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# RIGIDITY FOR VON NEUMANN ALGEBRAS OF GRAPH PRODUCT GROUPS I: STRUCTURE OF AUTOMORPHISMS

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We study various rigidity aspects of the von Neumann algebra  $L(\Gamma)$ , where  $\Gamma$  is a graph product group whose underlying graph is a certain cycle of cliques and the vertex groups are wreath-like product property (T) groups. Using an approach that combines methods from Popa's deformation/rigidity theory with new techniques pertaining to graph product algebras, we describe all symmetries of these von Neumann algebras and reduced  $C^*$ -algebras by establishing formulas in the spirit of Genevois and Martin's results on automorphisms of graph product groups.

## 1. Introduction

Graph product groups were introduced by E. Green [1990] in her Ph.D. thesis as natural generalizations of classical right-angled Artin and Coxeter groups. Their study has become a trendy subject over the years as they play key roles in various branches of topology and group theory. For example, over the last decade graph product groups have been intensively studied through the lens of geometric group theory resulting in many new important discoveries—[Agol 2013; Antolín and Minasyan 2015; Haglund and Wise 2008; Minasyan and Osin 2015; Wise 2009], just to enumerate a few.

In a different direction, by using techniques from measured group theory, interesting orbit equivalence rigidity results have been obtained for measure-preserving actions on probability spaces of specific classes of graph product groups, including many right-angled Artin groups [Horbez and Huang 2022; Horbez et al. 2023].

General graph product groups were considered in the analytic framework of von Neumann algebras for the first time in [Caspers and Fima 2017]. Since then several structural results such as strong solidity, absence/uniqueness of Cartan subalgebras, and classification of their tensor decompositions have been established in [Caspers 2020; Caspers and Fima 2017; Chifan and Kunnawalkam Elayavalli 2024; Chifan et al. 2018; Ding and Kunnawalkam Elayavalli 2024] for von Neumann algebras arising from these groups and their actions on probability spaces. Since general graph product groups display such a rich combinatorial structure, much remains to be done in this area, and understanding how this complexity is reflected in the von Neumann algebras remains mysterious.

This paper is the first of two which will investigate new rigidity aspects for von Neumann algebras of graph product groups through the powerful deformation/rigidity theory of Popa [2007]. This theory provides a novel conceptual framework through which a large number of impressive structural and rigidity

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results for von Neumann algebras have been discovered over the last two decades; see the surveys [Ioana 2013; 2018; Popa 2007; Vaes 2013]. These two papers will analyze new inputs in this theory from the perspective of graph product algebras. In the first paper, we completely describe the structure of all  $*$ -isomorphisms between von Neumann algebras arising from a large class of graph product groups; see Section 4. In the second paper [Chifan et al. 2025], we investigate superrigidity aspects of these von Neumann algebras.

**1.1. Statements of the main results.** To properly introduce our results, we briefly recall the construction of graph product groups. Let  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$  be a finite *simple graph* (i.e.,  $\mathcal{G}$  does not admit more than one edge between any two vertices, and no edge of  $\mathcal{G}$  starts and ends at the same vertex). The *graph product group*  $\Gamma = \mathcal{G}\{\Gamma_v\}$  of a given family of *vertex groups*  $\{\Gamma_v\}_{v \in \mathcal{V}}$  is the quotient of the free product  $*_{v \in \mathcal{V}} \Gamma_v$  by the relations  $[\Gamma_u, \Gamma_v] = 1$  whenever  $u$  and  $v$  are connected by an edge,  $(u, v) \in \mathcal{E}$ . Thus, graph products can be thought of as groups that “interpolate” between the direct product  $\times_{v \in \mathcal{V}} \Gamma_v$  (when  $\mathcal{G}$  is complete) and the free product  $*_{v \in \mathcal{V}} \Gamma_v$  (when  $\mathcal{G}$  has  $n$ ).

For any subgraph  $\mathcal{H} = (\mathcal{U}, \mathcal{F})$  of  $\mathcal{G}$ , we denote by  $\Gamma_{\mathcal{H}}$  the subgroup generated by  $\Gamma_{\mathcal{H}} = \langle \Gamma_u : u \in \mathcal{U} \rangle$ , and we call it the *full subgroup* of  $\mathcal{G}\{\Gamma_v\}$  corresponding to  $\mathcal{H}$ . A *clique*  $\mathcal{C}$  of  $\mathcal{G}$  is a maximal, complete subgraph of  $\mathcal{G}$ . The set of cliques of  $\mathcal{G}$  will be denoted by  $\text{cliq}(\mathcal{G})$ . The full subgroups  $\Gamma_{\mathcal{C}}$  for  $\mathcal{C} \in \text{cliq}(\mathcal{G})$  are called the *clique subgroups* of  $\mathcal{G}\{\Gamma_v\}$ .

In this paper we are interested in graph product groups arising from a specific class of graphs which we introduce next. A graph  $\mathcal{G}$  is called a *simple cycle of cliques* (the collection of such graphs we abbreviate  $\text{CC}_1$ ) if there is an enumeration of its clique set  $\text{cliq}(\mathcal{G}) = \{\mathcal{C}_1, \dots, \mathcal{C}_n\}$  with  $n \geq 4$  such that the subgraphs  $\mathcal{C}_{i,j} := \mathcal{C}_i \cap \mathcal{C}_j$  satisfy

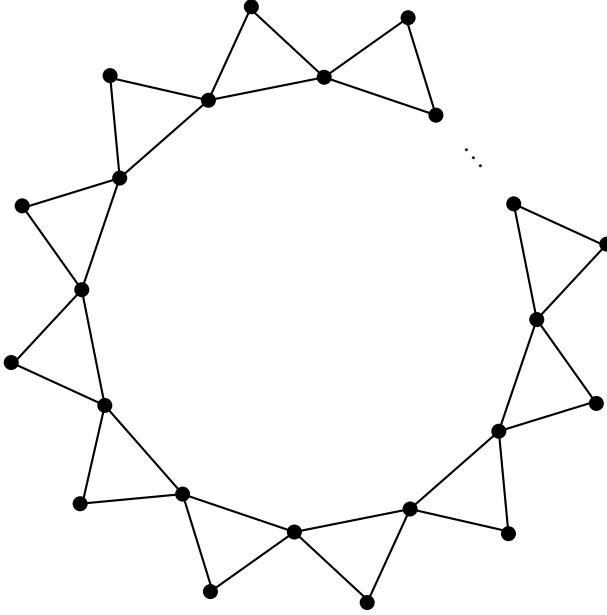
$$\mathcal{C}_{i,j} = \begin{cases} \emptyset & \text{if } \hat{i} - \hat{j} \in \mathbb{Z}_n \setminus \{\hat{1}, \hat{n-1}\}, \\ \neq \emptyset & \text{if } \hat{i} - \hat{j} \in \{\hat{1}, \hat{n-1}\}, \end{cases} \quad (1-1)$$

$$\mathcal{C}_i^{\text{int}} := \mathcal{C}_i \setminus (\mathcal{C}_{i-1,i} \cup \mathcal{C}_{i,i+1}) \neq \emptyset \quad \text{for all } i \in \overline{1, n}, \text{ with conventions } 0 = n \text{ and } n+1 = 1.$$

Note this automatically implies the cardinality  $|\mathcal{C}_i| \geq 3$  for all  $i$ . Also such an enumeration  $\text{cliq}(\mathcal{G}) = \{\mathcal{C}_1, \dots, \mathcal{C}_n\}$  is called a *consecutive cliques enumeration*. A basic example of such a graph is any simple, length  $n$ , cycle of triangles  $\mathcal{F}_n = (\mathcal{V}_n, \mathcal{E}_n)$ , which essentially looks like a flower-shaped graph with  $n$  petals, shown in Figure 1. In fact any graph in  $\text{CC}_1$  is a two-level clustered graph that is a specific retraction of  $\mathcal{F}_n$ ; for more details the reader may consult Section 4.

The goal of this paper is to describe the structure of all  $*$ -isomorphisms between *graph product group von Neumann algebras* (i.e., group von Neumann algebras arising from graph product groups), where the underlying graphs belong to  $\text{CC}_1$ . To introduce our results, we first highlight a canonical family of  $*$ -isomorphisms between these algebras that are analogous to the graph product groups situation. Let  $\mathcal{G}, \mathcal{H} \in \text{CC}_1$  be isomorphic graphs, and fix  $\sigma : \mathcal{G} \rightarrow \mathcal{H}$  an isometry. Let  $\text{cliq}(\mathcal{G}) = \{\mathcal{C}_1, \dots, \mathcal{C}_n\}$  be a consecutive cliques enumeration. Let  $\Gamma_{\mathcal{G}}$  and  $\Lambda_{\mathcal{H}}$  be graph product groups and assume that, for every  $i \in \overline{1, n}$ , there are  $*$ -isomorphisms

$$\theta_{i-1,i} : \mathcal{L}(\Gamma_{\mathcal{C}_{i-1,i}}) \rightarrow \mathcal{L}(\Lambda_{\sigma(\mathcal{C}_{i-1,i})}), \quad \xi_i : \mathcal{L}(\Gamma_{\mathcal{C}_i^{\text{int}}}) \rightarrow \mathcal{L}(\Lambda_{\sigma(\mathcal{C}_i^{\text{int}})}), \quad \theta_{i,i+1} : \mathcal{L}(\Gamma_{\mathcal{C}_{i,i+1}}) \rightarrow \mathcal{L}(\Lambda_{\sigma(\mathcal{C}_{i,i+1})});$$



**Figure 1.** A simple, length  $n$  cycle of triangles, which is an example of a graph that is in  $\text{CC}_1$ .

here and in what follows we use the convention as before that  $n = 0$  and  $n + 1 = 1$ . Results in [Section 7.1](#) show these  $*$ -isomorphisms induce a unique  $*$ -isomorphism  $\phi_{\theta, \xi, \sigma} : \mathcal{L}(\Gamma_{\mathcal{G}}) \rightarrow \mathcal{L}(\Lambda_{\mathcal{H}})$  defined as

$$\phi_{\theta, \xi, \sigma}(x) = \begin{cases} \theta_{i-1, i}(x) & \text{if } x \in \mathcal{L}(\Gamma_{\mathcal{C}_{i-1, i}}), \\ \xi_i(x) & \text{if } x \in \mathcal{L}(\Gamma_{\mathcal{C}_i^{\text{int}}}) \end{cases} \quad \text{for all } i \in \overline{1, n}. \quad (1-2)$$

When  $\Gamma_{\mathcal{G}} = \Lambda_{\mathcal{H}}$ , this construction yields a group of  $*$ -automorphisms of  $\mathcal{L}(\Gamma_{\mathcal{G}})$ , which we denote by  $\text{Loc}_{c,g}(\mathcal{L}(\Gamma_{\mathcal{G}}))$ . We also denote by  $\text{Loc}_c(\mathcal{L}(\Gamma_{\mathcal{G}}))$  the subgroup of all local automorphisms satisfying  $\sigma = \text{Id}$ . Notice that

$$\text{Loc}_c(\mathcal{L}(\Gamma_{\mathcal{G}})) \cong \bigoplus_i \text{Aut}(\mathcal{L}(\Gamma_{\mathcal{C}_{i-1, i}})) \oplus \text{Aut}(\mathcal{L}(\Gamma_{\mathcal{C}_i^{\text{int}}})),$$

and also  $\text{Loc}_c(\mathcal{L}(\Gamma_{\mathcal{G}})) \leq \text{Loc}_{c,g}(\mathcal{L}(\Gamma_{\mathcal{G}}))$  has finite index.

Next, we highlight a class of automorphisms in  $\text{Loc}_c(\mathcal{L}(\Gamma_{\mathcal{G}}))$  needed to state our main results. Consider  $n$ -tuples  $a = (a_{i, i+1})_i$  and  $b = (b_i)_i$  of nontrivial unitaries  $a_{i, i+1} \in \mathcal{L}(\Gamma_{\mathcal{C}_{i-1, i}})$  and  $b_i \in \mathcal{L}(\Gamma_{\mathcal{C}_i^{\text{int}}})$  for every  $i \in \overline{1, n}$ . If in (1-2), we let  $\theta_{i, i+1} = \text{ad}(a_{i, i+1})$  and  $\xi_i = \text{ad}(b_i)$ , and then the corresponding local automorphism  $\phi_{\theta, \xi, \text{Id}}$  is most of the time an outer automorphism of  $\mathcal{L}(\Gamma)$  and will be denoted by  $\phi_{a, b}$  throughout. These automorphisms form a normal subgroup denoted by  $\text{Loc}_{c,i}(\mathcal{L}(\Gamma_{\mathcal{G}})) \triangleleft \text{Loc}_c(\mathcal{L}(\Gamma_{\mathcal{G}}))$ ; see [Section 7.1](#) for more details.

Developing an approach which combines outgrowths of prior methods in Popa's deformation/rigidity theory [[Ioana et al. 2008](#)] with a new technique on analyzing cancellation in cyclic relations of graph von Neumann algebras ([Section 5](#)), we are able to describe all  $*$ -isomorphisms between these algebras solely in terms of the aforementioned local isomorphisms. This can be viewed as a von Neumann algebra

counterpart of very general and deep results of Genevois and Martin [2019, Corollary C] from geometric group theory describing the structure of the automorphisms of graph product groups.

**Theorem A.** *Let  $\mathcal{G}, \mathcal{H} \in \text{CC}_1$ , and let  $\Gamma = \mathcal{G}\{\Gamma_v\}$  and  $\Lambda = \mathcal{H}\{\Lambda_w\}$  be graph products such that*

- (1)  *$\Gamma_v$  and  $\Lambda_w$  are icc property (T) groups for all  $v \in \mathcal{V}$  and  $w \in \mathcal{W}$ ,*
- (2) *there is a class  $\mathcal{C}$  of countable groups which satisfies the s-unique prime factorization property (see Definition 7.6) for which  $\Gamma_v$  and  $\Lambda_w$  belong to  $\mathcal{C}$  for all  $v \in \mathcal{V}$  and  $w \in \mathcal{W}$ .*

Let  $t > 0$ , and let  $\Theta : \mathcal{L}(\Gamma)^t \rightarrow \mathcal{L}(\Lambda)$  be any  $*$ -isomorphism. Then  $t = 1$  and one can find an isometry  $\sigma : \mathcal{G} \rightarrow \mathcal{H}$ ,  $*$ -isomorphisms  $\theta_{i-1,i} : \mathcal{L}(\Gamma_{\mathcal{C}_{i-1,i}}) \rightarrow \mathcal{L}(\Gamma_{\mathcal{C}_{\sigma(\mathcal{C}_{i-1,i})}})$  and  $\xi_i : \mathcal{L}(\Gamma_{\mathcal{C}_i^{\text{int}}}) \rightarrow \mathcal{L}(\Gamma_{\sigma(\mathcal{C}_i^{\text{int}})})$  for all  $i \in \overline{1, n}$ , and a unitary  $u \in \mathcal{L}(\Lambda)$  such that  $\Theta = \text{ad}(u) \circ \phi_{\theta, \xi, \sigma}$ .

This theorem applies to fairly large classes of property (T) vertex groups, including: all fibered Rips constructions considered in [Chifan et al. 2023a; 2024], and all wreath-like product groups  $\mathcal{WR}(A, B \curvearrowright I)$ , where  $A$  is either abelian or icc,  $B$  is an icc subgroup of a hyperbolic group, and the action  $B \curvearrowright I$  has amenable stabilizers [Chifan et al. 2023b]. The result also implies that the fundamental group [Murray and von Neumann 1936] of these graph product group  $\text{II}_1$ -factors is always trivial; this means that if  $\Gamma$  is a graph product group as in Theorem A, then  $\{t > 0 : \mathcal{L}(\Gamma)^t \cong \mathcal{L}(\Gamma)\} = 1$ . Recall that Popa [2006a] used his deformation/rigidity theory for obtaining the first examples of  $\text{II}_1$ -factors with trivial fundamental group, hence answering a longstanding open problem of Kadison; see [Ge 2003]. Subsequently, a large number of striking results on computations of fundamental groups of  $\text{II}_1$ -factors were obtained; see the introduction of [Chifan et al. 2024]. To our knowledge, Theorem A provides the first instance of computing the fundamental group for nontrivial graph product von Neumann algebras which is not a tensor product.

Specializing Theorem A to the case when the vertex groups  $\Gamma_v$  and  $\Lambda_w$  are the property (T) wreath-like product groups as in [Chifan et al. 2023c, Theorem 7.5], we obtain a fairly concrete description of all such isomorphisms between these graph product group von Neumann algebras; namely, they appear as compositions between the canonical group-like isomorphisms and the clique-inner local automorphisms of  $\mathcal{L}(\Lambda)$  described above.

**Theorem B.** *Let  $\mathcal{G}, \mathcal{H} \in \text{CC}_1$ , and let  $\Gamma = \mathcal{G}\{\Gamma_v\}$  and  $\Lambda = \mathcal{H}\{\Lambda_w\}$  be graph product groups where all vertex groups  $\Gamma_v, \Lambda_w$  are property (T) wreath-like product groups of the form  $\mathcal{WR}(A, B \curvearrowright I)$ , where  $A$  is abelian,  $B$  is an icc subgroup of a hyperbolic group, and  $B \curvearrowright I$  has infinite orbits.*

*Then, for any  $t > 0$  and  $*$ -isomorphism  $\Theta : \mathcal{L}(\Gamma)^t \rightarrow \mathcal{L}(\Lambda)$ , we have  $t = 1$  and one can find a character  $\eta \in \text{Char}(\Gamma)$ , a group isomorphism  $\delta \in \text{Isom}(\Gamma, \Lambda)$ , a  $*$ -automorphism  $\phi_{a,b} \in \text{Loc}_{c,i}(\mathcal{L}(\Lambda))$ , and a unitary  $u \in \mathcal{L}(\Lambda)$  such that  $\Theta = \text{ad}(u) \circ \phi_{a,b} \circ \Psi_{\eta, \delta}$ .*

In the statement of Theorem B and also throughout the paper, given a character  $\eta \in \text{Char}(\Gamma)$  and a group isomorphism  $\delta \in \text{Isom}(\Gamma, \Lambda)$ , we denote by  $\Psi_{\eta, \delta}$  the  $*$ -isomorphism from  $\mathcal{L}(\Gamma)$  to  $\mathcal{L}(\Lambda)$  given by  $\Psi_{\eta, \delta}(u_g) = \eta(g)v_{\delta(g)}$  for any  $g \in \Gamma$ . Here,  $\{u_g : g \in \Gamma\}$  and  $\{v_h : h \in \Lambda\}$  are the canonical group unitaries of  $\mathcal{L}(\Gamma)$  and  $\mathcal{L}(\Lambda)$ , respectively.

To this end we recall that in [Chifan et al. 2023c, Corollary 2.12] it was shown that the property (T) regular wreath-like products covered by the previous theorem can be chosen to have trivial abelianization and prescribed finitely presented outer automorphism groups. Using this, **Theorem A** yields the following.

**Corollary C.** *Let  $\mathcal{G} \in \text{CC}_1$ , and fix  $\text{cliq}(\mathcal{G}) = \{\mathcal{C}_1, \dots, \mathcal{C}_n\}$  a consecutive enumeration of its cliques. Let  $\Gamma = \mathcal{G}\{\Gamma_v\}$  be any graph product groups (as in **Theorem B**). Assume in addition that its vertex groups are pairwise nonisomorphic and have trivial abelianization and trivial outer automorphisms. Then the outer automorphisms satisfy the formula*

$$\text{Out}(\mathcal{L}(\Gamma)) \cong \bigoplus_{i=1}^n \mathcal{U}(\mathcal{L}(\Gamma_{\mathcal{C}_{i-1,i}})) \oplus \mathcal{U}(\mathcal{L}(\Gamma_{\mathcal{C}_i^{\text{int}}})).$$

By applying **Corollary C** to the case when the underlying graph  $\mathcal{G}$  is the  $n$ -petals flower-shaped  $\mathcal{F}_n = (\mathcal{V}_n, \mathcal{E}_n)$ , see [Figure 1](#), we obtain the slimmest types of outer automorphisms groups one could have in this setup. Namely, we deduce that  $\text{Out}(\mathcal{L}(\Gamma)) \cong \bigoplus_{v \in \mathcal{V}_n} \mathcal{U}(\mathcal{L}(\Gamma_v))$ .

We conclude our introduction with **Corollary D**, where we describe all  $*$ -isomorphisms of the reduced  $C^*$ -algebras of graph product groups that we considered in **Theorem B**. This result can be seen as a  $C^*$ -algebraic version of [Genevois and Martin 2019, Corollary C].

