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GLOBAL WELL-POSEDNESS FOR TWO-DIMENSIONAL INHOMOGENEOUS VISCOUS FLOWS WITH ROUGH DATA VIA DYNAMIC INTERPOLATION

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We consider the evolution of two-dimensional incompressible flows with variable density, only bounded and bounded away from zero. Assuming that the initial velocity belongs to a suitable critical subspace of L^2, we prove a global-in-time existence and stability result for the initial (boundary) value problem.

Our proof relies on new time decay estimates for finite energy weak solutions and on a "dynamic interpolation" argument. We show that the constructed solutions have a uniformly C^1 flow, which ensures the propagation of geometrical structures in the fluid and guarantees that the Eulerian and Lagrangian formulations of the equations are equivalent. By adopting this latter formulation, we establish the uniqueness of the solutions for prescribed data and the continuity of the flow map in an energy-like functional framework.

In contrast with prior works, our results hold in the critical regularity setting *without any smallness assumption*. Our approach uses only elementary tools and applies indistinctly to the cases where the fluid domain is the whole plane, a smooth two-dimensional bounded domain, or the torus.

Introduction

An extensive literature has been devoted to the mathematical analysis of the Navier–Stokes equations that govern the evolution of the velocity field $u = u(t, x)$ and pressure function $P = P(t, x)$ of homogeneous incompressible viscous flows in a domain Ω of $\mathbb{R}^d$. Recall that these equations read as

$$\begin{cases} u_t + \operatorname{div}(u \otimes u) - \mu\Delta u + \nabla P = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\ u|_{t=0} = u_0 & \text{in } \Omega \end{cases} \tag{NS}$$

and, if Ω has a boundary, are supplemented with homogeneous Dirichlet boundary conditions for the velocity.

The global existence theory for (NS) originated in the paper by J. Leray [1934b]. In the case $\Omega = \mathbb{R}^3$, by combining the energy balance associated to (NS),

$$\frac{1}{2}\|u(t)\|_{L^2}^2 + \mu\int_0^t \|\nabla u\|_{L^2}^2\, d\tau = \frac{1}{2}\|u_0\|_{L^2}^2, \tag{0-1}$$

with compactness arguments, he constructed, for any divergence-free u_0 in $L^2(\mathbb{R}^3; \mathbb{R}^3)$, a global distributional solution of (NS) satisfying (0-1) *with an inequality* (viz. the left-hand side is bounded by the right-hand side).

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It is by now well understood that Leray's result is true in any open subset Ω of $\mathbb{R}^d$ with $d = 2, 3$; see for instance the first part of [Chemin et al. 2006]. However, despite the numerous papers devoted to the topic and significant recent progresses, the question of uniqueness of finite energy solutions in the case $d = 3$ has not been completely solved yet. The two-dimensional situation is much better understood: finite energy solutions are unique and do satisfy (0-1) with an equality. Although uniqueness in dimension 2 could be hinted from [Leray 1934a], it has been established only by O. A. Ladyzhenskaya [1959] and J.-L. Lions and G. Prodi [Lions and Prodi 1959].

In the present paper, we are concerned with *inhomogeneous*, that is, with variable density, incompressible viscous flows. The evolution of these flows, which can be encountered in models of geophysics or mixtures, is often described by the following *inhomogeneous incompressible Navier–Stokes equations*:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) - \mu \Delta u + \nabla P = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}_+ \times \Omega. \end{cases} \tag{INS}$$

Above, u and P still denote the velocity and the pressure, respectively, and $\rho = \rho(t, x)$ stands for the density, which for obvious physical reasons has to be nonnegative. If we supplement (INS) with initial data and boundary conditions

$$\rho|_{t=0} = \rho_0, \quad u|_{t=0} = u_0 \quad \text{and} \quad u|_{\partial\Omega} = 0, \tag{0-2}$$

then the energy balance associated to (INS) reads as

$$\frac{1}{2}\|(\sqrt{\rho}u)(t)\|_{L^2}^2 + \mu \int_0^t \|\nabla u\|_{L^2}^2 \, d\tau = \frac{1}{2}\|\sqrt{\rho_0}u_0\|_{L^2}^2. \tag{0-3}$$

The divergence-free condition ensures that the Lebesgue norms of ρ are conserved and that,

$$\text{for all } t \in \mathbb{R}_+, \quad \inf_{x\in\Omega} \rho(t, x) = \inf_{x\in\Omega} \rho_0(x) \quad \text{and} \quad \sup_{x\in\Omega} \rho(t, x) = \sup_{x\in\Omega} \rho_0(x). \tag{0-4}$$

In the torus case, we have in addition the conservation of total momentum

$$\int_{\mathbb{T}^2} (\rho u)(t, x) \, dx = \int_{\mathbb{T}^2} (\rho_0 u_0)(x) \, dx. \tag{0-5}$$

Like (NS), equations (INS) have a scaling invariance (if Ω is stable by dilation): they are invariant for all $\lambda > 0$ by the transform

$$(\rho, u, P)(t, x) \rightsquigarrow (\rho, \lambda u, \lambda^2 P)(\lambda^2 t, \lambda x). \tag{0-6}$$

Although (INS) is of hyperbolic-parabolic-type while (NS) is parabolic, similar results hold for the initial value (or boundary value) problem. For instance:

- In any dimension, provided ρ_0 is bounded and nonnegative and $\sqrt{\rho_0}u_0$ is in L^2, there exists a global weak solution satisfying (0-3) with inequality.[1]

[1] First proved by A. V. Kazhikhov [1974] if $\rho_0 > 0$, then for general $\rho_0 \geq 0$ by J. Simon [1990]. P.-L. Lions [1996] pointed out that the density is a renormalized solution of the mass equation and treated density dependent viscosity coefficients. He also considered unbounded densities.

- Smooth enough data with density bounded and bounded away from zero generate a unique local-in-time smooth solution, which is global in the two-dimensional case and also in higher dimensions if the initial velocity is small.[2]

In dimension 2, the quantities that come into play in the energy balance (0-3) are scaling invariant in the sense of (0-6). However, unlike the case with constant density, it is not known whether finite energy two-dimensional weak solutions with bounded density, albeit having critical regularity, are unique.

In order to explain the difference between the variable and constant density cases and to motivate the assumptions that will be made in this paper, let us sketch the proof of the uniqueness of finite energy solutions for (NS) in dimension 2. Assume that we are given two solutions (u, P) and $(\tilde{u}, \widetilde{P})$ pertaining to the same finite energy initial velocity u_0. Then, $\delta u := \tilde{u} - u$ and $\delta P := \widetilde{P} - P$ satisfy

$$\begin{cases} \delta u_t + \operatorname{div}(u \otimes \delta u) - \mu \Delta \delta u + \nabla \delta P = -\operatorname{div}(\delta u \otimes \tilde{u}) & \text{in } \mathbb{R}_+ \times \Omega, \\ \operatorname{div} \delta u = 0 & \text{in } \mathbb{R}_+ \times \Omega. \end{cases}$$

Taking the $L^2(\Omega; \mathbb{R}^2)$ scalar product with δu, integrating by parts where needed and using the Hölder inequality to bound the right-hand side yields

$$\frac{1}{2}\frac{d}{dt}\|\delta u\|_{L^2}^2 + \mu\|\nabla \delta u\|_{L^2}^2 \leq \|\nabla \tilde{u}\|_{L^2}\|\delta u\|_{L^4}^2,$$

which, in light of the celebrated Ladyzhenskaya inequality

$$\|z\|_{L^4}^2 \leq C\|z\|_{L^2}\|\nabla z\|_{L^2}, \tag{0-7}$$

leads to

$$\frac{1}{2}\frac{d}{dt}\|\delta u\|_{L^2}^2 + \mu\|\nabla \delta u\|_{L^2}^2 \leq C\|\nabla \tilde{u}\|_{L^2}\|\delta u\|_{L^2}\|\nabla \delta u\|_{L^2} \leq \frac{\mu}{2}\|\nabla \delta u\|_{L^2}^2 + \frac{C^2}{2\mu}\|\nabla \tilde{u}\|_{L^2}^2\|\delta u\|_{L^2}^2.$$

At this stage, Gronwall's lemma allows us to conclude that

$$\|\delta u(t)\|_{L^2}^2 + \mu \int_0^t \|\nabla \delta u\|_{L^2}^2 \, d\tau \leq e^{(C^2/\mu)\int_0^t \|\nabla \tilde{u}\|_{L^2}^2 \, d\tau}\|\delta u(0)\|_{L^2}^2.$$

Owing to (0-1), the exponential term is finite. Hence we have $\delta u \equiv 0$ if $\tilde{u}(0) = u(0)$.

In contrast, when comparing two finite energy solutions (ρ, u, P) and $(\tilde{\rho}, \tilde{u}, \widetilde{P})$ of (INS), we get the following system for $\delta\rho := \tilde{\rho} - \rho$, δu, and δP:

$$\begin{cases} \delta\rho_t + \operatorname{div}(\delta\rho u) = -\operatorname{div}(\tilde{\rho}\delta u), \\ (\rho\delta u)_t + \operatorname{div}(\rho u \otimes \nabla \delta u) - \mu\Delta\delta u + \nabla\delta P = -(\delta\rho\tilde{u})_t - \operatorname{div}(\rho u \otimes \delta u) - \operatorname{div}(\rho \delta u \otimes \tilde{u}), \\ \operatorname{div} \delta u = 0. \end{cases}$$

Since $\tilde{\rho}$ is only bounded, the first line is a transport equation by the divergence-free vector field u, with a source term that has (at most) the regularity C^{-1} with respect to the space variable. Now, in order to control the propagation of negative regularity in a transport equation, we need

$$\nabla u \in L^1_{\mathrm{loc}}(\mathbb{R}_+; L^\infty). \tag{0-8}$$

[2]First established by O. A. Ladyzhenskaya and V. A. Solonnikov [1975].

However, this property generally fails for finite energy solutions of (INS) and even for the two-dimensional heat equation. In fact, the set of functions u_0 such that the solution u to the free heat equation with initial data u_0 satisfies $\nabla u \in L^1(\mathbb{R}_+; L^\infty)$ is the homogeneous Besov space $\dot B^{-1}_{\infty,1}$, and L^2 is not embedded in this space.

To avoid working in spaces with negative regularity, one can recast (INS) in the Lagrangian coordinate system as in [Danchin and Mucha 2019]. Then, the density becomes time-independent and the velocity equation keeps its parabolicity (at least for small time). However, the equivalence between the Eulerian and Lagrangian formulations of (INS) in our low-regularity context still requires (0-8), a property that cannot be expected if u_0 is only in L^2 since it fails for the heat flow.

To make a long story short, it is not clear that uniqueness holds for (INS) in the framework of just finite energy solutions.

Before describing in more detail the main objective of the article, let us recall some recent results on the well-posedness theory for (INS). A number of works have been devoted to this issue under weaker assumptions than in [Ladyzhenskaya and Solonnikov 1975]. This is mainly to relax the positivity condition on the density or the regularity assumptions on the initial data. Regarding the first question, it has been observed by Y. Cho and H. Kim [2004] that (INS) is well-posed for smooth enough data and, possibly, vanishing densities satisfying a suitable compatibility condition. Recently, J. Li [2017] discovered that this condition is no longer needed if one considers H^1 regularity for the velocity, and the full well-posedness theory for general only bounded (not necessarily positive) initial densities and H^1 velocities has been carried out in a joint work with P. B. Mucha [Danchin and Mucha 2019].

Regarding the minimal regularity requirement of the velocity for well-posedness, the scaling invariance of (INS) pointed out in (0-6) suggests (if $\Omega = \mathbb{R}^d$) that one should take $\rho_0 \in L^\infty(\mathbb{R}^d)$ and $u_0 \in \dot H^{d/2-1}(\mathbb{R}^d)$. In the constant density case and for $d = 3$, this assumption is in accordance with the well-known Fujita and Kato theorem [1964]. However, as, again, $\nabla e^{t\Delta} u_0$ need not be in $L^1_{\mathrm{loc}}(\mathbb{R}_+; L^\infty)$ if $u_0 \in \dot H^{d/2-1}(\mathbb{R}^d)$, it is not clear that uniqueness may be achieved if there is no additional regularity in the variable density case. In this direction, it has been proved in [Danchin 2003; 2004] that if u_0 belongs to the homogeneous Besov space $\dot B^{d/2-1}_{2,1}(\mathbb{R}^d)$, a large subspace of $\dot H^{d/2-1}(\mathbb{R}^d)$ with the same scaling invariance, then (INS) is globally well-posed in dimension 2 (or in higher dimensions if u_0 is small) *provided ρ_0 is close to some positive constant in the homogeneous Besov space* $\dot B^{d/2}_{2,1}(\mathbb{R}^d)$. This result is satisfactory as regards the regularity requirement for the velocity, since it is critical and closely related to the L^2 space, but the condition on the density is rather restrictive both because ρ_0 has to be almost constant and since it has to be continuous (the space $\dot B^{d/2}_{2,1}(\mathbb{R}^d)$ is embedded in the set $\mathcal{C}_b(\mathbb{R}^d)$ of bounded and continuous functions on $\mathbb{R}^d$). The result of [Danchin 2003] has been significantly improved recently in the two-dimensional case: H. Abidi and G. Gui [2021] established the global well-posedness without any smallness condition on the data if $\rho_0 - 1$ is in $\dot B^1_{2,1}(\mathbb{R}^2)$ and u_0 belongs to $\dot B^0_{2,1}(\mathbb{R}^2)$. The corresponding result in dimension 3 has been obtained with completely different techniques by H. Xu [2022] (for small u_0 of course). As said before, works based on the use of critical Besov spaces for the density precludes considering the case of densities that are discontinuous along an interface, a situation which is of particular interest if one believes (INS) to be a relevant model for mixtures of incompressible viscous flows with different densities.

This very situation — sometimes called *the density patch problem* — has been extensively studied lately, see, e.g., [Danchin and Mucha 2019; Gancedo and García-Juárez 2018; Liao and Zhang 2019].

Well-posedness results for only bounded initial density, bounded away from zero, and smooth enough velocity have been obtained in a joint work with P. B. Mucha [Danchin and Mucha 2013b], then improved by M. Paicu, P. Zhang and Z. Zhang in [Paicu et al. 2013] (there, u_0 is in $H^s(\mathbb{R}^2)$ for some $s > 0$ if $d = 2$, and in $H^1(\mathbb{R}^3)$ if $d = 3$). In the whole space case, the critical regularity index has been reached in an intriguing work by P. Zhang [2020]. He established the global existence for any small enough divergence-free u_0 with coefficients in $\dot{B}^{1/2}_{2,1}(\mathbb{R}^3)$ while ρ_0 is only bounded and bounded away from zero. It has been observed recently in a joint work with S. Wang [Danchin and Wang 2023] that Zhang's solutions actually satisfy (0-8) and are thus unique.

The main goal of the present paper is to investigate the counterpart *in dimension* 2 *and for large initial data* of Zhang's result recalled just above: we want to establish a global well-posedness result for general divergence-free velocity fields u_0 with critical regularity of L^2-type and densities ρ_0 simply satisfying

$$\begin{aligned} \rho_* &:= \operatorname*{ess\,inf}_{x\in\Omega} \rho_0(x) > 0, \\ \rho^* &:= \operatorname*{ess\,sup}_{x\in\Omega} \rho_0(x) < \infty. \end{aligned} \tag{0-9}$$

According to [Abidi and Gui 2021], a good candidate to achieve the Lipschitz property within a critical regularity framework of L^2-type is the space $\dot{B}^0_{2,1}$. However, owing to the use of Fourier analysis techniques, rather strong regularity assumptions on the density were made in that work. Here, since we want to consider only bounded densities, we shall adopt a completely different approach. In fact, we shall combine real interpolation and three levels of time decay estimates (corresponding to $\dot{H}^{-1}$, L^2, and $\dot{H}^1$ data, respectively) for a linearized version of (INS) that can be obtained just by energy arguments and basic properties of the Stokes system, so as to work out a space for u_0 that coincides with $\dot{B}^0_{2,1}$ if ρ_0 is smooth (but that might depend on it if it is not). The overall strategy is so robust that it can be adapted to other systems.

The rest of the paper is structured as follows: in the next section we state our main results and explain the key steps of the proof. Then, in Section 2, we establish a first family of time decay estimates pertaining to the case where u_0 is just in L^2 and construct corresponding global finite energy weak solutions for (INS). Section 3 is devoted to proving more a priori decay estimates. The final goal is to establish that, under a slightly stronger assumption on the initial velocity very close to the regularity $\dot{B}^0_{2,1}$, the Lipschitz property (0-8) is satisfied. Finally, we establish in Section 4 the existence and uniqueness of a solution under this assumption, assuming only (0-9) and that the velocity belongs to the aforementioned space. The same method also provides stability estimates for the flow map in the energy space.

Notation. In the rest of the paper, Ω will be either a C^2 bounded domain of $\mathbb{R}^2$, a two-dimensional torus, or $\mathbb{R}^2$. It will be convenient to use the same notation $\dot{H}^s(\Omega)$ to designate:

- the classical homogeneous Sobolev space if $\Omega = \mathbb{R}^2$,
- the subset of functions of H^s with mean value 0 if $\Omega = \mathbb{T}^2$,

- the space $H^s_0(\Omega)$ (that is the completion of $\mathcal{C}^\infty_c(\Omega)$ for the $H^s(\mathbb{R}^2)$ norm) if Ω is a bounded domain and $s\in[0,1]$,
- the dual of $H_0^{-s}(\Omega)$ if Ω is a bounded domain and $s\in[-1,0]$.

We designate by $L^2_\sigma(\Omega)$ the set of divergence-free vector fields with coefficients in $L^2(\Omega)$ (such that $u_0\cdot n=0$ at $\partial\Omega$ in the bounded domain case, with n being the unit exterior normal vector to $\partial\Omega$), and denote by $\mathcal{P}$ the orthogonal projector from $L^2(\Omega;\mathbb{R}^2)$ to $L^2_\sigma(\Omega)$.

For any normed space X, Lebesgue index $q\in[1,\infty]$, and time $T\in[0,\infty]$, we shall define

$$\|z\|_{L^q_T(X)}:=\big\|\|z(t)\|_X\big\|_{L^q(0,T)},$$

omitting T if it is ∞. In the case where z has several components in X, we keep the same notation for the norm.

As usual, C designates harmless positive real numbers, and we shall often write $A\lesssim B$ instead of $A\le CB$. To emphasize the dependency with respect to parameters $a_1,\dots,a_n$, we adopt the notation $C_{a_1,\dots,a_n}$. The notation $C_{\rho,v}$ stands for various "constants" that only depend (algebraically) on the infimum and supremum of ρ and on "energy-like" norms of v, that is, *on norms that could be eventually bounded by* $\|u_0\|_{L^2}$ *if* (ρ,v) *were a solution to* (INS). Obvious examples are $\|v\|_{L^\infty(L^2)}$ or $\|\nabla v\|_{L^2(L^2)}$ (remember (0-3)) but also $\|v\|_{L^4(L^4)}$ (use (0-7)) and so on.

1. Results and strategy

The first step is to exhibit time decay estimates for finite energy solutions. More precisely, we shall establish the following statement.

Theorem 1.1. *Let u_0 be in $L^2_\sigma(\Omega)$ and ρ_0 satisfy* (0-9). *Then,* (INS) *supplemented with* (0-2) *admits a global solution (ρ,u,P) satisfying* (0-4) (*and* (0-5) *if* $\Omega=\mathbb{T}^2$), $u\in L^\infty(\mathbb{R}_+;L^2_\sigma)$, $\nabla u\in L^2(\mathbb{R}_+\times\Omega)$, *and*

$$\frac12\|(\sqrt{\rho}u)(t)\|^2_{L^2}+\mu\int_0^t\|\nabla u\|^2_{L^2}\,d\tau\le\frac12\|\sqrt{\rho_0}u_0\|^2_{L^2},\quad t>0. \tag{1-1}$$

Furthermore, there exists a constant C depending only on Ω, ρ_*, *and* ρ^* *such that, for all* $t>0$, *we have*

$$\begin{aligned}\|\nabla^k u(t)\|_{L^2}&\le C(\mu t)^{-k/2}\|u_0\|_{L^2}\quad &&\text{for } k=0,1,2,\\ \|\nabla^k(u_t,\dot u)(t)\|_{L^2}&\le C(\mu t)^{-1-k/2}\|u_0\|_{L^2}\quad &&\text{for } k=0,1,\\ \|\nabla P(t)\|_{L^2}&\le Ct^{-1}\|u_0\|_{L^2},\end{aligned}$$

where $\dot u$ denotes the convective derivative of u; that is, $\dot u:=u_t+u\cdot\nabla u$.

