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DOUBLE DUALS AND HILBERT MODULES

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Let A be a C^*-algebra, H be a Hilbert A-module and $K(H)$ be the closure of the set of finite-rank module maps. We show that the W^*-algebra of all bounded A^{**}-module maps on the smallest self-dual Hilbert A^{**}-module containing H is isomorphic to $K(H)^{**}$ as W^*-algebras. We also show that the unit ball of H is closed in $H^\sharp$, the dual of H in an A-weak topology of $H^\sharp$, and the unit ball of H is also dense in the unit ball of $H^\sharp$ in a weak* topology. Some versions of the Kaplansky density theorem for Hilbert C^*-modules are also presented.

1. Introduction

Hilbert C^*-modules as a generalization of Hilbert spaces were first introduced by I. Kaplansky [1953] in special cases and later by W. Paschke [1973] for general C^*-algebras. Hilbert C^*-modules are crucial to Kasparov's formulation of KK-theory [1980]. Early applications also include C^*-algebraic quantum group theory; see [Baaj and Skandalis 1993]. Later, in the study of Cuntz semigroups in connection with the classification of amenable C^*-algebras, Hilbert C^*-modules play an important role; see, for example, [Brown and Ciuperca 2009; Brown and Lin 2025; Coward et al. 2008; Ortega et al. 2011].

Let A be a C^*-algebra. Unlike Hilbert spaces, bounded module maps on a Hilbert A-module H may not have adjoints and the dual module $H^\sharp$, i.e., the Banach A-module of all bounded module maps from H to A, may not be identified as elements in H. Moreover, the C^*-algebra $L(H)$ of all bounded module maps with adjoints may not be a W^*-algebra. If $H_0 \subset H$ is a Hilbert A-submodule, a bounded module map $\varphi : H_0 \to A$ may not be extended to a bounded module map from H to A. In general, one should not expect that H can be decomposed into an orthogonal direct sum of H_0 and its orthogonal complement. In fact, H_0 may not even have an orthogonal complement. Study of these phenomena may be found, for example, in [Lin 1991a; 1992] and more recently in [Brown and Lin 2025].

However, Paschke [1973] found that, if A is a W^*-algebra, then the dual module $H^\sharp$ of a Hilbert A-module H can be made into a Hilbert A-module in a natural way which extends H, and $H^\sharp$ is a self-dual Hilbert A-module. Even if A is not a W^*-algebra, one can extend H into an A^{**}-module $H \bullet A^{**}$ naturally. Then its dual $H^\sim := (H \bullet A^{**})^\sharp$ becomes a self-dual Hilbert A^{**}-module containing H. In fact, $H^\sim$ is the smallest self-dual Hilbert A^{**}-module containing H as a Hilbert A-submodule; see Proposition 3.2. Paschke showed that the Banach algebra of all bounded module maps on $H^\sim$ becomes a W^*-algebra.

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For a Hilbert A-module H, the rank-1 module maps are the module maps T of the form $T(h)=x\langle y,h\rangle$ for all $h\in H$ (and fixed $x,y\in H$, where $\langle\cdot,\cdot\rangle$ is the A-valued inner product). Denote by $F(H)$ the linear span of rank-1 module maps and denote by $K(H)$ the norm closure of $F(H)$. $K(H)$ is a C^*-algebra and an important algebra related to the Hilbert module H. It was proved by Kasparov [1980, Theorem 1] that the C^*-algebra $L(H)$ may be identified with $M(K(H))$, the multiplier algebra of $K(H)$, and it was proved in [Lin 1991a] that the Banach algebra of all bounded module maps on H is identified with the left multipliers of $K(H)$. (All Hilbert A-modules considered in this paper are right A-modules.) Over the decades, we eventually realized that it is rather convenient to work in $B(H^\sim)$ in many occasions as we study module maps on a Hilbert module H. It is not difficult to establish a natural normal homomorphism $\Psi: K(H)^{**}\to B(H^\sim)$ which extends beyond $M(K(H))$ and $LM(K(H))$. It remained unknown for many years whether Ψ is an isomorphism. The original motivation of this paper is to show that indeed Ψ is an isomorphism between W^*-algebras $K(H)^{**}$ and $B(H^\sim)$.

As we study the relation among Hilbert modules H, $H\bullet A^{**}$ and $H^\sim$, naturally we ask: how dense is H in $H\bullet A^{**}$ and in $H^\sim$? Since $H^\sim=(H\bullet A^{**})^\sharp$, the dual of $H\bullet A^{**}$, one may also ask about the density of H in $H^\sharp$ in general.

We first note that it was shown (Theorem 6.1 of [Brown and Lin 2025]) that H is dense in $H^\sharp$ in an A-weak topology. More precisely, for any $\xi\in H^\sharp$, there is a net $\{x_\alpha\}$ in H with $\|x_\alpha\|\le\|\xi\|$ for all α such that $\lim_\alpha\|\xi(x)-\langle x_\alpha,x\rangle\|=0$ for all $x\in H$. However, we show here that the unit ball of H is closed in $H^\sharp$ in the topology where $x_\alpha\to\xi$ if and only if $\lim_\alpha\|\langle\xi-x_\alpha,\zeta\rangle\|=0$ for all $\zeta\in H^\sharp$, and where the inner product is extended to $H^\sim$.