**Corollary D.** *Let  $\mathcal{G}, \mathcal{H} \in \text{CC}_1$ , and let  $\Gamma = \mathcal{G}\{\Gamma_v\}$  and  $\Lambda = \mathcal{H}\{\Lambda_w\}$  be graph product groups (as in **Theorem B**). Then, for any  $*$ -isomorphism  $\Theta : C_r^*(\Gamma) \rightarrow C_r^*(\Lambda)$ , there exist a character  $\eta \in \text{Char}(\Gamma)$ , a group isomorphism  $\delta \in \text{Isom}(\Gamma, \Lambda)$ , a  $*$ -automorphism  $\phi_{a,b} \in \text{Loc}_{c,i}(\mathcal{L}(\Lambda))$ , and a unitary  $u \in \mathcal{L}(\Lambda)$  such that  $\Theta = \text{ad}(u) \circ \phi_{a,b} \circ \Psi_{\eta, \delta}$ .*

In fact, this result is a consequence of **Theorem B** since the graph product groups that we consider have trivial amenable radical (see [Lemma 4.3](#)) and, consequently, their reduced  $C^*$ -algebras have unique trace [[Breuillard et al. 2017](#)].

## 2. Preliminaries

**2.1. Terminology.** Throughout this document all von Neumann algebras are denoted by calligraphic letters, e.g.,  $\mathcal{M}$ ,  $\mathcal{N}$ ,  $\mathcal{P}$ ,  $\mathcal{Q}$ , etc. All von Neumann algebras  $\mathcal{M}$  considered in this document will be tracial, i.e., endowed with a unital, faithful, normal linear functional  $\tau : \mathcal{M} \rightarrow \mathbb{C}$  satisfying  $\tau(xy) = \tau(yx)$  for all  $x, y \in \mathcal{M}$ . This induces a norm on  $\mathcal{M}$  with the formula  $\|x\|_2 = \tau(x^*x)^{1/2}$  for all  $x \in \mathcal{M}$ . The  $\|\cdot\|_2$ -completion of  $\mathcal{M}$  will be denoted by  $L^2(\mathcal{M})$ .

Given a von Neumann algebra  $\mathcal{M}$ , we will denote by  $\mathcal{U}(\mathcal{M})$  its unitary group and by  $\mathcal{Z}(\mathcal{M})$  its center. Given a unital inclusion  $\mathcal{N} \subset \mathcal{M}$  of von Neumann algebras, we denote by  $\mathcal{N}' \cap \mathcal{M} = \{x \in \mathcal{M} : [x, \mathcal{N}] = 0\}$  the relative commutant of  $\mathcal{N}$  inside  $\mathcal{M}$ , and by  $\mathcal{N}_{\mathcal{M}}(\mathcal{N}) = \{u \in \mathcal{U}(\mathcal{M}) : u\mathcal{N}u^* = \mathcal{N}\}$  the normalizer of  $\mathcal{N}$  inside  $\mathcal{M}$ . We say that the inclusion  $\mathcal{N}$  is regular in  $\mathcal{M}$  if  $\mathcal{N}_{\mathcal{M}}(\mathcal{N})'' = \mathcal{M}$  and irreducible if  $\mathcal{N}' \cap \mathcal{M} = \mathbb{C}1$ .

**2.2. Graph product groups.** In this preliminary section we briefly recall the notion of graph product groups introduced by E. Green [[1990](#)] while also highlighting some of its features that are relevant to this article. Let  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$  be a finite simple graph, where  $\mathcal{V}$  and  $\mathcal{E}$  denote its vertex and edge sets,

respectively. Let  $\{\Gamma_v\}_{v \in \mathcal{V}}$  be a family of groups called vertex groups. The graph product group associated with this data, denoted by  $\mathcal{G}\{\Gamma_v, v \in \mathcal{V}\}$  or simply  $\mathcal{G}\{\Gamma_v\}$ , is the group generated by  $\Gamma_v$ ,  $v \in \mathcal{V}$ , with the only relations being  $[\Gamma_u, \Gamma_v] = 1$  whenever  $(u, v) \in \mathcal{E}$ . Given any subset  $\mathcal{U} \subset \mathcal{V}$ , the subgroup  $\Gamma_{\mathcal{U}} = \langle \Gamma_u : u \in \mathcal{U} \rangle$  of  $\mathcal{G}\{\Gamma_v, v \in \mathcal{V}\}$  is called a *full subgroup*. This can be identified with the graph product  $\mathcal{G}_{\mathcal{U}}\{\Gamma_u, u \in \mathcal{U}\}$  corresponding to the subgraph  $\mathcal{G}_{\mathcal{U}}$  of  $\mathcal{G}$ , spanned by the vertices of  $\mathcal{U}$ . For every  $v \in \mathcal{V}$ , we denote by  $\text{lk}(v)$  the subset of vertices  $w \neq v$  such that  $(w, v) \in \mathcal{E}$ . Similarly, for every  $\mathcal{U} \subseteq \mathcal{V}$ , we define  $\text{lk}(\mathcal{U}) = \bigcap_{u \in \mathcal{U}} \text{lk}(u)$ . We also use the convention that  $\text{lk}(\emptyset) = \mathcal{V}$ . Notice that  $\mathcal{U} \cap \text{lk}(\mathcal{U}) = \emptyset$ .

Graph product groups naturally admit many amalgamated free product decompositions. One such decomposition—which is essential for deriving our main results— involves full subgroup factors in [Green 1990, Lemma 3.20] as follows. For any  $w \in \mathcal{V}$ , we have

$$\mathcal{G}\{\Gamma_v\} = \Gamma_{\mathcal{V} \setminus \{w\}} \underset{\Gamma_{\text{lk}(w)}}{\ast} \Gamma_{\text{st}(w)}, \quad (2-1)$$

where  $\text{st}(w) = \{w\} \cup \text{lk}(w)$ . Notice that  $\Gamma_{\text{lk}(w)} \not\leq \Gamma_{\text{st}(w)}$ , but it could be the case that  $\Gamma_{\text{lk}(w)} = \Gamma_{\mathcal{V} \setminus \{w\}}$  when  $\mathcal{V} = \text{st}(w)$ . In this case the amalgam decomposition is called degenerate.

Similarly, for every subgraph  $\mathcal{U} \subset \mathcal{G}$ , we write  $\text{st}(\mathcal{U}) = \mathcal{U} \cup \text{lk}(\mathcal{U})$ . A maximal complete subgraph  $\mathcal{C} \subseteq \mathcal{G}$  is called a *clique* and the collections of all cliques of  $\mathcal{G}$  will be denoted by  $\text{cliq}(\mathcal{G})$ . Below we highlight various properties of full subgroups that will be useful in this paper. The first is [Antolín and Minasyan 2015, Lemma 3.7], the second is [Antolín and Minasyan 2015, Proposition 3.13], while the third is [Antolín and Minasyan 2015, Proposition 3.4].

**Proposition 2.1** [Antolín and Minasyan 2015]. *Let  $\Gamma = \mathcal{G}\{\Gamma_v\}$  be any graph product of groups with  $g \in \Gamma$ , and let  $\mathcal{S}, \mathcal{T} \subseteq \mathcal{G}$  be any subgraphs. Then the following hold:*

- (1) *If  $g\Gamma_{\mathcal{T}}g^{-1} \subset \Gamma_{\mathcal{S}}$ , then there is  $h \in \Gamma_{\mathcal{S}}$  such that  $g\Gamma_{\mathcal{T}}g^{-1} = h\Gamma_{\mathcal{T} \cap \mathcal{S}}h^{-1}$ . In particular, if  $\mathcal{S} = \mathcal{T}$ , then  $g\Gamma_{\mathcal{T}}g^{-1} = \Gamma_{\mathcal{S}}$ .*
- (2) *The normalizer of  $\Gamma_{\mathcal{T}}$  inside  $\Gamma$  satisfies  $N_{\Gamma}(\Gamma_{\mathcal{T}}) = \Gamma_{\mathcal{T} \cup \text{link}(\mathcal{T})}$ .*
- (3) *There exist  $\mathcal{D} \subseteq \mathcal{S} \cap \mathcal{T}$  and  $h \in \Gamma_{\mathcal{T}}$  such that  $g\Gamma_{\mathcal{S}}g^{-1} \cap \Gamma_{\mathcal{T}} = h\Gamma_{\mathcal{D}}h^{-1}$ .*

**2.3. Popa's intertwining-by-bimodules techniques.** We next recall the *intertwining-by-bimodules* technique of Popa [2006b, Theorem 2.1 and Corollary 2.3], which is a powerful criterion for identifying intertwiners between arbitrary subalgebras of tracial von Neumann algebras.

**Theorem 2.2** [Popa 2006b]. *Let  $(\mathcal{M}, \tau)$  be a tracial von Neumann algebra and  $\mathcal{P} \subset p\mathcal{M}p$ ,  $\mathcal{Q} \subset q\mathcal{M}q$  be von Neumann subalgebras. Then the following are equivalent:*

- (1) *There exist projections  $p_0 \in \mathcal{P}$ ,  $q_0 \in \mathcal{Q}$ , a  $*$ -homomorphism  $\theta : p_0\mathcal{P}p_0 \rightarrow q_0\mathcal{Q}q_0$ , and a nonzero partial isometry  $v \in q_0\mathcal{M}p_0$  such that  $\theta(x)v = vx$  for all  $x \in p_0\mathcal{P}p_0$ .*
- (2) *There is no sequence  $(u_n)_{n \geq 1} \subset \mathcal{U}(\mathcal{P})$  satisfying  $\|E_{\mathcal{Q}}(x^*u_n y)\|_2 \rightarrow 0$  for all  $x, y \in p\mathcal{M}$ .*

*If one of these equivalent conditions holds, we write  $\mathcal{P} \prec_{\mathcal{M}} \mathcal{Q}$  and say that **a corner of  $\mathcal{P}$  embeds into  $\mathcal{Q}$  inside  $\mathcal{M}$** . Moreover, if  $\mathcal{P}p' \prec_{\mathcal{M}} \mathcal{Q}$  for any nonzero projection  $p' \in \mathcal{P}' \cap p\mathcal{M}p$ , then write  $\mathcal{P} \prec_{\mathcal{M}}^s \mathcal{Q}$ .*

Given an arbitrary graph product group, our next lemma clarifies the intertwining of subalgebras of full subgroups in the associated graph product group von Neumann algebra.

**Lemma 2.3.** *Let  $\Gamma = \mathcal{G}\{\Gamma_v\}$  be any graph product of infinite groups, and let  $\mathcal{S}, \mathcal{T} \subseteq \mathcal{G}$  be any subgraphs. If  $\mathcal{L}(\Gamma_{\mathcal{S}}) \prec_{\mathcal{L}(\Gamma)} \mathcal{L}(\Gamma_{\mathcal{T}})$ , then  $\mathcal{S} \subseteq \mathcal{T}$ .*

*Proof.* By applying [Chifan and Ioana 2018, Lemma 2.2], there is  $g \in \Gamma$  such that  $[\Gamma_{\mathcal{S}} : \Gamma_{\mathcal{S}} \cap g\Gamma_{\mathcal{T}}g^{-1}] < \infty$ . By Proposition 2.1, one can find a subgraph  $\mathcal{P} \subseteq \mathcal{S} \cap \mathcal{T}$  and  $k \in \Gamma_{\mathcal{S}}$  such that  $\Gamma_{\mathcal{S}} \cap g\Gamma_{\mathcal{T}}g^{-1} = k\Gamma_{\mathcal{P}}k^{-1}$ . Thus  $k\Gamma_{\mathcal{P}}k^{-1} < \Gamma_{\mathcal{S}}$  is a finite index subgroup. Since  $k \in \Gamma_{\mathcal{S}}$ , it follows that  $\Gamma_{\mathcal{P}} < \Gamma_{\mathcal{S}}$  has finite index as well. Since  $|\Gamma_v| = \infty$ , for all  $v \in \mathcal{G}$ , we must have that  $[\Gamma_{\mathcal{S}} : \Gamma_{\mathcal{P}}] = 1$ , and hence  $\Gamma_{\mathcal{S}} = \Gamma_{\mathcal{P}}$ . Thus,  $\mathcal{S} = \mathcal{P} \subset \mathcal{S} \cap \mathcal{T}$ , and hence  $\mathcal{S} \subseteq \mathcal{T}$ .  $\square$

**Remark 2.4.** The proof of Lemma 2.3 shows that if  $\Gamma = \mathcal{G}\{\Gamma_v\}$  is a graph product of infinite groups and  $\mathcal{S}, \mathcal{T} \subseteq \mathcal{G}$  are subgraphs such that  $[\Gamma_{\mathcal{S}} : \Gamma_{\mathcal{S}} \cap g\Gamma_{\mathcal{T}}g^{-1}] < \infty$  for some  $g \in \Gamma$ , then  $\mathcal{S} \subseteq \mathcal{T}$ .

**2.4. Quasinormalizers of von Neumann algebras.** Given an inclusion  $\mathcal{P} \subset \mathcal{M}$  of tracial von Neumann algebras, we define the quasinormalizer  $\mathcal{QN}_{\mathcal{M}}(\mathcal{P})$  as the subgroup of all elements  $x \in \mathcal{M}$  for which there exist  $x_1, \dots, x_n \in \mathcal{M}$  such that  $\mathcal{P}x \subseteq \sum x_i \mathcal{P}$  and  $x\mathcal{P} \subseteq \sum \mathcal{P}x_i$ ; see [Popa 1999, Definition 4.8].

**Lemma 2.5** [Fang et al. 2011; Popa 2006b]. *Let  $\mathcal{P} \subset \mathcal{M}$  be tracial von Neumann algebras. For any projection  $p \in \mathcal{P}$ , we have that  $W^*(\mathcal{QN}_{p\mathcal{M}p}(p\mathcal{P}p)) = pW^*(\mathcal{QN}_{\mathcal{M}}(\mathcal{P}))p$ .*

Given a group inclusion  $H < G$ , the quasinormalizer  $\text{QN}_G(H)$  is the group of all  $g \in G$  for which there exists a finite set  $F \subset G$  such that  $Hg \subset FH$  and  $gH \subset HF$ . The following result provides a relation between the group theoretical quasinormalizer and the von Neumann algebraic one.

**Lemma 2.6** [Fang et al. 2011, Corollary 5.2]. *Let  $\Lambda < \Gamma$  be countable groups. Then we have that  $W^*(\mathcal{QN}_{\mathcal{L}(\Gamma)}(\mathcal{L}(\Lambda))) = \mathcal{L}(\text{QN}_{\Gamma}(\Lambda))$ .*

We continue by computing the quasinormalizer of subalgebras of full subgroups in any graph product group von Neumann algebra. More generally, we show the following.

**Theorem 2.7.** *Let  $\Gamma = \mathcal{G}\{\Gamma_v\}$  be any graph product of infinite groups, and let  $\mathcal{S}, \mathcal{T} \subseteq \mathcal{G}$  be any subgraphs. Write  $\mathcal{M} = \mathcal{L}(\Gamma)$ , and assume there exist  $x, x_1, x_2, \dots, x_n \in \mathcal{M}$  such that  $\mathcal{L}(\Gamma_{\mathcal{S}})x \subseteq \sum_{k=1}^n x_k \mathcal{L}(\Gamma_{\mathcal{T}})$ . Thus  $\mathcal{S} \subseteq \mathcal{T}$  and  $x \in \mathcal{L}(\Gamma_{\mathcal{T} \cup \text{lk}(\mathcal{S})})$ .*

*Proof.* Using the proofs of [Chifan and Ioana 2018, Lemma 2.8 and Claim 2.3], we obtain that  $x$  belongs to the  $\|\cdot\|_2$ -closure of the linear span of  $\{u_g\}_{g \in S}$ . Here,  $S$  denotes the set of all elements  $g \in \Gamma$  for which  $[\Gamma_{\mathcal{S}} : \Gamma_{\mathcal{S}} \cap g\Gamma_{\mathcal{T}}g^{-1}] < \infty$ . By assuming that  $x \neq 0$ , it follows that  $S$  is nonempty. Fix  $g \in S$ . By using Remark 2.4, we derive that  $\mathcal{S} \subseteq \mathcal{T}$ , which gives the first part of the conclusion.

For proving the second part, note that by Proposition 2.1 one can find a subgraph  $\mathcal{P} \subseteq \mathcal{S}$  and  $k \in \Gamma_{\mathcal{S}}$  such that  $\Gamma_{\mathcal{S}} \cap g\Gamma_{\mathcal{T}}g^{-1} = k\Gamma_{\mathcal{P}}k^{-1}$ . Thus  $k\Gamma_{\mathcal{P}}k^{-1} < \Gamma_{\mathcal{S}}$  is a finite index subgroup. Since  $k \in \Gamma_{\mathcal{S}}$ , this further implies that  $\Gamma_{\mathcal{P}} < \Gamma_{\mathcal{S}}$  has finite index, and hence  $\mathcal{P} = \mathcal{S}$ . Using again that  $k \in \Gamma_{\mathcal{S}}$ , we get  $\Gamma_{\mathcal{S}} \cap g\Gamma_{\mathcal{T}}g^{-1} = k\Gamma_{\mathcal{P}}k^{-1} = \Gamma_{\mathcal{S}}$ , and thus  $g^{-1}\Gamma_{\mathcal{S}}g < \Gamma_{\mathcal{T}}$ . By Proposition 2.1, one can find  $r \in \Gamma_{\mathcal{T}}$  such that  $g^{-1}\Gamma_{\mathcal{S}}g = r\Gamma_{\mathcal{T}}r^{-1}$ . This relation implies in particular that  $gr \in N_{\Gamma}(\Gamma_{\mathcal{S}})$ , and since  $N_{\Gamma}(\Gamma_{\mathcal{S}}) = \Gamma_{\mathcal{S} \cup \text{lk}(\mathcal{S})}$  (see Proposition 2.1), we conclude that  $gr \in \Gamma_{\mathcal{S} \cup \text{lk}(\mathcal{S})}$ . Therefore,  $g \in \Gamma_{\mathcal{S} \cup \text{lk}(\mathcal{S})}\Gamma_{\mathcal{T}} \subset \Gamma_{\mathcal{T} \cup \text{lk}(\mathcal{S})}$ . This gives the desired conclusion.  $\square$

**Corollary 2.8.** *Let  $\Gamma = \mathcal{G}\{\Gamma_v\}$  be any graph product of infinite groups, and let  $\mathcal{C} \in \text{cliq}(\mathcal{G})$  be a clique with at least two vertices. Fix a vertex  $v \in \mathcal{C}$  such that  $\text{lk}(\mathcal{C} \setminus \{v\}) = \{v\}$ . Write  $\mathcal{M} = \mathcal{L}(\Gamma)$ , and assume there exist  $x, x_1, x_2, \dots, x_n \in \mathcal{M}$  such that  $\mathcal{L}(\Gamma_{\mathcal{C} \setminus \{v\}})x \subseteq \sum_{k=1}^n x_k \mathcal{L}(\Gamma_{\mathcal{C}})$ . Then  $x \in \mathcal{L}(\Gamma_{\mathcal{C}})$ .*

*Proof.* The result follows by applying [Theorem 2.7](#) for  $\mathcal{S} = \mathcal{C} \setminus \{v\}$  and  $\mathcal{T} = \mathcal{C}$ .  $\square$

**Lemma 2.9.** *Let  $\Gamma = \mathcal{G}\{\Gamma_v\}$  be a graph product of groups, and let  $\mathcal{C} \in \text{cliq}(\mathcal{G})$  be a clique. Let  $\mathcal{P} \subset p\mathcal{L}(\Gamma_{\mathcal{C}})p$  be a von Neumann subalgebra such that  $\mathcal{P} \not\prec_{\mathcal{L}(\Gamma_{\mathcal{C}})} L(\Gamma_{\mathcal{C}_v})$  for any  $v \in \mathcal{C}$ . If  $x \in \mathcal{L}(\Gamma)$  satisfies  $x\mathcal{P} \subset \sum_{i=1}^n \mathcal{L}(\Gamma_{\mathcal{C}})x_i$  for some  $x_1, \dots, x_n \in \mathcal{L}(\Gamma)$ , then  $xp \in \mathcal{L}(\Gamma_{\mathcal{C}})$ .*

*Proof.* Let  $g \in \Gamma \setminus \Gamma_{\mathcal{C}}$ . From [Proposition 2.1](#), there exist  $h \in \Gamma_{\mathcal{C}}$  and  $\mathcal{D} \subset \mathcal{C}$  such that  $\Gamma_{\mathcal{C}} \cap g\Gamma_{\mathcal{C}}g^{-1} = h\Gamma_{\mathcal{D}}h^{-1}$ . Note that [Theorem 2.7](#) shows  $\text{QN}_{\Gamma}^{(1)}(\Gamma_{\mathcal{C}}) = \Gamma_{\mathcal{C}}$  and therefore  $\mathcal{D} \neq \mathcal{C}$ ; otherwise, we would get  $g \in \text{QN}_{\Gamma}^{(1)}(\Gamma_{\mathcal{C}}) = \Gamma_{\mathcal{C}}$ , a contradiction. Thus, from the assumption we deduce  $\mathcal{P} \not\prec_{\mathcal{L}(\Gamma_{\mathcal{C}})} L(\Gamma_{\mathcal{C}} \cap g\Gamma_{\mathcal{C}}g^{-1})$  for any  $g \in \Gamma \setminus \Gamma_{\mathcal{C}}$ . The conclusion now follows from [\[Chifan and Ioana 2018, Lemma 2.7\]](#).  $\square$

**2.5. A result on normalizers in tensor product factors.** Our next proposition describes the normalizer of a  $\text{II}_1$ -factor  $\mathcal{N}$  inside the tensor product of  $\mathcal{N}$  with another  $\text{II}_1$ -factor.