Two remarks are in order:

- The constructed solutions satisfy more time decay estimates: see (2-11), (2-21), (2-26), Proposition 3.1 with $s'=0$, and Proposition 3.2 with $p=2$.

• As pointed out in [Danchin et al. 2024] for $H^1_0(\Omega)$ initial velocities, exponential time decay estimates hold if Ω is bounded. Following the proof of Lemma 5 therein, one can show that there exists a positive constant c_Ω depending only on Ω such that,

$$\text{for all } t \in \mathbb{R}_+, \quad \|(\sqrt{\rho}u)(t)\|_{L^2} \le e^{-c_\Omega \mu t/\rho^*}\|\sqrt{\rho_0}u_0\|_{L^2}.$$

From this inequality, one can deduce exponential decay for

$$\|t^{k/2}\nabla^k u\|_{L^2}, \quad \|t^{1+k/2}\nabla^k u_t\|_{L^2}, \quad \text{and} \quad \|t^{1+k/2}\nabla^k \dot{u}\|_{L^2}.$$

However, as exponential decay does not hold if $\Omega = \mathbb{R}^2$ and since we strive for a unified approach, we refrain from tracking it in the rest of the paper to simplify the presentation.

As underlined in the Introduction, in order to establish the uniqueness of solutions, we need a functional space that ensures (0-8). At the same time, we want our functional framework to be critical, to allow any initial density just bounded and bounded away from zero, and to be strongly related to the energy space L^2. Note that Theorem 1.1 ensures that ∇u belongs to the *weak* L^1 space for the time variable with values in the Sobolev space H^1. This latter space "almost" embeds in L^∞. A classical way to improve embeddings is to work out a space by means of *real interpolation with second parameter equal to* 1. In our context, since energy arguments play an important role, it is natural to interpolate from Sobolev spaces and to consider[3]

$$[\dot{H}^{-s}, \dot{H}^s]_{1/2,1} \quad \text{for some } s \in (0,1). \tag{1-2}$$

This definition gives the Besov space $\dot{B}^0_{2,1}$ (independently of the value of s).

Let us briefly explain why in the simpler situation where u is the solution of the free heat equation in $\mathbb{R}^2$, supplemented with initial data u_0 in $\dot{B}^0_{2,1}$, we do have (0-8). We start from the two inequalities

$$t\|\nabla u(t)\|_{L^\infty} \le C \min(t^{s/2}\|u_0\|_{\dot{H}^s}, t^{-s/2}\|u_0\|_{\dot{H}^{-s}}), \tag{1-3}$$

which may be easily derived by using the explicit formula for u in the Fourier space.

Then, we use the characterization of real interpolation spaces in terms of atomic decomposition like in, e.g., [Lions and Peetre 1964]. In our setting, it reads $z \in \dot{B}^0_{2,1}$ if and only if there exists a sequence $(z_j)_{j\in\mathbb{Z}}$ of $\dot{H}^{-s} \cap \dot{H}^s$ satisfying

$$z = \sum_{j\in\mathbb{Z}} z_j \quad \text{and} \quad \sum_{j\in\mathbb{Z}} (2^{-j/2}\|z_j\|_{\dot{H}^s} + 2^{j/2}\|z_j\|_{\dot{H}^{-s}}) < \infty.$$

The infimum of the above sum on all admissible decompositions of z defines a norm on $\dot{B}^0_{2,1}$. Now, take the decomposition

$$u_0 = \sum_{j\in\mathbb{Z}} u_{0,j}, \quad \text{with} \sum_{j\in\mathbb{Z}} (2^{-j/2}\|u_{0,j}\|_{\dot{H}^s} + 2^{j/2}\|u_{0,j}\|_{\dot{H}^{-s}}) \le 2\|u_0\|_{\dot{B}^0_{2,1}}, \tag{1-4}$$

[3] One could prefer to interpolate between *Lebesgue spaces* and consider the velocity in the Lorentz space $L^{2,1}$. However we do not know how to handle (INS) in this space. The reader is referred to [Danchin 2024] where the space $L^{2,1}$ is used for solving the two-dimensional system for pressureless gases.

and solve all the heat equations

$$(u_j)_t - \Delta u_j = 0, \quad u_j|_{t=0} = u_{0,j}.$$

As the heat equation is linear, we have $u = \sum_j u_j$, and thus

$$\int_0^\infty \|\nabla u\|_{L^\infty}\,dt \le \sum_{j\in\mathbb{Z}} \int_0^\infty \|\nabla u_j\|_{L^\infty}\,dt. \tag{1-5}$$

Now, for every j in $\mathbb{Z}$ and $A_j > 0$, we have, due to (1-3),

$$\begin{aligned}\int_0^\infty \|\nabla u_j\|_{L^\infty}\,dt &\le \int_0^{A_j} \|\nabla u_j\|_{L^\infty}\,dt + \int_{A_j}^\infty \|\nabla u_j\|_{L^\infty}\,dt\\ &\lesssim \|u_{0,j}\|_{\dot H^s}\int_0^{A_j} t^{-1+s/2}\,dt + \|u_{0,j}\|_{\dot H^{-s}}\int_{A_j}^\infty t^{-1-s/2}\,dt\\ &\lesssim \|u_{0,j}\|_{\dot H^s} A_j^{s/2} + \|u_{0,j}\|_{\dot H^{-s}} A_j^{-s/2}.\end{aligned}$$

Hence, choosing $A_j = 2^{-j/s}$ and remembering (1-4) and (1-5) gives (0-8) (globally in time).

This "dynamic interpolation approach" has been used before by T. Hmidi and S. Keraani [2008] for the transport equation and by Zhang [2020] for the velocity equation of (INS) (in dimension 3 and for small velocities). In both cases however, the initial data was decomposed according to a Littlewood–Paley decomposition. The additional flexibility that consists here in using general atomic decompositions enables us to do without Fourier analysis and to treat general domains.

As our aim is to prove (0-8) for (INS), we have to consider instead of the heat equation a linear system which captures both the effects of the density and of the convection. To this end, we consider

$$\begin{cases}(\rho u)_t + \operatorname{div}(v\otimes u) - \Delta u + \nabla P = 0 & \text{in } \mathbb{R}_+\times\Omega,\\ \operatorname{div} u = 0 & \text{in } \mathbb{R}_+\times\Omega,\\ u|_{t=0} = u_0 & \text{in } \Omega,\end{cases} \tag{1-6}$$

where the (smooth enough) triplet (ρ, v, u_0) is given with ρ bounded and bounded away from zero,

$$\rho_t + \operatorname{div}(\rho v) = 0, \quad \operatorname{div} v = 0, \quad \text{and} \quad v|_{\partial\Omega} = 0. \tag{1-7}$$

Clearly, if we succeed in proving (1-3) for (1-6) with a constant that only depends on ρ_*, ρ^*, and on energy-like norms of v, then repeating the above dynamic interpolation procedure will yield (0-8) for the solutions of (1-6) supplemented with initial data in $\dot B^0_{2,1}$, and then for (INS) if taking $v = u$.

The way to get (1-3) is to prove beforehand three families of time weighted estimates for (1-6) corresponding to initial data u_0 in L^2, $\dot H^1$, and $\dot H^{-1}$, respectively. The estimate in $\dot H^{-1}$ will be obtained by duality from the estimate in $\dot H^1$. This will lead us to consider the backward system associated with (1-6), and it is rather $\|\mathcal{P}(\rho u)(t)\|_{\dot H^{-1}}$ and, more generally, $\|\mathcal{P}(\rho u)(t)\|_{\dot H^{-s}}$ for $s\in(0,1)$ that can be estimated. In the end, combining the three families of inequalities with suitable Gagliardo–Nirenberg inequalities yields, instead of (1-3),

$$t\|\nabla u(t)\|_{L^\infty} \le C_{\rho,v}\min\big(t^{s/2}\|u_0\|_{\dot H^s},\, t^{-s/2}\|\mathcal{P}(\rho_0 u_0)\|_{\dot H^{-s}}\big). \tag{1-8}$$

Above, $C_{\rho,v}$ only depends on ρ_*, ρ^*, and on energy-like norms of v.

As a consequence, the suitable interpolation space to carry out our dynamic interpolation procedure for (1-6) is the one that is given in the following definition.

Definition 1.2. Let s be in $(0, 1)$ and a be a measurable function on Ω with positive lower bound. We denote by $\widetilde{B}^{0,s}_{a,1}(\Omega)$ the set of vector fields z in $L^2_\sigma(\Omega)$ such that there exists a sequence $(z_j)_{j\in\mathbb{Z}}$ of $L^2_\sigma(\Omega)$ satisfying:

- $z = \sum_{j\in\mathbb{Z}} z_j$ in the sense of distributions,
- for all $j \in \mathbb{Z}$, we have $\mathcal{P}(az_j) \in \dot{H}^{-s}(\Omega)$ and $z_j \in \dot{H}^s(\Omega)$,
- $\sum_{j\in\mathbb{Z}}(2^{-j/2}\|z_j\|_{\dot{H}^s} + 2^{j/2}\|\mathcal{P}(az_j)\|_{\dot{H}^{-s}})$ is finite.

The infimum on all admissible decompositions of z defines a norm on $\widetilde{B}^{0,s}_{a,1}(\Omega)$.

Let us highlight a few properties of these spaces.

- $(\widetilde{B}^{0,s}_{a,1}(\Omega))_{s\in(0,1)}$ is a family of nested Banach spaces: if $0 < s' < s < 1$, then $\widetilde{B}^{0,s}_{a,1}(\Omega) \hookrightarrow \widetilde{B}^{0,s'}_{a,1}(\Omega)$.
- Owing to (1-2), if a is a positive constant, then $\widetilde{B}^{0,s}_{a,1}$ is nothing other than $\dot{B}^0_{2,1}$, and if a has a positive lower bound a_*, then it embeds in L^2. Indeed, decomposing $z \in \widetilde{B}^{0,s}_{a,1}$ according to Definition 1.2 and using the fact that $\mathcal{P}$ is an L^2 orthogonal projector, one may write, for all $j \in \mathbb{Z}$,

$$\|z_j\|^2_{L^2} \le a_*^{-1}\int_\Omega \mathcal{P}(az_j)\cdot z_j\,dx \le a_*^{-1}(2^{j/2}\|\mathcal{P}(az_j)\|_{\dot{H}^{-1/2}})(2^{-j/2}\|z_j\|_{\dot{H}^{1/2}}), \tag{1-9}$$

which implies, by Young's inequality, that

$$\|z\|_{L^2} \le \frac{1}{2\sqrt{a_*}}\|z\|_{\widetilde{B}^{0,s}_{a,1}}.$$

- If a is bounded and $s = 2/p - 1$ for some $p \in (1, 2)$, then the critical Besov space

$$\dot{B}^{-1+2/p}_{p,1} := [L^p, \dot{W}^{2s}_p]_{1/2,1}$$

is embedded in $\widetilde{B}^{0,s}_{a,1}$. Indeed, if $z \in \dot{B}^{-1+2/p}_{p,1}$, then there exists a sequence $(z_j)_{j\in\mathbb{Z}}$ of the nonhomogeneous Sobolev space W^{2s}_p such that

$$z = \sum_{j\in\mathbb{Z}} z_j \quad\text{and}\quad \sum_{j\in\mathbb{Z}}(2^{-j/2}\|z_j\|_{W^{2s}_p} + 2^{j/2}\|z_j\|_{L^p}) \le 2\|z\|_{\dot{B}^{-1+2/p}_{p,1}}.$$

Now, the fact that $\mathcal{P} : L^p \to L^p$ and the embeddings $\dot{W}^{2s}_p \hookrightarrow \dot{H}^s$ and $L^p \hookrightarrow \dot{H}^{-s}$ allow us to write

$$\|z_j\|_{\dot{H}^s} \le C\|z_j\|_{\dot{W}^{2s}_p} \quad\text{and}\quad \|\mathcal{P}(az_j)\|_{\dot{H}^{-s}} \le C\|\mathcal{P}(az_j)\|_{L^p} \le C\|a\|_{L^\infty}\|z_j\|_{L^p},$$

which gives our claim.

- For general measurable functions a bounded and bounded away from zero, the space $\widetilde{B}^{0,s}_{a,1}$ might depend on s. However, in the case $s \in \left(0, \frac{1}{2}\right)$, if a is positive and piecewise constant along a finite number of Lipschitz curves, then it coincides with $\dot{B}^0_{2,1}$. Indeed, in this case the space $\dot{H}^{-s}$ is stable by multiplication by piecewise constant functions.

Our main global existence and uniqueness statement reads as follows.

Theorem 1.3. *Let ρ_0 satisfy* (0-9) *and u_0 be in $\widetilde{B}^{0,s}_{\rho_0,1}$ for some $s \in (0,1)$. Then*, (INS) *supplemented with* (0-2) *admits a unique global solution $(\rho, u, \nabla P)$ satisfying all the properties stated in Theorem 1.1* (*and the remarks that follow*) *and the energy balance* (0-3). *In addition, we have*

$$u \in \mathcal{C}(\mathbb{R}_+; L^2), \quad \nabla u \in L^1(\mathbb{R}_+; C_b \cap \dot{H}^1), \quad \sqrt{t}(\dot{u}, \nabla P, \nabla^2 u) \in L^{4/3}(\mathbb{R}_+; L^4)$$

and, *for all $t \in \mathbb{R}_+$, we have $u(t) \in \widetilde{B}^{0,s}_{\rho(t),1}$ with the inequality*

$$\|u(t)\|_{\widetilde{B}^{0,s}_{\rho(t),1}} \le C\|u_0\|_{\widetilde{B}^{0,s}_{\rho_0,1}}. \tag{1-10}$$

Remark 1.4. As a by-product of the proof of the uniqueness, we get a stability result with respect to the initial data in the energy space (see Theorem 4.2 below).

Remark 1.5. Owing to $\nabla u \in L^1(\mathbb{R}_+; C_b(\Omega))$, the flow of u has C^1 regularity with respect to the space variable, which means that the geometrical structures of the fluid during the evolution are conserved. For example, if ρ_0 takes two different positive values across a C^1 interface, then it remains so forever: the interface is just transported by the flow and keeps its C^1 regularity. Likewise, the (local) H^2 regularity of the interfaces is preserved since $\nabla^2 u \in L^1(\mathbb{R}_+; L^2(\Omega))$.

Remark 1.6. As said before, for $\Omega = \mathbb{R}^3$, a result in the same spirit has been obtained by Zhang [2020] in the small velocity case; see also [Danchin and Wang 2023]. An important difference with our situation is that, in dimension 3, the critical space for the velocity is $\dot{B}^{1/2}_{2,1} := [L^2, \dot{H}^1]_{1/2,1}$. Hence, it is enough to prove time weighted energy estimates in L^2 and $\dot{H}^1$, and the relevant critical space for u_0 does not depend on ρ_0.

To simplify the presentation, we assume hereafter that $s = \frac{1}{2}$. We use the short notation $\widetilde{B}^0_{\rho_0,1}$ for $\widetilde{B}^{0,1/2}_{\rho_0,1}$.

Let us briefly present the main steps of the proof of Theorem 1.3. The global existence of a solution being ensured by prior results, the main point is to exhibit enough regularity of the solution to ensure uniqueness. As already explained at length in the Introduction, the key is to establish (0-8), and this will be actually performed on the linear system (1-6).

The first step is to prove energy-type weighted estimates for (1-6) that require only u_0 to be in L^2 and the density to be bounded and bounded away from zero. The three principles guiding our search for estimates are:

- taking *convective derivatives* $D_t := \partial_t + v \cdot \nabla$ (since $D_t\rho = 0$) rather than space derivatives, since ρ has no regularity,
- using differential operators $\sqrt{t}\nabla,\ t\partial_t$, and tD_t (that are of order 0 in the parabolic scaling),
- transferring time regularity to space regularity by means of the maximal regularity properties of the Stokes system (see the Appendix), observing that

$$\mu\Delta u - \nabla P = \rho\dot{u} \quad \text{and} \quad \operatorname{div} u = 0 \quad \text{in } \Omega, \quad \text{with } \dot{u} := \partial_t u + v \cdot \nabla u. \tag{1-11}$$

In the end, this allows us to control quantities like $\|\sqrt{t}\nabla u(t)\|_{L^2},\ \|t\partial_t u(t)\|_{L^2},\ \|t\dot{u}(t)\|_{L^2}$, or $\|t\nabla^2 u(t)\|_{L^2}$ in terms of $\|u_0\|_{L^2},\ \rho_*,\ \rho^*$, and energy-like norms of v.

The second step is to propagate the $\dot{H}^1$ and the $\dot{H}^{-1}$ norms. On the one hand, $\dot{H}^1$ estimates for (INS) have been known since [Ladyzhenskaya and Solonnikov 1975] (we shall also derive time weighted versions of these estimates). On the other hand, propagating *negative* Sobolev regularity seems to be new. This will be achieved by duality after observing that the backward system associated with (1-6) satisfies the same family of estimates in $\dot{H}^s$. However, owing to the density dependent structure of the latter system, we will have only access to $\|\mathcal{P}(\rho u)(t)\|_{\dot{H}^{-s}}$, whence the “weighted” definition of the interpolation space $\widetilde{B}^{0,s}_{\rho,1}$.

The third step is devoted to propagating the regularity $\widetilde{B}^{0}_{\rho,1}$ and to bounding ∇u in $L^1(\mathbb{R}_+; L^\infty)$ in terms of the data only. In passing, we exhibit some controls of other critical norms (like, e.g., that of $\dot{u}$ in $L^1(\mathbb{R}_+; L^2)$) that will be needed in the proof of uniqueness and stability. All these bounds rely on the dynamic interpolation method that has been described above for the heat equation. In the end, we get

$$\int_0^\infty \|\nabla u\|_{L^\infty}\,dt + \int_0^\infty \|\dot{u}\|_{L^2}\,dt + \left(\int_0^\infty t^{2/3}\|\dot{u}\|_{L^4}^{4/3}\,dt\right)^{3/4} \le C\|u_0\|_{\widetilde{B}^0_{\rho_0,1}}.$$

The fourth step is the proof of existence of a global solution corresponding to the assumptions of Theorems 1.1 or 1.3. For Theorem 1.1, the overall strategy is standard: we smooth out the data, resort to classical results that ensure the existence of a sequence of global smooth solutions for (INS), and use the aforementioned estimates and compactness to pass to the limit. For Theorem 1.3, it is a bit the same, except that one has to be careful when smoothing out the velocity, owing to the “exotic” definition of the space $\widetilde{B}^0_{\rho_0,1}$. The easiest way is to truncate a decomposition of u_0 so as to have an approximate initial velocity in the smoother space $H^{1/2}$.

The last step is devoted to uniqueness and stability for (INS). As in [Danchin and Mucha 2019], we reformulate (INS) in Lagrangian coordinates. The properties of the solutions provided by Theorem 1.3, in particular (0-8), ensure that the two formulations are equivalent. The gain is that we do not have to worry about the density as it is time-independent. As for the difference of the two velocities in Lagrangian coordinates, it satisfies a parabolic-type equation and may be estimated in

$$L^\infty(\mathbb{R}_+; L^2) \cap L^2(\mathbb{R}_+; \dot{H}^1).$$

The computations are in the spirit of those of [Danchin et al. 2024]. However, in our case the velocity is less regular by one derivative, which requires some care.

As a concluding remark, we want to point out that, in contrast with numerous recent works dedicated to the inhomogeneous incompressible Navier–Stokes equations, our approach does not use Fourier analysis at all. It just relies on very basic energy arguments, interpolation, embedding, and on the classical regularity theory for the Stokes system (this is the only place where some assumptions have to be made on the fluid domain). For simplicity here we considered $\mathbb{R}^2$, $\mathbb{T}^2$, or C^2 bounded domains, but more general domains could be treated in the same way.

Hereafter we shall focus on the case $\mu = 1$ for simplicity. The general case follows due to the rescaling

$$\begin{aligned} \rho(t,x) &:= \tilde{\rho}(\mu t, x),\\ u(t,x) &:= \mu\tilde{u}(\mu t, x),\\ P(t,x) &:= \mu^2\widetilde{P}(\mu t, x). \end{aligned}$$

2. Weak solutions with time decay

This section is devoted to proving Theorem 1.1: we here construct finite energy weak solutions satisfying algebraic time decay estimates of different orders, without requiring more regularity on u_0 than L^2. The exponential decay that can be expected in the bounded domain case (see [Danchin et al. 2024]), is not addressed to simplify the presentation, as it is not needed for achieving the main result of the paper.

