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EXISTENCE OF SOLUTIONS TO A FRACTIONAL SEMILINEAR HEAT EQUATION IN UNIFORMLY LOCAL WEAK ZYGMUND-TYPE SPACES

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We introduce uniformly local weak Zygmund-type spaces and obtain an optimal sufficient condition for the existence of solutions to the critical fractional semilinear heat equation.

1. Introduction

Consider the Cauchy problem for the fractional semilinear heat equation

$$\begin{cases} \partial_t u + (-\Delta)^{\frac{\theta}{2}} u = |u|^{p-1}u, & x \in \mathbb{R}^n, \ t > 0, \\ u(x,0) = \varphi(x), & x \in \mathbb{R}^n, \end{cases} \tag{P}$$

where $n \geq 1$, $\partial_t := \partial/\partial t$, $\theta \in (0,2]$, $p > 1$, and φ is a locally integrable function in $\mathbb{R}^n$. Here $(-\Delta)^{\theta/2}$ denotes the fractional power of the Laplace operator $-\Delta$ in $\mathbb{R}^n$. In this paper we establish the local-in-time existence of solutions to problem (P) in the critical case

$$p = p_\theta := 1 + \frac{\theta}{n}$$

in the framework of uniformly local weak Zygmund-type spaces.

The solvability of the Cauchy problem for semilinear heat equations has fascinated many mathematicians since the pioneering work by Fujita [1966]. The literature is very large, and we refer the reader to the comprehensive monograph [Quittner and Souplet 2007] and the papers [Andreucci and DiBenedetto 1991; Baras and Pierre 1985; Brezis and Cazenave 1996; Fujishima et al. 2023; 2024; Fujishima and Ioku 2021; 2022; Giraudon and Miyamoto 2022; Hisa and Ishige 2018; Hisa et al. 2023; Ishige et al. 2014; 2020; 2022; Kozono and Yamazaki 1994; Laister and Sierżęga 2020; 2021; Laister et al. 2016; Miyamoto 2021; Robinson and Sierżęga 2013; Sugitani 1975; Weissler 1981; Zhanpeisov 2023], some of which are closely related to this paper, while the others include recent developments in this subject. The study of the solvability of problem (P) is divided into the following three cases:

$$1 < p < p_\theta \ \text{(subcritical case)}, \quad p > p_\theta \ \text{(supercritical case)}, \quad p = p_\theta \ \text{(critical case)}.$$

We collect some known results on necessary conditions and sufficient conditions for the existence of solutions to problem (P).

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(1) Subcritical case ($1 < p < p_\theta$).

(a) Necessity: There exists $C_1 = C_1(n, \theta, p) > 0$ such that if problem (P) possesses a nonnegative solution in $\mathbb{R}^n \times (0, T)$ for some $T > 0$, then

$$\sup_{x\in\mathbb{R}^n} \int_{B(x,T^{1/\theta})} \varphi(y)\, dy \le C_1 T^{\frac{n}{\theta}-\frac{1}{p-1}}.$$

See [Andreucci and DiBenedetto 1991; Baras and Pierre 1985] for $\theta = 2$ and [Hisa and Ishige 2018] for $\theta \in (0, 2]$.

(b) Sufficiency: There exists $\epsilon_1 = \epsilon_1(n, \theta, p) > 0$ such that if

$$\sup_{x\in\mathbb{R}^n} \int_{B(x,T^{1/\theta})} |\varphi(y)|\, dy \le \epsilon_1 T^{\frac{n}{\theta}-\frac{1}{p-1}}$$

for some $T \in (0, \infty)$, then problem (P) possesses a solution in $\mathbb{R}^n \times (0, T)$. See, e.g., [Andreucci and DiBenedetto 1991; Hisa and Ishige 2018; Weissler 1981].

The results in (a) and (b) (see also (1-4)) imply that, for any nonnegative measurable initial function φ in $\mathbb{R}^n$, problem (P) possesses a local-in-time nonnegative solution if and only if

$$\sup_{x\in\mathbb{R}^n} \int_{B(x,1)} \varphi(y)\, dy < \infty.$$

(2) Supercritical case ($p > p_\theta$).

(a) Necessity: There exists $C_2 = C_2(n, \theta, p) > 0$ such that if problem (P) possesses a nonnegative solution in $\mathbb{R}^n \times (0, T)$ for some $T > 0$, then

$$\sup_{x\in\mathbb{R}^n} \sup_{\sigma\in(0,T^{1/\theta})} |B(x,\sigma)|^{\frac{\theta}{n(p-1)}-1} \int_{B(x,\sigma)} \varphi(y)\, dy \le C_2.$$

See [Andreucci and DiBenedetto 1991; Baras and Pierre 1985] for $\theta = 2$ and [Hisa and Ishige 2018] for $\theta \in (0, 2]$.

(b) Sufficiency: For any $r \in (1, \infty)$, there exists $\epsilon_2 = \epsilon_2(n, \theta, p, r) > 0$ such that if

$$\sup_{x\in\mathbb{R}^n} \sup_{\sigma\in(0,T^{1/\theta})} |B(x,\sigma)|^{\frac{\theta}{n(p-1)}-\frac{1}{r}} \left[\int_{B(x,\sigma)} |\varphi(y)|^r\, dy\right]^{\frac{1}{r}} \le \epsilon_2$$

for some $T \in (0, \infty]$, then problem (P) possesses a solution in $\mathbb{R}^n \times (0, T)$. See [Kozono and Yamazaki 1994; Robinson and Sierżęga 2013] for $\theta = 2$ and [Hisa and Ishige 2018; Ishige et al. 2020; 2022; Zhanpeisov 2023] for $\theta \in (0, 2]$. See, e.g., [Andreucci and DiBenedetto 1991; Ishige et al. 2014; Weissler 1981] for related results.

(3) Critical case ($p = p_\theta$).

(a) Necessity: There exists $C_3 = C_3(n, \theta) > 0$ such that if problem (P) possesses a nonnegative solution in $\mathbb{R}^n \times (0, T)$ for some $T > 0$, then

$$\sup_{x\in\mathbb{R}^n} \int_{B(x,\sigma)} \varphi(y)\, dy \le C_3 \left[\log\left(e + \frac{T^{1/\theta}}{\sigma}\right)\right]^{-\frac{n}{\theta}}, \quad \sigma \in (0, T^{\frac{1}{\theta}}).$$

See [Baras and Pierre 1985] for $\theta = 2$ and [Hisa and Ishige 2018] for $\theta \in (0, 2]$.

(b) Sufficiency: For any $\alpha > 0$, there exists $\epsilon_3 = \epsilon_3(n, \theta, \alpha) > 0$ such that if

$$\sup_{x\in\mathbb{R}^n} \Psi_\alpha^{-1}\left[\frac{1}{|B(x,\sigma)|}\int_{B(x,\sigma)} \Psi_\alpha(T^{\frac{1}{p-1}}|\varphi(y)|)\,dy\right] \le \epsilon_3\rho(\sigma T^{-\frac{1}{\theta}}), \quad \sigma \in (0, T^{\frac{1}{\theta}}),$$

for some $T > 0$, then problem (P) possesses a solution in $\mathbb{R}^n \times (0, T)$, where

$$\Psi_\alpha(s) := s[\log(e+s)]^\alpha, \quad \rho(s) := s^{-n}\left[\log\left(e+\frac{1}{s}\right)\right]^{-\frac{n}{\theta}}.$$

See [Hisa and Ishige 2018; Ishige et al. 2020; 2022].

Furthermore, the results in (2) and (3) imply the following results.

(4) Let $p \ge p_\theta$, and set

$$\varphi_c(x) := \begin{cases} |x|^{-n}\left[\log\left(e+\frac{1}{|x|}\right)\right]^{-\frac{n}{\theta}-1} & \text{if } p = p_\theta, \\ |x|^{-\frac{\theta}{p-1}} & \text{if } p > p_\theta \end{cases} \quad \text{for } x \in \mathbb{R}^n. \tag{1-1}$$

(a) There exists $C_4 = C_4(n, \theta, p) > 0$ such that if

$$\varphi(x) \ge C_4\varphi_c(x)$$

for almost all x in a neighborhood of the origin, then problem (P) possesses no local-in-time nonnegative solutions.

(b) There exists $\epsilon_4 = \epsilon_4(n, \theta, p) > 0$ such that if

$$|\varphi(x)| \le \epsilon_4\varphi_c(x) + K, \quad \text{a.a. } x \in \mathbb{R}^n,$$

for some $K \ge 0$, then problem (P) possesses a local-in-time solution.

The results in (4) show that the "strength" of the singularity at the origin of the function φ_c is the critical threshold for the local-in-time solvability of problem (P). The function φ_c is quite useful for identifying optimal function spaces to which initial functions belong from the view of the solvability of problem (P). We remark that assertion (2b) with $r = 1$ and assertion (3b) with $\alpha = 0$ do not hold. (See [Takahashi 2016, Theorem 1 and Proposition 1], which treat only the case of $\theta = 2$ but which is also applicable to the case of $\theta \in (0, 2)$. See also [Kan and Takahashi 2017, Section 4].)

There are (at least) two useful strategies for the proof of the existence of solutions to problem (P). One is the supersolution method (SSM) and the other is the contraction mapping theorem (CMT). SSM depends on the following principle: if there exists a nonnegative supersolution v to problem (P) in $\mathbb{R}^n \times (0, T)$ for some $T > 0$, then problem (P) possesses a nonnegative solution u in $\mathbb{R}^n \times (0, T)$ such that $u \le v$ in $\mathbb{R}^n \times (0, T)$. In our problem (P) with nonnegative initial function φ, the following functions have been used as supersolutions in $\mathbb{R}^n \times (0, T)$ for some $T > 0$:

$$2S_\theta(t)\varphi \ \ (1 < p < p_\theta), \quad 2(S_\theta(t)\varphi^r)^{\frac{1}{r}} \ \ (p > p_\theta), \quad 2\Psi_\alpha^{-1}(S_\theta(t)\Psi_\alpha(\varphi)) \ \ (p = p_\theta),$$

where $S_\theta(t)\varphi$ is a solution to the fractional heat equation (see (1-5)), $r > 1$, and Ψ_α is as in assertion (3b). (See, e.g., [Weissler 1981] for $1 < p < p_\theta$; [Hisa and Ishige 2018; Robinson and Sierżęga 2013] for $p > p_\theta$; [Hisa and Ishige 2018] for $p = p_\theta$.) Furthermore, thanks to the arguments in [Tayachi and Weissler 2014], SSM is also applicable to the study of the existence of sign-changing solutions to problem (P) (see [Ishige et al. 2020; 2022]), however we require additional arguments in which sense the solution converges to the initial function. On the other hand, CMT is widely used in the proof of the existence of solutions in various evolution equations, and the choice of function spaces is crucial. For our problem (P) with $p > p_\theta$, the existence of solutions has been proved by CMT in the framework of weak Lebesgue spaces (see [Fujishima and Ioku 2021; Ishige et al. 2014]) and Morrey spaces (see [Kozono and Yamazaki 1994; Zhanpeisov 2023]). The results in [Fujishima and Ioku 2021; Ishige et al. 2014; Kozono and Yamazaki 1994; Zhanpeisov 2023] cover the result in (4b) with $p > p_\theta$. However, in the critical case $p = p_\theta$, the arguments in [Fujishima and Ioku 2021; Ishige et al. 2014; Kozono and Yamazaki 1994; Zhanpeisov 2023] are not applicable to the proof of assertion (4b) by the logarithmic singularity of φ_c.

	supersolution method (SSM)	weak spaces (CMT)	Morrey spaces (CMT)
$p > p_\theta$	[Hisa and Ishige 2018] [Robinson and Sierżęga 2013]	[Fujishima and Ioku 2021] [Ishige et al. 2014]	[Kozono and Yamazaki 1994] [Zhanpeisov 2023]
$p = p_\theta$	[Hisa and Ishige 2018]	open	not applicable (see [Takahashi 2016])

The aim of this paper is to establish a sharp sufficient condition on the existence of solutions to problem (P) in the critical case $p = p_\theta$ in the framework of Banach spaces. For the critical case $p = p_\theta$, the weak Zygmund space $L^{1,\infty}(\log L)^{1+n/\theta}$ seems a reasonable Banach space since $\varphi_c \in L^{1,\infty}(\log L)^{1+n/\theta}$. (See Remark 4.4 (i) for the definition of the weak Zygmund spaces $L^{q,\infty}(\log L)^\alpha$, where $1 \le q < \infty$ and $\alpha \ge 0$.) Then we require sharp decay estimates of solutions to the fractional heat equation in the weak Zygmund spaces $L^{q,\infty}(\log L)^\alpha$; however, by the peculiarity of $L^{1,\infty}(\log L)^\alpha$, it seems difficult to obtain our desired sharp decay estimates. (See Remark 4.4 (ii) for further details.)

In this paper we introduce new weak Zygmund-type spaces $\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha$ and uniformly local weak Zygmund-type spaces $\mathfrak{L}^{q,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^\alpha$. Then we establish sharp decay estimates of solutions to the fractional heat equation in the spaces $\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha$ and $\mathfrak{L}^{q,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^\alpha$, and obtain a sufficient condition on the existence of solutions to problem (P) with $p = p_\theta$ in the framework of the space $\mathfrak{L}^{q,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^\alpha$. Our sufficient condition is simpler than that of assertion (3b) and covers assertion (4b) with $p = p_\theta$.

We introduce some notation and define the weak Zygmund-type spaces $\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha$ and the uniformly local weak Zygmund-type spaces $\mathfrak{L}^{q,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^\alpha$. We also formulate the definition of solutions to problem (P). Let $\mathcal{M}$ be the set of Lebesgue measurable sets in $\mathbb{R}^n$. For any $E \in \mathcal{M}$, we denote by $|E|$ and χ_E the n-dimensional Lebesgue measure of E and the characteristic function of E, respectively. Let L^1_{loc} be the set of locally integrable functions in $\mathbb{R}^n$. For any $q \in [1,\infty]$, we denote by L^q and $\|\cdot\|_{L^q}$ the usual L^q-space on $\mathbb{R}^n$ and its norm, respectively.

Let $q \in [1,\infty]$ and $\alpha \in [0,\infty)$. We define the weak Zygmund-type space $\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha$ by

$$\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha := \{f \in L^1_{\mathrm{loc}} : \|f\|_{\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha} < \infty\},$$

where

$$\|f\|_{\mathfrak{L}^{q,\infty}(\log\mathfrak{L})^\alpha} := \begin{cases} \sup_{s>0}\Big\{\big[\log\big(e+\frac{1}{s}\big)\big]^\alpha \sup_{|E|=s} \fint_E |f(x)|^q\,dx\Big\}^{\frac{1}{q}} & \text{if } q\in[1,\infty), \\ \|f\|_{L^\infty} & \text{if } q=\infty. \end{cases} \tag{1-2}$$

Then $\mathfrak{L}^{q,\infty}(\log\mathfrak{L})^\alpha$ is a Banach space equipped with the norm $\|\cdot\|_{\mathfrak{L}^{q,\infty}(\log\mathfrak{L})^\alpha}$ (see Lemma 2.1). See (2-9) for different expressions of the norm $\|\cdot\|_{\mathfrak{L}^{q,\infty}(\log\mathfrak{L})^\alpha}$. We remark that

$$L^q = \mathfrak{L}^{q,\infty}(\log\mathfrak{L})^0 \supset \mathfrak{L}^{q,\infty}(\log\mathfrak{L})^\alpha \quad \text{for } \alpha\ge 0. \tag{1-3}$$

Next, we introduce the uniformly local weak Zygmund-type space $\mathfrak{L}^{q,\infty}_{\mathrm{ul}}(\log\mathfrak{L})^\alpha$ by

$$\mathfrak{L}^{q,\infty}_{\mathrm{ul}}(\log\mathfrak{L})^\alpha := \{f\in L^1_{\mathrm{loc}} : \|f\|_{\mathfrak{L}^{q,\infty}_{\mathrm{ul}}(\log\mathfrak{L})^\alpha} < \infty\},$$

where

$$\|f\|_{\mathfrak{L}^{q,\infty}_{\mathrm{ul}}(\log\mathfrak{L})^\alpha} := \sup_{z\in\mathbb{R}^n} \|f\chi_{B(z,1)}\|_{\mathfrak{L}^{q,\infty}(\log\mathfrak{L})^\alpha}.$$

Then $\mathfrak{L}^{q,\infty}_{\mathrm{ul}}(\log\mathfrak{L})^\alpha$ is also a Banach space equipped with the norm $\|\cdot\|_{\mathfrak{L}^{q,\infty}_{\mathrm{ul}}(\log\mathfrak{L})^\alpha}$. We often write, for any $f\in\mathfrak{L}^{q,\infty}_{\mathrm{ul}}(\log\mathfrak{L})^\alpha$ and $\rho>0$,

$$|||f|||_{q,\alpha;\rho} := \sup_{z\in\mathbb{R}^n} \|f\chi_{B(z,\rho)}\|_{\mathfrak{L}^{q,\infty}(\log\mathfrak{L})^\alpha}$$

for simplicity. We remark that $\mathfrak{L}^{\infty,\infty}_{\mathrm{ul}}(\log\mathfrak{L})^\alpha = L^\infty$ and $|||\cdot|||_{\infty,\alpha;\rho} = \|\cdot\|_{L^\infty}$ for all $\alpha\in[0,\infty)$. Notice that, for any $k\ge 1$, there exists $C=C(n,k)>0$ such that

$$|||f|||_{q,\alpha;k\rho} \le C\,|||f|||_{q,\alpha;\rho} \tag{1-4}$$

for $f\in\mathfrak{L}^{q,\infty}_{\mathrm{ul}}(\log\mathfrak{L})^\alpha$ and $\rho>0$.

