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QUANTITATIVE STABILITY FOR COMPLEX MONGE–AMPÈRE EQUATIONS, I

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We generalize several known stability estimates for complex Monge–Ampère equations to the setting of low (or high) energy potentials. We apply our estimates to obtain, among other things, a quantitative domination principle, and metric properties of the space of potentials of finite energy. Further applications will be given in subsequent papers.

1. Introduction

Let (X, ω) be a compact Kähler manifold of dimension n and let α be a big cohomology $(1, 1)$-class in X. Let θ be a closed smooth real $(1, 1)$-form in α. For $u \in \mathrm{PSH}(X, \theta)$, we put $\theta_u := dd^c u + \theta$. Let $\phi \in \mathrm{PSH}(X, \theta)$ such that $\phi \le 0$ and $\int_X \theta_\phi^n > 0$, where θ_ϕ^n denotes the non-pluripolar self-product of θ_ϕ (see [Bedford and Taylor 1987; Boucksom et al. 2010]). Denote by $\mathrm{PSH}(X, \theta, \phi)$ the set of θ-psh functions u with $u \le \phi$. Note that it is slightly different from the usual definition of $\mathrm{PSH}(X, \theta, \phi)$ in which u is only required to be more singular than ϕ. This difference is not essential. We say that ϕ is a *model θ-psh function* (see [Darvas et al. 2018b; Ross and Witt Nyström 2014]) if $\phi = P_\theta[\phi]$ and $\int_X \theta_\phi^n > 0$, where

$$P_\theta[\phi] := \big(\sup\{\psi \in \mathrm{PSH}(X, \theta) : \psi \le 0,\ \psi \le \phi + O(1)\}\big)^*.$$

The function $P_\theta[\phi]$ is called a roof-top envelope in [Darvas et al. 2018b]. By [Darvas et al. 2018b], the function $P_\theta[u]$ is a model one for every $u \in \mathrm{PSH}(X, \theta)$ with $\int_X \theta_u^n > 0$, and for every $u \in \mathrm{PSH}(X, \theta, \phi)$ with $\int_X \theta_u^n = \int_X \theta_\phi^n$ we have $P_\theta[u] = P_\theta[\phi]$.

Let ϕ be now a model θ-psh function. Let $\mathcal{E}(X, \theta, \phi)$ be the space of θ-psh functions $u \le \phi$ with $\int_X \theta_u^n = \int_X \theta_\phi^n$. Let μ be a non-pluripolar measure with $\mu(X) = \int_X \theta_\phi^n$. It was proved in [Darvas et al. 2021a] (see also [Darvas et al. 2018b; Do and Vu 2022a]) that the Monge–Ampère equation with prescribed singularities

$$(dd^c u + \theta)^n = \mu, \quad u \in \mathrm{PSH}(X, \theta, \phi), \tag{1-1}$$

admits a unique solution $u \in \mathcal{E}(X, \theta, \phi)$ and $\sup_X (u - \phi) = 0$. We note that the left-hand side of (1-1) denotes the non-pluripolar self-product of θ_u (see [Bedford and Taylor 1987; Boucksom et al. 2010; Guedj and Zeriahi 2007; Vu 2021]). We refer to [Boucksom et al. 2010; Cegrell 1998; Dinew 2009; Kołodziej 1998; Yau 1978], to cite a few, for the well-known case where α is big and ϕ is a potential of minimal singularities in α.

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The aim of this paper is to study the following stability question for (1-1).

Problem 1.1. *Let θ, ϕ be as above. Let $u_j \in \mathcal{E}(X, \theta, \phi)$ for $j = 1, 2$ and $\mu_j := \theta^n_{u_j}$ for $j = 1, 2$. Compare u_1 with u_2 in terms of a suitable "distance" between μ_1, μ_2.*

To our best knowledge, there has been no available quantitative comparison between potentials of finite energy in general, even in the case where α is Kähler and $\phi \equiv 0$. The closest result that we know of is the uniqueness property (by [Dinew 2009] in the Kähler case and by [Boucksom et al. 2010; Darvas et al. 2021a] in the present setting) which says that $u_1 = u_2$ if $\mu_1 = \mu_2$. There were however some concrete estimates for the distance between u_1, u_2 in terms of μ_1, μ_2 but one had to assume some extra assumption (i.e., $u_1, u_2 \in \mathcal{E}^1(X, \theta, \phi)$); see [Błocki 2003; Guedj and Zeriahi 2012]. We will explain details below.

The goal of this paper is to solve Problem 1.1 for any potential of high or low energy. As one will see in our applications later in this paper or in our subsequent paper, it is crucial to consider Problem 1.1 for potentials in low energy.

Let $\widetilde{\mathcal{W}}^-$ be the set of convex, nondecreasing functions $\chi : \mathbb{R}_{\le 0} \to \mathbb{R}_{\le 0}$ such that $\chi(0) = 0$ and $\chi \not\equiv 0$. Let $\mathcal{W}^-$ be the subset of $\chi \in \widetilde{\mathcal{W}}^-$ such that $\chi(-\infty) = -\infty$. Note that in general $\chi \in \widetilde{\mathcal{W}}^-$ can be bounded. It is crucial in our method that we consider also bounded weights $\chi \in \widetilde{\mathcal{W}}^-$. Let $M \ge 1$ be a constant and $\mathcal{W}^+_M$ the usual space of increasing concave functions $\chi : \mathbb{R}_{\le 0} \to \mathbb{R}_{\le 0}$ such that $\chi(0) = 0$, $\chi < 0$ on $(-\infty, 0)$, and $|t\chi'(t)| \le M|\chi(t)|$ for every $t \le 0$.

Let $\varrho := \int_X \theta^n_\phi$. For $\chi \in \widetilde{\mathcal{W}}^- \cup \mathcal{W}^+_M$ and $u \in \mathrm{PSH}(X, \theta, \phi)$, let

$$E^0_{\chi,\theta,\phi}(u) := -\varrho^{-1} \int_X \chi(u - \phi)\theta^n_u,$$

which is called *the* (*normalized*) *χ-energy* of u (with respect to θ, ϕ). We define

$$\mathcal{E}_\chi(X, \theta, \phi) := \{u \in \mathcal{E}(X, \theta, \phi) : E^0_{\chi,\theta,\phi}(u) < \infty\}.$$

Certainly if χ is bounded, then $\mathcal{E}_\chi(X, \theta, \phi) = \mathcal{E}(X, \theta, \phi)$. We would like to point out however that our method is not about the finiteness of $E^0_{\chi,\theta,\phi}(u)$ but estimating the size of that quantity. Thus whether χ is bounded or not does not make much difference for our later arguments. Put

$$I^0_\chi(u, v) := \varrho^{-1} \int_{\{u<v\}} \chi(u - v)(\theta^n_v - \theta^n_u) + \varrho^{-1} \int_{\{u>v\}} \chi(v - u)(\theta^n_u - \theta^n_v)$$

for $u, v \in \mathcal{E}_\chi(X, \theta, \phi)$. The factor ϱ^{-1} in the defining formulae for $E^0_{\chi,\theta,\phi}(u)$ and $I^0_\chi(u, v)$ plays the role of a normalizing constant. In geometric applications it is important to treat the case where $\varrho \to 0$, i.e., to obtain estimates uniformly as $\varrho \to 0$.

Clearly if $\theta^n_u = \theta^n_v$, then $I^0_\chi(u, v) = 0$. We will see later that each term in the sum defining $I^0_\chi(u, v)$ is nonnegative. We recall that there is a natural (quasi)metric on the space $\mathcal{E}_\chi(X, \theta, \phi)$ constructed in [Darvas 2019; 2024; Gupta 2023], and see [Darvas et al. 2018a; Di Nezza and Lu 2020; Trusiani 2022; Xia 2023] as well. The functional $I^0_\chi(u, v)$ has an intimate relation with these quasimetrics. We refer to the end of Section 3 for details on this connection. Here is our main result.

Theorem 1.2. *Let θ be a closed smooth real $(1,1)$-form and ϕ be a negative θ-psh function such that*

$$\varrho := \int_X \theta_\phi^n > 0.$$

Let $\chi, \tilde{\chi} \in \widetilde{\mathcal{W}}^- \cup \mathcal{W}_M^+$ $(M \geq 1)$ such that $\tilde{\chi} \leq \chi$, and if $\chi \in \widetilde{\mathcal{W}}^-$, then $\lim_{t\to-\infty}(\chi(t)/\tilde{\chi}(t)) = 0$. Let $B \geq 1$ be a constant and let $u_j, \psi_j \in \mathcal{E}(X,\theta,\phi)$ satisfy $u_1 \leq u_2$ and

$$E^0_{\tilde{\chi},\theta,\phi}(u_j) + E^0_{\tilde{\chi},\theta,\phi}(\psi_j) \leq B$$

for $j = 1, 2$. Then there exist a constant $C > 0$ depending only on n, $\tilde{\chi}(-1)$ and M, and a continuous increasing function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ depending only on $\chi, \tilde{\chi}$ such that $f(0) = 0$ and

$$\int_X -\chi(u_1 - u_2)(\theta_{\psi_1}^n - \theta_{\psi_2}^n) \leq C\varrho B^2 f^{\circ n}(I^0_\chi(u_1, u_2)), \tag{1-2}$$

where $f^{\circ n} := f \circ f \circ \cdots \circ f$ (n-iterate of f). Moreover, if $\phi = P_\theta[\phi]$ and $\sup_X u_1 = \sup_X u_2$ then

$$\int_X -\chi(u_1 - u_2)(\theta_{\psi_1}^n + \theta_{\psi_2}^n) \leq \varrho\, g(I^0_\chi(u_1, u_2)), \tag{1-3}$$

where $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a continuous increasing function depending only on n, M, X, ω, θ, χ, $\tilde{\chi}$ and B such that $g(0) = 0$.

If $\chi \in \mathcal{W}_M^+$, then one can certainly apply Theorem 1.2 to $\tilde{\chi} = \chi$. Nevertheless, we underline that in applications it is of crucial importance to consider $\chi \in \widetilde{\mathcal{W}}^-$. In this case in order to have (1-3), it is necessary to require an upper bound for $\tilde{\chi}$-energy of u_j, where $\tilde{\chi}$ "dominates" χ as in the statement of Theorem 1.2. We refer to Section 3.4 for details.

One sees that (1-3) implies, in particular, that if $I^0_\chi(u_1, u_2) \to 0$, then the expression in the left-hand side also converges to 0. Theorem 1.2 follows from Theorems 3.1 and 3.2 below, where the functions f and g are given explicitly. We note that *the single Theorem 1.2 contains the following three important results in pluripotential theory*: *uniqueness of solutions of complex Monge–Ampère equations*, *domination principle*, *and comparison of capacities*. We obtain indeed quantitative (hence stronger) versions of these results for which we refer to Section 4. The quantitative version of uniqueness theorem (see Theorem 4.2 below) provides an answer to Problem 1.1. Readers can also find, in Section 4, a quantitative version of the fact that the convergence in Darvas's metric in $\mathcal{E}_\chi(X,\theta,\phi)$ implies the convergence in capacity. Notice that such an estimate seems to be not reachable by using the usual plurisubharmonic envelope method.

The main novelty of Theorem 1.2 is that it deals with *arbitrary* weights. Similar statements was already known for $\chi(t) = t$ (see [Berman et al. 2019; Błocki 2003; Guedj and Zeriahi 2012; Trusiani 2023]). However the proof there only work *exclusively* for this case. One should notice that the weight $\chi(t) = t$ is very special: it is linear and lies in the middle between higher energy weights and lower energy weights. As to the proof of Theorem 1.2, going up to the space of higher energy weights or going down to the space of lower energy weights are equally difficult. We will explain this point in more details in the paragraph after Theorem 1.3 below.

The key in the proof of Theorem 1.2 is Proposition 3.5 in Section 3, a simplified version of which we state here for readers' convenience.

Theorem 1.3. *Let $\chi, \tilde{\chi} \in \widetilde{\mathcal{W}}^- \cup \mathcal{W}^+_M$ such that $\tilde{\chi} \le \chi$ and $\chi \in \mathscr{C}^1(\mathbb{R})$. Let $u_1, u_2, u_3 \in \mathcal{E}(X, \theta, \phi)$ such that $u_1 \le u_2$ and $u_j - \phi$ is bounded ($j = 1, 2, 3$), where ϕ is a negative θ-psh function satisfying $\varrho := \mathrm{vol}(\theta_\phi) > 0$. Then there exist a constant $C > 0$ depending only on n, $\tilde{\chi}(-1)$ and M such that*

$$\int_X \chi'(u_1 - u_2)\, d(u_1 - u_2) \wedge d^c(u_1 - u_2) \wedge \theta^{n-1}_{u_3} \le C\varrho B^2 f^{\circ(n-1)}(I^0_\chi(u_1, u_2)),$$

where $B := \sum_{j=1}^3 \max\{E^0_{\tilde{\chi},\theta,\phi}(u_j), 1\}$ and $f : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is a continuous function such that $f(0) = 0$ if one has either $\chi \in \mathcal{W}^+_M$ or $\chi \in \widetilde{\mathcal{W}}^-$ and $\lim_{t\to-\infty}(\chi(t)/\tilde{\chi}(t)) = 0$.

As far as we know, all of previous works related to Theorem 1.3 only concern $\chi(t) = t$. In this case, Theorem 1.3 is known with an explicit f and without $\tilde{\chi}$ if ϕ is of minimal singularity in the cohomology class of θ, by [Błocki 2003; Guedj and Zeriahi 2012].

The key ingredients in previous versions of Theorem 1.3 for $\chi(t) = t$ are integration by parts arguments. Direct generalization of such reasoning immediately break down if $\chi \ne \mathrm{id}$: in a more precise but technical level, the integration by parts arguments give terms like $\chi'(u_1 - u_2)d(u_1 - u_3) \wedge d^c(u_1 - u_3)$, such a quantity is easy to bound if $\chi = \mathrm{id}$ (hence $\chi' \equiv 1$), but not if $\chi \ne \mathrm{id}$.

In order to prove Theorem 1.3, we still use this strategy but need to use a so-called "monotonicity argument" from [Do and Vu 2022a; Vu 2021; 2022] to deal with general χ. In a nutshell it is about using intensively the plurilocality of Monge–Ampère operators together with the monotonicity of pluricomplex energy which allows one to bound from above "Monge–Ampère quantities" of bad potentials by that of nicer potentials. This method is a flexible tool to deal with "low regularity", and was a key in the proof of the convexity of the class of potentials of finite χ-energy in [Vu 2022], as well as giving a characterization of the class of Monge–Ampère measures with potentials of finite χ-energy in [Do and Vu 2022a].

We refer to the end of the paper for some applications of our main results. Furthermore, the quantitative domination principle obtained in Section 4 was used crucially in [Dang and Vu 2023] to describe the degeneration of conic Kähler–Einstein metrics. We note also that the present paper is the first part of the manuscript [Do and Vu 2022b], in which we give a more or less satisfactory treatment for a much more general question than Problem 1.1: precisely, we establish quantitative stability when both the cohomology class and the singularity type vary. The second part of [Do and Vu 2022b], where this generalization is treated, will be submitted separately due to the length constraint.

The paper is organized as follows. In Section 2, we recall the integration by parts formula from [Vu 2022], auxiliary facts about weights are also collected there. Theorems 1.2 and 1.3 are proved in Section 3. Applications will be given in Section 4.

2. Preliminaries

2.1. ***Integration by parts.*** In this subsection, we recall the integration by parts formula obtained in [Vu 2022, Theorem 2.6]. This formula will play a key role in our proof of main results later.

Let X be a compact Kähler manifold. Let $T_1, \dots, T_m$ be closed positive $(1, 1)$-currents on X. Let T be a closed positive current of bidegree (p, p) on X. The *T-relative non-pluripolar product* $\langle \bigwedge_{j=1}^m T_j \dot{\wedge} T \rangle$ is defined in a way similar to that of the usual non-pluripolar product (see [Vu 2021]). The product $\langle \bigwedge_{j=1}^m T_j \dot{\wedge} T \rangle$

is a closed positive current of bidegree $(m+p, m+p)$; and the wedge product $\langle \bigwedge_{j=1}^{m} T_j \dot\wedge T\rangle$ as an operator on currents is symmetric with respect to $T_1, \dots, T_m$ and is homogeneous. In latter applications, we will only use the case where T is the non-pluripolar product of some closed positive (1, 1)-currents, say, $T = \langle T_{m+1} \wedge \cdots \wedge T_{m+l}\rangle$, where T_j are (1, 1)-currents for $m+1 \le j \le m+l$. In this case, $\langle T_1 \wedge \cdots \wedge T_m \dot\wedge T\rangle$ is simply equal to $\langle \bigwedge_{j=1}^{m+l} T_j\rangle$. We usually remove the bracket $\langle\ \rangle$ in the non-pluripolar product to ease the notation.

Recall that a *dsh* function on X is the difference of two quasi-plurisubharmonic (quasi-psh for short) functions on X (see [Dinh and Sibony 2006]). These functions are well-defined outside pluripolar sets. Let v be a dsh function on X. Let T be a closed positive current on X. We say that v is *T-admissible* if there exist quasi-psh functions φ_1, φ_2 such that $v = \varphi_1 - \varphi_2$ and T has no mass on $\{\varphi_j = -\infty\}$ for $j = 1, 2$. In particular, if T has no mass on pluripolar sets, then every dsh function is T-admissible.

Assume now that v is T-admissible. Let φ_1, φ_2 be quasi-psh functions such that $v = \varphi_1 - \varphi_2$ and T has no mass on $\{\varphi_j = -\infty\}$ for $j = 1, 2$. Let

$$\varphi_{j,k} := \max\{\varphi_j, -k\}$$

for every $j = 1, 2$ and $k \in \mathbb{N}$. Put $v_k := \varphi_{1,k} - \varphi_{2,k}$. Put

$$Q_k := dv_k \wedge d^c v_k \wedge T = \tfrac{1}{2} dd^c v_k^2 \wedge T - v_k dd^c v_k \wedge T.$$

By the plurifine locality with respect to T (see [Vu 2021, Theorem 2.9]) applied to the right-hand side of the last equality, we have

$$\mathbf{1}_{\bigcap_{j=1}^{2}\{\varphi_j > -k\}} Q_k = \mathbf{1}_{\bigcap_{j=1}^{2}\{\varphi_j > -k\}} Q_s \tag{2-1}$$

for every $s \ge k$. We say that $\langle dv \wedge d^c v \dot\wedge T\rangle$ is *well-defined* if the mass of $\mathbf{1}_{\bigcap_{j=1}^{2}\{\varphi_j > -k\}} Q_k$ is uniformly bounded on k. In this case, using (2-1) implies that there exists a positive current Q on X such that for every bounded Borel form Φ with compact support on X such that

$$\langle Q, \Phi\rangle = \lim_{k\to\infty} \langle \mathbf{1}_{\bigcap_{j=1}^{2}\{\varphi_j > -k\}} Q_k, \Phi\rangle,$$

and we define $\langle dv \wedge d^c v \dot\wedge T\rangle$ to be the current Q. This agrees with the classical definition if v is the difference of two bounded quasi-psh functions. One can check that this definition is independent of the choice of φ_1, φ_2. By [Vu 2022, Lemma 2.5], if v is bounded, then $\langle dv \wedge d^c v \dot\wedge T\rangle$ is well-defined.