On the other hand, it is easy to see that, for any $\xi\in H\bullet A^{**}$, there is a net $\{x_\lambda\}$ in H such that $\lim_\lambda\pi_U(\langle x_\lambda,y\rangle)(v)=\pi_U(\langle\xi,y\rangle)(v)$ for all $y\in H\bullet A^{**}$ and $v\in H_U$, where H_U is the Hilbert space corresponding to the universal representation π_U of A. To be a more useful approximation, one may ask whether the net can be chosen to be bounded (by $\|\xi\|$). We will present a Kaplansky-style density theorem. Perhaps a more interesting question is: how dense is H in $H^\sim=(H\bullet A^{**})^\sharp$? Since $H^\sim$ is the dual of $H\bullet A^{**}$, it is relatively easy to show that, for any $\zeta\in H^\sim$, there is a net $\{z_\alpha\}$ in H such that

$$\lim_\lambda f(\langle z_\alpha,y\rangle)=f(\langle\zeta,y\rangle)\quad\text{for all } y\in H\bullet A^{**}\text{ and } f\in A^*.$$

It is more challenging to show that y can be replaced by any element in $H^\sim=(H\bullet A^{**})^\sharp$. We show that the unit ball of H is actually dense in the unit ball of $H^\sim$ in the weak* topology (as $H^\sim$ is a conjugate space), another Kaplansky-style density theorem. In fact, we show a stronger density theorem that, for any $\xi\in H^\sim$, there is a net $\{x_\alpha\}$ in H with $\|x_\alpha\|\le\|\xi\|$ such that

$$\lim_\alpha f(\langle\xi-x_\alpha,\xi-x_\alpha\rangle)=0\quad\text{for all } f\in A^*.$$

2. Self-duals

Definition 2.1. Let A be a C^*-algebra. Denote by $\tilde A$ the minimum unitization of A. We use the following convention: if A is a C^*-subalgebra of a unital C^*-algebra B, we write $1_{\tilde A}=1_B$ if either A is unital and $1_A=1_{\tilde A}=1_B$, or $A^\perp=\{b\in B: ba=ab=0\}=\{0\}$, and we unitize A by adjoining 1_B to form $\tilde A\subset B$.

Definition 2.2. Let X be a Hilbert space and $B(X)$ be the C^*-algebra of all bounded linear operators on X. Suppose that $A \subset B(X)$. Then $\bar{A}^{\mathrm{SOT}}$ is the closure of A in the strong operator topology. Note that if $\{e_\alpha\}$ is an approximate identity for A, then $e_\alpha \nearrow 1_M$, i.e., e_α increasingly converges to the identity of $M = \bar{A}^{\mathrm{SOT}}$ in the strong operator topology as well as in the weak* topology (of M). In particular, we may write $1_{\tilde{A}} = 1_M$.

This works particularly for the pair A and A^{**} (where X is H_u, the Hilbert space corresponding to the universal representation of A).

In general, if M is a W^*-algebra, we denote by M_* the predual of M.

Definition 2.3. Let A be a C^*-algebra. In this paper, we use the formal definition of Hilbert modules in [Paschke 1973] and consider only right A-modules. Recall that a linear space H is a pre-Hilbert module if it is also a right A-module with an inner product $H \times H \to A$ satisfying the following properties: for any $x, y, z \in H$, $a \in A$ and $\lambda \in \mathbb{C}$,

(1) $\langle x, \lambda y + z\rangle = \langle x, y\rangle + \lambda\langle x, y\rangle$,

(2) $\langle x, ya\rangle = \langle x, y\rangle a$,

(3) $\langle x, y\rangle^* = \langle y, x\rangle$,

(4) $\langle x, x\rangle \geq 0$; if $\langle x, x\rangle = 0$, then $x = 0$.

Define $\|x\| = \|\langle x, x\rangle\|^{1/2}$ for $x \in H$. Then H becomes a normed space. H is a Hilbert A-module if H is complete with this norm.

Denote by $H^\sharp$ the Banach space of all bounded module maps from H into A. A Hilbert A-module is said to be self-dual if, for every $f \in H^\sharp$, there is $x \in H$ such that

$$f(y) = \langle x, y\rangle \quad \text{for all } y \in H.$$

Denote by $B(H)$ the Banach algebra of all bounded module maps from H into itself, and by $L(H)$ the C^*-algebra of all those bounded module maps T with an adjoint T^* in $L(H)$ defined by

$$\langle T(x), y\rangle = \langle x, T^*(y)\rangle \quad \text{for all } x, y \in H.$$

Let $F(H)$ be the algebra of all finite-rank module maps, i.e., the linear span of all bounded module maps of the form $\theta_{x,y} : H \to H$ defined by

$$\theta_{x,y}(\xi) = x\langle y, \xi\rangle$$

for all $\xi \in H$ and $x, y \in H$. Denote by $K(H)$ the norm closure of $F(H)$, which is a C^*-algebra.

By Theorem 1 of [Kasparov 1980], we identify $L(H)$ with $M(K(H))$, the multiplier algebra of $K(H)$ and, by Theorem 1.5 of [Lin 1991a], $B(H)$ with $LM(K(H))$, the Banach algebra of left multipliers of $K(H)$ (in $K(H)^{**}$). If H is self-dual, then $B(H) = L(H)$.