2.1. ***Time decay estimates for the linearized momentum equation.*** We here aim at proving time weighted energy estimates for the linear system (1-6) in the case where the (smooth enough) given pair (ρ, v) satisfies (1-7) and

$$\begin{aligned}\rho_* &= \inf_{(t,x)\in\mathbb{R}_+\times\Omega} \rho(t,x) > 0,\\ \rho^* &= \sup_{(t,x)\in\mathbb{R}_+\times\Omega} \rho(t,x) < \infty.\end{aligned} \tag{2-1}$$

System (1-6) is supplemented with a divergence-free initial velocity field u_0, vanishing at the boundary in the bounded domain case and, in the torus case, such that

$$\int_{\mathbb{T}^2} (\rho_0 u_0)(x)\,dx = 0.$$

This latter assumption is not restrictive owing to the Galilean invariance of the system and will enable us to use freely the Gagliardo–Nirenberg inequality (A-2).

We aim at proving energy estimates for the solution with time weights $t^{k/2}$ for $k \in \{0, 1, 2, 3\}$. We strive for bounds depending only on ρ_*, ρ^*, $\|u_0\|_{L^2}$, and on *energy-type norms of* v in the meaning given at the end of the Introduction of the paper. This latter point is fundamental for getting not only Theorem 1.1 but also Theorem 1.3.

Before proceeding, let us warn the reader that we unfortunately did not find a way to avoid the tedious calculations that will follow, since it is has to be checked with the greatest care that only "energy-type norms" come into play.

The basic energy balance. Taking the L^2 scalar product of (1-6) with u yields

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 = 0. \tag{2-2}$$

From this, we get, for all $t \in \mathbb{R}_+$,

$$\|(\sqrt{\rho}u)(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u\|_{L^2}^2\,d\tau = \|\sqrt{\rho}_0 u_0\|_{L^2}^2. \tag{2-3}$$

As $\rho_* > 0$, combining (2-3) with the Gagliardo–Nirenberg inequality (A-1) recalled in the Appendix yields, for all $2 \le p < \infty$,

$$\|u\|_{L^q(L^p)} \le C_p \rho_*^{-1/2} \|\sqrt{\rho_0} u_0\|_{L^2}, \quad \text{with } \frac{1}{p} + \frac{1}{q} = \frac{1}{2}. \tag{2-4}$$

Estimates with weight $\sqrt{t}$. Let us rewrite (1-6) as

$$\Delta u - \nabla P = \rho \dot{u} \quad \text{and} \quad \operatorname{div} u = 0 \quad \text{in } \Omega, \quad \text{with } \dot{u} := u_t + v\cdot\nabla u. \tag{2-5}$$

Taking the $L^2(\Omega;\mathbb{R}^2)$ scalar product of (2-5) with $t\dot{u}$ yields, for all $t \geq 0$,

$$\int_\Omega \rho t|\dot{u}|^2\,dx = t\int_\Omega \Delta u\cdot u_t\,dx - t\int_\Omega \nabla P\cdot u_t\,dx + t\int_\Omega (\Delta u - \nabla P)\cdot(v\cdot\nabla u)\,dx.$$

As $\operatorname{div} u = 0$, integrating by parts and using again (2-5) yields

$$\frac{1}{2}\frac{d}{dt}\int_\Omega t|\nabla u|^2\,dx - \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \int_\Omega \rho t|\dot{u}|^2\,dx = \int_\Omega \rho t\dot{u}\cdot(v\cdot\nabla u)\,dx. \tag{2-6}$$

Remembering (2-2) and performing a time integration, we get, for all $t \geq 0$,

$$\frac{1}{4}\int_\Omega \rho(t)|u(t)|^2\,dx + \frac{t}{2}\int_\Omega |\nabla u(t)|^2\,dx + \int_0^t\int_\Omega \tau\rho|\dot{u}|^2\,dx\,d\tau \\ = \frac{1}{4}\int_\Omega \rho_0|u_0|^2\,dx + \int_0^t\int_\Omega \tau\rho\dot{u}\cdot(v\cdot\nabla u)\,dx\,d\tau. \tag{2-7}$$

Of course, since $u_t = \dot{u} - v\cdot\nabla u$, one can write

$$\tfrac{1}{4}\|\sqrt{\rho}u_t\|_{L^2}^2 \leq \tfrac{1}{2}\|\sqrt{\rho}\dot{u}\|_{L^2}^2 + \tfrac{1}{2}\|\sqrt{\rho}v\cdot\nabla u\|_{L^2}^2.$$

Hence, adding up this inequality multiplied by t with (2-7) and using Young's inequality to bound the last term of (2-7), we discover that

$$\|\sqrt{\rho(t)}u(t)\|_{L^2}^2 + 2\|\sqrt{t}\nabla u(t)\|_{L^2}^2 + \int_0^t (\|\sqrt{\rho\tau}\dot{u}\|_{L^2}^2 + \|\sqrt{\rho\tau}u_\tau\|_{L^2}^2)\,d\tau \\ \leq \|\sqrt{\rho_0}u_0\|_{L^2}^2 + 6\int_0^t \|\sqrt{\rho\tau}v\cdot\nabla u\|_{L^2}^2\,d\tau. \tag{2-8}$$

Combining Hölder's inequality, Ladyzhenskaya's inequality (0-7), and Young's inequality yields

$$\|\sqrt{\rho}v\cdot\nabla u\|_{L^2}^2 \leq \frac{\varepsilon}{\rho^*}\|\nabla^2 u\|_{L^2}^2 + \frac{\rho^*}{\varepsilon}\|\sqrt{\rho}v\|_{L^4}^4\|\nabla u\|_{L^2}^2, \quad \varepsilon > 0, \tag{2-9}$$

and taking advantage of the regularity theory of the Stokes system (recalled in the Appendix) gives

$$\|\nabla^2 u\|_{L^2}^2 + \|\nabla P\|_{L^2}^2 \leq C_\Omega\rho^*\|\sqrt{\rho}\dot{u}\|_{L^2}^2. \tag{2-10}$$

Hence, choosing $\varepsilon > 0$ suitably small in (2-9), using (2-10), then reverting to (2-8) and applying Gronwall's lemma allows us to conclude that there exist positive constants c_Ω and C_Ω, depending only on Ω, such that

$$X_1(t) \leq \|\sqrt{\rho}_0 u_0\|_{L^2}^2 e^{C_1^v(t)}, \quad \text{with } C_1^v(t) := C_\Omega\rho^*\int_0^t \|\sqrt{\rho}v\|_{L^4}^4\,d\tau, \tag{2-11}$$

where

$$X_1(t) := \|(\sqrt{\rho}u)(t)\|_{L^2}^2 + 2\|\sqrt{t}\nabla u(t)\|_{L^2}^2 + \frac{1}{2}\int_0^t \left(\|\sqrt{\rho\tau}\dot{u}\|_{L^2}^2 + \|\sqrt{\rho\tau}u_\tau\|_{L^2}^2 + \frac{c_\Omega}{\rho^*}\|\sqrt{\tau}(\nabla^2 u, \nabla P)\|_{L^2}^2\right)d\tau.$$

Estimates with weight t. Applying ∂_t to (1-6) gives

$$\rho u_{tt}+\rho v\cdot\nabla u_t-\Delta u_t+\nabla P_t=-\rho_t\dot u-\rho v_t\cdot\nabla u. \tag{2-12}$$

As $\operatorname{div} u_t=0$, testing (2-12) by t^2u_t then observing that

$$\rho_t=-\operatorname{div}(\rho v)\quad\text{and}\quad |u_t|^2=|\dot u|^2-2\dot u\cdot(v\cdot\nabla u)+|v\cdot\nabla u|^2$$

gives, after performing a few integration by parts,

$$\frac12\frac{d}{dt}\int_\Omega\rho t^2|u_t|^2\,dx+\int_\Omega t^2|\nabla u_t|^2\,dx=\int_\Omega t\rho|\dot u|^2\,dx-2\int_\Omega\rho t\dot u\cdot(v\cdot\nabla u)\,dx+\int_\Omega t\rho|v\cdot\nabla u|^2\,dx\\+\int_\Omega t^2\operatorname{div}(\rho v)\dot u\cdot u_t\,dx-\int_\Omega t^2\rho(v_t\cdot\nabla u)\cdot u_t\,dx.$$

Adding up twice (2-2) and (2-6) to this latter inequality, we obtain

$$\frac{d}{dt}\int_\Omega\Big(\rho|u|^2+t|\nabla u|^2+\frac{\rho t^2}{2}|u_t|^2\Big)\,dx+\int_\Omega(|\nabla u|^2+\rho t|\dot u|^2+t^2|\nabla u_t|^2)\,dx\\=\int_\Omega\rho t|v\cdot\nabla u|^2\,dx+\int_\Omega t^2\operatorname{div}(\rho v)\dot u\cdot u_t\,dx-\int_\Omega t^2\rho(v_t\cdot\nabla u)\cdot u_t\,dx=:I_1+I_2+I_3. \tag{2-13}$$

Thanks to (2-9), (2-10) and Young's inequality, we have

$$I_1\le\tfrac12\|\sqrt{\rho t}\dot u\|_{L^2}^2+C\rho^*\|\sqrt\rho v\|_{L^4}^4\|\sqrt t\nabla u\|_{L^2}^2. \tag{2-14}$$

For the term I_2, an integration by parts yields

$$I_2=-\int_\Omega t^2(\rho v\cdot\nabla\dot u)\cdot u_t\,dx-\int_\Omega t^2(\rho v\cdot\nabla u_t)\cdot\dot u\,dx=:I_{21}+I_{22}.$$

By (0-7), Hölder's and Young's inequalities, and (2-1), we have, for some constant C depending only on ρ_*, ρ^*, and Ω,

$$I_{21}\le C\|t\nabla\dot u\|_{L^2}\|\sqrt\rho v\|_{L^4}\|tu_t\|_{L^2}^{1/2}\|t\nabla u_t\|_{L^2}^{1/2}\\\le\tfrac1{10}(\|t\nabla u_t\|_{L^2}^2+\|t\nabla\dot u\|_{L^2}^2)+C\|\sqrt\rho v\|_{L^4}^4\|\sqrt\rho tu_t\|_{L^2}^2. \tag{2-15}$$

The same arguments lead to

$$I_{22}\le\tfrac1{10}(\|t\nabla u_t\|_{L^2}^2+\|t\nabla\dot u\|_{L^2}^2)+C\|\sqrt\rho v\|_{L^4}^4\|\sqrt\rho t\dot u\|_{L^2}^2. \tag{2-16}$$

For I_3, one has, still owing to Hölder's and Young's inequalities and (A-1) or (A-2),

$$I_3\le\|\sqrt{\rho t}\,v_t\|_{L^2}\|t\sqrt\rho\,u_t\|_{L^4}\|\sqrt t\nabla u\|_{L^4}\\\le\tfrac1{10}\|t\nabla u_t\|_{L^2}\|\nabla u\|_{L^2}+C\|\sqrt{\rho t}v_t\|_{L^2}^2\|t\sqrt\rho u_t\|_{L^2}\|t\nabla^2u\|_{L^2}. \tag{2-17}$$

Hence, inserting (2-14)–(2-17) in (2-13) gives

$$\frac{d}{dt}\big(\|\sqrt\rho u\|_{L^2}^2+\|\sqrt t\nabla u\|_{L^2}^2+\tfrac12\|\sqrt\rho tu_t\|_{L^2}^2\big)+\tfrac12(\|\nabla u\|_{L^2}^2+\|\sqrt{\rho t}\dot u\|_{L^2}^2+\|t\nabla u_t\|_{L^2}^2)-\tfrac14\|t\nabla\dot u\|_{L^2}^2\\\lesssim\|\sqrt\rho v\|_{L^4}^4(\|\sqrt\rho t(\dot u,u_t)\|_{L^2}^2+\|\sqrt t\nabla u\|_{L^2}^2)+\|\sqrt{\rho t}\,v_t\|_{L^2}^2\|t\sqrt\rho u_t\|_{L^2}\|t\nabla^2u\|_{L^2}. \tag{2-18}$$

To close the estimate, we have to bound $\|\sqrt{\rho}t\dot{u}\|_{L^2}$, $\|t\nabla^2 u\|_{L^2}$, and $\|t\nabla\dot{u}\|_{L^2}$. For the first two terms, one may use (0-7), (2-10) and the definition of $\dot{u}$ to get

$$\begin{aligned}\|t(\nabla^2 u, \nabla P)\|_{L^2} &\le C_\Omega(\sqrt{\rho^*}\|t\sqrt{\rho}u_t\|_{L^2} + \|\rho t^{1/4}v\|_{L^4}\|\sqrt{t}\nabla u\|_{L^2}^{1/2}\|t\nabla^2 u\|_{L^2}^{1/2})\\ &\le \tfrac{1}{2}\|t\nabla^2 u\|_{L^2} + C_\Omega(\sqrt{\rho^*}\|t\sqrt{\rho}u_t\|_{L^2} + \|\rho t^{1/4}v\|_{L^4}^2\|\sqrt{t}\nabla u\|_{L^2}).\end{aligned}$$

This, in the end, implies that

$$\frac{1}{4}\|\sqrt{\rho}t\dot{u}\|_{L^2} + \frac{c_\Omega}{\sqrt{\rho^*}}\|t\nabla^2 u, t\nabla P\|_{L^2} \le C(\|t\sqrt{\rho}u_t\|_{L^2} + \|t^{1/4}v\|_{L^4}^2\|\sqrt{t}\nabla u\|_{L^2}). \tag{2-19}$$

Finally, from the definition of $\dot{u}$, Hölder's inequality and (0-7), we may write

$$\begin{aligned}\|t\nabla\dot{u}\|_{L^2} &\le \|t\nabla u_t\|_{L^2} + \|t\nabla v\cdot\nabla u\|_{L^2} + \|tv\cdot\nabla^2 u\|_{L^2}\\ &\le \|t\nabla u_t\|_{L^2} + \|\sqrt{t}\nabla v\|_{L^4}\|\nabla u\|_{L^2}^{1/2}\|t\nabla^2 u\|_{L^2}^{1/2} + C\|v\|_{L^4}\|t\dot{u}\|_{L^2}^{1/2}\|t\nabla\dot{u}\|_{L^2}^{1/2},\end{aligned}$$

which implies that

$$\|t\nabla\dot{u}\|_{L^2} \le 2\|t\nabla u_t\|_{L^2} + \tfrac{1}{4}\|\nabla u\|_{L^2} + C(\|\sqrt{t}\nabla v\|_{L^4}^2\|t\nabla^2 u\|_{L^2} + \|v\|_{L^4}^2\|\sqrt{\rho}t\dot{u}\|_{L^2}). \tag{2-20}$$

Let us set

$$\begin{aligned}X_2(t) := \|(\sqrt{\rho}u)(t)\|_{L^2}^2 + \|\sqrt{t}\nabla u(t)\|_{L^2}^2 + \frac{1}{4}\|\sqrt{\rho}tu_t\|_{L^2}^2 + \frac{1}{16}\|\sqrt{\rho}t\dot{u}\|_{L^2}^2 + \frac{c_\Omega}{\rho^*}\|t(\nabla^2 u, \nabla P)\|_{L^2}^2&\\ + \frac{1}{16}\int_0^t (\|\nabla u\|_{L^2}^2 + \|\sqrt{\rho}\tau\dot{u}\|_{L^2}^2 + \|\tau\nabla u_\tau\|_{L^2}^2 + \|\tau\nabla\dot{u}\|_{L^2}^2)\,d\tau.&\end{aligned}$$

Integrating (2-18) on $[0, t]$, taking advantage of (2-19) and (2-20), and then, finally, using Gronwall's lemma, we conclude that there exists a constant C depending only on Ω, ρ_*, and ρ^* such that

$$\begin{aligned}&X_2(t) \le \|u_0\|_{L^2}^2 e^{C_2^v(t)},\\ &\quad\text{with } C_2^v(t) := C\biggl(\sup_{\tau\in[0,t]}\|\tau^{1/4}v(\tau)\|_{L^4}^4 + \int_0^t (\|\sqrt{\rho}v\|_{L^4}^4 + \|\sqrt{\tau}\nabla v\|_{L^4}^4 + \|\sqrt{\rho\tau}v_\tau\|_{L^2}^2)\,d\tau\biggr).\end{aligned} \tag{2-21}$$

Estimates with weight $t^{3/2}$. Let $D_t := \partial_t + v\cdot\nabla$ and $\ddot{u} := D_t\dot{u}$. We have[4]

$$\rho\ddot{u} - \Delta\dot{u} + \nabla\dot{P} = F := \nabla v\cdot\nabla P - \Delta v\cdot\nabla u - 2\nabla^2 u\cdot\nabla v. \tag{2-22}$$

Taking the $L^2(\Omega; \mathbb{R}^2)$ scalar product with $t^3\ddot{u}$, we readily get

$$\frac{1}{2}\frac{d}{dt}\|t^{3/2}\nabla\dot{u}(t)\|_{L^2}^2 + \|t^{3/2}\sqrt{\rho}\ddot{u}\|_{L^2}^2 = \frac{3}{2}\|t\nabla\dot{u}\|_{L^2}^2 + \sum_{i=1}^5 J_i, \tag{2-23}$$

with

$$J_1 := \int_\Omega \Delta\dot{u}\cdot(t^3 v\cdot\nabla\dot{u})\,dx, \quad J_2 := -\int_\Omega \nabla\dot{P}\cdot(t^3 v\cdot(\nabla v\cdot\nabla u))\,dx,$$

$$J_3 := \int_\Omega \nabla\dot{P}\cdot(t^3 v_t\cdot\nabla u)\,dx, \quad J_4 := \int_\Omega \nabla\dot{P}\cdot(t^3 v\cdot(v\cdot\nabla^2 u))\,dx, \quad J_5 := \int_\Omega F\cdot t^3\ddot{u}\,dx.$$

[4]Here we use the notation $(\nabla^2 u\cdot\nabla v)^i := \sum_{1\le j,k\le d}\partial_k v^j\partial_j\partial_k u^i$.

