \mathbf{1}_{\omega_R\times\mathbb{R}^d}(\phi^t(0,\eta e))\,dt \le \frac{N}{\nu}\int_0^{2\pi} \mathbf{1}_{\omega_R}\Big(\frac{\eta}{\nu}\sin(t)e\Big)\,dt = \frac{2N}{\nu}\int_{-\pi/2}^{\pi/2} \mathbf{1}_{\omega_R}\Big(\frac{\eta}{\nu}\sin(t)e\Big)\,dt.$$

Provided $\eta \neq 0$, we make the change of variables $s = \eta\sin t$, for which we have $dt = (\eta^2 - s^2)^{-1/2}\,ds$; this leads to

$$\int_0^T \mathbf{1}_{\omega_R\times\mathbb{R}^d}(\phi^t(0,\eta e))\,dt \le \frac{2N}{\nu}\int_{-|\eta|}^{|\eta|} \mathbf{1}_{\omega_R}\Big(\frac{s}{\nu}e\Big)\frac{ds}{\sqrt{\eta^2-s^2}}.$$

From Lemma 4.1 above, we conclude that, for any η large enough,

$$\int_0^T \mathbf{1}_{\omega_R\times\mathbb{R}^d}(\phi^t(0,\eta e))\,dt \le \frac{2N}{\nu}\int_{-cR\nu}^{cR\nu} \frac{ds}{\sqrt{\eta^2-s^2}}.$$

An extra change of variables yields

$$\int_0^T \mathbf{1}_{\omega_R\times\mathbb{R}^d}(\phi^t(0,\eta e))\,dt \le \frac{2N}{\nu}\int_{-cR\nu/\eta}^{cR\nu/\eta} \frac{ds}{\sqrt{1-s^2}} = O\Big(\frac{R}{\eta}\Big)$$

as η tends to infinity and R is fixed. We deduce that, for any $R > 0$,

$$\liminf_{\rho\to\infty} \int_0^T \mathbf{1}_{\omega_R\times\mathbb{R}^d}(\phi^t(\rho))\,dt = 0.$$

The necessary condition of Theorem 1.3 then proves that observability cannot hold from ω in time T. □

4.2. *Proof of Proposition 1.5.* We first reduce to the case where the matrix A is diagonal in the canonical basis of $\mathbb{R}^2$. Then we investigate the isotropic and anisotropic cases separately.

Step 1: *Reduction to positive cones containing half coordinate axes.* Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear symplectic mapping. Then $\nabla(p\circ S) = S^*(\nabla p)\circ S$, and we observe that

$$\frac{d}{dt}S^{-1}\phi^t(S\rho) = S^{-1}J(S^{-1})^*S^*\nabla p(\phi^t(S\rho)) = J\nabla(p\circ S)(S^{-1}\phi^t(S\rho)).$$

This means that the conjugation of the Hamiltonian flow of p by S is the Hamiltonian flow of $p\circ S$. Thus, for any measurable set $C \subset \mathbb{R}^2\times\mathbb{R}^2$,

$$\int_0^T \mathbf{1}_C(\phi^t(\rho))\,dt = \int_0^T \mathbf{1}_{S^{-1}C}((S^{-1}\phi^t S)(S^{-1}\rho))\,dt,$$

and finally, since $S^{-1}\rho \to \infty$ if and only if $\rho \to \infty$, we deduce that

$$\liminf_{\rho\to\infty} \int_0^T \mathbf{1}_C(\phi^t(\rho))\,dt = \liminf_{\rho\to\infty} \int_0^T \mathbf{1}_{S^{-1}C}((S^{-1}\phi^t S)(\rho))\,dt. \tag{4-1}$$

Denote by Q the orthogonal matrix that diagonalizes A as follows:

$$Q^{-1}AQ = \begin{pmatrix} \nu_-^2 & 0 \\ 0 & \nu_+^2 \end{pmatrix}, \quad \text{with } Q\begin{pmatrix}1\\0\end{pmatrix} = e_- \text{ and } Q\begin{pmatrix}0\\1\end{pmatrix} = e_+.$$

We apply the above observation (4-1) to the map

$$S = \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix}.$$

It is indeed symplectic since $Q = (Q^{-1})^*$ is an orthogonal matrix. When the subset of the phase space C is of the form $\omega(\varepsilon)$ given in the statement, the resulting set $S^{-1}C$ is $\tilde{\omega}(\varepsilon) = C_\varepsilon^1 \cup C_\varepsilon^2$, where

$$C_\varepsilon^1 = \{(x_1, x_2) \in \mathbb{R}^2 : |x_2| < \tan(\tfrac{1}{2}\varepsilon)x_1\} \quad \text{and} \quad C_\varepsilon^2 = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| < \tan(\tfrac{1}{2}\varepsilon)x_2\}.$$

The corresponding Hamiltonian is

$$(p \circ S)(x, \xi) = \tfrac{1}{2}(Qx \cdot AQx + |Q\xi|^2) = \tfrac{1}{2}(\nu_-^2 {x_1}^2 + \nu_+^2 {x_2}^2 + |\xi|^2).$$

That is to say, we have reduced the problem to the study of observability from $\tilde{\omega}(\varepsilon)$ for the above Hamiltonian: the Schrödinger equation is observable from $\omega(\varepsilon)$ in time T for the Hamiltonian p is and only if it is observable from $\tilde{\omega}(\varepsilon)$ in time T for the Hamiltonian $p \circ S$. From now on, we write $\omega(\varepsilon)$ instead of $\tilde{\omega}(\varepsilon)$, p instead of $p \circ S$ respectively, and $(\nu_1, \nu_2) = (\nu_-, \nu_+)$.

Step 2: *Isotropic case.* The case where $\nu_+ = \nu_- = \nu$ follows from Proposition 1.4. Indeed, since $\varepsilon < \pi/2$, one has $\overline{\omega(\varepsilon)} \cap L_\pm = \{0\}$, where $L_\pm = \{x_2 = \pm x_1\}$ are eigenspaces of $A = \nu^2\,\mathrm{Id}$. Therefore, isotropic oscillators are not observable from $\omega(\varepsilon)$.

Anisotropic case. We assume that the harmonic oscillator is anisotropic, i.e., $\nu_1 < \nu_2$, and we want to show that $\omega(\varepsilon)$ observes the Schrödinger equation. Anticipating the use of Theorem 1.3 where the observation set has to be enlarged, we will rather prove that the dynamical condition in (1-7) is satisfied by the smaller set $\omega(\varepsilon/2) = C_{\varepsilon/2}^1 \cup C_{\varepsilon/2}^2$. We fix an initial point $\rho^0 = (x_1^0, x_2^0; \xi_1^0, \xi_2^0) \in \mathbb{R}^2 \times \mathbb{R}^2$. We write the space components of the flow as follows:

$$x_j^t = \cos(\nu_j t)x_j^0 + \frac{1}{\nu_j}\sin(\nu_j t)\xi_j^0 = A_j \sin(\nu_j t + \theta_j), \quad j \in \{1, 2\}, t \in \mathbb{R},$$

$$\text{with } A_j = \sqrt{(x_j^0)^2 + \left(\frac{\xi_j^0}{\nu_j}\right)^2} \quad \text{and} \quad \cos\theta_j = \frac{\xi_j^0/\nu_j}{A_j}, \quad \sin\theta_j = \frac{x_j^0}{A_j}.$$

Our first goal will be to prove that the dynamical condition (1-7) is satisfied in the time interval $[0, T_0]$, where T_0 is given in (1-18). We can consider ρ^0 to be nonzero since we are interested in what happens at infinity. Therefore $A_1 > 0$ or $A_2 > 0$. Also keep in mind that $\rho^0 \to \infty$ if and only if $|(A_1, A_2)| \to +\infty$.

Step 3: *Time spent in* $C^2_{\varepsilon/2}$. First we look at the possibility to be in the cone $C^2_{\varepsilon/2}$. This will certainly be the case provided A_1 is very small compared to A_2, that is to say the projected trajectory (x_1^t, x_2^t) is almost contained in the ordinate axis. We prove

$$\int_0^{T_0} \mathbf{1}_{C^2_{\varepsilon/2}}(x_1^t, x_2^t)\,dt \geq \frac{\pi}{\nu_2}\left(1 - \frac{A_1/A_2}{\tan(\varepsilon/4)}\right). \tag{4-2}$$

Suppose $t \in [0, T_0]$ is such that $\sin(\nu_2 t + \theta_2) \geq \delta$, namely $x_2^t \geq A_2\delta$. Assuming that $A_2 > 0$, one has

$$|x_1^t| \leq A_1 \leq \frac{A_1}{A_2\delta} x_2^t. \tag{4-3}$$

We want to quantify the amount of t such that this holds. In the following estimate, we use the fact that $T_0 \geq 2\pi/\nu_2$ and the classical concavity inequality $\sin x \geq 2x/\pi$ for all $x \in [0, \pi/2]$:

$$\int_0^{T_0} \mathbf{1}_{\sin(\nu_2 t+\theta_2)\geq\delta}\,dt \geq \int_0^{2\pi/\nu_2} \mathbf{1}_{\sin(\nu_2 t)\geq\delta}\,dt \geq \frac{1}{\nu_2}\int_0^{2\pi} \mathbf{1}_{\sin t\geq\delta}\,dt \geq \frac{2}{\nu_2}\int_0^{\pi/2} \mathbf{1}_{2t/\pi\geq\delta}\,dt = \frac{\pi}{\nu_2}(1-\delta).$$

Now in (4-2), we wish $A_1/(A_2\delta)$ to be strictly less than $\tan(\varepsilon/4)$, that is to say $\delta > A_1/(A_2\tan(\varepsilon/4))$. Therefore, for any δ satisfying this condition, the time spent by the trajectory in $C^2_{\varepsilon/2}$ can be bounded from below by

$$\int_0^{T_0} \mathbf{1}_{x_2^t\geq A_2\delta}\,dt \geq \int_0^{T_0} \mathbf{1}_{\sin(\nu_2 t+\theta_2)\geq\delta}\,dt \geq \frac{\pi}{\nu_2}(1-\delta),$$

so that, maximizing the right-hand side with respect to δ, one obtains (4-3). Notice that this inequality is useful only if A_1/A_2 is small enough. In the opposite case where A_1/A_2 is large, we use another argument (ν_1 and ν_2 do not play a symmetric role here).

Step 4: *Time spent in* $C^1_{\varepsilon/2}$. Let us now consider the times when the trajectory is in the other cone $C^1_{\varepsilon/2}$. Set $\eta = \lfloor \nu_2/\nu_1 \rfloor + 1 - \nu_2/\nu_1 \in (0, 1]$. The main claim in this step of the proof is

$$\exists t_2 \in [0, T_0]: \quad x_2^{t_2} = 0 \quad \text{and} \quad x_1^{t_2} \geq A_1\delta_1, \quad \text{where } \delta_1 = \min\left(\frac{\nu_1}{\nu_2}\eta, 1 - \frac{\nu_1}{\nu_2}\right). \tag{4-4}$$

Denote by t_1 the first zero of $\sin(\nu_1 t + \theta_1)$ in $[0, T_0]$. It exists since by definition, $T_0 \geq \pi/\nu_2(1 + \nu_2/\nu_1) \geq \pi/\nu_1$. It turns out that t_1 is given by

$$t_1 = \frac{\pi}{\nu_1}\left(\left\lceil \frac{\theta_1}{\pi} \right\rceil - \frac{\theta_1}{\pi}\right).$$

Then $t_1 \in [0, \pi/\nu_1)$, and we know that $\sin(\nu_1 t + \theta_1)$ has constant sign on $I_1 := [0, t_1]$, on $I_2 := [t_1, t_1 + \pi/\nu_1] \cap [0, T_0]$ and on $I_3 := [t_1 + \pi/\nu_1, t_1 + 2\pi/\nu_1] \cap [0, T_0]$. Observe that I_1 is possibly reduced to a singleton, I_2 is always nontrivial, and I_3 is possibly empty. One can check this from the fact that T_0 can be rewritten

$$T_0 = \frac{\pi}{\nu_1} + (1+\eta)\frac{\pi}{\nu_2}. \tag{4-5}$$

Because $T_0 \geq \pi/\nu_1$, we know that $\sin(\nu_1 t + \theta_1)$ vanishes at least once in $[0, T_0]$. We first distinguish cases according to whether there are a single one or more than two of these zeroes in this interval.

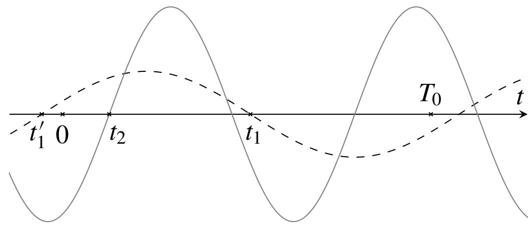


Figure 5. The dashed line is $t \mapsto A_1 \sin(\nu_1 t + \theta_1)$, the gray line is $t \mapsto A_2 \sin(\nu_2 t + \theta_2)$, with $\nu_2/\nu_1 = 1.7$.

Case 1: Assume first t_1 is the only zero in $[0, T_0]$. This case is illustrated in Figure 5. In view of (4-5), t_1 lies at distance $> (1+\eta)\pi/\nu_2$ from the boundary of $[0, T_0]$, otherwise $t_1 + \pi/\nu_1$ or $t_1 - \pi/\nu_1$ is another zero in $[0, T_0]$. In particular, the intervals $[0, t_1]$ and $[t_1, T_0]$ have length $\geq (1+\eta)\pi/\nu_2$. We know that $\sin(\nu_1 t + \theta_1) \geq 0$ on one of these intervals, that we denote by $\tilde{I}$. Given that $\tilde{I}$ has length $\geq (1+\eta)\pi/\nu_2$, it contains a zero of $\sin(\nu_2 t + \theta_2)$, lying at distance $\geq (\pi/\nu_2)(\eta/2)$ from the boundary of $\tilde{I}$. We denote such a zero by t_2. Given that the only zero of $\sin(\nu_1 t + \theta_1)$ in $\tilde{I}$ is t_1, we deduce that the distance between t_2 and the closest zero t_1' of $\sin(\nu_1 t + \theta_1)$ is at least $(\pi/\nu_2)(\eta/2)$. Then the inequality $\sin x \geq 2x/\pi$ on $x \in [0, \pi/2]$ yields

$$\sin(\nu_1 t_2 + \theta_1) = \sin(\nu_1 (t_2 - t_1') + \nu_1 t_1' + \theta_1) = \sin(\nu_1 |t_2 - t_1'|) \geq \frac{2\nu_1}{\pi} |t_2 - t_1'| \geq \frac{\nu_1}{\nu_2} \eta. \tag{4-6}$$

The absolute value resulting from the second inequality is due to the fact that we chose t_2 in an interval where $\sin(\nu_1 t + \theta_1) \geq 0$, or equivalently, $\nu_1 t_1 + \theta_1$ is an even or odd multiple of π according to the sign of $t_2 - t_1$. We conclude that $x_2^{t_2} = 0$ by definition of t_2 and that we have $x_1^{t_2} \geq A_1 \nu_1 \eta / \nu_2$ in virtue of (4-6), hence the claim (4-4).

Case 2: Now we treat the case where $t_1 + \pi/\nu_1$ also lies in $[0, T_0]$. The situation is illustrated in Figure 6. The interval $J_1 := [t_1, t_1 + \pi/\nu_1]$ is contained in $[0, T_0]$. As we already mentioned, $\sin(\nu_1 t + \theta_1)$ has constant sign on J_1.

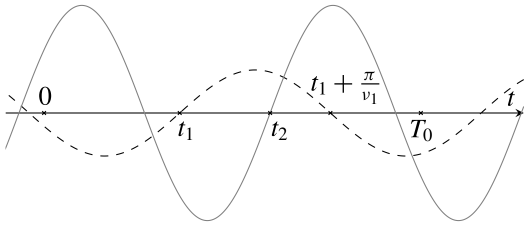
Subcase 2a: If $\sin(\nu_1 t + \theta_1) \geq 0$ on J_1, since J_1 has length $\pi/\nu_1 > \pi/\nu_2$, then $t \mapsto \sin(\nu_2 t + \theta_2)$ vanishes in J_1, and we can choose a zero t_2 at distance $\geq (\pi/2)(1/\nu_1 - 1/\nu_2)$ from the boundary of J_1. This is illustrated in Figure 6, left. Reproducing the previous argument with the concavity inequality for the sine function, we deduce that

$$\sin(\nu_1 t_2 + \theta_1) \geq \frac{2\nu_1}{\pi} \times \frac{\pi}{2} \left(\frac{1}{\nu_1} - \frac{1}{\nu_2} \right) = 1 - \frac{\nu_1}{\nu_2}.$$

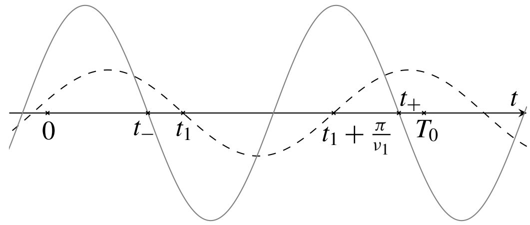
Therefore in this case, there is $t_2 \in [0, T_0]$ with $x_2^{t_2} = 0$ and $x_1^{t_2} \geq (1 - \nu_1/\nu_2) A_1$, hence the claim (4-4).

Subcase 2b: In the remaining case where $\sin(\nu_1 t + \theta_1) \leq 0$ on J_1, we introduce some additional notation; see Figure 6, right. We denote by t_- (resp. t_+) the largest (resp. smallest) zero of $\sin(\nu_2 t + \theta_2)$ which is $< t_1$ (resp. $> t_1 + \pi/\nu_1$), given respectively by

$$t_- = \frac{\pi}{\nu_2} \left(\left\lceil \frac{\nu_2 t_1 + \theta_2}{\pi} \right\rceil - 1 - \frac{\theta_2}{\pi} \right) \quad \text{and} \quad t_+ = \frac{\pi}{\nu_2} \left(\left\lfloor \frac{\nu_2 (t_1 + \pi/\nu_1) + \theta_2}{\pi} \right\rfloor + 1 - \frac{\theta_2}{\pi} \right).$$



$\sin(\nu_1 t+\theta_1)\geq 0$ for $t\in J_1=[t_1, t_1+\pi/\nu_1]$.



$\sin(\nu_1 t+\theta_1)\leq 0$ for $t\in J_1=[t_1, t_1+\pi/\nu_1]$.

Figure 6. The dashed line is $t\mapsto A_1\sin(\nu_1 t+\theta_1)$, the gray line is $t\mapsto A_2\sin(\nu_2 t+\theta_2)$, with $\nu_2/\nu_1=1.2$.

They both have the property that $\sin(\nu_1 t_\pm+\theta_1)>0$, but we wish to quantify this statement in order to have a uniform lower bound. We observe that we can write

$$t_+-t_-=\frac{\pi}{\nu_2}\Big(k+1+\Big\lfloor\frac{\nu_2}{\nu_1}\Big\rfloor\Big),$$

with $k\in\{0,1\}$. Indeed, from the definition of t_+ and t_- and the properties of the floor and ceiling functions, we see that

$$k=\Big\lfloor\frac{\nu_2(t_1+\pi/\nu_1)+\theta_2}{\pi}\Big\rfloor-\Big\lceil\frac{\nu_2 t_1+\theta_2}{\pi}\Big\rceil+1-\Big\lfloor\frac{\nu_2}{\nu_1}\Big\rfloor$$

is an integer satisfying

$$-1\leq\frac{\nu_2}{\nu_1}-1-\Big\lfloor\frac{\nu_2}{\nu_1}\Big\rfloor<k\leq 1+\frac{\nu_2}{\nu_1}-\Big\lfloor\frac{\nu_2}{\nu_1}\Big\rfloor<2,$$

whence $k=0$ or 1. In particular, we remark that the distance between t_- and t_+ is always less than T_0. This implies that either t_- or t_+ belongs to $[0, T_0]$.

Subcase 2b(i): Suppose t_- and t_+ both belong to $[0, T_0]$. We have

$$(t_+-t_-)-\frac{\pi}{\nu_1}=\frac{\pi}{\nu_2}\Big(k+1+\Big\lfloor\frac{\nu_2}{\nu_1}\Big\rfloor-\frac{\nu_2}{\nu_1}\Big)=\frac{\pi}{\nu_2}(k+\eta)\geq\frac{\pi}{\nu_2}\eta.$$

Recalling that $t_-<t_1$ and $t_+>t_1+\pi/\nu_1$, we deduce that either $t_1-t_-\geq\pi\eta/2\nu_2$ or $t_+-(t_1+\pi/\nu_1)\geq\pi\eta/2\nu_2$. We call t_2 the zero, among t_- and t_+, that satisfies this property. Then, the concavity inequality for the sine function allows to conclude that $x_1^{t_2}\geq A_1\nu_1\eta/\nu_2$ again.

Subcase 2b(ii): If $t_-\notin[0, T_0]$, so that $t_+\in[0, T_0]$, we can estimate the distance of t_+ from $t_1+\pi/\nu_1$ and T_0 as

$$t_+-\Big(t_1+\frac{\pi}{\nu_1}\Big)=t_+-t_--\frac{\pi}{\nu_1}-(t_1-t_-)=\frac{\pi}{\nu_2}(k+\eta)-(t_1-t_-)\geq\frac{\pi}{\nu_2}\eta-\frac{\pi}{\nu_2}(1-k), \tag{4-7}$$

where we used the fact that $|t_1-t_-|\leq\pi/\nu_2$ by construction in the last inequality; and

$$T_0-t_+=T_0-(t_+-t_-)-t_-=\frac{\pi}{\nu_2}(1-k)-t_-\geq\frac{\pi}{\nu_2}(1-k), \tag{4-8}$$

where this time we have used that $t_-<0$ by assumption.

Now observe that t_+ satisfies by definition

$$t_1+\frac{2\pi}{\nu_1}-t_+=\frac{\pi}{\nu_1}-\Big(t_+-\Big(t_1+\frac{\pi}{\nu_1}\Big)\Big)\geq\frac{\pi}{\nu_1}-\frac{\pi}{\nu_2}.$$

Thus, if $|t_+-(t_1+\pi/\nu_1)|\geq(\pi/2)\min(\eta/\nu_2,\,1/\nu_1-1/\nu_2)$, then $t_2=t_+$ lies at distance greater or equal to $(\pi/2)\min(\eta/\nu_2,\,1/\nu_1-1/\nu_2)$ from the boundary of the interval $[t_1+\pi/\nu_1,\,t_1+2\pi/\nu_1]$, to which it belongs. This allows us to deduce that $x_1^{t_2}\geq A_1\delta_1$ using the inequality $\sin x\geq 2x/\pi$ on $[0,\pi/2]$ again, and $x_2^{t_2}=0$ by definition of $t_2=t_+$. If on the contrary $|t_+-(t_1+\pi/\nu_1)|\leq(\pi/2)\min(\eta/\nu_2,\,1/\nu_1-1/\nu_2)$, then from (4-7), it follows that $k=0$, so $t_2=t_++\pi/\nu_2\leq T_0$ from (4-8). Then

$$t_1+\frac{2\pi}{\nu_1}-t_2=\frac{\pi}{\nu_1}-\frac{\pi}{\nu_2}-\Big(t_+-\Big(t_1+\frac{\pi}{\nu_1}\Big)\Big)\geq\frac{\pi}{2}\Big(\frac{1}{\nu_1}-\frac{1}{\nu_2}\Big).$$

In particular, t_2 lies again at large enough distance of the boundary of $[t_1+\pi/\nu_1,\,t_1+2\pi/\nu_1]$. We deduce as before that $x_1^{t_2}\geq A_1\delta_1$ and $x_2^{t_2}=0$.

Subcase 2b(iii): It remains to deal with the case where $t_+\notin[0,T_0]$, hence $t_-\in[0,T_0]$, which is symmetrical. We write

$$t_1-t_-=-\Big(t_+-t_1-\frac{\pi}{\nu_1}\Big)+t_+-t_--\frac{\pi}{\nu_1}\geq-\frac{\pi}{\nu_2}+\frac{\pi}{\nu_2}(k+\eta)=\frac{\pi}{\nu_2}\eta-\frac{\pi}{\nu_2}(1-k), \tag{4-9}$$

$$t_-=T_0-(t_+-t_-)+t_+-T_0\geq\frac{\pi}{\nu_2}(1-k), \tag{4-10}$$

using respectively that $|t_1+\pi/\nu_1-t_+|\leq\pi/\nu_2$ by construction of t_+, and $t_+>T_0$ by assumption. By definition of t_- we have

$$t_--\Big(t_1-\frac{\pi}{\nu_1}\Big)=\frac{\pi}{\nu_1}-(t_1-t_-)\geq\frac{\pi}{\nu_1}-\frac{\pi}{\nu_2},$$

so $t_2=t_-$ satisfies $x_1^{t_2}\geq A_1\delta_1$ and $x_2^{t_2}=0$ provided $|t_1-t_-|\geq(\pi/2)\min(\eta/\nu_2,\,1/\nu_1-1/\nu_2)$. Otherwise, $k=0$ in virtue of (4-9), so (4-10) ensures that $t_2=t_--\pi/\nu_2\geq 0$. Then we check that

$$t_2-\Big(t_1-\frac{\pi}{\nu_1}\Big)=\frac{\pi}{\nu_1}-\frac{\pi}{\nu_2}-(t_1-t_-)\geq\frac{\pi}{2}\Big(\frac{1}{\nu_1}-\frac{1}{\nu_2}\Big),$$

and we conclude similarly to the previous case.

The discussion above shows that (4-4) is true. In particular, $(x_1^{t_2},x_2^{t_2})$ is in the cone $C^1_{\varepsilon/2}$. Using that the sine function is 1-Lipschitz, we know that for t in a neighborhood of 0, we have

$$|x_2^{t_2+t}|\leq A_2\nu_2|t|\quad\text{and}\quad x_1^{t_2+t}\geq A_1(\delta_1-\nu_1|t|).$$

So for t small enough, $(x_1^{t_2+t},x_2^{t_2+t})$ will remain in the cone $C^1_{\varepsilon/2}$. Quantitatively, as soon as t fulfills the condition

$$|t|<\frac{\dfrac{\delta_1}{\nu_1}}{1+\dfrac{\nu_2}{\nu_1}\dfrac{A_2/A_1}{\tan(\varepsilon/4)}}, \tag{4-11}$$

we compute that

$$x_1^{t_2+t}>A_1\frac{\delta_1\dfrac{\nu_2}{\nu_1}\dfrac{A_2/A_1}{\tan(\varepsilon/4)}}{1+\dfrac{\nu_2}{\nu_1}\dfrac{A_2/A_1}{\tan(\varepsilon/4)}}>\frac{A_2\nu_2}{\tan(\varepsilon/4)}|t|\geq\frac{|x_2^{t_2+t}|}{\tan(\varepsilon/4)}.$$

This means that for t satisfying (4-11), the point $(x_1^{t_2+t}, x_2^{t_2+t})$ belongs indeed to the cone $C^1_{\varepsilon/2}$. In the case where $t_2 = 0$ or $t_2 = T_0$, we may restrict ourselves to times t satisfying $t \geq 0$ or $t \leq 0$ in addition to (4-11), so that in the end, we obtain

$$\int_0^{T_0} \mathbf{1}_{C^1_{\varepsilon/2}}(x_1^t, x_2^t)\, dt \geq \min\left(T_0, \frac{\frac{\delta_1}{\nu_1}}{1 + \frac{\nu_2}{\nu_1} \frac{A_2/A_1}{\tan(\varepsilon/4)}} \right). \tag{4-12}$$

Step 5: *Upper bound on the optimal observation time.* Now that we have (4-3) and (4-12) at hand, we can obtain a lower bound independent of the values of A_1 and A_2. If on the one hand $A_1/A_2 \leq \tan(\varepsilon/4)/2$, then (4-3) yields

$$\int_0^{T_0} \mathbf{1}_{(x_1^t, x_2^t) \in \omega(\varepsilon/2)}\, dt \geq \frac{\pi}{2\nu_2},$$

while on the other hand, if $A_2/A_1 \leq 2/\tan(\varepsilon/4)$, then (4-12) leads to

$$\int_0^{T_0} \mathbf{1}_{(x_1^t, x_2^t) \in \omega(\varepsilon/2)}\, dt \geq \min\left(T_0, \frac{\frac{\delta_1}{\nu_1}}{1 + \frac{\nu_2}{\nu_1} \frac{2}{\tan^2(\varepsilon/4)}} \right) \geq \frac{\varepsilon^2}{16} \min\left(T_0, \frac{\delta_1}{\nu_1 + 2\nu_2} \right) \tag{4-13}$$

(to get the second inequality, use that $\varepsilon/4 \leq \tan(\varepsilon/4) \leq 1$ since $\varepsilon \leq \pi/2$ by assumption). On the whole, we have

$$\int_0^{T_0} \mathbf{1}_{(x_1^t, x_2^t) \in \omega(\varepsilon/2)}\, dt \geq \frac{\varepsilon^2}{32} \min\left(\frac{\pi}{\nu_2}, \frac{\delta_1}{\nu_1 + \nu_2} \right) = c\varepsilon^2, \tag{4-14}$$

and setting $T_\varepsilon = T_0 - c\varepsilon^2/2$, we deduce

$$\int_0^{T_\varepsilon} \mathbf{1}_{(x_1^t, x_2^t) \in \omega(\varepsilon/2)}\, dt \geq \int_0^{T_0} \mathbf{1}_{(x_1^t, x_2^t) \in \omega(\varepsilon/2)}\, dt - \frac{c}{2}\varepsilon^2 \geq \frac{c}{2}\varepsilon^2.$$

Therefore the dynamical condition (1-7) holds in time T_ε. Setting $T = T_0 - c\varepsilon^2/4 > T_\varepsilon$, we use Theorem 1.3 to conclude that observability is true on $[0, T]$ from $\omega(\varepsilon/2)_R \setminus K$, for some $R > 0$ and for any compact set K. We can take K to be a ball with radius large enough so that $\omega(\varepsilon/2)_R \setminus K \subset \omega(\varepsilon)$ (this can be justified by an argument similar to Lemma 4.1). We conclude that observability holds from $\omega(\varepsilon)$ in time T. This proves the upper bound in (1-18).

Step 6: *Lower bound on the optimal observation time.* Fix $\varepsilon \in (0, \pi/4)$. We recall that $\nu_2 > \nu_1$. Our objective is to exhibit trajectories (x_1^t, x_2^t) that do not meet the set $\omega(2\varepsilon)$. They typically look like the one shown in Figure 3. Take $\delta > 0$ a small parameter to be chosen later. These trajectories we look for are of the form

$$x_1^t = A_1 \sin\left(\pi \frac{\nu_1}{\nu_2}(1 - \delta) - \nu_1 t \right) \quad \text{and} \quad x_2^t = A_2 s \sin(\pi\delta + \nu_2 t), \tag{4-15}$$

with $s \in \{+1, -1\}$, and $A_1, A_2 > 0$ to be tuned properly later on as well.

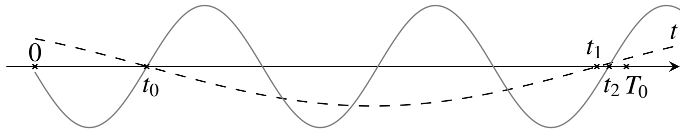


Figure 7. The dashed line is $t \mapsto x_1^t$, the gray line is $t \mapsto x_2^t$, with $\nu_2/\nu_1 = 3.9$, as defined in (4-15).

Let us introduce three remarkable times t_0, t_1 and t_2: provided $\delta < \frac{1}{2}$, the first zeroes of x_1^t and x_2^t in the interval $[0, T_0]$ coincide and are given by

$$t_0 = \frac{\pi}{\nu_2}(1-\delta).$$

The next zero of x_1^t is

$$t_1 = t_0 + \frac{\pi}{\nu_1}.$$

As for x_2^t, its first zero that is strictly larger than t_1 is given by

$$t_2 = \frac{\pi}{\nu_2}\Big(1 + \Big\lfloor \delta + \frac{\nu_2}{\pi} t_1 \Big\rfloor - \delta\Big) = \frac{\pi}{\nu_2}\Big(1 + \Big\lfloor 1 + \frac{\nu_2}{\nu_1}\Big\rfloor - \delta\Big) = T_0 - \frac{\pi}{\nu_2}\delta. \tag{4-16}$$

This is illustrated in Figure 7. Notice that $t_2 \le T_0$. By construction, the interval $[t_1, t_2]$ has length $t_2 - t_1 \in (0, \pi/\nu_2]$, and x_2^t has constant sign on this interval. We choose the sign s involved in the definition (4-15) of x_2^t in such a way that $x_2^t \le 0$ on $[t_1, t_2]$. In particular, the projected trajectory (x_1^t, x_2^t) cannot cross $C_{2\varepsilon}^2$ in the time interval $[t_1, t_2]$. Likewise, since $x_1^0 > 0$, it follows that $x_1^t \le 0$ on $[t_0, t_1]$, by definition of t_0, t_1. In particular, the curve (x_1^t, x_2^t) cannot be in $C_{2\varepsilon}^1$ for $t \in [t_0, t_1]$.

Set $T = t_2 - \pi\delta/\nu_2$. In each interval $[0, t_0]$, $[t_0, t_1]$ and $[t_1, T]$, we want to exclude the possibility for the trajectory to be in $C_{2\varepsilon}^1$ or $C_{2\varepsilon}^2$ by suitably choosing the parameters δ, A_1, A_2.

To achieve this goal, we are interested in estimating from above and from below x_1^t and x_2^t in these intervals. We first deal with x_1^t. Recalling that the sine function is 1-Lipschitz, we know that

$$|x_1^t| \le A_1 \nu_1 \min(|t - t_0|, |t - t_1|) \quad \forall t \in \mathbb{R}. \tag{4-17}$$

We obtain lower estimates by roughly bounding from below $\sin x$ on $[0, \pi]$ by the "triangle" function $(2/\pi)\min(x, \pi - x)$. For $t \in [0, t_0]$, that leads to

$$\begin{aligned} |x_1^t| &\ge A_1 \frac{2}{\pi} \min\big(\nu_1 |t_0 - t|, |\pi - \nu_1(t_0 - t)|\big) \ge A_1 \frac{2\nu_1}{\pi} \min\Big(|t_0 - t|, \Big|\frac{\pi}{\nu_1} - \frac{\pi}{\nu_2} + \delta\frac{\pi}{\nu_2} + t\Big|\Big) \\ &\ge A_1 \frac{2\nu_1}{\pi} \min\Big(|t_0 - t|, \frac{\pi}{\nu_1} - \frac{\pi}{\nu_2}\Big), \end{aligned} \tag{4-18}$$

for $t \in [t_0, t_1]$ we obtain

$$|x_1^t| \ge A_1 \frac{2\nu_1}{\pi} \min(|t - t_0|, |t - t_1|), \tag{4-19}$$

while for $t \in [t_1, T]$, we obtain

$$\begin{aligned} |x_1^t| &\ge A_1 \frac{2}{\pi} \min\big(\nu_1 |t - t_1|, |\pi - \nu_1(t - t_1)|\big) \ge A_1 \frac{2\nu_1}{\pi} \min\Big(|t - t_1|, \Big|\frac{\pi}{\nu_1} + t_1 - t\Big|\Big) \\ &\ge A_1 \frac{2\nu_1}{\pi} \min\Big(|t_1 - t|, \frac{\pi}{\nu_1} + t_1 - T\Big). \end{aligned}$$

The last inequality rests on the fact that $\pi/\nu_1 + t_1 \geq T$. More quantitatively, we have

$$\begin{aligned}\frac{\pi}{\nu_2} + t_1 &= T_0 + \frac{\pi}{\nu_2}\Big(1 + \frac{\nu_2}{\nu_1} + 1 - \delta - 2 - \Big\lfloor \frac{\nu_2}{\nu_1} \Big\rfloor\Big) \\ &= T + \frac{\pi}{\nu_2}\delta + \frac{\pi}{\nu_2}\Big(\frac{\nu_2}{\nu_1} - \Big\lfloor \frac{\nu_2}{\nu_1} \Big\rfloor\Big). \end{aligned} \tag{4-20}$$

In particular,

$$\frac{\pi}{\nu_1} + t_1 - T = \pi\Big(\frac{1}{\nu_1} - \frac{1}{\nu_2}\Big) + \Big(\frac{\pi}{\nu_2} + t_1 - T\Big) \geq \pi\Big(\frac{1}{\nu_1} - \frac{1}{\nu_2}\Big) + \frac{\pi}{\nu_2}\delta, \tag{4-21}$$

which leads to

$$|x_1^t| \geq A_1 \frac{2\nu_1}{\pi} \min\Big(|t_1 - t|, \pi\Big(\frac{1}{\nu_1} - \frac{1}{\nu_2}\Big)\Big) \quad \forall t \in [t_1, T].$$

We obtain a similar estimate for x_2^t: it vanishes at t_0 and t_2, so using again that the sine function is 1-Lipschitz we get

$$|x_2^t| \leq A_2 \nu_2 \min(|t_0 - t|, |t_2 - t|) \quad \forall t \in \mathbb{R}. \tag{4-22}$$

Near, t_1, we want an accurate upper bound using the fact that $x_2^{t_1} \leq 0$ (recall that we chose the sign s in (4-15) so that this is true): for any $t \in \mathbb{R}$, we have

$$x_2^t \leq x_2^t - x_2^{t_1} \leq A_2 \nu_2 |t - t_1|. \tag{4-23}$$

As for a lower bound, we obtain, for $t \in [0, t_0]$,

$$\begin{aligned}|x_2^t| &\geq A_2 \frac{2}{\pi} \min\big(\nu_2|t_0 - t|, |\pi - \nu_2(t_0 - t)|\big) \geq A_2 \frac{2\nu_2}{\pi} \min\Big(|t_0 - t|, \delta\frac{\pi}{\nu_2} + t\Big) \\ &\geq A_2 \frac{2\nu_2}{\pi} \min\Big(|t_0 - t|, \delta\frac{\pi}{\nu_2}\Big), \end{aligned} \tag{4-24}$$

and, for $t \in [t_1, T]$,

$$\begin{aligned}|x_2^t| &\geq A_2 \frac{2}{\pi} \min\big(|\pi - \nu_2(t_2 - t)|, \nu_2|t_2 - t|\big) \geq A_2 \frac{2\nu_2}{\pi} \min\Big(\Big|\frac{\pi}{\nu_2} - (t_2 - t)\Big|, |t_2 - t|\Big) \\ &\geq A_2 \frac{2\nu_2}{\pi} \min\Big(|t - t_1|, \frac{\pi}{\nu_2}\delta\Big). \end{aligned} \tag{4-25}$$

This time, the last inequality holds true since on the one hand, $t_2 - t \geq t_2 - T = \pi\delta/\nu_2$, and on the other hand, thanks to (4-20) and (4-16), we check that, for any $t \in [t_1, T]$,

$$\frac{\pi}{\nu_2} - (t_2 - t) = (t - t_1) + \Big(\frac{\pi}{\nu_2} + t_1\Big) - t_2 = (t - t_1) + \frac{\pi}{\nu_2}\Big(\frac{\nu_2}{\nu_1} - \Big\lfloor \frac{\nu_2}{\nu_1} \Big\rfloor\Big) \geq t - t_1.$$

Now we show that the two conditions

$$2\frac{\nu_1}{\nu_2}\varepsilon \leq \frac{A_2}{A_1}\delta, \tag{4-26}$$

$$2\frac{\nu_2}{\nu_1}\varepsilon \leq \frac{A_1}{A_2} \min\Big(1, \frac{\nu_2}{\nu_1} - 1\Big) \tag{4-27}$$

imply that the curve (x_1^t, x_2^t) does not cross the set $\omega(2\varepsilon)$ in the interval $[0, T]$. We study the three intervals $[0, t_0]$, $[t_0, t_1]$ and $[t_1, T]$ separately.

- Let $t \in [0, t_0]$. On the one hand, the condition (4-26) implies that

$$A_1 \nu_1 |t - t_0| \frac{4\varepsilon}{\pi} \le A_2 \frac{2\nu_2}{\pi} \min\Bigl(|t - t_0|, \delta \frac{\pi}{\nu_2}\Bigr)$$

(recall that $t_0 \le \pi/\nu_2$ and $\delta \le 1/2$). Using that $\tan\varepsilon \le 4\varepsilon/\pi$ for $\varepsilon \in [0, \pi/4]$, we obtain

$$A_1 \nu_1 |t - t_0| \tan\varepsilon \le A_2 \frac{2\nu_2}{\pi} \min\Bigl(|t - t_0|, \delta \frac{\pi}{\nu_2}\Bigr),$$

which leads to $\tan(\varepsilon)|x_1^t| \le |x_2^t|$ in virtue of (4-17) and (4-24). Therefore $(x_1^t, x_2^t) \notin C^1_{2\varepsilon}$. On the other hand, the condition (4-27) implies that

$$A_2 \nu_2 |t_0 - t| \frac{4\varepsilon}{\pi} \le A_1 \frac{2\nu_1}{\pi} \min\Bigl(|t_0 - t|, \frac{\pi}{\nu_1} - \frac{\pi}{\nu_2}\Bigr)$$

(recall again that $t_0 \le \pi/\nu_2$). Using that $\tan\varepsilon \le 4\varepsilon/\pi$ for $\varepsilon \in [0, \pi/4]$, we obtain

$$A_2 \nu_2 |t_0 - t| \tan\varepsilon \le A_1 \frac{2\nu_1}{\pi} \min\Bigl(|t_0 - t|, \frac{\pi}{\nu_1} - \frac{\pi}{\nu_2}\Bigr),$$

which leads to $\tan(\varepsilon)|x_2^t| \le |x_1^t|$ in virtue of (4-22) and (4-18). Therefore $(x_1^t, x_2^t) \notin C^2_{2\varepsilon}$.

- On $[t_0, t_1]$, the situation is slightly simpler because we already know that $x_1^t \le 0$ on this interval, which means that the trajectory does not cross $C^1_{2\varepsilon}$ by construction. In addition, condition (4-27) implies that

$$A_2 \nu_2 \min(|t_0 - t|, |t_1 - t|) \frac{4\varepsilon}{\pi} \le A_1 \frac{2\nu_1}{\pi} \min(|t - t_0|, |t_1 - t|).$$

Then (4-22), (4-23) and (4-19) yield $\tan(\varepsilon) x_2^t \le |x_1^t|$; hence $(x_1^t, x_2^t) \notin C^2_{2\varepsilon}$.

- We finally consider $t \in [t_1, T]$. Notice that by construction, $x_2^t \le 0$ on $[t_1, T]$, so that the trajectory does not enter $C^2_{2\varepsilon}$. To disprove the fact that it meets $C^1_{2\varepsilon}$, we check that the condition (4-26) implies

$$A_1 \nu_1 |t - t_1| \frac{4\varepsilon}{\pi} \le A_2 \frac{2\nu_2}{\pi} \min\Bigl(|t - t_1|, \frac{\pi}{\nu_2}\delta\Bigr),$$

owing to the fact that $\pi/\nu_2 \ge T - t_1 \ge t - t_1$ (this can be deduced from (4-21)). Then (4-17) and (4-25) lead to $\tan(\varepsilon)|x_1^t| \le |x_2^t|$, which shows indeed that $(x_1^t, x_2^t) \notin C^1_{2\varepsilon}$.

To sum up, in order to ensure that $t \mapsto (x_1^t, x_2^t)$ does not cross $\omega(2\varepsilon)$, it suffices to choose A_1/A_2 properly, as well as δ, so that (4-26) and (4-27) are fulfilled. If we set

$$\delta = \frac{4\varepsilon^2}{\min(1, \nu_2/\nu_1 - 1)} \quad \text{and} \quad \frac{A_1}{A_2} = 2\varepsilon \frac{\nu_2/\nu_1}{\min(1, \nu_2/\nu_1 - 1)}, \tag{4-28}$$

we can check that these two conditions are indeed satisfied.

The conclusion is as follows: we consider a sequence of initial data of the form

$$\rho_n = \Bigl(A_{1,n} \sin\Bigl(\pi \frac{\nu_1}{\nu_2}(1 - \delta)\Bigr), A_{2,n} s \sin(\pi\delta)\Bigr),$$

with $A_{1,n}/A_{2,n}$ as in (4-28) and $A_{1,n}, A_{2,n} \to \infty$ as $n \to \infty$. The x component of the trajectory $t \mapsto \phi^t(\rho_n)$ is then of the same form as the projected trajectory (x_1^t, x_2^t) that we studied. Given that these trajectories do not cross $\omega(2\varepsilon)$, we conclude that the observability condition of Theorem 1.3 is not true in time T, namely

$$\mathfrak{K}_{p_0}^{\infty}(\omega(2\varepsilon), T) = 0.$$

Yet for any $R > 0$, as we have already seen earlier, $\omega(\varepsilon)_R$ is contained in $\omega(2\varepsilon)$ modulo a compact set. Thus for any $R > 0$, there exists a compact set $K(R) \subset \mathbb{R}^d$ such that

$$\mathfrak{K}_{p_0}^{\infty}(\omega(\varepsilon)_R \setminus K(R), T) \leq \mathfrak{K}_{p_0}^{\infty}(\omega(2\varepsilon), T) = 0.$$

We conclude thanks to the necessary condition in Theorem 1.3 that observability cannot hold in $\omega(\varepsilon)$ in time T. It remains to see that by definition (recall (4-16) and (4-28)), we have

$$T = t_2 - \frac{\pi}{\nu_2}\delta = T_0 - 2\frac{\pi}{\nu_2}\delta = T_0 - C\varepsilon^2. \tag{4-29}$$

This ends the proof of the lower bound of the optimal observation time. □

4.3. *Proof of Proposition 1.7.* The aim of this proposition is to study observability from measurable conical sets for the (exact) isotropic harmonic oscillator. We first simplify the situation owing to periodicity properties of the isotropic quantum harmonic oscillator.

Step 1: *Upper bound of the optimal observation time.* First recall that there exists a complex number z of modulus 1 such that

$$e^{i\pi P/\nu} u = z u(-\bullet) \quad \forall u \in L^2(\mathbb{R}^d). \tag{4-30}$$

See for instance[11] [Zworski 2012, Section 11.3.1] or [Folland 1989, (4.26)]. In particular, the propagator e^{-itP} is $2\pi/\nu$-periodic modulo multiplication by z^2. This enables us to show that observability holds in some time T if and only if it holds in time $2\pi/\nu$: assume the Schrödinger equation is observable from $\omega \subset \mathbb{R}^d$ in some time T; let k be an integer such that $2\pi k/\nu \geq T$. The aforementioned $2\pi/\nu$-periodicity of the harmonic oscillator leads to

$$\begin{aligned} \|u\|_{L^2(\mathbb{R}^d)}^2 \leq C \int_0^T \|e^{-itP} u\|_{L^2(\omega)}^2 \, dt \leq C \int_0^{2\pi k/\nu} \|e^{-itP} u\|_{L^2(\omega)}^2 \, dt \\ = Ck \int_0^{2\pi/\nu} \|e^{-itP} u\|_{L^2(\omega)}^2 \, dt \end{aligned} \tag{4-31}$$

for any $u \in L^2$ so that observability holds in ω in time $2\pi/\nu$. In particular, the optimal observation time is always $\leq 2\pi/\nu$. We can further reduce the observation time by $(2Ck)^{-1}$ (see Lemma A.3), so that the optimal observation time is in fact $T_\star < 2\pi/\nu$. Incidentally, the property (4-30) yields

$$\int_0^{\pi/\nu} \|e^{-itP} u\|_{L^2(\omega \cup -\omega)}^2 \, dt \leq \int_0^{2\pi/\nu} \|e^{-itP} u\|_{L^2(\omega)}^2 \, dt \leq 2 \int_0^{\pi/\nu} \|e^{-itP} u\|_{L^2(\omega \cup -\omega)}^2 \, dt, \tag{4-32}$$

which will be useful later on.

[11] The property (4-30) can be derived from the fact that the spectrum of P is made of half integer multiples of ν, together with parity properties of eigenfunctions.

Step 2: *Necessary condition.* Assume observability holds from $\omega = \omega(\Sigma)$ in some time T. Let k be a positive integer such that $2\pi k/\nu \geq T$. Using (4-31) and (4-32), we obtain

$$\forall u \in L^2(\mathbb{R}^d), \quad \|u\|^2_{L^2(\mathbb{R}^d)} \leq C \int_0^T \|e^{-itP} u\|^2_{L^2(\omega)}\, dt \leq 2Ck \int_0^{\pi/\nu} \|e^{-itP} u\|^2_{L^2(\omega \cup -\omega)}\, dt.$$

We choose for u a particular coherent state. Following [Combescure and Robert 2012], for any $\rho_0 = (x_0, \xi_0)$, we set

$$\varphi_{\rho_0}(x) = \Big(\frac{\nu}{\pi}\Big)^{d/4} e^{-(i/2)\xi_0\cdot x_0 + i\xi_0\cdot x} \exp\Big(-\frac{\nu}{2}|x - x_0|^2\Big).$$

Then

$$e^{-itP}\varphi_{\rho_0} = e^{-(i/2)t\nu d}\varphi_{\rho_t}, \tag{4-33}$$

where $\rho_t = \phi^t(\rho_0)$ is the evolution of ρ_0 in phase space along the Hamiltonian flow associated with $p(x, \xi) = \frac{1}{2}(\nu^2|x|^2 + |\xi|^2)$, that is to say,

$$\rho_t = \Big(\cos(\nu t)x_0 + \sin(\nu t)\frac{\xi_0}{\nu}, -\nu\sin(\nu t)x_0 + \cos(\nu t)\xi_0\Big).$$

Equation (4-33) can be checked by observing that the derivative of both sides agree, or by applying [Combescure and Robert 2012, Proposition 3]. Selecting an initial datum of the form $\rho_0 = (0, \xi_0)$ with a nonzero ξ_0, the observability inequality implies

$$\begin{aligned} 1 = \|\varphi_{\rho_0}\|^2_{L^2(\mathbb{R}^d)} &\leq C \int_0^T \|\varphi_{\rho_t}\|^2_{L^2(\omega)}\, dt \leq 2kC \int_0^{\pi/\nu} \|\varphi_{\rho_t}\|^2_{L^2(\omega\cup-\omega)}\, dt \\ &= 2kC\Big(\frac{\pi}{\nu}\Big)^{d/2} \int_0^{\pi/\nu}\int_{\omega\cup-\omega} \Big|\exp\Big(-\frac{\nu}{2}\Big|x - \sin(\nu t)\frac{\xi_0}{\nu}\Big|^2\Big)\Big|^2\, dx\, dt \\ &= 4k\frac{C}{\nu}\Big(\frac{\pi}{\nu}\Big)^{d/2} \int_0^{\pi/2}\int_{\omega\cup-\omega} \exp\Big(-\nu\Big|x - \sin(t)\frac{\xi_0}{\nu}\Big|^2\Big)\, dx\, dt. \end{aligned} \tag{4-34}$$

We used a change of variables in the integral over t and the fact that $\sin(x) = \sin(\pi - x)$ to obtain the last equality. Next we truncate the integrals in t and in x using respectively a small parameter $\delta > 0$ and a large parameter $R > 0$:

$$\begin{aligned} &\int_0^{\pi/2}\Big(\frac{\pi}{\nu}\Big)^{d/2}\int_{\omega\cup-\omega}\exp\Big(-\nu\Big|x-\sin(t)\frac{\xi_0}{\nu}\Big|^2\Big)\, dx\, dt \\ &\leq \pi\delta + \int_{\pi/2\delta}^{(\pi/2)(1-\delta)}\Big(\frac{\pi}{\nu}\Big)^{d/2}\Big(\int_{\omega\cup-\omega}\exp\Big(-\nu\Big|x-\sin(t)\frac{\xi_0}{\nu}\Big|^2\Big)\mathbf{1}_{B_R(\sin(t)\xi_0/\nu)}(x)\, dx + \int_{\mathbb{R}^d\setminus B_R(0)} e^{-\nu|x|^2}\, dx\Big)\, dt. \end{aligned}$$

The rightmost integral is controlled by c/R for some constant $c > 0$. Therefore

$$\begin{aligned} &\int_0^{\pi/2}\Big(\frac{\pi}{\nu}\Big)^{d/2}\int_{\omega\cup-\omega}\exp\Big(-\nu\Big|x - \sin(t)\frac{\xi_0}{\nu}\Big|^2\Big)\, dx\, dt \\ &\qquad\qquad \leq \pi\delta + \frac{c}{R} + \Big(\frac{\pi}{\nu}\Big)^{d/2}\int_{\pi\delta/2}^{(\pi/2)(1-\delta)}\Big|(\omega\cup-\omega)\cap B_R\Big(\sin(t)\frac{\xi_0}{\nu}\Big)\Big|\, dt. \end{aligned}$$

We get rid of the sine in the right-hand side by noting that $\cos t \geq 1-2t/\pi \geq \delta$ for any $t \in [\pi\delta/2, \pi(1-\delta)/2]$, and changing variables:

$$\int_{\pi\delta/2}^{\pi(1-\delta)/2} \left|(\omega \cup -\omega) \cap B_R\Big(\sin(t)\frac{\xi_0}{\nu}\Big)\right| dt \leq \int_{\pi\delta/2}^{\pi(1-\delta)/2} \left|(\omega \cup -\omega) \cap B_R\Big(\sin(t)\frac{\xi_0}{\nu}\Big)\right| \frac{|\cos t|}{\delta}\, dt$$
$$= \frac{1}{\delta}\int_{\sin(\pi\delta/2)}^{\sin(\pi(1-\delta)/2)} \left|(\omega \cup -\omega) \cap B_R\Big(s\frac{\xi_0}{\nu}\Big)\right| ds.$$

Using that $\sin x \geq 2x/\pi$ on $[0, \pi/2]$, we finally deduce that

$$\int_0^{\pi/2} \Big(\frac{\pi}{\nu}\Big)^{d/2} \int_{\omega\cup-\omega} \exp\Big(-\nu\Big|x - \sin(t)\frac{\xi_0}{\nu}\Big|^2\Big)\, dx\, dt$$
$$\leq \pi\delta + \frac{c}{R} + \frac{1}{\delta}\Big(\frac{\pi}{\nu}\Big)^{d/2} \int_\delta^1 \left|(\omega \cup -\omega) \cap B_R\Big(s\frac{\xi_0}{\nu}\Big)\right| ds. \tag{4-35}$$

We plug this into (4-34) to obtain

$$\frac{1}{2} = \frac{1}{2}\|\varphi_{\rho_0}\|^2_{L^2(\mathbb{R}^d)} \leq 4k\frac{C}{\delta\nu}\Big(\frac{\pi}{\nu}\Big)^{d/2} \int_\delta^1 \left|(\omega \cup -\omega) \cap B_R\Big(s\frac{\xi_0}{\nu}\Big)\right| ds, \tag{4-36}$$

where we absorbed the remainder terms of (4-35) in the left-hand side by choosing δ sufficiently small and R sufficiently large. We now use a scaling argument in the right-hand side, which is possible since the set $\omega \cup -\omega$ is conical: for any $s \in [\delta, 1]$, writing

$$\theta_0 = \frac{\xi_0}{|\xi_0|} \quad \text{and} \quad r = \frac{\nu R}{\delta|\xi_0|}, \tag{4-37}$$

we have

$$\left|(\omega \cup -\omega) \cap B_R\Big(s\frac{\xi_0}{\nu}\Big)\right| = \Big(s\frac{|\xi_0|}{\nu}\Big)^d \left|(\omega \cup -\omega) \cap B_{\nu R/s|\xi_0|}(\theta_0)\right|$$
$$\leq \Big(\frac{R}{\delta}\Big)^d r^{-d} |(\omega \cup -\omega) \cap B_r(\theta_0)|.$$

After integrating over the s variable, the estimate (4-36) becomes

$$1 = \|\varphi_{\rho_0}\|^2_{L^2(\mathbb{R}^d)} \leq 8k\frac{C}{\delta\nu}\Big(\frac{\pi}{\nu}\Big)^{d/2}\Big(\frac{R}{\delta}\Big)^d r^{-d} \left|(\omega \cup -\omega) \cap B_r(\theta_0)\right|. \tag{4-38}$$

We now reformulate the right-hand side in terms of the lower density $\Theta^-_{\widehat{\Sigma}}$ defined in (1-20). To do so, we observe that the triangle inequality yields, for $r \in (0, 1)$,

$$\forall x \in B_r(\theta_0), \quad ||x| - 1| \leq |x - \theta_0| \quad \text{and} \quad \left|\frac{x}{|x|} - \theta_0\right| = \left|\frac{x - \theta_0}{|x|} + \frac{\theta_0}{|x|}(1 - |x|)\right| \leq \frac{2r}{1-r},$$

which in turn leads to

$$B_r(\theta_0) \subset \left\{x \in \mathbb{R}^d : 1 - r \leq |x| \leq 1 + r \ \text{ and } \left|\frac{x}{|x|} - \theta_0\right| \leq \frac{2r}{1-r}\right\}, \quad r \in (0, 1).$$

Recall that if $|\xi_0|$ is large enough, then (4-37) implies $r \in (0, 1)$. We conclude by a spherical change of coordinates that

$$\begin{aligned} |(\omega \cup -\omega) \cap B_r(\theta_0)| &\le \int_{1-r}^{1+r} \int_{\mathbb{S}^{d-1}} \mathbf{1}_{|\theta-\theta_0|\le 2r/(1-r)} \mathbf{1}_{\omega\cup-\omega}(\tilde{r}\theta) c_d \tilde{r}^{d-1}\, d\sigma(\theta)\, d\tilde{r} \\ &\le \int_{1-r}^{1+r} \int_{\mathbb{S}^{d-1}\cap B_{2r/(1-r)}(\theta_0)} \mathbf{1}_{\widehat{\Sigma}}(\theta) c_d 2^{d-1}\, d\sigma(\theta)\, d\tilde{r} \\ &= c_d 2^{d-1} \times 2r\sigma(\widehat{\Sigma} \cap B_{2r/(1-r)}(\theta_0)). \end{aligned} \tag{4-39}$$

In addition, one has

$$\sigma(B_r(\theta_0)) \le c_d' r^{d-1}. \tag{4-40}$$

(In the above estimates, c_d and c_d' are constants depending only on the dimension.) Combining (4-38), (4-39) and (4-40), we obtain

$$\begin{aligned} 1 = \|\varphi_{\rho_0}\|^2_{L^2(\mathbb{R}^d)} &\le c_d c_d' 2^{d+3} k \frac{C}{\delta\nu}\Big(\frac{\pi}{\nu}\Big)^{d/2}\Big(\frac{R}{\delta}\Big)^d \times \Big(\frac{2}{1-r}\Big)^{d-1} \frac{1}{c_d'\Big(\dfrac{2r}{1-r}\Big)^{d-1}}\, \sigma(\widehat{\Sigma} \cap B_{2r/(1-r)}(\theta_0)) \\ &\le c_d c_d' 2^{d+3} k \frac{C}{\delta\nu}\Big(\frac{\pi}{\nu}\Big)^{d/2}\Big(\frac{R}{\delta}\Big)^d \Big(\frac{2}{1-r}\Big)^d \frac{\sigma(\widehat{\Sigma}\cap B_{2r/(1-r)}(\theta_0))}{\sigma(B_{2r/(1-r)}(\theta_0))}. \end{aligned}$$

Recalling that r behaves as $1/|\xi_0|$, it remains to let $\xi_0 \to \infty$ with $\xi_0/|\xi_0| = \theta_0$ arbitrary, to deduce that

$$1 \le c_d c_d' 2^{d+3} k \frac{C}{\delta\nu}\Big(\frac{\pi}{\nu}\Big)^{d/2}\Big(\frac{2R}{\delta}\Big)^d \Theta^-_{\widehat{\Sigma}}(\theta_0) \quad \forall \theta_0 \in \mathbb{S}^{d-1}.$$

This concludes the proof of the necessary condition.