We refer to [Kasparov 1980; Lin 1991a; 1992; Paschke 1973] for common terminologies related to Hilbert C^*-modules.

Definition 2.4. Let A be a C^*-algebra and H a Hilbert A-module. Let us give the definition of a self-dual Hilbert A^{**}-module $H^\sim$; see Definition 1.3 of [Lin 1991a].

We may view H as a Hilbert $\tilde{A}$-module. Let B be a unital C^*-algebra containing A and $1_{\tilde{A}} = 1_B$ (see the convention in Definition 2.1). The algebraical tensor product $H \otimes B$ becomes a right B-module if we set $(h \otimes a) \cdot b = h \otimes ab$ for any $h \in H$ and $a, b \in B$. Define $\langle -, - \rangle : H \otimes B \times H \otimes B \to B$ by

$$\left\langle \sum_i h_i \otimes a_i, \sum_j x_j \otimes b_j \right\rangle = \sum_{i,j} a_i^* \langle h_i, x_j \rangle b_j$$

and $N = \{z \in H \otimes A^{**} : \langle z, z \rangle = 0\}$. Then $(H \otimes B)/N$ becomes a pre-Hilbert B-module (see Section 4 of [Paschke 1973], but exchange B with A). Denote by $H \bullet B := ((H \otimes B)/N)^-$ (the completion of) the Hilbert B-module.

We are particularly interested in the case that $B = A^{**}$. We view $\tilde{A}$ as a C^*-subalgebra of A^{**}. Then $H^\sim := (H \bullet A^{**})^\sharp$ is a self-dual Hilbert A^{**}-module.

Note that $\tilde{A}$ is ultraweakly dense in A^{**} (since A is). By applying the result [Paschke 1973, Theorem 4.2] to the pair A^{**} (as A in that result) and $\tilde{A}$ (as B in that result, see also the remark right after the proof of that result), we obtain an isometric (surjective) isomorphism $\iota : H^\sim := (H \bullet A^{**})^\sharp \to B(H, A^{**})$, with $B(H, A^{**})$ the Banach space of all bounded A-module maps from H to A^{**} (written as $M(H, A^{**})$ in that same result).

Let $x \in H$ and $b \in B$. Then

$$\|(x \otimes b)/N\|^2 = \|b^* \langle x, x \rangle b\| \le \|x\|^2 \|b^* b\|.$$

Hence

$$\|(x \otimes b)/N\| \le \|x\| \|b\|.$$

In what follows, for $x \in H$ and $b \in B$, we write $x \bullet b := (x \otimes b)/N$.

In general, if E is a self-dual Hilbert module, then $B(E) = L(E)$; see [Paschke 1973, Corollary 3.5]. If in addition A is a W^*-algebra, $B(E)$ is also a W^* -algebra; see [Paschke 1973, Proposition 3.11].

Let us recall the description of the predual of $B(E)$ in this case. Denote by $E_\sim$ the linear space E with the "twisted" scalar multiplication (i.e., $\lambda x = \bar{\lambda} x$ for $x \in E$ and $\lambda \in \mathbb{C}$) and consider $E_\sim \otimes E \otimes A_*$ with the greatest cross-norm, where A_* is the usual predual of the W^*-algebra A. For each $T \in B(E)$, define a linear functional $\check{T}$ on $E_\sim \otimes E \otimes A_*$ by

$$\check{T}\left(\sum_{j=1}^n x_j \otimes y_j \otimes g_j\right) = \sum_{j=1}^n g_j(\langle T(x_j), y_j \rangle)$$

for $x_j, y_j \in E$ and $g_j \in A_*$, $1 \le j \le n$. The map $T \to \check{T}$ is a linear isometry of $B(E) = L(E)$ into $(E_\sim \otimes E \otimes A_*)^*$. It was shown [Paschke 1973, Proposition 3.10] that $\check{B(E)}$ is weak*-closed in $E_\sim \otimes E \otimes A_*$. A bounded net $\{T_\alpha\}$ in $B(E)$ converges to $T \in B(E)$ in the weak* topology if and only if

$$f(\langle T_\alpha(x), y \rangle) \to f(\langle T(x), y \rangle) \quad \text{for all } x, y \in E \text{ and } f \in A_*$$

[Paschke 1973, Remark 3.9 and Proposition 3.10]. In particular, $B(H^\sim)$ is a W^*-algebra.

Definition 2.5. Keep the notation in Definition 2.4. Recall that H is a Hilbert A-module and $\tilde{A} \subset B$. Then $\iota : H \to H \bullet B$ defined by $x \to x \otimes 1$ is an injective map. Note that, for all $a \in A$,

$$\begin{aligned}\langle (x \cdot a) \otimes 1 - x \otimes a, (x \cdot a) \otimes 1 - x \otimes a \rangle &= \langle x \cdot a, x \cdot a \rangle - \langle x \cdot a, x \rangle a - a^* \langle x, x \cdot a \rangle + a^* \langle x, x \rangle a \\ &= a^* \langle x, x \rangle a - a^* \langle x, x \rangle a - a^* \langle x, x \rangle a + a^* \langle x, x \rangle a = 0.\end{aligned}$$

Hence $\iota(x \cdot a) = x \otimes a / N$ for all $a \in \tilde{A}$. In the case $B = A^{**}$, we then extend ι from $H^\sharp$ to $(H \bullet A^{**})^\sharp$ by

$$\iota(f)(x \bullet b) = f(x) b \quad \text{for all } x \in H \text{ and } b \in A^{**}$$

and $f \in H^\sharp$. Note that the map is a module map from $H^\sharp$ to $(H^\sim)^\sharp$, which is conjugate module isomorphic to $H^\sim$.