We formulate the definition of solutions to problem (P). Let $\theta\in(0,2]$. Let G_θ be the fundamental solution to the fractional heat equation

$$\partial_t v + (-\Delta)^{\frac{\theta}{2}} v = 0 \quad \text{in } \mathbb{R}^n\times(0,\infty).$$

For any φ in L^1_{loc}, we write

$$(S_\theta(t)\varphi)(x) := \int_{\mathbb{R}^n} G_\theta(x-y,t)\varphi(y)\,dy, \quad (x,t)\in\mathbb{R}^n\times(0,\infty), \tag{1-5}$$

for simplicity.

Definition 1.1. Let $\theta\in(0,2]$, $p>1$, and $T>0$. Set $F_p(s) := |s|^{p-1}s$ for $s\in\mathbb{R}$. Let u be a measurable and finite almost everywhere function in $\mathbb{R}^n\times(0,T)$. We say that u is a solution to problem (P) in $\mathbb{R}^n\times(0,T)$ if, for almost all $(x,t)\in\mathbb{R}^n\times(0,T)$,

- $G_\theta(x-y,t)\varphi(y)$ is integrable in $\mathbb{R}^n$ with respect to $y\in\mathbb{R}^n$,
- $G_\theta(x-y,t-s)F_p(u(y,s))$ is integrable in $\mathbb{R}^n\times(0,t)$ with respect to $(y,s)\in\mathbb{R}^n\times(0,t)$,
- u satisfies

$$u(x,t) = [S_\theta(t)\varphi](x) + \int_0^t [S_\theta(t-s)F_p(u(s))](x)\,ds.$$

We are ready to state our main results.

Theorem 1.2. *Let $\theta \in (0,2]$, $p = p_\theta = 1+\theta/n$, and $T_* \in (0,\infty)$. Then there exists $\epsilon > 0$ such that if $\varphi \in \mathfrak{L}^{1,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^{n/\theta}$ satisfies*

$$|||\varphi|||_{1,\frac{n}{\theta};T^{1/\theta}} \le \epsilon \quad \textit{for some } T \in (0, T_*],$$

then problem (P) possesses a solution $u \in C((0,T) : \mathfrak{L}^{1,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^{n/\theta}) \cap L^\infty_{\mathrm{loc}}((0,T) : L^\infty)$ in $\mathbb{R}^n \times (0,T)$, with u satisfying

$$\sup_{t\in(0,T)} |||u(t)|||_{1,\frac{n}{\theta};T^{1/\theta}} + \sup_{t\in(0,T)} t^{\frac{n}{\theta}} \left[\log\left(e+\frac{1}{t}\right)\right]^{\frac{n}{\theta}} \|u(t)\|_{L^\infty} < \infty. \tag{1-6}$$

Furthermore, the solution u satisfies

$$\begin{aligned} \lim_{t\to+0} \|u(t) - S_\theta(t)\varphi\|_{\mathfrak{L}^{1,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^\gamma} &= 0 \quad \textit{for any } \gamma \in [0, n/\theta), \\ \lim_{t\to+0} u(t) &= \varphi \quad \textit{in the sense of distributions.} \end{aligned} \tag{1-7}$$

We remark that Theorem 1.2 with $T_* = \infty$ does not hold since problem (P) possesses no global-in-time positive solutions (see [Sugitani 1975]). As a direct consequence of Theorem 1.2, we obtain assertion (4b).

Corollary 1.3. *Let $\theta \in (0,2]$ and $p = p_\theta$. Let φ_c be as in (1-1). Then there exists $\epsilon > 0$ such that if*

$$|\varphi(x)| \le \epsilon\varphi_c(x) + K, \quad \textit{a.a. } x \in \mathbb{R}^n,$$

for some $K \ge 0$, then problem (P) possesses a local-in-time solution.

Furthermore, as a consequence of Theorem 1.2, we have the following.

Theorem 1.4. *Let $\theta \in (0,2]$ and $p = p_\theta$. If $\varphi \in \mathfrak{L}^{1,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^\alpha$ for some $\alpha > n/\theta$, then problem (P) possesses a solution u in $\mathbb{R}^n \times (0,T)$ for some $T > 0$, with u satisfying (1-6) and (1-7).*

The rest of this paper is organized as follows. In Section 2 we collect some properties of nonincreasing rearrangements of measurable functions and prove some lemmas in $\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha$ and $\mathfrak{L}^{q,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^\alpha$. Furthermore, we recall Hardy's inequalities and some properties of $S_\theta(t)\varphi$. In Section 3 we establish decay estimates of $S_\theta(t)\varphi$ in weak Zygmund-type spaces (see Proposition 3.1). Furthermore, we obtain decay estimates of $S_\theta(t)\varphi$ in $\mathfrak{L}^{q,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^\alpha$ using Besicovitch's covering lemma. In Section 4 we apply the contraction mapping theorem in $\mathfrak{L}^{q,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^\alpha$ to prove Theorems 1.2 and 1.4. In the Appendix we give two propositions on relations among the weak Zygmund-type spaces $\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha$, the Zygmund spaces $L^q(\log L)^\alpha$, and the weak Zygmund spaces $L^{q,\infty}(\log L)^\alpha$.

2. Preliminaries

In this section we introduce some notation and give some lemmas on our weak Zygmund-type spaces. Furthermore, we recall some lemmas on Hardy's inequalities. In all that follows, we will use C to denote generic positive constants and point out that C may take different values within a calculation. For any positive functions f_1 and f_2 in $(0,\infty)$, we write

$$f_1 \asymp f_2 \;\text{ for } s > 0 \quad \text{if } C^{-1} f_2(s) \le f_1(s) \le C f_2(s) \;\text{ for } s > 0.$$

2.1. ***Weak Zygmund-type spaces.*** For any (Lebesgue) measurable function f in $\mathbb{R}^n$, we denote by μ_f the distribution function of f, that is,

$$\mu_f(\lambda) := |\{x : |f(x)| > \lambda\}| \quad \text{for } \lambda > 0.$$

We define the nonincreasing rearrangement f^* of f by

$$f^*(s) := \inf\{\lambda > 0 : \mu_f(\lambda) \le s\} \quad \text{for } s \in [0, \infty).$$

Here we adopt the convention $\inf \varnothing = \infty$. Then f^* is nonincreasing and right-continuous in $[0, \infty)$, and it has the following properties (see [Grafakos 2008, Proposition 1.4.5]):

$$(kf)^* = |k| f^*, \quad (|f|^q)^* = (f^*)^q, \quad \int_{\mathbb{R}^n} |f(x)|^q \, dx = \int_0^\infty f^*(s)^q \, ds, \quad f^*(0) = \|f\|_{L^\infty}, \tag{2-1}$$

where $q \in (0, \infty)$ and $k \in \mathbb{R}$. We remark that if $E \in \mathcal{M}$ with $|E| < \infty$, then

$$(\chi_E)^*(s) = \chi_{[0,|E|)}(s) \quad \text{for } s \ge 0. \tag{2-2}$$

Define

$$f^{**}(s) := \frac{1}{s} \int_0^s f^*(\tau) \, d\tau \quad \text{for } s \in (0, \infty). \tag{2-3}$$

Here we collect properties of f^* and f^{**} used in the paper.

(a) Since f^* is nonincreasing in $(0, \infty)$, it follows that

$$f^{**}(s) \ge f^*(s) \quad \text{for } s \in (0, \infty). \tag{2-4}$$

(b) For any $q \in [1, \infty)$, Jensen's inequality together with (2-1) yields

$$(f^{**}(s))^q \le \frac{1}{s} \int_0^s f^*(\tau)^q \, d\tau = \frac{1}{s} \int_0^s (|f|^q)^*(\tau) \, d\tau = (|f|^q)^{**}(s) \quad \text{for } s \in (0, \infty). \tag{2-5}$$

(c) It follows from [Bennett and Sharpley 1988, Chapter 2, Proposition 3.3] that

$$f^{**}(s) = \frac{1}{s} \int_0^s f^*(\tau) \, d\tau = \frac{1}{s} \sup_{|E|=s} \int_E |f(x)| \, dx \quad \text{for } s \in (0, \infty). \tag{2-6}$$

(d) (O'Neil's inequality) For any $f, g \in L^1$, it follows from [O'Neil 1963, Lemma 1.6] that

$$(f * g)^{**}(s) \le \int_s^\infty f^{**}(\tau) g^{**}(\tau) \, d\tau \quad \text{for } s \in (0, \infty), \tag{2-7}$$

where

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x - y) g(y) \, dy.$$

(e) For any $f_1, f_2 \in L^1_{\mathrm{loc}}$, it follows from [O'Neil 1963, Theorem 3.3] that

$$(f_1 f_2)^{**}(s) \le \frac{1}{s} \int_0^s f_1^*(\tau) f_2^*(\tau) \, d\tau \quad \text{for } s \in (0, \infty). \tag{2-8}$$

Let $q \in [1,\infty)$ and $\alpha \ge 0$. For any L^1_{loc}-function f, by (1-2), (2-1), and (2-6), we have

$$\|f\|_{\mathfrak{L}^{q,\infty}(\log\mathfrak{L})^\alpha} = \sup_{s>0}\biggl\{\biggl[\log\biggl(e+\frac{1}{s}\biggr)\biggr]^\alpha s(|f|^q)^{**}(s)\biggr\}^{\frac{1}{q}} = \sup_{s>0}\biggl\{\biggl[\log\biggl(e+\frac{1}{s}\biggr)\biggr]^\alpha \int_0^s (|f|^q)^*(\tau)\,d\tau\biggr\}^{\frac{1}{q}}$$
$$= \sup_{s>0}\biggl\{\biggl[\log\biggl(e+\frac{1}{s}\biggr)\biggr]^\alpha \int_0^s f^*(\tau)^q\,d\tau\biggr\}^{\frac{1}{q}}. \tag{2-9}$$

Furthermore, for any $E \in \mathcal{M}$ with $|E| < \infty$, it follows from (2-2) and (2-8) that

$$(f\chi_E)^{**}(s) = (f\chi_E\chi_E)^{**}(s) \le \frac{1}{s}\int_0^s (f\chi_E)^*(\tau)(\chi_E)^*(\tau)\,d\tau = \frac{1}{s}\int_0^{\min\{s,|E|\}} (f\chi_E)^*(\tau)\,d\tau. \tag{2-10}$$

For any $\beta \in [\alpha,\infty)$, since

$$\text{the function } (0,\infty) \ni \tau \mapsto \biggl[\log\biggl(e+\frac{1}{\tau}\biggr)\biggr]^{\alpha-\beta} \in \mathbb{R} \text{ is nondecreasing,} \tag{2-11}$$

by (2-1), (2-9), and (2-10), we have

$$\begin{aligned}
\|f\chi_E\|_{\mathfrak{L}^{1,\infty}(\log\mathfrak{L})^\alpha} &= \sup_{s>0}\biggl\{\biggl[\log\biggl(e+\frac{1}{s}\biggr)\biggr]^\alpha s(f\chi_E)^{**}(s)\biggr\}\\
&\le \sup_{s>0}\biggl\{\biggl[\log\biggl(e+\frac{1}{s}\biggr)\biggr]^\alpha \int_0^{\min\{s,|E|\}} (f\chi_E)^*(\tau)\,d\tau\biggr\}\\
&= \sup_{0<s\le|E|}\biggl\{\biggl[\log\biggl(e+\frac{1}{s}\biggr)\biggr]^{\beta+\alpha-\beta} \int_0^{\min\{s,|E|\}} (f\chi_E)^*(\tau)\,d\tau\biggr\}\\
&\le \biggl[\log\biggl(e+\frac{1}{|E|}\biggr)\biggr]^{\alpha-\beta} \sup_{0<s\le|E|}\biggl\{\biggl[\log\biggl(e+\frac{1}{s}\biggr)\biggr]^{\beta} \int_0^{\min\{s,|E|\}} (f\chi_E)^*(\tau)\,d\tau\biggr\}\\
&\le \biggl[\log\biggl(e+\frac{1}{|E|}\biggr)\biggr]^{\alpha-\beta} \sup_{s>0}\biggl\{\biggl[\log\biggl(e+\frac{1}{s}\biggr)\biggr]^{\beta} \int_0^{s} (f\chi_E)^*(\tau)\,d\tau\biggr\}\\
&= \biggl[\log\biggl(e+\frac{1}{|E|}\biggr)\biggr]^{\alpha-\beta} \|f\chi_E\|_{\mathfrak{L}^{1,\infty}(\log\mathfrak{L})^\beta}.
\end{aligned}$$

In particular,

$$|||f|||_{1,\alpha;\rho} \le C\biggl[\log\biggl(e+\frac{1}{\rho}\biggr)\biggr]^{\alpha-\beta} |||f|||_{1,\beta;\rho} \tag{2-12}$$

for $f \in \mathfrak{L}^{1,\infty}_{\mathrm{ul}}(\log\mathfrak{L})^\beta$, $0 \le \alpha \le \beta$, and $\rho > 0$. Here we show that $\mathfrak{L}^{q,\infty}(\log\mathfrak{L})^\alpha$ and $\mathfrak{L}^{q,\infty}_{\mathrm{ul}}(\log\mathfrak{L})^\alpha$ are Banach spaces.

Lemma 2.1. *For any $1 \le q < \infty$ and $\alpha \ge 0$, the weak Zygmund-type space $\mathfrak{L}^{q,\infty}(\log\mathfrak{L})^\alpha$ and the uniformly local weak Zygmund-type space $\mathfrak{L}^{q,\infty}_{\mathrm{ul}}(\log\mathfrak{L})^\alpha$ are Banach spaces.*

Proof. Let $1 \le q < \infty$ and $\alpha \ge 0$. It suffices to prove that $\mathfrak{L}^{q,\infty}(\log\mathfrak{L})^\alpha$ (resp. $\mathfrak{L}^{q,\infty}_{\mathrm{ul}}(\log\mathfrak{L})^\alpha$) is a complete metric space with the norm $\|\cdot\|_{\mathfrak{L}^{q,\infty}(\log\mathfrak{L})^\alpha}$ (resp. $\|\cdot\|_{\mathfrak{L}^{q,\infty}_{\mathrm{ul}}(\log\mathfrak{L})^\alpha}$). Let $\{f_n\}$ be a Cauchy sequence in $\mathfrak{L}^{q,\infty}(\log\mathfrak{L})^\alpha$. It follows from (1-3) that $\{f_n\}$ is a Cauchy sequence in L^q, and hence there exists $f \in L^q$

such that $f_n \to f$ as $n \to \infty$ in L^q. Since the Cauchy sequence $\{f_n\}$ is bounded in $\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha$, we observe from (1-2) that $f \in \mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha$. It remains to prove that $f_n \to f$ as $n \to \infty$ in $\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha$. For this aim, we take a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}\}$ converges almost everywhere to f. Then Fatou's lemma gives us that

$$\begin{aligned}\|f - f_{n_j}\|_{\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha} &\le \sup_{s>0}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^\alpha \sup_{|E|=s} \liminf_{k\to\infty} \int_E |f_{n_k}(x) - f_{n_j}(x)|^q\,dx\right\}^{\frac{1}{q}} \\ &\le \liminf_{k\to\infty} \|f_{n_k} - f_{n_j}\|_{\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha}.\end{aligned}$$

This implies that f_{n_j} converges to f in $\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha$. Thus $\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha$ is a complete metric space. Similarly, we see that $\mathfrak{L}^{q,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^\alpha$ is a complete metric space. □

Next, we prove two lemmas on our weak Zygmund-type spaces and uniformly local weak Zygmund-type spaces.

Lemma 2.2. *Let $q_1, q_2 \in [1,\infty]$ and $\alpha_1, \alpha_2 \ge 0$ be such that*

$$1 = \frac{1}{q_1} + \frac{1}{q_2}, \quad \alpha = \frac{\alpha_1}{q_1} + \frac{\alpha_2}{q_2}. \tag{2-13}$$

Then

$$\|f_1 f_2\|_{\mathfrak{L}^{1,\infty}(\log \mathfrak{L})^\alpha} \le \|f_1\|_{\mathfrak{L}^{q_1,\infty}(\log \mathfrak{L})^{\alpha_1}} \|f_2\|_{\mathfrak{L}^{q_2,\infty}(\log \mathfrak{L})^{\alpha_2}} \tag{2-14}$$

for $f_1 \in \mathfrak{L}^{q_1,\infty}(\log \mathfrak{L})^{\alpha_1}$ and $f_2 \in \mathfrak{L}^{q_2,\infty}(\log \mathfrak{L})^{\alpha_2}$. Furthermore,

$$|||\tilde f_1 \tilde f_2|||_{1,\alpha;\rho} \le |||\tilde f_1|||_{q_1,\alpha_1;\rho} |||\tilde f_2|||_{q_2,\alpha_2;\rho} \tag{2-15}$$

for $\tilde f_1 \in \mathfrak{L}^{q_1,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^{\alpha_1}$, $\tilde f_2 \in \mathfrak{L}^{q_2,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^{\alpha_2}$, and $\rho > 0$.