Step 3: *Sufficient condition.* Write for short $\omega = \omega(\Sigma)$ again. The fact that $\widehat{\Sigma} = \Sigma \cup -\Sigma$ has full measure, namely $\sigma(\mathbb{S}^{d-1} \setminus \widehat{\Sigma}) = 0$, implies that $\mathbb{R}^d \setminus (\omega \cup -\omega)$ is Lebesgue negligible (recall the definition of $\omega(\Sigma)$ in (1-19)). Therefore the left-hand side of (4-32) with $k = 1$ yields

$$\int_0^{2\pi/\nu} \|e^{-itP}u\|^2_{L^2(\omega)}\, dt \ge \int_0^{\pi/\nu} \|e^{-itP}u\|^2_{L^2(\omega\cup-\omega)}\, dt = \int_0^{\pi/\nu} \|e^{-itP}u\|^2_{L^2(\mathbb{R}^d)}\, dt = \frac{\pi}{\nu}\|u\|^2_{L^2(\mathbb{R}^d)},$$

where we used the fact that the propagator is an isometry. □

5. Proofs of observability results from spherical sets

In this section, we give proofs of the results presented in Section 1.4.3, which concern observation sets that are spherical in the sense of (1-23). Propositions 1.9 and 1.11 are proved in Sections 5.1 and 5.3 respectively. Section 5.2 is dedicated to the proof of Lemma 1.12.

5.1. ***Proof of Proposition 1.9.*** The rotation S_θ of angle θ reads

$$S_\theta y = \big(\cos\theta y_1 + \sin\theta y_2, -\sin\theta y_1 + \cos\theta y_2, y_3, \dots, y_d\big), \quad y = (y_1, y_2, \dots, y_d) \in \mathbb{R}^d.$$

In the sequel, we set L_0 to be the two-dimensional plane spanned by the vectors

$$e_1 = M(1,0,0,\dots,0) \quad \text{and} \quad e_2 = M(0,1,0,0,\dots 0).$$

The two linear maps

$$\Pi_{L_0} = \tfrac{1}{2}M(\mathrm{Id} - S_\pi)M^{-1} \quad \text{and} \quad \Pi_{L_0^\perp} = \tfrac{1}{2}M(\mathrm{Id} + S_\pi)M^{-1}$$

are the orthogonal projectors on L_0 and $L_0^\perp$ respectively, since M is orthogonal. With the notation of (iii), we can write, with a slight abuse of notation,

$$V(x_0) = \widetilde{V}_0(|M^{-1}x_0|) \quad \forall x_0 \in L_0. \tag{5-1}$$

Let us investigate the properties of the gradient of V on L_0.

Lemma 5.1. *Let $x_0 \in L_0$. Then*

$$\nabla V(x_0) \in L_0 \quad \textit{and} \quad \exists c = c(|x_0|) \geq 0 : \quad \nabla V(x_0) = c x_0.$$

Proof. Assumptions (i) and (ii) (with $\theta = \pi$) yield, for any $x \in \mathbb{R}^d$,

$$-\nabla V(-x) = \nabla V(x) \quad \text{and} \quad M S_{-\pi} M^{-1} \nabla V(M S_\pi M^{-1} x) = \nabla V(x). \tag{5-2}$$

Yet since $x_0 \in L_0$, we have $\Pi_{L_0} x_0 = x_0$ so that

$$x_0 = -M S_\pi M^{-1} x_0, \tag{5-3}$$

and noticing that $S_\pi = S_{-\pi}$, we obtain combining the two equations (5-2):

$$\nabla V(x_0) = -\nabla V(-x_0) = -M S_\pi M^{-1} \nabla V(-M S_\pi M^{-1} x_0) = -M S_\pi M^{-1} \nabla V(x_0).$$

That means exactly that $\Pi_{L_0^\perp} \nabla V(x_0) = 0$, or in other words, $\nabla V(x_0) \in L_0$.

Next we prove that $\nabla V(x_0)$ is collinear with x_0. We first get rid of the case $x_0 = 0$: the first equation in (5-2) implies that $\nabla V(0) = 0$. From now on, we assume that $x_0 \neq 0$. We compute

$$\frac{d}{d\theta} M S_\theta M^{-1} = \frac{d}{d\theta}\big(M S_\theta M^{-1} \Pi_{L_0} + M S_\theta M^{-1} \Pi_{L_0^\perp}\big) = M S_{\theta+\pi/2} M^{-1} \Pi_{L_0}.$$

This is true because $M S_\theta M^{-1} \Pi_{L_0^\perp}$ is independent of θ ($M S_\theta M^{-1}$ is the identity in $L_0^\perp$). Therefore, differentiating the equality $V(x) = V(M S_\theta M^{-1} x)$ at $\theta = 0$, we obtain

$$\begin{aligned} 0 = \frac{d}{d\theta} V(x_0)_{|\theta=0} = \frac{d}{d\theta} V(M S_\theta M^{-1} x_0)_{|\theta=0} &= \nabla V(x_0) \cdot M S_{\pi/2} M^{-1} \Pi_{L_0} x_0 \\ = \nabla V(x_0) \cdot M S_{\pi/2} M^{-1} x_0. \end{aligned}$$

This means that $\nabla V(x_0)$ is orthogonal to $M S_{\pi/2} M^{-1} x_0$. Yet the plane L_0 is invariant by $M S_\theta M^{-1}$ and $x_0 \perp M S_{\pi/2} M^{-1} x_0$. Since $\nabla V(x_0) \in L_0$ and L_0 has dimension 2, we deduce that $\nabla V(x_0) = c x_0$ for some $c \in \mathbb{R}$. We claim that $c \geq 0$ as a consequence of (iii) that $\widetilde{V}_0$ is nondecreasing. Indeed for $t > 0$ close to zero, using (5-1), the Taylor formula at order 1 yields

$$0 \leq \widetilde{V}_0((1+t)|M^{-1}x_0|) - \widetilde{V}_0(|M^{-1}x_0|) = V(x_0 + t x_0) - V(x_0) = t \nabla V(x_0) \cdot x_0 + o(t).$$

Dividing by $t > 0$, we find that $\nabla V(x_0)\cdot x_0 = c|x_0|^2 \geq 0$. Thus $c = \nabla V(x_0)\cdot x_0/|x_0|^2$ depends only on $|x_0|$ since V restricted to L_0 is radial. □

This lemma allows us to exhibit periodic circular orbits of the Hamiltonian flow of p. For any $x_0 \in L_0$, denoting by c the scalar such that $\nabla V(x_0) = cx_0$, the phase space curve

$$x^t = MS_{\sqrt{c}t}M^{-1}x_0, \quad \xi^t = \sqrt{c}MS_{\sqrt{c}t+\pi/2}M^{-1}x_0 \tag{5-4}$$

is the trajectory of the Hamiltonian flow with initial data $(x_0, \sqrt{c}MS_{\pi/2}M^{-1}x_0)$. This follows from uniqueness in the Picard–Lindelöf theorem, since the above curve solves on the one hand

$$\frac{d}{dt}x^t = \sqrt{c}MS_{\sqrt{c}t+\pi/2}M^{-1}\Pi_{L_0}x_0 = \xi^t,$$

and on the other hand, in view of (5-3) and observing that $|x^t| = |x_0|$ for any t,

$$\frac{d}{dt}\xi^t = cMS_{\sqrt{c}t+\pi}M^{-1}\Pi_{L_0}x_0 = cMS_{\sqrt{c}t}M^{-1}(\Pi_{L_0^\perp} - \Pi_{L_0})x_0 = -cx^t = -\nabla V(x^t).$$

To conclude, we argue as follows: since by assumption observability holds from $\omega(I)$ in time $T > 0$, the necessary condition of Theorem 1.3 implies that there exists $R > 0$ such that

$$\exists \epsilon > 0, \exists A > 0 : \forall |\rho| \geq A, \quad \int_0^T \mathbf{1}_{\omega(I)_R\times\mathbb{R}^d}(\phi^t(\rho))\,dt \geq \epsilon.$$

Let $x_0 \in L_0$ be such that $|x_0| \geq A$. We consider the Hamiltonian trajectory issued from the point $(x_0, \sqrt{c(x_0)}M^{-1}S_{\pi/2}Mx_0)$ constructed in (5-4). Then $|x^t|$ is constant over time, which implies that

$$\epsilon \leq \int_0^T \mathbf{1}_{\omega(I)_R\times\mathbb{R}^d}(\phi^t(\rho))\,dt = \int_0^T \mathbf{1}_{\omega(I)_R}(x^t)\,dt = T\mathbf{1}_{I_R}(|x_0|),$$

whence $|x_0| \in I_R$. We deduce that

$$\forall s \in \mathbb{R}_+, \quad I_R \cap [s, s+A] \neq \varnothing,$$

which implies the desired result (1-24) with $r = A + 2R$. □

5.2. ***Proof of Lemma 1.12.*** Firstly we assume that ν_2/ν_1 is rational: we write it as an irreducible fraction p/q. The number $T = 2\pi p/\nu_2 = 2\pi q/\nu_1$ is the period of the Hamiltonian flow associated with $\frac{1}{2}(x\cdot Ax + |\xi|^2)$. Without loss of generality, we can assume that A is diagonal, and that the eigenvectors associated with ν_1^2 and ν_2^2 are the vectors $(1, 0)$ and $(0, 1)$ of the canonical basis of $\mathbb{R}^2$.

We want to prove that $\Lambda(\nu_2/\nu_1)$ defined in (1-26) is equal to the quantity

$$\Lambda_0 = \sup_{\rho_0\in\mathbb{R}^4\setminus\{0\}} \frac{\min_{t\in[0,T]}|(\pi\circ\phi^t)(\rho_0)|}{\max_{t\in[0,T]}|(\pi\circ\phi^t)(\rho_0)|}, \tag{5-5}$$

where we recall that $\pi : (x, \xi) \mapsto x$.

We start with two remarks, related to explicit expressions of the Hamiltonian flow. First we can replace the supremum on $\mathbb{R}^4$ by a maximum on a compact set parametrizing trajectories, e.g., the unit sphere $\mathbb{S}^3$, because the Hamiltonian flow is homogeneous of degree 1, that is $\phi^t(\lambda\rho_0) = \lambda\phi^t(\rho_0)$ for any scalar $\lambda \in \mathbb{R}$

(it fact ϕ^t is a linear map for all t). Second, since $|x^t|^2 = |x_1^t|^2 + |x_2^t|^2$, it will be easier to compute Λ_0^2. In view of these remarks, and writing the Hamiltonian trajectories in action-angle coordinates:

$$x_1^t = A_1 \sin(\nu_1 t + \theta_1) \quad \text{and} \quad x_2^t = A_2 \sin(\nu_2 t + \theta_2), \tag{5-6}$$

we want to study

$$\begin{aligned}
\Lambda_0^2 &= \sup_{\substack{A_1^2+A_2^2=1 \\ \theta_1,\theta_2\in\mathbb{R}}} \frac{\min_{t\in[0,T]}\big(A_1^2\sin^2(\nu_1 t+\theta_1)+A_2^2\sin^2(\nu_2 t+\theta_2)\big)}{\max_{t\in[0,T]}\big(A_1^2\sin^2(\nu_1 t+\theta_1)+A_2^2\sin^2(\nu_2 t+\theta_2)\big)} \\
&= \sup_{\substack{\lambda\in[0,1] \\ \theta_1,\theta_2\in\mathbb{R}}} \frac{\min_{t_1\in[0,T]}\big((1-\lambda)\sin^2(\nu_1 t_1+\theta_1)+\lambda\sin^2(\nu_2 t_1+\theta_2)\big)}{\max_{t_2\in[0,T]}\big((1-\lambda)\sin^2(\nu_1 t_2+\theta_1)+\lambda\sin^2(\nu_2 t_2+\theta_2)\big)} \\
&= \sup_{\substack{\lambda\in[0,1] \\ \theta_1,\theta_2\in\mathbb{R}}} \frac{\min_{t_1\in[0,T]}\big((1-\lambda)\sin^2(\nu_1 t_1+\theta_1)+\lambda\sin^2(\nu_2 t_1+\theta_2)\big)}{1-\min_{t_2\in[0,T]}\big((1-\lambda)\cos^2(\nu_1 t_2+\theta_1)+\lambda\cos^2(\nu_2 t_2+\theta_2)\big)}.
\end{aligned}$$

In view of the periodicity in the variables θ_1 and θ_2, the supremum in the variables $\lambda, \theta_1, \theta_2$ is in fact a supremum over $(\lambda, \theta_1, \theta_2) \in [0, 1] \times [0, 2\pi] \times [0, 2\pi]$. A compactness and continuity argument shows that this supremum is attained for some triple $(\lambda, \theta_1, \theta_2)$. Furthermore, one can check that $\max_{\lambda,\theta_1,\theta_2} = \max_{\theta_1,\theta_2} \max_\lambda$. Thus we should simplify the problem first by considering fixed values for θ_1 and θ_2, and maximizing with respect to these variables ultimately. Therefore our objective is to compute

$$\Lambda_{\theta_1,\theta_2}^2 = \max_{\lambda\in[0,1]} \frac{\min_{t_1\in[0,T]}\big((1-\lambda)\sin^2(\nu_1 t_1+\theta_1)+\lambda\sin^2(\nu_2 t_1+\theta_2)\big)}{1-\min_{t_2\in[0,T]}\big((1-\lambda)\cos^2(\nu_1 t_2+\theta_1)+\lambda\cos^2(\nu_2 t_2+\theta_2)\big)}. \tag{5-7}$$

We can further simplify this by rewriting in more pleasant terms the minima in the numerator and the denominator. It relies on the following fact.

Step I: *Simplification of the optimization problem.* The minimum we want to estimate involves a sum of two squared sine functions that oscillate at different frequencies. Intuitively, it looks reasonable that the minimum of such a sum is attained between two zeroes that achieve the minimal distance between a zero of the first sine function, and a zero of the second. This is a motivation to introduce

$$d_0 = d_0(\theta_1, \theta_2) = \frac{4pq}{T} \min_{\substack{\sin(\nu_j t_j+\theta_j)=0 \\ j=1,2}} |t_1 - t_2|. \tag{5-8}$$

It is indeed a minimum, and not only an infimum, thanks to the rational ratio between ν_1 and ν_2, or equivalently, thanks to the periodicity of the Hamiltonian flow. We can give an explicit expression of this quantity reasoning as follows: the numbers t_1 and t_2 are such that $\sin(\nu_j t_j + \theta_j) = 0$, $j = 1, 2$, if and only if there exist two integers k_1 and k_2 such that

$$\nu_j t_j + \theta_j = k_j \pi.$$

Therefore

$$|t_1 - t_2| = \left|\pi\Big(\frac{k_1}{\nu_1} - \frac{k_2}{\nu_2}\Big) - \Big(\frac{\theta_1}{\nu_1} - \frac{\theta_2}{\nu_2}\Big)\right| = \frac{T}{2pq}\left|(k_1 p - k_2 q) - \Big(p\frac{\theta_1}{\pi} - q\frac{\theta_2}{\pi}\Big)\right|.$$

Yet since p and q are coprime integers, it follows from Bézout's identity that $k_1 p - k_2 q$ can take any value in $\mathbb{Z}$ when we vary k_1 and k_2. We deduce that

$$d_0 = 2\,\mathrm{dist}\Big(p\frac{\theta_1}{\pi} - q\frac{\theta_2}{\pi}, \mathbb{Z}\Big) = \mathrm{dist}\Big(p\frac{\theta_1}{\pi/2} - q\frac{\theta_2}{\pi/2}, 2\mathbb{Z}\Big).$$

Incidentally, this expression implies that $d_0 \in [0, 1]$. Now we claim that

$$\min_{t\in[0,T]}\big((1-\lambda)\sin^2(\nu_1 t+\theta_1)+\lambda\sin^2(\nu_2 t+\theta_2)\big) = \min_{s\in[0,1]}\Big((1-\lambda)\sin^2\Big(\frac{\pi/2}{p}sd_0\Big)+\lambda\sin^2\Big(\frac{\pi/2}{q}(1-s)d_0\Big)\Big). \tag{5-9}$$

This amounts to proving that the minimum in t in the left-hand side of (5-9) is attained between two zeros t_1, t_2 of $\sin(\nu_1 t+\theta_1)$ and $\sin(\nu_2 t+\theta_2)$ such that $|t_2 - t_1| = Td_0/4pq$. We first show that the minimum in s (in the right-hand side) is less than the minimum in t (in the left-hand side). To do so, we pick $t_0 \in [0, T]$ that attains the minimum in t. We choose t_j two zeroes of $\sin(\nu_j t+\theta_j)$ respectively, $j = 1, 2$, that are the closest possible to t_0. Due to periodicity, they satisfy $|t_j - t_0| \le \pi/(2\nu_j)$. That t_0 attains the minimum means that it is a critical point of the function

$$F : t \mapsto (1-\lambda)\sin^2(\nu_1 t+\theta_1)+\lambda\sin^2(\nu_2 t+\theta_2) = (1-\lambda)\sin^2(\nu_1(t-t_1))+\lambda\sin^2(\nu_2(t-t_2)). \tag{5-10}$$

Classical trigonometry formulae then yield

$$(1-\lambda)\nu_1\sin(2\nu_1(t_0-t_1))+\lambda\nu_2\sin(2\nu_2(t_0-t_2)) = F'(t_0) = 0. \tag{5-11}$$

Recalling that $|2\nu_j(t_0-t_j)| \le \pi$, we see that $\sin(2\nu_j(t_0-t_j))$ is of the same sign as $t_0 - t_j$, thus leading to the condition that

$$(t_0-t_1)(t_0-t_2) \le 0,$$

or in other words, t_0 lies between t_1 and t_2. Let $s_0 \in [0, 1]$ be such that $t_0 = (1-s_0)t_1 + s_0 t_2$. We obtain

$$\begin{aligned} F(t_0) &= (1-\lambda)\sin^2(\nu_1(t_0-t_1))+\lambda\sin^2(\nu_2(t_0-t_2)) \\ &= (1-\lambda)\sin^2(\nu_1 s_0(t_2-t_1))+\lambda\sin^2(\nu_2(1-s_0)(t_1-t_2)). \end{aligned}$$

We finally use that $|t_1 - t_2| \ge Td_0/4pq$ and the monotonicity of the sine function on $[0, \pi/2]$ to deduce one inequality in (5-9), namely:

$$\min_{t\in[0,T]} F(t) \ge \min_{s\in[0,1]}\Big((1-\lambda)\sin^2\Big(\frac{\pi/2}{p}sd_0\Big)+\lambda\sin^2\Big(\frac{\pi/2}{q}(1-s)d_0\Big)\Big). \tag{5-12}$$

To check the converse inequality, we proceed as follows: we pick t_1 and t_2, zeroes of $\sin(\nu_j t+\theta_j)$ respectively, that satisfy $|t_1 - t_2| = Td_0/4pq$. Denote by J the closed interval with endpoints t_1, t_2. Let $t_0 \in J$ be a point where F restricted to J attains its minimum. Then introducing a parameter $s \in [0, 1]$ such that $t = (1-s)t_1 + st_2$, we obtain

$$F(t_0) \le F(t) = (1-\lambda)\sin^2\Big(\frac{\pi/2}{p}sd_0\Big)+\lambda\sin^2\Big(\frac{\pi/2}{q}(1-s)d_0\Big)$$

for all $s \in [0, 1]$. This results in

$$\min_{t\in[0,T]} F(t) \le \min_{s\in[0,1]}\Big((1-\lambda)\sin^2\Big(\frac{\pi/2}{p}sd_0\Big)+\lambda\sin^2\Big(\frac{\pi/2}{q}(1-s)d_0\Big)\Big),$$

which shows together with (5-12) that (5-9) is true. We observe in the definition of $\Lambda_{\theta_1,\theta_2}$ (see (5-7)) that a similar minimum is involved with cosine functions instead of sine functions. To reduce to the case of sine functions and use (5-9), we simply recall that $\cos(x) = \sin(x+\pi/2)$. We obtain

$$\begin{aligned}\min_{t\in[0,T]}\bigl((1-\lambda)\cos^2(\nu_1 t+\theta_1)+\lambda\cos^2(\nu_2 t+\theta_2)\bigr)\\ &= \min_{t\in[0,T]}\Bigl((1-\lambda)\sin^2\Bigl(\nu_1 t+\theta_1+\frac{\pi}{2}\Bigr)+\lambda\sin^2\Bigl(\nu_2 t+\theta_2+\frac{\pi}{2}\Bigr)\Bigr)\\ &= \min_{s\in[0,1]}\Bigl((1-\lambda)\sin^2\Bigl(\frac{\pi/2}{p}sd_{\pi/2}\Bigr)+\lambda\sin^2\Bigl(\frac{\pi/2}{q}(1-s)d_{\pi/2}\Bigr)\Bigr),\end{aligned}$$

where we set (recall the definition of d_0 in (5-8))

$$d_{\pi/2}=d_{\pi/2}(\theta_1,\theta_2)=d_0\Bigl(\theta_1+\frac{\pi}{2},\theta_2+\frac{\pi}{2}\Bigr)=\operatorname{dist}\Bigl(p\frac{\theta_1}{\pi/2}-q\frac{\theta_2}{\pi/2}+p-q,2\mathbb{Z}\Bigr).$$

Depending on whether p and q have the same parity, we can state that

$$d_{\pi/2}=\begin{cases} d_0 & \text{if } p-q\equiv 0 \pmod 2,\\ 1-d_0 & \text{if } p-q\equiv 1 \pmod 2.\end{cases}$$

With this at hand, we can rewrite $\Lambda^2_{\theta_1,\theta_2}$ defined in (5-7) as

$$\Lambda^2_{\theta_1,\theta_2}=\max_{\lambda\in[0,1]}\frac{\min_{s_1\in[0,1]}\Bigl((1-\lambda)\sin^2\Bigl(\frac{\pi/2}{p}s_1d_0\Bigr)+\lambda\sin^2\Bigl(\frac{\pi/2}{q}(1-s_1)d_0\Bigr)\Bigr)}{1-\min_{s_2\in[0,1]}\Bigl((1-\lambda)\sin^2\Bigl(\frac{\pi/2}{p}s_2d_{\pi/2}\Bigr)+\lambda\sin^2\Bigl(\frac{\pi/2}{q}(1-s_2)d_{\pi/2}\Bigr)\Bigr)}.$$

Step II: *Computation of* $\Lambda^2_{\theta_1,\theta_2}$. We set, for any $\lambda\in[0,1]$ and $s\in[0,1]$,

$$g_\lambda(s)=g_{\lambda,d_0}(s)=(1-\lambda)\sin^2\Bigl(\frac{\pi/2}{p}sd_0\Bigr)+\lambda\sin^2\Bigl(\frac{\pi/2}{q}(1-s)d_0\Bigr).$$

In the perspective of computing $\Lambda^2_{\theta_1,\theta_2}$, we first show the following result.

Lemma 5.2. *One has*

$$\max_{\lambda\in[0,1]}\min_{s\in[0,1]}g_\lambda(s)=g_{\lambda_0}(s_0)=\sin^2\Bigl(\frac{\pi/2}{p+q}d_0\Bigr),$$

where $s_0=p/(p+q)$ *and* $\lambda_0=q/(p+q)$.

Proof. Firstly, we observe that $g_\lambda(s_0)$ is independent of λ, since it solves

$$\sin^2\Bigl(\frac{\pi/2}{p}s_0d_0\Bigr)=\sin^2\Bigl(\frac{\pi/2}{q}(1-s_0)d_0\Bigr).$$

This remarkable property implies that for any $\lambda\in[0,1]$, we have

$$\forall\lambda'\in[0,1],\qquad \min_{s\in[0,1]}g_{\lambda'}(s)\le g_{\lambda'}(s_0)=g_\lambda(s_0),$$

which results in

$$\max_{\lambda'\in[0,1]}\min_{s\in[0,1]}g_{\lambda'}(s)\le g_\lambda(s_0)\quad\forall\lambda\in[0,1].$$

Now we to show that the equality is reached when $\lambda = \lambda_0$ introduced in the statement. Noticing that $(1-\lambda_0)/p = \lambda_0/q = 1/(p+q)$, we obtain, using classical trigonometry formulae,

$$\begin{aligned}
g'_{\lambda_0}(s) &= \frac{\pi}{2} d_0 \Big(2\frac{1}{p}(1-\lambda_0)\cos\Big(\frac{\pi/2}{p} s d_0\Big)\sin\Big(\frac{\pi/2}{p} s d_0\Big) - 2\frac{1}{q}\lambda_0 \cos\Big(\frac{\pi/2}{q}(1-s)d_0\Big)\sin\Big(\frac{\pi/2}{q}(1-s)d_0\Big)\Big) \\
&= \frac{\pi/2}{p+q} d_0 \Big(\sin\Big(\frac{\pi}{p} s d_0\Big) - \sin\Big(\frac{\pi}{q}(1-s)d_0\Big)\Big) \\
&= \frac{\pi}{p+q} d_0 \cos\Big(\frac{\pi}{2} d_0\Big(\frac{s}{p} + \frac{1-s}{q}\Big)\Big) \sin\Big(\frac{\pi}{2} d_0\Big(\frac{s}{p} - \frac{1-s}{q}\Big)\Big) \\
&= \frac{\pi}{p+q} d_0 \cos\Big(\frac{\pi}{2} d_0\Big(\frac{s}{p} + \frac{1-s}{q}\Big)\Big) \sin\Big(\frac{\pi}{2} d_0\Big(\frac{1}{p} + \frac{1}{q}\Big)(s-s_0)\Big).
\end{aligned}$$

We observe that the cosine is always nonnegative for any $s \in [0, 1]$, because $d_0 \le 1$. As for the sine, it is nonpositive for $s \le s_0$ and nonnegative for $s \ge s_0$. We deduce that $g'_{\lambda_0}(s) \le 0$ on $[0, s_0]$ and $g'_{\lambda_0}(s) \ge 0$ on $[s_0, 1]$. Therefore, the minimum of g_{λ_0} is attained at s_0. □

Regarding the denominator in the definition of $\Lambda^2_{\theta_1,\theta_2}$, observing that λ_0 and s_0 in the above lemma do not dependent on d_0 or $d_{\pi/2}$, we find

$$\begin{aligned}
\min_{\lambda\in[0,1]}\Big(1 - \min_{s\in[0,1]}\Big((1-\lambda)\sin^2\Big(\frac{\pi/2}{q} s d_{\pi/2}\Big) + \lambda \sin^2\Big(\frac{\pi/2}{p}(1-s)d_{\pi/2}\Big)\Big)\Big) &= 1 - \max_{\lambda\in[0,1]}\min_{s\in[0,1]} g_{\lambda,d_{\pi/2}}(s) \\
&= 1 - g_{\lambda_0,d_{\pi/2}}(s_0) \\
&= \cos^2\Big(\frac{\pi/2}{p+q} d_{\pi/2}\Big).
\end{aligned}$$

This implies that λ_0 maximizes the minimum of the numerator and minimizes the maximum of the denominator at once. Moreover, when $\lambda = \lambda_0$, the minimum of the denominator and the maximum of the numerator are reached at a common value s_0. Therefore

$$\Lambda^2_{\theta_1,\theta_2} = \frac{\sin^2\Big(\frac{\pi/2}{p+q} d_0\Big)}{\cos^2\Big(\frac{\pi/2}{p+q} d_{\pi/2}\Big)}.$$

When p and q have the same parity, we have $d_{\pi/2} = d_0$, so that

$$\Lambda_{\theta_1,\theta_2} = \tan\Big(\frac{\pi/2}{p+q} d_0\Big). \tag{5-13}$$

When they do not have the same parity, then $d_{\pi/2} = 1 - d_0$ and we obtain

$$\begin{aligned}
\Lambda_{\theta_1,\theta_2} = \frac{\sin\Big(\frac{\pi/2}{p+q} d_0\Big)}{\cos\Big(\frac{\pi/2}{p+q}(1-d_0)\Big)} &= \sin\Big(\frac{\pi/2}{p+q}\Big) - \cos\Big(\frac{\pi/2}{p+q}\Big)\frac{\sin\Big(\frac{\pi/2}{p+q}(1-d_0)\Big)}{\cos\Big(\frac{\pi/2}{p+q}(1-d_0)\Big)} \\
&= \sin\Big(\frac{\pi/2}{p+q}\Big) - \cos\Big(\frac{\pi/2}{p+q}\Big)\tan\Big(\frac{\pi/2}{p+q}(1-d_0)\Big).
\end{aligned} \tag{5-14}$$

Recall that in the above formulae, the dependence on the phase shifts θ_1 and θ_2 is hidden in d_0. Thus it remains to optimize over these parameters θ_1, θ_2 to compute the quantity Λ_0 defined in (5-5). In the first case (5-13), we notice that $d_0 \le 1$, and that the equality is achieved for $\theta_1 = \pi/(2p)$ and $\theta_2 = 0$, for instance, so that

$$\Lambda_0 = \tan\Big(\frac{\pi/2}{p+q}\Big).$$

In the second case (5-14), the maximum is reached for $d_0 = 1$ as well, so that

$$\Lambda_0 = \sin\Big(\frac{\pi/2}{p+q}\Big).$$

The conclusion is that $\Lambda_0 = \Lambda(p/q)$, where the function Λ is the one defined in (1-26).

Step III: *Irrational case.* We now consider the case where $\nu_2/\nu_1 \notin \mathbb{Q}$. To obtain the sought result, it suffices to show that for any nonzero initial data $(A_1, A_2, \theta_1, \theta_2)$ of the flow (in action-angle coordinates), the projected trajectory

$$x_1^t = A_1 \sin(\nu_1 t + \theta_1), \quad x_2^t = A_2 \sin(\nu_2 t + \theta_2),$$

satisfies $\inf_{t\in\mathbb{R}} |x^t| = 0$. Let us consider the convergents $(p_j/q_j)_{j\in\mathbb{N}}$ of ν_2/ν_1, in irreducible form (see Remark 1.14). In view of (1-31), we introduce $\epsilon_j \in [-1, 1]$ such that

$$\frac{\nu_2}{\nu_1} = \frac{p_j}{q_j} + \frac{\epsilon_j}{q_j^2}. \tag{5-15}$$

We exhibit a sequence of times $(t_j)_{j\in\mathbb{N}}$ such that $x_1^{t_j} = 0$ and $x_2^{t_j} \to 0$ as $j \to \infty$. Since p_j and q_j are coprime integers, we can fix for each $j \in \mathbb{N}$ a pair of Bézout coefficients $(k_j, l_j) \in \mathbb{Z}^2$ such that

$$k_j p_j - l_j q_j = 1, \quad \text{with } |k_j| \le q_j. \tag{5-16}$$

Set

$$a_j = \left\lfloor \frac{\frac{\theta_1}{\pi} p_j - \frac{\theta_2}{\pi} q_j}{1 + \epsilon_j \frac{k_j}{q_j}} \right\rfloor \quad \text{and} \quad t_j = \frac{a_j k_j \pi - \theta_1}{\nu_1}. \tag{5-17}$$

By definition of t_j, we have $x_1^{t_j} = 0$. Thus it remains to check that $x_2^{t_j} \to 0$ as $j \to \infty$. We have

$$\begin{aligned}
\nu_2 t_j + \theta_2 - a_j l_j \pi &= \Big(\frac{p_j}{q_j} + \frac{\epsilon_j}{q_j^2}\Big) \times \pi \Big(a_j k_j - \frac{\theta_1}{\pi}\Big) + \theta_2 - a_j l_j \pi \\
&= \frac{p_j}{q_j} a_j k_j \pi - a_j l_j \pi + \frac{\epsilon_j}{q_j^2} \times \pi \Big(a_j k_j - \frac{\theta_1}{\pi}\Big) + \frac{\pi}{q_j}\Big(\frac{\theta_2}{\pi} q_j - \frac{\theta_1}{\pi} p_j\Big) \\
&= a_j \frac{\pi}{q_j} + a_j \epsilon_j \frac{k_j}{q_j^2}\pi - \frac{\theta_1 \epsilon_j}{q_j^2} - \frac{\pi}{q_j}\Big(\frac{\theta_1}{\pi} p_j - \frac{\theta_2}{\pi} q_j\Big) \\
&= \frac{\pi}{q_j}\Big(1 + \epsilon_j \frac{k_j}{q_j}\Big)\Big(a_j - \frac{\frac{\theta_1}{\pi} p_j - \frac{\theta_2}{\pi} q_j}{1 + \epsilon_j \frac{k_j}{q_j}}\Big) - \frac{\theta_1 \epsilon_j}{q_j^2}.
\end{aligned}$$

We used (5-15) to obtain the first equality and (5-16) for the third. In the last line, the two factors between parentheses are bounded. Indeed, the first factor is bounded by 2 since we chose k_j such that $|k_j| \leq q_j$ in (5-16) and because $|\epsilon_j| \leq 1$ from (5-15). The second factor is bounded by 1 due to the definition of a_j (5-17). Recalling that $q_j \to \infty$ since ν_2/ν_1 is irrational, we obtain $\nu_2 t_j + \theta_2 = a_j l_j \pi + o(1)$ as $j \to \infty$, hence $x_2^{t_j} \to 0$, and the proof of Lemma 1.12 is complete. □

5.3. *Proof of Proposition 1.11.* As we did in Section 5.2, we assume without loss of generality that A is diagonal, with eigenvalues ν_1^2 and ν_2^2 associated with the eigenvectors $(1, 0)$ and $(0, 1)$ in $\mathbb{R}^2$.

Step 1: *Construction of an equivalent shrunk observation set.* Recall that the sufficient condition of Theorem 1.3 implies observability from an "enlarged" observation set. This leads us to construct a shrunk set $\tilde{I} \subset I$, such that $\tilde{I}_R = \tilde{I} + (-R, R)$ is contained in I up to a bounded set, so that the same is true for the sets $\omega(\tilde{I})$ and $\omega(I)$. In the lemma below, when $I \subset \mathbb{R}_+$, we use the notation $I_R := \bigcup_{s \in I}(s - R, s + R)$.

Lemma 5.3 (shrunk observation set). *Let $I = \bigcup_n I_n$, where $I_n \subset \mathbb{R}_+$ are open intervals, with $|I_n| \to +\infty$ if the union is infinite. Then there exists a family of disjoint open intervals $(\tilde{J}_n)_n$ in $\mathbb{R}_+$ (with $|\tilde{J}_n| \to +\infty$ if there are infinitely many of them) such that the set $\tilde{I} = \bigcup_n \tilde{J}_n$ satisfies the following*:

(i) $\tilde{I} \subset I$.

(ii) *For any $R > 0$, the set $\tilde{I}_R \setminus I$ is bounded.*

(iii) *For any $R > 0$, one has $\kappa_\star(\tilde{I}) = \kappa_\star(\tilde{I}_R) = \kappa_\star(I) = \kappa_\star(I_R)$.*

Proof. Recall the definition of $\kappa_\star$ in (1-25). We write the open set I as a union of disjoint open intervals $I = \bigcup_n J_n$. Let us fix $R > 0$. We first deal with the case where there are only finitely many J_n's. If I is bounded, one has $\kappa_\star(I) = \kappa_\star(I_R) = 0$ and $\tilde{I} = \varnothing$ satisfies the conclusions of the lemma. If I is not bounded, then there is an index n_0 for which J_{n_0} is of the form $J_{n_0} = (a, +\infty)$. Then for any $R > 0$ the equality $\kappa_\star(I) = \kappa_\star(I_R) = 1$ and $\tilde{I} = J_{n_0}$ satisfies the conclusions of the lemma as well.

We now consider the case where there are infinitely many J_n's. By assumption, one has $|J_n| \to +\infty$ as $n \to \infty$. Writing $J_n = (a_n, b_n)$, with $a_n < b_n < \infty$, we define for any index n the interval

$$\tilde{J}_n = \left(a_n + \tfrac{1}{2}\sqrt{4 + \delta_n},\, b_n - \tfrac{1}{2}\sqrt{4 + \delta_n}\right), \quad \text{where } \delta_n = \min(a_n, b_n - a_n).$$

Since the J_n's are disjoint and $|J_n| = b_n - a_n \to +\infty$, we also have $a_n \to +\infty$, so $\delta_n \to +\infty$ too. Incidentally, one readily checks that $|\tilde{J}_n| \to +\infty$ as well. Thus, defining

$$\tilde{I} = \bigcup_n \tilde{J}_n,$$

we have $\tilde{I} \subset I$, namely the property (i), and given any $R > 0$, there are finitely many n's such that $R \geq \sqrt{\delta_n}/2$. This implies that the thickened set $\tilde{I}_R$ is contained in I modulo a bounded set, and hence we obtain (ii). The crucial point of this construction is claim (iii). As a consequence of the inclusions $\tilde{I} \subset \tilde{I}_R$ and $I \subset I_R$, we have $\kappa_\star(\tilde{I}) \leq \kappa_\star(\tilde{I}_R)$ and $\kappa_\star(I) \leq \kappa_\star(I_R)$. Moreover, in virtue of (ii), we can write $\tilde{I}_R = (\tilde{I}_R \cap I) \cup A$, where $A = \tilde{I}_R \setminus I$ is bounded. Since $(1/r)|A \cap [0, r]| \leq (1/r)|A| \to 0$ as $r \to +\infty$, one can check that $\kappa_\star(\tilde{I}_R) \leq \kappa_\star(\tilde{I}_R \cap I) \leq \kappa_\star(I)$. To sum up, we have proved so far that $\kappa_\star(\tilde{I}) \leq \kappa_\star(\tilde{I}_R) \leq \kappa_\star(I) \leq \kappa_\star(I_R)$.

Thus, in order to prove (iii), it remains to check that $\kappa_\star(I_R) \le \kappa_\star(\tilde{I})$. Unless we are in the straightforward case $\kappa_\star(I_R) = 0$, we pick $\kappa \in (0, \kappa_\star(I_R))$, so that by definition of $\kappa_\star$, we have

$$\exists c > 0, \exists r_0 > 0 : \forall r \ge r_0, \quad \frac{1}{r}|I_R \cap [\kappa r, r]| \ge c. \tag{5-18}$$

In the sequel, to simplify notation, we write $J_n^R = (J_n)_R$. Up to enlarging r_0, we can assume that for any index n such that $J_n^R \cap [\kappa r_0, +\infty) \ne \varnothing$, we have $\delta_n \ge 5 + 8R$ (recall that $\delta_n \to +\infty$). Fix an $r \ge r_0$. Then there is a finite (possibly empty) set of indices $\{n_k\}_k$ such that $J_{n_k}^R \subset [\kappa r, r]$. Assume first that

$$\frac{1}{r}\left|\bigcup_k J_{n_k}^R \cap [\kappa r, r]\right| = \frac{1}{r}\sum_k |J_{n_k}^R| \ge \frac{c}{2}. \tag{5-19}$$

Then

$$\begin{aligned}\frac{1}{r}|\tilde{I} \cap [\kappa r, r]| &\ge \frac{1}{r}\sum_k |\tilde{J}_{n_k}| = \frac{1}{r}\sum_k \bigl(|J_{n_k}^R| - \bigl(\sqrt{4+\delta_{n_k}} + 2R\bigr)\bigr) \\ &\ge \frac{1}{r}\sum_k \left(1 - \frac{\sqrt{4+\delta_{n_k}} + 2R}{\delta_{n_k} + 2R}\right)|J_{n_k}^R| \ge \frac{1}{r}\sum_k \left(1 - \sqrt{\frac{4}{\delta_n^2} + \frac{1}{\delta_{n_k}}} - \frac{2R}{\delta_n + 2R}\right)|J_{n_k}^R|.\end{aligned}$$

To obtain the second to last inequality, we used the fact that by definition of δ_n, we have $|J_n| \ge \delta_n$, which implies in particular that $|J_n^R| \ge \delta_n + 2R$. Using in the last line that $\delta_{n_k} \ge 5 + 8R$, together with (5-19), we obtain

$$\frac{1}{r}|\tilde{I} \cap [\kappa r, r]| \ge \left(1 - \sqrt{\frac{9}{25}} - \frac{1}{5}\right)\frac{1}{r}\sum_k |J_{n_k}^R| \ge \frac{1}{5} \times \frac{c}{2}. \tag{5-20}$$

Otherwise, if now (5-19) is not satisfied, then recalling (5-18), we have

$$\frac{1}{r}\left|\left(I_R \setminus \bigcup_k J_{n_k}^R\right) \cap [\kappa r, r]\right| \ge \frac{c}{2}.$$

Any interval $J_n^R \subset I_R \setminus \bigcup_k J_{n_k}^R$ intersecting $[\kappa r, r]$ must contain κr or r, otherwise it would satisfy $J_n^R \cap [\kappa r, r] = \varnothing$, or $J_n^R \subset (\kappa r, r)$ (the latter would imply that $n \in \{n_k\}_k$). Therefore, there are at most two such intervals. We deduce that there is an index $n_\star$ such that $J_{n_\star}^R \not\subset [\kappa r, r]$ but $J_{n_\star}^R \cap [\kappa r, r] \ne \varnothing$, with

$$\frac{1}{r}|J_{n_\star}^R \cap [\kappa r, r]| \ge \frac{c}{4}.$$

Writing $J_{n_\star}^R = (a_{n_\star} - R, b_{n_\star} + R)$, the fact that $J_{n_\star}^R \cap [\kappa r, r] \ne \varnothing$ imposes that $a_{n_\star} - R \le r$; hence $a_{n_\star} \le r + R$. Thus we obtain

$$\begin{aligned}\frac{1}{r}|\tilde{I} \cap [\kappa r, r]| \ge \frac{1}{r}|\tilde{J}_{n_\star} \cap [\kappa r, r]| &\ge \frac{1}{r}\bigl(|J_{n_\star}^R \cap [\kappa r, r]| - \sqrt{4+\delta_{n_\star}} - 2R\bigr) \\ &\ge \frac{c}{4} - \frac{\sqrt{4+r+R} + 2R}{r}.\end{aligned} \tag{5-21}$$

We used the fact that $\delta_{n_\star} \le a_{n_\star} \le r + R$ to obtain the last inequality. In view of the estimates (5-20) and (5-21), in any case we have

$$\frac{1}{r}|\tilde{I} \cap [\kappa r, r]| \ge \min\left(\frac{c}{10}, \frac{c}{4} - \frac{\sqrt{4+r+R} + 2R}{r}\right).$$

We conclude that

$$\liminf_{r\to+\infty} \frac{1}{r}|\tilde{I}\cap[\kappa r, r]| > 0.$$

Recalling that κ is any arbitrary number $< \kappa_\star(I_R)$, we finally get the desired converse inequality $\kappa_\star(\tilde{I}) \geq \kappa_\star(I_R)$. Thus (iii) is proved, which concludes the proof of the lemma. □

In the sequel, we will proceed as follows: to prove that $\kappa_\star(I) > \Lambda(\nu_2/\nu_1)$ is a sufficient condition to have observability from $\omega(I)$, we will check that the dynamical condition (1-7) of Theorem 1.3 is true in the smaller set $\omega(\tilde{I})$, where $\tilde{I}$ is given by Lemma 5.3. To show that it is also necessary, we will check that the condition (1-7) is violated in the larger set $\omega(I)_R = \omega(I_R)$ for any $R > 0$.

Step 2: *Geometric condition of observability for rationally dependent characteristic frequencies.* We investigate the validity of the dynamical condition (1-7) of Theorem 1.3. In the case where $\nu_2/\nu_1 \in \mathbb{Q}$, writing $\nu_2/\nu_1 = p/q$ as an irreducible fraction, the period of the Hamiltonian flow is given by $T_0 = 2\pi p/\nu_2 = 2\pi q/\nu_1$. We write for short $\Lambda = \Lambda(\nu_2/\nu_1)$ and $\kappa_\star = \kappa_\star(I)$. Our goal now is to reformulate the dynamical condition (1-7) using the area formula.

Proposition 5.4 (area formula [Evans and Gariepy 2015, Theorem 3.9]). *Let $J \subset \mathbb{R}$ be a bounded interval and let $\gamma : J \to \mathbb{R}^n$ be a Lipschitz curve. Then γ is differentiable at Lebesgue-almost every point in J and for any Borel set $E \subset \mathbb{R}^n$, one has*

$$\int_J \mathbf{1}_E(\gamma(t))|\gamma'(t)|\,dt = \int_{\operatorname{Im}\gamma\cap E} \#\gamma^{-1}(\{x\})\,d\mathcal{H}^1(x).$$

Here, $\operatorname{Im}\gamma = \{\gamma(t) : t \in J\} \subset \mathbb{R}^n$, $\#\gamma^{-1}(\{x\})$ *stands for the cardinality of the set* $\{t \in J : \gamma(t) = x\}$, *and* $\mathcal{H}^1$ *is the one-dimensional Hausdorff measure.*

We will apply this formula to a curve of the form $\gamma : t \mapsto |x^t| \in \mathbb{R}$ defined on $J = (0, T)$, where $t \mapsto (x^t, \xi^t)$ is a trajectory of the Hamiltonian flow. Calculations will involve the inverse Jacobian $|\gamma'(t)|^{-1}$. Using anisotropy[12] of the harmonic oscillator ($p \neq q$), we can check that the Jacobian vanishes only at a finite number of points.

Lemma 5.5. *Let $t \mapsto (x^t, \xi^t)$ be a trajectory of the Hamiltonian flow of an anisotropic harmonic oscillator, with initial datum $\rho_0 = (x_0, \xi_0)$. Then the curve $\gamma : \mathbb{R} \ni t \mapsto |x^t| \in \mathbb{R}_+$ is Lipschitz with constant $\sqrt{2p(\rho_0)}$. If $\rho_0 \neq 0$, then γ is of class C^∞ in $\mathbb{R}\setminus\{\gamma = 0\}$. Moreover, the set*

$$S_\gamma := \{t \in \mathbb{R} : \gamma(t) = 0 \ \text{ or } \ \gamma'(t) = 0\}$$

is locally finite, namely for any bounded interval $I \subset \mathbb{R}$, the set $S_\gamma \cap I$ is finite. In addition, for any bounded interval $I \subset \mathbb{R}$, one has

$$\exists k = k(I) \in \mathbb{N} : \forall s \in \mathbb{R}_+, \quad \#\gamma^{-1}(\{s\}) \cap I \leq k. \tag{5-22}$$

[12]In the excluded isotropic case ($p = q = 1$), one can choose (x^0, ξ^0) so that $|x^t|$ is constant, as we did in the proof of Proposition 1.9 (see Section 5.1). In such a situation, the set $\operatorname{Im}\gamma \subset \mathbb{R}_+$ is reduced to a point. This is a very singular situation, since the Jacobian $|\gamma'(t)|$ is identically zero.

Proof. That γ is Lipschitz follows from the inverse triangle inequality, the Hamilton equations (1-6) and the fact that $p(x,\xi)=V(x)+\frac{1}{2}|\xi|^2$ is preserved by the flow:

$$|\gamma(t_2)-\gamma(t_1)|\leq|x^{t_2}-x^{t_1}|\leq|t_2-t_1|\sup_{t\in\mathbb{R}}|\xi^t|\leq|t_2-t_1|\sqrt{2p(\rho_0)}.$$

From now on, we assume that $\rho_0\neq 0$. First notice that the set $\{\gamma=0\}$ is closed since γ is continuous. Given that $t\mapsto x^t$ is smooth, the curve γ is smooth in a neighborhood of any point $t\in\mathbb{R}\setminus\{\gamma=0\}$, so that $\gamma\in C^\infty(\mathbb{R}\setminus\{\gamma=0\})$. To show that S_γ is locally finite, it is sufficient to prove that it is closed and also discrete, namely that it is made of isolated points.[13]

We first check that it is closed by observing that the map $f:t\mapsto\gamma^2(t)=|x^t|^2$ belongs to $C^\infty(\mathbb{R})$ and that

$$S_\gamma=\{t\in\mathbb{R}:f'(t)=0\}. \tag{5-23}$$

To check this equality, we use the fact that $f'(t)=2\gamma(t)\gamma'(t)$ for all $t\in\mathbb{R}\setminus S_\gamma$. If $t\notin S_\gamma$, then it follows that $\gamma(t)\gamma'(t)\neq 0$. Conversely, if $t\in S_\gamma$, either $\gamma(t)\neq 0$, so that $\gamma'(t)=0$, in which case $f'(t)=2\gamma(t)\gamma'(t)=0$; or $\gamma(t)=0$, which implies that $x^t=0$, hence $f'(t)=2x^t\cdot\xi^t=0$. This justifies (5-23).

Thus it remains to show that S_γ is discrete. Let us compute the derivatives of f up to order 4:

$$f'(t)=2x^t\cdot\xi^t, \tag{5-24}$$

$$f^{(2)}(t)=2|\xi^t|^2-2x^t\cdot Ax^t, \tag{5-25}$$

$$f^{(3)}(t)=-4\xi^t\cdot Ax^t-2\xi^t\cdot Ax^t-2x^t\cdot A\xi^t=-8\xi^t\cdot Ax^t, \tag{5-26}$$

$$f^{(4)}(t)=8(|Ax^t|^2-\xi^t\cdot A\xi^t). \tag{5-27}$$

Let us write the Taylor expansion of f' at order 3 near $t_0\in\mathbb{R}$:

$$f'(t)=f'(t_0)+(t-t_0)f^{(2)}(t_0)+\frac{(t-t_0)^2}{2}f^{(3)}(t_0)+\frac{(t-t_0)^3}{6}f^{(4)}(t_0)+o((t-t_0)^3). \tag{5-28}$$

Suppose that $t_0\in S_\gamma$. Then $f'(t_0)=0$ in virtue of (5-23). If $f^{(2)}(t_0)\neq 0$, then (5-28) yields

$$|f'(t)|\geq\frac{|f^{(2)}(t_0)|}{2}|t-t_0|$$

for all t in a neighborhood U of t_0. In particular, $S_\gamma\cap U=\{t_0\}$, meaning that t_0 is isolated. Likewise, if $f^{(2)}(t_0)=0$ but $f^{(3)}(t_0)\neq 0$, then (5-28) leads to

$$|f'(t)|\geq\frac{|f^{(3)}(t_0)|}{4}|t-t_0|^2$$

in a neighborhood of t_0, so that t_0 is isolated again.

Now, if $f^{(2)}(t_0)=f^{(3)}(t_0)=0$, we show that necessarily $f^{(4)}(t_0)\neq 0$. In view of (5-24), (5-25), and (5-26), we have

$$x^{t_0}\cdot\xi^{t_0}=0,\quad |\xi^{t_0}|^2=x^{t_0}\cdot Ax^{t_0},\quad\text{and}\quad \xi^{t_0}\cdot Ax^{t_0}=0. \tag{5-29}$$

[13] If $S\subset\mathbb{R}$ is closed and discrete, the for any compact interval $I\subset\mathbb{R}$, the set $S\cap I$ is compact. Since S is discrete, the set $S\cap I$ can be covered by open sets containing at most one element of S. Then, extracting a finite subcovering shows that $S\cap I$ is finite.

The first and third equalities mean that $\xi^{t_0} \perp x^{t_0}$ and $\xi^{t_0} \perp Ax^{t_0}$. Moreover, the second equality ensures that $x^{t_0} \neq 0$ and $\xi^{t_0} \neq 0$, otherwise $(x^{t_0}, \xi^{t_0}) = (0,0)$; hence $\rho_0 = 0$. Since we are in two dimensions, we deduce that Ax^{t_0} and x^{t_0} are parallel, and therefore x^{t_0} is an eigenvector of A. Since $\xi^{t_0} \perp x^{t_0}$ and $\xi^{t_0} \neq 0$, we deduce that ξ^{t_0} is also an eigenvector, associated with a different eigenvalue since A has two distinct eigenvalues by assumption. We relabel ν_1 and ν_2 so that $Ax^{t_0} = \nu_x^2 x^{t_0}$ and $A\xi^{t_0} = \nu_\xi^2 \xi^{t_0}$. Plugging this into the second equality in (5-29) yields $|\xi^{t_0}|^2 = \nu_x^2 |x^{t_0}|^2$, from which we deduce that the fourth derivative (5-27) cannot vanish at t_0, given that the oscillator is anisotropic ($\nu_x \neq \nu_\xi$):

$$|Ax^{t_0}|^2 - \xi^{t_0} \cdot A\xi^{t_0} = \nu_x^4 |x^{t_0}|^2 - \nu_\xi^2 |\xi^{t_0}|^2 = \nu_x^2(\nu_x^2 - \nu_\xi^2)|x^{t_0}|^2 \neq 0.$$

Therefore (5-28) implies that

$$|f'(t)| \geq \frac{|f^{(4)}(t_0)|}{12} |t - t_0|^3$$

in a neighborhood of t_0, that is to say the critical point t_0 is again isolated. To sum up, the above argument shows that there exists a neighborhood U of t_0 such that $U \cap S_\gamma = \{t_0\}$, so S_γ is indeed a discrete set.

Now fix $I \subset \mathbb{R}$ a bounded interval. We have just shown that $n = \#(S_\gamma \cap I)$ is finite. To prove (5-22), we observe that the complement of S_γ in I is a union of at most $n+1$ open intervals in I, on which γ' does not vanish and has constant sign (use the intermediate value theorem). Therefore γ is one-to-one in each of these intervals. We infer that

$$\forall s \in \mathbb{R}, \quad \#\{t \in I : \gamma(t) = s\} \leq n + 1 + \#(S_\gamma \cap I) = 2n + 1. \qquad \square$$

Let us assume that $\kappa_\star \leq \Lambda$ and fix $R > 0$. Recalling that $\kappa_\star = \kappa_\star(I) = \kappa_\star(I_R)$ from (iii) in Lemma 5.3, we know that there exists a sequence $(r_n)_{n \in \mathbb{N}}$ tending to $+\infty$ along which

$$\frac{1}{r_n} |I_R \cap [\kappa_\star r_n, r_n]| \xrightarrow[n \to \infty]{} 0. \tag{5-30}$$

According to Step II of Section 5.2, considering actions

$$(A_1, A_2) = (\sqrt{1 - \lambda_0}, \sqrt{\lambda_0}) = \left(\sqrt{\frac{p}{p+q}}, \sqrt{\frac{q}{p+q}} \right)$$

and initial angles $(\theta_1, \theta_2) = (\pi/(2p), 0)$, one obtains a trajectory of the Hamiltonian flow $t \mapsto (x^t, \xi^t)$ such that

$$\min_{t \in [0,T]} |x^t| = \Lambda \max_{t \in [0,T]} |x^t|, \tag{5-31}$$

that is to say a trajectory that attains the supremum (5-5). Here T is any real number larger than the period of the flow T_0. In view of the homogeneity of degree 1 of the Hamiltonian flow, we know that $t \mapsto (cx^t, c\xi^t)$ is still a trajectory of the Hamiltonian flow, for any scalar $c \in \mathbb{R}$. Note that (5-31) above ensures that $|x^t|$ is bounded from below by a positive constant for all times. Therefore, Lemma 5.5 implies that the curve $\gamma : (0, T) \ni t \mapsto |x^t|$ is smooth. The corresponding set S_γ of Lemma 5.5 is nothing but $S_\gamma = \{\gamma' \neq 0\}$. A consequence of this lemma is that S_γ has vanishing measure. Thus we write

$$(0, T) \setminus S_\gamma = \bigcup_{N \in \mathbb{N}} B_N, \quad \text{where } B_N = \{t \in (0, T) : |\gamma'(t)| \geq 2^{-N}\}. \tag{5-32}$$

Fix an arbitrary $N \in \mathbb{N}$ and a scalar $c > 0$. Then we obtain

$$\int_{B_N} \mathbf{1}_{I_R}(c|x^t|)\,dt \le \frac{2^N}{c}\int_{B_N} \mathbf{1}_{I_R}(c\gamma(t))|c\gamma'(t)|\,dt = \frac{2^N}{c}\int_{c\gamma(B_N)} \mathbf{1}_{I_R}(s)\#\Big\{t \in (0,T) : \gamma(t) = \frac{s}{c}\Big\}\,ds$$
$$\le \frac{2^N k}{c}\int_{c\gamma(B_N)} \mathbf{1}_{I_R}(s)\,ds \le \frac{2^N k}{c}\int_{c\Lambda \max\gamma}^{c\max\gamma} \mathbf{1}_{I_R}(s)\,ds, \tag{5-33}$$

where the equality results from the area formula (Proposition 5.4) applied to $E = I_R$. The integer k is the one from Lemma 5.5 (5-22). The last inequality follows from the fact that $|x^t|$ spans the interval $[\Lambda \max\gamma, \max\gamma]$ by construction (recall (5-31)). Thus taking $c = c_n = r_n/\max\gamma$, with $(r_n)_{n\in\mathbb{N}}$ the sequence from (5-30), we obtain

$$\int_{B_N} \mathbf{1}_{I_R}(c_n\gamma(t))\,dt \le 2^N k(\max\gamma) \times \frac{1}{r_n}|I_R \cap [\Lambda r_n, r_n]| \xrightarrow[n\to\infty]{} 0,$$

by (5-30), since $\Lambda \ge \kappa_\star$. Now going back to (5-32), since the set S_γ is negligible, monotone convergence ensures that $|B_N| \to T$ as $N \to \infty$. We finally obtain that

$$\int_0^T \mathbf{1}_{\omega(I_R)}(c_n x^t)\,dt \le |(0,T)\setminus B_N| + \int_{B_N} \mathbf{1}_{I_R}(c_n|x^t|)\,dt = T - |B_N| + o(1)$$

as $n \to \infty$. We let $N \to \infty$ to conclude that the dynamical condition (1-7) is not fulfilled, namely

$$\liminf_{\rho\to\infty} \int_0^T \mathbf{1}_{\omega(I)_R\times\mathbb{R}^d}(\phi^t(\rho))\,dt = 0.$$

The parameter $R > 0$ is arbitrary. Therefore the necessary condition of Theorem 1.3 tells us that observability from $\omega(I)$ in time T does not hold, and $T \ge T_0$ itself is arbitrary.