**Proposition 2.10.** *Let  $\mathcal{N}$  and  $\mathcal{P}$  be  $\text{II}_1$ -factors and write  $\mathcal{M} = \mathcal{N} \bar{\otimes} \mathcal{P}$ . If  $u \in \mathcal{U}(\mathcal{M})$  satisfies  $u\mathcal{N}u^* = \mathcal{N}$ , then one can find  $a \in \mathcal{U}(\mathcal{N})$  and  $b \in \mathcal{U}(\mathcal{P})$  such that  $u = a \otimes b$ .*

*Proof.* Let  $(\xi_i)_{i \in I} \subset L^2(\mathcal{P})$  be a Pimsner–Popa basis for the inclusion  $\mathcal{N} \subset \mathcal{M}$ , let  $u = \sum_i E_{\mathcal{N}}(u\xi_i^*) \otimes \xi_i$ , and write  $\eta_i = E_{\mathcal{N}}(u\xi_i^*)$ . If  $\theta : \mathcal{N} \rightarrow \mathcal{N}$  denotes the  $*$ -isomorphism  $\theta = \text{ad}(u)$ , then we have  $\theta(x)u = ux$  for all  $x \in \mathcal{N}$ . This combined with the above formula yields  $\theta(x)\eta_i \otimes \xi_i = \theta(x)u = ux = \eta_i x \otimes \xi_i$ . Hence, for all  $x \in \mathcal{N}$  and all  $i$ , we have

$$\theta(x)\eta_i = \eta_i x. \quad (2-2)$$

Let  $u_i \in \mathcal{N}$  be the partial isometry in the polar decomposition of  $\eta_i$ . Thus  $\theta(x)u_i = u_i x$  for all  $x \in \mathcal{N}$  and all  $i$ . In particular, we get  $u_i^*u_i \in \mathcal{N}' \cap \mathcal{N} = \mathbb{C}1$ , and hence  $u_i \in \mathcal{U}(\mathcal{N})$  for all  $i$ . The prior relations also imply that  $u_i^*xu_i = \theta(x) = u_j^*xu_j$  for all  $i, j \in I$ . In particular, we have  $u_i u_j^* \in \mathcal{N}' \cap \mathcal{N} = \mathbb{C}1$ , and thus one can find scalars  $c_{i,j} \in \mathbb{T}$  such that  $u_i = c_{i,j}u_j$  for all  $i, j \in I$ . Relation (2-2) also implies that  $|\eta_i| \in \mathcal{N}' \cap L^2(\mathcal{N})$  and, since  $\mathcal{N}$  is a  $\text{II}_1$ -factor, we get  $|\eta_i| \in \mathbb{C}1$ . In conclusion,  $\eta_i \in \mathbb{C}\mathcal{U}(\mathcal{N})$  for all  $i$ , and one can find  $d_{i,j} \in \mathbb{C}$  such that  $\eta_i = d_{i,j}\eta_j$  for all  $i, j \in I$ . Fix  $j \in I$  with  $\eta_j \neq 0$ . Using the above relations, we have  $u = \sum_i \eta_i \otimes \xi_i = \sum_i d_{i,j}\eta_j \otimes \xi_i = \eta_j \otimes (\sum_i d_{i,j}\xi_i) = \eta_j \otimes b$ , where we write  $b = \sum_i d_{i,j}\xi_i \in L^2(\mathcal{P})$ . Since  $\eta_j \in \mathbb{C}\mathcal{U}(\mathcal{N})$ , we get the desired conclusion.  $\square$

### 3. Wreath-like product groups

A new category of groups called *wreath-like product groups* were introduced in the previous work [\[Chifan et al. 2023b\]](#). To briefly recall their construction, let  $A$  and  $B$  be any countable groups, and assume that  $B \curvearrowright I$  is an action on a countable set. One says  $W$  is a wreath-like product of  $A$  and  $B \curvearrowright I$  if it can be realized as a group extension

$$1 \rightarrow \bigoplus_{i \in I} A_i \hookrightarrow W \xrightarrow{\varepsilon} B \rightarrow 1 \quad (3-1)$$

which satisfies the following properties:

- (a)  $A_i \cong A$  for all  $i \in I$ .
- (b) The action by conjugation of  $W$  on  $\bigoplus_{i \in I} A_i$  permutes the direct summands according to the rule

$$wA_iw^{-1} = A_{\varepsilon(w)i} \quad \text{for all } w \in W, \quad i \in I.$$

The class of all such wreath-like groups is denoted by  $\mathcal{WR}(A, B \curvearrowright I)$ . When  $I = B$  and the action  $B \curvearrowright I$  is by translation, this consists of so-called regular wreath-like product groups and we simply denote their class by  $\mathcal{WR}(A, B)$ .

Notice that every classical generalized wreath product  $A \wr I B$  belongs to  $\mathcal{WR}(A, B \curvearrowright I)$ . However, building examples of nonsplit wreath-like products is a far more involved problem. One way to approach this is through the use of the so-called Magnus embedding [1939]: these are quotients groups of the form  $\Gamma/[\Lambda, \Lambda]$ , where  $\Lambda \triangleleft \Gamma$  is a normal subgroup. Methods of this type were used by Cohen and Lyndon to produce many such quotients in the context of one-relator groups. The following result is a particular case of [Chifan et al. 2023b, Corollary 4.6] and relies on the prior works [Dahmani et al. 2017; Osin 2007; Sun 2020].

**Corollary 3.1.** *Let  $G$  be an icc hyperbolic group. For every infinite order element  $g \in G$ , there exists  $d \in \mathbb{N}$  such that, for every  $k \in \mathbb{N}$  divisible by  $d$ , we have the following:*

- (a)  $G/[\langle\langle g^k \rangle\rangle, \langle\langle g^k \rangle\rangle] \in \mathcal{WR}(\mathbb{Z}, G/\langle\langle g^k \rangle\rangle \curvearrowright I)$ , where  $\langle g^k \rangle$  is normal in  $E_G(g)$ , the action  $G/\langle\langle g^k \rangle\rangle \curvearrowright I$  is transitive, and all the stabilizers of elements of  $I$  are isomorphic to the finite group  $E_G(g)/\langle g^k \rangle$ . Here,  $E_G(g)$  denotes the elementary subgroup generated by  $g$ ,  $\langle g^k \rangle$  denotes the subgroup generated by  $g^k$  and  $\langle\langle g^k \rangle\rangle$  denotes the smallest normal subgroup that contains  $g^k$ .
- (b)  $G/\langle\langle g^k \rangle\rangle$  is an icc hyperbolic group.

Developing a new quotienting method in the context of Cohen–Lyndon triples, [Chifan et al. 2023b, Theorem 2.5] constructed many examples of property (T) regular wreath-like product groups as follows.

**Theorem 3.2** [Chifan et al. 2023b]. *Let  $G$  be a hyperbolic group. For every finitely generated group  $A$ , there exists a quotient  $W$  of  $G$  such that  $W \in \mathcal{WR}(A, B)$  for some hyperbolic group  $B$ .*

For further use we also recall the following result on prescribed outer automorphisms of property (T) regular wreath-like product groups established in [Chifan et al. 2023b, Theorem 6.9].

**Theorem 3.3** [Chifan et al. 2023b]. *For every finitely presented group  $Q$  and every finitely generated group  $A_0$ , there exist groups  $A, B$  and a regular wreath-like product  $W \in \mathcal{WR}(A, B)$  with the following properties:*

- (a)  $W$  has property (T) and has no nontrivial characters.
- (b)  $A$  is the direct sum of  $|Q|$  copies of  $A_0$ . In particular,  $A = A_0$  if  $Q = \{1\}$ .
- (c)  $B$  is an icc normal subgroup of a hyperbolic group  $H$  and  $H/B \cong Q$ . In particular,  $B$  is hyperbolic whenever  $Q$  is finite.
- (d)  $\text{Out}(W) \cong Q$ .

**Remark 3.4.** Since  $A_0$  can be any finitely generated group, it follows that if we fix the group  $Q$  there are infinitely many pairwise nonisomorphic regular wreath-like product groups  $W \in \mathcal{WR}(A, B)$  which satisfy (a)–(d) in the prior theorem.

**3.1. Unique prime factorization for von Neumann algebras of wreath-like product groups.** In this subsection, more precisely in [Theorem 3.6](#), we show that von Neumann algebras of certain wreath-like product groups satisfy the unique prime factorization of Ozawa and Popa [\[2010\]](#). First, we point out the following structural result for commuting property (T) von Neumann subalgebras of von Neumann algebras that arise from trace-preserving actions of certain wreath-like product groups.

**Lemma 3.5.** *Let  $\Gamma$  be a wreath-like product group of the form  $\mathcal{WR}(A, B \curvearrowright I)$ , where  $A$  is abelian and  $B$  is an icc subgroup of a hyperbolic group. Let  $\Gamma \curvearrowright \mathcal{N}$  be a trace-preserving action and write  $\mathcal{M} = \mathcal{N} \rtimes \Gamma$ .*

*If  $\mathcal{A}, \mathcal{B} \subset p\mathcal{M}p$  are commuting property (T) von Neumann subalgebras, then  $\mathcal{A} \prec_{\mathcal{M}} \mathcal{N}$  or  $\mathcal{B} \prec_{\mathcal{M}} \mathcal{N}$ .*

The proof of [Lemma 3.5](#) follows from [\[Chifan et al. 2023c, Theorem 6.4\]](#), and the main ingredient of its proof is Popa and Vaes' structure theorem [\[2014\]](#) for normalizers in crossed products arising from actions of hyperbolic group.

**Theorem 3.6.** *Let  $\Gamma_i$  be a property (T) wreath-like product group of the form  $\mathcal{WR}(A, B \curvearrowright I)$  for any  $i \in \overline{1, n}$ , where  $A$  is abelian and  $B$  is an icc subgroup of a hyperbolic group.*

*If  $\mathcal{M} := \mathcal{L}(\Gamma_1) \bar{\otimes} \cdots \bar{\otimes} \mathcal{L}(\Gamma_n) = \mathcal{P}_1 \bar{\otimes} \mathcal{P}_2$  is a tensor product decomposition into  $\text{II}_1$ -factors, then there exist a unitary  $u \in \mathcal{M}$ , a decomposition  $\mathcal{M} = \mathcal{P}_1^t \bar{\otimes} \mathcal{P}_2^{1/t}$ , and a partition  $T_1 \sqcup T_2 = \overline{1, n}$  such that  $\mathcal{L}(\times_{k \in S_j} \Gamma_k) = u\mathcal{P}_j^{t_j} u^*$  for any  $j \in \{1, 2\}$ .*

*Proof.* To fix some notation, we have that  $\Gamma_i$  belongs to  $\mathcal{WR}(A_i, B_i \curvearrowright I_i)$  for any  $i \in \overline{1, n}$ , where  $A_i$  is abelian and  $B_i$  is an icc subgroup of a hyperbolic group. Note that, since  $\mathcal{P}_1$  and  $\mathcal{P}_2$  have property (T), by applying [Lemma 3.5](#), we obtain a map  $\phi : \overline{1, n} \rightarrow \overline{1, 2}$  such that

$$\mathcal{P}_{\phi(i)} \prec_{\mathcal{M}} \overline{\bigotimes_{k \neq i} \mathcal{L}(\Gamma_k)} \quad \text{for any } i \in \overline{1, n}.$$

By [\[Drimbe et al. 2019, Lemma 2.8 \(2\)\]](#), there exists a partition  $\overline{1, n} = S_1 \sqcup S_2$  such that  $\mathcal{P}_j \prec_{\mathcal{M}} \mathcal{L}(\times_{k \in S_j} \Gamma_k)$  for any  $j$ . By passing to relative commutants, we get  $\mathcal{L}(\times_{k \in S_j} \Gamma_k) \prec_{\mathcal{M}} \mathcal{P}_j$ , for any  $j$ . The conclusion of the theorem follows by using standards arguments that rely on [\[Ozawa and Popa 2004, Proposition 12\]](#) and [\[Ge 1996, Theorem A\]](#).  $\square$

**Corollary 3.7.** *Let  $\Gamma_1, \dots, \Gamma_n$  and  $\Lambda_1, \dots, \Lambda_m$  be property (T) wreath-like product groups of the form  $\mathcal{WR}(A, B \curvearrowright I)$ , where  $A$  is abelian,  $B$  is an icc subgroup of a hyperbolic group, and  $B \curvearrowright I$  has infinite stabilizers.*

*If there exists  $t > 0$  such that  $\mathcal{L}(\Gamma_1 \times \cdots \times \Gamma_n)^t = \mathcal{L}(\Lambda_1 \times \cdots \times \Lambda_m)$ , then  $t = 1$ ,  $n = m$ , and there is a unitary  $u \in \mathcal{L}(\Gamma_1 \times \cdots \times \Gamma_n)$  such that  $u\mathbb{T}(\Gamma_1 \times \cdots \times \Gamma_n)u^* = \mathbb{T}(\Lambda_1 \times \cdots \times \Lambda_m)$ .*

*Proof.* The result follows directly by combining [Theorem 3.6](#) and [\[Chifan et al. 2023b, Theorem 1.3\]](#).  $\square$

#### 4. Graph product groups associated with cycles of cliques graphs

In this section we highlight a class of graphs considered to have good clustering properties. Specifically, a graph  $\mathcal{G}$  is called a *simple cycle of cliques* (and belongs to the class  $\text{CC}$ ) if there is an enumeration of its cliques set  $\text{cliq}(\mathcal{G}) = \{\mathcal{C}_1, \dots, \mathcal{C}_n\}$  with  $n \geq 4$  such that the subgraphs  $\mathcal{C}_{i,j} := \mathcal{C}_i \cap \mathcal{C}_j$  satisfy the conditions

$$\mathcal{C}_{i,j} = \begin{cases} \emptyset & \text{if } \hat{i} - \hat{j} \in \mathbb{Z}_n \setminus \{\hat{1}, \hat{n-1}\}, \\ \neq \emptyset & \text{if } \hat{i} - \hat{j} \in \{\hat{1}, \hat{n-1}\}. \end{cases} \quad (4-1)$$

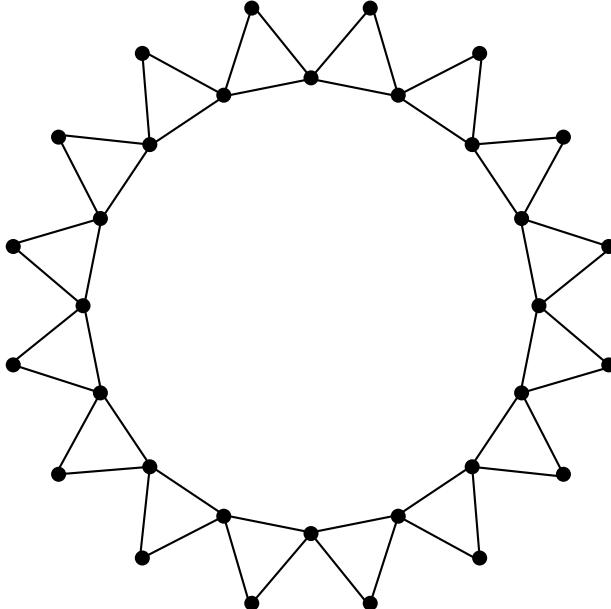
Here, the classes  $\hat{i}$  and  $\hat{j}$  belong to  $\mathbb{Z}/n\mathbb{Z}$ . We will also refer to  $\text{cliq}(\mathcal{G}) = \{\mathcal{C}_1, \dots, \mathcal{C}_n\}$  satisfying the previous properties as the *the consecutive enumeration* of the cliques of  $\mathcal{G}$ .

For every  $i \in \overline{1, n}$ , we define  $\mathcal{C}_i^{\text{int}} := \mathcal{C}_i \setminus (\mathcal{C}_{i-1,i} \cup \mathcal{C}_{i,i+1})$ , where we declare that  $0 = n$  and  $n+1 = 1$ . When  $\mathcal{C}_i^{\text{int}} \neq \emptyset$  for all  $i \in \overline{1, n}$ , one says that  $\mathcal{G}$  belongs to the class  $\text{CC}_1$ . Most of our main results will involve graphs of this form. Throughout this article we will use all these notations consistently.

A basic example of a graph in the class  $\text{CC}_1$  is a simple cycle of triangles called  $\mathcal{F}_n$ , where  $n$  is the number of cliques; see [Figure 2](#) below for  $n = 16$ .

In fact every graph  $\mathcal{G} \in \text{CC}_1$  appears as a two-level clustered graph which is a specific retraction of  $\mathcal{F}_n$  as follows. There exists a graph projection map  $\Phi : \mathcal{G} \rightarrow \mathcal{F}_n$  such that, for every vertex  $v \in \mathcal{F}_n$ , the cluster  $\Phi^{-1}(v) \subset \mathcal{G}$  is a complete subgraph of  $\mathcal{G}$ . In addition, whenever  $v, w \in \mathcal{F}_n$  are connected in  $\mathcal{F}_n$ , there are edges in  $\mathcal{G}$  between all vertices of the corresponding clusters  $\Phi^{-1}(v)$  and  $\Phi^{-1}(w)$ .

We continue by recording some elementary combinatorial properties of graph product groups associated with graphs that are simple cycles of cliques. The proof of the following lemma is straightforward and we leave it to the reader.