Proof. Let $q_1, q_2 \in [1,\infty)$ and $\alpha_1, \alpha_2 \ge 0$ satisfy (2-13). Let

$$f_1 \in \mathfrak{L}^{q_1,\infty}(\log \mathfrak{L})^{\alpha_1} \quad \text{and} \quad f_2 \in \mathfrak{L}^{q_2,\infty}(\log \mathfrak{L})^{\alpha_2}.$$

It follows from Hölder's inequality, (2-8), and (2-9) that

$$\begin{aligned}\|f_1 f_2\|_{\mathfrak{L}^{1,\infty}(\log \mathfrak{L})^\alpha} &= \sup_{s>0}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^\alpha s (f_1 f_2)^{**}(s)\right\} \\ &\le \sup_{s>0}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^\alpha \int_0^s f_1^*(\tau) f_2^*(\tau)\,d\tau\right\} \\ &\le \sup_{s>0}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^\alpha \left(\int_0^s f_1^*(\tau)^{q_1}\,d\tau\right)^{\frac{1}{q_1}} \left(\int_0^s f_2^*(\tau)^{q_2}\,d\tau\right)^{\frac{1}{q_2}}\right\} \\ &\le \sup_{s>0}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^{\alpha_1} \int_0^s f_1^*(\tau)^{q_1}\,d\tau\right\}^{\frac{1}{q_1}} \sup_{s>0}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^{\alpha_2} \int_0^s f_2^*(\tau)^{q_2}\,d\tau\right\}^{\frac{1}{q_2}} \\ &= \|f_1\|_{\mathfrak{L}^{q_1,\infty}(\log \mathfrak{L})^{\alpha_1}} \|f_2\|_{\mathfrak{L}^{q_2,\infty}(\log \mathfrak{L})^{\alpha_2}}.\end{aligned}$$

Thus (2-14) holds. Furthermore, for any $\tilde{f}_1 \in \mathfrak{L}^{q_1,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^{\alpha_1}$, $\tilde{f}_2 \in \mathfrak{L}^{q_2,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^{\alpha_2}$, and $\rho > 0$, by (2-14) we have

$$\begin{aligned}
|||\tilde{f}_1 \tilde{f}_2|||_{1,\alpha;\rho} &= \sup_{x\in\mathbb{R}^n} \|\tilde{f}_1 \tilde{f}_2 \chi_{B(x,\rho)}\|_{\mathfrak{L}^{1,\infty}(\log \mathfrak{L})^\alpha} \\
&\le \sup_{x\in\mathbb{R}^n} \{\|\tilde{f}_1 \chi_{B(x,\rho)}\|_{\mathfrak{L}^{q_1,\infty}(\log \mathfrak{L})^{\alpha_1}} \|\tilde{f}_2 \chi_{B(x,\rho)}\|_{\mathfrak{L}^{q_2,\infty}(\log \mathfrak{L})^{\alpha_2}}\} \\
&\le \sup_{x\in\mathbb{R}^n} \|\tilde{f}_1 \chi_{B(x,\rho)}\|_{\mathfrak{L}^{q_1,\infty}(\log \mathfrak{L})^{\alpha_1}} \cdot \sup_{x\in\mathbb{R}^n} \|\tilde{f}_2 \chi_{B(x,\rho)}\|_{\mathfrak{L}^{q_2,\infty}(\log \mathfrak{L})^{\alpha_2}} \\
&= |||\tilde{f}_1|||_{q_1,\alpha_1;\rho} |||\tilde{f}_2|||_{q_2,\alpha_2;\rho}.
\end{aligned}$$

Thus (2-15) holds, and Lemma 2.2 follows for $q_1, q_2 \in [1,\infty)$. If $q_1 = \infty$ or $q_2 = \infty$, the conclusion follows from (1-2). □

Lemma 2.3. *Let $q \in [1,\infty)$ and $\alpha \ge 0$. Then, for any $r > 0$ with $rq \ge 1$,*

$$\begin{aligned}
\| |f|^r \|_{\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha} &= \|f\|^r_{\mathfrak{L}^{rq,\infty}(\log \mathfrak{L})^\alpha} && \text{for } f \in \mathfrak{L}^{rq,\infty}(\log \mathfrak{L})^\alpha, \\
||| |\tilde{f}|^r |||_{q,\alpha;\rho} &= |||\tilde{f}|||^r_{rq,\alpha;\rho} && \text{for } \tilde{f} \in \mathfrak{L}^{q,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^\alpha \text{ and } \rho > 0.
\end{aligned}$$

Proof. It follows from (2-9) that

$$\begin{aligned}
\| |f|^r \|_{\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha} &= \sup_{s>0} \left\{ \left[\log\left(e + \frac{1}{s}\right)\right]^\alpha \int_0^s ((|f|^r)^q)^*(\tau)\, d\tau \right\}^{\frac{1}{q}} \\
&= \sup_{s>0} \left\{ \left[\log\left(e + \frac{1}{s}\right)\right]^\alpha \int_0^s (|f|^{rq})^*(\tau)\, d\tau \right\}^{\frac{r}{rq}} \\
&= \|f\|^r_{\mathfrak{L}^{rq,\infty}(\log \mathfrak{L})^\alpha}
\end{aligned}$$

for $f \in \mathfrak{L}^{rq,\infty}(\log \mathfrak{L})^\alpha$. Then

$$\begin{aligned}
||| |\tilde{f}|^r |||_{q,\alpha;\rho} &= \sup_{x\in\mathbb{R}^n} \| |\tilde{f}|^r \chi_{B(x,\rho)} \|_{\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha} \\
&= \sup_{x\in\mathbb{R}^n} \| |\tilde{f}| \chi_{B(x,\rho)} \|^r_{\mathfrak{L}^{rq,\infty}(\log \mathfrak{L})^\alpha} \\
&= |||\tilde{f}|||^r_{rq,\alpha;\rho}
\end{aligned}$$

for $\tilde{f} \in \mathfrak{L}^{rq,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^\alpha$ and $\rho > 0$. Thus Lemma 2.3 follows. □

2.2. *Hardy's inequalities.* We recall the following two lemmas on Hardy's inequality. (See [Muckenhoupt 1972, Theorems 1 and 2].) Throughout this paper, for any $q \in [1,\infty]$, we denote by q' the Hölder conjugate of q, that is, $q' = q/(q-1)$ if $q \in (1,\infty)$, $q' = \infty$ if $q = 1$, and $q' = 1$ if $q = \infty$.

Lemma 2.4. *Let $q \in [1,\infty]$. Let U and V be locally integrable functions in $[0,\infty)$. Then there exists $C > 0$ such that*

$$\|U\tilde{F}\|_{L^q((0,\infty))} \le C\|Vf\|_{L^q((0,\infty))}, \quad \text{with } \tilde{F}(s) := \int_0^s f(\tau)\, d\tau,$$

holds for all locally integrable functions f in $[0,\infty)$ if and only if

$$\sup_{s>0}\{\|U\|_{L^q((s,\infty))}\|V^{-1}\|_{L^{q'}((0,s))}\} < \infty.$$

Lemma 2.5. *Let $q \in [1,\infty]$. Let U and V be locally integrable functions in $[0,\infty)$. Then there exists $C > 0$ such that*

$$\|U \hat{F}\|_{L^q((0,\infty))} \le C\|Vf\|_{L^q((0,\infty))}, \quad \text{with } \hat{F}(s) := \int_s^\infty f(\tau)\,d\tau,$$

holds for all locally integrable functions f in $(0,\infty)$, with $f \in L^1((1,\infty))$, if and only if

$$\sup_{s>0}\{\|U\|_{L^q((0,s))}\|V^{-1}\|_{L^{q'}((s,\infty))}\} < \infty.$$

2.3. *Fundamental solutions.* Let $\theta \in (0,2]$. Let G_θ be the fundamental solution to the fractional heat equation

$$\partial_t v + (-\Delta)^{\frac{\theta}{2}} v = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty).$$

The function G_θ is positive and smooth in $\mathbb{R}^n \times (0,\infty)$, and it satisfies

$$\begin{aligned} &G_\theta(x,t) = (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{|x|^2}{4t}\right) \le C h_{\theta,t}(x) && \text{if } \theta = 2, \\ &G_\theta(x,t) \asymp h_{\theta,t}(x) && \text{if } 0 < \theta < 2 \end{aligned} \tag{2-16}$$

for $(x,t) \in \mathbb{R}^n \times (0,\infty)$, where

$$h_{\theta,t}(x) := t^{-\frac{n}{\theta}}(1 + t^{-\frac{1}{\theta}}|x|)^{-n-\theta}. \tag{2-17}$$

Furthermore,

- $G_\theta(x,t) = t^{-\frac{n}{\theta}} G_\theta(t^{-\frac{1}{\theta}}x, 1)$, $\int_{\mathbb{R}^n} G_\theta(x,t)\,dx = 1$,
- $G_\theta(\cdot,1)$ is radially symmetric and $G_\theta(x,1) \le G_\theta(y,1)$ if $|x| \ge |y|$,
- $G_\theta(x,t) = \int_{\mathbb{R}^n} G_\theta(x-y,t-s) G_\theta(y,s)\,dy$

for x, $y \in \mathbb{R}^n$ and $0 < s < t$ (see, e.g., [Bogdan and Jakubowski 2007; Brandolese and Karch 2008; Sugitani 1975]), and

$$\lim_{t\to+0} \|S_\theta(t)\eta - \eta\|_{L^\infty} = 0 \quad \text{for } \eta \in C_0(\mathbb{R}^n). \tag{2-18}$$

In addition, it follows from Young's inequality that

$$\|S_\theta(t)\eta\|_{L^q} \le C t^{-\frac{n}{\theta}\left(\frac{1}{r}-\frac{1}{q}\right)} \|\eta\|_{L^r} \tag{2-19}$$

for $\eta \in L^r$, $1 \le r \le q \le \infty$, and $t > 0$.

3. Decay estimates of $S_\theta(t)\varphi$

In this section we obtain decay estimates of $S_\theta(t)\varphi$ in our weak Zygmund-type spaces and uniformly local weak Zygmund-type spaces. For simplicity we write $g_t := G_\theta(\cdot,t)$ and $h_t := h_{\theta,t}$.

Proposition 3.1. *Let $\theta \in (0,2]$, $1 \le r \le q \le \infty$, and α, $\beta \ge 0$. Assume that $\alpha \le \beta$ if $r = q$. Then there exists $C > 0$ such that*

$$\|S_\theta(t)\varphi\|_{\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\beta} \le C t^{-\frac{n}{\theta}\left(\frac{1}{r}-\frac{1}{q}\right)} \left[\log\left(e + \frac{1}{t}\right)\right]^{-\frac{\alpha}{r}+\frac{\beta}{q}} \|\varphi\|_{\mathfrak{L}^{r,\infty}(\log \mathfrak{L})^\alpha}$$

for $\varphi \in \mathfrak{L}^{r,\infty}(\log \mathfrak{L})^\alpha$ and $t > 0$.

Before starting the proof, we recall the following relations on logarithmic functions: for any fixed $L > 1$ and $k > 0$,

$$\log\left(e+\frac{1}{s}\right) \asymp \log\left(L+\frac{1}{s}\right) \asymp \log\left(e+\frac{k}{s}\right) \asymp \log\left(e+\frac{1}{s^k}\right) \quad \text{for } s > 0. \tag{3-1}$$

Furthermore, we have the following results.

Lemma 3.2. (1) *Let $q > -1$ and $\alpha \in \mathbb{R}$. Then there exists $C_1 > 0$ such that*

$$\int_0^s \tau^q \left[\log\left(e+\frac{1}{\tau}\right)\right]^\alpha d\tau \le C_1 s^{q+1} \left[\log\left(e+\frac{1}{s}\right)\right]^\alpha \quad \textit{for } s > 0.$$

(2) *Let $S > 0$ and $\alpha < -1$. Then there exists $C_2 > 0$ such that*

$$\int_0^s \tau^{-1} \left[\log\left(e+\frac{1}{\tau}\right)\right]^\alpha d\tau \le C_2 \left[\log\left(e+\frac{1}{s}\right)\right]^{\alpha+1} \quad \textit{for } s \in (0, S).$$

(3) *Let $q < -1$ and $\alpha \in \mathbb{R}$. Then there exists $C_3 > 0$ such that*

$$\int_s^\infty \tau^q \left[\log\left(e+\frac{1}{\tau}\right)\right]^\alpha d\tau \le C_3 s^{q+1} \left[\log\left(e+\frac{1}{s}\right)\right]^\alpha \quad \textit{for } s > 0.$$

Proof. We prove assertion (1). Let $\delta > 0$ be such that $q - \delta > -1$. Then there exists $L \in [e, \infty)$ such that

$$\text{the function } (0, \infty) \ni \tau \mapsto \tau^\delta \left[\log\left(L+\frac{1}{\tau}\right)\right]^\alpha \text{ is nondecreasing.} \tag{3-2}$$

This together with (3-1) implies that

$$\begin{aligned}\int_0^s \tau^q \left[\log\left(e+\frac{1}{\tau}\right)\right]^\alpha d\tau &\le C \int_0^s \tau^{q-\delta} \cdot \tau^\delta \left[\log\left(L+\frac{1}{\tau}\right)\right]^\alpha d\tau \\ &\le C s^\delta \left[\log\left(L+\frac{1}{s}\right)\right]^\alpha \int_0^s \tau^{q-\delta}\, d\tau \le C s^{q+1} \left[\log\left(e+\frac{1}{s}\right)\right]^\alpha\end{aligned}$$

for $s > 0$. Thus assertion (1) follows.

We prove assertion (2). Let $S > 0$. It follows that

$$\int_0^s \tau^{-1} \left[\log\left(e+\frac{1}{\tau}\right)\right]^\alpha d\tau \le C \int_0^s \tau^{-1} |\log\tau|^\alpha\, d\tau \le C|\log s|^{\alpha+1} \le C\left[\log\left(e+\frac{1}{s}\right)\right]^{\alpha+1}$$

for $s \in \left(0, \frac{1}{2}\right)$. If $S \ge \frac{1}{2}$, then

$$\int_0^s \tau^{-1} \left[\log\left(e+\frac{1}{\tau}\right)\right]^\alpha d\tau \le \int_{\frac{1}{4}}^S \tau^{-1} \left[\log\left(e+\frac{1}{\tau}\right)\right]^\alpha d\tau + C \le C \le C\left[\log\left(e+\frac{1}{s}\right)\right]^{\alpha+1}$$

for $s \in \left[\frac{1}{2}, S\right)$. Thus assertion (2) follows.

It remains to prove assertion (3). Let $\delta' > 0$ be such that $q + \delta' < -1$. Then there exists $L' \in [e, \infty)$ such that

$$\text{the function } (0, \infty) \ni \tau \mapsto \tau^{-\delta'} \left[\log\left(L'+\frac{1}{\tau}\right)\right]^\alpha \text{ is nonincreasing.}$$

This together with (3-1) implies that

$$\begin{aligned}\int_s^\infty \tau^q\left[\log\left(e+\frac{1}{\tau}\right)\right]^\alpha d\tau &\le C\int_s^\infty \tau^{q+\delta'}\cdot\tau^{-\delta'}\left[\log\left(L'+\frac{1}{\tau}\right)\right]^\alpha d\tau\\ &\le Cs^{-\delta'}\left[\log\left(L'+\frac{1}{s}\right)\right]^\alpha\int_s^\infty \tau^{q+\delta'}\,d\tau \le Cs^{q+1}\left[\log\left(e+\frac{1}{s}\right)\right]^\alpha\end{aligned}$$

for $s>0$. Thus assertion (3) follows. □

Next, we prepare the following lemma on h_t^*, where $h_t=h_{\theta,t}$ is as in (2-17).

Lemma 3.3. *Let $1\le r\le q<\infty$ and $\gamma\in\mathbb{R}$. Assume that $\gamma\ge 0$ if $r=q$. Then there exists $C>0$ such that*

$$\int_0^\infty \tau^{q(1-\frac{1}{r})}\left[\log\left(e+\frac{1}{\tau}\right)\right]^\gamma (h_t^*(\tau))^q\,d\tau \le Ct^{-\frac{nq}{\theta}(\frac{1}{r}-\frac{1}{q})}\left[\log\left(e+\frac{1}{t}\right)\right]^\gamma \tag{3-3}$$

for $t>0$.

Proof. It follows from (2-17) that

$$(h_t)^*(s)=h_t((\omega_n^{-1}s)^{\frac{1}{n}}e_1)\le Ct^{-\frac{n}{\theta}}(1+t^{-\frac{1}{\theta}}s^{\frac{1}{n}})^{-n-\theta}$$

for $s\in[0,\infty)$ and $t\in(0,\infty)$, where ω_n is the volume of the n-dimensional unit ball $B(0,1)$ and $e_1:=(1,0,\dots,0)\in\mathbb{R}^n$. Then

$$\begin{aligned} I &:= \int_0^\infty \tau^{q(1-\frac{1}{r})}\left[\log\left(e+\frac{1}{\tau}\right)\right]^\gamma (h_t^*(\tau))^q\,d\tau\\ &\le Ct^{-\frac{nq}{\theta}}\int_0^\infty \tau^{q(1-\frac{1}{r})}\left[\log\left(e+\frac{1}{\tau}\right)\right]^\gamma (1+t^{-\frac{1}{\theta}}\tau^{\frac{1}{n}})^{-q(n+\theta)}\,d\tau\\ &\le Ct^{-\frac{nq}{\theta}(\frac{1}{r}-\frac{1}{q})}\int_0^\infty \xi^{nq(1-\frac{1}{r})+n-1}(1+\xi)^{-q(n+\theta)}\left[\log\left(e+\frac{1}{(t^{1/\theta}\xi)^n}\right)\right]^\gamma d\xi \end{aligned}\tag{3-4}$$

for $t>0$.