For any $\varepsilon > 0$, the terms J_1 through J_5 may be bounded as follows by combining Hölder's inequality, Young's inequality, and (A-1) with $p = 4$ or $p = 6$ (and (A-4) for J_4):

$$\begin{aligned}
J_1 &\le \|t^{3/2}\nabla^2\dot u\|_{L^2}\|v\|_{L^4}\|t^{3/2}\nabla\dot u\|_{L^4}\\
&\le \varepsilon\|t^{3/2}\nabla^2\dot u\|_{L^2}^2 + C_\varepsilon\|v\|_{L^4}^4\|t^{3/2}\nabla\dot u\|_{L^2}^2,\\
J_2 &\le \|t^{3/2}\nabla\dot P\|_{L^2}\|t^{1/6}v\|_{L^6}\|\sqrt{t}\nabla v\|_{L^6}\|t^{5/6}\nabla u\|_{L^6}\\
&\le C\|t^{3/2}\nabla\dot P\|_{L^2}\|t^{1/6}v\|_{L^6}\|\sqrt{t}\nabla v\|_{L^6}\|\sqrt{t}\nabla u\|_{L^2}^{1/3}\|t\nabla^2 u\|_{L^6}^{2/3}\\
&\le \varepsilon\|t^{3/2}\nabla\dot P\|_{L^2}^2 + C_\varepsilon\|t^{1/6}v\|_{L^6}^2\|\sqrt{t}\nabla v\|_{L^6}^2\|\sqrt{t}\nabla u\|_{L^2}^{2/3}\|t\nabla^2 u\|_{L^2}^{4/3},\\
J_3 &\le \|t^{3/2}\nabla\dot P\|_{L^2}\|tv_t\|_{L^4}\|t^{1/2}\nabla u\|_{L^4}\\
&\le \varepsilon\|t^{3/2}\nabla\dot P\|_{L^2}^2 + C_\varepsilon\|tv_t\|_{L^4}^4\|t^{1/2}\nabla u\|_{L^2}^2 + \|t^{1/2}\nabla^2 u\|_{L^2}^2,\\
J_4 &\le \|t^{3/2}\nabla\dot P\|_{L^2}\|t^{1/6}v\|_{L^6}^2\|t^{7/6}\nabla^2 u\|_{L^6}\\
&\le C\|t^{3/2}\nabla\dot P\|_{L^2}\|t^{1/6}v\|_{L^6}^2\|\sqrt{\rho t}\dot u\|_{L^2}^{1/3}\|t^{3/2}\nabla\dot u\|_{L^2}^{2/3}\\
&\le \varepsilon\|t^{3/2}\nabla\dot P\|_{L^2}^2 + C_\varepsilon\|\sqrt{\rho t}\dot u\|_{L^2}^2 + C_\varepsilon\|t^{1/6}v\|_{L^6}^6\|t^{3/2}\nabla\dot u\|_{L^2}^2,\\
J_5 &\le \varepsilon\|t^{3/2}\sqrt{\rho}\ddot u\|_{L^2}^2 + \frac{C_\varepsilon}{\rho^*}\|t^{3/2}F\|_{L^2}^2.
\end{aligned}$$

Thanks to Hölder's inequality, (0-7), and (A-4), we have

$$\begin{aligned}
\|t^{3/2}F\|_{L^2}^2 &\le \|\sqrt{t}\nabla v\|_{L^4}^2\|t(\nabla P,\nabla^2 u)\|_{L^4}^2 + \|t\nabla^2 v\|_{L^4}^2\|\sqrt{t}\nabla u\|_{L^4}^2\\
&\lesssim \|\sqrt{t}\nabla v\|_{L^4}^2\|\sqrt{\rho t}\dot u\|_{L^2}\|t^{3/2}\nabla\dot u\|_{L^2} + \|t\nabla^2 v\|_{L^4}^2\|\sqrt{t}\nabla u\|_{L^2}\|\sqrt{t}\nabla^2 u\|_{L^2}\\
&\lesssim \|\sqrt{\rho t}\dot u\|_{L^2}^2 + \|\sqrt{t}\nabla^2 u\|_{L^2}^2 + \|\sqrt{t}\nabla v\|_{L^4}^4\|t^{3/2}\nabla\dot u\|_{L^2}^2 + \|t\nabla^2 v\|_{L^4}^4\|\sqrt{t}\nabla u\|_{L^2}^2.
\end{aligned}$$

To close the estimates, we need to bound

$$t^{3/2}\nabla\dot P \quad\text{and}\quad t^{3/2}\nabla^2\dot u \quad\text{in } L^2(\mathbb{R}_+\times\Omega).$$

Now, we observe that the couple $(\dot u, \nabla\dot P)$ satisfies the inhomogeneous Stokes system

$$-\Delta\dot u + \nabla\dot P = F - \rho\ddot u \quad\text{and}\quad \operatorname{div}\dot u = \operatorname{Tr}(\nabla v\cdot\nabla u) \quad\text{in } \Omega, \tag{2-24}$$

with boundary condition $\dot u|_{\partial\Omega} = 0$ if Ω is a bounded domain, $\dot u(t) \to 0$ at infinity (due to $\dot u(t) \in L^2$ for all $t > 0$) in the case $\Omega = \mathbb{R}^2$, and

$$\int_{\mathbb{T}^2}\rho\dot u\,dx = 0 \quad\text{if } \Omega = \mathbb{T}^2.$$

Hence, applying (A-4) with $p = 2$ guarantees that

$$\|\nabla^2\dot u, \nabla\dot P\|_{L^2}^2 \lesssim \|F\|_{L^2}^2 + \|\rho\ddot u\|_{L^2}^2 + \|\nabla^2 v\otimes\nabla u\|_{L^2}^2 + \|\nabla v\otimes\nabla^2 u\|_{L^2}^2. \tag{2-25}$$

The last two terms are parts of F. Hence bounding $\|t^{3/2}F\|_{L^2}$ as above and putting this together with the previous inequalities, we conclude after time integration that

$$\begin{aligned}
X_3(t) &:= \|t^{3/2}\nabla\dot u(t)\|_{L^2}^2 + \int_0^t \|\tau^{3/2}(\sqrt{\rho}\ddot u, \nabla\dot P, \nabla^2\dot u)\|_{L^2}^2\,d\tau \\
&\lesssim \int_0^t (\|v\|_{L^4}^4 + \|\tau^{1/6}v\|_{L^6}^6 + \|\tau^{1/2}\nabla v\|_{L^4}^4)\|\tau^{3/2}\nabla\dot u\|_{L^2}^2\,d\tau \\
&\qquad + \int_0^t \|\tau^{1/2}\nabla^2 u, \sqrt{\rho\tau}\dot u\|_{L^2}^2\,d\tau + \int_0^t (\|\tau v_\tau\|_{L^4}^4 + \|\tau\nabla^2 v\|_{L^4}^4)\|\tau^{1/2}\nabla u\|_{L^2}^2\,d\tau \\
&\qquad + \int_0^t \|\tau^{1/6}v\|_{L^6}^2\|\sqrt{\tau}\nabla v\|_{L^6}^2\|\sqrt{\tau}\nabla u\|_{L^2}^{2/3}\|\tau\nabla^2 u\|_{L^2}^{4/3}\,d\tau.
\end{aligned}$$

After using Gronwall's lemma and the inequalities of the previous steps, we get

$$\begin{aligned}
&X_3(t) \le C\|u_0\|_{L^2}^2 e^{C_3^v(t)}, \\
&\quad\text{with } C_3^v(t) := C\int_0^t \big(\|v\|_{L^4}^4 + (1+\|\tau^{1/4}v\|_{L^4}^4)\|v\|_{L^6}^3 + \|\tau^{1/6}v\|_{L^6}^6 + \|\sqrt{\tau}\nabla v\|_{L^6}^3 \\
&\qquad\qquad + \|\tau^{1/2}v_\tau\|_{L^2}^2 + \|\tau^{1/2}\nabla v\|_{L^4}^4 + \|\tau\nabla^2 v\|_{L^4}^4 + \|\tau v_\tau\|_{L^4}^4\big)\,d\tau. \qquad (2\text{-}26)
\end{aligned}$$

2.2. *The proof of Theorem 1.1.* Let us fix some data (ρ_0, u_0) such that $u_0 \in L^2$ and $0 < \rho_* \le \rho_0 \le \rho^* < \infty$. Then we smooth out the velocity so as to get a sequence $(u_0^n)_{n\in\mathbb{N}}$ of H^1 divergence-free vector fields (vanishing at $\partial\Omega$ in the bounded domain case) that converges strongly to u_0 in L^2. It is known (see [Danchin and Mucha 2019] for the bounded domain or torus cases and [Paicu et al. 2013] for the $\mathbb{R}^2$ case) that such data generate a unique global solution $(\rho^n, u^n, \nabla P^n)$ with relatively smooth velocity. In particular, the computations leading to the estimates of the previous subsection may be justified for $\rho = \rho^n$, $u = v = u^n$, and we get, for all $t \ge 0$ for some constant depending only on ρ_*, ρ^*, and Ω,

$$X_0^n(t) := \|(\sqrt{\rho^n}u^n)(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u^n\|_{L^2}^2\,d\tau \le \|\sqrt{\rho_0}u_0^n\|_{L^2}^2, \qquad (2\text{-}27)$$

$$X_1^n(t) \le \|\sqrt{\rho_0}u_0^n\|_{L^2}^2 e^{C_1^n(t)}, \quad \text{with } C_1^n(t) := C\int_0^t \|u^n\|_{L^4}^4\,d\tau, \qquad (2\text{-}28)$$

$$\begin{aligned}
&X_2^n(t) \le \|\sqrt{\rho_0}u_0^n\|_{L^2}^2 e^{C_2^n(t)}, \\
&\quad\text{with } C_2^n(t) := C\Big(\sup_{\tau\in[0,t]}\|\tau^{1/4}u^n(\tau)\|_{L^4}^4 + \int_0^t (\|u^n\|_{L^4}^4 + \|\sqrt{\tau}\nabla u^n\|_{L^4}^4 + \|\sqrt{\tau}u_\tau^n\|_{L^2}^2)\,d\tau\Big), \qquad (2\text{-}29)
\end{aligned}$$

$$\begin{aligned}
&X_3^n(t) \le C\|u_0^n\|_{L^2}^2 e^{C_3^n(t)}, \\
&\quad\text{with } C_3^n(t) := C\int_0^t \big((1+\|\tau^{1/4}u^n\|_{L^4}^4)\|u^n\|_{L^6}^3 + \|\tau^{1/6}u^n\|_{L^6}^6 + \|\sqrt{\tau}\nabla v^n\|_{L^6}^3 \\
&\qquad\qquad + \|\tau^{1/2}v_\tau^n\|_{L^2}^2 + \|u^n, \tau^{1/2}\nabla u^n, \tau\nabla^2 u^n, \tau u_\tau^n\|_{L^4}^4\big)\,d\tau. \qquad (2\text{-}30)
\end{aligned}$$

Above, X_j^n for $j \in \{1, 2, 3\}$ are the quantities defined in (2-11), (2-21), and (2-26), respectively, pertaining to $(\rho^n, u^n, \nabla P^n)$.

The fundamental point is that all the norms coming into play in C_1^n, C_2^n, and C_3^n may be bounded by means of $M := \sup_{n\in\mathbb{N}} \|u_0^n\|_{L^2}$, ρ_*, and ρ^*. For C_1^n, this just stems from (2-4) with $p=4$. Hence we have, for some $C_M := C(\rho_*, \rho^*, M)$,

$$\sup_{t\in\mathbb{R}_+} X_1^n(t) \le C_M.$$

Combining with (0-7) and (2-27), we thus get

$$\sup_{t\in\mathbb{R}_+} \|t^{1/4}u^n(t)\|_{L^4}^4 \lesssim \|u^n\|_{L^\infty(L^2)}^2 \|\sqrt{t}\nabla u^n\|_{L^\infty(L^2)}^2 \lesssim M^2 C_M, \tag{2-31}$$

$$\|\sqrt{t}\nabla u^n\|_{L^4(L^4)}^4 \lesssim \|\sqrt{t}\nabla u^n\|_{L^\infty(L^2)}^2 \|\sqrt{t}\nabla^2 u^n\|_{L^2(L^2)}^2 \lesssim C_M^2, \tag{2-32}$$

$$\|\sqrt{\rho t}\,u_t^n\|_{L^2(L^2)}^2 \lesssim C_M; \tag{2-33}$$

whence, remembering (2-29), we have, up to a change of C_M,

$$X_2^n(t) \le C_M \quad \text{for all } t \ge 0.$$

Finally, one has to bound the terms of C_3^n independently of n. Let us just treat the third term as an example. We write that, owing to (A-1) with $p=6$,

$$\begin{aligned}\int_0^\infty \|t^{1/6}u^n\|_{L^6}^6\,dt &\lesssim \int_0^\infty \|u^n\|_{L^2}^2 \|\sqrt{t}\nabla u^n\|_{L^2}^2 \|\nabla u^n\|_{L^2}^2\,dt \\ &\le \|u^n\|_{L^\infty(L^2)}^2 \|\sqrt{t}\nabla u^n\|_{L^\infty(L^2)}^2 \|\nabla u^n\|_{L^2(L^2)}^2 \lesssim M^4 C_M.\end{aligned}$$

As a conclusion, we deduce that there exists a constant, still denoted by C_M, such that, for all $n \in \mathbb{N}$, we have

$$\sup_{t\in\mathbb{R}_+} (X_0^n(t) + X_1^n(t) + X_2^n(t) + X_3^n(t)) \le C_M.$$

Regarding the density, the divergence-free property of u^n clearly ensures that,

$$\text{for all } n \in \mathbb{N}, \quad \text{for all } t \in \mathbb{R}_+, \qquad \rho_* \le \rho^n(t) \le \rho^*.$$

At this point, arguing as in the classical proofs of global existence of weak solutions for (INS) (see, e.g., [Boyer and Fabrie 2013; Lions 1996]), one can conclude that $(\rho^n, u^n, \nabla P^n)_{n\in\mathbb{N}}$ converges weakly, up to a subsequence, to a global distributional solution of (INS) satisfying not only (2-1) and the usual energy inequality (0-3), but also

$$\sup_{t\in\mathbb{R}_+} (X_1(t) + X_2(t) + X_3(t)) \le C_{\rho_*, \rho^*, \|u_0\|_{L^2}}.$$

3. More decay estimates

The goal of this section is to prove that the solutions to the linearized momentum equation (1-6), with ρ satisfying (2-1) and v verifying the regularity properties listed in Theorem 1.1 supplemented with divergence-free u_0 in $\widetilde{B}^0_{\rho_0,1}$, satisfy (0-8). Achieving this requires several steps. The cornerstones are estimates in $\dot{H}^1$ and $\dot{H}^{-1}$ for the solution to (1-6) (in addition to the estimates that have been proved hitherto) and the interpolation method that has been described in Section 1.

3.1. ***A priori estimates involving $\dot H^1$ regularity of u_0.*** In this section, we consider system (1-6) with some source term g. Our aim is to prove estimates of u in $\dot H^1$ in terms of $\nabla u_0 \in L^2$ and g in $L^2(L^2)$. Considering here a source term will be needed when proving estimates in $\dot H^{-1}$ by means of a duality method.

Basic estimates in $\dot H^1$. Let $f := g/\rho$. Taking the L^2 scalar product of the first line of (1-6) with u_t yields, after integrating by parts in the term with Δu,

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \|\sqrt{\rho}\,u_t\|_{L^2}^2 = \int_\Omega \sqrt{\rho}(f - v\cdot\nabla u)\cdot(\sqrt{\rho}u_t)\,dx. \tag{3-1}$$

By virtue of Young's and Hölder's inequality, we have

$$\int_\Omega \sqrt{\rho}(f - v\cdot\nabla u)\cdot(\sqrt{\rho}u_t)\,dx \le \frac{1}{2}\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\sqrt{\rho}f\|_{L^2}^2 + \|\sqrt{\rho}v\cdot\nabla u\|_{L^2}^2.$$

Since $\dot u = u_t + v\cdot\nabla u$, we may write

$$\|\sqrt{\rho}\dot u\|_{L^2} \le \|\sqrt{\rho}u_t\|_{L^2} + \|\sqrt{\rho}v\cdot\nabla u\|_{L^2}.$$

Remembering (2-9), this yields, for some constant c_Ω depending only on Ω,

$$\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \frac{1}{4}\|\sqrt{\rho}(u_t,\dot u)\|_{L^2}^2 + \frac{c_\Omega}{\rho^*}\|\nabla^2 u, \nabla P\|_{L^2}^2 \le 4\|\sqrt{\rho}f\|_{L^2}^2. \tag{3-2}$$

In the end, combining with Gronwall's lemma and remembering that $f = g/\rho$, we get

$$\begin{aligned}\|\nabla u(t)\|_{L^2}^2 &+ \frac{1}{4}\int_0^t \|\sqrt{\rho}(u_t,\dot u)\|_{L^2}^2\,d\tau + \frac{c_\Omega}{\rho^*}\int_0^t \|\nabla^2 u, \nabla P\|_{L^2}^2\,d\tau \\ &\le e^{C\rho^*\int_0^t \|\sqrt{\rho}v\|_{L^4}^4\,d\tau}\left(\|\nabla u_0\|_{L^2}^2 + 4\int_0^t e^{-C\rho^*\int_0^\tau \|\sqrt{\rho}v\|_{L^4}^4\,d\tau'}\left\|\frac{g}{\sqrt{\rho}}\right\|_{L^2}^2 d\tau\right).\end{aligned} \tag{3-3}$$

Decay estimates with weight $\sqrt{t}$. Assuming in the rest of this section that $g \equiv 0$, we proceed as when proving (2-21) except that we take the L^2 scalar product of (2-12) with tu_t instead of t^2u_t. In this way, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big(\|\sqrt{\rho t}u_t\|_{L^2}^2 &+ \frac{1}{2}\|\nabla u\|_{L^2}^2\Big) + \|\sqrt{t}\nabla u_t\|_{L^2}^2 \\ &= \int_\Omega t\,\mathrm{div}(\rho v)\dot u\cdot u_t\,dx - \int_\Omega t\rho(v_t\cdot\nabla u)\cdot u_t\,dx - \int_\Omega \rho(v\cdot\nabla u)\cdot u_t\,dx.\end{aligned} \tag{3-4}$$

Combining (A-1), Young's inequality, and (2-9) gives

$$-2\int_\Omega \rho(v\cdot\nabla u)\cdot u_t\,dx \le \frac{1}{2}\|\sqrt{\rho}u_t\|_{L^2}^2 + \frac{c_\Omega}{\rho^*}\|\nabla^2 u\|_{L^2}^2 + C\rho^*\|\sqrt{\rho}v\|_{L^4}^4\|\nabla u\|_{L^2}^2.$$

Hence, adding half (3-2) to (3-4) yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\sqrt{\rho t}u_t\|_{L^2}^2 &+ \|\nabla u\|_{L^2}^2) + \|\sqrt{t}\nabla u_t\|_{L^2}^2 + \frac{1}{6}\|\sqrt{\rho}(u_t,\dot u)\|_{L^2}^2 + c_\Omega\|\nabla^2 u, \nabla P\|_{L^2}^2 \\ &\le C\|\sqrt{\rho}v\|_{L^4}^4\|\nabla u\|_{L^2}^2 + \int_\Omega t\,\mathrm{div}(\rho v)\dot u\cdot u_t\,dx - \int_\Omega t\rho(v_t\cdot\nabla u)\cdot u_t\,dx.\end{aligned} \tag{3-5}$$

We integrate by parts in the second term of the right-hand side, which gives

$$\int_\Omega t\,\mathrm{div}(\rho v)\dot u\cdot u_t\,dx = -\int_\Omega t(\rho v\cdot\nabla\dot u)\cdot u_t\,dx - \int_\Omega t(\rho v\cdot\nabla u_t)\cdot\dot u\,dx.$$

The two integrals may be handled as when proving (2-21). We get

$$\int_\Omega t\,\mathrm{div}(\rho v)\dot u\cdot u_t\,dx \le \frac14\|\sqrt t(\nabla\dot u,\nabla u_t)\|_{L^2}^2 + C\|\sqrt\rho v\|_{L^4}^4\|\sqrt{\rho t}(\dot u,u_t)\|_{L^2}^2.$$

To bound the last term of (3-5), we proceed as follows (for all $\varepsilon>0$):

$$\begin{aligned}\int_\Omega t\rho(v_t\cdot\nabla u)\cdot u_t\,dx &\le \|\sqrt{\rho t}v_t\|_{L^2}\|\sqrt{\rho t}u_t\|_{L^4}\|\nabla u\|_{L^4}\\ &\le \varepsilon\|\nabla^2u\|_{L^2}^2+\varepsilon\|\sqrt t\nabla u_t\|_{L^2}^2 + C_\varepsilon\|\sqrt{\rho t}v_t\|_{L^2}^2\|\sqrt{\rho t}u_t\|_{L^2}\|\nabla u\|_{L^2}.\end{aligned}$$

From the definition of $\dot u$ and (2-10), it is easy to get

$$\|\sqrt t(\nabla^2 u,\nabla P,\sqrt\rho\dot u)\|_{L^2}\le C(\|\sqrt{\rho t}u_t\|_{L^2}+\|\sqrt\rho v\|_{L^4}^2\|\sqrt t\nabla u\|_{L^2}). \tag{3-6}$$

By Hölder's inequality, (A-1), and (A-4) with $p=4$, we also notice that

$$\|\sqrt t\nabla\dot u\|_{L^2}-\|\sqrt t\nabla u_t\|_{L^2}\lesssim\|\sqrt t\nabla v\|_{L^4}\|\nabla u\|_{L^2}^{1/2}\|\nabla^2u\|_{L^2}^{1/2}+\|v\|_{L^4}\|\sqrt{\rho t}\dot u\|_{L^2}^{1/2}\|\sqrt t\nabla\dot u\|_{L^2}^{1/2}$$

which implies that

$$\|\sqrt t\nabla\dot u\|_{L^2}\le 2\|\sqrt t\nabla u_t\|_{L^2}+\tfrac14\|\nabla^2u\|_{L^2}+C(\|\sqrt t\nabla v\|_{L^4}^2\|\nabla u\|_{L^2}+\|v\|_{L^4}^2\|\sqrt{\rho t}\dot u\|_{L^2}).$$

Inserting all the above inequalities in (3-5) then using Gronwall's lemma and (2-11), we discover that

$$Y_1(t)\lesssim\|\nabla u_0\|_{L^2}^2e^{\widetilde C_1^v(t)},\quad \text{with } \widetilde C_1^v(t):=C\int_0^t(\|\sqrt\tau\nabla v,v\|_{L^4}^4+\|\sqrt{\rho\tau}v_\tau\|_{L^2}^2)\,d\tau, \tag{3-7}$$

where

$$\begin{aligned}Y_1(t):=\|\sqrt{\rho t}(u_t,\dot u)\|_{L^2}^2+\|\nabla u\|_{L^2}^2+\frac{c_\Omega}{\rho^*}\|\sqrt t(\nabla^2u,\nabla P)\|_{L^2}^2\\ +\int_0^t\left(\|\sqrt\tau(\nabla u_\tau,\nabla\dot u)\|_{L^2}^2+\|\sqrt\rho(u_\tau,\dot u)\|_{L^2}^2+\frac{c_\Omega}{\rho^*}\|\nabla^2u,\nabla P\|_{L^2}^2\right)d\tau.\end{aligned}$$