**Figure 2.** A cycle of 16 triangles is a simple example of a graph in  $\text{CC}_1$ .

**Lemma 4.1.** *Let  $\mathcal{G} \in \text{CC}_1$ , and let  $\mathcal{C}_1, \dots, \mathcal{C}_n$  be an enumeration of its consecutive cliques. Let  $\{\Gamma_v, v \in \mathcal{V}\}$  be a collection of groups, and let  $\Gamma_{\mathcal{G}}$  be the corresponding graph product group. We denote by  $\{w_i\}_{i=1}^n$  the petal outer vertices of  $\mathcal{F}_n$  and by  $\{b_i\}_{i=1}^n$  the petal base vertices of  $\mathcal{F}_n$ .*

*Then  $\Gamma_{\mathcal{G}}$  can be realized as a graph product  $\Gamma'_{\mathcal{F}_n}$  associated to the graph  $\mathcal{F}_n$ , where the vertex groups are defined by*

$$\Gamma'_{w_i} = \bigoplus_{v \in \mathcal{C}_i^{\text{int}}} \Gamma_v \quad \text{and} \quad \Gamma'_{b_i} = \bigoplus_{v \in \mathcal{C}_{i-1,i}} \Gamma_v \quad \text{for every } i \in \overline{1, n}.$$

**Proposition 4.2.** *Let  $\mathcal{G} \in \text{CC}_1$ , and let  $\mathcal{C}_1, \dots, \mathcal{C}_n$  be an enumeration of its consecutive cliques. Let  $\{\Gamma_v, v \in \mathcal{V}\}$  be a collection of infinite groups, and let  $\Gamma_{\mathcal{G}}$  be the corresponding graph product group. Then the following properties hold:*

- (1) *If  $g \in \Gamma_{\mathcal{C}_{i-1} \Delta \mathcal{C}_i}$  and  $h \in \Gamma_{\mathcal{C}_i \Delta \mathcal{C}_{i+1}}$  satisfy  $gh \in \Gamma_{\mathcal{G} \setminus \mathcal{C}_i^{\text{int}}}$ , then one can find  $a \in \Gamma_{(\mathcal{C}_{i-1} \setminus \mathcal{C}_{i-1,i}) \cup \mathcal{C}_{i,i+1}}$ ,  $s \in \Gamma_{\mathcal{C}_i^{\text{int}}}$ , and  $b \in \Gamma_{(\mathcal{C}_{i+1} \setminus \mathcal{C}_{i,i+1}) \cup \mathcal{C}_{i-1,i}}$  such that  $g = as$  and  $h = s^{-1}b$ .*
- (2) *Let  $g \in \Gamma_{\mathcal{C}_{i-2,i-1} \cup \mathcal{C}_{i,i+1}}$ ,  $h \in \Gamma_{\mathcal{C}_{i-1,i} \cup \mathcal{C}_{i+1,i+2}}$ , and  $k \in \Gamma_{\mathcal{C}_{i,i+1} \cup \mathcal{C}_{i+2,i+3}}$  such that*

$$ghk \in \Gamma_{\mathcal{C}_{i-2,i-1} \cup \mathcal{C}_{i-1,i} \cup \mathcal{C}_{i+1,i+2} \cup \mathcal{C}_{i+2,i+3}}.$$

*Then one can find  $a \in \Gamma_{\mathcal{C}_{i-2,i-1}}$ ,  $b \in \Gamma_{\mathcal{C}_{i+2,i+3}}$ , and  $s \in \Gamma_{\mathcal{C}_{i,i+1}}$  such that  $g = as$  and  $k = s^{-1}b$ .*

- (3) *For each  $i \in \overline{1, n}$ , let  $x_{i,i+1} \in \Gamma_{\mathcal{C}_i \cup \mathcal{C}_{i+1}}$  such that  $x_{1,2}x_{2,3} \cdots x_{n-1,n}x_{n,1} = 1$ . Then, for each  $i \in \overline{1, n}$ , one can find  $a_i \in \Gamma_{\mathcal{C}_{i-1,i}}$ ,  $b_i \in \Gamma_{\mathcal{C}_i^{\text{int}}}$ , and  $c_i \in \Gamma_{\mathcal{C}_{i,i+1}}$  such that  $x_{i,i+1} = a_i b_i c_i b_{i+1}^{-1} a_{i+2}^{-1} c_{i+1}^{-1}$ . Here we use the convention that  $n+1 = 1$ ,  $n+2 = 2$ , etc.*

*Proof.* Here  $\Delta$  is the symmetric difference operation defined by  $A \Delta B = (A \cup B) \setminus (A \cap B)$ . We recall the *normal form* [Green 1990, Theorem 3.9], which in graph product groups plays the role that reduced words play in free product groups. If  $1 \neq g \in \Gamma_{\mathcal{G}}$  is expressed as  $g = g_1 \cdots g_n$ , we say  $g$  is in normal form if each  $g_i$  is a nonidentity element of some vertex group (called a *syllable*) and if it is impossible, through repeated swapping of syllables (corresponding to adjacent vertices in  $\mathcal{G}$ ), to bring together two syllables from the same vertex group. By [Green 1990, Theorem 3.9], every  $1 \neq g \in \Gamma_{\mathcal{G}}$  has a normal form  $g = g_1 \cdots g_n$  and it is unique up to a finite number of consecutive syllable shuffles. Moreover, given any sequence of syllables  $g_1 \cdots g_n$ , there is an inductive procedure for putting this sequence into normal form: if  $h_1 \cdots h_r$  is the normal form of  $g_1 \cdots g_m$ , then the normal form of  $g_1 \cdots g_{m+1}$  is either

- (i)  $h_1 \cdots h_r$  if  $g_{m+1} = 1$ ,
- (ii)  $h_1 \cdots h_{j-1} h_{j+1} \cdots h_r$  if  $h_j$  shuffles to the end and  $g_{m+1} = h_j^{-1}$ ,
- (iii)  $h_1 \cdots h_{j-1} h_{j+1} \cdots h_r (h_j g_{m+1})$  if  $h_j$  shuffles to the end,  $g_{m+1} \neq h_j^{-1}$ , and  $g_{m+1}, h_j$  belong to the same vertex group, or
- (iv)  $h_1 \cdots h_r g_{m+1}$  if  $g_{m+1}$  is in a different vertex group from that of every syllable which can be shuffled down.

Note that the normal form of an element  $g \in \Gamma_{\mathcal{G}}$  has minimal syllable length with respect to all the sequences of syllables representing  $g$ .

We are now ready to prove the three assertions of the proposition. For (1), let  $g = g_1 \cdots g_n$  and  $h = h_1 \cdots h_m$  be the normal forms of  $g$  and  $h$ . Then  $gh$  has a normal form  $gh = k_1 \cdots k_r$ , determined by the procedure described in the previous paragraph. By assumption,  $k_j \notin \bigcup_{v \in \mathcal{C}_i^{\text{int}}} \{\Gamma_v\}$  for all  $j \in \overline{1, r}$ . Now, if  $g_j \notin \bigcup_{v \in \mathcal{C}_i^{\text{int}}} \{\Gamma_v\}$  for all  $j \in \overline{1, n}$ , then each  $h_i \in \bigcup_{v \in \mathcal{C}_i^{\text{int}}} \{\Gamma_v\}$  is one of the syllables occurring in the normal form of  $gh$ . Since this cannot happen, we have  $h_i \notin \bigcup_{v \in \mathcal{C}_i^{\text{int}}} \{\Gamma_v\}$  for all  $i \in \overline{1, m}$ , and hence we can take  $a = g$ ,  $b = h$ , and  $s$  the empty word. Assume that  $g_j \in \bigcup_{v \in \mathcal{C}_i^{\text{int}}} \{\Gamma_v\}$  for some  $j \in \overline{1, n}$ . We note that we may assume  $j = n$  since, if  $g_i \in \bigcup_{v \in \mathcal{C}_{i-1} \setminus \mathcal{C}_i} \{\Gamma_v\}$  for some  $i \in \overline{j+1, n}$ , then  $g_j$  would be a syllable in  $gh$  since it cannot be shuffled past  $g_i$ , which shows that  $g_{j+1} \cdots g_n \in \bigcup_{v \in \mathcal{C}_i} \{\Gamma_v\}$ . This implies that  $g_n^{-1} = h_i$  for some  $i \in \overline{1, m}$ . Choosing the smallest such  $i$  and noting that  $h_1, \dots, h_{i-1} \in \bigcup_{v \in \mathcal{C}_i} \{\Gamma_v\}$  (since it must be possible to shuffle  $h_i$  up to  $g_n$  as in (ii) of the previous paragraph), we may assume that  $h_1 = g_n^{-1}$ . Continuing in this way we see that we can take  $a = g_1 \cdots g_{k-1}$ ,  $b = h_{n-k+2} \cdots h_m$ , and  $s = g_k \cdots g_n$ , where  $g_j \notin \bigcup_{v \in \mathcal{C}_i^{\text{int}}} \{\Gamma_v\}$  for all  $j \in \overline{1, k-1}$  and  $h = g_n^{-1}, \dots, g_k^{-1} h_{n-k+2}, \dots, h_m$ . Notice too that none of the syllables  $h_{n-k+2}, \dots, h_m$  can belong to  $\bigcup_{v \in \mathcal{C}_i^{\text{int}}} \{\Gamma_v\}$  since the inverse of such a syllable cannot be any of the syllables  $g_1, \dots, g_{k-1}$ . This proves (1).

For (2), let  $g = g_1 \cdots g_n$ ,  $h = h_1 \cdots h_m$ , and  $k = k_1 \cdots k_r$  be normal forms. If  $g_i \notin \bigcup_{v \in \mathcal{C}_{i,i+1}} \{\Gamma_v\}$  for all  $i \in \overline{1, n}$ , then  $k_j \notin \bigcup_{v \in \mathcal{C}_{i,i+1}} \{\Gamma_v\}$  for all  $j \in \overline{1, n}$  since neither  $h$  nor  $ghk$  have normal forms with syllables in  $\bigcup_{v \in \mathcal{C}_{i,i+1}} \{\Gamma_v\}$  by assumption, and hence we can take  $a = g$ ,  $b = k$ , and  $s$  the empty word. Otherwise we must have  $g_j \in \Gamma_v$  for some  $v \in \mathcal{C}_{i,i+1}$ , and as in the proof of part (1) we can assume  $j = n$  and  $k_1 = g_n^{-1}$  (note that  $g_j$  commutes with each syllable in the normal form of  $h$ ). Continuing, we see that we can take  $a = g_1 \cdots g_{l-1}$ ,  $b = k_{n-l+2} \cdots k_r$ , and  $s = g_l \cdots g_n$ , where  $g_j \notin \bigcup_{v \in \mathcal{C}_{i,i+1}} \{\Gamma_v\}$  for all  $j \in \overline{1, l-1}$  and  $k = g_n^{-1} \cdots g_l^{-1} k_{n-l+2} \cdots k_r$ . This proves (2).

For (3), observe first that every  $x_{i,i+1} \in \Gamma_{\mathcal{C}_i \cup \mathcal{C}_{i+1}} = \Gamma_{\mathcal{C}_{i,i+1}} \times (\Gamma_{\mathcal{C}_i \setminus \mathcal{C}_{i+1}} * \Gamma_{\mathcal{C}_{i+1} \setminus \mathcal{C}_i})$  can be written in the form  $\tilde{a}_i \tilde{b}_i \tilde{c}_i \tilde{d}_i \tilde{e}_i \tilde{f}_i$ , where

$$\tilde{a}_i \in \Gamma_{\mathcal{C}_{i-1,i}}, \quad \tilde{b}_i \in \Gamma_{\mathcal{C}_i^{\text{int}}}, \quad \tilde{c}_i \in \Gamma_{\mathcal{C}_{i,i+1}}, \quad \tilde{d}_i \in \Gamma_{\mathcal{C}_{i+1}^{\text{int}}}, \quad \tilde{e}_i \in \Gamma_{\mathcal{C}_{i+1,i+2}}, \quad \tilde{f}_i \in \Gamma_{\mathcal{C}_i \cup \mathcal{C}_{i+1}}.$$

Moreover, we can assume that the normal form of  $x_{i,i+1}$  is the sequence obtained by concatenating the normal forms of  $\tilde{a}_i$ ,  $\tilde{b}_i$ ,  $\tilde{c}_i$ ,  $\tilde{d}_i$ ,  $\tilde{e}_i$ ,  $\tilde{f}_i$ , and if  $\tilde{f}_i = f_1 \cdots f_n$  is the normal form of  $\tilde{f}_i$ , then  $f_1$  belongs to a group  $\Gamma_v$ , where  $v$  is vertex in  $\mathcal{C}_i \setminus \mathcal{C}_{i+1}$ .

We continue by showing that we can assume  $\tilde{f}_i = 1$ . Notice that this is the case if there is no syllable  $g$  occurring in the normal form of  $x_{i,i+1}$  belonging to  $\bigcup_{v \in \mathcal{C}_{i+1} \setminus \mathcal{C}_i} \{\Gamma_v\}$ ; indeed, in this case  $\tilde{d}_i, \tilde{e}_i = 1$  and all the syllables occurring in the normal form of  $\tilde{f}_i$  can be shuffled up to the normal forms of  $\tilde{a}_i$ ,  $\tilde{b}_i$ ,  $\tilde{c}_i$ . So it remains to assume that there is such a syllable  $g$  and assume by contradiction that  $\tilde{f}_i \neq 1$ . Notice that our hypotheses imply that  $f_1^{-1}$  is a syllable in the normal form of  $x_{i-2,i-1} x_{i-1,i}$ , and  $g^{-1}$  is a syllable in the normal form of  $x_{i+1,i+2} x_{i+2,i+3}$ . This implies that the normal form of  $x_{1,2} x_{2,3} \cdots x_{i-1,i} x_{i,i+1}$  must still contain  $f_1$  as  $f_1^{-1}$  cannot shuffle past  $g$  to cancel with  $f_1$ . Consequently, the normal form of  $x_{1,2} x_{2,3} \cdots x_{n-1,n} x_{n,1}$  must still contain  $f_1$  as  $f_1^{-1}$  cannot shuffle past  $g$  or  $g^{-1}$  to cancel with  $f_1$ . This gives a contradiction, and hence we can assume  $\tilde{f}_i = 1$ .

Next, we observe that  $\tilde{b}_i = \tilde{d}_{i-1}^{-1}$  for each  $i$  since our hypotheses imply that all the syllables occurring in the normal form of  $\tilde{b}_i^{-1}$  occur in the one for  $x_{i-1,i}$ , and only  $\tilde{d}_{i-1}$  has normal form with

syllables coming from  $\bigcup_{v \in \mathcal{C}_i^{\text{int}}} \{\Gamma_v\}$ . To finish the proof, set  $a_i = \tilde{a}_i$ ,  $b_i = \tilde{b}_i$ , and  $c_i = \tilde{c}_i$  and note that, since  $\tilde{e}_i \tilde{c}_{i+1} \tilde{a}_{i+2} = 1$  (being the only elements in our decompositions belonging to  $\Gamma_{\mathcal{C}_{i+1,i+2}}$ ), we have  $x_{i,i+1} = \tilde{a}_i \tilde{b}_i \tilde{c}_i \tilde{d}_i \tilde{e}_i = a_i b_i c_i b_{i+1}^{-1} a_{i+2}^{-1} (a_{i+2} \tilde{e}_i) = a_i b_i c_i b_{i+1}^{-1} a_{i+2}^{-1} c_{i+1}^{-1}$ .  $\square$

**Lemma 4.3.** *Let  $\Gamma = \mathcal{G}\{\Gamma_v, v \in \mathcal{V}\}$  be a graph product of groups such that  $\mathcal{G} \in \text{CC}_1$ . Then  $\Gamma$  has trivial amenable radical.*

*Proof.* Assume by contradiction that there exists a nontrivial amenable normal subgroup  $A$  of  $\Gamma$ . Since  $\Gamma$  is icc, we get that  $A$  is an infinite group. For any  $w \in \mathcal{V}$ , note that  $\text{st}(w) \neq \mathcal{V}$  and  $\mathcal{G}\{\Gamma_v\} = \Gamma_{\mathcal{V} \setminus \{w\}} *_{\Gamma_{\text{lk}(w)}} \Gamma_{\text{st}(w)}$ . Since  $A$  is an amenable, normal subgroup of  $\Gamma$ , it follows that  $\mathcal{L}(A) \prec_{\mathcal{L}(\Gamma)} \mathcal{L}(\Gamma_{\text{lk}(w)})$  [Vaes 2014, Theorem A]. In particular, by using [Drimbe et al. 2019, Lemma 2.4], it follows that  $\mathcal{L}(A) \prec_{\mathcal{L}(\Gamma)}^s \mathcal{L}(\Gamma_{\mathcal{C}})$  for any  $\mathcal{C} \in \text{cliq}(\mathcal{G})$ . Let  $\mathcal{C}, \mathcal{D} \in \text{cliq}(\mathcal{G})$  such that  $\mathcal{C} \cap \mathcal{D} = \emptyset$ . Using [Vaes 2013, Lemma 2.7], there is  $g \in \Gamma$  such that  $\mathcal{L}(A) \prec_{\mathcal{L}(\Gamma)}^s \mathcal{L}(\Gamma_{\mathcal{C}} \cap g \Gamma_{\mathcal{D}} g^{-1})$ . Note however that Proposition 2.1 implies that  $\Gamma_{\mathcal{C}} \cap g \Gamma_{\mathcal{D}} g^{-1} = 1$ . This shows that  $\mathcal{L}(A) \prec_{\mathcal{L}(\Gamma)} \mathbb{C}1$ , and thus gives the contradiction that  $A$  is finite.  $\square$

We end this section by recording a result describing all automorphisms of graph product groups  $\Gamma_{\mathcal{G}}$  associated with graphs in the class  $\text{CC}_1$ . This is a particular case of a powerful theorem in geometric group theory established recently by Genevois and Martin [2019, Corollary C].

To state the result we briefly recall a special class of automorphisms of graph product groups. For any isometry  $\sigma : \mathcal{G} \rightarrow \mathcal{G}$  and any collection of group isomorphisms  $\Phi = \{\phi_v : \Gamma_v \rightarrow \Gamma_{\sigma(v)} : v \in \mathcal{V}\}$ , define the *local automorphism*  $(\sigma, \Phi)$  to be the canonical automorphism of  $\Gamma_{\mathcal{G}}$  induced by the maps  $\bigcup_{v \in \mathcal{V}} \Gamma_v \ni g \rightarrow \phi_{\sigma(v)}(g) \in \Gamma_{\mathcal{G}}$ . One can easily observe that, under composition, these form a subgroup of  $\text{Aut}(\Gamma_{\mathcal{G}})$  which is denoted by  $\text{Loc}(\Gamma_{\mathcal{G}})$ . We denote by  $\text{Loc}_0(\Gamma_{\mathcal{G}})$  the subgroup of local automorphisms satisfying  $\sigma = \text{Id}$ . Notice that  $\text{Loc}_0(\Gamma_{\mathcal{G}})$  is naturally isomorphic to  $\bigoplus_{v \in \mathcal{V}} \text{Aut}(\Gamma_v)$ . Moreover, the inclusion  $\text{Loc}_0(\Gamma_{\mathcal{G}}) \leq \text{Loc}(\Gamma_{\mathcal{G}})$  has finite index.

**Theorem 4.4** [Genevois and Martin 2019]. *Let  $\Gamma_{\mathcal{G}}$  be a graph product associated with a graph  $\mathcal{G} \in \text{CC}_1$ . Then its automorphism group  $\text{Aut}(\Gamma_{\mathcal{G}})$  is generated by the inner and the local automorphisms of  $\Gamma_{\mathcal{G}}$ . In fact we have  $\text{Aut}(\Gamma_{\mathcal{G}}) = \text{Inn}(\Gamma_{\mathcal{G}}) \rtimes \text{Loc}(\Gamma_{\mathcal{G}})$ , and therefore*

$$\begin{aligned} \text{Aut}(\Gamma_{\mathcal{G}}) &\cong \Gamma_{\mathcal{G}} \rtimes \left( \left( \bigoplus_{v \in \mathcal{V}} \text{Aut}(\Gamma_v) \right) \rtimes \text{Sym}(\Gamma_{\mathcal{G}}) \right), \\ \text{Out}(\Gamma_{\mathcal{G}}) &\cong \left( \bigoplus_{v \in \mathcal{V}} \text{Aut}(\Gamma_v) \right) \rtimes \text{Sym}(\Gamma_{\mathcal{G}}). \end{aligned} \tag{4-2}$$

Here,  $\text{Sym}(\Gamma_{\mathcal{G}})$  is an explicit finite subgroup of automorphisms of  $\Gamma_{\mathcal{G}}$ .

*Proof.* One can easily check that the graphs in  $\text{CC}_1$  are atomic and therefore the conclusion follows immediately from [Genevois and Martin 2019, Corollary C].  $\square$

**Remark 4.5.** (1) If in the hypothesis of Theorem 4.4 we assume in addition that  $\{\Gamma_v\}_{v \in \mathcal{V}}$  are pairwise nonisomorphic, then we have  $\text{Sym}(\Gamma_{\mathcal{G}}) = 1$  in the automorphism group formulae (4-2). The same holds if instead we assume that any two cliques of  $\mathcal{G}$  have different cardinalities and for any  $\mathcal{C} \in \text{cliq}(\mathcal{G})$  the set  $\{\Gamma_v\}_{v \in \mathcal{C}}$  consists of pairwise nonisomorphic subgroups.

(2) One of the main goals of this paper is to establish both von Neumann algebraic and  $C^*$ -algebraic analogs of [Theorem 4.4](#), under various assumptions on the vertex groups. For the specific statements in this direction, the reader may consult Corollaries [7.11](#) and [7.14](#) in [Section 7](#).

## 5. Von Neumann algebraic cancellation in cyclic relations

In this section we establish a von Neumann algebraic analog of [Proposition 4.2](#) (3) describing the structure of all unitaries that satisfy a similar cyclic relation ([Lemma 5.1](#)). We start by first proving the following von Neumann algebraic counterpart of [Proposition 4.2](#) (1).