We first consider the case of $\gamma\ge 0$. It follows from (3-1) that

$$\begin{aligned}\left[\log\left(e+\frac{1}{(t^{1/\theta}\xi)^n}\right)\right]^\gamma &\le C\left[\log\left(e+\frac{1}{t^{1/\theta}\xi}\right)\right]^\gamma\\ &\le C\left[\log\left(e+\frac{1}{t^{1/\theta}}\right)+\log\left(e+\frac{1}{\xi}\right)\right]^\gamma\\ &\le C\left[\log\left(e+\frac{1}{t}\right)\right]^\gamma + C\left[\log\left(e+\frac{1}{\xi}\right)\right]^\gamma\end{aligned}\tag{3-5}$$

for $t>0$ and $\xi\in\left(0,\frac{1}{2}\right)$. Similarly, by (3-1), we have

$$\left[\log\left(e+\frac{1}{(t^{1/\theta}\xi)^n}\right)\right]^\gamma \le \left[\log\left(e+\frac{2^n}{(t^{1/\theta})^n}\right)\right]^\gamma \le C\left[\log\left(e+\frac{1}{t}\right)\right]^\gamma \tag{3-6}$$

for $t > 0$ and $\xi \in \left[\frac{1}{2}, \infty\right)$. Since

$$nq\left(1-\frac{1}{r}\right)+n-1-q(n+\theta) = -\frac{nq}{r}+n-1-q\theta = -nq\left(\frac{1}{r}-\frac{1}{q}\right)-1-q\theta < -1, \tag{3-7}$$

by Lemma 3.2, (3-4), (3-5), and (3-6), we obtain

$$\begin{aligned}
I &\le Ct^{-\frac{nq}{\theta}(\frac{1}{r}-\frac{1}{q})}\int_0^{\frac{1}{2}} \xi^{nq(1-\frac{1}{r})+n-1}\left(\left[\log\left(e+\frac{1}{t}\right)\right]^{\gamma}+\left[\log\left(e+\frac{1}{\xi}\right)\right]^{\gamma}\right)d\xi \\
&\qquad + Ct^{-\frac{nq}{\theta}(\frac{1}{r}-\frac{1}{q})}\int_{\frac{1}{2}}^{\infty} \xi^{nq(1-\frac{1}{r})+n-1}(1+\xi)^{-q(n+\theta)}\left[\log\left(e+\frac{1}{t}\right)\right]^{\gamma}d\xi \\
&\le Ct^{-\frac{nq}{\theta}(\frac{1}{r}-\frac{1}{q})}\left(1+\left[\log\left(e+\frac{1}{t}\right)\right]^{\gamma}\right) \\
&\le Ct^{-\frac{nq}{\theta}(\frac{1}{r}-\frac{1}{q})}\left[\log\left(e+\frac{1}{t}\right)\right]^{\gamma}
\end{aligned}$$

for $t > 0$. This implies (3-3) in the case of $\gamma \ge 0$.

Consider the case of $\gamma < 0$. Then, by (3-1), we have

$$\left[\log\left(e+\frac{1}{(t^{1/\theta}\xi)^n}\right)\right]^{\gamma} \le \left[\log\left(e+\frac{2^n}{(t^{1/\theta})^n}\right)\right]^{\gamma} \le C\left[\log\left(e+\frac{1}{t}\right)\right]^{\gamma} \tag{3-8}$$

for $t > 0$ and $\xi \in \left(0, \frac{1}{2}\right)$. Let $0 < \delta < \theta q/|\gamma|$. We find $L \in [e, \infty)$ such that the function f in $(0, \infty)$ defined by

$$f(z) := z^{\delta}\log\left(L+\frac{1}{z^n}\right)$$

is nondecreasing in $(0, \infty)$. Since $\gamma < 0$, by (3-1), we obtain

$$\begin{aligned}
\left[\log\left(e+\frac{1}{(t^{1/\theta}\xi)^n}\right)\right]^{\gamma} &\le C\left[\log\left(L+\frac{1}{(t^{1/\theta}\xi)^n}\right)\right]^{\gamma} = C[z^{-\delta\gamma}f(z)^{\gamma}]|_{z=t^{1/\theta}\xi} \\
&\le C(t^{\frac{1}{\theta}}\xi)^{-\delta\gamma}f(z)^{\gamma}|_{z=\frac{t^{1/\theta}}{2}} \le C\xi^{-\delta\gamma}\left[\log\left(e+\frac{2^n}{t^{n/\theta}}\right)\right]^{\gamma} \\
&\le C\xi^{-\delta\gamma}\left[\log\left(e+\frac{1}{t}\right)\right]^{\gamma}
\end{aligned}$$

for $t > 0$ and $\xi \in \left[\frac{1}{2}, \infty\right)$. This together with (3-7) and (3-8) implies that

$$\begin{aligned}
I &\le Ct^{-\frac{nq}{\theta}(\frac{1}{r}-\frac{1}{q})}\int_0^{\frac{1}{2}} \xi^{nq(1-\frac{1}{r})+n-1}\left[\log\left(e+\frac{1}{t}\right)\right]^{\gamma}d\xi \\
&\qquad + Ct^{-\frac{nq}{\theta}(\frac{1}{r}-\frac{1}{q})}\int_{\frac{1}{2}}^{\infty} \xi^{nq(1-\frac{1}{r})+n-1-\delta\gamma}(1+\xi)^{-q(n+\theta)}\left[\log\left(e+\frac{1}{t}\right)\right]^{\gamma}d\xi \\
&\le Ct^{-\frac{nq}{\theta}(\frac{1}{r}-\frac{1}{q})}\left[\log\left(e+\frac{1}{t}\right)\right]^{\gamma}
\end{aligned}$$

for $t > 0$. This implies (3-3) in the case of $\gamma < 0$. Thus Lemma 3.3 follows. □

Proof of Proposition 3.1. The proof is divided into the following three cases:

$$1 \le r < q < \infty, \quad 1 \le r = q < \infty, \quad 1 \le r \le q = \infty.$$

Step 1. Consider the case of $1 \le r < q < \infty$. It follows from (2-4), (2-7), (2-9), and (2-16) that

$$\begin{aligned}\|S_\theta(t)\varphi\|^q_{\mathfrak{L}^{q,\infty}(\log\mathfrak{L})^\beta} &= \sup_{s>0}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^\beta \int_0^s ((S_\theta(t)\varphi)^*(\tau))^q\,d\tau\right\}\\ &\le \sup_{s>0}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^\beta \int_0^s ((S_\theta(t)\varphi)^{**}(\tau))^q\,d\tau\right\}\\ &\le \sup_{s>0}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^\beta \int_0^s \left(\int_\tau^\infty g_t^{**}(\eta)\varphi^{**}(\eta)\,d\eta\right)^q d\tau\right\}\\ &\le C\sup_{s>0}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^\beta \int_0^s \left(\int_\tau^\infty h_t^{**}(\eta)\varphi^{**}(\eta)\,d\eta\right)^q d\tau\right\}\end{aligned}$$

for $t>0$. Furthermore, thanks to (2-11), we have

$$\|S_\theta(t)\varphi\|^q_{\mathfrak{L}^{q,\infty}(\log\mathfrak{L})^\beta} \le C\int_0^\infty \left(\left[\log\left(e+\frac{1}{\tau}\right)\right]^{\frac{\beta}{q}} \int_\tau^\infty h_t^{**}(\eta)\varphi^{**}(\eta)\,d\eta\right)^q d\tau \tag{3-9}$$

for $t>0$. On the other hand, set

$$U(\tau) := \left[\log\left(e+\frac{1}{\tau}\right)\right]^{\frac{\beta}{q}}, \quad V(\tau) := \tau\left[\log\left(e+\frac{1}{\tau}\right)\right]^{\frac{\beta}{q}}$$

for $\tau>0$. It follows from Lemma 3.2 (1) and (3) that

$$\begin{aligned}&\sup_{s>0}\left(\int_0^s |U(\tau)|^q\,d\tau\right)^{\frac{1}{q}}\left(\int_s^\infty |V(\tau)|^{-q'}\,d\tau\right)^{\frac{1}{q'}}\\ &\qquad\le \sup_{s>0}\left\{Cs^{\frac{1}{q}}\left[\log\left(e+\frac{1}{s}\right)\right]^{\frac{\beta}{q}}\cdot Cs^{-1+\frac{1}{q'}}\left[\log\left(e+\frac{1}{s}\right)\right]^{-\frac{\beta}{q}}\right\} < \infty.\end{aligned}$$

Then, by Lemma 2.5, (2-3), and (3-9), we have

$$\begin{aligned}\|S_\theta(t)\varphi\|^q_{\mathfrak{L}^{q,\infty}(\log\mathfrak{L})^\beta} &\le C\int_0^\infty \left(\tau\left[\log\left(e+\frac{1}{\tau}\right)\right]^{\frac{\beta}{q}} h_t^{**}(\tau)\varphi^{**}(\tau)\right)^q d\tau\\ &\le C\sup_{s>0}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^\alpha s(\varphi^{**}(s))^r\right\}^{\frac{q}{r}}\int_0^\infty\left(\tau^{1-\frac{1}{r}}\left[\log\left(e+\frac{1}{\tau}\right)\right]^{-\frac{\alpha}{r}+\frac{\beta}{q}} h_t^{**}(\tau)\right)^q d\tau\end{aligned}$$

for $t>0$. This together with (2-5) and (2-9) implies that

$$\|S_\theta(t)\varphi\|^q_{\mathfrak{L}^{q,\infty}(\log\mathfrak{L})^\beta} \le C\|\varphi\|^q_{\mathfrak{L}^{r,\infty}(\log\mathfrak{L})^\alpha}\int_0^\infty\left(\tau^{-\frac{1}{r}}\left[\log\left(e+\frac{1}{\tau}\right)\right]^\gamma \int_0^\tau h_t^*(\xi)\,d\xi\right)^q d\tau \tag{3-10}$$

for $t > 0$, where

$$\gamma := -\frac{\alpha}{r} + \frac{\beta}{q}.$$

Set

$$\widetilde{U}(\tau) = \tau^{-\frac{1}{r}}\left[\log\left(e + \frac{1}{\tau}\right)\right]^{\gamma}, \quad \widetilde{V}(\tau) = \tau^{1-\frac{1}{r}}\left[\log\left(e + \frac{1}{\tau}\right)\right]^{\gamma}.$$

Since $q > r$ and $q' < r'$, by Lemma 3.2 (1) and (3), we have

$$\begin{aligned}\sup_{s>0}\left(\int_s^\infty |\widetilde{U}(\tau)|^q \, d\tau\right)^{\frac{1}{q}}\left(\int_0^s |\widetilde{V}(\tau)|^{-q'} \, d\tau\right)^{\frac{1}{q'}} \\ = \sup_{s>0}\left(\int_s^\infty \tau^{-\frac{q}{r}}\left[\log\left(e + \frac{1}{\tau}\right)\right]^{q\gamma} d\tau\right)^{\frac{1}{q}}\left(\int_0^s \tau^{-\frac{q'}{r'}}\left[\log\left(e + \frac{1}{\tau}\right)\right]^{-q'\gamma} d\tau\right)^{\frac{1}{q'}} \\ \le \sup_{s>0}\left\{C s^{\frac{1}{q}-\frac{1}{r}}\left[\log\left(e + \frac{1}{s}\right)\right]^{\gamma} \cdot C s^{\frac{1}{q'}-\frac{1}{r'}}\left[\log\left(e + \frac{1}{s}\right)\right]^{-\gamma}\right\} < \infty. \qquad (3\text{-}11)\end{aligned}$$

Applying Lemma 2.4 to (3-10), by (3-11), we obtain

$$\|S_\theta(t)\varphi\|^q_{\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\beta} \le C\|\varphi\|^q_{\mathfrak{L}^{r,\infty}(\log \mathfrak{L})^\alpha} \int_0^\infty \left(\tau^{1-\frac{1}{r}}\left[\log\left(e + \frac{1}{\tau}\right)\right]^{\gamma} h_t^*(\tau)\right)^q d\tau$$

for $t > 0$. This together with Lemma 3.3 implies that

$$\|S_\theta(t)\varphi\|^q_{\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\beta} \le C t^{-\frac{nq}{\theta}\left(\frac{1}{r}-\frac{1}{q}\right)}\left[\log\left(e + \frac{1}{t}\right)\right]^{q\gamma}\|\varphi\|^q_{\mathfrak{L}^{r,\infty}(\log \mathfrak{L})^\alpha}$$

for $t > 0$. Thus Proposition 3.1 follows in the case of $1 \le r < q < \infty$.

Step 2. Consider the case of $1 \le r = q < \infty$. It follows from Hölder's inequality and (2-16) that

$$\begin{aligned}|[S_\theta(t)\varphi](x)|^r &\le C\left(\int_{\mathbb{R}^n} |h_t(x-y)|\,|\varphi(y)|\,dy\right)^r \\ &\le C\left(\int_{\mathbb{R}^n} |h_t(x-y)|\,dy\right)^{r-1}\int_{\mathbb{R}^n} |h_t(x-y)|\,|\varphi(y)|^r\,dy \\ &\le C\int_{\mathbb{R}^n} |h_t(x-y)|\,|\varphi(y)|^r\,dy.\end{aligned}$$

Then it follows from (2-7) and (2-9) that

$$\begin{aligned}\|S_\theta(t)\varphi\|^r_{\mathfrak{L}^{r,\infty}(\log \mathfrak{L})^\beta} &= \sup_{s>0}\left\{\left[\log\left(e + \frac{1}{s}\right)\right]^{\beta} s(|S_\theta(t)\varphi|^r)^{**}(s)\right\} \\ &\le C\sup_{s>0}\left\{\left[\log\left(e + \frac{1}{s}\right)\right]^{\beta} s\int_s^\infty (h_t)^{**}(\tau)(|\varphi|^r)^{**}(\tau)\,d\tau\right\} \qquad (3\text{-}12)\end{aligned}$$

for $t > 0$. Set

$$\widehat{U}(r) = r\left[\log\left(e+\frac{1}{r}\right)\right]^{\beta}, \quad \widehat{V}(r) = r^2\left[\log\left(e+\frac{1}{r}\right)\right]^{\beta}.$$

Similarly to (3-2), we find $L \in [e, \infty)$ such that

$$\text{the function } (0,\infty) \ni r \mapsto r\left[\log\left(L+\frac{1}{r}\right)\right]^{\beta} \text{ is nondecreasing.}$$

Then, by (3-1), we have

$$\begin{aligned}\|\widehat{U}\|_{L^\infty(0,s)} &\le C \sup_{r\in(0,s)}\left\{r\left[\log\left(L+\frac{1}{r}\right)\right]^{\beta}\right\}\\ &\le Cs\left[\log\left(L+\frac{1}{s}\right)\right]^{\beta}\\ &\le Cs\left[\log\left(e+\frac{1}{s}\right)\right]^{\beta}\end{aligned}$$

for $s > 0$. This together with Lemma 3.2 (3) implies that

$$\sup_{s>0}\left\{\|\widehat{U}\|_{L^\infty((0,s))}\int_s^\infty |\widehat{V}(\tau)|^{-1}\,d\tau\right\} \le \sup_{s>0}\left\{Cs\left[\log\left(e+\frac{1}{s}\right)\right]^{\beta}\cdot Cs^{-1}\left[\log\left(e+\frac{1}{s}\right)\right]^{-\beta}\right\} < \infty. \tag{3-13}$$

Applying Lemma 2.5 with $q = \infty$, by (2-9), (3-12) and (3-13), we obtain

$$\begin{aligned}\|S_\theta(t)\varphi\|^r_{\mathfrak{L}^{r,\infty}(\log\mathfrak{L})^{\beta}} &\le C\sup_{s>0}\left\{s^2\left[\log\left(e+\frac{1}{s}\right)\right]^{\beta}(h_t)^{**}(s)(|\varphi|^r)^{**}(s)\right\}\\ &\le C\sup_{s>0}\left\{s\left[\log\left(e+\frac{1}{s}\right)\right]^{\beta-\alpha}(h_t)^{**}(s)\right\}\cdot\sup_{s>0}\left\{s\left[\log\left(e+\frac{1}{s}\right)\right]^{\alpha}(|\varphi|^r)^{**}(s)\right\}\\ &= C\|h_t\|_{\mathfrak{L}^{1,\infty}(\log\mathfrak{L})^{\beta-\alpha}}\|\varphi\|^r_{\mathfrak{L}^{r,\infty}(\log\mathfrak{L})^{\alpha}}\end{aligned} \tag{3-14}$$

for $t > 0$. Furthermore, since $\alpha \le \beta$, by Lemma 3.3, (2-9), and (2-11), we have

$$\begin{aligned}\|h_t\|_{\mathfrak{L}^{1,\infty}(\log\mathfrak{L})^{\beta-\alpha}} &= \sup_{s>0}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^{\beta-\alpha}\int_0^s (h_t)^*(\tau)\,d\tau\right\}\\ &\le \int_0^\infty\left[\log\left(e+\frac{1}{\tau}\right)\right]^{\beta-\alpha}(h_t)^*(\tau)\,d\tau\\ &\le C\left[\log\left(e+\frac{1}{t}\right)\right]^{\beta-\alpha}\end{aligned}$$

for $t > 0$. This together with (3-14) implies that

$$\|S_\theta(t)\varphi\|^r_{\mathfrak{L}^{r,\infty}(\log\mathfrak{L})^{\beta}} \le C\left[\log\left(e+\frac{1}{t}\right)\right]^{\beta-\alpha}\|\varphi\|^r_{\mathfrak{L}^{r,\infty}(\log\mathfrak{L})^{\alpha}} \quad \text{for } t > 0.$$

Thus Proposition 3.1 follows in the case of $1 \le r = q < \infty$.

Step 3. It remains to consider the case of $1 \le r \le q = \infty$. If $r = q = \infty$, then it follows from (2-16) that

$$\|S_\theta(t)\varphi\|_{L^\infty} \le C\|\varphi\|_{L^\infty} \int_{\mathbb{R}^n} h_t(y)\,dy \le C\|\varphi\|_{L^\infty}$$

for $t > 0$, and Proposition 3.1 follows. On the other hand, in the case of $1 \le r < q = \infty$, let $\tilde{q} \in (r, \infty)$. Then, by Proposition 3.1 with $q = \tilde{q} > r$, (1-3), (2-19), and (3-1), we have

$$\begin{aligned}\|S_\theta(t)\varphi\|_{L^\infty} &= \Big\|S_\theta\Big(\frac{t}{2}\Big)S_\theta\Big(\frac{t}{2}\Big)\varphi\Big\|_{L^\infty} \le Ct^{-\frac{n}{\theta\tilde{q}}}\Big\|S_\theta\Big(\frac{t}{2}\Big)\varphi\Big\|_{L^{\tilde{q}}} = C^{-\frac{n}{\theta\tilde{q}}}\Big\|S_\theta\Big(\frac{t}{2}\Big)\varphi\Big\|_{\mathfrak{L}^{\tilde{q},\infty}(\log\mathfrak{L})^0}\\ &\le Ct^{-\frac{n}{\theta\tilde{q}}}\cdot Ct^{-\frac{n}{\theta}(\frac{1}{r}-\frac{1}{\tilde{q}})}\Big[\log\Big(e+\frac{2}{t}\Big)\Big]^{-\frac{\alpha}{r}}\|\varphi\|_{\mathfrak{L}^{r,\infty}(\log\mathfrak{L})^\alpha}\\ &= Ct^{-\frac{n}{\theta r}}\Big[\log\Big(e+\frac{1}{t}\Big)\Big]^{-\frac{\alpha}{r}}\|\varphi\|_{\mathfrak{L}^{r,\infty}(\log\mathfrak{L})^\alpha}\end{aligned}$$

for $t > 0$. Thus Proposition 3.1 follows in the case of $1 \le r < q = \infty$. The proof of Proposition 3.1 is complete. □

Furthermore, by Proposition 3.1, we employ the arguments in the proof of [Hisa and Ishige 2018, Lemma 2.1] to obtain decay estimates of $S_\theta(t)\varphi$ in uniformly local weak Zygmund-type spaces.

