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# STABILITY AND LORENTZIAN GEOMETRY FOR AN INVERSE PROBLEM OF A SEMILINEAR WAVE EQUATION

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This paper concerns an inverse boundary value problem for a semilinear wave equation on a globally hyperbolic Lorentzian manifold. We prove a Hölder stability result for recovering an unknown potential $q$ of the nonlinear wave equation $\square_g u + qu^m = 0,\ m \geq 4$, from the Dirichlet-to-Neumann map. Our proof is based on the recent higher-order linearization method and use of Gaussian beams. We also extend earlier uniqueness results by removing the assumptions of convex boundary and that pairs of light-like geodesics can intersect only once. For this, we construct special light-like geodesics and other general constructions in Lorentzian geometry. We expect these constructions to be applicable in studies of related problems as well.



## 1. Introduction

We consider the stability and uniqueness of an inverse problem for the nonlinear wave equation on an $(n+1)$-dimensional, $n \geq 2$, globally hyperbolic Lorentzian manifold. As is well known, any globally hyperbolic Lorentzian manifold $N$ is isometric to a product manifold $\mathbb{R} \times M$ equipped with the product metric

$$g = -\beta(t,x)\,dt^2 + h(t,x). \tag{1}$$

Here $\beta > 0$ is a smooth function and $h(t,\cdot),\ t \in \mathbb{R}$, is a smooth one-parameter family of Riemannian metrics on an $n$-dimensional manifold $M$; see, e.g., [Bernal and Sánchez 2005]. Let $\Omega \subset M$ be a smooth submanifold of dimension $n$ with smooth boundary and let us denote the lateral boundary of $[0,T] \times \Omega \subset N$ by

$$\Sigma := [0,T] \times \partial\Omega.$$

In local coordinates $(x^a)$ the d'Alembertian wave operator $\square_g$ of $g$ has the form

$$\square_g u = -\sum_{a,b=0}^{n} \frac{1}{\sqrt{|\det(g)|}} \frac{\partial}{\partial x^a}\Big(\sqrt{|\det(g)|}\, g^{ab} \frac{\partial u}{\partial x^b}\Big).$$

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*Keywords:* Lorentzian geometry, nonlinear wave equations, inverse problems.

Here we write $(g^{-1})_{ab} = (g^{ab})$, $a, b = 0, \dots, n$, as usual. We consider the nonlinear wave equation

$$\begin{cases} \square_g u(t,x) + q(t,x)u(t,x)^m = 0 & \text{in } [0,T]\times\Omega, \\ u = f & \text{on } [0,T]\times\partial\Omega, \\ u(0,x) = \partial_t u(0,x) = 0 & \text{on } \Omega, \end{cases} \tag{2}$$

where we assume that the exponent $m$ is an integer greater than or equal to 4. The inverse problem we study is the stability of recovery of the potential $q$ from the Dirichlet-to-Neumann (DN) map

$$\Lambda : H_0^{s+1}(\Sigma) \to H^s(\Sigma), \quad f \mapsto \partial_\nu u_f|_\Sigma,$$

where $u_f$ is the unique small solution of (2) and $\partial_\nu$ is the normal derivative on $\Sigma$. Here also $H_0^{s+1}$ and $H^s$ refer to Sobolev spaces, where $s \in \mathbb{N}$ will be specified later. See Section 1.4 for details about Sobolev spaces and Section 2 for details about the well-posedness of the forward problem. The present work is a continuation of the authors' earlier work [Lassas et al. 2022], which considered the stability of a recovery of the potential $q$ of (2) in the Minkowski space of $\mathbb{R}^{n+1}$. We describe our main results in Section 1.1.

Studies of uniqueness and stability of the recovery of unknown parameters in inverse problems are motivated by practical applications. Let us mention some results on inverse problems for linear wave type equations. First results in this direction for the linear wave equation with vanishing initial data were obtained in [Belishev 1987; Belishev and Kurylev 1992]. The approach there is called the boundary control method and it combines both the wave propagation and controllability results [Katchalov et al. 2001]. The boundary control method allows also an effective numerical algorithm [de Hoop et al. 2018]. Recently, there have been several results on determining a Riemannian manifold from partial data boundary measurements for the linear wave equation and related equations such as the ones in [Anderson et al. 2004; Helin et al. 2018; Isozaki et al. 2017; Kian et al. 2019; Krupchyk et al. 2008; Kurylev et al. 2018b; Lassas 2018; Lassas and Oksanen 2014]. However, the boundary control method has been applicable only in the cases where the coefficients of the equation are time-independent, or when the lower-order terms are real analytic in the time variable [Eskin 2007]. In a geometric setting it has been studied if it is possible to recover a Riemannian metric $g$ from the Dirichlet-to-Neumann map of the equation $(\partial_t^2 - \Delta_g)u = 0$ in a stable way. Earlier results for recovery of the metric are based on Tataru's unique continuation principle, which yields stability estimates of logarithmic type; see, e.g., [Bosi et al. 2022]. Later these results have been improved by using different techniques and different assumptions. For example, in [Stefanov and Uhlmann 2005] it was shown that a simple Riemannian metric $g$ can be recovered in a Hölder stable way from the DN map. For examples of instability of inverse problems for a wide class of equations; see [Koch et al. 2021].

Concerning the unique recovery of potentials for a linear counterpart of (2) with lower-order terms we mention [Feizmohammadi et al. 2021; Stefanov 1989; Stefanov and Yang 2018]. These works make use of propagation of singularities along bicharacteristics to determine integrals of the unknown coefficients along light rays. In these results, the Dirichlet-to-Neumann or scattering operator needs to be known over all of the lateral boundary $\Sigma$.

Moving on to inverse problems for nonlinear wave equations, Kurylev, Lassas and Uhlmann [Kurylev et al. 2018a] observed that nonlinearity can be used as a beneficial tool in inverse problems for nonlinear wave equations. By exploiting the nonlinearity, some still unsolved inverse problems for linear hyperbolic equations have recently been solved for their nonlinear counterparts. The first results in [Kurylev et al. 2018a], for the scalar wave equation with a quadratic nonlinearity, already showed that local measurements of solutions of the nonlinear wave equation determine the global topology, differentiable structure and the conformal class of the metric $g$ on a globally hyperbolic $(3+1)$-dimensional Lorentzian manifold. The results of [Kurylev et al. 2018a] use the so-called *higher-order linearization method*, which has made inverse problems for nonlinear equations more approachable. The method has given rise to many new results on inverse problems for nonlinear equations. We will explain the method later in this Introduction.

The authors of [Lassas et al. 2018] studied inverse problems for general semilinear wave equations on Lorentzian manifolds, and in [Lassas et al. 2017] they studied the analogous problem for the Einstein–Maxwell equations. The papers [Hintz et al. 2022a; 2022b] are closely related to this work. They use the higher-order linearization method to study uniqueness for the inverse problem of (2). However, these works have additional assumptions that the domain $\Omega$ of the time cylinder $[0, T] \times \Omega$ is convex and that light-like geodesics can only intersect once. These conditions are removed in the present work. Our results will in particular improve results in [Hintz et al. 2022b].

The research of inverse problems for nonlinear equations is expanding fast. By using the higher-order linearization method, inverse problems for nonlinear models have been studied for example in [Balehowsky et al. 2022; Cârstea et al. 2019; Chen et al. 2021; 2022; de Hoop et al. 2019; 2020; Feizmohammadi and Oksanen 2020; 2022; Kang and Nakamura 2002; Krupchyk and Uhlmann 2020a; 2020b; Kurylev et al. 2022; Lai et al. 2021; Lassas et al. 2021a; 2021b; Oksanen et al. 2024; Sun and Uhlmann 1997; Uhlmann and Wang 2020; Wang and Zhou 2019].

**1.1.** ***Main results.*** The present work is a continuation of [Lassas et al. 2022] to the setting of globally hyperbolic Lorentzian manifolds. In that work we considered a stability result for a recovery of the potential $q$ of (2) in $\mathbb{R}^{n+1}$. We denote by $(N, g)$ a globally hyperbolic manifold. We assume that the dimension of $N$ is $n+1$, where $n \geq 2$. As explained earlier, we view $N$ as the product manifold $\mathbb{R} \times M$ equipped with the product metric (1) and where $M$ is an $n$-dimensional manifold. For $T > 0$, we fix a time-interval $[0, T]$. We assume that $\Omega \subset M$ is an $n$-dimensional submanifold of $M$ and that $\Omega$ has a smooth nonempty boundary $\partial\Omega$.

The finite propagation speed of solutions to the wave equation and the causal structure of $(N, g)$ cause natural limitations on the parts of $[0, T] \times \Omega$ where we can obtain information about the potential in the inverse problem. Let $W$ be a compact set belonging to both the chronological future $I^+(\Sigma)$ and past $I^-(\Sigma)$ of the lateral boundary $\Sigma = [0, T] \times \partial\Omega$:

$$W \subset I^-(\Sigma) \cap I^+(\Sigma) \cap ([0, T] \times \Omega). \tag{3}$$

(See Section 1.2 for the definitions of $I^\pm(\Sigma)$ and other basic Lorentzian geometry concepts.) This is the domain which can be reached by sending waves from $\Sigma$ so that the possible signals generated by a nonlinear interaction of the waves can also be detected on $\Sigma$. We do not assume that $[0, T] \times \partial\Omega$ is convex or that light-like geodesics of $(N, g)$ can only intersect once.

Below we use the notation $H_0^s$ for the closure of the space of compactly supported smooth functions, with respect to the Sobolev $H^s$ norm. The main result of this work is the following:

**Theorem 1** (stability estimate). *Suppose $(N,g)$, $N=\mathbb{R}\times M$, is an $(n+1)$-dimensional globally hyperbolic Lorentzian manifold. Let $T>0$ and let $\Omega\subset M$ be a submanifold with smooth nonempty boundary. Let $m\geq 4$ be an integer, $s\in\mathbb{N}$ with $s+1>\frac{n+1}{2}$ and $r\in\mathbb{R}$ with $r\leq s$. Let $j=1,2$. Assume that $q_j\in C^{s+1}(\mathbb{R}\times\Omega)$ satisfy $\|q_j\|_{C^{s+1}}\leq c$, $j=1,2$, for some $c>0$. Let $\Lambda_j: H_0^{s+1}(\Sigma)\to H^r(\Sigma)$ be the corresponding Dirichlet-to-Neumann maps of the nonlinear wave equation (2).*

*Let $\varepsilon_0>0$, $L>0$ and $\delta\in(0,L)$ be such that*

$$\|\Lambda_1(f)-\Lambda_2(f)\|_{H^r(\Sigma)}\leq\delta$$

*for all $f\in H_0^{s+1}(\Sigma)$ with $\|f\|_{H^{s+1}(\Sigma)}\leq\varepsilon_0$. Then there exists a constant $C>0$, independent of $q_1,q_2$ and $\delta>0$, such that*

$$\|q_1-q_2\|_{L^\infty(W)}\leq C\delta^{\sigma(s,m)}, \tag{4}$$

*where*

$$\sigma(s,m)=\frac{8(m-1)}{2m(m-1)(8s-n+13)+2m-1}.$$

A corollary of the theorem is a uniqueness result, which improves the main result of [Hintz et al. 2022b] by allowing nonconvex boundary and light-like geodesics to intersect more than once.

**Corollary 2** (uniqueness). *Adopt the notation and assumptions of Theorem 1. Then the Dirichlet-to-Neumann map $\Lambda$ uniquely determines the potential $q$ within the set $W$.*

We only consider the case $m\geq 4$ in this work as the other natural cases $m=2$ or $m=3$ would lead to additional considerations. The reason is that our method leads to a density problem for products of $m+1$ solutions of the wave equation. The solutions we use do not yield density in the case $m=2$, and not even in the case $m=3$, when light-like geodesics can intersect several times. We mention that the authors of [Hintz et al. 2022b] needed to use different types of solutions in their uniqueness proof when $m=2$ than in the cases $m\geq 3$. We expect that both the cases $m=2$ and $m=3$ can be handled by a method developed in [Feizmohammadi et al. 2023] for an elliptic equation with quadratic nonlinearity transferred to the current hyperbolic setting. We consider the cases $m=2,3$ in a future work.

We explain next how our results are proved and how we are able to consider nonconvex boundaries and the case where light-like geodesics can intersect more than once.

**1.2. *Sketch of the proof of Theorem 1.*** Let us discuss the main ideas behind the proof of Theorem 1. We first discuss how to recover $q$ uniquely from the DN map $\Lambda$ associated with (2). To avoid technical details, the presentation here is slightly formal. We also only consider here the case $m=4$ for simplicity, while the case $m>4$ is similar.

We first recall some notation and definitions in Lorentzian geometry following the books [Beem et al. 1996; O'Neill 1983]. Let $(N,g)$ be a Lorentzian manifold. A smooth path $\mu:(a,b)\to N$ is said to be time-like if $g(\dot\mu(s),\dot\mu(s))<0$ for all $s\in(a,b)$. The path $\mu$ is causal if $g(\dot\mu(s),\dot\mu(s))\leq 0$ and $\dot\mu(s)\neq 0$ for all $s\in(a,b)$. For $p,q\in N$ we write $p\ll q$ if $p\neq q$ and there is a future-pointing time-like path from

$p$ to $q$. Similarly, $p < q$ if $p \neq q$ and there is a future-pointing causal path from $p$ to $q$, and $p \leq q$ when $p = q$ or $p < q$. The chronological future of $p \in N$ is the set $I^+(p) = \{q \in N \mid p \ll q\}$ and the causal future of $p$ is $J^+(p) = \{q \in N \mid p \leq q\}$. The chronological past $I^-(q)$ and causal past $J^-(q)$ of $q \in N$ are defined similarly. If $A \subset N$, then we define $J^\pm(A) = \bigcup_{p \in A} J^\pm(p)$. The sets $I^\pm(p)$ are always open. If $(N, g)$ is in addition globally hyperbolic, then the sets $J^\pm(p)$ are closed, and the sets $I^\pm(p)$ and $J^\pm(p)$ are related by $\mathrm{cl}(I^\pm(p)) = J^\pm(p)$; see [O'Neill 1983, Lemmas 14.6 and 14.22]. Finally, a geodesic from $p \in N$ with initial direction $\xi \in T_pN$ is denoted by $\gamma_{p,\xi}(t) = \exp_p(t\xi)$.

Consider $f_j \in H_0^{s+1}(\Sigma)$, $j = 1, 2, 3, 4$, with $\|f_j\|_{H^{s+1}(\Sigma)} \leq c_0$ for some constant $c_0 > 0$. Let us denote by $u_{\varepsilon_1 f_1 + \cdots + \varepsilon_4 f_4}$ the solution to (2) with boundary data $\varepsilon_1 f_1 + \cdots + \varepsilon_4 f_4$, where $\varepsilon_j > 0$ are sufficiently small parameters. We abbreviate the notation by writing $\vec{\varepsilon} = 0$ when referring to $\varepsilon_1 = \cdots = \varepsilon_4 = 0$. By taking the mixed derivative $\partial^4_{\varepsilon_1 \cdots \varepsilon_4}|_{\vec{\varepsilon}=0}$ of the solution $u_{\varepsilon_1 f_1 + \cdots + \varepsilon_4 f_4}$ to (2) with respect to the parameters $\varepsilon_1, \ldots, \varepsilon_4$, we see that the function

$$w := \frac{\partial}{\partial \varepsilon_1} \cdots \frac{\partial}{\partial \varepsilon_4}\bigg|_{\vec{\varepsilon}=0} u_{\varepsilon_1 f_1 + \cdots + \varepsilon_4 f_4}$$

solves the equation

$$\square_g w = -16 q v_1 v_2 v_3 v_4 \quad \text{in } [0, T] \times \Omega \tag{5}$$

with vanishing Cauchy and boundary data. Here the functions $v_j$, $j = 1, \ldots, 4$, satisfy

$$\begin{cases} \square_g v_j = 0 & \text{in } [0, T] \times \Omega, \\ v_j = f_j & \text{on } [0, T] \times \partial\Omega, \\ v_j|_{t=0} = \partial_t v_j|_{t=0} = 0 & \text{in } \Omega. \end{cases} \tag{6}$$

This way we have produced new linear equations from the nonlinear equation (2). If the DN map $\Lambda$ is known, then the normal derivative of $w$ is also known on $\Sigma$. This is true, because

$$\partial_\nu w = \partial^4_{\varepsilon_1 \cdots \varepsilon_4}|_{\vec{\varepsilon}=0} \Lambda(\varepsilon_1 f_1 + \cdots + \varepsilon_4 f_4).$$

Let $v_0$ be an auxiliary smooth function solving $\square_g v = 0$ in $[0, T] \times \Omega$, with $v_0|_{t=T} = \partial_t v_0|_{t=T} = 0$ in $\Omega$. The function $v_0$ will compensate for the fact that $\partial_\nu w$ is known only on the lateral boundary $\Sigma$, but not on $\{t = T\}$. The normal derivative $\partial_\nu w$ is known on $\{t = 0\}$ due to the initial conditions. Multiplying (5) by $v_0$ and integrating by parts on $[0, T] \times \Omega$, we arrive at the useful integral identity

$$\begin{aligned} \int_\Sigma v_0 \partial^4_{\varepsilon_1 \cdots \varepsilon_4}|_{\vec{\varepsilon}=0} \Lambda(\varepsilon_1 f_1 + \cdots + \varepsilon_4 f_4)\, dS &= \int_{[0,T]\times\Omega} v_0 \square_g w \, dV_g \\ &= -16 \int_{[0,T]\times\Omega} q v_0 v_1 v_2 v_3 v_4 \, dV_g. \end{aligned} \tag{7}$$

This means that the quantity

$$\int_{[0,T]\times\Omega} q v_0 v_1 v_2 v_3 v_4 \, dV_g \tag{8}$$

is known from the knowledge of the DN map $\Lambda$. Since the functions $v_j$, $j = 1, \ldots, 4$, were arbitrary solutions to (6), we are able to choose suitable solutions $v_j$ so that the products of the form $v_0 v_1 v_2 v_3 v_4$ become dense in $L^1([0, T] \times \Omega)$. This recovers the potential $q$ uniquely. The procedure we have now explained results in new equations, and an integral identity relating the DN map and the unknown $q$, by differentiating solutions to the nonlinear equation (2) depending on several parameters. This procedure in general is called the *higher-order linearization method*.

The earlier work [Lassas et al. 2022] by the authors studied an analogous stability problem in the Minkowski space. There $v_j$ were chosen to be approximate plane waves so that the product $v_1v_2v_3v_4$ in the integral (8) essentially becomes a delta function of a hyperplane. Hence the integral (8) in that work became the Radon transformation of $qv_0$ in $\mathbb{R}^n$. Since the Radon transformation in $\mathbb{R}^n$ is invertible, this recovered $q$. In 1+1 dimensions, the integral (8) becomes an integral of $qv_0$ against a delta distribution, in which case the recovery of pointwise values of $qv_0$ is trivial. The auxiliary function $v_0$ in the product $qv_0$ can be eliminated by choosing $v_0$ suitably.

Motivated by the above explanation, in the present work we shall consider the so-called *Gaussian beam* solutions $v_j$ to (6). One can think of Gaussian beams as wave packets traveling on light-like geodesics. In Sections 3 and 5 we will show that by using the nonlinearity of (2) and Gaussian beams, one can produce *approximate delta distributions* from the product $v_1v_2v_3v_4$ in (8). This uses the fact that Gaussian beams are solutions to the linear wave equation (6) with exponential concentration to a neighborhood of a given light-like geodesics up to a small error term. Thus, if two different geodesics intersect, then the product of the corresponding Gaussian beams concentrates near the intersection points of the geodesics. The product of four, instead of two, Gaussian beams is required to cancel oscillations of the product of the solutions. (If oscillations would not be canceled, one would expect not to be able to recover $q$ due to the nonstationary phase.)

Let us explain how we use four Gaussian beams in (7) in more detail. Let us consider $p_0 \in W \subset I^-(\Sigma) \cap I^+(\Sigma) \cap ([0,T] \times \Omega)$. We show that there exist two different geodesics $\gamma_1$ and $\gamma_2$ that pass through $p_0$ and that intersect $\Sigma$ in a suitable manner. We distinguish two cases depending on whether $\gamma_1$ and $\gamma_2$ intersect only once or multiple times. Let us explain first the simpler case, where the geodesics $\gamma_1$ and $\gamma_2$ intersect only at the point $p_0$. Let $v_1$ and $v_2$ be Gaussian beam solutions to (6) with respect to $\gamma_1$ and $\gamma_2$. Making the choice $v_3 = \bar{v}_1$ and $v_4 = \bar{v}_2$ yields $v_1v_2v_3v_4 = |v_1|^2|v_2|^2$. Evaluating this product, one finds that the product $|v_1|^2|v_2|^2$ is an approximation of the delta distribution concentrated at $p_0$. Therefore, by using the integral identity (7) for this specific product $v_1v_2v_3v_4$, and the knowledge of the DN map, we can recover $qv_0$ at $p_0$. We take $v_0$ to be another Gaussian beam that is nonzero at $p_0$. This way we have recovered $q$ at $p_0$. Repeating the argument for all points of $W$ recovers $q$ on $W$.

Suppose next that $\gamma_1$ and $\gamma_2$ intersect at points $x_1 \le \cdots \le x_P$, $P \ge 2$. Using arguments similar to those above, the integral (8) reduces to an integral of $qv_0$ against a sum of approximate delta functions located at the intersection points $x_1, \dots, x_P$. That is, by using (7), we know from the DN map $\Lambda$ the quantity

$$\sum_{k=1}^{P} q(x_k)v_0(x_k) \tag{9}$$

up to an error, which can be made arbitrarily small by taking a parameter associated to the Gaussian beams large enough. The task is then to decouple the information about $qv_0$ at each single point $x_k$ from the sum above.

To decouple the information, the choice of $v_0$ plays a crucial role. Recall that the only requirement from $v_0$ was that it satisfies the wave equation $\Box_g v_0 = 0$ with Cauchy data vanishing at $t = T$. We show that there is a family $(v_0^{(k)})_{k=1}^P$ of $P$ functions, satisfying the required conditions for $v_0$, with the property that the matrix

$$\mathcal{V} := \begin{pmatrix} v_0^{(1)}(x_1) & v_0^{(1)}(x_2) & \cdots & v_0^{(1)}(x_P) \\ v_0^{(2)}(x_1) & v_0^{(2)}(x_2) & \cdots & v_0^{(2)}(x_P) \\ \vdots & & \ddots & \vdots \\ v_0^{(P)}(x_1) & v_0^{(P)}(x_2) & \cdots & v_0^{(P)}(x_P) \end{pmatrix}$$

is invertible. Thus, by using (9) for each $v_0^{(k)}$ in place of $v_0$ separately we know the quantity

$$\mathcal{V} \begin{pmatrix} q(x_1) \\ \vdots \\ q(x_P) \end{pmatrix}$$

from the DN map $\Lambda$. Since $\mathcal{V}$ is a known invertible matrix, this uniquely recovers the values of the unknown potential $q$ at the points $x_1, \dots, x_P$. We explain in Section 1.3 the idea of how the matrix $\mathcal{V}$ is constructed, while complete statements and proofs about the matter are in Section 5.6. The matrix $\mathcal{V}$ is called a *separation matrix*.

So far, we have sketched the proof of unique recovery of $q$ from the DN map $\Lambda$ associated with (2). We briefly discuss how to quantify the uniqueness result and thus to prove a stability estimate. To obtain a stability estimate for $q$ in terms of $\Lambda$, instead of differentiating (2) with respect to $\varepsilon_1, \dots, \varepsilon_4$, we take the mixed finite difference $D^4_{\varepsilon_1 \cdots \varepsilon_4}$ of $u_{\varepsilon_1 f_1 + \cdots + \varepsilon_2 f_4}$ at $\vec{\varepsilon} = 0$. (Recall that $\vec{\varepsilon} = 0$ stands for $\varepsilon_1 = \cdots = \varepsilon_4 = 0$.) In this case, we obtain a slightly different version of the integral identity (7) given by

$$-16 \int_{[0,T]\times\Omega} q v_0 v_1 v_2 v_3 v_4 \, dV_g = \int_\Sigma v_0 D^4_{\varepsilon_1 \cdots \varepsilon_4}\Big|_{\vec{\varepsilon}=0} \Lambda(\varepsilon_1 f_1 + \cdots + \varepsilon_4 f_4) \, dS + \frac{1}{\varepsilon_1 \cdots \varepsilon_4} \int_{[0,T]\times\Omega} v_0 \Box_g \widetilde{\mathcal{R}} \, dV_g .$$

Here the second integral on the right is a small error term, where $\widetilde{\mathcal{R}}$ is of the size $\mathcal{O}(\langle \varepsilon_1, \dots, \varepsilon_4 \rangle^7)$ in an energy space norm. For details, see (11) and (23)–(24). Here we also denote by $\langle \varepsilon_1, \dots, \varepsilon_4 \rangle^7$ an unspecified homogeneous polynomial of order 7 in $\varepsilon_1, \dots, \varepsilon_4$. If $p_0 \in W$ is fixed, a stability result for $q$ at $p_0$ follows by using Gaussian beams associated to the light-like geodesics $\gamma_1$ and $\gamma_2$ described above, optimizing with respect to the parameters $\varepsilon_1, \dots, \varepsilon_4$ and the parameters related to the Gaussian beams $v_1$, $v_2$, $v_3$ and $v_4$. The implied constant of the stability estimate at the fixed-point estimate depends on $p_0$. To show that the constant can in fact be taken to be independent of $p_0$ we must vary the geodesics $\gamma_1$ and $\gamma_2$ and the corresponding Gaussian beams smoothly. This requires some work, which is done in Section 3. In addition, we must also use different separation matrices for different points in $W$. These separation matrices will be constructed with respect to a suitable finite collection of solutions to $\Box_g v = 0$. The finite collection will called a *separation filter*, which is explained in the next section.