Let w be another T-admissible dsh function. If T is of bidegree $(n-1, n-1)$, we can also define the current $\langle dv \wedge d^c w \dot\wedge T\rangle$ by a similar procedure as above. More precisely, we say $\langle dv \wedge d^c w \dot\wedge T\rangle$ is *well-defined* if $\langle dv \wedge d^c v \dot\wedge T\rangle$, $\langle dw \wedge d^c w \dot\wedge T\rangle$, and $\langle d(v+w) \wedge d^c (v+w) \dot\wedge T\rangle$ are well-defined. In this case, as in the classical case of bounded potentials, the defining formula for $\langle dv \wedge d^c w \dot\wedge T\rangle$ is obvious:

$$2\langle dv \wedge d^c w \dot\wedge T\rangle = \langle d(v+w) \wedge d^c (v+w) \dot\wedge T\rangle - \langle dv \wedge d^c v \dot\wedge T\rangle - \langle dw \wedge d^c w \dot\wedge T\rangle.$$

As above, if v, w are bounded T-admissible, then $\langle dv \wedge d^c w \dot\wedge T\rangle$ is well-defined and given by the above formula. The following Cauchy–Schwarz inequality is clear from definition.

Lemma 2.1. *Assume that $\langle dv \wedge d^c w \dot\wedge T\rangle$ is well-defined. Then, for every positive Borel function χ, we have*

$$\int_X \chi \langle dv \wedge d^c w \dot\wedge T\rangle \le \Big(\int_X \chi \langle dv \wedge d^c v \dot\wedge T\rangle\Big)^{1/2} \Big(\int_X \chi \langle dw \wedge d^c w \dot\wedge T\rangle\Big)^{1/2}.$$

We put

$$\langle dd^c v \dot{\wedge} T\rangle := \langle dd^c\varphi_1 \dot{\wedge} T\rangle - \langle dd^c\varphi_2 \dot{\wedge} T\rangle,$$

which is independent of the choice of φ_1, φ_2. The following integration by parts formula is crucial for us later.

Theorem 2.2 ([Vu 2022, Theorem 2.6] or [Do and Vu 2022a, Theorem 3.1]). *Let T be a closed positive current of bidegree $(n-1, n-1)$ on X. Let v, w be bounded T-admissible dsh functions on X. If $\chi : \mathbb{R} \to \mathbb{R}$ is a $\mathscr{C}^3$ function then*

$$\begin{aligned}\int_X \chi(w)\langle dd^c v \dot{\wedge} T\rangle &= \int_X v\chi''(w)\langle dw \wedge d^c w \dot{\wedge} T\rangle + \int_X v\chi'(w)\langle dd^c w \dot{\wedge} T\rangle \\ &= -\int_X \chi'(w)\langle dw \wedge d^c v \dot{\wedge} T\rangle. \end{aligned} \tag{2-2}$$

Since the case where T is a non-pluripolar product of $(1, 1)$-currents plays an important role in the study of complex Monge–Ampère equations, we present below an equivalent natural way to define the current $\langle d\varphi \wedge d^c\varphi \dot{\wedge} T\rangle$ in this setting. It is just for the purpose of clarification.

Lemma 2.3. *Let $u_1, \dots, u_m$ be negative psh functions on an open subset U in $\mathbb{C}^n$ such that $T := \langle dd^c u_1 \wedge \cdots \wedge dd^c u_m\rangle$ is well-defined. Let v be the difference of two bounded psh functions on U. For $k \in \mathbb{N}$, put $u_{j,k} := \max\{u_j, -k\}$ and*

$$T_k := dd^c u_{1,k} \wedge \cdots \wedge dd^c u_{m,k}.$$

Then

$$dv \wedge d^c v \wedge T = dv \wedge d^c v \wedge T_k$$

on $\bigcap_{j=1}^m \{u_j > -k\}$.

Proof. Put

$$\psi_k := k^{-1}\max\{u_1 + \cdots + u_m, -k\} + 1.$$

Observe $\psi_k T_k = \psi_k T$. Now regularizing v and using the continuity of Monge–Ampère operators of bounded potentials, we obtain

$$\psi_k dv \wedge d^c v \wedge T = \psi_k dv \wedge d^c v \wedge T_k.$$

Hence

$$dv \wedge d^c v \wedge T = dv \wedge d^c v \wedge T_k$$

on $U := \bigcap_{j=1}^m \{u_j > -k/(2m)\}$ (for $\psi_k \geq \frac{1}{2}$ on U). Note that $dv \wedge d^c v \wedge T_k = dv \wedge d^c v \wedge T_{k/(2m)}$ on U by the plurifine locality. Thus the desired assertion follows. □

Let $T_1, \dots, T_m$ be closed positive $(1, 1)$-currents on X. Let $n := \dim X$. Consider now

$$T := \langle T_1 \wedge \cdots \wedge T_m\rangle.$$

Note that T has no mass on pluripolar sets. Let φ_1, φ_2 be negative quasi-psh function on X. Let $\varphi_{j,k}$ $(j = 1, 2)$ be as before and $v := \varphi_1 - \varphi_2$. In the moment, we work locally. Let U be an open small enough local chart (biholomorphic to a polydisk in $\mathbb{C}^n$) in X such that $T_j = dd^c u_j$ for $j = 1, \dots, m$, where u_j are negative psh functions on U. Put $u_{j,k} := \max\{u_j, -k\}$ for $k \in \mathbb{N}$, and

$$T_k := dd^c u_{1,k} \wedge \cdots \wedge dd^c u_{m,k}, \quad Q'_k := dv_k \wedge d^c v_k \wedge T_k.$$

Put $A_k := \bigcap_{j=1}^2 \{\varphi_j > -k\} \cap \bigcap_{j=1}^m \{u_j > -k\}$. By plurifine properties of Monge–Ampère operators,

$$\mathbf{1}_{A_k} Q'_k = \mathbf{1}_{A_k} Q'_s$$

for every $s \geq k$. One can check that the condition that $(\mathbf{1}_{A_k} Q'_k)_k$ is of mass bounded uniformly (on compact subsets in U) in k is independent of the choice of potentials.

Proposition 2.4. *The current $\mathbf{1}_{A_k} Q'_k$ is of mass bounded uniformly in k on compact subsets in U for every U (small enough biholomorphic to a polydisk in $\mathbb{C}^n$) if and only if the current $\langle dv \wedge d^c v \dot{\wedge} T\rangle$ is well-defined. In this case*

$$\langle dv \wedge d^c v \dot{\wedge} T\rangle = \lim_{k\to\infty} \mathbf{1}_{A_k} Q'_k. \tag{2-3}$$

Proof. By writing a smooth form of bidegree $(n-m-1, n-m-1)$ as the difference of two smooth positive forms, we can assume without loss of generality that T is of bidegree $(n-1, n-1)$ (hence $m = n-1$). Assume that $\langle dv \wedge d^c v \dot{\wedge} T\rangle$ is well-defined. We will check that $\mathbf{1}_{A_k} Q'_k$ is of mass bounded uniformly in k on compact subsets in U. Let χ be a nonnegative smooth function compactly supported on U. Put

$$\psi := \varphi_1 + \varphi_2 + u_1 + \cdots + u_m, \quad \psi_k := k^{-1} \max\{\psi, -k\} + 1,$$

and $\varphi_{jk} := \max\{\varphi_j, -k\}$ for $1 \leq j \leq 2$. Observe that $0 \leq \psi_k \leq 1$ and if $\psi_k > 0$, then $\varphi_j > -k$ for $1 \leq j \leq 2$; and

$$\psi_k(x) \geq 1 - s/k \tag{2-4}$$

for every $x \in A_{s/(m+2)}$ and $1 \leq s \leq k$. Recall $v_k := \varphi_{1k} - \varphi_{2k}$ which is bounded (but not necessarily uniformly in k). Observe that $\langle dv \wedge d^c v \dot{\wedge} T\rangle$ has no mass on pluripolar sets because T is so (see, for example, [Vu 2021, Lemma 2.1]). Put $Q''_k := \psi_k Q_k = \psi_k \mathbf{1}_{A_k} Q'_k$. By (2-4) and Lemma 2.3, we have

$$\begin{aligned}\langle dv \wedge d^c v \dot{\wedge} T\rangle &= \lim_{k\to\infty} \psi_k dv_k \wedge d^c v_k \wedge T \\ &= \lim_{k\to\infty} \psi_k dv_k \wedge d^c v_k \wedge T_k = \lim_{k\to\infty} Q''_k \end{aligned} \tag{2-5}$$

on U. On the other hand, by (2-4) again, we see that the claim that Q''_k is of mass uniformly bounded on compact subsets in U is equivalent to that $\mathbf{1}_{A_k} Q'_k$ is so. This together with (2-5) yields the desired assertion.

Conversely, suppose now that $\mathbf{1}_{A_k} Q'_k$ is of mass bounded uniformly in k on compact subsets in U for every U. Thus there exists a positive current R on U such that $\mathbf{1}_{A_k} R = \mathbf{1}_{A_k} Q'_k$ for every k and U. Set

$$\tilde{\psi} := \varphi_1 + \varphi_2, \quad \tilde{\psi}_k := k^{-1} \max\{\tilde{\psi}, -k\} + 1.$$

Let $s \in \mathbb{N}$ with $s \geq k$. Observe

$$\psi_s R = \tilde{\psi}_k \psi_s R + (1 - \tilde{\psi}_k)\psi_s R.$$

The second term in the right-hand side of the last inequality tends to 0 (uniformly in s) because $\tilde{\psi}_k$ converges pointwise to 1 outside a pluripolar set and R has no mass on pluripolar sets. Using Lemma 2.3, we have

$$\begin{aligned}\tilde{\psi}_k\psi_s R &= \tilde{\psi}_k\psi_s dv_s\wedge d^c v_s\wedge T_s\\ &= \tilde{\psi}_k\psi_s dv_s\wedge d^c v_s\wedge T = \tilde{\psi}_k\psi_s dv_k\wedge d^c v_k\wedge T.\end{aligned}$$

Here we used the plurifine topology properties with respect to T (see [Vu 2021, Theorem 2.9]), thanks to the fact that $\varphi_{j,k}=\varphi_{j,s}$ on $\{\tilde{\psi}_k\neq 0\}$ for $j=1,2$ (recall $s\geq k$), and they are bounded psh functions. Letting $s\to\infty$ gives

$$\tilde{\psi}_k R = \tilde{\psi}_k \mathbf{1}_{\bigcup_{j=1}^m\{u_j>-\infty\}} dv_k\wedge d^c v_k\wedge T = \tilde{\psi}_k dv_k\wedge d^c v_k\wedge T$$

because the current $dv_k\wedge d^c v_k\wedge T$ has no mass on pluripolar sets. Now letting $k\to\infty$ gives the desired assertion. □

Thanks to Proposition 2.4, we can use the right-hand side of (2-3) to define $\langle dv\wedge d^c v\dot{\wedge}T\rangle$ in the case where T is the non-pluripolar product of some closed positive $(1,1)$-currents. By the same reason, in this case, we will use the expression $dv\wedge d^c w\wedge T_1\wedge\ldots\wedge T_{n-1}$ to denote $\big\langle dv\wedge d^c w\dot{\wedge}\langle T_1\wedge\cdots\wedge T_{n-1}\rangle\big\rangle$ whenever it is well-defined.

2.2. ***Auxiliary facts on weights.*** In this subsection, we present some facts about weights needed for the proofs of main results.

Recall that $\widetilde{\mathcal{W}}^-$ is the set of all convex, nondecreasing functions $\chi:\mathbb{R}_{\leq 0}\to\mathbb{R}_{\leq 0}$ such that $\chi(0)=0$ and $\chi\not\equiv 0$. Let $M\geq 1$ be a constant and $\mathcal{W}_M^+$ the usual space of increasing concave functions $\chi:\mathbb{R}_{\leq 0}\to\mathbb{R}_{\leq 0}$ such that $\chi(0)=0$, $\chi<0$ on $(-\infty,0)$, and $|t\chi'(t)|\leq M|\chi(t)|$ for every $t\leq 0$. We have the following lemmas.

Lemma 2.5. *Let $c>0$, $0<\delta<1$ and $\chi:\mathbb{R}\to\mathbb{R}$ such that $\chi(t)=ct$ for every $t\geq-\delta$ and $\chi|_{(-\infty,0]}\in\widetilde{\mathcal{W}}^-\cup\mathcal{W}_M^+$ $(M\geq 1)$. Let g be a smooth radial cut-off function supported in $[-1,1]$ on $\mathbb{R}$, i.e., $g(t)=g(-t)$ for $t\in\mathbb{R}$, $0\leq g\leq 1$ and $\int_{\mathbb{R}}g(t)\,dt=1$. Put $g_\epsilon(t):=\epsilon^{-1}g(\epsilon t)$ for every constant $\epsilon>0$ and $\chi_\epsilon:=\chi*g_\epsilon$ (the convolution of χ with g_ϵ).*

(i) *If $\chi|_{(-\infty,0]}\in\widetilde{\mathcal{W}}^-$, then $\chi_\epsilon|_{(-\infty,0]}\in\widetilde{\mathcal{W}}^-$ for every $0<\epsilon<\delta$, $\chi_\epsilon\searrow\chi$ as $\epsilon\searrow 0$ and $\sup(\chi_\epsilon-\chi)\leq c\epsilon$.*

(ii) *If $\chi|_{(-\infty,0]}\in\mathcal{W}_M^+$ and $0<\epsilon<\delta^2/2$ then $\chi_\epsilon|_{(-\infty,0]}\in\mathcal{W}^+_{M/(1-\delta)}$. Moreover, if $0<\epsilon<\delta^2/8$ then*

$$\overline{\chi}_\epsilon:=\chi_\epsilon(\cdot+\epsilon)-c\epsilon\in\mathcal{W}^+_{M/(1-\delta)^2},\qquad \overline{\chi}_\epsilon\geq\chi-c\epsilon,$$

and $\overline{\chi}_\epsilon$ converges uniformly to χ as $\epsilon\to 0$ on compact subsets in $\mathbb{R}$.

Proof. Part (i) follows from [Do and Vu 2022a, Lemma 2.1]. Part (ii) can be obtained more or less by similar arguments as in the last reference. We provide details for readers' convenience. It is clear that χ_ϵ is a concave, increasing function with $\chi_\epsilon(0)=0$. We will show that

$$\chi_\epsilon'(t)\leq\frac{M}{1-\delta}\frac{\chi_\epsilon(t)}{t}\tag{2-6}$$

for every $t<0$ and $0<\epsilon<\delta^2/2$.

If $t < -\delta/2$ then

$$\chi_\epsilon'(t) = \int_{-\epsilon}^{\epsilon} \chi'(t-s) g_\epsilon(s)\, ds \le \int_{-\epsilon}^{\epsilon} \frac{M\chi(t-s)}{t-s} g_\epsilon(s)\, ds \\ \le \int_{-\epsilon}^{\epsilon} \frac{M\chi(t-s)}{t+\epsilon} g_\epsilon(s)\, ds = \frac{M\chi_\epsilon(t)}{t+\epsilon} = \frac{Mt}{t+\epsilon}\frac{\chi_\epsilon(t)}{t} \le \frac{M}{1-\delta}\frac{\chi_\epsilon(t)}{t}$$

for every $0 < \epsilon < \delta^2/2$.

On the other hand, if $t \ge -\delta/2$, then $\chi_\epsilon(t) = \chi(t) = ct$ for every $0 < \epsilon < \delta^2/2$. As a consequence,

$$\chi_\epsilon'(t) = \chi'(t) \le \frac{M\chi(t)}{t} = M\frac{\chi_\epsilon(t)}{t}.$$

Thus, (2-6) follows. Hence, $\chi_\epsilon|_{(-\infty,0]} \in \mathcal{W}^+_{M/(1-\delta)}$.

Now, we consider $\overline{\chi}_\epsilon$. Since χ is increasing, one sees that $\overline{\chi}_\epsilon \ge \chi - c\epsilon$ and $\overline{\chi}_\epsilon$ converges uniformly to χ as $\epsilon \to 0$ on compact subsets in $\mathbb{R}$. It remains to show that $\overline{\chi}_\epsilon \in \mathcal{W}^+_{M(1+\delta)/(1-\delta)}$ for every $0 < \epsilon < \delta^2/8$. Note that

$$\overline{\chi}_\epsilon = h_\epsilon * g_\epsilon,$$

where $h_\epsilon(t) = \chi(t+\epsilon) - c\epsilon$. The function $\overline{\chi}_\epsilon(t)$ is concave, increasing and $\overline{\chi} + \epsilon(0) = 0$.

If $-\delta/2 \le t < 0$ then $h_\epsilon(t) = \chi(t) = ct$ for every $0 < \epsilon < \delta^2/2$. Therefore

$$h_\epsilon'(t) = \chi'(t) \le \frac{M\chi(t)}{t} = M\frac{h_\epsilon(t)}{t}.$$

If $t < -\delta/2$ then

$$h_\epsilon'(t) = \chi'(t+\epsilon) \le M\frac{\chi(t+\epsilon)}{t+\epsilon} \\ \le M\frac{\chi(t+\epsilon) - c\epsilon}{t+\epsilon} = M\frac{h_\epsilon(t)}{t+\epsilon} = \frac{Mt}{t+\epsilon}\frac{h_\epsilon(t)}{t} \le \frac{M}{1-\delta}\frac{h_\epsilon(t)}{t}$$

for every $0 < \epsilon < \delta^2/2$.