Proposition 3.4. *Let $\theta \in (0, 2]$, $1 \le r \le q \le \infty$, and $\alpha, \beta \ge 0$. Assume that $\alpha \le \beta$ if $r = q$. There exists $C > 0$ such that, for any $T > 0$,*

$$|||S_\theta(t)\varphi|||_{q,\beta;T^{1/\theta}} \le Ct^{-\frac{n}{\theta}(\frac{1}{r}-\frac{1}{q})}\Big[\log\Big(e+\frac{1}{t}\Big)\Big]^{-\frac{\alpha}{r}+\frac{\beta}{q}}|||\varphi|||_{r,\alpha;T^{1/\theta}} \tag{3-15}$$

for $\varphi \in \mathfrak{L}^{r,\infty}_{\mathrm{ul}}(\log\mathfrak{L})^\alpha$ and $t \in (0, T]$.

Proof. We first consider the case of $\theta \in (0, 2)$. It suffices to prove

$$t^{\frac{n}{\theta}(\frac{1}{r}-\frac{1}{q})}\Big[\log\Big(e+\frac{1}{t}\Big)\Big]^{\frac{\alpha}{r}-\frac{\beta}{q}}\|\chi_{B(z,T^{1/\theta})}S_\theta(t)\varphi\|_{\mathfrak{L}^{q,\infty}(\log\mathfrak{L})^\beta} \le C|||\varphi|||_{r,\alpha;T^{1/\theta}} \tag{3-16}$$

for $z \in \mathbb{R}^n$ and $0 < t \le T$. For the proof, by translating if necessary, we have only to consider the case of $z = 0$.

By Besicovitch's covering lemma, we can find an integer m depending only on n and a set

$$\{x_{k,i}\}_{k=1,\dots,m,\,i\in\mathbb{N}} \subset \mathbb{R}^n \setminus B(0, 10T^{\frac{1}{\theta}})$$

such that

$$B_{k,i} \cap B_{k,j} = \varnothing \ \text{ if } i \ne j \quad \text{and} \quad \mathbb{R}^n \setminus B(0, 10T^{\frac{1}{\theta}}) \subset \bigcup_{k=1}^m \bigcup_{i=1}^\infty B_{k,i}, \tag{3-17}$$

where $B_{k,i} := \overline{B(x_{k,i}, T^{1/\theta})}$. Then

$$|[S_\theta(t)\varphi](x)| \le |u_0(x,t)| + \sum_{k=1}^m\sum_{i=1}^\infty |u_{k,i}(x,t)|, \quad (x,t) \in \mathbb{R}^n \times (0,T), \tag{3-18}$$

where

$$u_0(x,t) := [S_\theta(t)(\varphi\chi_{B(0,10T^{1/\theta})})](x), \quad u_{k,i}(x,t) := [S_\theta(t)(\varphi\chi_{B_{k,i}})](x).$$

By Proposition 3.1 and (1-4), we have

$$\begin{aligned}\|u_0(t)\chi_{B(0,T^{1/\theta})}\|_{\mathfrak{L}^{q,\infty}(\log\mathfrak{L})^\beta} &\le \|u_0(t)\|_{\mathfrak{L}^{q,\infty}(\log\mathfrak{L})^\beta}\\ &\le Ct^{-\frac{n}{\theta}(\frac{1}{r}-\frac{1}{q})}\left[\log\left(e+\frac{1}{t}\right)\right]^{-\frac{\alpha}{r}+\frac{\beta}{q}}\|\varphi\chi_{B(0,10T^{1/\theta})}\|_{\mathfrak{L}^{r,\infty}(\log\mathfrak{L})^\alpha}\\ &\le Ct^{-\frac{n}{\theta}(\frac{1}{r}-\frac{1}{q})}\left[\log\left(e+\frac{1}{t}\right)\right]^{-\frac{\alpha}{r}+\frac{\beta}{q}}|||\varphi|||_{r,\alpha;10T^{1/\theta}}\\ &\le Ct^{-\frac{n}{\theta}(\frac{1}{r}-\frac{1}{q})}\left[\log\left(e+\frac{1}{t}\right)\right]^{-\frac{\alpha}{r}+\frac{\beta}{q}}|||\varphi|||_{r,\alpha;T^{1/\theta}} \qquad (3\text{-}19)\end{aligned}$$

for $t\in(0,T]$.

Let $k=1,\dots,m$ and $i\in\mathbb{N}$. By (2-16), we have

$$|u_{k,i}(x,t)|\le C\int_{B(x_{k,i},T^{1/\theta})}h_t(x-y)|\varphi(y)|\,dy = C\int_{\mathbb{R}^n}h_t(x-z-x_{k,i})\varphi_{k,i}(z)\,dz \qquad (3\text{-}20)$$

for $(x,t)\in\mathbb{R}^n\times(0,\infty)$, where $\varphi_{k,i}(x)=|\varphi(x+x_{k,i})|\chi_{B(0,T^{1/\theta})}$. Since $|x_{k,i}|\ge 10T^{1/\theta}$, it follows that

$$\begin{aligned}(1+T^{-\frac{1}{\theta}}|x_{k,i}|)(1+t^{-\frac{1}{\theta}}|x-z|) &= 1+T^{-\frac{1}{\theta}}|x_{k,i}|+t^{-\frac{1}{\theta}}|x-z|+t^{-\frac{1}{\theta}}T^{-\frac{1}{\theta}}|x_{k,i}||x-z|\\ &\le 1+3t^{-\frac{1}{\theta}}|x_{k,i}|+t^{-\frac{1}{\theta}}|x-z|\\ &= 1+4t^{-\frac{1}{\theta}}(|x_{k,i}|-|x-z|)-t^{-\frac{1}{\theta}}|x_{k,i}|+5t^{-\frac{1}{\theta}}|x-z|\\ &\le 4(1+t^{-\frac{1}{\theta}}(|x_{k,i}|-|x-z|))\le 4(1+t^{-\frac{1}{\theta}}|x-z-x_{k,i}|)\end{aligned}$$

for $x,z\in B(0,T^{1/\theta})$ and $t\in(0,T)$. This together with (2-16) implies that

$$\begin{aligned}h_t(x-z-x_{k,i}) &\le Ct^{-\frac{n}{\theta}}(1+T^{-\frac{1}{\theta}}|x_{k,i}|)^{-n-\theta}(1+t^{-\frac{1}{\theta}}|x-z|)^{-n-\theta}\\ &\le C(1+T^{-\frac{1}{\theta}}|x_{k,i}|)^{-n-\theta}g_t(x-z) \qquad (3\text{-}21)\end{aligned}$$

for $x,z\in B(0,T^{1/\theta})$ and $t\in(0,T)$. We observe from (3-20) and (3-21) that

$$|u_{k,i}(x,t)|\le C(1+T^{-\frac{1}{\theta}}|x_{k,i}|)^{-n-\theta}[S_\theta(t)\varphi_{k,i}](x)$$

for $x\in B(0,T^{1/\theta})$ and $t\in(0,T)$. Then, by Proposition 3.1, we obtain

$$\begin{aligned}&\|u_{k,i}(t)\chi_{B(0,T^{1/\theta})}\|_{\mathfrak{L}^{q,\infty}(\log\mathfrak{L})^\beta}\\ &\quad\le C(1+T^{-\frac{1}{\theta}}|x_{k,i}|)^{-n-\theta}\|S_\theta(t)\varphi_{k,i}\|_{\mathfrak{L}^{q,\infty}(\log\mathfrak{L})^\beta}\\ &\quad\le C(1+T^{-\frac{1}{\theta}}|x_{k,i}|)^{-n-\theta}t^{-\frac{n}{\theta}(\frac{1}{r}-\frac{1}{q})}\left[\log\left(e+\frac{1}{t}\right)\right]^{-\frac{\alpha}{r}+\frac{\beta}{q}}\|\varphi_{k,i}\|_{\mathfrak{L}^{r,\infty}(\log\mathfrak{L})^\alpha}\\ &\quad= C(1+T^{-\frac{1}{\theta}}|x_{k,i}|)^{-n-\theta}t^{-\frac{n}{\theta}(\frac{1}{r}-\frac{1}{q})}\left[\log\left(e+\frac{1}{t}\right)\right]^{-\frac{\alpha}{r}+\frac{\beta}{q}}\|\varphi\chi_{B(x_{k,i},T^{1/\theta})}\|_{\mathfrak{L}^{r,\infty}(\log\mathfrak{L})^\alpha}\\ &\quad\le C(1+T^{-\frac{1}{\theta}}|x_{k,i}|)^{-n-\theta}t^{-\frac{n}{\theta}(\frac{1}{r}-\frac{1}{q})}\left[\log\left(e+\frac{1}{t}\right)\right]^{-\frac{\alpha}{r}+\frac{\beta}{q}}|||\varphi|||_{r,\alpha;T^{1/\theta}} \qquad (3\text{-}22)\end{aligned}$$

for $t\in(0,T)$.

On the other hand, since

$$\tfrac{1}{2}|y| \le \tfrac{1}{2}(|x_{k,i}| + T^{\frac{1}{\theta}}) \le |x_{k,i}| \quad \text{for } y \in B_{k,i},$$

we have

$$\frac{1}{|B_{k,i}|}\int_{B_{k,i}} \bigl(1+\tfrac{1}{2}T^{-\frac{1}{\theta}}|y|\bigr)^{-n-\theta}\,dy \ge (1+T^{-\frac{1}{\theta}}|x_{k,i}|)^{-n-\theta}.$$

Then, by (3-17), we see that

$$\begin{aligned}\sum_{i=1}^{\infty}(1+T^{-\frac{1}{\theta}}|x_{k,i}|)^{-n-\theta} &\le CT^{-\frac{n}{\theta}}\sum_{i=1}^{\infty}\int_{B_{k,i}}\bigl(1+\tfrac{1}{2}T^{-\frac{1}{\theta}}|y|\bigr)^{-n-\theta}\,dy \\ &\le CT^{-\frac{n}{\theta}}\int_{\mathbb{R}^n}\bigl(1+\tfrac{1}{2}T^{-\frac{1}{\theta}}|y|\bigr)^{-n-\theta}\,dy \le C \qquad (3\text{-}23)\end{aligned}$$

for $T>0$. Combining (3-18), (3-19), (3-22), and (3-23), we obtain

$$\begin{aligned}&t^{\frac{n}{\theta}(\frac{1}{r}-\frac{1}{q})}\Bigl[\log\Bigl(e+\frac{1}{t}\Bigr)\Bigr]^{\frac{\alpha}{r}-\frac{\beta}{q}}\|\chi_{B(0,T^{1/\theta})}S_\theta(t)\varphi\|_{\mathfrak{L}^{q,\infty}(\log\mathfrak{L})^\beta} \\ &\qquad\le C\,|||\varphi|||_{r,\alpha;T^{1/\theta}} + C\,|||\varphi|||_{r,\alpha;T^{1/\theta}}\sum_{k=1}^{m}\sum_{i=1}^{\infty}(1+T^{-\frac{1}{\theta}}|x_{k,i}|)^{-n-\theta} \\ &\qquad\le C\,|||\varphi|||_{r,\alpha;T^{1/\theta}}\end{aligned}$$

for $t\in(0,T)$. This implies (3-16) with $z=0$; that is, (3-15) holds. Thus Proposition 3.4 follows in the case of $0<\theta<2$.

Consider the case of $\theta=2$; that is, $S_\theta(t)=e^{t\Delta}$. Let $\tau=t^{1/2}$. It follows from (2-16) that

$$\begin{aligned}|[e^{t\Delta}\varphi](x)| &\le C\int_{\mathbb{R}^n} t^{-\frac{n}{2}}(1+t^{-\frac{1}{2}}|x-y|)^{-n-1}|\varphi(y)|\,dy \\ &\le C[S_1(\tau)|\varphi|](x)\end{aligned}$$

for $(x,t)\in\mathbb{R}^n\times(0,\infty)$. This together with Proposition 3.4 in the case of $\theta=1$ implies that

$$\begin{aligned}|||e^{t\Delta}\varphi|||_{q,\beta;T^{1/2}} &\le C\,|||S_1(\tau)|\varphi|\,|||_{q,\beta;T^{1/2}} \\ &\le C\tau^{-n(\frac{1}{r}-\frac{1}{q})}\Bigl[\log\Bigl(e+\frac{1}{\tau}\Bigr)\Bigr]^{-\frac{\alpha}{r}+\frac{\beta}{q}}|||\varphi|||_{r,\alpha;T^{1/2}} \\ &\le Ct^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\Bigl[\log\Bigl(e+\frac{1}{t}\Bigr)\Bigr]^{-\frac{\alpha}{r}+\frac{\beta}{q}}|||\varphi|||_{r,\alpha;T^{1/2}} \quad \text{for } t\in(0,T).\end{aligned}$$

Thus Proposition 3.4 follows in the case of $\theta=2$. The proof of Proposition 3.4 is complete. □

4. Proof of Theorems 1.2 and 1.4

We apply the contraction mapping theorem to problem (P) in uniformly local weak Zygmund-type spaces and prove Theorems 1.2 and 1.4. We also prove Corollary 1.3. We first prove the following proposition.

Proposition 4.1. *Let $p = p_\theta$, $T_* \in (0,\infty)$, and $\gamma \in [0, n/\theta)$. Then there exists $\epsilon > 0$ such that if $\varphi \in \mathfrak{L}^{1,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^{n/\theta}$ satisfies*

$$|||\varphi|||_{1,\frac{n}{\theta};T^{1/\theta}} \le \epsilon \quad \textit{for some } T \in (0, T_*], \tag{4-1}$$

then problem (P) possesses a solution

$$u \in C((0,T) : \mathfrak{L}^{1,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^{n/\theta}) \cap L^\infty_{\mathrm{loc}}(0,T : L^\infty) \quad \textit{in } \mathbb{R}^n \times (0,T),$$

with u satisfying

$$\begin{aligned} |||u(t)|||_{1,\frac{n}{\theta};T^{1/\theta}} &\le C|||\varphi|||_{1,\frac{n}{\theta};T^{1/\theta}}, \\ |||u(t)|||_{p,\gamma;T^{1/\theta}} &\le Ct^{-\frac{n}{\theta}(1-\frac{1}{p})}\left[\log\left(e+\frac{1}{t}\right)\right]^{-\frac{n}{\theta}+\frac{\gamma}{p}} |||\varphi|||_{1,\frac{n}{\theta};T^{1/\theta}}, \\ \|u(t)\|_{L^\infty} &\le Ct^{-\frac{n}{\theta}}\left[\log\left(e+\frac{1}{t}\right)\right]^{-\frac{n}{\theta}} |||\varphi|||_{1,\frac{n}{\theta};T^{1/\theta}} \end{aligned} \tag{4-2}$$

for $t \in (0,T)$. Here C is a positive constant depending only on T_*, n, θ, and γ.

Throughout this section, we set

$$T_* \in (0,\infty), \quad T \in (0,T_*], \quad p := p_\theta = 1 + \frac{\theta}{n}, \quad \alpha := \frac{n}{\theta}, \quad 0 \le \gamma < \alpha, \quad \varphi \in \mathfrak{L}^{1,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^{\alpha}.$$

Let $\epsilon > 0$, and assume (4-1). By Proposition 3.4, we find $C_* > 0$ such that

$$\begin{aligned} \sup_{0<t<T} |||S_\theta(t)\varphi|||_{1,\alpha;T^{1/\theta}} &\le C_*|||\varphi|||_{1,\alpha;T^{1/\theta}} \le C_*\epsilon, \\ \sup_{0<t<T} t^{\frac{n}{\theta}(1-\frac{1}{p})}\left[\log\left(e+\frac{1}{t}\right)\right]^{-\frac{\gamma}{p}+\alpha} |||S_\theta(t)\varphi|||_{p,\gamma;T^{1/\theta}} &\le C_*|||\varphi|||_{1,\alpha;T^{1/\theta}} \le C_*\epsilon, \\ \sup_{0<t<T} t^{\frac{n}{\theta}}\left[\log\left(e+\frac{1}{t}\right)\right]^{\alpha} \|S_\theta(t)\varphi\|_{L^\infty} &\le C_*|||\varphi|||_{1,\alpha;T^{1/\theta}} \le C_*\epsilon. \end{aligned} \tag{4-3}$$

Define

$$X_T := C((0,T) : \mathfrak{L}^{1,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^{\alpha}) \cap L^\infty_{\mathrm{loc}}((0,T) : \mathfrak{L}^{p,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^{\gamma}) \cap L^\infty_{\mathrm{loc}}((0,T) : L^\infty).$$

Setting $C^* = 2C_*$, for any $u \in X_T$, we say that $u \in X_T(C^*\epsilon)$ if u satisfies

$$\sup_{0<t<T} |||u(t)|||_{1,\alpha;T^{1/\theta}} + \sup_{0<t<T} t^{\frac{n}{\theta}(1-\frac{1}{p})}\left[\log\left(e+\frac{1}{t}\right)\right]^{-\frac{\gamma}{p}+\alpha} |||u(t)|||_{p,\gamma;T^{1/\theta}} + \sup_{0<t<T} t^{\frac{n}{\theta}}\left[\log\left(e+\frac{1}{t}\right)\right]^{\alpha} \|u(t)\|_{L^\infty} \le C^*\epsilon. \tag{4-4}$$

For any $u, v \in X_T(C^*\epsilon)$, set

$$d_X(u,v) := d^1_X(u,v) + d^2_X(u,v) + d^3_X(u,v),$$

where

$$d^1_X(u,v) := \sup_{0<t<T} |||u(t) - v(t)|||_{1,\alpha;T^{1/\theta}},$$

$$d_X^2(u,v) := \sup_{0<t<T} t^{\frac{n}{\theta}(1-\frac{1}{p})}\left[\log\left(e+\frac{1}{t}\right)\right]^{-\frac{\gamma}{p}+\alpha} |||u(t)-v(t)|||_{p,\gamma;T^{1/\theta}},$$
$$d_X^3(u,v) := \sup_{0<t<T} t^{\frac{n}{\theta}}\left[\log\left(e+\frac{1}{t}\right)\right]^{\alpha} \|u(t)-v(t)\|_{L^\infty}.$$

Then (X_T, d_X) is a Banach space and $X_T(C^*\epsilon)$ is closed in (X_T, d_X). Define

$$\Phi(u) := S_\theta(t)\varphi + \int_0^t S_\theta(t-s)F_p(u(s))\,ds \quad \text{for } u \in X_T(C^*\epsilon),$$

where $F_p(s) = |s|^{p-1}s$ for $s \in \mathbb{R}$. For the proof of Proposition 4.1 we prepare the following two lemmas.