Decay estimates with weight t. Still assuming $f\equiv 0$, we now take the L^2 scalar product of (2-22) with $tD_t(t\dot u)$ and get

$$\frac12\frac{d}{dt}\|\nabla(t\dot u)\|_{L^2}^2+\|\sqrt\rho D_t(t\dot u)\|_{L^2}^2=\int_\Omega(tF-t\nabla\dot P+\rho\dot u)\cdot D_t(t\dot u)\,dx+\int_\Omega\Delta(t\dot u)\cdot(v\cdot\nabla(t\dot u))\,dx.$$

Hence, for all $\varepsilon>0$,

$$\begin{aligned}\frac12\frac{d}{dt}&\|\nabla(t\dot u(t))\|_{L^2}^2+\|\sqrt\rho D_t(t\dot u)\|_{L^2}^2\\ &\le\varepsilon(\|\nabla^2(t\dot u)\|_{L^2}^2+\|\sqrt\rho D_t(t\dot u)\|_{L^2}^2)+\frac1\varepsilon\left(\|v\cdot\nabla(t\dot u)\|_{L^2}^2+\|\sqrt\rho\dot u\|_{L^2}^2+\left\|\frac{tF-t\dot\nabla P}{\sqrt\rho}\right\|_{L^2}^2\right).\end{aligned}\tag{3-8}$$

To continue, we must estimate $t\dot P$ and $t\nabla^2\dot u$. To this end, we recall inequality (2-25) and observe that

$$\|\sqrt\rho t\ddot u\|_{L^2}\le\|\sqrt\rho D_t(t\dot u)\|_{L^2}+\|\sqrt\rho\dot u\|_{L^2}.$$

Hence, taking ε small enough in (3-8) yields

$$\|\nabla(t\dot{u}(t))\|_{L^2}^2+\|\sqrt{\rho}D_t(t\dot{u}),\nabla(t\dot{P}),\nabla^2(t\dot{u})\|_{L^2}^2 \lesssim \|\sqrt{\rho}\dot{u}\|_{L^2}^2+\|v\cdot\nabla(t\dot{u})\|_{L^2}^2+\|t\nabla^2 v\otimes\nabla u\|_{L^2}^2+\|t\nabla^2 u\otimes\nabla v\|_{L^2}^2+\|t\nabla v\cdot\nabla P\|_{L^2}^2. \tag{3-9}$$

We can bound the first term of the right-hand side according to (3-3). To bound the other terms, we have

$$\|v\cdot\nabla(t\dot{u})\|_{L^2}^2\le\frac{C}{\varepsilon}\|v\|_{L^4}^4\|\nabla(t\dot{u})\|_{L^2}^2+\varepsilon\|\nabla^2(t\dot{u})\|_{L^2}^2,$$

$$\|t\nabla^2 v\otimes\nabla u\|_{L^2}^2\lesssim\|t\nabla^2 v\|_{L^4}^2(\|\nabla u\|_{L^2}^2\|\nabla^2 u\|_{L^2}^2)^{1/2},$$

$$\|t\nabla^2 u\otimes\nabla v\|_{L^2}^2+\|t\nabla v\cdot\nabla P\|_{L^2}^2\lesssim\|\sqrt{t}(\nabla^2 u,\nabla P)\|_{L^4}^2\|\sqrt{t}\nabla v\|_{L^4}^2.$$

Using regularity estimates for (2-5) and (0-7) yields

$$\|\sqrt{t}(\nabla^2 u,\nabla P)\|_{L^4}^2\lesssim\|\sqrt{t}\dot{u}\|_{L^4}^2\lesssim\|\dot{u}\|_{L^2}\|t\nabla\dot{u}\|_{L^2}.$$

Hence

$$\|t\nabla^2 u\otimes\nabla v\|_{L^2}^2+\|t\nabla v\cdot\nabla P\|_{L^2}^2\lesssim\|\sqrt{t}\nabla v\|_{L^4}^2\|\dot{u}\|_{L^2}\|t\nabla\dot{u}\|_{L^2}\lesssim\|\dot{u}\|_{L^2}^2+\|\sqrt{t}\nabla v\|_{L^4}^4\|t\nabla\dot{u}\|_{L^2}^2.$$

Plugging all these inequalities in (3-8), using (3-3), and integrating on $[0,t]$ gives

$$Y_2(t):=\|\nabla(t\dot{u}(t))\|_{L^2}^2+\int_0^t\|\sqrt{\rho}D_\tau(\tau\dot{u}),\nabla(\tau\dot{P}),\nabla^2(\tau\dot{u})\|_{L^2}^2\,d\tau \lesssim\int_0^t(\|v\|_{L^4}^4+\|\sqrt{\tau}\nabla v\|_{L^4}^4)\|\tau\nabla\dot{u}\|_{L^2}^2\,d\tau+\|\nabla u_0\|_{L^2}^2 e^{C\int_0^t\|v\|_{L^4}^4\,d\tau}(1+\|\tau\nabla^2 v\|_{L_t^4(L^4)}^4).$$

At this stage, Gronwall's lemma enables us to conclude

$$Y_2(t)\le C\|\nabla u_0\|_{L^2}^2 e^{\widetilde{C}_2^v(t)},\quad\text{with }\widetilde{C}_2^v(t):=C\int_0^t\|v,\sqrt{\tau}\nabla v,\tau\nabla^2 v\|_{L^4}^4\,d\tau. \tag{3-10}$$

Estimates in $\dot{H}^s$ for $s\in(0,1)$. If we denote by E the linear operator that associates to (u_0,g) the solution u to (1-6) on $\mathbb{R}_+\times\Omega$, then the previous inequalities (2-3) and (3-3) and the fact that the norms in $L^2(\rho\,dx)$ or $L^2(dx)$ are equivalent (recall (0-4)) ensure that

- E maps $L^2(\Omega)\times L^2(\mathbb{R}_+;\dot{H}^{-1}(\Omega))$ to $L^\infty(\mathbb{R}_+;L^2(\Omega))\cap L^2(\mathbb{R}_+;\dot{H}^1(\Omega))$,
- E maps $\dot{H}^1(\Omega)\times L^2(\mathbb{R}_+;L^2(\Omega))$ to $L^\infty(\mathbb{R}_+;\dot{H}^1(\Omega))\cap L^2(\mathbb{R}_+;\dot{H}^2(\Omega))$.

Consequently, the complex interpolation theory ensures that, for all $s\in[0,1]$,

$$E:\dot{H}^s(\Omega)\times L^2(\mathbb{R}_+;\dot{H}^{s-1}(\Omega))\to L^\infty(\mathbb{R}_+;\dot{H}^s(\Omega))\cap L^2(\mathbb{R}_+;\dot{H}^{s+1}(\Omega)),$$

with, for some constant C_ρ depending only on ρ_* and ρ^*, we have the bound

$$\sup_{t\in[0,T]}\|u(t)\|_{\dot{H}^s}^2+\int_0^T\|u\|_{\dot{H}^{s+1}}^2\,dt\le C_\rho e^{Cs\rho^*\int_0^T\|\sqrt{\rho}v\|_{L^4}^4\,dt}\biggl(\|u_0\|_{\dot{H}^s}^2+\int_0^T\|g\|_{\dot{H}^{s-1}}^2\,dt\biggr). \tag{3-11}$$

For $g\equiv 0$, due to (2-21) and (3-10), for all $t>0$, the linear operator that associates to u_0 the function $t\dot{u}(t)$ — with u being the solution to (1-6) with no source term — maps L^2 to L^2 and $\dot{H}^1$ to $\dot{H}^1$. Hence it maps $\dot{H}^s$ to $\dot{H}^s$ for all $s\in[0,1]$, and we have

$$\|t\dot{u}(t)\|_{\dot{H}^s}\le Ce^{(s/2)\widetilde{C}_2^v(t)}\|u_0\|_{\dot{H}^s}\quad\text{for all }t>0. \tag{3-12}$$

3.2. ***Estimates in negative Sobolev spaces.*** We here prove estimates for (1-6) in the case of initial data in Sobolev space with negative regularity.

Data in $\dot H^{-1}$. To estimate $\sqrt{\rho}u$ in $L^2(0,T\times\Omega)$, we consider the *backward* parabolic system

$$\begin{cases}\rho w_t+\rho v\cdot\nabla w+\Delta w+\nabla Q=\rho u,\\ \operatorname{div} w=0,\\ w|_{t=T}=0.\end{cases}\tag{3-13}$$

By definition of w, we have

$$\int_0^T\!\!\int_\Omega u\cdot(\rho u)\,dx\,dt=\int_0^T\!\!\int_\Omega u\cdot(\rho w_t+\rho v\cdot\nabla w+\Delta w+\nabla Q)\,dx\,dt.$$

Integrating by parts and remembering that $\partial_t\rho+\operatorname{div}(\rho v)=0$ and $\operatorname{div} w=0$ yields

$$\int_0^T\!\!\int_\Omega \rho|u|^2\,dx\,dt=-\int_0^T\!\!\int_\Omega(\rho\dot u-\Delta u+\nabla P)\cdot w\,dx\,dt+\int_\Omega((\rho u)(T)\cdot w(T)-\rho_0u_0\cdot w(0))\,dx.$$

As $w(T)=0$ and u satisfies (1-6), we conclude that

$$\int_0^T\!\!\int_\Omega \rho|u|^2\,dx\,dt=-\int_\Omega\rho_0u_0\cdot w(0)\,dx\le\|\rho_0u_0\|_{\dot H^{-1}}\|\nabla w(0)\|_{L^2}.$$

Adapting the proof of (3-3) to (3-13) yields

$$\|\nabla w(0)\|_{L^2}^2\le e^{\rho^*\int_0^T\|\sqrt{\rho}\,v\|_{L^4}^4\,dt}\|\sqrt{\rho}u\|_{L^2(0,T\times\Omega)}^2.$$

Hence we have

$$\|\sqrt{\rho}u\|_{L^2(0,T\times\Omega)}\le\|\rho_0u_0\|_{\dot H^{-1}}e^{(\rho^*/2)\int_0^T\|\sqrt{\rho}v\|_{L^4}^4\,dt}.\tag{3-14}$$

In order to bound $\mathcal{P}(\rho u)(T)$ in $\dot H^{-1}$, we start from

$$\|\mathcal{P}(\rho u)(T)\|_{\dot H^{-1}}=\sup_{\substack{\|w_T\|_{\dot H^1}=1\\ \operatorname{div} w=0}}\int_\Omega(\rho u)(T)\cdot w_T\,dx$$

and solve (3-13) with no source term and data w_T at time $t=T$. Hence,

$$\begin{aligned}0&=\int_0^T\!\!\int_\Omega(\rho w_t+\rho v\cdot\nabla w+\Delta w+\nabla Q)\cdot u\,dx\,dt\\&=-\int_0^T\!\!\int_\Omega\rho(\partial_tu+v\cdot\nabla u-\Delta u)\cdot w\,dx\,dt+\int_\Omega(\rho(T)u(T)\cdot w_T-\rho_0u_0\cdot w(0))\,dx.\end{aligned}$$

Since u satisfies (1-6) and $\operatorname{div} w=0$, we get

$$\int_\Omega(\rho u)(T)\cdot w_T\,dx=\int_\Omega\rho_0u_0\cdot w(0)\,dx.\tag{3-15}$$

As

$$\|\nabla w(0)\|_{L^2}\le e^{(\rho^*/2)\int_0^T\|\sqrt{\rho}v\|_{L^4}^4\,dt}\|\nabla w_T\|_{L^2},$$

we conclude that

$$\|\mathcal{P}(\rho u)(T)\|_{\dot H^{-1}}\le\|\mathcal{P}(\rho_0u_0)\|_{\dot H^{-1}}e^{(\rho^*/2)\int_0^T\|\sqrt{\rho}v\|_{L^4}^4\,dt}.\tag{3-16}$$

Estimates in $\dot H^{-s}$ for $s\in(0,1)$. We start from

$$\|\mathcal{P}(\rho u)(T)\|_{\dot H^{-s}}=\sup_{\substack{\|w_T\|_{\dot H^s}=1\\ \operatorname{div} w=0}}\int_\Omega(\rho u)(T)\cdot w_T\,dx.$$

Using (3-15), we get, for any divergence-free $w_T\in\dot H^s$ with norm equal to 1,

$$\left|\int_\Omega(\rho u)(T)\cdot w_T\,dx\right|\le\|\mathcal{P}(\rho_0u_0)\|_{\dot H^{-s}}\|w(0)\|_{\dot H^s},$$

where w is the solution of (3-13) with no source term and data w_T at time T.

Keeping (3-11) in mind, we easily conclude that

$$\|\mathcal{P}(\rho u)(T)\|_{\dot H^{-s}}\le C\|\mathcal{P}(\rho_0u_0)\|_{\dot H^{-s}}e^{(Cs/2)\rho^*\int_0^T\|\sqrt{\rho}\,v\|_{L^4}^4\,d\tau}. \tag{3-17}$$

3.3. *More time decay estimates.* In this section, we point out a number of time decay estimates for (1-6) in Sobolev and Lebesgue spaces that may be deduced from what we proved hitherto and basic interpolation results.

Sobolev decay estimates. These are summarized in the following proposition.

Proposition 3.1. *The following estimates hold*:

- *For any* $0\le s\le2$ *and* $0\le s'\le1$, *we have*

$$\|u(t)\|_{\dot H^s}\le C_{\rho,v}t^{-(s+s')/2}\|\mathcal{P}(\rho_0u_0)\|_{\dot H^{-s'}},\quad t>0. \tag{3-18}$$

- *For any* $0\le s,s'\le1$,

$$\|tu_t(t)\|_{\dot H^s}+\|t\dot u(t)\|_{\dot H^s}\le C_{\rho,v}\,t^{-(s+s')/2}\|\mathcal{P}(\rho_0u_0)\|_{\dot H^{-s'}},\quad t>0. \tag{3-19}$$

- *For any* $0\le s\le1$,

$$\|t\dot u(t),u(t)\|_{\dot H^1}\le Ce^{\widetilde C_2^v(t)+\widetilde C_3^v(t)}t^{(s-1)/2}\|u_0\|_{\dot H^s}, \tag{3-20}$$

$$\|\dot u(t),u_t(t)\|_{L^2}\le Ce^{\widetilde C_2^v(t)+\widetilde C_3^v(t)}t^{-(2-s)/2}\|u_0\|_{\dot H^s}, \tag{3-21}$$

$$\|\dot u(t)\|_{\dot H^s}\le Ce^{\widetilde C_2^v(t)+\widetilde C_3^v(t)}t^{-(1+s)/2}\|u_0\|_{\dot H^1}. \tag{3-22}$$

Proof. The previous sections guarantee that

$$t^{k/2}\|\nabla^ku(t)\|_{L^2}\le C_{\rho,v}\|u_0\|_{L^2}\quad\text{for } k=0,1,2, \tag{3-23}$$

$$t^{1+k/2}\|\nabla^k(u_t,\dot u)(t)\|_{L^2}\le C_{\rho,v}\|u_0\|_{L^2}\quad\text{for } k=0,1. \tag{3-24}$$

The key observation for proving (3-18) is that having the density bounded and bounded away from zero ensures that

$$\|\mathcal{P}(\rho z)\|_{L^2}\simeq\|z\|_{L^2}\quad\text{for all } z\in L^2_\sigma. \tag{3-25}$$

Indeed, since $\mathcal{P}$ is an L^2 orthogonal projector, we may write

$$\|\mathcal{P}(\rho z)\|_{L^2}\le\|\rho z\|_{L^2}\le\rho^*\|z\|_{L^2}$$

and

$$\rho_*\|z\|_{L^2}^2 \le \int_\Omega \rho|z|^2\,dx = \int_\Omega \mathcal{P}(\rho z)\cdot z\,dx$$
$$\le \|\mathcal{P}(\rho z)\|_{L^2}\|z\|_{L^2}.$$

Inequality (3-18) in the case $s'=0$ thus follows from (3-23), with $k=0, 2$, and complex interpolation. In order to attain negative values of s', we use again (3-25) then argue by duality as follows for all $t>0$:

$$\|\mathcal{P}(\rho u)(t)\|_{L^2} = \sup_{\|w\|_{L^2_\sigma}=1}\int_\Omega (\rho u)(t)\cdot w\,dx = \sup_{\|w\|_{L^2_\sigma}=1}\int_\Omega \rho_0 u_0\cdot w(0)\,dx$$
$$\le \|\mathcal{P}(\rho_0 u_0)\|_{\dot H^{-s'}} \sup_{\|w\|_{L^2_\sigma}=1}\|w(0)\|_{\dot H^{s'}},$$

where $w(0)$ stands for the solution at time $t=0$ of the backward Stokes system (3-13) with no source term and data w at time t. Now, using the inequality we have just proved (that, obviously, also holds for (3-13)), we discover that

$$\|w(0)\|_{\dot H^{s'}} \le Ct^{-s'/2}\|w\|_{L^2},$$

whence

$$\|\rho(t)u(t)\|_{L^2} \le Ct^{-s'/2}\|\mathcal{P}(\rho_0 u_0)\|_{\dot H^{-s'}}. \tag{3-26}$$

Since inequality (3-23) is valid on any interval $[t_0, t]$ (if replacing u_0 by $u(t_0)$ and t by $t-t_0$, of course), one can assert that, for all $s\in[0,2]$, we have

$$\|u(t)\|_{\dot H^s} \le Ct^{-s/2}\left\|(\rho u)\left(\tfrac12 t\right)\right\|_{L^2},$$

which, combined with (3-26) $\left(\text{at time } \tfrac12 t\right)$ completes the proof of (3-18) for all $0\le s\le 2$ and $0\le s'\le 1$.

Next, using (3-24), with $k=0, 1$, and complex interpolation yields (3-19) for $s'=0$ and all $s\in[0,1]$. Since the inequality also holds if u_0 is replaced with $u\left(\tfrac12 t\right)$, using again (3-26) yields the desired inequality for all $s'\in[0,1]$.

By the same token, combining the above result with the continuity properties resulting from inequalities (2-26), (3-3), (3-7) and (3-10) gives the last three inequalities of the statement. The details are left to the reader. □

Decay estimates in Lebesgue spaces. Inequalities (3-23) and (3-24) also imply the following result.

Proposition 3.2. *The following inequalities hold*:

- *If* $1<p\le 2\le q\le\infty$ *then*

$$\|u(t)\|_{L^q} + \|\sqrt{t}\nabla u(t)\|_{L^q} \le C_{\rho,v}t^{1/q-1/p}\|u_0\|_{L^p}. \tag{3-27}$$

- *If* $1<p\le 2\le q<\infty$ *then*

$$\|t(\dot u, u_t, \nabla^2 u, \nabla P)(t)\|_{L^q} \le C_{\rho,v}t^{1/q-1/p}\|u_0\|_{L^p}. \tag{3-28}$$

Proof. Combining the Gagliardo–Nirenberg inequality (A-1) and (3-23) with $k = 0, 1, 2$, it is easy to get

$$\|u(t)\|_{L^q} + \|\sqrt{t}\nabla u(t)\|_{L^q} \le C_{\rho,v} t^{1/q-1/2}\|u_0\|_{L^2}, \quad 2 \le q < \infty, \tag{3-29}$$

while (3-24) ensures that

$$\|u_t(t), \dot{u}(t)\|_{L^q} \le C_{\rho,v} t^{1/q-3/2}\|u_0\|_{L^2}. \tag{3-30}$$

Since $(u, \nabla P)$ satisfies the Stokes system (2-5), inequality (A-4) gives

$$\|\nabla^2 u(t)\|_{L^q} + \|\nabla P(t)\|_{L^q} \le C_{\rho,v} t^{1/q-3/2}\|u_0\|_{L^2}, \quad 2 \le q < \infty. \tag{3-31}$$

Remember that[5]

$$\|z\|_{L^\infty} \le C\|z\|_{L^4}^{1/2}\|\nabla z\|_{L^4}^{1/2}. \tag{3-32}$$

Taking first $z = u$ and using (3-29) with $p = 4$, then $z = \nabla u$ and using (3-31) with $p = 4$ allows us to reach the index $q = \infty$ in (3-29).