Then, for every $0 < \epsilon < \delta^2/2$, we have $h_\epsilon \in \mathcal{W}^+_{M/(1-\delta)}$ and $h_\epsilon = ct$ for every $t \ge -\delta/2$. Hence, for every $0 < \epsilon < \delta^2/8$, we have

$$\overline{\chi}_\epsilon = h_\epsilon * g_\epsilon \in \mathcal{W}^+_{M/((1-\delta)(1-\delta/2))} \subset \mathcal{W}^+_{M/(1-\delta)^2}. \qquad \square$$

Lemma 2.6. *Let $\chi, \tilde{\chi} \in \widetilde{\mathcal{W}}^- \cup \mathcal{W}^+_M$ ($M \ge 1$) such that $\tilde{\chi} \le \chi$. Then there exist sequences of functions $\chi_j, \tilde{\chi}_j \in \widetilde{\mathcal{W}}^- \cup \mathcal{W}^+_{M_j}$ (with $M_j \searrow M$ as $j \to \infty$) satisfying the following conditions*:

- $\chi_j \in \mathscr{C}^\infty(\mathbb{R})$ *for every j.*
- $\chi_j \ge \tilde{\chi}_j$ *and* $\chi_j \ge \chi - 2^{-j}$ *for every j big enough.*
- $\tilde{\chi} - 2^{-j} \le \tilde{\chi}_j \le \tilde{\chi}$ *on* $(-\infty, -1]$ *for every j big enough.*
- χ_j *converges uniformly to χ on compact subsets in* $\mathbb{R}_{\le 0}$.

Proof. We split the proof into two cases.

Case 1: $\chi \in \widetilde{\mathcal{W}}^-$. For every $j \ge 1$, we set

$$\overline{\chi}_j(t) = \begin{cases} \max\{\chi(t), c_j t\} & \text{if } t < 0, \\ c_j t & \text{if } t \ge 0, \end{cases}$$

where

$$c_j := \frac{-\chi(-2^{-j})}{2^{-j}}.$$

Then $\overline{\chi}_j$ satisfies the hypothesis of Lemma 2.5 for $\delta := 2^{-j}$. Let g be a smooth radial cut-off function supported in $[-1, 1]$ on $\mathbb{R}$, i.e., $g(t) = g(-t)$ for $t \in \mathbb{R}$, $0 \le g \le 1$ and $\int_{\mathbb{R}} g(t)\, dt = 1$. For every $j \ge 1$, we define

$$\chi_j = \overline{\chi}_j * g_{4^{-j-1}} \quad \text{and} \quad \tilde{\chi}_j = \tilde{\chi}.$$

By Lemma 2.5, χ_j and $\tilde{\chi}_j$ satisfy the desired conditions.

Case 2: $\chi \in \mathcal{W}^+_M$. Since $\chi \ge \tilde{\chi}$, we also have $\tilde{\chi} \in \mathcal{W}^+_M$. Assume that g and c_j are as in Case 1. For every $j \ge 1$, we define

$$\overline{\chi}_j(t) = \begin{cases} \min\{\chi(t), c_j t\} & \text{if } t < 0, \\ c_j t & \text{if } t \ge 0, \end{cases}$$

and

$$\chi_j(t) = (\overline{\chi}_j(\cdot + 4^{-j-1}) * g_{4^{-j-1}})(t) - c_j 4^{-j-1}.$$

We also set $\tilde{\chi}_j(t) = \min\{\tilde{\chi}(t), \chi_j(t)\}$. By Lemma 2.5, χ_j and $\tilde{\chi}_j$ satisfy the desired conditions. □

Let ϕ be a negative θ-psh function. We denote by $\mathrm{PSH}(X, \theta, \phi)$ the set of θ-psh functions $u \le \phi$. Recall that by monotonicity, we always have $\int_X \theta_u^n \le \int_X \theta_\phi^n$, where for every θ-psh function v, we put $\theta_v := dd^c v + \theta$. We also define by $\mathcal{E}(X, \theta, \phi)$ the set of $u \in \mathrm{PSH}(X, \theta, \phi)$ of full Monge–Ampère mass with respect to ϕ, i.e., $\int_X \theta_u^n = \int_X \theta_\phi^n$.

Let $\chi \in \widetilde{\mathcal{W}}^- \cup \mathcal{W}^+_M$, and $u \in \mathrm{PSH}(X, \theta, \phi)$. We put

$$E_{\chi,\theta,\phi}(u) := \int_X -\chi(u - \phi)\theta_u^n.$$

We also define by $\mathcal{E}_\chi(X, \theta, \phi)$ the set of $u \in \mathcal{E}(X, \theta, \phi)$ with $E_{\chi,\theta,\phi}(u) < \infty$.

Lemma 2.7. *Let $\chi \in \widetilde{\mathcal{W}}^- \cup \mathcal{W}^+_M$ and $u_1, u_2 \in \mathcal{E}_\chi(X, \theta, \phi)$. Then there exists a constant $C_1 > 0$ depending only on n and M such that*

$$-\int_X \chi(u_1 - \phi)\theta_{u_2}^n \le C_1 \sum_{j=1}^{2} E_{\chi,\phi,\theta}(u_j),$$

and

$$E_{\chi,\theta,\phi}(au_1 + (1-a)u_2) \le C_1 \sum_{j=1}^{2} E_{\chi,\theta,\phi}(u_j)$$

for every $0 < a < 1$. Furthermore if $u_1 \ge u_2$, then

$$E_{\chi,\phi,\theta}(u_1) \le C_2 E_{\chi,\phi,\theta}(u_2)$$

for some constant C_2 depending only on n and M.

Proof. The first and third inequalities are from [Do and Vu 2022a, Lemma 3.2] (see also [Guedj and Zeriahi 2007, Propositions 2.3, 2.5] for the case where $\phi = 0$ and θ is a Kähler form). The second desired inequality was implicitly proved in the proof of convexity of finite energy classes in [Vu 2022, Proposition 3.3] (in a much broader context). Alternatively one can use properties of envelopes in [Darvas et al. 2018b] to get the same conclusion. We prove here the second desired inequality using ideas from [Vu 2022] for readers' convenience. We only consider $\chi \in \widetilde{\mathcal{W}}^-$. The case where $\chi \in \mathcal{W}^+_M$ is done similarly.

Considering $u_j - \epsilon$ for $\epsilon > 0$ instead of u_j, and taking $\epsilon \to 0$ later, without loss of generality, we can assume that $u_j < \phi \le 0$ for $j = 1, 2$. By replacing u_j, θ by $u_j - \phi, \theta_\phi$ respectively, we can assume that $\phi = 0$, but θ is no longer a smooth form but a closed positive $(1, 1)$-current. This change causes no trouble for us. Let $v := au_1 + (1-a)u_2$. Observe that $X \subset \{u_1 < u_2\} \cup \{u_1 > 2u_2\}$. Hence

$$\begin{aligned}
E_{\chi,\theta}(v) &\le \int_{\{u_1<u_2\}} -\chi(v)\theta_v^n + \int_{\{u_1>2u_2\}} -\chi(v)\theta_v^n \\
&\le \sum_{k=0}^n \biggl(\int_{\{u_1<u_2\}} -\chi(u_1)\theta_{u_1}^k \wedge \theta_{u_2}^{n-k} + \int_{\{u_1>2u_2\}} -\chi((1+a)u_2)\theta_{u_1}^k \wedge \theta_{u_2}^{n-k} \biggr) \\
&\le \sum_{k=0}^n \int_{\{u_1<u_2\}} -\chi(u_1)\theta_{u_1}^k \wedge \theta_{\max\{u_1,u_2\}}^{n-k} + \sum_{k=0}^n \int_{\{u_1>2u_2\}} -2^{k+1}\chi(u_2)\theta_{\max\{u_1/2,u_2\}}^k \wedge \theta_{u_2}^{n-k} \\
&\le \sum_{k=0}^n \biggl(\int_X -\chi(u_1)\theta_{u_1}^k \wedge \theta_{\max\{u_1,u_2\}}^{n-k} + 2^{k+1} \int_X -\chi(u_2)\theta_{\max\{u_1/2,u_2\}}^k \wedge \theta_{u_2}^{n-k} \biggr) \\
&\lesssim E_{\chi,\theta}(u_1) + E_{\chi,\theta}(\max\{u_1, (u_1+u_2)/2\}) + E_{\chi,\theta}(u_2) + E_{\chi,\theta}(\max\{u_1/4 + u_2/2, u_2\}) \\
&\lesssim E_{\chi,\theta}(u_1) + E_{\chi,\theta}(u_2),
\end{aligned}$$

where the two last estimates hold due to the first and third inequalities of the lemma. ☐

Lemma 2.8. *Let $\chi, \tilde\chi \in \widetilde{\mathcal{W}}^- \cup \mathcal{W}^+_M$ such that $\tilde\chi \le \chi$ and let $u_1, u_2, \dots, u_{n+1} \in \mathcal{E}(X, \theta, \phi)$. Define $\varrho := \operatorname{vol}(\theta_\phi)$. Then there exists a constant $C > 0$ depending only on n and M such that*

$$-\int_X \chi(\epsilon(u_1-\phi))\theta_{u_2} \wedge \cdots \wedge \theta_{u_{n+1}} \le CB\varrho(1-\tilde\chi(-1))Q_0(\epsilon)$$

for every $0 < \epsilon \le 1$, where

$$B = 1 + \max_{1\le j\le n+1} E_{\tilde\chi,\theta,\phi}(u_j)/\varrho \quad \textit{and} \quad Q_0(\epsilon) := \sup_{\{t\le -1\}} \frac{\chi(\epsilon t)}{\tilde\chi(t)}.$$

Proof. Let L be the left-hand side of the desired inequality. We have

$$\begin{aligned}
L &\le -\int_{\{u_1\ge\phi-1\}} \chi(\epsilon(u_1-\phi))\theta_{u_2} \wedge \cdots \wedge \theta_{u_{n+1}} - \int_{\{u_1<\phi-1\}} \chi(\epsilon(u_1-\phi))\theta_{u_2} \wedge \cdots \wedge \theta_{u_{n+1}} \\
&\le -\chi(-\epsilon)\varrho - Q_0(\epsilon) \int_{\{u_1<\phi-1\}} \tilde\chi(u_1-\phi)\theta_{u_2} \wedge \cdots \wedge \theta_{u_{n+1}} \\
&\le -\varrho Q_0(\epsilon)\tilde\chi(-1) - Q_0(\epsilon) \int_X \tilde\chi(u_1-\phi)\theta_{u_2} \wedge \cdots \wedge \theta_{u_{n+1}} \\
&\le -\varrho Q_0(\epsilon)\tilde\chi(-1) + CQ_0(\epsilon) \max_{1\le j\le n+1} E_{\tilde\chi,\theta,\phi}(u_j),
\end{aligned}$$

where $C > 0$ depends only on n and M. The last estimate holds due to Lemma 2.7. Thus the desired inequality follows. ☐

By the convexity/concavity and by the assumption $\tilde\chi \le \chi$, we have

$$\begin{cases} Q_0(\epsilon) \ge \epsilon Q_0(1) & \text{if } \chi \in \widetilde{\mathcal{W}}^-, \\ Q_0(\epsilon) \le \epsilon Q_0(1) & \text{if } \chi \in \mathcal{W}^+_M \end{cases} \tag{2-7}$$

for every $0 < \epsilon \le 1$. Moreover, if $\chi \in \widetilde{\mathcal{W}}^-$ and $\chi(t)/\tilde\chi(t) \to 0$ as $t \to -\infty$, then by the definition of Q_0, we also have

$$Q_0(\epsilon) \le \frac{\chi(-\epsilon^{1/2})}{\tilde\chi(-1)} + \sup_{\{t \le -\epsilon^{-1/2}\}} \frac{\chi(t)}{\tilde\chi(t)} \xrightarrow{\epsilon \to 0^+} 0. \tag{2-8}$$

Let $u_1, u_2 \in \mathcal{E}_\chi(X, \theta, \phi)$, and $v := \max\{u_1, u_2\}$. Put

$$\nu(u_1, u_2) := \chi(-|u_1 - u_2|)(\theta^n_{u_2} - \theta^n_{u_1})$$

and

$$\begin{aligned} I_\chi(u_1, u_2) &:= \int_{\{u_1 < u_2\}} \nu(u_1, u_2) + \int_{\{u_1 > u_2\}} \nu(u_2, u_1) \\ &= \int_X \nu(u_1, v) + \int_X \nu(u_2, v). \end{aligned} \tag{2-9}$$

Proposition 2.9. *Let $\chi \in \widetilde{\mathcal{W}}^- \cup \mathcal{W}^+_M$. Let ϕ is a negative θ-psh function and $u_1, u_2 \in \mathcal{E}_\chi(X, \theta, \phi)$. Then*

$$I_\chi(u_1, u_2) \ge 0.$$

Proof. Define $\mu = \theta^n_{u_2} - \theta^n_{u_1}$. Since χ is absolutely continuous, we have χ is differentiable almost everywhere and $-\chi(t) = \int_t^0 \chi'(s)\,ds$ for every $t < 0$. Hence

$$\begin{aligned} \int_{\{u_1<u_2\}} \nu(u_1, u_2) &= -\int_{\{u_1<u_2\}} \left(\int_{u_1-u_2}^0 \chi'(t)\,dt \right) d\mu \\ &= -\int_{\{u_1<u_2\}} \left(\int_{-\infty}^0 \chi'(t)\mathbf{1}_{\{u_1<u_2+t\}}\,dt \right) d\mu \\ &= -\int_{-\infty}^0 \chi'(t)\mu\{u_1 < u_2 + t\}\,dt. \end{aligned}$$

Moreover, it follows from [Darvas et al. 2021a, Lemma 2.3] that $\mu\{u_1 < u_2 + t\} \le 0$ for every $t \le 0$. Hence

$$\int_{\{u_1<u_2\}} \nu(u_1, u_2) = -\int_{-\infty}^0 \chi'(t)\mu\{u_1 < u_2 + t\}\,dt \ge 0.$$

Similarly,

$$\int_{\{u_2<u_1\}} \nu(u_2, u_1) \ge 0.$$

Thus

$$I_\chi(u_1, u_2) = \int_{\{u_1<u_2\}} \nu(u_1, u_2) + \int_{\{u_2<u_1\}} \nu(u_2, u_1) \ge 0. \qquad \square$$

3. Stability for weighted potentials

3.1. ***Main results.*** Let $\chi, \tilde{\chi} \in \widetilde{\mathcal{W}}^- \cup \mathcal{W}_M^+$ ($M \geq 1$) such that $\tilde{\chi} \leq \chi$. For each constant $t \geq 0$, we let

$$Q(t) = Q_{\chi,\tilde{\chi}}(t) := \begin{cases} 1 & \text{if } t \geq 1, \\ (Q_0(t)/Q_0(1))^{1/2} & \text{if } 0 < t < 1 \text{ and } \chi \in \widetilde{\mathcal{W}}^-, \\ t^{1/2} & \text{if } 0 < t < 1 \text{ and } \chi \in \mathcal{W}_M^+, \\ \lim_{s\to 0^+} Q(s) & \text{if } t = 0, \end{cases} \tag{3-1}$$

where Q_0 is defined as in Lemma 2.8. We remove the subscript $\chi, \tilde{\chi}$ from $Q_{\chi,\tilde{\chi}}$ if $\chi, \tilde{\chi}$ are clear from the context. Note that Q is increasing continuous function in t and

$$Q(0) = 0 \quad \text{if either} \quad \chi \in \mathcal{W}_M^+ \quad \text{or} \quad \lim_{t\to -\infty} \frac{\chi(t)}{\tilde{\chi}(t)} = 0. \tag{3-2}$$

For the convenience, we normalize energies with respect to $\varrho := \int_X \theta_\phi^n$ as

$$E^0_{\tilde{\chi},\theta,\phi} := \varrho^{-1} E_{\tilde{\chi},\theta,\phi}, \quad I^0_\chi(u_1, u_2) = \varrho^{-1} I_\chi(u_1, u_2).$$

Theorem 3.1. *Let θ be a closed smooth real $(1, 1)$-form and ϕ be a negative θ-psh function such that $\varrho := \operatorname{vol}(\theta_\phi) > 0$. Let $\chi, \tilde{\chi} \in \widetilde{\mathcal{W}}^- \cup \mathcal{W}_M^+$ ($M \geq 1$) such that $\tilde{\chi} \leq \chi$. Let $B \geq 1$ be a constant and let $u_j, \psi_j \in \mathcal{E}(X, \theta, \phi)$ satisfy $u_1 \leq u_2$ and*

$$E^0_{\tilde{\chi},\theta,\phi}(u_j) + E^0_{\tilde{\chi},\theta,\phi}(\psi_j) \leq B$$

for $j = 1, 2$. Then there exists a constant $C_n > 0$ depending only on n and M such that

$$\int_X -\chi(u_1 - u_2)(\theta^n_{\psi_1} - \theta^n_{\psi_2}) \leq C_n \varrho B^2 (1 - \tilde{\chi}(-1))^2 Q^{\circ n}(I^0_\chi(u_1, u_2)), \tag{3-3}$$

where Q is defined by (3-1), *and $Q^{\circ n} := Q \circ Q \circ \cdots \circ Q$ (n-iterate of Q).*

Since the measure $\theta^n_{\psi_1} - \theta^n_{\psi_2}$ is not positive, we need the following consequence of the above theorem for later applications on stability estimates.

Theorem 3.2. *Let θ be a closed smooth real $(1, 1)$-form and ϕ be a negative θ-psh function such that $\phi = P_\theta[\phi]$, $\varrho := \operatorname{vol}(\theta_\phi) > 0$ and $\theta \leq A\omega$ for some constant $A \geq 1$. Let $\chi, \tilde{\chi} \in \widetilde{\mathcal{W}}^- \cup \mathcal{W}_M^+$ ($M \geq 1$) such that $\tilde{\chi} \leq \chi$. Let $B \geq 1$ be a constant and $u_1, u_2, \psi \in \mathcal{E}(X, \theta, \phi)$ satisfying*

$$E^0_{\tilde{\chi},\theta,\phi}(u_1) + E^0_{\tilde{\chi},\theta,\phi}(u_2) + E^0_{\tilde{\chi},\theta,\phi}(\psi) \leq B$$

for $j = 1, 2$. Then, for every constant $m > 0$ and $0 < \gamma < 1$, there exists a constant $C > 0$ depending on n, M, X, ω, m and γ such that

$$\int_X -\chi(-|u_1 - u_2|)\theta^n_\psi \leq -\varrho\chi(-|a_1 - a_2| - \lambda^m) + C\varrho A^{(1-\gamma)/m}(B - \tilde{\chi}(-A))^2(1 - \tilde{\chi}(-1))^2 \lambda^\gamma,$$

where $a_j := \sup_X u_j$ and $\lambda = Q^{\circ n}(I^0_\chi(u_1, u_2))$.

