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CHIAN YEONG CHUAH, ZHEN-CHUAN LIU AND TAO MEI

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A MARCINKIEWICZ MULTIPLIER THEORY FOR SCHUR MULTIPLIERS

CHIAN YEONG CHUAH, ZHEN-CHUAN LIU AND TAO MEI

We prove a Marcinkiewicz-type multiplier theory for the boundedness of Schur multipliers on the Schatten p-classes. This generalizes a previous result of J. Bourgain for Toeplitz-type Schur multipliers and complements a recent result by J. Conde-Alonso et al. (*Ann. of Math.* (2) **198**:3 (2023), 1229–1260). As a corollary, we obtain a new unconditional decomposition for the Schatten p-classes, $1 < p < \infty$. We extend our main result to the $\mathbb{Z}^d$ and $\mathbb{R}^d$ cases, and include an operator-valued version of it using Pisier's noncommutative $L^\infty(\ell_1)$-norm.

1. Introduction

Let $A \in B(H)$ be a bounded operator on a (separable) Hilbert space H. We can write A in its matrix representation

$$A = (a_{k,j})_{k,j\in\mathbb{Z}},$$

with $a_{k,j} = \langle Ae_k, e_j\rangle$ for a given orthonormal basis $\{e_k\}_{k\in\mathbb{Z}}$ of H. Given a bounded function m on $\mathbb{Z}\times\mathbb{Z}$, we call the map

$$M_m : (a_{k,j}) \mapsto (m(k, j)a_{k,j}) \tag{1-1}$$

a Schur multiplier with symbol m. The study of the boundedness of Schur multipliers with respect to the Schatten p-norms has a rich history [Bennett 1977; Arazy 1982; Bożejko and Fendler 1984; Berkson and Gillespie 1994; Pisier 1998; 2001; Harcharras 1999; Clément et al. 2000; Aleksandrov and Peller 2002; Doust and Gillespie 2005]. The recent study of noncommutative analysis on the approximation properties of operator functions and operator algebras [Haagerup et al. 2010; Neuwirth and Ricard 2011; Caspers and de la Salle 2015; Potapov et al. 2015; 2017; de Laat and de la Salle 2018; Caspers et al. 2019; Parcet et al. 2022; Mei et al. 2022; Conde-Alonso et al. 2023], especially the work [Lafforgue and de la Salle 2011] on the approximation property of higher-rank Lie groups, draws a lot of attention to the boundedness of Schur multipliers for the case where $1 < p \neq 2 < \infty$. Conde-Alonso, González–Pérez, Parcet and Tablate [Conde-Alonso et al. 2023] recently proved a Hörmander–Mikhlin-type Schur multiplier theory for S^p, $1 < p \neq 2 < \infty$, in their remarkable work.

In this article, we prove a Marcinkiewicz-type Schur multiplier theory. Hörmander–Mikhlin-type multipliers and Marcinkiewicz-type multipliers are rooted in classical Fourier analysis. Like their counterpart in Fourier analysis, Marcinkiewicz-type Schur multipliers are a larger class of multipliers

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and their p-boundedness is more subtle; it was shown in [Tao and Wright 2001] that the L^p-bounds of Marcinkiewicz Fourier multipliers are of order $p^{3/2}$ as $p \to \infty$.

When m is of Toeplitz-type, i.e., $m(k, j) = \dot{m}(k-j)$ for some function $\dot{m} : \mathbb{Z} \to \mathbb{C}$, one may apply a well-known transference method and obtain bounded Schur multipliers from the classical Fourier multiplier theory. J. Bourgain's work [1986, Theorem 4, Corollary 20] on scalar-valued Fourier multipliers acting on Schatten p-valued functions implies that the following Marcinkiewicz-type condition is sufficient for the boundedness of M_m on the Schatten p-classes for all $1 < p < \infty$:

$$\sum_{2^{n-1} \le |k| < 2^n} |\dot{m}(k+1) - \dot{m}(k)| < C \tag{1-2}$$

for all $n \in \mathbb{N}$. Let $\dot{m}_\varepsilon(k) = \varepsilon_n$ for $2^{n-1} \le |k| < 2^n$. Then the associated multiplier M_{m_ε} is bounded for any sequence $\varepsilon_n = \pm 1$.

To extend Bourgain's result to general non-Toeplitz-type Schur multipliers, one may ask whether the condition that

$$\sum_{2^{n-1} \le |k| < 2^n} |m(k+j+1, j) - m(k+j, j)| < C \tag{1-3}$$

for all $n \in \mathbb{N}$, $j \in \mathbb{Z}$ implies the S^p boundedness of general Schur multipliers. The answer is yes if m is Toeplitz since condition (1-3) reduces to Bourgain's condition (1-2) in that case. The answer would be yes for general Schur multipliers as well if the family M_{m_ε}, defined after condition (1-2), is R-bounded for any family of sequences $\varepsilon = (\varepsilon_k)_k$ valued in $\{\pm 1\}$. This implication was proved in the works of Berkson and Gillespie [1994], Doust and Gillespie [2005] and Clément, de Pagter, Sukochev and Witvliet [Clément et al. 2000], in which they studied the connection between vector-valued Littlewood–Paley theory and Marcinkiewicz multiplier theory. We show in Section 5.1 that this is not true in general and the condition (1-3) is not sufficient for the S^p-boundedness of the associated Schur multiplier.[1]

The main result of this article is the following.

Theorem 1.1. *M_m defined in (1-1) extends to a bounded map on the Schatten p-classes S^p for all $1 < p < \infty$ with bounds $\lesssim (p^2/(p-1))^3$ if m is bounded and there exists a constant C such that*

$$\sum_{2^{n-1} \le |k| < 2^n} |m(k+j+1, j) - m(k+j, j)| < C, \tag{1-4}$$

$$\sum_{2^{n-1} \le |k| < 2^n} |m(j, k+j+1) - m(j, k+j)| < C \tag{1-5}$$

for all $n \in \mathbb{N}$, $j \in \mathbb{Z}$.

The writing of this article was motivated by the recent article [Conde-Alonso et al. 2023], although the third author had known Theorem 1.1 previously. The authors of [loc. cit.] further studied Schatten-p-classes indexed in d-dimensional Euclidean spaces, aiming for possible applications to the approximation properties of higher-rank Lie groups. Following this trend, we extend Theorem 1.1 to the higher-dimensional

[1] This also shows that the family of "Littlewood–Paley operators" M_{m_ε} mentioned above is *not* R-bounded.

cases as well. The proof of Theorem 1.1 relies on the crucial property that a Schur multiplier is an operator-valued Fourier multiplier multiplying from the left, and is simultaneously an operator-valued Fourier multiplier multiplying from the right. This property was already used by the authors of [Conde-Alonso et al. 2023] in proving a Hörmander–Mikhlin-type criterion for the boundedness of Schur multipliers.

Theorem 1.1 implies a new unconditional decomposition for Schatten p classes. For $(n, \ell) \in \mathbb{Z} \times \mathbb{Z}$, let $E_{0,\ell} = \{(\ell, \ell)\} \subset \mathbb{Z} \times \mathbb{Z}$. Let

$$E_{n,\ell} = \{(k, j) \in \mathbb{Z} \times \mathbb{Z} : 2^{n-1} \le k - j < 2^n,\ \ell 2^n \le k < (\ell+1)2^n\}$$

for $n > 0$, and

$$E_{n,\ell} = \{(k, j) \in \mathbb{Z} \times \mathbb{Z} : -2^{|n|} < k - j \le -2^{|n|-1},\ \ell 2^{|n|} \le k < (\ell+1)2^{|n|}\}$$

for $n < 0$. We then have the decomposition

$$\mathbb{Z} \times \mathbb{Z} = \bigcup_{(n,\ell)\in\mathbb{Z}\times\mathbb{Z}} E_{n,\ell}.$$

Let $P_{n,\ell}$ be the projection onto $\mathrm{span}\{e_{k,j}, (k, j) \in E_{n,\ell}\}$. It is easy to see that $\sum_{n,\ell} \varepsilon_{n,\ell} P_{n,\ell}$ is a Schur multiplier satisfying the assumptions of Theorem 1.1 for any bounded sequence $\varepsilon_{n,\ell}$. We then obtain an unconditional decomposition of S^p.

Corollary 1.2. *$\sum_{n,\ell} \varepsilon_{n,\ell} P_{n,\ell}$ extends to a bounded map on S^p for all $1 < p < \infty$ for any bounded sequence $\varepsilon_{n,\ell}$.*

We will prove Theorem 1.1 in Section 3. We will explain how to extend Theorem 1.1 to the higher-dimensional case in Section 4 and explain that the ball-type Schur multipliers remain bounded on $S^p(\ell^2(\mathbb{Z}^d))$ for $d > 1$ (Example 4.4), contrary to the behavior of Fourier multipliers. We will show that the condition (1-4) alone is not sufficient in Section 5.1 and explain an operator-valued version of Theorem 1.1 in Section 5.2.