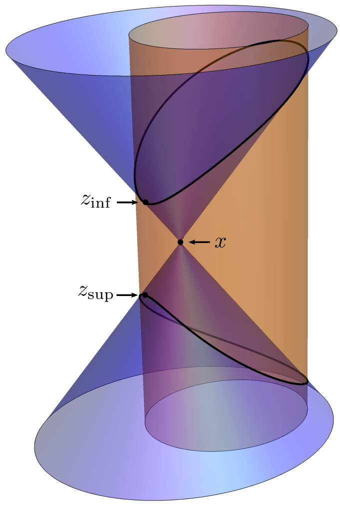


**Figure 1.** The lateral boundary $\Sigma$ (orange cylinder) intersects the lightcone (blue cone) of a point $x$ (apex of the cone) along the black curves. The point $z_{\text{sup}}$ is the latest and $z_{\text{inf}}$ the earliest point on $\Sigma$ which can be reached from $x$ by an optimal geodesics. We call these optimal geodesics boundary optimal geodesics.

**1.3.** ***Lorentzian geometry tools.*** To prove our main results, we make some constructions in Lorentzian geometry. The main constructions we develop are *boundary optimal geodesics* and *separation matrices*. We explain briefly what these are next. Since we expect the constructions to have applications in related inverse problems as well, and they might also be of interest in Lorentzian geometry in general, this section is written to be independent of the inverse problem we consider. We follow the terminology of the book [O'Neill 1983], and we have included the used concepts of causality in Section 1.4 for an easy access.

***Boundary optimal geodesics.*** Let us first explain what is a boundary optimal geodesic. As before we consider the subset $[0,T]\times\Omega$ of a globally hyperbolic smooth Lorentzian manifold $\mathbb{R}\times M$, $\dim(M)=n\geq 2$, equipped with the metric (1) and where $\Omega$ is a smooth submanifold of $M$ with boundary and of dimension $n$. The lateral boundary $\Sigma$ refers to the set $[0,T]\times\partial\Omega$ as before. As is by now quite standard, see, e.g., [Kurylev et al. 2018a; O'Neill 1983], we say that a geodesic connecting the points $x, y \in N$, $x \leq y$, is optimal if the time separation function $\tau$ of these points vanishes, $\tau(x,y)=0$. The time separation function is the supremum of lengths of piecewise smooth future-directed causal paths from $x$ to $y$; see (49) or [O'Neill 1983] for details. An optimal geodesic is always light-like.

Let us then consider a point $x \in I^-(\Sigma)\cap([0,T]\times\Omega)$. In the inverse problem of this paper, we consider Gaussian beams that vanish on a neighborhood of $\{t=T\}$. For this, it is required to find past-directed light-like geodesics of $[0,T]\times\Omega$ from $\Sigma$ to $x\in[0,T]\times\Omega$, which do not intersect the set $\{t=T\}$. In Lemma 15, we show that we may find a point $z_{\text{inf}}$ of the lateral boundary $\Sigma$ and an optimal past-directed geodesic $\gamma$ from $z_{\text{inf}}$ to $x$. The situation is illustrated in Figure 1. In the figure, the point $z_{\text{inf}}\in\Sigma$, is the point which has the smallest time coordinate in the intersection of the light-like future of $x$ (the upper cone) and $\Sigma$. The light-like geodesic $\gamma$ from $z_{\inf}$ to $x$ is not only optimal, i.e., $\tau(x, z_{\inf}) = 0$, but it also necessarily intersects $\Sigma$ transversally even if $\Sigma$ would be nonconvex. We call the geodesic $\gamma$ a boundary optimal geodesic.

Note that by deforming $\Sigma$ in the figure to a nonconvex manifold, it is possible to find optimal geodesics from $x$ to points in $\Sigma$, which intersect $\Sigma$ tangentially. Therefore, not all optimal geodesics are boundary optimal geodesics. Similarly, for $x \in I^+(\Sigma) \cap ([0, T] \times \Omega)$, we also prove in Lemma 15 that we may find a future-directed boundary optimal geodesic from $z_{\sup} \in \Sigma$ to $x$ also presented in Figure 1.

We remark that in inverse problems related to the one studied in this paper, convexity of the lateral boundary is assumed to have light-like geodesics that intersect the boundary transversally; see, e.g., [Hintz et al. 2022b]. By using boundary optimal geodesics of this paper, the convexity assumption in that work can be dropped. We expect this to be true also in related inverse problems.

We make the notion of boundary optimal geodesics precise in the form of the following definition. Below, the time coordinate, or the time function, of $N$ is $t$.

**Definition 3** (boundary optimal geodesic). Let $(N, g)$ be globally hyperbolic, $N = \mathbb{R} \times M$, $\Omega \subset M$ a manifold with boundary and $\Sigma = [0, T] \times \partial\Omega$. We call a geodesic $\gamma : [0, 1] \to [0, T] \times \Omega$ a past-directed boundary optimal geodesic to $x \in J^-(\Sigma)$ if

(1) $\gamma(0) \in \Sigma$ and $\gamma(1) = x$,

(2) the time coordinate of $\gamma(0)$ equals

$$t_{\inf} = \inf\{\tilde{d} \in [0, T] \mid \text{there is } \tilde{z} \in \Sigma \text{ such that } t(\tilde{z}) = \tilde{d} \text{ and } \tau(x, \tilde{z}) > 0\},$$

(3) $\gamma$ is an optimal geodesic connecting the points $x$ and $\gamma(0)$.

Similarly, we call $\gamma$ a future-directed boundary optimal geodesic to $x \in J^+(\Sigma)$ if the time coordinate of $\gamma(0)$ equals instead

$$t_{\sup} = \sup\{\tilde{d} \in [0, T] \mid \text{there is } \tilde{z} \in \Sigma \text{ such that } t(\tilde{z}) = \tilde{d} \text{ and } \tau(\tilde{z}, x) > 0\}.$$

We refer to both past- and future-directed boundary optimal geodesics to $x$ respectively belonging to $J^-(\Sigma)$ and $J^+(\Sigma)$ collectively as boundary optimal geodesics.

**Remark 4.** Boundary optimal geodesics are related to a recently introduced concept of null distance [Allen and Burtscher 2022; Sormani and Vega 2016]. A null distance turns a Lorentzian manifold admitting a suitable time function into a metric space in a conformally invariant way. In particular, a globally hyperbolic manifold $N$ becomes a metric space with a metric $d : N \times N \to [0, \infty)$. We wish to state here the following facts, even though we do not use them.

If $\gamma$ is a boundary optimal geodesic connecting $z \in \Sigma$ to $x$, then $|t(x) - t(z)| = d(x, z)$. Moreover, a boundary optimal geodesic minimizes the distance between $x$ and its future causal lateral boundary $\Sigma \cap J^+(x)$ in the sense that

$$d(\Sigma \cap J^+(x), x) = d(z_{\inf}, x),$$

where $z_{\inf} \in \Sigma \cap J^+(x)$ is the starting point of a past-directed boundary optimal geodesic to $x$. We have similarly for the past causal lateral boundary $\Sigma \cap J^-(x)$. In this sense, boundary optimal geodesics are an analogue to Riemannian geodesics that minimize the distance to a boundary.

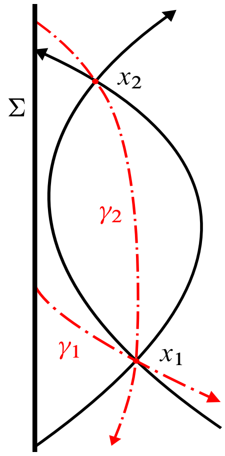


**Figure 2.** Past-directed light-like geodesics (red dashed lines) that separate the intersection points $x_1$ and $x_2$ of future-directed light-like geodesics (black). The geodesics in red and black intersect $\Sigma$ at times $t < T$ and $t > 0$ respectively.

***Separation matrices.*** Having explained what optimal and boundary optimal geodesics are, we are ready to present what a separation matrix is and how it is constructed.

**Definition 5** (separation matrix). Let $x_1, \dots, x_P \in [0, T] \times \Omega$ and $v_1, \dots, v_P$ be solutions to $\square_g v = 0$ in $[0, T] \times \Omega$. If the matrix

$$\begin{pmatrix} v_1(x_1) & v_2(x_1) & \cdots & v_P(x_1) \\ v_1(x_2) & v_2(x_2) & \cdots & v_P(x_2) \\ \vdots & & \ddots & \vdots \\ v_1(x_P) & v_2(x_P) & \cdots & v_P(x_P) \end{pmatrix} \tag{10}$$

is invertible, we call it a separation matrix.

In general, if $x_1, \dots, x_P \in I^-(\Sigma) \cap ([0, T] \times \partial\Omega)$ satisfy $x_1 < \cdots < x_P$ we show in Lemma 17 that there are $P$ solutions $v_k$, $k = 1, \dots, P$, to the wave equation $\square_g v = 0$ whose Cauchy data vanish on $\{t = T\}$ such that the corresponding matrix (10) is invertible and thus a separation matrix.

Let us consider here the simplest nontrivial case $P = 2$ and assume that $x_1, x_2 \in I^-(\Sigma) \cap ([0, T] \times \partial\Omega)$ satisfy $x_1 < x_2$. To construct suitable solutions $v_1$ and $v_2$ in this case, we proceed by first choosing two light-like geodesics as follows. The choice is illustrated in Figure 2, where the points $x_1$ and $x_2$ are the intersection points of the black curves. (In our inverse problem the black curves are also geodesics, but that is not important for the present discussion.) By the discussion above, we may find a boundary optimal geodesic $\gamma_1$ between $x_1$ and $x_{1,\mathrm{inf}} \in \Sigma$ and another boundary optimal geodesic $\gamma_2$ connecting $x_2$ to $\Sigma$. Next we note that if $\gamma_1$ also meets $x_2$, then we can perturb the initial direction of $\gamma_1$ at $x_1$ to have a new light-like geodesic that does not meet $x_2$. Indeed, if the new perturbed geodesic would still meet $x_2$, then it is a fact that there would be a shortcut path from $x_1$ to $\Sigma$ which has positive length. This would contradict the condition $\tau(x_1, x_{1,\mathrm{inf}}) = 0$. We refer to the proof of Lemma 17 for the details. We also note that it is possible that $\gamma_2$ meets $x_1$.

By the above discussion, we have the light-like geodesic $\gamma_1$ from $x_1$ to $\Sigma$ which does not meet $x_2$ and another light-like geodesic from $x_2$ to $\Sigma$. Corresponding to these two geodesics there are respective Gaussian beam solutions $v_1$ and $v_2$ to $\Box_g v = 0$ with vanishing Cauchy data at $\{t = T\}$. By using the properties of Gaussian beams, we know that $v_1$ and $v_2$ are concentrated to small neighborhoods of the corresponding geodesics, respectively. See Section 3 for details. Thus we have for $k, l = 1, 2$ that

$$\begin{aligned} |v_k(x_l)| &\approx 1, \quad k = l, \\ |v_k(x_l)| &\ll 1, \quad k > l, \\ |v_k(x_l)| &\le c_0, \quad k < l, \end{aligned}$$

where $c_0 > 0$ is a constant. Therefore the matrix $\mathcal{V}$ in (10) in this case is approximately a lower triangular matrix with ones on the diagonal. Thus $\mathcal{V}$ is invertible, and hence a separation matrix in our terminology. Vaguely speaking, we can separate points by solutions to the wave equation $\Box_g v = 0$. We mention that a similar condition has been used in the study of inverse problems for elliptic equations in [Guillarmou et al. 2019; Lassas et al. 2020].

Finally, we mention that when proving our stability result in this paper, we can only use finitely many separation matrices. For this, we show that there are finitely many solutions $v$ to $\Box_g v = 0$ with vanishing Cauchy data at $\{t = T\}$ such that the separation matrices made out of these solutions can separate any fixed number of points in $I^-(\Sigma) \cap ([0, T] \times \Omega)$ that are distinct in a precise sense. In the definition below, $\bar{g}$ is an auxiliary Riemannian metric on $[0, T] \times \Omega$.

**Definition 6** (separation filter). Let $K \subset [0, T] \times \Omega$ be compact and $P \in \mathbb{N}$. A finite collection $\mathcal{M} \subset C^\infty([0, T] \times \Omega)$ of solutions to $\Box_g v = 0$ is called a separation filter if the following holds: For any points $x_1, \dots, x_P \in K$ such that $x_1 < x_2 < \cdots < x_P$ and $d_{\bar{g}}(x_k, x_l) > \delta$ for $x_k \neq x_l$, $k, l = 1, \dots, P$, there are $v_1, \dots, v_P \in \mathcal{M}$ such that the matrix $(v_k(x_l))_{k,l=1}^P$ in (53) is invertible (and thus a separation matrix).

In Lemma 18 we show that if $K \subset I^-(\Sigma) \cap I^+(\Sigma) \cap ([0, T] \times \Omega)$, then a separation filter exists.

**1.4.** ***Preliminary definitions.*** The Sobolev spaces $H^s$ on a compact smooth manifold can be defined in several ways (up to equivalent norms). We define Sobolev spaces first on the manifold $N = \mathbb{R} \times M$ using partition of unity on charts; see, e.g., [Hörmander 1983; Roe 1988; Taylor 2011]. Sobolev spaces on the time cylinder $[0, T] \times \Omega$ are then defined by restriction:

$$H^s([0, T] \times \Omega) := \{f|_{[0,T]\times\Omega} \mid f \in H^s(\mathbb{R} \times M)\}.$$

As usual, the dual space of $H^r([0, T] \times \Omega)$, $r \ge 0$, is defined as

$$\widetilde{H}^{-r}([0, T] \times \Omega) := \{f \in H^{-r}(\mathbb{R} \times M) \mid \operatorname{supp} f \subset [0, T] \times \overline{\Omega}\}.$$

It is endowed with the norm

$$\|g\|_{\widetilde{H}^{-r}([0,T]\times\Omega)} := \sup \frac{|g(v)|}{\|v\|_{H^r([0,T]\times M)}},$$

where the supremum is over all $v \in H^r([0,T]\times M)$, $v \neq 0$, with $\operatorname{supp} v \subset [0,T]\times\overline{\Omega}$. By the Riesz representation theorem, one can always find $f_0 \in H^r(\mathbb{R}\times M)$ so that for all $v \in H^r(\mathbb{R}\times M)$

$$\|f\|_{\widetilde{H}^{-r}([0,T]\times\Omega)} = \|f_0\|_{H^r(\mathbb{R}\times M)}, \quad f(v) = \langle f_0, v\rangle.$$

Additionally, if $\operatorname{supp} v \subset [0,T]\times\overline{\Omega}$, then we have for all $v \in H^r([0,T]\times M)$ the estimate

$$|f(v)| = |\langle f_0, v\rangle| \le \|f\|_{\widetilde{H}^{-r}([0,T]\times\Omega)}\|v\|_{H^r([0,T]\times\Omega)}.$$

Sobolev spaces of the manifold $\Omega$ with boundary are defined similarly. By the notation $H_0^s$ we mean the closure of the space of compactly supported smooth functions with respect to the Sobolev $H^s$ norm.

***Structure of the paper.*** This paper is organized as follows. In Section 1.1 we present our main results and explain briefly the structure of the proofs. Section 2 studies the forward problem of the nonlinear equation (2). Most of the proofs of Section 2 are included in the Appendix. Section 3 concerns the construction of Gaussian beams in Lorentzian manifolds. In Section 4 we construct the tools of Lorentzian geometry which we use in our inverse problem. This section in particular shows it is possible distinguish different points of a Lorentzian manifold by using solutions to the wave equation. The section introduces the concepts of *boundary optimal geodesics* and *separation matrices*. Finally, in Section 5 we collect the results we have obtained until that point to give a proof for our main theorem. For clarity, the proof is split into several parts.

## 2. Well-posedness of the forward problem

To prove existence of small solutions for the nonlinear wave equation (2), we start by recalling the corresponding results for the linear initial-boundary value problem

$$\begin{cases} \square_g u = F & \text{in } [0,T]\times\Omega, \\ \quad u = f & \text{on } [0,T]\times\partial\Omega, \\ u|_{t=0} = u_0, \ \ \partial_t u|_{t=0} = u_1 & \text{in } \Omega. \end{cases}$$

Let $s \in \mathbb{N}$. Convenient spaces for solutions of the wave equation are called *energy spaces* $E^s$, defined as

$$E^s = \bigcap_{0\le k\le s} C^k([0,T]; H^{s-k}(\Omega)).$$

These spaces are equipped with the norm

$$\|u\|_{E^s} = \sup_{0<t<T} \sum_{0\le k\le s} \|\partial_t^k u(\,\cdot\,,t)\|_{H^{s-k}(\Omega)}. \tag{11}$$

As is the case with the Sobolev spaces $H^s(\Omega)$, the space $E^s$ is an algebra if $s > \frac{n}{2}$ and we have the norm estimate

$$\|uv\|_{E^s} \le C_s\|u\|_{E^s}\|v\|_{E^s} \quad \text{for all } u, v \in E^s.$$

The above facts are well known, see, e.g., [Choquet-Bruhat 2009, Appendix III, Definitions 3.4(2) and 3.5], but for completeness of our presentation, we sketch a proof for them here for the case $s \in \mathbb{N}$. For this, we let $u, v \in E^s$ and show that the pointwise product $uv$ is in $E^s$. Since

$$\|uv\|_{E^s} = \sup_{0<t<T} \sum_{k=0}^{s} \|\partial_t^k (uv)\|_{H^{s-k}(\Omega)},$$

it suffices to show that each term of the form

$$\sup_{0<t<T} \|\partial_t^a u\, \partial_t^b v\|_{H^{s-k}(\Omega)}$$

is finite for $a + b = k$ and for each $k = 0, \dots, s$. By using [Choquet-Bruhat 2009, Appendix III, Definition 3.4(2)] or [Behzadan and Holst 2021, Corollary 6.3 or Theorem 7.4], we see that when $s_1, s_2 \geq s \geq 0$ and $s_1 + s_2 > s + \frac{n}{2}$ the following multiplication property holds in Lipschitz domains:

$$H^{s_1}(\Omega) \times H^{s_2}(\Omega) \subset H^s(\Omega).$$

Since $u, v \in E^s$ we find $\partial_t^a u \in H^{s-a}(\Omega)$ and $\partial_t^a v \in H^{s-b}(\Omega)$ for all fixed $t \in [0, T]$ and the implied norms are uniformly bounded in $t$. We have $s-a,\ s-b \geq s-k \geq 0$ and $(s-a)+(s-b) > (s-k)+\frac{n}{2}$, since $s > \frac{n}{2}$ and $a + b = k$. This implies $\partial_t^a u \partial_t^b v \in H^{s-k}(\Omega)$ for all $t \in [0, T]$ with the implied norm uniformly bounded in $t$ as required.

**Remark 7.** We note that $E^s \subset H^s([0, T] \times \Omega)$. Conversely, due to the standard Sobolev embedding $H^s([0, T] \times \Omega) \subset C^k([0, T] \times \Omega)$, when $s > k + \frac{n+1}{2}$, we have that $H^{s'}([0, T] \times \Omega) \subset E^s$, when $s' > s + \frac{n+1}{2}$. In particular,

$$\|u\|_{H^s([0,T]\times\Omega)} \lesssim \|u\|_{E^s} \lesssim \|u\|_{H^{s'}([0,T]\times\Omega)}. \tag{12}$$

For the wave equations we consider, we need to assume certain compatibility conditions between the boundary values and the initial data. The compatibility conditions for (2) to order 2 are given by

$$\begin{aligned} f|_{t=0} = u_0|_{\partial\Omega}, \quad \partial_t f|_{t=0} = \partial_t u|_{\{0\}\times\partial\Omega} = u_1|_{\partial\Omega}, \\ \partial_t^2 f|_{t=0} = \partial_t^2 u|_{\{0\}\times\partial\Omega} = \beta^{-1}|_{\{0\}\times\partial\Omega}(\Delta_h u_0|_{\partial\Omega} + F|_{\{0\}\times\partial\Omega}). \end{aligned} \tag{13}$$

Here the smooth function $\beta$ and $g$ are related by (1). The compatibility conditions up to general order $s$ are obtained by setting $\partial_t^k f|_{t=0} = \partial_t^k u|_{\{0\}\times\partial\Omega}$, for $k = 0, \dots, s$, and then solving for $\partial_t^k u|_{\{0\}\times\partial\Omega}$ in terms of the initial data by using the equation $\Box_g u = F$. These conditions guarantee that at the boundary $\partial\Omega$ the initial data $(u_0, u_1)$ is compatible with the corresponding boundary condition $f$. These conditions have been discussed for example in [Katchalov et al. 2001, Section 2.3.7] in the simpler case where the metric is time-independent. Especially, if $\partial_t^k f|_{t=0} = 0$ for all $k = 0, \dots, s$, or if $f$ is supported away from the Cauchy surface $\{t = 0\}$, and $F \equiv 0$ and $u_0 \equiv u_1 \equiv 0$, then the compatibility conditions of order $s$ hold.

**Proposition 8** (existence and estimates for the linear equation [Ikawa 1968; Lasiecka et al. 1986]). *Assume that $(\mathbb{R}\times M, g)$ is a globally hyperbolic Lorentzian manifold as in (1) and $\Omega \subset M$ is a compact submanifold with nonempty boundary. Let $s \in \mathbb{N}$ be a positive integer and assume that $F \in E^s$, $f \in H^{s+1}(\Sigma)$,*

*$u_0 \in H^{s+1}(\Omega)$ and $u_1 \in H^s(\Omega)$ satisfy the compatibility conditions. Then the equation*

$$\begin{cases} \Box_g u = F & \text{in } [0,T]\times\Omega, \\ u = f & \text{on } \Sigma, \\ u = u_0, \quad \partial_t u = u_1 & \text{in } \{t=0\}\times\Omega \end{cases} \tag{14}$$

*has a unique solution $u \in E^{s+1}$ satisfying*

$$\|u\|_{E^{s+1}} \le C(\|F\|_{E^s} + \|f\|_{H^{s+1}(\Sigma)} + \|u_0\|_{H^{s+1}(\Omega)} + \|u_1\|_{H^s(\Omega)}) \tag{15}$$

*and $\partial_\nu u|_\Sigma \in H^s(\Sigma)$.*

As we could not find a proof for Proposition 8 in general for globally hyperbolic Lorentzian manifolds, we have included one in the Appendix. The energy estimates of the linear problem (14) directly allow us to conclude that the nonlinear problem (2) has a unique small solution in $E^{s+1}$. The proof of the following lemma is similar to the one in [Lassas et al. 2022, Proof of Lemma 1, Appendix A]. We omit the proof.

**Lemma 9.** *Let $m \ge 2$ be an integer and $\Omega \subset M$ be a compact submanifold, $\dim(\Omega) = \dim(M)$, with nonempty boundary. Assume $s \in \mathbb{N}$ is such that $s+1 > \frac{n+1}{2}$. Suppose that $q \in C^{s+1}([0,T]\times\Omega)$ satisfies the a priori bound $\|q\|_{C^{s+1}} \le c$ for some $c > 0$. Then there are $\kappa > 0$ and $\rho > 0$ such that if $f \in H^{s+1}(\Sigma)$ satisfies $\|f\|_{H^{s+1}(\Sigma)} \le \kappa$, and $\partial_t^\alpha f|_{t=0} = 0$ for all $\alpha = 0, \dots, s$ on $[0,T]\times\partial\Omega$, then there is a unique solution to*

$$\begin{cases} \Box_g u + q u^m = 0 & \text{in } [0,T]\times\Omega, \\ u = f & \text{on } [0,T]\times\partial\Omega, \\ u|_{t=0} = \partial_t u|_{t=0} = 0 & \text{in } \Omega \end{cases} \tag{16}$$

*in the ball*

$$B_\rho(0) := \{u \in E^{s+1} \mid \|u\|_{E^{s+1}} < \rho\} \subset E^{s+1}.$$

*Furthermore, the solution satisfies the estimate*

$$\|u\|_{E^{s+1}} \le C_0 \|f\|_{H^{s+1}(\Sigma)},$$

*where $C_0 > 0$ is a constant independent of $f$ and $q$.*

If the boundary data of the nonlinear equation (16) depends on small parameters, we may expand the corresponding solution $u$ in terms of the small parameters. Indeed, let $\varepsilon_1, \dots, \varepsilon_m > 0$ be small parameters and define

$$\vec{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_m).$$

Consider the boundary value in (16)

$$f(x) = \sum_{j=1}^m \varepsilon_j f_j(x),$$

where $f_j \in H^{s+1}(\Sigma)$, $j = 1, \dots, m$, satisfies the compatibility conditions to order $s$ and $\|f\|_{H^{s+1}(\Sigma)} \le \kappa$ for some $\kappa > 0$. Let us denote in the usual multi-index notation

$$\vec{k} = (k_1, \dots, k_m),$$

where $k_j \in \{0, \dots, m\}$. Then by repeating the proof of Proposition 1 in [Lassas et al. 2022], we find that $u$ can be expanded as

$$u = \sum_{j=1}^{m} \varepsilon_j v_j + \sum_{|\bar{k}|=m} \binom{m}{k_1, \dots, k_m} \varepsilon_1^{k_1} \cdots \varepsilon_m^{k_m} w_{\bar{k}} + \mathcal{R}. \tag{17}$$

The functions $v_j,\ j = 1, \dots, m$, satisfy

$$\begin{cases} \square_g v_j = 0 & \text{in } [0,T] \times \Omega, \\ v_j = f_j & \text{on } [0,T] \times \partial\Omega, \\ v_j|_{t=0} = 0, \quad \partial_t v_j|_{t=0} = 0 & \text{in } \Omega \end{cases} \tag{18}$$

and the functions $w_{\bar{k}}$ satisfy

$$\begin{cases} \square_g w_{\bar{k}} + q v_1^{k_1} \cdots v_m^{k_m} = 0 & \text{in } [0,T] \times \Omega, \\ w_{\bar{k}} = 0 & \text{on } [0,T] \times \partial\Omega, \\ w_{\bar{k}}|_{t=0} = 0, \quad \partial_t w_{\bar{k}}|_{t=0} = 0 & \text{in } \Omega. \end{cases} \tag{19}$$

The remainder $\mathcal{R}$ is bounded in the energy spaces as follows:

$$\begin{aligned} \|\mathcal{R}\|_{E^{s+2}} &\le c(s,T) \|q\|_{E^{s+1}}^2 \left\| \sum_{j=1}^{m} \varepsilon_j f_j \right\|_{H^{s+1}(\Sigma)}^{2m-1}, \\ \|\square \mathcal{R}\|_{E^{s+1}} &\le C(s,T) \|q\|_{E^{s+1}}^2 \left\| \sum_{j=1}^{m} \varepsilon_j f_j \right\|_{H^{s+1}(\Sigma)}^{2m-1}. \end{aligned} \tag{20}$$

By using the expansion formula (17), we will next derive an integral equation which relates the potential $q$ to the DN map $\Lambda$. In general, relating an unknown parameter/function in an inverse problem for a nonlinear equation to a formula for solutions to linear equations is called a *higher-order linearization* method. See for example [Kurylev et al. 2018a; Lassas et al. 2018; 2021b], where solutions are differentiated with respect to small parameters. However, as we are interested in stability of our inverse problem, we need accurate control on the remainder terms. For this reason, following [Lassas et al. 2022], instead of differentiating we use finite differences $D^m_{\vec{\varepsilon}}$. The mixed finite difference of $u$ at $\vec{\varepsilon} = 0$, that is, $\varepsilon_1 = \cdots = \varepsilon_m = 0$, is defined by the formula

$$D^m_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0} u_{\varepsilon_1 f_1 + \cdots + \varepsilon_m f_m} = \frac{1}{\varepsilon_1 \cdots \varepsilon_m} \sum_{\sigma \in \{0,1\}^m} (-1)^{|\sigma|+m} u_{\sigma_1 \varepsilon_1 f_1 + \cdots + \sigma_m \varepsilon_m f_m}, \tag{21}$$

where $u_{\varepsilon_1 f_1 + \cdots + \varepsilon_m f_m}$ is the unique solution to (16) with $f$ replaced by $\varepsilon_1 f_1 + \cdots + \varepsilon_m f_m$. Then the mixed finite difference $D^m_{\vec{\varepsilon}}$ of the solution $u$ of (16) takes the form

$$D^m_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0} u = m!\, w_{1,1,\dots,1} + D^m_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_m}|_{\vec{\varepsilon}=0} \overline{\mathcal{R}}, \tag{22}$$

where $\overline{\mathcal{R}}$ is a sum of the remainders of the solutions $u_{\sigma_1 \varepsilon_1 f_1 + \cdots + \sigma_m \varepsilon_m f_m}$ in (21).