In (3-29) and (3-31), the term $\|u_0\|_{L^2}$ may be replaced with $\|u(\frac{1}{2}t)\|_{L^2}$. Consequently, using (2-1), (3-26), embedding $L^p \hookrightarrow \dot{H}^{-1+2/p}$ for all $1 < p \le 2$, and the fact that $\mathcal{P} : L^p \to L^p$ ensures that

$$\begin{aligned}\|u(t)\|_{L^2} \simeq \|\mathcal{P}(\rho u)(t)\|_{L^2} &\le C_{\rho,v} t^{1/2-1/p}\|\mathcal{P}(\rho_0 u_0)\|_{\dot{H}^{1-2/p}} \\ &\le C_{\rho,v} t^{1/2-1/p}\|\mathcal{P}(\rho_0 u_0)\|_{L^p} \le C_{\rho,v} t^{1/2-1/p}\|u_0\|_{L^p},\end{aligned}$$

which, plugged into (3-29) and (A-4), completes the proofs of (3-27) and (3-28) for all admissible values of p and q. □

Decay estimates for L^2-in-time norms. Putting together (2-3), (2-11), (2-21), and (2-26), we see that

$$\int_0^t \big(\|\nabla u\|_{L^2}^2 + \|\sqrt{\tau}(\nabla^2 u, \nabla P)\|_{L^2}^2 + \|\sqrt{\tau}(\dot{u}, u_\tau)\|_{L^2}^2 + \|\tau(\nabla u_\tau, \nabla\dot{u})\|_{L^2}^2 + \|\tau^{3/2}\ddot{u}\|_{L^2}^2 + \|\tau^{3/2}(\nabla^2\dot{u}, \nabla\dot{P})\|_{L^2}^2\big)\, d\tau \le C_{\rho,v}\|u_0\|_{L^2}^2. \tag{3-33}$$

This will enable us to prove the following family of decay estimates.

Proposition 3.3. *The following inequalities hold*:

$$\|\tau^{1/2-1/q}\nabla u\|_{L_t^2(L^q)} \le C_{\rho,v}\|u_0\|_{L^2} \quad \text{for all } 2 \le q \le \infty, \tag{3-34}$$

$$\|\tau^{1-1/q}(\dot{u}, u_t)\|_{L_t^2(L^q)} \le C_{\rho,v}\|u_0\|_{L^2} \quad \text{for all } 2 \le q \le \infty, \tag{3-35}$$

$$\|\tau^{1-1/q}(\nabla^2 u, \nabla P)\|_{L_t^2(L^q)} \le C_{\rho,v}\|u_0\|_{L^2} \quad \text{for all } 2 \le q < \infty, \tag{3-36}$$

$$\|\tau^{3/2-1/q}\nabla\dot{u}\|_{L_t^2(L^q)} \le C_{\rho,v}\|u_0\|_{L^2} \quad \text{for all } 2 \le q < \infty. \tag{3-37}$$

Proof. Except for $q = \infty$, inequality (3-34) follows from the Gagliardo–Nirenberg inequality (A-1) and the fact that

$$\|\nabla u\|_{L_t^2(L^2)} + \|\sqrt{\tau}\nabla^2 u\|_{L_t^2(L^2)} \le C_{\rho,v}\|u_0\|_{L^2}.$$

[5] In the torus case, this inequality holds under the assumption $\int_{\mathbb{T}^2} az\, dx = 0$ for some nonnegative function a with mean value 1. The idea of the proof is similar to that of (A-2).

Similarly, except for the case $q=\infty$, inequality (3-35) for $\dot u$ stems from (A-1) and

$$\|\tau\nabla\dot u\|_{L^2_t(L^2)}+\|\sqrt{\tau}\dot u\|_{L^2_t(L^2)}\le C_{\rho,v}\|u_0\|_{L^2}.$$

Now, since (u,P) satisfies (2-5), the regularity properties of the Stokes system pointed out in (A-4) and (3-35) guarantee that

$$\|\tau^{1-1/q}(\nabla^2u,\nabla P)\|_{L^2_t(L^q)}\le C_{\rho,v}\|u_0\|_{L^2}\quad\text{for all }2\le q<\infty.$$

Putting together this latter inequality and (3-34) with $q=4$ and remembering (3-32) yields (3-34) for $q=\infty$.

Note that (3-33) also implies that

$$\|\tau^{3/2}\nabla^2\dot u\|_{L^2_t(L^2)}+\|\tau\nabla\dot u\|_{L^2_t(L^2)}\le C_{\rho,v}\|u_0\|_{L^2},$$

and thus (3-37) by (A-1). Using it with $q=4$ as well as (3-35) (also with $q=4$) and (3-32) gives (3-35) for $\dot u$ and $q=\infty$.

To prove that u_t satisfies (3-35), it suffices to check that

$$\|\tau^{1-1/q}v\cdot\nabla u\|_{L^2_t(L^q)}\le C_{\rho,v}\|u_0\|_{L^2}\quad\text{for all }2\le q\le\infty.$$

Now, by Hölder's inequality, we have

$$\|\tau^{1-1/q}v\cdot\nabla u\|_{L^2_t(L^q)}\le\|\tau^{1/2}v\|_{L^\infty_t(L^\infty)}\|\tau^{1/2-1/q}\nabla u\|_{L^2_t(L^q)}.$$

The term with v is energy-like (see (3-27)), which completes the proof. □

3.4. ***The Lipschitz control and other properties needed for stability.*** In the present subsection, we point out some additional properties of the velocity field that are valid in the case where u_0 is in $\widetilde B^0_{\rho_0,1}$. The most important one is the Lipschitz control. We shall also prove that the regularity $\widetilde B^0_{\rho_0,1}$ is preserved by the flow, and that other norms that will be needed in the proof of uniqueness and stability are finite.

These results follow from the Sobolev estimates we proved in the previous section and on the dynamic interpolation argument presented for the heat equation in Section 1.

Now, fix some u_0 in $\widetilde B^0_{\rho_0,1}$ and a sequence $(u_{0,j})_{j\in\mathbb Z}$ of L^2_σ such that

$$u_0=\sum_{j\in\mathbb Z}u_{0,j},\quad\text{with }\mathcal P(\rho_0u_{0,j})\in\dot H^{-1/2},\ \ u_{0,j}\in\dot H^{1/2}\quad\text{for all }j\in\mathbb Z$$
$$\text{and}\quad\sum_{j\in\mathbb Z}(2^{-j/2}\|u_{0,j}\|_{\dot H^{1/2}}+2^{j/2}\|\mathcal P(\rho_0u_{0,j})\|_{\dot H^{-1/2}})\le 2\|u_0\|_{\widetilde B^0_{\rho_0,1}}.\tag{3-38}$$

Then, for each $j\in\mathbb Z$, we solve the linear system

$$\begin{cases}\rho\partial_tu_j+\rho v\cdot\nabla u_j-\Delta u_j+\nabla P_j=0,\\ \operatorname{div}u_j=0,\\ u_j|_{t=0}=u_{0,j}.\end{cases}\tag{3-39}$$

From (3-38) and the uniqueness properties of system (1-6) in the energy space, we deduce that

$$u=\sum_{j\in\mathbb Z}u_j.\tag{3-40}$$

The Lipschitz bound. Recall the Gagliardo–Nirenberg inequality

$$\|\nabla z\|_{L^\infty} \le C\|z\|_{L^4}^{1/4}\|\nabla^2 z\|_{L^4}^{3/4}. \tag{3-41}$$

Combined with the elliptic estimates for the Stokes system and Sobolev embedding, this implies that, for all $t > 0$ and $j \in \mathbb{Z}$,

$$\|\nabla u_j(t)\|_{L^\infty} \le Ct^{-3/4}\|u_j(t)\|_{L^4}^{1/4}\|t\dot{u}_j(t)\|_{L^4}^{3/4} \le Ct^{-3/4}\|u_j(t)\|_{\dot{H}^{1/2}}^{1/4}\|t\dot{u}_j(t)\|_{\dot{H}^{1/2}}^{3/4}.$$

Hence, taking advantage of (3-11) and (3-12) gives

$$\|\nabla u_j(t)\|_{L^\infty} \le C_{\rho,v}t^{-3/4}\|u_{0,j}\|_{\dot{H}^{1/2}}.$$

Since we also have

$$\|\nabla u_j(t)\|_{L^\infty} \le C_{\rho,v}t^{-3/4}\left\|u_j\left(\tfrac{1}{2}t\right)\right\|_{\dot{H}^{1/2}},$$

we conclude in light of (3-18) that

$$\|\nabla u_j(t)\|_{L^\infty} \le C_{\rho,v}t^{-5/4}\|\mathcal{P}(\rho_0 u_{0,j})\|_{\dot{H}^{-1/2}}.$$

Hence, arguing as in Section 1, we conclude that

$$\int_0^\infty \|\nabla u\|_{L^\infty}\,dt \le C_{\rho,v}\|u_0\|_{\widetilde{B}^0_{\rho_0,1}}. \tag{3-42}$$

Remark 3.4. Recall the more accurate interpolation inequality

$$\|\nabla z\|_{\dot{B}^{1/2}_{4,1}} \le C\|z\|_{L^4}^{1/2}\|\nabla^2 z\|_{L^4}^{3/4}. \tag{3-43}$$

Repeating the above dynamic interpolation procedure thus actually gives

$$\int_0^\infty \|\nabla u\|_{\dot{B}^{1/2}_{4,1}}\,dt \le C_{\rho,v}\;\|u_0\|_{\widetilde{B}^0_{\rho_0,1}}.$$

Since $\dot{B}^{1/2}_{4,1} \hookrightarrow \mathcal{C}_b$, this ensures that the flow of the velocity field is uniformly C^1 with respect to the space variable.

Propagating the initial regularity. Owing to (3-11) and to (3-17) with $s = \frac{1}{2}$, we have, for all $j \in \mathbb{Z}$ and $t \ge 0$,

$$\|u_j(t)\|_{\dot{H}^{1/2}} \le C_{\rho,v}\|u_{0,j}\|_{\dot{H}^{1/2}} \quad \text{and} \quad \|\mathcal{P}(\rho u_j)(t)\|_{\dot{H}^{-1/2}} \le C_{\rho,v}\|\mathcal{P}(\rho_0 u_{0,j})\|_{\dot{H}^{-1/2}}.$$

Hence, multiplying the first (resp. second) inequality by $2^{-j/2}$ (resp. $2^{j/2}$) then summing over $j \in \mathbb{Z}$ yields

$$\|u(t)\|_{\widetilde{B}^0_{\rho(t),1}} \le C_{\rho,v}\|u_0\|_{\widetilde{B}^0_{\rho_0,1}}.$$

Additional bounds for the pressure and the time derivative of the velocity. In addition to the Lipschitz bound on velocity, our proof of uniqueness will require that $\sqrt{t}\dot{u}$ and $\sqrt{t}\nabla P$ are in $L^{4/3}(\mathbb{R}_+; L^4)$, and we will also need the property that $\dot{u}$ and $\sqrt{t}D\dot{u}$ are in $L^1(\mathbb{R}_+; L^2)$ to prove the stability of the flow map.

Again, in light of the decomposition (3-40) and of the triangle inequality, in order to prove that $\sqrt{t}\dot u$ is in $L^{4/3}(\mathbb{R}_+;L^4)$, it suffices to estimate $t\dot u_j$ for all $j\in\mathbb{Z}$. Now, owing to the Sobolev embedding and the inequalities (that stem from (3-12) and (3-19) with $s=s'=\frac12$)

$$\|\dot u_j(t)\|_{\dot H^{1/2}}\le C_{\rho,v}t^{-1}\|u_{0,j}\|_{\dot H^{1/2}}\quad\text{and}\quad \|\dot u_j(t)\|_{\dot H^{1/2}}\le C_{\rho,v}t^{-3/2}\|\mathcal{P}(\rho_0u_{0,j})\|_{\dot H^{-1/2}},$$

we may write, for all $A_j>0$,

$$\begin{aligned}\|\sqrt{t}\dot u_j\|^{4/3}_{L^{4/3}(\mathbb{R}_+;L^4)}&\le C\int_0^\infty t^{2/3}\|\dot u_j\|^{4/3}_{\dot H^{1/2}}\,dt\\&\le C_{\rho,v}\biggl(\int_0^{A_j}t^{2/3}(t^{-1}\|u_{0,j}\|_{\dot H^{1/2}})^{4/3}\,dt+\int_{A_j}^\infty t^{2/3}(t^{-3/2}\|\mathcal{P}(\rho_0u_{0,j})\|_{\dot H^{-1/2}})^{4/3}\,dt\biggr)\\&\le C_{\rho,v}(A_j^{1/3}\|u_{0,j}\|^{4/3}_{\dot H^{1/2}}+A_j^{-1/3}\|\mathcal{P}(\rho_0u_{0,j})\|^{4/3}_{\dot H^{-1/2}}),\end{aligned}\tag{3-44}$$

which gives, if taking $A_j=2^{-2j}$ and using (A-4),

$$\|(\sqrt{t}\dot u,\sqrt{t}\nabla^2u,\sqrt{t}\nabla P)\|_{L^{4/3}(\mathbb{R}_+;L^4)}\le C_{\rho,v}\|u_0\|_{\widetilde B^0_{\rho_0,1}}.\tag{3-45}$$

Similarly, in order to bound $\dot u$ in $L^1(\mathbb{R}_+;L^2)$, it suffices to get appropriate bounds in terms of the data for $\dot u_j$ in $L^1(\mathbb{R}_+;L^2)$ and for all $j\in\mathbb{Z}$. The inequalities (that stem from (2-21) and (3-7))

$$\|\dot u_j(t)\|_{L^2}\le C_{\rho,v}t^{-1}\|u_{0,j}\|_{L^2}\quad\text{and}\quad\|\dot u_j(t)\|_{L^2}\le C_{\rho,v}t^{-1/2}\|\nabla u_{0,j}\|_{L^2}$$

and complex interpolation give

$$\|\dot u_j(t)\|_{L^2}\le C_{\rho,v}t^{-3/4}\|u_{0,j}\|_{\dot H^{1/2}}.$$

Furthermore, combining with (3-19), we discover that, for all $j\in\mathbb{Z}$,

$$\|\dot u_j(t)\|_{L^2}\le C_{\rho,v}t^{-5/4}\|\mathcal{P}(\rho_0u_{0,j})\|_{\dot H^{-1/2}}.$$

Hence we have, for all $j\in\mathbb{Z}$ and $A_j>0$,

$$\begin{aligned}\int_0^\infty\|\dot u_j(t)\|_{L^2}\,dt&\le\int_0^{A_j}\|\dot u_j(t)\|_{L^2}\,dt+\int_{A_j}^\infty\|\dot u_j(t)\|_{L^2}\,dt\\&\le C_{\rho,v}\biggl(\int_0^{A_j}(t^{-3/4}\|u_{0,j}\|_{\dot H^{1/2}})\,dt+\int_{A_j}^\infty(t^{-5/4}\|\mathcal{P}(\rho_0u_{0,j})\|_{\dot H^{-1/2}})\,dt\biggr)\\&\le C_{\rho,v}(A_j^{1/4}\|u_{0,j}\|_{\dot H^{1/2}}+A_j^{-1/4}\|\mathcal{P}(\rho_0u_{0,j})\|_{\dot H^{-1/2}}).\end{aligned}$$

Taking $A_j=2^{-2j}$, summing over j, then using the regularity properties of the Stokes system thus gives

$$\|\nabla^2u,\nabla P,\dot u\|_{L^1(\mathbb{R}_+;L^2)}\le C_{\rho,v}\|u_0\|_{\widetilde B^0_{\rho_0,1}}.\tag{3-46}$$

In the same way, one can prove that

$$\|\sqrt{t}D\dot u\|_{L^1(\mathbb{R}_+;L^2)}\le C_{\rho,v}\,\|u_0\|_{\widetilde B^0_{\rho_0,1}}.\tag{3-47}$$

It suffices to use, as a consequence of (3-19) and (3-20), that

$$\|\sqrt{t}\nabla\dot u_j(t)\|_{L^2}\le C_{\rho,v}t^{-3/4}\|u_{0,j}\|_{\dot H^{1/2}}\quad\text{and}\quad\|\sqrt{t}\nabla\dot u_j(t)\|_{L^2}\le C_{\rho,v}t^{-5/4}\|\mathcal{P}(\rho_0u_{0,j})\|_{\dot H^{-1/2}}.$$

4. A global well-posedness result for large data

This section is devoted to the proof of Theorem 1.3 and of stability estimates.

4.1. ***The proof of existence.*** Consider data (ρ_0, u_0) satisfying the hypotheses of Theorem 1.3. Since the space $\widetilde{B}^0_{\rho_0,1}$ is embedded in L^2_σ, Theorem 1.1 provides us with a global weak solution $(\rho, u, \nabla P)$ satisfying the properties therein, and it is only a matter of checking that this solution has the additional properties that are listed in Theorem 1.3. To do so, we fix some decomposition $\sum_j u_{0,j}$ of u_0 given by Definition 1.2 and look, for all $j \in \mathbb{Z}$, at the solution u_j to the linear system (1-6) with density ρ, transport field u, and initial data $u_{0,j}$. Since each $u_{0,j}$ is in $L^2_\sigma \cap \dot{H}^{1/2}$ and $\mathcal{P}(\rho_0 u_{0,j}) \in \dot{H}^{-1/2}$, standard techniques yield a unique global solution $(u_j, \nabla P_j)$ that satisfies, for all $t \geq 0$,

$$\frac{1}{2}\|\sqrt{\rho(t)}u_j(t)\|^2_{L^2} + \int_0^t \|\nabla u_j\|^2_{L^2}\,d\tau = \frac{1}{2}\|\sqrt{\rho_0}u_{0,j}\|^2_{L^2}, \tag{4-1}$$

$$\|\mathcal{P}(\rho u_j)(t)\|_{\dot{H}^{-1/2}} \leq C(\rho_*, \rho^*, \|u_0\|_{L^2})\|\mathcal{P}(\rho_0 u_{0,j})\|_{\dot{H}^{-1/2}}, \tag{4-2}$$

$$\|u_j(t)\|_{\dot{H}^{1/2}} \leq C(\rho_*, \rho^*, \|u_0\|_{L^2})\|u_{0,j}\|_{\dot{H}^{1/2}}. \tag{4-3}$$

Remembering (1-9), this ensures that the L^2-valued series $\sum_j u_j$ converges normally on $\mathbb{R}_+$. Its sum $\tilde{u}$ thus also belongs to the energy space. Furthermore, as for each $j \in \mathbb{Z}$, we have $u_j \in \mathcal{C}(\mathbb{R}_+; L^2)$ (observe that $t^{3/4}u^j_t$ is in $L^\infty(\mathbb{R}_+; L^2)$ owing to (3-21)), and we deduce that $\tilde{u} \in \mathcal{C}(\mathbb{R}_+; L^2)$. Next, if we define $u^n := \sum_{|j|\leq n} u_j$, then we see that, for all $n \in \mathbb{N}$,

$$\partial_t(\rho(u^n - \tilde{u})) + \operatorname{div}(\rho u \otimes (u^n - \tilde{u})) - \Delta(u^n - \tilde{u}) + \nabla(P^n - \widetilde{P}) = 0, \quad \operatorname{div}(u^n - \tilde{u}) = 0,$$

which implies

$$\frac{1}{2}\|\sqrt{\rho(t)}(u^n - \tilde{u})(t)\|^2_{L^2} + \int_0^t \|\nabla(u^n - \tilde{u})\|^2_{L^2}\,d\tau = \frac{1}{2}\|\sqrt{\rho_0}(u^n(0) - u(0))\|^2_{L^2}.$$

As the right-hand side tends to 0 for n going to 0, the velocity field $\tilde{u}$ satisfies the energy balance (0-3), and it is also easy to conclude that, like u, it satisfies (1-6) with density ρ, transport field u, and initial data u_0. In particular,

$$\partial_t(\rho(u - \tilde{u})) + \operatorname{div}(\rho u \otimes (u - \tilde{u})) - \Delta(u - \tilde{u}) + \nabla(P - \widetilde{P}) = 0, \quad \operatorname{div}(u - \tilde{u}) = 0.$$

As $(u-\tilde{u})(0) = 0$ and the two solutions are in the energy space, they must coincide. Now, inequalities (4-2) and (4-3) ensure that one can propagate the regularity $\widetilde{B}^0_{\rho_0,1}$, getting (1-10). Likewise, the justification that u satisfies (0-8), that $(\dot{u}, \sqrt{t}D\dot{u}, D^2u, \nabla P) \in L^1(\mathbb{R}_+; L^2)$, and that $\sqrt{t}\dot{u} \in L^{4/3}(\mathbb{R}_+; L^4)$ may be achieved by following the arguments of the previous section. The fundamental point is that all the bounds that are needed for the u_j in the process only depend on ρ_*, ρ^*, $\|u_0\|_{L^2}$, $\|\mathcal{P}(\rho_0 u_{0,j})\|_{\dot{H}^{-1/2}}$, and $\|u_{0,j}\|_{\dot{H}^{1/2}}$.