2. Preliminaries

Given $d \in \mathbb{N}$, denote by $\mathcal{B}(\ell^2(\mathbb{Z}^d))$ the set of bounded linear operators on $\ell^2(\mathbb{Z}^d)$. We represent the operator $A \in \mathcal{B}(\ell^2(\mathbb{Z}^d))$ as $A = (a_{i,j})_{(i,j)\in\mathbb{Z}^d\times\mathbb{Z}^d}$ with $a_{i,j} = \langle Ae_i, e_j\rangle$ for the canonical basis $\{e_i\}$ of $\ell^2(\mathbb{Z}^d)$. Given a bounded function m on $\mathbb{Z}^d \times \mathbb{Z}^d$, the associated Schur multiplier

$$M_m(A) = (m(i, j)a_{ij})$$

extends to a bounded operator on the Hilbert–Schmidt class $S^2(\ell^2(\mathbb{Z}^d))$. We call m the symbol of M_m. Recall that the Schatten p-class S^p, $1 \le p < \infty$, is the collection of all compact operators A with a finite Schatten-p norm, which is defined as

$$\|A\|_p = (\mathrm{tr}[(A^*A)^{\frac{p}{2}}])^{\frac{1}{p}} = \Big(\sum_i s_i^p\Big)^{\frac{1}{p}} \tag{2-1}$$

for $1 \le p < \infty$, where s_i is the i-th singular value of A. The Schatten p norm is unitary invariant and does not depend on the choice of the orthonormal basis. The Schatten-class S^p, $1 \le p < \infty$, and

$B(\ell^2(\mathbb{Z}^d))$ share many similar properties with ℓ^p, $1 \le p \le \infty$. In particular, the dual space of S^1 (resp. S^p, $1 < p < \infty$) is isomorphic to $B(\ell^2)$ (resp. $S^{p/(p-1)}$). The family forms an interpolation scale

$$[B(\ell^2), S^1]_{\frac{1}{p}} = S^p$$

for $1 < p < \infty$. However, S^p does not admit an unconditional basis whenever $p \neq 2$. We will prove that, for m satisfying additional conditions (1-4) and (1-5), M_m extends to a bounded map on S^p for all $1 < p < \infty$, which immediately implies the unconditional decomposition for S^p as stated in Corollary 1.2.

Given $d \in \mathbb{N}$, we denote by $L^p(\mathbb{T}^d; S^p)$ the space of S^p-valued Bochner integrable functions f such that

$$\|f\|_{L^p} = \left(\mathrm{tr}\left[\int_{[0,1)^d} |f|^p(z)\, d\theta\right]\right)^{\frac{1}{p}} < \infty.$$

Here we let $z = e^{i2\pi\theta}$, with $\theta \in [0, 1)^d$, and $|f|^p = (f^* f)^{p/2}$ is defined via the functional calculus.

For $f \in L^p(\mathbb{T}^d; S^p)$ with $1 < p < \infty$, we have the Fourier expansion

$$f(z) \sim \sum_{k \in \mathbb{Z}^d} \hat{f}(k) z^k,$$

with $\hat{f}(k) = \int_{[0,1]^d} f(z)\bar{z}^k\, d\theta \in S^p$. Given R a finite subset of $\mathbb{Z}^d$, denote by $S_R f$ the partial Fourier sum

$$S_R f(z) = \sum_{k \in R} \hat{f}(k) z^k. \tag{2-2}$$

Choose $\delta \in C^\infty(\mathbb{R})$ such that $0 \le \delta \le 1$, $\mathrm{supp}(\delta) \subset \left[-2\sqrt{d}, -\frac{1}{4}\right] \cup \left[\frac{1}{4}, 2\sqrt{d}\right]$ and $\delta(x) = 1$ when $\frac{1}{2} \le |x| \le \sqrt{d}$. For $j \ge 0$, define $\delta_j(x) := \delta(2^{-j}x)$. For $f \in L^2(\mathbb{T}^d; S^2)$, we define

$$S_j f(z) = \sum_{k \in \mathbb{Z}^d} \delta_j(|k|_2) \hat{f}(k) z^k.$$

We denote by $|k|_2$ the ℓ_2 norm of $k \in \mathbb{Z}^d$ in the formula above and will denote by $|k|_\infty$ the ℓ_∞ norm of k. Let $(E_j)_{j \ge 0}$ be the cubes with squared holes in $\mathbb{Z}^d$ given by

$$E_j = \begin{cases} \{k \in \mathbb{Z}^d : 2^{j-1} \le |k|_\infty < 2^j\}, & j > 0, \\ \{0\}, & j = 0. \end{cases} \tag{2-3}$$

Note our construction implies $S_{E_j} S_j = S_{E_j}$, which we will need later.

For a sequence (f_k) in $L^p(\mathbb{T}^d; S^p)$, we use the classical notation

$$\|(f_k)\|_{L^p(\ell_2^c)} = \left\| \left(\sum_k |f_k|^2\right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{T}^d; S^p)}, \quad \|(f_k)\|_{L^p(\ell_2^r)} = \left\| \left(\sum_k |f_k^*|^2\right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{T}^d; S^p)},$$

and

$$\|(f_k)\|_{L^p(\ell_2)} = \begin{cases} \max\{\|(f_k)\|_{L^p(\ell_2^c)}, \|(f_k^*)\|_{L^p(\ell_2^c)}\} & \text{if } 2 \le p \le \infty, \\ \inf_{y_k + z_k = f_k} \|(y_k)\|_{L^p(\ell_2^c)} + \|(z_k)\|_{L^p(\ell_2^r)} & \text{if } 0 < p < 2. \end{cases}$$

The above definition is justified by the following noncommutative Khintchine inequality:

Lemma 2.1 [Lust-Piquard 1986; Lust-Piquard and Pisier 1991]. *Let (ε_k) be a sequence of independent Rademacher random variables. Then, for $1 \le p < \infty$,*

$$\alpha_p^{-1} E_\varepsilon \Big\| \sum_k \varepsilon_k \otimes f_k \Big\|_{L^p(\mathbb{T}^d;S^p)} \le \|(f_k)\|_{L^p(\ell_2)} \le \beta_p E_\varepsilon \Big\| \sum_k \varepsilon_k \otimes f_k \Big\|_{L^p(\mathbb{T}^d;S^p)}. \tag{2-4}$$

The optimal constant β_p is no greater than $\sqrt{3}$ for $1 \le p \le 2$ and is 1 for $p \ge 2$ (see [Haagerup and Musat 2007]); α_p is 1 for $1 \le p \le 2$ and is of order $\sqrt{p}$ as $p \to \infty$. Inequality (2-4) was pushed further to the case where $0 < p < 1$ (see [Pisier and Ricard 2017]).

We will need the following noncommutative Littlewood–Paley theorem on $\mathbb{Z}^d$.

Lemma 2.2. *There is a constant $C_d > 0$ that depends only on d such that*

$$\|(S_j f)_{j\ge 0}\|_{L^p(\ell_2)} \le C_d \frac{p^2}{p-1} \|f\|_{L^p(\mathbb{T}^d;S^p)}, \tag{2-5}$$

$$\|f\|_{L^p(\mathbb{T}^d;S^p)} \le C_d \frac{p^2}{p-1} \|(S_{E_j} f)_{j\ge 0}\|_{L^p(\ell_2)} \tag{2-6}$$

for all $f \in L^p(\mathbb{T}^d; S^p)$ and $1 < p < \infty$.

Proof. This lemma is well known. We explain here that the dependence of the constants on p is in the order of $p^2/(p-1)$. Given $\varepsilon_j = \pm 1$, let $M_\varepsilon = \sum_{j\ge 0} \varepsilon_j S_j$. Our choice of S_j's makes M_ε a so-called Hörmander–Mikhlin multiplier, which in particular is a Calderón–Zygmund operator. So it is bounded from the classical Hardy space H^1 to L^1. Moreover, it is from H^1 to H^1 since it commutes with the classical Hilbert transform. By [Mei 2007, Theorem 6.4], it extends to a bounded operator on the semicommutative BMO space $\mathrm{BMO}_{\mathrm{cr}}(L^\infty(\mathbb{T}^d) \bar{\otimes} \mathcal{B}(\ell^2(\mathbb{Z}^d)))$. Inequality (2-5) then follows from the interpolation result [Mei 2007, Theorem 6.2] and the Khintchine inequality (2-4). Inequality (2-6) follows from (2-5) by duality because of the identity $\langle f, g\rangle = \sum_j \langle S_{E_j} f, S_{E_j} g\rangle = \sum_j \langle S_j f, S_{E_j} g\rangle$. □

Lemma 2.3. *Suppose R_j is a family of boxes with sides parallel to the axes in $\mathbb{R}^d$. Then there is a constant $C_d > 0$ that depends only on d such that, for all $1 < p < \infty$ and for all families of measurable functions f_j on $\mathbb{R}^d$, we have*

$$\Big\| \Big(\sum_j |S_{R_j}(f_j)|^2 \Big)^{\frac12} \Big\|_{L^p(\mathbb{T}^d;S^p)} \le C_d \Big(\frac{p^2}{p-1} \Big)^d \Big\| \Big(\sum_j |f_j|^2 \Big)^{\frac12} \Big\|_{L^p(\mathbb{T}^d;S^p)}. \tag{2-7}$$

Proof. Assume $d = 1$ and $R_j = [a_j, b_j]$. Let T_a to be the operator that sends $f(\cdot)$ to $f(\cdot)e^{i2\pi a_j(\cdot)}$. Let P_+ be the analytic projection. Then

$$S_{R_j} = T_{a_j} P_+ T_{-a_j} - T_{b_j} P_+ T_{-b_j}.$$

Note that $|T_{a_j} f_j|^2 = |f_j|^2$ and we obtain the inequality for $d = 1$ by the boundedness of P_+. The case $d > 1$ holds due to Fubini's theorem. □

Lemma 2.4. *For sequences $(a_n), (c_n) \in B(H)$, we have*

$$\Big| \sum_n a_n^* c_n \Big|^2 \le \Big\| \sum_n a_n^* a_n \Big\| \Big(\sum_n c_n^* c_n \Big). \tag{2-8}$$