We turn to the case where $\kappa_\star > \Lambda$. This time, we take $T = T_0$ to be the period of the Hamiltonian flow and check that the observability condition (1-7) holds in $\omega(\tilde{I})$. We pick $\kappa \in (\Lambda, \kappa_\star)$. In virtue of Lemma 5.3(iii), we have $\kappa_\star = \kappa_\star(I) = \kappa_\star(\tilde{I})$ so that

$$\exists c > 0, \exists r_0 > 0 : \forall r \ge r_0, \quad \frac{1}{r}|\tilde{I} \cap [\kappa r, r]| \ge c. \tag{5-34}$$

Let (x^t, ξ^t) be a trajectory of the Hamiltonian flow with initial datum ρ_0. One can estimate $\tilde{r} = \max|x^t|$ from below as follows: since the time t_0 at which the maximum is reached is also a (local) maximum of $|x^t|^2$, the second derivative satisfies

$$\frac{d^2}{dt^2}|x^t|^2{}_{|t=t_0} = 2|\xi^{t_0}|^2 - 2x^{t_0}\cdot Ax^{t_0} \le 0.$$

Thus

$$\tilde{r}^2 := |x^{t_0}|^2 \ge x^{t_0}\cdot\frac{A}{\|A\|}x^{t_0} \ge \frac{1}{\|A\|}\Big(\frac{1}{2}x^{t_0}\cdot Ax^{t_0} + \frac{1}{2}|\xi^{t_0}|^2\Big) = \frac{1}{\|A\|}p(\rho_0). \tag{5-35}$$

Provided $|\rho_0|$ is large enough so that $p(\rho_0) \ge \|A\|r_0^2$, we see in particular that $\tilde{r} \ge r_0$. Introduce $\gamma : (0,T) \ni t \mapsto |x^t|$. We know from Lemma 5.5 that γ is Lipschitz with constant $\sqrt{2p(\rho_0)} \le \sqrt{2\|A\|}\tilde{r}$

(this inequality is a consequence of (5-35) above). In particular, we have $|\gamma'(t)| \le \sqrt{2\|A\|}\tilde{r}$ outside the set S_γ from Lemma 5.5. Thus we can apply again the area formula (Proposition 5.4):

$$\begin{aligned}\int_0^T \mathbf{1}_{\omega(\tilde{I})}(x^t)\,dt = \int_0^T \mathbf{1}_{\tilde{I}}(|x^t|)\,dt &\ge (2\|A\|)^{-1/2}\frac{1}{\tilde{r}}\int_0^T \mathbf{1}_{\tilde{I}}(\gamma(t))|\gamma'(t)|\,dt \\ &= (2\|A\|)^{-1/2}\frac{1}{\tilde{r}}\int_{\gamma((0,T))} \mathbf{1}_{\tilde{I}}(s)\#\{t\in(0,T):\gamma(t)=s\}\,ds \\ &\ge (2\|A\|)^{-1/2}\frac{1}{\tilde{r}}\int_{\gamma((0,T))} \mathbf{1}_{\tilde{I}}(s)\,ds. \end{aligned} \tag{5-36}$$

This time, one has $\gamma((0,T)) \supset [\Lambda\tilde{r},\tilde{r}] \supset [\kappa\tilde{r},\tilde{r}]$ (by definition of Λ; see (5-5)). This means that

$$\int_0^T \mathbf{1}_{\omega(\tilde{I})}(x^t)\,dt \ge (2\|A\|)^{-1/2}\frac{1}{\tilde{r}}\int_{\kappa\tilde{r}}^{\tilde{r}} \mathbf{1}_{\tilde{I}}(s)\,ds \ge (2\|A\|)^{-1/2}c, \tag{5-37}$$

where the last inequality is due to (5-34) (recall that $\tilde{r} \ge r_0$). Therefore the dynamical condition (1-7) of Theorem 1.3 is satisfied. In fact, the explicit expression of the Hamiltonian flow in action-angle coordinates (5-6) shows that $|x^t|^2$ is $(T_0/2)$-periodic.[14] Therefore, setting $\tilde{c} := (2\|A\|)^{-1/2}c$, the dynamical condition (1-7) is equivalently satisfied in time $T_0/2 - \tilde{c}/4$:

$$\int_0^{T_0/2-\tilde{c}/4} \mathbf{1}_{\omega(\tilde{I})}(x^t)\,dt \ge \frac{1}{2}\int_0^{T_0} \mathbf{1}_{\omega(\tilde{I})}(x^t)\,dt - \frac{\tilde{c}}{4} \ge (2\|A\|)^{-1/2}\frac{c}{4}.$$

By Theorem 1.3, this implies that observability holds from $\omega(\tilde{I})_R \setminus K$ for some $R > 0$ and any compact set $K \subset \mathbb{R}^d$ in any time $> T_0/2 - \tilde{c}/4$, which in turn implies observability from $\omega(I)$ in virtue of Lemma 5.3(ii). Incidentally, the optimal observation time is strictly smaller than $T_0/2$.

Step 3: *Diophantine approximation in the irrational case.* We assume that $\nu_2/\nu_1 \in \mathbb{R}\setminus\mathbb{Q}$ and denote by p_j/q_j the reduced fraction expression of its convergents (see Remark 1.14). We investigate the validity of the dynamical condition (1-7) by approximating the trajectories of the "irrational" Hamiltonian flow by the trajectories of the "rational" Hamiltonian flow obtained by replacing ν_2/ν_1 with its convergent p_j/q_j. For instance, a projected trajectory of the irrational harmonic oscillator of the form

$$x_1^t = A_1\sin(\nu_1 t+\theta_1), \quad x_2^t = A_2\sin(\nu_2 t+\theta_2), \tag{5-38}$$

should be compared to

$$x_{j,1}^t = A_1\sin(\nu_1 t+\theta_1), \quad x_{j,2}^t = A_2\sin\left(\frac{p_j}{q_j}\nu_1 t+\theta_2\right), \tag{5-39}$$

which is a trajectory of the Hamiltonian flow of the (rational) harmonic oscillator with characteristic frequencies ν_1 and $p_j\nu_1/q_j$, whose classical Hamiltonian is

$$p_j(x,\xi) = \frac{1}{2}\left(\nu_1^2 x_1^2 + \frac{p_j^2}{q_j^2}\nu_1^2 x_2^2\right) + \frac{1}{2}(\xi_1^2+\xi_2^2).$$

[14] One can check that the projected trajectories of rational harmonic oscillators are invariant by point reflection with respect to the origin or axial symmetry with respect to some coordinate axis, depending on whether p and q have the same parity or not.

The distance between these two trajectories is

$$|x^t - x_j^t| = |x_2^t - x_{j,2}^t| \le A_2|\nu_2 - \frac{p_j}{q_j}\nu_1||t| \le A_2\frac{\nu_1|t|}{q_j^2}, \tag{5-40}$$

owing to the fact that the sine function is 1-Lipschitz and to the Diophantine approximation result (1-31). We already know from Lemma 1.12 that

$$\min_{t\in[0,T_j]}|x_j^t| \le \Lambda_j \max_{t\in[0,T_j]}|x_j^t|, \quad \text{where } T_j = \frac{2\pi}{\nu_1}q_j, \ \ \Lambda_j = \Lambda\left(\frac{p_j}{q_j}\right).$$

The time T_j is the period of the flow of the rational harmonic oscillator with characteristic frequencies ν_1 and $p_j\nu_1/q_j$. Let us set

$$m_j = \min_{t\in\mathbb{R}}|x_j^t| \quad \text{and} \quad M_j = \max_{t\in\mathbb{R}}|x_j^t|. \tag{5-41}$$

Although the trajectory $t \mapsto x_j^t$ is T_j-periodic, it will be convenient to compare x_j^t and x^t on smaller times. Then in view of (5-40), on the time interval $[0, \eta T_j]$, where $\eta \in (0, 1]$, the norm $|x^t|$ spans an interval J_j^η such that

$$J_j^\eta \subset \left[m_j - A_2\eta\frac{2\pi}{q_j}, M_j + A_2\eta\frac{2\pi}{q_j}\right], \tag{5-42}$$

and if $\eta = 1$, since $|x_j^t|$ attains m_j and M_j on the time interval $[0, T_j]$, we have

$$\left[m_j + A_2\frac{2\pi}{q_j}, M_j - A_2\frac{2\pi}{q_j}\right] \subset J_j^1. \tag{5-43}$$

So now, according to the value of $\kappa_\star$, we check whether the dynamical condition (1-7) of Theorem 1.3 is satisfied, using the area formula.

Step 4: *Geometric condition of observability for rationally independent characteristic frequencies.* Take $\eta = 1$, that is, we consider a whole period of the rational Hamiltonian flow. We first establish a lower bound on the time spent by $t \mapsto x^t$ in $\omega(\tilde{I})$. We consider $\kappa_\star > 0$ here. From Lemma 5.5, we know that $\gamma : (0, T_j) \ni t \mapsto |x^t|$ is Lipschitz with constant $\sqrt{2p(\rho_0)}$. Yet, similarly to (5-35), we have

$$p(\rho_0) \le \|A\|\widetilde{M}_j^2, \quad \text{where } \widetilde{M}_j = \max_{t\in[0,T_j]}|x^t|,$$

so that γ is Lipschitz with constant $\sqrt{2\|A\|}\widetilde{M}_j$. Applying the area formula (Proposition 5.4), we obtain as in (5-36) the lower bound

$$\int_0^{T_j} \mathbf{1}_{\omega(\tilde{I})}(x^t)\,dt \ge (2\|A\|)^{-1/2}\frac{1}{\widetilde{M}_j}\int_{J_j^1}\mathbf{1}_{\tilde{I}}(s)\,ds,$$

and in view of (5-43) and (5-42) with $\eta = 1$, we deduce that

$$\int_0^{T_j} \mathbf{1}_{\omega(\tilde{I})}(x^t)\,dt \ge \frac{(2\|A\|)^{-1/2}}{M_j + A_2\frac{2\pi}{q_j}}\int_{m_j+2\pi A_2/q_j}^{M_j-2\pi A_2/q_j}\mathbf{1}_{\tilde{I}}(s)\,ds.$$

Observing that $A_2 \leq M_j$ and that $m_j \leq \Lambda_j M_j$, we obtain

$$\int_0^{T_j} \mathbf{1}_{\omega(\tilde{I})}(x^t)\,dt \geq \frac{(2\|A\|)^{-1/2}}{M_j\big(1+\frac{2\pi}{q_j}\big)} \int_{M_j(\Lambda_j+2\pi/q_j)}^{M_j(1-2\pi/q_j)} \mathbf{1}_{\tilde{I}}(s)\,ds. \tag{5-44}$$

Setting

$$r = M_j\left(1-\frac{2\pi}{q_j}\right) \quad \text{and} \quad \tilde{\Lambda}_j = \frac{\Lambda_j+\frac{2\pi}{q_j}}{1-\frac{2\pi}{q_j}}, \tag{5-45}$$

we can write the lower bound in (5-44) under the form

$$\int_0^{T_j} \mathbf{1}_{\omega(\tilde{I})}(x^t)\,dt \geq (2\|A\|)^{-1/2}\frac{1-\frac{2\pi}{q_j}}{1+\frac{2\pi}{q_j}} \times \frac{1}{r}\int_{\tilde{\Lambda}_j r}^{r} \mathbf{1}_{\tilde{I}}(s)\,ds. \tag{5-46}$$

We assume that $q_j > 2\pi$ so that $r > 0$, which is the case for j large enough since $q_j \to \infty$. The above estimate (5-46) is valid for any trajectory of the (irrational) Hamiltonian flow with initial datum $\rho_0 \neq 0$. In addition, we remark that M_j defined in (5-41) tends to infinity as $\rho_0 \to \infty$, so that r defined in (5-45) tends to $+\infty$ as $\rho_0 \to \infty$ too. Thus (5-46) leads to

$$\liminf_{\rho\to\infty}\int_0^{T_j} \mathbf{1}_{\omega(\tilde{I})\times\mathbb{R}^d}(\phi^t(\rho))\,dt \geq (2\|A\|)^{-1/2}\frac{1-\frac{2\pi}{q_j}}{1+\frac{2\pi}{q_j}} \times \liminf_{r\to+\infty}\frac{1}{r}|\tilde{I}\cap[\tilde{\Lambda}_j r, r]|. \tag{5-47}$$

In order to deduce a positive lower bound, it suffices that $\tilde{\Lambda}_j < \kappa_\star = \kappa_\star(\tilde{I}) = \kappa_\star(I)$. This is achieved provided $q_j \geq 6\pi/\kappa_\star \geq 6\pi$. Indeed, under this condition, we have on the one hand

$$\frac{1-\frac{2\pi}{q_j}}{1+\frac{2\pi}{q_j}} \geq \frac{1-\frac{2\pi}{6\pi}}{1+\frac{2\pi}{6\pi}} = \frac{1}{2}, \tag{5-48}$$

and on the other hand, recalling the definition of $\tilde{\Lambda}_j$ in (5-45), the formula (1-26) for Λ_j, and using that $\sin x \leq x$ and $\tan x \leq 4x/\pi$ for $x \in [0, \pi/4]$, we obtain

$$\tilde{\Lambda}_j \leq \frac{\frac{4}{\pi}\times\frac{\pi/2}{p_j+q_j}+\frac{2\pi}{q_j}}{1-\frac{2\pi}{6\pi}} \leq 3\frac{1+\pi}{q_j} \leq \frac{1}{2}\left(\frac{1}{\pi}+1\right)\kappa_\star < \kappa_\star. \tag{5-49}$$

Now we turn to the upper bound on the time spent by projected trajectories of the (irrational) Hamiltonian flow in $\omega(I)_R$, for a fixed $R > 0$. We consider $\kappa_\star \in [0,1]$ arbitrary now, with the convention $1/\kappa_\star = +\infty$ if $\kappa_\star = 0$. We go back to $\eta \in (0,1]$. We select a curve $t \mapsto (x_j^t, \xi_j^t)$ of the rational flow that maximizes the ratio $\min_t |x_j^t| / \max_t |x_j^t|$, namely that satisfies

$$m_j = \min_{t\in[0,T_j]}|x_j^t| = \Lambda_j \max_{t\in[0,T_j]}|x_j^t| = \Lambda_j M_j.$$

This curve is of the form (5-39) for well-chosen action and angle variables. We consider $t \mapsto (x^t, \xi^t)$ the corresponding trajectory of the irrational flow given by (5-38), that is the integral curve obtained by substituting ν_2 for $p_j\nu_1/q_j$ in (5-39). Notice that this trajectory depends on j. We still write

$\gamma(t)=|x^t|$. By Lemma 5.5, we know that it is a Lipschitz map and that there exists an integer k_0 such that $\#\gamma^{-1}(s)\cap[0,T_j]\le k_0$ for all $s\in\mathbb{R}_+$. Reproducing the computation (5-33), we find

$$\int_{B_N}\mathbf{1}_{I_R}(c|x^t|)\,dt\le\frac{2^Nk_0}{c}\int_{c\Lambda_j\max\gamma}^{c\max\gamma}\mathbf{1}_{I_R}(s)\,ds\le\frac{2^Nk_0}{c}\int_{cJ_j^\eta}\mathbf{1}_{I_R}(s)\,ds,$$

where we recall that the parameter $c>0$ is an arbitrary scaling factor, and B_N is defined similarly to (5-32) by

$$B_N=\{t\in[0,\eta T_j]:|\gamma'(t)|\ge 2^{-N}\}.$$

In view of (5-42), this leads to

$$\int_{B_N}\mathbf{1}_{I_R}(c|x^t|)\,dt\le\frac{2^Nk_0}{c}\int_{c(m_j-2\pi A_2\eta/q_j)}^{c(M_j+2\pi A_2\eta/q_j)}\mathbf{1}_{I_R}(s)\,ds.$$

As we did before in (5-44), we use the fact that $A_2\le M_j$, together with $m_j=\Lambda_jM_j$ (the equality is important here) to obtain

$$\int_{B_N}\mathbf{1}_{I_R}(c|x^t|)\,dt\le\frac{2^Nk_0}{c}\int_{cM_j(\Lambda_j-2\pi\eta/q_j)}^{cM_j(1+2\pi\eta/q_j)}\mathbf{1}_{I_R}(s)\,ds.$$

Defining now

$$r=M_j\left(1+\eta\frac{2\pi}{q_j}\right)\quad\text{and}\quad\tilde\Lambda_j=\frac{\Lambda_j-\eta\frac{2\pi}{q_j}}{1+\eta\frac{2\pi}{q_j}},$$

we end up with

$$\int_{B_N}\mathbf{1}_{I_R}(c|x^t|)\,dt\le 2^Nk_0M_j\left(1+\eta\frac{2\pi}{q_j}\right)\frac{1}{cr}\int_{\tilde\Lambda_jcr}^{cr}\mathbf{1}_{I_R}(s)\,ds.$$

We finally prove that this upper bound tends to zero along a well-chosen sequence of parameters c provided $\tilde\Lambda_j\ge\kappa_\star$. This is fulfilled whenever $q_j\le\delta/\kappa_\star$, for a small enough constant δ. To see this, we can use that $\tan x\ge x$ and $\sin x\ge 2x/\pi$ on $[0,\pi/2]$ to control Λ_j from below by $1/(p_j+q_j)$. Then (1-31) leads to $p_j/q_j\le\nu_2/\nu_1+1$, which yields

$$\Lambda_j\ge\frac{1}{p_j+q_j}\ge\frac{1}{q_j\left(2+\frac{\nu_2}{\nu_1}\right)}=:\frac{C}{q_j}.$$

Assuming that $\eta<C/(2\pi)\le 1$, we obtain

$$\tilde\Lambda_j\ge\frac{\frac{C-2\pi\eta}{q_j}}{1+\eta\frac{2\pi}{q_j}}\ge\frac{1}{q_j}\times\frac{C-2\pi\eta}{1+2\pi\eta}\ge\frac{C-2\pi\eta}{\delta(1+2\pi\eta)}\kappa_\star.$$

This yields $\tilde\Lambda_j\ge\kappa_\star$ if δ is small enough, so that by definition of $\kappa_\star$, letting $c\to+\infty$, we obtain

$$\begin{aligned}\liminf_{c\to+\infty}\int_0^{\eta T_j}\mathbf{1}_{\omega(I_R)}(cx^t)\,dt&\le|[0,\eta T_j]\setminus B_N|+2^Nk_0M_j\left(1+\eta\frac{2\pi}{q_j}\right)\liminf_{c\to+\infty}\frac{1}{cr}\int_{\tilde\Lambda_jcr}^{cr}\mathbf{1}_{I_R}(s)\,ds\\&=\eta T_j-|B_N|,\end{aligned}\tag{5-50}$$

which tends to zero as $N\to\infty$.

The general conclusion is the following: if $\kappa_\star>0$ and $j\in\mathbb{N}$ is such that $q_j\ge 6\pi/\kappa_\star$, we know by (5-49) that $\tilde\Lambda_j<\kappa_\star$, so that by definition of $\kappa_\star$, the estimate (5-47), together with (5-48), proves that the dynamical condition (1-7) of Theorem 1.3 holds for $\omega(\tilde{I})$ in time $T_j = 2\pi q_j/\nu_1$. If on the contrary $\kappa_\star \in [0, 1]$, and $q_j \leq \delta/\kappa_\star$ for some $\delta > 0$ depending only on ν_2/ν_1, then, from (5-50), the dynamical condition (1-7) is violated in $\omega(I)_R$ for any $R > 0$ on the time interval $[0, \eta T_j]$, where $\eta > 0$ depends only on ν_2/ν_1 again. Theorem 1.3 then implies that the Schrödinger equation is observable from $\omega(I)$ if and only if $\kappa_\star > 0$. If indeed $\kappa_\star > 0$, then the optimal observation time $T_\star = T_\star(\omega(I))$ is controlled as follows: there exist constants $C, c > 0$ such that

$$c q_{j_1} \leq T_\star \leq C q_{j_2}, \tag{5-51}$$

where j_1 is the largest index such that $q_j \leq \delta/\kappa_\star$ and j_2 is the smallest index such that $q_j \geq 6\pi/\kappa_\star$.

To go from (5-51) to the desired estimate (1-29) in the case where ν_2/ν_1 is Diophantine, we use the fact that the irrationality exponent τ, defined in (1-27), is related to the growth of the q_j's. This comes from the formula

$$\tau(\mu) = 1 + \limsup_{j\to\infty} \frac{\log q_{j+1}}{\log q_j}$$

(see [Durand 2015, Proposition 1.8] or [Sondow 2004, Theorem 1]). When τ is finite, we deduce in particular that for any $\varepsilon > 0$, we have, for any j large enough,

$$\frac{\log q_{j+1}}{\log q_j} \leq \tau - 1 + \varepsilon,$$

which leads to the existence of a constant $C_\varepsilon > 0$ such that

$$q_{j+1} \leq C_\varepsilon {q_j}^{\tau-1+\varepsilon} \quad \forall j \in \mathbb{N}.$$

By definition of the indices j_1 and j_2, we obtain

$$\frac{\delta}{\kappa_\star} \leq q_{j_1+1} \leq C_\varepsilon {q_{j_1}}^{\tau-1+\varepsilon} \quad \text{and} \quad q_{j_2} \leq C_\varepsilon {q_{j_2-1}}^{\tau-1+\varepsilon} \leq C_\varepsilon \Big(\frac{6\pi}{\kappa_\star}\Big)^{\tau-1+\varepsilon}.$$

Plugging this into (5-51), we finally deduce (1-29). This concludes the proof of Proposition 1.11. □

Appendix A: Reduction to a weaker observability inequality

The following proposition shows that $\mathrm{Obs}(\omega, T)$ is equivalent to a similar inequality with a remainder involving a compact operator. The argument goes back to Bardos, Lebeau and Rauch [Bardos et al. 1992]. This reformulation of the problem paves the way for the use of microlocal analysis: we are interested in the propagation of high-energy modes through the Schrödinger evolution, discarding anything that is microlocalized near a fixed energy sublevel $\{p \leq \mathrm{cst}\}$. An alternative route could be to slice the phase space according to energy layers of the Hamiltonian $p(x, \xi) = V(x) + \frac{1}{2}|\xi|^2$; see [Lebeau 1992; Burq and Zworski 2012; Anantharaman and Macià 2014].

Proposition A.1. *Suppose P is a self-adjoint operator with compact resolvent, and let B be a bounded operator on $L^2(\mathbb{R}^d)$ satisfying the unique continuation property*:

$$\textit{for any eigenfunction } u \textit{ of } P, \quad Bu = 0 \implies u = 0. \tag{A-1}$$

Let $T_0 > 0$ and assume there exists a compact self-adjoint operator K such that

$$\exists C_0 > 0 : \forall u \in L^2(\mathbb{R}^d), \quad \|u\|_{L^2}^2 \le C_0 \int_0^{T_0} \|Be^{-itP}u\|_{L^2}^2 \, dt + (u, Ku)_{L^2}. \tag{A-2}$$

Then for every $T > T_0$, there exists $C > 0$ such that

$$\forall u \in L^2(\mathbb{R}^d), \quad \|u\|_{L^2}^2 \le C \int_0^{T} \|Be^{-itP}u\|_{L^2}^2 \, dt.$$

Remark A.2. The operators of the form $P = V(x) - \frac{1}{2}\Delta$ that we consider, with V subject to Assumption 1.1, satisfy the unique continuation property of the statement when B is the multiplication by the indicator function of a nonempty open set. See [Le Rousseau et al. 2022, Theorem 5.2].

Proof. Let us introduce, for any $S \in \mathbb{R}$,

$$A_S = \int_0^S e^{itP} B^* B e^{-itP} \, dt,$$

and denote by $\mathscr{I}_S$ its kernel (the space of so-called invisible solutions). One can check that

$$\mathscr{I}_S = \bigcap_{t \in [0,S]} \ker Be^{-itP} = \{u \in L^2(\mathbb{R}^d) : \forall t \in [0, S],\ Be^{-itP}u = 0\},$$

using the fact that $e^{itP} B^* B e^{-itP} \ge 0$ for all $t \in \mathbb{R}$ as operators, and that the map $t \mapsto Be^{-itP}$ is strongly continuous. The space $\mathscr{I}_S$ is a closed linear subspace of $L^2(\mathbb{R}^d)$, both for the strong and the weak topology (use for instance that A_S is a bounded operator). Moreover, one has the property that $S_1 \le S_2$ yields $\mathscr{I}_{S_1} \supset \mathscr{I}_{S_2}$. It implies that for any S, the set

$$\mathscr{I}_S^- = \bigcup_{S' > S} \mathscr{I}_{S'}$$

is also a linear subspace, contained in $\mathscr{I}_S$.

Step 1: $\mathscr{I}_{T_0}$ is finite-dimensional. This assertion is a consequence of the fact that K is coercive on $\mathscr{I}_{T_0}$, namely

$$\forall u \in \mathscr{I}_{T_0}, \quad \|u\|_{L^2} \le \|Ku\|_{L^2},$$

which follows directly from assumption (A-2) and the Cauchy–Schwarz inequality. Setting $W = \operatorname{Ran} K_{|\mathscr{I}_{T_0}}$, we deduce that $K : \mathscr{I}_{T_0} \to W$ is one-to-one and its inverse K^{-1} is bounded as an operator in $\mathscr{L}(W, \mathscr{I}_{T_0})$. Now denote by $\overline{B}_{\mathscr{I}_{T_0}}$ the closed unit ball of $\mathscr{I}_{T_0}$. Since $\mathscr{I}_{T_0}$ is strongly and weakly closed, the same holds for its closed unit ball as a subset of $L^2(\mathbb{R}^d)$. We deduce that $\overline{B}_{\mathscr{I}_{T_0}}$ is weakly compact. The compactness of K implies that $K(\overline{B}_{\mathscr{I}_{T_0}})$ is (strongly) compact in $L^2(\mathbb{R}^d)$. Since it is contained in W, it is compact in W. Therefore the fact that $K^{-1} : W \to \mathscr{I}_{T_0}$ is bounded implies that $\overline{B}_{\mathscr{I}_{T_0}} = K^{-1}(K(\overline{B}_{\mathscr{I}_{T_0}}))$ is compact. We deduce by the Riesz theorem that $\mathscr{I}_{T_0}$ is finite-dimensional.

Step 2: $\mathscr{I}_{T_0}^-$ is stable by P. Let us check that $\mathscr{I}_{T_0}^- \subset \operatorname{Dom} P$. Let $u \in \mathscr{I}_{T_0}^-$ and set

$$u_\epsilon = \frac{e^{-i\epsilon P}u - u}{\epsilon} \quad \forall \epsilon \ne 0.$$

By definition of $\mathcal{I}_{T_0}^-$, the function u belongs to $\mathcal{I}_{T_0+\epsilon_0}$ for some $\epsilon_0 > 0$, so that $u_\epsilon \in \mathcal{I}_{T_0}^-$ for any $\epsilon \in (0, \epsilon_0)$. Recall from the previous step that $\mathcal{I}_{T_0} \supset \mathcal{I}_{T_0}^-$ is finite-dimensional. We observe that $v \mapsto \|(P-i)^{-1}v\|_{L^2}$ is a norm on $\mathcal{I}_{T_0}$, so it is equivalent to the L^2 norm. Yet we see that

$$(P-i)^{-1}u_\epsilon = \frac{e^{-i\epsilon P}(P-i)^{-1}u - (P-i)^{-1}u}{\epsilon},$$

with $(P-i)^{-1}u \in \operatorname{Dom} P$, so that $(P-i)^{-1}u_\epsilon$ converges as $\epsilon \to 0$. Since

$$\forall \epsilon_1, \epsilon_2 \in (0, \epsilon_0), \quad \|u_{\epsilon_2} - u_{\epsilon_1}\|_{L^2} \le C\|(P-i)^{-1}u_{\epsilon_2} - (P-i)^{-1}u_{\epsilon_1}\|_{L^2},$$

we deduce that $(u_\epsilon)_\epsilon$ is a Cauchy sequence, hence it converges, which means that $u \in \operatorname{Dom} P$. Thus $\mathcal{I}_{T_0}^- \subset \operatorname{Dom} P$. It remains to see that $\lim_{\epsilon\to 0} u_\epsilon = -iPu$ belongs to $\mathcal{I}_{T_0}^-$, which is a consequence of the fact that $\mathcal{I}_{T_0}^-$ is finite-dimensional, hence closed.

Step 3: $\mathcal{I}_{T_0}^- = \{0\}$. This results from the unique continuation property (A-1). Indeed, we can argue as follows: from the previous steps, $\mathcal{I}_{T_0}^-$ is a finite-dimensional linear subspace of $L^2(\mathbb{R}^d)$ which is stable by the self-adjoint operator P. Therefore there exists a basis $(u_1, u_2, \dots, u_n)$ of $\mathcal{I}_{T_0}^-$ made of eigenvectors of P. By definition of $\mathcal{I}_S$, these eigenvectors satisfy in particular $Bu_j = 0$. So by the unique continuation result (A-1), we find that $\mathcal{I}_{T_0}^-$ must be trivial.

Step 4: Conclusion. Let $T > T_0$. We want to show that $A_T \ge c$ for some $c > 0$. To do this, it suffices to prove that A_T is invertible, because A_T is self-adjoint and $A_T \ge 0$. The assumption (A-2) implies that the self-adjoint operator $A_T + K$ is invertible, meaning that zero does not belong to its spectrum. Since K is compact and self-adjoint, we classically know that A_T has the same essential spectrum as $A_T + K$, so in particular zero is not in the essential spectrum of A_T. It is not an eigenvalue neither since $\ker A_T \subset \mathcal{I}_{T_0}^- = \{0\}$. Therefore A_T is invertible, and the conclusion follows. □

The following lemma is not related to the previous proposition. Still, it is worth stating it properly since we use it on several occasions throughout the article.

Lemma A.3. *Let $\omega \subset \mathbb{R}^d$ be measurable. Assume* $\operatorname{Obs}(\omega, T)$ *holds in some time $T > 0$ with a cost $C > 0$, namely*

$$\forall u \in L^2(\mathbb{R}^d), \quad \|u\|_{L^2(\mathbb{R}^d)}^2 \le C \int_0^T \|e^{-itP}u\|_{L^2(\omega)}^2 \, dt.$$

Then $\operatorname{Obs}(\omega, T-\varepsilon)$ *holds for any $\varepsilon < 1/C$.*

Proof. We use the fact that the propagator e^{-itP} is an isometry on $L^2(\mathbb{R}^d)$ to get

$$C\int_{T-\varepsilon}^T \|e^{-itP}u\|_{L^2(\omega)}^2 \, dt \le C\int_{T-\varepsilon}^T \|e^{-itP}u\|_{L^2(\mathbb{R}^d)}^2 \, dt = C\varepsilon\|u\|_{L^2(\mathbb{R}^d)}^2.$$

Thus we can absorb this term in the left-hand side of the observability inequality provided $C\varepsilon < 1$:

$$(1 - C\varepsilon)\|u\|_{L^2(\mathbb{R}^d)}^2 \le C\int_0^{T-\varepsilon} \|e^{-itP}u\|_{L^2(\omega)}^2 \, dt,$$

namely $\operatorname{Obs}(\omega, T-\varepsilon)$ holds with cost $C(1 - C\varepsilon)^{-1}$. □

Appendix B: Pseudodifferential operators

We recall below basics of the theory of pseudodifferential operators (see the textbooks [Hörmander 1985; Lerner 2010; Martinez 2002; Zworski 2012] for further details). We will also need a precise bound on the remainder of the pseudodifferential calculus and of the sharp Gårding inequality. This is why we reproduce the proofs of these results below.

B.1. ***Weyl quantization.*** Let $a \in \mathscr{S}(\mathbb{R}^{2d})$. We define the operator $\mathrm{Op}(a)$ acting on the Schwartz class $\mathscr{S}(\mathbb{R}^d)$ by

$$[\mathrm{Op}(a)u](x) = (2\pi)^{-d} \int_{\mathbb{R}^{2d}} e^{i(x-y)\cdot\xi} a\Big(\frac{x+y}{2}, \xi\Big) u(y)\, dy\, d\xi, \quad u \in \mathscr{S}(\mathbb{R}^d),\ x \in \mathbb{R}^d.$$

It is known that $\mathrm{Op}(a) : \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d)$ is continuous. The quantization Op extends to tempered distributions: for any $a \in \mathscr{S}'(\mathbb{R}^{2d})$, the operator $\mathrm{Op}(a) : \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d)$ is continuous.

B.2. ***Symbol classes.***

Definition B.1 (symbol classes). Let f be an order function.[15] Then the symbol class $S(f)$ is the set of functions $a \in C^\infty(\mathbb{R}^{2d})$ satisfying

$$\forall \alpha \in \mathbb{N}^{2d}, \exists C_\alpha > 0 : \forall \rho \in \mathbb{R}^{2d}, \quad |\partial^\alpha a(\rho)| \le C_\alpha f(\rho).$$

Collecting the best constants C_α for each α, the quantities

$$|a|^\ell_{S(f)} = \max_{|\alpha| \le \ell} C_\alpha, \quad \ell \in \mathbb{N},$$

are seminorms that turn the vector space $S(f)$ into a Fréchet space.

Any $a \in S(f)$ is a tempered distribution and yields a continuous linear operator $\mathrm{Op}(a) : \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d)$.

B.3. ***L^2-boundedness of pseudodifferential operators.***

Theorem B.2 (Calderón–Vaillancourt). *There exist constants $C_d, k_d > 0$ depending only on the dimension d such that, for any $a \in S(1)$, the operator $\mathrm{Op}(a)$ can be extended to a bounded operator on $L^2(\mathbb{R}^d)$ with the bound*

$$\|\mathrm{Op}(a)\|_{L^2 \to L^2} \le C_d |a|^{k_d}_{S(1)}.$$

B.4. ***Refined estimate in the pseudodifferential calculus.*** Let a_1, a_2 be two symbols. We have seen previously that the composition $\mathrm{Op}(a_1)\,\mathrm{Op}(a_2)$ makes sense as an operator on the Schwartz space. This operator is also a pseudodifferential operator, whose symbol is denoted by $a_1 \# a_2$, called the Moyal product of a_1 and a_2, and satisfies

$$\mathrm{Op}(a_1)\,\mathrm{Op}(a_2) = \mathrm{Op}(a_1 \# a_2). \tag{B-1}$$

[15] A positive function f on the phase space is said to be an order function if

$$\exists C > 0, \exists N > 0 : \forall \rho, \rho_0 \in \mathbb{R}^{2d}, \quad f(\rho) \le C \langle \rho - \rho_0 \rangle^N f(\rho_0).$$

More generally, one can define the h-Moyal product, depending on a parameter $h \in (0, 1]$, as

$$(a_1 \#_h a_2)(\rho) = e^{-\frac{1}{2}ih\sigma(\partial_{\rho_1},\partial_{\rho_2})} a_1(\rho_1)a_2(\rho_2)_{|\rho_1=\rho_2=\rho},$$

where σ is the canonical symplectic form on $\mathbb{R}^{2d}$. Taking $h = 1$, one gets a formula for the Moyal product in (B-1) above. The h-Moyal product is known to be a bilinear continuous map between symbol classes; see [Zworski 2012, Theorem 4.17] or [Lerner 2010, Theorem 2.3.7] for instance.

Proposition B.3 (continuity of Moyal product). *Let f_1, f_2 be two order functions. Then the map*

$$S(f_1) \times S(f_2) \to S(f_1 f_2), \quad (a_1, a_2) \mapsto a_1 \#_h a_2$$

is bilinear continuous, with constants independent of $h \in (0, 1]$. More precisely, for any $\ell \in \mathbb{N}$, there exist $k \in \mathbb{N}$ and $C_\ell > 0$ such that

$$|a_1 \#_h a_2|^{\ell}_{S(f_1 f_2)} \le C_\ell |a_1|^k_{S(f_1)} |a_2|^k_{S(f_2)} \quad \forall h \in (0, 1], \forall (a_1, a_2) \in S(f_1) \times S(f_2).$$

A stationary phase argument leads to an asymptotic expansion of the Moyal product

$$a_1 \# a_2 \sim \sum_j \frac{(-i/2)^j}{j!} \sigma(\partial_{\rho_1}, \partial_{\rho_2})^j a_1(\rho_1) a_2(\rho_2)_{|\rho=\rho_1=\rho_2}.$$

In the sequel, we denote by $\mathcal{R}_{j_0}(a_1, a_2)$ the remainder of order j_0 in this asymptotic expansion, namely

$$\mathcal{R}_{j_0}(a_1, a_2)(\rho) = a_1 \# a_2 - \sum_{j=0}^{j_0-1} \frac{(-i/2)^j}{j!} \sigma(\partial_{\rho_1}, \partial_{\rho_2})^j a_1(\rho_1) a_2(\rho_2)_{|\rho_1=\rho_2=\rho}.$$

Estimates on this remainder term are usually stated as follows.

Proposition B.4 (pseudodifferential calculus). *Let f_1, f_2 be two order functions. Then for any integer $j_0 \ge 1$, the map*

$$S(f_1) \times S(f_2) \to S(f_1 f_2), \quad (a_1, a_2) \mapsto \mathcal{R}_{j_0}(a_1, a_2),$$

is bilinear continuous.

In our study, it will be convenient to have a slightly more precise statement. Actually, the explicit formula for the remainder allows to prove that its seminorms are controlled not only by the seminorms of a_1 and a_2 but more precisely by the seminorms of the derivatives $d^{j_0} a_1$ and $d^{j_0} a_2$.

Proposition B.5 (refined estimate). *Let f_1, f_2 be two order functions. Then, for any $j_0 \ge 1$,*

$$\forall \ell \in \mathbb{N}, \exists k \in \mathbb{N}, \exists C_\ell > 0 : \quad |\mathcal{R}_{j_0}(a_1, a_2)|^{\ell}_{S(f_1 f_2)} \le C_\ell |d^{j_0} a_1|^k_{S(f_1)} |d^{j_0} a_2|^k_{S(f_2)},$$

for all $(a_1, a_2) \in S(f_1) \times S(f_2)$.

Proof. We outline the arguments of the proof, which are classical, trying to keep track of constants carefully. The starting point of this result is the explicit expression of the remainder (see [Zworski 2012, Theorem 4.11] for instance):

$$\mathcal{R}_{j_0}(a_1, a_2)(\rho) = \left(\frac{-i}{2}\right)^{j_0} \int_0^1 \frac{(1-t)^{j_0-1}}{(j_0-1)!} e^{-\frac{1}{2}it\sigma(\partial_{\rho_1},\partial_{\rho_2})} \sigma(\partial_{\rho_1}, \partial_{\rho_2})^{j_0} a_1(\rho_1) a_2(\rho_2)_{|\rho_1=\rho_2=\rho}\, dt.$$

The binomial expansion of $\sigma(\partial_{\rho_1}, \partial_{\rho_2})^{j_0}$ exhibits a particular structure: we observe that the integrand of the integral over t can be written as a sum of terms of the form

$$e^{-\frac{1}{2}it\sigma(\partial_{\rho_1},\partial_{\rho_2})}(\partial^{\alpha_1}a_1)(\rho_1)(\partial^{\alpha_2}a_2)(\rho_2)_{|\rho_1=\rho_2=\rho}$$

with $|\alpha_1| = |\alpha_2| = j_0$, which corresponds exactly to $\partial^{\alpha_1}a_1 \#_t \partial^{\alpha_2}a_2$. By Proposition B.3, we know that the Moyal product is a bilinear continuous map $S(f_1) \times S(f_2) \to S(f_1 f_2)$ with respect to the Fréchet space topology, with seminorm estimates independent of $t \in (0, 1]$. This yields

$$|\mathcal{R}_{j_0}(a_1, a_2)|^0_{S(f_1 f_2)} \le C_0 |d^{j_0}a_1|^k_{S(f_1)} |d^{j_0}a_2|^k_{S(f_2)}. \tag{B-2}$$

In order to handle seminorms of order $\ell \ge 0$, we use the Leibniz formula:

$$\partial\mathcal{R}_{j_0}(a_1, a_2) = \mathcal{R}_{j_0}(\partial a_1, a_2) + \mathcal{R}_{j_0}(a_1, \partial a_2),$$

and we apply (B-2). The result follows. □

B.5. ***Positivity.*** Heuristically, the quantization of a nonnegative symbol is an almost-nonnegative operator. The formal statement, known as the Gårding inequality, says that the negative part of the operator is controlled in terms of the Planck parameter in semiclassical analysis, or exhibits some decay at infinity in the phase space in microlocal analysis. In the main part of the article, we need to apply the Gårding inequality to a symbol in $S(1)$ whose derivatives, of any order, behave like $1/R$, where R is a large parameter. Unfortunately, such a symbol does not fit in the semiclassical framework, in which derivatives of order j behave like $1/R^j$. Thus we provide in this paragraph a refined statement of the sharp Gårding inequality that keeps track of the dependence of the remainder term on the seminorms of the derivatives of the symbol.

Proposition B.6 (sharp Gårding inequality). *There exists a constant $c_d > 0$ and an integer $k_d \ge 0$ depending only on the dimension d such that the following holds. For any real-valued symbol $a \in S(1)$ satisfying $a \ge 0$, one has*

$$\mathrm{Op}(a) \ge -c_d |\mathrm{Hess}\, a|^{k_d}_{S(1)} \,\mathrm{Id}\,.$$

Proof. We redo the usual proof (see for instance [Zworski 2012]) using the refined estimate on the remainder in the pseudodifferential calculus (Proposition B.5). Let us prove that for z sufficiently negative, the operator $\mathrm{Op}(a - z)$ is invertible, which in turn shows that it is nonnegative by classical arguments.

Step 1: Estimate of the derivatives of $(a - z)^{-1}$. Using the assumption that $a \ge 0$, we classically have

$$|\nabla a(\rho)| \le \sqrt{2|\mathrm{Hess}\, a|_\infty a(\rho)} \quad \forall \rho \in \mathbb{R}^{2d} \tag{B-3}$$

(see [Zworski 2012, Lemma 4.31] for instance). Besides, the Faà di Bruno formula tells us that, for any nonzero $\alpha \in \mathbb{N}^{2d}$, the partial derivative $\partial^\alpha (a - z)^{-1}$ can be computed as a sum of terms of the form

$$\frac{1}{(a - z)^{1+\ell}} \prod_{j=1}^{\ell} \partial^{\alpha_j} a,$$

with $1 \le \ell \le |\alpha|$, $\sum_{j=1}^{\ell} \alpha_j = \alpha$, $|\alpha_j| \neq 0$ for all j. Denote by ℓ' the number indices j such that $|\alpha_j| = 1$. We apply (B-3) to the ℓ' factors of the form $\partial^{\alpha_j} a$ corresponding to these indices, and we bound the $\ell - \ell'$ other ones by seminorms of the Hessian of a (recall that $|\alpha_j| \ge 2$ for those remaining indices). We obtain

$$\left| \frac{1}{(a-z)^{1+\ell}} \prod_{j=1}^{\ell} \partial^{\alpha_j} a \right| \le \frac{1}{|a-z|^{1+\ell}} (2|\operatorname{Hess} a|_\infty a(\rho))^{\ell'/2} (|\operatorname{Hess} a|_{S(1)}^{|\alpha|})^{\ell-\ell'}.$$

We deduce that

$$\begin{aligned} \left| \frac{1}{(a-z)^{1+\ell}} \prod_{j=1}^{\ell} \partial^{\alpha_j} a \right| &\le \frac{1}{|a-z|^{1+\ell}} 2^{\ell'/2} |a(\rho)|^{\ell'/2} (|\operatorname{Hess} a|_{S(1)}^{|\alpha|})^{\ell-\ell'/2} \\ &\le \frac{1}{|a-z|^{1+\ell}} 2^{\ell'} (|a-z|^{\ell'/2} + |z|^{\ell'/2}) (|\operatorname{Hess} a|_{S(1)}^{|\alpha|})^{\ell-\ell'/2}. \end{aligned}$$

Putting together all the terms in the Faà di Bruno formula, and using that $a - z \ge |z|$ (since $z \le 0$), we finally get that there exists a constant $C > 0$ (depending on $|\alpha|$) such that

$$\left| \partial^\alpha \frac{1}{a-z} \right| \le \frac{C}{|z|} \max_{\substack{1 \le \ell \le |\alpha| \\ 0 \le \ell' \le \ell}} \left(\frac{|\operatorname{Hess} a|_{S(1)}^{|\alpha|}}{|z|} \right)^{\ell-\ell'/2}.$$

Assuming that $|z| \ge |\operatorname{Hess} a|_{S(1)}^{|\alpha|}$, we arrive at

$$\left| \partial^\alpha \frac{1}{a-z} \right| \le \frac{C}{|z|} \sqrt{\frac{|\operatorname{Hess} a|_{S(1)}^{|\alpha|}}{|z|}}. \tag{B-4}$$

Step 2: Invertibility of $\operatorname{Op}(a-z)$. From the previous step, we know that $a - z$ and $(a-z)^{-1}$ are in $S(1)$ with explicit seminorm estimates, provided $|z|$ is large enough. We perform the pseudodifferential calculus,

$$\operatorname{Op}(a-z) \operatorname{Op}\left(\frac{1}{a-z} \right) = \operatorname{Id} + 0 + \operatorname{Op}(\mathscr{R}_2),$$

keeping in mind that the second term in the asymptotic expansion vanishes because both symbols are functions of the same symbol. According to the Calderón–Vaillancourt theorem (Theorem B.2), our refined estimate on the remainder (Proposition B.5), and finally to (B-4), we obtain

$$\|\operatorname{Op}(\mathscr{R}_2)\|_{L^2 \to L^2} \le C_d |\mathscr{R}_2|_{S(1)}^{k_d} \le C_d |\operatorname{Hess} a|_{S(1)}^{k_1'} |\operatorname{Hess}(a-z)^{-1}|_{S(1)}^{k_2'} \le C \left(\frac{|\operatorname{Hess} a|_{S(1)}^{k}}{|z|} \right)^{3/2}$$

for some constant C and some integer k independent of a and z, and provided z is negative enough. Actually when $z \le -(2C)^{2/3} |\operatorname{Hess} a|_{S(1)}^{k}$, we obtain that $\|\operatorname{Op}(\mathscr{R}_2)\| \le \frac{1}{2}$, so that $\operatorname{Id} + \operatorname{Op}(\mathscr{R}_2)$ is invertible by Neumann series. This leads classically to the invertibility of $\operatorname{Op}(a-z)$. ☐

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GLOBAL WELL-POSEDNESS FOR TWO-DIMENSIONAL INHOMOGENEOUS VISCOUS FLOWS WITH ROUGH DATA VIA DYNAMIC INTERPOLATION

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We consider the evolution of two-dimensional incompressible flows with variable density, only bounded and bounded away from zero. Assuming that the initial velocity belongs to a suitable critical subspace of L^2, we prove a global-in-time existence and stability result for the initial (boundary) value problem.

Our proof relies on new time decay estimates for finite energy weak solutions and on a "dynamic interpolation" argument. We show that the constructed solutions have a uniformly C^1 flow, which ensures the propagation of geometrical structures in the fluid and guarantees that the Eulerian and Lagrangian formulations of the equations are equivalent. By adopting this latter formulation, we establish the uniqueness of the solutions for prescribed data and the continuity of the flow map in an energy-like functional framework.

In contrast with prior works, our results hold in the critical regularity setting *without any smallness assumption*. Our approach uses only elementary tools and applies indistinctly to the cases where the fluid domain is the whole plane, a smooth two-dimensional bounded domain, or the torus.

Introduction

An extensive literature has been devoted to the mathematical analysis of the Navier–Stokes equations that govern the evolution of the velocity field $u = u(t, x)$ and pressure function $P = P(t, x)$ of homogeneous incompressible viscous flows in a domain Ω of $\mathbb{R}^d$. Recall that these equations read as

$$\begin{cases} u_t + \operatorname{div}(u \otimes u) - \mu \Delta u + \nabla P = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\ u|_{t=0} = u_0 & \text{in } \Omega \end{cases} \tag{NS}$$

and, if Ω has a boundary, are supplemented with homogeneous Dirichlet boundary conditions for the velocity.

The global existence theory for (NS) originated in the paper by J. Leray [1934b]. In the case $\Omega = \mathbb{R}^3$, by combining the energy balance associated to (NS),

$$\frac{1}{2}\|u(t)\|_{L^2}^2 + \mu \int_0^t \|\nabla u\|_{L^2}^2 \, d\tau = \frac{1}{2}\|u_0\|_{L^2}^2, \tag{0-1}$$

with compactness arguments, he constructed, for any divergence-free u_0 in $L^2(\mathbb{R}^3; \mathbb{R}^3)$, a global distributional solution of (NS) satisfying (0-1) *with an inequality* (viz. the left-hand side is bounded by the right-hand side).

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It is by now well understood that Leray's result is true in any open subset Ω of $\mathbb{R}^d$ with $d = 2, 3$; see for instance the first part of [Chemin et al. 2006]. However, despite the numerous papers devoted to the topic and significant recent progresses, the question of uniqueness of finite energy solutions in the case $d = 3$ has not been completely solved yet. The two-dimensional situation is much better understood: finite energy solutions are unique and do satisfy (0-1) with an equality. Although uniqueness in dimension 2 could be hinted from [Leray 1934a], it has been established only by O. A. Ladyzhenskaya [1959] and J.-L. Lions and G. Prodi [Lions and Prodi 1959].

In the present paper, we are concerned with *inhomogeneous*, that is, with variable density, incompressible viscous flows. The evolution of these flows, which can be encountered in models of geophysics or mixtures, is often described by the following *inhomogeneous incompressible Navier–Stokes equations*:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) - \mu\Delta u + \nabla P = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}_+ \times \Omega. \end{cases} \tag{INS}$$

Above, u and P still denote the velocity and the pressure, respectively, and $\rho = \rho(t, x)$ stands for the density, which for obvious physical reasons has to be nonnegative. If we supplement (INS) with initial data and boundary conditions

$$\rho|_{t=0} = \rho_0, \quad u|_{t=0} = u_0 \quad \text{and} \quad u|_{\partial\Omega} = 0, \tag{0-2}$$

then the energy balance associated to (INS) reads as

$$\frac{1}{2}\|(\sqrt{\rho}u)(t)\|_{L^2}^2 + \mu\int_0^t \|\nabla u\|_{L^2}^2\,d\tau = \frac{1}{2}\|\sqrt{\rho_0}u_0\|_{L^2}^2. \tag{0-3}$$

The divergence-free condition ensures that the Lebesgue norms of ρ are conserved and that,

$$\text{for all } t \in \mathbb{R}_+, \quad \inf_{x\in\Omega} \rho(t, x) = \inf_{x\in\Omega} \rho_0(x) \quad \text{and} \quad \sup_{x\in\Omega} \rho(t, x) = \sup_{x\in\Omega} \rho_0(x). \tag{0-4}$$

In the torus case, we have in addition the conservation of total momentum

$$\int_{\mathbb{T}^2} (\rho u)(t, x)\,dx = \int_{\mathbb{T}^2} (\rho_0 u_0)(x)\,dx. \tag{0-5}$$

Like (NS), equations (INS) have a scaling invariance (if Ω is stable by dilation): they are invariant for all $\lambda > 0$ by the transform

$$(\rho, u, P)(t, x) \rightsquigarrow (\rho, \lambda u, \lambda^2 P)(\lambda^2 t, \lambda x). \tag{0-6}$$

Although (INS) is of hyperbolic-parabolic-type while (NS) is parabolic, similar results hold for the initial value (or boundary value) problem. For instance:

- In any dimension, provided ρ_0 is bounded and nonnegative and $\sqrt{\rho_0}u_0$ is in L^2, there exists a global weak solution satisfying (0-3) with inequality.[1]

[1] First proved by A. V. Kazhikhov [1974] if $\rho_0 > 0$, then for general $\rho_0 \geq 0$ by J. Simon [1990]. P.-L. Lions [1996] pointed out that the density is a renormalized solution of the mass equation and treated density dependent viscosity coefficients. He also considered unbounded densities.

- Smooth enough data with density bounded and bounded away from zero generate a unique local-in-time smooth solution, which is global in the two-dimensional case and also in higher dimensions if the initial velocity is small.[2]

In dimension 2, the quantities that come into play in the energy balance (0-3) are scaling invariant in the sense of (0-6). However, unlike the case with constant density, it is not known whether finite energy two-dimensional weak solutions with bounded density, albeit having critical regularity, are unique.

In order to explain the difference between the variable and constant density cases and to motivate the assumptions that will be made in this paper, let us sketch the proof of the uniqueness of finite energy solutions for (NS) in dimension 2. Assume that we are given two solutions (u, P) and $(\tilde{u}, \widetilde{P})$ pertaining to the same finite energy initial velocity u_0. Then, $\delta u := \tilde{u} - u$ and $\delta P := \widetilde{P} - P$ satisfy

$$\begin{cases} \delta u_t + \operatorname{div}(u \otimes \delta u) - \mu \Delta \delta u + \nabla \delta P = -\operatorname{div}(\delta u \otimes \tilde{u}) & \text{in } \mathbb{R}_+ \times \Omega, \\ \operatorname{div} \delta u = 0 & \text{in } \mathbb{R}_+ \times \Omega. \end{cases}$$

Taking the $L^2(\Omega; \mathbb{R}^2)$ scalar product with δu, integrating by parts where needed and using the Hölder inequality to bound the right-hand side yields

$$\frac{1}{2}\frac{d}{dt}\|\delta u\|_{L^2}^2 + \mu \|\nabla \delta u\|_{L^2}^2 \le \|\nabla \tilde{u}\|_{L^2} \|\delta u\|_{L^4}^2,$$

which, in light of the celebrated Ladyzhenskaya inequality

$$\|z\|_{L^4}^2 \le C \|z\|_{L^2} \|\nabla z\|_{L^2}, \tag{0-7}$$

leads to

$$\frac{1}{2}\frac{d}{dt}\|\delta u\|_{L^2}^2 + \mu \|\nabla \delta u\|_{L^2}^2 \le C \|\nabla \tilde{u}\|_{L^2} \|\delta u\|_{L^2} \|\nabla \delta u\|_{L^2} \le \frac{\mu}{2} \|\nabla \delta u\|_{L^2}^2 + \frac{C^2}{2\mu} \|\nabla \tilde{u}\|_{L^2}^2 \|\delta u\|_{L^2}^2.$$

At this stage, Gronwall's lemma allows us to conclude that

$$\|\delta u(t)\|_{L^2}^2 + \mu \int_0^t \|\nabla \delta u\|_{L^2}^2 \, d\tau \le e^{(C^2/\mu) \int_0^t \|\nabla \tilde{u}\|_{L^2}^2 \, d\tau} \|\delta u(0)\|_{L^2}^2.$$

Owing to (0-1), the exponential term is finite. Hence we have $\delta u \equiv 0$ if $\tilde{u}(0) = u(0)$.

In contrast, when comparing two finite energy solutions (ρ, u, P) and $(\tilde{\rho}, \tilde{u}, \widetilde{P})$ of (INS), we get the following system for $\delta\rho := \tilde{\rho} - \rho$, δu, and δP:

$$\begin{cases} \delta \rho_t + \operatorname{div}(\delta \rho u) = -\operatorname{div}(\tilde{\rho} \delta u), \\ (\rho \delta u)_t + \operatorname{div}(\rho u \otimes \nabla \delta u) - \mu \Delta \delta u + \nabla \delta P = -(\delta \rho \tilde{u})_t - \operatorname{div}(\rho u \otimes \delta u) - \operatorname{div}(\rho \delta u \otimes \tilde{u}), \\ \operatorname{div} \delta u = 0. \end{cases}$$

Since $\tilde{\rho}$ is only bounded, the first line is a transport equation by the divergence-free vector field u, with a source term that has (at most) the regularity C^{-1} with respect to the space variable. Now, in order to control the propagation of negative regularity in a transport equation, we need

$$\nabla u \in L^1_{\text{loc}}(\mathbb{R}_+; L^\infty). \tag{0-8}$$

[2] First established by O. A. Ladyzhenskaya and V. A. Solonnikov [1975].

However, this property generally fails for finite energy solutions of (INS) and even for the two-dimensional heat equation. In fact, the set of functions u_0 such that the solution u to the free heat equation with initial data u_0 satisfies $\nabla u \in L^1(\mathbb{R}_+; L^\infty)$ is the homogeneous Besov space $\dot{B}^{-1}_{\infty,1}$, and L^2 is not embedded in this space.

To avoid working in spaces with negative regularity, one can recast (INS) in the Lagrangian coordinate system as in [Danchin and Mucha 2019]. Then, the density becomes time-independent and the velocity equation keeps its parabolicity (at least for small time). However, the equivalence between the Eulerian and Lagrangian formulations of (INS) in our low-regularity context still requires (0-8), a property that cannot be expected if u_0 is only in L^2 since it fails for the heat flow.

To make a long story short, it is not clear that uniqueness holds for (INS) in the framework of just finite energy solutions.

Before describing in more detail the main objective of the article, let us recall some recent results on the well-posedness theory for (INS). A number of works have been devoted to this issue under weaker assumptions than in [Ladyzhenskaya and Solonnikov 1975]. This is mainly to relax the positivity condition on the density or the regularity assumptions on the initial data. Regarding the first question, it has been observed by Y. Cho and H. Kim [2004] that (INS) is well-posed for smooth enough data and, possibly, vanishing densities satisfying a suitable compatibility condition. Recently, J. Li [2017] discovered that this condition is no longer needed if one considers H^1 regularity for the velocity, and the full well-posedness theory for general only bounded (not necessarily positive) initial densities and H^1 velocities has been carried out in a joint work with P. B. Mucha [Danchin and Mucha 2019].

Regarding the minimal regularity requirement of the velocity for well-posedness, the scaling invariance of (INS) pointed out in (0-6) suggests (if $\Omega = \mathbb{R}^d$) that one should take $\rho_0 \in L^\infty(\mathbb{R}^d)$ and $u_0 \in \dot{H}^{d/2-1}(\mathbb{R}^d)$. In the constant density case and for $d = 3$, this assumption is in accordance with the well-known Fujita and Kato theorem [1964]. However, as, again, $\nabla e^{t\Delta} u_0$ need not be in $L^1_{\text{loc}}(\mathbb{R}_+; L^\infty)$ if $u_0 \in \dot{H}^{d/2-1}(\mathbb{R}^d)$, it is not clear that uniqueness may be achieved if there is no additional regularity in the variable density case. In this direction, it has been proved in [Danchin 2003; 2004] that if u_0 belongs to the homogeneous Besov space $\dot{B}^{d/2-1}_{2,1}(\mathbb{R}^d)$, a large subspace of $\dot{H}^{d/2-1}(\mathbb{R}^d)$ with the same scaling invariance, then (INS) is globally well-posed in dimension 2 (or in higher dimensions if u_0 is small) *provided ρ_0 is close to some positive constant in the homogeneous Besov space* $\dot{B}^{d/2}_{2,1}(\mathbb{R}^d)$. This result is satisfactory as regards the regularity requirement for the velocity, since it is critical and closely related to the L^2 space, but the condition on the density is rather restrictive both because ρ_0 has to be almost constant and since it has to be continuous (the space $\dot{B}^{d/2}_{2,1}(\mathbb{R}^d)$ is embedded in the set $\mathcal{C}_b(\mathbb{R}^d)$ of bounded and continuous functions on $\mathbb{R}^d$). The result of [Danchin 2003] has been significantly improved recently in the two-dimensional case: H. Abidi and G. Gui [2021] established the global well-posedness without any smallness condition on the data if $\rho_0 - 1$ is in $\dot{B}^1_{2,1}(\mathbb{R}^2)$ and u_0 belongs to $\dot{B}^0_{2,1}(\mathbb{R}^2)$. The corresponding result in dimension 3 has been obtained with completely different techniques by H. Xu [2022] (for small u_0 of course). As said before, works based on the use of critical Besov spaces for the density precludes considering the case of densities that are discontinuous along an interface, a situation which is of particular interest if one believes (INS) to be a relevant model for mixtures of incompressible viscous flows with different densities.

This very situation — sometimes called *the density patch problem* — has been extensively studied lately, see, e.g., [Danchin and Mucha 2019; Gancedo and García-Juárez 2018; Liao and Zhang 2019].