From now on, we may view H as a submodule of $H^\sim$ and, sometimes, omit the map ι.

The following result provides a convenient and easy fact that $H \bullet B$ is the smallest Hilbert B-module containing H as a Hilbert A-module.

Proposition 2.6. *Let A and B be a pair of C^*-algebras such that $A \subset B$, B is unital and $1_{\tilde{A}} = 1_B$. Suppose that H is a Hilbert A-module, H_1 is a Hilbert B-module and there is an embedding $\iota : H \to H_1$ as Hilbert modules, i.e., ι is a linear and A-module map such that*

$$\langle \iota(x), \iota(y) \rangle = \langle x, y \rangle \quad \text{for all } x, y \in H.$$

Then there is a unique B-module embedding $\tilde{\iota} : H \bullet B \to H_1$ such that

$$\tilde{\iota}(x \bullet b) = \iota(x) b \quad \text{for all } x \in H \text{ and } b \in B, \qquad \langle \tilde{\iota}(\xi), \tilde{\iota}(\zeta) \rangle = \langle \xi, \zeta \rangle \quad \text{for all } \xi, \zeta \in H \bullet B.$$

Proof. For any $\xi = \sum_{i=1}^n x_i \bullet a_i$, where $x_i \in H$ and $a_i \in B$ $(1 \le i \le n)$, define

$$\tilde{\iota}(\xi) = \sum_{i=1}^n \iota(x_i) a_i.$$

Then, for $\zeta = \sum_{i=1}^n y_i \bullet b_i$,

$$\langle \tilde{\iota}(\xi), \tilde{\iota}(\zeta) \rangle = \sum_{i,j}^n a_i^* \langle x_i, y_j \rangle b_j = \langle \xi, \zeta \rangle.$$

In particular,

$$\| \tilde{\iota}(\xi), \tilde{\iota}(\xi) \rangle \| = \left\| \sum_{i,j}^n a_i^* \langle x_i, x_j \rangle b_j \right\| = \|\xi\|^2.$$

Therefore $\|\tilde{\iota}\| \le 1$ on $(H \otimes B)/N$. So $\tilde{\iota}$ is uniquely extended to a contractive linear map from $H \bullet B$ into H_1. It is a B-module map. Since $(H \otimes B)/N$ is dense in $H \bullet B$,

$$\langle \tilde{\iota}(x), \tilde{\iota}(y) \rangle = \langle x, y \rangle \quad \text{for all } x, y \in H \bullet B.$$

To see this embedding is unique, let $\tilde{\iota}_1$ be another such embedding. Then $(\tilde{\iota} - \tilde{\iota}_1)|_H = 0$. For any $\xi = \sum_{i=1}^n x_i \bullet a_i$, where $x_i \in H$ and $a_i \in B$,

$$(\tilde{\iota} - \tilde{\iota}_1)(\xi) = \sum_{i=1}^n (\iota(x_i) - \iota(x_i)) \bullet a_i = 0.$$

In other words, $\tilde{\iota}_1 = \tilde{\iota}$. □

Definition 2.7. Keep the notation in Definitions 2.3, 2.4 and 2.5. Recall that $F(H)$ is the algebra of all finite-rank module maps. Define $\Psi_0 : F(H) \to F(H \bullet B) \subset B(H \bullet B)$ by

$$\Psi_0(\theta_{x,y})(\zeta) = \iota(x)\langle \iota(y), \zeta\rangle$$

for all $\zeta \in H \bullet B,\ x, y \in H$. Ψ is a $*$-preserving homomorphism from the $*$-algebra $F(H)$ into $F(H \bullet B)$. Moreover, Ψ_0 is an isometry on $F(H)$. In particular, $\|\Psi_0\| = 1$. Therefore it extends uniquely to a C^*-algebra homomorphism from $K(H)$ to $K(H \bullet B)$, which preserves the norm. It has to be an isometry as $F(H)$ is dense in $K(H)$.

In the case that $B = A^{**}$, we may define $\widetilde{\Psi}_0 : F(H) \to F(H^\sim) \subset B(H^\sim)$ by

$$\widetilde{\Psi}_0(\theta_{x,y})(\zeta) = \iota(x)\langle \iota(y), \zeta\rangle$$

for all $\zeta \in H^\sim,\ x, y \in H$. Then $\widetilde{\Psi}_0$ is a $*$-preserving homomorphism from the $*$-algebra $F(H)$ into $F(H^\sim)$ and it extends uniquely to a C^*-algebra homomorphism $\widetilde{\Psi}_0$ from $K(H)$ to $K(H^\sim)$, which preserves the norm. Recall that $\iota(H^\sharp) \subset H^\sim$.