Lemma 4.2. *Let $\epsilon > 0$, and assume that (4-1) holds for some $T \in (0, T_*]$. Then there exists $C = C(n,\theta,C_*,T_*) > 0$ such that*

$$d_X^1(\Phi(u),\Phi(v)) + d_X^2(\Phi(u),\Phi(v)) \le C\epsilon^{p-1}\, d_X^2(u,v) \quad \text{for } u, v \in X_T(C^*\epsilon).$$

Proof. Let $u, v \in X_T(C^*\epsilon)$. Let $0 < s < t < T$. It follows that

$$|F_p(u(x,s)) - F_p(v(x,s))| \le w(x,s)|u(x,s)-v(x,s)| \quad \text{for } x \in \mathbb{R}^n, \tag{4-5}$$

where $w(x,s) := p(|u(x,s)|^{p-1} + |v(x,s)|^{p-1})$. Then, by Lemmas 2.2 and 2.3, we have

$$\begin{aligned} |||F_p(u(s)) - F_p(v(s))|||_{1,\gamma;T^{1/\theta}} &\le |||w(s)|||_{\frac{p}{(p-1)},\gamma;T^{1/\theta}} |||u(s)-v(s)|||_{p,\gamma;T^{1/\theta}} \\ &\le p(|||u(s)|||^{p-1}_{p,\gamma;T^{1/\theta}} + |||v(s)|||^{p-1}_{p,\gamma;T^{1/\theta}})|||u(s)-v(s)|||_{p,\gamma;T^{1/\theta}}. \end{aligned} \tag{4-6}$$

Since $u, v \in X_T(C^*\epsilon)$, by (4-4), we obtain

$$\begin{aligned} &|||F_p(u(s)) - F_p(v(s))|||_{1,\gamma;T^{1/\theta}} \\ &\le Cs^{-\frac{n(p-1)}{\theta}(1-\frac{1}{p})}\left[\log\left(e+\frac{1}{s}\right)\right]^{\frac{\gamma(p-1)}{p}-\alpha(p-1)} (C^*\epsilon)^{p-1} C s^{-\frac{n}{\theta}(1-\frac{1}{p})}\left[\log\left(e+\frac{1}{s}\right)\right]^{\frac{\gamma}{p}-\alpha} d_X^2(u,v) \\ &= C\epsilon^{p-1} s^{-\frac{n(p-1)}{\theta}}\left[\log\left(e+\frac{1}{s}\right)\right]^{\gamma-\alpha p} d_X^2(u,v). \end{aligned} \tag{4-7}$$

This together with Proposition 3.4 implies that

$$\begin{aligned} &\left|\left|\left|\int_0^t S_\theta(t-s)[F_p(u(s)) - F_p(v(s))]\,ds\right|\right|\right|_{q,\beta;T^{1/\theta}} \\ &\le \int_0^t |||S_\theta(t-s)[F_p(u(s)) - F_p(v(s))]|||_{q,\beta;T^{1/\theta}}\,ds \\ &\le C\int_0^t (t-s)^{-\frac{n}{\theta}(1-\frac{1}{q})}\left[\log\left(e+\frac{1}{t-s}\right)\right]^{-\gamma+\frac{\beta}{q}} |||F_p(u(s)) - F_p(v(s))|||_{1,\gamma;T^{1/\theta}}\,ds \\ &\le C\epsilon^{p-1} d_X^2(u,v)\int_0^t (t-s)^{-\frac{n}{\theta}(1-\frac{1}{q})}\left[\log\left(e+\frac{1}{t-s}\right)\right]^{-\gamma+\frac{\beta}{q}} s^{-\frac{n}{\theta}(p-1)}\left[\log\left(e+\frac{1}{s}\right)\right]^{\gamma-\alpha p} ds \end{aligned} \tag{4-8}$$

for $q \in [1, p]$ and $\beta \in [\gamma, \alpha]$.

On the other hand, since

$$\gamma-\alpha p=\gamma-\frac{n}{\theta}\Big(1+\frac{\theta}{n}\Big)=\gamma-\frac{n}{\theta}-1=\gamma-\alpha-1<-1, \tag{4-9}$$

by Lemma 3.2 (2) and (3-1), we have

$$\begin{aligned}
&\int_0^{\frac{t}{2}}(t-s)^{-\frac{n}{\theta}(1-\frac{1}{q})}\Big[\log\Big(e+\frac{1}{t-s}\Big)\Big]^{-\gamma+\frac{\beta}{q}}s^{-\frac{n}{\theta}(p-1)}\Big[\log\Big(e+\frac{1}{s}\Big)\Big]^{\gamma-\alpha p}\,ds\\
&\quad\le Ct^{-\frac{n}{\theta}(1-\frac{1}{q})}\Big[\log\Big(e+\frac{1}{t}\Big)\Big]^{-\gamma+\frac{\beta}{q}}\int_0^{\frac{t}{2}}s^{-1}\Big[\log\Big(e+\frac{1}{s}\Big)\Big]^{\gamma-\alpha p}\,ds\\
&\quad\le Ct^{-\frac{n}{\theta}(1-\frac{1}{q})}\Big[\log\Big(e+\frac{1}{t}\Big)\Big]^{-\gamma+\frac{\beta}{q}}\cdot C\Big[\log\Big(e+\frac{1}{t}\Big)\Big]^{\gamma-\frac{n}{\theta}}=Ct^{-\frac{n}{\theta}(1-\frac{1}{q})}\Big[\log\Big(e+\frac{1}{t}\Big)\Big]^{\frac{\beta}{q}-\alpha}
\end{aligned}\tag{4-10}$$

for $t\in(0,T)$. Similarly, since

$$-\frac{n}{\theta}\Big(1-\frac{1}{q}\Big)\ge-\frac{n(p-1)}{\theta p}=-\frac{1}{p}>-1,$$

by Lemma 3.2 (1) and (3-1), we obtain

$$\begin{aligned}
&\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{n}{\theta}(1-\frac{1}{q})}\Big[\log\Big(e+\frac{1}{t-s}\Big)\Big]^{-\gamma+\frac{\beta}{q}}s^{-\frac{n}{\theta}(p-1)}\Big[\log\Big(e+\frac{1}{s}\Big)\Big]^{\gamma-\alpha p}\,ds\\
&\qquad\le Ct^{-\frac{n}{\theta}(p-1)}\Big[\log\Big(e+\frac{1}{t}\Big)\Big]^{\gamma-\alpha p}\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{n}{\theta}(1-\frac{1}{q})}\Big[\log\Big(e+\frac{1}{t-s}\Big)\Big]^{-\gamma+\frac{\beta}{q}}\,ds\\
&\qquad\le Ct^{-1}\Big[\log\Big(e+\frac{1}{t}\Big)\Big]^{\gamma-\alpha p}\cdot Ct^{-\frac{n}{\theta}(1-\frac{1}{q})+1}\Big[\log\Big(e+\frac{1}{t}\Big)\Big]^{-\gamma+\frac{\beta}{q}}\\
&\qquad=Ct^{-\frac{n}{\theta}(1-\frac{1}{q})}\Big[\log\Big(e+\frac{1}{t}\Big)\Big]^{\frac{\beta}{q}-\alpha p}\le Ct^{-\frac{n}{\theta}(1-\frac{1}{q})}\Big[\log\Big(e+\frac{1}{t}\Big)\Big]^{\frac{\beta}{q}-\alpha}
\end{aligned}\tag{4-11}$$

for $t\in(0,T)$. Combining (4-8), (4-10), and (4-11) with $(q,\beta)=(1,\alpha)$ and (p,γ), we deduce that

$$\begin{aligned}
&d_X^1(\Phi(u),\Phi(v))+d_X^2(\Phi(u),\Phi(v))\\
&\quad=\sup_{0<t<T}\Big|\Big|\Big|\int_0^t S_\theta(t-s)[F_p(u(s))-F_p(v(s))]\,ds\Big|\Big|\Big|_{1,\alpha;T^{1/\theta}}\\
&\qquad\qquad+\sup_{0<t<T}t^{\frac{n}{\theta}(1-\frac{1}{p})}\Big[\log\Big(e+\frac{1}{t}\Big)\Big]^{-\frac{\gamma}{p}+\alpha}\Big|\Big|\Big|\int_0^t S_\theta(t-s)[F_p(u(s))-F_p(v(s))]\,ds\Big|\Big|\Big|_{p,\gamma;T^{1/\theta}}\\
&\quad\le C\epsilon^{p-1}d_X^2(u,v)
\end{aligned}$$

for $u,v\in X_T(C^*\epsilon)$. Thus Lemma 4.2 follows. □

Lemma 4.3. *Let $\epsilon>0$, and assume that (4-1) holds for some $T\in(0,T_*]$. Then there exists $C=C(n,\theta,C_*,T_*)>0$ such that*

$$d_X^3(\Phi(u),\Phi(v))\le C\epsilon^{p-1}(d_X^2(u,v)+d_X^3(u,v))\quad\textit{for } u,v\in X_T(C^*\epsilon).$$

Proof. Let $u, v \in X_T(C^*\epsilon)$. Let $0 < s < t < T$. Similarly to (4-6), we have

$$\|F_p(u(s)) - F_p(v(s))\|_{L^\infty} \le \|w(s)\|_{L^\infty}\|u(s) - v(s)\|_{L^\infty} \le p(\|u(s)\|_{L^\infty}^{p-1} + \|v(s)\|_{L^\infty}^{p-1})\|u(s) - v(s)\|_{L^\infty}.$$

Since $u, v \in X_T(C^*\epsilon)$, by (4-4), we obtain

$$\|F_p(u(s))-F_p(v(s))\|_{L^\infty} \le C s^{-\frac{n(p-1)}{\theta}}\left[\log\left(e+\frac{1}{s}\right)\right]^{-\alpha(p-1)}(C^*\epsilon)^{p-1}\cdot s^{-\frac{n}{\theta}}\left[\log\left(e+\frac{1}{s}\right)\right]^{-\alpha} d_X^3(u,v) = C\epsilon^{p-1}s^{-\frac{np}{\theta}}\left[\log\left(e+\frac{1}{s}\right)\right]^{-\alpha p} d_X^3(u,v).$$

This together with Proposition 3.4 and (4-7) implies that

$$\begin{aligned}
&\left\|\int_0^t S_\theta(t-s)[F_p(u(s)) - F_p(v(s))]\,ds\right\|_{L^\infty} \\
&\le \int_0^t \|S_\theta(t-s)[F_p(u(s)) - F_p(v(s))]\|_{L^\infty}\,ds \\
&\le C\int_0^{\frac{t}{2}} (t-s)^{-\frac{n}{\theta}}\left[\log\left(e+\frac{1}{t-s}\right)\right]^{-\gamma} |||F_p(u(s)) - F_p(v(s))|||_{1,\gamma;T^{1/\theta}}\,ds \\
&\qquad + C\int_{\frac{t}{2}}^t \|F_p(u(s)) - F_p(v(s))\|_{L^\infty}\,ds \\
&\le C\epsilon^{p-1}d_X^2(u,v)\int_0^{\frac{t}{2}} (t-s)^{-\frac{n}{\theta}}\left[\log\left(e+\frac{1}{t-s}\right)\right]^{-\gamma} s^{-\frac{n}{\theta}(p-1)}\left[\log\left(e+\frac{1}{s}\right)\right]^{\gamma-\alpha p} ds \\
&\qquad + C\epsilon^{p-1}d_X^3(u,v)\int_{\frac{t}{2}}^t s^{-\frac{np}{\theta}}\left[\log\left(e+\frac{1}{s}\right)\right]^{-\alpha p} ds \\
&\le C\epsilon^{p-1}t^{-\frac{n}{\theta}}\left[\log\left(e+\frac{1}{t}\right)\right]^{-\gamma} d_X^2(u,v)\int_0^{\frac{t}{2}} s^{-1}\left[\log\left(e+\frac{1}{s}\right)\right]^{\gamma-\alpha p} ds \\
&\qquad + C\epsilon^{p-1}t^{-\frac{np}{\theta}+1}\left[\log\left(e+\frac{1}{t}\right)\right]^{-\alpha p} d_X^3(u,v).
\end{aligned}$$

Since $np = n + \theta$ and $\alpha p > \alpha$, we have

$$\begin{aligned}
&\left\|\int_0^t S_\theta(t-s)[F_p(u(s)) - F_p(v(s))]\,ds\right\|_{L^\infty} \\
&\le C\epsilon^{p-1}t^{-\frac{n}{\theta}}\left[\log\left(e+\frac{1}{t}\right)\right]^{-\gamma} d_X^2(u,v)\int_0^{\frac{t}{2}} s^{-1}\left[\log\left(e+\frac{1}{s}\right)\right]^{\gamma-\alpha p} ds \\
&\qquad + C\epsilon^{p-1}t^{-\frac{n}{\theta}}\left[\log\left(e+\frac{1}{t}\right)\right]^{-\alpha} d_X^3(u,v).
\end{aligned} \tag{4-12}$$

Furthermore, by Lemma 3.2 (2) and (4-9) we see that

$$\int_0^{\frac{t}{2}} s^{-1}\left[\log\left(e+\frac{1}{s}\right)\right]^{\gamma-\alpha p} ds \le C\left[\log\left(e+\frac{1}{t}\right)\right]^{\gamma-\alpha} \tag{4-13}$$

for $t \in (0, T)$. Combining (4-12) and (4-13), we deduce that

$$\begin{aligned} d_X^3(\Phi(u), \Phi(v)) &= \sup_{0<t<T} t^{\frac{n}{\theta}} \left[\log\left(e+\frac{1}{t}\right)\right]^{\alpha} \left\| \int_0^t S_\theta(t-s)[F_p(u(s)) - F_p(v(s))]\, ds \right\|_{L^\infty} \\ &\le C\epsilon^{p-1}(d_X^2(u,v) + d_X^3(u,v)) \end{aligned}$$

for $u, v \in X_T(C^*\epsilon)$. Thus Lemma 4.3 follows. □

Proof of Proposition 4.1. Let $T_* > 0$. Let $\epsilon > 0$ be small enough. Let $\varphi \in \mathfrak{L}^{1,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^\alpha$ be such that $|||\varphi|||_{1,\alpha;T^{1/\theta}} < \epsilon$ for some $T \in (0, T_*]$. By (4-3), (4-4), and Lemma 4.2, we have

$$\begin{aligned} &\sup_{t\in(0,T)} |||\Phi(u(t))|||_{1,\alpha;T^{1/\theta}} + \sup_{0<t<T} \left\{ t^{\frac{n}{\theta}(1-\frac{1}{p})} \left[\log\left(e+\frac{1}{t}\right)\right]^{-\frac{\gamma}{p}+\alpha} |||\Phi(u(t))|||_{p,\gamma;T^{1/\theta}} \right\} \\ &\le |||S_\theta(t)\varphi|||_{1,\alpha,T^{1/\theta}} + \sup_{0<t<T} \left\{ t^{\frac{n}{\theta}(1-\frac{1}{p})} \left[\log\left(e+\frac{1}{t}\right)\right]^{-\frac{\gamma}{p}+\alpha} |||S_\theta(t)\varphi|||_{p,\gamma;T^{1/\theta}} \right\} \\ &\qquad\qquad + d_X^1(\Phi(u), \Phi(0)) + d_X^2(\Phi(u), \Phi(0)) \\ &\le C_*\epsilon + C\epsilon^{p-1} d_X^2(u, 0) \\ &\le C_*\epsilon + C\epsilon^{p-1}\cdot C^*\epsilon \\ &\le C^*\epsilon \end{aligned} \tag{4-14}$$

for $u \in X_T(C^*\epsilon)$. Similarly, we observe from Lemma 4.3, (4-3), and (4-4) that

$$\begin{aligned} \sup_{0<t<T} \left\{ t^{\frac{n}{\theta}} \left[\log\left(e+\frac{1}{t}\right)\right]^{\alpha} \|\Phi(u(t))\|_{L^\infty} \right\} &\le \sup_{0<t<T} \left\{ t^{\frac{n}{\theta}} \left[\log\left(e+\frac{1}{t}\right)\right]^{\alpha} \|S_\theta(t)\varphi\|_{L^\infty} \right\} + d_X^3(\Phi(u), \Phi(0)) \\ &\le C_*\epsilon + C\epsilon^{p-1}(d_X^2(u,0) + d_X^3(u,0)) \\ &\le C_*\epsilon + C\epsilon^{p-1}\cdot 2C^*\epsilon \\ &\le C^*\epsilon \end{aligned} \tag{4-15}$$

for $u \in X_T(C^*\epsilon)$. By (4-14) and (4-15), we see that $\Phi(u) \in X_T(C^*\epsilon)$ for $u \in X_T(C^*\epsilon)$. Furthermore, taking small enough $\epsilon > 0$ if necessary, by Lemmas 4.2 and 4.3, we have

$$\begin{aligned} d_X(\Phi(u), \Phi(v)) &= d_X^1(\Phi(u), \Phi(v)) + d_X^2(\Phi(u), \Phi(v)) + d_X^3(\Phi(u), \Phi(v)) \\ &\le C\epsilon^{p-1}(d_X^2(u,v) + d_X^3(u,v)) \\ &\le \tfrac{1}{2} d_X(u,v) \end{aligned}$$

for $u, v \in X_T(C^*\epsilon)$. Then we apply the contraction mapping theorem to find a unique $u_* \in X_T(C^*\epsilon)$ such that $\Phi(u_*) = u_*$ in $X_T(C^*\epsilon)$. The function u_* is a solution to problem (P) in $\mathbb{R}^n \times (0, T)$, with u_* satisfying (4-2). Thus Proposition 4.1 follows. □