Well-posedness results for only bounded initial density, bounded away from zero, and smooth enough velocity have been obtained in a joint work with P. B. Mucha [Danchin and Mucha 2013b], then improved by M. Paicu, P. Zhang and Z. Zhang in [Paicu et al. 2013] (there, u_0 is in $H^s(\mathbb{R}^2)$ for some $s > 0$ if $d = 2$, and in $H^1(\mathbb{R}^3)$ if $d = 3$). In the whole space case, the critical regularity index has been reached in an intriguing work by P. Zhang [2020]. He established the global existence for any small enough divergence-free u_0 with coefficients in $\dot{B}^{1/2}_{2,1}(\mathbb{R}^3)$ while ρ_0 is only bounded and bounded away from zero. It has been observed recently in a joint work with S. Wang [Danchin and Wang 2023] that Zhang's solutions actually satisfy (0-8) and are thus unique.

The main goal of the present paper is to investigate the counterpart *in dimension* 2 *and for large initial data* of Zhang's result recalled just above: we want to establish a global well-posedness result for general divergence-free velocity fields u_0 with critical regularity of L^2-type and densities ρ_0 simply satisfying

$$\rho_* := \operatorname*{ess\,inf}_{x\in\Omega} \rho_0(x) > 0, \qquad \rho^* := \operatorname*{ess\,sup}_{x\in\Omega} \rho_0(x) < \infty. \tag{0-9}$$

According to [Abidi and Gui 2021], a good candidate to achieve the Lipschitz property within a critical regularity framework of L^2-type is the space $\dot{B}^0_{2,1}$. However, owing to the use of Fourier analysis techniques, rather strong regularity assumptions on the density were made in that work. Here, since we want to consider only bounded densities, we shall adopt a completely different approach. In fact, we shall combine real interpolation and three levels of time decay estimates (corresponding to $\dot{H}^{-1}$, L^2, and $\dot{H}^1$ data, respectively) for a linearized version of (INS) that can be obtained just by energy arguments and basic properties of the Stokes system, so as to work out a space for u_0 that coincides with $\dot{B}^0_{2,1}$ if ρ_0 is smooth (but that might depend on it if it is not). The overall strategy is so robust that it can be adapted to other systems.

The rest of the paper is structured as follows: in the next section we state our main results and explain the key steps of the proof. Then, in Section 2, we establish a first family of time decay estimates pertaining to the case where u_0 is just in L^2 and construct corresponding global finite energy weak solutions for (INS). Section 3 is devoted to proving more a priori decay estimates. The final goal is to establish that, under a slightly stronger assumption on the initial velocity very close to the regularity $\dot{B}^0_{2,1}$, the Lipschitz property (0-8) is satisfied. Finally, we establish in Section 4 the existence and uniqueness of a solution under this assumption, assuming only (0-9) and that the velocity belongs to the aforementioned space. The same method also provides stability estimates for the flow map in the energy space.

Notation. In the rest of the paper, Ω will be either a C^2 bounded domain of $\mathbb{R}^2$, a two-dimensional torus, or $\mathbb{R}^2$. It will be convenient to use the same notation $\dot{H}^s(\Omega)$ to designate:

- the classical homogeneous Sobolev space if $\Omega = \mathbb{R}^2$,
- the subset of functions of H^s with mean value 0 if $\Omega = \mathbb{T}^2$,

- the space $H_0^s(\Omega)$ (that is the completion of $\mathcal{C}_c^\infty(\Omega)$ for the $H^s(\mathbb{R}^2)$ norm) if Ω is a bounded domain and $s \in [0, 1]$,
- the dual of $H_0^{-s}(\Omega)$ if Ω is a bounded domain and $s \in [-1, 0]$.

We designate by $L^2_\sigma(\Omega)$ the set of divergence-free vector fields with coefficients in $L^2(\Omega)$ (such that $u_0 \cdot n = 0$ at $\partial\Omega$ in the bounded domain case, with n being the unit exterior normal vector to $\partial\Omega$), and denote by $\mathcal{P}$ the orthogonal projector from $L^2(\Omega; \mathbb{R}^2)$ to $L^2_\sigma(\Omega)$.

For any normed space X, Lebesgue index $q \in [1, \infty]$, and time $T \in [0, \infty]$, we shall define

$$\|z\|_{L^q_T(X)} := \big\| \|z(t)\|_X \big\|_{L^q(0,T)},$$

omitting T if it is ∞. In the case where z has several components in X, we keep the same notation for the norm.

As usual, C designates harmless positive real numbers, and we shall often write $A \lesssim B$ instead of $A \le CB$. To emphasize the dependency with respect to parameters $a_1, \dots, a_n$, we adopt the notation $C_{a_1,\dots,a_n}$. The notation $C_{\rho,v}$ stands for various "constants" that only depend (algebraically) on the infimum and supremum of ρ and on "energy-like" norms of v, that is, *on norms that could be eventually bounded by* $\|u_0\|_{L^2}$ *if* (ρ, v) *were a solution to* (INS). Obvious examples are $\|v\|_{L^\infty(L^2)}$ or $\|\nabla v\|_{L^2(L^2)}$ (remember (0-3)) but also $\|v\|_{L^4(L^4)}$ (use (0-7)) and so on.

1. Results and strategy

The first step is to exhibit time decay estimates for finite energy solutions. More precisely, we shall establish the following statement.

Theorem 1.1. *Let u_0 be in $L^2_\sigma(\Omega)$ and ρ_0 satisfy* (0-9). *Then*, (INS) *supplemented with* (0-2) *admits a global solution (ρ, u, P) satisfying* (0-4) (*and* (0-5) *if* $\Omega = \mathbb{T}^2$), $u \in L^\infty(\mathbb{R}_+; L^2_\sigma)$, $\nabla u \in L^2(\mathbb{R}_+ \times \Omega)$, *and*

$$\frac{1}{2}\|(\sqrt{\rho}u)(t)\|^2_{L^2} + \mu \int_0^t \|\nabla u\|^2_{L^2}\, d\tau \le \frac{1}{2}\|\sqrt{\rho_0}u_0\|^2_{L^2}, \quad t > 0. \tag{1-1}$$

Furthermore, there exists a constant C depending only on Ω, ρ_*, and ρ^* such that, for all $t > 0$, we have

$$\begin{aligned} \|\nabla^k u(t)\|_{L^2} &\le C(\mu t)^{-k/2}\|u_0\|_{L^2} \quad \textit{for } k = 0, 1, 2,\\ \|\nabla^k (u_t, \dot u)(t)\|_{L^2} &\le C(\mu t)^{-1-k/2}\|u_0\|_{L^2} \quad \textit{for } k = 0, 1,\\ \|\nabla P(t)\|_{L^2} &\le Ct^{-1}\|u_0\|_{L^2}, \end{aligned}$$

where $\dot u$ denotes the convective derivative of u; that is, $\dot u := u_t + u \cdot \nabla u$.

Two remarks are in order:

- The constructed solutions satisfy more time decay estimates: see (2-11), (2-21), (2-26), Proposition 3.1 with $s' = 0$, and Proposition 3.2 with $p = 2$.

• As pointed out in [Danchin et al. 2024] for $H^1_0(\Omega)$ initial velocities, exponential time decay estimates hold if Ω is bounded. Following the proof of Lemma 5 therein, one can show that there exists a positive constant c_Ω depending only on Ω such that,

$$\text{for all } t \in \mathbb{R}_+, \quad \|(\sqrt{\rho}u)(t)\|_{L^2} \le e^{-c_\Omega \mu t/\rho^*}\|\sqrt{\rho_0}u_0\|_{L^2}.$$

From this inequality, one can deduce exponential decay for

$$\|t^{k/2}\nabla^k u\|_{L^2}, \quad \|t^{1+k/2}\nabla^k u_t\|_{L^2}, \quad \text{and} \quad \|t^{1+k/2}\nabla^k \dot{u}\|_{L^2}.$$

However, as exponential decay does not hold if $\Omega = \mathbb{R}^2$ and since we strive for a unified approach, we refrain from tracking it in the rest of the paper to simplify the presentation.

As underlined in the Introduction, in order to establish the uniqueness of solutions, we need a functional space that ensures (0-8). At the same time, we want our functional framework to be critical, to allow any initial density just bounded and bounded away from zero, and to be strongly related to the energy space L^2. Note that Theorem 1.1 ensures that ∇u belongs to the *weak* L^1 space for the time variable with values in the Sobolev space H^1. This latter space "almost" embeds in L^∞. A classical way to improve embeddings is to work out a space by means of *real interpolation with second parameter equal to* 1. In our context, since energy arguments play an important role, it is natural to interpolate from Sobolev spaces and to consider[3]

$$[\dot{H}^{-s}, \dot{H}^s]_{1/2,1} \quad \text{for some } s \in (0,1). \tag{1-2}$$

This definition gives the Besov space $\dot{B}^0_{2,1}$ (independently of the value of s).

Let us briefly explain why in the simpler situation where u is the solution of the free heat equation in $\mathbb{R}^2$, supplemented with initial data u_0 in $\dot{B}^0_{2,1}$, we do have (0-8). We start from the two inequalities

$$t\|\nabla u(t)\|_{L^\infty} \le C\min(t^{s/2}\|u_0\|_{\dot{H}^s}, t^{-s/2}\|u_0\|_{\dot{H}^{-s}}), \tag{1-3}$$

which may be easily derived by using the explicit formula for u in the Fourier space.

Then, we use the characterization of real interpolation spaces in terms of atomic decomposition like in, e.g., [Lions and Peetre 1964]. In our setting, it reads $z \in \dot{B}^0_{2,1}$ if and only if there exists a sequence $(z_j)_{j\in\mathbb{Z}}$ of $\dot{H}^{-s} \cap \dot{H}^s$ satisfying

$$z = \sum_{j\in\mathbb{Z}} z_j \quad \text{and} \quad \sum_{j\in\mathbb{Z}}(2^{-j/2}\|z_j\|_{\dot{H}^s} + 2^{j/2}\|z_j\|_{\dot{H}^{-s}}) < \infty.$$

The infimum of the above sum on all admissible decompositions of z defines a norm on $\dot{B}^0_{2,1}$. Now, take the decomposition

$$u_0 = \sum_{j\in\mathbb{Z}} u_{0,j}, \quad \text{with } \sum_{j\in\mathbb{Z}}(2^{-j/2}\|u_{0,j}\|_{\dot{H}^s} + 2^{j/2}\|u_{0,j}\|_{\dot{H}^{-s}}) \le 2\|u_0\|_{\dot{B}^0_{2,1}}, \tag{1-4}$$

[3] One could prefer to interpolate between *Lebesgue spaces* and consider the velocity in the Lorentz space $L^{2,1}$. However we do not know how to handle (INS) in this space. The reader is referred to [Danchin 2024] where the space $L^{2,1}$ is used for solving the two-dimensional system for pressureless gases.

and solve all the heat equations

$$(u_j)_t - \Delta u_j = 0, \quad u_j|_{t=0} = u_{0,j}.$$

As the heat equation is linear, we have $u = \sum_j u_j$, and thus

$$\int_0^\infty \|\nabla u\|_{L^\infty}\,dt \le \sum_{j\in\mathbb{Z}} \int_0^\infty \|\nabla u_j\|_{L^\infty}\,dt. \tag{1-5}$$

Now, for every j in $\mathbb{Z}$ and $A_j > 0$, we have, due to (1-3),

$$\begin{aligned}\int_0^\infty \|\nabla u_j\|_{L^\infty}\,dt &\le \int_0^{A_j} \|\nabla u_j\|_{L^\infty}\,dt + \int_{A_j}^\infty \|\nabla u_j\|_{L^\infty}\,dt\\ &\lesssim \|u_{0,j}\|_{\dot H^s} \int_0^{A_j} t^{-1+s/2}\,dt + \|u_{0,j}\|_{\dot H^{-s}} \int_{A_j}^\infty t^{-1-s/2}\,dt\\ &\lesssim \|u_{0,j}\|_{\dot H^s} A_j^{s/2} + \|u_{0,j}\|_{\dot H^{-s}} A_j^{-s/2}.\end{aligned}$$

Hence, choosing $A_j = 2^{-j/s}$ and remembering (1-4) and (1-5) gives (0-8) (globally in time).

This “dynamic interpolation approach” has been used before by T. Hmidi and S. Keraani [2008] for the transport equation and by Zhang [2020] for the velocity equation of (INS) (in dimension 3 and for small velocities). In both cases however, the initial data was decomposed according to a Littlewood–Paley decomposition. The additional flexibility that consists here in using general atomic decompositions enables us to do without Fourier analysis and to treat general domains.

As our aim is to prove (0-8) for (INS), we have to consider instead of the heat equation a linear system which captures both the effects of the density and of the convection. To this end, we consider

$$\begin{cases}(\rho u)_t + \operatorname{div}(v\otimes u) - \Delta u + \nabla P = 0 & \text{in } \mathbb{R}_+\times\Omega,\\ \operatorname{div} u = 0 & \text{in } \mathbb{R}_+\times\Omega,\\ u|_{t=0} = u_0 & \text{in } \Omega,\end{cases} \tag{1-6}$$

where the (smooth enough) triplet (ρ, v, u_0) is given with ρ bounded and bounded away from zero,

$$\rho_t + \operatorname{div}(\rho v) = 0, \quad \operatorname{div} v = 0, \quad \text{and} \quad v|_{\partial\Omega} = 0. \tag{1-7}$$

Clearly, if we succeed in proving (1-3) for (1-6) with a constant that only depends on ρ_*, ρ^*, and on energy-like norms of v, then repeating the above dynamic interpolation procedure will yield (0-8) for the solutions of (1-6) supplemented with initial data in $\dot B^0_{2,1}$, and then for (INS) if taking $v = u$.

The way to get (1-3) is to prove beforehand three families of time weighted estimates for (1-6) corresponding to initial data u_0 in L^2, $\dot H^1$, and $\dot H^{-1}$, respectively. The estimate in $\dot H^{-1}$ will be obtained by duality from the estimate in $\dot H^1$. This will lead us to consider the backward system associated with (1-6), and it is rather $\|\mathcal{P}(\rho u)(t)\|_{\dot H^{-1}}$ and, more generally, $\|\mathcal{P}(\rho u)(t)\|_{\dot H^{-s}}$ for $s\in(0,1)$ that can be estimated. In the end, combining the three families of inequalities with suitable Gagliardo–Nirenberg inequalities yields, instead of (1-3),

$$t\|\nabla u(t)\|_{L^\infty} \le C_{\rho,v} \min(t^{s/2}\|u_0\|_{\dot H^s},\, t^{-s/2}\|\mathcal{P}(\rho_0 u_0)\|_{\dot H^{-s}}). \tag{1-8}$$

Above, $C_{\rho,v}$ only depends on ρ_*, ρ^*, and on energy-like norms of v.

As a consequence, the suitable interpolation space to carry out our dynamic interpolation procedure for (1-6) is the one that is given in the following definition.

Definition 1.2. Let s be in $(0,1)$ and a be a measurable function on Ω with positive lower bound. We denote by $\widetilde{B}^{0,s}_{a,1}(\Omega)$ the set of vector fields z in $L^2_\sigma(\Omega)$ such that there exists a sequence $(z_j)_{j\in\mathbb{Z}}$ of $L^2_\sigma(\Omega)$ satisfying:

- $z=\sum_{j\in\mathbb{Z}} z_j$ in the sense of distributions,
- for all $j\in\mathbb{Z}$, we have $\mathcal{P}(az_j)\in \dot H^{-s}(\Omega)$ and $z_j\in\dot H^s(\Omega)$,
- $\sum_{j\in\mathbb{Z}}(2^{-j/2}\|z_j\|_{\dot H^s}+2^{j/2}\|\mathcal{P}(az_j)\|_{\dot H^{-s}})$ is finite.

The infimum on all admissible decompositions of z defines a norm on $\widetilde{B}^{0,s}_{a,1}(\Omega)$.

Let us highlight a few properties of these spaces.

- $(\widetilde{B}^{0,s}_{a,1}(\Omega))_{s\in(0,1)}$ is a family of nested Banach spaces: if $0<s'<s<1$, then $\widetilde{B}^{0,s}_{a,1}(\Omega)\hookrightarrow\widetilde{B}^{0,s'}_{a,1}(\Omega)$.
- Owing to (1-2), if a is a positive constant, then $\widetilde{B}^{0,s}_{a,1}$ is nothing other than $\dot B^0_{2,1}$, and if a has a positive lower bound a_*, then it embeds in L^2. Indeed, decomposing $z\in\widetilde{B}^{0,s}_{a,1}$ according to Definition 1.2 and using the fact that $\mathcal{P}$ is an L^2 orthogonal projector, one may write, for all $j\in\mathbb{Z}$,

$$\|z_j\|^2_{L^2}\le a_*^{-1}\int_\Omega \mathcal{P}(az_j)\cdot z_j\,dx\le a_*^{-1}(2^{j/2}\|\mathcal{P}(az_j)\|_{\dot H^{-1/2}})(2^{-j/2}\|z_j\|_{\dot H^{1/2}}),\tag{1-9}$$

which implies, by Young's inequality, that

$$\|z\|_{L^2}\le\frac{1}{2\sqrt{a_*}}\|z\|_{\widetilde{B}^{0,s}_{a,1}}.$$

- If a is bounded and $s=2/p-1$ for some $p\in(1,2)$, then the critical Besov space

$$\dot B^{-1+2/p}_{p,1}:=[L^p,\dot W^{2s}_p]_{1/2,1}$$

is embedded in $\widetilde{B}^{0,s}_{a,1}$. Indeed, if $z\in\dot B^{-1+2/p}_{p,1}$, then there exists a sequence $(z_j)_{j\in\mathbb{Z}}$ of the nonhomogeneous Sobolev space W^{2s}_p such that

$$z=\sum_{j\in\mathbb{Z}} z_j\quad\text{and}\quad\sum_{j\in\mathbb{Z}}(2^{-j/2}\|z_j\|_{W^{2s}_p}+2^{j/2}\|z_j\|_{L^p})\le 2\|z\|_{\dot B^{-1+2/p}_{p,1}}.$$

Now, the fact that $\mathcal{P}:L^p\to L^p$ and the embeddings $\dot W^{2s}_p\hookrightarrow\dot H^s$ and $L^p\hookrightarrow\dot H^{-s}$ allow us to write

$$\|z_j\|_{\dot H^s}\le C\|z_j\|_{\dot W^{2s}_p}\quad\text{and}\quad\|\mathcal{P}(az_j)\|_{\dot H^{-s}}\le C\|\mathcal{P}(az_j)\|_{L^p}\le C\|a\|_{L^\infty}\|z_j\|_{L^p},$$

which gives our claim.

- For general measurable functions a bounded and bounded away from zero, the space $\widetilde{B}^{0,s}_{a,1}$ might depend on s. However, in the case $s\in\left(0,\frac12\right)$, if a is positive and piecewise constant along a finite number of Lipschitz curves, then it coincides with $\dot B^0_{2,1}$. Indeed, in this case the space $\dot H^{-s}$ is stable by multiplication by piecewise constant functions.

Our main global existence and uniqueness statement reads as follows.

Theorem 1.3. *Let* ρ_0 *satisfy* (0-9) *and* u_0 *be in* $\widetilde{B}^{0,s}_{\rho_0,1}$ *for some* $s \in (0,1)$. *Then*, (INS) *supplemented with* (0-2) *admits a unique global solution* $(\rho, u, \nabla P)$ *satisfying all the properties stated in Theorem 1.1* (*and the remarks that follow*) *and the energy balance* (0-3). *In addition, we have*

$$u \in \mathcal{C}(\mathbb{R}_+; L^2), \quad \nabla u \in L^1(\mathbb{R}_+; C_b \cap \dot{H}^1), \quad \sqrt{t}(\dot{u}, \nabla P, \nabla^2 u) \in L^{4/3}(\mathbb{R}_+; L^4)$$

and, *for all* $t \in \mathbb{R}_+$, *we have* $u(t) \in \widetilde{B}^{0,s}_{\rho(t),1}$ *with the inequality*

$$\|u(t)\|_{\widetilde{B}^{0,s}_{\rho(t),1}} \le C\|u_0\|_{\widetilde{B}^{0,s}_{\rho_0,1}}. \tag{1-10}$$

Remark 1.4. As a by-product of the proof of the uniqueness, we get a stability result with respect to the initial data in the energy space (see Theorem 4.2 below).

Remark 1.5. Owing to $\nabla u \in L^1(\mathbb{R}_+; C_b(\Omega))$, the flow of u has C^1 regularity with respect to the space variable, which means that the geometrical structures of the fluid during the evolution are conserved. For example, if ρ_0 takes two different positive values across a C^1 interface, then it remains so forever: the interface is just transported by the flow and keeps its C^1 regularity. Likewise, the (local) H^2 regularity of the interfaces is preserved since $\nabla^2 u \in L^1(\mathbb{R}_+; L^2(\Omega))$.

Remark 1.6. As said before, for $\Omega = \mathbb{R}^3$, a result in the same spirit has been obtained by Zhang [2020] in the small velocity case; see also [Danchin and Wang 2023]. An important difference with our situation is that, in dimension 3, the critical space for the velocity is $\dot{B}^{1/2}_{2,1} := [L^2, \dot{H}^1]_{1/2,1}$. Hence, it is enough to prove time weighted energy estimates in L^2 and $\dot{H}^1$, and the relevant critical space for u_0 does not depend on ρ_0.

To simplify the presentation, we assume hereafter that $s = \frac{1}{2}$. We use the short notation $\widetilde{B}^0_{\rho_0,1}$ for $\widetilde{B}^{0,1/2}_{\rho_0,1}$.

Let us briefly present the main steps of the proof of Theorem 1.3. The global existence of a solution being ensured by prior results, the main point is to exhibit enough regularity of the solution to ensure uniqueness. As already explained at length in the Introduction, the key is to establish (0-8), and this will be actually performed on the linear system (1-6).

The first step is to prove energy-type weighted estimates for (1-6) that require only u_0 to be in L^2 and the density to be bounded and bounded away from zero. The three principles guiding our search for estimates are:

- taking *convective derivatives* $D_t := \partial_t + v \cdot \nabla$ (since $D_t\rho = 0$) rather than space derivatives, since ρ has no regularity,
- using differential operators $\sqrt{t}\nabla,\ t\partial_t$, and tD_t (that are of order 0 in the parabolic scaling),
- transferring time regularity to space regularity by means of the maximal regularity properties of the Stokes system (see the Appendix), observing that

$$\mu\Delta u - \nabla P = \rho\dot{u} \quad \text{and} \quad \operatorname{div} u = 0 \quad \text{in } \Omega, \quad \text{with } \dot{u} := \partial_t u + v \cdot \nabla u. \tag{1-11}$$

In the end, this allows us to control quantities like $\|\sqrt{t}\nabla u(t)\|_{L^2},\ \|t\partial_t u(t)\|_{L^2},\ \|t\dot{u}(t)\|_{L^2}$, or $\|t\nabla^2 u(t)\|_{L^2}$ in terms of $\|u_0\|_{L^2},\ \rho_*,\ \rho^*$, and energy-like norms of v.

The second step is to propagate the $\dot H^1$ and the $\dot H^{-1}$ norms. On the one hand, $\dot H^1$ estimates for (INS) have been known since [Ladyzhenskaya and Solonnikov 1975] (we shall also derive time weighted versions of these estimates). On the other hand, propagating *negative* Sobolev regularity seems to be new. This will be achieved by duality after observing that the backward system associated with (1-6) satisfies the same family of estimates in $\dot H^s$. However, owing to the density dependent structure of the latter system, we will have only access to $\|\mathcal{P}(\rho u)(t)\|_{\dot H^{-s}}$, whence the "weighted" definition of the interpolation space $\widetilde{B}^{0,s}_{\rho,1}$.

The third step is devoted to propagating the regularity $\widetilde{B}^{0}_{\rho,1}$ and to bounding ∇u in $L^1(\mathbb{R}_+;L^\infty)$ in terms of the data only. In passing, we exhibit some controls of other critical norms (like, e.g., that of $\dot u$ in $L^1(\mathbb{R}_+;L^2)$) that will be needed in the proof of uniqueness and stability. All these bounds rely on the dynamic interpolation method that has been described above for the heat equation. In the end, we get

$$\int_0^\infty \|\nabla u\|_{L^\infty}\,dt+\int_0^\infty \|\dot u\|_{L^2}\,dt+\left(\int_0^\infty t^{2/3}\|\dot u\|_{L^4}^{4/3}\,dt\right)^{3/4}\le C\|u_0\|_{\widetilde{B}^{0}_{\rho_0,1}}.$$

The fourth step is the proof of existence of a global solution corresponding to the assumptions of Theorems 1.1 or 1.3. For Theorem 1.1, the overall strategy is standard: we smooth out the data, resort to classical results that ensure the existence of a sequence of global smooth solutions for (INS), and use the aforementioned estimates and compactness to pass to the limit. For Theorem 1.3, it is a bit the same, except that one has to be careful when smoothing out the velocity, owing to the "exotic" definition of the space $\widetilde{B}^{0}_{\rho_0,1}$. The easiest way is to truncate a decomposition of u_0 so as to have an approximate initial velocity in the smoother space $H^{1/2}$.

The last step is devoted to uniqueness and stability for (INS). As in [Danchin and Mucha 2019], we reformulate (INS) in Lagrangian coordinates. The properties of the solutions provided by Theorem 1.3, in particular (0-8), ensure that the two formulations are equivalent. The gain is that we do not have to worry about the density as it is time-independent. As for the difference of the two velocities in Lagrangian coordinates, it satisfies a parabolic-type equation and may be estimated in

$$L^\infty(\mathbb{R}_+;L^2)\cap L^2(\mathbb{R}_+;\dot H^1).$$

The computations are in the spirit of those of [Danchin et al. 2024]. However, in our case the velocity is less regular by one derivative, which requires some care.

As a concluding remark, we want to point out that, in contrast with numerous recent works dedicated to the inhomogeneous incompressible Navier–Stokes equations, our approach does not use Fourier analysis at all. It just relies on very basic energy arguments, interpolation, embedding, and on the classical regularity theory for the Stokes system (this is the only place where some assumptions have to be made on the fluid domain). For simplicity here we considered $\mathbb{R}^2$, $\mathbb{T}^2$, or C^2 bounded domains, but more general domains could be treated in the same way.

Hereafter we shall focus on the case $\mu=1$ for simplicity. The general case follows due to the rescaling

$$\begin{aligned}\rho(t,x)&:=\tilde\rho(\mu t,x),\\ u(t,x)&:=\mu\tilde u(\mu t,x),\\ P(t,x)&:=\mu^2\widetilde{P}(\mu t,x).\end{aligned}$$

2. Weak solutions with time decay

This section is devoted to proving Theorem 1.1: we here construct finite energy weak solutions satisfying algebraic time decay estimates of different orders, without requiring more regularity on u_0 than L^2. The exponential decay that can be expected in the bounded domain case (see [Danchin et al. 2024]), is not addressed to simplify the presentation, as it is not needed for achieving the main result of the paper.

2.1. ***Time decay estimates for the linearized momentum equation.*** We here aim at proving time weighted energy estimates for the linear system (1-6) in the case where the (smooth enough) given pair (ρ, v) satisfies (1-7) and

$$\rho_* = \inf_{(t,x)\in\mathbb{R}_+\times\Omega} \rho(t,x) > 0,\qquad \rho^* = \sup_{(t,x)\in\mathbb{R}_+\times\Omega} \rho(t,x) < \infty. \tag{2-1}$$

System (1-6) is supplemented with a divergence-free initial velocity field u_0, vanishing at the boundary in the bounded domain case and, in the torus case, such that

$$\int_{\mathbb{T}^2} (\rho_0 u_0)(x)\,dx = 0.$$

This latter assumption is not restrictive owing to the Galilean invariance of the system and will enable us to use freely the Gagliardo–Nirenberg inequality (A-2).

We aim at proving energy estimates for the solution with time weights $t^{k/2}$ for $k \in \{0, 1, 2, 3\}$. We strive for bounds depending only on ρ_*, ρ^*, $\|u_0\|_{L^2}$, and on *energy-type norms of* v in the meaning given at the end of the Introduction of the paper. This latter point is fundamental for getting not only Theorem 1.1 but also Theorem 1.3.

Before proceeding, let us warn the reader that we unfortunately did not find a way to avoid the tedious calculations that will follow, since it is has to be checked with the greatest care that only "energy-type norms" come into play.

The basic energy balance. Taking the L^2 scalar product of (1-6) with u yields

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 = 0. \tag{2-2}$$

From this, we get, for all $t \in \mathbb{R}_+$,

$$\|(\sqrt{\rho}u)(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u\|_{L^2}^2\,d\tau = \|\sqrt{\rho}_0 u_0\|_{L^2}^2. \tag{2-3}$$

As $\rho_* > 0$, combining (2-3) with the Gagliardo–Nirenberg inequality (A-1) recalled in the Appendix yields, for all $2 \le p < \infty$,

$$\|u\|_{L^q(L^p)} \le C_p \rho_*^{-1/2} \|\sqrt{\rho_0}u_0\|_{L^2}, \quad \text{with } \frac{1}{p} + \frac{1}{q} = \frac{1}{2}. \tag{2-4}$$

Estimates with weight $\sqrt{t}$. Let us rewrite (1-6) as

$$\Delta u-\nabla P=\rho\dot u \quad\text{and}\quad \operatorname{div} u=0 \quad \text{in } \Omega, \quad \text{with } \dot u:=u_t+v\cdot\nabla u. \tag{2-5}$$

Taking the $L^2(\Omega;\mathbb{R}^2)$ scalar product of (2-5) with $t\dot u$ yields, for all $t\ge 0$,

$$\int_\Omega \rho t|\dot u|^2\,dx=t\int_\Omega \Delta u\cdot u_t\,dx-t\int_\Omega \nabla P\cdot u_t\,dx+t\int_\Omega(\Delta u-\nabla P)\cdot(v\cdot\nabla u)\,dx.$$

As $\operatorname{div} u=0$, integrating by parts and using again (2-5) yields

$$\frac12\frac{d}{dt}\int_\Omega t|\nabla u|^2\,dx-\frac12\int_\Omega|\nabla u|^2\,dx+\int_\Omega \rho t|\dot u|^2\,dx=\int_\Omega \rho t\dot u\cdot(v\cdot\nabla u)\,dx. \tag{2-6}$$

Remembering (2-2) and performing a time integration, we get, for all $t\ge 0$,

$$\frac14\int_\Omega \rho(t)|u(t)|^2\,dx+\frac t2\int_\Omega|\nabla u(t)|^2\,dx+\int_0^t\!\!\int_\Omega \tau\rho|\dot u|^2\,dx\,d\tau \\ =\frac14\int_\Omega \rho_0|u_0|^2\,dx+\int_0^t\!\!\int_\Omega \tau\rho\dot u\cdot(v\cdot\nabla u)\,dx\,d\tau. \tag{2-7}$$

Of course, since $u_t=\dot u-v\cdot\nabla u$, one can write

$$\tfrac14\|\sqrt{\rho}u_t\|_{L^2}^2\le\tfrac12\|\sqrt{\rho}\dot u\|_{L^2}^2+\tfrac12\|\sqrt{\rho}v\cdot\nabla u\|_{L^2}^2.$$

Hence, adding up this inequality multiplied by t with (2-7) and using Young's inequality to bound the last term of (2-7), we discover that

$$\|\sqrt{\rho(t)}u(t)\|_{L^2}^2+2\|\sqrt{t}\nabla u(t)\|_{L^2}^2+\int_0^t(\|\sqrt{\rho\tau}\dot u\|_{L^2}^2+\|\sqrt{\rho\tau}u_\tau\|_{L^2}^2)\,d\tau \\ \le\|\sqrt{\rho_0}u_0\|_{L^2}^2+6\int_0^t\|\sqrt{\rho\tau}v\cdot\nabla u\|_{L^2}^2\,d\tau. \tag{2-8}$$

Combining Hölder's inequality, Ladyzhenskaya's inequality (0-7), and Young's inequality yields

$$\|\sqrt{\rho}v\cdot\nabla u\|_{L^2}^2\le\frac{\varepsilon}{\rho^*}\|\nabla^2u\|_{L^2}^2+\frac{\rho^*}{\varepsilon}\|\sqrt{\rho}v\|_{L^4}^4\|\nabla u\|_{L^2}^2,\quad \varepsilon>0, \tag{2-9}$$

and taking advantage of the regularity theory of the Stokes system (recalled in the Appendix) gives

$$\|\nabla^2u\|_{L^2}^2+\|\nabla P\|_{L^2}^2\le C_\Omega\rho^*\|\sqrt{\rho}\dot u\|_{L^2}^2. \tag{2-10}$$

Hence, choosing $\varepsilon>0$ suitably small in (2-9), using (2-10), then reverting to (2-8) and applying Gronwall's lemma allows us to conclude that there exist positive constants c_Ω and C_Ω, depending only on Ω, such that

$$X_1(t)\le\|\sqrt{\rho}_0u_0\|_{L^2}^2e^{C_1^v(t)},\quad \text{with } C_1^v(t):=C_\Omega\rho^*\int_0^t\|\sqrt{\rho}v\|_{L^4}^4\,d\tau, \tag{2-11}$$

where

$$X_1(t):=\|(\sqrt{\rho}u)(t)\|_{L^2}^2+2\|\sqrt{t}\nabla u(t)\|_{L^2}^2+\frac12\int_0^t\Big(\|\sqrt{\rho\tau}\dot u\|_{L^2}^2+\|\sqrt{\rho\tau}u_\tau\|_{L^2}^2+\frac{c_\Omega}{\rho^*}\|\sqrt{\tau}(\nabla^2u,\nabla P)\|_{L^2}^2\Big)\,d\tau.$$

Estimates with weight t. Applying ∂_t to (1-6) gives

$$\rho u_{tt} + \rho v\cdot\nabla u_t - \Delta u_t + \nabla P_t = -\rho_t \dot u - \rho v_t\cdot\nabla u. \tag{2-12}$$

As $\operatorname{div} u_t = 0$, testing (2-12) by $t^2 u_t$ then observing that

$$\rho_t = -\operatorname{div}(\rho v) \quad\text{and}\quad |u_t|^2 = |\dot u|^2 - 2\dot u\cdot(v\cdot\nabla u) + |v\cdot\nabla u|^2$$

gives, after performing a few integration by parts,

$$\frac12\frac{d}{dt}\int_\Omega \rho t^2|u_t|^2\,dx + \int_\Omega t^2|\nabla u_t|^2\,dx = \int_\Omega t\rho|\dot u|^2\,dx - 2\int_\Omega \rho t\dot u\cdot(v\cdot\nabla u)\,dx + \int_\Omega t\rho|v\cdot\nabla u|^2\,dx \\ + \int_\Omega t^2\operatorname{div}(\rho v)\dot u\cdot u_t\,dx - \int_\Omega t^2\rho(v_t\cdot\nabla u)\cdot u_t\,dx.$$

Adding up twice (2-2) and (2-6) to this latter inequality, we obtain

$$\frac{d}{dt}\int_\Omega\biggl(\rho|u|^2 + t|\nabla u|^2 + \frac{\rho t^2}{2}|u_t|^2\biggr)dx + \int_\Omega(|\nabla u|^2 + \rho t|\dot u|^2 + t^2|\nabla u_t|^2)\,dx \\ = \int_\Omega \rho t|v\cdot\nabla u|^2\,dx + \int_\Omega t^2\operatorname{div}(\rho v)\dot u\cdot u_t\,dx - \int_\Omega t^2\rho(v_t\cdot\nabla u)\cdot u_t\,dx =: I_1 + I_2 + I_3. \tag{2-13}$$

Thanks to (2-9), (2-10) and Young's inequality, we have

$$I_1 \le \tfrac12\|\sqrt{\rho t}\dot u\|_{L^2}^2 + C\rho^*\|\sqrt\rho v\|_{L^4}^4\|\sqrt t\nabla u\|_{L^2}^2. \tag{2-14}$$

For the term I_2, an integration by parts yields

$$I_2 = -\int_\Omega t^2(\rho v\cdot\nabla\dot u)\cdot u_t\,dx - \int_\Omega t^2(\rho v\cdot\nabla u_t)\cdot\dot u\,dx =: I_{21} + I_{22}.$$

By (0-7), Hölder's and Young's inequalities, and (2-1), we have, for some constant C depending only on $\rho_*,\ \rho^*$, and Ω,

$$I_{21} \le C\|t\nabla\dot u\|_{L^2}\|\sqrt\rho v\|_{L^4}\|tu_t\|_{L^2}^{1/2}\|t\nabla u_t\|_{L^2}^{1/2} \\ \le \tfrac1{10}(\|t\nabla u_t\|_{L^2}^2 + \|t\nabla\dot u\|_{L^2}^2) + C\|\sqrt\rho v\|_{L^4}^4\|\sqrt\rho tu_t\|_{L^2}^2. \tag{2-15}$$

The same arguments lead to

$$I_{22} \le \tfrac1{10}(\|t\nabla u_t\|_{L^2}^2 + \|t\nabla\dot u\|_{L^2}^2) + C\|\sqrt\rho v\|_{L^4}^4\|\sqrt\rho t\dot u\|_{L^2}^2. \tag{2-16}$$

For I_3, one has, still owing to Hölder's and Young's inequalities and (A-1) or (A-2),

$$I_3 \le \|\sqrt{\rho t}\,v_t\|_{L^2}\|t\sqrt\rho\,u_t\|_{L^4}\|\sqrt t\nabla u\|_{L^4} \\ \le \tfrac1{10}\|t\nabla u_t\|_{L^2}\|\nabla u\|_{L^2} + C\|\sqrt{\rho t}v_t\|_{L^2}^2\|t\sqrt\rho u_t\|_{L^2}\|t\nabla^2u\|_{L^2}. \tag{2-17}$$

Hence, inserting (2-14)–(2-17) in (2-13) gives

$$\frac{d}{dt}\bigl(\|\sqrt\rho u\|_{L^2}^2 + \|\sqrt t\nabla u\|_{L^2}^2 + \tfrac12\|\sqrt\rho tu_t\|_{L^2}^2\bigr) + \tfrac12(\|\nabla u\|_{L^2}^2 + \|\sqrt{\rho t}\dot u\|_{L^2}^2 + \|t\nabla u_t\|_{L^2}^2) - \tfrac14\|t\nabla\dot u\|_{L^2}^2 \\ \lesssim \|\sqrt\rho v\|_{L^4}^4(\|\sqrt\rho t(\dot u, u_t)\|_{L^2}^2 + \|\sqrt t\nabla u\|_{L^2}^2) + \|\sqrt{\rho t}\,v_t\|_{L^2}^2\|t\sqrt\rho u_t\|_{L^2}\|t\nabla^2u\|_{L^2}. \tag{2-18}$$

To close the estimate, we have to bound $\|\sqrt{\rho}t\dot{u}\|_{L^2}$, $\|t\nabla^2 u\|_{L^2}$, and $\|t\nabla\dot{u}\|_{L^2}$. For the first two terms, one may use (0-7), (2-10) and the definition of $\dot{u}$ to get

$$\begin{aligned}\|t(\nabla^2 u,\nabla P)\|_{L^2} &\le C_\Omega(\sqrt{\rho^*}\|t\sqrt{\rho}u_t\|_{L^2}+\|\rho t^{1/4}v\|_{L^4}\|\sqrt{t}\nabla u\|_{L^2}^{1/2}\|t\nabla^2 u\|_{L^2}^{1/2})\\ &\le \tfrac12\|t\nabla^2 u\|_{L^2}+C_\Omega(\sqrt{\rho^*}\|t\sqrt{\rho}u_t\|_{L^2}+\|\rho t^{1/4}v\|_{L^4}^2\|\sqrt{t}\nabla u\|_{L^2}).\end{aligned}$$

This, in the end, implies that

$$\frac14\|\sqrt{\rho}t\dot{u}\|_{L^2}+\frac{c_\Omega}{\sqrt{\rho^*}}\|t\nabla^2 u,t\nabla P\|_{L^2}\le C(\|t\sqrt{\rho}u_t\|_{L^2}+\|t^{1/4}v\|_{L^4}^2\|\sqrt{t}\nabla u\|_{L^2}).\tag{2-19}$$

Finally, from the definition of $\dot{u}$, Hölder's inequality and (0-7), we may write

$$\begin{aligned}\|t\nabla\dot{u}\|_{L^2} &\le \|t\nabla u_t\|_{L^2}+\|t\nabla v\cdot\nabla u\|_{L^2}+\|tv\cdot\nabla^2 u\|_{L^2}\\ &\le \|t\nabla u_t\|_{L^2}+\|\sqrt{t}\nabla v\|_{L^4}\|\nabla u\|_{L^2}^{1/2}\|t\nabla^2 u\|_{L^2}^{1/2}+C\|v\|_{L^4}\|t\dot{u}\|_{L^2}^{1/2}\|t\nabla\dot{u}\|_{L^2}^{1/2},\end{aligned}$$

which implies that

$$\|t\nabla\dot{u}\|_{L^2}\le 2\|t\nabla u_t\|_{L^2}+\tfrac14\|\nabla u\|_{L^2}+C(\|\sqrt{t}\nabla v\|_{L^4}^2\|t\nabla^2 u\|_{L^2}+\|v\|_{L^4}^2\|\sqrt{\rho}t\dot{u}\|_{L^2}).\tag{2-20}$$

Let us set

$$\begin{aligned}X_2(t):=\|(\sqrt{\rho}u)(t)\|_{L^2}^2+\|\sqrt{t}\nabla u(t)\|_{L^2}^2+\frac14\|\sqrt{\rho}tu_t\|_{L^2}^2+\frac{1}{16}\|\sqrt{\rho}t\dot{u}\|_{L^2}^2+\frac{c_\Omega}{\rho^*}\|t(\nabla^2 u,\nabla P)\|_{L^2}^2\\ +\frac{1}{16}\int_0^t(\|\nabla u\|_{L^2}^2+\|\sqrt{\rho}\tau\dot{u}\|_{L^2}^2+\|\tau\nabla u_\tau\|_{L^2}^2+\|\tau\nabla\dot{u}\|_{L^2}^2)\,d\tau.\end{aligned}$$

Integrating (2-18) on $[0,t]$, taking advantage of (2-19) and (2-20), and then, finally, using Gronwall's lemma, we conclude that there exists a constant C depending only on Ω, ρ_*, and ρ^* such that

$$\begin{aligned}&X_2(t)\le\|u_0\|_{L^2}^2e^{C_2^v(t)},\\ &\quad\text{with } C_2^v(t):=C\biggl(\sup_{\tau\in[0,t]}\|\tau^{1/4}v(\tau)\|_{L^4}^4+\int_0^t(\|\sqrt{\rho}v\|_{L^4}^4+\|\sqrt{\tau}\nabla v\|_{L^4}^4+\|\sqrt{\rho\tau}v_\tau\|_{L^2}^2)\,d\tau\biggr).\end{aligned}\tag{2-21}$$

Estimates with weight $t^{3/2}$. Let $D_t:=\partial_t+v\cdot\nabla$ and $\ddot{u}:=D_t\dot{u}$. We have[4]

$$\rho\ddot{u}-\Delta\dot{u}+\nabla\dot{P}=F:=\nabla v\cdot\nabla P-\Delta v\cdot\nabla u-2\nabla^2 u\cdot\nabla v.\tag{2-22}$$

Taking the $L^2(\Omega;\mathbb{R}^2)$ scalar product with $t^3\ddot{u}$, we readily get

$$\frac12\frac{d}{dt}\|t^{3/2}\nabla\dot{u}(t)\|_{L^2}^2+\|t^{3/2}\sqrt{\rho}\ddot{u}\|_{L^2}^2=\frac32\|t\nabla\dot{u}\|_{L^2}^2+\sum_{i=1}^5 J_i,\tag{2-23}$$

with

$$J_1:=\int_\Omega\Delta\dot{u}\cdot(t^3v\cdot\nabla\dot{u})\,dx,\quad J_2:=-\int_\Omega\nabla\dot{P}\cdot(t^3v\cdot(\nabla v\cdot\nabla u))\,dx,$$

$$J_3:=\int_\Omega\nabla\dot{P}\cdot(t^3v_t\cdot\nabla u)\,dx,\quad J_4:=\int_\Omega\nabla\dot{P}\cdot(t^3v\cdot(v\cdot\nabla^2 u))\,dx,\quad J_5:=\int_\Omega F\cdot t^3\ddot{u}\,dx.$$

[4] Here we use the notation $(\nabla^2 u\cdot\nabla v)^i:=\sum_{1\le j,k\le d}\partial_k v^j\partial_j\partial_k u^i$.

For any $\varepsilon > 0$, the terms J_1 through J_5 may be bounded as follows by combining Hölder's inequality, Young's inequality, and (A-1) with $p=4$ or $p=6$ (and (A-4) for J_4):

$$\begin{aligned}
J_1 &\le \|t^{3/2}\nabla^2\dot u\|_{L^2}\|v\|_{L^4}\|t^{3/2}\nabla\dot u\|_{L^4}\\
&\le \varepsilon\|t^{3/2}\nabla^2\dot u\|_{L^2}^2 + C_\varepsilon\|v\|_{L^4}^4\|t^{3/2}\nabla\dot u\|_{L^2}^2,\\
J_2 &\le \|t^{3/2}\nabla\dot P\|_{L^2}\|t^{1/6}v\|_{L^6}\|\sqrt{t}\nabla v\|_{L^6}\|t^{5/6}\nabla u\|_{L^6}\\
&\le C\|t^{3/2}\nabla\dot P\|_{L^2}\|t^{1/6}v\|_{L^6}\|\sqrt{t}\nabla v\|_{L^6}\|\sqrt{t}\nabla u\|_{L^2}^{1/3}\|t\nabla^2 u\|_{L^6}^{2/3}\\
&\le \varepsilon\|t^{3/2}\nabla\dot P\|_{L^2}^2 + C_\varepsilon\|t^{1/6}v\|_{L^6}^2\|\sqrt{t}\nabla v\|_{L^6}^2\|\sqrt{t}\nabla u\|_{L^2}^{2/3}\|t\nabla^2 u\|_{L^2}^{4/3},\\
J_3 &\le \|t^{3/2}\nabla\dot P\|_{L^2}\|tv_t\|_{L^4}\|t^{1/2}\nabla u\|_{L^4}\\
&\le \varepsilon\|t^{3/2}\nabla\dot P\|_{L^2}^2 + C_\varepsilon\|tv_t\|_{L^4}^4\|t^{1/2}\nabla u\|_{L^2}^2 + \|t^{1/2}\nabla^2 u\|_{L^2}^2,\\
J_4 &\le \|t^{3/2}\nabla\dot P\|_{L^2}\|t^{1/6}v\|_{L^6}^2\|t^{7/6}\nabla^2 u\|_{L^6}\\
&\le C\|t^{3/2}\nabla\dot P\|_{L^2}\|t^{1/6}v\|_{L^6}^2\|\sqrt{\rho t}\dot u\|_{L^2}^{1/3}\|t^{3/2}\nabla\dot u\|_{L^2}^{2/3}\\
&\le \varepsilon\|t^{3/2}\nabla\dot P\|_{L^2}^2 + C_\varepsilon\|\sqrt{\rho t}\dot u\|_{L^2}^2 + C_\varepsilon\|t^{1/6}v\|_{L^6}^6\|t^{3/2}\nabla\dot u\|_{L^2}^2,\\
J_5 &\le \varepsilon\|t^{3/2}\sqrt{\rho}\ddot u\|_{L^2}^2 + \frac{C_\varepsilon}{\rho^*}\|t^{3/2}F\|_{L^2}^2.
\end{aligned}$$

Thanks to Hölder's inequality, (0-7), and (A-4), we have

$$\begin{aligned}
\|t^{3/2}F\|_{L^2}^2 &\le \|\sqrt{t}\nabla v\|_{L^4}^2\|t(\nabla P,\nabla^2 u)\|_{L^4}^2 + \|t\nabla^2 v\|_{L^4}^2\|\sqrt{t}\nabla u\|_{L^4}^2\\
&\lesssim \|\sqrt{t}\nabla v\|_{L^4}^2\|\sqrt{\rho t}\dot u\|_{L^2}\|t^{3/2}\nabla\dot u\|_{L^2} + \|t\nabla^2 v\|_{L^4}^2\|\sqrt{t}\nabla u\|_{L^2}\|\sqrt{t}\nabla^2 u\|_{L^2}\\
&\lesssim \|\sqrt{\rho t}\dot u\|_{L^2}^2 + \|\sqrt{t}\nabla^2 u\|_{L^2}^2 + \|\sqrt{t}\nabla v\|_{L^4}^4\|t^{3/2}\nabla\dot u\|_{L^2}^2 + \|t\nabla^2 v\|_{L^4}^4\|\sqrt{t}\nabla u\|_{L^2}^2.
\end{aligned}$$

To close the estimates, we need to bound

$$t^{3/2}\nabla\dot P \quad\text{and}\quad t^{3/2}\nabla^2\dot u \quad \text{in } L^2(\mathbb{R}_+\times\Omega).$$

Now, we observe that the couple $(\dot u, \nabla\dot P)$ satisfies the inhomogeneous Stokes system

$$-\Delta\dot u + \nabla\dot P = F - \rho\ddot u \quad\text{and}\quad \operatorname{div}\dot u = \operatorname{Tr}(\nabla v\cdot\nabla u) \quad \text{in } \Omega, \tag{2-24}$$

with boundary condition $\dot u|_{\partial\Omega} = 0$ if Ω is a bounded domain, $\dot u(t)\to 0$ at infinity (due to $\dot u(t)\in L^2$ for all $t>0$) in the case $\Omega=\mathbb{R}^2$, and

$$\int_{\mathbb{T}^2}\rho\dot u\,dx = 0 \quad \text{if } \Omega = \mathbb{T}^2.$$

Hence, applying (A-4) with $p=2$ guarantees that

$$\|\nabla^2\dot u, \nabla\dot P\|_{L^2}^2 \lesssim \|F\|_{L^2}^2 + \|\rho\ddot u\|_{L^2}^2 + \|\nabla^2 v\otimes\nabla u\|_{L^2}^2 + \|\nabla v\otimes\nabla^2 u\|_{L^2}^2. \tag{2-25}$$

The last two terms are parts of F. Hence bounding $\|t^{3/2}F\|_{L^2}$ as above and putting this together with the previous inequalities, we conclude after time integration that

$$
\begin{aligned}
X_3(t) &:= \|t^{3/2}\nabla\dot u(t)\|_{L^2}^2 + \int_0^t \|\tau^{3/2}(\sqrt{\rho}\ddot u, \nabla\dot P, \nabla^2\dot u)\|_{L^2}^2\,d\tau \\
&\lesssim \int_0^t (\|v\|_{L^4}^4 + \|\tau^{1/6}v\|_{L^6}^6 + \|\tau^{1/2}\nabla v\|_{L^4}^4)\|\tau^{3/2}\nabla\dot u\|_{L^2}^2\,d\tau \\
&\qquad + \int_0^t \|\tau^{1/2}\nabla^2 u, \sqrt{\rho\tau}\dot u\|_{L^2}^2\,d\tau + \int_0^t (\|\tau v_\tau\|_{L^4}^4 + \|\tau\nabla^2 v\|_{L^4}^4)\|\tau^{1/2}\nabla u\|_{L^2}^2\,d\tau \\
&\qquad + \int_0^t \|\tau^{1/6}v\|_{L^6}^2\|\sqrt{\tau}\nabla v\|_{L^6}^2\|\sqrt{\tau}\nabla u\|_{L^2}^{2/3}\|\tau\nabla^2 u\|_{L^2}^{4/3}\,d\tau.
\end{aligned}
$$

After using Gronwall's lemma and the inequalities of the previous steps, we get

$$
\begin{aligned}
&X_3(t) \le C\|u_0\|_{L^2}^2 e^{C_3^v(t)}, \\
&\quad \text{with } C_3^v(t) := C\int_0^t \big(\|v\|_{L^4}^4 + (1+\|\tau^{1/4}v\|_{L^4}^4)\|v\|_{L^6}^3 + \|\tau^{1/6}v\|_{L^6}^6 + \|\sqrt{\tau}\nabla v\|_{L^6}^3 \\
&\qquad\qquad + \|\tau^{1/2}v_\tau\|_{L^2}^2 + \|\tau^{1/2}\nabla v\|_{L^4}^4 + \|\tau\nabla^2 v\|_{L^4}^4 + \|\tau v_\tau\|_{L^4}^4\big)\,d\tau. \qquad (2\text{-}26)
\end{aligned}
$$

2.2. ***The proof of Theorem 1.1.*** Let us fix some data (ρ_0, u_0) such that $u_0 \in L^2$ and $0 < \rho_* \le \rho_0 \le \rho^* < \infty$. Then we smooth out the velocity so as to get a sequence $(u_0^n)_{n\in\mathbb{N}}$ of H^1 divergence-free vector fields (vanishing at $\partial\Omega$ in the bounded domain case) that converges strongly to u_0 in L^2. It is known (see [Danchin and Mucha 2019] for the bounded domain or torus cases and [Paicu et al. 2013] for the $\mathbb{R}^2$ case) that such data generate a unique global solution $(\rho^n, u^n, \nabla P^n)$ with relatively smooth velocity. In particular, the computations leading to the estimates of the previous subsection may be justified for $\rho = \rho^n$, $u = v = u^n$, and we get, for all $t \ge 0$ for some constant depending only on ρ_*, ρ^*, and Ω,

$$
X_0^n(t) := \|(\sqrt{\rho^n}u^n)(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u^n\|_{L^2}^2\,d\tau \le \|\sqrt{\rho_0}u_0^n\|_{L^2}^2, \qquad (2\text{-}27)
$$

$$
X_1^n(t) \le \|\sqrt{\rho_0}u_0^n\|_{L^2}^2 e^{C_1^n(t)}, \quad \text{with } C_1^n(t) := C\int_0^t \|u^n\|_{L^4}^4\,d\tau, \qquad (2\text{-}28)
$$

$$
\begin{aligned}
&X_2^n(t) \le \|\sqrt{\rho_0}u_0^n\|_{L^2}^2 e^{C_2^n(t)}, \\
&\quad \text{with } C_2^n(t) := C\Big(\sup_{\tau\in[0,t]}\|\tau^{1/4}u^n(\tau)\|_{L^4}^4 + \int_0^t (\|u^n\|_{L^4}^4 + \|\sqrt{\tau}\nabla u^n\|_{L^4}^4 + \|\sqrt{\tau}u_\tau^n\|_{L^2}^2)\,d\tau\Big), \qquad (2\text{-}29)
\end{aligned}
$$

$$
\begin{aligned}
&X_3^n(t) \le C\|u_0^n\|_{L^2}^2 e^{C_3^n(t)}, \\
&\quad \text{with } C_3^n(t) := C\int_0^t \big((1+\|\tau^{1/4}u^n\|_{L^4}^4)\|u^n\|_{L^6}^3 + \|\tau^{1/6}u^n\|_{L^6}^6 + \|\sqrt{\tau}\nabla v^n\|_{L^6}^3 \\
&\qquad\qquad + \|\tau^{1/2}v_\tau^n\|_{L^2}^2 + \|u^n, \tau^{1/2}\nabla u^n, \tau\nabla^2 u^n, \tau u_\tau^n\|_{L^4}^4\big)\,d\tau. \qquad (2\text{-}30)
\end{aligned}
$$

Above, X_j^n for $j \in \{1, 2, 3\}$ are the quantities defined in (2-11), (2-21), and (2-26), respectively, pertaining to $(\rho^n, u^n, \nabla P^n)$.

The fundamental point is that all the norms coming into play in C_1^n, C_2^n, and C_3^n may be bounded by means of $M := \sup_{n\in\mathbb{N}} \|u_0^n\|_{L^2}$, ρ_*, and ρ^*. For C_1^n, this just stems from (2-4) with $p=4$. Hence we have, for some $C_M := C(\rho_*, \rho^*, M)$,

$$\sup_{t\in\mathbb{R}_+} X_1^n(t) \le C_M.$$

Combining with (0-7) and (2-27), we thus get

$$\sup_{t\in\mathbb{R}_+} \|t^{1/4}u^n(t)\|_{L^4}^4 \lesssim \|u^n\|_{L^\infty(L^2)}^2 \|\sqrt{t}\nabla u^n\|_{L^\infty(L^2)}^2 \lesssim M^2 C_M, \tag{2-31}$$

$$\|\sqrt{t}\nabla u^n\|_{L^4(L^4)}^4 \lesssim \|\sqrt{t}\nabla u^n\|_{L^\infty(L^2)}^2 \|\sqrt{t}\nabla^2 u^n\|_{L^2(L^2)}^2 \lesssim C_M^2, \tag{2-32}$$

$$\|\sqrt{\rho t}u_t^n\|_{L^2(L^2)}^2 \lesssim C_M; \tag{2-33}$$

whence, remembering (2-29), we have, up to a change of C_M,

$$X_2^n(t) \le C_M \quad \text{for all } t \ge 0.$$

Finally, one has to bound the terms of C_3^n independently of n. Let us just treat the third term as an example. We write that, owing to (A-1) with $p=6$,

$$\begin{aligned}\int_0^\infty \|t^{1/6}u^n\|_{L^6}^6\,dt &\lesssim \int_0^\infty \|u^n\|_{L^2}^2 \|\sqrt{t}\nabla u^n\|_{L^2}^2 \|\nabla u^n\|_{L^2}^2\,dt \\ &\le \|u^n\|_{L^\infty(L^2)}^2 \|\sqrt{t}\nabla u^n\|_{L^\infty(L^2)}^2 \|\nabla u^n\|_{L^2(L^2)}^2 \lesssim M^4 C_M.\end{aligned}$$

As a conclusion, we deduce that there exists a constant, still denoted by C_M, such that, for all $n\in\mathbb{N}$, we have

$$\sup_{t\in\mathbb{R}_+} (X_0^n(t) + X_1^n(t) + X_2^n(t) + X_3^n(t)) \le C_M.$$

Regarding the density, the divergence-free property of u^n clearly ensures that,

$$\text{for all } n\in\mathbb{N}, \quad \text{for all } t\in\mathbb{R}_+, \qquad \rho_* \le \rho^n(t) \le \rho^*.$$

At this point, arguing as in the classical proofs of global existence of weak solutions for (INS) (see, e.g., [Boyer and Fabrie 2013; Lions 1996]), one can conclude that $(\rho^n, u^n, \nabla P^n)_{n\in\mathbb{N}}$ converges weakly, up to a subsequence, to a global distributional solution of (INS) satisfying not only (2-1) and the usual energy inequality (0-3), but also

$$\sup_{t\in\mathbb{R}_+} (X_1(t) + X_2(t) + X_3(t)) \le C_{\rho_*,\rho^*,\|u_0\|_{L^2}}.$$

3. More decay estimates

The goal of this section is to prove that the solutions to the linearized momentum equation (1-6), with ρ satisfying (2-1) and v verifying the regularity properties listed in Theorem 1.1 supplemented with divergence-free u_0 in $\widetilde{B}^0_{\rho_0,1}$, satisfy (0-8). Achieving this requires several steps. The cornerstones are estimates in $\dot{H}^1$ and $\dot{H}^{-1}$ for the solution to (1-6) (in addition to the estimates that have been proved hitherto) and the interpolation method that has been described in Section 1.