Proposition 2.8. *Let $A \subset B$ be a pair of C^*-algebras, where B is unital and $1_B = 1_{\tilde{A}}$. Let $T \in K(H)$. Then $\Psi_0(T)(x \bullet b) = T(x) \bullet b$ for all $x \in H$ and $b \in B$.*

Proof. From the definition, for any $S \in F(H)$, any $x \in H$ and any $b \in B$,

$$\Psi_0(S)(x \otimes b) = S(x) \otimes b \pmod{\mathbb{N}}.$$

Fix $T \in K(H)$, and let $\epsilon > 0$. There exists $S \in F(H)$ such that

$$\|T - S\| < \tfrac{1}{4}\epsilon(1 + \|x \bullet b\| + \|x\|\,\|b\|).$$

Then

$$\|\Psi_0(T) - \Psi_0(S)\| < \tfrac{1}{4}\epsilon(1 + \|x \otimes b\| + \|x\|\,\|b\|) \quad \text{and} \quad \|T(x) \bullet b - S(x) \bullet b\| \le \tfrac{1}{2}\epsilon.$$

Hence

$$\|\Psi_0(T)(x \bullet b) - T(x) \bullet b\| < \epsilon.$$

Since this holds for all $\epsilon > 0$, we conclude that

$$\Psi_0(x \bullet b) = T(x) \bullet b. \qquad \square$$

Lemma 2.9. *Let A and B be as in Proposition 2.8 and H be a Hilbert A-module. Suppose that $\{E_\lambda\}$ is an approximate identity for $K(H)$. Then $\{\Psi_0(E_\lambda)\}$ forms an approximate identity for $K(H \bullet B)$. Moreover*

$$\lim_\lambda \|\Psi_0(E_\lambda)(x) - x\| = 0 \quad \textit{for all } x \in H \bullet B.$$

Proof. By Lemma 3.1 of [Brown and Lin 2025],

$$\lim_\lambda \|E_\lambda(x) - x\| = 0 \quad \text{for all } x \in H. \tag{2-1}$$

Let $S = \sum_{i=1}^n \theta_{x_i,y_i}$, where $x_i, y_i \in (H \otimes B)/N,\ 1 \le i \le n$. Write $x_i = \sum_{j=1}^{k(i)} \xi_{j,i} \bullet b_{j,i}$, where $\xi_{j,i} \in H$ and $b_{j,i} \in B,\ j = 1, 2, \ldots, k(i),\ i = 1, 2, \ldots, n$. By Proposition 2.8,

$$\Psi_0(E_\lambda)(\xi_{j,i} \bullet b_{j,i}) = E_\lambda(\xi_{j,i}) \bullet b_{j,i}.$$

By (2-1),

$$\lim_{\lambda} \|\Psi_0(E_\lambda)(\xi_{j,i} \bullet b_{j,i}) - (\xi_{j,i} \bullet b_{j,i})\| = 0 \tag{2-2}$$

for $j = 1, 2, \ldots, k(i)$, $i = 1, 2, \ldots, n$. It follows that

$$\lim_{\lambda} \|\Psi_0(E_\lambda)(x_i) - x_i\| = 0, \quad i = 1, 2, \ldots, n.$$

For any $z \in H \bullet B$,

$$\Psi_0(E_\lambda)\theta_{x_i,y_i}(z) = (\Psi_0(E_\lambda)x_i)\langle y_i, z\rangle = E_\lambda(x_i)\langle y_i, z\rangle.$$

It follows that, for $1 \le i \le n$,

$$\lim_{\lambda} \|\Psi_0(E_\lambda)\theta_{x_i,y_i} - \theta_{x_i,y_i}\| = 0.$$

Hence

$$\lim_{\lambda} \|\Psi_0(E_\lambda)S - S\| = 0.$$

The set of those module maps with the form of S is norm-dense in $K(H \bullet B)$. Therefore we conclude that

$$\lim_{\lambda} \|\Psi_0(E_\lambda)S - S\| = 0 \quad \text{for all } S \in K(H \bullet B).$$

It follows that $\{\Psi_0(E_\lambda)\}$ forms an approximate identity for $K(H \bullet B)$. □

2.10. Let A be a C^*-algebra and H be a Hilbert A-module. Then $H^\sharp$ is a Banach A-module in general. Recall that, for each $T \in B(H)$, one may define a bounded conjugate module map $T^* : H \to H^\sharp$ as follows: for $x, y \in H$, define

$$T^*(x)(y) = \langle x, T(y)\rangle.$$

So, for a fixed x, we have that $T^*(x)$ gives an element in $H^\sharp$. Moreover, T^* is a bounded conjugate module map from H to $H^\sharp$ with $\|T^*\| = \|T\|$. However, if we view H as a submodule of $H^\sharp$, then T^* is a bounded module map. Note that, if $T \in L(H)$, then $T^* \in L(H)$ and $T^*(H) \subset H$.

If A is a W^*-algebra, by Theorem 3.2 of [Paschke 1973], $H^\sharp$ becomes a Hilbert A module in a natural way. For $T \in B(H)$ and $f \in H^\sharp$, define, for each $x \in H$,

$$\widetilde{T}(f)(x) = \langle f, T^*(x)\rangle, \tag{2-3}$$

where T^* is defined above. Thus $\widetilde{T}(f)$ is a bounded linear module map from H to A with $\|\widetilde{T}(f)\| \le \|T\|\|f\|$. Hence we extend T to a bounded (conjugate) module map from $H^\sharp$ to $H^\sharp$. As we view $H^\sharp$ as a Hilbert A-submodule in this case, T is in fact a bounded module map on $H^\sharp$ (we will take the conjugate as Hilbert space cases). By Corollary 3.7 of [Paschke 1973], such an extension is unique.