Proof. Given any $v \in S^2$, we have by the Cauchy–Schwarz inequality that

$$\operatorname{tr}\Big(v^*\Big|\sum_n a_n^* c_n\Big|^2 v\Big) = \operatorname{tr}\Big(\Big|\sum_n a_n^* c_n v\Big|^2\Big) \le \Big\|\sum_n a_n^* a_n\Big\| \operatorname{tr}\Big[v^*\Big(\sum_n c_n^* c_n\Big)v\Big].$$

Since v is arbitrary, we obtain (2-8). □

3. Proof of Theorem 1.1

For $z \in \mathbb{T}^d$ given, let Π_z be the *-homomorphism on $\mathcal{B}(\ell^2(\mathbb{Z}^d))$ defined as

$$\Pi_z(A) = U_z A U_z^*,$$

with U_z the unitary sending e_k to $z^k e_k$. It is easy to see that Π_z has the presentation

$$\Pi_z : A = (a_{k,j}) \longmapsto (a_{k,j} z^{k-j}), \tag{3-1}$$

with $k, j \in \mathbb{Z}^d$. Π_z defines an isometric isomorphism on $\mathcal{B}(\ell^2(\mathbb{Z}^d))$ and $S^p(\ell^2(\mathbb{Z}^d))$ for all $1 \le p < \infty$ because all these norms are unitary invariant. Considering z as a variable on $\mathbb{T}^d$, define $\Pi : \mathcal{B}(\ell^2(\mathbb{Z}^d)) \to L^\infty(\mathbb{T}^d) \otimes \mathcal{B}(\ell^2(\mathbb{Z}^d))$ as

$$\Pi(A)(z) = \Pi_z(A). \tag{3-2}$$

Then Π is an isometric isomorphism from $\mathcal{B}(\ell^2(\mathbb{Z}^d))$ to $L^\infty(\mathbb{T}^d) \otimes \mathcal{B}(\ell^2(\mathbb{Z}^d))$ and from $S^p(\ell^2(\mathbb{Z}^d))$ to $L^p(\mathbb{T}^d; S^p(\ell^2(\mathbb{Z}^d)))$ for all $1 \le p < \infty$.

Given a symbol $m = (m(i, j))_{i,j\in\mathbb{Z}^d}$ and $A = (a_{i,j})_{i,j\in\mathbb{Z}^d} \in S^p(\ell^2(\mathbb{Z}^d))$, set

$$\begin{aligned} M_l(n) = \sum_{s\in\mathbb{Z}^d} m(s, s-n)\, e_{s,s}, \quad M_r(n) = \sum_{s\in\mathbb{Z}^d} m(s+n, s)\, e_{s,s}, \\ A(n) = \sum_{s\in\mathbb{Z}^d} a_{s,s-n} e_{s,s-n} = \sum_{s\in\mathbb{Z}^d} a_{s+n,s} e_{s+n,s} \end{aligned} \tag{3-3}$$

for $n \in \mathbb{Z}^d$. Here $e_{s,t}$ denotes the operator on $\ell^2(\mathbb{Z}^d)$ sending e_t to e_s. Then $M_l(n), M_r(n) \in \mathcal{B}(\ell^2(\mathbb{Z}^d))$ with norm bounded by C, $A(n) \in S^p(\ell^2(\mathbb{Z}^d))$ for all $n \in \mathbb{Z}^d$, and

$$\Pi(A)(z) = \sum_n A(n) z^n, \quad M_m(A) = \sum_n M_l(n)A(n) = \sum_n A(n)M_r(n).$$

Here $M_l(n)A(n)$ and $A(n)M_r(n)$ denote the products of operators in $\mathcal{B}(\ell^2(\mathbb{Z}^d))$. Let $f = \Pi(A)$, i.e.,

$$f(z) = \sum_{n\in\mathbb{Z}^d} A(n) z^n. \tag{3-4}$$

Denote by $\Pi(S^p)$ the image of Π, i.e., the subspace of $L_p(\mathbb{T}^d; S^p)$ consisting of all f in the form of (3-4). Define the operator-valued Fourier multiplier T_M on $\Pi(S^p)$ as

$$T_M f(z) = \sum_{n\in\mathbb{Z}^d} M_l(n) A(n) z^n. \tag{3-5}$$

Note that $M_l(n)A(n) = A(n)M_r(n)$ for all $n \in \mathbb{Z}^d$; we can represent T_M as a multiplier from the right:

$$T_M f(z) = \sum_{n\in\mathbb{Z}^d} A(n)M_r(n)z^n. \tag{3-6}$$

T_M is defined so that the following identity holds:

$$T_M f = \Pi(M_m(A)).$$

Since Π is a trace preserving $*$-homomorphism, we have

$$\|A\|_{S^p} = \|f\|_{L^p(\mathbb{T}^d;S^p)}, \quad \|M_m A\|_{S^p} = \|T_M(f)\|_{L^p(\mathbb{T}^d;S^p)}. \tag{3-7}$$

In order to prove M_m's boundedness on S^p, we only need to prove that T_M is bounded on $\Pi(S^p)$, i.e., the subspace of $L_p(\mathbb{T}^d; S^p)$ consisting all f in the form of (3-4). By Lemma 2.2 and the transference relation (3-7), it is sufficient to show the inequality

$$\Big\|\Big(\sum_{j\geq 0}|S_{E_j}(T_M f)|^2\Big)^{\frac12}\Big\|_{L_p(\mathbb{T}^d;S^p)} \leq C_d\Big(\frac{p^2}{p-1}\Big)^d\Big\|\Big(\sum_{j\geq 0}|S_j f|^2\Big)^{\frac12}\Big\|_{L_p(\mathbb{T}^d;S^p)} \tag{3-8}$$

and its adjoint form

$$\Big\|\Big(\sum_{j\geq 0}|(S_{E_j}(T_M f))^*|^2\Big)^{\frac12}\Big\|_{L_p(\mathbb{T}^d;S^p)} \leq C_d\Big(\frac{p^2}{p-1}\Big)^d\Big\|\Big(\sum_{j\geq 0}|(S_j f)^*|^2\Big)^{\frac12}\Big\|_{L_p(\mathbb{T}^d;S^p)} \tag{3-9}$$

for $p \geq 2$. By duality, we will obtain M_m's boundedness on S^p for $1 < p < 2$ as well.

We will use (3-5) as the presentation of T_M to prove (3-8) and will use the presentation (3-6) to prove (3-9). Note that E_j is symmetric so $(S_{E_j}(T_M f))^* = S_{E_j}(T_M f)^*$ and

$$(T_M f)^* = \Big(\sum_{n\in\mathbb{Z}^d} A(n)M_r(n)z^n\Big)^* = \sum_{n\in\mathbb{Z}^d}(M_r(-n))^*(A(-n))^*z^n.$$

So, both $S_{E_j}(T_M f)$ and $(S_{E_j}(T_m f))^*$ have the multiplier symbols on the left. This allows us to write the corresponding squares in the forms with M_r or M_l sitting in the middle for both $S_{E_j}(T_M f)$ and $(S_{E_j}(T_m f))^*$ and avoid the usual trouble caused by the noncommutativity of the operator products. After noting these facts, the argument for the case $d = 1$ is rather standard, which we record below.

Proof of Theorem 1.1. We now set $d = 1$. By the notation in (3-3), the conditions (1-4) and (1-5) are equivalent to

$$\sup_{j\geq 0}\Big\|\sum_{n\in E_j}|M_l(n+1) - M_l(n)|\Big\|_\infty < C, \quad \sup_{j\geq 0}\Big\|\sum_{n\in E_j}|M_r(n+1) - M_r(n)|\Big\|_\infty < C, \tag{3-10}$$

where E_j is defined as in (2-3). Following the definition (2-2), we define $S_{(a,b)}$ as

$$S_{(a,b)}g(z) = \sum_{a<n<b}\hat{g}(n)z^n \tag{3-11}$$

for $g(z) = \sum_{n\in\mathbb{Z}}\hat{g}(n)z^n \in L^p(\mathbb{T}; S^p)$. We will deliberately extend the use of this notation and set

$$S_{(a,b)}g = -S_{(b,a)}g$$

when $a > b$. For $j \in \mathbb{N}$, write $E_{j,1} = (-2^j, -2^{j-1}]$, $E_{j,2} = [2^{j-1}, 2^j)$. Let $2_1^{(j)} = -2^j$, $2_2^{(j)} = 2^j$ and

$$\Delta M_l(n) = \begin{cases} M_l(n) - M_l(n-1), & n < 0, \\ M_l(n+1) - M_l(n), & n > 0. \end{cases} \tag{3-12}$$

By applying summation by parts and the presentation of $T_M f$ in (3-5) and (3-6), we obtain

$$S_{E_j}(T_M f) = \sum_{i=1,2} S_{E_{j,i}}(T_M f) = \sum_{i=1,2} \Big(M_l(2_i^{(j-1)})(S_{E_{j,i}} f) + \sum_{n \in E_{j,i}} \Delta M_l(n)(S_{(n,2_i^{(j)})} f) \Big) \tag{3-13}$$

$$= \sum_{i=1,2} \Big((S_{E_{j,i}} f) M_r(2_i^{(j-1)}) + \sum_{n \in E_{j,i}} (S_{(n,2_i^{(j)})} f) \Delta M_r(n) \Big), \tag{3-14}$$

with $\Delta M_r(n)$ defined similarly. We will use the presentation (3-13) to prove (3-8) and will use (3-14) to prove (3-9). The arguments are similar. So we will only give the argument for (3-8). We will ignore the term $j = 0$ in (3-8) because $\|S_{E_0}(T_M f)\|_{L^p(\mathbb{T};S^p)} \le C\|f\|_{L^p(\mathbb{T};S^p)}$.