3.1. ***A priori estimates involving $\dot H^1$ regularity of u_0.*** In this section, we consider system (1-6) with some source term g. Our aim is to prove estimates of u in $\dot H^1$ in terms of $\nabla u_0 \in L^2$ and g in $L^2(L^2)$. Considering here a source term will be needed when proving estimates in $\dot H^{-1}$ by means of a duality method.

Basic estimates in $\dot H^1$. Let $f := g/\rho$. Taking the L^2 scalar product of the first line of (1-6) with u_t yields, after integrating by parts in the term with Δu,

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \|\sqrt{\rho}\,u_t\|_{L^2}^2 = \int_\Omega \sqrt{\rho}(f - v\cdot\nabla u)\cdot(\sqrt{\rho}u_t)\,dx. \tag{3-1}$$

By virtue of Young's and Hölder's inequality, we have

$$\int_\Omega \sqrt{\rho}(f - v\cdot\nabla u)\cdot(\sqrt{\rho}u_t)\,dx \le \frac{1}{2}\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\sqrt{\rho}f\|_{L^2}^2 + \|\sqrt{\rho}v\cdot\nabla u\|_{L^2}^2.$$

Since $\dot u = u_t + v\cdot\nabla u$, we may write

$$\|\sqrt{\rho}\dot u\|_{L^2} \le \|\sqrt{\rho}u_t\|_{L^2} + \|\sqrt{\rho}v\cdot\nabla u\|_{L^2}.$$

Remembering (2-9), this yields, for some constant c_Ω depending only on Ω,

$$\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \frac{1}{4}\|\sqrt{\rho}(u_t,\dot u)\|_{L^2}^2 + \frac{c_\Omega}{\rho^*}\|\nabla^2 u, \nabla P\|_{L^2}^2 \le 4\|\sqrt{\rho}f\|_{L^2}^2. \tag{3-2}$$

In the end, combining with Gronwall's lemma and remembering that $f = g/\rho$, we get

$$\begin{aligned}\|\nabla u(t)\|_{L^2}^2 &+ \frac{1}{4}\int_0^t \|\sqrt{\rho}(u_t,\dot u)\|_{L^2}^2\,d\tau + \frac{c_\Omega}{\rho^*}\int_0^t \|\nabla^2 u, \nabla P\|_{L^2}^2\,d\tau \\ &\le e^{C\rho^*\int_0^t \|\sqrt{\rho}v\|_{L^4}^4\,d\tau}\biggl(\|\nabla u_0\|_{L^2}^2 + 4\int_0^t e^{-C\rho^*\int_0^\tau \|\sqrt{\rho}v\|_{L^4}^4\,d\tau'}\Bigl\|\frac{g}{\sqrt{\rho}}\Bigr\|_{L^2}^2\,d\tau\biggr).\end{aligned} \tag{3-3}$$

Decay estimates with weight $\sqrt{t}$. Assuming in the rest of this section that $g \equiv 0$, we proceed as when proving (2-21) except that we take the L^2 scalar product of (2-12) with tu_t instead of $t^2 u_t$. In this way, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Bigl(\|\sqrt{\rho t}u_t\|_{L^2}^2 &+ \frac{1}{2}\|\nabla u\|_{L^2}^2\Bigr) + \|\sqrt{t}\nabla u_t\|_{L^2}^2 \\ &= \int_\Omega t\,\mathrm{div}(\rho v)\dot u\cdot u_t\,dx - \int_\Omega t\rho(v_t\cdot\nabla u)\cdot u_t\,dx - \int_\Omega \rho(v\cdot\nabla u)\cdot u_t\,dx.\end{aligned} \tag{3-4}$$

Combining (A-1), Young's inequality, and (2-9) gives

$$-2\int_\Omega \rho(v\cdot\nabla u)\cdot u_t\,dx \le \frac{1}{2}\|\sqrt{\rho}u_t\|_{L^2}^2 + \frac{c_\Omega}{\rho^*}\|\nabla^2 u\|_{L^2}^2 + C\rho^*\|\sqrt{\rho}v\|_{L^4}^4\|\nabla u\|_{L^2}^2.$$

Hence, adding half (3-2) to (3-4) yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\sqrt{\rho t}u_t\|_{L^2}^2 &+ \|\nabla u\|_{L^2}^2) + \|\sqrt{t}\nabla u_t\|_{L^2}^2 + \frac{1}{6}\|\sqrt{\rho}(u_t,\dot u)\|_{L^2}^2 + c_\Omega\|\nabla^2 u, \nabla P\|_{L^2}^2 \\ &\le C\|\sqrt{\rho}v\|_{L^4}^4\|\nabla u\|_{L^2}^2 + \int_\Omega t\,\mathrm{div}(\rho v)\dot u\cdot u_t\,dx - \int_\Omega t\rho(v_t\cdot\nabla u)\cdot u_t\,dx.\end{aligned} \tag{3-5}$$

We integrate by parts in the second term of the right-hand side, which gives

$$\int_\Omega t\,\mathrm{div}(\rho v)\dot u\cdot u_t\,dx=-\int_\Omega t(\rho v\cdot\nabla\dot u)\cdot u_t\,dx-\int_\Omega t(\rho v\cdot\nabla u_t)\cdot\dot u\,dx.$$

The two integrals may be handled as when proving (2-21). We get

$$\int_\Omega t\,\mathrm{div}(\rho v)\dot u\cdot u_t\,dx\le\frac14\|\sqrt t(\nabla\dot u,\nabla u_t)\|_{L^2}^2+C\|\sqrt\rho v\|_{L^4}^4\|\sqrt{\rho t}(\dot u,u_t)\|_{L^2}^2.$$

To bound the last term of (3-5), we proceed as follows (for all $\varepsilon>0$):

$$\begin{aligned}\int_\Omega t\rho(v_t\cdot\nabla u)\cdot u_t\,dx&\le\|\sqrt{\rho t}v_t\|_{L^2}\|\sqrt{\rho t}u_t\|_{L^4}\|\nabla u\|_{L^4}\\&\le\varepsilon\|\nabla^2u\|_{L^2}^2+\varepsilon\|\sqrt t\nabla u_t\|_{L^2}^2+C_\varepsilon\|\sqrt{\rho t}v_t\|_{L^2}^2\|\sqrt{\rho t}u_t\|_{L^2}\|\nabla u\|_{L^2}.\end{aligned}$$

From the definition of $\dot u$ and (2-10), it is easy to get

$$\|\sqrt t(\nabla^2u,\nabla P,\sqrt\rho\dot u)\|_{L^2}\le C(\|\sqrt{\rho t}u_t\|_{L^2}+\|\sqrt\rho v\|_{L^4}^2\|\sqrt t\nabla u\|_{L^2}).\tag{3-6}$$

By Hölder's inequality, (A-1), and (A-4) with $p=4$, we also notice that

$$\|\sqrt t\nabla\dot u\|_{L^2}-\|\sqrt t\nabla u_t\|_{L^2}\lesssim\|\sqrt t\nabla v\|_{L^4}\|\nabla u\|_{L^2}^{1/2}\|\nabla^2u\|_{L^2}^{1/2}+\|v\|_{L^4}\|\sqrt{\rho t}\dot u\|_{L^2}^{1/2}\|\sqrt t\nabla\dot u\|_{L^2}^{1/2}$$

which implies that

$$\|\sqrt t\nabla\dot u\|_{L^2}\le2\|\sqrt t\nabla u_t\|_{L^2}+\tfrac14\|\nabla^2u\|_{L^2}+C(\|\sqrt t\nabla v\|_{L^4}^2\|\nabla u\|_{L^2}+\|v\|_{L^4}^2\|\sqrt{\rho t}\dot u\|_{L^2}).$$

Inserting all the above inequalities in (3-5) then using Gronwall's lemma and (2-11), we discover that

$$Y_1(t)\lesssim\|\nabla u_0\|_{L^2}^2e^{\widetilde C_1^v(t)},\quad\text{with }\widetilde C_1^v(t):=C\int_0^t(\|\sqrt\tau\nabla v,v\|_{L^4}^4+\|\sqrt{\rho\tau}v_\tau\|_{L^2}^2)\,d\tau,\tag{3-7}$$

where

$$\begin{aligned}Y_1(t):=\|\sqrt{\rho t}(u_t,\dot u)\|_{L^2}^2+\|\nabla u\|_{L^2}^2+\frac{c_\Omega}{\rho^*}\|\sqrt t(\nabla^2u,\nabla P)\|_{L^2}^2&\\+\int_0^t\Big(\|\sqrt\tau(\nabla u_\tau,\nabla\dot u)\|_{L^2}^2+\|\sqrt\rho(u_\tau,\dot u)\|_{L^2}^2+\frac{c_\Omega}{\rho^*}\|\nabla^2u,\nabla P\|_{L^2}^2\Big)\,d\tau.&\end{aligned}$$

Decay estimates with weight t. Still assuming $f\equiv0$, we now take the L^2 scalar product of (2-22) with $tD_t(t\dot u)$ and get

$$\frac12\frac{d}{dt}\|\nabla(t\dot u)\|_{L^2}^2+\|\sqrt\rho D_t(t\dot u)\|_{L^2}^2=\int_\Omega(tF-t\nabla\dot P+\rho\dot u)\cdot D_t(t\dot u)\,dx+\int_\Omega\Delta(t\dot u)\cdot(v\cdot\nabla(t\dot u))\,dx.$$

Hence, for all $\varepsilon>0$,

$$\begin{aligned}\frac12\frac{d}{dt}\|\nabla(t\dot u(t))\|_{L^2}^2+\|\sqrt\rho D_t(t\dot u)\|_{L^2}^2&\\\le\varepsilon(\|\nabla^2(t\dot u)\|_{L^2}^2+\|\sqrt\rho D_t(t\dot u)\|_{L^2}^2)+\frac1\varepsilon\bigg(\|v\cdot\nabla(t\dot u)\|_{L^2}^2+\|\sqrt\rho\dot u\|_{L^2}^2+\bigg\|\frac{tF-t\dot\nabla P}{\sqrt\rho}\bigg\|_{L^2}^2\bigg).&\end{aligned}\tag{3-8}$$

To continue, we must estimate $t\dot P$ and $t\nabla^2\dot u$. To this end, we recall inequality (2-25) and observe that

$$\|\sqrt\rho t\ddot u\|_{L^2}\le\|\sqrt\rho D_t(t\dot u)\|_{L^2}+\|\sqrt\rho\dot u\|_{L^2}.$$

Hence, taking ε small enough in (3-8) yields

$$\|\nabla(t\dot u(t))\|_{L^2}^2+\|\sqrt{\rho}D_t(t\dot u),\nabla(t\dot P),\nabla^2(t\dot u)\|_{L^2}^2 \lesssim \|\sqrt{\rho}\dot u\|_{L^2}^2+\|v\cdot\nabla(t\dot u)\|_{L^2}^2+\|t\nabla^2 v\otimes\nabla u\|_{L^2}^2+\|t\nabla^2 u\otimes\nabla v\|_{L^2}^2+\|t\nabla v\cdot\nabla P\|_{L^2}^2. \tag{3-9}$$

We can bound the first term of the right-hand side according to (3-3). To bound the other terms, we have

$$\|v\cdot\nabla(t\dot u)\|_{L^2}^2\le\frac{C}{\varepsilon}\|v\|_{L^4}^4\|\nabla(t\dot u)\|_{L^2}^2+\varepsilon\|\nabla^2(t\dot u)\|_{L^2}^2,$$
$$\|t\nabla^2 v\otimes\nabla u\|_{L^2}^2\lesssim\|t\nabla^2 v\|_{L^4}^2(\|\nabla u\|_{L^2}^2\|\nabla^2 u\|_{L^2}^2)^{1/2},$$
$$\|t\nabla^2 u\otimes\nabla v\|_{L^2}^2+\|t\nabla v\cdot\nabla P\|_{L^2}^2\lesssim\|\sqrt{t}(\nabla^2 u,\nabla P)\|_{L^4}^2\|\sqrt{t}\nabla v\|_{L^4}^2.$$

Using regularity estimates for (2-5) and (0-7) yields

$$\|\sqrt{t}(\nabla^2 u,\nabla P)\|_{L^4}^2\lesssim\|\sqrt{t}\dot u\|_{L^4}^2\lesssim\|\dot u\|_{L^2}\|t\nabla\dot u\|_{L^2}.$$

Hence

$$\|t\nabla^2 u\otimes\nabla v\|_{L^2}^2+\|t\nabla v\cdot\nabla P\|_{L^2}^2\lesssim\|\sqrt{t}\nabla v\|_{L^4}^2\|\dot u\|_{L^2}\|t\nabla\dot u\|_{L^2}\lesssim\|\dot u\|_{L^2}^2+\|\sqrt{t}\nabla v\|_{L^4}^4\|t\nabla\dot u\|_{L^2}^2.$$

Plugging all these inequalities in (3-8), using (3-3), and integrating on $[0,t]$ gives

$$Y_2(t):=\|\nabla(t\dot u(t))\|_{L^2}^2+\int_0^t\|\sqrt{\rho}D_\tau(\tau\dot u),\nabla(\tau\dot P),\nabla^2(\tau\dot u)\|_{L^2}^2\,d\tau$$
$$\lesssim\int_0^t(\|v\|_{L^4}^4+\|\sqrt{\tau}\nabla v\|_{L^4}^4)\|\tau\nabla\dot u\|_{L^2}^2\,d\tau+\|\nabla u_0\|_{L^2}^2e^{C\int_0^t\|v\|_{L^4}^4\,d\tau}(1+\|\tau\nabla^2 v\|_{L^4_t(L^4)}^4).$$

At this stage, Gronwall's lemma enables us to conclude

$$Y_2(t)\le C\|\nabla u_0\|_{L^2}^2e^{\widetilde{C}_2^v(t)},\quad\text{with }\widetilde{C}_2^v(t):=C\int_0^t\|v,\sqrt{\tau}\nabla v,\tau\nabla^2 v\|_{L^4}^4\,d\tau. \tag{3-10}$$

Estimates in $\dot H^s$ for $s\in(0,1)$. If we denote by E the linear operator that associates to (u_0,g) the solution u to (1-6) on $\mathbb{R}_+\times\Omega$, then the previous inequalities (2-3) and (3-3) and the fact that the norms in $L^2(\rho\,dx)$ or $L^2(dx)$ are equivalent (recall (0-4)) ensure that

- E maps $L^2(\Omega)\times L^2(\mathbb{R}_+;\dot H^{-1}(\Omega))$ to $L^\infty(\mathbb{R}_+;L^2(\Omega))\cap L^2(\mathbb{R}_+;\dot H^1(\Omega))$,
- E maps $\dot H^1(\Omega)\times L^2(\mathbb{R}_+;L^2(\Omega))$ to $L^\infty(\mathbb{R}_+;\dot H^1(\Omega))\cap L^2(\mathbb{R}_+;\dot H^2(\Omega))$.

Consequently, the complex interpolation theory ensures that, for all $s\in[0,1]$,

$$E:\dot H^s(\Omega)\times L^2(\mathbb{R}_+;\dot H^{s-1}(\Omega))\to L^\infty(\mathbb{R}_+;\dot H^s(\Omega))\cap L^2(\mathbb{R}_+;\dot H^{s+1}(\Omega)),$$

with, for some constant C_ρ depending only on ρ_* and ρ^*, we have the bound

$$\sup_{t\in[0,T]}\|u(t)\|_{\dot H^s}^2+\int_0^T\|u\|_{\dot H^{s+1}}^2\,dt\le C_\rho e^{Cs\rho^*\int_0^T\|\sqrt{\rho}v\|_{L^4}^4\,dt}\left(\|u_0\|_{\dot H^s}^2+\int_0^T\|g\|_{\dot H^{s-1}}^2\,dt\right). \tag{3-11}$$

For $g\equiv 0$, due to (2-21) and (3-10), for all $t>0$, the linear operator that associates to u_0 the function $t\dot u(t)$ — with u being the solution to (1-6) with no source term — maps L^2 to L^2 and $\dot H^1$ to $\dot H^1$. Hence it maps $\dot H^s$ to $\dot H^s$ for all $s\in[0,1]$, and we have

$$\|t\dot u(t)\|_{\dot H^s}\le Ce^{(s/2)\widetilde{C}_2^v(t)}\|u_0\|_{\dot H^s}\quad\text{for all }t>0. \tag{3-12}$$

3.2. ***Estimates in negative Sobolev spaces.*** We here prove estimates for (1-6) in the case of initial data in Sobolev space with negative regularity.

Data in $\dot H^{-1}$. To estimate $\sqrt{\rho}u$ in $L^2(0,T\times\Omega)$, we consider the *backward* parabolic system

$$\begin{cases} \rho w_t+\rho v\cdot\nabla w+\Delta w+\nabla Q=\rho u,\\ \operatorname{div} w=0,\\ w|_{t=T}=0.\end{cases}\tag{3-13}$$

By definition of w, we have

$$\int_0^T\!\!\int_\Omega u\cdot(\rho u)\,dx\,dt=\int_0^T\!\!\int_\Omega u\cdot(\rho w_t+\rho v\cdot\nabla w+\Delta w+\nabla Q)\,dx\,dt.$$

Integrating by parts and remembering that $\partial_t\rho+\operatorname{div}(\rho v)=0$ and $\operatorname{div} w=0$ yields

$$\int_0^T\!\!\int_\Omega \rho|u|^2\,dx\,dt=-\int_0^T\!\!\int_\Omega(\rho\dot u-\Delta u+\nabla P)\cdot w\,dx\,dt+\int_\Omega((\rho u)(T)\cdot w(T)-\rho_0u_0\cdot w(0))\,dx.$$

As $w(T)=0$ and u satisfies (1-6), we conclude that

$$\int_0^T\!\!\int_\Omega \rho|u|^2\,dx\,dt=-\int_\Omega\rho_0u_0\cdot w(0)\,dx\le\|\rho_0u_0\|_{\dot H^{-1}}\|\nabla w(0)\|_{L^2}.$$

Adapting the proof of (3-3) to (3-13) yields

$$\|\nabla w(0)\|_{L^2}^2\le e^{\rho^*\int_0^T\|\sqrt{\rho}\,v\|_{L^4}^4\,dt}\|\sqrt{\rho}u\|^2_{L^2(0,T\times\Omega)}.$$

Hence we have

$$\|\sqrt{\rho}u\|_{L^2(0,T\times\Omega)}\le\|\rho_0u_0\|_{\dot H^{-1}}e^{(\rho^*/2)\int_0^T\|\sqrt{\rho}v\|_{L^4}^4\,dt}.\tag{3-14}$$

In order to bound $\mathcal{P}(\rho u)(T)$ in $\dot H^{-1}$, we start from

$$\|\mathcal{P}(\rho u)(T)\|_{\dot H^{-1}}=\sup_{\substack{\|w_T\|_{\dot H^1}=1\\ \operatorname{div} w=0}}\int_\Omega(\rho u)(T)\cdot w_T\,dx$$

and solve (3-13) with no source term and data w_T at time $t=T$. Hence,

$$\begin{aligned}0&=\int_0^T\!\!\int_\Omega(\rho w_t+\rho v\cdot\nabla w+\Delta w+\nabla Q)\cdot u\,dx\,dt\\ &=-\int_0^T\!\!\int_\Omega\rho(\partial_t u+v\cdot\nabla u-\Delta u)\cdot w\,dx\,dt+\int_\Omega(\rho(T)u(T)\cdot w_T-\rho_0u_0\cdot w(0))\,dx.\end{aligned}$$

Since u satisfies (1-6) and $\operatorname{div} w=0$, we get

$$\int_\Omega(\rho u)(T)\cdot w_T\,dx=\int_\Omega\rho_0u_0\cdot w(0)\,dx.\tag{3-15}$$

As

$$\|\nabla w(0)\|_{L^2}\le e^{(\rho^*/2)\int_0^T\|\sqrt{\rho}v\|_{L^4}^4\,dt}\|\nabla w_T\|_{L^2},$$

we conclude that

$$\|\mathcal{P}(\rho u)(T)\|_{\dot H^{-1}}\le\|\mathcal{P}(\rho_0u_0)\|_{\dot H^{-1}}e^{(\rho^*/2)\int_0^T\|\sqrt{\rho}v\|_{L^4}^4\,dt}.\tag{3-16}$$

Estimates in $\dot H^{-s}$ for $s\in(0,1)$. We start from

$$\|\mathcal{P}(\rho u)(T)\|_{\dot H^{-s}}=\sup_{\substack{\|w_T\|_{\dot H^s}=1\\ \operatorname{div} w=0}}\int_\Omega(\rho u)(T)\cdot w_T\,dx.$$

Using (3-15), we get, for any divergence-free $w_T\in\dot H^s$ with norm equal to 1,

$$\left|\int_\Omega(\rho u)(T)\cdot w_T\,dx\right|\le\|\mathcal{P}(\rho_0u_0)\|_{\dot H^{-s}}\|w(0)\|_{\dot H^s},$$

where w is the solution of (3-13) with no source term and data w_T at time T.

Keeping (3-11) in mind, we easily conclude that

$$\|\mathcal{P}(\rho u)(T)\|_{\dot H^{-s}}\le C\|\mathcal{P}(\rho_0u_0)\|_{\dot H^{-s}}e^{(Cs/2)\rho^*\int_0^T\|\sqrt{\rho}\,v\|_{L^4}^4\,d\tau}. \tag{3-17}$$

3.3. *More time decay estimates.* In this section, we point out a number of time decay estimates for (1-6) in Sobolev and Lebesgue spaces that may be deduced from what we proved hitherto and basic interpolation results.

Sobolev decay estimates. These are summarized in the following proposition.

Proposition 3.1. *The following estimates hold*:

- *For any* $0\le s\le 2$ *and* $0\le s'\le 1$, *we have*

$$\|u(t)\|_{\dot H^s}\le C_{\rho,v}t^{-(s+s')/2}\|\mathcal{P}(\rho_0u_0)\|_{\dot H^{-s'}},\quad t>0. \tag{3-18}$$

- *For any* $0\le s,s'\le 1$,

$$\|tu_t(t)\|_{\dot H^s}+\|t\dot u(t)\|_{\dot H^s}\le C_{\rho,v}\,t^{-(s+s')/2}\|\mathcal{P}(\rho_0u_0)\|_{\dot H^{-s'}},\quad t>0. \tag{3-19}$$

- *For any* $0\le s\le 1$,

$$\|t\dot u(t),u(t)\|_{\dot H^1}\le Ce^{\widetilde C_2^v(t)+\widetilde C_3^v(t)}t^{(s-1)/2}\|u_0\|_{\dot H^s}, \tag{3-20}$$

$$\|\dot u(t),u_t(t)\|_{L^2}\le Ce^{\widetilde C_2^v(t)+\widetilde C_3^v(t)}t^{-(2-s)/2}\|u_0\|_{\dot H^s}, \tag{3-21}$$

$$\|\dot u(t)\|_{\dot H^s}\le Ce^{\widetilde C_2^v(t)+\widetilde C_3^v(t)}t^{-(1+s)/2}\|u_0\|_{\dot H^1}. \tag{3-22}$$

Proof. The previous sections guarantee that

$$t^{k/2}\|\nabla^ku(t)\|_{L^2}\le C_{\rho,v}\|u_0\|_{L^2}\quad\text{for }k=0,1,2, \tag{3-23}$$

$$t^{1+k/2}\|\nabla^k(u_t,\dot u)(t)\|_{L^2}\le C_{\rho,v}\|u_0\|_{L^2}\quad\text{for }k=0,1. \tag{3-24}$$

The key observation for proving (3-18) is that having the density bounded and bounded away from zero ensures that

$$\|\mathcal{P}(\rho z)\|_{L^2}\simeq\|z\|_{L^2}\quad\text{for all }z\in L^2_\sigma. \tag{3-25}$$

Indeed, since $\mathcal{P}$ is an L^2 orthogonal projector, we may write

$$\|\mathcal{P}(\rho z)\|_{L^2}\le\|\rho z\|_{L^2}\le\rho^*\|z\|_{L^2}$$

and

$$\rho_*\|z\|_{L^2}^2 \le \int_\Omega \rho|z|^2\,dx = \int_\Omega \mathcal{P}(\rho z)\cdot z\,dx \le \|\mathcal{P}(\rho z)\|_{L^2}\|z\|_{L^2}.$$

Inequality (3-18) in the case $s'=0$ thus follows from (3-23), with $k=0,2$, and complex interpolation. In order to attain negative values of s', we use again (3-25) then argue by duality as follows for all $t>0$:

$$\|\mathcal{P}(\rho u)(t)\|_{L^2} = \sup_{\|w\|_{L^2_\sigma}=1}\int_\Omega (\rho u)(t)\cdot w\,dx = \sup_{\|w\|_{L^2_\sigma}=1}\int_\Omega \rho_0 u_0\cdot w(0)\,dx \le \|\mathcal{P}(\rho_0 u_0)\|_{\dot H^{-s'}} \sup_{\|w\|_{L^2_\sigma}=1}\|w(0)\|_{\dot H^{s'}},$$

where $w(0)$ stands for the solution at time $t=0$ of the backward Stokes system (3-13) with no source term and data w at time t. Now, using the inequality we have just proved (that, obviously, also holds for (3-13)), we discover that

$$\|w(0)\|_{\dot H^{s'}} \le Ct^{-s'/2}\|w\|_{L^2},$$

whence

$$\|\rho(t)u(t)\|_{L^2} \le Ct^{-s'/2}\|\mathcal{P}(\rho_0 u_0)\|_{\dot H^{-s'}}. \tag{3-26}$$

Since inequality (3-23) is valid on any interval $[t_0, t]$ (if replacing u_0 by $u(t_0)$ and t by $t-t_0$, of course), one can assert that, for all $s\in[0,2]$, we have

$$\|u(t)\|_{\dot H^s} \le Ct^{-s/2}\left\|(\rho u)\left(\tfrac12 t\right)\right\|_{L^2},$$

which, combined with (3-26) $\left(\text{at time } \tfrac12 t\right)$ completes the proof of (3-18) for all $0\le s\le 2$ and $0\le s'\le 1$.

Next, using (3-24), with $k=0,1$, and complex interpolation yields (3-19) for $s'=0$ and all $s\in[0,1]$. Since the inequality also holds if u_0 is replaced with $u\left(\tfrac12 t\right)$, using again (3-26) yields the desired inequality for all $s'\in[0,1]$.

By the same token, combining the above result with the continuity properties resulting from inequalities (2-26), (3-3), (3-7) and (3-10) gives the last three inequalities of the statement. The details are left to the reader. □

Decay estimates in Lebesgue spaces. Inequalities (3-23) and (3-24) also imply the following result.

Proposition 3.2. *The following inequalities hold*:

- *If* $1<p\le 2\le q\le\infty$ *then*

$$\|u(t)\|_{L^q} + \|\sqrt{t}\nabla u(t)\|_{L^q} \le C_{\rho,v} t^{1/q-1/p}\|u_0\|_{L^p}. \tag{3-27}$$

- *If* $1<p\le 2\le q<\infty$ *then*

$$\|t(\dot u, u_t, \nabla^2 u, \nabla P)(t)\|_{L^q} \le C_{\rho,v} t^{1/q-1/p}\|u_0\|_{L^p}. \tag{3-28}$$

Proof. Combining the Gagliardo–Nirenberg inequality (A-1) and (3-23) with $k = 0, 1, 2$, it is easy to get

$$\|u(t)\|_{L^q} + \|\sqrt{t}\nabla u(t)\|_{L^q} \le C_{\rho,v} t^{1/q-1/2}\|u_0\|_{L^2}, \quad 2 \le q < \infty, \tag{3-29}$$

while (3-24) ensures that

$$\|u_t(t), \dot{u}(t)\|_{L^q} \le C_{\rho,v} t^{1/q-3/2}\|u_0\|_{L^2}. \tag{3-30}$$

Since $(u, \nabla P)$ satisfies the Stokes system (2-5), inequality (A-4) gives

$$\|\nabla^2 u(t)\|_{L^q} + \|\nabla P(t)\|_{L^q} \le C_{\rho,v} t^{1/q-3/2}\|u_0\|_{L^2}, \quad 2 \le q < \infty. \tag{3-31}$$

Remember that[5]

$$\|z\|_{L^\infty} \le C\|z\|_{L^4}^{1/2}\|\nabla z\|_{L^4}^{1/2}. \tag{3-32}$$

Taking first $z = u$ and using (3-29) with $p = 4$, then $z = \nabla u$ and using (3-31) with $p = 4$ allows us to reach the index $q = \infty$ in (3-29).

In (3-29) and (3-31), the term $\|u_0\|_{L^2}$ may be replaced with $\|u(\frac{1}{2}t)\|_{L^2}$. Consequently, using (2-1), (3-26), embedding $L^p \hookrightarrow \dot{H}^{-1+2/p}$ for all $1 < p \le 2$, and the fact that $\mathcal{P} : L^p \to L^p$ ensures that

$$\begin{aligned}\|u(t)\|_{L^2} \simeq \|\mathcal{P}(\rho u)(t)\|_{L^2} &\le C_{\rho,v} t^{1/2-1/p}\|\mathcal{P}(\rho_0 u_0)\|_{\dot{H}^{1-2/p}} \\ &\le C_{\rho,v} t^{1/2-1/p}\|\mathcal{P}(\rho_0 u_0)\|_{L^p} \le C_{\rho,v} t^{1/2-1/p}\|u_0\|_{L^p},\end{aligned}$$

which, plugged into (3-29) and (A-4), completes the proofs of (3-27) and (3-28) for all admissible values of p and q. □

Decay estimates for L^2-in-time norms. Putting together (2-3), (2-11), (2-21), and (2-26), we see that

$$\begin{aligned}\int_0^t \big(\|\nabla u\|_{L^2}^2 + \|\sqrt{\tau}(\nabla^2 u, \nabla P)\|_{L^2}^2 + \|\sqrt{\tau}(\dot{u}, u_\tau)\|_{L^2}^2 \\ + \|\tau(\nabla u_\tau, \nabla\dot{u})\|_{L^2}^2 + \|\tau^{3/2}\ddot{u}\|_{L^2}^2 + \|\tau^{3/2}(\nabla^2\dot{u}, \nabla\dot{P})\|_{L^2}^2\big)\, d\tau \le C_{\rho,v}\|u_0\|_{L^2}^2.\end{aligned} \tag{3-33}$$

This will enable us to prove the following family of decay estimates.

Proposition 3.3. *The following inequalities hold*:

$$\|\tau^{1/2-1/q}\nabla u\|_{L_t^2(L^q)} \le C_{\rho,v}\|u_0\|_{L^2} \quad \text{for all } 2 \le q \le \infty, \tag{3-34}$$

$$\|\tau^{1-1/q}(\dot{u}, u_t)\|_{L_t^2(L^q)} \le C_{\rho,v}\|u_0\|_{L^2} \quad \text{for all } 2 \le q \le \infty, \tag{3-35}$$

$$\|\tau^{1-1/q}(\nabla^2 u, \nabla P)\|_{L_t^2(L^q)} \le C_{\rho,v}\|u_0\|_{L^2} \quad \text{for all } 2 \le q < \infty, \tag{3-36}$$

$$\|\tau^{3/2-1/q}\nabla\dot{u}\|_{L_t^2(L^q)} \le C_{\rho,v}\|u_0\|_{L^2} \quad \text{for all } 2 \le q < \infty. \tag{3-37}$$

Proof. Except for $q = \infty$, inequality (3-34) follows from the Gagliardo–Nirenberg inequality (A-1) and the fact that

$$\|\nabla u\|_{L_t^2(L^2)} + \|\sqrt{\tau}\nabla^2 u\|_{L_t^2(L^2)} \le C_{\rho,v}\|u_0\|_{L^2}.$$

[5] In the torus case, this inequality holds under the assumption $\int_{\mathbb{T}^2} az\, dx = 0$ for some nonnegative function a with mean value 1. The idea of the proof is similar to that of (A-2).

Similarly, except for the case $q=\infty$, inequality (3-35) for $\dot u$ stems from (A-1) and

$$\|\tau\nabla\dot u\|_{L^2_t(L^2)}+\|\sqrt{\tau}\dot u\|_{L^2_t(L^2)}\le C_{\rho,v}\|u_0\|_{L^2}.$$

Now, since (u,P) satisfies (2-5), the regularity properties of the Stokes system pointed out in (A-4) and (3-35) guarantee that

$$\|\tau^{1-1/q}(\nabla^2u,\nabla P)\|_{L^2_t(L^q)}\le C_{\rho,v}\|u_0\|_{L^2}\quad\text{for all } 2\le q<\infty.$$

Putting together this latter inequality and (3-34) with $q=4$ and remembering (3-32) yields (3-34) for $q=\infty$.

Note that (3-33) also implies that

$$\|\tau^{3/2}\nabla^2\dot u\|_{L^2_t(L^2)}+\|\tau\nabla\dot u\|_{L^2_t(L^2)}\le C_{\rho,v}\|u_0\|_{L^2},$$

and thus (3-37) by (A-1). Using it with $q=4$ as well as (3-35) (also with $q=4$) and (3-32) gives (3-35) for $\dot u$ and $q=\infty$.

To prove that u_t satisfies (3-35), it suffices to check that

$$\|\tau^{1-1/q}v\cdot\nabla u\|_{L^2_t(L^q)}\le C_{\rho,v}\|u_0\|_{L^2}\quad\text{for all } 2\le q\le\infty.$$

Now, by Hölder's inequality, we have

$$\|\tau^{1-1/q}v\cdot\nabla u\|_{L^2_t(L^q)}\le\|\tau^{1/2}v\|_{L^\infty_t(L^\infty)}\|\tau^{1/2-1/q}\nabla u\|_{L^2_t(L^q)}.$$

The term with v is energy-like (see (3-27)), which completes the proof. □

3.4. *The Lipschitz control and other properties needed for stability.* In the present subsection, we point out some additional properties of the velocity field that are valid in the case where u_0 is in $\widetilde B^0_{\rho_0,1}$. The most important one is the Lipschitz control. We shall also prove that the regularity $\widetilde B^0_{\rho_0,1}$ is preserved by the flow, and that other norms that will be needed in the proof of uniqueness and stability are finite.

These results follow from the Sobolev estimates we proved in the previous section and on the dynamic interpolation argument presented for the heat equation in Section 1.

Now, fix some u_0 in $\widetilde B^0_{\rho_0,1}$ and a sequence $(u_{0,j})_{j\in\mathbb Z}$ of L^2_σ such that

$$u_0=\sum_{j\in\mathbb Z}u_{0,j},\quad\text{with } \mathcal P(\rho_0u_{0,j})\in\dot H^{-1/2},\ \ u_{0,j}\in\dot H^{1/2}\quad\text{for all } j\in\mathbb Z$$
$$\text{and}\quad\sum_{j\in\mathbb Z}(2^{-j/2}\|u_{0,j}\|_{\dot H^{1/2}}+2^{j/2}\|\mathcal P(\rho_0u_{0,j})\|_{\dot H^{-1/2}})\le 2\|u_0\|_{\widetilde B^0_{\rho_0,1}}.\tag{3-38}$$

Then, for each $j\in\mathbb Z$, we solve the linear system

$$\begin{cases}\rho\partial_tu_j+\rho v\cdot\nabla u_j-\Delta u_j+\nabla P_j=0,\\ \operatorname{div}u_j=0,\\ u_j|_{t=0}=u_{0,j}.\end{cases}\tag{3-39}$$

From (3-38) and the uniqueness properties of system (1-6) in the energy space, we deduce that

$$u=\sum_{j\in\mathbb Z}u_j.\tag{3-40}$$

The Lipschitz bound. Recall the Gagliardo–Nirenberg inequality

$$\|\nabla z\|_{L^\infty} \le C\|z\|_{L^4}^{1/4}\|\nabla^2 z\|_{L^4}^{3/4}. \tag{3-41}$$

Combined with the elliptic estimates for the Stokes system and Sobolev embedding, this implies that, for all $t > 0$ and $j \in \mathbb{Z}$,

$$\|\nabla u_j(t)\|_{L^\infty} \le Ct^{-3/4}\|u_j(t)\|_{L^4}^{1/4}\|t\dot{u}_j(t)\|_{L^4}^{3/4} \le Ct^{-3/4}\|u_j(t)\|_{\dot{H}^{1/2}}^{1/4}\|t\dot{u}_j(t)\|_{\dot{H}^{1/2}}^{3/4}.$$

Hence, taking advantage of (3-11) and (3-12) gives

$$\|\nabla u_j(t)\|_{L^\infty} \le C_{\rho,v}t^{-3/4}\|u_{0,j}\|_{\dot{H}^{1/2}}.$$

Since we also have

$$\|\nabla u_j(t)\|_{L^\infty} \le C_{\rho,v}t^{-3/4}\left\|u_j\left(\tfrac{1}{2}t\right)\right\|_{\dot{H}^{1/2}},$$

we conclude in light of (3-18) that

$$\|\nabla u_j(t)\|_{L^\infty} \le C_{\rho,v}t^{-5/4}\|\mathcal{P}(\rho_0 u_{0,j})\|_{\dot{H}^{-1/2}}.$$

Hence, arguing as in Section 1, we conclude that

$$\int_0^\infty \|\nabla u\|_{L^\infty}\,dt \le C_{\rho,v}\|u_0\|_{\widetilde{B}^0_{\rho_0,1}}. \tag{3-42}$$

Remark 3.4. Recall the more accurate interpolation inequality

$$\|\nabla z\|_{\dot{B}^{1/2}_{4,1}} \le C\|z\|_{L^4}^{1/2}\|\nabla^2 z\|_{L^4}^{3/4}. \tag{3-43}$$

Repeating the above dynamic interpolation procedure thus actually gives

$$\int_0^\infty \|\nabla u\|_{\dot{B}^{1/2}_{4,1}}\,dt \le C_{\rho,v}\,\|u_0\|_{\widetilde{B}^0_{\rho_0,1}}.$$

Since $\dot{B}^{1/2}_{4,1} \hookrightarrow \mathcal{C}_b$, this ensures that the flow of the velocity field is uniformly C^1 with respect to the space variable.

Propagating the initial regularity. Owing to (3-11) and to (3-17) with $s = \frac{1}{2}$, we have, for all $j \in \mathbb{Z}$ and $t \ge 0$,

$$\|u_j(t)\|_{\dot{H}^{1/2}} \le C_{\rho,v}\|u_{0,j}\|_{\dot{H}^{1/2}} \quad \text{and} \quad \|\mathcal{P}(\rho u_j)(t)\|_{\dot{H}^{-1/2}} \le C_{\rho,v}\|\mathcal{P}(\rho_0 u_{0,j})\|_{\dot{H}^{-1/2}}.$$

Hence, multiplying the first (resp. second) inequality by $2^{-j/2}$ (resp. $2^{j/2}$) then summing over $j \in \mathbb{Z}$ yields

$$\|u(t)\|_{\widetilde{B}^0_{\rho(t),1}} \le C_{\rho,v}\|u_0\|_{\widetilde{B}^0_{\rho_0,1}}.$$

Additional bounds for the pressure and the time derivative of the velocity. In addition to the Lipschitz bound on velocity, our proof of uniqueness will require that $\sqrt{t}\dot{u}$ and $\sqrt{t}\nabla P$ are in $L^{4/3}(\mathbb{R}_+; L^4)$, and we will also need the property that $\dot{u}$ and $\sqrt{t}D\dot{u}$ are in $L^1(\mathbb{R}_+; L^2)$ to prove the stability of the flow map.

Again, in light of the decomposition (3-40) and of the triangle inequality, in order to prove that $\sqrt{t}\dot{u}$ is in $L^{4/3}(\mathbb{R}_+; L^4)$, it suffices to estimate $t\dot{u}_j$ for all $j \in \mathbb{Z}$. Now, owing to the Sobolev embedding and the inequalities (that stem from (3-12) and (3-19) with $s = s' = \frac{1}{2}$)

$$\|\dot{u}_j(t)\|_{\dot{H}^{1/2}} \le C_{\rho,v} t^{-1} \|u_{0,j}\|_{\dot{H}^{1/2}} \quad \text{and} \quad \|\dot{u}_j(t)\|_{\dot{H}^{1/2}} \le C_{\rho,v} t^{-3/2} \|\mathcal{P}(\rho_0 u_{0,j})\|_{\dot{H}^{-1/2}},$$

we may write, for all $A_j > 0$,

$$\begin{aligned}\|\sqrt{t}\dot{u}_j\|^{4/3}_{L^{4/3}(\mathbb{R}_+;L^4)} &\le C \int_0^\infty t^{2/3} \|\dot{u}_j\|^{4/3}_{\dot{H}^{1/2}}\,dt \\ &\le C_{\rho,v}\left(\int_0^{A_j} t^{2/3}(t^{-1}\|u_{0,j}\|_{\dot{H}^{1/2}})^{4/3}\,dt + \int_{A_j}^\infty t^{2/3}(t^{-3/2}\|\mathcal{P}(\rho_0 u_{0,j})\|_{\dot{H}^{-1/2}})^{4/3}\,dt\right) \\ &\le C_{\rho,v}(A_j^{1/3}\|u_{0,j}\|^{4/3}_{\dot{H}^{1/2}} + A_j^{-1/3}\|\mathcal{P}(\rho_0 u_{0,j})\|^{4/3}_{\dot{H}^{-1/2}}),\end{aligned} \tag{3-44}$$

which gives, if taking $A_j = 2^{-2j}$ and using (A-4),

$$\|(\sqrt{t}\dot{u}, \sqrt{t}\nabla^2 u, \sqrt{t}\nabla P)\|_{L^{4/3}(\mathbb{R}_+;L^4)} \le C_{\rho,v} \|u_0\|_{\widetilde{B}^0_{\rho_0,1}}. \tag{3-45}$$

Similarly, in order to bound $\dot{u}$ in $L^1(\mathbb{R}_+; L^2)$, it suffices to get appropriate bounds in terms of the data for $\dot{u}_j$ in $L^1(\mathbb{R}_+; L^2)$ and for all $j \in \mathbb{Z}$. The inequalities (that stem from (2-21) and (3-7))

$$\|\dot{u}_j(t)\|_{L^2} \le C_{\rho,v} t^{-1} \|u_{0,j}\|_{L^2} \quad \text{and} \quad \|\dot{u}_j(t)\|_{L^2} \le C_{\rho,v} t^{-1/2} \|\nabla u_{0,j}\|_{L^2}$$

and complex interpolation give

$$\|\dot{u}_j(t)\|_{L^2} \le C_{\rho,v} t^{-3/4} \|u_{0,j}\|_{\dot{H}^{1/2}}.$$

Furthermore, combining with (3-19), we discover that, for all $j \in \mathbb{Z}$,

$$\|\dot{u}_j(t)\|_{L^2} \le C_{\rho,v} t^{-5/4} \|\mathcal{P}(\rho_0 u_{0,j})\|_{\dot{H}^{-1/2}}.$$

Hence we have, for all $j \in \mathbb{Z}$ and $A_j > 0$,

$$\begin{aligned}\int_0^\infty \|\dot{u}_j(t)\|_{L^2}\,dt &\le \int_0^{A_j} \|\dot{u}_j(t)\|_{L^2}\,dt + \int_{A_j}^\infty \|\dot{u}_j(t)\|_{L^2}\,dt \\ &\le C_{\rho,v}\left(\int_0^{A_j} (t^{-3/4}\|u_{0,j}\|_{\dot{H}^{1/2}})\,dt + \int_{A_j}^\infty (t^{-5/4}\|\mathcal{P}(\rho_0 u_{0,j})\|_{\dot{H}^{-1/2}})\,dt\right) \\ &\le C_{\rho,v}(A_j^{1/4}\|u_{0,j}\|_{\dot{H}^{1/2}} + A_j^{-1/4}\|\mathcal{P}(\rho_0 u_{0,j})\|_{\dot{H}^{-1/2}}).\end{aligned}$$

Taking $A_j = 2^{-2j}$, summing over j, then using the regularity properties of the Stokes system thus gives

$$\|\nabla^2 u, \nabla P, \dot{u}\|_{L^1(\mathbb{R}_+;L^2)} \le C_{\rho,v} \|u_0\|_{\widetilde{B}^0_{\rho_0,1}}. \tag{3-46}$$

In the same way, one can prove that

$$\|\sqrt{t} D\dot{u}\|_{L^1(\mathbb{R}_+;L^2)} \le C_{\rho,v}\, \|u_0\|_{\widetilde{B}^0_{\rho_0,1}}. \tag{3-47}$$

It suffices to use, as a consequence of (3-19) and (3-20), that

$$\|\sqrt{t}\nabla\dot{u}_j(t)\|_{L^2} \le C_{\rho,v} t^{-3/4} \|u_{0,j}\|_{\dot{H}^{1/2}} \quad \text{and} \quad \|\sqrt{t}\nabla\dot{u}_j(t)\|_{L^2} \le C_{\rho,v} t^{-5/4} \|\mathcal{P}(\rho_0 u_{0,j})\|_{\dot{H}^{-1/2}}.$$

4. A global well-posedness result for large data

This section is devoted to the proof of Theorem 1.3 and of stability estimates.

4.1. ***The proof of existence.*** Consider data (ρ_0, u_0) satisfying the hypotheses of Theorem 1.3. Since the space $\widetilde{B}^0_{\rho_0,1}$ is embedded in L^2_σ, Theorem 1.1 provides us with a global weak solution $(\rho, u, \nabla P)$ satisfying the properties therein, and it is only a matter of checking that this solution has the additional properties that are listed in Theorem 1.3. To do so, we fix some decomposition $\sum_j u_{0,j}$ of u_0 given by Definition 1.2 and look, for all $j \in \mathbb{Z}$, at the solution u_j to the linear system (1-6) with density ρ, transport field u, and initial data $u_{0,j}$. Since each $u_{0,j}$ is in $L^2_\sigma \cap \dot{H}^{1/2}$ and $\mathcal{P}(\rho_0 u_{0,j}) \in \dot{H}^{-1/2}$, standard techniques yield a unique global solution $(u_j, \nabla P_j)$ that satisfies, for all $t \geq 0$,

$$\frac{1}{2}\|\sqrt{\rho(t)}u_j(t)\|^2_{L^2} + \int_0^t \|\nabla u_j\|^2_{L^2}\, d\tau = \frac{1}{2}\|\sqrt{\rho_0}u_{0,j}\|^2_{L^2}, \tag{4-1}$$

$$\|\mathcal{P}(\rho u_j)(t)\|_{\dot{H}^{-1/2}} \leq C(\rho_*, \rho^*, \|u_0\|_{L^2})\|\mathcal{P}(\rho_0 u_{0,j})\|_{\dot{H}^{-1/2}}, \tag{4-2}$$

$$\|u_j(t)\|_{\dot{H}^{1/2}} \leq C(\rho_*, \rho^*, \|u_0\|_{L^2})\|u_{0,j}\|_{\dot{H}^{1/2}}. \tag{4-3}$$

Remembering (1-9), this ensures that the L^2-valued series $\sum_j u_j$ converges normally on $\mathbb{R}_+$. Its sum $\tilde{u}$ thus also belongs to the energy space. Furthermore, as for each $j \in \mathbb{Z}$, we have $u_j \in \mathcal{C}(\mathbb{R}_+; L^2)$ (observe that $t^{3/4}u^j_t$ is in $L^\infty(\mathbb{R}_+; L^2)$ owing to (3-21)), and we deduce that $\tilde{u} \in \mathcal{C}(\mathbb{R}_+; L^2)$. Next, if we define $u^n := \sum_{|j|\leq n} u_j$, then we see that, for all $n \in \mathbb{N}$,

$$\partial_t(\rho(u^n - \tilde{u})) + \operatorname{div}(\rho u \otimes (u^n - \tilde{u})) - \Delta(u^n - \tilde{u}) + \nabla(P^n - \widetilde{P}) = 0, \quad \operatorname{div}(u^n - \tilde{u}) = 0,$$

which implies

$$\frac{1}{2}\|\sqrt{\rho(t)}(u^n - \tilde{u})(t)\|^2_{L^2} + \int_0^t \|\nabla(u^n - \tilde{u})\|^2_{L^2}\, d\tau = \frac{1}{2}\|\sqrt{\rho_0}(u^n(0) - u(0))\|^2_{L^2}.$$

As the right-hand side tends to 0 for n going to 0, the velocity field $\tilde{u}$ satisfies the energy balance (0-3), and it is also easy to conclude that, like u, it satisfies (1-6) with density ρ, transport field u, and initial data u_0. In particular,

$$\partial_t(\rho(u - \tilde{u})) + \operatorname{div}(\rho u \otimes (u - \tilde{u})) - \Delta(u - \tilde{u}) + \nabla(P - \widetilde{P}) = 0, \quad \operatorname{div}(u - \tilde{u}) = 0.$$

As $(u - \tilde{u})(0) = 0$ and the two solutions are in the energy space, they must coincide. Now, inequalities (4-2) and (4-3) ensure that one can propagate the regularity $\widetilde{B}^0_{\rho_0,1}$, getting (1-10). Likewise, the justification that u satisfies (0-8), that $(\dot{u}, \sqrt{t}D\dot{u}, D^2u, \nabla P) \in L^1(\mathbb{R}_+; L^2)$, and that $\sqrt{t}\dot{u} \in L^{4/3}(\mathbb{R}_+; L^4)$ may be achieved by following the arguments of the previous section. The fundamental point is that all the bounds that are needed for the u_j in the process only depend on ρ_*, ρ^*, $\|u_0\|_{L^2}$, $\|\mathcal{P}(\rho_0 u_{0,j})\|_{\dot{H}^{-1/2}}$, and $\|u_{0,j}\|_{\dot{H}^{1/2}}$.

4.2. ***The proof of uniqueness.*** Let $(\rho^1, u^1, \nabla P^1)$ and $(\rho^2, u^2, \nabla P^2)$ be two solutions fulfilling the properties listed in Theorem 1.3 and corresponding to data (ρ^1_0, u^1_0) and (ρ^2_0, u^2_0), respectively. As in [Danchin and Mucha 2019], in order to prove that

$$(\rho^1, u^1, \nabla P^1) \equiv (\rho^2, u^2, \nabla P^2)$$

in the case where the two initial data coincide, we shall compare the solutions at the level of their own Lagrangian coordinates. To do so, we consider, for $i=1,2$, the flow X^i of u^i that is defined by the (integrated) ODE

$$X^i(t,y)=y+\int_0^t u^i(\tau,X^i(\tau,y))\,d\tau. \tag{4-4}$$

Since ∇u^i is in $L^1(\mathbb{R}_+;L^\infty)$ and $\sqrt{t}u^i$ is in $L^\infty(0,T\times\Omega)$ (see (3-27) with $p=2$ and $q=\infty$), there exists a unique continuous flow X^i on $(0,T)\times\Omega$ that is Lipschitz with respect to the space variable.

In Lagrangian coordinates the density is equal to the initial density. As for the velocity and the pressure defined by

$$Q^i(t,y)=P^i(t,X^i(t,y)) \quad\text{and}\quad v^i(t,y)=u^i(t,X^i(t,y)), \tag{4-5}$$

they satisfy

$$\begin{cases}\rho_0^i v_t^i-\operatorname{div}_{v^i}\nabla_{v^i}v^i+\nabla_{v^i}Q^i=0,\\ \operatorname{div}_{v^i}v^i=0,\end{cases} \tag{4-6}$$

where

$$\nabla_{v^i}:=(A^i)^\top\nabla_y \quad\text{and}\quad \operatorname{div}_{v^i}:=\operatorname{div}_y(A^i\,\cdot\,)=(A^i)^\top:\nabla_y,\quad\text{with } A^i:=(DX^i)^{-1}.$$

The fact that ∇u^i is in $L^1(\mathbb{R}_+;L^\infty)$ and the other properties of regularity ensure that (INS) and (4-6) (with time-independent density) are equivalent.

Observe that, due to (4-4) and the definition of v^i, we have

$$DX^i(t,y)=\mathrm{Id}+\int_0^t Dv^i(\tau,y)\,d\tau. \tag{4-7}$$

Hence, since $\det DX^i\equiv 1$ (owing to $\operatorname{div} v^i=0$), we have, for $i=1,2$,

$$A^i(t)=\mathrm{Id}+\begin{pmatrix}\int_0^t\partial_2 v^{i,2}\,d\tau & -\int_0^t\partial_2 v^{i,1}\,d\tau\\ -\int_0^t\partial_1 v^{i,2}\,d\tau & \int_0^t\partial_1 v^{i,1}\,d\tau\end{pmatrix}. \tag{4-8}$$

Hence $\delta A:=A^2-A^1$ depends linearly on $\nabla\delta v$ (with $\delta v:=v^2-v^1$) as follows:

$$\delta A(t)=\begin{pmatrix}\int_0^t\partial_2\delta v^2\,d\tau & -\int_0^t\partial_2\delta v^1\,d\tau\\ -\int_0^t\partial_1\delta v^2\,d\tau & \int_0^t\partial_1\delta v^1\,d\tau\end{pmatrix}. \tag{4-9}$$

Now, setting $\Delta_{v^i}:=\operatorname{div}_{v^i}\nabla_{v^i}$ and $\delta Q:=Q^2-Q^1$, we discover that $(\delta v,\delta Q)$ satisfies

$$\begin{cases}\rho_0^1\delta v_t-\Delta_{v^1}\delta v+\nabla_{v^1}\delta Q=(\Delta_{v^2}-\Delta_{v^1})v^2-(\nabla_{v^2}-\nabla_{v^1})Q^2-\delta\rho_0 v_t^2,\\ \operatorname{div}_{v^1}\delta v=(\operatorname{div}_{v^1}-\operatorname{div}_{v^2})v^2=-\operatorname{div}(\delta A v^2).\end{cases} \tag{4-10}$$

In order to prove uniqueness in the case where the initial data are the same and, more generally, stability estimates with respect to the initial data, using the basic energy method — which consists of taking the L^2 scalar product of (4-10) with δv — is not appropriate, since one cannot eliminate the pressure term (there is no reason why we should have $\operatorname{div}_{v^1}\delta v=0$). To overcome the difficulty, we proceed as in [Danchin and Mucha 2019], solving first the equation

$$\operatorname{div}_{v^1}w=-\operatorname{div}(\delta A v^2)=-\delta A^\top:\nabla v^2,\quad\text{with } \delta A:=A^2-A^1. \tag{4-11}$$

Then, we look at the system for $z := \delta v - w$, namely

$$\begin{cases}\rho_0^1 z_t - \Delta_{v^1} z + \nabla_{v^1}\delta Q = (\Delta_{v^2} - \Delta_{v^1})v^2 - (\nabla_{v^2} - \nabla_{v^1})Q^2 - \rho_0^1 w_t + \Delta_{v^1} w - \delta\rho_0 v_t^2,\\ \operatorname{div}_{v^1} z = 0,\end{cases} \tag{4-12}$$

supplemented with $z|_{t=0} = \delta v_0$.

Solving (4-11) relies on the following lemma.