For more details about the finite differences of $u$, we refer the reader to [Lassas et al. 2022, Appendix C].

Let $v_0$ be an auxiliary function solving $\square_g v_0 = 0$ with $v_0|_{t=T} = \partial_t v_0|_{t=T} = 0$ in $\Omega$. By multiplying the DN-map $\Lambda$ associated with (2) by $v_0$ and integrating by parts over $[0, T] \times \Omega$, we obtain

$$\int_\Sigma v_0 D^m_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0} \Lambda(\varepsilon_1 f_1 + \cdots + \varepsilon_m f_m)\, dS = \int_\Sigma v_0 D^m_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0} \partial_\nu u_{\varepsilon_1 f_1 + \cdots + \varepsilon_m f_m}\, dS = m! \int_{[0,T]\times\Omega} v_0 \square_g w_{1,1,\dots,1}\, dV_g + \frac{1}{\varepsilon_1 \cdots \varepsilon_m} \int_{[0,T]\times\Omega} v_0 \square_g \widetilde{\mathcal{R}}\, dV_g.$$

Here we defined

$$\widetilde{\mathcal{R}} := \varepsilon_1 \varepsilon_2 \cdots \varepsilon_m D^m_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0} \overline{\mathcal{R}} \tag{23}$$

and $\widetilde{\mathcal{R}}$ satisfies

$$\begin{aligned} \|\widetilde{\mathcal{R}}\|_{E^{s+2}} &\le c(s, T) \|q\|^2_{E^{s+1}} \sum_{\sigma \in \{0,1\}^m} \|\sigma_1 \varepsilon_1 f_1 + \cdots + \sigma_m \varepsilon_m f_m\|^{2m-1}_{H^{s+1}(\Sigma)}, \\ \|\square \widetilde{\mathcal{R}}\|_{E^{s+1}} &\le C(s, T) \|q\|^2_{E^{s+1}} \sum_{\sigma \in \{0,1\}^m} \|\sigma_1 \varepsilon_1 f_1 + \cdots + \sigma_m \varepsilon_m f_m\|^{2m-1}_{H^{s+1}(\Sigma)}. \end{aligned} \tag{24}$$

We have arrived at the following integral identity which connects the potential $q$ with the DN-map $\Lambda$.

**Integral identity.**

$$-m! \int_{[0,T]\times\Omega} q v_0 v_1 v_2 \cdots v_m\, dV_g = \int_\Sigma v_0 D^m_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0} \Lambda(\varepsilon_1 f_1 + \cdots + \varepsilon_m f_m)\, dS + \frac{1}{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_m} \int_{[0,T]\times\Omega} v_0 \square \widetilde{\mathcal{R}}\, dV_g. \tag{25}$$

Our analysis of the inverse problem is based on this formula.

## 3. Gaussian beams

In this section we record some facts about Gaussian beams. Gaussian beams on a Lorentzian manifold $(N, g)$, $\dim(N) = n + 1 \ge 3$, are approximate solutions to the wave equation $\square_g v = 0$. If $s$ is a geodesic parameter of a light-like geodesic $\gamma : [s_1, s_2] \to N$ and $(s, y)$, $y = (y_1, \dots, y_n) \in \mathbb{R}^n$, are suitable Fermi coordinates (see (26) below) on a neighborhood of the graph $\Gamma$ of $\gamma$, then a Gaussian beam in the coordinates $(s, y)$ looks roughly like

$$e^{i y_1 \tau - a\tau |y|^2},$$

up to a normalization. By graph of $\gamma$ we mean the image set

$$\Gamma := \gamma([s_1, s_2]).$$

Here $a > 0$ and $\tau$ is a large parameter. Therefore, the qualitative behavior of a Gaussian beam is oscillation in a direction $y_1$ transversal to the geodesic $\gamma$ and Gaussian concentration around the graph of $\gamma$.

The construction of Gaussian beams is well known; see, e.g., [Babich et al. 1985; Feizmohammadi and Oksanen 2022; Ralston 1982]. We include details about the construction since we wish to keep track of the constants that will be implicit in our stability estimate of Theorem 1. Our presentation of the construction follows closely [Feizmohammadi and Oksanen 2022, Section 4] to which we refer for omitted details. We mention here the recent work [Krupchyk et al. 2022], which constructs related Gaussian beam quasimodes in a Riemannian setting by using more sophisticated methods, which lead to better estimates.

Fermi coordinates are constructed by inverting the map

$$(s, y) \mapsto \exp_{\gamma(s)}\Big(\sum_{k=1}^{n} y^k e_k(s)\Big) \in N. \tag{26}$$

Here $e_k(s)$ are the parallel transportations along a light-like geodesic $\gamma$ of the last $n$ vectors of a frame $\{e_0, e_1, \dots, e_n\}$ of $T_{\gamma(0)}$ with

$$e_0 = \dot\gamma(0).$$

The other vectors of the frame are chosen so that, for $j, k = 2, \dots n$, it holds

$$g(e_0, e_0) = 0, \quad g(e_1, e_1) = 0, \quad g(e_0, e_1) = -2, \quad g(e_j, e_k) = \delta_{jk}. \tag{27}$$

The frame $\{e_0, e_1, \dots, e_n\}$ is called a pseudo-orthonormal frame. (Due to relation to the usual light-cone coordinates, we could also call it a lightcone frame.) Since the frame $\{e_0(s), e_1(s), \dots, e_n(s)\}$ is the parallel transportation of $\{e_0, e_1, \dots, e_n\}$ along $\gamma$, the conditions (27) hold for $e_j$, $j = 0, \dots, n$, replaced with $e_j(s)$ and $e_0(s) = \dot\gamma(s)$.

We work in the Fermi coordinates described above. In the Fermi coordinates $(s, y)$, the geodesic $\gamma$ corresponds to $(s, 0)$ and the coordinate representation $g|_\gamma = g(s, 0)$ of the metric $g$ restricted to $\gamma$ satisfies

$$g|_\gamma = -2\,ds\,dy_1 + \sum_{k=2}^{n} dy_k\,dy_k.$$

Gaussian beams are constructed by using a WKB ansatz $e^{i\tau\Theta(s,y)}a(s, y)$ to approximately solve the equation $\Box_g v = 0$ in the Fermi coordinates $(s, y)$. We have

$$\Box_g(e^{i\tau\Theta}a) = e^{i\tau\Theta}(\tau^2 g(d\Theta, d\Theta) - 2i\tau g(d\Theta, da) + i\tau(\Box_g\Theta)a + \Box_g a). \tag{28}$$

We will choose a *phase function* $\Theta$ and an *amplitude function* $a$ so that the right-hand side of (28) is $\mathcal{O}(\tau^{-K})$ in $H^k([0, T]\times\Omega)$ for given $k \ge 0$ and $K \in \mathbb{N}$. To do so, we first approximately solve the eikonal equation

$$g(d\Theta, d\Theta) = 0. \tag{29}$$

After finding an (approximate) solution $\Theta$ to the eikonal equation, we equate the last three terms of (28) by inserting $\Theta$ into

$$-2i\tau g(d\Theta, da) + i\tau(\Box_g\Theta)a + \Box_g a = 0.$$

By assuming an expansion of the form

$$a = a_0 + \tau^{-1}a_1 + \tau^{-2}a_2 + \dots + \tau^{-N}a_N$$

for the amplitude $a$, where $N \in \mathbb{N}$ is to be chosen later, we are led by equating the powers of $\tau$ to a family of $N+1$ equations

$$-2i g(d\Theta, da_0) + i(\Box_g \Theta)a_0 = 0, \tag{30}$$

$$-2i g(d\Theta, da_j) + i(\Box_g \Theta)a_j - \Box_g a_{j-1} = 0, \tag{31}$$

$j = 1, \dots, N$. We solve these equations approximately and recursively in $j$ starting from $a_0$. Equations (30) and (31) are called transport equations.

In what follows, we refer to [Feizmohammadi and Oksanen 2022] for omitted details. To solve the eikonal equation (29) approximately, one sets

$$\Theta = \sum_{j=0}^{N} \Theta_j(s, y),$$

where $\Theta_j(s, y)$ is a homogeneous polynomial of order $j$ in $y \in \mathbb{R}^n$. We say that $g(d\Theta, d\Theta)$ vanishes to order $N$ on $\Gamma$, or that $g(d\Theta, d\Theta) = 0$ is satisfied to order $N$ on $\Gamma$, if

$$(\partial_y^\alpha g(d\Theta, d\Theta))(s, 0) = 0,$$

where $\alpha$ is any multi-index with $|\alpha| \le N$. We set

$$\Theta_0 = 0 \text{ and } \Theta_1 = y_1. \tag{32}$$

It follows that

$$g(d\Theta, d\Theta)(s, 0) = 0 \quad \text{and} \quad (\partial_{y_l} g(d\Theta, d\Theta))(s, 0) = 0,$$

where $l = 1, \dots, n$. That is, the eikonal equation (29) is satisfied to order 1 on $\Gamma$. The conditions (32) imply the invariantly written conditions

$$\Theta(\gamma(s)) = 0 \quad \text{and} \quad \nabla\Theta(\gamma(s)) = e_1(s).$$

To have that $g(d\Theta, d\Theta) = 0$ is satisfied to order 2 on $\Gamma$ is more complicated. For this, one uses the quadratic ansatz

$$\Theta_2(s, y) = y \cdot H(s) y,$$

where $H(s)$ is a complex $n \times n$ matrix and "$\cdot$" refers to the usual $\mathbb{R}^n$ inner product and $y \in \mathbb{R}^n$. This ansatz leads to the Riccati equation, which is a first-order matrix-valued ODE. For our purposes, the form of the Riccati equation is not important and it suffices to say that one can find a complex solution $H(s)$ to the equation with $\mathrm{Im}(H(s)) > 0$. The conditions $\mathrm{Im}(H(s)) > 0$ and $\Theta_0 = 0$ together imply the invariantly written conditions

$$\mathrm{Im}(\nabla^2\Theta(\gamma(s))) \ge 0 \quad \text{and} \quad \mathrm{Im}(\nabla^2\Theta)(\gamma(s))|_{\dot{\gamma}(s)^\perp} > 0.$$

Here we use the notation $\dot{\gamma}(s)^\perp$ to denote the algebraic complement to $\dot{\gamma}(s)$ in $T_{\gamma(s)}N$. That is, $\mathbb{R}\dot{\gamma}(s) \oplus \dot{\gamma}(s)^\perp = T_{\gamma(s)}N$.

Solving the eikonal equation to order 2 is enough to understand the qualitative properties of the phase function $\Theta$ needed in our inverse problem. However, we wish to have that

$$\Box_g(e^{is\Theta(x)}a(x)) = \mathcal{O}_{H^k([0,T]\times\Omega)}(\tau^{-K}).$$

For this, we solve the eikonal equation to an order $N$, which depends on $k$ and $K$. This can be done by solving additional ODEs, but we omit the details. After finding $\Theta$ so that $g(d\Theta, d\Theta)$ vanishes to order $N$ on $\Gamma$, the term $\tau^2 g(d\Theta, d\Theta)$ in the expansion (28) of $\Box_g(e^{i\tau\Theta}a)$ satisfies

$$\tau^2 g(d\Theta, d\Theta) \le C_0\tau^2|y|^{N+1}. \tag{33}$$

We choose a specific $N$ later.

Next we insert the phase function $\Theta$ that we have constructed into the transport equations (30) and (31) to find an amplitude function $a$. To solve the transport equations, we write

$$a_k = \chi\left(\frac{|y|}{\delta'}\right)b_k, \tag{34}$$

so that

$$a = \chi\left(\frac{|y|}{\delta'}\right)\sum_{k=0}^{N}\tau^{-k}b_k.$$

Here $\chi \in C_c^\infty(\mathbb{R})$ is a fixed cutoff function, which is identically 1 on a neighborhood of $0 \in \mathbb{R}$ and $\delta' > 0$ is chosen small enough so that $\chi(|y|/\delta')$ is compactly supported in the domain of the Fermi coordinates.

We seek the $b_k$, $k = 1, \dots, N$, in the form

$$b_k = \sum_{j=0}^{N} b_{k,j}(s, y), \tag{35}$$

where $b_{k,j}(s, y)$ is a complex-valued homogeneous polynomial of order $j$ in $y$. We are interested in the specific form only of the leading term $b_{0,0}$. The transport equation concerning $b_0$ is

$$-2g(d\Theta, da_0) + (\Box_g\Theta)a_0 = 0,$$

which is satisfied to order 0 if

$$-2g(d\Theta, db_{0,0})(s, 0) + (\Box_g\Theta)b_{0,0}(s, 0) = 0.$$

Here we used that $\chi(|y|/\delta') = 1$ to order 1 at $y = 0$. We have $d\Theta(s, 0) = dy^1$ and $g^{01}(s, 0) = -1$. It is calculated in [Feizmohammadi and Oksanen 2022, Section 4.2] that $(\Box_g\Theta)(s, 0) = \frac{d}{ds}\log\det(Y(s))$, where $Y(s)$ is a one-parameter nondegenerate matrix field which solves an ODE with the initial condition $Y(0) = I_{n\times n}$, the $n \times n$ identity matrix. Thus we have that the equation for $b_{0,0}(s)$ is solved by

$$b_{0,0}(s) = \det(Y(s))^{-\frac{1}{2}}, \tag{36}$$

with

$$b_{0,0}(0) = 1. \tag{37}$$

Recall that the terms $a_0$, $b_0$ and $b_{0,j}$, $j = 1, 2, \dots, N$, are related by (34)–(35). The terms $b_{0,j}$, $j = 1, 2, \dots, N$, are constructed by solving linear ODEs so that $-2g(d\Theta, da_0) + (\Box_g\Theta)a_0 = 0$ is satisfied to order $N$. The higher-order transport equations (31) concerning $b_k$, $k \ge 1$, can be solved recursively to order $N$ by using similar arguments. We omit the details, and only conclude that there is $C_1 > 0$ so that

$$|-2ig(d\Theta, da_0) + i(\Box_g\Theta)a_0| \le C_1|y|^{N+1},$$
$$|-2ig(d\Theta, da_k) + i(\Box_g\Theta)a_k - \Box_g a_{k-1}| \le C_1|y|^{N+1},$$

$k = 1, \dots, N$. Since $a = a_0 + \tau^{-1}a_1 + \tau^{-2}a_2 + \cdots + \tau^{-N}a_N$, we have that

$$\begin{aligned}
&-2i\tau g(d\Theta, da) + i\tau(\Box_g\Theta)a + \Box_g a \\
&= \tau\sum_{k=0}^{N}\tau^{-k}(-2ig(d\Theta, da_k) + i(\Box_g\Theta)a_k) + \sum_{k=0}^{N}\tau^{-k}\Box_g a_k \\
&= \tau\sum_{k=1}^{N}\tau^{-k}(-2ig(d\Theta, da_k) + i(\Box_g\Theta)a_k + \Box_g a_{k-1}) + \tau(-2ig(d\Theta, da_0) + i(\Box_g\Theta)a_0) + \tau^{-N}\Box_g a_N \\
&= \tau\mathcal{O}_{L^\infty}(|y|^{N+1}) + \mathcal{O}(\tau^{-N}).
\end{aligned}$$

By additionally recalling from (33) that $\tau^2 g(d\Theta, d\Theta) \le C_0\tau^2|y|^{N+1}$, we have

$$\begin{aligned}
e^{-i\tau\Theta}\Box_g(e^{i\tau\Theta}a) &= \tau^2 g(d\Theta, d\Theta) - 2i\tau g(d\Theta, da) + i\tau(\Box_g\Theta)a + \Box_g a \\
&\le C_0\tau^2|y|^{N+1} + C_1\tau|y|^{N+1} + C_2\tau^{-N}.
\end{aligned}$$

By redefining $\delta' > 0$ smaller, if necessary, we have that

$$|e^{i\tau\Theta(s,y)}| \le Ce^{-c\tau|y|^2}$$

for $(s, y)$ in the support of $a$. Recall that our aim is to show that

$$\|\Box_g(e^{i\tau\Theta(s,y)}a(s, y))\|_{H^k([0,T]\times\Omega)} = \mathcal{O}(\tau^{-K}). \tag{38}$$

Taking $k$ derivatives of $\Box_g(e^{i\tau\Theta(s,y)}a(s, y))$ gives

$$|\nabla^k\Box_g(e^{i\tau\Theta(s,y)}a(s, y))| \le C_3 e^{-\tau c|y|^2}\sum_{l=0}^{k}\tau^{k-l}(\tau^2|y|^{N+1-l} + \tau|y|^{N+1-l} + \tau^{-N}). \tag{39}$$

We calculate the integral of (39) squared using polar coordinates for the $y$-variable and the standard formula $\int_0^\infty r^l e^{-\tau c r^2}\,\mathrm{d}r \sim \tau^{-(l+1)/2}$ for $l \ge 0$. Note that since the light-like geodesic $\gamma$ of $(N, g)$ is causal, $[0, T] \times \Omega$ compact and $(N, g)$ globally hyperbolic, the geodesic $\gamma = \gamma(s)$ exits $[0, T] \times \Omega$ after a finite parameter time $r_0$. Thus the integration in the coordinate $s$ will be over a finite interval $[0, r_0]$. The above discussion implies the estimate

$$\begin{aligned}
\|\Box_g(e^{i\tau\Theta(s,y)}a(s, y))\|^2_{H^k(M)} &\lesssim \sum_{l=0}^{k}\tau^{2(k-l)}\left(\int_0^{r_0} e^{-2\tau c r^2} r^{n-1}(\tau^4 r^{2N+2-2l} + \tau^{-2N})\,dr\right) \\
&\lesssim \sum_{l=0}^{k}\tau^{2(k-l)}(\tau^4\tau^{-\frac{n+2N+2-2l}{2}} + \tau^{\frac{-2N-n}{2}}) \\
&\lesssim \tau^{2k+4-\frac{n+2+2N}{2}} = \tau^{2k+3-\frac{n}{2}-N}
\end{aligned}$$

for $\tau$ and $N$ large enough. (Here we have relaxed the notation and written $A \lesssim B$ if there is a constant $\widetilde{C}$ independent of $\tau$ such that $A \le \widetilde{C}B$.) If $p > 1$, we may $L^p$-normalize the function $e^{i\tau\Theta}a$ so that

$$\int_M |\tau^{\frac{n}{2p}} e^{i\tau\Theta} a|^p \lesssim \tau^{\frac{n}{2}} \int_0^\infty r^{n-1} e^{-\tau c r^2} \lesssim 1,$$

in which case we also have

$$\int_M |\nabla^l (\tau^{\frac{n}{2p}} e^{i\tau\Theta} a)|^2 \lesssim \tau^{\frac{n}{p}} \tau^{2l} \tau^{-\frac{n}{2}}.$$

Therefore, if we define $N = N(n, k, K, p)$ so that it satisfies

$$-2K = 2k + 3 - \frac{n}{2} - N + \frac{n}{p}, \tag{40}$$

we have (38). (If $N$ above is not an integer, we redefine it as $\lfloor N+1 \rfloor$.)

By collecting the details of the construction and by defining

$$v_\tau = \tau^{\frac{n}{2p}} e^{i\tau\Theta} a$$

we have:

**Proposition 10** (Gaussian beams). *Let $(N, g)$ be a globally hyperbolic Lorentzian manifold, $N = \mathbb{R} \times M$ and $\dim(N) = n + 1 \ge 3$. Let $\Omega$ be a compact submanifold of $M$ with boundary, and $\dim(\Omega) = n$. Let $T > 0$ and let $\gamma$ be a light-like geodesic of $(N, g)$. Let $k, K, l \in \mathbb{N}$ and $p \ge 2$. There is $\tau_0 \ge 1$ and a family of functions $(v_\tau) \subset C^\infty([0, T] \times \Omega)$ such that for $\tau \ge \tau_0$*

$$\begin{aligned} \|\square_g v_\tau\|_{H^k([0,T]\times\Omega)} &= \mathcal{O}(\tau^{-K}), \\ \|v_\tau\|_{L^p([0,T]\times\Omega)} &= \mathcal{O}(1), \\ \|v_\tau\|_{H^l([0,T]\times\Omega)} &= \mathcal{O}(\tau^{\frac{n}{2p}-\frac{n}{4}+l}) \end{aligned} \tag{41}$$

*as $\tau \to \infty$. The function $v_\tau$ is called a Gaussian beam and it has the form*

$$v_\tau = \tau^{\frac{n}{2p}} e^{i\tau\Theta(x)} a(x),$$

*where $\Theta$ is a smooth complex function (independent of $\tau$) on a neighborhood of $\gamma([0, L])$ satisfying*

$$\begin{aligned} &\Theta(\gamma(s)) = 0, \quad \nabla\Theta(\gamma(s)) = e_1(s), \\ &\operatorname{Im}(\nabla^2\Theta(\gamma(s))) \ge 0, \quad \operatorname{Im}(\nabla^2\Theta)(\gamma(s))|_{\dot{\gamma}(s)^\perp} > 0. \end{aligned} \tag{42}$$

*Here also*

$$a(\gamma(s)) = a_0(\gamma(s)) + \mathcal{O}(\tau^{-1}),$$

*where*

$$a_0(\gamma(s)) = \det(Y(s))^{-\frac{1}{2}}$$

*is nonvanishing and independent of $\tau$. Here $Y(s)$ is a nondegenerate $n \times n$ matrix-valued function. The support of $a$ can be taken to be in any small neighborhood $U$ of $\gamma([0, L])$ chosen beforehand. If $s_0 \in [0, L]$, we may arrange so that $a_0(\gamma(s_0)) = 1$.*

The Gaussian beams can be corrected to be exact solutions to $\square v = 0$.

**Corollary 11.** *Let us adopt the assumptions and notation of Proposition 10. Assume in addition that the light-like geodesic $\gamma$ does not intersect $\{t=0\}$. Assume that $l' \in \mathbb{N}$ satisfies $k > l'-1+\frac{n+1}{2}$. Then there are Gaussian beams $v_\tau$ satisfying the conditions of Proposition 10 and functions $r_\tau \in C^\infty([0,T]\times\Omega)$ such that*

$$v := v_\tau + r_\tau$$

*is a solution to*

$$\begin{cases} \Box_g v = 0 & \text{in } [0,T]\times\Omega, \\ v = v_\tau & \text{on } [0,T]\times\partial\Omega, \\ v|_{t=0} = \partial_t v|_{t=0} = 0 & \text{in } \Omega. \end{cases} \tag{43}$$

*The functions $r_\tau$ satisfy*

$$\|r_\tau\|_{H^{l'}([0,T]\times\Omega)} = \mathcal{O}(\tau^{-K}). \tag{44}$$

*Proof.* By assumption the graph of $\gamma$ has a neighborhood $U$ which does not intersect a neighborhood of $\{t=0\}$. Let $v_\tau$ be Gaussian beams which are supported in $U$ and satisfy the conditions of Proposition 10. By Proposition 8, there exists a solution to

$$\begin{cases} \Box_g r_\tau = -\Box_g v_\tau & \text{in } [0,T]\times\Omega, \\ r_\tau = 0 & \text{on } [0,T]\times\partial\Omega, \\ r_\tau|_{t=0} = \partial_t r_\tau|_{t=0} = 0 & \text{in } \Omega. \end{cases}$$

Then $v = v_\tau + r_\tau$ solves (43).