4.2. ***The proof of uniqueness.*** Let $(\rho^1, u^1, \nabla P^1)$ and $(\rho^2, u^2, \nabla P^2)$ be two solutions fulfilling the properties listed in Theorem 1.3 and corresponding to data (ρ^1_0, u^1_0) and (ρ^2_0, u^2_0), respectively. As in [Danchin and Mucha 2019], in order to prove that

$$(\rho^1, u^1, \nabla P^1) \equiv (\rho^2, u^2, \nabla P^2)$$

in the case where the two initial data coincide, we shall compare the solutions at the level of their own Lagrangian coordinates. To do so, we consider, for $i=1,2$, the flow X^i of u^i that is defined by the (integrated) ODE

$$X^i(t,y)=y+\int_0^t u^i(\tau,X^i(\tau,y))\,d\tau. \tag{4-4}$$

Since ∇u^i is in $L^1(\mathbb{R}_+;L^\infty)$ and $\sqrt{t}u^i$ is in $L^\infty(0,T\times\Omega)$ (see (3-27) with $p=2$ and $q=\infty$), there exists a unique continuous flow X^i on $(0,T)\times\Omega$ that is Lipschitz with respect to the space variable.

In Lagrangian coordinates the density is equal to the initial density. As for the velocity and the pressure defined by

$$Q^i(t,y)=P^i(t,X^i(t,y)) \quad\text{and}\quad v^i(t,y)=u^i(t,X^i(t,y)), \tag{4-5}$$

they satisfy

$$\begin{cases}\rho_0^i v_t^i-\operatorname{div}_{v^i}\nabla_{v^i}v^i+\nabla_{v^i}Q^i=0,\\ \operatorname{div}_{v^i}v^i=0,\end{cases} \tag{4-6}$$

where

$$\nabla_{v^i}:=(A^i)^\top\nabla_y \quad\text{and}\quad \operatorname{div}_{v^i}:=\operatorname{div}_y(A^i\,\cdot\,)=(A^i)^\top:\nabla_y, \quad\text{with } A^i:=(DX^i)^{-1}.$$

The fact that ∇u^i is in $L^1(\mathbb{R}_+;L^\infty)$ and the other properties of regularity ensure that (INS) and (4-6) (with time-independent density) are equivalent.

Observe that, due to (4-4) and the definition of v^i, we have

$$DX^i(t,y)=\mathrm{Id}+\int_0^t Dv^i(\tau,y)\,d\tau. \tag{4-7}$$

Hence, since $\det DX^i\equiv 1$ (owing to $\operatorname{div} v^i=0$), we have, for $i=1,2$,

$$A^i(t)=\mathrm{Id}+\begin{pmatrix}\int_0^t\partial_2 v^{i,2}\,d\tau & -\int_0^t\partial_2 v^{i,1}\,d\tau\\ -\int_0^t\partial_1 v^{i,2}\,d\tau & \int_0^t\partial_1 v^{i,1}\,d\tau\end{pmatrix}. \tag{4-8}$$

Hence $\delta A:=A^2-A^1$ depends linearly on $\nabla\delta v$ (with $\delta v:=v^2-v^1$) as follows:

$$\delta A(t)=\begin{pmatrix}\int_0^t\partial_2\delta v^2\,d\tau & -\int_0^t\partial_2\delta v^1\,d\tau\\ -\int_0^t\partial_1\delta v^2\,d\tau & \int_0^t\partial_1\delta v^1\,d\tau\end{pmatrix}. \tag{4-9}$$

Now, setting $\Delta_{v^i}:=\operatorname{div}_{v^i}\nabla_{v^i}$ and $\delta Q:=Q^2-Q^1$, we discover that $(\delta v,\delta Q)$ satisfies

$$\begin{cases}\rho_0^1\delta v_t-\Delta_{v^1}\delta v+\nabla_{v^1}\delta Q=(\Delta_{v^2}-\Delta_{v^1})v^2-(\nabla_{v^2}-\nabla_{v^1})Q^2-\delta\rho_0 v_t^2,\\ \operatorname{div}_{v^1}\delta v=(\operatorname{div}_{v^1}-\operatorname{div}_{v^2})v^2=-\operatorname{div}(\delta A v^2).\end{cases} \tag{4-10}$$

In order to prove uniqueness in the case where the initial data are the same and, more generally, stability estimates with respect to the initial data, using the basic energy method — which consists of taking the L^2 scalar product of (4-10) with δv — is not appropriate, since one cannot eliminate the pressure term (there is no reason why we should have $\operatorname{div}_{v^1}\delta v=0$). To overcome the difficulty, we proceed as in [Danchin and Mucha 2019], solving first the equation

$$\operatorname{div}_{v^1}w=-\operatorname{div}(\delta A v^2)=-\delta A^\top:\nabla v^2, \quad\text{with } \delta A:=A^2-A^1. \tag{4-11}$$

Then, we look at the system for $z := \delta v - w$, namely

$$\begin{cases}\rho_0^1 z_t - \Delta_{v^1} z + \nabla_{v^1}\delta Q = (\Delta_{v^2} - \Delta_{v^1})v^2 - (\nabla_{v^2} - \nabla_{v^1})Q^2 - \rho_0^1 w_t + \Delta_{v^1} w - \delta\rho_0 v_t^2,\\ \operatorname{div}_{v^1} z = 0,\end{cases} \tag{4-12}$$

supplemented with $z|_{t=0} = \delta v_0$.

Solving (4-11) relies on the following lemma.

Lemma 4.1. *Assume that Ω is a C^2 bounded domain, the torus, or the whole space. Fix $T > 0$ and define*

$$E_T := \{w \in \mathcal{C}([0,T]; L^2),\ \nabla w \in L^2(0, T\times\Omega),\ w|_{\partial\Omega} = 0 \text{ and } w_t \in L^{4/3}(0, T\times\Omega)\}.$$

There exists a constant c depending only on Ω such that, whenever the divergence-free vector field u satisfies

$$\|\nabla u\|_{L^2(0,T\times\Omega)} + \|\nabla u\|_{L^1(0,T;L^\infty)} \le c \tag{4-13}$$

then, for all vector fields $k \in \mathcal{C}([0,T]; L^2)$ such that $\operatorname{div} k \in L^2(0, T\times\Omega)$ *and $k_t \in L^{4/3}(0, T\times\Omega)$, there exists a vector field w in the space E_T satisfying*

$$\operatorname{div}(Aw) = \operatorname{div} k$$

(*where A is defined from u as in* (4-8)) *and the inequalities*

$$\|w(t)\|_{L^2} \le C\|k(t)\|_{L^2} \qquad \text{for all } t \in [0,T], \tag{4-14}$$

$$\|\nabla w\|_{L^2_T(L^2)} \le C\|\operatorname{div} k\|_{L^2_T(L^2)}, \tag{4-15}$$

$$\|w_t\|_{L^{4/3}_T(L^{4/3})} \le C(\|k_t\|_{L^{4/3}_T(L^{4/3})} + \|\nabla u\|_{L^2_T(L^2)}\|w\|_{L^4_T(L^4)}). \tag{4-16}$$

Proof. With the notation of Lemma A.1 in the Appendix, we introduce the map

$$\Phi : w \mapsto z := \mathcal{B}(k + (\mathrm{Id} - A)w).$$

It is only a matter of proving that Φ admits a fixed point. That Φ maps E_T to E_T follows from Lemma A.1 and easy modifications of the computations below. Hence, as E_T is a Banach space, it suffices to show that the linear map Φ is strictly contractive. To do so, take two elements w^1 and w^2 of E_T. Then, we have

$$\Phi(w^2) - \Phi(w^1) = \mathcal{B}((\mathrm{Id} - A)\delta w), \quad \text{with } \delta w := w^2 - w^1.$$

Remembering (4-8) and that $\mathcal{B} : L^2 \to L^2$, we thus have

$$\|\Phi(w^2) - \Phi(w^1)\|_{L^\infty_T(L^2)} \le C\|\nabla u\|_{L^1_T(L^\infty)}\|\delta w\|_{L^\infty_T(L^2)}. \tag{4-17}$$

Next, using again (4-8) and the fact that

$$\operatorname{div}((\mathrm{Id} - A)\delta w) = (\mathrm{Id} - A^\top) : \nabla\delta w,$$

we readily get

$$\|\nabla(\Phi(w^2) - \Phi(w^1))\|_{L^2_T(L^2)} \le C\|\nabla u\|_{L^1_T(L^\infty)}\|\nabla\delta w\|_{L^2_T(L^2)}. \tag{4-18}$$

Finally, using

$$((\mathrm{Id} - A)\delta w)_t = (\mathrm{Id} - A)\delta w_t - A_t\delta w$$

yields, for almost every $t \in [0, T]$,

$$\begin{aligned}\|(\Phi(w^2)-\Phi(w^1))_t(t)\|_{L^{4/3}} &\lesssim \|(\mathrm{Id}-A(t))\delta w_t(t)\|_{L^{4/3}} + \|A_t(t)\delta w(t)\|_{L^{4/3}}\\ &\lesssim \|\nabla u\|_{L^1_t(L^\infty)}\|\delta w_t(t)\|_{L^{4/3}} + \|\nabla u(t)\|_{L^2}\|\delta w(t)\|_{L^4}\\ &\lesssim \|\nabla u\|_{L^1_t(L^\infty)}\|\delta w_t(t)\|_{L^{4/3}} + \|\nabla u(t)\|_{L^2}\|\delta w(t)\|_{L^2}^{1/2}\|\nabla\delta w(t)\|_{L^2}^{1/2}. \end{aligned} \tag{4-19}$$

Combining (4-17)–(4-19), we conclude

$$\|(\Phi(w^2)-\Phi(w^1)\|_{E_T} \le C(\|\nabla u\|_{L^1_T(L^\infty)} + \|\nabla u\|_{L^2_T(L^2)})\|\delta w\|_{E_T}.$$

Hence, if (4-13) is satisfied with a suitable small $c > 0$ then Φ is contractive, which ensures the existence of w in E_T satisfying the desired equation. Finally, using the fact that we thus have $w = \mathcal{B}k + \mathcal{B}((\mathrm{Id}-A)w)$ and that

$$\mathrm{div}((\mathrm{Id}-A)w) = (\mathrm{Id}-A^\top) : \nabla w,$$
$$((\mathrm{Id}-A)w)_t = (\mathrm{Id}-A)w_t - A_t w,$$

mimicking the above calculations gives (4-14), (4-15), and (4-16). □

In what follows, we assume that T has been chosen such that (4-13) is satisfied for u^1 and u^2, and we define w on $[0, T] \times \Omega$ according to the above lemma with $k = -\delta A v^2$. We shall use repeatedly that, owing to (4-9) and the Cauchy–Schwarz inequality, we have

$$\max(\|t^{-1/2}\delta A\|_{L^\infty_T(L^2)}, \|(\delta A)_t\|_{L^2(0,T\times\Omega)}) \le \|\nabla\delta v\|_{L^2(0,T\times\Omega)}. \tag{4-20}$$

Hence, thanks to (4-14), we have, for all $t \in [0, T]$,

$$\|w(t)\|_{L^2} \le C\|\sqrt{t}v^2(t)\|_{L^\infty}\|\nabla\delta v\|_{L^2(0,t\times\Omega)}. \tag{4-21}$$

Next, as

$$(\delta A v^2)_t = \delta A_t v^2 + \delta A v^2_t,$$

inequality (4-16) (before time integration) and (4-9) guarantee that

$$\|w_t\|_{L^{4/3}} \le C(\|\nabla v^1\|_{L^2}\|w\|_{L^4} + \|\nabla\delta v\|_{L^2}\|v^2\|_{L^4} + \|\delta A\|_{L^2}\|v^2_t\|_{L^4}). \tag{4-22}$$

Finally, using $\mathrm{div}(\delta A v^2) = \delta A^\top : \nabla v^2$, inequalities (4-15) and (4-20) yield

$$\|Dw(t)\|_{L^2} \le C\|\nabla\delta v\|_{L^2_t(L^2)}\|\sqrt{t}\nabla v^2\|_{L^\infty_t(L^\infty)}. \tag{4-23}$$

Now, taking the $L^2(0, t \times \Omega)$ scalar product of the first equation of (4-12) with z and integrating by parts in some terms yields

$$\frac{1}{2}\|\sqrt{\rho_0^1}z\|^2_{L^\infty(0,t;L^2)} + \int_0^t \|\nabla_{v^1}z\|^2_{L^2}\,d\tau = \frac{1}{2}\|\sqrt{\rho_0^1}\delta u_0\|^2_{L^2} + \sum_{j=1}^5 I_j(t), \tag{4-24}$$

with

$$I_1(t) := -\int_0^t\int_\Omega (\delta A(A^2)^\top + A^1\delta A^\top)\nabla v^2 : \nabla z\,dx\,d\tau,$$

$$I_2(t) := -\int_0^t\int_\Omega \delta A^\top\nabla Q^2\cdot z\,dx\,d\tau, \qquad I_3(t) := -\int_0^t\int_\Omega \rho_0^1 w_\tau\cdot z\,dx\,d\tau,$$

$$I_4(t) := -\int_0^t\int_\Omega (A^1)^\top\nabla w : (A^1)^\top\nabla z\,dx\,d\tau, \qquad I_5(t) := -\int_0^t\int_\Omega \delta\rho_0 v_t^2\cdot z\,dx\,d\tau. \tag{4-25}$$

We shall often use that, due to (4-8),

$$\|\nabla z\|_{L^2(0,T\times\Omega)} \simeq \|\nabla_{v^1} z\|_{L^2(0,T\times\Omega)}. \tag{4-26}$$

From this we easily get

$$I_1(t) \le C\int_0^t \|\tau^{-1/2}\delta A(\tau)\|_{L^2}\|\sqrt{\tau}\nabla v^2(\tau)\|_{L^\infty}\|\nabla_{v^1} z(\tau)\|_{L^2}\,d\tau.$$

Hence, using (4-20) and Young's inequality,

$$I_1 \le C\|\sqrt{\tau}\nabla v^2\|^2_{L^2_t(L^\infty)}\|\nabla\delta v\|^2_{L^2(0,t\times\Omega)} + \frac{1}{8}\int_0^t \|\nabla_{v^1} z\|^2_{L^2}\,d\tau. \tag{4-27}$$

Next, by (4-20), (4-26), Hölder's inequality, and (0-7), we have

$$\begin{aligned} I_2 &\le C\int_0^t \|\tau^{-1/2}\delta A\|_{L^2}\|\sqrt{\tau}\nabla Q^2\|_{L^4}\|z\|^{1/2}_{L^2}\|\nabla z\|^{1/2}_{L^2}\,d\tau \\ &\le \frac{1}{8}\int_0^t \|\nabla_{v^1} z\|^2_{L^2}\,d\tau + C\|\tau^{-1/2}\delta A\|^{4/3}_{L^\infty_t(L^2)}\|z\|^{2/3}_{L^\infty_t(L^2)}\int_0^t \|\sqrt{\tau}\nabla Q^2\|^{4/3}_{L^4}\,d\tau. \end{aligned}$$

Hence, in light of (4-20), Young's inequality, and (0-9), we have

$$I_2 \le \frac{1}{8}\int_0^t \Big(\|\nabla_{v^1} z\|^2_{L^2} + \frac{1}{4}\|\nabla\delta v\|^2_{L^2}\Big)\,d\tau + C\|\sqrt{\rho_0^1}z\|^2_{L^\infty_t(L^2)}\|\sqrt{\tau}\nabla Q^2\|^4_{L^{4/3}_t(L^4)}. \tag{4-28}$$

In order to bound I_3, we start with the inequality

$$I_3 \le \rho^*\int_0^t \|w_\tau\|_{L^{4/3}}\|z\|_{L^4}\,d\tau.$$

Taking advantage of (4-22) to bound w_τ and of the Gagliardo–Nirenberg and Young inequalities yields

$$\begin{aligned} I_3 &\lesssim \int_0^t \|z\|^{1/2}_{L^2}\|\nabla z\|^{1/2}_{L^2}(\|\nabla v^1\|_{L^2}\|w\|_{L^4} + \|v^2\|_{L^4}\|\nabla\delta v\|_{L^2} + \|\delta A\|_{L^2}\|v^2_\tau\|_{L^4})\,d\tau \\ &\le \frac{1}{8}\int_0^t \|\nabla_{v^1} z\|^2_{L^2}\,d\tau + \frac{1}{32}\int_0^t \|\nabla\delta v\|^2_{L^2}\,d\tau + C\int_0^t \|v^2\|^4_{L^4}\|z\|^2_{L^2}\,d\tau + I_{31} + I_{32}, \end{aligned}$$

with

$$I_{31} := C\int_0^t \|z\|^{2/3}_{L^2}\|\nabla v^1\|^{4/3}_{L^2}\|w\|^{2/3}_{L^2}\|\nabla w\|^{2/3}_{L^2}\,d\tau \quad\text{and}\quad I_{32} := C\int_0^t \|z\|^{2/3}_{L^2}\|\delta A\|^{4/3}_{L^2}\|v^2_\tau\|^{4/3}_{L^4}\,d\tau.$$

Just using (4-20) yields

$$I_{32} \le \|\nabla \delta v\|_{L_t^2(L^2)}^{4/3} \|z\|_{L_t^\infty(L^2)}^{2/3} \|\sqrt{\tau}\, v_\tau^2\|_{L_t^{4/3}(L^4)}^{4/3}.$$

In order to bound I_{31}, one has to use (4-21) and (4-23), which yields

$$\begin{aligned} I_{31} &\le C \int_0^t \|z\|_{L^2}^{2/3} \|\nabla v^1\|_{L^2}^{4/3} \|\sqrt{\tau}\, v^2\|_{L^\infty}^{2/3} \|\nabla \delta v\|_{L_\tau^2(L^2)}^{2/3} \|\tau^{-1/2} \delta A(\tau)\|_{L^2}^{2/3} \|\sqrt{\tau} \nabla v^2\|_{L^\infty}^{2/3}\, d\tau \\ &\le C \|\nabla \delta v\|_{L_t^2(L^2)}^{4/3} \|z\|_{L_t^\infty(L^2)}^{2/3} \int_0^t \|\sqrt{\tau}\, v^2\|_{L^\infty}^{2/3} \|\nabla v^1\|_{L^2}^{4/3} \|\sqrt{\tau} \nabla v^2\|_{L^\infty}^{2/3}\, d\tau. \end{aligned}$$

This enables us to get the following bound for I_3:

$$\begin{aligned} &I_3(t) \\ &\le \frac{1}{8} \|\nabla_{v^1} z\|_{L_t^2(L^2)}^2 + \frac{1}{16} \|\nabla \delta v\|_{L_t^2(L^2)}^2 \\ &+ C\bigg(\|v^2\|_{L_t^4(L^4)}^4 + \bigg(\int_0^t \|\sqrt{\tau}\, v^2\|_{L^\infty}^{2/3} \|\nabla v^1\|_{L^2}^{4/3} \|\sqrt{\tau} \nabla v^2\|_{L^\infty}^{2/3}\, d\tau \bigg)^3 + \|\sqrt{\tau}\, v_\tau^2\|_{L_t^{4/3}(L^4)}^4 \bigg) \|\sqrt{\rho_0^1} z\|_{L_t^\infty(L^2)}^2. \end{aligned} \tag{4-29}$$

Next, thanks to (4-23), (4-20), and the Cauchy–Schwarz and Young inequalities,

$$\begin{aligned} I_4 \le C \int_0^t \|\nabla w\|_{L^2} \|\nabla_{v^1} z\|_{L^2}\, d\tau &\le C \int_0^t \|\tau^{-1/2} \delta A\|_{L^2} \|\sqrt{\tau} \nabla v^2\|_{L^\infty} \|\nabla_{v^1} z\|_{L^2}\, d\tau \\ &\le \frac{1}{8} \int_0^t \|\nabla_{v^1} z\|_{L^2}^2\, d\tau + C \|\sqrt{\tau} \nabla v^2\|_{L^2(0,t;L^\infty)}^2 \|\nabla \delta v\|_{L^2(0,t\times\Omega)}^2. \end{aligned} \tag{4-30}$$

Finally, it is obvious that

$$I_5(t) \le \left\| \frac{\delta\rho_0}{\sqrt{\rho_0^1}} \right\|_{L^\infty} \|\sqrt{\rho_0^1} z\|_{L_t^\infty(L^2)} \|v_t^2\|_{L_t^1(L^2)}. \tag{4-31}$$