3.2. *Proof of Theorem 3.1.* Here is the first step in the proof of Theorem 3.1.

Lemma 3.3. *If Theorem 3.1 holds for u_j, ψ_j of the same singularity type as ϕ, then it holds for the general case.*

Proof. Let u_j, ψ_j ($j = 1, 2$) be as in the statement of Theorem 3.1. For every $k > 0$, we define $u_{j,k} := \max\{u_j, \phi - k\}$ and $\psi_{j,k} = \max\{\psi_j, \phi - k\}$. By Lemma 2.7, there exists a constant $C_1 > 0$ depending only on n and M such that

$$E^0_{\tilde{\chi},\theta,\phi}(u_{j,k}) + E^0_{\tilde{\chi},\theta,\phi}(\psi_{j,k}) \le C_1 B$$

for $j = 1, 2$ and for every $k > 0$. Therefore, by the assumption, there exists a constant $C_2 > 0$ depending only on n and M such that

$$\int_X -\chi(u_{1,k} - u_{2,k})(\theta^n_{\psi_{1,l}} - \theta^n_{\psi_{2,l}}) \le C_2 \varrho B^2 (1 - \tilde{\chi}(-1))^2 Q^{\circ(n)}(I^0_\chi(u_{1,k}, u_{2,k}))$$

for every $k, l > 0$. Letting $l \to \infty$ and using [Darvas et al. 2021b, Theorem 2.2], we get

$$\int_X -\chi(u_{1,k} - u_{2,k})(\theta^n_{\psi_1} - \theta^n_{\psi_2}) \le C_2 \varrho B^2 (1 - \tilde{\chi}(-1))^2 Q^{\circ(n)}(I^0_\chi(u_{1,k}, u_{2,k})) \tag{3-4}$$

for every $k > 0$. We will show that

$$I_\chi(u_1, u_2) = \lim_{k\to\infty} I_\chi(u_{1,k}, u_{2,k}). \tag{3-5}$$

Define

$$f := \chi(u_1 - u_2)(\theta^n_{u_2} - \theta^n_{u_1}), \quad f_k := \chi(u_{1,k} - u_{2,k})(\theta^n_{u_{2,k}} - \theta^n_{u_{1,k}}).$$

We have

$$\begin{aligned} I_\chi(u_{1,k}, u_{2,k}) = \int_X f_k &= \int_{\{u_1 > \phi - k\}} f_k + \int_{\{u_1 \le \phi - k\}} f_k \\ &= \int_{\{u_1 > \phi - k\}} f + \int_{\{u_1 \le \phi - k\}} f_k \\ &= I_\chi(u_1, u_2) - \int_{\{u_1 \le \phi - k\}} f + \int_{\{u_1 \le \phi - k\}} f_k. \end{aligned}$$

Then

$$\begin{aligned} |I_\chi(u_{1,k}, u_{2,k}) - I_\chi(u_1, u_2)| &= \left| \int_{\{u_1 \le \phi - k\}} f - \int_{\{u_1 \le \phi - k\}} f_k \right| \\ &\le \int_{\{u_1 \le \phi - k\}} \mu + \int_{\{u_1 \le \phi - k\}} -\chi(u_{1,k} - u_{2,k})(\theta^n_{u_{2,k}} + \theta^n_{u_{1,k}}) \\ &\le \int_{\{u_1 \le \phi - k\}} \mu + \int_{\{u_1 \le \phi - k\}} -\chi(-k)(\theta^n_{u_{2,k}} + \theta^n_{u_{1,k}}), \end{aligned}$$

where $\mu = -\chi(u_1 - \phi)(\theta^n_{u_1} + \theta^n_{u_2})$. By Lemma 2.7, we have $\int_X \mu < \infty$. Then it follows from Lebesgue's dominated convergence theorem that $\lim_{k\to\infty} \int_{\{u_1 \le \phi - k\}} \mu = 0$. Therefore,

$$\limsup_{k\to\infty} |I_\chi(u_{1,k}, u_{2,k}) - I_\chi(u_1, u_2)| \le \limsup_{k\to\infty} \int_{\{u_1 \le \phi - k\}} -\chi(-k)(\theta^n_{u_{1,k}} + \theta^n_{u_{2,k}}). \tag{3-6}$$

By the fact that

$$\int_X \theta^n_{u_{1,k}} = \int_X \theta^n_{u_{2,k}} = \int_X \theta^n_{\phi}, \quad \mathbf{1}_{\{u_1>\phi-k\}}\theta^n_{u_{j,k}} = \mathbf{1}_{\{u_1>\phi-k\}}\theta^n_{u_j} \quad (j=1,2),$$

we have

$$-\chi(-k)\int_{\{u_1\le\phi-k\}}(\theta^n_{u_{1,k}}+\theta^n_{u_{2,k}}) = -\chi(-k)\int_{\{u_1\le\phi-k\}}(\theta^n_{u_1}+\theta^n_{u_2}) \le \int_{\{u_1\le\phi-k\}}\mu. \tag{3-7}$$

By using (3-6), (3-7) and the fact $\lim_{k\to\infty}\int_{\{u_1\le\phi-k\}}\mu=0$, we get (3-5). Now, combining (3-4) and (3-5), we obtain

$$\int_X -\chi(u_1-u_2)(\theta^n_{\psi_1}-\theta^n_{\psi_2}) \le C_2\varrho B^2(1-\tilde\chi(-1))^2 Q^{\circ(n)}(I^0_\chi(u_1,u_2)). \qquad \square$$

Lemma 3.4. *Let $M\ge 1$ and $\chi,\tilde\chi\in\widetilde{\mathcal{W}}^-\cup\mathcal{W}^+_M$ such that $\tilde\chi\le\chi$ and $\chi\in\mathscr{C}^1(\mathbb{R})$. Let $u_1,u_2,\dots,u_{n+2}\in\mathcal{E}(X,\theta,\phi)$ such that $u_1\le u_2$ and $u_j-\phi$ is bounded $(j=1,2,\dots,n+2)$, where ϕ is a negative θ-psh function satisfying $\varrho:=\mathrm{vol}(\theta_\phi)>0$. Set*

$$T=\theta_{u_4}\wedge\cdots\wedge\theta_{u_{n+2}}, \quad I=\left|\int_X \chi'(u_1-u_2)\,d(u_1-u_2)\wedge d^c(u_1-u_3)\wedge T\right|,$$

and

$$J=\int_X \chi'(u_1-u_2)\,d(u_1-u_2)\wedge d^c(u_1-u_2)\wedge T.$$

Then there exists $C>0$ depending only on n and M such that

$$I\le C\varrho B(1-\tilde\chi(-1))Q(J/\varrho),$$

where $B:=\sum_{j=1}^{n+2}\max\{E^0_{\tilde\chi,\theta,\phi}(u_j),1\}$ and Q is defined by (3-1).

Clearly if $\chi,\tilde\chi\in\widetilde{\mathcal{W}}^-$, then the above constant C does not depend on M.

Proof. In this proof, we use the symbols $\lesssim$ and $\gtrsim$ for inequalities modulo a constant depending only on n and M. By Theorem 2.2 and Lemma 2.7, we have

$$I=\left|\int_X -\chi(u_1-u_2)\,dd^c(u_1-u_3)\wedge T\right| \lesssim \varrho B=\varrho B Q(1).$$

Therefore, without loss of generality, we can assume that $J/\varrho<1$. Approximating u_3 by $u_3-\delta$ with $\delta\searrow 0$, we can assume that $u_3<\phi$ on X.

For each $0<\epsilon<\frac12$ we let

$$U(\epsilon)=\{u_1-u_2<\epsilon(u_1+u_3-2\phi)\}, \quad V(\epsilon)=\{u_1-u_2>\epsilon(u_1+u_3-2\phi)\},$$

and $\Gamma(\epsilon)=\{u_1-u_2=\epsilon(u_1+u_3-2\phi)\}$. Since $\Gamma(\epsilon_1)\cap\Gamma(\epsilon_2)=\varnothing$ for every $\epsilon_1\ne\epsilon_2$ (note $u_3<\phi$), we have

$$\int_{\Gamma(\epsilon)} d(u_1-u_3)\wedge d^c(u_1-u_3)\wedge T=0 \tag{3-8}$$

for almost everywhere $\epsilon\in\left(0,\frac12\right)$.

Let $0 < \epsilon < \frac{1}{2}$ be a constant satisfying (3-8). To simplify the notation, from now on, we write U, V, Γ for $U(\epsilon)$, $V(\epsilon)$, $\Gamma(\epsilon)$ respectively. Define

$$\tilde{u}_1 = \frac{u_1 + \epsilon u_3}{1+\epsilon}, \quad \tilde{u}_2 = \max\left\{\frac{u_2 + \epsilon u_3}{1+\epsilon}, \ \frac{(1-\epsilon)u_1 + 2\epsilon\phi}{1+\epsilon}\right\} \quad \text{and} \quad \tilde{\varphi} = \tilde{u}_1 - \tilde{u}_2.$$

Then $\varphi := (u_1 - u_2) = (1+\epsilon)\tilde{\varphi}$ on U. Hence

$$\begin{aligned} I &= \left|\int_X -\chi(\varphi)\, dd^c(u_1 - u_3) \wedge T\right| \\ &\le \left|\int_U -\chi(\varphi)\, dd^c(u_1 - u_3) \wedge T\right| + \left|\int_{X\backslash U} -\chi(\varphi)\, dd^c(u_1 - u_3) \wedge T\right| \\ &\le \left|\int_U -\chi((1+\epsilon)\tilde{\varphi})\, dd^c(u_1 - u_3) \wedge T\right| + \left|\int_{X\backslash U} -\chi(\varphi)(\theta_{u_1} + \theta_{u_3}) \wedge T\right| \\ &\le \left|\int_U -\chi((1+\epsilon)\tilde{\varphi})\, dd^c(u_1 - u_3) \wedge T\right| + \left|\int_{X\backslash U} -\chi(\epsilon(u_1 + u_3 - 2\phi))(\theta_{u_1} + \theta_{u_3}) \wedge T\right| \\ &:= I_1 + I_2, \end{aligned}$$

where in the last inequality we used the fact that χ is increasing and $\varphi \ge \epsilon(u_1 + u_2 - 2\phi)$ on $X\backslash U$. By Lemma 2.7, we have $E^0_{\tilde{\chi},\theta,\phi}\big(\frac{1}{2}(u_1 + u_3)\big) \lesssim B$. Therefore, it follows from Lemma 2.8 that

$$I_2 \le 2\int_X -\chi\big(2\epsilon\big(\tfrac{1}{2}(u_1+u_3) - \phi\big)\big)\theta_{(u_1+u_3)/2} \wedge T \lesssim B\varrho(1 - \tilde{\chi}(-1))Q_0(2\epsilon). \tag{3-9}$$

In order to estimate I_1, we divide it into two terms

$$\begin{aligned} I_1 &\le \left|\int_X -\chi((1+\epsilon)\tilde{\varphi})\, dd^c(u_1 - u_3) \wedge T\right| + \left|\int_{X\backslash U} -\chi((1+\epsilon)\tilde{\varphi})\, dd^c(u_1 - u_3) \wedge T\right| \\ &:= I_3 + I_4. \end{aligned}$$

Note that $\tilde{u}_1 - \tilde{u}_2 = \epsilon(u_1 + u_3 - 2\phi)/(1+\epsilon)$ on $X \setminus U$. Hence

$$I_4 \le \int_{X\backslash U} -\chi((1+\epsilon)\tilde{\varphi})(\theta_{u_1} + \theta_{u_3}) \wedge T \le \int_{X\backslash U} -\chi(\epsilon(u_1 + u_2 - 2\phi))(\theta_{u_1} + \theta_{u_3}) \wedge T.$$

Using Lemma 2.8 again, we get

$$I_4 \lesssim B\varrho(1 - \tilde{\chi}(-1))Q_0(2\epsilon). \tag{3-10}$$

Using integration by parts, we have

$$I_3 = (1+\epsilon)\left|\int_X \chi'((1+\epsilon)\tilde{\varphi})\, d\tilde{\varphi} \wedge d^c(u_1 - u_3) \wedge T\right|.$$

Moreover, by the Cauchy–Schwarz inequality and by the choice of ϵ (see (3-8)), we get

$$\int_\Gamma \chi'((1+\epsilon)\tilde{\varphi})\, d\tilde{\varphi} \wedge d^c(u_1 - u_3) \wedge T = 0.$$

Hence

$$I_3 = (1+\epsilon)\left|\int_{U\cup V} \chi'((1+\epsilon)\tilde{\varphi})d\tilde{\varphi}\wedge d^c(u_1-u_3)\wedge T\right| \le (1+\epsilon)(I_5 I_6)^{1/2}, \tag{3-11}$$

where

$$I_5 = \int_{U\cup V} \chi'((1+\epsilon)\tilde{\varphi})\, d(u_1-u_3)\wedge d^c(u_1-u_3)\wedge T,$$
$$I_6 = \int_{U\cup V} \chi'((1+\epsilon)\tilde{\varphi})\, d\tilde{\varphi}\wedge d^c\tilde{\varphi}\wedge T.$$

Since $(1+\epsilon)\tilde{\varphi} \le \epsilon(u_1+u_3-2\phi)$, if $\chi\in\widetilde{\mathcal{W}}^-$ (hence χ' is nonnegative and increasing on $\mathbb{R}_{\le 0}$) then

$$\begin{aligned}
I_5 &\le \int_X \chi'(\epsilon(u_1+u_3-2\phi))\, d(u_1-u_3)\wedge d^c(u_1-u_3)\wedge T\\
&\lesssim \int_X \chi'(\epsilon(u_1+u_3-2\phi))\, d(u_1-\phi)\wedge d^c(u_1-\phi)\wedge T\\
&\qquad\qquad + \int_X \chi'(\epsilon(u_1+u_3-2\phi))\, d(u_3-\phi)\wedge d^c(u_3-\phi)\wedge T\\
&\le \int_X \chi'(\epsilon(u_1-\phi))\, d(u_1-\phi)\wedge d^c(u_1-\phi)\wedge T + \int_X \chi'(\epsilon(u_3-\phi))\, d(u_3-\phi)\wedge d^c(u_3-\phi)\wedge T\\
&= \epsilon^{-1}\int_X -\chi(\epsilon(u_1-\phi))\, dd^c(u_1-\phi)\wedge T + \epsilon^{-1}\int_X -\chi(\epsilon(u_3-\phi))\, dd^c(u_3-\phi)\wedge T\\
&\lesssim B\varrho(1-\tilde{\chi}(-1))\epsilon^{-1}Q_0(\epsilon),
\end{aligned}$$

where the last estimate holds due to Lemma 2.8.

Define $v_1 := (u_1+2u_3)/3$ and $v_2 := (2u_1+u_3)/3$. Since

$$(1+\epsilon)(\tilde{u}_1-\tilde{u}_2) \ge u_1+u_3-2\phi, \quad u_1-u_3 = -3(v_1-v_2),$$

one sees that if $\chi\in\mathcal{W}^+_M$ (hence χ' is nonnegative and decreasing in $\mathbb{R}_{\le 0}$) then

$$\begin{aligned}
I_5 &\le \int_X \chi'((u_1+u_3-2\phi))\, d(u_1-u_3)\wedge d^c(u_1-u_3)\wedge T\\
&\lesssim \int_X \chi'((u_1+u_3-2\phi))\big(d(v_1-\phi)\wedge d^c(v_1-\phi) + d(v_2-\phi)\wedge d^c(v_2-\phi)\big)\wedge T\\
&\le \int_X \chi'(3(v_1-\phi))\, d(v_1-\phi)\wedge d^c(v_1-\phi)\wedge T + \int_X \chi'(3(v_2-\phi))\, d(v_2-\phi)\wedge d^c(v_2-\phi)\wedge T\\
&= \frac{1}{3}\int_X -\chi(3(v_1-\phi))\, dd^c(v_1-\phi)\wedge T + \frac{1}{3}\int_X -\chi(3(v_2-\phi))\, dd^c(v_2-\phi)\wedge T\\
&\le \int_X -\chi(3(v_1-\phi))(\theta_{v_1}+\theta_\phi)\wedge T + \int_X -\chi(3(v_2-\phi))(\theta_{v_2}+\theta_\phi)\wedge T\\
&\le 3^M\int_X -\chi(v_1-\phi)(\theta_{v_1}+\theta_\phi)\wedge T + 3^M\int_X -\chi(v_2-\phi)(\theta_{v_2}+\theta_\phi)\wedge T\\
&\lesssim B\varrho,
\end{aligned}$$

where the two last estimates hold due to Lemma 2.7 and the fact

$$\log(-\chi(3t)) - \log(-\chi(t)) = \int_t^{3t} \frac{\chi'(s)}{\chi(s)}\, ds \le \int_t^{3t} \frac{M}{s}\, ds = M \log 3$$

for every $\chi \in \mathcal{W}_M^+$ and $t \le 0$. Combining the estimates in both cases, we obtain

$$I_5 \lesssim B\varrho(1-\tilde{\chi}(-1))\frac{Q(\epsilon)^2}{\epsilon}, \tag{3-12}$$

where we used the inequality $Q(\epsilon) \ge \epsilon^{1/2}$ if $\chi \in \widetilde{W}_M^+$. Now, we estimate I_6. Since U, V are open in the plurifine topology and

$$(1+\epsilon)\tilde{\varphi} = \begin{cases} \varphi & \text{on } U, \\ \epsilon(u_1+u_3-2\phi) & \text{on } V, \end{cases}$$

we have

$$\begin{aligned} I_6 &= \int_U \chi'((1+\epsilon)\tilde{\varphi})\, d\tilde{\varphi} \wedge d^c\tilde{\varphi} \wedge T + \int_V \chi'((1+\epsilon)\tilde{\varphi})\, d\tilde{\varphi} \wedge d^c\tilde{\varphi} \wedge T \\ &= (1+\epsilon)^{-2} \int_U \chi'(\varphi)\, d\varphi \wedge d^c\varphi \wedge T \\ &\qquad + \frac{\epsilon^2}{(1+\epsilon)^2} \int_V \chi'(\epsilon(u_1+u_3-2\phi))\, d(u_1+u_3-2\phi) \wedge d^c(u_1+u_3-2\phi) \wedge T \\ &\le J + \epsilon^2 \int_X \chi'(\epsilon(u_1+u_3-2\phi))\, d(u_1+u_3-2\phi) \wedge d^c(u_1+u_3-2\phi) \wedge T \\ &= J + \epsilon \int_X -\chi(\epsilon(u_1+u_3-2\phi))\, dd^c(u_1+u_3-2\phi) \wedge T. \end{aligned}$$

Therefore, it follows from Lemma 2.8 that

$$I_6 \lesssim J + B\varrho(1-\tilde{\chi}(-1))\epsilon Q_0(2\epsilon). \tag{3-13}$$

Combining (3-9)–(3-13), we get

$$\begin{aligned} I \le I_1 + I_2 \le I_3 + I_4 + I_2 &\lesssim (I_5 I_6)^{1/2} + I_4 + I_2 \\ &\lesssim (B\varrho(1-\tilde{\chi}(-1))\epsilon^{-1}J)^{1/2} Q(\epsilon) + B\varrho(1-\tilde{\chi}(-1))Q(2\epsilon)^2. \end{aligned}$$

Letting $\epsilon \searrow J/(2\varrho)$ (and supposing ϵ satisfies (3-8)), we obtain

$$I \lesssim B\varrho(1-\tilde{\chi}(-1))Q(J/\varrho). \qquad \square$$

Proposition 3.5. *Let $\chi, \tilde{\chi} \in \widetilde{\mathcal{W}}^- \cup \mathcal{W}_M^+$ such that $\tilde{\chi} \le \chi$ and $\chi \in \mathscr{C}^1(\mathbb{R})$. Let $u_1, u_2, u_3 \in \mathcal{E}(X, \theta, \phi)$ such that $u_1 \le u_2$ and $u_j - \phi$ is bounded* ($j = 1, 2, 3$), *where ϕ is a negative θ-psh function satisfying $\varrho := \mathrm{vol}(\theta_\phi) > 0$. Then there exists a constant $C_n > 0$ depending only on n and M such that*

$$\int_X \chi'(u_1-u_2)\, d(u_1-u_2) \wedge d^c(u_1-u_2) \wedge \theta_{u_3}^{n-1} \le C_n \varrho B^2 (1-\tilde{\chi}(-1))^2 Q^{\circ(n-1)}(I_\chi^0(u_1,u_2)), \tag{3-14}$$

where $B := \sum_{j=1}^3 \max\{E^0_{\tilde{\chi},\theta,\phi}(u_j), 1\}$ and Q is defined by (3-1).