**Lemma 5.1.** *Let  $\Lambda_1, \Lambda_2, \Sigma < \Gamma$  be groups satisfying the following properties:*

- (1)  $\Lambda_1 \cap \Lambda_2 = \Lambda_1 \cap \Sigma = \Lambda_2 \cap \Sigma = 1$ .
- (2) *For any  $g_1 \in \Lambda_1 \vee \Sigma$  and  $g_2 \in \Lambda_2 \vee \Sigma$  satisfying  $g_1 g_2 \in \Lambda_1 \vee \Lambda_2$ , one can find  $a_1 \in \Lambda_1$ ,  $a_2 \in \Lambda_2$  and  $s \in \Sigma$  such that  $g_1 = a_1 s$  and  $g_2 = s^{-1} a_2$ .*

*Then, for any  $y_1 \in \mathcal{U}(\mathcal{L}(\Lambda_1 \vee \Sigma))$  and  $y_2 \in \mathcal{U}(\mathcal{L}(\Lambda_2 \vee \Sigma))$  satisfying  $y_1 y_2 \in \mathcal{U}(\mathcal{L}(\Lambda_1 \vee \Lambda_2))$ , one can find  $v_1 \in \mathcal{U}(\mathcal{L}(\Lambda_1))$ ,  $v_2 \in \mathcal{U}(\mathcal{L}(\Lambda_2))$ , and  $x \in \mathcal{U}(\mathcal{L}(\Sigma))$  such that  $y_1 = v_1 x$  and  $y_2 = x^* v_2$ .*

Above we used the notation that if  $\Gamma_1, \Gamma_2 < \Gamma$  are groups, then we denote by  $\Gamma_1 \vee \Gamma_2$  the subgroup of  $\Gamma$  generated by  $\Gamma_1$  and  $\Gamma_2$ .

*Proof of Lemma 5.1.* For each  $i = 1, 2$ , consider the Fourier expansion of  $y_i = \sum_{g_i \in \Lambda_i \vee \Sigma} (y_i)_{g_i} u_{g_i}$ . Since  $y_1 y_2 \in \mathcal{L}(\Lambda_1 \vee \Lambda_2)$  using condition (2), we have that

$$y = y_1 y_2 = \sum_{\substack{g_1 \in \Lambda_1 \vee \Sigma \\ g_2 \in \Lambda_2 \vee \Sigma \\ g_1 g_2 \in \Lambda_1 \vee \Lambda_2}} (y_1)_{g_1} (y_1)_{g_2} u_{g_1 g_2} = \sum_{\substack{a_1 \in \Lambda_1 \\ a_2 \in \Lambda_2 \\ s \in \Sigma}} (y_1)_{a_1 s} (y_2)_{s^{-1} a_2} u_{a_1 a_2}.$$

The above formula and basic approximations show that

$$1 = \sum_{s \in \Sigma} E_{\mathcal{L}(\Lambda_1)}(y_1 u_{s^{-1}}) E_{\mathcal{L}(\Lambda_2)}(u_s y_2) y^*,$$

where the right-hand side is only  $\|\cdot\|_1$ -summable. Using this in combination with the Cauchy–Schwarz inequality, we further get

$$\begin{aligned} 1 &= \tau \left( \sum_{s \in \Sigma} E_{\mathcal{L}(\Lambda_1)}(y_1 u_{s^{-1}}) E_{\mathcal{L}(\Lambda_2)}(u_s y_2) y^* \right) \leq \sum_{s \in \Sigma} |\tau(E_{\mathcal{L}(\Lambda_1)}(y_1 u_{s^{-1}}) E_{\mathcal{L}(\Lambda_2)}(u_s y_2) y^*)| \\ &\leq \sum_{s \in \Sigma} \|E_{\mathcal{L}(\Lambda_1)}(y_1 u_{s^{-1}})\|_2 \|E_{\mathcal{L}(\Lambda_2)}(u_s y_2)\|_2 \\ &\leq \left( \sum_{s \in \Sigma} \|E_{\mathcal{L}(\Lambda_1)}(y_1 u_{s^{-1}})\|_2^2 \right)^{1/2} \left( \sum_{s \in \Sigma} \|E_{\mathcal{L}(\Lambda_2)}(u_s y_2)\|_2^2 \right)^{1/2} \leq \|y_1\|_2 \|y_2\|_2 = 1. \end{aligned}$$

Thus, we must have equality in the Cauchy–Schwarz inequality, and hence, for every  $s$ , there is  $c_s \in \mathbb{C}$  satisfying

$$E_{\mathcal{L}(\Lambda_1)}(y_1 u_{s^{-1}}) = c_s y E_{\mathcal{L}(\Lambda_2)}(y_2^* u_{s^{-1}}). \quad (5-1)$$

Taking absolute values, we get  $|E_{\mathcal{L}(\Lambda_1)}(y_1 u_{s^{-1}})| = |c_s| |E_{\mathcal{L}(\Lambda_2)}(y_2^* u_{s^{-1}})|$ , and since  $\Lambda_1 \cap \Lambda_2 = 1$  we conclude  $|E_{\mathcal{L}(\Lambda_1)}(y_1 u_{s^{-1}})| = |c_s| |E_{\mathcal{L}(\Lambda_2)}(y_2^* u_{s^{-1}})| \in \mathbb{C}1$ . Using the polar decomposition formula one can find  $d_s, e_s \in \mathbb{C}$  and unitaries  $x_s \in \mathcal{L}(\Lambda_1)$ ,  $z_s \in \mathcal{L}(\Lambda_2)$  satisfying

$$E_{\mathcal{L}(\Lambda_1)}(y_1 u_{s^{-1}}) = d_s x_s \quad \text{and} \quad E_{\mathcal{L}(\Lambda_2)}(y_2^* u_{s^{-1}}) = e_s z_s.$$

Combining these with (5-1), we get  $d_s x_s = c_s e_s y z_s$  for all  $s \in \Sigma$ ; in particular, for every  $d_s \neq 0$ , we have  $x_s = (e_s c_s / d_s) y z_s$ . Hence, for all  $s, t \in \Sigma$  with  $d_s, d_t \neq 0$ , we have  $x_t^* x_s = (e_s c_s \overline{e_t c_t} / d_s \hat{d}_t) z_t^* z_s$ . Again, as  $\Lambda_1 \cap \Lambda_2 = 1$ , one can find  $c_{s,t}, d_{s,t} \in \mathbb{C}$  such that

$$x_s = c_{s,t} x_t \quad \text{and} \quad z_s = d_{s,t} z_t. \quad (5-2)$$

Fix  $t \in \Sigma$ . Using the prior relations, we see that

$$y_1 = \sum_{s \in \Sigma} E_{\mathcal{L}(\Lambda_1)}(y_1 u_{s^{-1}}) u_s = \sum_{s \in \Sigma} d_s x_s u_s = \sum_{s \in \Sigma} d_s c_{s,t} x_t u_s = x_t \left( \sum_{s \in \Sigma} d_s c_{s,t} u_s \right).$$

In particular, this shows there are  $v_1 \in \mathcal{U}(\mathcal{L}(\Lambda_1))$  and  $x \in \mathcal{U}(\mathcal{L}(\Sigma))$  such that  $y_1 = v_1 x$ . Similarly, the prior relations also imply that  $y_2 = x^* v_2$  for some  $v_2 \in \mathcal{U}(\mathcal{L}(\Lambda_2))$ .  $\square$

**Theorem 5.2.** *Let  $\mathcal{G}$  be a graph in the class  $\text{CC}_1$ , and let  $\mathcal{C}_1, \dots, \mathcal{C}_n$  be an enumeration of its consecutive cliques. Let  $\Gamma_v$ ,  $v \in \mathcal{V}$ , be a collection of icc groups, and let  $\Gamma_{\mathcal{G}}$  be the corresponding graph product group. For each  $i \in \overline{1, n}$ , assume  $x_{i,i+1} = a_{i,i+1} b_{i,i+1}$ , where  $a_{i,i+1} \in \mathcal{U}(\mathcal{L}(\Gamma_{\mathcal{C}_{i,i+1}}))$  and  $b_{i,i+1} \in \mathcal{U}(\mathcal{L}(\Gamma_{\mathcal{C}_i \cup \mathcal{C}_{i+1} \setminus \mathcal{C}_{i,i+1}}))$ . If  $x_{1,2} x_{2,3} \cdots x_{n-1,n} x_{n,1} = 1$ , then for each  $i \in \overline{1, n}$  one can find  $a_i \in \mathcal{U}(\mathcal{L}(\Gamma_{\mathcal{C}_{i-1,i}}))$ ,  $b_i \in \mathcal{U}(\mathcal{L}(\Gamma_{\mathcal{C}_i^{\text{int}}}))$ , and  $c_i \in \mathcal{U}(\mathcal{L}(\Gamma_{\mathcal{C}_{i,i+1}}))$  such that  $x_{i,i+1} = a_i b_i c_i b_{i+1}^* a_{i+2}^* c_{i+1}^*$ . Here, we use the convention that  $n+1 = 1$  and  $n+2 = 2$ .*

*Proof.* Fix an arbitrary  $i \in \overline{1, n}$ . Using  $x_{1,2} x_{2,3} \cdots x_{n-1,n} x_{n,1} = 1$ , it follows that

$$b_{i-1,i} b_{i,i+1} = a_{i-1,i}^* x_{i-2,i-1}^* \cdots x_{1,2}^* x_{n,1}^* \cdots x_{i+1,i+2}^* a_{i,i+1}^*.$$

Since  $a_{i-1,i}, a_{i,i+1} \in \mathcal{L}(\Gamma_{\mathcal{G} \setminus \mathcal{C}_i^{\text{int}}})$  and  $x_{j,j+1}, a_{j,j+1} \in \mathcal{L}(\Gamma_{\mathcal{C}_j \cup \mathcal{C}_{j+1}})$ , for any  $j \in \overline{1, n}$ , we get that

$$b_{i-1,i} b_{i,i+1} = a_{i-1,i}^* x_{i-2,i-1}^* \cdots x_{1,2}^* x_{n,1}^* \cdots x_{i+1,i+2}^* a_{i,i+1}^* \in \mathcal{L}(\Gamma_{\mathcal{G} \setminus \mathcal{C}_i^{\text{int}}}).$$

Since  $b_{i-1,i} b_{i,i+1} \in \mathcal{L}(\Gamma_{\mathcal{C}_{i-1} \cup \mathcal{C}_i \cup \mathcal{C}_{i+1}})$ , we deduce that

$$b_{i-1,i} b_{i,i+1} \in \mathcal{U}(\mathcal{L}(\Gamma_{\mathcal{C}_{i-1} \cup \mathcal{C}_{i+1}})). \quad (5-3)$$

Now, fix two words  $g_1 \in \Gamma_{(\mathcal{C}_{i-1} \cup \mathcal{C}_i) \setminus \mathcal{C}_{i-1,i}}$  and  $g_2 \in \Gamma_{(\mathcal{C}_i \cup \mathcal{C}_{i+1}) \setminus \mathcal{C}_{i,i+1}}$  such that  $g_1 g_2 \in \Gamma_{\mathcal{C}_{i-1} \cup \mathcal{C}_{i+1}}$ . Using Proposition 4.2 (1), there exist  $a_1 \in \Gamma_{(\mathcal{C}_{i-1} \setminus \mathcal{C}_i) \cup \mathcal{C}_{i,i+1}}$ ,  $b \in \Gamma_{(\mathcal{C}_{i+1} \setminus \mathcal{C}_i) \cup \mathcal{C}_{i-1,i}}$ , and  $s \in \Gamma_{\mathcal{C}_i^{\text{int}}}$  such that  $g_1 = a s$  and  $g_2 = s^{-1} b$ . Thus, applying Lemma 5.1 for  $\Lambda_1 = \Gamma_{(\mathcal{C}_{i-1} \setminus \mathcal{C}_i) \cup \mathcal{C}_{i,i+1}}$ ,  $\Lambda_2 = \Gamma_{(\mathcal{C}_{i+1} \setminus \mathcal{C}_i) \cup \mathcal{C}_{i-1,i}}$ , and  $\Sigma = \Gamma_{\mathcal{C}_i^{\text{int}}}$ , we derive from (5-3) that one can find unitaries  $x_{i-1} \in \mathcal{L}(\Gamma_{(\mathcal{C}_{i-1} \setminus \mathcal{C}_i) \cup \mathcal{C}_{i,i+1}})$  and  $z_{i-1} \in \mathcal{L}(\Gamma_{\mathcal{C}_i^{\text{int}}})$  such that  $b_{i-1,i} = x_{i-1} z_{i-1}$ .

Lemma 5.1 also implies from (5-3) that  $b_{i,i+1} = z_{i-1}^* y_i$  for some  $y_i \in \mathcal{U}(\mathcal{L}(\Gamma_{(\mathcal{C}_{i+1} \setminus \mathcal{C}_i) \cup \mathcal{C}_{i-1,i}}))$ . Using this with  $b_{i,i+1} = x_i z_i$ , we get that  $b_{i,i+1} = z_{i-1}^* y_i = x_i z_i$ . Hence,  $x_i^* z_{i-1}^* = z_i y_i^* =: t_{i,i+1}$  and note

that  $t_{i,i+1} \in \mathcal{L}(\Gamma_{\mathcal{C}_{i-1,i} \cup \mathcal{C}_{i+1,i+2}})$ . Thus,  $x_{i,i+1} = a_{i,i+1}b_{i,i+1} = a_{i,i+1}z_{i-1}^*t_{i,i+1}^*z_i = z_{i-1}^*a_{i,i+1}t_{i,i+1}^*z_i$ . In conclusion, we showed that, for any  $i \in \overline{1, n}$ , we have

$$x_{i,i+1} = z_{i-1}^*a_{i,i+1}t_{i,i+1}z_i. \quad (5-4)$$

Now, we note that, since  $x_{1,2}x_{2,3} \cdots x_{n-1,n}x_{n,1} = 1$ , we obviously have

$$a_{1,2}t_{1,2}a_{2,3}t_{2,3} \cdots a_{n-1,n}t_{n-1,n}a_{n,1}t_{n,1} = 1.$$

Again we will use this relation together with the same argument from the proof of [Lemma 5.1](#) to show that  $x_{i,i+1}$  has the form described in the conclusion of the theorem. First, observe the cyclic relation and use a similar argument as in the beginning of the proof to show that

$$\begin{aligned} w &:= t_{i-1,i}a_{i,i+1}t_{i,i+1}a_{i+1,i+2}t_{i+1,i+2}a_{i+2,i+3} \\ &= a_{i-1,i}^* \cdots t_{1,2}^*a_{1,2}^*t_{n,1}^*a_{n,1}^* \cdots t_{i+2,i+3}^* \in \mathcal{L}(\Gamma_{\mathcal{C}_{i-2,i-1} \cup \mathcal{C}_{i-1,i} \cup \mathcal{C}_{i+1,i+2} \cup \mathcal{C}_{i+2,i+3}}). \end{aligned}$$

Now, fix three words  $w_1 \in \Gamma_{\mathcal{C}_{i-2,i-1} \cup \mathcal{C}_{i,i+1}}$ ,  $w_2 \in \Gamma_{\mathcal{C}_{i-1,i} \cup \mathcal{C}_{i+1,i+2}}$ , and  $w_3 \in \Gamma_{\mathcal{C}_{i,i+1} \cup \mathcal{C}_{i+2,i+3}}$  satisfying  $w_1w_2w_3 \in \Gamma_{\mathcal{C}_{i-2,i-1} \cup \mathcal{C}_{i-1,i} \cup \mathcal{C}_{i+1,i+2} \cup \mathcal{C}_{i+2,i+3}}$ . Using [Proposition 4.2](#)(2), we have  $w_1 = as$  and  $w_3 = s^{-1}b$ , where  $a \in \Gamma_{\mathcal{C}_{i-2,i-1}}$ ,  $b \in \Gamma_{\mathcal{C}_{i+2,i+3}}$ , and  $s \in \Gamma_{\mathcal{C}_{i,i+1}}$ . Since  $t_{i,i+1}a_{i+1,i+2} \in \mathcal{L}(\Gamma_{\mathcal{C}_{i-1,i} \cup \mathcal{C}_{i+1,i+2}})$ , we can write the Fourier expansions

$$\begin{aligned} t_{i-1,i}a_{i,i+1} &= \sum_{w_1 \in \mathcal{C}_{i-2,i-1} \cup \mathcal{C}_{i,i+1}} (t_{i-1,i}a_{i,i+1})_{w_1}u_{w_1}, \\ t_{i+1,i+2}a_{i+2,i+3} &= \sum_{w_3 \in \mathcal{C}_{i,i+1} \cup \mathcal{C}_{i+2,i+3}} (t_{i+1,i+2}a_{i+2,i+3})_{w_3}u_{w_3}. \end{aligned}$$

All these observations imply that

$$\begin{aligned} w &= (t_{i-1,i}a_{i,i+1})(t_{i,i+1}a_{i+1,i+2})(t_{i+1,i+2}a_{i+2,i+3}) \\ &= \sum_{s \in \Gamma_{\mathcal{C}_{i,i+1}}} E_{\mathcal{L}(\Gamma_{\mathcal{C}_{i-2,i-1}})}(t_{i-1,i}a_{i,i+1}u_{s^{-1}})t_{i,i+1}a_{i+1,i+2}E_{\mathcal{L}(\Gamma_{\mathcal{C}_{i+2,i+3}})}(u_s t_{i+1,i+2}a_{i+2,i+3}), \end{aligned}$$

where again the convergence is in  $\|\cdot\|_1$ . Using this and the Cauchy–Schwarz inequality, we get

$$\begin{aligned} 1 &= \sum_{s \in \Gamma_{\mathcal{C}_{i,i+1}}} |\tau(E_{\mathcal{L}(\Gamma_{\mathcal{C}_{i-2,i-1}})}(t_{i-1,i}a_{i,i+1}u_{s^{-1}})t_{i,i+1}a_{i+1,i+2}E_{\mathcal{L}(\Gamma_{\mathcal{C}_{i+2,i+3}})}(u_s t_{i+1,i+2}a_{i+2,i+3})w^*)| \\ &\leq \sum_{s \in \Gamma_{\mathcal{C}_{i,i+1}}} \|E_{\mathcal{L}(\Gamma_{\mathcal{C}_{i-2,i-1}})}(t_{i-1,i}a_{i,i+1}u_{s^{-1}})\|_2 \|E_{\mathcal{L}(\Gamma_{\mathcal{C}_{i+2,i+3}})}(a_{i+2,i+3}^*t_{i+1,i+2}u_{s^{-1}})\|_2 \\ &\leq \left( \sum_{s \in \Gamma_{\mathcal{C}_{i,i+1}}} \|E_{\mathcal{L}(\Gamma_{\mathcal{C}_{i-2,i-1}})}(t_{i-1,i}a_{i,i+1}u_{s^{-1}})\|_2^2 \right)^{1/2} \left( \sum_{s \in \Gamma_{\mathcal{C}_{i,i+1}}} \|E_{\mathcal{L}(\Gamma_{\mathcal{C}_{i+2,i+3}})}(a_{i+2,i+3}^*t_{i+1,i+2}u_{s^{-1}})\|_2^2 \right)^{1/2} \\ &\leq \|t_{i-1,i}a_{i,i+1}\|_2 \|a_{i+2,i+3}^*t_{i+1,i+2}\|_2 = 1. \end{aligned}$$

Therefore, one can get scalars  $c_s$  such that

$$E_{\mathcal{L}(\Gamma_{\mathcal{C}_{i-2,i-1}})}(t_{i-1,i}a_{i,i+1}u_{s^{-1}}) = c_s w E_{\mathcal{L}(\Gamma_{\mathcal{C}_{i+2,i+3}})}(a_{i+2,i+3}^*t_{i+1,i+2}u_{s^{-1}})a_{i+1,i+2}^*t_{i,i+1}. \quad (5-5)$$

Thus, proceeding in the same fashion as in the proof of [Lemma 5.1](#), one can find  $d_s, e_s \in \mathbb{C}$ ,  $g_s \in \mathcal{U}(\mathcal{L}(\Gamma_{\mathcal{C}_{i-2,i-1}}))$ , and  $h_s \in \mathcal{U}(\mathcal{L}(\Gamma_{\mathcal{C}_{i+2,i+3}}))$  such that

$$\begin{aligned} E_{\mathcal{L}(\Gamma_{\mathcal{C}_{i-2,i-1}})}(t_{i-1,i}a_{i,i+1}u_{s^{-1}}) &= d_s g_s, \\ E_{\mathcal{L}(\Gamma_{\mathcal{C}_{i+2,i+3}})}(a_{i+2,i+3}^*t_{i+1,i+2}^*u_{s^{-1}})a_{i+1,i+2}^*t_{i,i+1}^* &= e_s h_s. \end{aligned} \quad (5-6)$$

Hence, [\(5-5\)](#) gives that  $d_s g_s = c_s e_s h_s$  for all  $s \in \Gamma_{\mathcal{C}_{i,i+1}}$ , and finally employing the same arguments as in the first part one can find scalars  $c_{s,t}, d_{s,t}$  such that  $g_s = c_{s,t} g_t$  and  $h_s = d_{s,t} h_t$  for all  $s, t \in \Gamma_{\mathcal{C}_{i,i+1}}$ . Using [\(5-6\)](#), we derive that

$$t_{i-1,i}a_{i,i+1} = \sum_{s \in \Gamma_{\mathcal{C}_{i,i+1}}} d_s g_s u_s = g_e \sum_{s \in \Gamma_{\mathcal{C}_{i,i+1}}} d_s c_{s,e} u_s.$$

This further implies that one can find unitaries  $r_{i-1} \in \mathcal{U}(\mathcal{L}(\Gamma_{\mathcal{C}_{i-2,i-1}}))$  and  $p_{i-1} \in \mathcal{U}(\mathcal{L}(\Gamma_{\mathcal{C}_{i,i+1}}))$  such that  $t_{i-1,i}a_{i,i+1} = r_{i-1}p_{i-1}$ , and hence  $t_{i-1,i} = r_{i-1}p_{i-1}a_{i,i+1}^*$ . Similarly, we get  $t_{i,i+1} = r_i p_i a_{i+1,i+2}^*$ , and hence, from [\(5-4\)](#), we deduce that

$$x_{i,i+1} = z_{i-1}^*a_{i,i+1}t_{i,i+1}z_i = z_{i-1}^*a_{i,i+1}r_i p_i a_{i+1,i+2}^*z_i = r_i z_{i-1}^*a_{i,i+1}z_i p_i a_{i+1,i+2}^*. \quad (5-7)$$

Now, one can see that, using the cyclic relation  $x_{1,2} \cdots x_{n-1,n}x_{n,1} = 1$ , we get that  $p_i = r_{i+2}^*$ . This together with [\(5-7\)](#) gives the desired conclusion by taking  $a_i = r_i$ ,  $b_i = z_{i-1}^*$ , and  $c_i = a_{i,i+1}$ .  $\square$

## 6. Rigid subalgebras of graph product group von Neumann algebras

In this section we classify all rigid subalgebras of von Neumann algebras associated with graph product groups. This should be viewed as a counterpart of [\[Ioana et al. 2008, Theorem 4.3\]](#) for amalgamated free product von Neumann algebras. In fact, the latter plays an essential role in deriving our result. For convenience, we include a detailed proof of how it follows from this.