By Proposition 10 we have that $\|\Box_g v_\tau\|_{H^k([0,T]\times\Omega)} = \mathcal{O}(\tau^{-K})$, where $k, K$ can be chosen freely. By Remark 7 for $k > l'-1+\frac{n+1}{2}$ it holds that $H^k([0,T]\times\Omega) \subset E^{l'-1}$. Choosing $k > l'-1+\frac{n+1}{2}$ in Proposition 8 and using (12) shows that

$$\|r_\tau\|_{H^{l'}([0,T]\times\Omega)} \lesssim \|r_\tau\|_{E^{l'}} \lesssim \|\Box_g v_\tau\|_{E^{l'-1}} \lesssim \|\Box_g v_\tau\|_{H^k([0,T]\times\Omega)} = \mathcal{O}(\tau^{-K})$$

as claimed. ☐

**Remark 12.** We shall also need solutions to the wave equation

$$\begin{cases} \Box_g v = 0 & \text{in } [0,T]\times\Omega, \\ v = f & \text{on } [0,T]\times\partial\Omega, \\ v|_{t=T} = \partial_t v|_{t=T} = 0 & \text{in } \Omega, \end{cases} \tag{45}$$

where the Cauchy data of $v$ vanishes at the top of the time cylinder. Solutions to (45) can be found as follows. Consider the isometry $h$ given by $t \mapsto T-t$ and let $\tilde{g} = h^* g$. Let $\tilde{f} = f(T-t,x)$ and let $\tilde{v}$ be the unique solution to

$$\begin{cases} \Box_{\tilde{g}} \tilde{v} = 0 & \text{in } [0,T]\times\Omega, \\ \tilde{v} = \tilde{f} & \text{on } [0,T]\times\partial\Omega, \\ \tilde{v}|_{t=0} = \partial_t \tilde{v}|_{t=0} = 0 & \text{in } \Omega. \end{cases}$$

Because the wave operator is invariant under isometries we have

$$h^*(\Box_{\tilde{g}} \tilde{v}) = \Box_g (h^* \tilde{v}),$$

whence $v(t,x) := (h^*\tilde{v})(t,x) = \tilde{v}(T-t,x)$ solves (45).

We next vary the initial point and direction of a light-like geodesic to construct a family of Gaussian beams. The Gaussian beams will be constructed so that the implied constants of the family of Gaussian beams are uniformly bounded. This uniformity of constants is essential when proving stability estimates. We mention here a similar consideration in the Riemannian setting [Dos Santos Ferreira et al. 2020, Section 4.1].

To obtain such Gaussian beams, we start with a lemma. We define the set $\mathrm{PSO}(N)$ of pseudo-orthonormal frames as

$$\mathrm{PSO}(N) := \{(e_0,\dots,e_n)\in (TN)^{n+1} \mid g(e_0,e_0)=0,\ g(e_1,e_1)=0,\ g(e_0,e_1)=-2,\\ g(e_j,e_k)=\delta_{jk} \text{ for } j,k=2,3,\dots,n\}.$$

The lemma especially says that on a neighborhood of any point of $N$ there is a local pseudo-orthonormal frame.

**Lemma 13.** *Let $z_0\in N$ and let $V_0\in T_{z_0}N$ be a light-like vector. The set of pseudo-orthonormal frames admits a local section $E:\mathcal{U}\to \mathrm{PSO}(N)$ such that the first component $(E(z_0))_0$ of $E$ at $z_0$ is $V_0$. Here $\mathcal{U}$ is an open neighborhood of $z_0$.*

*Proof.* The existence of a pseudo-orthonormal frame $e=(e_0,e_1,\dots,e_n)$ of the tangent space $T_{z_0}N$ over the single point $z_0$ with $e_0=V_0$ was shown in [Feizmohammadi and Oksanen 2022]. By using local coordinates $(x^k)$ on a neighborhood $\mathcal{U}\subset M$ of $z_0$ let us define the mapping

$$F(x,E): x(\mathcal{U})\times\mathbb{R}^{(n+1)\times(n+1)}\to\mathbb{R}^{(n+1)\times(n+1)},$$

where $x(\mathcal{U})\subset\mathbb{R}^{n+1}$, by the conditions

$$F(x,E)_{jk}=g_x(E_j,E_k)-g_{z_0}(e_j,e_k)\quad \text{if } j\ge k,$$
$$F(x,E)_{jk}=g_x(e_j,E_k)-g_{z_0}(e_j,e_k)\quad \text{if } j<k.$$

Here $E_j$ is the $j$-th column vector of the $(n+1)\times(n+1)$ matrix $E$. Here also $g_x(E_j,E_k)=\langle E_j,g(x)E_k\rangle$ and $g_x(e_j,E_k)=\langle e_j,g(x)E_k\rangle$, where $g(x)$ is the coordinate representation matrix of $g$ in the coordinates $(x^k)$. The perhaps ad hoc looking conditions for $F(x,E)_{jk}$ for $j<k$ are related to the fact that local sections $E$ of $\mathrm{PSO}(M)$ satisfying $(E(z_0))_0=V_0$ (should they exist) are not unique without additional conditions. The conditions for $F(x,E)_{jk}$ for $j<k$ remove this ambiguity.

We apply the implicit function theorem (see, e.g., [Renardy and Rogers 2004, Theorem 10.6]) to show that there is a smooth mapping $x\mapsto E(x)$ such that $F(x,E(x))=0$. In this case $E$ is a smooth section of $\mathrm{PSO}(N)$ by the conditions for $F(x,E)_{jk}$ for $j\ge k$ and by the symmetry of $g$. To apply the implicit function theorem, note that $F(z_0,e)=0$ and that

$$(D_EF|_{x=z_0,E=e}(v))_{jk}=g_{z_0}(v_j,e_k)+g_{z_0}(e_j,v_k)\quad \text{if } j\ge k, \tag{46}$$
$$(D_EF|_{x=z_0,E=e}(v))_{jk}=g_{z_0}(e_j,v_k)\quad \text{if } j<k, \tag{47}$$

where $j,k=0,1,\dots,n$ and $v=(v_0,v_1,\dots,v_n)\in\mathbb{R}^{(n+1)\times(n+1)}$. Assume that $(D_EF|_{x=z_0,E=e}(v))=0$. Since $g$ is symmetric, the condition (47) implies that $g_{z_0}(v_j,e_k)=0$ for $j>k$. Substituting this into (46) then implies that $g_{z_0}(e_j, v_k) = 0$ for $j \geq k$. Thus we actually have that $g_{z_0}(e_j, v_k) = 0$ for all $j, k = 0, 1, \dots, n$. Since $g$ is nondegenerate and $e$ is a frame, it follows that each $v_k \in \mathbb{R}^{n+1}$ is the zero vector of $\mathbb{R}^{n+1}$. Thus $D_E F|_{x=z_0, E=e}$ is injective, and also surjective by dimensionality. Thus, by the implicit function theorem, and by redefining $\mathcal{U}$ smaller if necessary, there is a smooth mapping $E : \mathcal{U} \to \mathrm{PSO}(N)$. This is our desired section. ☐

We remark that it is likely that another proof of the above lemma can be obtained by generalizing the Gram–Schmidt procedure to the current situation. We also mention the similar construction [Dos Santos Ferreira et al. 2020, Lemma 6.1] in the Riemannian setting.

In the next result $|V_0 - \dot{\gamma}_x(s_0)|$ is defined by using local coordinates.

**Corollary 14.** *Let $\gamma$ be a light-like geodesic of $(N, g)$. Assume as in Proposition 10 and adopt its notation. Let $s_0$ be in the domain of $\gamma$ and let us define $\gamma(s_0) = z_0$ and $\dot{\gamma}(t_0) = V_0$. Let also $\delta > 0$. Then there is $\tau_0 \geq 1$ and a neighborhood $U$ of $z_0$ and a family of Gaussian beams*

$$v_\tau(x, \cdot)$$

*solving $\Box_g v_\tau(x, \cdot) = 0$ in $[0, T] \times \Omega$ (including the correction term) parametrized by $x \in \mathcal{U}$. Here "$\cdot$" refers to points in $N$ and $\tau \geq \tau_0$. The geodesics $\gamma_x$ corresponding to the Gaussian beams $v_\tau(x, \cdot)$ satisfy $|V_0 - \dot{\gamma}_x(s_0)| \leq \delta$ and the implied constants of $v_\tau(x, \cdot)$ in Proposition 10 and Corollary 11 are uniformly bounded in $x$.*

*Proof.* The proof is based on inspecting the construction of the Gaussian beams at the beginning of this section that lead to Proposition 10, and by using Corollary 11 and Lemma 13.

Let $v_\tau$ be a Gaussian beam without the error term corresponding to the geodesic $\gamma$ as in Proposition 10. Note that this implies that we have chosen initial data for the certain ODEs used in the construction (such as the Riccati equation). Let us record these initial data and also define

$$v_\tau(z_0, \cdot) := v_\tau(\cdot).$$

By Lemma 13 there is a local section $E$ of $\mathrm{PSO}(M)$ such that $(E(z_0))_0 = V_0$. We define a local vector field $V$ by

$$V(x) = (E(x))_0.$$

By redefining the domain of $E$ smaller, if necessary, we have that $|V(x) - \dot{\gamma}(0)| < \delta$. The section $E$ also defines a family of Fermi coordinates by the formula (26) parametrized by $x$. Since $E$ is smooth, the corresponding Fermi coordinates depend smoothly on $x$ (say in any $C^k$ norm in the Fréchet sense). Also the domain of the Fermi coordinates is uniformly bounded by the same reason. Let $x \in \mathcal{U}$ and let us pass to the Fermi coordinates determined by $E(x)$. We construct a Gaussian beam

$$v_\tau(x, \cdot)$$

with the following properties: (a) It corresponds to the geodesic $\gamma_{x,V(x)}$ with initial data $x \in M$ and $V(x) \in T_x M$. (b) It is constructed by exactly the same method described in the beginning of this section by using the same initial data for the corresponding ODEs that we used for $v_\tau$. Since the coefficients of the ODEs are determined by the smooth metric $g$ and the initial data are the same as for $v_\tau$, the Gaussian beam $v_\tau(x,\cdot)$ differs boundedly and uniformly in $x$ from $v_\tau(\cdot)$ (say in any $C^k(M)$ norm). In particular, the implied constants in Proposition 10 are uniform in $x$.

Finally, we use Corollary 11 to find correction terms for $v_\tau(x,\cdot)$ such that the implied constants in (44) are uniform in $x$. □

## 4. Separation of points

In this section $(N, g)$ is a globally hyperbolic smooth Lorentzian manifold without boundary. The length of a piecewise smooth causal path $\alpha : [a,b] \to N$ is defined as

$$l(\alpha) := \sum_{j=0}^{k-1} \int_{a_j}^{a_{j+1}} \sqrt{-g(\dot\alpha(s),\dot\alpha(s))}\, ds, \tag{48}$$

where $a_0 < a_1 < \cdots < a_{k-1} < a_k$ are chosen such that $\alpha$ is smooth on each interval $(a_j, a_{j+1})$ for $j = 0,\dots,k-1$. The time separation function, see, e.g., [O'Neill 1983], is denoted by $\tau : N \times N \to [0,\infty)$ and defined as

$$\tau(x,y) := \begin{cases} \sup l(\alpha), & y \in J^+(x), \\ 0, & y \notin J^+(x), \end{cases} \tag{49}$$

where the supremum is taken over all piecewise smooth future-directed causal curves $\alpha : [0,1] \to N$ that satisfy $\alpha(0) = x$ and $\alpha(1) = y$. By [O'Neill 1983, Chapter 14, Lemma 16], we have that

$$\tau(x,z) > 0 \quad \text{if and only if} \quad x \ll z. \tag{50}$$

As before, we view $N$ as the product manifold $\mathbb{R} \times M$ and assume that $\Omega \subset M$, $\dim(\Omega) = \dim(M)$, is a smooth compact manifold with boundary. As before, let $\Sigma$ denote the lateral boundary $[0,T] \times \partial\Omega$. Let us consider $x \in I^+(\Sigma) \cap I^-(\Sigma)$. We say that $\gamma_1 : [0,1] \to [0,T] \times \Omega$ is a future-directed *optimal geodesic* connecting $\Sigma$ to $x$ if there is

$$z_1 \in J^-(x) \cap \Sigma \quad \text{such that} \quad \gamma_1(0) = z_1,\ \gamma_1(1) = x \text{ and } \tau(z_1, x) = 0.$$

Similarly, we say that $\gamma_2 : [0,1] \to [0,T] \times \Omega$ is a past-directed optimal geodesic connecting $\Sigma$ to $x$ if there is

$$z_2 \in J^+(x) \cap \Sigma \quad \text{such that} \quad \gamma_2(0) = z_2,\ \gamma_2(1) = x \text{ and } \tau(x, z_2) = 0.$$

We always understand optimal geodesics as their maximal extensions. Note that by definition future/past-directed optimal geodesics are always light-like. The next lemma says that such optimal geodesics always exist. We assume the notation and assumptions used earlier in this section. The situation of the lemma is illustrated in Figure 1, which can be found in Section 1.3 in the Introduction.

In the lemma we consider intersection times of geodesics and $\Sigma$. This means that if the geodesic is denoted by $\gamma : [0,1] \to [0,T] \times \Omega$, then the first intersection time is the smallest $s_0 \in [0,1]$ such that $\gamma(s_0) \in \Sigma$. Typically $s_0$ will be 0. That the intersection in the lemma is transverse means that $\dot\gamma(s_0)$ is transversal to the tangent space $T_{\gamma(s_0)}\Sigma$. We do not claim anything about possible other intersections of $\gamma$ and $\Sigma$.

**Lemma 15** (boundary optimal geodesics). *Let $(N, g)$ be globally hyperbolic, $N = \mathbb{R} \times M$. If $x \in I^+(\Sigma) \cap ([0, T] \times \Omega)$, there exists a future-directed optimal geodesic $\gamma : [0, 1] \to [0, T] \times \Omega$ from $\Sigma$ to $x$ and the first intersection of $\gamma$ and $\Sigma$ is transverse. Similarly, if $x \in I^-(\Sigma) \cap ([0, T] \times \Omega)$, there exists a past-directed optimal geodesic $\gamma : [0, 1] \to [0, T] \times \Omega$ from $\Sigma$ to $x$ and the first intersection of $\gamma$ and $\Sigma$ is transverse.*

*Proof.* Existence: Let us first consider the claim about the existence of future-directed optimal geodesic. For this, let us define

$$t_{\sup} = \sup\{\tilde{d} \in [0, T] \mid \text{there is } \tilde{z} \in \Sigma \text{ such that } t(\tilde{z}) = \tilde{d} \text{ and } \tau(\tilde{z}, x) > 0\}. \tag{51}$$

Here $\tau$ is defined on $N \times N$. The number $t_{\sup}$ will be the time coordinate of $z_{\sup}$ in Figure 1. By assumption $x \in I^+(\Sigma)$ and thus there is $\tilde{z} \in \Sigma$ such that $x \in I^+(\tilde{z})$ with $\tau(\tilde{z}, x) > 0$ by (50). We also have $t(\tilde{z}) \in [0, T]$. Consequently the supremum in (51) exists and $t_{\sup} \in [0, T]$. Let $z_k \in \Sigma$ and $t(z_k) = t_k$ be such that $t_k \to t_{\sup}$ as $k \to \infty$. Since $z_k \in \Sigma$ and $\Sigma$ is compact, we may pass to a subsequence so that $z_k \to z_{\sup} \in \Sigma$. We also have $t(z_{\sup}) = t_{\sup}$ by continuity of the time function $t$.

We claim that $\tau(z_{\sup}, x) = 0$. We argue by contradiction and assume the opposite that $\tau(z_{\sup}, x) > 0$. Then there is a timelike future-directed path $\eta : [0, 1] \to N$ connecting $z_{\sup}$ to $x$ by (50). Since $\eta$ is timelike and $I^-(x)$ is open, we may deform $\eta$ slightly on a neighborhood of $z_{\sup}$ to a future-directed timelike path that connects $z' \in \Sigma$ to $x$ so that $t(z') > t_{\sup}$. Thus $x \in I^+(z')$ and we still have $\tau(z', x) > 0$ by (50). This is a contradiction to the definition of $t_{\sup}$. We conclude that $\tau(z, x) = 0$. Since $(N, g)$ is globally hyperbolic, there is a future-directed light-like geodesic $\gamma_1 : [0, 1] \to N$ from $z_{\sup}$ to $x$ of length $\tau(z_{\sup}, x) = 0$; see [O'Neill 1983, Chapter 14, Proposition 19].

We note that $\gamma_1$ is actually a path $[0, 1] \to [0, T] \times \Omega$. Indeed, if $\gamma_1$ meets the complement of $[0, T] \times \Omega$, then $\gamma_1$ necessarily intersects $\Sigma$ at a parameter time $s_0 < 1$ before it meets $z_{\sup}$ at the parameter time 1. Since $\gamma_1$ is causal, it follows that $t(\gamma_1(s_0)) > t_{\sup} = t(z_{\sup})$, where $\gamma_1(s_0) \in \Sigma$. Since $\Sigma$ is timelike, there is point $\hat{z} \in \Sigma$ with $t_{\sup} < t(\hat{z}) < t(\gamma_1(s_0))$ and a future-directed timelike path $\hat{\eta}$ connecting $\hat{z}$ to $\gamma_1(s_0)$. Thus, a path achieved by composing the paths $\hat{\eta}$ and $\gamma_1$ has positive length by the definition (48). It follows that $\tau(\hat{z}, x) > 0$ by the definition (49). We have arrived to a contradiction with the definition of $z_{\sup}$, since $t(\hat{z}) > t_{\sup}$.

Transversality: We next show that the optimal geodesic $\gamma$ constructed above intersects the lateral boundary $\Sigma$ transversally. Assume that $\gamma$ is parametrized so that $\gamma(0) = z_{\sup}$. Let $S_{t_{\sup}} = \{t_{\sup}\} \times M$ be the Cauchy level surface at $t = t_{\sup}$. Let $T = (T_1, \dots, T_{n-1})$ be a basis for the tangent space $T_{z_{\sup}}\partial\Omega$. Then $\{T, \nu\}$, where $\nu$ is the normal vector to $\partial\Omega$ at $z_{\sup}$ in $S_{t_{\sup}}$, is a basis for $T_{z_{\sup}} S_{t_{\sup}}$. Consequently, the tangent space $T_{z_{\sup}} N$ is spanned by $\{\partial_t, T, \nu\}$, where $\partial_t$ is the coordinate vector of $[0, T]$. Let us write $\dot{\gamma}(0) \in T_{z_{\sup}} N$ in the form

$$\dot{\gamma}(0) = (\dot{\gamma}^t(0), \dot{\gamma}^T(0), \dot{\gamma}^\nu(0)).$$

Suppose now to the contrary that $\gamma$ does not intersect $\Sigma$ transversally. Then it follows that $\dot{\gamma}^\nu(0) = 0$. Indeed, if this is not the case, then $T_{z_{\sup}}\Sigma + T_{z_{\sup}} \operatorname{graph}(\gamma)$ would be equal to $T_{z_{\sup}} N$. Let us check whether $\dot{\gamma}(0)$ is normal to $\Sigma_{t_{\sup}} := \Sigma \cap \{t = t_{\sup}\}$. Since $\Sigma_{t_{\sup}}$ is space-like, the normal space

$$N_{z_{\sup}} \Sigma_{t_{\sup}} := \{v \in T_{z_{\sup}} N \mid \langle v, w \rangle_g = 0 \text{ for all } w \in T_{z_{\sup}} \Sigma_{t_{\sup}}\}$$

(see [O'Neill 1983, p. 98 or p. 198]) is spanned by $\partial_t$ and $\nu$. To see this, note that a vector $X \in T_{z_{\sup}}N$, $X = a\partial_t + b \cdot T + c\nu$, is in $N_{z_{\sup}}\Sigma_{t_{\sup}}$ if and only if $b \in \mathbb{R}^{n-1}$ is zero. Note $\dot{\gamma}^{\nu}(0) = 0$; then if $\dot{\gamma}(0) \in N_{z_{\sup}}\Sigma_{t_{\sup}}$, we must have $\dot{\gamma}(0) = (\dot{\gamma}^t(0), 0, 0)$. But this is not possible, since $\gamma$ is light-like. So $\dot{\gamma}(0)$ is not normal to $\Sigma_{t_{\sup}}$ and by [O'Neill 1983, Chapter 10, Lemma 50] there exists a time-like curve $\sigma$ from $x$ to $\Sigma_{t_{\sup}}$. By slightly deforming $\sigma$ we obtain another time-like curve $\tilde{\sigma}$ connecting $x$ to $z' \in \Sigma$ with $t(z') > t_{\sup}$. This contradicts the definition of $t_{\sup}$.

The claim about the past-directed optimal geodesic follows by defining

$$t_{\inf} = \inf\{\tilde{d} \in [0, T] \mid \text{there is } \tilde{z} \in \Sigma \text{ such that } t(\tilde{z}) = \tilde{d} \text{ and } \tau(x, \tilde{z}) > 0\}$$

and by using arguments analogous to the ones above to find $z_{\inf} \in \Sigma$ with $\tau(x, z_{\inf}) = 0$. □

By using boundary optimal geodesics and related Gaussian beams we may separate points of $[0, T] \times \Omega$ by solutions to $\square_g v = 0$. We mention here that separation of points by solutions has been beneficial in the study of inverse problems for elliptic equations [Guillarmou et al. 2019; Lassas et al. 2020].

**Proposition 16** (separation of points). *Let $(N, g)$ be globally hyperbolic, $N = \mathbb{R} \times M$. Let $x \in I^-(\Sigma) \cap ([0, T] \times \Omega)$ and $y \in N$ be such that $y \notin J^-(x)$. Denote by $v_f$ the solution to $\square_g v = 0$ in $N$ with $v|_\Sigma = f$ and whose Cauchy data vanishes at $t = T$. Then there is $f \in C^\infty(\Sigma)$ such that*

$$v_f(x) \neq v_f(y).$$

*If $x \in I^+(\Sigma) \cap ([0, T] \times \Omega)$ and $x \notin J^-(y)$, we have the same claim for solutions of $\square_g v = 0$ in $N$ with $v|_\Sigma = f$ whose Cauchy data instead vanishes at $t = 0$.*

*Proof.* We first claim that there is a past-directed light-like geodesic from $\Sigma$ that meets the point $x$ but not $y$. We argue by contradiction and assume the opposite that all past-directed light-like geodesics from $\Sigma$ to $x$ meet both $x$ and $y$. Since $x \in I^-(\Sigma) \cap ([0, T] \times \Omega)$, by Lemma 15 we have that there is a past-directed boundary optimal geodesic $\gamma_1 : [0, 1] \to [0, T] \times \Omega$ with $\gamma_1(0) = z \in \Sigma$ and $\gamma_1(1) = x$. The first intersection of $\gamma_1$ with $\Sigma$ is transverse. If $x \notin J^-(y)$, then by the assumption $y \notin J^-(x)$ we have that $x$ and $y$ are not causally connected. Thus $\gamma_1$ cannot pass through $y$ and we have found our light-like geodesic. Therefore, we may assume that $y \geq x$.

Let $\tilde{\gamma}_1 = \tilde{\gamma}_1(s)$ be a past-directed light-like geodesic with $\tilde{\gamma}_1(0) \in \Sigma$ such that $\tilde{\gamma}_1$ intersects $\Sigma$ transversally at $s = 0$, and which satisfies $\tilde{\gamma}_1(\tilde{s}) = x$ for some $\tilde{s} \geq 0$. The geodesic $\tilde{\gamma}_1$ can be obtained by perturbing the tangent vector of $\gamma_1$ at $\gamma_1(1) = x$ slightly. Note that the condition of transversal intersection is an open condition. By assumption $\tilde{\gamma}_1$ meets $y$. In this case we have a shortcut path, which has timelike portion, obtained by traveling along $\tilde{\gamma}_1$ from $x$ to a point $y'$ close to $y$, doing a shortcut from $y'$ to $\gamma_1$ and then by continuing along $\gamma_1$ to $z$; see [O'Neill 1983, Chapter 10, Proposition 46]. Since the shortcut path has timelike portion, it has positive length. Since $y \geq x$, the shortcut path is also future-directed. It follows that $\tau(x, z) > 0$. This contradicts the optimality of $\gamma_1$. We conclude that $\tilde{\gamma}_1$ is a past-directed light-like geodesic from $\Sigma$ that meets $x$ but not $y$.

To conclude the proof, we use Proposition 10 and choose a Gaussian beam $v_\tau = \tau^{n/4} e^{i\tau\Theta} a$ corresponding to $\tilde{\gamma}_1$ with $k > n$, $K = 1$ and $p, l = 2$. We also choose the support of the amplitude $a$ be so small that $y \notin \operatorname{supp}(a)$ and $\operatorname{supp}(a) \cap \{t = T\} = \varnothing$. We will use the Sobolev embedding $H^{l'} \subset L^\infty$, which holds for $l' > \frac{n+1}{2}$. Since $k > n$, we have $k - \frac{n-1}{2} > \frac{n+1}{2}$, which shows that we can take $l'$ such that $\frac{n+1}{2} < l' < k - \frac{n-1}{2}$. Applying Corollary 11 with these $k$ and $l'$ shows that there is $r_\tau \in C^\infty(N)$ such that

$$v_f := \tau^{-\frac{n}{4}} v = \tau^{-\frac{n}{4}} v_\tau + \tau^{-\frac{n}{4}} r_\tau$$

satisfies $\square_g v_f = 0$ and

$$\tau^{-\frac{n}{4}} v_\tau(x) = 1 \quad \text{and} \quad \tau^{-\frac{n}{4}} v_\tau(y) = 0 \quad \text{for all } \tau \geq \tau_0$$

and

$$\|\tau^{-\frac{n}{4}} r_\tau\|_{L^\infty(N)} \leq C\tau^{-\frac{n}{4}} \|r_\tau\|_{H^{l'}(N)} = \tau^{-\frac{n}{4}} \mathcal{O}(\tau^{-1}).$$

We mention for future reference that at any other point $z \in [0,T] \times \Omega$ we have

$$|v_f(z)| \leq |\tau^{-\frac{n}{4}} v_\tau(z)| + |\tau^{-\frac{n}{4}} r_\tau(z)| \leq C' + |\tau^{-\frac{n}{4}} r_\tau(z)| \leq C \tag{52}$$

for all $\tau$ large enough. Here we used the above Sobolev embedding. Taking $\tau$ large enough shows that

$$v_f(x) \neq v_f(y).$$

The claim about the case $x \in I^+(\Sigma)$ and $x \notin J^-(y)$ can be proved in similar way. □

We next consider the case where we have multiple points of $[0,T] \times \Omega$, which we wish to separate by solutions of the wave equation $\square_g v = 0$. The points will correspond to the intersection points of pairs of geodesics we use for our inverse problem. The matrix (53) below will be a *separation matrix* in the sense of Definition 5.