By Lemma 3.7 of [Lin 1992], one may ease the assumption that A is a W^*-algebra to the assumption that A is a monotone complete C^*-algebra.

Proposition 2.11. *Let A and B be as in Proposition 2.8, H be a Hilbert A-module and $\{E_\lambda\}$ an approximate identity for $K(H)$. Then*

$$\lim_{\lambda}(\sup\{\|\widetilde{\Psi}_0(E_\lambda)(f)(x) - f(x)\| : f \in (H \bullet B)^\sharp,\ \|f\| \le 1\}) = 0 \quad \textit{for all } x \in H \bullet B. \tag{2-4}$$

Moreover, suppose that $(H \bullet B)^\sharp$ extends $H \bullet B$ as a Hilbert B-module, then, for any $T \in B((H \bullet B)^\sharp)$,

$$\lim_{\lambda} \|\langle \widetilde{\Psi}_0(E_\lambda)T\widetilde{\Psi}_0(E_\lambda)(x), y\rangle - \langle T(x), y\rangle\| = 0 \quad \textit{for all } x, y \in H \bullet B.$$

Proof. Fix $f \in H^\sharp$. For any $x \in H \bullet B$, by Lemma 2.9,

$$\|\widetilde{\Psi}_0(E_\lambda)(f)(x) - f(x)\| = \|f(E_\lambda(x)) - f(x)\| \le \|f\|\|E_\lambda(x) - x\| \to 0.$$

Hence (2-4) holds.

To see the "moreover" part of the lemma, let $T \in B((H \bullet B)^\sharp)$. Then, for any $x, y \in H \bullet B$,

$$\begin{aligned}\|\langle \widetilde{\Psi}_0(E_\lambda)T\widetilde{\Psi}_0(E_\lambda)(x), y\rangle - \langle T(x), y\rangle\| \\ \le \|\langle T\widetilde{\Psi}_0(E_\lambda)(x), \widetilde{\Psi}_0(E_\lambda)(y)\rangle - \langle T(x), \widetilde{\Psi}_0(E_\lambda)(y)\rangle\| + \|\langle T(x), \widetilde{\Psi}_0(E_\lambda)(y)\rangle - \langle T(x), y\rangle\| \\ \le \|y\|\|T\|\|\Psi_0(E_\lambda)(x) - x\| + \|T\|\|x\|\|\Psi_0(E_\lambda)(y) - y\|\end{aligned}$$

By applying Lemma 2.9 to the two terms of the last inequality above, we conclude that

$$\lim_\lambda \|\langle \widetilde{\Psi}_0(E_\lambda)T\widetilde{\Psi}_0(E_\lambda)(x), y\rangle - \langle T(x), y\rangle\| = 0 \quad \text{for all } x, y \in H \bullet B. \qquad \square$$

Definition 2.12. Let A be a C^*-algebra and H be a Hilbert A-module. Recall [Lin 1991a, Theorem 1.5] that we identify $B(H)$ with $LM(K(H))$, the Banach algebra of left multipliers of $K(H)$ (in $K(H)^{**}$).

By Lemma 2.9, Ψ_0 maps $K(H)$ into $K(H \bullet B)$ which maps approximate identities to approximate identities. We may then extend a homomorphism $\Psi_0 : B(H) = LM(K(H)) \to LM(K(H \bullet B)) = B(H \bullet B)$ by

$$\Psi_0(T) = \lim_\lambda \Psi_0(TE_\lambda),$$

where the convergence is in the left strict topology of $LM(K(H \bullet B))$. Since $\Psi_0|_{K(H)}$ is an isometry, so is Ψ_0.

We are mostly interested in the case that $B = A^{**}$. By Theorem 3.2 of [Paschke 1973], $(H \bullet A^{**})^\sharp$ is a self-dual Hilbert A^{**}-module. Therefore, by Section 2.10, for each $T \in B(H)$, the extension $\widetilde{\Psi}_0(T)$ is unique. Hence Ψ_0 may be extended to a Banach algebra isomorphism $\widetilde{\Psi}_0$ from $B(H)$ into $B(H^\sim)$ such that

$$\widetilde{\Psi}_0(T)|_{H \bullet A^{**}} = \Psi_0(T) \quad \text{for all } T \in B(H). \tag{2-5}$$

We will visualize the map Ψ_0 a bit more.

Proposition 2.13. *Let A and B be a pair of C^*-algebras as in Proposition 2.8 and H be a Hilbert A-module. Then, for any $T \in B(H)$,*

$$\lim_\lambda \|\Psi_0(T)\Psi_0(E_\lambda)(x) - \Psi_0(T)(x)\| = 0 \quad \textit{for all } x \in H \bullet B. \tag{2-6}$$

Moreover

$$\Psi_0(T)(x \bullet b) = T(x) \bullet b \quad \textit{for all } x \in H \textit{ and } b \in B.$$

Consequently, $\widetilde{\Psi}_0(\mathrm{id}_H) = \mathrm{id}_{H^\sim}$.