Note $\Delta M_l(n)$ is a diagonal operator; we can write $\Delta M_l(n) = a_n^* b_n$, with a_n, b_n diagonal operators and $|a_n|^2 = |b_n|^2 = |\Delta M_l(n)|$. Then by Lemma 2.4, we have, for $i = 1, 2$,

$$\begin{aligned} \Big| \sum_{n \in E_{j,i}} \Delta M_\ell(n) S_{(n,2_i^{(j)})} f \Big|^2 &\le \Big\| \sum_{n \in E_{j,i}} |\Delta M_l(n)| \Big\|_\infty \Big(\sum_{n \in E_{j,i}} |b_n S_{(n,2_i^{(j)})} f|^2 \Big) \\ &\le C \Big(\sum_{n \in E_{j,i}} |b_n S_{(n,2_i^{(j)})} f|^2 \Big) \qquad (3\text{-}15) \\ &= C \Big(\sum_{n \in E_{j,i}} |S_{(n,2_i^{(j)})} (b_n S_j f)|^2 \Big). \qquad (3\text{-}16) \end{aligned}$$

Thus,

$$\Big\| \Big(\sum_{j \in \mathbb{N}} \Big| \sum_{n \in E_{j,i}} \Delta M_\ell(n) S_{(n,2_i^{(j)})} f \Big|^2 \Big)^{\frac12} \Big\|_{L^p(\mathbb{T};S^p)} \le C^{\frac12} \Big\| \Big(\sum_{j \in \mathbb{N}} \sum_{n \in E_{j,i}} |S_{(n,2_i^{(j)})} (b_n S_j f)|^2 \Big)^{\frac12} \Big\|_{L^p(\mathbb{T};S^p)}.$$

By Lemma 2.3, we get

$$\begin{aligned} &\Big\| \Big(\sum_{j \in \mathbb{N}} \Big| \sum_{n \in E_{j,i}} \Delta M_\ell(n) S_{(n,2_i^{(j)})} f \Big|^2 \Big)^{\frac12} \Big\|_{L^p(\mathbb{T};S^p)} \\ &\le C_2 \frac{p^2}{p-1} C^{\frac12} \Big\| \Big(\sum_{j \in \mathbb{N}} \sum_{n \in E_{j,i}} |(b_n S_j f)|^2 \Big)^{\frac12} \Big\|_{L^p(\mathbb{T};S^p)} \\ &= C_2 \frac{p^2}{p-1} C^{\frac12} \Big\| \Big(\sum_{j \in \mathbb{N}} (S_j f)^* \Big(\sum_{n \in E_{j,i}} |b_n|^2 \Big) S_j f \Big)^{\frac12} \Big\|_{L^p(\mathbb{T};S^p)} \\ &= C_2 \frac{p^2}{p-1} C^{\frac12} \Big\| \Big(\sum_{j \in \mathbb{N}} (S_j f)^* \Big(\sum_{n \in E_{j,i}} |\Delta M_l(n)| \Big) S_j f \Big)^{\frac12} \Big\|_{L^p(\mathbb{T};S^p)} \\ &\le C_2 \frac{p^2}{p-1} C \Big\| \Big(\sum_{j \in \mathbb{N}} |S_j f|^2 \Big)^{\frac12} \Big\|_{L^p(\mathbb{T};S^p)}. \qquad (3\text{-}17) \end{aligned}$$

Hence, by (3-13), (3-17), and Lemma 2.3

$$
\begin{aligned}
&\Big\| \Big(\sum_{j\in\mathbb{N}} |S_{E_{j,i}}(T_M f)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\mathbb{T};S^p)} \\
&\qquad \le \Big\| \Big(\sum_{j\in\mathbb{N}} |M_l(2_i^{(j-1)})(S_{E_{j,i}} f)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\mathbb{T};S^p)} + C\frac{p^2}{p-1} C_2 \Big\| \Big(\sum_{j\in\mathbb{N}} |S_j f|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\mathbb{T};S^p)} \\
&\qquad \le \Big\| C\Big(\sum_{j\in\mathbb{N}} |(S_{E_{j,i}} f)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\mathbb{T};S^p)} + C\frac{p^2}{p-1} C_2 \Big\| \Big(\sum_{j\in\mathbb{N}} |S_j f|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\mathbb{T};S^p)} \\
&\qquad = \Big\| C\Big(\sum_{j\in\mathbb{N}} |(S_{E_{j,i}} S_j f)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\mathbb{T};S^p)} + C\frac{p^2}{p-1} C_2 \Big\| \Big(\sum_{j\in\mathbb{N}} |S_j f|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\mathbb{T};S^p)} \\
&\qquad \le C\frac{p^2}{p-1} \Big\| \Big(\sum_{j\in\mathbb{N}} |S_j f|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\mathbb{T};S^p)}
\end{aligned}
$$

for $i = 1, 2$. Therefore we finish the proof of (3-8). The arguments for the adjoint version of (3-9) are similar. We then complete the proof of Theorem 1.1. □

Corollary 3.1 [Conde-Alonso et al. 2023, Corollary 3.5]. *The following Mikhlin conditions imply the boundedness of M_m on S^p for all $1 < p < \infty$:*

$$|m(s, s+k) - m(s, s+k+1)| \le \frac{C}{|k|}, \tag{3-18}$$

$$|m(s+k, s) - m(s+k+1, s)| \le \frac{C}{|k|}. \tag{3-19}$$

Proof. It is clear that the Mikhlin conditions (3-18), (3-19) imply the Marcinkiewicz-type conditions (1-4), (1-5). □

4. The case $d > 1$

In this part, we generalize Theorem 1.1 to the d-dimensional case. Before we proceed to the main statement of the theorem, we need to borrow some notation from the calculus of finite differences.

Definition 4.1. Let $\sigma : \mathbb{Z}^d \to \mathbb{C}$ and $j = (j_1, \dots, j_d) \in \mathbb{Z}^d$. Let $\{e_j\}_{j=1}^d$ be standard basis of $\mathbb{Z}^d$, i.e., the j-th entry of e_j is 1 and all other entries are 0 for $j = 1, \dots, d$. We define the forward partial difference operators Δ_{t_j} by

$$\Delta_{t_j}\sigma(t) := \sigma(t+e_j) - \sigma(t), \tag{4-1}$$

and for $\alpha \in \{0, 1\}^d$, define

$$\Delta_t^\alpha := \Delta_{t_1}^{\alpha_1} \cdots \Delta_{t_d}^{\alpha_d}.$$

For $\alpha = (1, \dots, 1) \in \{0, 1\}^d$, we simplify the notation Δ_t^α as Δ_t. Readers can find more information on the calculus of finite differences in Chapter 3 of [Ruzhansky and Turunen 2010].

4.1. *The case $d = 2$.* Recall that we have the partition $\mathbb{Z}^2 = \bigcup_{j\ge 0} E_j$ with E_j defined as

$$E_j = \begin{cases} \{(0,0)\}, & j = 0, \\ \{(n_1, n_2) \in \mathbb{Z}^2 : 2^{j-1} \le |(n_1, n_2)|_\infty < 2^j\}, & j \ge 1. \end{cases} \tag{4-2}$$

Theorem 4.2. *Given $m=(m_{s,t})_{s,t\in\mathbb{Z}^2}\in\mathcal{B}(\ell^2(\mathbb{Z}^2))$, suppose m satisfies*:

(i) $\sup_{s,t\in\mathbb{Z}^2}|m_{s,t}|<C_1$.

(ii) *For any $k\in\mathbb{N}$, $s\in\mathbb{Z}^2$, there are constants C_2, C_3 such that*

$$\Biggl(\sum_{t=(t_1,\pm 2^{k-1})\in E_k}|\Delta_{t_1}m_{s,s+t}|+\sum_{t=(\pm 2^{k-1},t_2)\in E_k}|\Delta_{t_2}m_{s,s+t}|\Biggr)<C_2, \tag{4-3}$$

$$\sum_{t=(t_1,t_2)\in E_k}|\Delta_t m_{s,s+t}|<C_3, \tag{4-4}$$

and

$$\Biggl(\sum_{t=(t_1,\pm 2^{k-1})\in E_k}|\Delta_{t_1}m_{s+t,s}|+\sum_{t=(\pm 2^{k-1},t_2)\in E_k}|\Delta_{t_2}m_{s+t,s}|\Biggr)<C_2, \tag{4-5}$$

$$\sum_{t=(t_1,t_2)\in E_k}|\Delta_t m_{s+t,s}|<C_3. \tag{4-6}$$

Then M_m is a bounded Schur multiplier on $S^p(\ell^2(\mathbb{Z}^2))$ for $p\in(1,\infty)$ with an upper bound $\lesssim(p^2/(p-1))^4$. Here C_1, C_2 and C_3 are positive absolute constants.

Now we come to the proof of Theorem 4.2. As explained at the beginning of Section 3, we only need to prove (3-8) and its adjoint version. Recall that S_{E_j} is the partial sum projection on $L^p(\mathbb{T}^2; S^p(\mathbb{Z}^2))$ given by $S_{E_j}f(z)=\sum_{n\in E_j}\hat{f}(n)z^n$, where $z\in\mathbb{T}^2$.