**Lemma 17** (existence of separation matrix). *Let $(N,g)$ be globally hyperbolic, $N = \mathbb{R} \times M$. Let $x_1, \dots, x_P \in I^-(\Sigma) \cap ([0,T] \times \Omega)$ be such that $x_1 < x_2 < \cdots < x_P$. Denote by $v_f$ the solution of $\square_g v = 0$ in $[0,T] \times \Omega$ with $v|_\Sigma = f$ and whose Cauchy data vanishes at $t = T$. Then there are boundary values $f_k \in C^\infty(\Sigma)$ such that the matrix*

$$\begin{pmatrix} v_{f_1}(x_1) & v_{f_2}(x_1) & \cdots & v_{f_P}(x_1) \\ v_{f_1}(x_2) & v_{f_2}(x_2) & \cdots & v_{f_P}(x_2) \\ \vdots & & \ddots & \vdots \\ v_{f_1}(x_P) & v_{f_2}(x_P) & \cdots & v_{f_P}(x_P) \end{pmatrix} \tag{53}$$

*is invertible.*

*If $x_k \in I^+(\Sigma) \cap ([0,T] \times \Omega)$, we have the similar claim for solutions of $\square_g v = 0$ in $[0,T] \times \Omega$ with $v|_\Sigma = f$ whose Cauchy data instead vanishes at $t = 0$.*

*Proof.* The proof is an iteration of the proof Proposition 16. First we let $\gamma_1$ be a past-directed boundary optimal geodesic that connects a point $z \in \Sigma$ to the point $x_1$. By the shortcut argument in the proof of Proposition 16, we deduce after possibly redefining $\gamma_1$ as its small perturbation that $\gamma_1$ does not meet any of the other points $x_k$, $k = 2, \dots, P$. Let $v_{f_1}$ be a Gaussian beam solution (including the correction term) as in the proof of Proposition 16, where $f_1 \in C^\infty(\Sigma)$. Then there is $\tau_1 > 0$ such that for $\tau \geq \tau_1$ we have

$$v_{f_1}(x_1) = 1 \quad \text{and} \quad v_{f_1}(x_k) = \mathcal{O}(\tau^{-1-\frac{n}{4}}), \quad k = 2, \dots, P.$$

Next, let $\gamma_2$ be a past-directed boundary optimal geodesic that connects $z \in \Sigma$ to the point $x_2$. By repeating the above argument we find a boundary value $f_2 \in C^\infty(\Sigma)$ and a solution $v_{f_2}$ such that

$$v_{f_2}(x_2) = 1 \quad \text{and} \quad v_{f_2}(x_k) = \mathcal{O}(\tau^{-1-\frac{n}{4}}), \quad k = 3, \dots, P,$$

for all $\tau \geq \tau_2$. Note that we do not claim that we have much control on the value of $v_{f_2}$ at $x_1$ and it might be that $\gamma_2$ meets also the point $x_1$. However, by (52) we know that $|v_{f_2}|$ at $x_1$ is bounded by $C$ (possibly by defining $\tau_2$ larger). This is illustrated in Figure 2, which can be found in Section 1.3 in the Introduction. By repeating the above arguments, we find other solutions $v_{f_k}$, $k = 3, \dots, P$, such that the matrix (53) becomes of the form

$$\mathcal{V}_\tau = \begin{pmatrix} 1 & \mathcal{O}(\tau^{-1-\frac{n}{4}}) & \mathcal{O}(\tau^{-1-\frac{n}{4}}) \\ \# & \ddots & \mathcal{O}(\tau^{-1-\frac{n}{4}}) \\ \# & \# & 1 \end{pmatrix}.$$

Here # means unspecified complex numbers bounded by some fixed constant. The determinant of this matrix tends to 1 as $\tau \to \infty$. Therefore, there is $\tau_0 \geq 1$ such that the matrix (53) is invertible for all $\tau \geq \tau_0$. □

The previous lemma shows that if we are given a set of points $x_1 < \cdots < x_k$, one can find a set of Gaussian beams separating these points. However, for the proof of the stability estimate in Theorem 1, we need a finite collection of Gaussian beams that separate any sufficiently distinct $P \in \mathbb{N}$ points. The collection will be a *separation filter* in the sense of Definition 6. Existence of such a collection is the content of the next lemma.

Let $\bar{g}$ be an auxiliary Riemannian metric on $\mathbb{R} \times M$.

**Lemma 18** (existence of separation filter). *Let $P \geq 1$ be an integer and let $\delta > 0$. Suppose $K \subset I^-(\Sigma) \cap I^+(\Sigma) \cap ([0, T] \times \Omega)$ is a compact set. There exists a finite collection $\mathcal{M} \subset C^\infty([0, T] \times \Omega)$ of solutions to $\Box_g v_f = 0$ with the following properties: Assume that $x_1, \dots, x_P \in K$ are any points such that $x_1 < x_2 < \cdots < x_P$ and $d_{\bar{g}}(x_k, x_l) > \delta$ for $x_k \neq x_l$, $k, l = 1, \dots, P$. Then there are solutions $v_{f_1}, \dots, v_{f_P} \in \mathcal{M}$ corresponding to boundary values $f_k \in C^\infty(\Sigma)$, and which have vanishing Cauchy data at $t = T$, such that the matrix $(v_{f_k}(x_l))_{k,l=1}^P$ in (53) is invertible. Thus $\mathcal{M}$ is a separation filter.*

*Proof.* Case 1: If $P = 1$, then the situation is similar to the proof of Proposition 16. Applying Lemma 15 to $x_1$, we find a past-directed boundary optimal geodesic $\gamma$ from $\Sigma$ to $x_1$, whose first intersection with $\Sigma$ is transverse. Using Corollary 11 we can then construct a Gaussian beam $v$ (including the correction term and with vanishing Cauchy data at $\{t = T\}$) corresponding to $\gamma$ such that

$$v(x_1) = 1.$$

By continuity of $v$, the point $x_1$ has a neighborhood $B(x_1)$ such that

$$|v(z)| > \tfrac{2}{3} \quad \text{for all } z \in B(x_1).$$

Doing this for all points $x \in K$ we find an open cover of $K$ of the form

$$\bigcup_{x \in K} B(x)$$

and for each $B(x)$ the corresponding optimal geodesic and the respective Gaussian beam $v$. Because $K$ is compact, there is a finite subcover

$$\bigcup_{j=1}^{R} B(x^j)$$

of $K$ and the corresponding finite collection of Gaussian beams. Denoting this collection by $\mathcal{M}$ completes the proof for $P = 1$.

Case 2: Suppose now $P \geq 2$. To begin, consider a complex matrix of the form

$$\begin{pmatrix} d_1 & & \mathcal{O} \\ & \ddots & \\ \# & & d_P \end{pmatrix}, \tag{54}$$

where all entries # are bounded by a fixed constant $C > 0$ and the diagonal entries satisfy $|d_j| > \frac{2}{3}$, $j = 1, \dots, P$. When the elements of the upper triangular part $\mathcal{O}$ are of the size $\varepsilon > 0$, the determinant of the matrix in (54) equals

$$d_1 \cdots d_P + O(\varepsilon).$$

This can be seen by considering the definition of the determinant in terms of minors. Thus the matrix in (54) is invertible when $\varepsilon$ is small enough.

We construct an open cover of $K$ as follows. Let $\widetilde{K} \subset J^+(\Sigma) \cap J^-(\Sigma) \cap ([0, T] \times \Omega)$ be an open neighborhood of $K$. Let us fix $x \in \widetilde{K}$ and let $B_{\delta/2}(x)$ denote a $\frac{\delta}{2}$-radius ball centered at $x$ with respect to the metric $\bar{g}$. Let us also define

$$\mathcal{V}(x) := (J^+(x) \setminus B_{\frac{\delta}{2}}(x)) \cap ([0, T] \times \Omega).$$

Since $J^+(x)$ is closed, the set $\mathcal{V}(x)$ is compact for all $x \in \widetilde{K}$. We define the subset of $\mathcal{V}(x)^{P-1}$ of ordered points by

$$\mathcal{T}(x) := \{(x_2, \dots, x_P) \in \mathcal{V}(x)^{P-1} : x \leq x_2 \leq \cdots \leq x_P\}.$$

Because the relation $\leq$ is closed (see, e.g., [O'Neill 1983, Section 14, Lemma 22]), the set $\mathcal{T}(x)$ is compact as a closed subset of the compact set $\mathcal{V}(x)^{P-1}$.

Let $\varepsilon > 0$ and let $X = (x_2, \dots, x_P) \in \mathcal{T}(x)$. Recall that when constructing a Gaussian beam $v$, we can bound its size in absolute value by using the estimate (52). Since $x \in I^-(\Sigma) \cap ([0, T] \times \Omega)$, there is $f_X \in C^\infty(\Sigma)$ and a Gaussian beam $v_X$ (including the correction term and with vanishing Cauchy data at $\{t = T\}$) and $\tau_X > 0$ such that there is a neighborhood $U_\varepsilon(x) \subset B_{\delta/3}(x)$ of $x$ and neighborhoods $B(x_k)$ of $x_k$ such that

$$\begin{aligned} |v_{f_X}| &\geq \tfrac{2}{3} && \text{on } U_\varepsilon(x), \\ |v_{f_X}| &< \varepsilon && \text{on } B(x_k), \quad k = 2, 3, \dots, P, \\ |v_{f_X}| &\leq C && \text{on } [0, T] \times \Omega, \end{aligned} \tag{55}$$

where $C > 0$ is independent of $\varepsilon > 0$. Here we have first normalized so that $v_{f_X}(x) = 1$. Then we have chosen the $\tau_X$ large enough, so that the condition $|v_{f_X}| < \varepsilon$ holds on $B(x_k)$, and $|v_{f_X}| \leq C$ on $[0, T] \times \Omega$.

These conditions can be obtained since the correction term of a Gaussian beam can be made arbitrarily small by taking the corresponding $\tau$ large enough. Then, by the continuity of $v_{f_X}$ and $v_{f_X}(x) = 1$, we have chosen the neighborhood $U_\varepsilon(x)$ so that $|v_{f_X}| \geq \frac{2}{3}$. Note that since here $\tau_X$ depends on $\varepsilon$ and $v_{f_X}$ depends on $\tau_X$, the neighborhood $U_\varepsilon(x)$ depends on $\varepsilon$ as indicated in the notation. See the argument in the proof of Proposition 16 for more details.

We now modify the open sets $U_\varepsilon(x)$ slightly. Let us define

$$\widetilde{U}_\varepsilon(x) := I^+(x) \cap U_\varepsilon(x).$$

We have that

$$|v_{f_X}| \geq \tfrac{2}{3} \quad \text{on } \widetilde{U}_\varepsilon(x).$$

Moreover, we have

$$x \leq z \quad \text{for all } z \in \widetilde{U}_\varepsilon(x). \tag{56}$$

We then have an open cover of $\mathcal{T}(x)$ given by

$$\bigcup_{X \in \mathcal{T}(x)} B(x_2) \times \cdots \times B(x_P).$$

Since $\mathcal{T}(x)$ is compact, we may pass to a finite open subcover

$$\bigcup_{X \in \mathcal{J}_\uparrow(x)} B(x_2) \times \cdots \times B(x_P),$$

where $\mathcal{J}_\uparrow(x)$ is a finite subset of $\mathcal{T}(x)$ and which depends on $\varepsilon$. Note that for each $X = (x_2, \ldots, x_P) \in \mathcal{J}_\uparrow(x)$ there are associated neighborhoods $B(x_2), \ldots, B(x_P)$ of the points $x_2, \ldots, x_P$ and an open set $\widetilde{U}_\varepsilon(x)$. This shows that to each point $x \in \widetilde{K}$ we can attach a finite collection

$$\mathcal{M}_\varepsilon(x) \subset C^\infty([0, T] \times \Omega)$$

of solutions with the following property: for any $X \in \mathcal{T}(x)$ there is some Gaussian beam solution $v_{f_X} \in \mathcal{M}_\varepsilon(x)$ corresponding to a boundary value $f_X$ with the property (55) with $U_\varepsilon(x)$ replaced by $\widetilde{U}_\varepsilon(x)$. We repeat the above argument for all $x \in \widetilde{K}$. Note that if $x \in K$, then there is $\tilde{x} \in \widetilde{K} \cap J^-(x)$ so that $x \in \widetilde{U}_\varepsilon(\tilde{x})$. Thus, our construction yields an open cover of $K \subset [0, T] \times \Omega$ by the sets $\widetilde{U}_\varepsilon(x)$ described above. By compactness, finitely many sets $\widetilde{U}_\varepsilon(x)$ suffice to cover $K$. Let $x^{(j)} \in [0, T] \times \Omega$ be the corresponding points, such that

$$\bigcup_{j=1}^{R_\varepsilon} \widetilde{U}_\varepsilon(x^{(j)}) \tag{57}$$

is a finite subcover of $K$, where $R_\varepsilon \in \mathbb{N}$. To each of these finitely many points $x^{(j)}$ there is also attached a finite subset $\mathcal{J}_\varepsilon(x^{(j)}) \subset \mathcal{T}(x^{(j)})$, $j = 1, \ldots, R_\varepsilon$. Corresponding to this finite cover, we take as the collection of boundary values $\mathcal{M}_\varepsilon$ the set

$$\mathcal{M}_\uparrow := \bigcup_{j=1}^{R_\varepsilon} \mathcal{M}_\varepsilon(x^{(j)}).$$

Let then $x_1, x_2, \dots, x_P \in K$ with $x_1 < x_2 < \cdots < x_P$ and $d_{\bar{g}}(x_l, x_k) > \delta$ for $k \neq l$ with $k, l = 1, \dots, P$. Let us consider first the point $x_1 \in K$. Corresponding to $x_1$, there is an index $j_1 \in \{1, \dots, R_\varepsilon\}$ and a neighborhood $\widetilde{U}_\varepsilon(x^{(j_1)})$ of $x_1$, where $\widetilde{U}_\varepsilon(x^{(j_1)})$ belongs to the finite subcover (57) of $K$. The radius of $\widetilde{U}_\varepsilon(x^{(j_1)})$ is less than $\frac{\delta}{3}$. Note that $d_{\bar{g}}(x^{(j_1)}, x_k) > \frac{\delta}{2}$ for $k = 2, 3, \dots, P$. Indeed, we have that

$$d_{\bar{g}}(x^{(j_1)}, x_k) \geq d_{\bar{g}}(x_1, x_k) - d_{\bar{g}}(x^{(j_1)}, x_1) > \delta - \frac{\delta}{3} = \frac{2\delta}{3} > \frac{\delta}{2}. \tag{58}$$

Moreover, (56) implies $x^{(j_1)} \leq x_1$. Thus $x^{(j_1)} \leq x_2 \leq x_3 \leq \cdots \leq x_P$. Using this and (58), we obtain

$$(x_2, x_3, \dots, x_P) \in \mathcal{T}(x^{(j_1)}).$$

Consequently, using the definition of $\mathcal{J}_\varepsilon(x^{(j_1)})$, we find $X = (x_2^{(j_1)}, \dots, x_P^{(j_1)}) \in \mathcal{J}_\varepsilon(x^{(j_1)})$ with the associated neighborhoods $B(x_k^{(j_1)})$ of $x_k$, $k = 2, 3, \dots, P$, satisfying the following property: there is a Gaussian beam solution $v_{f_1} \in \mathcal{M}_\varepsilon$ corresponding to a boundary value $f_1$ such that

$$\begin{aligned} |v_{f_1}| &\geq \tfrac{2}{3} && \text{on } \widetilde{U}_\varepsilon(x^{(j_1)}), \\ |v_{f_1}| &< \varepsilon && \text{on } B(x_k^{(j_1)}), \quad k = 2, 3, \dots, P, \\ |v_{f_1}| &\leq C && \text{on } [0, T] \times \Omega. \end{aligned}$$

Let us then proceed to the point $x_2$. Much as above, regarding this point there is $j_2 \in \{1, \dots, R_\varepsilon\}$, $x^{(j_2)} \in [0, T] \times \Omega$ and neighborhoods $\widetilde{U}_\varepsilon(x^{(j_2)})$ of $x_2$ and neighborhoods $B(x_k^{(j_2)})$ of $x_k$, $k = 3, 4 \dots, x_P$, and a Gaussian beam $v_{f_2}$, such that

$$\begin{aligned} |v_{f_2}| &\geq \tfrac{2}{3} && \text{on } \widetilde{U}_\varepsilon(x^{(j_2)}), \\ |v_{f_2}| &< \varepsilon && \text{on } B(x_k^{(j_2)}), \quad k = 3, 4, \dots, P, \\ |v_{f_2}| &\leq C && \text{on } [0, T] \times \Omega. \end{aligned}$$

Continuing in this manner, we have indices $j_1, j_2, \dots, j_P$ and a set of Gaussian beams $v_{f_k}$, $k = 1, \dots, P$, such that $|v_{f_k}| \geq \frac{2}{3}$ on a neighborhood $\widetilde{U}_\varepsilon(x^{(j_k)})$ of $x_k$ and $|v_{f_k}| < \varepsilon$ on a neighborhood $B(x_l^{(j_k)})$ of $x_l$ for $l > k$ and $|v_{f_k}| < C$ on $[0, T] \times \Omega$.

The separation matrix (53) corresponding to the functions $v_{f_k}$ and points $x_k$ is invertible for $\varepsilon \leq \varepsilon_0$ for $\varepsilon_0$ small enough. We set $\mathcal{M} := \mathcal{M}_{\varepsilon_0}$. Finally, we note that the number of Gaussian beams used is

$$\#\mathcal{M} = \#\left(\bigcup_{j=1}^{R_{\varepsilon_0}} \mathcal{M}_{\varepsilon_0}(x^{(j)})\right) \leq \sum_{j=1}^{R_{\varepsilon_0}} \#(\mathcal{M}_{\varepsilon_0}(x^{(j)})) = \sum_{j=1}^{R_{\varepsilon_0}} \#\mathcal{J}_{\varepsilon_0}(x^{(j)}),$$

which is finite. □

**Remark 19.** We will apply Lemma 17 as follows. Suppose the points $x_1 < \cdots < x_P$ are the intersection points of two light-like geodesics $\gamma_1$ and $\gamma_2$. We will use Lemma 17 to show that there is a choice of $P$ solutions $v_{f_1}, \dots, v_{f_P} \in \mathcal{M}$ which separate the points $x_1, \dots, x_P$. Moreover, these solutions have zero Cauchy data at $t = T$.

We also mention that we have a result similar to Lemma 18 for solutions that have vanishing Cauchy data at $\{t = 0\}$. The result is obtained, for example, from Lemma 18 by considering the isometry $t \mapsto T - t$ as in Remark 12.

## 5. Proof of the stability estimate: Theorem 1

Assume the conditions from Theorem 1, especially that

$$\|\Lambda_1(f)-\Lambda_2(f)\|_{H^r(\Sigma)} \le \delta,$$

where $r \le s+1$ and $s+1 > \frac{n+1}{2}$, and $\delta > 0$. Here $\Lambda_1$ and $\Lambda_2$ are the DN maps of the nonlinear wave equation (2) corresponding to the potentials $q_1$ and $q_2$, respectively. We show that we have explicit control on the $L^\infty$ norm of $q_1 - q_2$ in terms of $\delta$. The proof will be divided into several steps.

**5.1.** ***Step 1*: *integral identity from finite differences.*** Let $j = 1, \dots, m$ and $\varepsilon_j > 0$ be small parameters. Let $\kappa$ be as in Lemma 9. Assume that $f_j \in H^{s+1}(\Sigma)$ is a family of functions satisfying $\partial_t^\alpha f_j|_{t=0} = 0$ on $[0,T]\times\partial\Omega$, $\alpha = 0, \dots, s$, and that

$$\|\varepsilon_1 f_1 + \cdots + \varepsilon_m f_m\|_{H^{s+1}([0,T]\times\Omega)} \le \kappa.$$

For $l = 1, 2$, we have that the boundary value problems

$$\begin{cases} \Box_g u_l + q_l u_l^m = 0 & \text{in } [0,T]\times\Omega, \\ u_l = \varepsilon_1 f_1 + \cdots + \varepsilon_m f_m & \text{on } [0,T]\times\partial\Omega, \\ u_l|_{t=0} = 0, \quad \partial_t u_l|_{t=0} = 0 & \text{in } \Omega \end{cases}$$

have unique small solutions $u_l = u_{\varepsilon_1 f_1 + \cdots + \varepsilon_m f_m}$ as described in Lemma 9. According to (17), the solutions $u_l$ have expansions of the form

$$u_l = \varepsilon_1 v_{l,1} + \cdots + \varepsilon_m v_{l,m} + \sum_{|\vec{k}|=m} \binom{m}{k_1, \dots, k_m} \varepsilon_1^{k_1} \cdots \varepsilon_m^{k_m} w_{l,\vec{k}} + \mathcal{R}_l,$$

where $v_{l,j}$ satisfy (18) and $w_{l,\vec{k}}$ satisfy (19) with $q$ replaced by $q_l$. We also used the notation $\vec{k} = (k_1, \dots, k_m)$. In particular, we know by (19) that

$$w_{l,\vec{1}} := w_{l,(1,\dots,1)}$$

satisfy

$$\begin{cases} \Box_g w_{l,\vec{1}} + q_l v_{l,1} \cdots v_{l,m} = 0 & \text{in } [0,T]\times\Omega, \\ w_{l,\vec{1}} = 0 & \text{on } [0,T]\times\partial\Omega, \\ w_{l,\vec{1}}|_{t=0} = 0, \quad \partial_t w_{l,\vec{1}}|_{t=0} = 0 & \text{in } \Omega. \end{cases} \tag{59}$$

Note that since (18) for $v_{l,j}$ are independent of $q_l$, we have by the uniqueness of solutions that

$$v_{1,j} = v_{2,j} =: v_j, \quad j = 1, \dots, m. \tag{60}$$

Moreover, according to (20), the correction terms $\mathcal{R}_l$ for $l = 1, 2$ satisfy

$$\|\mathcal{R}_l\|_{E^{s+2}} + \|\Box_g \mathcal{R}_l\|_{E^{s+1}} \le C(s,T)\|q_l\|^2_{E^{s+1}} \|\varepsilon_1 f_1 + \cdots + \varepsilon_m f_m\|^{2m-1}_{H^{s+1}(\Sigma)}.$$

We apply the finite difference operator $D^m_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0}$ of order $m$, defined in (21), to $u_l$. By (22), we have

$$D^m_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0} u_l = m!\, w_{l,\vec{1}} + \frac{1}{\varepsilon_1 \cdots \varepsilon_m} \widetilde{\mathcal{R}}_l,$$

where $\overline{\mathcal{R}}_l$ contains sum of the remainder terms $\mathcal{R}_l$ appearing in the finite differences. Consequently, by taking into account (59) and (60), we obtain

$$\Box_g D^m_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0} u_l = -m!\, q_l v_1 \cdots v_m + \frac{1}{\varepsilon_1 \cdots \varepsilon_m} \Box_g \widetilde{\mathcal{R}}_l,$$

where $\widetilde{\mathcal{R}}_l = \varepsilon_1 \cdots \varepsilon_m \overline{\mathcal{R}}_l$, $l = 1, 2$.