Proof. The identity (2-6) follows immediately from Lemma 2.9.

Since

$$\Psi_0(TE_\lambda)(x \bullet b) = TE_\lambda(x) \bullet b,$$

by (2-6) and by Lemma 3.1 of [Brown and Lin 2025],

$$\Psi_0(T)(x \bullet b) = T(x) \bullet b$$

for all $x \in H$ and $b \in B$.

For the last part of the proposition, we note that, by considering the pair A and A^{**}, and by the "moreover" part of the proposition, $\Psi_0(\mathrm{id}_H) = \mathrm{id}_{H\bullet A^{**}}$. Therefore, since the extension $\widetilde{\Psi}_0(\mathrm{id}_{H\bullet A^{**}})$ is unique (Corollary 3.7 of [Paschke 1973], see Section 2.10 for convenience), we must have that $\widetilde{\Psi}_0(\mathrm{id}_H) = \mathrm{id}_{H^\sim}$. □

The following is a slightly strengthened restatement of [Brown and Lin 2025, Proposition 2.3].

Proposition 2.14. *Let A be a C^*-algebra and H a Hilbert A-module. Then there is a homomorphism Ψ from $K(H)^{**}$ into $B(H^\sim)$ such that $\Psi|_{B(H)} = \widetilde{\Psi}_0$. Moreover, if $T \in K(H)^{**}$ and $T_\lambda \in K(H)^{**}$ such that $T_\lambda \to T$ in the weak* topology, then*

$$\lim_\lambda f(\langle \Psi(T_\lambda)(x), y\rangle) = f(\langle \Psi(T)(x), y\rangle) \quad \textit{for all } x, y \in H^\sim \textit{ and } f \in A^*.$$

Proof. By Definition 2.4, $B(H^\sim) = L(H^\sim)$ is a W^*-algebra; see [Paschke 1973, Proposition 3.11]. Let $\pi : B(H^\sim) \to B(H_\pi)$ be a faithful normal representation such that $\pi(B(H^\sim))$ is weakly closed in $B(H_\pi)$. Then, by, for example, [Pedersen 1979, Theorem 3.7.7] and [Conway 2000, Corollary 46.5], there is a normal homomorphism $\Phi : K(H)^{**} \to B(H_\pi)$ such that $\Phi|_{K(H)} = \pi \circ \widetilde{\Psi}_0|_{K(H)}$ and $\pi \circ \widetilde{\Psi}_0(K(H))$ is weakly dense in $\Phi(K(H)^{**})$. Since $\pi(B(H^\sim))$ is a von Neumann algebra, $\Phi(K(H)^{**}) \subset \pi(B(H^\sim))$. Since π is injective, we may define $\Psi = \pi^{-1} \circ \Phi$. Recall that π^{-1} is an isomorphism between W^*-algebras $\pi(B(H^\sim))$ and $B(H^\sim)$. It follows that Ψ is weak*-continuous. Then, $\Psi|_{K(H)} = \pi^{-1} \circ \pi \circ \widetilde{\Psi}_0|_{K(H)} = \widetilde{\Psi}_0|_{K(H)}$.

Let $V = B(H^\sim)_*$ be the predual (as Banach spaces). Then Ψ induces a map $\Psi^* : V \to K(H)^*$, the predual of $K(H)^{**}$, by $L(\Psi^*(v)) = \Psi(L)(v)$ for all $L \in (K(H)^*)^*$ and $v \in V$. Thus if $T_\lambda \in K(H)^{**}$ such that $T_\lambda \to T$ in the weak* topology in $K(H)^{**}$, then $\Psi(T_\lambda)(v) = T_\lambda(\Psi^*(v))$ converges to $T(\Psi^*(v)) = \Psi(T)(v)$ for all $v \in V$. In other words, $\Psi(T_\lambda) \to \Psi(T)$ in the weak* topology in $V^* = B(H^\sim)$. By Definition 2.4 (see Remark 3.9 and proof of Theorem 3.10 of [Paschke 1973]), this implies, in particular, for any $f \in A^*$, $x, y \in H^\sim$, that $f(\langle \Psi(T_\lambda)(x), y\rangle) \to f(\langle \Psi(T)(x), y\rangle)$.

By Theorem 1.5 of [Lin 1991a], $B(H) = LM(K(H))$. Let $\{E_\lambda\}$ be an approximate identity for $K(H)$. Then $TE_\lambda \in K(H)$ for all $T \in B(H)$. It follows from Proposition 2.13 that, for $T \in B(H)$,

$$\lim_\lambda \|\Psi(TE_\lambda)(f)(x) - \Psi(T)(f)(x)\| = \lim_\lambda \|\Psi(T)\Psi(E_\lambda)(f)(x) - \Psi(T)(f)(x)\| = 0$$

for all $x \in H \bullet A^{**}$ and $f \in (H \bullet A^{**})^\sharp$. On the other hand, by Lemma 2.9,

$$\lim_\lambda \|\Psi_0(TE_\lambda)(x) - \Psi_0(T)(x)\| = 0 \quad \text{for all } x \in H \bullet A^{**}.$$