Proof. Let

$$\varphi := u_1 - u_2, \quad T := \sum_{j=1}^{n-1} \theta_{u_1}^j \wedge \theta_{u_2}^{n-1-j}$$

and

$$T_{k,l} := \theta_{u_1}^k \wedge \theta_{u_2}^l \wedge \theta_{u_3}^{n-k-l-1}, \quad L_{k,l} := \int_X \chi'(\varphi)\, d\varphi \wedge d^c\varphi \wedge T_{k,l}.$$

Observe

$$\theta_{u_2}^n - \theta_{u_1}^n = -dd^c\varphi \wedge T$$

and

$$L_{k,n-1-k} \le \int_X \chi'(\varphi)\, d\varphi \wedge d^c\varphi \wedge T = \varrho I_\chi^0(u_1, u_2) \tag{3-15}$$

by integration by parts. We now prove by inverse induction on $m := k + l$ that

$$L_{k,l} \le C_{m,n} \varrho B^2 (1 - \tilde{\chi}(-1))^2 Q^{\circ(n-1-k-l)}(I_\chi^0(u_1, u_2)) \tag{3-16}$$

for some constant $C_{m,n} > 1$ depending only on m, n and M. The desired assertion (3-14) is the case where $k = l = 0$. In what follows we use the symbols $\lesssim$ and $\gtrsim$ for inequalities modulo a constant depending only on n and M. We have checked (3-16) for $k+l = n-1$. Suppose that (3-16) holds for $k+l = m$ with $0 < m \le n-1$. We will verify it for $L_{k-1,l}$, where $k+l = m$ and $k > 1$. The case $L_{k,l-1}$ is done similarly.

Denote $S_{k-1,l} = \theta_{u_1}^{k-1} \wedge \theta_{u_2}^l \wedge \theta_{u_3}^{n-k-l-1}$. Then

$$L_{k-1,l} - L_{k,l} = \int_X \chi'(\varphi)\, d\varphi \wedge d^c\varphi \wedge dd^c(u_3 - u_1) \wedge S_{k-1,l}.$$

Using integration by parts, we have

$$\begin{aligned} L_{k-1,l} - L_{k,l} &= \int_X -\chi(\varphi)\, dd^c(\varphi) \wedge dd^c(u_3 - u_1) \wedge S_{k-1,l} \\ &= \int_X -\chi(\varphi)\, dd^c(u_3 - u_1) \wedge T_{k,l} - \int_X -\chi(\varphi)\, dd^c(u_3 - u_1) \wedge T_{k-1,l+1} \\ &= \int_X \chi'(\varphi)\, d\varphi \wedge d^c(u_3 - u_1) \wedge T_{k,l} - \int_X \chi'(\varphi)\, d\varphi \wedge d^c(u_3 - u_1) \wedge T_{k-1,l+1}. \end{aligned}$$

Therefore, it follows from Lemma 3.4 that

$$L_{k-1,l} - L_{k,l} \lesssim \varrho B (1 - \tilde{\chi}(-1)) \big(Q(L_{k,l}/\varrho) + Q(L_{k-1,l+1}/\varrho) \big).$$

Hence, by using the inductive hypothesis, we get

$$\begin{aligned} L_{k-1,l} &\lesssim \varrho B^2 (1 - \tilde{\chi}(-1))^2 Q^{\circ(n-1-m)}(I_\chi^0(u_1, u_2)) \\ &\qquad\qquad + \varrho B (1 - \tilde{\chi}(-1)) Q\big(C_{m,n} B^2 (1 - \tilde{\chi}(-1))^2 Q^{\circ(n-1-m)}(I_\chi^0(u_1, u_2)) \big) \\ &\lesssim \varrho B^2 (1 - \tilde{\chi}(-1))^2 Q^{\circ(n-m)}(I_\chi^0(u_1, u_2)). \end{aligned}$$

Here we use the fact $Q(t_1) \le (t_1/t_2)^{1/2} Q(t_2)$ for every $t_1 > t_2 > 0$ (see Lemma 3.6).

Thus (3-16) holds for $L_{k-1,l}$. □

Lemma 3.6. *The function $h(t) = (Q(t))^2/t$ is nonincreasing in $\mathbb{R}_{>0}$.*

Proof. If $\chi \in \mathcal{W}_M^+$ then

$$h(t) = \begin{cases} \dfrac{1}{t} & \text{if } t \geq 1, \\ 1 & \text{if } 0 < t < 1 \end{cases}$$

is a nonincreasing function.

We consider the case $\chi \in \widetilde{\mathcal{W}}^-$. We have

$$h(t) = \begin{cases} \dfrac{1}{t} & \text{if } t \geq 1, \\ \dfrac{Q_0(t)}{t Q_0(1)} & \text{if } 0 < t < 1. \end{cases}$$

It is clear that h is decreasing in $[1, \infty)$. We need to show that h is nonincreasing in $(0, 1)$. Since χ is convex, we have

$$\frac{\chi(t_1 s)}{t_1 s} \leq \frac{\chi(t_2 s)}{t_2 s}$$

for every $0 < t_2 < t_1 < 1$ and $s < 0$. Dividing both sides of the last estimate by $\tilde{\chi}(s)/s$, we get

$$\frac{\chi(t_1 s)}{t_1 \tilde{\chi}(s)} \leq \frac{\chi(t_2 s)}{t_2 \tilde{\chi}(s)}.$$

Taking the supremum of both sides, we obtain

$$\frac{Q_0(t_1)}{t_1} = \sup_{s \leq -1} \frac{\chi(t_1 s)}{t_1 \tilde{\chi}(s)} \leq \sup_{s \leq -1} \frac{\chi(t_2 s)}{t_2 \tilde{\chi}(s)} = \frac{Q_0(t_2)}{t_2}.$$

Then $h(t_1) \leq h(t_2)$. Hence h is nonincreasing in $(0, 1)$. □

End of the proof of Theorem 3.1. By Lemma 3.3, we can assume that u_j, ψ_j are of the same singularity type as ϕ. Now let $(\chi_j)_{j \in \mathbb{N}}$, $(\tilde{\chi}_j)_{j \in \mathbb{N}}$ be the sequences approximating χ, $\tilde{\chi}$ respectively in Lemma 2.6. By Lebesgue's dominated convergence theorem, observe that

$$\lim_{j \to \infty} I^0_{\chi_j}(u_1, u_2) = I^0_{\chi}(u_1, u_2)$$

and

$$\lim_{j \to \infty} \int_X -\chi_j(u_1 - u_2)(\theta^n_{\psi_1} - \theta^n_{\psi_2}) = \int_X -\chi(u_1 - u_2)(\theta^n_{\psi_1} - \theta^n_{\psi_2}).$$

On the other hand, for $\epsilon \in (0, 1]$ we also get

$$\lim_{j \to \infty} Q_{\chi_j, \tilde{\chi}_j}(\epsilon) = Q_{\chi, \tilde{\chi}}(\epsilon),$$

because for $t \leq -1$, one has

$$\frac{\chi_j(\epsilon t)}{\tilde{\chi}_j(t)} \leq \frac{\chi_j(\epsilon t)}{\tilde{\chi}(t)} \leq \frac{\chi(\epsilon t) - 2^{-j}}{\tilde{\chi}(t)} \quad \text{and} \quad \frac{\chi_j(\epsilon t)}{\tilde{\chi}_j(t)} \geq \frac{\chi_j(\epsilon t)}{\tilde{\chi}(t) - 2^{-j}},$$

which converges to $\chi(\epsilon t)/\tilde{\chi}(t)$ (by Lemma 2.6). Hence, by considering χ_j, $\tilde{\chi}_j$ instead of χ, $\tilde{\chi}$, we can further assume that $\chi \in \mathscr{C}^1(\mathbb{R})$.

Let L be the left-hand side of the desired inequality. We have

$$\begin{aligned} L &= \int_X -\chi(u_1-u_2)(\theta^n_{\psi_1}-\theta^n_{u_1}) - \int_X -\chi(u_1-u_2)(\theta^n_{\psi_2}-\theta^n_{u_1}) \\ &= \int_X -\chi(u_1-u_2)\, dd^c(\psi_1-u_1)\wedge T_1 - \int_X -\chi(u_1-u_2)\, dd^c(\psi_2-u_1)\wedge T_2 \\ &= L_1 - L_2, \end{aligned}$$

where $T_j = \sum_{l=0}^{n-1} \theta^l_{\psi_j}\wedge\theta^{n-l-1}_{u_1}$. Using integration by parts and Lemma 3.4, we get

$$\begin{aligned} L_1 &= \int_X \chi'(u_1-u_2)\, d(u_1-u_2)\wedge d^c(\psi_1-u_1)\wedge T_1 \\ &\le C_1\varrho B(1-\tilde\chi(-1))\, Q\biggl(\varrho^{-1}\int_X \chi'(u_1-u_2)\, d(u_1-u_2)\wedge d^c(u_1-u_2)\wedge T_1\biggr), \end{aligned}$$

where $C_1>0$ depends only on n and M. Observe that there is a dimensional constant C_1' such that

$$T_1 \le C_1'\theta^{n-1}_{(\psi_1+u_1)/2}.$$

Moreover one has

$$E_{\tilde\chi,\theta,\phi}((\psi_1+u_1)/2) \lesssim E_{\tilde\chi,\theta,\phi}(\psi_1) + E_{\tilde\chi,\theta,\phi}(u_1)$$

by Lemma 2.7. Hence, it follows from Proposition 3.5 (applied to $u_3 := (\psi_1+u_1)/2$) that

$$\varrho^{-1}\int_X \chi'(u_1-u_2)\, d(u_1-u_2)\wedge d^c(u_1-u_2)\wedge T_1 \le C_2B^2(1-\tilde\chi(-1))^2 Q^{\circ(n-1)}(I^0_\chi(u_1,u_2)),$$

where $C_2>1$ depends only on n and M. Then

$$L_1 \le C_3\varrho B^2(1-\tilde\chi(-1))^2 Q^{\circ n}(I^0_\chi(u_1,u_2)),$$

where $C_3>0$ depends only on n and M. Here we use the fact $Q(t_1)\le (t_1/t_2)^{1/2}Q(t_2)$ for every $t_1>t_2>0$ (Lemma 3.6).

By the same arguments, we also have

$$-L_2 \le C_4\varrho B^2(1-\tilde\chi(-1))^2 Q^{\circ n}(I^0_\chi(u_1,u_2)),$$

where $C_4>0$ depends only on n and M.

Hence

$$L = L_1 - L_2 \le (C_3+C_4)\varrho B^2(1-\tilde\chi(-1))^2 Q^{\circ n}(I^0_\chi(u_1,u_2)). \qquad \square$$

3.3. ***Proof of Theorem 3.2.*** Recall that for every Borel set E in X, we define

$$\operatorname{cap}_{\theta,\phi}(E) := \sup\biggl\{\int_E \theta^n_h : h\in \operatorname{PSH}(X,\theta),\ \phi-1\le h\le\phi\biggr\}.$$

The following is an improvement of results from [Darvas et al. 2018b; 2021a] (see also [Boucksom et al. 2010; Kołodziej 2003]).

Theorem 3.7. *Let $A \geq 1$ be a constant and let θ be a closed smooth real $(1,1)$-form such that $\theta \leq A\omega$. Let $\phi \in \mathrm{PSH}(X,\theta)$ and $0 \leq f \in L^p(X)$ for some constant $p > 1$ such that $\phi = P[\phi]$ and $0 < \int_X f\omega^n = \int_X \theta_\phi^n := \varrho$. Assume $u \in \mathcal{E}(X,\theta,\phi)$ satisfies $\sup_X(u-\phi) = 0$ and $\theta_u^n = f\omega^n$. Then, there exists a constant $C \geq 1$ depending only on X, ω, n and p such that*

$$u \geq \phi - CA\big(\log \|f \,\mathrm{vol}_\omega(X)^q/\varrho\|_{L^p} + \log A + 1\big), \tag{3-17}$$

where $\mathrm{vol}_\omega(X) := \int_X \omega^n$ and $q = p/(p-1)$.

By Hölder inequalities, one sees that

$$1 = \int_X \frac{f}{\varrho}\omega^n \leq \|f/\varrho\|_{L^p}(\mathrm{vol}_\omega(X))^q,$$

and then $\log \|f \,\mathrm{vol}_\omega(X)^q/\varrho\|_{L^p} \geq 0$.

Proof. Without loss of generality, we can assume that $\mathrm{vol}_\omega(X) = 1$. Recall that there exists a constant $\nu > 0$ depending only on X, ω such that

$$\int_X \exp(-\psi/\nu)\omega^n \leq C_0^2$$

for every $\psi \in \mathrm{PSH}(X,\omega)$ with $\sup_X \psi = 0$, where $C_0 \geq 1$ is a constant depending only on X and ω. Consequently, one gets

$$\int_X \exp(-\psi/(A\nu))\omega^n \leq C_0^2$$

for every $\psi \in \mathrm{PSH}(X,\theta) \subset \mathrm{PSH}(X,A\omega)$ with $\sup_X \psi = 0$. By the same arguments as in the proof of [Darvas et al. 2018b, Proposition 4.30] (use [Darvas et al. 2021a, Lemma 3.9] instead of [Darvas et al. 2018b, Lemma 4.9]), we have

$$\int_E \omega^n \leq C_0^2 \exp\bigg(-\frac{1}{2A\nu}\bigg(\frac{\mathrm{cap}_{\theta,\phi}(E)}{\varrho}\bigg)^{-1/n}\bigg)$$

for every Borel set $E \subset X$. Therefore, by the Hölder inequality and the fact that $e^{-1/t} \leq m!\,t^m$ for every $m \in \mathbb{N}$ and every $t > 0$, there exists $A_0 > 0$ depending only on X, ω, n and p such that

$$\varrho^{-1}\int_E \theta_u^n = \int_E (f/\varrho)\omega^n \leq \|f/\varrho\|_{L^p}\bigg(\int_E \omega^n\bigg)^{1/q} \leq A_0 A^{2n}\|f/\varrho\|_{L^p}\frac{\mathrm{cap}_{\theta,\phi}(E)^2}{\varrho^2} \tag{3-18}$$

for every Borel set $E \subset X$, where $1/p + 1/q = 1$. On the other hand, letting $b = (A\nu q)^{-1}$ and $B_0 = (C_0^2)^{1/q}$, we have

$$\varrho^{-1}\int_X e^{-bw}\theta_u^n \leq \|f/\varrho\|_{L^p}\bigg(\int_X e^{-bqw}\omega^n\bigg)^{1/q} \leq B_0\|f/\varrho\|_{L^p} \tag{3-19}$$

for every $w \in \mathrm{PSH}(X,\theta)$ with $\sup_X w = 0$.