We manipulate the integral identity (25) to relate the difference of the DN maps $\Lambda_1$ and $\Lambda_2$ to the difference of the unknown potentials $q_1$ and $q_2$ in terms of $v_j$. For this, consider an auxiliary function $v_0$ which satisfies $\Box_g v_0 = 0$ in $[0,T]\times\Omega$, with $v_0|_{t=T} = \partial_t v_0|_{t=T} = 0$ in $\Omega$. Applying (25) to the difference of the DN maps yields

$$\begin{aligned} -m! &\int_{[0,T]\times\Omega} (q_1 - q_2) v_0 v_1 \cdots v_m \, dV_g \\ &= \frac{1}{\varepsilon_1 \cdots \varepsilon_m} \int_{[0,T]\times\Omega} v_0 \Box_g (\widetilde{\mathcal{R}}_1 - \widetilde{\mathcal{R}}_2) \, dV_g + \int_\Sigma v_0 D^m_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0} (\Lambda_1 - \Lambda_2)(\varepsilon_1 f_1 + \cdots + \varepsilon_m f_m) \, dS. \end{aligned} \tag{61}$$

The finite difference $D^m_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0}$ of $u_l$ is a sum of $2^m$ terms. By using (61), we calculate

$$\begin{aligned} m!\, &|\langle v_0(q_1 - q_2), v_1 \cdots v_m \rangle_{L^2([0,T]\times\Omega)}| \\ &\le |\langle v_0, D^m_{\vec{\varepsilon}=0} [(\Lambda_1 - \Lambda_2)(\varepsilon_1 f_1 + \cdots + \varepsilon_m f_m)] \rangle_{L^2(\Sigma)}| + (\varepsilon_1 \cdots \varepsilon_m)^{-1} |\langle v_0, \Box_g (\widetilde{R}_1 - \widetilde{R}_2) \rangle_{L^2([0,T]\times\Omega)}| \\ &\le 2^m (\varepsilon_1 \cdots \varepsilon_m)^{-1} |\langle v_0, (\Lambda_1 - \Lambda_2)(\varepsilon_1 f_1 + \cdots + \varepsilon_m f_m) \rangle_{L^2(\Sigma)}| \\ &\qquad\qquad + (\varepsilon_1 \cdots \varepsilon_m)^{-1} |\langle v_0, \Box_g (\widetilde{R}_1 - \widetilde{R}_2) \rangle_{L^2([0,T]\times\Omega)}| \\ &\le 2^m \delta (\varepsilon_1 \cdots \varepsilon_m)^{-1} \|v_0\|_{\widetilde{H}^{-r}(\Sigma)} + (\varepsilon_1 \cdots \varepsilon_m)^{-1} \|\Box_g (\widetilde{R}_1 - \widetilde{R}_2)\|_{H^{s+1}([0,T]\times\Omega)} \|v_0\|_{\widetilde{H}^{-(s+1)}([0,T]\times\Omega)} \\ &\le 2^m \delta (\varepsilon_1 \cdots \varepsilon_m)^{-1} \|v_0\|_{\widetilde{H}^{-r}(\Sigma)} + C_{s+1} (\varepsilon_1 \cdots \varepsilon_m)^{-1} \|\Box_g (\widetilde{R}_1 - \widetilde{R}_2)\|_{E^{s+1}} \|v_0\|_{\widetilde{H}^{-(s+1)}([0,T]\times\Omega)} \\ &\le C_{m,s+1} (\varepsilon_1 \cdots \varepsilon_m)^{-1} (\|v_0\|_{\widetilde{H}^{-r}(\Sigma)} + \|v_0\|_{\widetilde{H}^{-(s+1)}([0,T]\times\Omega)}) \\ &\qquad\qquad \times \left[ 2^m \delta + C(s,T)(\|q_1\|^2_{E^{s+1}} + \|q_2\|^2_{E^{s+1}}) \left( \sum_{j=1}^m \varepsilon_j \|f_j\|_{H^{s+1}(\Sigma)} \right)^{2m-1} \right] \\ &\le C (\varepsilon_1 \cdots \varepsilon_m)^{-1} \left[ \delta + \left( \sum_{j=1}^m \varepsilon_j \|f_j\|_{H^{s+1}(\Sigma)} \right)^{2m-1} \right]. \end{aligned} \tag{62}$$

Here we used the assumption $\|\Lambda_1(f) - \Lambda_2(f)\|_{H^r(\Sigma)} \le \delta$ for $f = \varepsilon_1 f_1 + \cdots + \varepsilon_m f_m$. We also used that the norm in $H^{s+1}([0,T]\times\Omega)$ is bounded by the norm in $E^{s+1}$ up to a multiplicative factor $C_{s+1}$ as noticed in Remark 7. The final constant $C$ is given by

$$C = \max\{C_{m,s+1}, C(s,T)(\|q_1\|^2_{E^{s+1}} + \|q_2\|^2_{E^{s+1}})\} (\|v_0\|_{\widetilde{H}^{-r}(\Sigma)} + \|v_0\|_{\widetilde{H}^{-(s+1)}([0,T]\times\Omega)}).$$

Here we have respectively denoted by $\widetilde{H}^{-r}(\Sigma)$ and $\widetilde{H}^{-(s+1)}([0,T]\times\Omega)$ the dual spaces of $H^r(\Sigma)$ and $H^{s+1}([0,T]\times\Omega)$.

**5.2. *Step 2*: *approximation of a delta distribution by a product of Gaussian beams.*** Recall that $(v_j)_{j=1}^m$ is a family solutions to $\Box_g v_j = 0$ as in (18). The second step of the proof of Theorem 1 is to choose the solutions $v_j$ so that they allow us to obtain information about $q_1 - q_2$ on the left-hand side of the estimate (62). The boundary values corresponding to $v_j$ will be denoted by $f_j$. We use the Gaussian beam construction of Section 3 to produce approximate delta functions from products of Gaussian beams. We shall need the following elementary results. Let $f : \mathbb{R}^d \to \mathbb{C}$ be a Lipschitz function. We define the Lipschitz seminorm of $f$ as

$$\|f\|_{\mathrm{Lip}} := \inf\{c \geq 0 \mid |f(x) - f(y)| \leq c|x - y|\}.$$

**Lemma 20.** *Let $d \in \mathbb{N}$, $\tau > 0$ and $b$ be Lipschitz. The estimate*

$$\left| b(z_0) - \left(\frac{\tau}{\pi}\right)^{\frac{d}{2}} \int_{\mathbb{R}^d} b(z) e^{-\tau|z-z_0|^2}\, dz \right| \leq c_d \|b\|_{\mathrm{Lip}} \tau^{-\frac{1}{2}}$$

*holds true for all $z_0 \in \mathbb{R}^d$. In particular, the integral on the left converges uniformly to $b(z_0)$ when $\tau \to \infty$. Here $c_d := \Gamma\left(\frac{d+1}{2}\right) / \Gamma\left(\frac{d}{2}\right)$.*

We omit the proof of Lemma 20 as it can be proved similarly to the following more general result:

**Lemma 21.** *Let $\tau > 0$, $x \in \mathbb{R}^d_+$, $d \geq 2$, and assume $x = (x_1, \ldots, x_d)$, where $x_1 \geq 0$. Let $b : \mathbb{R}^d_+ \to \mathbb{C}$ be Lipschitz. Define a map $\Phi : (-\infty, 0] \to \left[\frac{1}{2}, 1\right]$ by*

$$\Phi(s) := \frac{1}{\sqrt{\pi}} \int_s^{\infty} e^{-t^2}\, dt. \tag{63}$$

*The estimate*

$$\left| b(x) - \frac{1}{\Phi(-\sqrt{\tau} x_1)} \left(\frac{\tau}{\pi}\right)^{\frac{d}{2}} \int_{\mathbb{R}^d \cap \{z_1 \geq 0\}} b(z) e^{-\tau|z-x|^2}\, dz \right| \leq 2 c_d \|b\|_{\mathrm{Lip}} \tau^{-\frac{1}{2}}$$

*holds true for all $x \in \mathbb{R}^d \cap \{x_1 \geq 0\}$. In particular, the integral on the left converges uniformly to $b$ as $\tau \to \infty$. Here $c_d = \Gamma\left(\frac{d+1}{2}\right) / \Gamma\left(\frac{d}{2}\right)$.*

*Proof.* Let us write $x = (x_1, x')$ and assume without loss of generality that $x' = 0$. To begin, recall the identities

$$\int_{\mathbb{R}^d} e^{-|z|^2}\, dz = \pi^{\frac{d}{2}} \quad \text{and} \quad \int_{\mathbb{R}^d} |z| e^{-|z|^2}\, dz = c_d \pi^{\frac{d}{2}}.$$

Note also that

$$\int_{\mathbb{R}^d \cap \{z_1 \geq 0\}} e^{-\tau|z-x|^2}\, dz = \int_0^{\infty} \int_{\mathbb{R}^{d-1}} e^{-\tau((s-x_1)^2 + |z'|^2)}\, dz'\, ds = \left(\frac{\pi}{\tau}\right)^{\frac{d}{2}} \Phi(-\sqrt{\tau} x_1).$$

Since $b$ is Lipschitz in $\mathbb{R}^d_+$, we have

$$|b(\tau^{-\frac{1}{2}} s + x_1, \tau^{-\frac{1}{2}} z') - b(x_1, 0)| \leq \|b\|_{\mathrm{Lip}} \tau^{-\frac{1}{2}} |(s, z')|$$

for any $x_1 \geq 0$ and$(s, z') \in \mathbb{R}^d$, $s > -\sqrt{\tau}x_1$. Thus we may calculate

$$\begin{aligned}
&\left|\Phi(-\sqrt{\tau}x_1)b(x_1,0) - \left(\frac{\tau}{\pi}\right)^{\frac{d}{2}} \int_{\mathbb{R}^d \cap \{z_1 \geq 0\}} b(z) e^{-\tau|z-x|^2}\, dz\right| \\
&\qquad = \left(\frac{\tau}{\pi}\right)^{\frac{d}{2}} \int_{\mathbb{R}^d \cap \{z_1 \geq 0\}} (b(x_1,0) - b(z)) e^{-\tau|z-x|^2}\, dz \\
&\qquad = \pi^{-\frac{d}{2}} \int_{-\sqrt{\tau}x_1}^{\infty} \int_{\mathbb{R}^{d-1}} (b(x_1,0) - b(\tau^{-\frac{1}{2}}s + x_1, \tau^{-\frac{1}{2}}z')) e^{-|z|^2}\, dz'\, ds \\
&\qquad \leq \|b\|_{\mathrm{Lip}} \pi^{-\frac{d}{2}} \int_{-\sqrt{\tau}x_1}^{\infty} \int_{\mathbb{R}^{d-1}} |(s,z')| e^{-|z|^2}\, dz'\, ds \\
&\qquad \leq \|b\|_{\mathrm{Lip}} \tau^{-\frac{1}{2}} \pi^{-\frac{d}{2}} \int_{\mathbb{R}^d} |z| e^{-|z|^2}\, dz = c_d \|b\|_{\mathrm{Lip}} \tau^{-\frac{1}{2}}.
\end{aligned}$$

Finally, dividing the above inequality by $\Phi(-\sqrt{\tau}x_1)$, and observing that $\Phi$ is monotone and satisfies $\Phi(0) = \frac{1}{2}$ and $\Phi(s) \to 1$ as $s \to -\infty$, we have the claim. □

We will apply Lemmas 20 and 21 with $d = n+1$ and the function $b$ will be a multiple of $q_1 - q_2$. Lemma 20 will be applied for recovery of points that lie in $W \setminus \Sigma$ and Lemma 21 for recovery of points on $\Sigma$.

To achieve the factor $\tau^{d/2} = \tau^{(n+1)/2}$ appearing in Lemmas 20 and 21, we use the solutions of Corollary 11 with $p = 4$ and scale them by a constant $\tau^{1/8}$. This change amounts to scaling the boundary values $f_j$ by $\tau^{1/8}$. The estimates (41) and (44) still hold by taking $k$, $l$, $K$ and $N$ large enough. Moreover, when applying Lemma 21, we modify the functions of Corollary 11 by multiplying them by $\Phi(-\sqrt{\tau}x_1)$ with a suitable number $x_1 \geq 0$.

Recall that Gaussian beams concentrate on light-like geodesics. We show that at the intersection points of geodesics, the corresponding product of Gaussian beams approximates the delta function of the intersection point. Taking this approach, one can recover information about the difference of the unknown potentials $q_1$ and $q_2$ at points where the geodesics intersect. When the geodesics intersect only once, the proof is simpler and instructive. For this reason, we first analyze the case where the geodesics intersect only once and prove the general case after that.

**5.3.** ***Proof in the case of a single intersection point.*** Let $p_0 \in W$, where $W$ is as in (3). In this case $p_0 \in I^+(\Sigma)$ by assumption and by Lemma 15 there is a future-directed optimal geodesic $\gamma_1$ from $\Sigma$ to $p_0$ that does not intersect $\{t = 0\}$. By making a small perturbation of $\gamma_1$, we have another geodesic $\gamma_2$ that intersects $\gamma_1$ at $p_0$ and does not intersect $\{t = 0\}$. Since the geodesics are causal, they exit the compact set $[0, T] \times \Omega$ in finite parameter time. By the assumption of this simplified case, $\gamma_1$ and $\gamma_2$ intersect only at $p_0$. Let $\delta' > 0$ be small parameter so that the Fermi coordinates (26), associated to $\gamma_1$ and $\gamma_2$, are defined for $|y| < \delta'$.

By Proposition 10 and Corollary 11 there is $\tau_0 > 0$ such that, for $j = 1, 2$ and $\tau \geq \tau_0$, we may choose

$$v_j = \tau^{\frac{1}{8}}(v_{\tau,j} + r_j) \quad \text{and} \quad f_j = v_j|_{\Sigma}, \quad j = 1, 2, \tag{64}$$

so that $\Box_g(v_{\tau,j}+r_j)=0$ in $[0,T]\times\Omega$. Here the function $v_{\tau,j}$ stands for the Gaussian beam described in Section 3 corresponding to the geodesic $\gamma_j$. We also have that the correction term $r_j$ satisfies

$$r_j|_\Sigma = 0, \quad j=1,2. \tag{65}$$

By (34) and (35) and Proposition 10 applied with $p=4$, we have for $\tau\ge\tau_0$

$$\begin{aligned} v_{\tau,j}(s,y) &= \tau^{\frac{n}{8}} e^{i\tau\Theta_j(s,y)} a^{(j)}(s,y), && \tau\ge\tau_0,\\ a^{(j)}(s,y) &= \chi\left(\frac{|y|}{\delta'}\right)\sum_{k'=0}^{N}\tau^{-k'} b_{k'}^{(j)}(s,y), && \tau\ge\tau_0,\\ b_{k'}^{(j)}(s,y) &= \sum_{k''=0}^{N} b_{k',k''}^{(j)}(s,y), \end{aligned} \tag{66}$$

where $b_{k',k''}^{(j)}(s,y)$ is a family of complex-valued homogeneous polynomials of order $k''$ in the variable $y$. We emphasize that all functions on the right-hand sides of (66) are independent of $\tau$. Thanks to Proposition 10, see also (36) and (37), we also have

$$b_{k'}^{(j)}(0,0) = b_{0,0}^{(j)}(0,0) = 1, \quad j=1,2. \tag{67}$$

In addition, by (40), (41) and (44), we get for $j=1,2$ and $k>l+\frac{1}{2}(n-1)$

$$\begin{aligned} \|v_{\tau,j}\|_{H^l([0,T]\times\Omega)} &= O(\tau^{-\frac{n}{8}+l}), && \tau\ge\tau_0,\\ \|r_j\|_{H^l([0,T]\times\Omega)} &= O(\tau^{-K}), && \tau\ge\tau_0, \end{aligned} \tag{68}$$

if $N$ satisfies $K=\frac{1}{2}(N+1-k)-1$. (If $N$ defined this way is not an integer, we redefine it as $\lfloor N+1\rfloor$.) We imposed the condition $k>l+\frac{1}{2}(n-1)$ to embed the energy space $E^l$ into $H^k([0,T]\times\Omega)$; see Remark 7. This condition is needed to control certain Sobolev norms in the following computations. Furthermore, by (41) and assuming that $l>\frac{1}{4}(n+1)$ (to embed $H^l([0,T]\times\Omega)$ into $L^4([0,T]\times\Omega)$) we get

$$\begin{aligned} \|v_{\tau,j}\|_{L^4([0,T]\times\Omega)} &= O(1), && j=1,2,\ \tau\ge\tau_0,\\ \|r_j\|_{L^4([0,T]\times\Omega)} &= O(\tau^{-K}), && j=1,2,\ \tau\ge\tau_0. \end{aligned} \tag{69}$$

Since $\Box_g$ is a linear operator, the complex conjugates of $v_1$ and $v_2$, denoted by $\bar{v}_1$ and $\bar{v}_2$, also solve $\Box_g v=0$. We set

$$v_j := \bar{v}_{j-2} \quad\text{and}\quad f_j := v_j|_\Sigma, \quad j=3,4.$$

Combining the trace theorem with (65) and (68) in the case $l=s+\frac{3}{2}$, we obtain an estimate for the boundary values $f_j$ for $j=1,2,3,4$ and $\tau\ge\tau_0$, as

$$\begin{aligned} \|f_j\|_{H^{s+1}(\Sigma)} &= \|v_j|_\Sigma\|_{H^{s+1}(\Sigma)} = \tau^{\frac{1}{8}}\|(v_{\tau,j}+r_j)|_\Sigma\|_{H^{s+1}(\Sigma)}\\ &\le \tau^{\frac{1}{8}}\|v_{\tau,j}\|_{H^{s+3/2}([0,T]\times\Omega)} \le C\tau^{s-\frac{n}{8}+\frac{13}{8}}. \end{aligned} \tag{70}$$

For $j=5,\dots,m$, we choose Gaussian beams at fixed $\tau=\tau_0$ as

$$v_j = \tau_0^{-\frac{n+1}{8}} v_1|_{\tau=\tau_0} \quad\text{and}\quad f_j = v_j|_\Sigma, \quad j=5,\dots,m. \tag{71}$$

Let us write

$$\hat{v} = v_5\cdots v_m.$$

**Remark 22.** We remark that by making $\tau_0 > 0$ large enough, there exists $c > 0$ such that

$$|\hat{v}(s, y)| > c \tag{72}$$

in a neighborhood of $(s, y) = (0, 0)$. Indeed, by taking $l > \frac{n+1}{2}$ and combining Morrey's inequality with (68), we deduce that both $v_{\tau,1}$ and $r_1$ are continuous functions for $\tau \geq \tau_0$. In particular, the function $v_1$ is continuous according to (64). Proposition 10 ensures that $\Theta_1(0, 0) = 0$ and $b^{(1)}_{0,0}(0, 0) = 1$. Looking at (66) one has

$$a_1(0, 0) = 1 + O(\tau^{-1}), \quad \tau \geq \tau_0.$$

Hence

$$\tau^{-\frac{n+1}{8}} v_1(0, 0) = 1 + \tau^{-\frac{n}{8}} r_1(0, 0) = 1 + O(\tau^{-\frac{n}{8}}), \quad \tau \geq \tau_0,$$

where in the last equality, we have used (68) to deduce $\|r_1\|_{L^\infty([0,T]\times\Omega)} = O(1)$. Thus we have, by redefining $\tau_0$ if necessary,

$$|\hat{v}(0, 0)| = (\tau^{-\frac{n+1}{8}} |v_1(0, 0)|)^{m-4} > \tfrac{1}{2}$$

for all $\tau \geq \tau_0$. By the continuity of $\hat{v}$, we have (72) on a neighborhood of $(0, 0)$.

With these choices, we now analyze the left-hand side of (62). We decompose the product $v_1 \cdots v_m$ as the sum of a leading term plus lower-order terms. A straightforward computation holding for $\tau \geq \tau_0$ yields

$$\begin{aligned} v_1 \cdots v_m &= |v_1|^2 |v_2|^2 \hat{v} \\ &= \tau^{\frac{1}{2}} |v_{\tau,1} + r_1|^2 |v_{\tau,2} + r_2|^2 \hat{v} \\ &= \tau^{\frac{1}{2}} (|v_{\tau,1}|^2 + v_{\tau,1}\bar{r}_1 + r_1\bar{v}_{\tau,1} + |r_1|^2)(|v_{\tau,2}|^2 + v_{\tau,2}\bar{r}_2 + r_2\bar{v}_{\tau,2} + |r_2|^2)\hat{v} \\ &= \tau^{\frac{1}{2}} |v_{\tau,1}|^2 |v_{\tau,2}|^2 \hat{v} + \mathcal{L}_1, \end{aligned} \tag{73}$$

where $\mathcal{L}_1$ is a sum of products of terms each containing $r_1$ or $r_2$, or their complex conjugates, as well as $\hat{v}$ as a factor. Consequently, we can choose $(N, k, l, K)$ in (68) so that together with the Cauchy–Schwarz inequality, we obtain

$$\|\mathcal{L}_1\|_{L^1([0,T]\times\Omega)} = O(\tau^{-R}) \tag{74}$$

for some $R > 1$. Indeed, let us analyze one term of $\mathcal{L}_1$, say $\tau^{1/2} v_{\tau,1} |v_{\tau,2}|^2 \bar{r}_1 \hat{v}$. As $\hat{v}$ is continuous, it is bounded in $[0, T] \times \Omega$. Using (69), we have for $\tau \geq \tau_0$

$$\begin{aligned} \tau^{\frac{1}{2}} \|v_{\tau,1} |v_{\tau,2}|^2 \bar{r}_1 \hat{v}\|_{L^1([0,T]\times\Omega)} &\lesssim \tau^{\frac{1}{2}} \|v_{\tau,1} |v_{\tau,2}|^2 \bar{r}_1\|_{L^1([0,T]\times\Omega)} \\ &\lesssim \tau^{\frac{1}{2}} \|v_{\tau,1}\|_{L^4([0,T]\times\Omega)} \|v_{\tau,2}\|^2_{L^4([0,T]\times\Omega)} \|r_1\|_{L^4([0,T]\times\Omega)} = O(\tau^{\frac{1}{2}-K}). \end{aligned}$$

A similar analysis allows us to deduce that the $L^1([0, T] \times \Omega)$ norms of the other terms of $\mathcal{L}_1$ are $O(\tau^{1/2-K})$. Therefore

$$\|\mathcal{L}_1\|_{L^1([0,T]\times\Omega)} = O(\tau^{\frac{1}{2}-K}), \quad \tau \geq \tau_0.$$

Thus we can take $R = K - \frac{1}{2}$ in (74). Note that we can always find suitable parameters $l$, $k$, $N$ and $K$ satisfying $K = \frac{N+1-k}{2} - 1$, $k > l + \frac{n-1}{2}$ and $l > \frac{n+1}{2} > \frac{n+1}{4}$. One possible choice is

$$l = n+1, \quad k = 3n+1, \quad K = 2, \quad N = 3(n+1).$$

Let us now analyze the leading term in the expansion (73):

$$\tau^{\frac{1}{2}} |v_{\tau,1}|^2 |v_{\tau,2}|^2 \hat{v} = \tau^{\frac{n+1}{2}} e^{i\tau\Theta_1(x)} e^{-i\tau\overline{\Theta}_1(x)} e^{i\tau\Theta_2(x)} e^{-i\tau\overline{\Theta}_2(x)} |a^{(1)}(x)|^2 |a^{(2)}(x)|^2 \hat{v}(x).$$

For technical convenience, we consider a normal coordinate system $(x^a)_{a=0}^n$ centered at the point $p_0$, which is the unique intersection of the geodesics $\gamma_1$ and $\gamma_2$. At the center of the normal coordinates the metric is the identity matrix and all Christoffel symbols vanish; see, e.g., [O'Neill 1983, Section 3]. At the point $p_0$ both the phase functions $\Theta_1$ and $\Theta_2$ vanish and their gradients are real. Using the properties (42), we have the following Taylor expansion around $p_0$:

$$\Theta_1(x) - \overline{\Theta}_1(x) + \Theta_2(x) - \overline{\Theta}_2(x) = 2i\, x \cdot \nabla^2 \operatorname{Im}(\Theta_1 + \Theta_2)|_{x=0} x + O(|x|^3).$$

Here $\nabla^2 \operatorname{Im}(\Theta_1 + \Theta_2)$ is a positive definite matrix at $p_0$ (i.e., at $x = 0$ in normal coordinates) by the last two conditions of (42), because $\Theta_1$ and $\Theta_2$ are positive semidefinite and positive definite in directions transversal to $\dot{\gamma}_1$ and $\dot{\gamma}_2$ respectively.

Recall from (66) that the amplitude $a^{(j)}$, $j = 1, 2$, has the cut-off function $\chi$ as a factor. Therefore, we may redefine $\delta' > 0$ smaller, if necessary, so that at the intersection $U_1 \cap U_2$ of the supports

$$U_j := \operatorname{supp}(a^{(j)}) = \operatorname{supp}(v_{j,\tau})$$

we have $\operatorname{Im}(\Theta_1 + \Theta_2) > 0$. Let us write

$$\mathcal{H} := 2\nabla^2 \operatorname{Im}(\Theta_1 + \Theta_2)|_{x=0} > 0 \tag{75}$$

so that in the normal coordinates

$$\Theta_1(x) - \overline{\Theta}_1(x) + \Theta_2(x) - \overline{\Theta}_2(x) = i\, x \cdot \mathcal{H} x + \widehat{\Theta}(x), \tag{76}$$

where $\widehat{\Theta}(x) = O(|x|^3)$. Using the precise expressions in (66) for $a^{(j)}$, $j = 1, 2$, we see that

$$|a^{(1)}(x)|^2 \, |a^{(2)}(x)|^2 = |b_0^{(1)}(x)|^2 \, |b_0^{(2)}(x)|^2 + \tau^{-1} \mathcal{L}_2(x),$$

where

$$\|\mathcal{L}_2\|_{L^1([0,T]\times\Omega)} = O(1). \tag{77}$$

Via a calculation similar to the one done in deriving (73), we deduce in the coordinates $(x^a)_{a=0}^n$ that

$$\tau^{\frac{1}{2}} |v_{\tau,1}|^2 |v_{\tau,2}|^2 \hat{v} = \tau^{\frac{n+1}{2}} |\chi_1(x)|^2 |\chi_2(x)|^2 |b_0^{(1)}(x)|^2 |b_0^{(2)}(x)|^2 \hat{v}(x) e^{i\tau\widehat{\Theta}(x)} e^{-\tau x \cdot \mathcal{H} x} + \underbrace{\tau^{-1} \tau^{\frac{n+1}{2}} |\chi_1(x)|^2 |\chi_2(x)|^2 \hat{v}(x) e^{i\tau\widehat{\Theta}(x)} e^{-\tau x \cdot \mathcal{H} x} \mathcal{L}_2(x)}_{:=\widehat{\mathcal{L}}_2(x)}. \tag{78}$$

Here the functions $\chi_j$, $j = 1, 2$, stand for the normal coordinate representations of $\chi_j$, which in Fermi coordinates $(s, y)$ corresponding to the geodesics $\gamma_j$ take the form $\chi(|y|/\delta')$. Note that $\chi_j(0) = 1$. Recall that $\widehat{\Theta}(x) = O(|x|^3)$. By using (77), making the change of variables $x \mapsto \tau^{-1/2}x$ and using the fact $\tau\widehat{\Theta}(\tau^{-1/2}x) = \tau^{-1/2}O(|x|^3) = O(|x|^3)$ one calculates that

$$\|\widehat{\mathcal{L}}_2\|_{L^1([0,T]\times\Omega)} = O(\tau^{-1}). \tag{79}$$

(See (84) below for a similar calculation.)