Applying the definition of M_l (3-3), we see that (4-3) and (4-4) imply

$$\Bigl\|\sum_{n=(n_1,\pm 2^{j-1})\in E_j}|\Delta_{n_1}M_l(n)|\Bigr\|_\infty+\Bigl\|\sum_{n=(\pm 2^{j-1},n_2)\in E_j}|\Delta_{n_2}M_l(n)|\Bigr\|_\infty<C_2, \tag{4-7}$$

$$\Bigl\|\sum_{n=(n_1,n_2)\in E_j}|\Delta_n M_l(n)|\Bigr\|_\infty<C_3. \tag{4-8}$$

To prove (3-8), we will cut E_j into four rectangles $E_{j,k}$, $k=1,\dots,4$, for $j\ge 1$. Let $I_j=[2^{j-1},2^j)\cap\mathbb{Z}$, $J_j=[-2^{j-1},2^j)\cap\mathbb{Z}$, and set

$$\begin{aligned} E_{j,1}&=J_j\times I_j, & E_{j,2}&=(-I_j)\times J_j,\\ E_{j,3}&=I_j\times(-J_j), & E_{j,4}&=(-J_j)\times(-I_j).\end{aligned}$$

Thus, we have

$$S_{E_j}T_M f=\sum_{i=1}^4 S_{E_{j,i}}T_M f. \tag{4-9}$$

To prove (3-8), it is sufficient to prove

$$\Bigl\|\Bigl(\sum_{j=0}^\infty|S_{E_{j,i}}T_M f|^2\Bigr)^{\frac12}\Bigr\|_{L^p(\mathbb{T}^2;S^p)}\le C'\Bigl(\frac{p^2}{p-1}\Bigr)^2\Bigl\|\Bigl(\sum_{j=0}^\infty|S_j f|^2\Bigr)^{\frac12}\Bigr\|_{L^p(\mathbb{T}^2;S^p)} \tag{4-10}$$

for $p\ge 2$, $i=1,2,3,4$. The arguments for $i=1,2,3,4$ are similar. We will give the argument for $i=1$ only. By the fundamental theorem of calculus,

$$\begin{aligned}S_{E_{j,1}}T_M f &= M_l(-2^{j-1},2^{j-1})S_{E_{j,1}}f+\sum_{n_1\in J_j}\Delta_{n_1}M_l(n_1,2^{j-1})S_{(n_1,2^j)\times I_j}f\\ &\qquad+\sum_{n_2\in E_j}\Delta_{n_2}M_l(-2^{j-1},n_2)S_{J_j\times(n_2,2^j)}f+\sum_{n=(n_1,n_2)\in E_{j,1}}\Delta_n M_l(n_1,n_2)S_{(n_1,2^j)\times(n_2,2^j)}f\\ &=: P_j^1+P_j^2+P_j^3+P_j^4. \end{aligned}\tag{4-11}$$

By the operator inequality $\left|\sum_{k=1}^n a_k\right|^2\le n\sum_{k=1}^n|a_k|^2$, we have

$$|S_{E_{j,1}}T_M f|^2=|P_j^1+P_j^2+P_j^3+P_j^4|^2\le 4(|P_j^1|^2+|P_j^2|^2+|P_j^3|^2+|P_j^4|^2).\tag{4-12}$$

For part P_j^1, by assumption (i) of Theorem 4.2, we have

$$|P_j^1|^2=|M_l(-2^{j-1},2^{j-1})\,S_{E_{j,1}}f|^2\le C_1^2|S_{E_{j,1}}f|^2=C_1^2|S_{E_{j,1}}S_jf|^2.\tag{4-13}$$

By Lemma 2.3,

$$\left\|\left(\sum_{j\ge0}|P_j^1|^2\right)^{\frac12}\right\|_{L^p(\mathbb{T}^2;S^p(\mathbb{Z}^2))}\le C\left(\frac{p^2}{p-1}\right)^2\left\|\left(\sum_{j\ge0}|S_jf|^2\right)^{\frac12}\right\|_{L^p(\mathbb{T}^2;S^p(\mathbb{Z}^2))}.\tag{4-14}$$

For part P_j^2, we follow the arguments similar to (3-16) and (3-17) in the one-dimensional case and write $\Delta_{n_1}M_l(n_1,2^{j-1})=a_n^*b_n$, with $|a_n|^2=|b_n|^2=|\Delta_{n_1}M_l(n_1,2^{j-1})|$. Letting $R_{n_1,j}=(n_1,2^j)\times I_j$, we have

$$|P_j^2|^2=\left|\sum_{n_1\in J_j}\Delta_{n_1}M_l(n_1,2^{j-1})S_{(n_1,2^j)\times I_j}f\right|^2\le\left\|\sum_{n_1\in J_j}|\Delta_{n_1}M_l(n_1,2^{j-1})|\right\|_\infty\left(\sum_{n_1\in J_j}|S_{R_{n_1,j}}(b_nS_jf)|^2\right).\tag{4-15}$$

Thus, by (4-15), (4-7) and Lemma 2.3 and following the arguments similar to the case $d=1$, we get

$$\left\|\left(\sum_{j\ge0}|P_j^2|^2\right)^{\frac12}\right\|_{L^p(\mathbb{T}^2;S^p(\mathbb{Z}^2))}\le C_2\left(\frac{p^2}{p-1}\right)^2\left\|\left(\sum_{j\ge0}|S_jf|^2\right)^{\frac12}\right\|_{L^p(\mathbb{T}^2;S^p)}.\tag{4-16}$$

Similarly, we have

$$\left\|\left(\sum_{j\ge0}|P_j^3|^2\right)^{\frac12}\right\|_{L^p(\mathbb{T}^2;S^p)}\le C_2\left(\frac{p^2}{p-1}\right)^2\left\|\left(\sum_{j\ge0}|S_jf|^2\right)^{\frac12}\right\|_{L^p(\mathbb{T}^2;S^p)}.\tag{4-17}$$

Now we come to the estimate of part P_j^4. Define $R_{n,j}=(n_1,2^j)\times(n_2,2^j)$. Similarly,

$$\begin{aligned}\left\|\left(\sum_{j\ge0}|P_j^4|^2\right)^{\frac12}\right\|_{L^p(\mathbb{T}^2;S^p)}&\le C_3^{\frac12}\left\|\left(\sum_{j\ge0}\sum_{n=(n_1,n_2)\in E_{j,1}}|S_{R_{n,j}}S_j|\Delta_nM_l(n)|^{\frac12}f|^2\right)^{\frac12}\right\|_{L^p(\mathbb{T}^2;S^p)}\\ &\le C_3^{\frac12}\left(\frac{p^2}{p-1}\right)^2\left\|\left(\sum_{j\ge0}\sum_{n=(n_1,n_2)\in E_{j,1}}|S_j|\Delta_nM_l(n)|^{\frac12}f|^2\right)^{\frac12}\right\|_{L^p(\mathbb{T}^2;S^p)}\\ &\le C_3\left(\frac{p^2}{p-1}\right)^2\left\|\left(\sum_{j\ge0}|S_jf|^2\right)^{\frac12}\right\|_{L^p(\mathbb{T}^2;S^p)}.\end{aligned}\tag{4-18}$$

Therefore, by (4-9), (4-11) and (4-15)–(4-18), we have

$$\left\| \left(\sum_{j=0}^{\infty} |S_{E_{j,1}} T_M f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{T}^2; S^p)} \le C' \left(\frac{p^2}{p-1} \right)^2 \left\| \left(\sum_{j=0}^{\infty} |S_j f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{T}^2; S^p)}. \tag{4-19}$$

Thus, (4-10) is proved. Hence, we finish the proof of Theorem 4.2.

4.2. *Higher-dimensional case.* We need some additional notation to deal with the case $d > 2$. Borrowing the notation from [Hytönen et al. 2016], we denote by $\mathbb{Z}^\alpha$ the space

$$\mathbb{Z}^\alpha := \{(n_i)_{i:\alpha_i=1} : n_i \in \mathbb{Z}\}$$

for $\alpha \in \{0,1\}^d$. For any $n \in \mathbb{Z}^d$ and $E = I_1 \times \cdots \times I_d \subseteq \mathbb{Z}^d$, let

$$n_\alpha := (n_i)_{i:\alpha_i=1} \in \mathbb{Z}^\alpha, \qquad E_\alpha := \prod_{i:\alpha_i=1} I_i \subseteq \mathbb{Z}^\alpha$$

be their natural projections onto $\mathbb{Z}^\alpha$. In particular, we will use the splittings $n = (n_\alpha, n_{1-\alpha}) \in \mathbb{Z}^\alpha \times \mathbb{Z}^{\mathbf{1}-\alpha}$ and $E = E_\alpha \times E_{\mathbf{1}-\alpha}$, where $\mathbf{1} = (1, \dots, 1)$. Suppose $s, t \in \mathbb{Z}^d$ and we abbreviate the interval notation $[s,t) \cap \mathbb{Z}^d$ as $[s,t)$.

Similarly, denote by $\mathcal{J}^d$ the partition $\mathcal{J}^d := \{E_j : j \ge 0\}$ of $\mathbb{Z}^d$, where

$$E_j = \begin{cases} \{(0,\dots,0)\}, & j = 0, \\ \{(n_1,\dots,n_d) \in \mathbb{Z}^d : 2^{j-1} \le |(n_1,\dots,n_d)|_\infty < 2^j\}, & j \ge 1. \end{cases} \tag{4-20}$$

Each E_j can be further decomposed into $2^d(2^d-1)$ subsets and each of the subsets can be obtained by translation of the cube $F_j = [2^{j-1}, 2^j) \times \cdots \times [2^{j-1}, 2^j)$. Following similar procedures to those in the two-dimensional case and using the discrete fundamental theorem formula,

$$\begin{aligned} \chi_{[s,t)}(n) m(n) &= \chi_{[s,t)} \sum_{\alpha \in \{0,1\}^d} \sum_{k_\alpha \in [s,n)_\alpha} \Delta^\alpha m(s_{\mathbf{1}-\alpha}, k_\alpha) \\ &= \sum_{\alpha \in \{0,1\}^d} \sum_{k_\alpha \in [s,t)_\alpha} \chi_{[k,t)_\alpha \times [s,t)_{\mathbf{1}-\alpha}}(n) \Delta^\alpha m(s_{\mathbf{1}-\alpha}, k_\alpha), \end{aligned} \tag{4-21}$$

we can obtain the following theorem. The details are left to the interested reader.