Lemma 4.1. *Assume that Ω is a C^2 bounded domain, the torus, or the whole space. Fix $T > 0$ and define*

$$E_T := \{w \in \mathcal{C}([0, T]; L^2),\ \nabla w \in L^2(0, T \times \Omega),\ w|_{\partial\Omega} = 0 \text{ and } w_t \in L^{4/3}(0, T \times \Omega)\}.$$

There exists a constant c depending only on Ω such that, whenever the divergence-free vector field u satisfies

$$\|\nabla u\|_{L^2(0,T\times\Omega)} + \|\nabla u\|_{L^1(0,T;L^\infty)} \le c \tag{4-13}$$

then, for all vector fields $k \in \mathcal{C}([0, T]; L^2)$ such that $\operatorname{div} k \in L^2(0, T \times \Omega)$ *and $k_t \in L^{4/3}(0, T \times \Omega)$, there exists a vector field w in the space E_T satisfying*

$$\operatorname{div}(Aw) = \operatorname{div} k$$

(*where A is defined from u as in* (4-8)) *and the inequalities*

$$\|w(t)\|_{L^2} \le C\|k(t)\|_{L^2} \qquad \text{for all } t \in [0, T], \tag{4-14}$$

$$\|\nabla w\|_{L^2_T(L^2)} \le C\|\operatorname{div} k\|_{L^2_T(L^2)}, \tag{4-15}$$

$$\|w_t\|_{L^{4/3}_T(L^{4/3})} \le C(\|k_t\|_{L^{4/3}_T(L^{4/3})} + \|\nabla u\|_{L^2_T(L^2)}\|w\|_{L^4_T(L^4)}). \tag{4-16}$$

Proof. With the notation of Lemma A.1 in the Appendix, we introduce the map

$$\Phi : w \mapsto z := \mathcal{B}(k + (\mathrm{Id} - A)w).$$

It is only a matter of proving that Φ admits a fixed point. That Φ maps E_T to E_T follows from Lemma A.1 and easy modifications of the computations below. Hence, as E_T is a Banach space, it suffices to show that the linear map Φ is strictly contractive. To do so, take two elements w^1 and w^2 of E_T. Then, we have

$$\Phi(w^2) - \Phi(w^1) = \mathcal{B}((\mathrm{Id} - A)\delta w), \quad \text{with } \delta w := w^2 - w^1.$$

Remembering (4-8) and that $\mathcal{B} : L^2 \to L^2$, we thus have

$$\|\Phi(w^2) - \Phi(w^1)\|_{L^\infty_T(L^2)} \le C\|\nabla u\|_{L^1_T(L^\infty)}\|\delta w\|_{L^\infty_T(L^2)}. \tag{4-17}$$

Next, using again (4-8) and the fact that

$$\operatorname{div}((\mathrm{Id} - A)\delta w) = (\mathrm{Id} - A^\top) : \nabla\delta w,$$

we readily get

$$\|\nabla(\Phi(w^2) - \Phi(w^1))\|_{L^2_T(L^2)} \le C\|\nabla u\|_{L^1_T(L^\infty)}\|\nabla\delta w\|_{L^2_T(L^2)}. \tag{4-18}$$

Finally, using

$$((\mathrm{Id} - A)\delta w)_t = (\mathrm{Id} - A)\delta w_t - A_t\delta w$$

yields, for almost every $t \in [0, T]$,

$$\begin{aligned}\|(\Phi(w^2)-\Phi(w^1))_t(t)\|_{L^{4/3}} &\lesssim \|(\mathrm{Id}-A(t))\delta w_t(t)\|_{L^{4/3}} + \|A_t(t)\delta w(t)\|_{L^{4/3}} \\ &\lesssim \|\nabla u\|_{L^1_t(L^\infty)}\|\delta w_t(t)\|_{L^{4/3}} + \|\nabla u(t)\|_{L^2}\|\delta w(t)\|_{L^4} \\ &\lesssim \|\nabla u\|_{L^1_t(L^\infty)}\|\delta w_t(t)\|_{L^{4/3}} + \|\nabla u(t)\|_{L^2}\|\delta w(t)\|_{L^2}^{1/2}\|\nabla\delta w(t)\|_{L^2}^{1/2}. \end{aligned} \tag{4-19}$$

Combining (4-17)–(4-19), we conclude

$$\|(\Phi(w^2)-\Phi(w^1)\|_{E_T} \le C(\|\nabla u\|_{L^1_T(L^\infty)} + \|\nabla u\|_{L^2_T(L^2)})\|\delta w\|_{E_T}.$$

Hence, if (4-13) is satisfied with a suitable small $c > 0$ then Φ is contractive, which ensures the existence of w in E_T satisfying the desired equation. Finally, using the fact that we thus have $w = \mathcal{B}k + \mathcal{B}((\mathrm{Id}-A)w)$ and that

$$\begin{aligned}\mathrm{div}((\mathrm{Id}-A)w) &= (\mathrm{Id}-A^\top) : \nabla w, \\ ((\mathrm{Id}-A)w)_t &= (\mathrm{Id}-A)w_t - A_t w,\end{aligned}$$

mimicking the above calculations gives (4-14), (4-15), and (4-16). □

In what follows, we assume that T has been chosen such that (4-13) is satisfied for u^1 and u^2, and we define w on $[0, T]\times\Omega$ according to the above lemma with $k = -\delta A v^2$. We shall use repeatedly that, owing to (4-9) and the Cauchy–Schwarz inequality, we have

$$\max(\|t^{-1/2}\delta A\|_{L^\infty_T(L^2)}, \|(\delta A)_t\|_{L^2(0,T\times\Omega)}) \le \|\nabla\delta v\|_{L^2(0,T\times\Omega)}. \tag{4-20}$$

Hence, thanks to (4-14), we have, for all $t \in [0, T]$,

$$\|w(t)\|_{L^2} \le C\|\sqrt{t}v^2(t)\|_{L^\infty}\|\nabla\delta v\|_{L^2(0,t\times\Omega)}. \tag{4-21}$$

Next, as

$$(\delta A v^2)_t = \delta A_t v^2 + \delta A v_t^2,$$

inequality (4-16) (before time integration) and (4-9) guarantee that

$$\|w_t\|_{L^{4/3}} \le C(\|\nabla v^1\|_{L^2}\|w\|_{L^4} + \|\nabla\delta v\|_{L^2}\|v^2\|_{L^4} + \|\delta A\|_{L^2}\|v_t^2\|_{L^4}). \tag{4-22}$$

Finally, using $\mathrm{div}(\delta A v^2) = \delta A^\top : \nabla v^2$, inequalities (4-15) and (4-20) yield

$$\|Dw(t)\|_{L^2} \le C\|\nabla\delta v\|_{L^2_t(L^2)}\|\sqrt{t}\nabla v^2\|_{L^\infty_t(L^\infty)}. \tag{4-23}$$

Now, taking the $L^2(0, t\times\Omega)$ scalar product of the first equation of (4-12) with z and integrating by parts in some terms yields

$$\frac12\|\sqrt{\rho_0^1}z\|^2_{L^\infty(0,t;L^2)} + \int_0^t \|\nabla_{v^1} z\|^2_{L^2}\,d\tau = \frac12\|\sqrt{\rho_0^1}\delta u_0\|^2_{L^2} + \sum_{j=1}^5 I_j(t), \tag{4-24}$$

with

$$I_1(t) := -\int_0^t\int_\Omega (\delta A(A^2)^\top + A^1\delta A^\top)\nabla v^2 : \nabla z\,dx\,d\tau,$$

$$I_2(t) := -\int_0^t\int_\Omega \delta A^\top\nabla Q^2\cdot z\,dx\,d\tau, \qquad I_3(t) := -\int_0^t\int_\Omega \rho_0^1 w_\tau\cdot z\,dx\,d\tau,$$

$$I_4(t) := -\int_0^t\int_\Omega (A^1)^\top\nabla w : (A^1)^\top\nabla z\,dx\,d\tau, \qquad I_5(t) := -\int_0^t\int_\Omega \delta\rho_0 v_t^2\cdot z\,dx\,d\tau. \tag{4-25}$$

We shall often use that, due to (4-8),

$$\|\nabla z\|_{L^2(0,T\times\Omega)} \simeq \|\nabla_{v^1} z\|_{L^2(0,T\times\Omega)}. \tag{4-26}$$

From this we easily get

$$I_1(t) \le C\int_0^t \|\tau^{-1/2}\delta A(\tau)\|_{L^2}\|\sqrt{\tau}\nabla v^2(\tau)\|_{L^\infty}\|\nabla_{v^1} z(\tau)\|_{L^2}\,d\tau.$$

Hence, using (4-20) and Young's inequality,

$$I_1 \le C\|\sqrt{\tau}\nabla v^2\|_{L_t^2(L^\infty)}^2\|\nabla\delta v\|_{L^2(0,t\times\Omega)}^2 + \frac{1}{8}\int_0^t \|\nabla_{v^1} z\|_{L^2}^2\,d\tau. \tag{4-27}$$

Next, by (4-20), (4-26), Hölder's inequality, and (0-7), we have

$$\begin{aligned} I_2 &\le C\int_0^t \|\tau^{-1/2}\delta A\|_{L^2}\|\sqrt{\tau}\nabla Q^2\|_{L^4}\|z\|_{L^2}^{1/2}\|\nabla z\|_{L^2}^{1/2}\,d\tau \\ &\le \frac{1}{8}\int_0^t \|\nabla_{v^1} z\|_{L^2}^2\,d\tau + C\|\tau^{-1/2}\delta A\|_{L_t^\infty(L^2)}^{4/3}\|z\|_{L_t^\infty(L^2)}^{2/3}\int_0^t \|\sqrt{\tau}\nabla Q^2\|_{L^4}^{4/3}\,d\tau. \end{aligned}$$

Hence, in light of (4-20), Young's inequality, and (0-9), we have

$$I_2 \le \frac{1}{8}\int_0^t \Big(\|\nabla_{v^1} z\|_{L^2}^2 + \frac{1}{4}\|\nabla\delta v\|_{L^2}^2\Big)\,d\tau + C\|\sqrt{\rho_0^1}z\|_{L_t^\infty(L^2)}^2\|\sqrt{\tau}\nabla Q^2\|_{L_t^{4/3}(L^4)}^4. \tag{4-28}$$

In order to bound I_3, we start with the inequality

$$I_3 \le \rho^*\int_0^t \|w_\tau\|_{L^{4/3}}\|z\|_{L^4}\,d\tau.$$

Taking advantage of (4-22) to bound w_τ and of the Gagliardo–Nirenberg and Young inequalities yields

$$\begin{aligned} I_3 &\lesssim \int_0^t \|z\|_{L^2}^{1/2}\|\nabla z\|_{L^2}^{1/2}(\|\nabla v^1\|_{L^2}\|w\|_{L^4} + \|v^2\|_{L^4}\|\nabla\delta v\|_{L^2} + \|\delta A\|_{L^2}\|v_\tau^2\|_{L^4})\,d\tau \\ &\le \frac{1}{8}\int_0^t \|\nabla_{v^1} z\|_{L^2}^2\,d\tau + \frac{1}{32}\int_0^t \|\nabla\delta v\|_{L^2}^2\,d\tau + C\int_0^t \|v^2\|_{L^4}^4\|z\|_{L^2}^2\,d\tau + I_{31} + I_{32}, \end{aligned}$$

with

$$I_{31} := C\int_0^t \|z\|_{L^2}^{2/3}\|\nabla v^1\|_{L^2}^{4/3}\|w\|_{L^2}^{2/3}\|\nabla w\|_{L^2}^{2/3}\,d\tau \quad\text{and}\quad I_{32} := C\int_0^t \|z\|_{L^2}^{2/3}\|\delta A\|_{L^2}^{4/3}\|v_\tau^2\|_{L^4}^{4/3}\,d\tau.$$

Just using (4-20) yields

$$I_{32}\le \|\nabla\delta v\|^{4/3}_{L^2_t(L^2)}\|z\|^{2/3}_{L^\infty_t(L^2)}\|\sqrt{\tau}v^2_\tau\|^{4/3}_{L^{4/3}_t(L^4)}.$$

In order to bound I_{31}, one has to use (4-21) and (4-23), which yields

$$\begin{aligned}I_{31}&\le C\int_0^t \|z\|^{2/3}_{L^2}\|\nabla v^1\|^{4/3}_{L^2}\|\sqrt{\tau}v^2\|^{2/3}_{L^\infty}\|\nabla\delta v\|^{2/3}_{L^2_\tau(L^2)}\|\tau^{-1/2}\delta A(\tau)\|^{2/3}_{L^2}\|\sqrt{\tau}\nabla v^2\|^{2/3}_{L^\infty}\,d\tau\\ &\le C\|\nabla\delta v\|^{4/3}_{L^2_t(L^2)}\|z\|^{2/3}_{L^\infty_t(L^2)}\int_0^t\|\sqrt{\tau}v^2\|^{2/3}_{L^\infty}\|\nabla v^1\|^{4/3}_{L^2}\|\sqrt{\tau}\nabla v^2\|^{2/3}_{L^\infty}\,d\tau.\end{aligned}$$

This enables us to get the following bound for I_3:

$$\begin{aligned}&I_3(t)\\ &\le \frac18\|\nabla_{v^1}z\|^2_{L^2_t(L^2)}+\frac1{16}\|\nabla\delta v\|^2_{L^2_t(L^2)}\\ &+C\biggl(\|v^2\|^4_{L^4_t(L^4)}+\biggl(\int_0^t\|\sqrt{\tau}v^2\|^{2/3}_{L^\infty}\|\nabla v^1\|^{4/3}_{L^2}\|\sqrt{\tau}\nabla v^2\|^{2/3}_{L^\infty}\,d\tau\biggr)^3+\|\sqrt{\tau}v^2_\tau\|^4_{L^{4/3}_t(L^4)}\biggr)\|\sqrt{\rho^1_0}z\|^2_{L^\infty_t(L^2)}.\end{aligned}\tag{4-29}$$

Next, thanks to (4-23), (4-20), and the Cauchy–Schwarz and Young inequalities,

$$\begin{aligned}I_4\le C\int_0^t\|\nabla w\|_{L^2}\|\nabla_{v^1}z\|_{L^2}\,d\tau&\le C\int_0^t\|\tau^{-1/2}\delta A\|_{L^2}\|\sqrt{\tau}\nabla v^2\|_{L^\infty}\|\nabla_{v^1}z\|_{L^2}\,d\tau\\ &\le\frac18\int_0^t\|\nabla_{v^1}z\|^2_{L^2}\,d\tau+C\|\sqrt{\tau}\nabla v^2\|^2_{L^2(0,t;L^\infty)}\|\nabla\delta v\|^2_{L^2(0,t\times\Omega)}.\end{aligned}\tag{4-30}$$

Finally, it is obvious that

$$I_5(t)\le\left\|\frac{\delta\rho_0}{\sqrt{\rho^1_0}}\right\|_{L^\infty}\|\sqrt{\rho^1_0}z\|_{L^\infty_t(L^2)}\|v^2_t\|_{L^1_t(L^2)}.\tag{4-31}$$

So plugging (4-27)–(4-31) into (4-24) and taking $t=T$ yields

$$\begin{aligned}\|\sqrt{\rho^1_0}\,z\|^2_{L^\infty_T(L^2)}+\|\nabla_{v^1}z\|^2_{L^2_T(L^2)}&\le\|\sqrt{\rho^1_0}\delta u_0\|^2_{L^2}+A(T)\|\sqrt{\rho^1_0}z\|^2_{L^\infty_T(L^2)}\\ &\quad+\Bigl(\frac18+C\|\sqrt{t}\nabla v^2\|^2_{L^2_T(L^\infty)}\Bigr)\|\nabla\delta v\|^2_{L^2_T(L^2)}+2\left\|\frac{\delta\rho_0}{\sqrt{\rho^1_0}}\right\|^2_{L^\infty}\|v^2_t\|^2_{L^1_T(L^2)},\end{aligned}$$

with

$$\begin{aligned}A(T):=C\biggl(&\|v^2\|^4_{L^4_T(L^4)}+\|\sqrt{t}v^2_t\|^4_{L^{4/3}_T(L^4)}+\|\sqrt{\tau}\nabla Q^2\|^4_{L^{4/3}_T(L^4)}\\ &+\biggl(\int_0^t\|\sqrt{\tau}v^2\|^{2/3}_{L^\infty}\|\nabla v^1\|^{4/3}_{L^2}\|\sqrt{\tau}\nabla v^2\|^{2/3}_{L^\infty}\,d\tau\biggr)^3\biggr).\end{aligned}$$

The regularity properties of the constructed solutions guarantee that $A(\infty)$ is finite, and the Lebesgue dominated convergence theorem thus ensures that if T is small enough then

$$\max(8C\|\sqrt{t}\nabla v^2\|^2_{L^2_T(L^\infty)},2A(T))\le1.\tag{4-32}$$

Under this hypothesis, the above inequality becomes

$$\tfrac12\|\sqrt{\rho^1_0}\,z\|^2_{L^\infty_T(L^2)}+\|\nabla_{v^1}z\|^2_{L^2_T(L^2)}\le\|\sqrt{\rho^1_0}\delta u_0\|^2_{L^2}+\tfrac14\|\nabla\delta v\|^2_{L^2_T(L^2)}+C\|\delta\rho_0\|^2_{L^\infty}\|v^2_t\|^2_{L^1_T(L^2)}.\tag{4-33}$$

Since $\nabla\delta v = \nabla z + \nabla w$ and owing to (4-20), (4-23), and (4-26), we may write

$$\|\nabla\delta v\|^2_{L^2_T(L^2)} \le 2\|\nabla z\|^2_{L^2_T(L^2)} + 2\|\nabla w\|^2_{L^2_T(L^2)} \le \tfrac{5}{2}\|\nabla_{v^1} z\|^2_{L^2_T(L^2)} s + C\|\sqrt{t}\nabla v^2\|^2_{L^2_T(L^\infty)}\|\nabla\delta v\|^2_{L^2_T(L^2)}.$$

Hence, under assumption (4-32) (up to a change of C if needed), we have

$$\|\nabla\delta v\|^2_{L^2(0,T\times\Omega)} \le 3\|\nabla_{v^1} z\|^2_{L^2(0,T\times\Omega)}. \tag{4-34}$$

Plugging this inequality into (4-33) gives

$$\tfrac{1}{2}\|\sqrt{\rho_0^1}\, z\|^2_{L^\infty_T(L^2)} + \tfrac{1}{4}\|\nabla_{v^1} z\|^2_{L^2_T(L^2)} \le C(\|\sqrt{\rho_0^1}\delta u_0\|^2_{L^2} + \|\delta\rho_0\|^2_{L^\infty}\|v_t^2\|^2_{L^1_T(L^2)}). \tag{4-35}$$

In the case where the two solutions correspond to the same initial data, this ensures that $z \equiv 0$ on $[0, T]$. Remembering (4-34) and (4-21), one can conclude uniqueness on $[0, T]$ and then on $\mathbb{R}_+$ by a standard bootstrap argument.

4.3. ***Continuity of the flow map.*** We consider here the case where the two previous solutions correspond to possibly different data. To begin with, we have to observe that (4-34) and (4-35) together imply that if

$$\|\sqrt{t}v^2\|_{L^\infty(\mathbb{R}_+\times\Omega)} \le K, \tag{4-36}$$

then, in light of inequalities (4-21), (4-34) and (4-35), there exists some constant $c > 0$ such that if $\tilde{A}(T_0) \le c$, then we have

$$\|\sqrt{\rho_0^1}\,\delta v\|_{L^\infty_{T_0}(L^2)} + \|\nabla_{v^1}\delta v\|_{L^2_{T_0}(L^2)} \le C(1+K)(\|\sqrt{\rho_0^1}\delta u_0\|_{L^2} + \|\delta\rho_0\|_{L^\infty}), \tag{4-37}$$

where we define, for all $T \in [0, \infty]$,

$$\tilde{A}(T) := \|v^2\|^4_{L^4_T(L^4)} + \|\sqrt{t}(v_t^2, \nabla Q^2)\|^{4/3}_{L^{4/3}_T(L^4)} + (1+K)(\|\nabla v^1\|^2_{L^2_T(L^2)} + \|\sqrt{\tau}\nabla v^2\|^2_{L^2_T(L^\infty)}) + \|v_t^2\|_{L^1_T(L^2)}.$$

Now, if we consider data that belong to a bounded subset of $\widetilde{B}^0_{\rho_0,1}$, then K in (4-36) and $\tilde{A}(\infty)$ can be uniformly bounded. By iterating the procedure that led to (4-37), this allows us to get in the end

$$\|\sqrt{\rho_0^1}\,\delta v\|_{L^\infty_T(L^2)} + \|\nabla_{v^1}\delta v\|_{L^2_T(L^2)} \le Ce^{C\tilde{A}(\infty)}(\|\sqrt{\rho_0^1}\delta u_0\|_{L^2} + \|\delta\rho_0\|_{L^\infty}). \tag{4-38}$$

Then, reverting to the Eulerian coordinates gives the following stability statement.

Theorem 4.2. *Consider two solutions (ρ^1, u^1, P^1) and (ρ^2, u^2, P^2) corresponding to initial data (ρ_0^1, u_0^1) and (ρ_0^2, u_0^2) given by Theorem 1.3. Assume that*

$$0 < \rho_* \le \rho_0^1, \rho_0^2 \le \rho^* \quad \textit{and} \quad \max(\|u_0^1\|_{\widetilde{B}^0_{\rho_0^1,1}}, \|u_0^2\|_{\widetilde{B}^0_{\rho_0^2,1}}) \le M.$$

Then we have

$$\|\sqrt{\rho_0^1}\,\delta u\|_{L^\infty_T(L^2)} + \|\nabla\delta u\|_{L^2_T(L^2)} \le C_{\rho_*,\rho^*,M}(\|\sqrt{\rho_0^1}\delta u_0\|_{L^2} + \|\delta\rho_0\|_{L^\infty}) \tag{4-39}$$

and, for all $p \in [2, \infty)$,

$$\|\delta\rho(t)\|_{\dot{W}^{-1,p}} \le C_{p,\rho_*,\rho^*,M}(\|\delta\rho_0\|_{\dot{W}^{-1,p}} + t^{1/2+1/p}(\|\sqrt{\rho_0^1}\delta u_0\|_{L^2} + \|\delta\rho_0\|_{L^\infty})). \tag{4-40}$$

Proof. Although our regularity assumptions are weaker, we shall follow [Danchin et al. 2024] to bound the difference of the velocities. The starting point is the relation

$$\begin{aligned}\nabla_y\delta v = K_1+K_2+K_3,\quad \text{with } K_1(t,y) &:= \nabla_y\delta X(t,y)\cdot\nabla_x u^2(t,X^2(t,y)),\\ K_2(t,y) &:= \nabla_y X^1(t,y)\cdot\nabla_x\delta u(t,X^2(t,y)),\\ K_3(t,y) &:= \nabla_y X^1(t,y)\cdot(\nabla_x u^1(t,X^2(t,y))-\nabla_x u^1(t,X^2(t,y))).\end{aligned}$$

Since $\nabla\delta u(t,X^2(t,y)) = A_1^\top(t,y)K_2(t,y)$ and the flow X^2 is measure-preserving, the above decomposition implies that

$$\|\nabla\delta u\|_{L^2} \le \|A_1\|_{L^\infty}(\|\nabla\delta v\|_{L^2}+\|K_1\|_{L^2}+\|K_3\|_{L^2}).$$

Bounding K_1 may be done as in [Danchin et al. 2024]. We get, for all $t\ge 0$,

$$\|K_1(t)\|_{L^2} \le C\|\sqrt{t}\nabla u^2(t)\|_{L^\infty}\|\nabla\delta v\|_{L^2_t(L^2)}.$$

For bounding K_3, we use the relation

$$K_3(t,y) = \nabla X_1(t,y)\cdot\left(\int_1^2(\nabla^2 u^1(t,X^s(t,y)))\cdot\left(\frac{dX^s}{ds}(t,y)\right)ds\right),$$

where the "interpolating flow" X^s stands for the solution to

$$X^s(t,y) = y+\int_0^t((2-s)u^1(\tau,X^s(\tau,y))+(s-1)u^2(\tau,X^s(\tau,y)))\,d\tau.$$

As $X^s(t,\cdot)$ is also measure-preserving, it is easy to prove that (again, see [Danchin et al. 2024])

$$\left\|\frac{dX^s}{ds}(t,\cdot)\right\|_{L^4} \le C\|\delta u\|_{L^1_t(L^4)}.$$

Thanks to that and to Hölder's inequality, we deduce that

$$\|K_3(t)\|_{L^2} \le C(1+\|\nabla u^1\|_{L^1_t(L^\infty)})\|t^{3/4}\nabla^2 u^1(t)\|_{L^4}\|\delta u\|_{L^4_t(L^4)}.$$

Hence, in the end, if T is chosen such that

$$\max\left(\int_0^T\|\nabla u^1(t)\|_{L^\infty}\,dt,\ \int_0^T\|\nabla u^2(t)\|_{L^\infty}\,dt\right)\le 1,$$

then we have, using also (A-4),

$$\|\nabla\delta u\|_{L^2_T(L^2)} \lesssim (1+\|\sqrt{t}\nabla u^2\|_{L^2_T(L^\infty)})\|\nabla\delta v\|_{L^2_T(L^2)}+\|t^{3/4}\dot u^1\|_{L^2_T(L^4)}\|\delta u\|_{L^4_T(L^4)}.$$

The last term may be handled by means of (0-7), and one ends up with

$$\|\nabla\delta u\|_{L^2_T(L^2)} \lesssim (1+\|\sqrt{t}\nabla u^2\|_{L^2_T(L^\infty)})\|\nabla\delta v\|_{L^2_T(L^2)}+\|t^{3/4}\dot u^1\|^2_{L^2_T(L^4)}\|\sqrt{\rho^1}\delta u\|_{L^\infty_T(L^2)}. \tag{4-41}$$

Remember that the constructed solutions satisfy $\sqrt{t}\nabla u^2\in L^2(\mathbb{R}_+;L^\infty)$ and note that, since

$$\|t^{3/4}\dot u^1\|_{L^2_T(L^4)} \le C\|t\dot u^1\|^{1/2}_{L^\infty_T(L^2)}\|\sqrt{t}D\dot u^1\|^{1/2}_{L^1_T(L^2)},$$

inequalities (2-21) and (3-47) guarantee that $t^{3/4}\dot{u}^1$ is in $L^2(\mathbb{R}_+; L^4)$. So we are left with bounding $\sqrt{\rho^1}\delta u$ in $L^\infty(0, T; L^2)$. To do so, we use, as in [Danchin et al. 2024], the relation

$$\sqrt{\rho_0^1(y)}\delta v(t, y) = \sqrt{\rho^1}(t, X^1(t, y))\left(\delta u(t, X^1(t, y)) + \int_1^2 Du^2(t, X^s(t, y))\frac{dX^s}{ds}(t, y)\, ds\right).$$

Hence, as all the flows X^s are measure-preserving and ρ_1 is bounded from below,

$$\begin{aligned}\|\sqrt{\rho^1(t)}\delta u(t)\|_{L^2} &\le \|\sqrt{\rho_0^1}\delta v(t)\|_{L^2} + C\sqrt{\rho^*}\|Du^2(t)\|_{L^4}\|\delta u\|_{L^1_t(L^4)}\\ &\le \|\sqrt{\rho_0^1}\delta v(t)\|_{L^2} + C\|t^{3/4}Du^2(t)\|_{L^4}\|\delta u\|_{L^4_t(L^4)}\\ &\le \|\sqrt{\rho_0^1}\delta v(t)\|_{L^2} + C\|\sqrt{t}Du^2(t)\|_{L^2}^{1/2}\|tD^2u^2(t)\|_{L^2}^{1/2}\|\nabla\delta u\|_{L^2_t(L^2)}^{1/2}\|\sqrt{\rho^1(t)}\delta u\|_{L^\infty_t(L^2)}^{1/2}.\end{aligned}$$

Since both the terms with $\sqrt{t}Du^2$ and with tD^2u^2 may be bounded in terms of ρ_*, ρ^*, and $\|u_0^2\|_{L^2}$ only, we end up with

$$\|\sqrt{\rho^1}\delta u\|_{L^\infty_T(L^2)} \le 2\|\sqrt{\rho_0^1}\delta v\|_{L^\infty_T(L^2)} + C(\rho_*, \rho^*, \|u_0^2\|_{L^2})\|\nabla\delta u\|_{L^2_T(L^2)}.$$

Putting this inequality together with (4-41) and remembering (4-38) allows us to conclude that there exists an absolute constant C such that, for small enough T, we have

$$\|\sqrt{\rho_0^1}\,\delta u\|_{L^\infty_T(L^2)} + \|\nabla\delta u\|_{L^2_T(L^2)} \le C(\|\sqrt{\rho_0^1}\delta u_0\|_{L^2} + \|\delta\rho_0\|_{L^\infty}),$$

then arguing by induction and using the bounds on u^1 and u^2 in terms of the data yields (4-39).

Finally, the difference between the (Eulerian) densities may be bounded by resorting to the classical theory of transport equation. Indeed, we have

$$\partial_t\delta\rho + \operatorname{div}(\delta\rho u^2) = -\operatorname{div}(\rho^1\delta u).$$

Hence, we may write, for all $p \in [1, \infty]$ and $t \ge 0$,

$$\begin{aligned}\|\delta\rho(t)\|_{\dot{W}^{-1,p}} &\le \left(\|\delta\rho_0\|_{\dot{W}^{-1,p}} + \int_0^t e^{-\int_0^\tau \|\nabla u^2\|_{L^\infty}\,d\tau'}\|\rho_1\delta u\|_{L^p}\,d\tau\right)e^{\int_0^t \|\nabla u^2\|_{L^\infty}\,d\tau}\\ &\le (\|\delta\rho_0\|_{\dot{W}^{-1,p}} + \rho^* t^{1/2+1/p}\|\delta u\|_{L_t^{2p/(p-2)}(L^p)})e^{\int_0^t \|\nabla u^2\|_{L^\infty}\,d\tau}.\end{aligned}$$

Combining inequality (4-39) with the Gagliardo–Nirenberg inequality provides us with a control of δu in $L^{2p/(p-2)}(\mathbb{R}_+; L^p)$ for all $p \in [2, \infty)$. In the end, we get (4-40). □

Remark 4.3. In the bounded or torus cases, one can take advantage of exponential decay to get a time-independent bound. The details are left to the reader.

Appendix

Here we recall some results that played a key role throughout the paper. The first one is the following Gagliardo–Nirenberg inequality that extends (0-7):

$$\|z\|_{L^p} \le C_p\|z\|_{L^2}^{2/p}\|\nabla z\|_{L^2}^{1-2/p}, \quad 2 \le p < \infty. \tag{A-1}$$

It holds with the same constant in $\mathbb{R}^2$ and for any $z \in H^1_0(\Omega)$ in a general domain Ω, or in the torus $\mathbb{T}^2$ *provided the mean value of z is zero.* In the torus case, however, we rather are in situations where

$$\int_{\mathbb{T}^2} az\,dx = 0$$

for some nonnegative measurable function a with positive mean value (say 1 with no loss of generality). Then, we claim that

$$\|z\|_{L^p} \le C_{p,a} \|z\|_{L^2}^{2/p} \|\nabla z\|_{L^2}^{1-2/p}, \quad \text{with } C_{p,a} := C_p \log^{(p-2)/p}(e + \|a\|_{L^2}). \tag{A-2}$$

Indeed, decomposing z into $z = \bar{z} + \tilde{z}$ with $\bar{z} := \fint_{\mathbb{T}^2} z\,dx$, we have

$$\begin{aligned}\int_{\mathbb{T}^2} |z|^p\,dx &= \int_{\mathbb{T}^2} |z|^2 |\tilde{z} + \bar{z}|^{p-2}\,dx \\ &\lesssim |\bar{z}|^{p-2} \|z\|_{L^2}^2 + \int_{\mathbb{T}^2} |z|^2 |\tilde{z}|^{p-2}\,dx \\ &\lesssim |\bar{z}|^{p-2} \|z\|_{L^2}^2 + \|z\|_{L^p}^2 \|\tilde{z}\|_{L^p}^{p-2}.\end{aligned}$$

Now, $\tilde{z}$ is mean-free and thus satisfies (A-1). Besides, according to [Danchin and Mucha 2019, (A.2)],

$$|\bar{z}| \le C \log(e + \|a\|_{L^2}) \|\nabla z\|_{L^2}.$$

Hence,

$$\|z\|_{L^p}^p \le C \log(e + \|a\|_{L^2}) \|\nabla z\|_{L^2}^{p-2} \|z\|_{L^2}^2 + C_p \|z\|_{L^p}^2 (\|\tilde{z}\|_{L^2}^{2/p} \|\nabla z_{L^2}^{1-2/p})^{p-2}.$$

Then, (A-2) follows from $\|\tilde{z}\|_{L^2} \le \|z\|_{L^2}$. □

Next, we recall a well-known result for the inhomogeneous Stokes equations

$$-\Delta w + \nabla Q = f \quad \text{and} \quad \operatorname{div} w = g \quad \text{in } \Omega, \tag{A-3}$$

with data $f \in L^p(\Omega)$ and $g \in \dot{W}^{1,p}(\Omega)$, $1 < p < \infty$.

In the bounded domain case (with g having mean value 0), it is known (see, e.g., [Galdi 2011]) that (A-3) admits a unique solution $(w, \nabla Q) \in W^{2,p}(\Omega) \times L^p(\Omega)$ such that $w|_{\partial\Omega} = 0$, and that the following bound holds:

$$\|\nabla^2 w, \nabla Q\|_{L^p} \le C(\|f\|_{L^p} + \|\nabla g\|_{L^p}). \tag{A-4}$$

A similar result holds in $\Omega = \mathbb{R}^2$ or $\Omega = \mathbb{T}^2$ provided we consider only solutions such that $w \to 0$ at infinity ($\mathbb{R}^2$ case) or $\fint_{\mathbb{T}^2} aw\,dx = 0$ for some nonnegative bounded function a with mean value 1 (torus case). Indeed, one can set

$$\nabla Q = \mathcal{Q} f, \quad \text{with } \mathcal{Q} := -(-\Delta)^{-1} \nabla \operatorname{div},$$

then solve the Poisson equation $-\Delta w = f + \nabla Q$. Uniqueness is given by the supplementary conditions that are prescribed above.

Finally, in the proof of stability and uniqueness, we used the following result.

Lemma A.1. *Assume that* Ω *is a* C^2 *bounded domain, the torus, or the whole space. Then, there exists a linear operator* $\mathcal{B}$ *that maps* L^p *to* L^p *for all* $p \in (1,\infty)$ *such that, for all* $k \in L^p(\Omega;\mathbb{R}^d)$ *(with mean value* 0 *in the case* $\Omega = \mathbb{T}^d$*), we have*

$$\operatorname{div}(\mathcal{B}k) = \operatorname{div} k.$$

Furthermore, if $\operatorname{div} k \in L^q(\Omega)$ *for some* $q \in (1,\infty)$*, then we have* $\mathcal{B}k \in W_0^{1,q}(\Omega;\mathbb{R}^n)$ *with* $\|\nabla \mathcal{B}k\|_{L^q} \le C\|\operatorname{div} k\|_{L^q}$*, and if* k *(seen as a function from* $\mathbb{R}_+$ *to some space* L^r *with* $1 < r < \infty$*) is differentiable for almost every* $t \in \mathbb{R}_+$*, then so is* $\mathcal{B}k$*, and we have* $\|(\mathcal{B}k)_t\|_{L^r} \le C\|k_t\|_{L^r}$ *for almost every* $t \in \mathbb{R}_+$*.*

Proof. Whenever Ω is a C^2 bounded domain, the existence of $\mathcal{B}$ as well as the first two properties have been established in [Danchin and Mucha 2013a]. The third one stems from the fact that, owing to the continuity and linearity of $\mathcal{B}$, we may write in the L^r meaning

$$(\mathcal{B}k)_t(t) = \lim_{h\to 0} \frac{\mathcal{B}k(t+h) - \mathcal{B}k(t)}{h} = \lim_{h\to 0} \mathcal{B}\left(\frac{k(t+h)-k(t)}{h}\right) = \mathcal{B}k_t.$$

If Ω is the torus or the whole space, then one can just set $\mathcal{B} := -(-\Delta)^{-1}\nabla \operatorname{div}$. □

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QUANTITATIVE STABILITY FOR COMPLEX MONGE–AMPÈRE EQUATIONS, I

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We generalize several known stability estimates for complex Monge–Ampère equations to the setting of low (or high) energy potentials. We apply our estimates to obtain, among other things, a quantitative domination principle, and metric properties of the space of potentials of finite energy. Further applications will be given in subsequent papers.

1. Introduction

Let (X, ω) be a compact Kähler manifold of dimension n and let α be a big cohomology $(1,1)$-class in X. Let θ be a closed smooth real $(1,1)$-form in α. For $u \in \mathrm{PSH}(X,\theta)$, we put $\theta_u := dd^c u + \theta$. Let $\phi \in \mathrm{PSH}(X,\theta)$ such that $\phi \le 0$ and $\int_X \theta_\phi^n > 0$, where θ_ϕ^n denotes the non-pluripolar self-product of θ_ϕ (see [Bedford and Taylor 1987; Boucksom et al. 2010]). Denote by $\mathrm{PSH}(X,\theta,\phi)$ the set of θ-psh functions u with $u \le \phi$. Note that it is slightly different from the usual definition of $\mathrm{PSH}(X,\theta,\phi)$ in which u is only required to be more singular than ϕ. This difference is not essential. We say that ϕ is a *model θ-psh function* (see [Darvas et al. 2018b; Ross and Witt Nyström 2014]) if $\phi = P_\theta[\phi]$ and $\int_X \theta_\phi^n > 0$, where

$$P_\theta[\phi] := \big(\sup\{\psi \in \mathrm{PSH}(X,\theta) : \psi \le 0,\ \psi \le \phi + O(1)\}\big)^*.$$

The function $P_\theta[\phi]$ is called a roof-top envelope in [Darvas et al. 2018b]. By [Darvas et al. 2018b], the function $P_\theta[u]$ is a model one for every $u \in \mathrm{PSH}(X,\theta)$ with $\int_X \theta_u^n > 0$, and for every $u \in \mathrm{PSH}(X,\theta,\phi)$ with $\int_X \theta_u^n = \int_X \theta_\phi^n$ we have $P_\theta[u] = P_\theta[\phi]$.

Let ϕ be now a model θ-psh function. Let $\mathcal{E}(X,\theta,\phi)$ be the space of θ-psh functions $u \le \phi$ with $\int_X \theta_u^n = \int_X \theta_\phi^n$. Let μ be a non-pluripolar measure with $\mu(X) = \int_X \theta_\phi^n$. It was proved in [Darvas et al. 2021a] (see also [Darvas et al. 2018b; Do and Vu 2022a]) that the Monge–Ampère equation with prescribed singularities

$$(dd^c u + \theta)^n = \mu, \quad u \in \mathrm{PSH}(X,\theta,\phi), \tag{1-1}$$

admits a unique solution $u \in \mathcal{E}(X,\theta,\phi)$ and $\sup_X(u-\phi) = 0$. We note that the left-hand side of (1-1) denotes the non-pluripolar self-product of θ_u (see [Bedford and Taylor 1987; Boucksom et al. 2010; Guedj and Zeriahi 2007; Vu 2021]). We refer to [Boucksom et al. 2010; Cegrell 1998; Dinew 2009; Kołodziej 1998; Yau 1978], to cite a few, for the well-known case where α is big and ϕ is a potential of minimal singularities in α.

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The aim of this paper is to study the following stability question for (1-1).

Problem 1.1. *Let θ, ϕ be as above. Let $u_j \in \mathcal{E}(X, \theta, \phi)$ for $j = 1, 2$ and $\mu_j := \theta^n_{u_j}$ for $j = 1, 2$. Compare u_1 with u_2 in terms of a suitable "distance" between μ_1, μ_2.*

To our best knowledge, there has been no available quantitative comparison between potentials of finite energy in general, even in the case where α is Kähler and $\phi \equiv 0$. The closest result that we know of is the uniqueness property (by [Dinew 2009] in the Kähler case and by [Boucksom et al. 2010; Darvas et al. 2021a] in the present setting) which says that $u_1 = u_2$ if $\mu_1 = \mu_2$. There were however some concrete estimates for the distance between u_1, u_2 in terms of μ_1, μ_2 but one had to assume some extra assumption (i.e., $u_1, u_2 \in \mathcal{E}^1(X, \theta, \phi)$); see [Błocki 2003; Guedj and Zeriahi 2012]. We will explain details below.

The goal of this paper is to solve Problem 1.1 for any potential of high or low energy. As one will see in our applications later in this paper or in our subsequent paper, it is crucial to consider Problem 1.1 for potentials in low energy.

Let $\widetilde{\mathcal{W}}^-$ be the set of convex, nondecreasing functions $\chi : \mathbb{R}_{\le 0} \to \mathbb{R}_{\le 0}$ such that $\chi(0) = 0$ and $\chi \not\equiv 0$. Let $\mathcal{W}^-$ be the subset of $\chi \in \widetilde{\mathcal{W}}^-$ such that $\chi(-\infty) = -\infty$. Note that in general $\chi \in \widetilde{\mathcal{W}}^-$ can be bounded. It is crucial in our method that we consider also bounded weights $\chi \in \widetilde{\mathcal{W}}^-$. Let $M \ge 1$ be a constant and $\mathcal{W}^+_M$ the usual space of increasing concave functions $\chi : \mathbb{R}_{\le 0} \to \mathbb{R}_{\le 0}$ such that $\chi(0) = 0$, $\chi < 0$ on $(-\infty, 0)$, and $|t\chi'(t)| \le M|\chi(t)|$ for every $t \le 0$.

Let $\varrho := \int_X \theta^n_\phi$. For $\chi \in \widetilde{\mathcal{W}}^- \cup \mathcal{W}^+_M$ and $u \in \mathrm{PSH}(X, \theta, \phi)$, let

$$E^0_{\chi,\theta,\phi}(u) := -\varrho^{-1} \int_X \chi(u - \phi)\theta^n_u,$$

which is called *the* (*normalized*) *χ-energy* of u (with respect to θ, ϕ). We define

$$\mathcal{E}_\chi(X, \theta, \phi) := \{u \in \mathcal{E}(X, \theta, \phi) : E^0_{\chi,\theta,\phi}(u) < \infty\}.$$

Certainly if χ is bounded, then $\mathcal{E}_\chi(X, \theta, \phi) = \mathcal{E}(X, \theta, \phi)$. We would like to point out however that our method is not about the finiteness of $E^0_{\chi,\theta,\phi}(u)$ but estimating the size of that quantity. Thus whether χ is bounded or not does not make much difference for our later arguments. Put

$$I^0_\chi(u, v) := \varrho^{-1} \int_{\{u<v\}} \chi(u - v)(\theta^n_v - \theta^n_u) + \varrho^{-1} \int_{\{u>v\}} \chi(v - u)(\theta^n_u - \theta^n_v)$$

for $u, v \in \mathcal{E}_\chi(X, \theta, \phi)$. The factor ϱ^{-1} in the defining formulae for $E^0_{\chi,\theta,\phi}(u)$ and $I^0_\chi(u, v)$ plays the role of a normalizing constant. In geometric applications it is important to treat the case where $\varrho \to 0$, i.e., to obtain estimates uniformly as $\varrho \to 0$.

Clearly if $\theta^n_u = \theta^n_v$, then $I^0_\chi(u, v) = 0$. We will see later that each term in the sum defining $I^0_\chi(u, v)$ is nonnegative. We recall that there is a natural (quasi)metric on the space $\mathcal{E}_\chi(X, \theta, \phi)$ constructed in [Darvas 2019; 2024; Gupta 2023], and see [Darvas et al. 2018a; Di Nezza and Lu 2020; Trusiani 2022; Xia 2023] as well. The functional $I^0_\chi(u, v)$ has an intimate relation with these quasimetrics. We refer to the end of Section 3 for details on this connection. Here is our main result.

Theorem 1.2. *Let θ be a closed smooth real $(1,1)$-form and ϕ be a negative θ-psh function such that*

$$\varrho := \int_X \theta_\phi^n > 0.$$

Let $\chi, \tilde\chi \in \widetilde{\mathcal{W}}^- \cup \mathcal{W}_M^+$ $(M \geq 1)$ such that $\tilde\chi \leq \chi$, and if $\chi \in \widetilde{\mathcal{W}}^-$, then $\lim_{t\to-\infty}(\chi(t)/\tilde\chi(t)) = 0$. Let $B \geq 1$ be a constant and let $u_j, \psi_j \in \mathcal{E}(X,\theta,\phi)$ satisfy $u_1 \leq u_2$ and

$$E^0_{\tilde\chi,\theta,\phi}(u_j) + E^0_{\tilde\chi,\theta,\phi}(\psi_j) \leq B$$

for $j = 1, 2$. Then there exist a constant $C > 0$ depending only on n, $\tilde\chi(-1)$ and M, and a continuous increasing function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ depending only on $\chi, \tilde\chi$ such that $f(0) = 0$ and

$$\int_X -\chi(u_1 - u_2)(\theta^n_{\psi_1} - \theta^n_{\psi_2}) \leq C\varrho B^2 f^{\circ n}(I^0_\chi(u_1, u_2)), \tag{1-2}$$

where $f^{\circ n} := f \circ f \circ \cdots \circ f$ (n-iterate of f). Moreover, if $\phi = P_\theta[\phi]$ and $\sup_X u_1 = \sup_X u_2$ then

$$\int_X -\chi(u_1 - u_2)(\theta^n_{\psi_1} + \theta^n_{\psi_2}) \leq \varrho\, g(I^0_\chi(u_1, u_2)), \tag{1-3}$$

where $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a continuous increasing function depending only on n, M, X, ω, θ, χ, $\tilde\chi$ and B such that $g(0) = 0$.

If $\chi \in \mathcal{W}_M^+$, then one can certainly apply Theorem 1.2 to $\tilde\chi = \chi$. Nevertheless, we underline that in applications it is of crucial importance to consider $\chi \in \widetilde{\mathcal{W}}^-$. In this case in order to have (1-3), it is necessary to require an upper bound for $\tilde\chi$-energy of u_j, where $\tilde\chi$ "dominates" χ as in the statement of Theorem 1.2. We refer to Section 3.4 for details.

One sees that (1-3) implies, in particular, that if $I^0_\chi(u_1, u_2) \to 0$, then the expression in the left-hand side also converges to 0. Theorem 1.2 follows from Theorems 3.1 and 3.2 below, where the functions f and g are given explicitly. We note that *the single Theorem 1.2 contains the following three important results in pluripotential theory*: *uniqueness of solutions of complex Monge–Ampère equations*, *domination principle*, *and comparison of capacities*. We obtain indeed quantitative (hence stronger) versions of these results for which we refer to Section 4. The quantitative version of uniqueness theorem (see Theorem 4.2 below) provides an answer to Problem 1.1. Readers can also find, in Section 4, a quantitative version of the fact that the convergence in Darvas's metric in $\mathcal{E}_\chi(X,\theta,\phi)$ implies the convergence in capacity. Notice that such an estimate seems to be not reachable by using the usual plurisubharmonic envelope method.

The main novelty of Theorem 1.2 is that it deals with *arbitrary* weights. Similar statements was already known for $\chi(t) = t$ (see [Berman et al. 2019; Błocki 2003; Guedj and Zeriahi 2012; Trusiani 2023]). However the proof there only work *exclusively* for this case. One should notice that the weight $\chi(t) = t$ is very special: it is linear and lies in the middle between higher energy weights and lower energy weights. As to the proof of Theorem 1.2, going up to the space of higher energy weights or going down to the space of lower energy weights are equally difficult. We will explain this point in more details in the paragraph after Theorem 1.3 below.

The key in the proof of Theorem 1.2 is Proposition 3.5 in Section 3, a simplified version of which we state here for readers' convenience.

Theorem 1.3. *Let* $\chi, \tilde{\chi} \in \widetilde{\mathcal{W}}^- \cup \mathcal{W}^+_M$ *such that* $\tilde{\chi} \le \chi$ *and* $\chi \in \mathscr{C}^1(\mathbb{R})$. *Let* $u_1, u_2, u_3 \in \mathcal{E}(X, \theta, \phi)$ *such that* $u_1 \le u_2$ *and* $u_j - \phi$ *is bounded* $(j = 1, 2, 3)$, *where* ϕ *is a negative* θ*-psh function satisfying* $\varrho := \mathrm{vol}(\theta_\phi) > 0$. *Then there exist a constant* $C > 0$ *depending only on* n, $\tilde{\chi}(-1)$ *and* M *such that*

$$\int_X \chi'(u_1 - u_2)\, d(u_1 - u_2) \wedge d^c(u_1 - u_2) \wedge \theta^{n-1}_{u_3} \le C\varrho B^2 f^{\circ(n-1)}(I^0_\chi(u_1, u_2)),$$

where $B := \sum_{j=1}^3 \max\{E^0_{\tilde{\chi},\theta,\phi}(u_j), 1\}$ *and* $f : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ *is a continuous function such that* $f(0) = 0$ *if one has either* $\chi \in \mathcal{W}^+_M$ *or* $\chi \in \widetilde{\mathcal{W}}^-$ *and* $\lim_{t\to-\infty}(\chi(t)/\tilde{\chi}(t)) = 0$.

As far as we know, all of previous works related to Theorem 1.3 only concern $\chi(t) = t$. In this case, Theorem 1.3 is known with an explicit f and without $\tilde{\chi}$ if ϕ is of minimal singularity in the cohomology class of θ, by [Błocki 2003; Guedj and Zeriahi 2012].

The key ingredients in previous versions of Theorem 1.3 for $\chi(t) = t$ are integration by parts arguments. Direct generalization of such reasoning immediately break down if $\chi \ne \mathrm{id}$: in a more precise but technical level, the integration by parts arguments give terms like $\chi'(u_1 - u_2)d(u_1 - u_3) \wedge d^c(u_1 - u_3)$, such a quantity is easy to bound if $\chi = \mathrm{id}$ (hence $\chi' \equiv 1$), but not if $\chi \ne \mathrm{id}$.

In order to prove Theorem 1.3, we still use this strategy but need to use a so-called “monotonicity argument” from [Do and Vu 2022a; Vu 2021; 2022] to deal with general χ. In a nutshell it is about using intensively the plurilocality of Monge–Ampère operators together with the monotonicity of pluricomplex energy which allows one to bound from above “Monge–Ampère quantities” of bad potentials by that of nicer potentials. This method is a flexible tool to deal with “low regularity”, and was a key in the proof of the convexity of the class of potentials of finite χ-energy in [Vu 2022], as well as giving a characterization of the class of Monge–Ampère measures with potentials of finite χ-energy in [Do and Vu 2022a].

We refer to the end of the paper for some applications of our main results. Furthermore, the quantitative domination principle obtained in Section 4 was used crucially in [Dang and Vu 2023] to describe the degeneration of conic Kähler–Einstein metrics. We note also that the present paper is the first part of the manuscript [Do and Vu 2022b], in which we give a more or less satisfactory treatment for a much more general question than Problem 1.1: precisely, we establish quantitative stability when both the cohomology class and the singularity type vary. The second part of [Do and Vu 2022b], where this generalization is treated, will be submitted separately due to the length constraint.

The paper is organized as follows. In Section 2, we recall the integration by parts formula from [Vu 2022], auxiliary facts about weights are also collected there. Theorems 1.2 and 1.3 are proved in Section 3. Applications will be given in Section 4.

2. Preliminaries

2.1. ***Integration by parts.*** In this subsection, we recall the integration by parts formula obtained in [Vu 2022, Theorem 2.6]. This formula will play a key role in our proof of main results later.

Let X be a compact Kähler manifold. Let $T_1, \dots, T_m$ be closed positive $(1, 1)$-currents on X. Let T be a closed positive current of bidegree (p, p) on X. The T*-relative non-pluripolar product* $\langle \bigwedge_{j=1}^m T_j \dot{\wedge} T \rangle$ is defined in a way similar to that of the usual non-pluripolar product (see [Vu 2021]). The product $\langle \bigwedge_{j=1}^m T_j \dot{\wedge} T \rangle$

is a closed positive current of bidegree $(m+p, m+p)$; and the wedge product $\langle \bigwedge_{j=1}^m T_j \dot\wedge T \rangle$ as an operator on currents is symmetric with respect to $T_1, \dots, T_m$ and is homogeneous. In latter applications, we will only use the case where T is the non-pluripolar product of some closed positive (1, 1)-currents, say, $T = \langle T_{m+1} \wedge \cdots \wedge T_{m+l} \rangle$, where T_j are (1, 1)-currents for $m+1 \le j \le m+l$. In this case, $\langle T_1 \wedge \cdots \wedge T_m \dot\wedge T \rangle$ is simply equal to $\langle \bigwedge_{j=1}^{m+l} T_j \rangle$. We usually remove the bracket $\langle\ \rangle$ in the non-pluripolar product to ease the notation.

Recall that a *dsh* function on X is the difference of two quasi-plurisubharmonic (quasi-psh for short) functions on X (see [Dinh and Sibony 2006]). These functions are well-defined outside pluripolar sets. Let v be a dsh function on X. Let T be a closed positive current on X. We say that v is *T-admissible* if there exist quasi-psh functions φ_1, φ_2 such that $v = \varphi_1 - \varphi_2$ and T has no mass on $\{\varphi_j = -\infty\}$ for $j = 1, 2$. In particular, if T has no mass on pluripolar sets, then every dsh function is T-admissible.

Assume now that v is T-admissible. Let φ_1, φ_2 be quasi-psh functions such that $v = \varphi_1 - \varphi_2$ and T has no mass on $\{\varphi_j = -\infty\}$ for $j = 1, 2$. Let

$$\varphi_{j,k} := \max\{\varphi_j, -k\}$$

for every $j = 1, 2$ and $k \in \mathbb{N}$. Put $v_k := \varphi_{1,k} - \varphi_{2,k}$. Put

$$Q_k := dv_k \wedge d^c v_k \wedge T = \tfrac{1}{2} dd^c v_k^2 \wedge T - v_k dd^c v_k \wedge T.$$

By the plurifine locality with respect to T (see [Vu 2021, Theorem 2.9]) applied to the right-hand side of the last equality, we have

$$\mathbf{1}_{\bigcap_{j=1}^2 \{\varphi_j > -k\}} Q_k = \mathbf{1}_{\bigcap_{j=1}^2 \{\varphi_j > -k\}} Q_s \tag{2-1}$$

for every $s \ge k$. We say that $\langle dv \wedge d^c v \dot\wedge T \rangle$ is *well-defined* if the mass of $\mathbf{1}_{\bigcap_{j=1}^2 \{\varphi_j > -k\}} Q_k$ is uniformly bounded on k. In this case, using (2-1) implies that there exists a positive current Q on X such that for every bounded Borel form Φ with compact support on X such that

$$\langle Q, \Phi \rangle = \lim_{k \to \infty} \langle \mathbf{1}_{\bigcap_{j=1}^2 \{\varphi_j > -k\}} Q_k, \Phi \rangle,$$

and we define $\langle dv \wedge d^c v \dot\wedge T \rangle$ to be the current Q. This agrees with the classical definition if v is the difference of two bounded quasi-psh functions. One can check that this definition is independent of the choice of φ_1, φ_2. By [Vu 2022, Lemma 2.5], if v is bounded, then $\langle dv \wedge d^c v \dot\wedge T \rangle$ is well-defined.

Let w be another T-admissible dsh function. If T is of bidegree $(n-1, n-1)$, we can also define the current $\langle dv \wedge d^c w \dot\wedge T \rangle$ by a similar procedure as above. More precisely, we say $\langle dv \wedge d^c w \dot\wedge T \rangle$ is *well-defined* if $\langle dv \wedge d^c v \dot\wedge T \rangle$, $\langle dw \wedge d^c w \dot\wedge T \rangle$, and $\langle d(v+w) \wedge d^c (v+w) \dot\wedge T \rangle$ are well-defined. In this case, as in the classical case of bounded potentials, the defining formula for $\langle dv \wedge d^c w \dot\wedge T \rangle$ is obvious:

$$2\langle dv \wedge d^c w \dot\wedge T \rangle = \langle d(v+w) \wedge d^c (v+w) \dot\wedge T \rangle - \langle dv \wedge d^c v \dot\wedge T \rangle - \langle dw \wedge d^c w \dot\wedge T \rangle.$$

As above, if v, w are bounded T-admissible, then $\langle dv \wedge d^c w \dot\wedge T \rangle$ is well-defined and given by the above formula. The following Cauchy–Schwarz inequality is clear from definition.

Lemma 2.1. *Assume that $\langle dv \wedge d^c w \dot\wedge T \rangle$ is well-defined. Then, for every positive Borel function χ, we have*

$$\int_X \chi \langle dv \wedge d^c w \dot\wedge T \rangle \le \left(\int_X \chi \langle dv \wedge d^c v \dot\wedge T \rangle \right)^{1/2} \left(\int_X \chi \langle dw \wedge d^c w \dot\wedge T \rangle \right)^{1/2}.$$

We put

$$\langle dd^c v \dot{\wedge} T\rangle := \langle dd^c \varphi_1 \dot{\wedge} T\rangle - \langle dd^c \varphi_2 \dot{\wedge} T\rangle,$$

which is independent of the choice of φ_1, φ_2. The following integration by parts formula is crucial for us later.

Theorem 2.2 ([Vu 2022, Theorem 2.6] or [Do and Vu 2022a, Theorem 3.1]). *Let T be a closed positive current of bidegree $(n-1, n-1)$ on X. Let v, w be bounded T-admissible dsh functions on X. If $\chi : \mathbb{R} \to \mathbb{R}$ is a $\mathscr{C}^3$ function then*

$$\int_X \chi(w)\langle dd^c v \dot{\wedge} T\rangle = \int_X v\chi''(w)\langle dw \wedge d^c w \dot{\wedge} T\rangle + \int_X v\chi'(w)\langle dd^c w \dot{\wedge} T\rangle$$
$$= -\int_X \chi'(w)\langle dw \wedge d^c v \dot{\wedge} T\rangle. \tag{2-2}$$

Since the case where T is a non-pluripolar product of (1, 1)-currents plays an important role in the study of complex Monge–Ampère equations, we present below an equivalent natural way to define the current $\langle d\varphi \wedge d^c\varphi \dot{\wedge} T\rangle$ in this setting. It is just for the purpose of clarification.

Lemma 2.3. *Let $u_1, \dots, u_m$ be negative psh functions on an open subset U in $\mathbb{C}^n$ such that $T := \langle dd^c u_1 \wedge \cdots \wedge dd^c u_m\rangle$ is well-defined. Let v be the difference of two bounded psh functions on U. For $k \in \mathbb{N}$, put $u_{j,k} := \max\{u_j, -k\}$ and*

$$T_k := dd^c u_{1,k} \wedge \cdots \wedge dd^c u_{m,k}.$$

Then

$$dv \wedge d^c v \wedge T = dv \wedge d^c v \wedge T_k$$

on $\bigcap_{j=1}^m \{u_j > -k\}$.

Proof. Put

$$\psi_k := k^{-1} \max\{u_1 + \cdots + u_m, -k\} + 1.$$

Observe $\psi_k T_k = \psi_k T$. Now regularizing v and using the continuity of Monge–Ampère operators of bounded potentials, we obtain

$$\psi_k dv \wedge d^c v \wedge T = \psi_k dv \wedge d^c v \wedge T_k.$$

Hence

$$dv \wedge d^c v \wedge T = dv \wedge d^c v \wedge T_k$$

on $U := \bigcap_{j=1}^m \{u_j > -k/(2m)\}$ $\big($for $\psi_k \geq \frac{1}{2}$ on $U\big)$. Note that $dv \wedge d^c v \wedge T_k = dv \wedge d^c v \wedge T_{k/(2m)}$ on U by the plurifine locality. Thus the desired assertion follows. □

Let $T_1, \dots, T_m$ be closed positive (1, 1)-currents on X. Let $n := \dim X$. Consider now

$$T := \langle T_1 \wedge \cdots \wedge T_m\rangle.$$

Note that T has no mass on pluripolar sets. Let φ_1, φ_2 be negative quasi-psh function on X. Let $\varphi_{j,k}$ $(j = 1, 2)$ be as before and $v := \varphi_1 - \varphi_2$. In the moment, we work locally. Let U be an open small enough local chart (biholomorphic to a polydisk in $\mathbb{C}^n$) in X such that $T_j = dd^c u_j$ for $j = 1, \dots, m$, where u_j are negative psh functions on U. Put $u_{j,k} := \max\{u_j, -k\}$ for $k \in \mathbb{N}$, and

$$T_k := dd^c u_{1,k} \wedge \cdots \wedge dd^c u_{m,k}, \quad Q'_k := dv_k \wedge d^c v_k \wedge T_k.$$

Put $A_k := \bigcap_{j=1}^{2}\{\varphi_j > -k\} \cap \bigcap_{j=1}^{m}\{u_j > -k\}$. By plurifine properties of Monge–Ampère operators,

$$\mathbf{1}_{A_k} Q'_k = \mathbf{1}_{A_k} Q'_s$$

for every $s \geq k$. One can check that the condition that $(\mathbf{1}_{A_k} Q'_k)_k$ is of mass bounded uniformly (on compact subsets in U) in k is independent of the choice of potentials.

Proposition 2.4. *The current $\mathbf{1}_{A_k} Q'_k$ is of mass bounded uniformly in k on compact subsets in U for every U (small enough biholomorphic to a polydisk in $\mathbb{C}^n$) if and only if the current $\langle dv \wedge d^c v \dot{\wedge} T\rangle$ is well-defined. In this case*

$$\langle dv \wedge d^c v \dot{\wedge} T\rangle = \lim_{k\to\infty} \mathbf{1}_{A_k} Q'_k. \tag{2-3}$$

Proof. By writing a smooth form of bidegree $(n-m-1, n-m-1)$ as the difference of two smooth positive forms, we can assume without loss of generality that T is of bidegree $(n-1, n-1)$ (hence $m = n-1$). Assume that $\langle dv \wedge d^c v \dot{\wedge} T\rangle$ is well-defined. We will check that $\mathbf{1}_{A_k} Q'_k$ is of mass bounded uniformly in k on compact subsets in U. Let χ be a nonnegative smooth function compactly supported on U. Put

$$\psi := \varphi_1 + \varphi_2 + u_1 + \cdots + u_m, \quad \psi_k := k^{-1}\max\{\psi, -k\} + 1,$$

and $\varphi_{jk} := \max\{\varphi_j, -k\}$ for $1 \leq j \leq 2$. Observe that $0 \leq \psi_k \leq 1$ and if $\psi_k > 0$, then $\varphi_j > -k$ for $1 \leq j \leq 2$; and

$$\psi_k(x) \geq 1 - s/k \tag{2-4}$$

for every $x \in A_{s/(m+2)}$ and $1 \leq s \leq k$. Recall $v_k := \varphi_{1k} - \varphi_{2k}$ which is bounded (but not necessarily uniformly in k). Observe that $\langle dv \wedge d^c v \dot{\wedge} T\rangle$ has no mass on pluripolar sets because T is so (see, for example, [Vu 2021, Lemma 2.1]). Put $Q''_k := \psi_k Q_k = \psi_k \mathbf{1}_{A_k} Q'_k$. By (2-4) and Lemma 2.3, we have

$$\begin{aligned}\langle dv \wedge d^c v \dot{\wedge} T\rangle &= \lim_{k\to\infty} \psi_k dv_k \wedge d^c v_k \wedge T \\ &= \lim_{k\to\infty} \psi_k dv_k \wedge d^c v_k \wedge T_k = \lim_{k\to\infty} Q''_k\end{aligned} \tag{2-5}$$

on U. On the other hand, by (2-4) again, we see that the claim that Q''_k is of mass uniformly bounded on compact subsets in U is equivalent to that $\mathbf{1}_{A_k} Q'_k$ is so. This together with (2-5) yields the desired assertion.