For every $h \in \mathrm{PSH}(X,\theta)$ with $\phi - 1 \le h \le \phi$, for each $0 \le t \le 1$ and $s > 0$, we have

$$t^n \int_{\{u<\phi-t-s\}} \theta_h^n \le \int_{\{u<(1-t)\phi+th-s\}} \theta_{(1-t)\phi+th}^n \le \int_{\{u<(1-t)\phi+th-s\}} \theta_u^n \le \int_{\{u<\phi-s\}} \theta_u^n,$$

where the third estimate holds due to the comparison principle [Darvas et al. 2021a, Lemma 2.3]. Then

$$t^n \mathrm{cap}_{\theta,\phi}(u<\phi-t-s) \le \int_{\{u<\phi-s\}} \theta_u^n \tag{3-20}$$

for every $0 \le t \le 1,\ s > 0$. Therefore, it follows from (3-18) that

$$t^n \varrho^{-1} \mathrm{cap}_{\theta,\phi}(u<\phi-t-s) \le A_1 \varrho^{-2} \mathrm{cap}_{\theta,\phi}(u<\phi-s)^2,$$

where $A_1 = A_0 A^{2n} \|f/\varrho\|_{L^p}$. Putting $g(s) = \varrho^{-1/n} \mathrm{cap}_{\theta,\phi}(u<\phi-s)^{1/n}$, the above inequality becomes

$$t g(t+s) \le A_1^{1/n} g(s)^2.$$

Hence, it follows from [Eyssidieux et al. 2009, Lemma 2.4 and Remark 2.5] that if $g(s_0) < 1/(2A_1^{1/n})$ then $g(s) = 0$ for all $s \ge s_0 + 2$. Moreover, by (3-20) and the condition (3-19), we have

$$g(s+1)^n \le \varrho^{-1} \int_{\{u<\phi-s\}} \theta_u^n \le \varrho^{-1} \int_X e^{b(\phi-u-s)} \theta_u^n \le B_1 e^{-bs}$$

for every $s > 0$, where $B_1 = B_0 \|f/\varrho\|_{L^p}$. Then $g(s+1) < 1/(2A_1^{1/n})$ provided that

$$s > \frac{n \log 2 + \log A_1}{b} + \frac{\log B_1}{b}.$$

Hence $g(s) = 0$ for every

$$s \ge \frac{n \log 2 + \log A_1}{b} + \frac{\log B_1}{b} + 4.$$

Thus

$$u \ge \phi - \left(\frac{n \log 2 + \log A_1}{b} + \frac{\log B_1}{b} + 4 \right) = \phi - C_1 \log \|f/\varrho\|_{L^p} - C_2,$$

where $C_1 = 2/b = 2\nu q A$ and

$$\begin{aligned} C_2 &= 4 + \frac{n \log 2 + \log A_0 + \log B_0 + 2n \log A}{b} \\ &= 4 + \nu q (n \log 2 + \log A_0 + \log B_0 + 2n \log A) A. \end{aligned}$$

□

Lemma 3.8. *There exists a constant $C > 0$ depending only on n, X and ω such that for every $u \in \mathrm{PSH}(X,\omega)$ satisfying $\sup_X u = 0$ and for every constant $0 < t \le 1$, one has*

$$\int_{\{u>-t\}} \omega^n \ge C t^{2n}. \tag{3-21}$$

Proof. Let $(U_j, \varphi_j)_{j=1}^m$ be such that $U_j \subset X$ are open, $\varphi_j : 4\mathbb{B} \to U_j$ are biholomorphic and $\bigcup_{j=1}^m \varphi_j(\mathbb{B}) = X$ (where $\mathbb{B}$ is the open unit ball in $\mathbb{C}^n$), and there is a smooth psh function ρ_j in U_j such that $dd^c \rho_j = \omega$ for $1 \leq j \leq m$. Let

$$C_\rho = \sup_{1\leq j\leq m} \sup_{2\mathbb{B}} \|\nabla(\rho_j \circ \varphi_j)\|.$$

Assume $u(z_0) = 0$. Then there exists $1 \leq j_0 \leq m$ such that $z_0 \in \varphi_{j_0}(\mathbb{B})$. Let $w_0 = \varphi_{j_0}^{-1}(z_0)$, $\hat{u}(w) = u \circ \varphi_{j_0}(w)$ and $\hat{\rho}(w) = \rho_{j_0} \circ \varphi_{j_0}(w) - \rho_{j_0} \circ \varphi_{j_0}(w_0)$. By the plurisubharmonicity of $\hat{u} + \hat{\rho}$, for every $t > 0$ and $0 < r < 1$, we have

$$\begin{aligned} 0 = (\hat{u} + \hat{\rho})(w_0) &\leq \frac{1}{\mathrm{vol}_{\mathbb{C}^n}(r\mathbb{B})} \int_{r\mathbb{B}} (\hat{u} + \hat{\rho})\, dV_{2n} \\ &\leq C_\rho r + \frac{1}{c_{2n} r^{2n}} \int_{r\mathbb{B}} \hat{u}\, dV_{2n} \\ &\leq C_\rho r - \frac{t}{c_{2n} r^{2n}} \int_{r\mathbb{B}\cap\{\hat{u}\leq -t\}} dV_{2n} \\ &\leq C_\rho r - t + \frac{t}{c_{2n} r^{2n}} \int_{r\mathbb{B}\cap\{\hat{u}> -t\}} dV_{2n} \\ &\leq C_\rho r - t + \frac{C_\omega t}{r^{2n}} \mathrm{vol}_\omega(\{u > -t\}), \end{aligned}$$

where $c_{2n} = \mathrm{vol}_{\mathbb{C}^n}(\mathbb{B})$ and $C_\omega > 0$ is a constant depending only on n, X, ω. It follows that

$$\mathrm{vol}_\omega(\{u > -t\}) \geq \frac{r^{2n}}{C_\omega}\left(1 - \frac{C_\rho r}{t}\right).$$

Hence, for every $0 < t < 1$, by choosing $r = t/(1 + C_\rho)$, we have

$$\mathrm{vol}_\omega(\{u > -t\}) \geq C t^{2n},$$

where $C = 1/C_\omega(1 + C_\rho)^{2n+1}$ depends only on n, X and ω. □

End of the proof of Theorem 3.2. Without loss of generality, we can assume that $u_1 \leq u_2$. Let $W_t = \{u_1 > a_1 - t\}$ for $0 < t \leq 1$. We have

$$\int_{W_t} -\chi(u_1 - u_2)\omega^n \leq \int_{W_t} -\chi(u_1 - a_2)\omega^n \leq -b_t \chi(a_1 - a_2 - t), \tag{3-22}$$

where $b_t := \mathrm{vol}(W_t)$.

It follows from Lemma 3.8 that $W_t \neq \varnothing$. Moreover,

$$b_t := \int_{W_t} \omega^n \geq C_1 \left(\frac{t}{A}\right)^{2n}, \tag{3-23}$$

where $C_1 > 0$ is a constant depending only on n, X and ω. By [Darvas et al. 2021a, Theorem A] (see also [Do and Vu 2022a, Theorem 3]), there exists a unique $\varphi \in \mathcal{E}(X, \theta, \phi)$ with $\sup_X(\varphi - \phi) = 0$ such that

$$\theta_\varphi^n = \frac{\varrho}{b_t} \mathbf{1}_{W_t} \omega^n.$$

It follows from Theorem 3.7 that

$$\phi - C_2 A(-\log t + \log A + 1) \le \varphi \le \phi \tag{3-24}$$

for some constant $C_2 \ge 1$ depending only on n, X and ω. Thus, we have

$$E^0_{\tilde\chi,\theta,\phi}(\varphi) \le -\tilde\chi\big(-C_2 A(-\log t + \log A + 1)\big) \le -C_3\Big(\log \frac{Ae}{t}\Big)^M \tilde\chi(-A),$$

where $C_3 > 0$ depends only on n, X, ω and M.

Hence, it follows from Theorem 3.1 that

$$\int_X -\chi(u_1-u_2)(\theta^n_\psi - \theta^n_\varphi) \le C_4 \varrho \Big(\log \frac{Ae}{t}\Big)^{2M} (B - \tilde\chi(-A))^2 (1-\tilde\chi(-1))^2 \lambda, \tag{3-25}$$

where $\lambda = Q^{\circ(n)}(I^0_\chi(u_1,u_2))$ and $C_4 > 0$ depends only on n, X, ω and M.

Combining (3-22) and (3-25), we get

$$\int_X -\chi(u_1-u_2)\theta^n_\psi \le -\varrho\chi(a_1-a_2-t) + C_4\varrho\Big(\log\frac{Ae}{t}\Big)^{2M}(B-\tilde\chi(-A))^2(1-\tilde\chi(-1))^2\lambda.$$

Letting $t \to \lambda^m$, we get

$$\begin{aligned}\int_X -\chi(u_1-u_2)\theta^n_\psi &\le -\varrho\chi(a_1-a_2-\lambda^m) + C_4\varrho\Big(\log\frac{Ae}{\lambda^m}\Big)^{2M}(B-\tilde\chi(-A))^2(1-\tilde\chi(-1))^2\lambda \\ &\le -\varrho\chi(a_1-a_2-\lambda^m) + C_5\varrho\frac{A^{(1-\gamma)/m}}{\lambda^{1-\gamma}}(B-\tilde\chi(-A))^2(1-\tilde\chi(-1))^2\lambda \\ &\le -\varrho\chi(a_1-a_2-\lambda^m) + C_5\varrho A^{(1-\gamma)/m}(B-\tilde\chi(-A))^2(1-\tilde\chi(-1))^2\lambda^\gamma,\end{aligned}$$

where $C_5 > 0$ depends only on n, X, ω, M, m and γ. □

Remark 3.9. The hypothesis that $\tilde\chi \le \chi$ in Theorems 3.1 and 3.2 can be slightly relaxed: the same statement remains true if $\tilde\chi \le \chi$ on $(-\infty,-1]$ and $\chi(-1) = -1$. Indeed, we only need the inequality $\tilde\chi \le \chi$ to guarantee that $E_{\chi,\theta,\phi}(u) \le E_{\tilde\chi,\theta,\phi}(u)$ for $u \in \mathrm{PSH}(X,\theta,\phi)$. If we only have $\tilde\chi \le \chi$ on $(-\infty,-1]$, then

$$E_{\chi,\theta,\phi}(u) \le E_{\tilde\chi,\theta,\phi}(u) - \chi(-1)\,\mathrm{vol}(\theta_\phi).$$

This is still sufficient for the proof of Theorems 3.1 and 3.2.

Later we will apply Theorem 3.2 to the special case where $\chi(t) = \max\{t,-1\}$ and $\tilde\chi \in \widetilde{\mathcal{W}}^-$ with $\tilde\chi(-1) = -1$. In this case, we can compute explicitly $Q_{0,\chi,\tilde\chi}(\epsilon) = \sup_{\{t\le -1\}} \chi(\epsilon t)/\tilde\chi(t)$ as follows. Observe that

$$Q_{0,\chi,\tilde\chi}(\epsilon) = \max\bigg\{\sup_{-\epsilon^{-1}\le t\le -1}\frac{\chi(\epsilon t)}{\tilde\chi(t)},\ \sup_{t\le -\epsilon^{-1}}\frac{\chi(\epsilon t)}{\tilde\chi(t)}\bigg\} = \max\bigg\{\sup_{-\epsilon^{-1}\le t\le -1}\frac{\epsilon t}{\tilde\chi(t)},\ \frac{-1}{\tilde\chi(-\epsilon^{-1})}\bigg\}.$$

Since $\tilde\chi \in \widetilde{\mathcal{W}}^-$, the function $t/\tilde\chi(t)$ is decreasing; hence $Q_{0,\chi,\tilde\chi}(\epsilon) = (-\tilde\chi(-\epsilon^{-1}))^{-1}$.

If $\chi(t) = \tilde\chi(t) = -(-t)^p$ for some constant $p > 0$, then one sees directly that $Q_{0,\chi,\tilde\chi}(\epsilon) = \epsilon^p$. However we will not use this special case in applications.

3.4. ***A counterexample.*** Let

$$\chi(t) := -\log(-t+1) \in \mathcal{W}^-.$$

In this subsection, to simplify the notation, we define $E_\chi(u) := E_{\chi,\omega,0}(u)$, where by 0 we mean the constant function equal to 0. Our goal in this subsection is to construct sequences of functions $u_m, v_m \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$ such that

(i) $0 \geq u_m \geq v_m$, $\sup_X u_m = \sup_X v_m = 0$,

(ii) $u_m, v_m \to 0$ in L^1 as $m \to \infty$,

(iii) $\sup_m (E_\chi(u_m) + E_\chi(v_m)) < \infty$ and $\lim_{m\to\infty} I_\chi(u_m, v_m) = 0$ but

(iv) $\inf_m \int_X -\chi(u_m - v_m)(dd^c v_m + \omega)^n > 0$.

As a consequence of our construction of u_m, v_m below, we see that Theorem 1.2 (and Theorem 1.3) does not hold in general if $\chi = \tilde{\chi} \in \mathcal{W}^-$. Here is our construction. On the unit ball $\mathbb{B}$ of $\mathbb{C}^n$, we define

$$\varphi_m = \max\{\log|z|, -e^m\} \quad \text{and} \quad F_m = \{z \in \mathbb{B} : \log|z| = -e^m\}, \quad m > 0.$$

Lemma 3.10. *We have*

$$\int_{F_m} (dd^c\varphi_m)^k \wedge (dd^c|z|^2)^{n-k} = \begin{cases} O(e^{-e^m}) & \text{if } k < n, \\ c & \text{if } k = n, \end{cases} \tag{3-26}$$

where $c := \int_{\{z=0\}} (dd^c \log|z|)^n > 0$.

Proof. The case $k = n$ follows from Stokes' theorem. We consider now $k < n$. Let $\mathbb{B}_r$ be the ball of radius $r > 0$ centered at 0 in $\mathbb{C}^n$. Observe that $\varphi_m = \log|z|$ on an open neighborhood of $\partial\mathbb{B}_{2e^{-e^m}}$. Using this and Stokes' theorem, we obtain

$$\begin{aligned}\int_{F_m} (dd^c\varphi_m)^k \wedge (dd^c|z|^2)^{n-k} &\leq \int_{\mathbb{B}_{2e^{-e^m}}} (dd^c\varphi_m)^k \wedge (dd^c|z|^2)^{n-k} \\ &= \int_{\mathbb{B}_{2e^{-e^m}}} (dd^c \log|z|)^k \wedge (dd^c|z|^2)^{n-k}.\end{aligned}$$

By direct computations (and approximating $\log|z|$ by $\frac{1}{2}\log(|z|^2+\epsilon)$ as $\epsilon \to 0$), we see that

$$\int_{\mathbb{B}_{2e^{-e^m}}} (dd^c \log|z|)^k \wedge (dd^c|z|^2)^{n-k} = O(e^{-e^m}).$$

Hence the desired assertion for $k < n$ follows. □

Let $g : \mathbb{B} \to U$ be a biholomorphic mapping from $\mathbb{B}$ to an open subset U of X. Let $\psi \in C_0^\infty(\mathbb{B})$ such that $0 \leq \psi \leq 1$ and $\psi|_{\mathbb{B}_{1/2}} = 1$. Let

$$\tilde{\varphi}_m = (\varphi_m \psi) \circ g^{-1}.$$

Then there exists a constant $A \geq 1$ such that $\tilde{\varphi}_m$ is $A\omega$-psh for every $m > 0$. Now, for all $m > A^{-n}$, we define

$$u_m = \frac{\tilde{\varphi}_m}{\sqrt[n]{m+1}} \quad \text{and} \quad v_m = \frac{\tilde{\varphi}_{m+1}}{\sqrt[n]{m}}.$$

We have $u_m, v_m \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$ with $\sup_X u_m = \sup_X v_m = 0$ and $0 \geq u_m \geq v_m \xrightarrow{L^1} 0$ as $m \to \infty$.

Put $\mu_m := (dd^c u_m + \omega)^n$ and $\nu_m = (dd^c v_m + \omega)^n$. We have

$$\mathbf{1}_{X \setminus g(F_m)} \mu_m + \mathbf{1}_{X \setminus g(F_{m+1})} \nu_m \leq C_1 \omega^n \tag{3-27}$$

for every m, where $C_1 > 0$ is a constant. By (3-26), we also have

$$\mu_m(g(F_m)) = \frac{c}{m+1} + O(e^{-e^m}) \quad \text{and} \quad \nu_m(g(F_{m+1})) = \frac{c}{m} + O(e^{-e^m}). \tag{3-28}$$

By (3-27), (3-28) and by the fact $v_m \xrightarrow{L^1} 0$, there exists $C_2 > 0$ such that

$$E_\chi(v_m) \leq C_1 \int_{X \setminus g(F_{m+1})} -\chi(v_m)\omega^n - \chi\left(\frac{-e^{m+1}}{\sqrt[n]{m}}\right)\left(\frac{c}{m} + O(e^{-e^m})\right) \leq C_2$$

for every $m \gg 1$. Hence, $\sup_m E_\chi(v_m) < \infty$. Since $v_m \leq u_m \leq 0$, we also have $\sup_m E_\chi(u_m) < \infty$. On the other hand,

$$\begin{aligned}\int_X -\chi(v_m - u_m)(dd^c v_m + \omega)^n &\geq \int_{g(F_{m+1})} -\chi(v_m - u_m)(dd^c v_m + \omega)^n \\ &\geq \frac{c}{m}\log\left(\frac{e^{m+1}}{\sqrt[n]{m}} - \frac{e^m}{\sqrt[n]{m+1}} + 1\right) \\ &\geq \frac{c}{m}\log\left(\frac{(e-1)e^m}{\sqrt[n]{m+1}}\right) \geq \frac{c}{2}\end{aligned}$$

for $m \gg 1$. It remains to show that $\lim_{m\to\infty} I_\chi(u_m, v_m) = 0$. By (3-27) and (3-28), we have

$$\begin{aligned}I_\chi(u_m, v_m) = \int_{X \setminus g(F_m \cup F_{m+1})} & -\chi(v_m - u_m)(\nu_m - \mu_m) \\ & - \chi((v_m - u_m)(e^{-e^{m+1}}))\nu_m(g(F_{m+1})) + \chi((v_m - u_m)(e^{-e^m}))\mu_m(g(F_m)) \\ \leq C_1 \int_X |v_m|\omega^n &+ \frac{c}{m}\log\left(\frac{e^{m+1}}{\sqrt[n]{m}} - \frac{e^m}{\sqrt[n]{m+1}} + 1\right) \\ & - \frac{c}{m+1}\log\left(\frac{e^m}{\sqrt[n]{m}} - \frac{e^m}{\sqrt[n]{m+1}} + 1\right) + O(e^{-e^m}) \\ \leq C_1 \int_X |v_m|\omega^n &+ \frac{c}{m}\log\left(\frac{e}{\sqrt[n]{m}} - \frac{1}{\sqrt[n]{m+1}} + e^{-m}\right) \\ & - \frac{c}{m+1}\log\left(\frac{1}{\sqrt[n]{m}} - \frac{1}{\sqrt[n]{m+1}} + e^{-m}\right) + \frac{c}{m+1} + O(e^{-e^m}) \\ \xrightarrow{m\to\infty} 0.\end{aligned}$$

Hence we get $\lim_{m\to\infty} I_\chi(u_m, v_m) = 0$.