Theorem 4.3. *Given $m = (m_{s,t})_{s,t \in \mathbb{Z}^d} \in \mathcal{B}(\ell^2(\mathbb{Z}^d))$. Suppose m satisfies that, for some $C > 0$:*

(i) $\sup_{s,t \in \mathbb{Z}^d} |m_{s,t}| < C$.

(ii) *For any $n \in \mathbb{N}$, $s \in \mathbb{Z}^d$, $\alpha \in \{0,1\}^d$, $\alpha \ne 0$, and any $r^{(n)} \in \mathbb{Z}^d$ satisfying $|r_i^{(n)}| = 2^{n-1}$ for all $i = 1, \dots, d$,*

$$\sum_{t = (t_\alpha, r^{(n)}_{\mathbf{1}-\alpha}) \in E_n} |\Delta_t^\alpha m_{s,s+t}| < C, \qquad \sum_{t = (t_\alpha, r^{(n)}_{\mathbf{1}-\alpha}) \in E_n} |\Delta_t^\alpha m_{s+t,s}| < C. \tag{4-22}$$

Then M_m extends to a bounded Schur multiplier on $S^p(\ell^2(\mathbb{Z}^d))$ for $p \in (1,\infty)$ with an upper bound $C_d(p^2/(p-1))^{d+2}$. Here C_d is a constant dependent only on the dimension d.

Note that we cannot hope for an analogue of Theorem 4.3 with E_n defined by the ℓ^2-metric instead of the ℓ^∞ metric because the ball-type Fourier multipliers are not uniformly bounded on $L^p(\mathbb{T}^d)$ for any $d > 1$. Doust and Gillespie [2005, Theorem 6.2] gave an example of ball-type Schur multipliers for the case $d = 1$. Their argument does not seem to extend to the case $d > 1$.

Example 4.4 (ball Schur multipliers). Let $X_0 = \{(0, 0)\} \subset \mathbb{Z}^d \times \mathbb{Z}^d$. For $i \in \mathbb{N}$, let

$$X_i = \{(k, j) \in \mathbb{Z}^d \times \mathbb{Z}^d : 2^{i-1} \le |(k, j)|_2 < 2^i\}.$$

Let $m_X = \sum_i \varepsilon_i \mathbb{1}_{X_i}$, with $|\varepsilon_i| \le 1$. Then $|\Delta_t^\alpha m| \le 2^{|\alpha|}$. Note that

$$|(k, j)|_2 \simeq |k|_2 + |j|_2 \simeq |k - j|_2 + |j|_2 \simeq |k - j|_\infty + |j|_\infty \simeq |k - j|_\infty + |k|_\infty.$$

We can find a constant K_d which only depends on d such that the set

$$\{(s_0, s_0 + t) : t \in E_n\} \cup \{(s_0 + t, s_0) : t \in E_n\}$$

intersects with at most K_d many X_i's for any fixed s_0. Since $\bigcup_{0\le i\le n} X_i$ is convex for all n, we conclude that there are at most $2^d K_d$ many nonzero terms in the two summations in (4-22), and the summations are bounded by $2^d K_d$. So (4-22) is satisfied and $M_m = \sum_i \varepsilon_i P_{X_i}$ is bounded on S^p for any $1 < p < \infty$.

4.3. ***The case of continuous indices.*** We explain in this section that Theorems 1.1 and 4.3 extend to the continuous case by approximation. Let $S^p(\mathbb{R}^d)$ be the space of Schatten p-class operators acting on the Hilbert space $L^2(\mathbb{R}^d)$. We identify $S^2(\mathbb{R}^d)$ as $L^2(\mathbb{R}^d \times \mathbb{R}^d)$, so for $A \in S^2(\mathbb{R}^d)$ we can talk about its pointwise value $a_{s,t}$. For $m \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, we consider the Schur-multiplier-type map

$$M_m(A) = (m(s, t)a_{s,t})_{s,t\in\mathbb{R}^d}.$$

Motivated by the work of [Lafforgue and de la Salle 2011; Conde-Alonso et al. 2023], we wish to find sufficient conditions on m so that M_m extends to a bounded map with respect to the S^p-norm for $1 < p < \infty$.

Theorem 4.5. *For $p \in (1, \infty)$, consider the Schur multiplier M_m on $S^p(\mathbb{R})$ with symbol $m(\cdot, \cdot)$ in $L^\infty(\mathbb{R}^2)$ whose partial derivatives are continuous on $(-2^{j+1}, -2^j) \cup (2^j, 2^{j+1})$ for all $j \in \mathbb{Z}$. Suppose there exists an absolute constant C such that, for all $j \in \mathbb{Z}$ and $x, y \in \mathbb{R}$,*

$$\int_{-2^{j+1}}^{-2^j} |\partial_1 m(y + t, y)|\, dt + \int_{2^j}^{2^{j+1}} |\partial_1 m(y + t, y)|\, dt \le C, \tag{4-23}$$

$$\int_{-2^{j+1}}^{-2^j} |\partial_2 m(x, x + t)|\, dt + \int_{2^j}^{2^{j+1}} |\partial_2 m(x, x + t)|\, dt \le C. \tag{4-24}$$

Then, the Schur multiplier M_m extends to a bounded map on $S^p(\mathbb{R})$ with $\|M_m\| \le C \max\{p^3, 1/(p - 1)^3\}$.

Proof. Let $\mathcal{D}_k$ be the σ-algebra generated by dyadic cubes

$$Q_{k,s,t} = \left(\frac{s}{2^k}, \frac{s+1}{2^k}\right] \times \left(\frac{t}{2^k}, \frac{t+1}{2^k}\right], \quad s, t \in \mathbb{Z}.$$

Then $(\mathcal{D}_k)_{k=1}^{\infty}$ is the usual dyadic filtration for $\mathbb{R}^2$. Given $m \in L^\infty(\mathbb{R}^2)$, let $m_k = \mathbb{E}_k(m)$ be the conditional expectation of m with respect to the σ-algebra $\mathcal{D}_k$. That is to say

$$m_k(x) = \sum_{Q \in \mathcal{D}_k} \frac{1}{|Q|}\left[\int_Q m(y)\,dy\right]\chi_Q(x) \quad \text{for all } x \in \mathbb{R}^2.$$

Let $L^2(\mathbb{R}, \mathcal{D}_k)$ be the L^2 space of all $\mathcal{D}_k$-measurable functions. Let $\tilde{m}_k(s,t) = m(s/2^k, t/2^k)$ for $s, t \in \mathbb{Z}$. Note that $S^p(L^2(\mathbb{R}, \mathcal{D}_k))$ is isometrically isomorphic to $S^p(\ell^2(\mathbb{Z}))$. We see that $M_{\tilde{m}_k}$ extends to a bounded Schur multiplier on $S^p(\ell^2(\mathbb{Z}))$ with the same norm if M_{m_k} extends to a bounded Schur multiplier on $S^p(L^2(\mathbb{R}, \mathcal{D}_k))$ and vice versa. By Lemma 1.11 of [Lafforgue and de la Salle 2011],

$$\|M_m\| = \overline{\lim}_{k\to\infty}\|M_{m_k}\| = \overline{\lim}_{k\to\infty}\|M_{\tilde{m}_k}\|.$$

So, we need to show that $M_{\tilde{m}_k}$ satisfies conditions (1-4), (1-5). First, we verify condition (1-5). For each $j \in \mathbb{N}$ and $s \in \mathbb{Z}$,