Conversely, suppose now that $\mathbf{1}_{A_k} Q'_k$ is of mass bounded uniformly in k on compact subsets in U for every U. Thus there exists a positive current R on U such that $\mathbf{1}_{A_k} R = \mathbf{1}_{A_k} Q'_k$ for every k and U. Set

$$\tilde{\psi} := \varphi_1 + \varphi_2, \quad \tilde{\psi}_k := k^{-1}\max\{\tilde{\psi}, -k\} + 1.$$

Let $s \in \mathbb{N}$ with $s \geq k$. Observe

$$\psi_s R = \tilde{\psi}_k \psi_s R + (1 - \tilde{\psi}_k)\psi_s R.$$

The second term in the right-hand side of the last inequality tends to 0 (uniformly in s) because $\tilde{\psi}_k$ converges pointwise to 1 outside a pluripolar set and R has no mass on pluripolar sets. Using Lemma 2.3, we have

$$\begin{aligned}\tilde{\psi}_k \psi_s R &= \tilde{\psi}_k \psi_s dv_s \wedge d^c v_s \wedge T_s \\ &= \tilde{\psi}_k \psi_s dv_s \wedge d^c v_s \wedge T = \tilde{\psi}_k \psi_s dv_k \wedge d^c v_k \wedge T.\end{aligned}$$

Here we used the plurifine topology properties with respect to T (see [Vu 2021, Theorem 2.9]), thanks to the fact that $\varphi_{j,k} = \varphi_{j,s}$ on $\{\tilde{\psi}_k \neq 0\}$ for $j = 1, 2$ (recall $s \geq k$), and they are bounded psh functions. Letting $s \to \infty$ gives

$$\tilde{\psi}_k R = \tilde{\psi}_k \mathbf{1}_{\bigcup_{j=1}^m \{u_j > -\infty\}} dv_k \wedge d^c v_k \wedge T = \tilde{\psi}_k dv_k \wedge d^c v_k \wedge T$$

because the current $dv_k \wedge d^c v_k \wedge T$ has no mass on pluripolar sets. Now letting $k \to \infty$ gives the desired assertion. □

Thanks to Proposition 2.4, we can use the right-hand side of (2-3) to define $\langle dv \wedge d^c v \dot{\wedge} T \rangle$ in the case where T is the non-pluripolar product of some closed positive $(1, 1)$-currents. By the same reason, in this case, we will use the expression $dv \wedge d^c w \wedge T_1 \wedge \ldots \wedge T_{n-1}$ to denote $\big\langle dv \wedge d^c w \dot{\wedge} \langle T_1 \wedge \cdots \wedge T_{n-1} \rangle \big\rangle$ whenever it is well-defined.

2.2. *Auxiliary facts on weights.* In this subsection, we present some facts about weights needed for the proofs of main results.

Recall that $\widetilde{\mathcal{W}}^-$ is the set of all convex, nondecreasing functions $\chi : \mathbb{R}_{\leq 0} \to \mathbb{R}_{\leq 0}$ such that $\chi(0) = 0$ and $\chi \not\equiv 0$. Let $M \geq 1$ be a constant and $\mathcal{W}_M^+$ the usual space of increasing concave functions $\chi : \mathbb{R}_{\leq 0} \to \mathbb{R}_{\leq 0}$ such that $\chi(0) = 0$, $\chi < 0$ on $(-\infty, 0)$, and $|t\chi'(t)| \leq M|\chi(t)|$ for every $t \leq 0$. We have the following lemmas.

Lemma 2.5. *Let $c > 0$, $0 < \delta < 1$ and $\chi : \mathbb{R} \to \mathbb{R}$ such that $\chi(t) = ct$ for every $t \geq -\delta$ and $\chi|_{(-\infty,0]} \in \widetilde{\mathcal{W}}^- \cup \mathcal{W}_M^+$ $(M \geq 1)$. Let g be a smooth radial cut-off function supported in $[-1, 1]$ on $\mathbb{R}$, i.e., $g(t) = g(-t)$ for $t \in \mathbb{R}$, $0 \leq g \leq 1$ and $\int_{\mathbb{R}} g(t)\,dt = 1$. Put $g_\epsilon(t) := \epsilon^{-1} g(\epsilon t)$ for every constant $\epsilon > 0$ and $\chi_\epsilon := \chi * g_\epsilon$ (the convolution of χ with g_ϵ).*

(i) *If $\chi|_{(-\infty,0]} \in \widetilde{\mathcal{W}}^-$, then $\chi_\epsilon|_{(-\infty,0]} \in \widetilde{\mathcal{W}}^-$ for every $0 < \epsilon < \delta$, $\chi_\epsilon \searrow \chi$ as $\epsilon \searrow 0$ and $\sup(\chi_\epsilon - \chi) \leq c\epsilon$.*

(ii) *If $\chi|_{(-\infty,0]} \in \mathcal{W}_M^+$ and $0 < \epsilon < \delta^2/2$ then $\chi_\epsilon|_{(-\infty,0]} \in \mathcal{W}^+_{M/(1-\delta)}$. Moreover, if $0 < \epsilon < \delta^2/8$ then*

$$\overline{\chi}_\epsilon := \chi_\epsilon(\cdot + \epsilon) - c\epsilon \in \mathcal{W}^+_{M/(1-\delta)^2}, \qquad \overline{\chi}_\epsilon \geq \chi - c\epsilon,$$

and $\overline{\chi}_\epsilon$ converges uniformly to χ as $\epsilon \to 0$ on compact subsets in $\mathbb{R}$.

Proof. Part (i) follows from [Do and Vu 2022a, Lemma 2.1]. Part (ii) can be obtained more or less by similar arguments as in the last reference. We provide details for readers' convenience. It is clear that χ_ϵ is a concave, increasing function with $\chi_\epsilon(0) = 0$. We will show that

$$\chi_\epsilon'(t) \leq \frac{M}{1-\delta} \frac{\chi_\epsilon(t)}{t} \tag{2-6}$$

for every $t < 0$ and $0 < \epsilon < \delta^2/2$.

If $t < -\delta/2$ then

$$\chi_\epsilon'(t) = \int_{-\epsilon}^{\epsilon} \chi'(t-s) g_\epsilon(s)\, ds \le \int_{-\epsilon}^{\epsilon} \frac{M\chi(t-s)}{t-s} g_\epsilon(s)\, ds \le \int_{-\epsilon}^{\epsilon} \frac{M\chi(t-s)}{t+\epsilon} g_\epsilon(s)\, ds = \frac{M\chi_\epsilon(t)}{t+\epsilon} = \frac{Mt}{t+\epsilon}\frac{\chi_\epsilon(t)}{t} \le \frac{M}{1-\delta}\frac{\chi_\epsilon(t)}{t}$$

for every $0 < \epsilon < \delta^2/2$.

On the other hand, if $t \ge -\delta/2$, then $\chi_\epsilon(t) = \chi(t) = ct$ for every $0 < \epsilon < \delta^2/2$. As a consequence,

$$\chi_\epsilon'(t) = \chi'(t) \le \frac{M\chi(t)}{t} = M\frac{\chi_\epsilon(t)}{t}.$$

Thus, (2-6) follows. Hence, $\chi_\epsilon|_{(-\infty,0]} \in \mathcal{W}^+_{M/(1-\delta)}$.

Now, we consider $\overline{\chi}_\epsilon$. Since χ is increasing, one sees that $\overline{\chi}_\epsilon \ge \chi - c\epsilon$ and $\overline{\chi}_\epsilon$ converges uniformly to χ as $\epsilon \to 0$ on compact subsets in $\mathbb{R}$. It remains to show that $\overline{\chi}_\epsilon \in \mathcal{W}^+_{M(1+\delta)/(1-\delta)}$ for every $0 < \epsilon < \delta^2/8$. Note that

$$\overline{\chi}_\epsilon = h_\epsilon * g_\epsilon,$$

where $h_\epsilon(t) = \chi(t+\epsilon) - c\epsilon$. The function $\overline{\chi}_\epsilon(t)$ is concave, increasing and $\overline{\chi} + \epsilon(0) = 0$.

If $-\delta/2 \le t < 0$ then $h_\epsilon(t) = \chi(t) = ct$ for every $0 < \epsilon < \delta^2/2$. Therefore

$$h_\epsilon'(t) = \chi'(t) \le \frac{M\chi(t)}{t} = M\frac{h_\epsilon(t)}{t}.$$

If $t < -\delta/2$ then

$$h_\epsilon'(t) = \chi'(t+\epsilon) \le M\frac{\chi(t+\epsilon)}{t+\epsilon} \le M\frac{\chi(t+\epsilon) - c\epsilon}{t+\epsilon} = M\frac{h_\epsilon(t)}{t+\epsilon} = \frac{Mt}{t+\epsilon}\frac{h_\epsilon(t)}{t} \le \frac{M}{1-\delta}\frac{h_\epsilon(t)}{t}$$

for every $0 < \epsilon < \delta^2/2$.

Then, for every $0 < \epsilon < \delta^2/2$, we have $h_\epsilon \in \mathcal{W}^+_{M/(1-\delta)}$ and $h_\epsilon = ct$ for every $t \ge -\delta/2$. Hence, for every $0 < \epsilon < \delta^2/8$, we have

$$\overline{\chi}_\epsilon = h_\epsilon * g_\epsilon \in \mathcal{W}^+_{M/((1-\delta)(1-\delta/2))} \subset \mathcal{W}^+_{M/(1-\delta)^2}. \qquad \square$$

Lemma 2.6. *Let $\chi, \tilde{\chi} \in \widetilde{\mathcal{W}}^- \cup \mathcal{W}^+_M$ $(M \ge 1)$ such that $\tilde{\chi} \le \chi$. Then there exist sequences of functions $\chi_j, \tilde{\chi}_j \in \widetilde{\mathcal{W}}^- \cup \mathcal{W}^+_{M_j}$ (with $M_j \searrow M$ as $j \to \infty$) satisfying the following conditions*:

- $\chi_j \in \mathscr{C}^\infty(\mathbb{R})$ *for every j.*
- $\chi_j \ge \tilde{\chi}_j$ *and* $\chi_j \ge \chi - 2^{-j}$ *for every j big enough.*
- $\tilde{\chi} - 2^{-j} \le \tilde{\chi}_j \le \tilde{\chi}$ *on* $(-\infty, -1]$ *for every j big enough.*
- χ_j *converges uniformly to χ on compact subsets in* $\mathbb{R}_{\le 0}$.

Proof. We split the proof into two cases.

Case 1: $\chi \in \widetilde{\mathcal{W}}^-$. For every $j \ge 1$, we set

$$\overline{\chi}_j(t) = \begin{cases} \max\{\chi(t), c_j t\} & \text{if } t < 0, \\ c_j t & \text{if } t \ge 0, \end{cases}$$

where

$$c_j := \frac{-\chi(-2^{-j})}{2^{-j}}.$$

Then $\overline{\chi}_j$ satisfies the hypothesis of Lemma 2.5 for $\delta := 2^{-j}$. Let g be a smooth radial cut-off function supported in $[-1, 1]$ on $\mathbb{R}$, i.e., $g(t) = g(-t)$ for $t \in \mathbb{R}$, $0 \le g \le 1$ and $\int_{\mathbb{R}} g(t)\,dt = 1$. For every $j \ge 1$, we define

$$\chi_j = \overline{\chi}_j * g_{4^{-j-1}} \quad \text{and} \quad \tilde{\chi}_j = \tilde{\chi}.$$

By Lemma 2.5, χ_j and $\tilde{\chi}_j$ satisfy the desired conditions.

Case 2: $\chi \in \mathcal{W}^+_M$. Since $\chi \ge \tilde{\chi}$, we also have $\tilde{\chi} \in \mathcal{W}^+_M$. Assume that g and c_j are as in Case 1. For every $j \ge 1$, we define

$$\overline{\chi}_j(t) = \begin{cases} \min\{\chi(t), c_j t\} & \text{if } t < 0, \\ c_j t & \text{if } t \ge 0, \end{cases}$$

and

$$\chi_j(t) = (\overline{\chi}_j(\cdot + 4^{-j-1}) * g_{4^{-j-1}})(t) - c_j 4^{-j-1}.$$

We also set $\tilde{\chi}_j(t) = \min\{\tilde{\chi}(t), \chi_j(t)\}$. By Lemma 2.5, χ_j and $\tilde{\chi}_j$ satisfy the desired conditions. □

Let ϕ be a negative θ-psh function. We denote by $\mathrm{PSH}(X, \theta, \phi)$ the set of θ-psh functions $u \le \phi$. Recall that by monotonicity, we always have $\int_X \theta_u^n \le \int_X \theta_\phi^n$, where for every θ-psh function v, we put $\theta_v := dd^c v + \theta$. We also define by $\mathcal{E}(X, \theta, \phi)$ the set of $u \in \mathrm{PSH}(X, \theta, \phi)$ of full Monge–Ampère mass with respect to ϕ, i.e., $\int_X \theta_u^n = \int_X \theta_\phi^n$.

Let $\chi \in \widetilde{\mathcal{W}}^- \cup \mathcal{W}^+_M$, and $u \in \mathrm{PSH}(X, \theta, \phi)$. We put

$$E_{\chi,\theta,\phi}(u) := \int_X -\chi(u - \phi)\theta_u^n.$$

We also define by $\mathcal{E}_\chi(X, \theta, \phi)$ the set of $u \in \mathcal{E}(X, \theta, \phi)$ with $E_{\chi,\theta,\phi}(u) < \infty$.

Lemma 2.7. *Let $\chi \in \widetilde{\mathcal{W}}^- \cup \mathcal{W}^+_M$ and $u_1, u_2 \in \mathcal{E}_\chi(X, \theta, \phi)$. Then there exists a constant $C_1 > 0$ depending only on n and M such that*

$$-\int_X \chi(u_1 - \phi)\theta_{u_2}^n \le C_1 \sum_{j=1}^{2} E_{\chi,\phi,\theta}(u_j),$$

and

$$E_{\chi,\theta,\phi}(au_1 + (1-a)u_2) \le C_1 \sum_{j=1}^{2} E_{\chi,\theta,\phi}(u_j)$$

for every $0 < a < 1$. Furthermore if $u_1 \ge u_2$, then

$$E_{\chi,\phi,\theta}(u_1) \le C_2 E_{\chi,\phi,\theta}(u_2)$$

for some constant C_2 depending only on n and M.

Proof. The first and third inequalities are from [Do and Vu 2022a, Lemma 3.2] (see also [Guedj and Zeriahi 2007, Propositions 2.3, 2.5] for the case where $\phi = 0$ and θ is a Kähler form). The second desired inequality was implicitly proved in the proof of convexity of finite energy classes in [Vu 2022, Proposition 3.3] (in a much broader context). Alternatively one can use properties of envelopes in [Darvas et al. 2018b] to get the same conclusion. We prove here the second desired inequality using ideas from [Vu 2022] for readers' convenience. We only consider $\chi \in \widetilde{\mathcal{W}}^-$. The case where $\chi \in \mathcal{W}_M^+$ is done similarly.

Considering $u_j - \epsilon$ for $\epsilon > 0$ instead of u_j, and taking $\epsilon \to 0$ later, without loss of generality, we can assume that $u_j < \phi \leq 0$ for $j = 1, 2$. By replacing u_j, θ by $u_j - \phi$, θ_ϕ respectively, we can assume that $\phi = 0$, but θ is no longer a smooth form but a closed positive $(1, 1)$-current. This change causes no trouble for us. Let $v := au_1 + (1-a)u_2$. Observe that $X \subset \{u_1 < u_2\} \cup \{u_1 > 2u_2\}$. Hence

$$
\begin{aligned}
E_{\chi,\theta}(v) &\leq \int_{\{u_1<u_2\}} -\chi(v)\theta_v^n + \int_{\{u_1>2u_2\}} -\chi(v)\theta_v^n \\
&\leq \sum_{k=0}^{n}\left(\int_{\{u_1<u_2\}} -\chi(u_1)\theta_{u_1}^k \wedge \theta_{u_2}^{n-k} + \int_{\{u_1>2u_2\}} -\chi((1+a)u_2)\theta_{u_1}^k \wedge \theta_{u_2}^{n-k}\right) \\
&\leq \sum_{k=0}^{n} \int_{\{u_1<u_2\}} -\chi(u_1)\theta_{u_1}^k \wedge \theta_{\max\{u_1,u_2\}}^{n-k} + \sum_{k=0}^{n} \int_{\{u_1>2u_2\}} -2^{k+1}\chi(u_2)\theta_{\max\{u_1/2,u_2\}}^k \wedge \theta_{u_2}^{n-k} \\
&\leq \sum_{k=0}^{n}\left(\int_X -\chi(u_1)\theta_{u_1}^k \wedge \theta_{\max\{u_1,u_2\}}^{n-k} + 2^{k+1}\int_X -\chi(u_2)\theta_{\max\{u_1/2,u_2\}}^k \wedge \theta_{u_2}^{n-k}\right) \\
&\lesssim E_{\chi,\theta}(u_1) + E_{\chi,\theta}(\max\{u_1, (u_1+u_2)/2\}) + E_{\chi,\theta}(u_2) + E_{\chi,\theta}(\max\{u_1/4 + u_2/2, u_2\}) \\
&\lesssim E_{\chi,\theta}(u_1) + E_{\chi,\theta}(u_2),
\end{aligned}
$$

where the two last estimates hold due to the first and third inequalities of the lemma. □

Lemma 2.8. *Let $\chi, \tilde{\chi} \in \widetilde{\mathcal{W}}^- \cup \mathcal{W}_M^+$ such that $\tilde{\chi} \leq \chi$ and let $u_1, u_2, \ldots, u_{n+1} \in \mathcal{E}(X, \theta, \phi)$. Define $\varrho := \mathrm{vol}(\theta_\phi)$. Then there exists a constant $C > 0$ depending only on n and M such that*

$$
-\int_X \chi(\epsilon(u_1 - \phi))\theta_{u_2} \wedge \cdots \wedge \theta_{u_{n+1}} \leq CB\varrho(1 - \tilde{\chi}(-1))Q_0(\epsilon)
$$

for every $0 < \epsilon \leq 1$, where

$$
B = 1 + \max_{1\leq j\leq n+1} E_{\tilde{\chi},\theta,\phi}(u_j)/\varrho \quad \text{and} \quad Q_0(\epsilon) := \sup_{\{t\leq -1\}} \frac{\chi(\epsilon t)}{\tilde{\chi}(t)}.
$$

Proof. Let L be the left-hand side of the desired inequality. We have

$$
\begin{aligned}
L &\leq -\int_{\{u_1\geq\phi-1\}} \chi(\epsilon(u_1 - \phi))\theta_{u_2} \wedge \cdots \wedge \theta_{u_{n+1}} - \int_{\{u_1<\phi-1\}} \chi(\epsilon(u_1 - \phi))\theta_{u_2} \wedge \cdots \wedge \theta_{u_{n+1}} \\
&\leq -\chi(-\epsilon)\varrho - Q_0(\epsilon)\int_{\{u_1<\phi-1\}} \tilde{\chi}(u_1 - \phi)\theta_{u_2} \wedge \cdots \wedge \theta_{u_{n+1}} \\
&\leq -\varrho Q_0(\epsilon)\tilde{\chi}(-1) - Q_0(\epsilon)\int_X \tilde{\chi}(u_1 - \phi)\theta_{u_2} \wedge \cdots \wedge \theta_{u_{n+1}} \\
&\leq -\varrho Q_0(\epsilon)\tilde{\chi}(-1) + CQ_0(\epsilon) \max_{1\leq j\leq n+1} E_{\tilde{\chi},\theta,\phi}(u_j),
\end{aligned}
$$

where $C > 0$ depends only on n and M. The last estimate holds due to Lemma 2.7. Thus the desired inequality follows. □

By the convexity/concavity and by the assumption $\tilde{\chi} \le \chi$, we have

$$\begin{cases} Q_0(\epsilon) \ge \epsilon Q_0(1) & \text{if } \chi \in \widetilde{\mathcal{W}}^-, \\ Q_0(\epsilon) \le \epsilon Q_0(1) & \text{if } \chi \in \mathcal{W}_M^+ \end{cases} \tag{2-7}$$

for every $0 < \epsilon \le 1$. Moreover, if $\chi \in \widetilde{\mathcal{W}}^-$ and $\chi(t)/\tilde{\chi}(t) \to 0$ as $t \to -\infty$, then by the definition of Q_0, we also have

$$Q_0(\epsilon) \le \frac{\chi(-\epsilon^{1/2})}{\tilde{\chi}(-1)} + \sup_{\{t \le -\epsilon^{-1/2}\}} \frac{\chi(t)}{\tilde{\chi}(t)} \xrightarrow{\epsilon \to 0^+} 0. \tag{2-8}$$

Let $u_1, u_2 \in \mathcal{E}_\chi(X, \theta, \phi)$, and $v := \max\{u_1, u_2\}$. Put

$$\nu(u_1, u_2) := \chi(-|u_1 - u_2|)(\theta_{u_2}^n - \theta_{u_1}^n)$$

and

$$\begin{aligned} I_\chi(u_1, u_2) &:= \int_{\{u_1 < u_2\}} \nu(u_1, u_2) + \int_{\{u_1 > u_2\}} \nu(u_2, u_1) \\ &= \int_X \nu(u_1, v) + \int_X \nu(u_2, v). \end{aligned} \tag{2-9}$$

Proposition 2.9. *Let $\chi \in \widetilde{\mathcal{W}}^- \cup \mathcal{W}_M^+$. Let ϕ is a negative θ-psh function and $u_1, u_2 \in \mathcal{E}_\chi(X, \theta, \phi)$. Then*

$$I_\chi(u_1, u_2) \ge 0.$$

Proof. Define $\mu = \theta_{u_2}^n - \theta_{u_1}^n$. Since χ is absolutely continuous, we have χ is differentiable almost everywhere and $-\chi(t) = \int_t^0 \chi'(s)\,ds$ for every $t < 0$. Hence

$$\begin{aligned} \int_{\{u_1 < u_2\}} \nu(u_1, u_2) &= -\int_{\{u_1 < u_2\}} \left(\int_{u_1 - u_2}^0 \chi'(t)\,dt \right) d\mu \\ &= -\int_{\{u_1 < u_2\}} \left(\int_{-\infty}^0 \chi'(t) \mathbf{1}_{\{u_1 < u_2 + t\}}\,dt \right) d\mu \\ &= -\int_{-\infty}^0 \chi'(t) \mu\{u_1 < u_2 + t\}\,dt. \end{aligned}$$

Moreover, it follows from [Darvas et al. 2021a, Lemma 2.3] that $\mu\{u_1 < u_2 + t\} \le 0$ for every $t \le 0$. Hence

$$\int_{\{u_1 < u_2\}} \nu(u_1, u_2) = -\int_{-\infty}^0 \chi'(t) \mu\{u_1 < u_2 + t\}\,dt \ge 0.$$

Similarly,

$$\int_{\{u_2 < u_1\}} \nu(u_2, u_1) \ge 0.$$

Thus

$$I_\chi(u_1, u_2) = \int_{\{u_1 < u_2\}} \nu(u_1, u_2) + \int_{\{u_2 < u_1\}} \nu(u_2, u_1) \ge 0. \qquad \square$$

3. Stability for weighted potentials

3.1. *Main results.* Let $\chi, \tilde{\chi} \in \widetilde{\mathcal{W}}^- \cup \mathcal{W}_M^+$ ($M \geq 1$) such that $\tilde{\chi} \leq \chi$. For each constant $t \geq 0$, we let

$$Q(t) = Q_{\chi,\tilde{\chi}}(t) := \begin{cases} 1 & \text{if } t \geq 1, \\ (Q_0(t)/Q_0(1))^{1/2} & \text{if } 0 < t < 1 \text{ and } \chi \in \widetilde{\mathcal{W}}^-, \\ t^{1/2} & \text{if } 0 < t < 1 \text{ and } \chi \in \mathcal{W}_M^+, \\ \lim_{s \to 0^+} Q(s) & \text{if } t = 0, \end{cases} \tag{3-1}$$

where Q_0 is defined as in Lemma 2.8. We remove the subscript $\chi, \tilde{\chi}$ from $Q_{\chi,\tilde{\chi}}$ if $\chi, \tilde{\chi}$ are clear from the context. Note that Q is increasing continuous function in t and

$$Q(0) = 0 \quad \text{if either} \quad \chi \in \mathcal{W}_M^+ \quad \text{or} \quad \lim_{t \to -\infty} \frac{\chi(t)}{\tilde{\chi}(t)} = 0. \tag{3-2}$$

For the convenience, we normalize energies with respect to $\varrho := \int_X \theta_\phi^n$ as

$$E^0_{\tilde{\chi},\theta,\phi} := \varrho^{-1} E_{\tilde{\chi},\theta,\phi}, \quad I^0_\chi(u_1, u_2) = \varrho^{-1} I_\chi(u_1, u_2).$$

Theorem 3.1. *Let θ be a closed smooth real $(1, 1)$-form and ϕ be a negative θ-psh function such that $\varrho := \mathrm{vol}(\theta_\phi) > 0$. Let $\chi, \tilde{\chi} \in \widetilde{\mathcal{W}}^- \cup \mathcal{W}_M^+$ ($M \geq 1$) such that $\tilde{\chi} \leq \chi$. Let $B \geq 1$ be a constant and let $u_j, \psi_j \in \mathcal{E}(X, \theta, \phi)$ satisfy $u_1 \leq u_2$ and*

$$E^0_{\tilde{\chi},\theta,\phi}(u_j) + E^0_{\tilde{\chi},\theta,\phi}(\psi_j) \leq B$$

for $j = 1, 2$. Then there exists a constant $C_n > 0$ depending only on n and M such that

$$\int_X -\chi(u_1 - u_2)(\theta^n_{\psi_1} - \theta^n_{\psi_2}) \leq C_n \varrho B^2 (1 - \tilde{\chi}(-1))^2 Q^{\circ n}(I^0_\chi(u_1, u_2)), \tag{3-3}$$

where Q is defined by (3-1), *and $Q^{\circ n} := Q \circ Q \circ \cdots \circ Q$ (n-iterate of Q).*

Since the measure $\theta^n_{\psi_1} - \theta^n_{\psi_2}$ is not positive, we need the following consequence of the above theorem for later applications on stability estimates.

Theorem 3.2. *Let θ be a closed smooth real $(1, 1)$-form and ϕ be a negative θ-psh function such that $\phi = P_\theta[\phi]$, $\varrho := \mathrm{vol}(\theta_\phi) > 0$ and $\theta \leq A\omega$ for some constant $A \geq 1$. Let $\chi, \tilde{\chi} \in \widetilde{\mathcal{W}}^- \cup \mathcal{W}_M^+$ ($M \geq 1$) such that $\tilde{\chi} \leq \chi$. Let $B \geq 1$ be a constant and $u_1, u_2, \psi \in \mathcal{E}(X, \theta, \phi)$ satisfying*

$$E^0_{\tilde{\chi},\theta,\phi}(u_1) + E^0_{\tilde{\chi},\theta,\phi}(u_2) + E^0_{\tilde{\chi},\theta,\phi}(\psi) \leq B$$

for $j = 1, 2$. Then, for every constant $m > 0$ and $0 < \gamma < 1$, there exists a constant $C > 0$ depending on n, M, X, ω, m and γ such that

$$\int_X -\chi(-|u_1 - u_2|)\theta^n_\psi \leq -\varrho\chi(-|a_1 - a_2| - \lambda^m) + C\varrho A^{(1-\gamma)/m}(B - \tilde{\chi}(-A))^2(1 - \tilde{\chi}(-1))^2 \lambda^\gamma,$$

where $a_j := \sup_X u_j$ and $\lambda = Q^{\circ n}(I^0_\chi(u_1, u_2))$.

3.2. ***Proof of Theorem 3.1.*** Here is the first step in the proof of Theorem 3.1.

Lemma 3.3. *If Theorem 3.1 holds for u_j, ψ_j of the same singularity type as ϕ, then it holds for the general case.*

Proof. Let u_j, ψ_j $(j = 1, 2)$ be as in the statement of Theorem 3.1. For every $k > 0$, we define $u_{j,k} := \max\{u_j, \phi - k\}$ and $\psi_{j,k} = \max\{\psi_j, \phi - k\}$. By Lemma 2.7, there exists a constant $C_1 > 0$ depending only on n and M such that

$$E^0_{\tilde{\chi},\theta,\phi}(u_{j,k}) + E^0_{\tilde{\chi},\theta,\phi}(\psi_{j,k}) \le C_1 B$$

for $j = 1, 2$ and for every $k > 0$. Therefore, by the assumption, there exists a constant $C_2 > 0$ depending only on n and M such that

$$\int_X -\chi(u_{1,k} - u_{2,k})(\theta^n_{\psi_{1,l}} - \theta^n_{\psi_{2,l}}) \le C_2 \varrho B^2 (1 - \tilde{\chi}(-1))^2 Q^{\circ(n)}(I^0_\chi(u_{1,k}, u_{2,k}))$$

for every $k, l > 0$. Letting $l \to \infty$ and using [Darvas et al. 2021b, Theorem 2.2], we get

$$\int_X -\chi(u_{1,k} - u_{2,k})(\theta^n_{\psi_1} - \theta^n_{\psi_2}) \le C_2 \varrho B^2 (1 - \tilde{\chi}(-1))^2 Q^{\circ(n)}(I^0_\chi(u_{1,k}, u_{2,k})) \tag{3-4}$$

for every $k > 0$. We will show that

$$I_\chi(u_1, u_2) = \lim_{k\to\infty} I_\chi(u_{1,k}, u_{2,k}). \tag{3-5}$$

Define

$$f := \chi(u_1 - u_2)(\theta^n_{u_2} - \theta^n_{u_1}), \quad f_k := \chi(u_{1,k} - u_{2,k})(\theta^n_{u_{2,k}} - \theta^n_{u_{1,k}}).$$

We have

$$\begin{aligned} I_\chi(u_{1,k}, u_{2,k}) = \int_X f_k &= \int_{\{u_1 > \phi - k\}} f_k + \int_{\{u_1 \le \phi - k\}} f_k \\ &= \int_{\{u_1 > \phi - k\}} f + \int_{\{u_1 \le \phi - k\}} f_k \\ &= I_\chi(u_1, u_2) - \int_{\{u_1 \le \phi - k\}} f + \int_{\{u_1 \le \phi - k\}} f_k. \end{aligned}$$

Then

$$\begin{aligned} |I_\chi(u_{1,k}, u_{2,k}) - I_\chi(u_1, u_2)| &= \left| \int_{\{u_1 \le \phi - k\}} f - \int_{\{u_1 \le \phi - k\}} f_k \right| \\ &\le \int_{\{u_1 \le \phi - k\}} \mu + \int_{\{u_1 \le \phi - k\}} -\chi(u_{1,k} - u_{2,k})(\theta^n_{u_{2,k}} + \theta^n_{u_{1,k}}) \\ &\le \int_{\{u_1 \le \phi - k\}} \mu + \int_{\{u_1 \le \phi - k\}} -\chi(-k)(\theta^n_{u_{2,k}} + \theta^n_{u_{1,k}}), \end{aligned}$$

where $\mu = -\chi(u_1 - \phi)(\theta^n_{u_1} + \theta^n_{u_2})$. By Lemma 2.7, we have $\int_X \mu < \infty$. Then it follows from Lebesgue's dominated convergence theorem that $\lim_{k\to\infty} \int_{\{u_1 \le \phi - k\}} \mu = 0$. Therefore,

$$\limsup_{k\to\infty} |I_\chi(u_{1,k}, u_{2,k}) - I_\chi(u_1, u_2)| \le \limsup_{k\to\infty} \int_{\{u_1 \le \phi - k\}} -\chi(-k)(\theta^n_{u_{1,k}} + \theta^n_{u_{2,k}}). \tag{3-6}$$

By the fact that

$$\int_X \theta^n_{u_{1,k}} = \int_X \theta^n_{u_{2,k}} = \int_X \theta^n_\phi, \quad \mathbf{1}_{\{u_1>\phi-k\}}\theta^n_{u_{j,k}} = \mathbf{1}_{\{u_1>\phi-k\}}\theta^n_{u_j} \quad (j=1,2),$$

we have

$$-\chi(-k)\int_{\{u_1\le\phi-k\}}(\theta^n_{u_{1,k}}+\theta^n_{u_{2,k}}) = -\chi(-k)\int_{\{u_1\le\phi-k\}}(\theta^n_{u_1}+\theta^n_{u_2}) \le \int_{\{u_1\le\phi-k\}}\mu. \tag{3-7}$$

By using (3-6), (3-7) and the fact $\lim_{k\to\infty}\int_{\{u_1\le\phi-k\}}\mu=0$, we get (3-5). Now, combining (3-4) and (3-5), we obtain

$$\int_X -\chi(u_1-u_2)(\theta^n_{\psi_1}-\theta^n_{\psi_2}) \le C_2\varrho B^2(1-\tilde\chi(-1))^2 Q^{\circ(n)}(I^0_\chi(u_1,u_2)). \qquad \square$$

Lemma 3.4. *Let $M\ge 1$ and $\chi,\tilde\chi\in\widetilde{\mathcal{W}}^-\cup\mathcal{W}^+_M$ such that $\tilde\chi\le\chi$ and $\chi\in\mathscr{C}^1(\mathbb{R})$. Let $u_1,u_2,\dots,u_{n+2}\in\mathcal{E}(X,\theta,\phi)$ such that $u_1\le u_2$ and $u_j-\phi$ is bounded $(j=1,2,\dots,n+2)$, where ϕ is a negative θ-psh function satisfying $\varrho:=\operatorname{vol}(\theta_\phi)>0$. Set*

$$T=\theta_{u_4}\wedge\cdots\wedge\theta_{u_{n+2}}, \quad I=\left|\int_X \chi'(u_1-u_2)\,d(u_1-u_2)\wedge d^c(u_1-u_3)\wedge T\right|,$$

and

$$J=\int_X \chi'(u_1-u_2)\,d(u_1-u_2)\wedge d^c(u_1-u_2)\wedge T.$$

Then there exists $C>0$ depending only on n and M such that

$$I\le C\varrho B(1-\tilde\chi(-1))Q(J/\varrho),$$

where $B:=\sum_{j=1}^{n+2}\max\{E^0_{\tilde\chi,\theta,\phi}(u_j),1\}$ and Q is defined by (3-1).

Clearly if $\chi,\tilde\chi\in\widetilde{\mathcal{W}}^-$, then the above constant C does not depend on M.

Proof. In this proof, we use the symbols $\lesssim$ and $\gtrsim$ for inequalities modulo a constant depending only on n and M. By Theorem 2.2 and Lemma 2.7, we have

$$I=\left|\int_X -\chi(u_1-u_2)\,dd^c(u_1-u_3)\wedge T\right| \lesssim \varrho B=\varrho B Q(1).$$

Therefore, without loss of generality, we can assume that $J/\varrho<1$. Approximating u_3 by $u_3-\delta$ with $\delta\searrow 0$, we can assume that $u_3<\phi$ on X.

For each $0<\epsilon<\frac12$ we let

$$U(\epsilon)=\{u_1-u_2<\epsilon(u_1+u_3-2\phi)\}, \quad V(\epsilon)=\{u_1-u_2>\epsilon(u_1+u_3-2\phi)\},$$

and $\Gamma(\epsilon)=\{u_1-u_2=\epsilon(u_1+u_3-2\phi)\}$. Since $\Gamma(\epsilon_1)\cap\Gamma(\epsilon_2)=\varnothing$ for every $\epsilon_1\ne\epsilon_2$ (note $u_3<\phi$), we have

$$\int_{\Gamma(\epsilon)} d(u_1-u_3)\wedge d^c(u_1-u_3)\wedge T=0 \tag{3-8}$$

for almost everywhere $\epsilon\in\left(0,\frac12\right)$.

Let $0 < \epsilon < \frac{1}{2}$ be a constant satisfying (3-8). To simplify the notation, from now on, we write U, V, Γ for $U(\epsilon)$, $V(\epsilon)$, $\Gamma(\epsilon)$ respectively. Define

$$\tilde{u}_1 = \frac{u_1 + \epsilon u_3}{1+\epsilon}, \quad \tilde{u}_2 = \max\left\{\frac{u_2 + \epsilon u_3}{1+\epsilon}, \ \frac{(1-\epsilon)u_1 + 2\epsilon\phi}{1+\epsilon}\right\} \quad \text{and} \quad \tilde{\varphi} = \tilde{u}_1 - \tilde{u}_2.$$

Then $\varphi := (u_1 - u_2) = (1+\epsilon)\tilde{\varphi}$ on U. Hence

$$\begin{aligned} I &= \left|\int_X -\chi(\varphi)\, dd^c(u_1 - u_3) \wedge T\right| \\ &\le \left|\int_U -\chi(\varphi)\, dd^c(u_1 - u_3) \wedge T\right| + \left|\int_{X\setminus U} -\chi(\varphi)\, dd^c(u_1 - u_3) \wedge T\right| \\ &\le \left|\int_U -\chi((1+\epsilon)\tilde{\varphi})\, dd^c(u_1 - u_3) \wedge T\right| + \left|\int_{X\setminus U} -\chi(\varphi)(\theta_{u_1} + \theta_{u_3}) \wedge T\right| \\ &\le \left|\int_U -\chi((1+\epsilon)\tilde{\varphi})\, dd^c(u_1 - u_3) \wedge T\right| + \left|\int_{X\setminus U} -\chi(\epsilon(u_1 + u_3 - 2\phi))(\theta_{u_1} + \theta_{u_3}) \wedge T\right| \\ &:= I_1 + I_2, \end{aligned}$$

where in the last inequality we used the fact that χ is increasing and $\varphi \ge \epsilon(u_1 + u_2 - 2\phi)$ on $X\setminus U$. By Lemma 2.7, we have $E^0_{\tilde{\chi},\theta,\phi}\big(\frac{1}{2}(u_1 + u_3)\big) \lesssim B$. Therefore, it follows from Lemma 2.8 that

$$I_2 \le 2\int_X -\chi\big(2\epsilon\big(\tfrac{1}{2}(u_1 + u_3) - \phi\big)\big)\theta_{(u_1+u_3)/2} \wedge T \lesssim B\varrho(1 - \tilde{\chi}(-1))Q_0(2\epsilon). \tag{3-9}$$

In order to estimate I_1, we divide it into two terms

$$\begin{aligned} I_1 &\le \left|\int_X -\chi((1+\epsilon)\tilde{\varphi})\, dd^c(u_1 - u_3) \wedge T\right| + \left|\int_{X\setminus U} -\chi((1+\epsilon)\tilde{\varphi})\, dd^c(u_1 - u_3) \wedge T\right| \\ &:= I_3 + I_4. \end{aligned}$$

Note that $\tilde{u}_1 - \tilde{u}_2 = \epsilon(u_1 + u_3 - 2\phi)/(1+\epsilon)$ on $X \setminus U$. Hence

$$I_4 \le \int_{X\setminus U} -\chi((1+\epsilon)\tilde{\varphi})(\theta_{u_1} + \theta_{u_3}) \wedge T \le \int_{X\setminus U} -\chi(\epsilon(u_1 + u_2 - 2\phi))(\theta_{u_1} + \theta_{u_3}) \wedge T.$$

Using Lemma 2.8 again, we get

$$I_4 \lesssim B\varrho(1 - \tilde{\chi}(-1))Q_0(2\epsilon). \tag{3-10}$$

Using integration by parts, we have

$$I_3 = (1+\epsilon)\left|\int_X \chi'((1+\epsilon)\tilde{\varphi})\, d\tilde{\varphi} \wedge d^c(u_1 - u_3) \wedge T\right|.$$

Moreover, by the Cauchy–Schwarz inequality and by the choice of ϵ (see (3-8)), we get

$$\int_\Gamma \chi'((1+\epsilon)\tilde{\varphi})\, d\tilde{\varphi} \wedge d^c(u_1 - u_3) \wedge T = 0.$$

Hence

$$I_3 = (1+\epsilon)\left|\int_{U\cup V} \chi'((1+\epsilon)\tilde{\varphi})d\tilde{\varphi}\wedge d^c(u_1-u_3)\wedge T\right| \le (1+\epsilon)(I_5I_6)^{1/2}, \tag{3-11}$$

where

$$I_5 = \int_{U\cup V} \chi'((1+\epsilon)\tilde{\varphi})\, d(u_1-u_3)\wedge d^c(u_1-u_3)\wedge T,$$
$$I_6 = \int_{U\cup V} \chi'((1+\epsilon)\tilde{\varphi})\, d\tilde{\varphi}\wedge d^c\tilde{\varphi}\wedge T.$$

Since $(1+\epsilon)\tilde{\varphi} \le \epsilon(u_1+u_3-2\phi)$, if $\chi \in \widetilde{\mathcal{W}}^-$ (hence χ' is nonnegative and increasing on $\mathbb{R}_{\le 0}$) then

$$\begin{aligned}
I_5 &\le \int_X \chi'(\epsilon(u_1+u_3-2\phi))\, d(u_1-u_3)\wedge d^c(u_1-u_3)\wedge T\\
&\lesssim \int_X \chi'(\epsilon(u_1+u_3-2\phi))\, d(u_1-\phi)\wedge d^c(u_1-\phi)\wedge T\\
&\qquad\qquad + \int_X \chi'(\epsilon(u_1+u_3-2\phi))\, d(u_3-\phi)\wedge d^c(u_3-\phi)\wedge T\\
&\le \int_X \chi'(\epsilon(u_1-\phi))\, d(u_1-\phi)\wedge d^c(u_1-\phi)\wedge T + \int_X \chi'(\epsilon(u_3-\phi))\, d(u_3-\phi)\wedge d^c(u_3-\phi)\wedge T\\
&= \epsilon^{-1}\int_X -\chi(\epsilon(u_1-\phi))\, dd^c(u_1-\phi)\wedge T + \epsilon^{-1}\int_X -\chi(\epsilon(u_3-\phi))\, dd^c(u_3-\phi)\wedge T\\
&\lesssim B\varrho(1-\tilde{\chi}(-1))\epsilon^{-1}Q_0(\epsilon),
\end{aligned}$$

where the last estimate holds due to Lemma 2.8.

Define $v_1 := (u_1+2u_3)/3$ and $v_2 := (2u_1+u_3)/3$. Since

$$(1+\epsilon)(\tilde{u}_1-\tilde{u}_2) \ge u_1+u_3-2\phi, \quad u_1-u_3 = -3(v_1-v_2),$$

one sees that if $\chi \in \mathcal{W}_M^+$ (hence χ' is nonnegative and decreasing in $\mathbb{R}_{\le 0}$) then

$$\begin{aligned}
I_5 &\le \int_X \chi'((u_1+u_3-2\phi))\, d(u_1-u_3)\wedge d^c(u_1-u_3)\wedge T\\
&\lesssim \int_X \chi'((u_1+u_3-2\phi))\big(d(v_1-\phi)\wedge d^c(v_1-\phi) + d(v_2-\phi)\wedge d^c(v_2-\phi)\big)\wedge T\\
&\le \int_X \chi'(3(v_1-\phi))\, d(v_1-\phi)\wedge d^c(v_1-\phi)\wedge T + \int_X \chi'(3(v_2-\phi))\, d(v_2-\phi)\wedge d^c(v_2-\phi)\wedge T\\
&= \frac{1}{3}\int_X -\chi(3(v_1-\phi))\, dd^c(v_1-\phi)\wedge T + \frac{1}{3}\int_X -\chi(3(v_2-\phi))\, dd^c(v_2-\phi)\wedge T\\
&\le \int_X -\chi(3(v_1-\phi))(\theta_{v_1}+\theta_\phi)\wedge T + \int_X -\chi(3(v_2-\phi))(\theta_{v_2}+\theta_\phi)\wedge T\\
&\le 3^M\int_X -\chi(v_1-\phi)(\theta_{v_1}+\theta_\phi)\wedge T + 3^M\int_X -\chi(v_2-\phi)(\theta_{v_2}+\theta_\phi)\wedge T\\
&\lesssim B\varrho,
\end{aligned}$$

where the two last estimates hold due to Lemma 2.7 and the fact

$$\log(-\chi(3t)) - \log(-\chi(t)) = \int_t^{3t} \frac{\chi'(s)}{\chi(s)}\, ds \le \int_t^{3t} \frac{M}{s}\, ds = M \log 3$$

for every $\chi \in \mathcal{W}_M^+$ and $t \le 0$. Combining the estimates in both cases, we obtain

$$I_5 \lesssim B\varrho(1-\tilde{\chi}(-1))\frac{Q(\epsilon)^2}{\epsilon}, \tag{3-12}$$

where we used the inequality $Q(\epsilon) \ge \epsilon^{1/2}$ if $\chi \in \widetilde{W}_M^+$. Now, we estimate I_6. Since U, V are open in the plurifine topology and

$$(1+\epsilon)\tilde{\varphi} = \begin{cases} \varphi & \text{on } U, \\ \epsilon(u_1+u_3-2\phi) & \text{on } V, \end{cases}$$

we have

$$\begin{aligned} I_6 &= \int_U \chi'((1+\epsilon)\tilde{\varphi})\, d\tilde{\varphi} \wedge d^c\tilde{\varphi} \wedge T + \int_V \chi'((1+\epsilon)\tilde{\varphi})\, d\tilde{\varphi} \wedge d^c\tilde{\varphi} \wedge T \\ &= (1+\epsilon)^{-2} \int_U \chi'(\varphi)\, d\varphi \wedge d^c\varphi \wedge T \\ &\qquad\qquad + \frac{\epsilon^2}{(1+\epsilon)^2} \int_V \chi'(\epsilon(u_1+u_3-2\phi))\, d(u_1+u_3-2\phi) \wedge d^c(u_1+u_3-2\phi) \wedge T \\ &\le J + \epsilon^2 \int_X \chi'(\epsilon(u_1+u_3-2\phi))\, d(u_1+u_3-2\phi) \wedge d^c(u_1+u_3-2\phi) \wedge T \\ &= J + \epsilon \int_X -\chi(\epsilon(u_1+u_3-2\phi))\, dd^c(u_1+u_3-2\phi) \wedge T. \end{aligned}$$

Therefore, it follows from Lemma 2.8 that

$$I_6 \lesssim J + B\varrho(1-\tilde{\chi}(-1))\epsilon Q_0(2\epsilon). \tag{3-13}$$

Combining (3-9)–(3-13), we get

$$\begin{aligned} I \le I_1 + I_2 &\le I_3 + I_4 + I_2 \\ &\lesssim (I_5 I_6)^{1/2} + I_4 + I_2 \\ &\lesssim (B\varrho(1-\tilde{\chi}(-1))\epsilon^{-1}J)^{1/2} Q(\epsilon) + B\varrho(1-\tilde{\chi}(-1))Q(2\epsilon)^2. \end{aligned}$$

Letting $\epsilon \searrow J/(2\varrho)$ (and supposing ϵ satisfies (3-8)), we obtain

$$I \lesssim B\varrho(1-\tilde{\chi}(-1))Q(J/\varrho). \qquad \square$$

Proposition 3.5. *Let $\chi, \tilde{\chi} \in \widetilde{\mathcal{W}}^- \cup \mathcal{W}_M^+$ such that $\tilde{\chi} \le \chi$ and $\chi \in \mathscr{C}^1(\mathbb{R})$. Let $u_1, u_2, u_3 \in \mathcal{E}(X, \theta, \phi)$ such that $u_1 \le u_2$ and $u_j - \phi$ is bounded* ($j = 1, 2, 3$), *where ϕ is a negative θ-psh function satisfying $\varrho := \mathrm{vol}(\theta_\phi) > 0$. Then there exists a constant $C_n > 0$ depending only on n and M such that*

$$\int_X \chi'(u_1-u_2)\, d(u_1-u_2) \wedge d^c(u_1-u_2) \wedge \theta_{u_3}^{n-1} \le C_n \varrho B^2 (1-\tilde{\chi}(-1))^2 Q^{\circ(n-1)}(I_\chi^0(u_1,u_2)), \tag{3-14}$$

where $B := \sum_{j=1}^3 \max\{E^0_{\tilde{\chi},\theta,\phi}(u_j), 1\}$ and Q is defined by (3-1).

Proof. Let
$$\varphi := u_1 - u_2, \quad T := \sum_{j=1}^{n-1} \theta_{u_1}^j \wedge \theta_{u_2}^{n-1-j}$$
and
$$T_{k,l} := \theta_{u_1}^k \wedge \theta_{u_2}^l \wedge \theta_{u_3}^{n-k-l-1}, \quad L_{k,l} := \int_X \chi'(\varphi)\, d\varphi \wedge d^c\varphi \wedge T_{k,l}.$$
Observe
$$\theta_{u_2}^n - \theta_{u_1}^n = -dd^c\varphi \wedge T$$
and
$$L_{k,n-1-k} \le \int_X \chi'(\varphi)\, d\varphi \wedge d^c\varphi \wedge T = \varrho I_\chi^0(u_1, u_2) \tag{3-15}$$
by integration by parts. We now prove by inverse induction on $m := k + l$ that
$$L_{k,l} \le C_{m,n} \varrho B^2 (1 - \tilde{\chi}(-1))^2 Q^{\circ(n-1-k-l)}(I_\chi^0(u_1, u_2)) \tag{3-16}$$
for some constant $C_{m,n} > 1$ depending only on m, n and M. The desired assertion (3-14) is the case where $k = l = 0$. In what follows we use the symbols $\lesssim$ and $\gtrsim$ for inequalities modulo a constant depending only on n and M. We have checked (3-16) for $k + l = n - 1$. Suppose that (3-16) holds for $k + l = m$ with $0 < m \le n - 1$. We will verify it for $L_{k-1,l}$, where $k + l = m$ and $k > 1$. The case $L_{k,l-1}$ is done similarly.

Denote $S_{k-1,l} = \theta_{u_1}^{k-1} \wedge \theta_{u_2}^l \wedge \theta_{u_3}^{n-k-l-1}$. Then
$$L_{k-1,l} - L_{k,l} = \int_X \chi'(\varphi)\, d\varphi \wedge d^c\varphi \wedge dd^c(u_3 - u_1) \wedge S_{k-1,l}.$$
Using integration by parts, we have
$$\begin{aligned} L_{k-1,l} - L_{k,l} &= \int_X -\chi(\varphi)\, dd^c(\varphi) \wedge dd^c(u_3 - u_1) \wedge S_{k-1,l} \\ &= \int_X -\chi(\varphi)\, dd^c(u_3 - u_1) \wedge T_{k,l} - \int_X -\chi(\varphi)\, dd^c(u_3 - u_1) \wedge T_{k-1,l+1} \\ &= \int_X \chi'(\varphi)\, d\varphi \wedge d^c(u_3 - u_1) \wedge T_{k,l} - \int_X \chi'(\varphi)\, d\varphi \wedge d^c(u_3 - u_1) \wedge T_{k-1,l+1}. \end{aligned}$$
Therefore, it follows from Lemma 3.4 that
$$L_{k-1,l} - L_{k,l} \lesssim \varrho B (1 - \tilde{\chi}(-1)) \big(Q(L_{k,l}/\varrho) + Q(L_{k-1,l+1}/\varrho) \big).$$
Hence, by using the inductive hypothesis, we get
$$\begin{aligned} L_{k-1,l} &\lesssim \varrho B^2 (1 - \tilde{\chi}(-1))^2 Q^{\circ(n-1-m)}(I_\chi^0(u_1, u_2)) \\ &\qquad + \varrho B (1 - \tilde{\chi}(-1)) Q\big(C_{m,n} B^2 (1 - \tilde{\chi}(-1))^2 Q^{\circ(n-1-m)}(I_\chi^0(u_1, u_2)) \big) \\ &\lesssim \varrho B^2 (1 - \tilde{\chi}(-1))^2 Q^{\circ(n-m)}(I_\chi^0(u_1, u_2)). \end{aligned}$$
Here we use the fact $Q(t_1) \le (t_1/t_2)^{1/2} Q(t_2)$ for every $t_1 > t_2 > 0$ (see Lemma 3.6).

Thus (3-16) holds for $L_{k-1,l}$. □

Lemma 3.6. *The function $h(t) = (Q(t))^2/t$ is nonincreasing in $\mathbb{R}_{>0}$.*

Proof. If $\chi \in \mathcal{W}_M^+$ then

$$h(t) = \begin{cases} \dfrac{1}{t} & \text{if } t \geq 1, \\ 1 & \text{if } 0 < t < 1 \end{cases}$$

is a nonincreasing function.

We consider the case $\chi \in \widetilde{\mathcal{W}}^-$. We have

$$h(t) = \begin{cases} \dfrac{1}{t} & \text{if } t \geq 1, \\ \dfrac{Q_0(t)}{t Q_0(1)} & \text{if } 0 < t < 1. \end{cases}$$

It is clear that h is decreasing in $[1, \infty)$. We need to show that h is nonincreasing in $(0, 1)$. Since χ is convex, we have

$$\frac{\chi(t_1 s)}{t_1 s} \leq \frac{\chi(t_2 s)}{t_2 s}$$

for every $0 < t_2 < t_1 < 1$ and $s < 0$. Dividing both sides of the last estimate by $\tilde{\chi}(s)/s$, we get

$$\frac{\chi(t_1 s)}{t_1 \tilde{\chi}(s)} \leq \frac{\chi(t_2 s)}{t_2 \tilde{\chi}(s)}.$$

Taking the supremum of both sides, we obtain

$$\frac{Q_0(t_1)}{t_1} = \sup_{s \leq -1} \frac{\chi(t_1 s)}{t_1 \tilde{\chi}(s)} \leq \sup_{s \leq -1} \frac{\chi(t_2 s)}{t_2 \tilde{\chi}(s)} = \frac{Q_0(t_2)}{t_2}.$$

Then $h(t_1) \leq h(t_2)$. Hence h is nonincreasing in $(0, 1)$. □

End of the proof of Theorem 3.1. By Lemma 3.3, we can assume that u_j, ψ_j are of the same singularity type as ϕ. Now let $(\chi_j)_{j \in \mathbb{N}}$, $(\tilde{\chi}_j)_{j \in \mathbb{N}}$ be the sequences approximating χ, $\tilde{\chi}$ respectively in Lemma 2.6. By Lebesgue's dominated convergence theorem, observe that

$$\lim_{j \to \infty} I^0_{\chi_j}(u_1, u_2) = I^0_{\chi}(u_1, u_2)$$

and

$$\lim_{j \to \infty} \int_X -\chi_j(u_1 - u_2)(\theta^n_{\psi_1} - \theta^n_{\psi_2}) = \int_X -\chi(u_1 - u_2)(\theta^n_{\psi_1} - \theta^n_{\psi_2}).$$

On the other hand, for $\epsilon \in (0, 1]$ we also get

$$\lim_{j \to \infty} Q_{\chi_j, \tilde{\chi}_j}(\epsilon) = Q_{\chi, \tilde{\chi}}(\epsilon),$$

because for $t \leq -1$, one has

$$\frac{\chi_j(\epsilon t)}{\tilde{\chi}_j(t)} \leq \frac{\chi_j(\epsilon t)}{\tilde{\chi}(t)} \leq \frac{\chi(\epsilon t) - 2^{-j}}{\tilde{\chi}(t)} \quad \text{and} \quad \frac{\chi_j(\epsilon t)}{\tilde{\chi}_j(t)} \geq \frac{\chi_j(\epsilon t)}{\tilde{\chi}(t) - 2^{-j}},$$

which converges to $\chi(\epsilon t)/\tilde{\chi}(t)$ (by Lemma 2.6). Hence, by considering χ_j, $\tilde{\chi}_j$ instead of χ, $\tilde{\chi}$, we can further assume that $\chi \in \mathscr{C}^1(\mathbb{R})$.

Let L be the left-hand side of the desired inequality. We have

$$\begin{aligned} L &= \int_X -\chi(u_1-u_2)(\theta_{\psi_1}^n - \theta_{u_1}^n) - \int_X -\chi(u_1-u_2)(\theta_{\psi_2}^n - \theta_{u_1}^n) \\ &= \int_X -\chi(u_1-u_2)\, dd^c(\psi_1-u_1)\wedge T_1 - \int_X -\chi(u_1-u_2)\, dd^c(\psi_2-u_1)\wedge T_2 \\ &= L_1 - L_2, \end{aligned}$$

where $T_j = \sum_{l=0}^{n-1} \theta_{\psi_j}^l \wedge \theta_{u_1}^{n-l-1}$. Using integration by parts and Lemma 3.4, we get

$$\begin{aligned} L_1 &= \int_X \chi'(u_1-u_2)\, d(u_1-u_2)\wedge d^c(\psi_1-u_1)\wedge T_1 \\ &\le C_1\varrho B(1-\tilde\chi(-1))Q\biggl(\varrho^{-1}\int_X \chi'(u_1-u_2)\, d(u_1-u_2)\wedge d^c(u_1-u_2)\wedge T_1\biggr), \end{aligned}$$

where $C_1 > 0$ depends only on n and M. Observe that there is a dimensional constant C_1' such that

$$T_1 \le C_1' \theta^{n-1}_{(\psi_1+u_1)/2}.$$

Moreover one has

$$E_{\tilde\chi,\theta,\phi}((\psi_1+u_1)/2) \lesssim E_{\tilde\chi,\theta,\phi}(\psi_1) + E_{\tilde\chi,\theta,\phi}(u_1)$$

by Lemma 2.7. Hence, it follows from Proposition 3.5 (applied to $u_3 := (\psi_1+u_1)/2$) that

$$\varrho^{-1}\int_X \chi'(u_1-u_2)\, d(u_1-u_2)\wedge d^c(u_1-u_2)\wedge T_1 \le C_2B^2(1-\tilde\chi(-1))^2 Q^{\circ(n-1)}(I_\chi^0(u_1,u_2)),$$

where $C_2 > 1$ depends only on n and M. Then

$$L_1 \le C_3\varrho B^2(1-\tilde\chi(-1))^2 Q^{\circ n}(I_\chi^0(u_1,u_2)),$$

where $C_3 > 0$ depends only on n and M. Here we use the fact $Q(t_1) \le (t_1/t_2)^{1/2}Q(t_2)$ for every $t_1 > t_2 > 0$ (Lemma 3.6).

By the same arguments, we also have

$$-L_2 \le C_4\varrho B^2(1-\tilde\chi(-1))^2 Q^{\circ n}(I_\chi^0(u_1,u_2)),$$

where $C_4 > 0$ depends only on n and M.

Hence

$$L = L_1 - L_2 \le (C_3+C_4)\varrho B^2(1-\tilde\chi(-1))^2 Q^{\circ n}(I_\chi^0(u_1,u_2)). \qquad \square$$

3.3. *Proof of Theorem 3.2.* Recall that for every Borel set E in X, we define

$$\mathrm{cap}_{\theta,\phi}(E) := \sup\biggl\{\int_E \theta_h^n : h \in \mathrm{PSH}(X,\theta),\ \phi - 1 \le h \le \phi\biggr\}.$$

The following is an improvement of results from [Darvas et al. 2018b; 2021a] (see also [Boucksom et al. 2010; Kołodziej 2003]).