So plugging (4-27)–(4-31) into (4-24) and taking $t = T$ yields

$$\begin{aligned} \|\sqrt{\rho_0^1}\, z\|_{L_T^\infty(L^2)}^2 + \|\nabla_{v^1} z\|_{L_T^2(L^2)}^2 &\le \|\sqrt{\rho_0^1} \delta u_0\|_{L^2}^2 + A(T) \|\sqrt{\rho_0^1} z\|_{L_T^\infty(L^2)}^2 \\ &\quad + \Big(\frac{1}{8} + C \|\sqrt{t} \nabla v^2\|_{L_T^2(L^\infty)}^2 \Big) \|\nabla \delta v\|_{L_T^2(L^2)}^2 + 2 \left\| \frac{\delta\rho_0}{\sqrt{\rho_0^1}} \right\|_{L^\infty}^2 \|v_t^2\|_{L_T^1(L^2)}^2, \end{aligned}$$

with

$$\begin{aligned} A(T) := C\bigg(&\|v^2\|_{L_T^4(L^4)}^4 + \|\sqrt{t}\, v_t^2\|_{L_T^{4/3}(L^4)}^4 + \|\sqrt{\tau} \nabla Q^2\|_{L_T^{4/3}(L^4)}^4 \\ &+ \bigg(\int_0^t \|\sqrt{\tau}\, v^2\|_{L^\infty}^{2/3} \|\nabla v^1\|_{L^2}^{4/3} \|\sqrt{\tau} \nabla v^2\|_{L^\infty}^{2/3}\, d\tau \bigg)^3 \bigg). \end{aligned}$$

The regularity properties of the constructed solutions guarantee that $A(\infty)$ is finite, and the Lebesgue dominated convergence theorem thus ensures that if T is small enough then

$$\max(8C \|\sqrt{t} \nabla v^2\|_{L_T^2(L^\infty)}^2, 2A(T)) \le 1. \tag{4-32}$$

Under this hypothesis, the above inequality becomes

$$\tfrac{1}{2} \|\sqrt{\rho_0^1}\, z\|_{L_T^\infty(L^2)}^2 + \|\nabla_{v^1} z\|_{L_T^2(L^2)}^2 \le \|\sqrt{\rho_0^1} \delta u_0\|_{L^2}^2 + \tfrac{1}{4} \|\nabla \delta v\|_{L_T^2(L^2)}^2 + C \|\delta\rho_0\|_{L^\infty}^2 \|v_t^2\|_{L_T^1(L^2)}^2. \tag{4-33}$$

Since $\nabla\delta v = \nabla z + \nabla w$ and owing to (4-20), (4-23), and (4-26), we may write

$$\|\nabla\delta v\|^2_{L^2_T(L^2)} \le 2\|\nabla z\|^2_{L^2_T(L^2)} + 2\|\nabla w\|^2_{L^2_T(L^2)} \le \tfrac{5}{2}\|\nabla_{v^1} z\|^2_{L^2_T(L^2)} s + C\|\sqrt{t}\nabla v^2\|^2_{L^2_T(L^\infty)}\|\nabla\delta v\|^2_{L^2_T(L^2)}.$$

Hence, under assumption (4-32) (up to a change of C if needed), we have

$$\|\nabla\delta v\|^2_{L^2(0,T\times\Omega)} \le 3\|\nabla_{v^1} z\|^2_{L^2(0,T\times\Omega)}. \tag{4-34}$$

Plugging this inequality into (4-33) gives

$$\tfrac{1}{2}\|\sqrt{\rho_0^1}\, z\|^2_{L^\infty_T(L^2)} + \tfrac{1}{4}\|\nabla_{v^1} z\|^2_{L^2_T(L^2)} \le C(\|\sqrt{\rho_0^1}\delta u_0\|^2_{L^2} + \|\delta\rho_0\|^2_{L^\infty}\|v_t^2\|^2_{L^1_T(L^2)}). \tag{4-35}$$

In the case where the two solutions correspond to the same initial data, this ensures that $z \equiv 0$ on $[0, T]$. Remembering (4-34) and (4-21), one can conclude uniqueness on $[0, T]$ and then on $\mathbb{R}_+$ by a standard bootstrap argument.

4.3. ***Continuity of the flow map.*** We consider here the case where the two previous solutions correspond to possibly different data. To begin with, we have to observe that (4-34) and (4-35) together imply that if

$$\|\sqrt{t}v^2\|_{L^\infty(\mathbb{R}_+\times\Omega)} \le K, \tag{4-36}$$

then, in light of inequalities (4-21), (4-34) and (4-35), there exists some constant $c > 0$ such that if $\tilde{A}(T_0) \le c$, then we have

$$\|\sqrt{\rho_0^1}\,\delta v\|_{L^\infty_{T_0}(L^2)} + \|\nabla_{v^1}\delta v\|_{L^2_{T_0}(L^2)} \le C(1+K)(\|\sqrt{\rho_0^1}\delta u_0\|_{L^2} + \|\delta\rho_0\|_{L^\infty}), \tag{4-37}$$

where we define, for all $T \in [0, \infty]$,

$$\tilde{A}(T) := \|v^2\|^4_{L^4_T(L^4)} + \|\sqrt{t}(v_t^2, \nabla Q^2)\|^{4/3}_{L^{4/3}_T(L^4)} + (1+K)(\|\nabla v^1\|^2_{L^2_T(L^2)} + \|\sqrt{\tau}\nabla v^2\|^2_{L^2_T(L^\infty)}) + \|v_t^2\|_{L^1_T(L^2)}.$$

Now, if we consider data that belong to a bounded subset of $\widetilde{B}^0_{\rho_0,1}$, then K in (4-36) and $\tilde{A}(\infty)$ can be uniformly bounded. By iterating the procedure that led to (4-37), this allows us to get in the end

$$\|\sqrt{\rho_0^1}\,\delta v\|_{L^\infty_T(L^2)} + \|\nabla_{v^1}\delta v\|_{L^2_T(L^2)} \le Ce^{C\tilde{A}(\infty)}(\|\sqrt{\rho_0^1}\delta u_0\|_{L^2} + \|\delta\rho_0\|_{L^\infty}). \tag{4-38}$$

Then, reverting to the Eulerian coordinates gives the following stability statement.

Theorem 4.2. *Consider two solutions (ρ^1, u^1, P^1) and (ρ^2, u^2, P^2) corresponding to initial data (ρ_0^1, u_0^1) and (ρ_0^2, u_0^2) given by Theorem 1.3. Assume that*

$$0 < \rho_* \le \rho_0^1, \rho_0^2 \le \rho^* \quad \text{and} \quad \max(\|u_0^1\|_{\widetilde{B}^0_{\rho_0^1,1}}, \|u_0^2\|_{\widetilde{B}^0_{\rho_0^2,1}}) \le M.$$

Then we have

$$\|\sqrt{\rho_0^1}\,\delta u\|_{L^\infty_T(L^2)} + \|\nabla\delta u\|_{L^2_T(L^2)} \le C_{\rho_*,\rho^*,M}(\|\sqrt{\rho_0^1}\delta u_0\|_{L^2} + \|\delta\rho_0\|_{L^\infty}) \tag{4-39}$$

and, for all $p \in [2, \infty)$,

$$\|\delta\rho(t)\|_{\dot{W}^{-1,p}} \le C_{p,\rho_*,\rho^*,M}(\|\delta\rho_0\|_{\dot{W}^{-1,p}} + t^{1/2+1/p}(\|\sqrt{\rho_0^1}\delta u_0\|_{L^2} + \|\delta\rho_0\|_{L^\infty})). \tag{4-40}$$

Proof. Although our regularity assumptions are weaker, we shall follow [Danchin et al. 2024] to bound the difference of the velocities. The starting point is the relation

$$\nabla_y\delta v = K_1+K_2+K_3,\quad \text{with } K_1(t,y):=\nabla_y\delta X(t,y)\cdot\nabla_x u^2(t,X^2(t,y)),$$
$$K_2(t,y):=\nabla_y X^1(t,y)\cdot\nabla_x\delta u(t,X^2(t,y)),$$
$$K_3(t,y):=\nabla_y X^1(t,y)\cdot(\nabla_x u^1(t,X^2(t,y))-\nabla_x u^1(t,X^2(t,y))).$$

Since $\nabla\delta u(t,X^2(t,y))=A_1^\top(t,y)K_2(t,y)$ and the flow X^2 is measure-preserving, the above decomposition implies that

$$\|\nabla\delta u\|_{L^2}\le\|A_1\|_{L^\infty}(\|\nabla\delta v\|_{L^2}+\|K_1\|_{L^2}+\|K_3\|_{L^2}).$$

Bounding K_1 may be done as in [Danchin et al. 2024]. We get, for all $t\ge 0$,

$$\|K_1(t)\|_{L^2}\le C\|\sqrt{t}\nabla u^2(t)\|_{L^\infty}\|\nabla\delta v\|_{L^2_t(L^2)}.$$

For bounding K_3, we use the relation

$$K_3(t,y)=\nabla X_1(t,y)\cdot\left(\int_1^2(\nabla^2u^1(t,X^s(t,y)))\cdot\left(\frac{dX^s}{ds}(t,y)\right)ds\right),$$

where the “interpolating flow” X^s stands for the solution to

$$X^s(t,y)=y+\int_0^t((2-s)u^1(\tau,X^s(\tau,y))+(s-1)u^2(\tau,X^s(\tau,y)))\,d\tau.$$

As $X^s(t,\cdot)$ is also measure-preserving, it is easy to prove that (again, see [Danchin et al. 2024])

$$\left\|\frac{dX^s}{ds}(t,\cdot)\right\|_{L^4}\le C\|\delta u\|_{L^1_t(L^4)}.$$

Thanks to that and to Hölder’s inequality, we deduce that

$$\|K_3(t)\|_{L^2}\le C(1+\|\nabla u^1\|_{L^1_t(L^\infty)})\|t^{3/4}\nabla^2u^1(t)\|_{L^4}\|\delta u\|_{L^4_t(L^4)}.$$

Hence, in the end, if T is chosen such that

$$\max\left(\int_0^T\|\nabla u^1(t)\|_{L^\infty}\,dt,\int_0^T\|\nabla u^2(t)\|_{L^\infty}\,dt\right)\le 1,$$

then we have, using also (A-4),

$$\|\nabla\delta u\|_{L^2_T(L^2)}\lesssim(1+\|\sqrt{t}\nabla u^2\|_{L^2_T(L^\infty)})\|\nabla\delta v\|_{L^2_T(L^2)}+\|t^{3/4}\dot u^1\|_{L^2_T(L^4)}\|\delta u\|_{L^4_T(L^4)}.$$

The last term may be handled by means of (0-7), and one ends up with

$$\|\nabla\delta u\|_{L^2_T(L^2)}\lesssim(1+\|\sqrt{t}\nabla u^2\|_{L^2_T(L^\infty)})\|\nabla\delta v\|_{L^2_T(L^2)}+\|t^{3/4}\dot u^1\|^2_{L^2_T(L^4)}\|\sqrt{\rho^1}\delta u\|_{L^\infty_T(L^2)}.\tag{4-41}$$

Remember that the constructed solutions satisfy $\sqrt{t}\nabla u^2\in L^2(\mathbb{R}_+;L^\infty)$ and note that, since

$$\|t^{3/4}\dot u^1\|_{L^2_T(L^4)}\le C\|t\dot u^1\|^{1/2}_{L^\infty_T(L^2)}\|\sqrt{t}D\dot u^1\|^{1/2}_{L^1_T(L^2)},$$

inequalities (2-21) and (3-47) guarantee that $t^{3/4}\dot{u}^1$ is in $L^2(\mathbb{R}_+; L^4)$. So we are left with bounding $\sqrt{\rho^1}\delta u$ in $L^\infty(0, T; L^2)$. To do so, we use, as in [Danchin et al. 2024], the relation

$$\sqrt{\rho_0^1(y)}\delta v(t, y) = \sqrt{\rho^1}(t, X^1(t, y))\left(\delta u(t, X^1(t, y)) + \int_1^2 Du^2(t, X^s(t, y))\frac{dX^s}{ds}(t, y)\, ds\right).$$

Hence, as all the flows X^s are measure-preserving and ρ_1 is bounded from below,

$$\begin{aligned}\|\sqrt{\rho^1(t)}\delta u(t)\|_{L^2} &\le \|\sqrt{\rho_0^1}\delta v(t)\|_{L^2} + C\sqrt{\rho^*}\|Du^2(t)\|_{L^4}\|\delta u\|_{L^1_t(L^4)}\\ &\le \|\sqrt{\rho_0^1}\delta v(t)\|_{L^2} + C\|t^{3/4}Du^2(t)\|_{L^4}\|\delta u\|_{L^4_t(L^4)}\\ &\le \|\sqrt{\rho_0^1}\delta v(t)\|_{L^2} + C\|\sqrt{t}Du^2(t)\|_{L^2}^{1/2}\|tD^2u^2(t)\|_{L^2}^{1/2}\|\nabla\delta u\|_{L^2_t(L^2)}^{1/2}\|\sqrt{\rho^1(t)}\delta u\|_{L^\infty_t(L^2)}^{1/2}.\end{aligned}$$

Since both the terms with $\sqrt{t}Du^2$ and with tD^2u^2 may be bounded in terms of ρ_*, ρ^*, and $\|u_0^2\|_{L^2}$ only, we end up with

$$\|\sqrt{\rho^1}\delta u\|_{L^\infty_T(L^2)} \le 2\|\sqrt{\rho_0^1}\delta v\|_{L^\infty_T(L^2)} + C(\rho_*, \rho^*, \|u_0^2\|_{L^2})\|\nabla\delta u\|_{L^2_T(L^2)}.$$

Putting this inequality together with (4-41) and remembering (4-38) allows us to conclude that there exists an absolute constant C such that, for small enough T, we have

$$\|\sqrt{\rho_0^1}\,\delta u\|_{L^\infty_T(L^2)} + \|\nabla\delta u\|_{L^2_T(L^2)} \le C(\|\sqrt{\rho_0^1}\delta u_0\|_{L^2} + \|\delta\rho_0\|_{L^\infty}),$$

then arguing by induction and using the bounds on u^1 and u^2 in terms of the data yields (4-39).

Finally, the difference between the (Eulerian) densities may be bounded by resorting to the classical theory of transport equation. Indeed, we have

$$\partial_t\delta\rho + \operatorname{div}(\delta\rho u^2) = -\operatorname{div}(\rho^1\delta u).$$

Hence, we may write, for all $p \in [1, \infty]$ and $t \ge 0$,

$$\begin{aligned}\|\delta\rho(t)\|_{\dot{W}^{-1,p}} &\le \left(\|\delta\rho_0\|_{\dot{W}^{-1,p}} + \int_0^t e^{-\int_0^\tau \|\nabla u^2\|_{L^\infty}\,d\tau'}\|\rho_1\delta u\|_{L^p}\,d\tau\right)e^{\int_0^t \|\nabla u^2\|_{L^\infty}\,d\tau}\\ &\le (\|\delta\rho_0\|_{\dot{W}^{-1,p}} + \rho^* t^{1/2+1/p}\|\delta u\|_{L_t^{2p/(p-2)}(L^p)})e^{\int_0^t \|\nabla u^2\|_{L^\infty}\,d\tau}.\end{aligned}$$

Combining inequality (4-39) with the Gagliardo–Nirenberg inequality provides us with a control of δu in $L^{2p/(p-2)}(\mathbb{R}_+; L^p)$ for all $p \in [2, \infty)$. In the end, we get (4-40). □

Remark 4.3. In the bounded or torus cases, one can take advantage of exponential decay to get a time-independent bound. The details are left to the reader.

Appendix

Here we recall some results that played a key role throughout the paper. The first one is the following Gagliardo–Nirenberg inequality that extends (0-7):

$$\|z\|_{L^p} \le C_p\|z\|_{L^2}^{2/p}\|\nabla z\|_{L^2}^{1-2/p}, \quad 2 \le p < \infty. \tag{A-1}$$

It holds with the same constant in $\mathbb{R}^2$ and for any $z \in H_0^1(\Omega)$ in a general domain Ω, or in the torus $\mathbb{T}^2$ *provided the mean value of z is zero.* In the torus case, however, we rather are in situations where

$$\int_{\mathbb{T}^2} az\,dx = 0$$

for some nonnegative measurable function a with positive mean value (say 1 with no loss of generality). Then, we claim that

$$\|z\|_{L^p} \le C_{p,a}\|z\|_{L^2}^{2/p}\|\nabla z\|_{L^2}^{1-2/p}, \quad \text{with } C_{p,a} := C_p \log^{(p-2)/p}(e + \|a\|_{L^2}). \tag{A-2}$$

Indeed, decomposing z into $z = \bar{z} + \tilde{z}$ with $\bar{z} := \fint_{\mathbb{T}^2} z\,dx$, we have

$$\begin{aligned}\int_{\mathbb{T}^2} |z|^p\,dx &= \int_{\mathbb{T}^2} |z|^2|\tilde{z}+\bar{z}|^{p-2}\,dx\\ &\lesssim |\bar{z}|^{p-2}\|z\|_{L^2}^2 + \int_{\mathbb{T}^2} |z|^2|\tilde{z}|^{p-2}\,dx\\ &\lesssim |\bar{z}|^{p-2}\|z\|_{L^2}^2 + \|z\|_{L^p}^2\|\tilde{z}\|_{L^p}^{p-2}.\end{aligned}$$

Now, $\tilde{z}$ is mean-free and thus satisfies (A-1). Besides, according to [Danchin and Mucha 2019, (A.2)],

$$|\bar{z}| \le C\log(e + \|a\|_{L^2})\|\nabla z\|_{L^2}.$$

Hence,

$$\|z\|_{L^p}^p \le C\log(e+\|a\|_{L^2})\|\nabla z\|_{L^2}^{p-2}\|z\|_{L^2}^2 + C_p\|z\|_{L^p}^2(\|\tilde{z}\|_{L^2}^{2/p}\|\nabla z_{L^2}^{1-2/p})^{p-2}.$$

Then, (A-2) follows from $\|\tilde{z}\|_{L^2} \le \|z\|_{L^2}$. □

Next, we recall a well-known result for the inhomogeneous Stokes equations

$$-\Delta w + \nabla Q = f \quad \text{and} \quad \operatorname{div} w = g \quad \text{in } \Omega, \tag{A-3}$$

with data $f \in L^p(\Omega)$ and $g \in \dot{W}^{1,p}(\Omega)$, $1 < p < \infty$.

In the bounded domain case (with g having mean value 0), it is known (see, e.g., [Galdi 2011]) that (A-3) admits a unique solution $(w, \nabla Q) \in W^{2,p}(\Omega) \times L^p(\Omega)$ such that $w|_{\partial\Omega} = 0$, and that the following bound holds:

$$\|\nabla^2 w, \nabla Q\|_{L^p} \le C(\|f\|_{L^p} + \|\nabla g\|_{L^p}). \tag{A-4}$$

A similar result holds in $\Omega = \mathbb{R}^2$ or $\Omega = \mathbb{T}^2$ provided we consider only solutions such that $w \to 0$ at infinity ($\mathbb{R}^2$ case) or $\int_{\mathbb{T}^2} aw\,dx = 0$ for some nonnegative bounded function a with mean value 1 (torus case). Indeed, one can set

$$\nabla Q = \mathcal{Q}f, \quad \text{with } \mathcal{Q} := -(-\Delta)^{-1}\nabla\operatorname{div},$$

then solve the Poisson equation $-\Delta w = f + \nabla Q$. Uniqueness is given by the supplementary conditions that are prescribed above.

Finally, in the proof of stability and uniqueness, we used the following result.

Lemma A.1. *Assume that Ω is a C^2 bounded domain, the torus, or the whole space. Then, there exists a linear operator $\mathcal{B}$ that maps L^p to L^p for all $p \in (1, \infty)$ such that, for all $k \in L^p(\Omega; \mathbb{R}^d)$ (with mean value 0 in the case $\Omega = \mathbb{T}^d$), we have*

$$\operatorname{div}(\mathcal{B}k) = \operatorname{div} k.$$

Furthermore, if $\operatorname{div} k \in L^q(\Omega)$ for some $q \in (1, \infty)$, then we have $\mathcal{B}k \in W_0^{1,q}(\Omega; \mathbb{R}^n)$ with $\|\nabla \mathcal{B}k\|_{L^q} \leq C\|\operatorname{div} k\|_{L^q}$, and if k (seen as a function from $\mathbb{R}_+$ to some space L^r with $1 < r < \infty$) is differentiable for almost every $t \in \mathbb{R}_+$, then so is $\mathcal{B}k$, and we have $\|(\mathcal{B}k)_t\|_{L^r} \leq C\|k_t\|_{L^r}$ for almost every $t \in \mathbb{R}_+$.

Proof. Whenever Ω is a C^2 bounded domain, the existence of $\mathcal{B}$ as well as the first two properties have been established in [Danchin and Mucha 2013a]. The third one stems from the fact that, owing to the continuity and linearity of $\mathcal{B}$, we may write in the L^r meaning

$$(\mathcal{B}k)_t(t) = \lim_{h\to 0} \frac{\mathcal{B}k(t+h) - \mathcal{B}k(t)}{h} = \lim_{h\to 0} \mathcal{B}\left(\frac{k(t+h) - k(t)}{h}\right) = \mathcal{B}k_t.$$

If Ω is the torus or the whole space, then one can just set $\mathcal{B} := -(-\Delta)^{-1}\nabla \operatorname{div}$. □

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