$$\begin{aligned}
&\sum_{\ell=0}^{2^j-2} |m_k(s, s+2^j+\ell+1) - m_k(s, s+2^j+\ell)| \\
&\quad = 2^{2k}\sum_{\ell=0}^{2^j-2}\left|\int_{\frac{s}{2^k}}^{\frac{s+1}{2^k}}\int_{\frac{2^j+\ell}{2^k}}^{\frac{2^j+\ell+1}{2^k}} M\Big(y, y+x+\frac{1}{2^k}\Big)\,dx\,dy - \int_{\frac{s}{2^k}}^{\frac{s+1}{2^k}}\int_{\frac{2^j+\ell}{2^k}}^{\frac{2^j+\ell+1}{2^k}} M(y, y+x)\,dx\,dy\right| \\
&\quad = 2^{2k}\sum_{\ell=0}^{2^j-2}\left|\int_{\frac{s}{2^k}}^{\frac{s+1}{2^k}}\int_{\frac{2^j+\ell}{2^k}}^{\frac{2^j+\ell+1}{2^k}}\int_{y+x}^{y+x+\frac{1}{2^k}} \partial_2 M(y,t)\,dt\,dx\,dy\right| \\
&\quad \le 2^{2k}\sum_{\ell=0}^{2^j-2}\int_{\frac{s}{2^k}}^{\frac{s+1}{2^k}}\int_{\frac{2^j+\ell}{2^k}}^{\frac{2^j+\ell+1}{2^k}}\int_{y+x}^{y+x+\frac{1}{2^k}} |\partial_2 M(y,t)|\,dt\,dx\,dy \\
&\quad = 2^{2k}\int_{\frac{s}{2^k}}^{\frac{s+1}{2^k}}\int_{\frac{2^j}{2^k}}^{\frac{2^{j+1}-1}{2^k}}\int_{y+x}^{y+x+\frac{1}{2^k}} |[\partial_2(M)](y,t)|\,dt\,dx\,dy \\
&\quad = 2^{2k}\int_{\frac{s}{2^k}}^{\frac{s+1}{2^k}}\int_{\frac{2^j}{2^k}}^{\frac{2^{j+1}-1}{2^k}}\int_{y+\frac{2^j}{2^k}}^{y+\frac{2^{j+1}}{2^k}} \chi_{(x,x+\frac{1}{2^k})}(t)|[\partial_2(M)](y,t)|\,dt\,dx\,dy \\
&\quad = 2^{2k}\int_{\frac{s}{2^k}}^{\frac{s+1}{2^k}}\int_{y+\frac{2^j}{2^k}}^{y+\frac{2^{j+1}}{2^k}}\int_{\frac{2^j}{2^k}}^{\frac{2^{j+1}-1}{2^k}} \chi_{(t-\frac{1}{2^k},t)}(x)\,dx|[\partial_2(M)](y,t)|\,dt\,dy \\
&\quad \le 2^{k}\int_{\frac{s}{2^k}}^{\frac{s+1}{2^k}}\int_{y+\frac{2^j}{2^k}}^{y+\frac{2^{j+1}}{2^k}} |\partial_2 M(y,t)|\,dt\,dy \le \sup_{y\in\mathbb{R}}\int_{\frac{2^j}{2^k}}^{\frac{2^{j+1}}{2^k}} |[\partial_2(M)](y, y+t)|\,dt \le A.
\end{aligned} \tag{4-25}$$

The last inequality follows from the assumption in (4-24). So, condition (1-5) is verified. Applying the same argument, we utilize the assumption in (4-23) to prove condition (1-4). Therefore, by Theorem 1.1, $\|M_m\| = \overline{\lim}_{k\to\infty}\|M_{\tilde{m}_k}\| \le C\max\{p^3, 1/(p-1)^3\}$. □

Similarly, Theorem 4.3 and Example 4.4 have analogues in the continuous case as well.

Theorem 4.6. *Define $E_j := \{t \in \mathbb{R}^d : 2^{j-1} \le |t|_\infty < 2^j\}$ for $j \in \mathbb{Z}$. For $p \in (1,\infty)$, consider the Schur multiplier $m \in L^\infty(\mathbb{R}^{2d})$ whose partial derivatives are continuous up to the boundary of E_k for all $k \in \mathbb{Z}$. Assume there exists a constant C such that*

$$\int_{(t_\alpha, r^{(j)}_{1-\alpha}) \in E_j} |\partial^\alpha m(s, s+t)|\, dt_\alpha \le C, \tag{4-26}$$

$$\int_{(t_\alpha, r^{(j)}_{1-\alpha}) \in E_j} |\partial^\alpha m(s+t, s)|\, dt_\alpha \le C \tag{4-27}$$

for any $j \in \mathbb{Z}$, $s \in \mathbb{R}^d$ and any $r^{(j)} \in \mathbb{R}^d$ with $|r_i^{(j)}| = 2^{j-1}$ for all $1 \le i \le d$. Then, the Schur multiplier M_m extends to a bounded operator on $S^p(\mathbb{R}^d)$ for all $1 < p < \infty$ with $\|M_m\| \le C_d \max\{p^{d+2}, 1/(p-1)^{d+2}\}$. Here t_α is defined as in Section 4.2.

Remark 4.7. The Schur multipliers in all theorems of this article are also completely bounded on S^p for $1 < p < \infty$; the arguments are exactly the same.

5. Discussions

5.1. *Counterexamples.*

(1) We show in the following that (1-4) alone is not sufficient for the boundedness of M_m.

Choose a large $K \in \mathbb{N}$. Let $m(s,t) = \exp(i2\pi kj/K)$ if $s = 2^k$, $t = 2^j$ for some $j, k \in \mathbb{N}$ satisfying $1 \le j < k \le K$, and $m(s,t) = 0$ for other $s, t \in \mathbb{N}$. Let $\tilde{m}(k,j) = \exp(i2\pi kj/K)$ if $1 \le j < k \le K$, and $\tilde{m}(k,j) = 0$ for other $k, j \in \mathbb{N}$. Let U be the partial isometry on $\ell_2(\mathbb{N})$ sending e_k to e_{2^k}. Then, we have $M_{\tilde{m}}(A) = U^* M_m(UAU^*)U$ for any $A \in S^p(\ell_2(\mathbb{N}))$ and $\|M_{\tilde{m}}\| \le \|M_m\|$.

Note that, for any N, j given, there exists at most one k (actually $k = N$) satisfying $k > j$ and

$$2^{N-1} - 1 \le |2^k - 2^j| < 2^N.$$

Using the fact that $|m(s,t)| \le 1$, for any N, t, we get

$$\sum_{2^{N-1} \le |r| < 2^N} |m(t+r+1, t) - m(t+r, t)| \le 2,$$

because there are at most two nonzero terms in the sum above. This means $(m(s,t))_{s,t}$ satisfies the row condition (1-4). On the other hand, if $s = 2^N$, then $|m(s,t) - m(s,t+1)|$ does not vanish if t or $t+1$ has the form of 2^j, $j = 1, \dots, N-1$, by the definition of $m(s,t)$. Hence we have

$$\sum_{2^{N-1} \le |t-s| < 2^N} |m(s,t+1) - m(s,t)| = \sum_{2^{N-1} \le s-t < 2^N} |m(s,t+1) - m(s,t)| = 2(N-1),$$

which shows that $(m(s,t))_{s,t}$ fails the column condition (1-5).

Let A be the $K \times K$ matrix $(\exp(-i2\pi kj/K))_{1 \le k, j \le K}$. Then A has S^p norm $K^{1/2+1/p}$. $M_{\tilde{m}}(A)$ is the lower triangular matrix with all nonzero coefficients being 1 which has S^p norm $\simeq K$ for any given p, $1 < p < \infty$. This shows that $K^{1/2-1/p} \lesssim \|M_{\tilde{m}}\| \le \|M_m\|$. We then conclude that (1-4) alone is not sufficient for the boundedness of M_m. By symmetry, (1-5) alone is not sufficient for the boundedness of M_m either.

(2) A smooth version of the example above implies that neither the assumption (3-18) nor the assumption (3-19) is removable in Corollary 3.1. Indeed, fix a large $K > 0$, let

$$m_1(s,t) = \exp\left(\frac{i2\pi \log_2 s \log_2 t}{K}\right)$$

for $1 \le s \le t \le 2^K,\ s,t \in \mathbb{N}$,

$$m_1(s,t) = \frac{2^{K+1}-t}{2^K}\exp(i2\pi \log_2 s)$$

for $1 \le s \le t,\ 2^K < t \le 2^{K+1},\ s,t \in \mathbb{N}$ and $m_1(s,t) = 0$ otherwise. Then m_1 satisfies (3-18) because

$$\left|\frac{\partial}{\partial t}\exp\left(\frac{i2\pi \log_2 s \log_2 t}{K}\right)\right| \lesssim \frac{1}{t} \le \frac{1}{t-s}$$

whenever $s < t \le 2^K$. Assuming the sufficiency of (3-18) would imply the uniform boundedness of M_{m_1} for all $1 < p < \infty$, which is wrong because $M_m(A) = M_{m_1}(VAV)$ for $A \in S^p(\ell^2(\mathbb{N}))$ and m, V defined above. We conclude that neither the assumption (3-18) nor the assumption (3-19) is removable.

(3) Let

$$F_{N,t} = \{(s,t) \in \mathbb{N}\times\mathbb{N} : 2^{N-1} \le |s-t| < 2^N\}$$

for $N, t \in \mathbb{N}$. Let $Q_{N,t}$ be the projection from $S^2(\ell_2(\mathbb{N}))$ onto the span of $\{e_{s,t} : (s,t) \in F_{N,t}\}$. One may wonder whether Corollary 1.2 can be improved so that the Schur multiplier $S_\varepsilon = \sum_{N,t\in\mathbb{N}} \varepsilon(N,t) Q_{N,t}$ is bounded for any sequence $|\varepsilon(N,t)| \le 1$. This is impossible as well.[2] To see this, let $\varepsilon(N,t) = \exp(i2\pi Nj/K)$ if $t = 2^j$ for some $j \in \mathbb{N}$ and $j < N \le K$. Let $\varepsilon(N,t) = 0$ otherwise. Let V be the projection on $\ell^2(\mathbb{N})$ such that $V(e_i) = e_i$ if $i = 2^k$ for some $k \in \mathbb{N}$, and $V(e_i) = 0$ otherwise. Then, for M_m defined in the first example and $A \in S^p(\ell_2(\mathbb{N}))$, we have

$$M_m(A) = S_\varepsilon(VAV).$$

Therefore, $K^{1/2-1/p} \lesssim \|S_\varepsilon\|$.

5.2. *Operator-valued symbol.* The Schatten p class has a natural operator space structure inherited from the operator space complex interpolation $S^p = (S^\infty, S^1)_{1/p}$, $1 < p < \infty$. Pisier [1998, Lemma 1.7] proved that, with respect to S^p's natural operator space structure, a map M on $S^p(\ell^2)$ is completely bounded if and only if $M \otimes \mathrm{id}_{S^p(H)}$ is bounded on $S^p(S^p) = S^p(\ell^2 \otimes H)$ for any separable Hilbert space H. We will explain an operator-valued version of Theorem 1.1 which particularly implies the complete boundedness of the Schur multipliers considered in Theorem 1.1. We will assume the readers are familiar with the terminology of operator spaces in this subsection.