Theorem 3.7. *Let $A \geq 1$ be a constant and let θ be a closed smooth real $(1,1)$-form such that $\theta \leq A\omega$. Let $\phi \in \mathrm{PSH}(X,\theta)$ and $0 \leq f \in L^p(X)$ for some constant $p > 1$ such that $\phi = P[\phi]$ and $0 < \int_X f\omega^n = \int_X \theta_\phi^n := \varrho$. Assume $u \in \mathcal{E}(X,\theta,\phi)$ satisfies $\sup_X(u-\phi) = 0$ and $\theta_u^n = f\omega^n$. Then, there exists a constant $C \geq 1$ depending only on X, ω, n and p such that*

$$u \geq \phi - CA\big(\log \|f \operatorname{vol}_\omega(X)^q/\varrho\|_{L^p} + \log A + 1\big), \tag{3-17}$$

where $\operatorname{vol}_\omega(X) := \int_X \omega^n$ *and* $q = p/(p-1)$.

By Hölder inequalities, one sees that

$$1 = \int_X \frac{f}{\varrho}\omega^n \leq \|f/\varrho\|_{L^p}(\operatorname{vol}_\omega(X))^q,$$

and then $\log \|f \operatorname{vol}_\omega(X)^q/\varrho\|_{L^p} \geq 0$.

Proof. Without loss of generality, we can assume that $\operatorname{vol}_\omega(X) = 1$. Recall that there exists a constant $\nu > 0$ depending only on X, ω such that

$$\int_X \exp(-\psi/\nu)\omega^n \leq C_0^2$$

for every $\psi \in \mathrm{PSH}(X,\omega)$ with $\sup_X \psi = 0$, where $C_0 \geq 1$ is a constant depending only on X and ω. Consequently, one gets

$$\int_X \exp(-\psi/(A\nu))\omega^n \leq C_0^2$$

for every $\psi \in \mathrm{PSH}(X,\theta) \subset \mathrm{PSH}(X, A\omega)$ with $\sup_X \psi = 0$. By the same arguments as in the proof of [Darvas et al. 2018b, Proposition 4.30] (use [Darvas et al. 2021a, Lemma 3.9] instead of [Darvas et al. 2018b, Lemma 4.9]), we have

$$\int_E \omega^n \leq C_0^2 \exp\biggl(-\frac{1}{2A\nu}\biggl(\frac{\operatorname{cap}_{\theta,\phi}(E)}{\varrho}\biggr)^{-1/n}\biggr)$$

for every Borel set $E \subset X$. Therefore, by the Hölder inequality and the fact that $e^{-1/t} \leq m!\,t^m$ for every $m \in \mathbb{N}$ and every $t > 0$, there exists $A_0 > 0$ depending only on X, ω, n and p such that

$$\varrho^{-1}\int_E \theta_u^n = \int_E (f/\varrho)\omega^n \leq \|f/\varrho\|_{L^p}\biggl(\int_E \omega^n\biggr)^{1/q} \leq A_0 A^{2n}\|f/\varrho\|_{L^p}\frac{\operatorname{cap}_{\theta,\phi}(E)^2}{\varrho^2} \tag{3-18}$$

for every Borel set $E \subset X$, where $1/p + 1/q = 1$. On the other hand, letting $b = (A\nu q)^{-1}$ and $B_0 = (C_0^2)^{1/q}$, we have

$$\varrho^{-1}\int_X e^{-bw}\theta_u^n \leq \|f/\varrho\|_{L^p}\biggl(\int_X e^{-bqw}\omega^n\biggr)^{1/q} \leq B_0\|f/\varrho\|_{L^p} \tag{3-19}$$

for every $w \in \mathrm{PSH}(X,\theta)$ with $\sup_X w = 0$.

For every $h \in \mathrm{PSH}(X,\theta)$ with $\phi - 1 \le h \le \phi$, for each $0 \le t \le 1$ and $s > 0$, we have

$$t^n \int_{\{u<\phi-t-s\}} \theta_h^n \le \int_{\{u<(1-t)\phi+th-s\}} \theta_{(1-t)\phi+th}^n \le \int_{\{u<(1-t)\phi+th-s\}} \theta_u^n \le \int_{\{u<\phi-s\}} \theta_u^n,$$

where the third estimate holds due to the comparison principle [Darvas et al. 2021a, Lemma 2.3]. Then

$$t^n \mathrm{cap}_{\theta,\phi}(u<\phi-t-s) \le \int_{\{u<\phi-s\}} \theta_u^n \tag{3-20}$$

for every $0 \le t \le 1,\ s > 0$. Therefore, it follows from (3-18) that

$$t^n \varrho^{-1} \mathrm{cap}_{\theta,\phi}(u<\phi-t-s) \le A_1 \varrho^{-2} \mathrm{cap}_{\theta,\phi}(u<\phi-s)^2,$$

where $A_1 = A_0 A^{2n} \|f/\varrho\|_{L^p}$. Putting $g(s) = \varrho^{-1/n} \mathrm{cap}_{\theta,\phi}(u<\phi-s)^{1/n}$, the above inequality becomes

$$t g(t+s) \le A_1^{1/n} g(s)^2.$$

Hence, it follows from [Eyssidieux et al. 2009, Lemma 2.4 and Remark 2.5] that if $g(s_0) < 1/(2A_1^{1/n})$ then $g(s) = 0$ for all $s \ge s_0 + 2$. Moreover, by (3-20) and the condition (3-19), we have

$$g(s+1)^n \le \varrho^{-1} \int_{\{u<\phi-s\}} \theta_u^n \le \varrho^{-1} \int_X e^{b(\phi-u-s)} \theta_u^n \le B_1 e^{-bs}$$

for every $s > 0$, where $B_1 = B_0 \|f/\varrho\|_{L^p}$. Then $g(s+1) < 1/(2A_1^{1/n})$ provided that

$$s > \frac{n\log 2 + \log A_1}{b} + \frac{\log B_1}{b}.$$

Hence $g(s) = 0$ for every

$$s \ge \frac{n\log 2 + \log A_1}{b} + \frac{\log B_1}{b} + 4.$$

Thus

$$u \ge \phi - \left(\frac{n\log 2 + \log A_1}{b} + \frac{\log B_1}{b} + 4\right) = \phi - C_1 \log \|f/\varrho\|_{L^p} - C_2,$$

where $C_1 = 2/b = 2\nu q A$ and

$$\begin{aligned} C_2 &= 4 + \frac{n\log 2 + \log A_0 + \log B_0 + 2n\log A}{b} \\ &= 4 + \nu q(n\log 2 + \log A_0 + \log B_0 + 2n\log A)A. \end{aligned}$$ □

Lemma 3.8. *There exists a constant $C > 0$ depending only on n, X and ω such that for every $u \in \mathrm{PSH}(X,\omega)$ satisfying $\sup_X u = 0$ and for every constant $0 < t \le 1$, one has*

$$\int_{\{u>-t\}} \omega^n \ge C t^{2n}. \tag{3-21}$$

Proof. Let $(U_j,\varphi_j)_{j=1}^m$ be such that $U_j\subset X$ are open, $\varphi_j:4\mathbb{B}\to U_j$ are biholomorphic and $\bigcup_{j=1}^m\varphi_j(\mathbb{B})=X$ (where $\mathbb{B}$ is the open unit ball in $\mathbb{C}^n$), and there is a smooth psh function ρ_j in U_j such that $dd^c\rho_j=\omega$ for $1\le j\le m$. Let

$$C_\rho=\sup_{1\le j\le m}\sup_{2\mathbb{B}}\|\nabla(\rho_j\circ\varphi_j)\|.$$

Assume $u(z_0)=0$. Then there exists $1\le j_0\le m$ such that $z_0\in\varphi_{j_0}(\mathbb{B})$. Let $w_0=\varphi_{j_0}^{-1}(z_0)$, $\hat{u}(w)=u\circ\varphi_{j_0}(w)$ and $\hat\rho(w)=\rho_{j_0}\circ\varphi_{j_0}(w)-\rho_{j_0}\circ\varphi_{j_0}(w_0)$. By the plurisubharmonicity of $\hat{u}+\hat\rho$, for every $t>0$ and $0<r<1$, we have

$$\begin{aligned}0=(\hat u+\hat\rho)(w_0)&\le\frac{1}{\mathrm{vol}_{\mathbb{C}^n}(r\mathbb{B})}\int_{r\mathbb{B}}(\hat u+\hat\rho)\,dV_{2n}\\&\le C_\rho r+\frac{1}{c_{2n}r^{2n}}\int_{r\mathbb{B}}\hat u\,dV_{2n}\\&\le C_\rho r-\frac{t}{c_{2n}r^{2n}}\int_{r\mathbb{B}\cap\{\hat u\le -t\}}dV_{2n}\\&\le C_\rho r-t+\frac{t}{c_{2n}r^{2n}}\int_{r\mathbb{B}\cap\{\hat u>-t\}}dV_{2n}\\&\le C_\rho r-t+\frac{C_\omega t}{r^{2n}}\mathrm{vol}_\omega(\{u>-t\}),\end{aligned}$$

where $c_{2n}=\mathrm{vol}_{\mathbb{C}^n}(\mathbb{B})$ and $C_\omega>0$ is a constant depending only on n, X, ω. It follows that

$$\mathrm{vol}_\omega(\{u>-t\})\ge\frac{r^{2n}}{C_\omega}\left(1-\frac{C_\rho r}{t}\right).$$

Hence, for every $0<t<1$, by choosing $r=t/(1+C_\rho)$, we have

$$\mathrm{vol}_\omega(\{u>-t\})\ge Ct^{2n},$$

where $C=1/C_\omega(1+C_\rho)^{2n+1}$ depends only on n, X and ω. □

End of the proof of Theorem 3.2. Without loss of generality, we can assume that $u_1\le u_2$. Let $W_t=\{u_1>a_1-t\}$ for $0<t\le 1$. We have

$$\int_{W_t}-\chi(u_1-u_2)\omega^n\le\int_{W_t}-\chi(u_1-a_2)\omega^n\le -b_t\chi(a_1-a_2-t),\tag{3-22}$$

where $b_t:=\mathrm{vol}(W_t)$.

It follows from Lemma 3.8 that $W_t\ne\varnothing$. Moreover,

$$b_t:=\int_{W_t}\omega^n\ge C_1\left(\frac{t}{A}\right)^{2n},\tag{3-23}$$

where $C_1>0$ is a constant depending only on n, X and ω. By [Darvas et al. 2021a, Theorem A] (see also [Do and Vu 2022a, Theorem 3]), there exists a unique $\varphi\in\mathcal{E}(X,\theta,\phi)$ with $\sup_X(\varphi-\phi)=0$ such that

$$\theta_\varphi^n=\frac{\varrho}{b_t}\mathbf{1}_{W_t}\omega^n.$$

It follows from Theorem 3.7 that

$$\phi - C_2 A(-\log t + \log A + 1) \le \varphi \le \phi \tag{3-24}$$

for some constant $C_2 \ge 1$ depending only on n, X and ω. Thus, we have

$$E^0_{\tilde\chi,\theta,\phi}(\varphi) \le -\tilde\chi\big(-C_2 A(-\log t + \log A + 1)\big) \le -C_3\Big(\log\frac{Ae}{t}\Big)^M \tilde\chi(-A),$$

where $C_3 > 0$ depends only on n, X, ω and M.

Hence, it follows from Theorem 3.1 that

$$\int_X -\chi(u_1 - u_2)(\theta^n_\psi - \theta^n_\varphi) \le C_4\varrho\Big(\log\frac{Ae}{t}\Big)^{2M}(B - \tilde\chi(-A))^2(1 - \tilde\chi(-1))^2\lambda, \tag{3-25}$$

where $\lambda = Q^{\circ(n)}(I^0_\chi(u_1, u_2))$ and $C_4 > 0$ depends only on n, X, ω and M.

Combining (3-22) and (3-25), we get

$$\int_X -\chi(u_1 - u_2)\theta^n_\psi \le -\varrho\chi(a_1 - a_2 - t) + C_4\varrho\Big(\log\frac{Ae}{t}\Big)^{2M}(B - \tilde\chi(-A))^2(1 - \tilde\chi(-1))^2\lambda.$$

Letting $t \to \lambda^m$, we get

$$\begin{aligned}\int_X -\chi(u_1 - u_2)\theta^n_\psi &\le -\varrho\chi(a_1 - a_2 - \lambda^m) + C_4\varrho\Big(\log\frac{Ae}{\lambda^m}\Big)^{2M}(B - \tilde\chi(-A))^2(1 - \tilde\chi(-1))^2\lambda \\ &\le -\varrho\chi(a_1 - a_2 - \lambda^m) + C_5\varrho\frac{A^{(1-\gamma)/m}}{\lambda^{1-\gamma}}(B - \tilde\chi(-A))^2(1 - \tilde\chi(-1))^2\lambda \\ &\le -\varrho\chi(a_1 - a_2 - \lambda^m) + C_5\varrho A^{(1-\gamma)/m}(B - \tilde\chi(-A))^2(1 - \tilde\chi(-1))^2\lambda^\gamma,\end{aligned}$$

where $C_5 > 0$ depends only on n, X, ω, M, m and γ. □

Remark 3.9. The hypothesis that $\tilde\chi \le \chi$ in Theorems 3.1 and 3.2 can be slightly relaxed: the same statement remains true if $\tilde\chi \le \chi$ on $(-\infty, -1]$ and $\chi(-1) = -1$. Indeed, we only need the inequality $\tilde\chi \le \chi$ to guarantee that $E_{\chi,\theta,\phi}(u) \le E_{\tilde\chi,\theta,\phi}(u)$ for $u \in \mathrm{PSH}(X, \theta, \phi)$. If we only have $\tilde\chi \le \chi$ on $(-\infty, -1]$, then

$$E_{\chi,\theta,\phi}(u) \le E_{\tilde\chi,\theta,\phi}(u) - \chi(-1)\operatorname{vol}(\theta_\phi).$$

This is still sufficient for the proof of Theorems 3.1 and 3.2.

Later we will apply Theorem 3.2 to the special case where $\chi(t) = \max\{t, -1\}$ and $\tilde\chi \in \widetilde{\mathcal{W}}^-$ with $\tilde\chi(-1) = -1$. In this case, we can compute explicitly $Q_{0,\chi,\tilde\chi}(\epsilon) = \sup_{\{t \le -1\}} \chi(\epsilon t)/\tilde\chi(t)$ as follows. Observe that

$$Q_{0,\chi,\tilde\chi}(\epsilon) = \max\bigg\{\sup_{-\epsilon^{-1}\le t\le -1}\frac{\chi(\epsilon t)}{\tilde\chi(t)}, \sup_{t \le -\epsilon^{-1}}\frac{\chi(\epsilon t)}{\tilde\chi(t)}\bigg\} = \max\bigg\{\sup_{-\epsilon^{-1}\le t\le -1}\frac{\epsilon t}{\tilde\chi(t)}, \frac{-1}{\tilde\chi(-\epsilon^{-1})}\bigg\}.$$

Since $\tilde\chi \in \widetilde{\mathcal{W}}^-$, the function $t/\tilde\chi(t)$ is decreasing; hence $Q_{0,\chi,\tilde\chi}(\epsilon) = (-\tilde\chi(-\epsilon^{-1}))^{-1}$.

If $\chi(t) = \tilde\chi(t) = -(-t)^p$ for some constant $p > 0$, then one sees directly that $Q_{0,\chi,\tilde\chi}(\epsilon) = \epsilon^p$. However we will not use this special case in applications.

3.4. *A counterexample.* Let

$$\chi(t) := -\log(-t+1) \in \mathcal{W}^-.$$

In this subsection, to simplify the notation, we define $E_\chi(u) := E_{\chi,\omega,0}(u)$, where by 0 we mean the constant function equal to 0. Our goal in this subsection is to construct sequences of functions $u_m, v_m \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$ such that

(i) $0 \geq u_m \geq v_m$, $\sup_X u_m = \sup_X v_m = 0$,

(ii) $u_m, v_m \to 0$ in L^1 as $m \to \infty$,

(iii) $\sup_m (E_\chi(u_m) + E_\chi(v_m)) < \infty$ and $\lim_{m\to\infty} I_\chi(u_m, v_m) = 0$ but

(iv) $\inf_m \int_X -\chi(u_m - v_m)(dd^c v_m + \omega)^n > 0.$

As a consequence of our construction of u_m, v_m below, we see that Theorem 1.2 (and Theorem 1.3) does not hold in general if $\chi = \tilde{\chi} \in \mathcal{W}^-$. Here is our construction. On the unit ball $\mathbb{B}$ of $\mathbb{C}^n$, we define

$$\varphi_m = \max\{\log|z|, -e^m\} \quad \text{and} \quad F_m = \{z \in \mathbb{B} : \log|z| = -e^m\}, \quad m > 0.$$

Lemma 3.10. *We have*

$$\int_{F_m} (dd^c\varphi_m)^k \wedge (dd^c|z|^2)^{n-k} = \begin{cases} O(e^{-e^m}) & \text{if } k < n, \\ c & \text{if } k = n, \end{cases} \tag{3-26}$$

where $c := \int_{\{z=0\}} (dd^c \log|z|)^n > 0$.

Proof. The case $k = n$ follows from Stokes' theorem. We consider now $k < n$. Let $\mathbb{B}_r$ be the ball of radius $r > 0$ centered at 0 in $\mathbb{C}^n$. Observe that $\varphi_m = \log|z|$ on an open neighborhood of $\partial\mathbb{B}_{2e^{-e^m}}$. Using this and Stokes' theorem, we obtain

$$\begin{aligned}\int_{F_m} (dd^c\varphi_m)^k \wedge (dd^c|z|^2)^{n-k} &\leq \int_{\mathbb{B}_{2e^{-e^m}}} (dd^c\varphi_m)^k \wedge (dd^c|z|^2)^{n-k} \\ &= \int_{\mathbb{B}_{2e^{-e^m}}} (dd^c \log|z|)^k \wedge (dd^c|z|^2)^{n-k}.\end{aligned}$$

By direct computations (and approximating $\log|z|$ by $\frac{1}{2}\log(|z|^2 + \epsilon)$ as $\epsilon \to 0$), we see that

$$\int_{\mathbb{B}_{2e^{-e^m}}} (dd^c \log|z|)^k \wedge (dd^c|z|^2)^{n-k} = O(e^{-e^m}).$$

Hence the desired assertion for $k < n$ follows. □

Let $g : \mathbb{B} \to U$ be a biholomorphic mapping from $\mathbb{B}$ to an open subset U of X. Let $\psi \in C_0^\infty(\mathbb{B})$ such that $0 \leq \psi \leq 1$ and $\psi|_{\mathbb{B}_{1/2}} = 1$. Let

$$\tilde{\varphi}_m = (\varphi_m \psi) \circ g^{-1}.$$

Then there exists a constant $A \geq 1$ such that $\tilde{\varphi}_m$ is $A\omega$-psh for every $m > 0$. Now, for all $m > A^{-n}$, we define

$$u_m = \frac{\tilde{\varphi}_m}{\sqrt[n]{m+1}} \quad \text{and} \quad v_m = \frac{\tilde{\varphi}_{m+1}}{\sqrt[n]{m}}.$$

We have $u_m, v_m \in \mathrm{PSH}(X, \omega) \cap L^\infty(X)$ with $\sup_X u_m = \sup_X v_m = 0$ and $0 \geq u_m \geq v_m \xrightarrow{L^1} 0$ as $m \to \infty$.

Put $\mu_m := (dd^c u_m + \omega)^n$ and $\nu_m = (dd^c v_m + \omega)^n$. We have

$$\mathbf{1}_{X \setminus g(F_m)} \mu_m + \mathbf{1}_{X \setminus g(F_{m+1})} \nu_m \leq C_1 \omega^n \tag{3-27}$$

for every m, where $C_1 > 0$ is a constant. By (3-26), we also have

$$\mu_m(g(F_m)) = \frac{c}{m+1} + O(e^{-e^m}) \quad \text{and} \quad \nu_m(g(F_{m+1})) = \frac{c}{m} + O(e^{-e^m}). \tag{3-28}$$

By (3-27), (3-28) and by the fact $v_m \xrightarrow{L^1} 0$, there exists $C_2 > 0$ such that

$$E_\chi(v_m) \leq C_1 \int_{X \setminus g(F_{m+1})} -\chi(v_m)\omega^n - \chi\left(\frac{-e^{m+1}}{\sqrt[n]{m}}\right)\left(\frac{c}{m} + O(e^{-e^m})\right) \leq C_2$$

for every $m \gg 1$. Hence, $\sup_m E_\chi(v_m) < \infty$. Since $v_m \leq u_m \leq 0$, we also have $\sup_m E_\chi(u_m) < \infty$. On the other hand,

$$\begin{aligned}
\int_X -\chi(v_m - u_m)(dd^c v_m + \omega)^n &\geq \int_{g(F_{m+1})} -\chi(v_m - u_m)(dd^c v_m + \omega)^n \\
&\geq \frac{c}{m} \log\left(\frac{e^{m+1}}{\sqrt[n]{m}} - \frac{e^m}{\sqrt[n]{m+1}} + 1\right) \\
&\geq \frac{c}{m} \log\left(\frac{(e-1)e^m}{\sqrt[n]{m+1}}\right) \geq \frac{c}{2}
\end{aligned}$$

for $m \gg 1$. It remains to show that $\lim_{m\to\infty} I_\chi(u_m, v_m) = 0$. By (3-27) and (3-28), we have

$$\begin{aligned}
I_\chi(u_m, v_m) = \int_{X \setminus g(F_m \cup F_{m+1})} & -\chi(v_m - u_m)(\nu_m - \mu_m) \\
& - \chi((v_m - u_m)(e^{-e^{m+1}}))\nu_m(g(F_{m+1})) + \chi((v_m - u_m)(e^{-e^m}))\mu_m(g(F_m)) \\
\leq C_1 \int_X |v_m|\omega^n &+ \frac{c}{m} \log\left(\frac{e^{m+1}}{\sqrt[n]{m}} - \frac{e^m}{\sqrt[n]{m+1}} + 1\right) \\
& - \frac{c}{m+1} \log\left(\frac{e^m}{\sqrt[n]{m}} - \frac{e^m}{\sqrt[n]{m+1}} + 1\right) + O(e^{-e^m}) \\
\leq C_1 \int_X |v_m|\omega^n &+ \frac{c}{m} \log\left(\frac{e}{\sqrt[n]{m}} - \frac{1}{\sqrt[n]{m+1}} + e^{-m}\right) \\
& - \frac{c}{m+1} \log\left(\frac{1}{\sqrt[n]{m}} - \frac{1}{\sqrt[n]{m+1}} + e^{-m}\right) + \frac{c}{m+1} + O(e^{-e^m}) \\
\xrightarrow{m\to\infty} 0. &
\end{aligned}$$

Hence we get $\lim_{m\to\infty} I_\chi(u_m, v_m) = 0$.

4. Applications

4.1. ***Quantitative version of Dinew's uniqueness theorem.*** For every Borel set E in X, recall that the capacity of E is given by

$$\operatorname{cap}(E)=\operatorname{cap}_\omega(E)=\sup_{\{w\in \mathrm{PSH}(X,\omega):0\le w\le 1\}}\int_E \omega_w^n.$$

We usually remove the subscript ω from $\operatorname{cap}_\omega$ if ω is clear from the context. There are generalizations of capacity in big cohomology classes, many of them are comparable; see Theorem 4.8 below and [Lu 2021]. Recall that a sequence of Borel functions $(u_j)_j$ is said to *converge to a Borel function u in capacity* if for every constant $\epsilon>0$, we have that $\operatorname{cap}(\{|u_j-u|\ge\epsilon\})$ converges to 0 as $j\to\infty$. Recall that for $u_j, u\in \mathrm{PSH}(X,\omega)$, if $u_j\to u$ in capacity, then $u_j\to u$ in L^1.

The convergence in capacity is of great importance in pluripotential theory in part because it implies the convergence of Monge–Ampère operators under reasonable circumstances. To study quantitatively the convergence in capacity, it is convenient to introduce the following distance function on $\mathrm{PSH}(X,\omega)$:

$$d_{\mathrm{cap}}(u,v):=\sup_{\{w\in \mathrm{PSH}(X,\omega):0\le w\le 1\}}\int_X |u-v|^{1/2}\omega_w^n$$

for every $u,v\in \mathrm{PSH}(X,\omega)$ (note that $d_{\mathrm{cap}}(u,v)<\infty$ thanks to the Chern–Levine–Nirenberg inequality). The number $\frac12$ in the definition of d_{cap} can be replaced by any constant in $(0,1)$. One can see that for $u_j,u\in\mathrm{PSH}(X,\omega)$ for $j\in\mathbb{N}$, $d_{\mathrm{cap}}(u_j,u)\to 0$ if and only if $|u_j-u|\to 0$ in capacity. Indeed, if $d_{\mathrm{cap}}(u_j,u)\to 0$, then it is clear that $|u_j-u|\to 0$ in capacity. For the converse statement, assume that $|u_j-u|$ converges to 0 in capacity, i.e., for every constant $\delta>0$, we have

$$\lim_{j\to\infty}\operatorname{cap}(\{|u_j-u|\ge\delta\})=0.$$

In particular, the L^1-norm of u_j is bounded uniformly in j. Consequently

$$\begin{aligned}\int_X |u_j-u|^{1/2}\omega_w^n &\le \int_{\{|u_j-u|\le\delta\}}|u_j-u|^{1/2}\omega_w^n+\int_{\{|u_j-u|\ge\delta\}}|u_j-u|^{1/2}\omega_w^n\\ &\le \delta^{1/2}\int_X\omega^n+\Big(\int_{\{|u_j-u|\ge\delta\}}\omega_w^n\Big)^{1/2}\Big(\int_{\{|u_j-u|\ge\delta\}}|u_j-u|\omega_w^n\Big)^{1/2}\quad\text{(Hölder's inequality)}\\ &\lesssim \delta^{1/2}\int_X\omega^n+\big(\operatorname{cap}(\{|u_j-u|\ge\delta\})\big)^{1/2},\end{aligned}$$

by Chern–Levine–Nirenberg inequality. Hence $d_{\mathrm{cap}}(u_j,u)\to 0$ if $|u_j-u|\to 0$ in capacity. The following result is an immediate consequence of the Chern–Levine–Nirenberg inequality.

Proposition 4.1. *Let $\theta\le A\omega$ be a closed smooth real $(1,1)$-form (where $A\ge 1$ is a constant) and ϕ be a model θ-psh function with $\varrho:=\int_X\theta_\phi^n>0$. Let $0\le w\le 1$ is an ω-psh function and ψ is the unique solution to the problem*

$$\begin{cases} u\in\mathcal{E}(X,\theta,\phi),\\ \theta_u^n=\dfrac{\varrho}{\operatorname{vol}(X)}(dd^c w+\omega)^n,\\ \sup_X u=0.\end{cases}\tag{4-1}$$

Then there exists a constant $C > 0$ depending only on X and ω such that

$$\int_X |\psi| \theta_\psi^n \leq C A \varrho.$$

Here is the main result of this subsection.

Theorem 4.2. *Let $\theta \leq A\omega$ be a closed smooth real $(1,1)$-form $(A \geq 1)$ and let ϕ be a model θ-psh function such that $\varrho := \mathrm{vol}(\theta_\phi) > 0$. Let $B \geq 1$, $\tilde{\chi} \in \widetilde{\mathcal{W}}^-$ and $u_1, u_2 \in \mathcal{E}(X, \theta, \phi)$ such that $\tilde{\chi}(-1) = -1$ and*

$$E^0_{\tilde{\chi},\theta,\phi}(u_1) + E^0_{\tilde{\chi},\theta,\phi}(u_2) \leq B.$$

Let $\chi(t) = \max\{t, -1\}$. Then, for every $0 < \gamma < 1$, there exists $C > 0$ depending only on n, X, ω and γ such that

$$d_{\mathrm{cap}}(u_1, u_2)^2 \leq C(A + |a_1 - a_2|)\big(|a_1 - a_2| + A(A+B)^2 \lambda^\gamma\big), \tag{4-2}$$

where

$$a_j := \sup_X u_j, \quad \lambda = \frac{1}{h^{\circ n}(1/I^0_\chi(u_1, u_2))} \quad \textit{and} \quad h(s) = (-\tilde{\chi}(-s))^{1/2}.$$

One sees that for $u_1, u_2 \in \mathcal{E}(X, \theta, \phi)$, we can find a common $\tilde{\chi} \in \mathcal{W}^-$ so that the assumption in Theorem 4.2 is satisfied. Thus if $\sup_X u_1 = \sup_X u_2 = 0$, and $\theta^n_{u_1} = \theta^n_{u_2}$, then the right-hand side of (4-2) vanishes; hence $u_1 = u_2$. We then recover Dinew's uniqueness theorem for prescribed singularities potentials [Boucksom et al. 2010; Darvas et al. 2018b; Dinew 2009].

Proof. Suppose that w is an arbitrary ω-psh function satisfying $0 \leq w \leq 1$ and ψ is the unique solution to the problem

$$\begin{cases} u \in \mathcal{E}(X, \theta, \phi), \\ \theta^n_u = \dfrac{\varrho}{\mathrm{vol}(X)} (dd^c w + \omega)^n, \\ \sup_X u = 0. \end{cases} \tag{4-3}$$

We split the proof into two cases.

Case 1: Assume now that $I^0_\chi(u_1, u_2) \leq 1$. Hence, we get $\lambda = Q^{\circ n}_{\chi,\tilde{\chi}}(I^0_\chi(u_1, u_2))$ (see Remark 3.9), and one has $-\tilde{\chi}(-A) \leq A$ because $\tilde{\chi}(-1) = -1$. It follows from Theorem 3.2 and Proposition 4.1 that, for every $0 < \gamma < 1$, there exists $C_1 > 0$ depending only on n, X, ω and γ such that

$$I := \int_X -\chi(-|u_1 - u_2|)\theta^n_\psi \leq -\varrho \chi(-|a_1 - a_2| - \lambda) + C_1 \varrho A (A+B)^2 \lambda^\gamma. \tag{4-4}$$

Moreover

$$\begin{aligned} \frac{\varrho}{\mathrm{vol}(X)} \int_X |u_1 - u_2|^{1/2} (dd^c w + \omega)^n &= \int_X |u_1 - u_2|^{1/2} \theta^n_\psi \\ &= \int_{\{|u_1 - u_2| \leq 1\}} |u_1 - u_2|^{1/2} \theta^n_\psi + \int_{\{|u_1 - u_2| > 1\}} |u_1 - u_2|^{1/2} \theta^n_\psi \\ &\leq I^{1/2} \bigg(\Big(\int_{\{|u_1 - u_2| \leq 1\}} \theta^n_\psi \Big)^{1/2} + \Big(\int_{\{|u_1 - u_2| > 1\}} |u_1 - u_2| \theta^n_\psi \Big)^{1/2} \bigg), \end{aligned}$$

where the last estimate holds due to the Cauchy–Schwarz inequality. Moreover, it follows from Chern–Levine–Nirenberg inequality [Kołodziej 2005] that

$$\begin{aligned}\int_X |u_1-a_1-u_2+a_2|\theta_\psi^n &= \frac{\varrho}{\mathrm{vol}(X)}\int_X |u_1-a_1-u_2+a_2|(dd^c w+\omega)^n \\ &\le C_2\varrho(\|u_1-a_1\|_{L^1(X)}+\|u_2-a_2\|_{L^1(X)}) \\ &\le \varrho C_3 A, \end{aligned} \tag{4-5}$$

where $C_2, C_3 > 0$ depend only on X and ω. Here, the last estimate holds due to the compactness of $\{u \in \mathrm{PSH}(X,\omega) : \sup_X u = 0\}$ in $L^1(X)$.

Hence

$$\frac{\varrho}{\mathrm{vol}(X)}\int_X |u_1-u_2|^{1/2}(dd^c w+\omega)^n \le C_4 I^{1/2}\varrho^{1/2}(A+|a_1-a_2|)^{1/2}, \tag{4-6}$$

where $C_4 > 0$ depends only on X and ω.

Combining (4-4) and (4-6), we get

$$\begin{aligned}\left(\int_X |u_1-u_2|^{1/2}(dd^c w+\omega)^n\right)^2 &\le C_5(A+|a_1-a_2|)\big(-\chi(-|a_1-a_2|-\lambda)+A(A+B)^2\lambda^\gamma\big) \\ &\le C_5(A+|a_1-a_2|)(|a_1-a_2|+\lambda+A(A+B)^2\lambda^\gamma) \\ &\le C_6(A+|a_1-a_2|)(|a_1-a_2|+A(A+B)^2\lambda^\gamma),\end{aligned}$$

where $C_5, C_6 > 0$ depend only on n, X, ω and γ. Since w is arbitrary, we obtain the desired inequality.

Case 2: We treat now the case where $I_\chi^0(u_1,u_2) \ge 1$.

Observe that $\lambda \ge 1$ in this case. Hence the right-hand side of (4-2) is greater than or equal to $C(A+|a_1-a_2|)$ because $A \ge 1$ and $\lambda \ge 1$. On the other hand, Hölder's inequality gives

$$\begin{aligned}\left(\int_X |u_1-u_2|^{1/2}(dd^c w+\omega)^n\right)^2 &\lesssim \int_X |u_1-u_2|(dd^c w+\omega)^n \\ &\le \int_X |u_1-a_1-u_2+a_2|(dd^c w+\omega)^n + |a_1-a_2|\int_X \omega^n \\ &\lesssim A+|a_1-a_2|\end{aligned}$$

by (4-5). Thus the desired estimate holds. □

Remark 4.3. If $B \ge A$ then the inequality (4-2) is equivalent to

$$d_{\mathrm{cap}}(u_1,u_2)^2 \le \widetilde{C}(A+|a_1-a_2|)(|a_1-a_2|+AB^2\lambda^\gamma),$$

where $\widetilde{C} > 0$ depends only on n, X, ω and γ.

4.2. *Quantitative version for the domination principle.*

Theorem 4.4. *Let $A \ge 1$ be a constant and let $\theta \le A\omega$ be a closed smooth real $(1,1)$-form and ϕ be a model θ-psh function, and $\varrho := \mathrm{vol}(\theta_\phi) > 0$. Let $B \ge 1$ be a constant, $\tilde{\chi} \in \widetilde{\mathcal{W}}^-$ and $u_1, u_2 \in \mathcal{E}(X,\theta,\phi)$ such that $\tilde{\chi}(-1) = -1$ and*

$$E^0_{\tilde{\chi},\theta,\phi}(u_1) + E^0_{\tilde{\chi},\theta,\phi}(u_2) \le B.$$

Assume that there exists a constant $0 \le c < 1$ *and a Radon measure* μ *on* X *satisfying* $\theta_{u_1}^n \le c\theta_{u_2}^n + \varrho\mu$ *on* $\{u_1 < u_2\}$ *and* $c_\mu := \int_{\{u_1<u_2\}} d\mu \le 1$. *Then there exists a constant* $C > 0$ *depending only on* n, X *and* ω *such that*

$$\operatorname{cap}_\omega\{u_1 < u_2 - \epsilon\} \le \frac{C \operatorname{vol}(X)(A+B)^2}{\epsilon(1-c)h^{\circ n}(1/c_\mu)}$$

for every $0 < \epsilon < 1$, *where* $h(s) = (-\tilde{\chi}(-s))^{1/2}$ *for every* $0 \le s \le \infty$.

In particular, *if* $c_\mu = 0$ *then* $\operatorname{cap}_\omega\{u_1 < u_2 - \epsilon\} = 0$ *for every* $\epsilon > 0$, *and then* $u_1 \ge u_2$ *on whole* X.

The standard domination principle corresponds to the case where $c = 0$ and $\mu := 0$. A non-quantitative version of this domination principle (i.e., for $\mu = 0$) in the non-Kähler setting was obtained in [Guedj and Lu 2023].

Proof of Theorem 4.4. Let w be an arbitrary ω-psh function satisfying $0 \le w \le 1$ and ψ is the unique solution to (4-1). Let $v = \max\{u_1, u_2\}$ and $\chi(t) = \max\{t, -1\} \ge \tilde{\chi}(t)$. By Theorem 3.1 and Proposition 4.1, there exists a constant $C_1 > 0$ depending only on n, X an ω such that

$$I_1 := \int_X -\chi(u_1 - v)(\theta_\psi^n - \theta_{u_1}^n) \le C_1\varrho(A+B)^2 Q^{\circ(n)}(I_\chi^0(u_1, v)), \tag{4-7}$$

$$I_2 := \int_X -\chi(u_1 - v)(\theta_{u_2}^n - \theta_{u_1}^n) \le C_1\varrho(A+B)^2 Q^{\circ(n)}(I_\chi^0(u_1, v)). \tag{4-8}$$

Moreover, by the fact $\theta_v^n = \theta_{u_2}^n$ on $\{u_1 < u_2\}$ and by the assumption $\theta_{u_1}^n \le c\theta_{u_2}^n + \varrho\mu$ on $\{u_1 < u_2\}$, we have

$$I_\chi^0(u_1, v) = \varrho^{-1}\int_{\{u_1<u_2\}} -\chi(u_1 - v)(\theta_{u_1}^n - \theta_{u_2}^n) \le \varrho^{-1}\int_{\{u_1<u_2\}} -\chi(u_1 - v)(\theta_{u_1}^n - c\theta_{u_2}^n) \le c_\mu. \tag{4-9}$$

Combining (4-7), (4-8) and (4-9), we get

$$\begin{aligned}(1-c)\int_X -\chi(u_1 - v)\theta_\psi^n &= \int_X -\chi(u_1 - v)(\theta_{u_1}^n - c\theta_{u_2}^n) + (1-c)I_1 + cI_2 \\ &\le \int_X -\chi(u_1 - v)(\theta_{u_1}^n - c\theta_{u_2}^n) + C_1\varrho(A+B)^2 Q^{\circ n}(c_\mu) \\ &\le \varrho c_\mu + C_1\varrho(A+B)^2 Q^{\circ n}(c_\mu) \\ &\le C\varrho(A+B)^2 Q^{\circ n}(c_\mu),\end{aligned}$$

where $C = C_1 + 1$. Hence

$$\int_{\{u_1<u_2-\epsilon\}} \omega_w^n = \frac{\operatorname{vol}(X)}{\varrho}\int_{\{u_1<u_2-\epsilon\}} \theta_\psi^n \le \frac{C \operatorname{vol}(X)(A+B)^2 Q^{\circ n}(c_\mu)}{(1-c)\epsilon}$$

for every $0 < \epsilon < 1$. Since w is arbitrary, it follows that

$$\operatorname{cap}_\omega\{u_1 < u_2 - \epsilon\} \le \frac{C \operatorname{vol}(X)(A+B)^2 Q^{\circ n}(c_\mu)}{(1-c)\epsilon}. \tag{4-10}$$

Moreover, by the definition of χ and the formula of Q, we have

$$Q(s) = \frac{1}{(-\tilde{\chi}(-1/s))^{1/2}} = \frac{1}{h(1/s)}$$

for every $0 < s \leq 1$, and $Q(0) = 0$. Then

$$Q^{\circ n}(s) = \frac{1}{h^{\circ n}(1/s)} \tag{4-11}$$

for every $0 \leq s \leq 1$. □

4.3. ***Relation to Darvas's metrics on the space of potentials of finite energy.*** Let $\chi \in \mathcal{W}^- \cup \mathcal{W}^+_M$. Let θ be a closed smooth real $(1,1)$-form in a big cohomology class. When θ is Kähler, it was proved in [Darvas 2015; 2017; 2024] that there is a natural metric d_χ on $\mathcal{E}_\chi(X,\theta)$ which makes the last space to be a complete metric space. When $\chi(t) = t$, such metrics have a long history and play an important role in the study of complex Monge–Ampère equations. We refer to these last references and [Berman et al. 2020; 2021] for more details. We now draw the connection between $I_\chi(u,v)$ and the metric on $\mathcal{E}_\chi(X,\theta)$. Let

$$\tilde{I}_\chi(u,v) = \int_{\{u<v\}} -\chi(u-v)(\theta_v^n + \theta_u^n) + \int_{\{u>v\}} -\chi(v-u)(\theta_u^n + \theta_v^n) \geq I_\chi(u,v).$$

By [Darvas 2015; 2017; 2024], there exists a constant $C > 0$ such that

$$C^{-1}\tilde{I}_\chi(u,v) \leq d_\chi(u,v) \leq C\tilde{I}_\chi(u,v)$$

for every $u, v \in \mathcal{E}_\chi(X,\theta)$ and θ is Kähler. It was proved in [Gupta 2023] (and also [Darvas 2015; Darvas et al. 2018a; Di Nezza and Lu 2020; Trusiani 2022; Xia 2023]) that $\tilde{I}_\chi(u,v)$ satisfies a quasitriangle inequality, and the convergence in $\tilde{I}_\chi(u,v)$ implies the convergence in capacity by using the plurisubharmonic envelope. Such a method is not quantitative. We present below quantitative version of this fact by using our approach.

Theorem 4.5. *Let $\theta \leq A\omega$ be a closed smooth real $(1,1)$-form ($A \geq 1$ is a constant) and ϕ be a model θ-psh function with $\varrho := \mathrm{vol}(\theta_\phi) > 0$. Let $B \geq 1$, $\tilde{\chi} \in \mathcal{W}^-$ and $u_1, u_2 \in \mathcal{E}(X,\theta,\phi)$ such that $\tilde{\chi}(-1) = -1$ and*

$$E^0_{\tilde{\chi},\theta,\phi}(u_1) + E^0_{\tilde{\chi},\theta,\phi}(u_2) \leq B.$$

Then there exist $C > 0$ depending only on n, X and ω such that

$$d_{\mathrm{cap}}(u_1,u_2)^2 \leq \frac{C\big(A + |\sup_X u_1 - \sup_X u_2|\big)(A+B)^2}{h^{\circ n}(\varrho/\tilde{I}_{\tilde{\chi}}(u_1,u_2))},$$

where $h(s) = (-\tilde{\chi}(-s))^{1/2}$ for every $0 \leq s \leq \infty$.

We note that the quantities $a_j := |\sup_X u_j|$ for $j = 1, 2$ (hence $|a_1 - a_2|$) can be bounded by a function of B and $\tilde{\chi}$ as follows. Since ϕ is a model, we have $-a_j = \sup_X(u_j - \phi)$. It follows that

$$B \geq E^0_{\tilde{\chi},\theta,\phi}(u_j) \geq -\tilde{\chi}(-a_j).$$

Consequently, we get $a_j \le -\tilde{\chi}^{-1}(-B)$ for $j = 1, 2$, where $\tilde{\chi}^{-1}$ denotes the inverse map of $\tilde{\chi} : \mathbb{R}_{\le 0} \to \mathbb{R}_{\le 0}$. Thus by Theorem 4.5, one sees that if $\tilde{I}_{\tilde{\chi}}(u_1, u_2)$ is small, then so is $d_{\text{cap}}(u_1, u_2)$ (uniformly in $u_1, u_2 \in \mathcal{E}(X, \theta, \phi)$ of $\tilde{\chi}$-energy bounded by a fixed constant).

Proof. Let $\chi(t) = \max\{t, -1\}$. Suppose that w is an arbitrary ω-psh function satisfying $0 \le w \le 1$. By the proof of Theorem 4.2 (see (4-6)), there exists $C_1 > 0$ depending only on X and ω such that

$$\left(\int_X |u_1 - u_2|^{1/2} (dd^c w + \omega)^n\right)^2 \le C_1\big(A + |\sup_X u_1 - \sup_X u_2|\big)\varrho^{-1} \int_X -\chi(-|u_1 - u_2|)\theta_\psi^n, \tag{4-12}$$

where ψ is defined by (4-3). Moreover, it follows from Theorem 3.1 (applied to $u_1, \max\{u_1, u_2\}$, $\psi_1 := \psi$, $\psi_2 := u_1$) and Proposition 4.1 that

$$\int_X -\chi(-|u_1 - u_2|)\theta_\psi^n \le \tilde{I}_\chi(u_1, u_2) + C_2\varrho(A + B)^2 Q^{\circ(n)}_{\chi,\tilde{\chi}}(I^0_\chi(u_1, u_2)),$$

where $C_2 > 0$ depends only on n. Therefore, since

$$Q^{\circ(n)}(s) = \frac{1}{h^{\circ(n)}(1/s)} \quad \text{and} \quad I_\chi(u_1, u_2) \le \tilde{I}_\chi(u_1, u_2) \le \tilde{I}_{\tilde{\chi}}(u_1, u_2)),$$

we obtain

$$\int_X -\chi(-|u_1 - u_2|)\theta_\psi^n \le \frac{C_3\varrho(A + B)^2}{h^{\circ(n)}(\varrho/\tilde{I}_{\tilde{\chi}}(u_1, u_2))}, \tag{4-13}$$

where $C_3 > 0$ depends only on n, X and ω. Combining (4-12) and (4-13), we get

$$\left(\int_X |u_1 - u_2|^{1/2} (dd^c w + \omega)^n\right)^2 \le \frac{C\left(A + |\sup_X u_1 - \sup_X u_2|\right)(A + B)^2}{h^{\circ(n)}(\varrho/\tilde{I}_{\tilde{\chi}}(u_1, u_2))},$$

where $C > 0$ depends only on n, X and ω. Since w is arbitrary, we get the desired inequality. □

Remark 4.6. Consider now a weight $\tilde{\chi} \in \mathcal{W}_M^+$ with $\tilde{\chi}(-1) = -1$. One sees that $\tilde{\chi}(t) \le (-t)^M \tilde{\chi}(-1) = -(-t)^M$ for $-1 \le t \le 0$, and $\tilde{\chi}(t) \le \tilde{\chi}_0(t) := t$ for $t \le -1$. Consequently, using Hölder's inequality, we get

$$\rho^{-1}\tilde{I}_{\tilde{\chi}_0}(u_1, u_2) \le 2(\rho^{-1}\tilde{I}_{\tilde{\chi}}(u_1, u_2))^{1/M} + \rho^{-1}\tilde{I}_{\tilde{\chi}}(u_1, u_2).$$

Hence, Theorem 4.5 applied to $\tilde{\chi}_0$ shows that if $\rho^{-1}\tilde{I}_{\tilde{\chi}}(u_1, u_2) \to 0$ and the normalized $\tilde{\chi}$-energies of u_1, u_2 are uniformly bounded, then $d_{\text{cap}}(u_1, u_2) \to 0$.

When $\tilde{\chi} \in \mathcal{W}_M^+$, we have another version of Theorem 4.5 which is more explicit.

Theorem 4.7. *Let $\theta \le A\omega$ be a closed smooth real $(1, 1)$-form ($A \ge 1$) and ϕ be a model θ-psh function such that $\varrho := \text{vol}(\theta_\phi) > 0$. Let $B \ge 1$, $\tilde{\chi} \in \mathcal{W}_M^+$ ($M \ge 1$) and $u_1, u_2 \in \mathcal{E}(X, \theta, \phi)$ be such that $\tilde{\chi}(-1) = -1$ and*

$$E^0_{\tilde{\chi},\theta,\phi}(u_1) + E^0_{\tilde{\chi},\theta,\phi}(u_2) \le B.$$

Then there exists $C > 0$ depending only on n and M such that

$$\int_X -\tilde{\chi}(-|u_1 - u_2|)\theta_\psi^n \le C\varrho B^2(\tilde{I}_{\tilde{\chi}}(u_1, u_2)/\varrho)^{2^{-n}} \tag{4-14}$$

for every $\psi \in \mathrm{PSH}(X,\theta)$ *with* $\phi - 1 \le \psi \le \phi$. *Moreover*, *if* $\sup_X u_1 = \sup_X u_2$ *then there exists* $C' > 0$ *depending on* n, X, ω, A *and* M *such that*

$$\tilde{I}_{\tilde{\chi}}(u_1,u_2) \le C' \varrho A^{1/2} B^2 (I^0_{\tilde{\chi}}(u_1,u_2))^{2^{-n-1}}.$$

Proof. The case $I^0_{\tilde{\chi}}(u_1,u_2) \ge 1$ is trivial because

$$\tilde{I}_{\tilde{\chi}}(u_1,u_2)/\varrho \ge I^0_{\tilde{\chi}}(u_1,u_2) \ge 1,$$

whereas the left-hand side of (4-14) is always bounded by a constant (depending on M) times B. Thus, from now on, it suffices to assume that $I^0_{\tilde{\chi}}(u_1,u_2) < 1$.

Denote $v = \max\{u_1,u_2\}$. By Lemma 2.7, we have $v \in \mathcal{E}(X,\theta,\phi)$ and $E^0_{\tilde{\chi},\theta,\phi}(v) \le C_1 B$, where $C_1 > 0$ depends only on n and M. Taking $\chi = \tilde{\chi}$ and using Theorem 3.1, we get

$$\int_X -\tilde{\chi}(u_j - v)\theta^n_\psi \le \int_X -\tilde{\chi}(u_j - v)\theta^n_{u_j} + C_2 \varrho B^2 (I^0_{\tilde{\chi}}(u_j,v))^{2^{-n}} \tag{4-15}$$

for $j = 1,2$, where $C_2 > 0$ depends on n and M. Note that

$$\int_X -\tilde{\chi}(u_1 - v)\theta^n_{u_1} + \int_X -\tilde{\chi}(u_2 - v)\theta^n_{u_2} \le \int_X -\tilde{\chi}(-|u_1-u_2|)(\theta^n_{u_1} + \theta^n_{u_2}) = \tilde{I}_{\tilde{\chi}}(u_1,u_2),$$

$$I^0_{\tilde{\chi}}(u_1,v) + I^0_{\tilde{\chi}}(u_2,v) = I^0_{\tilde{\chi}}(u_1,u_2) \le \varrho^{-1}\tilde{I}_{\tilde{\chi}}(u_1,u_2).$$

Hence, by (4-15), we get

$$\begin{aligned}
\int_X -\tilde{\chi}(-|u_1-u_2|)\theta^n_\psi &= \int_X -\tilde{\chi}(u_1-v)\theta^n_\psi + \int_X -\tilde{\chi}(u_2-v)\theta^n_\psi \\
&\le \int_X -\tilde{\chi}(u_1-v)\theta^n_{u_1} + \int_X -\tilde{\chi}(u_2-v)\theta^n_{u_2} + C_2\varrho B^2\big((I^0_{\tilde{\chi}}(u_1,v))^{2^{-n}} + (I^0_{\tilde{\chi}}(u_2,v))^{2^{-n}}\big) \\
&\le \tilde{I}_{\tilde{\chi}}(u_1,u_2) + 2C_2\varrho B^2 (\tilde{I}_{\tilde{\chi}}(u_1,u_2)/\varrho)^{2^{-n}} \\
&\le C_3 \varrho B^2 (\tilde{I}_{\tilde{\chi}}(u_1,u_2)/\varrho)^{2^{-n}},
\end{aligned}$$

where $C_3 > 0$ depends on n and M. Here, the last estimate holds due to the fact that $\tilde{I}_{\tilde{\chi}}(u_1,u_2) \le \varrho B$.

Now, we consider the case $\sup_X u_1 = \sup_X u_2$. By Theorem 3.2 $\big($choose $m = 1$ and $\gamma = \frac{1}{2}\big)$, there exists $C_4 > 0$ depending only on n, X, ω and M such that

$$\begin{aligned}
\tilde{I}_{\tilde{\chi}}(u_1,u_2) &\le \int_X -\tilde{\chi}(-|u_1-u_2|)(\theta^n_{u_1} + \theta^n_{u_2}) \\
&\le -2\varrho\,\tilde{\chi}(-(I^0_{\tilde{\chi}}(u_1,u_2))^{2^{-n}}) + C_4 \varrho A^{1/2} B^2 (I^0_{\tilde{\chi}}(u_1,u_2))^{2^{-n-1}}.
\end{aligned} \tag{4-16}$$

Moreover, since $\tilde{\chi}$ is concave, we have

$$\frac{\tilde{\chi}(t)}{t} \le \frac{\tilde{\chi}(-1)}{-1} = 1$$

for every $-1 < t < 0$. Hence, by (4-16), we have

$$\begin{aligned}
\tilde{I}_{\tilde{\chi}}(u_1,u_2) &\le 2\varrho (I^0_{\tilde{\chi}}(u_1,u_2))^{2^{-n}} + C_4\varrho A^{1/2} B^2 (I^0_{\tilde{\chi}}(u_1,u_2))^{2^{-n-1}} \\
&\le (2 + C_4)\varrho A^{1/2} B^2 (I^0_{\tilde{\chi}}(u_1,u_2))^{2^{-n-1}}.
\end{aligned}$$

□

4.4. ***Comparison of capacities.*** For every Borel subset E in X and for every $\varphi \in \mathrm{PSH}(X,\theta)$, we recall again that

$$\mathrm{cap}_{\theta,\varphi}(E) = \sup\left\{ \int_E \theta_\psi^n : \psi \in \mathrm{PSH}(X,\theta),\ \varphi - 1 \le \psi \le \varphi \right\}.$$

In [Lu 2021], it was shown that if φ_j $(j = 1, 2)$ is a θ_j-psh function with $\int_X (\theta_j + dd^c\varphi_j)^n > 0$ then there exists a continuous function $f : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ with $f(0) = 0$ such that $\mathrm{cap}_{\theta_1,\varphi_1}(E) \le f(\mathrm{cap}_{\theta_2,\varphi_2}(E))$ for every Borel set $E \subset X$. As an application of our main results, we obtain the following quantitative comparison of capacities for the case where φ_j is a model θ_j-psh function.

Theorem 4.8 (comparison of capacities). *Assume that $\theta_1, \theta_2 \le A\omega$ are closed smooth real $(1,1)$-forms representing big cohomology classes and, for $j = 1, 2$, that ϕ_j is a model θ_j-psh function satisfying $\int_X (dd^c\phi_j + \theta_j)^n = \varrho_j > 0$. Then, for every $0 < \gamma < 1$, there exists $C > 0$ depending only on n, X, ω, A and γ such that*

$$\frac{\mathrm{cap}_{\theta_1,\phi_1}(E)}{\varrho_1} \le C\left(\frac{\mathrm{cap}_{\theta_2,\phi_2}(E)}{\varrho_2}\right)^{2^{-n}\gamma}$$

for every Borel set $E \subset X$.

We now prove Theorem 4.8. First, we need the following lemma.

Lemma 4.9. *Let $A, B > 0$ be constants. Let θ be a closed smooth real $(1,1)$-form representing a big cohomology class such that $\theta \le A\omega$. Assume that u, v are θ-psh functions satisfying $v \le u \le v + B$. Then*

$$\int_X (-\psi)\theta_u^n \le \int_X (-\psi)\theta_v^n + nA^nB\int_X \omega^n$$

for every negative $A\omega$-psh function ψ.

Proof. Using approximations, we can assume that ψ is smooth. Let

$$T = \sum_{l=0}^{n-1} \theta_u^l \wedge \theta_v^{n-l-1}.$$

We have $\theta_u^n - \theta_v^n = dd^c(u - v) \wedge T$. Moreover, using integration by parts (Theorem 2.2), we get

$$\int_X (-\psi)dd^c(u-v)\wedge T = \int_X (u-v)dd^c(-\psi)\wedge T \le A\int_X (u-v)\omega\wedge T \le nA^nB\int_X \omega^n.$$

Hence

$$\int_X (-\psi)\theta_u^n \le \int_X (-\psi)\theta_v^n + nA^nB\int_X \omega^n. \qquad \square$$

Proof of Theorem 4.8. By the inner regularity of capacities (see [Darvas et al. 2018b, Lemma 4.2]), we only need consider the case where E is compact. Since the case $\mathrm{cap}_{\theta_2,\phi_2}(E) = \varrho_2$ is trivial, we can also assume that $\mathrm{cap}_{\theta_2,\phi_2}(E) < \varrho_2$. In particular, by Darvas et al. 2021a, Proposition 3.7; 2021b, Lemma 2.7], we have

$$\sup_X h^*_{E,\theta_2,\phi_2} = \sup_X (h^*_{E,\theta_2,\phi_2} - \phi_2) = 0,$$

where

$$h_{E,\theta_2,\phi_2} = \sup\{ w \in \mathrm{PSH}(X,\theta_2) : w|_E \le \phi_2 - 1,\ w \le \phi_2 \}.$$

Set $\chi(t) = \tilde{\chi}(t) = t$. We will use Theorem 3.2 for $u_1 = (h_{E,\theta_2,\phi_2})^*$ and $u_2 = \phi_2$. It is clear that $E^0_{\tilde{\chi},\theta_2,\phi_2}(u_2) = 0$ and $u_1 = u_2 - 1$ on $E \setminus N$, where N is a pluripolar set. Moreover, it follows from [Darvas et al. 2021a, Proposition 3.7] that

$$I^0_\chi(u_1, u_2) \le E^0_{\tilde{\chi},\theta_2,\phi_2}(u_1) = \varrho_2^{-1}\mathrm{cap}_{\theta_2,\phi_2}(E) \le 1.$$

By Theorem 3.2, for every $0 < \gamma < 1$ and $B \ge 1$, there exists $C > 0$ depending only on X, ω, n, A and γ such that

$$\int_E \theta^n_\psi \le \int_X \chi(-|u_1 - u_2|)\theta^n_\psi \le C\varrho_2 A(A+B)^2(\mathrm{cap}_{\theta_2,\phi_2}(E)/\varrho_2)^{2^{-n}\gamma}, \tag{4-17}$$

for every compact set E and for each $\psi \in \mathcal{E}(X, \theta_2, \phi_2)$ with $E^0_{\tilde{\chi},\theta_2,\phi_2}(\psi) \le B$. Let $\varphi \in \mathcal{E}(X, \theta_1, \phi_1)$ such that $\phi_1 - 1 \le \varphi \le \phi_1$ and $\int_E(\theta_1 + dd^c\varphi)^n \ge \frac{1}{2}\mathrm{cap}_{\theta_1,\phi_1}(E)$. By [Darvas et al. 2021a], there exists a unique function $\psi_0 \in \mathcal{E}(X, \theta_2, \phi_2)$ such that $\sup_X \psi_0 = 0$ and

$$(dd^c\psi_0 + \theta_2)^n = \frac{\varrho_2}{\varrho_1}(dd^c\varphi + \theta_1)^n.$$

When $\psi = \psi_0$, we have

$$\int_E \theta^n_\psi \ge \frac{\varrho_2}{2\varrho_1}\mathrm{cap}_{\theta_1,\phi_1}(E). \tag{4-18}$$

Moreover, by using Lemma 4.9 for φ, ϕ_1 and using the fact that $(dd^c\phi_2 + \theta_2)^n \le \mathbf{1}_{\{\phi_2=0\}}\theta_2^n$ (see [Darvas et al. 2018b, Theorem 3.8]), we have

$$\varrho_1 E^0_{\tilde{\chi},\theta_2,\phi_2}(\psi_0) = \int_X(\phi_2 - \psi_0)(dd^c\varphi + \theta_1)^n \le \int_X(-\psi_0)(dd^c\phi_1 + \theta_1)^n + nA^n\int_X\omega^n \le B, \tag{4-19}$$

where $B \ge 1$ depends only on A, X, ω, n. Combining (4-17), (4-18) and (4-19), we get

$$\begin{aligned}\mathrm{cap}_{\theta_1,\phi_1}(E) \le \frac{2\varrho_1}{\varrho_2}\int_E \theta^n_{\psi_0} &\le \frac{2\varrho_1}{\varrho_2}\int_X \chi(-|u_1 - u_2|)\theta^n_{\psi_0}\\ &\le 2C\varrho_1 A(A+B)^2(\mathrm{cap}_{\theta_2,\phi_2}(E)/\varrho_2)^{2^{-n}\gamma}.\end{aligned}$$

□

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