Proof of Theorem 1.2. Let $T > 0$. Let $\varphi \in \mathfrak{L}^{1,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^\alpha$ be such that $|||\varphi|||_{1,\alpha;T^{1/\theta}}$ is small enough. Then, by Proposition 4.1, we find a solution u to problem (P) in $\mathbb{R}^n \times (0, T)$, with u satisfying (4-2). Let

$\beta \in (\gamma, n/\theta)$. Then, by Proposition 3.4, Lemma 2.3, and (4-2), we obtain

$$\begin{aligned}
|||u(t)-S_\theta(t)\varphi|||_{1,\beta;T^{1/\theta}} &\le \int_0^t |||S_\theta(t-s)F_p(u(s))|||_{1,\beta;T^{1/\theta}}\,ds \\
&\le C\int_0^t \left[\log\left(e+\frac{1}{t-s}\right)\right]^{-\gamma+\beta} |||F_p(u(s))|||_{1,\gamma;T^{1/\theta}}\,ds \\
&= C\int_0^t \left[\log\left(e+\frac{1}{t-s}\right)\right]^{-\gamma+\beta} |||u(s)|||^p_{p,\gamma;T^{1/\theta}}\,ds \\
&\le C\,|||\varphi|||^p_{1,\alpha;T^{1/\theta}} \int_0^t \left[\log\left(e+\frac{1}{t-s}\right)\right]^{-\gamma+\beta} s^{-1}\left[\log\left(e+\frac{1}{s}\right)\right]^{\gamma-\alpha p} ds \qquad (4\text{-}16)
\end{aligned}$$

for $t \in (0, T)$. On the other hand, since $\beta < \theta/n$, by Lemma 3.2 (2) and (4-9), we have

$$\begin{aligned}
&\int_0^{\frac{t}{2}} \left[\log\left(e+\frac{1}{t-s}\right)\right]^{-\gamma+\beta} s^{-1}\left[\log\left(e+\frac{1}{s}\right)\right]^{\gamma-\alpha p} ds \\
&\qquad\le C\left[\log\left(e+\frac{1}{t}\right)\right]^{-\gamma+\beta} \int_0^{\frac{t}{2}} s^{-1}\left[\log\left(e+\frac{1}{s}\right)\right]^{\gamma-\alpha p} ds \\
&\qquad\le C\left[\log\left(e+\frac{1}{t}\right)\right]^{-\gamma+\beta} \cdot C\left[\log\left(e+\frac{1}{t}\right)\right]^{\gamma-\frac{n}{\theta}} \to 0 \qquad (4\text{-}17)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\frac{t}{2}}^{t} \left[\log\left(e+\frac{1}{t-s}\right)\right]^{-\gamma+\beta} s^{-1}\left[\log\left(e+\frac{1}{s}\right)\right]^{\gamma-\alpha p} ds \\
&\qquad\le Ct^{-1}\left[\log\left(e+\frac{1}{t}\right)\right]^{\gamma-\frac{n}{\theta}-1} \int_{\frac{t}{2}}^{t} \left[\log\left(e+\frac{1}{t-s}\right)\right]^{-\gamma+\beta} ds \\
&\qquad\le Ct^{-1}\left[\log\left(e+\frac{1}{t}\right)\right]^{\gamma-\frac{n}{\theta}-1} \cdot Ct\left[\log\left(e+\frac{1}{t}\right)\right]^{-\gamma+\beta} \to 0 \qquad (4\text{-}18)
\end{aligned}$$

as $t \to +0$. Combining (4-16), (4-17), and (4-18), we see that

$$\lim_{t\to+0} |||u(t)-S_\theta(t)\varphi|||_{1,\beta;T^{1/\theta}} = 0 \quad \text{for } \beta \in (\gamma, n/\theta).$$

This together with (2-12) implies that

$$\lim_{t\to+0} |||u(t)-S_\theta(t)\varphi|||_{1,\beta;T^{1/\theta}} = 0 \quad \text{for } \beta \in [0, n/\theta). \qquad (4\text{-}19)$$

It remains to prove that $u \to \varphi$ in the sense of distributions. Let $\eta \in C_0(\mathbb{R}^n)$. Let $R > 0$ be such that $\operatorname{supp}\eta \subset B(0, R)$. By (1-3), (1-4), and (4-19), we have

$$\begin{aligned}
\left|\int_{\mathbb{R}^n} (u(x,t)-[S_\theta(t)\varphi](x))\eta(x)\,dx\right| &\le C\,\|\eta\|_{L^\infty} \int_{B(0,R)} |u(x,t)-[S_\theta(t)\varphi](x)|\,dx \\
&\le C\,\|\eta\|_{L^\infty} |||u(t)-S_\theta(t)\varphi|||_{1,0;T^{1/\theta}} \to 0 \qquad (4\text{-}20)
\end{aligned}$$

as $t \to +0$. Set

$$\eta(x,t) := \int_{\mathbb{R}^n} G_\theta(x-y,t)\eta(y)\,dy \quad \text{for } (x,t) \in \mathbb{R}^n \times (0,\infty).$$

It follows from (2-18) that

$$\lim_{t\to+0} \|\eta(\cdot,t)-\eta\|_{L^\infty} = 0. \tag{4-21}$$

On the other hand, by (2-16), we have

$$|\eta(x,t)| \le Ct^{-\frac{n}{\theta}} \int_{B(0,R)} (1+t^{-\frac{1}{\theta}}|x-y|)^{-n-\theta}|\eta(y)|\,dy \le C\|\eta\|_{L^\infty} t^{-\frac{n}{\theta}} \cdot C(t^{-\frac{1}{\theta}}|x|)^{-n-\theta} \le T|x|^{-n-\theta}$$

for $x \in \mathbb{R}^n \setminus B(0,2R)$ and $t \in (0,T)$. Since $\|\eta(\cdot,t)\|_{L^\infty} \le \|\eta\|_{L^\infty}$ for $t > 0$, we obtain

$$|\eta(x,t)| \le C(1+|x|)^{-n-\theta} \quad \text{for } (x,t) \in \mathbb{R}^n \times (0,T). \tag{4-22}$$

Furthermore, it follows from Proposition 3.4 with $q = \infty$ that

$$[S_\theta(1)|\varphi|](0) = \int_{\mathbb{R}^n} G_\theta(y,1)|\varphi(y)|\,dy < \infty.$$

This together with (2-16) implies that

$$\int_{\mathbb{R}^n} (1+|y|)^{-n-\theta}|\varphi(y)|\,dy < \infty. \tag{4-23}$$

Therefore, by (4-21), (4-22), and (4-23), we apply the Fubini theorem and the Lebesgue convergence theorem to obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} [S_\theta(t)\varphi](x)\eta(x)\,dx \\
&\qquad = \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} G_\theta(x-y,t)\varphi(y)\,dy\right)\eta(x)\,dx = \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} G_\theta(x-y,t)\eta(x)\,dx\right)\varphi(y)\,dy \\
&\qquad = \int_{\mathbb{R}^n} \eta(y,t)\varphi(y)\,dy \to \int_{\mathbb{R}^n} \eta(y)\varphi(y)\,dy
\end{aligned}$$

as $t \to +0$. Then we deduce from (4-20) that

$$\lim_{t\to+0} \int_{\mathbb{R}^n} u(x,t)\eta(x)\,dx = \int_{\mathbb{R}^n} \varphi(x)\eta(x)\,dx \quad \text{for } \eta \in C_0(\mathbb{R}^n);$$

that is, $u(t) \to \varphi$ in the sense of distributions. The proof of Theorem 1.2 is complete. □

Proof of Corollary 1.3. Let φ_c be as in (1-1) with $p = p_\theta$. It follows from the definition of the nonincreasing rearrangements that

$$(\varphi_c)^*(s) \le Cs^{-1}\left[\log\left(e+\frac{1}{s}\right)\right]^{-\frac{n}{\theta}-1} \quad \text{for } s \in (0,\infty). \tag{4-24}$$

Let $S > 0$. Then, by Lemma 3.2 (2), (2-3), and (4-24), we see that

$$(\varphi_c)^{**}(s) \le Cs^{-1}\left[\log\left(e+\frac{1}{s}\right)\right]^{-\frac{n}{\theta}} \quad \text{for } s \in (0,S).$$

This implies that $\varphi_c \in \mathfrak{L}^{1,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^{n/\theta}$. Then Corollary 1.3 follows from Theorem 1.2. □

Proof of Theorem 1.4. Since $\alpha > n/\theta$, it follows from (2-12) that

$$|||\varphi|||_{1,\frac{n}{\theta};T^{1/\theta}} \le C\left[\log\left(e+\frac{1}{T^{1/\theta}}\right)\right]^{\frac{n}{\theta}-\alpha} |||\varphi|||_{1,\alpha;T^{1/\theta}} \to 0 \quad \text{as } T \to +0.$$

Then, by Theorem 1.2, we find a solution u to problem (P) in $\mathbb{R}^n \times (0,T)$ for some small enough $T > 0$, with u satisfying (1-6) and (1-7). Thus Theorem 1.4 follows. □

At the end of this paper we recall the definitions of the usual Zygmund space and the usual weak Zygmund space, and explain the advantage of our weak Zygmund-type spaces.

Remark 4.4. (i) We recall the Zygmund space $L^q(\log L)^\alpha$ and the weak Zygmund space $L^{q,\infty}(\log L)^\alpha$. For any $q \in [1,\infty]$ and $\alpha \ge 0$, set

$$L^q(\log L)^\alpha := \{f \in L^1_{\text{loc}}(\mathbb{R}^n) : \|f\|_{L^q(\log L)^\alpha} < \infty\},$$
$$L^{q,\infty}(\log L)^\alpha := \{f \in L^1_{\text{loc}}(\mathbb{R}^n) : \|f\|_{L^{q,\infty}(\log L)^\alpha} < \infty\},$$

where

$$\|f\|_{L^q(\log L)^\alpha} := \left(\int_0^\infty \left[\log\left(e+\frac{1}{s}\right)\right]^\alpha f^*(s)^q\,ds\right)^{\frac{1}{q}}, \tag{4-25}$$

$$\|f\|_{L^{q,\infty}(\log L)^\alpha} := \sup_{s>0}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^\alpha s f^*(s)^q\right\}^{\frac{1}{q}}. \tag{4-26}$$

See, e.g., [Bennett and Sharpley 1988, Chapter 4, Section 6] and [Wadade 2014]. For the case $q > 1$, as in the Lorentz space (see, e.g., [Grafakos 2008, Chapter 1, Exercises 1.4.3]), applying Hardy's inequality (see Lemma 2.4) and Lemma 3.2 with (2-3), for any $f \in L^1_{\text{loc}}$, we see that $f \in L^q(\log L)^\alpha$ if and only if

$$[f]_{L^q(\log L)^\alpha} := \left(\int_0^\infty \left[\log\left(e+\frac{1}{s}\right)\right]^\alpha f^{**}(s)^q\,ds\right)^{\frac{1}{q}} < \infty.$$

In contrast, the above relation does not hold for the case $q = 1$. In fact, applying integration by parts, we see that

$$\int_0^\infty \left[\log\left(e+\frac{1}{s}\right)\right]^\alpha f^*(s)\,ds = \alpha\int_0^\infty \left[\log\left(e+\frac{1}{s}\right)\right]^{\alpha-1} f^{**}(s)\frac{ds}{es+1} + \|f\|_{L^1}.$$

(ii) By O'Neil's inequality (2-7), we have the inequality

$$(G_\theta(\cdot,t) * \varphi)^{**}(s) \le \int_s^\infty (G_\theta(\cdot,t))^{**}(\tau)\varphi^{**}(\tau)\,d\tau, \quad s > 0,$$

which is crucial in the proof of our sharp decay estimates of $S_\theta(t)\varphi$. Our Zygmund-type spaces are defined by the average of the nonincreasing rearrangement, and they are effectively used in the proof of our sharp decay estimates of $S_\theta(t)\varphi$ (see the proof of Proposition 3.1). These sharp decay estimates of $S_\theta(t)\varphi$ in the spaces $\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha$ enable us to obtain Theorem 1.2.

On the other hand, since the weak Zygmund space $L^{q,\infty}(\log L)^\alpha$ is defined by the nonincreasing rearrangement, the inequality

$$(G_\theta(\cdot,t)*\varphi)^*(s) \le (G_\theta(\cdot,t)*\varphi)^{**}(s) \le \int_s^\infty (G_\theta(\cdot,t))^{**}(\tau)\varphi^{**}(\tau)\,d\tau, \quad s>0, \tag{4-27}$$

seems useful for the study of decay estimates of $S_\theta(t)\varphi$ in the space $L^{q,\infty}(\log L)^\alpha$. The first inequality in (4-27) follows from inequality (2-4). However, in general, inequality (2-4) is not sharp in $L^{1,\infty}(\log L)^\alpha$, where $\alpha>1$. Indeed, let $f\in L^1_{\mathrm{loc}}$ be such that

$$f^*(s) = s^{-1}\left[\log\left(e+\frac{1}{s}\right)\right]^{-\alpha}, \quad s>0,$$

where $\alpha>1$. Then $f\in L^{1,\infty}(\log L)^\alpha$ and

$$f^{**}(s) \asymp s^{-1}\left[\log\left(e+\frac{1}{s}\right)\right]^{-\alpha+1}$$

for small enough $s>0$. Then $f^*(s)/f^{**}(s)\to 0$ as $s\to +0$, and we see that inequality (2-4) is not sharp. This suggests that it is difficult to obtain sharp decay estimates of $S_\theta(t)\varphi$ in the usual weak Zygmund spaces.

(iii) In order to overcome the disadvantage of the usual weak Zygmund spaces, one might consider the weak Zygmund-type spaces

$$\mathbb{L}^{q,\infty}(\log\mathbb{L})^\alpha := \{f\in L^1_{\mathrm{loc}} : \|f\|_{\mathbb{L}^{q,\infty}(\log\mathbb{L})^\alpha} < \infty\},$$

where $1\le q<\infty$, $\alpha\ge 0$, and

$$\|f\|_{\mathbb{L}^{q,\infty}(\log\mathbb{L})^\alpha} := \sup_{s>0}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^\alpha s f^{**}(s)^q\right\}^{\frac{1}{q}}.$$

Indeed, applying the arguments to those in the proof of Proposition 3.4, we can obtain similar sharp decay estimates of $S_\theta(t)\varphi$ in the weak Zygmund-type space $\mathbb{L}^{q,\infty}(\log\mathbb{L})^\alpha$ to those in Proposition 3.4.

On the other hand, in the proof of Theorem 1.2, we used the inequality

$$\||f|^p\|_{\mathfrak{L}^{1,\infty}(\log\mathfrak{L})^\alpha} \le C\|f\|^p_{\mathfrak{L}^{p,\infty}(\log\mathfrak{L})^\alpha} \quad \text{for } f\in\mathfrak{L}^{p,\infty}(\log\mathfrak{L})^\alpha \tag{4-28}$$

in order to estimate the nonlinear term $|u|^{p-1}u$, where $p>1$ and $\alpha\ge 0$. Actually, (4-28) holds with $C=1$ and "$\le$" replace by "$=$" (see Lemma 2.3). In the case of $\mathbb{L}^{q,\infty}(\log\mathbb{L})^\alpha$, it follows from (2-5) that

$$\begin{aligned}\||f|^p\|_{\mathbb{L}^{1,\infty}(\log\mathbb{L})^\alpha} &= \sup_{s>0}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^\alpha s(|f|^p)^{**}(s)\right\}\\ &\ge \sup_{s>0}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^\alpha s(f^{**}(s))^p\right\} = \|f\|^p_{\mathbb{L}^{p,\infty}(\log\mathbb{L})^\alpha}\end{aligned}$$

for $f\in\mathbb{L}^{p,\infty}(\log\mathbb{L})^\alpha$; that is, the reverse to the desired inequality holds. This suggests that it is difficult to obtain a similar result to that of Theorem 1.2 in the framework of weak Zygmund-type spaces $\mathbb{L}^{q,\infty}(\log\mathbb{L})^\alpha$.