4. Applications

4.1. ***Quantitative version of Dinew's uniqueness theorem.*** For every Borel set E in X, recall that the capacity of E is given by

$$\operatorname{cap}(E)=\operatorname{cap}_\omega(E)=\sup_{\{w\in \operatorname{PSH}(X,\omega):0\le w\le 1\}}\int_E \omega_w^n.$$

We usually remove the subscript ω from $\operatorname{cap}_\omega$ if ω is clear from the context. There are generalizations of capacity in big cohomology classes, many of them are comparable; see Theorem 4.8 below and [Lu 2021]. Recall that a sequence of Borel functions $(u_j)_j$ is said to *converge to a Borel function u in capacity* if for every constant $\epsilon>0$, we have that $\operatorname{cap}(\{|u_j-u|\ge\epsilon\})$ converges to 0 as $j\to\infty$. Recall that for $u_j, u\in \operatorname{PSH}(X,\omega)$, if $u_j\to u$ in capacity, then $u_j\to u$ in L^1.

The convergence in capacity is of great importance in pluripotential theory in part because it implies the convergence of Monge–Ampère operators under reasonable circumstances. To study quantitatively the convergence in capacity, it is convenient to introduce the following distance function on $\operatorname{PSH}(X,\omega)$:

$$d_{\mathrm{cap}}(u,v):=\sup_{\{w\in \operatorname{PSH}(X,\omega):0\le w\le 1\}}\int_X |u-v|^{1/2}\omega_w^n$$

for every $u, v\in \operatorname{PSH}(X,\omega)$ (note that $d_{\mathrm{cap}}(u,v)<\infty$ thanks to the Chern–Levine–Nirenberg inequality). The number $\frac12$ in the definition of d_{cap} can be replaced by any constant in $(0,1)$. One can see that for $u_j, u\in \operatorname{PSH}(X,\omega)$ for $j\in\mathbb{N}$, $d_{\mathrm{cap}}(u_j,u)\to 0$ if and only if $|u_j-u|\to 0$ in capacity. Indeed, if $d_{\mathrm{cap}}(u_j,u)\to 0$, then it is clear that $|u_j-u|\to 0$ in capacity. For the converse statement, assume that $|u_j-u|$ converges to 0 in capacity, i.e., for every constant $\delta>0$, we have

$$\lim_{j\to\infty}\operatorname{cap}(\{|u_j-u|\ge\delta\})=0.$$

In particular, the L^1-norm of u_j is bounded uniformly in j. Consequently

$$\begin{aligned}\int_X |u_j-u|^{1/2}\omega_w^n &\le \int_{\{|u_j-u|\le\delta\}}|u_j-u|^{1/2}\omega_w^n+\int_{\{|u_j-u|\ge\delta\}}|u_j-u|^{1/2}\omega_w^n\\ &\le \delta^{1/2}\int_X\omega^n+\Big(\int_{\{|u_j-u|\ge\delta\}}\omega_w^n\Big)^{1/2}\Big(\int_{\{|u_j-u|\ge\delta\}}|u_j-u|\omega_w^n\Big)^{1/2}\quad\text{(Hölder's inequality)}\\ &\lesssim \delta^{1/2}\int_X\omega^n+\big(\operatorname{cap}(\{|u_j-u|\ge\delta\})\big)^{1/2},\end{aligned}$$

by Chern–Levine–Nirenberg inequality. Hence $d_{\mathrm{cap}}(u_j,u)\to 0$ if $|u_j-u|\to 0$ in capacity. The following result is an immediate consequence of the Chern–Levine–Nirenberg inequality.

Proposition 4.1. *Let $\theta\le A\omega$ be a closed smooth real $(1,1)$-form (where $A\ge 1$ is a constant) and ϕ be a model θ-psh function with $\varrho:=\int_X\theta_\phi^n>0$. Let $0\le w\le 1$ is an ω-psh function and ψ is the unique solution to the problem*

$$\begin{cases}u\in\mathcal{E}(X,\theta,\phi),\\ \theta_u^n=\dfrac{\varrho}{\operatorname{vol}(X)}(dd^cw+\omega)^n,\\ \sup_X u=0.\end{cases}\tag{4-1}$$

Then there exists a constant $C > 0$ depending only on X and ω such that

$$\int_X |\psi| \theta_\psi^n \le C A \varrho.$$

Here is the main result of this subsection.

Theorem 4.2. *Let $\theta \le A\omega$ be a closed smooth real $(1,1)$-form ($A \ge 1$) and let ϕ be a model θ-psh function such that $\varrho := \mathrm{vol}(\theta_\phi) > 0$. Let $B \ge 1$, $\tilde{\chi} \in \widetilde{\mathcal{W}}^-$ and $u_1, u_2 \in \mathcal{E}(X, \theta, \phi)$ such that $\tilde{\chi}(-1) = -1$ and*

$$E^0_{\tilde{\chi},\theta,\phi}(u_1) + E^0_{\tilde{\chi},\theta,\phi}(u_2) \le B.$$

Let $\chi(t) = \max\{t, -1\}$. Then, for every $0 < \gamma < 1$, there exists $C > 0$ depending only on n, X, ω and γ such that

$$d_{\mathrm{cap}}(u_1, u_2)^2 \le C(A + |a_1 - a_2|)\big(|a_1 - a_2| + A(A+B)^2 \lambda^\gamma\big), \tag{4-2}$$

where

$$a_j := \sup_X u_j, \quad \lambda = \frac{1}{h^{\circ n}(1/I^0_\chi(u_1, u_2))} \quad \textit{and} \quad h(s) = (-\tilde{\chi}(-s))^{1/2}.$$

One sees that for $u_1, u_2 \in \mathcal{E}(X, \theta, \phi)$, we can find a common $\tilde{\chi} \in \mathcal{W}^-$ so that the assumption in Theorem 4.2 is satisfied. Thus if $\sup_X u_1 = \sup_X u_2 = 0$, and $\theta^n_{u_1} = \theta^n_{u_2}$, then the right-hand side of (4-2) vanishes; hence $u_1 = u_2$. We then recover Dinew's uniqueness theorem for prescribed singularities potentials [Boucksom et al. 2010; Darvas et al. 2018b; Dinew 2009].

Proof. Suppose that w is an arbitrary ω-psh function satisfying $0 \le w \le 1$ and ψ is the unique solution to the problem

$$\begin{cases} u \in \mathcal{E}(X, \theta, \phi), \\ \theta_u^n = \dfrac{\varrho}{\mathrm{vol}(X)}(dd^c w + \omega)^n, \\ \sup_X u = 0. \end{cases} \tag{4-3}$$

We split the proof into two cases.

Case 1: Assume now that $I^0_\chi(u_1, u_2) \le 1$. Hence, we get $\lambda = Q^{\circ n}_{\chi,\tilde{\chi}}(I^0_\chi(u_1, u_2))$ (see Remark 3.9), and one has $-\tilde{\chi}(-A) \le A$ because $\tilde{\chi}(-1) = -1$. It follows from Theorem 3.2 and Proposition 4.1 that, for every $0 < \gamma < 1$, there exists $C_1 > 0$ depending only on n, X, ω and γ such that

$$I := \int_X -\chi(-|u_1 - u_2|)\theta_\psi^n \le -\varrho\chi(-|a_1 - a_2| - \lambda) + C_1 \varrho A(A+B)^2 \lambda^\gamma. \tag{4-4}$$

Moreover

$$\begin{aligned} \frac{\varrho}{\mathrm{vol}(X)} \int_X |u_1 - u_2|^{1/2}(dd^c w + \omega)^n &= \int_X |u_1 - u_2|^{1/2} \theta_\psi^n \\ &= \int_{\{|u_1 - u_2| \le 1\}} |u_1 - u_2|^{1/2}\theta_\psi^n + \int_{\{|u_1 - u_2| > 1\}} |u_1 - u_2|^{1/2}\theta_\psi^n \\ &\le I^{1/2}\bigg(\Big(\int_{\{|u_1 - u_2| \le 1\}} \theta_\psi^n\Big)^{1/2} + \Big(\int_{\{|u_1 - u_2| > 1\}} |u_1 - u_2|\theta_\psi^n\Big)^{1/2}\bigg), \end{aligned}$$

where the last estimate holds due to the Cauchy–Schwarz inequality. Moreover, it follows from Chern–Levine–Nirenberg inequality [Kołodziej 2005] that

$$\begin{aligned}\int_X |u_1-a_1-u_2+a_2|\theta_\psi^n &= \frac{\varrho}{\operatorname{vol}(X)}\int_X |u_1-a_1-u_2+a_2|(dd^c w+\omega)^n\\ &\le C_2\varrho(\|u_1-a_1\|_{L^1(X)}+\|u_2-a_2\|_{L^1(X)})\\ &\le \varrho C_3 A,\end{aligned} \tag{4-5}$$

where $C_2, C_3 > 0$ depend only on X and ω. Here, the last estimate holds due to the compactness of $\{u \in \operatorname{PSH}(X,\omega) : \sup_X u = 0\}$ in $L^1(X)$.

Hence

$$\frac{\varrho}{\operatorname{vol}(X)}\int_X |u_1-u_2|^{1/2}(dd^c w+\omega)^n \le C_4 I^{1/2}\varrho^{1/2}(A+|a_1-a_2|)^{1/2}, \tag{4-6}$$

where $C_4 > 0$ depends only on X and ω.

Combining (4-4) and (4-6), we get

$$\begin{aligned}\left(\int_X |u_1-u_2|^{1/2}(dd^c w+\omega)^n\right)^2 &\le C_5(A+|a_1-a_2|)\big(-\chi(-|a_1-a_2|-\lambda)+A(A+B)^2\lambda^\gamma\big)\\ &\le C_5(A+|a_1-a_2|)(|a_1-a_2|+\lambda+A(A+B)^2\lambda^\gamma)\\ &\le C_6(A+|a_1-a_2|)(|a_1-a_2|+A(A+B)^2\lambda^\gamma),\end{aligned}$$

where $C_5, C_6 > 0$ depend only on n, X, ω and γ. Since w is arbitrary, we obtain the desired inequality.

Case 2: We treat now the case where $I^0_\chi(u_1,u_2) \ge 1$.

Observe that $\lambda \ge 1$ in this case. Hence the right-hand side of (4-2) is greater than or equal to $C(A+|a_1-a_2|)$ because $A \ge 1$ and $\lambda \ge 1$. On the other hand, Hölder's inequality gives

$$\begin{aligned}\left(\int_X |u_1-u_2|^{1/2}(dd^c w+\omega)^n\right)^2 &\lesssim \int_X |u_1-u_2|(dd^c w+\omega)^n\\ &\le \int_X |u_1-a_1-u_2+a_2|(dd^c w+\omega)^n+|a_1-a_2|\int_X\omega^n\\ &\lesssim A+|a_1-a_2|\end{aligned}$$

by (4-5). Thus the desired estimate holds. □

Remark 4.3. If $B \ge A$ then the inequality (4-2) is equivalent to

$$d_{\mathrm{cap}}(u_1,u_2)^2 \le \widetilde{C}(A+|a_1-a_2|)(|a_1-a_2|+AB^2\lambda^\gamma),$$

where $\widetilde{C} > 0$ depends only on n, X, ω and γ.

4.2. *Quantitative version for the domination principle.*

Theorem 4.4. *Let $A \ge 1$ be a constant and let $\theta \le A\omega$ be a closed smooth real $(1,1)$-form and ϕ be a model θ-psh function, and $\varrho := \operatorname{vol}(\theta_\phi) > 0$. Let $B \ge 1$ be a constant, $\tilde\chi \in \widetilde{\mathcal{W}}^-$ and $u_1, u_2 \in \mathcal{E}(X,\theta,\phi)$ such that $\tilde\chi(-1) = -1$ and*

$$E^0_{\tilde\chi,\theta,\phi}(u_1)+E^0_{\tilde\chi,\theta,\phi}(u_2) \le B.$$

Assume that there exists a constant $0 \le c < 1$ *and a Radon measure* μ *on* X *satisfying* $\theta_{u_1}^n \le c\theta_{u_2}^n + \varrho\mu$ *on* $\{u_1 < u_2\}$ *and* $c_\mu := \int_{\{u_1<u_2\}} d\mu \le 1$. *Then there exists a constant* $C > 0$ *depending only on* n, X *and* ω *such that*

$$\mathrm{cap}_\omega\{u_1 < u_2 - \epsilon\} \le \frac{C\,\mathrm{vol}(X)(A+B)^2}{\epsilon(1-c)h^{\circ n}(1/c_\mu)}$$

for every $0 < \epsilon < 1$, *where* $h(s) = (-\tilde{\chi}(-s))^{1/2}$ *for every* $0 \le s \le \infty$.

In particular, *if* $c_\mu = 0$ *then* $\mathrm{cap}_\omega\{u_1 < u_2 - \epsilon\} = 0$ *for every* $\epsilon > 0$, *and then* $u_1 \ge u_2$ *on whole* X.

The standard domination principle corresponds to the case where $c = 0$ and $\mu := 0$. A non-quantitative version of this domination principle (i.e., for $\mu = 0$) in the non-Kähler setting was obtained in [Guedj and Lu 2023].

Proof of Theorem 4.4. Let w be an arbitrary ω-psh function satisfying $0 \le w \le 1$ and ψ is the unique solution to (4-1). Let $v = \max\{u_1, u_2\}$ and $\chi(t) = \max\{t, -1\} \ge \tilde{\chi}(t)$. By Theorem 3.1 and Proposition 4.1, there exists a constant $C_1 > 0$ depending only on n, X an ω such that

$$I_1 := \int_X -\chi(u_1 - v)(\theta_\psi^n - \theta_{u_1}^n) \le C_1\varrho(A+B)^2 Q^{\circ(n)}(I_\chi^0(u_1, v)), \tag{4-7}$$

$$I_2 := \int_X -\chi(u_1 - v)(\theta_{u_2}^n - \theta_{u_1}^n) \le C_1\varrho(A+B)^2 Q^{\circ(n)}(I_\chi^0(u_1, v)). \tag{4-8}$$

Moreover, by the fact $\theta_v^n = \theta_{u_2}^n$ on $\{u_1 < u_2\}$ and by the assumption $\theta_{u_1}^n \le c\theta_{u_2}^n + \varrho\mu$ on $\{u_1 < u_2\}$, we have

$$I_\chi^0(u_1, v) = \varrho^{-1}\int_{\{u_1<u_2\}} -\chi(u_1 - v)(\theta_{u_1}^n - \theta_{u_2}^n) \le \varrho^{-1}\int_{\{u_1<u_2\}} -\chi(u_1 - v)(\theta_{u_1}^n - c\theta_{u_2}^n) \le c_\mu. \tag{4-9}$$

Combining (4-7), (4-8) and (4-9), we get

$$\begin{aligned}(1-c)\int_X -\chi(u_1 - v)\theta_\psi^n &= \int_X -\chi(u_1 - v)(\theta_{u_1}^n - c\theta_{u_2}^n) + (1-c)I_1 + cI_2 \\ &\le \int_X -\chi(u_1 - v)(\theta_{u_1}^n - c\theta_{u_2}^n) + C_1\varrho(A+B)^2 Q^{\circ n}(c_\mu) \\ &\le \varrho c_\mu + C_1\varrho(A+B)^2 Q^{\circ n}(c_\mu) \\ &\le C\varrho(A+B)^2 Q^{\circ n}(c_\mu),\end{aligned}$$

where $C = C_1 + 1$. Hence

$$\int_{\{u_1<u_2-\epsilon\}} \omega_w^n = \frac{\mathrm{vol}(X)}{\varrho}\int_{\{u_1<u_2-\epsilon\}} \theta_\psi^n \le \frac{C\,\mathrm{vol}(X)(A+B)^2 Q^{\circ n}(c_\mu)}{(1-c)\epsilon}$$

for every $0 < \epsilon < 1$. Since w is arbitrary, it follows that

$$\mathrm{cap}_\omega\{u_1 < u_2 - \epsilon\} \le \frac{C\,\mathrm{vol}(X)(A+B)^2 Q^{\circ n}(c_\mu)}{(1-c)\epsilon}. \tag{4-10}$$

Moreover, by the definition of χ and the formula of Q, we have

$$Q(s) = \frac{1}{(-\tilde{\chi}(-1/s))^{1/2}} = \frac{1}{h(1/s)}$$

for every $0 < s \leq 1$, and $Q(0) = 0$. Then

$$Q^{\circ n}(s) = \frac{1}{h^{\circ n}(1/s)} \tag{4-11}$$

for every $0 \leq s \leq 1$. □

4.3. ***Relation to Darvas's metrics on the space of potentials of finite energy.*** Let $\chi \in \mathcal{W}^- \cup \mathcal{W}^+_M$. Let θ be a closed smooth real $(1,1)$-form in a big cohomology class. When θ is Kähler, it was proved in [Darvas 2015; 2017; 2024] that there is a natural metric d_χ on $\mathcal{E}_\chi(X,\theta)$ which makes the last space to be a complete metric space. When $\chi(t) = t$, such metrics have a long history and play an important role in the study of complex Monge–Ampère equations. We refer to these last references and [Berman et al. 2020; 2021] for more details. We now draw the connection between $I_\chi(u,v)$ and the metric on $\mathcal{E}_\chi(X,\theta)$. Let

$$\tilde{I}_\chi(u,v) = \int_{\{u<v\}} -\chi(u-v)(\theta_v^n + \theta_u^n) + \int_{\{u>v\}} -\chi(v-u)(\theta_u^n + \theta_v^n) \geq I_\chi(u,v).$$

By [Darvas 2015; 2017; 2024], there exists a constant $C > 0$ such that

$$C^{-1}\tilde{I}_\chi(u,v) \leq d_\chi(u,v) \leq C\tilde{I}_\chi(u,v)$$

for every $u, v \in \mathcal{E}_\chi(X,\theta)$ and θ is Kähler. It was proved in [Gupta 2023] (and also [Darvas 2015; Darvas et al. 2018a; Di Nezza and Lu 2020; Trusiani 2022; Xia 2023]) that $\tilde{I}_\chi(u,v)$ satisfies a quasitriangle inequality, and the convergence in $\tilde{I}_\chi(u,v)$ implies the convergence in capacity by using the plurisubharmonic envelope. Such a method is not quantitative. We present below quantitative version of this fact by using our approach.