For the sake of brevity, we set

$$q(x) = q_1(x) - q_2(x), \quad A(x) = |\chi_1(x)|^2 |\chi_2(x)|^2 |b_0^{(1)}(x)|^2 |b_0^{(2)}(x)|^2 \hat{v}(x). \tag{80}$$

By Proposition 10, see also (67), we have in the normal coordinates that $\Theta_j(0) = 0$ and $b_0^{(j)}(0) = 1$, $j = 1, 2$. Note also that $\widehat{\Theta}(0) = 0$. Thus one gets

$$A(0) = \hat{v}(0). \tag{81}$$

Integrating in the normal coordinates, and combining (73) and (78), we find

$$\begin{aligned}
\int_{[0,T]\times\Omega} & v_0(q_1 - q_2)v_1 \cdots v_m \, dV_g \\
&= \tau^{\frac{n+1}{2}} \int_{B(p_0)} v_0(x)q(x)A(x)e^{i\tau\widehat{\Theta}(x)}e^{-\tau x\cdot\mathcal{H}x}\,dx + \int_{B(p_0)} v_0(x)q(x)(\mathcal{L}_1(x) + \widehat{\mathcal{L}}_2(x))\,dx \\
&= \tau^{\frac{n+1}{2}} \int_{B(p_0)} v_0(x)q(x)A(x)e^{-\tau x\cdot\mathcal{H}x}\,dx + \int_{B(p_0)} v_0(x)q(x)(\mathcal{L}_1(x) + \widehat{\mathcal{L}}_2(x))dx \\
&\qquad + \tau^{\frac{n+1}{2}} \int_{B(p_0)} v_0(x)q(x)A(x)(e^{i\tau\widehat{\Theta}(x)} - 1)e^{-\tau x\cdot\mathcal{H}x}\,dx.
\end{aligned} \tag{82}$$

(Recall that $v_0$ is a function satisfying $\Box_g v_0 = 0$ with $v_0|_{t=T} = \partial_t v_0|_{t=T} = 0$ in $\Omega$.)

With slight abuse of notation, there are now two possible cases in the integral (82).

Case 1: If $U_1 \cap U_2 \cap \Sigma = \varnothing$, then $B(p_0)$ is a ball in $\mathbb{R}^{n+1}$ centered at $p_0$ such that $U_1 \cap U_2 \subset B(p_0)$ and we can proceed without changes.

Case 2: If $U_1 \cap U_2 \cap \Sigma \neq \varnothing$, then $B(p_0)$ is a ball in $\mathbb{R}^{n+1}_+$ centered at $p_0$ such that $U_1 \cap U_2 \subset B(p_0)$. In this case, we can similarly derive the identity (82) in boundary normal coordinates. As can be seen from Lemma 21, to obtain a proper normalization, we scale by the constant $1/\Phi(-\sqrt{\tau}x_1)$. This is achieved by multiplying the function $v_0$ by $1/\Phi(-\sqrt{\tau}x_1)$. Since $\Phi : (-\infty, 0] \to \left[\frac{1}{2}, 1\right]$, this scaling will contribute to redefining the constant of the stability estimate by a factor of at most 2. Here $\Phi(s) := \pi^{-1/2}\int_s^\infty e^{-t^2}\,dt$ is as in (63) and $x_1$ denotes the first coordinate of $p_0$ in local coordinates of $\mathbb{R}^{n+1}_+$.

We now analyze each term in (82) above. Thanks to (62), we can control the term on the left-hand side of (82) in terms of $\delta, \varepsilon_1, \ldots, \varepsilon_m$ and the size of $f_j$. The first term after the second equality in (82) contains information about $q_1 - q_2$ and will be analyzed last. At this point, the exponential function $e^{-\tau x\cdot\mathcal{H}x}$ will play a crucial role, as it will act as an approximate delta function. This is due to the fact that $\mathcal{H}$ is a positive definite matrix, see (75). By combining (74) and (79), and using the fact that both $v_0$ and $q$ are uniformly bounded, we have for $\tau \geq \tau_0$ that

$$\left|\int_{B(p_0)} v_0(x)q(x)(\mathcal{L}_1(x) + \widehat{\mathcal{L}}_2(x))\,dx\right| \lesssim \tau^{-1}. \tag{83}$$

Making the change of variables $x \mapsto \tau^{-1/2}x$, we obtain

$$\left|\tau^{\frac{n+1}{2}}\int_{B(p_0)} v_0(x)q(x)A(x)(e^{i\tau\widehat{\Theta}(x)}-1)e^{-\tau x\cdot\mathcal{H}x}\,dx\right| = \left|\int_{B(p_0)}(v_0qA)(\tau^{-\frac{1}{2}}x)(e^{i\tau\widehat{\Theta}(\tau^{-1/2}x)}-1)e^{-x\cdot\mathcal{H}x}\,dx\right| \lesssim \tau^{-\frac{1}{2}}. \quad (84)$$

In the last inequality we used that $|e^{z_1}-e^{z_2}|\le|z_1-z_2|e^{\max\{|z_1|,|z_2|\}}$ for all $z_1,z_2\in\mathbb{C}$ and $\widehat{\Theta}(x)=O(|x|^3)$ to deduce that

$$|e^{i\tau\widehat{\Theta}(\tau^{-\frac{1}{2}}x)}-1|\le\tau^{-\frac{1}{2}}|x|^3e^{\tau^{-\frac{1}{2}}|x|^3},\qquad \tau\ge\tau_0.$$

We also used that the functions $v_0qA$, $e^{-x\cdot\mathcal{H}x}$, $|x|^3$ and $e^{\tau^{-1/2}|x|^3}$ are uniformly bounded in $B(p_0)$.

Let us then analyze the first term after the second equality in (82). Since $\mathcal{H}$ is positive definite, there exists another positive definite matrix $B$ so that $B^2=\mathcal{H}$. Making the change of variables $x\mapsto Bx$, we deduce that in Case 1, where $U_1\cap U_2\cap\Sigma=\varnothing$, we have

$$\int_{B(p_0)} v_0(x)q(x)A(x)e^{-\tau x\cdot\mathcal{H}x}\,dx=\int_{\mathbb{R}^{n+1}} v_0(Bz)q(Bz)\,A(Bz)|g(z)|^{\frac{1}{2}}\,|\det B|^{-1}\,e^{-\tau|z|^2}\,dz. \quad (85)$$

For convenience, we set

$$b(z):=q(Bz)A(Bz)|g(z)|^{\frac{1}{2}}|\det B|^{-1}.$$

By using (81), we see that in normal coordinates

$$b(0)=(q_1(0)-q_2(0))\hat{v}(0)|\det\mathcal{H}|^{-\frac{1}{2}}. \quad (86)$$

The identities (82) and (85), combined with estimates (83) and (84) yield

$$\left|\left(\frac{\tau}{\pi}\right)^{\frac{n+1}{2}}\int_{\mathbb{R}^{n+1}} v_0(z)b(z)e^{-\tau|z|^2}\,dz\right|\lesssim\tau^{-\frac{1}{2}}+\left|\int_{[0,T]\times\Omega} v_0(q_1-q_2)v_1\cdots v_m\,dV_g\right|.$$

Thanks to (62), the second term on the right can be controlled in terms of $\delta$, $\varepsilon_1,\dots,\varepsilon_m$ and sizes of the functions $f_j$. Thereby, applying Lemma 20 with $z_0=0$ and $d=n+1$, we get

$$\begin{aligned}|b(0)|&\le\left|b(0)-\left(\frac{\tau}{\pi}\right)^{\frac{n+1}{2}}\int_{\mathbb{R}^{n+1}} v_0(z)b(z)e^{-\tau|z|^2}\,dz\right|+\left|\left(\frac{\tau}{\pi}\right)^{\frac{n+1}{2}}\int_{\mathbb{R}^{n+1}} v_0(z)b(z)e^{-\tau|z|^2}\,dz\right|\\
&\lesssim c_{n+1}\|v_0b\|_{C^1}\tau^{-\frac{1}{2}}+\tau^{-\frac{1}{2}}\\
&\qquad+\left[\delta\varepsilon_1^{-1}\cdots\varepsilon_m^{-1}+\varepsilon_1^{-1}\cdots\varepsilon_m^{-1}(\varepsilon_1\|f_1\|_{H^{s+1}(\Sigma)}+\dots+\varepsilon_m\|f_m\|_{H^{s+1}(\Sigma)})^{2m-1}\right]\\
&\lesssim\frac{C_{\Omega,m,T,q_j,\chi}M}{\kappa_0^{2m-1}}\left[2\tau^{-\frac{1}{2}}+\frac{\kappa_0^{2m-1}\delta}{mM}\varepsilon_1^{-1}\cdots\varepsilon_m^{-1}\right.\\
&\qquad\left.+\frac{1}{m-1}\varepsilon_1^{-1}\cdots\varepsilon_m^{-1}(\varepsilon_1\|f_1\|_{H^{s+1}(\Sigma)}+\dots+\varepsilon_m\|f_m\|_{H^{s+1}(\Sigma)})^{2m-1}\right],\end{aligned} \quad (87)$$

where $v_0$ can be chosen so that in normal coordinates $v_0(0)=1$. The above holds for any $M>0$ and $\kappa_0>0$. In the last step, we scaled $\delta$ by $\kappa_0^{2m-1}/(mM)$. The coefficients 2 and $1/(m-1)$ in front of $\tau^{-1/2}$ and $\varepsilon_1^{-1}\cdots\varepsilon_m^{-1}$ in (87) were included to simplify formulas later on. We will determine the constants $M$ and $\kappa_0$ later. Their role in obtaining a stability estimate will be clarified in Lemma 23 below.

In Case 2 we arrive at the same integral (82), but the integration is only over the half-space $\mathbb{R}^{n+1}_+$ and due to scaling of $v_0$ the integral is scaled by a constant $1/\Phi(-\sqrt{\tau}x_1))$. All other calculations after (82) remain similar, but one needs to apply Lemma 21 instead of Lemma 20 to obtain the estimate (87). We omit the details.

**5.4.** ***Step 3*: *optimizing the error terms.*** The last step of the proof of Theorem 1 (in this simplified setting) is to choose $\tau$ and $\varepsilon_1, \dots, \varepsilon_m$ in terms of $\delta$ to have the right-hand side of (87) as small as possible. We begin by setting

$$\varepsilon_1 = \dots = \varepsilon_m =: \varepsilon.$$

Note that by (70) and (71), we have for $\tau \geq \tau_0$ that

$$\begin{aligned} \varepsilon \| f_j \|_{H^{s+1}(\Sigma)} &\sim \varepsilon \tau^{s-\frac{n}{8}+\frac{13}{8}}, \quad j = 1, 2, 3, 4, \ \tau \geq \tau_0, \\ \varepsilon \| f_j \|_{H^{s+1}(\Sigma)} &\sim \varepsilon \tau_0^{s-\frac{n}{4}+\frac{3}{2}}, \quad j = 5, \dots, m. \end{aligned} \tag{88}$$

To guarantee the unique solvability of our nonlinear wave equation (16), we require the quantities on the right-hand sides of (88) to be bounded by $\kappa$, which was given by Lemma 9. Recall that $\tau_0 > 0$ is a fixed large parameter, which we chose at (71). The parameter was especially chosen so that the Gaussian beams $v_j$ for $j = 5, \dots, m$ have small enough correction terms.

Lemma 23 shows how to choose the parameters $\tau$ and $\varepsilon$ in (87) optimally given $\kappa > 0$ and $\delta \in (0, M)$. By choosing $\kappa_0 \leq \kappa$, we will see that the optimal value for $\tau$ is at least $\tau_0$ and we also have that $\varepsilon \| f_j \|_{H^{s+1}(\Sigma)} \leq \kappa$.

**Lemma 23.** *Let $C, M, s > 0$ and $m \in \mathbb{N}$. Let also $\tau_0 \geq 1$, $\delta \in (0, M)$ and $\kappa \in (0, 1)$. Then there are $\varepsilon > 0$, $\tau \geq \tau_0$ and $\kappa_0 \leq \kappa$ such that*

$$\begin{aligned} f(\varepsilon, \tau) &:= 2\tau^{-\frac{1}{2}} + \frac{\kappa_0^{2m-1}\delta}{mM}\varepsilon^{-m} + \frac{1}{m-1}\varepsilon^{m-1}\, \tau^{(2m-1)(s-\frac{n}{8}+\frac{13}{8})} \\ &\leq C_{s,m,M,\kappa_0}\, \delta^{\frac{8(m-1)}{2m(m-1)(8s-n+13)+2m-1}} \end{aligned}$$

*and we also have*

$$\varepsilon \tau^{s-\frac{n}{8}+\frac{13}{8}} \leq C\kappa.$$

*Proof.* To simplify the notation, let us write $\hat{s} := (2m-1)\big(s - \frac{n}{8} + \frac{13}{8}\big)$ and $\gamma_0 = \kappa_0^{2m-1}/M$. We take $\kappa_0 \leq \kappa$ to be so that $\gamma_0 < 1$. We will redefine $\kappa_0 > 0$ smaller later if necessary. A direct computation shows that

$$\partial_\varepsilon f = -(\gamma_0\delta)\varepsilon^{-m-1} + \varepsilon^{m-1}\tau^{\hat{s}}, \quad \partial_\tau f = -\tau^{-\frac{3}{2}} + \frac{\hat{s}}{m-1}\varepsilon^{m-1}\tau^{\hat{s}-1}.$$

Making $\partial_\varepsilon f = \partial_\tau f = 0$, we obtain the critical points of $f$, namely

$$\begin{aligned} \tau &= ((m-1)\hat{s}^{-1})^{\frac{2(2m-1)}{2\hat{s}m+2m-1}} (\gamma_0\delta)^{-\frac{2(m-1)}{2\hat{s}m+2m-1}}, \\ \varepsilon &= ((m-1)\hat{s}^{-1})^{-\frac{2\hat{s}}{2\hat{s}m+2m-1}} (\gamma_0\delta)^{\frac{4\hat{s}m+2m-1-2\hat{s}}{(2\hat{s}m+2m-1)(2m-1)}}. \end{aligned} \tag{89}$$

(One can also verify that the Hessian of $f$ at the critical point is positive definite, and hence the critical point is a local minimum.)

Note now that

$$\tau = ((m-1)\hat{s}^{-1})^{\frac{2(2m-1)}{2\hat{s}m+2m-1}}(\gamma_0\delta)^{-\frac{2(m-1)}{2\hat{s}m+2m-1}} \geq ((m-1)\hat{s}^{-1})^{\frac{2(2m-1)}{2\hat{s}m+2m-1}}\kappa_0^{-\frac{2(m-1)(2m-1)}{2\hat{s}m+2m-1}},$$

because by assumption $0<\delta<M$ and since $\gamma_0 = \kappa_0^{2m-1}/M$. Since the constant

$$((m-1)\hat{s}^{-1})^{\frac{2(2m-1)}{2\hat{s}m+2m-1}} > 0$$

and the exponent

$$-\frac{2(m-1)(2m-1)}{2\hat{s}m+2m-1} < 0$$

do not depend on $\kappa_0$, we may choose $\kappa_0$ so that $\kappa_0 < C\kappa$ and that $\tau$ in (90) satisfies

$$\tau = ((m-1)\hat{s}^{-1})^{\frac{2(2m-1)}{2\hat{s}m+2m-1}}(\gamma_0\delta)^{-\frac{2(m-1)}{2\hat{s}m+2m-1}} \geq \tau_0.$$

With these choices, we have at the critical point of $f(\varepsilon,\tau)$ given by (89)

$$\varepsilon\tau^{s-\frac{n}{8}+\frac{13}{8}} = \varepsilon\tau^{\frac{\hat{s}}{2m-1}} = (\gamma_0\delta)^{\frac{1}{(2m-1)}} = \left(\frac{\kappa_0^{2m-1}}{M}\delta\right)^{\frac{1}{2m-1}} \leq \kappa_0 < C\kappa$$

for all $0<\delta<M$. A straightforward calculation using (89) shows that $\tau^{-1/2}$, $(\gamma_0\delta)\varepsilon^{-m}$ and $\varepsilon^{m-1}\tau^{\hat{s}}$ are all bounded by $C_{s,m,M,\kappa_0}(\gamma_0\delta)^{(m-1)/(2\hat{s}m+2m-1)}$, where the constant $C_{s,m,M,\kappa_0}$ is independent of $\varepsilon$ and $\tau$. □

Recall (86) and (87). We set $\varepsilon_1 = \cdots = \varepsilon_m =: \varepsilon$ and apply Lemma 23 to obtain

$$\begin{aligned}|v_0(p_0)||q_1(p_0)-q_2(p_0)||\hat{v}(p_0)||\det\mathcal{H}|^{-\frac{1}{2}} &\lesssim \frac{C_{\Omega,T,q_j,\chi}M}{\kappa_0^{2m-1}}\left(2\tau^{-\frac{1}{2}} + \frac{\kappa_0^{2m-1}\delta}{mM}\varepsilon^{-m} + \frac{1}{m-1}\varepsilon^{m-1}\tau^{(2m-1)(s-\frac{n}{8}+\frac{13}{8})}\right)\\ &\leq C_0\delta^{\frac{8(m-1)}{2m(m-1)(8s-n+13)+2m-1}}.\end{aligned} \tag{90}$$

Since $p_0 \in I^-(\Sigma)\cap([0,T]\times\Omega)$, by Lemma 15 there exists a past-directed optimal geodesic from $\Sigma$ to $p_0$ such that the first intersection of the geodesic and $\Sigma$ is transverse. Since the intersection is transverse, the geodesic does not intersect $\{t=T\}$. Therefore, we may choose $v_0$ to be a Gaussian beam corresponding to the geodesic such that $v_0|_{t=T} = \partial_t v_0|_{t=T} = 0$. We may assume by normalizing that $v_0(p_0)=1$. Recall also that $\hat{v}(p_0) > c > 0$ and $|\det\mathcal{H}|>0$ by (72) and (75) respectively. Dividing (90) by the norm of $v_0(p_0)\hat{v}(p_0)|\det\mathcal{H}|^{-1/2}$, we have a stability estimate

$$|q_1(p_0)-q_2(p_0)| \leq C\delta^{\frac{8(m-1)}{2m(m-1)(8s-n+13)+2m-1}} \tag{91}$$

at the point $p_0$. We next show that the constant $C$ can be redefined to be independent of $p_0$.

**5.5.** ***Step 4*: *uniformity of the constant $C$*.** So far we have obtained the estimate (91) regarding the difference of $q_1$ and $q_2$ at the single point $p_0$. The constant $C$ may at this point depend on $p_0$. Next we argue that the constant $C$ can be redefined to be independent of $p_0$. This will then yield (4) and conclude the proof of Theorem 1 in the simplified setting, where we assumed that light-like geodesics can intersect only once.

To show that $C$ in (91) can be taken to be independent of $p_0$, we first construct an open cover of $W \subset I^+(\Sigma) \cap I^-(\Sigma)$ as follows. (Recall from (3) that $W$ is a compact set which we can reach and observe from $\Sigma$.) Let $z \in W$. By Lemma 15 there are optimal light-like geodesics $\gamma_1$ and $\gamma_2$ that intersect at $z$ and which do not intersect $\{t = 0\}$. We may reparametrize so that $\gamma_1(0) = \gamma_2(0) = z$. Let $\varepsilon = |\dot\gamma_1(0) - \dot\gamma_2(0)|$. Here and below $|\cdot|$ denotes the $\mathbb{R}^n$ norm of vectors in local coordinates.

By Corollary 14 there are open neighborhoods $\mathcal{U}_1$ and $\mathcal{U}_2$ of $z$ and families of Gaussian beams $v_\tau(x, l, \cdot)$ (including the correction term) parametrized by $x \in \mathcal{U}_l$, $l = 1, 2$, such that all the implied constants, such as $\tau_0$, in the construction of $v_\tau(x, l, \cdot)$ are uniform in $x$. Moreover, still by using Corollary 14, the geodesics $\gamma_{x,l}$ corresponding to the Gaussian beams $v_\tau(x, l, \cdot)$ satisfy $|\dot\gamma_l(0) - \dot\gamma_{x,l}(0)| \le \frac{\varepsilon}{3}$, $l = 1, 2$. Then, for $x \in \mathcal{U}_1 \cap \mathcal{U}_2$, we also have that

$$|\dot\gamma_{x,1}(0) - \dot\gamma_{x,2}(0)| \ge \frac{\varepsilon}{3} > 0. \tag{92}$$

We conclude that the geodesics $\gamma_{x,1}$ and $\gamma_{x,2}$ intersect at $x$ and do not have the same graph. We also set

$$\hat v_x(\cdot) = (v_\tau(x, l, \cdot))^{m-4}|_{\tau=\tau_0, l=1}$$

for $x \in \mathcal{U}_1 \cap \mathcal{U}_2$. By redefining $\tau_0$ larger, if necessary, we have that $|\hat v_x(x)| \ge d > 0$ for all $x \in \mathcal{U}_1 \cap \mathcal{U}_2$.

In deriving (91) in this Section 5, we used normal coordinates. Normal coordinates are uniquely defined by choosing an orthonormal basis at a point. By using a local orthonormal frame on a neighborhood $\mathcal{U}_3$ of $z$, we may find a family of normal coordinates smoothly parametrized by $x \in \mathcal{U}_3$. It follows that the contribution to $C$ in (91) coming from the use of normal coordinates may be taken to be uniformly bounded for all $x \in \mathcal{U}_3$. All things considered, by repeating the arguments in this Section 5, we may take the constant $C$ to be uniform for all $x \in \mathcal{U}_1 \cap \mathcal{U}_2 \cap \mathcal{U}_3$, where $\mathcal{U}_1 \cap \mathcal{U}_2 \cap \mathcal{U}_3$ is a neighborhood of $z$.

Recall that we aim to estimate the difference of $q_1$ and $q_2$ in the compact set $W \subset I^+(\Sigma) \cap I^-(\Sigma)$. By covering first the compact set $W$ by the sets $\mathcal{U}_1 \cap \mathcal{U}_2 \cap \mathcal{U}_3$ as described above and then passing to a finite subcover, we have that (91) holds for all $z \in W$. Finally, we apply Lemma 18 with $P = 1$ to deduce that there is a finite family of functions $v_{z,0}$ satisfying $\Box_g v_{z,0} = 0$ in $[0, T] \times \Omega$ and $v_{z,0}|_{t=T} = \partial_t v_{z,0}|_{t=T} = 0$ and such that $|v_{z,0}(z)| \ge c > 0$. (Only finitely many of the functions $v_{z,0}$ are actually distinct.) Combining everything yields the estimate

$$|(v_{z,0}(z)\hat v_z(z)(q_1 - q_2))(z)|\,|\det \mathcal{H}_z|^{-\frac{1}{2}} \le C\delta^{\frac{8(m-1)}{2m(m-1)(8s-n+13)+2m-1}}, \tag{93}$$

which holds for all $z \in W$. Here the point $z$ corresponds to the origin 0 of normal coordinates centered at $z$ and all the quantities are expressed in these coordinates. The point $z$ is also the point where the geodesics $\gamma_{z,1}$ and $\gamma_{z,2}$ corresponding to the Gaussian beams $v_\tau(z, 1, \cdot)$ and $v_\tau(z, 2, \cdot)$ intersect.

By Remark 22, we have that $|v_{z,0}(z)| \ge c > 0$ and hence $|\hat v_z(z)| \ge d > 0$ in (93). Let us estimate $|\det \mathcal{H}_z|$, where

$$\mathcal{H}_z = 2\nabla^2 \operatorname{Im}(\Theta_{z,1}(x) + \Theta_{z,2}(x))|_{x=z}.$$

Here $\Theta_{z,1}$ and $\Theta_{z,2}$ are the phase functions corresponding to the Gaussian beams $v_\tau(z, 1, \cdot)$ and $v_\tau(z, 2, \cdot)$ respectively. Here also $\nabla^2$ is the invariant Hessian. In the normal coordinates centered at $z$ we have that the geodesics $\gamma_{z,1}$ and $\gamma_{z,2}$ are rays emanating in from origin. Since $\gamma_{z,1}$ and $\gamma_{z,2}$ do not have the same graphs, the rays are not same and there is a positive angle (in the $\mathbb{R}^{n+1}$ metric) between the rays in the normal coordinates. Due to (92), the angle is uniformly bounded from below by a positive constant. Consequently, using also the facts that

$$\mathrm{Im}(\nabla^2\Theta_{z,l})(z) \geq 0, \quad \mathrm{Im}(\nabla^2\Theta_{z,l})(z)|_{\dot{\gamma}_{z,l}(0)^{\perp}} > 0$$

we conclude that there is $h > 0$ such that $|\det \mathcal{H}_z| > h$ for all $z \in W$. Dividing (93) by $|v_{z,0}(z)|$, $|\hat{v}_x(x)|$ and $|\det \mathcal{H}_z|^{-1/2}$, and redefining $C$ larger, if necessary, concludes the proof in the special case where we assumed that light-like geodesics can intersect only once.

**5.6.** ***Step 5*: *multiple intersections.*** We have proven Theorem 1 in the special case, which assumed that the used light-like geodesics intersect only once. In the case of multiple intersections, we can perform a similar analysis as in the special case, but this leads to an estimate for a sum of terms regarding the difference $q_1 - q_2$ at the intersection points. To separate the contributions coming from several intersection points, we will use separation matrices and a separation filter constructed in Lemmas 17 and 18. Most of the work needed to handle the case of several intersections was already done in proving these two lemmas.

Let $N$ be globally hyperbolic Lorentzian manifold. Let also $\bar{g}$ be an auxiliary Riemannian metric on $N$. The following lemma shows that given a compact set $K \subset N$ there is a bound on the number of possible intersections of pairs of causal geodesics in $K$. We will apply the lemma with $K = [0, T] \times \Omega$ and $N = \mathbb{R} \times M$. Let us recall some relevant facts. An open set $O$ of $N$ is convex if for every pair of points $p, q \in O$ with $p \neq q$ there is a unique geodesic $\gamma$ of $O$ connecting the points. Each point in $N$ has a neighborhood that is convex [O'Neill 1983, Section 5, Proposition 7]. Let $p \in N$ and let $U_p$ be its convex neighborhood. By [O'Neill 1983, Section 14, Exercise 10] (see also [Minguzzi 2019]), and the fact that $U_p$ is convex, it follows that $p$ has a neighborhood $V_p \subset U_p$ with two properties:

(i) Any causal curve starting in $V_p$ that leaves it never returns.

(ii) Two distinct geodesic segments in $V_p$ can intersect at most once.

We mention that in [O'Neill 1983] the sets $V_p$ are called causality neighborhoods. It follows from conditions (i) and (ii) that any two distinct causal geodesics can intersect at most once in $V_p$.