**Theorem 6.1.** *Let  $\Gamma = \mathcal{G}\{\Gamma_v\}$  be a graph product group, let  $\Gamma \curvearrowright \mathcal{P}$  be any trace-preserving action, and denote by  $\mathcal{M} = \mathcal{P} \rtimes \Gamma$  the corresponding crossed product von Neumann algebra. Let  $r \in \mathcal{M}$  be a projection, and let  $\mathcal{Q} \subset r\mathcal{M}r$  be a property (T) von Neumann subalgebra.*

*Then one can find a clique  $\mathcal{C} \in \text{cliq}(\mathcal{G})$  such that  $\mathcal{Q} \prec_{\mathcal{M}} \mathcal{P} \rtimes \Gamma_{\mathcal{C}}$ . Moreover, if  $\mathcal{Q} \not\prec \mathcal{P} \rtimes \Gamma_{\mathcal{C} \setminus \{c\}}$  for all  $c \in \mathcal{C}$ , then one can find projections  $q \in \mathcal{Q}$  and  $q' \in \mathcal{Q}' \cap r\mathcal{M}r$  with  $qq' \neq 0$  and a unitary  $u \in \mathcal{M}$  such that  $uq\mathcal{Q}q' u^* \subseteq \mathcal{P} \rtimes \Gamma_{\mathcal{C}}$ . In particular, if  $\mathcal{P} \rtimes \Gamma_{\mathcal{C}}$  is a factor, then one can take  $q = 1$  above.*

*Proof.* Let  $\mathcal{G}_0 = (\mathcal{V}_0, \mathcal{E}_0) \subseteq \mathcal{G} = (\mathcal{V}, \mathcal{E})$  be a subgraph with  $|\mathcal{V}_0|$  minimal such that  $\mathcal{Q} \prec \mathcal{P} \rtimes \Gamma_{\mathcal{G}_0}$ . In the remaining part we show that  $\mathcal{G}_0$  is complete, which proves the conclusion.

Write  $\mathcal{N} = \mathcal{P} \rtimes \Gamma_{\mathcal{G}_0}$ . Since  $\mathcal{Q} \prec_{\mathcal{M}} \mathcal{N}$ , one can find projections  $q \in \mathcal{Q}$  and  $p \in \mathcal{N}$ , a nonzero partial isometry  $v \in p\mathcal{M}q$ , and a  $*$ -isomorphism onto its image  $\theta : q\mathcal{Q}q \rightarrow \mathcal{R} := \theta(q\mathcal{Q}q) \subseteq p\mathcal{N}p$  such that  $\theta(x)v = vx$  for all  $x \in q\mathcal{Q}q$ . Notice that  $vv^* \in \mathcal{R}' \cap p\mathcal{M}p$  and  $v^*v \in (\mathcal{Q}' \cap \mathcal{M})q$ . Moreover, one can assume without any loss of generality that the support projection of  $E_{\mathcal{N}}(vv^*)$  equals  $p$ .

Assume by contradiction that  $\mathcal{G}_0$  is not complete. Thus, one can find  $v \in \mathcal{G}_0$  such that  $\Gamma_{\mathcal{G}_0}$  admits a noncanonical amalgam decomposition  $\Gamma_{\mathcal{G}_0} = \Gamma_{\mathcal{G}_0 \setminus \{v\}} *_{\Gamma_{\text{lk}(v)}} \Gamma_{\text{st}(v)}$ ; in particular, we have  $|\text{st}(v)| \leq |\mathcal{V}_0| - 1$ . Since  $\mathcal{Q}$  has property (T),  $\mathcal{R}$  has property (T) as well. Using [\[Ioana et al. 2008, Theorem 5.1\]](#), we have either (i)  $\mathcal{R} \prec_{\mathcal{N}} \mathcal{P} \rtimes \Gamma_{\mathcal{G}_0 \setminus \{v\}}$ , or (ii)  $\mathcal{R} \prec_{\mathcal{N}} \mathcal{P} \rtimes \Gamma_{\text{st}(v)}$ . Assume (i). Define  $\mathcal{X} := \mathcal{P} \rtimes \Gamma_{\mathcal{G}_0 \setminus \{v\}}$ . As  $\mathcal{R} \prec_{\mathcal{N}} \mathcal{X}$ ,

one can find projections  $e \in \mathcal{R}$  and  $f \in \mathcal{X}$ , a nonzero partial isometry  $w \in f\mathcal{N}e$ , and a  $*$ -isomorphism onto its image  $\psi : e\mathcal{R}e \rightarrow \mathcal{T} := \theta(e\mathcal{R}e) \subseteq f\mathcal{X}f$  such that  $\psi(x)w = wx$  for all  $x \in e\mathcal{R}e$ .

Next, we argue that  $wv \neq 0$ . Otherwise, we would have  $0 = wvv^*$ , and since  $w \in \mathcal{N}$ , we get  $0 = wE_{\mathcal{N}}(vv^*)$ . Therefore  $0 = ws(E_{\mathcal{N}}(vv^*)) = wp = w$ , which is a contradiction. Combining the previous intertwining relations, we get  $\phi(\theta(x))wv = w\theta(x)v = wvx$  for all  $x \in t\mathcal{Q}t$ ; here we write  $0 \neq t = \theta^{-1}(e)$ . Taking the polar decomposition of  $wv$  in the prior intertwining relation, we obtain that  $\mathcal{Q} \prec_{\mathcal{M}} \mathcal{X} = \mathcal{P} \rtimes \Gamma_{\mathcal{G} \setminus \{v\}}$ . However, since  $|\mathcal{V}_0 \setminus \{v\}| = |\mathcal{V}_0| - 1$ , this contradicts the minimality of  $|\mathcal{V}_0|$ . In a similar manner, one can show case (ii) also leads to a contradiction.

Next, we show the moreover part. Let  $\mathcal{S} = \mathcal{P} \rtimes \Gamma_{\mathcal{C}}$ . From the first part of the proof one can find projections  $q \in \mathcal{Q}$  and  $s \in \mathcal{S}$ , a nonzero partial isometry  $v_0 \in s\mathcal{M}q$ , and a  $*$ -isomorphism onto its image  $\theta : q\mathcal{Q}q \rightarrow \mathcal{Y} := \theta(q\mathcal{Q}q) \subseteq s\mathcal{S}s$  such that  $\theta(x)v_0 = v_0x$  for all  $x \in q\mathcal{Q}q$ . We note that  $v_0v_0^* \in \mathcal{Y}' \cap s\mathcal{M}s$  and  $v_0^*v_0 \in q\mathcal{Q}q' \cap q\mathcal{M}q$ . Moreover, one can assume without loss of generality that the support projection of  $E_{\mathcal{S}}(v_0v_0^*)$  equals  $s$ . Observe that we have an amalgamated free product decomposition  $\mathcal{M} = (\mathcal{P} \rtimes \Gamma_{\mathcal{V} \setminus \{c\}}) *_{\mathcal{P} \rtimes \Gamma_{\mathcal{C} \setminus \{c\}}} (\mathcal{P} \rtimes \Gamma_{\mathcal{C}})$ . Using the same argument as before, since  $\mathcal{Q} \not\prec_{\mathcal{M}} \mathcal{P} \rtimes \Gamma_{\mathcal{C} \setminus \{c\}}$ , we must have that  $\mathcal{Y} \not\prec_{\mathcal{S}} \mathcal{P} \rtimes \Gamma_{\mathcal{C} \setminus \{c\}}$ . Therefore, by [Ioana et al. 2008, Theorem 1.2.1], we have that  $v_0v_0^* \in \mathcal{S}$ , and hence the intertwining relation implies that  $v_0q\mathcal{Q}qv_0^* = \mathcal{Y}v_0v_0^* \subseteq \mathcal{S}$ . If  $u$  is a unitary extending  $v_0$ , we further see that  $uq\mathcal{Q}qv_0^*v_0u^* \subseteq \mathcal{S}$ . Letting  $q' = v_0^*v_0$ , we get the desired conclusion.

To see the last part, we note that, since  $\mathcal{P} \rtimes \Gamma_{\mathcal{C}}$  is a factor, after passing to a new unitary  $u$ , one can replace  $q$  above with its central support in  $\mathcal{Q}$ .  $\square$

## 7. Symmetries of graph product group von Neumann algebras

The main result of this section is a strong rigidity result describing all  $*$ -isomorphisms between factors associated with a fairly large family of graph product groups arising from finite graphs in the class  $\text{CC}_1$  ([Theorem 7.10](#)). As a by-product, we obtain concrete descriptions of all symmetries of these factors including such examples with trivial fundamental groups ([Corollaries 7.11](#) and [7.12](#)). However, to be able to state and prove these results, we first need to introduce some new terminology and establish a few preliminary results.

**7.1. Local isomorphisms of graph product group von Neumann algebras.** The isomorphism class of a von Neumann algebra associated with a graph product group tends to be fairly abundant. As in the group situation, a rich source of isomorphisms stems from both the isomorphism class of the underlying graph and the isomorphism classes of the von Neumann algebras of the vertex groups. By analogy with the group case, these are called local isomorphisms and we briefly explain their construction below.

Let  $\mathcal{G}$  and  $\mathcal{H}$  be simple finite graphs, and let  $\Gamma_{\mathcal{G}}$  and  $\Lambda_{\mathcal{H}}$  be graph product groups, where their vertex groups are  $\{\Gamma_v, v \in \mathcal{V}\}$  and  $\{\Lambda_w, w \in \mathcal{W}\}$ , respectively. Assume  $\mathcal{G}$  and  $\mathcal{H}$  are isometric and fix  $\sigma : \mathcal{G} \rightarrow \mathcal{H}$  an isometry. In addition, assume that  $\Phi = \{\Phi_v^\sigma, v \in \mathcal{V}\}$  is a collection of  $*$ -isomorphisms  $\Phi_v^\sigma : \mathcal{L}(\Gamma_v) \rightarrow \mathcal{L}(\Lambda_{\sigma(v)})$  for all  $v \in \mathcal{V}$ . Then the following holds.

**Theorem 7.1.** *There exists a unique  $*$ -isomorphism denoted by  $(\Phi, \sigma) : \mathcal{L}(\Gamma_{\mathcal{G}}) \rightarrow \mathcal{L}(\Lambda_{\mathcal{H}})$  which extends the maps  $\bigcup_{v \in \mathcal{V}} \mathcal{L}(\Gamma_v) \ni x \mapsto \Phi_v^\sigma(x) \in \mathcal{L}(\Lambda_{\mathcal{H}})$ .*

*Proof.* We recall that a word for  $\mathcal{G}$  is a finite sequence  $\mathbf{v} = (v_1, \dots, v_n)$  of elements in  $\mathcal{V}$  [Caspers and Fima 2017, Definition 1.2]. The word  $\mathbf{v}$  is called reduced if, whenever  $i < j$  and  $v_{i+1}, \dots, v_{j-1} \in \text{st}(v_j)$ , we have  $v_i \neq v_j$ . Following [Caspers and Fima 2017, Section 2.3],  $\mathcal{L}(\Gamma_{\mathcal{G}})$  can be presented alternatively as the graph product von Neumann algebra associated to the graph  $\mathcal{G}$  and vertex von Neumann algebras  $\{\mathcal{L}(\Gamma_v)\}_{v \in \mathcal{V}}$ .

We continue by proving the following claim: for any reduced word  $(v_1, \dots, v_n)$  in  $\mathcal{G}$  and elements  $a_i \in \mathcal{L}(\Gamma_{v_i})$  with  $\tau(a_i) = 0$ , we have  $\tau(\Phi_{v_1}^{\sigma}(a_1) \cdots \Phi_{v_n}^{\sigma}(a_n)) = 0$ . To show this, write  $w_i = \sigma(v_i) \in \mathcal{W}$  and  $b_i = \Phi_{v_i}^{\sigma}(a_i) \in \mathcal{L}(\Lambda_{w_i})$  for any  $i$ . Note that the word  $(w_1, \dots, w_n)$  is reduced in  $\mathcal{H}$  and  $\tau(b_i) = 0$  for any  $i$ . By considering the Fourier series of  $b_i$ , the claim follows by proving that, whenever  $h_i \in \Lambda_{w_i}$  with  $h_i \neq 1$ , we have  $h_1 \cdots h_n \neq 1$ . Since  $(w_1, \dots, w_n)$  is a reduced word in  $\mathcal{H}$ , it is easy to see that  $h_1 \cdots h_n$  is a reduced element of  $\Lambda_{\mathcal{H}}$  in the sense of [Green 1990, Definition 3.5]. Applying [Green 1990, Theorem 3.9] implies that  $h_1 \cdots h_n \neq 1$ , hence proving the claim.

Finally, our theorem follows directly by applying [Caspers and Fima 2017, Proposition 2.22].  $\square$

Hereafter,  $(\Phi, \sigma)$  will be called the *local isomorphism* induced by  $\sigma$  and  $\Phi = \{\Phi_v^{\sigma}, v \in \mathcal{V}\}$ . Whenever  $\mathcal{G} = \mathcal{H}$  and  $\Gamma_v = \Lambda_v$  for all  $v$ , these are called *local automorphisms* and they form a subgroup of  $\text{Aut}(\mathcal{L}(\Gamma_{\mathcal{G}}))$  under composition which will be denoted by  $\text{Loc}_{v,g}(\mathcal{L}(\Gamma_{\mathcal{G}}))$ . The subgroup of local automorphisms satisfying  $\sigma = \text{Id}$  is denoted by  $\text{Loc}_v(\mathcal{L}(\Gamma_{\mathcal{G}}))$ ; observe that it has finite index in  $\text{Loc}_{\mathcal{V}}(\mathcal{L}(\Gamma_{\mathcal{G}}))$ . Moreover, we have  $\text{uLoc}_{\mathcal{V}}(\mathcal{L}(\Gamma_{\mathcal{G}})) = \bigoplus_{v \in \mathcal{V}} \text{Aut}(\mathcal{L}(\Gamma_v))$ . Next, we observe that most of the time  $(\Phi, \sigma)$  is an outer automorphism.

**Proposition 7.2.** *Under the same assumptions as before, suppose in addition that  $\mathcal{G}$  is a graph satisfying  $\bigcap_{v \in \mathcal{V}} \text{star}(v) = \emptyset$ . Then  $(\Phi, \sigma)$  is inner if and only if  $\sigma = \text{Id}$  and  $\Phi_v^{\sigma} = \text{Id}$  for all  $v \in \mathcal{V}$ .*

*Proof.* Let  $\mathcal{M} = \mathcal{L}(\Gamma_{\mathcal{G}})$ , and let  $u \in \mathcal{U}(\mathcal{M})$  such that  $(\Phi, \sigma) = \text{ad}(u)$ . Fix  $v \in \mathcal{V}$ . From the definitions we have  $u\mathcal{L}(\Gamma_v)u^* = \mathcal{L}(\Gamma_{\sigma(v)})$ . Using Theorem 2.7, we get that  $v = \sigma(v)$  and  $u \in \mathcal{U}(\mathcal{L}(\Gamma_{\text{star}(v)}))$ . As this holds for all  $v \in \mathcal{V}$ , we get  $\sigma = \text{Id}$  and also  $u \in \bigcap_{v \in \mathcal{V}} \mathcal{L}(\Gamma_{\text{star}(v)}) = \mathbb{C}1$ . Hence  $(\Phi, \sigma) = \text{Id}$ , and also  $\Phi_v^{\sigma} = \text{Id}$  for all  $v \in \mathcal{V}$ .  $\square$

When  $\sigma = \text{Id}$ , let  $\text{Loc}_{v,i}(\mathcal{L}(\Gamma_{\mathcal{G}}))$  be the set of all local automorphisms  $(\Phi, \sigma)$  which satisfies the following: for any  $v \in \mathcal{V}$ , there exists a unitary  $u_v \in \mathcal{L}(\Gamma_v)$  such that  $\Phi_v^{\sigma} = \text{ad}(u_v)$ . It is easy to see that  $\text{Loc}_{v,i}(\mathcal{L}(\Gamma_{\mathcal{G}}))$  forms a normal subgroup of  $\text{Loc}_v(\mathcal{L}(\Gamma_{\mathcal{G}}))$  under composition. Thus, when there exists a  $v \in \mathcal{V}$  for which  $\Gamma_v$  is an icc group, it follows from Proposition 7.2 that  $\text{Loc}_{v,i}(\mathcal{L}(\Gamma_{\mathcal{G}}))$ , and hence  $\text{Out}(\mathcal{L}(\Gamma_{\mathcal{G}}))$ , is always an uncountable group. In conclusion, for this class of von Neumann algebras, in general, one cannot expect rigidity results and computations of their symmetries of the same precision level as the prior results [Popa and Vaes 2008; Vaes 2008].

**Remark 7.3.** It is worth mentioning that the class of local isomorphisms can be defined for all tracial graph products [Caspers and Fima 2017] (regardless if they come from groups or not) with essentially the same proofs.