Theorem 4.5. *Let $\theta \leq A\omega$ be a closed smooth real $(1,1)$-form ($A \geq 1$ is a constant) and ϕ be a model θ-psh function with $\varrho := \mathrm{vol}(\theta_\phi) > 0$. Let $B \geq 1$, $\tilde{\chi} \in \mathcal{W}^-$ and $u_1, u_2 \in \mathcal{E}(X,\theta,\phi)$ such that $\tilde{\chi}(-1) = -1$ and*

$$E^0_{\tilde{\chi},\theta,\phi}(u_1) + E^0_{\tilde{\chi},\theta,\phi}(u_2) \leq B.$$

Then there exist $C > 0$ depending only on n, X and ω such that

$$d_{\mathrm{cap}}(u_1,u_2)^2 \leq \frac{C\big(A + |\sup_X u_1 - \sup_X u_2|\big)(A+B)^2}{h^{\circ n}(\varrho/\tilde{I}_{\tilde{\chi}}(u_1,u_2))},$$

where $h(s) = (-\tilde{\chi}(-s))^{1/2}$ for every $0 \leq s \leq \infty$.

We note that the quantities $a_j := |\sup_X u_j|$ for $j = 1, 2$ (hence $|a_1 - a_2|$) can be bounded by a function of B and $\tilde{\chi}$ as follows. Since ϕ is a model, we have $-a_j = \sup_X(u_j - \phi)$. It follows that

$$B \geq E^0_{\tilde{\chi},\theta,\phi}(u_j) \geq -\tilde{\chi}(-a_j).$$

Consequently, we get $a_j \le -\tilde{\chi}^{-1}(-B)$ for $j = 1, 2$, where $\tilde{\chi}^{-1}$ denotes the inverse map of $\tilde{\chi} : \mathbb{R}_{\le 0} \to \mathbb{R}_{\le 0}$. Thus by Theorem 4.5, one sees that if $\tilde{I}_{\tilde{\chi}}(u_1, u_2)$ is small, then so is $d_{\mathrm{cap}}(u_1, u_2)$ (uniformly in $u_1, u_2 \in \mathcal{E}(X, \theta, \phi)$ of $\tilde{\chi}$-energy bounded by a fixed constant).

Proof. Let $\chi(t) = \max\{t, -1\}$. Suppose that w is an arbitrary ω-psh function satisfying $0 \le w \le 1$. By the proof of Theorem 4.2 (see (4-6)), there exists $C_1 > 0$ depending only on X and ω such that

$$\left(\int_X |u_1 - u_2|^{1/2}(dd^c w + \omega)^n\right)^2 \le C_1\big(A + |\sup_X u_1 - \sup_X u_2|\big)\varrho^{-1} \int_X -\chi(-|u_1 - u_2|)\theta_\psi^n, \tag{4-12}$$

where ψ is defined by (4-3). Moreover, it follows from Theorem 3.1 (applied to $u_1, \max\{u_1, u_2\}$, $\psi_1 := \psi$, $\psi_2 := u_1$) and Proposition 4.1 that

$$\int_X -\chi(-|u_1 - u_2|)\theta_\psi^n \le \tilde{I}_\chi(u_1, u_2) + C_2\varrho(A + B)^2 Q^{\circ(n)}_{\chi,\tilde{\chi}}(I^0_\chi(u_1, u_2)),$$

where $C_2 > 0$ depends only on n. Therefore, since

$$Q^{\circ(n)}(s) = \frac{1}{h^{\circ(n)}(1/s)} \quad \text{and} \quad I_\chi(u_1, u_2) \le \tilde{I}_\chi(u_1, u_2) \le \tilde{I}_{\tilde{\chi}}(u_1, u_2)),$$

we obtain

$$\int_X -\chi(-|u_1 - u_2|)\theta_\psi^n \le \frac{C_3\varrho(A + B)^2}{h^{\circ(n)}(\varrho/\tilde{I}_{\tilde{\chi}}(u_1, u_2))}, \tag{4-13}$$

where $C_3 > 0$ depends only on n, X and ω. Combining (4-12) and (4-13), we get

$$\left(\int_X |u_1 - u_2|^{1/2}(dd^c w + \omega)^n\right)^2 \le \frac{C\left(A + |\sup_X u_1 - \sup_X u_2|\right)(A + B)^2}{h^{\circ(n)}(\varrho/\tilde{I}_{\tilde{\chi}}(u_1, u_2))},$$

where $C > 0$ depends only on n, X and ω. Since w is arbitrary, we get the desired inequality. □

Remark 4.6. Consider now a weight $\tilde{\chi} \in \mathcal{W}^+_M$ with $\tilde{\chi}(-1) = -1$. One sees that $\tilde{\chi}(t) \le (-t)^M \tilde{\chi}(-1) = -(-t)^M$ for $-1 \le t \le 0$, and $\tilde{\chi}(t) \le \tilde{\chi}_0(t) := t$ for $t \le -1$. Consequently, using Hölder's inequality, we get

$$\rho^{-1}\tilde{I}_{\tilde{\chi}_0}(u_1, u_2) \le 2(\rho^{-1}\tilde{I}_{\tilde{\chi}}(u_1, u_2))^{1/M} + \rho^{-1}\tilde{I}_{\tilde{\chi}}(u_1, u_2).$$

Hence, Theorem 4.5 applied to $\tilde{\chi}_0$ shows that if $\rho^{-1}\tilde{I}_{\tilde{\chi}}(u_1, u_2) \to 0$ and the normalized $\tilde{\chi}$-energies of u_1, u_2 are uniformly bounded, then $d_{\mathrm{cap}}(u_1, u_2) \to 0$.

When $\tilde{\chi} \in \mathcal{W}^+_M$, we have another version of Theorem 4.5 which is more explicit.

Theorem 4.7. *Let $\theta \le A\omega$ be a closed smooth real $(1,1)$-form ($A \ge 1$) and ϕ be a model θ-psh function such that $\varrho := \mathrm{vol}(\theta_\phi) > 0$. Let $B \ge 1$, $\tilde{\chi} \in \mathcal{W}^+_M$ ($M \ge 1$) and $u_1, u_2 \in \mathcal{E}(X, \theta, \phi)$ be such that $\tilde{\chi}(-1) = -1$ and*

$$E^0_{\tilde{\chi},\theta,\phi}(u_1) + E^0_{\tilde{\chi},\theta,\phi}(u_2) \le B.$$

Then there exists $C > 0$ depending only on n and M such that

$$\int_X -\tilde{\chi}(-|u_1 - u_2|)\theta_\psi^n \le C\varrho B^2(\tilde{I}_{\tilde{\chi}}(u_1, u_2)/\varrho)^{2^{-n}} \tag{4-14}$$

for every $\psi \in \mathrm{PSH}(X, \theta)$ *with* $\phi - 1 \le \psi \le \phi$. *Moreover*, *if* $\sup_X u_1 = \sup_X u_2$ *then there exists* $C' > 0$ *depending on* n, X, ω, A *and* M *such that*

$$\tilde{I}_{\tilde{\chi}}(u_1, u_2) \le C' \varrho A^{1/2} B^2 (I^0_{\tilde{\chi}}(u_1, u_2))^{2^{-n-1}}.$$

Proof. The case $I^0_{\tilde{\chi}}(u_1, u_2) \ge 1$ is trivial because

$$\tilde{I}_{\tilde{\chi}}(u_1, u_2)/\varrho \ge I^0_{\tilde{\chi}}(u_1, u_2) \ge 1,$$

whereas the left-hand side of (4-14) is always bounded by a constant (depending on M) times B. Thus, from now on, it suffices to assume that $I^0_{\tilde{\chi}}(u_1, u_2) < 1$.

Denote $v = \max\{u_1, u_2\}$. By Lemma 2.7, we have $v \in \mathcal{E}(X, \theta, \phi)$ and $E^0_{\tilde{\chi},\theta,\phi}(v) \le C_1 B$, where $C_1 > 0$ depends only on n and M. Taking $\chi = \tilde{\chi}$ and using Theorem 3.1, we get

$$\int_X -\tilde{\chi}(u_j - v)\theta^n_\psi \le \int_X -\tilde{\chi}(u_j - v)\theta^n_{u_j} + C_2 \varrho B^2 (I^0_{\tilde{\chi}}(u_j, v))^{2^{-n}} \tag{4-15}$$

for $j = 1, 2$, where $C_2 > 0$ depends on n and M. Note that

$$\int_X -\tilde{\chi}(u_1 - v)\theta^n_{u_1} + \int_X -\tilde{\chi}(u_2 - v)\theta^n_{u_2} \le \int_X -\tilde{\chi}(-|u_1 - u_2|)(\theta^n_{u_1} + \theta^n_{u_2}) = \tilde{I}_{\tilde{\chi}}(u_1, u_2),$$

$$I^0_{\tilde{\chi}}(u_1, v) + I^0_{\tilde{\chi}}(u_2, v) = I^0_{\tilde{\chi}}(u_1, u_2) \le \varrho^{-1} \tilde{I}_{\tilde{\chi}}(u_1, u_2).$$

Hence, by (4-15), we get

$$\begin{aligned}
\int_X -\tilde{\chi}(-|u_1 - u_2|)\theta^n_\psi &= \int_X -\tilde{\chi}(u_1 - v)\theta^n_\psi + \int_X -\tilde{\chi}(u_2 - v)\theta^n_\psi \\
&\le \int_X -\tilde{\chi}(u_1 - v)\theta^n_{u_1} + \int_X -\tilde{\chi}(u_2 - v)\theta^n_{u_2} + C_2 \varrho B^2 \big((I^0_{\tilde{\chi}}(u_1, v))^{2^{-n}} + (I^0_{\tilde{\chi}}(u_2, v))^{2^{-n}}\big) \\
&\le \tilde{I}_{\tilde{\chi}}(u_1, u_2) + 2C_2 \varrho B^2 (\tilde{I}_{\tilde{\chi}}(u_1, u_2)/\varrho)^{2^{-n}} \\
&\le C_3 \varrho B^2 (\tilde{I}_{\tilde{\chi}}(u_1, u_2)/\varrho)^{2^{-n}},
\end{aligned}$$

where $C_3 > 0$ depends on n and M. Here, the last estimate holds due to the fact that $\tilde{I}_{\tilde{\chi}}(u_1, u_2) \le \varrho B$.

Now, we consider the case $\sup_X u_1 = \sup_X u_2$. By Theorem 3.2 (choose $m = 1$ and $\gamma = \frac{1}{2}$), there exists $C_4 > 0$ depending only on n, X, ω and M such that

$$\begin{aligned}
\tilde{I}_{\tilde{\chi}}(u_1, u_2) &\le \int_X -\tilde{\chi}(-|u_1 - u_2|)(\theta^n_{u_1} + \theta^n_{u_2}) \\
&\le -2\varrho\, \tilde{\chi}(-(I^0_{\tilde{\chi}}(u_1, u_2))^{2^{-n}}) + C_4 \varrho A^{1/2} B^2 (I^0_{\tilde{\chi}}(u_1, u_2))^{2^{-n-1}}.
\end{aligned} \tag{4-16}$$

Moreover, since $\tilde{\chi}$ is concave, we have

$$\frac{\tilde{\chi}(t)}{t} \le \frac{\tilde{\chi}(-1)}{-1} = 1$$

for every $-1 < t < 0$. Hence, by (4-16), we have

$$\begin{aligned}
\tilde{I}_{\tilde{\chi}}(u_1, u_2) &\le 2\varrho (I^0_{\tilde{\chi}}(u_1, u_2))^{2^{-n}} + C_4 \varrho A^{1/2} B^2 (I^0_{\tilde{\chi}}(u_1, u_2))^{2^{-n-1}} \\
&\le (2 + C_4)\varrho A^{1/2} B^2 (I^0_{\tilde{\chi}}(u_1, u_2))^{2^{-n-1}}.
\end{aligned}$$

□

4.4. ***Comparison of capacities.*** For every Borel subset E in X and for every $\varphi \in \mathrm{PSH}(X,\theta)$, we recall again that

$$\mathrm{cap}_{\theta,\varphi}(E) = \sup\left\{\int_E \theta_\psi^n : \psi \in \mathrm{PSH}(X,\theta),\ \varphi - 1 \le \psi \le \varphi\right\}.$$

In [Lu 2021], it was shown that if φ_j $(j=1,2)$ is a θ_j-psh function with $\int_X (\theta_j + dd^c\varphi_j)^n > 0$ then there exists a continuous function $f : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ with $f(0)=0$ such that $\mathrm{cap}_{\theta_1,\varphi_1}(E) \le f(\mathrm{cap}_{\theta_2,\varphi_2}(E))$ for every Borel set $E \subset X$. As an application of our main results, we obtain the following quantitative comparison of capacities for the case where φ_j is a model θ_j-psh function.

Theorem 4.8 (comparison of capacities). *Assume that $\theta_1, \theta_2 \le A\omega$ are closed smooth real $(1,1)$-forms representing big cohomology classes and, for $j=1,2$, that ϕ_j is a model θ_j-psh function satisfying $\int_X (dd^c\phi_j + \theta_j)^n = \varrho_j > 0$. Then, for every $0<\gamma<1$, there exists $C>0$ depending only on n, X, ω, A and γ such that*

$$\frac{\mathrm{cap}_{\theta_1,\phi_1}(E)}{\varrho_1} \le C\left(\frac{\mathrm{cap}_{\theta_2,\phi_2}(E)}{\varrho_2}\right)^{2^{-n}\gamma}$$

for every Borel set $E \subset X$.

We now prove Theorem 4.8. First, we need the following lemma.

Lemma 4.9. *Let $A, B > 0$ be constants. Let θ be a closed smooth real $(1,1)$-form representing a big cohomology class such that $\theta \le A\omega$. Assume that u, v are θ-psh functions satisfying $v \le u \le v + B$. Then*

$$\int_X (-\psi)\theta_u^n \le \int_X (-\psi)\theta_v^n + nA^nB\int_X \omega^n$$

for every negative $A\omega$-psh function ψ.

Proof. Using approximations, we can assume that ψ is smooth. Let

$$T = \sum_{l=0}^{n-1} \theta_u^l \wedge \theta_v^{n-l-1}.$$

We have $\theta_u^n - \theta_v^n = dd^c(u-v)\wedge T$. Moreover, using integration by parts (Theorem 2.2), we get

$$\int_X (-\psi)dd^c(u-v)\wedge T = \int_X (u-v)dd^c(-\psi)\wedge T \le A\int_X (u-v)\omega\wedge T \le nA^nB\int_X \omega^n.$$

Hence

$$\int_X (-\psi)\theta_u^n \le \int_X (-\psi)\theta_v^n + nA^nB\int_X \omega^n. \qquad \square$$

Proof of Theorem 4.8. By the inner regularity of capacities (see [Darvas et al. 2018b, Lemma 4.2]), we only need consider the case where E is compact. Since the case $\mathrm{cap}_{\theta_2,\phi_2}(E) = \varrho_2$ is trivial, we can also assume that $\mathrm{cap}_{\theta_2,\phi_2}(E) < \varrho_2$. In particular, by Darvas et al. 2021a, Proposition 3.7; 2021b, Lemma 2.7], we have

$$\sup_X h^*_{E,\theta_2,\phi_2} = \sup_X (h^*_{E,\theta_2,\phi_2} - \phi_2) = 0,$$

where

$$h_{E,\theta_2,\phi_2} = \sup\{w \in \mathrm{PSH}(X,\theta_2) : w|_E \le \phi_2 - 1,\ w \le \phi_2\}.$$

Set $\chi(t)=\tilde{\chi}(t)=t$. We will use Theorem 3.2 for $u_1=(h_{E,\theta_2,\phi_2})^*$ and $u_2=\phi_2$. It is clear that $E^0_{\tilde{\chi},\theta_2,\phi_2}(u_2)=0$ and $u_1=u_2-1$ on $E\setminus N$, where N is a pluripolar set. Moreover, it follows from [Darvas et al. 2021a, Proposition 3.7] that

$$I^0_\chi(u_1,u_2)\le E^0_{\tilde{\chi},\theta_2,\phi_2}(u_1)=\varrho_2^{-1}\mathrm{cap}_{\theta_2,\phi_2}(E)\le 1.$$

By Theorem 3.2, for every $0<\gamma<1$ and $B\ge 1$, there exists $C>0$ depending only on X,ω,n,A and γ such that

$$\int_E\theta_\psi^n\le\int_X\chi(-|u_1-u_2|)\theta_\psi^n\le C\varrho_2A(A+B)^2(\mathrm{cap}_{\theta_2,\phi_2}(E)/\varrho_2)^{2^{-n}\gamma},\tag{4-17}$$

for every compact set E and for each $\psi\in\mathcal{E}(X,\theta_2,\phi_2)$ with $E^0_{\tilde{\chi},\theta_2,\phi_2}(\psi)\le B$. Let $\varphi\in\mathcal{E}(X,\theta_1,\phi_1)$ such that $\phi_1-1\le\varphi\le\phi_1$ and $\int_E(\theta_1+dd^c\varphi)^n\ge\frac{1}{2}\mathrm{cap}_{\theta_1,\phi_1}(E)$. By [Darvas et al. 2021a], there exists a unique function $\psi_0\in\mathcal{E}(X,\theta_2,\phi_2)$ such that $\sup_X\psi_0=0$ and

$$(dd^c\psi_0+\theta_2)^n=\frac{\varrho_2}{\varrho_1}(dd^c\varphi+\theta_1)^n.$$

When $\psi=\psi_0$, we have

$$\int_E\theta_\psi^n\ge\frac{\varrho_2}{2\varrho_1}\mathrm{cap}_{\theta_1,\phi_1}(E).\tag{4-18}$$

Moreover, by using Lemma 4.9 for φ,ϕ_1 and using the fact that $(dd^c\phi_2+\theta_2)^n\le\mathbf{1}_{\{\phi_2=0\}}\theta_2^n$ (see [Darvas et al. 2018b, Theorem 3.8]), we have

$$\varrho_1E^0_{\tilde{\chi},\theta_2,\phi_2}(\psi_0)=\int_X(\phi_2-\psi_0)(dd^c\varphi+\theta_1)^n\le\int_X(-\psi_0)(dd^c\phi_1+\theta_1)^n+nA^n\int_X\omega^n\le B,\tag{4-19}$$

where $B\ge 1$ depends only on A,X,ω,n. Combining (4-17), (4-18) and (4-19), we get

$$\begin{aligned}\mathrm{cap}_{\theta_1,\phi_1}(E)&\le\frac{2\varrho_1}{\varrho_2}\int_E\theta_{\psi_0}^n\le\frac{2\varrho_1}{\varrho_2}\int_X\chi(-|u_1-u_2|)\theta_{\psi_0}^n\\&\le 2C\varrho_1A(A+B)^2(\mathrm{cap}_{\theta_2,\phi_2}(E)/\varrho_2)^{2^{-n}\gamma}.\end{aligned}$$

□

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