**Lemma 24.** *Let $(N, g)$ be a globally hyperbolic Lorentzian manifold and let $K \subset N$ be a compact set. There is $P \geq 1$ with the following property. Let $\gamma_1$ and $\gamma_2$ be two distinct causal geodesics. Then the number of intersection points of $\gamma_1$ and $\gamma_2$ is bounded by $P$,*

$$\#(\Gamma_1 \cap \Gamma_2) \leq P,$$

*where $\Gamma_j \subset N$ are the graphs of the geodesics $\gamma_j$, $j = 1, 2$.*

*Proof.* Let $\gamma_1$ and $\gamma_2$ be as in the statement of the lemma. Because $N$ is globally hyperbolic, every point $p \in N$ has a neighborhood $V_p$ satisfying the conditions (i) and (ii). Because $K$ is compact, there exists a finite subcover

$$\bigcup_{a=1}^{P} V_{p_a} \supset K$$

formed of sets $V_{p_a}$. Since a pair of distinct causal geodesics can intersect at most once within each $V_{p_a}$, it follows that the number of intersections of $\gamma_1$ and $\gamma_2$ is bounded by $P$. □

**Lemma 25.** *Let $\bar{g}$ be an auxiliary Riemannian metric on a globally hyperbolic Lorentzian manifold $(N, g)$ and let $K \subset N$ be compact. Then there exists $\tilde{\rho} > 0$ such that for any pair of distinct causal geodesics $\gamma_1$ and $\gamma_2$ intersecting at points $x_1, \dots, x_P$ we have*

$$d_{\bar{g}}(x_j, x_k) \geq \tilde{\rho}, \quad j \neq k,$$

*where $d_{\bar{g}}(x, y)$ is the distance induced by $\bar{g}$.*

*Proof.* Let $x_j$ and $x_k$, $x_j \neq x_k$, be intersection points of $\gamma_1$ and $\gamma_2$. Let $\{V_{p_a}\}_{a=1}^{P}$ be a finite open cover of $K$ consisting of sets with properties (i) and (ii). Let $\tilde{\rho} > 0$ be a Lebesgue number (see, e.g., [Munkres 1975, Lemma 27.5]) of $\{V_{p_a}\}_{a=1}^{P}$ with respect to the distance $d_{\bar{g}}$. It follows that the ball $B_{\bar{g}}(x_j, \tilde{\rho})$ belongs to $V_{p_a}$ for some $a \in \{1, \dots, P\}$. Since the geodesics $\gamma_1$ and $\gamma_2$ can intersect at most once in $V_{p_a}$, the point $x_k$ cannot belong to $V_{p_a}$. Consequently, $x_k \notin B_{\bar{g}}(x_j, \tilde{\rho})$ and thus $d_{\bar{g}}(x_j, x_k) \geq \tilde{\rho}$ as claimed. □

By Lemma 24 we know that there is $P \in \mathbb{N}$ such that light-like geodesics can intersect at most $P$ times in $[0, T] \times \Omega$. Let also $\bar{g}$ be an auxiliary Riemannian metric on $[0, T] \times \Omega$.

Let $\gamma_1$ and $\gamma_2$ be future-directed light-like geodesics starting from $\Sigma$ that intersect for the first time at $z$ and which do not intersect $\{t = 0\}$. Let

$$z_1, \dots, z_{P_0}$$

be the intersection points of $\gamma_1$ and $\gamma_2$ arranged as $z_1 \leq z_2 \leq \dots \leq z_{P_0}$, where $P_0 \leq P$ and

$$z = z_1.$$

As in (64), we choose

$$v_j = \tau^{\frac{1}{8}}(v_{\tau,j} + r_j), \quad j = 1, 2,$$

to be Gaussian beams associated to $\gamma_1$ and $\gamma_2$. We also choose

$$v_j = \bar{v}_{j-2}, \quad j = 3, 4, \qquad \text{and} \qquad \hat{v} = (v_1|_{\tau=\tau_0})^{m-4}$$

as before. Since the product $v_1 \cdots v_m$ is supported on neighborhoods of the intersection points, the term

$$\langle v_0(q_1 - q_2), v_1 \cdots v_m \rangle_{L^2([0,T]\times\Omega)} = \int_{[0,T]\times\Omega} v_0(q_1 - q_2) v_1 \cdots v_m \, dV_g$$

becomes a sum of terms

$$\sum_{j=1}^{P_0} \tau^{\frac{n+1}{2}} \int_{B(z_j)} v_0(x)(q_1 - q_2)(x) A(x) e^{i\tau\widehat{\Theta}(x)} e^{-\tau x \cdot \mathcal{H}_{z_j} x} \, dV_g, \tag{94}$$

where each set $B(z_j)$ is a neighborhood of $z_j$, $j = 1, \dots, P_0$. Here $\widehat{\Theta}(x)$ and $A(x)$ are defined similarly to (76) and (80) respectively and

$$\mathcal{H}_{z_j} = 2\nabla^2 \operatorname{Im}(\Theta_1(x) + \Theta_2(x))|_{x=z_j}, \quad j = 1, \dots, P_0$$

as before.

By Lemma 25 there is a uniform constant $\tilde{\rho} > 0$ independent of $z_1, \dots, z_{P_0}$ such that $d_{\tilde{g}}(z_i, z_j) \geq \tilde{\rho}$ for all $i \neq j$. This implies we can use Lemma 18 to find a separation filter on $[0, T] \times \Omega$. So, let $\mathcal{M} = \{v_{f_k}\}_{k \in \mathcal{K}}$ be a separation filter of $[0, T] \times \Omega$ given by Lemma 18 with the compact set $W$ as $K$ and $P_0$ as $P$. Here $f_k \in C^\infty(\Sigma)$ and $\mathcal{K}$ is a finite index set. According to Lemma 18, the corresponding solutions $v_{f_k}$ to $\Box_g v = 0$ in $[0, T] \times \Omega$ can be chosen so that the associated separation matrix $(v_{f_k}(z_j))_{k,j=1}^{P_0}$ is invertible.

We note that if $B(z_j) \cap \Sigma \neq \varnothing$ in (94), then the corresponding integrals can be taken over the half-space $\mathbb{R}^{n+1}_+$ in boundary normal coordinates. As indicated by Lemma 21 we need to use the scaling factor $1/\Phi(-\sqrt{\tau} z_{j,1})$ to recover the value of $q_1 - q_2$ at $z_j$. This can be achieved by scaling the functions $v_{f_k}$ of the separation matrices by $1/\Phi(-\sqrt{\tau} z_{j,1})$. This amounts to scaling the matrix element of the upper triangular parts of each of the separation matrices by $1/\Phi(-\sqrt{\tau} z_{j,1})$ if $B(z_j) \cap \Sigma \neq \varnothing$. Here $z_{j,1}$ is the first coordinate of $z_j$ in boundary normal coordinates. Recall from (63) that $\Phi : (-\infty, 0] \to \left[\frac{1}{2}, 1\right]$. Thus by choosing a larger $\tau_0$, if necessary, the separation matrices with scaled elements stay invertible. Much as in Step 4, it is possible to make the choices of the boundary normal coordinates so that the choices amount to redefining the constant $C$.

By repeating the calculation in (62) we have for each $k \in \mathcal{K}$ that

$$|\langle v_{f_k}(q_1 - q_2), v_1 \cdots v_m \rangle_{L^2([0,T]\times\Omega)}| \leq C_k\, (\varepsilon_1 \cdots \varepsilon_m)^{-1} \left[ \delta + \left( \sum_{j=1}^{m} \varepsilon_j \|f_j\|_{H^{s+1}(\Sigma)} \right)^{2m-1} \right].$$

We apply (94) with $v_{f_k}$ in place of $v_0$ and note that the integrals in (94) are the value of the integrand at $z_k$ plus a term of size $O(\tau^{-1/2})$ by calculations (75)–(87) and Lemmas 20 and 21. Optimizing as in Section 5.4 in $\tau$ and $\varepsilon_1, \dots, \varepsilon_m$ yields that

$$\left| \sum_{j=1}^{P_0} v_{f_k}(z_j)(q_1(z_j) - q_2(z_j))\hat{v}(z_j) |\det \mathcal{H}_{z_j}|^{-\frac{1}{2}} \right| \leq C\, \delta^{\frac{8(m-1)}{2m(m-1)(8s-n+13)+2m-1}}$$

for all $k = 1, \dots, P_0$. Let us define a matrix $A$ and a vector $\mathcal{Q}$ as

$$A_{kj} = v_{f_k}(z_j), \quad \mathcal{Q}_j = (q_1(z_j) - q_2(z_j))\hat{v}(z_j) |\det \mathcal{H}_{z_j}|^{-\frac{1}{2}},$$

where $j, k = 1, \dots, P_0$. Since the separation matrix $\{v_{f_k(x_j)}\}_{k,j=1}^{P_0}$ is invertible, we have that

$$|\mathcal{Q}_1| \leq \|\mathcal{Q}\| = \|A^{-1}(A\mathcal{Q})\| \leq \|A\|^{-1} \|A\mathcal{Q}\| \leq \|A\|^{-1} C\, \delta^{\frac{8(m-1)}{2m(m-1)(8s-n+13)+2m-1}}.$$

Recalling that $z_1 = z$, we thus have

$$\left| (q_1(z) - q_2(z))\hat{v}(z) |\det \mathcal{H}_z|^{-\frac{1}{2}} \right| \leq C\, \|A\|^{-1} \delta^{\frac{8(m-1)}{2m(m-1)(8s-n+13)+2m-1}}. \tag{95}$$

In (95), $\hat{v}_z$, $\det \mathcal{H}_z$, but also $\|A\|^{-1}$ depend on the point $z$. We argued in Section 5.5 that $\hat{v}_z$, $|\det \mathcal{H}_z|^{-1/2}$ have norms which are uniformly bounded from below with respect to $z$. Since the separation filter $\mathcal{M}$ is a finite collection, we may also bound $\|A\|^{-1}$ uniformly when we consider different points in $W$. Using these facts and by dividing by $|\hat{v}(z) \det \mathcal{H}_z|^{-1/2}$ and redefining $C$ shows that

$$\|q_1 - q_2\|_{L^\infty(W)} \leq C\, \delta^{\frac{8(m-1)}{2m(m-1)(8s-n+13)+2m-1}}.$$

This concludes the proof of Theorem 1.

## Appendix: Proof of Proposition 8

Before proceeding to the proof of Proposition 8, which concerns the well-posedness of the linear wave equation (14), we need the following lemma.

**Lemma 26.** *Let $(\mathbb{R}\times M, g)$ be a globally hyperbolic manifold. Let also $t_0 \in \mathbb{R}$ and let $S_{t_0} = \{t = t_0\}\times M$ be the corresponding Cauchy surface. Suppose $V \subset S_{t_0}$ is a compact set in $S_{t_0}$ and $W$ is an open neighborhood of $V$ in $\mathbb{R}\times M$. Then there exists $\varepsilon > 0$ such that $([t_0, t_0+\varepsilon]\times M)\cap J^+(V) \subset W$. In particular if $V \Subset U$, where $U$ is open in $S_{t_0}$, there exists $\varepsilon > 0$ such that $([t_0, t_0+\varepsilon]\times M)\cap J^+(V) \subset [t_0, t_0+\varepsilon]\times U$.*

*Proof.* For the first claim, assume that there is no such $\varepsilon > 0$. Then there are numbers $\varepsilon_k > 0$ with $\varepsilon_k \to 0$ as $k \to \infty$ and points $p_k \in ([t_0, t_0+\varepsilon_k]\times M)\cap J^+(V)$, but $p_k \notin W$. Since $W$ is open, any accumulation points of $p_k$, if existing, are not in $W$. As $\varepsilon_k \to 0$ there is $\varepsilon \geq \varepsilon_k$ for all sufficiently large $k \in \mathbb{N}$, say, $k \geq k_0$. It follows that $p_k \in ([t_0, t_0+\varepsilon]\times M)\cap J^+(V)$ for all $k \geq k_0$.

Because $\mathbb{R}\times M$ is foliated by the space-like Cauchy surfaces $S_t$, we have

$$[t_0, t_0+\varepsilon]\times M = \bigcup_{t\in[t_0,t_0+\varepsilon]} S_t.$$

Also $S_t \subset J^-(S_T)$ for all $t \leq T$, because if $\gamma$ is any nonextendible future-directed causal curve with $\gamma(s) \in S_t$ for some $s \in \mathbb{R}$, then this curve intersects $S_T$ in the future. By [Bär et al. 2007, Corollary A.5.4], the intersection $J^-(S_{t_0+\varepsilon})\cap J^+(V)$ is compact. So $[t_0, t_0+\varepsilon]\times M$ being a closed subset of $J^-(S_{t_0+\varepsilon})$ implies that $([t_0, t_0+\varepsilon]\times M)\cap J^+(V)$ is compact and there exists a convergent subsequence $p_{k_i} \to p \in ([t_0, t_0+\varepsilon]\times M)\cap J^+(V)$. Due to the construction, as $\varepsilon_{k_i} \to 0$ we have $p_{k_i} \to p \in \{t = t_0\}\times M \cap J^+(V) = V \subset W$. Thus $p \in W$, which is a contradiction.

Suppose now that $W = (a, b)\times U$ where $t_0 \in (a, b) \subset \mathbb{R}$. Then if $\varepsilon > 0$ is so small that $([t_0, t_0+\varepsilon]\times M)\cap J^+(V) \subset (a, b)\times U$, we have $([t_0, t_0+\varepsilon]\times M)\cap J^+(V) \subset [t_0, t_0+\varepsilon]\times U$. If not, we would have some $p = (t, x) \in ([t_0, t_0+\varepsilon]\times M)\cap J^+(V)$ with $t \notin [t_0, t_0+\varepsilon]$ or $x \notin U$. Both options are invalid, so also the second claim holds. □

*Proof of Proposition 8.* Let us first recall results in the special case where $\Omega$ is a domain $\Omega \subset \mathbb{R}^n$. From [Lasiecka et al. 1986] we know that there exists a unique solution $v \in E^{s+1}$ to the problem

$$\begin{cases} (\partial_t^2 - \Delta_h)v = F & \text{in } [0, T]\times\Omega, \\ v = f & \text{on } [0, T]\times\partial\Omega, \\ v = u_0, \quad \partial_t v = u_1 & \text{in } \{t = 0\}\times\Omega, \end{cases} \tag{96}$$

if $h(t,\cdot)$ is a smooth 1-parameter family of Riemannian metrics on $\mathbb{R}^n$ and if we assume that $F$, $f$, $u_0$ and $u_1$ satisfy the regularity and compatibility conditions of our proposition in $\mathbb{R}^n$. Under these assumptions, we also know from classical results such as [Ikawa 1968] that there exists a unique solution $w \in E^{s+1}$ to

$$\begin{cases} (\partial_t^2 - \Delta_h)w + Aw = G & \text{in } [0, T]\times\Omega, \\ w = 0 & \text{on } [0, T]\times\partial\Omega, \\ w = \partial_t w = 0 & \text{in } \{t = 0\}\times\Omega \end{cases} \tag{97}$$

when $A \in C^\infty([0,T]\times\Omega)$ and $G \in E^s$. By combining the mentioned results, we have that the problem

$$\begin{cases}(\partial_t^2-\Delta_h)u + Au = F & \text{in } [0,T]\times\Omega,\\ u = f & \text{on } [0,T]\times\partial\Omega,\\ u = u_0,\ \ \partial_t u = u_1 & \text{in } \{t=0\}\times\Omega\end{cases}\tag{98}$$

has a unique solution $u \in E^{s+1}$ and the regularity results of [Ikawa 1968; Lasiecka et al. 1986] also show that $\partial_\nu u \in H^s([0,T]\times\partial\Omega)$. Indeed, by solving first (96) for $v \in E^{s+1}$ and then defining $G := Av \in E^{s+1}$ for the problem (97) we find $w \in E^{s+1}$ (in fact $w \in E^{s+2}$) solving (97) and so that $u := v - w$ solves (98).

Let us then explain how these results translate to the case of a globally hyperbolic manifold $[0,T]\times M$ equipped with a Lorentzian metric $g = \beta(t,x)\,dt^2 - h(t,x)$. Here $\beta > 0$ is a smooth function and $h(t,\cdot)$ is a smooth 1-parameter family of Riemannian metrics on $M$. The function $\beta > 0$ is bounded from above and below by the compactness of $[0,T]\times\Omega$. Via a conformal change of variables we obtain a scaled metric $\tilde g = dt^2 - \beta^{-1}h$ for which the wave operator transforms as

$$\mathcal{P} := \beta^{\frac32}\Box_g\beta^{-\frac12} = \Box_{\tilde g} + V = \partial_t^2 - \Delta_{\beta^{-1}h} + V.$$

Here $V(t,x)$ is a smooth function and $\Delta_{\beta^{-1}h}$ for each $t \in [0,T]$ is the Laplace–Beltrami operator of the Riemannian metric $(\beta^{-1}h)(t,\cdot)$ on $M$. Then $u$ solving (14) is equivalent to $v := \beta^{1/2}u$ solving

$$\begin{cases}\mathcal{P}v = \beta^{\frac32}F & \text{in } [0,T]\times\Omega,\\ v = \beta^{\frac12}f & \text{on } \Sigma,\\ v = \beta^{\frac12}u_0,\ \ \partial_t v = \frac12\beta^{-\frac12}\partial_t\beta u_0 + \beta^{\frac12}u_1 & \text{in } \{t=0\}\times\Omega.\end{cases}\tag{99}$$

From [Hörmander 1983, Theorem 24.1.1] we know that there exists a unique solution to (99). (The result of that work is not however sufficient to us.) Also, in local coordinates in $\Omega$ this equation is of the form (98). Let us define

$$R = \beta^{\frac32}F,\quad r = \beta^{\frac12}f,\quad r_0 = \beta^{\frac12}u_0,\quad r_1 = \tfrac12\beta^{-\frac12}\partial_t\beta u_0 + \beta^{\frac12}u_1.$$

Note that $\{t=0\}\times M$ is a space-like Cauchy surface in $\mathbb{R}\times M$. Because $\Omega\subset M$ is a compact manifold, there exists a finite atlas $\{(U_j,\varphi_j)\}_{j=1}^k$ covering $\Omega$. Let $\chi_j$ be a partition of unity subordinate to $\{U_j\}_{j=1}^k$ and let us denote the support of $\chi_j$ as

$$V_j = \operatorname{supp}(\chi_j) \Subset U_j.$$

Let us also define

$$R_j = \chi_j R,\quad r_j = \chi_j|_\Sigma r,\quad r_{0,j} = \chi_j r_0,\quad r_{1,j} = \chi_j r_1,$$

denote the corresponding coordinate representations as

$$\tilde R_j = R_j\circ\varphi_j^{-1},\quad \tilde r_j = r\circ\varphi_j^{-1},\quad \tilde r_{0,j} = r_0\circ\varphi_j^{-1}\quad \tilde r_{1,j} = r_1\circ\varphi_j^{-1},$$

and let

$$\tilde U_j = \varphi_j(U_j).$$

We construct a solution to (14) by patching up local solutions following partly the proof of [Bär et al. 2007, Proposition 3.2.11]. As we will see, this is possible due to the finite speed of propagation of solutions to a wave equation. Let $K_j$ be an open set with compact closure such that $V_j \subset K_j$ and $\overline{K}_j \subset U_j$. If $t \in \mathbb{R}$, we may use Lemma 26 to deduce that there exists $\varepsilon > 0$ so that

$$((t, t+\varepsilon) \times \Omega) \cap J^+(V_j) \subset (t, t+\varepsilon) \times K_j \subset (t, t+\varepsilon) \times U_j$$

holds. (This is similar to [Bär et al. 2007, proof of Proposition 3.2.11].) Here $J^+$ is defined with respect to the conformal metric $\tilde{g}$. We remark that $J^+$ of a set is conformally invariant. By the compactness of $[0, T]$, there is a finite set of numbers $\varepsilon_i > 0$ and $t_i \in \mathbb{R}$ so that the intervals

$$I_i := (t_i, t_i + \varepsilon_i)$$

cover $[0, T]$. We are going to find a solution to our wave equation (14) iteratively in the index $i$ so that at each step of the iteration we have $(I_i \times \Omega) \cap J^+(V_j) \subset I_i \times U_j$, $j = 1, \dots, k$. Let us set $t_1 = 0 < t_2 < \dots < t_l$ and $t_l + \varepsilon_l = T$ and consider the set $((0, \varepsilon_1) \times \Omega) \cap J^+(V_j)$ first.

By the discussion around (98), we have that there is a unique solution $\tilde{u}_j \in E^{s+1}$ to

$$\begin{cases} \widetilde{\mathcal{P}}\tilde{u}_j = \widetilde{R}_j & \text{in } (0, \varepsilon_1) \times \widetilde{U}_j, \\ \tilde{u}_j = \tilde{r}_j & \text{on } (0, \varepsilon_1) \times \partial\widetilde{U}_j \cap \varphi_j(\partial\Omega), \\ \tilde{u}_j = 0 & \text{on } (0, \varepsilon_1) \times \partial\widetilde{U}_j \setminus \varphi_j(\partial\Omega), \\ \tilde{u}_j = \tilde{r}_{0,j}, \quad \partial_t \tilde{u}_j = \tilde{r}_{1,j} & \text{in } \{t = 0\} \times \widetilde{U}_j \end{cases} \tag{100}$$

in each coordinate chart $\widetilde{U}_j$, $j = 1, \dots, k$, in the time interval $(0, \varepsilon_1)$. (Here and below we understand $\varphi_j(\partial\Omega) = \varnothing$ if $U_j \cap \partial\Omega = \varnothing$.) Since our (14) satisfies the compatibility conditions (13), one can verify by a direct calculation that (100) satisfies the compatibility conditions of [Ikawa 1968; Lasiecka et al. 1986] that were needed for the unique solvability of (98). In particular, at the intersection of $\{t = 0\}$ and $\partial\widetilde{U}_j \cap \varphi_j(\partial\Omega)$ the compatibility conditions follow from the assumptions of the proposition we are proving. At the intersection of $\{t = 0\}$ and a neighborhood of $\partial\widetilde{U}_j \setminus \varphi_j(\partial\Omega)$ the initial values vanish due to the cut-off functions $\chi_j$. Thus (100) has a unique solution.

Next, let us define

$$u_j = \begin{cases} \tilde{u}_j \circ \varphi_j & \text{in } [0, \varepsilon_1] \times U_j, \\ 0 & \text{in } [0, \varepsilon_1] \times (\Omega \setminus U_j). \end{cases}$$

By the finite speed of propagation of solutions to a wave equation, see for example [Bär et al. 2007, Proposition 3.2.11], we have $\operatorname{supp}(u_j) \subset J^+(V_j)$, and by the condition $((0, \varepsilon_1) \times \Omega) \cap J^+(V_j) \subset (0, \varepsilon_1) \times K_j \subset (0, \varepsilon_1) \times U_j$, we have that

$$\tilde{u}_j = 0 \quad \text{in a neighborhood of } \partial\widetilde{U}_j \setminus \varphi_j(\partial\Omega).$$

Consequently, $u_j$ is the smooth continuation of $\tilde{u}_j \circ \varphi_j : U_j \to \mathbb{R}$ by zero and $u_j \in E^{s+1}$. We also continue $\tilde{u}_j$ smoothly by zero to $\mathbb{R}^n$ (or to $\mathbb{R}^+_n$ if $U_j$ is a boundary chart.)

We now patch up the functions $u_j$ as

$$u = \sum_{j=1}^{k} u_j \in E^{s+1}$$

to have a solution to (99) in the case $T=\varepsilon_1$. Indeed, we have on $((0,\varepsilon_1)\times U_j)$ that

$$\mathcal{P}u=\sum_{j=1}^{k}(\widetilde{\mathcal{P}}\tilde{u}_j)\circ\varphi_j=\sum_{j=1}^{k}\widetilde{R}_j\circ\varphi_j=\sum_{j=1}^{k}\chi_j R=R.$$

We also have that

$$\begin{cases} u=f & \text{on } [0,\varepsilon_1]\times\partial\Omega,\\ u=r_0,\quad \partial_t u=r_1 & \text{in } \{t=0\}\times\Omega,\end{cases}$$

which is (99) for $T=\varepsilon_1$.

We continue iteratively and extend $u$ to a solution of (14) in increasing time steps $t_i$. At each iteration step, which concerns the time-interval $I_i$, we use as the initial values $\tilde{u}|_{t=t_i}$ and $\partial_t u|_{t=t_i}$. These are well defined since $t_i<t_{i-1}+\varepsilon_{i-1}$. In this way, we found a unique solution $u\in E^{s+1}$ to (99) in $[0,T]\times\Omega$, and consequently a unique solution to (14) in the class $E^{s+1}$.

Next we show that the above regularity and unique existence results of solutions for (14) can be turned into the energy estimate (15) by using the closed graph theorem. Consider the Banach space $E^{s+1}$ and define a linear map

$$A: E^s\times H^{s+1}(\Sigma)\times H^{s+1}(\Omega)\times H^s(\Omega)\to E^{s+1}$$

by $A(F,f,u_0,u_1)=u$, where $u$ is the unique solution to (14). To have the energy estimate (15) it is sufficient to show that $A$ is continuous. By the closed graph theorem, this is in turn equivalent to showing that if

$$\begin{cases} (F_k,f_k,u_{0,k},u_{1,k})\to(F,f,u_0,u_1) & \text{in } E^s\times H^{s+1}(\Sigma)\times H^{s+1}(\Omega)\times H^s(\Omega),\\ A(F_k,f_k,u_{0,k},u_{1,k})\to u_\infty & \text{in } E^{s+1},\end{cases}$$

then

$$u_\infty=A(F,f,u_0,u_1).$$

Here $F_k\to\square_g u_\infty$ in $\mathcal{D}'([0,T]\times\Omega)$, $f_k\to u_\infty|_\Sigma$ in $\mathcal{D}'(\Sigma)$, and similarly for $t=0$, $u_{0,k}\to u_\infty$ and $u_{1,k}\to\partial_t u_\infty$ in $\mathcal{D}'(\Omega)$. Due to the uniqueness of limits, we have that $u_\infty$ solves (14). Therefore, by uniqueness of solutions to the wave equation, we have that $u=A(F,h,u_0,u_1)$. Hence $A$ is a bounded linear map and the energy estimate follows. □

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