Next, we highlight a family of  $*$ -isomorphisms between graph product von Neumann algebras that is specific to graphs in the class  $\text{CC}_1$  and is related more to the clique algebras structure than to the

vertex algebra structure as in the previous part. As before, let  $\mathcal{G}, \mathcal{H} \in \text{CC}_1$  be isomorphic graphs, and fix  $\sigma : \mathcal{G} \rightarrow \mathcal{H}$  an isometry. Let  $\text{cliq}(\mathcal{G}) = \{\mathcal{C}_1, \dots, \mathcal{C}_n\}$  be a consecutive enumeration of the cliques of  $\mathcal{G}$ . Let  $\Gamma_{\mathcal{G}}$  and  $\Lambda_{\mathcal{H}}$  be graph product groups and assume, for every  $i \in \overline{1, n}$ , there are  $*$ -isomorphisms  $\theta_{i-1, i} : \mathcal{L}(\Gamma_{\mathcal{C}_{i-1, i}}) \rightarrow \mathcal{L}(\Lambda_{\sigma(\mathcal{C}_{i-1, i})})$  and  $\xi_i : \mathcal{L}(\Gamma_{\mathcal{C}_i^{\text{int}}}) \rightarrow \mathcal{L}(\Lambda_{\sigma(\mathcal{C}_i^{\text{int}})})$ . Here, and afterwards, we use the notation  $\mathcal{C}_{0, 1} = \mathcal{C}_{n, 1}$ . Using [Lemma 4.1](#), we can view  $\Gamma_{\mathcal{G}}$  as a graph product group  $\Gamma'_{\mathcal{F}_n}$  over the graph  $\mathcal{F}_n$ , where the vertex groups satisfy  $\Gamma'_{w_i} = \bigoplus_{v \in \mathcal{C}_i^{\text{int}}} \Gamma_v$  and  $\Gamma'_{b_i} = \bigoplus_{v \in \mathcal{C}_{i-1, i}} \Gamma_v$ . Similarly,  $\Lambda_{\mathcal{H}} = \Lambda'_{\mathcal{F}_n}$ , where  $\Lambda'_{\sigma(w_i)} = \bigoplus_{v \in \mathcal{C}_i^{\text{int}}} \Lambda_{\sigma(v)}$  and  $\Lambda'_{\sigma(b_i)} = \bigoplus_{v \in \mathcal{C}_{i-1, i}} \Lambda_{\sigma(v)}$ . Therefore, using [Theorem 7.1](#), these isomorphisms induce a unique  $*$ -isomorphism  $\phi_{\theta, \xi, \sigma} : \mathcal{L}(\Gamma_{\mathcal{G}}) \rightarrow \mathcal{L}(\Lambda_{\mathcal{H}})$ :

$$\phi_{\theta, \xi, \sigma}(x) = \begin{cases} \theta_{i-1, i}(x) & \text{if } x \in \mathcal{L}(\Gamma_{\mathcal{C}_{i-1, i}}), \\ \xi_i(x) & \text{if } x \in \mathcal{L}(\Gamma_{\mathcal{C}_i^{\text{int}}}) \end{cases} \quad (7-1)$$

for all  $i \in \overline{1, n}$ .

When  $\Gamma_{\mathcal{G}} = \Lambda_{\mathcal{H}}$ , this construction yields a group of automorphisms of  $\mathcal{L}(\Gamma_{\mathcal{G}})$  that will be denoted by  $\text{Loc}_{c, g}(\mathcal{L}(\Gamma_{\mathcal{G}}))$ . We also denote by  $\text{Loc}_c(\mathcal{L}(\Gamma_{\mathcal{G}}))$  the subgroup of all automorphisms satisfying  $\sigma = \text{Id}$ . Note:  $\text{Loc}_c(\mathcal{L}(\Gamma_{\mathcal{G}})) \cong \bigoplus_i \text{Aut}(\mathcal{L}(\Gamma_{\mathcal{C}_{i-1, i}})) \oplus \text{Aut}(\mathcal{L}(\Gamma_{\mathcal{C}_i^{\text{int}}}))$  and also  $\text{Loc}_c(\mathcal{L}(\Gamma_{\mathcal{G}})) \leq \text{Loc}_{c, g}(\mathcal{L}(\Gamma_{\mathcal{G}}))$  has finite index.

Next, we highlight a subgroup of automorphisms in  $\text{Loc}_c(\mathcal{L}(\Gamma_{\mathcal{G}}))$  that will be useful in stating our main results. Namely, consider a family of nontrivial unitaries  $a_{i-1, i} \in \mathcal{L}(\Gamma_{\mathcal{C}_{i-1, i}})$  and  $b_i \in \mathcal{L}(\Gamma_{\mathcal{C}_i^{\text{int}}})$  for every  $i \in \overline{1, n}$ . If in the formula (7-1) we let  $\theta_{i-1, i} = \text{ad}(a_{i-1, i})$  and  $\xi_i = \text{ad}(b_i)$ , then the corresponding automorphism  $\phi_{\theta, \xi, \text{Id}}$  is an (most of the times outer) automorphism of  $\mathcal{L}(\Gamma)$  which we will denote by  $\phi_{a, b}$  throughout this section. The set of all such automorphisms forms a normal subgroup denoted by  $\text{Loc}_{c, i}(\mathcal{L}(\Gamma_{\mathcal{G}})) \triangleleft \text{Loc}_c(\mathcal{L}(\Gamma_{\mathcal{G}}))$ . From the definitions we also have  $\text{Loc}_{v, i}(\mathcal{L}(\Gamma_{\mathcal{G}})) < \text{Loc}_{c, i}(\mathcal{L}(\Gamma_{\mathcal{G}}))$  and  $\text{Loc}_v(\mathcal{L}(\Gamma_{\mathcal{G}})) < \text{Loc}_c(\mathcal{L}(\Gamma_{\mathcal{G}}))$ .

**Proposition 7.4.** *The automorphism  $\phi_{a, b} \in \text{Loc}_{c, i}(\mathcal{L}(\Gamma_{\mathcal{G}}))$  is inner if and only if  $a_{i, i+1} \in \mathcal{Z}(\mathcal{L}(\Gamma_{\mathcal{C}_{i-1, i}}))$  and  $b_i \in \mathcal{Z}(\mathcal{L}(\Gamma_{\mathcal{C}_i^{\text{int}}}))$  for all  $i \in \overline{1, n}$ .*

*Proof.* Assume that  $\phi_{a, b} \in \text{Loc}_{c, i}(\mathcal{L}(\Gamma_{\mathcal{G}}))$  is inner, and hence, there is a unitary  $u \in \mathcal{L}(\Gamma_{\mathcal{G}})$  such that  $\phi_{a, b}(x)u = ux$  for any  $x \in \mathcal{L}(\Gamma_{\mathcal{G}})$ . Fix an arbitrary  $i \in \overline{1, n}$ . Then, for any  $x \in \mathcal{L}(\Gamma_{\mathcal{C}_{i-1, i}})$ , we have  $a_{i-1, i}xa_{i, i-1}^* = uxu^*$ . Together with [Theorem 2.7](#), this yields  $u^*a_{i, i-1} \in \mathcal{L}(\Gamma_{\mathcal{C}_{i-1, i}})' \cap \mathcal{L}(\Gamma_{\mathcal{G}}) \subset \mathcal{L}(\Gamma_{\mathcal{C}_{i-1} \cup \mathcal{C}_i})$ . Since  $a_{i, i+1} \in \mathcal{L}(\Gamma_{\mathcal{C}_{i-1, i}})$ , it follows that  $u \in \bigcap_{i=1}^n \mathcal{L}(\Gamma_{\mathcal{C}_{i-1} \cup \mathcal{C}_i}) = \mathbb{C}1$ , and then it follows that  $a_{i, i+1} \in \mathcal{Z}(\mathcal{L}(\Gamma_{\mathcal{C}_{i-1, i}}))$ . Similarly, one can show that  $b_i \in \mathcal{Z}(\mathcal{L}(\Gamma_{\mathcal{C}_i^{\text{int}}}))$  for all  $i \in \overline{1, n}$ . This concludes one direction of the proof. As for the converse, note that we trivially have  $\phi_{a, b} = \text{Id}$ .  $\square$

**7.2. Computations of symmetries of graph product group von Neumann algebras.** Next, we introduce a few preliminary results needed to describe the isomorphisms between von Neumann algebras arising from graph product groups with property (T) vertex groups.

**Theorem 7.5.** *Let  $\Gamma = \mathcal{G}\{\Gamma_v\}$  and  $\Lambda = \mathcal{H}\{\Lambda_w\}$  be graph product groups, and assume that  $\Gamma_v$  and  $\Lambda_w$  are icc property (T) groups for all  $v \in \mathcal{V}$  and  $w \in \mathcal{W}$ . Let  $\theta : \mathcal{L}(\Gamma) \rightarrow \mathcal{L}(\Lambda)$  be any  $*$ -isomorphism. Then there is a bijection  $\sigma : \text{cliq}(\mathcal{G}) \rightarrow \text{cliq}(\mathcal{H})$  such that, for every  $\mathcal{C} \in \text{cliq}(\mathcal{G})$ , there is a unitary  $u_{\mathcal{C}} \in \mathcal{L}(\Lambda)$  such that  $\theta(\mathcal{L}(\Gamma_{\mathcal{C}})) = u_{\mathcal{C}}\mathcal{L}(\Lambda_{\sigma(\mathcal{C})})u_{\mathcal{C}}^*$ .*

*Proof.* Fix  $\mathcal{C} \in \text{cliq}(\mathcal{G})$ . Using the hypothesis and [Corollary 2.8](#), it follows that  $\theta(\mathcal{L}(\Gamma_{\mathcal{C}})) \subseteq \mathcal{L}(\Lambda)$  is a property (T) irreducible subfactor. By the first part of [Theorem 6.1](#), there must exist a clique  $\sigma(\mathcal{C}) \in \text{cliq}(\mathcal{H})$  such that  $\theta(\mathcal{L}(\Gamma_{\mathcal{C}})) \prec_{\mathcal{L}(\Lambda)} \mathcal{L}(\Gamma_{\sigma(\mathcal{C})})$ . Now, we argue that, for every  $c \in \sigma(\mathcal{C})$ , we have  $\theta(\mathcal{L}(\Gamma_{\mathcal{C}})) \not\prec_{\mathcal{L}(\Lambda)} \mathcal{L}(\Gamma_{\sigma(\mathcal{C}) \setminus \{c\}})$ . Indeed, if we assume  $\theta(\mathcal{L}(\Lambda_{\mathcal{C}})) \prec_{\mathcal{L}(\Lambda)} \mathcal{L}(\Gamma_{\sigma(\mathcal{C}) \setminus \{c\}})$ , then by passing to relative intertwining commutants we would get from [\[Vaes 2008, Lemma 3.5\]](#) that

$$\mathcal{L}(\Lambda_c) = \mathcal{L}(\Lambda_{\sigma(\mathcal{C}) \setminus \{c\}})' \cap \mathcal{L}(\Lambda) \prec_{\mathcal{L}(\Lambda)} \theta(\mathcal{L}(\Gamma_{\mathcal{C}}))' \cap \mathcal{L}(\Lambda) = \theta(\mathcal{L}(\Gamma_{\mathcal{C}})') \cap \mathcal{L}(\Lambda) = \mathbb{C}1,$$

which is a contradiction. Thus, by using that  $\mathcal{L}(\Lambda_{\sigma(\mathcal{C})})$  is a factor and  $\mathcal{L}(\Gamma_{\mathcal{C}}) \subset \mathcal{L}(\Gamma)$  is irreducible, it follows from the moreover part of [Theorem 6.1](#) that there exists a unitary  $u_{\mathcal{C}} \in \mathcal{L}(\Lambda)$  that satisfies  $u_{\mathcal{C}}\theta(\mathcal{L}(\Gamma_{\mathcal{C}}))u_{\mathcal{C}}^* \subseteq \mathcal{L}(\Lambda_{\sigma(\mathcal{C})})$ .

We now reverse the roles of  $\Gamma$  and  $\Lambda$ : in a similar manner for every  $\mathcal{D} \in \text{cliq}(\mathcal{H})$ , one can find  $\tau(\mathcal{D}) \in \text{cliq}(\mathcal{G})$  and a unitary  $w_{\mathcal{D}} \in \mathcal{L}(\Gamma)$  satisfying  $\mathcal{L}(\Lambda_{\mathcal{D}}) \subseteq w_{\mathcal{D}}\theta(\mathcal{L}(\Gamma_{\tau(\mathcal{D})}))w_{\mathcal{D}}^*$ . Altogether these show that

$$u_{\mathcal{C}}\theta(\mathcal{L}(\Gamma_{\mathcal{C}}))u_{\mathcal{C}}^* \subseteq \mathcal{L}(\Lambda_{\sigma(\mathcal{C})}) \subseteq w_{\sigma(\mathcal{C})}\theta(\mathcal{L}(\Gamma_{\tau(\sigma(\mathcal{C}))}))w_{\sigma(\mathcal{C})}^*.$$

In particular, [Theorem 2.7](#) implies that  $\mathcal{C} = \tau(\sigma(\mathcal{C}))$  and  $u_{\mathcal{C}}^*w_{\sigma(\mathcal{C})} \in \theta(\mathcal{L}(\Gamma_{\mathcal{C}}))$ . This combined with the prior containment implies that  $u_{\mathcal{C}}\theta(\mathcal{L}(\Gamma_{\mathcal{C}}))u_{\mathcal{C}}^* = \mathcal{L}(\Lambda_{\sigma(\mathcal{C})})$ . As  $\mathcal{C} = \tau(\sigma(\mathcal{C}))$  for any clique  $\mathcal{C}$  of  $\mathcal{G}$ , it follows in particular that  $\sigma$  is a bijection.  $\square$

**Remarks.** [Theorem 7.5](#) still holds under the more general assumption that each vertex group  $\Gamma_v$  possesses an infinite property (T) normal subgroup. The proof is essentially the same and is left to the reader.

We continue by recording a notion of unique prime factorization along with some examples that will be needed in the first main result.

**Definition 7.6.** A family  $\mathcal{C}$  of countable icc groups is said to satisfy the *s-unique prime factorization* if, whenever

$$\mathcal{M} = \mathcal{L}(\Gamma_1 \times \cdots \times \Gamma_m)^t = \mathcal{L}(\Lambda_1 \times \cdots \times \Lambda_n)$$

for some  $\Gamma_1, \dots, \Gamma_m, \Lambda_1, \dots, \Lambda_n$  that belong to  $\mathcal{C}$  and  $t > 0$ , we must have  $t = 1$  and  $m = n$ , and there exist a unitary  $u \in \mathcal{M}$  and a permutation  $\tau \in \mathfrak{S}_n$  such that  $u\mathcal{L}(\Gamma_i)u^* = \mathcal{L}(\Lambda_{\tau(i)})$  for all  $i \in \overline{1, n}$ .

There are several classes of natural examples of groups that satisfy this unique factorization condition in the literature, but for our paper only those which have property (T) will be relevant. Thus appealing to the results in [\[Chifan et al. 2023a; 2023b; 2024; Das 2020\]](#), we have the following.

**Corollary 7.7.** *Class  $\mathcal{C}$  satisfies the s-unique prime factorization whenever  $\mathcal{C}$  is one of the following:*

- (1) *The class of property (T) fibered Rips constructions* [\[Chifan et al. 2023a; 2024\]](#).
- (2) *The class of property (T) generalized wreath-like product groups  $WR(A, B \curvearrowright I)$ , where  $A$  is abelian,  $B$  is an icc subgroup of a hyperbolic group, and the action  $B \curvearrowright I$  has infinite stabilizers* [\[Chifan et al. 2023b; Chifan et al. 2023c\]](#).

*Proof.* Part (1) is a direct consequence of [\[Chifan et al. 2023a; 2024; Das 2020\]](#). Part (2) follows from [Theorem 3.6](#) and [Corollary 3.7](#).  $\square$

**Proposition 7.8.** *Let  $\Gamma = \mathcal{G}\{\Gamma_v\}$  and  $\Lambda = \mathcal{H}\{\Lambda_w\}$  be graph products such that*

- (1)  *$\Gamma_v$  and  $\Lambda_w$  are icc property (T) groups for all  $v \in \mathcal{V}$  and  $w \in \mathcal{W}$ ,*
- (2) *there is a class  $\mathcal{C}$  of countable groups which satisfies the s-unique prime factorization property (see Definition 7.6) for which  $\Gamma_v$  and  $\Lambda_w$  belong to  $\mathcal{C}$  for all  $v \in \mathcal{V}$  and  $w \in \mathcal{W}$ .*

Let  $0 < t < 1$  be a scalar and  $\Theta : \mathcal{L}(\Gamma)^t \rightarrow \mathcal{L}(\Lambda)$  be any  $*$ -isomorphism.

Then  $t = 1$  and there is a bijection  $\sigma : \text{cliq}(\mathcal{G}) \rightarrow \text{cliq}(\mathcal{H})$  such that, for every  $\mathcal{C} \in \text{cliq}(\mathcal{G})$ , there is a unitary  $u_{\mathcal{C}} \in \mathcal{L}(\Lambda)$  such that  $\Theta(\mathcal{L}(\Gamma_{\mathcal{C}})) = u_{\mathcal{C}} \mathcal{L}(\Lambda_{\sigma(\mathcal{C})}) u_{\mathcal{C}}^*$ .

*Proof.* First we observe that it suffices to show that  $t = 1$ , as the rest of the statement follows from Theorem 7.5.

Let  $\mathcal{D}$  be a clique in  $\mathcal{G}$ . Since  $\mathcal{L}(\Gamma_{\mathcal{D}})$  is a  $\text{II}_1$ -factor, there is a projection  $p \in \mathcal{L}(\Gamma_{\mathcal{D}})$  of trace  $\tau(p) = t$  with  $\mathcal{L}(\Gamma)^t = p\mathcal{L}(\Gamma)p$ . As  $\mathcal{L}(\Gamma_{\mathcal{D}})$  has property (T) then so does  $p\mathcal{L}(\Gamma_{\mathcal{D}})p$ . Since  $p\mathcal{L}(\Gamma_{\mathcal{D}})p \subset \Theta^{-1}(\mathcal{L}(\Lambda)) := \mathcal{N}$ , then by Theorem 6.1 one can find a clique  $\mathcal{F} \in \text{cliq}(\mathcal{H})$  such that  $p\mathcal{L}(\Gamma_{\mathcal{D}})p \prec_{\mathcal{N}} \Theta^{-1}(\mathcal{L}(\Lambda_{\mathcal{F}}))$ . Now, observe that since the inclusion  $p\mathcal{L}(\Gamma_{\mathcal{D}})p \subset \mathcal{N}$  is irreducible, we can proceed as in the proof of Theorem 7.5 to deduce that  $p\mathcal{L}(\Gamma_{\mathcal{D}})p \not\prec_{\mathcal{N}} \Theta^{-1}(\mathcal{L}(\Lambda_{\mathcal{F} \setminus \{c\}}))$  for every  $c \in \mathcal{F}$ . Thus, using the irreducibility condition and the moreover part of Theorem 6.1, there exists  $u \in \mathcal{U}(\Theta^{-1}(\mathcal{L}(\Lambda)))$  satisfying

$$up\mathcal{L}(\Gamma_{\mathcal{D}})pu^* \subset \Theta^{-1}(\mathcal{L}(\Lambda_{\mathcal{F}})). \quad (7-2)$$

Also observe that  $(up\mathcal{L}(\Gamma_{\mathcal{D}})pu^*)' \cap \Theta^{-1}(\mathcal{L}(\Lambda_{\mathcal{F}})) \subseteq up(\mathcal{L}(\Gamma_{\mathcal{D}}))' \cap \mathcal{L}(\Gamma_{\mathcal{D}})pu^* = \mathbb{C}p$ . Hence (7-2) is an irreducible inclusion of  $\text{II}_1$ -factors.