We will consider $A \in S^2(\ell^2(\mathbb{Z}) \otimes H)$ with H a separable Hilbert space. We present A in its matrix form $(a_{i,j})_{i,j\in\mathbb{Z}}$ with $a_{i,j} \in S^2(H)$. More precisely, denote by e_i the canonical basis of ℓ_2, let $e_{j,i}$ be the rank-1 operator on ℓ^2 sending e_i to e_j. Denote by tr (resp. τ) the canonical trace on $B(\ell^2(\mathbb{Z}))$ (resp. $B(H)$). We set

$$a_{i,j} = (\mathrm{tr} \otimes \mathrm{id})(A(e_{j,i} \otimes \mathrm{id}_H)).$$

[2] We can get the same conclusion for sequences $\varepsilon_k = \pm 1$ by choosing Hadamard orthogonal matrices instead of the matrices $(\exp((-i2\pi kj)/K))_{1\le k,j\le K}$.

Let $\mathcal{M}$ be a finite von Neumann algebra with a normal faithful tracial state τ. Given an $\mathcal{M}$-valued bounded function m on $\mathbb{Z}\times\mathbb{Z}$ and $A \in S^2(\ell^2(\mathbb{Z})\otimes H)$ in its matrix form $(a_{i,j})_{i,j\in\mathbb{Z}}$, we define $M_m(A)$ as the matrix

$$M_m(A) = (m_{i,j}\otimes a_{i,j})_{i,j}. \tag{5-1}$$

We will show that an analogue of Theorem 1.1 holds, that is, there exists $C_p \simeq (p^2/(p-1))^3$ for $1 < p < \infty$ such that

$$\|M_m(A)\|_{L^p(\mathcal{M}\otimes B(\ell^2(\mathbb{Z})\otimes H))} \le C_p \|A\|_{S^p(\ell^2\otimes H)}$$

for all $A \in S^2 \cap S^p$. By the density of $S^2\cap S^p$, M_m extends to a bounded operator from $S^p(\ell^2(\mathbb{Z})\otimes H)$ to $L^p(\mathcal{M}\otimes B(\ell^2(\mathbb{Z})\otimes H))$ when m satisfies Marcinkiewicz-type conditions. When $\mathcal{M}=\mathbb{C}$, this implies the complete boundedness of M_m in Theorem 1.1 by Pisier's result. We will need Pisier's $L_\infty(\ell_1)$ norm to express this Marcinkiewicz-type condition.

Definition 5.1 (Pisier's $L^\infty(\ell_1)$ norm). Given N-tuples $(x_1,\dots,x_N)$ in $\mathcal{M}$, set

$$\|x\|_{L^\infty(\mathcal{M};\ell_1)} = \inf\left\{\left\|\left(\sum a_j a_j^*\right)^{\frac12}\right\|\cdot\left\|\left(\sum b_j^* b_j\right)^{\frac12}\right\|\right\}, \tag{5-2}$$

where the infimum runs over all possible factorizations $x_j = a_j b_j$, with $a_j, b_j \in \mathcal{M}$.

When $x_k \ge 0$, we have $\|x\|_{L^\infty(\mathcal{M};\ell_1)} = \left\|\sum_k |x_k|\right\|$ but the two quantities are not comparable in general. Pisier showed that $\|x\|_{L^\infty(\mathcal{M};\ell_1)} < \infty$ if and only if there is a decomposition $x_k = x_{k,1}-x_{k,2}+ix_{k,3}-ix_{k,4}$ such that $x_{k,\ell}\ge 0$ and $\|(x_{k,\ell})_k\|_{L^\infty(\mathcal{M};\ell_1)} < \infty$ for all $\ell = 1,2,3,4$.

Given M_m defined as in (5-1), let

$$\Delta_s m(s,t) = m(s+1,t) - m(s,t), \qquad \Delta_t m(s,t) = m(s,t+1) - m(s,t)$$

for $s,t\in\mathbb{Z}$.

Theorem 5.2. *M_m defined as in (5-1) extends to a bounded map from Schatten p-classes $S^p(\ell^2\otimes H)$ to $L^p(\mathcal{M}\otimes B(\ell^2(\mathbb{Z})\otimes H))$ for all $1<p<\infty$ with bounds $\lesssim (p^2/(p-1))^3$ if m is bounded in $\mathcal{M}$ and there is a constant C such that,*

(i) *for any $n\in\mathbb{N}$, $t\in\mathbb{Z}$,*

$$\|(\Delta_s m(s+t,t))_{2^{n-1}\le|s|<2^n}\|_{L^\infty(\mathcal{M};\ell_1)} < C, \tag{5-3}$$

(ii) *for any $n\in\mathbb{N}$, $s\in\mathbb{Z}$,*

$$\|(\Delta_t m(s,s+t))_{2^{n-1}\le|t|<2^n}\|_{L^\infty(\mathcal{M};\ell_1)} < C. \tag{5-4}$$

Sketch of proof. Define $\tilde m(s,t) = m(s,t)\otimes 1_{\ell_2\otimes H}$ and $\tilde a_{s,t} = 1_{\mathcal{M}}\otimes a_{s,t}$. Then $\tilde m(s,t)$ commutes with $a_{s',t'}$ for any $s,t,s',t'\in\mathbb{Z}$. Let

$$\begin{aligned}\widetilde M_l(j) &= \sum_{s\in\mathbb{Z}} \tilde m(s,s-j)\otimes e_{s,s}, \qquad \widetilde M_r(j) = \sum_{s\in\mathbb{Z}} \tilde m(s+j,s)\otimes e_{s,s},\\ \tilde A(j) &= \sum_{s\in\mathbb{Z}} \tilde a_{s,s-j}\otimes e_{s,s-j} = \sum_{s\in\mathbb{Z}} \tilde a_{s+j,s}\otimes e_{s+j,s},\end{aligned} \tag{5-5}$$

with $e_{s,t}$ the canonical basis of $S^2(\ell_2(\mathbb{Z}))$. Let $f(z)=\sum_{j\in\mathbb{Z}}\tilde{A}(j)z^j$ and

$$T_{\widetilde{M}}f(z)=\sum_{j\in\mathbb{Z}}\widetilde{M}_l(j)\tilde{A}(j)z^j. \tag{5-6}$$

We still have

$$T_{\widetilde{M}}f(z)=\sum_{j\in\mathbb{Z}}\tilde{A}(j)\widetilde{M}_r(j)z^j \tag{5-7}$$

and the identities

$$\|f\|_{L^p(\mathcal{M}\otimes B(\ell^2(\mathbb{Z})\otimes H))}=\|A\|_{S^p(\ell_2\otimes H)},$$
$$|T_{\widetilde{M}}f\|_{L^p(\mathcal{M}\otimes B(\ell^2(\mathbb{Z})\otimes H))}=\|M_m(A)\|_{S^p(\ell_2\otimes H)}.$$

Moreover, the conditions (5-3) and (5-4) imply that

$$\|\Delta_l\widetilde{M}(j)_{2^{n-1}<|j|\le 2^n}\|_{L^\infty(\mathcal{M}\otimes B(\ell^2(\mathbb{Z})),\ell_1)},\ \|\Delta_r\widetilde{M}(j)_{2^{n-1}<|j|\le 2^n}\|_{L^\infty(M\otimes B(\ell^2(\mathbb{Z})),\ell_1)}<C$$

for

$$\Delta_l\widetilde{M}(j)=\widetilde{M}_l(j+1)-\widetilde{M}_l(j),\quad \Delta_r\widetilde{M}(j)=\widetilde{M}_r(j+1)-\widetilde{M}_r(j).$$

After these, it is not hard to check that the arguments for the proof of Theorem 1.1 work as well for the tensor case. □

Corollary 5.3. *The Schur multipliers considered in Theorem 1.1 are completely bounded on the Schatten classes S^p, $1<p<\infty$, with bounds $\lesssim (p^2/(p-1))^3$ with respect to their natural operator space structure.*

Remark 5.4. The optimal constant for the L^p bounds of the classical Marcinkiewicz Fourier multipliers is $p^{3/2}$ as $p\to\infty$ [Tao and Wright 2001]. It is unclear what is the optimal asymptotic order for the S^p-bounds of the Schur multipliers in Theorem 1.1.

Open Question. Assume m is a bounded map on $\mathbb{Z}\times\mathbb{Z}$ such that

$$\sum_s|m(k,j_s)-m(k,j_{s+1})|^2<C$$

for all possible increasing sequences $j_s\in\mathbb{Z}$. Does M_m extend to a bounded map on S^p for all $1<p<\infty$?

Remark 5.5. The authors heard this question from Potapov and Sukochev. They told the authors that it stems from the work of Birman and Solomyak on double operator integrals. The third author noticed Theorem 1.1 during his effort of attacking this question.

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CHIAN YEONG CHUAH: chuah.21@osu.edu
Department of Mathematics, The Ohio State University, Columbus, OH, United States

ZHEN-CHUAN LIU: liu.zhenchuan@uam.es
Departamento de Matemáticas, Universidad Autónoma de Madrid, Madrid, Spain

TAO MEI: tao_mei@baylor.edu
Department of Mathematics, Baylor University, Waco, TX, United States

and

Department of Mathematics, Texas A&M University, College Station, TX, United States

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