Appendix

Here we prove two propositions on relations between $L^q(\log L)^\alpha$, $L^{q,\infty}(\log L)^\alpha$, and $\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha$. We remark that the following relations hold for $\alpha = 0$:

$$L^q = L^q(\log L)^0 = \mathfrak{L}^{q,\infty}(\log \mathfrak{L})^0 \subsetneq L^{q,\infty} = L^{q,\infty}(\log L)^0 \quad \text{if } q \in [1,\infty).$$

Proposition A.1. *Let* $1 \le q < \infty$ *and* $\alpha \ge 0$. *Then*

$$\|f\|_{\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha} \le \|f\|_{L^q(\log L)^\alpha} \qquad \text{for } f \in L^q(\log L)^\alpha,$$
$$\|f\|_{L^{q,\infty}(\log L)^\alpha} \le \|f\|_{\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha} \quad \text{for } f \in \mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha.$$

Furthermore,

$$L^q(\log L)^\alpha \subsetneq \mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha \subsetneq L^{q,\infty}(\log L)^\alpha, \quad \alpha > 0.$$

Proof. By (2-9), (2-11), and (4-25), we see that

$$\begin{aligned}\|f\|_{\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha} &= \sup_{s>0}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^\alpha \int_0^s (f^*(\tau))^q\,d\tau\right\}^{\frac{1}{q}}\\ &\le \sup_{s>0}\left(\int_0^s \left[\log\left(e+\frac{1}{\tau}\right)\right]^\alpha (f^*(\tau))^q\,d\tau\right)^{\frac{1}{q}} = \|f\|_{L^q(\log L)^\alpha}\end{aligned}$$

for $f \in L^q(\log L)^\alpha$. This implies that $L^q(\log L)^\alpha \subset \mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha$. Let g be a function in $\mathbb{R}^n$ such that

$$g^*(s) = \frac{d}{ds}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^{-\alpha}\right\}\chi_{(0,\delta)}(s) = \frac{\alpha}{es^2+s}\left[\log\left(e+\frac{1}{s}\right)\right]^{-\alpha-1}\chi_{(0,\delta)}(s),$$

where $\delta > 0$ is chosen so that g^* is decreasing. Set $f(x) := |g(x)|^{1/q}$. It follows from (2-1) that $f^*(s)^q = g^*(s)$. Furthermore,

$$\|f\|^q_{\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha} = \sup_{s>0}\left[\log\left(e+\frac{1}{s}\right)\right]^\alpha \int_0^s g^*(\eta)\,d\eta = 1$$

and

$$\|f\|^q_{L^q(\log L)^\alpha} = \int_0^\infty \left[\log\left(e+\frac{1}{\eta}\right)\right]^\alpha g^*(\eta)\,d\eta = \int_0^\delta \frac{\alpha}{e\eta^2+\eta}\left[\log\left(e+\frac{1}{\eta}\right)\right]^{-1} d\eta = \infty.$$

Thus $L^q(\log L)^\alpha \subsetneq \mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha$.

On the other hand, it follows from (2-1), (2-4), (2-9), and (4-26) that

$$\begin{aligned}\|f\|_{\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha} &= \sup_{s>0}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^\alpha s(|f|^q)^{**}(s)\right\}^{\frac{1}{q}}\\ &\ge \sup_{s>0}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^\alpha s(|f|^q)^{*}(s)\right\}^{\frac{1}{q}}\\ &= \sup_{s>0}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^\alpha s(f^*(s))^q\right\}^{\frac{1}{q}} = \|f\|_{L^{q,\infty}(\log L)^\alpha}\end{aligned}$$

for $f \in \mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha$, and hence $\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha \subset L^{q,\infty}(\log L)^\alpha$. We finally show that the inclusion is strict. Let f be a function such that

$$f^*(s) = s^{-\frac{1}{q}}\left[\log\left(e+\frac{1}{s}\right)\right]^{-\frac{\alpha}{q}} \chi_{(0,\delta)}(s),$$

where $\delta > 0$ is chosen so that f^* is decreasing. Then $\|f\|_{L^{q,\infty}(\log L)^\alpha} = 1$. On the other hand, for the case $\alpha \le 1$, we see that

$$s(|f|^q)^{**}(s) = \int_0^s \eta^{-1}\left[\log\left(e+\frac{1}{\eta}\right)\right]^{-\alpha} d\eta \ge \int_0^s \eta^{-1}\left[\log\left(e+\frac{1}{\eta}\right)\right]^{-1} d\eta = \infty$$

for $s \in (0,\delta)$. This implies that $f \notin \mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha$. Furthermore, for the case $\alpha > 1$, there exists $C > 0$ such that

$$\begin{aligned} s(|f|^q)^{**}(s) &= \int_0^s \eta^{-1}\left[\log\left(e+\frac{1}{\eta}\right)\right]^{-\alpha} d\eta \\ &\ge C\int_0^s (e\eta^2+\eta)^{-1}\left[\log\left(e+\frac{1}{\eta}\right)\right]^{-\alpha} d\eta = \frac{C}{\alpha-1}\left[\log\left(e+\frac{1}{s}\right)\right]^{1-\alpha} \end{aligned}$$

for $s \in (0,\delta)$. In conclusion, there exists $C > 0$ such that

$$\|f\|_{\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha} \ge C \sup_{0<s<\delta}\left[\log\left(e+\frac{1}{s}\right)\right]^{\frac{1}{q}} = \infty.$$

Thus $\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha \subsetneq L^{q,\infty}(\log L)^\alpha$. The proof of Proposition A.1 is complete. □

Let f be a locally integrable function in $\mathbb{R}^n$ such that

$$\left[\log\left(e+\frac{1}{s}\right)\right]^{\alpha} s(|f|^q)^{**}(s) = 1, \quad s > 0, \tag{A-1}$$

which is a typical function in $\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha$. By (A-1), we see

$$f^*(s)^q = \frac{d}{ds}(s(|f|^q)^{**}(s)) = \frac{\alpha}{es^2+s}\left[\log\left(e+\frac{1}{s}\right)\right]^{-\alpha-1}, \quad s > 0.$$

Since

$$f^*(s) \asymp s^{-\frac{1}{q}}\left[\log\left(e+\frac{1}{s}\right)\right]^{-\frac{\alpha+1}{q}} \quad \text{for small enough } s > 0,$$

we see that f also has a typical singularity of functions in $L^{q,\infty}(\log L)^{\alpha+1}$. These arguments suggest that $\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha$ is closely related to $L^{q,\infty}(\log L)^{\alpha+1}$.

Proposition A.2. *Let $1 \le q < \infty$ and $\alpha > 0$. Then there exists $C > 0$ such that*

$$\|f\|_{\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha} \le C\|f\|_{L^{q,\infty}(\log L)^{\alpha+1}} \tag{A-2}$$

for $f \in L^{q,\infty}(\log L)^{\alpha+1}$. Furthermore,

$$\inf\left\{\frac{\|f\|_{\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha}}{\|f\|_{L^{q,\infty}(\log L)^{\alpha+1}}} : f \in L^{q,\infty}(\log L)^{\alpha+1}\right\} = 0. \tag{A-3}$$

Proof. By Lemma 2.3 and (2-1), it suffices to consider the case $q = 1$. We first prove (A-2) with $q = 1$. Let $f \in L^{1,\infty}(\log L)^{\alpha+1}$, where $\alpha > 0$. By Lemma 3.2 (2), for any $R > 0$, we have

$$\begin{aligned} sf^{**}(s) = \int_0^s f^*(\eta)\,d\eta &\le \left(\int_0^s \eta^{-1}\left[\log\left(e+\frac{1}{\eta}\right)\right]^{-\alpha-1} d\eta\right)\left(\sup_{\eta>0}\left[\log\left(e+\frac{1}{\eta}\right)\right]^{\alpha+1} \eta f^*(\eta)\right) \\ &\le C\left[\log\left(e+\frac{1}{s}\right)\right]^{-\alpha}\left(\sup_{\eta>0}\left[\log\left(e+\frac{1}{\eta}\right)\right]^{\alpha+1} \eta f^*(\eta)\right) \end{aligned}$$

for $s \in (0, R)$. This together with (2-9) implies that

$$\begin{aligned} \|f\|_{\mathfrak{L}^{1,\infty}(\log \mathfrak{L})^\alpha} &= \sup_{0<s<R}\left[\log\left(e+\frac{1}{s}\right)\right]^{\alpha} sf^{**}(s) \\ &\le C\sup_{\eta>0}\left[\log\left(e+\frac{1}{\eta}\right)\right]^{\alpha+1} \eta f^*(\eta) = C\|f\|_{L^{1,\infty}(\log L)^{\alpha+1}}. \end{aligned}$$

Thus (A-2) holds for $q = 1$, and the proof of (A-2) is complete.

Next, we prove (A-3) with $q = 1$. Let $\{f_n\}$ be a sequence in L^1_{loc} such that

$$f_n^*(s) = n[\log(e+n)]^{-\alpha-1}\chi_{(0,\frac{1}{n})}(s).$$

Since

$$f_n^{**}(s) = \begin{cases} n[\log(e+n)]^{-\alpha-1} & \text{for } s \in \left(0, \frac{1}{n}\right), \\ s^{-1}[\log(e+n)]^{-\alpha-1} & \text{for } s \in \left[\frac{1}{n}, \infty\right), \end{cases} \tag{A-4}$$

we have

$$\|f_n\|_{\mathfrak{L}^{1,\infty}(\log \mathfrak{L})^\alpha} = \sup_{s>0}\left[\log\left(e+\frac{1}{s}\right)\right]^{\alpha} sf_n^{**}(s) = \left[\log\left(e+\frac{1}{s}\right)\right]^{\alpha} sf_n^{**}(s)\bigg|_{s=\frac{1}{n}} = [\log(e+n)]^{-1}$$

for $n = 1, 2, \ldots$. On the other hand, similarly to (3-2), we find $L \in [e, \infty)$ such that

$$\text{the function } (0,\infty) \ni \tau \mapsto \tau\left[\log\left(L+\frac{1}{\tau}\right)\right]^{\alpha+1} \text{ is nondecreasing.}$$

Then, by (3-1) and (A-4), we have

$$\begin{aligned} \|f_n\|_{L^{1,\infty}(\log L)^{\alpha+1}} &= \sup_{s>0}\left[\log\left(e+\frac{1}{s}\right)\right]^{\alpha+1} sf_n^{*}(s) \\ &\ge C\sup_{s>0}\left[\log\left(L+\frac{1}{s}\right)\right]^{\alpha+1} sf_n^{*}(s) = C\left[\log\left(L+\frac{1}{s}\right)\right]^{\alpha+1} sf_n^{*}(s)\bigg|_{s=\frac{1}{n}} \ge C \end{aligned}$$

for $n = 1, 2, \ldots$. These imply (A-3). Thus Proposition A.2 follows. □

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References

[Andreucci and DiBenedetto 1991] D. Andreucci and E. DiBenedetto, "On the Cauchy problem and initial traces for a class of evolution equations with strongly nonlinear sources", *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **18**:3 (1991), 363–441. MR Zbl

[Baras and Pierre 1985] P. Baras and M. Pierre, "Critère d'existence de solutions positives pour des équations semi-linéaires non monotones", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **2**:3 (1985), 185–212. MR Zbl

[Bennett and Sharpley 1988] C. Bennett and R. Sharpley, *Interpolation of operators*, Pure Appl. Math. **129**, Academic Press, Boston, MA, 1988. MR Zbl

[Bogdan and Jakubowski 2007] K. Bogdan and T. Jakubowski, "Estimates of heat kernel of fractional Laplacian perturbed by gradient operators", *Comm. Math. Phys.* **271**:1 (2007), 179–198. MR Zbl

[Brandolese and Karch 2008] L. Brandolese and G. Karch, "Far field asymptotics of solutions to convection equation with anomalous diffusion", *J. Evol. Equ.* **8**:2 (2008), 307–326. MR Zbl

[Brezis and Cazenave 1996] H. Brezis and T. Cazenave, "A nonlinear heat equation with singular initial data", *J. Anal. Math.* **68** (1996), 277–304. MR Zbl

[Fujishima and Ioku 2021] Y. Fujishima and N. Ioku, "Solvability of a semilinear heat equation via a quasi scale invariance", pp. 79–101 in *Geometric properties for parabolic and elliptic PDEs*, Springer INdAM Ser. **47**, Springer, 2021. MR Zbl

[Fujishima and Ioku 2022] Y. Fujishima and N. Ioku, "Global in time solvability for a semilinear heat equation without the self-similar structure", *Partial Differ. Equ. Appl.* **3**:2 (2022), art. id. 23. MR Zbl

[Fujishima et al. 2023] Y. Fujishima, K. Hisa, K. Ishige, and R. Laister, "Solvability of superlinear fractional parabolic equations", *J. Evol. Equ.* **23**:1 (2023), art. id. 4. MR Zbl

[Fujishima et al. 2024] Y. Fujishima, K. Hisa, K. Ishige, and R. Laister, "Local solvability and dilation-critical singularities of supercritical fractional heat equations", *J. Math. Pures Appl.* (9) **186** (2024), 150–175. MR Zbl

[Fujita 1966] H. Fujita, "On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$", *J. Fac. Sci. Univ. Tokyo Sect. I* **13** (1966), 109–124. MR Zbl

[Giraudon and Miyamoto 2022] T. Giraudon and Y. Miyamoto, "Fractional semilinear heat equations with singular and nondecaying initial data", *Rev. Mat. Complut.* **35**:2 (2022), 415–445. MR Zbl

[Grafakos 2008] L. Grafakos, *Classical Fourier analysis*, 2nd ed., Grad. Texts in Math. **249**, Springer, 2008. MR Zbl

[Hisa and Ishige 2018] K. Hisa and K. Ishige, "Existence of solutions for a fractional semilinear parabolic equation with singular initial data", *Nonlinear Anal.* **175** (2018), 108–132. MR Zbl

[Hisa et al. 2023] K. Hisa, K. Ishige, and J. Takahashi, "Initial traces and solvability for a semilinear heat equation on a half space of $\mathbb{R}^N$", *Trans. Amer. Math. Soc.* **376**:8 (2023), 5731–5773. MR Zbl

[Ishige et al. 2014] K. Ishige, T. Kawakami, and K. Kobayashi, "Global solutions for a nonlinear integral equation with a generalized heat kernel", *Discrete Contin. Dyn. Syst. Ser. S* **7**:4 (2014), 767–783. MR Zbl

[Ishige et al. 2020] K. Ishige, T. Kawakami, and S. Okabe, "Existence of solutions for a higher-order semilinear parabolic equation with singular initial data", *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **37**:5 (2020), 1185–1209. MR Zbl

[Ishige et al. 2022] K. Ishige, T. Kawakami, and S. Okabe, "Existence of solutions to nonlinear parabolic equations via majorant integral kernel", *Nonlinear Anal.* **223** (2022), art. id. 113025. MR Zbl

[Kan and Takahashi 2017] T. Kan and J. Takahashi, "Time-dependent singularities in semilinear parabolic equations: existence of solutions", *J. Differential Equations* **263**:10 (2017), 6384–6426. MR Zbl

[Kozono and Yamazaki 1994] H. Kozono and M. Yamazaki, "Semilinear heat equations and the Navier–Stokes equation with distributions in new function spaces as initial data", *Comm. Partial Differential Equations* **19**:5-6 (1994), 959–1014. MR Zbl

[Laister and Sierżęga 2020] R. Laister and M. Sierżęga, "Well-posedness of semilinear heat equations in L^1", *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **37**:3 (2020), 709–725. MR Zbl

[Laister and Sierżęga 2021] R. Laister and M. Sierżęga, "A blow-up dichotomy for semilinear fractional heat equations", *Math. Ann.* **381**:1-2 (2021), 75–90. MR Zbl

[Laister et al. 2016] R. Laister, J. C. Robinson, M. Sierżęga, and A. Vidal-López, "A complete characterisation of local existence for semilinear heat equations in Lebesgue spaces", *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **33**:6 (2016), 1519–1538. MR Zbl

[Miyamoto 2021] Y. Miyamoto, "A doubly critical semilinear heat equation in the L^1 space", *J. Evol. Equ.* **21**:1 (2021), 151–166. MR Zbl

[Muckenhoupt 1972] B. Muckenhoupt, "Hardy's inequality with weights", *Studia Math.* **44** (1972), 31–38. MR Zbl

[O'Neil 1963] R. O'Neil, "Convolution operators and $L(p, q)$ spaces", *Duke Math. J.* **30** (1963), 129–142. MR Zbl

[Quittner and Souplet 2007] P. Quittner and P. Souplet, *Superlinear parabolic problems*: *blow-up*, *global existence and steady states*, Birkhäuser, Basel, 2007. MR Zbl

[Robinson and Sierżęga 2013] J. C. Robinson and M. Sierżęga, "Supersolutions for a class of semilinear heat equations", *Rev. Mat. Complut.* **26**:2 (2013), 341–360. MR Zbl

[Sugitani 1975] S. Sugitani, "On nonexistence of global solutions for some nonlinear integral equations", *Osaka Math. J.* **12** (1975), 45–51. MR Zbl

[Takahashi 2016] J. Takahashi, "Solvability of a semilinear parabolic equation with measures as initial data", pp. 257–276 in *Geometric properties for parabolic and elliptic PDE's*, Springer Proc. Math. Stat. **176**, Springer, 2016. MR Zbl

[Tayachi and Weissler 2014] S. Tayachi and F. B. Weissler, "The nonlinear heat equation with high order mixed derivatives of the Dirac delta as initial values", *Trans. Amer. Math. Soc.* **366**:1 (2014), 505–530. MR Zbl

[Wadade 2014] H. Wadade, "Optimal embeddings of critical Sobolev–Lorentz–Zygmund spaces", *Studia Math.* **223**:1 (2014), 77–96. MR Zbl

[Weissler 1981] F. B. Weissler, "Existence and nonexistence of global solutions for a semilinear heat equation", *Israel J. Math.* **38**:1-2 (1981), 29–40. MR Zbl

[Zhanpeisov 2023] E. Zhanpeisov, "Existence of solutions to fractional semilinear parabolic equations in Besov–Morrey spaces", *Discrete Contin. Dyn. Syst.* **43**:11 (2023), 3969–3986. MR Zbl

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