Next, since  $\Theta^{-1}(\mathcal{L}(\Lambda_{\mathcal{F}}))$  has property (T) and  $\Theta^{-1}(\mathcal{L}(\Lambda_{\mathcal{F}})) \subset p\mathcal{L}(\Gamma)p \subset \mathcal{L}(\Gamma) := \mathcal{M}$ , by Theorem 6.1 one can find  $\mathcal{D}' \in \text{cliq}(\mathcal{G})$  such that  $\Theta^{-1}(\mathcal{L}(\Lambda_{\mathcal{F}})) \prec_{\mathcal{M}} \mathcal{L}(\Gamma_{\mathcal{D}'})$ . Combining this with (7-2), we further get  $p\mathcal{L}(\Gamma_{\mathcal{D}})p \prec_{\mathcal{M}} \mathcal{L}(\Gamma_{\mathcal{D}'})$ , which further implies by Lemma 2.3 that  $\mathcal{D} \subseteq \mathcal{D}'$ , and since these are cliques we conclude that  $\mathcal{D} = \mathcal{D}'$ . In conclusion, the prior intertwining relation amounts to  $\Theta^{-1}(\mathcal{L}(\Lambda_{\mathcal{F}})) \prec_{\mathcal{M}} \mathcal{L}(\Gamma_{\mathcal{D}})$ . Since  $\mathcal{L}(\Gamma_{\mathcal{D}})$  is a  $\text{II}_1$ -factor, we further obtain  $\Theta^{-1}(\mathcal{L}(\Lambda_{\mathcal{F}})) \prec_{\mathcal{M}} up\mathcal{L}(\Gamma_{\mathcal{D}})pu^*$ . Since  $u \in p\mathcal{M}p$  is a unitary this further implies

$$\Theta^{-1}(\mathcal{L}(\Lambda_{\mathcal{F}})) \prec_{p\mathcal{M}p} up\mathcal{L}(\Gamma_{\mathcal{D}})pu^*. \quad (7-3)$$

By irreducibility, we have  $\Theta^{-1}(\mathcal{L}(\Lambda_{\mathcal{F}})) \not\prec_{p\mathcal{M}p} \Theta^{-1}(\mathcal{L}(\Lambda_{\mathcal{F} \setminus \{c\}}))$  for all  $c \in \mathcal{F}$ . Thus, (7-3), (7-2), and Lemma 2.9 further imply  $\Theta^{-1}(\mathcal{L}(\Lambda_{\mathcal{F}})) \prec_{\Theta^{-1}(\mathcal{L}(\Lambda_{\mathcal{F}}))} up\mathcal{L}(\Gamma_{\mathcal{D}})pu^*$ . Using [Chifan and Das 2018, Lemma 2.3], this requires that the inclusion (7-2) has finite index, and consequently, we have

$$\Theta^{-1}(\mathcal{L}(\Lambda_{\mathcal{F}})) \subset \mathcal{Q}\mathcal{N}_{p\mathcal{M}p}''(up\mathcal{L}(\Gamma_{\mathcal{D}})pu^*). \quad (7-4)$$

Since  $\mathcal{D}$  is a clique, Theorem 2.7 and Lemma 2.6 imply that  $QN_{\Gamma}(\Gamma_{\mathcal{D}}) = \Gamma_{\mathcal{D}}$ . Using this together with Lemmas 2.5 and 2.6, we obtain

$$\mathcal{Q}\mathcal{N}_{p\mathcal{M}p}''(up\mathcal{L}(\Gamma_{\mathcal{D}})pu^*) = up\mathcal{Q}\mathcal{N}_{\mathcal{M}}''(\mathcal{L}(\Gamma_{\mathcal{D}}))pu^* = up\mathcal{L}(QN_{\Gamma}(\Gamma_{\mathcal{D}}))pu^* = up\mathcal{L}(\Gamma_{\mathcal{D}})pu^*,$$

which together with (7-4) implies that  $\Theta^{-1}(\mathcal{L}(\Lambda_{\mathcal{F}})) \subset up\mathcal{L}(\Gamma_{\mathcal{D}})pu^*$ . Together with (7-2) it follows that  $\Theta^{-1}(\mathcal{L}(\Lambda_{\mathcal{F}})) = up\mathcal{L}(\Gamma_{\mathcal{D}})pu^*$ . Finally, the strong unique prime factorization property implies  $p = 1$  and thus  $t = 1$ , as desired.  $\square$

**7.3. Proofs of the main results.** With all the previous preparations at hand we are ready to prove the first main result, namely [Theorem A](#).

**Theorem 7.9.** *Let  $\mathcal{G}$  and  $\mathcal{H}$  be graphs in the class  $\text{CC}_1$ , and let  $\Gamma = \mathcal{G}\{\Gamma_v\}$  and  $\Lambda = \mathcal{H}\{\Lambda_w\}$  be graph product groups satisfying the following conditions:*

- (1)  *$\Gamma_v$  and  $\Lambda_w$  are icc property (T) groups for all  $v \in \mathcal{V}$  and  $w \in \mathcal{W}$ .*
- (2) *There is a class  $\mathcal{C}$  of countable groups which satisfies the s-unique prime factorization property (see [Definition 7.6](#)) for which  $\Gamma_v$  and  $\Lambda_w$  belong to  $\mathcal{C}$  for all  $v \in \mathcal{V}$  and  $w \in \mathcal{W}$ .*

Let  $t > 0$ , and let  $\Theta : \mathcal{L}(\Gamma)^t \rightarrow \mathcal{L}(\Lambda)$  be any  $*$ -isomorphism. Then  $t = 1$  and one can find an isometry  $\sigma : \mathcal{G} \rightarrow \mathcal{H}$ ,  $*$ -isomorphisms  $\theta_{i-1,i} : \mathcal{L}(\Gamma_{\mathcal{C}_{i-1,i}}) \rightarrow \mathcal{L}(\Gamma_{\sigma(\mathcal{C}_{i-1,i})})$  and  $\xi_i : \mathcal{L}(\Gamma_{\mathcal{C}_i^{\text{int}}}) \rightarrow \mathcal{L}(\Gamma_{\sigma(\mathcal{C}_i^{\text{int}})})$  for all  $i \in \overline{1, n}$ , and a unitary  $u \in \mathcal{L}(\Lambda)$  such that  $\Theta = \text{ad}(u) \circ \phi_{\theta, \xi}$ .

*Proof.* Without loss of generality, we can assume  $t \leq 1$  and from the prior theorem we have  $t = 1$ . Also for simplicity we will omit  $\Theta$  from the formulas. Using condition (2) in conjunction with [Theorem 7.5](#), one can find a bijection  $\sigma : \text{cliq}(\mathcal{G}) \rightarrow \text{cliq}(\mathcal{H})$  and unitaries  $u_1, \dots, u_n \in \mathcal{M}$  such that, for any  $i \in \overline{1, n}$ , we have

$$u_i \mathcal{L}(\Gamma_{\mathcal{C}_i}) u_i^* = \mathcal{L}(\Lambda_{\sigma(\mathcal{C}_i)}). \quad (7-5)$$

Next, condition (2) implies that, for any  $i \in \overline{1, n}$ , there exist a complete subgraph  $\mathcal{D}_i \subset \sigma(\mathcal{C}_i)$  and a unitary  $\tilde{u}_i \in \mathcal{L}(\Lambda_{\sigma(\mathcal{C}_i)})$  such that  $\tilde{u}_i \mathcal{L}(\Gamma_{\mathcal{C}_{i,i+1}}) \tilde{u}_i^* = \mathcal{L}(\Lambda_{\mathcal{D}_i})$ . Note that relation (7-5) still holds if we replace  $u_i$  by  $\tilde{u}_i$ . Therefore, for ease of notation, we denote  $\tilde{u}_i$  by  $u_i$ . Hence,  $\mathcal{L}(\Gamma_{\mathcal{C}_{i,i+1}}) = u_i^* \mathcal{L}(\Lambda_{\mathcal{D}_i}) u_i = u_{i+1}^* \mathcal{L}(\Lambda_{\mathcal{D}_{i+1}}) u_{i+1}$ , and therefore  $\mathcal{L}(\Lambda_{\mathcal{D}_i}) u_i u_{i+1}^* = u_i u_{i+1}^* \mathcal{L}(\Lambda_{\mathcal{D}_{i+1}})$ . By [Theorem 2.7](#) this further implies  $\mathcal{D}_i \subseteq \mathcal{D}_{i+1}$ , and similarly we get  $\mathcal{D}_i \supseteq \mathcal{D}_{i+1}$ ; thus,  $\mathcal{D}_i = \mathcal{D}_{i+1}$ . Furthermore, we see that  $u_i \mathcal{L}(\Gamma_{\mathcal{C}_{i,i+1}}) u_i^* = u_{i+1} \mathcal{L}(\Gamma_{\mathcal{C}_{i,i+1}}) u_{i+1}^* = \mathcal{L}(\Lambda_{\mathcal{D}_i})$ , and hence

$$u_i^* u_{i+1} \in \mathcal{N}_{\mathcal{M}}(\mathcal{L}(\Gamma_{\mathcal{C}_{i,i+1}})) = \mathcal{L}(\Gamma_{\mathcal{C}_{i,i+1} \sqcup \text{lk}(\mathcal{C}_{i,i+1})}) = \mathcal{L}(\Gamma_{\mathcal{C}_i \cup \mathcal{C}_{i+1}}).$$

Moreover, using [Proposition 2.10](#), we further have that  $u_i^* u_{i+1} = a_{i,i+1} b_{i,i+1}$ , where  $a_{i,i+1} \in \mathcal{U}(\mathcal{L}(\mathcal{C}_{i,i+1}))$  and  $b_{i,i+1} \in \mathcal{U}(\mathcal{L}(\Gamma_{(\mathcal{C}_i \cup \mathcal{C}_{i+1}) \setminus \mathcal{C}_{i,i+1}}))$ . To this end observe that if we let  $x_{i,i+1} := u_i^* u_{i+1}$  then we have  $x_{1,2} x_{2,3} \cdots x_{n,1} = 1$ . Thus, using [Theorem 5.2](#) for each  $i \in \overline{1, n}$ , one can find  $a_i \in \mathcal{U}(\mathcal{L}(\Gamma_{\mathcal{C}_{i-1,i}}))$ ,  $b_i \in \mathcal{U}(\mathcal{L}(\Gamma_{\mathcal{C}_i^{\text{int}}}))$ , and  $c_i \in \mathcal{U}(\mathcal{L}(\Gamma_{\mathcal{C}_{i,i+1}}))$  such that

$$u_i^* u_{i+1} = x_{i,i+1} = a_i b_i c_i b_{i+1}^* a_{i+2}^* c_{i+1}^*. \quad (7-6)$$

Using these relations recursively together with the commutation relations and performing the appropriate cancellations, we see that, for every  $i \in \overline{2, n}$ , we have

$$\begin{aligned} u_i &= u_1 (u_1^* u_2) (u_2^* u_3) \cdots (u_{i-2}^* u_{i-1}) (u_{i-1}^* u_i) \\ &= u_1 (a_1 b_1 c_1 b_2^* a_3^* c_2^*) (a_2 b_2 c_2 b_3^* a_4^* c_3^*) \cdots (a_{i-1} b_{i-1} c_{i-1} b_i^* a_{i+1}^* c_i^*) \\ &\quad \vdots \\ &= u_1 a_1 b_1 c_1 a_2 a_i^* b_i^* a_{i+1}^* c_i^*. \end{aligned} \quad (7-7)$$

Since  $a_i^* b_i^* a_{i+1}^* c_i^* \in \mathcal{U}(\mathcal{L}(\Gamma_{\mathcal{C}_i}))$ , we can see that by replacing each  $u_i$  by  $u = u_1 a_1 b_1 c_1 a_2$  the relations in (7-5) still hold. By writing  $\mathcal{F}_i = \sigma(\mathcal{C}_i)$  for all  $i$ , we observe that in particular these relations imply that  $u\mathcal{L}(\Gamma_{\mathcal{C}_{i,i+1}})u^* = \mathcal{L}(\Lambda_{\mathcal{F}_{i,i+1}})$  for all  $i$ . Passing to relative commutants in each clique algebra, we also have  $u\mathcal{L}(\Gamma_{\mathcal{C}_i^{\text{int}}})u^* = \mathcal{L}(\Lambda_{\mathcal{F}_i^{\text{int}}})$ . We now notice that the s-unique prime factorization property of the groups implies that the map  $\sigma$  arises from an isometry  $\sigma : \mathcal{G} \rightarrow \mathcal{H}$  still denoted by the same letter. Altogether these relations give the desired statement.  $\square$

Using the  $W^*$ -superrigid property (T) wreath-like product groups recently discovered in [Chifan et al. 2023b] as vertex groups in the previous result, one obtains an even more precise description of the  $*$ -isomorphisms between these von Neumann algebras; hence, we provide a proof for **Theorem B**.

**Theorem 7.10.** *Let  $\mathcal{G}$  and  $\mathcal{H}$  be graphs in the class  $\text{CC}_1$ , and let  $G = \mathcal{G}\{\Gamma_v\}$  and  $\Lambda = \mathcal{H}\{\Lambda_w\}$  be graph product groups where all vertex groups  $\Gamma_v$  and  $\Lambda_w$  are property (T) wreath-like product groups (as described in the second part of [Corollary 7.7](#)).*

*Then, for any  $t > 0$  and  $*$ -isomorphism  $\Theta : \mathcal{L}(\Gamma)^t \rightarrow \mathcal{L}(\Lambda)$ , we have  $t = 1$ , and one can find a character  $\eta \in \text{Char}(\Gamma)$ , a group isomorphism  $\delta \in \text{Isom}(\Gamma, \Lambda)$ , an automorphism of  $\mathcal{L}(\Lambda)$  of the form  $\phi_{a,b}$  (see the notation after (7-1)), and a unitary  $u \in \mathcal{L}(\Lambda)$  such that  $\Theta = \text{ad}(u^*) \circ \phi_{a,b} \circ \Psi_{\eta, \delta}$ .*

*Proof.* From the prior result we have  $t = 1$ . From [Theorem 7.9](#) one can find a graph isomorphism  $\sigma : \mathcal{G} \rightarrow \mathcal{H}$  and a unitary  $u \in \mathcal{L}(\Lambda)$  such that, for every clique  $\mathcal{C}_i \in \text{cliq}(\mathcal{G})$ , we have  $u\Theta(\mathcal{L}(\Gamma_{\mathcal{C}_i}))u^* = \mathcal{L}(\Lambda_{\sigma(\mathcal{C}_i)})$ . In particular, these relations imply that  $u\Theta(\mathcal{L}(\Gamma_{\mathcal{C}_{i,i+1}}))u^* = \mathcal{L}(\Lambda_{\sigma(\mathcal{C}_{i,i+1})})$  and also  $u\mathcal{L}(\Gamma_{\mathcal{C}_i^{\text{int}}})u^* = \mathcal{L}(\Lambda_{\sigma(\mathcal{C}_i^{\text{int}})})$  for all  $i \in \overline{1, n}$ . Furthermore, using [Corollary 3.7](#), one can find unitaries  $a_{i,i+1} \in \Theta(\mathcal{L}(\Gamma_{\mathcal{C}_{i,i+1}}))$  and  $b_i \in \Theta(\mathcal{L}(\Gamma_{\mathcal{C}_{i,i+1}}))$  such that

$$\mathbb{T}ua_{i,i+1}\Theta(\Gamma_{\mathcal{C}_{i,i+1}})a_{i,i+1}^*u^* = \mathbb{T}\Lambda_{\sigma(\mathcal{C}_{i,i+1})} \quad \text{and} \quad \mathbb{T}ub_i\Theta(\Gamma_{\mathcal{C}_i^{\text{int}}})b_i^*u^* = \mathbb{T}\Lambda_{\sigma(\mathcal{C}_i^{\text{int}})}.$$

Hence, there exists an automorphism of  $\mathcal{L}(\Lambda)$  of the form  $\phi_{a,b}$  such that, by letting  $\widetilde{\Theta} = \phi_{a,b}^{-1} \circ \text{ad}(u) \circ \Theta$ , we have

$$\mathbb{T}\widetilde{\Theta}(\Gamma_{\mathcal{C}_{i,i+1}}) = \mathbb{T}\Lambda_{\sigma(\mathcal{C}_{i,i+1})} \quad \text{and} \quad \mathbb{T}\widetilde{\Theta}(\Gamma_{\mathcal{C}_i^{\text{int}}}) = \mathbb{T}\Lambda_{\sigma(\mathcal{C}_i^{\text{int}})}$$

for any  $i \in \overline{1, n}$ . The conclusion trivially follows.  $\square$

Next, we record four immediate consequences of the prior result, and hence provide proofs to the other main results of the introduction.

**Corollary 7.11.** *Let  $\mathcal{G}$  be a graph in the class  $\text{CC}_1$ , and let  $\Gamma = \mathcal{G}\{\Gamma_v\}$  be the graph product groups where all vertex groups  $\Gamma_v$  are property (T) wreath-like product groups (as described in the second part of [Corollary 7.7](#)).*

*Then, for any automorphism  $\Theta \in \text{Aut}(\mathcal{L}(\Gamma))$ , one can find  $\eta \in \text{Char}(\Gamma)$ ,  $\delta \in \text{Aut}(\Gamma)$ , an automorphism of  $\mathcal{L}(\Gamma)$  of the form  $\phi_{a,b}$ , and a unitary  $u \in \mathcal{L}(\Gamma)$  such that  $\Theta = \text{ad}(u) \circ \phi_{a,b} \circ \Psi_{\eta, \delta}$ .*

**Corollary 7.12.** *Let  $\mathcal{G}$  be a graph in the class  $\text{CC}_1$ , and let  $\Gamma = \mathcal{G}\{\Gamma_v\}$  be the graph product groups where all vertex groups  $\Gamma_v$  are property (T) wreath-like product groups (as described in the second part of [Corollary 7.7](#)). Then the fundamental group  $\mathcal{F}(\mathcal{L}(\Gamma)) = \{1\}$ .*

In particular, combining these results with [Theorem 3.3](#) and [Remark 3.4](#), we obtain examples when the only outer automorphisms of von Neumann algebras of graph products are the only options discussed in relation (7-1).

**Corollary 7.13.** *Let  $\mathcal{G} \in \text{CC}_1$ , and fix  $\text{cliq}(\mathcal{G}) = \{\mathcal{C}_1, \dots, \mathcal{C}_n\}$  a consecutive enumeration of its cliques. Let  $\Gamma = \mathcal{G}\{\Gamma_v\}$  be the graph product groups where all vertex groups  $\Gamma_v$  are property (T) regular wreath-like product groups (as described in the second part of [Corollary 7.7](#)) which in addition are pairwise nonisomorphic, and have trivial abelianization and trivial outer automorphisms. Then*

$$\text{Out}(\mathcal{L}(\Gamma)) \cong \bigoplus_{i=1}^n \mathcal{U}(\mathcal{L}(\Gamma_{\mathcal{C}_{i-1,i}})) \oplus \mathcal{U}(\mathcal{L}(\Gamma_{\mathcal{C}_i^{\text{int}}})).$$

*Proof.* Let  $\Theta \in \text{Out}(\mathcal{L}(\Gamma))$ . By [Theorem 7.10](#), one can find a character  $\eta \in \text{Char}(\Gamma)$ , a group automorphism  $\delta \in \text{Aut}(\Gamma)$ , and an automorphism of  $\mathcal{L}(\Gamma)$  of the form  $\phi_{a,b}$  such that  $\Theta = \phi_{a,b} \circ \Psi_{\eta,\delta}$ . Note that, for any  $v \in \mathcal{V}$ , the restriction of  $\eta$  to  $\Gamma_v$  is a character of  $\Gamma_v$  and, by assumption, we get that  $\eta(g) = 1$  for any  $v \in \mathcal{V}$  and  $g \in \Gamma_v$ . Next, recall that by [Theorem 4.4](#) we have  $\text{Aut}(\Gamma) \cong \Gamma \rtimes ((\bigoplus_{v \in \mathcal{V}} \text{Aut}(\Gamma_v)) \rtimes \text{Sym}(\Gamma))$ . Now, because the vertex groups are pairwise nonisomorphic, then  $\text{Sym}(\Gamma) = 1$ . Moreover, since all automorphisms of the vertex groups are inner, it follows that  $\Psi_{\eta,\delta}$  is essentially an automorphism of the form  $\phi_{a',b'}$ , where  $a'$  and  $b'$  are collections of unitaries implemented by group elements. In conclusion, we have that  $\Theta = \phi_{c,d}$ , where  $c$  and  $d$  are some collections of unitaries, and the formula follows.  $\square$

**Corollary 7.14.** *Let  $\mathcal{G}$  and  $\mathcal{H}$  be graphs in the class  $\text{CC}_1$ , and let  $G = \mathcal{G}\{\Gamma_v\}$  and  $\Lambda = \mathcal{H}\{\Lambda_w\}$  be graph product groups where all vertex groups  $\Gamma_v$  and  $\Lambda_w$  are property (T) wreath-like product groups (as described in the second part of [Corollary 7.7](#)).*

*Then, for any  $*$ -isomorphism  $\Theta : C_r^*(\Gamma) \rightarrow C_r^*(\Lambda)$ , one can find a character  $\eta \in \text{Char}(\Gamma)$ , a group isomorphism  $\delta \in \text{Isom}(\Gamma, \Lambda)$ , an automorphism of  $\mathcal{L}(\Lambda)$  of the form  $\phi_{a,b}$ , and a unitary  $u \in \mathcal{L}(\Lambda)$  such that  $\Theta = \text{ad}(u^*) \circ \phi_{a,b} \circ \Psi_{\eta,\delta}$ .*

*Proof.* From [Lemma 4.3](#), we get that  $\Gamma$  has trivial amenable radical, and hence, by [\[Breuillard et al. 2017, Theorem 1.3\]](#), it follows that  $C_r^*(\Gamma)$  has unique trace. This implies that any  $*$ -isomorphism between  $C_r^*(\Gamma)$  and  $C_r^*(\Lambda)$  lifts to a  $*$ -isomorphism of the associated von Neumann algebras. Now the result follows from [Theorem 7.10](#).  $\square$

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