However, we have shown that $\Psi(TE_\lambda)(y) = \widetilde{\Psi}_0(TE_\lambda)(y) = \Psi_0(TE_\lambda)(y)$ for all $y \in H \bullet A^{**}$ (see also Definition 2.12). Therefore, combining these three facts, for $x, y \in H \bullet A^{**}$, we obtain

$$\langle \Psi(T)(x), y\rangle = \langle \Psi_0(T)(x), y\rangle.$$

It follows that $\Psi(T)|_{H\bullet A^{**}} = \Psi_0(T)$. Since the extension of $\Psi_0(T)$ to a bounded module map on $(H\bullet A^{**})^\sharp$ is unique (see the end of Section 2.10 and [Lin 1992, Lemma 3.5]), we have $\Psi(T) = \widetilde{\Psi}_0(T)$ for all $T \in B(H)$. Hence

$$\Psi|_{B(H)} = \widetilde{\Psi}_0.$$

□

Definition 2.15. Let M be a W^*-algebra and H be a Hilbert M-module. Then, $H^\sharp$ is a self-dual Hilbert M-module by [Paschke 1973, Theorem 3.2]. Let $F_0 : F(H) \to F(H^\sharp)$ be the homomorphism defined by

$$F_0(\theta_{x,y})(z) = x\langle y, z\rangle \quad \text{for all } z \in H^\sharp \text{ and } x, y \in H.$$

Clearly F_0 is an isometry. It extends uniquely to a homomorphism $F_0 : K(H) \to K(H^\sharp)$. We further extend $F : \widetilde{K(H)} \to \widetilde{K(H^\sharp)}$ by $F(\mathrm{id}_H) = \mathrm{id}_{H^\sharp}$.

Proposition 2.16. *Let M be a W^*-algebra and H a Hilbert M-module. Then there exists a unital normal homomorphism $F : K(H)^{**} \to B(H^\sharp)$ such that $F|_{K(H)} = F_0$ and, if $T_\lambda \to T$ in the weak* topology of $K(H)^{**}$, then*

$$\lim_\lambda f(\langle F(T_\lambda)(x), y\rangle) = f(\langle F(T)(x), y\rangle)$$

for all $x, y \in H^\sharp$ and $f \in M_*$, the predual of M. Moreover, $F(T) = \widetilde{T}$ for all $T \in B(H)$ as defined by (2-3).

Proof. Recall that $B(H^\sharp)$ is a W^*-algebra. We may assume that $B(H^\sharp)$ acts on a Hilbert space X as a von Neumann algebra with $1_{B(H^\sharp)} = \mathrm{id}_X$. Then, by [Lin 2001, Theorem 1.8.2] (see also [Pedersen 1979, Theorem 3.7.7]), there is a unital normal homomorphism $F : K(H)^{**} \to \overline{F_0(K(H))}^{\mathrm{SOT}} \subset B(H^\sharp)$ such that $F|_{K(H)} = F_0$. So F is weak*-continuous (see, for example, [Conway 2000, Corollary 46.5]).

Suppose that $T_\lambda \to T$ in the weak* topology of $K(H)^{**}$. Then $F(T_\lambda) \to F(T)$ in the weak* topology of $B(H^\sharp)$. Therefore (see the later part of Definition 2.4, also, Remark 3.9 and the proof of Proposition 3.9 of [Paschke 1973]),

$$f(\langle F(T_\lambda)(x), y\rangle) \to f(\langle F(T)(x), y\rangle) \quad \text{for all } x, y \in H^\sharp \text{ and } f \in M_*.$$

Let $\{E_\lambda\}$ be an approximate identity for $K(H)$. Then, for any $T \in B(H)$, by Lemma 3.1 of [Brown and Lin 2025],

$$\lim_\lambda \|F(T)F(E_\lambda)(x) - F(T)(x)\| = \lim_\lambda \|F(T)E_\lambda(x) - F(T)(x)\| = 0 \quad \text{for all } x \in H.$$

On the other hand, since $F(T)F(E_\lambda)|_H = F(TE_\lambda)|_H = TE_\lambda$ and (by [Brown and Lin 2025, Lemma 3.1])

$$\lim_\lambda \|TE_\lambda(x) - T(x)\| = 0,$$

we conclude that

$$T(x) = F(T)(x) \quad \text{for all } x \in H.$$

Since the extension of T to $H^\sharp$ is unique (by Proposition 3.6 of [Paschke 1973], see also Lemma 3.5 of [Lin 1992]), $\widetilde{T} = F(T)$. □

3. Isomorphism of $B(H^\sim)$ and $K(H)^{**}$

Let A be a monotone complete C^*-algebra and H be a Hilbert A-module. Then, by Lemma 3.7 of [Lin 1992], $H^\sharp$ becomes a self-dual Hilbert A-module such that $\langle \tau, x\rangle = \tau(x)$ for all $x \in H$ and $\tau \in H^\sharp$. Note that, if E is self-dual, we conjugate map $E^\sharp$ onto E just as in the case of Hilbert spaces.

We will apply the following lemma several times.

Proposition 3.1. *Let A be a monotone complete C^*-algebra and $H_1 \subset H_2$ be Hilbert A-modules such that H_2 is self-dual. Then $H_1^\sharp$ is an orthogonal summand of $H_2^\sharp$ and the embedding $H_1^\sharp \to H_2^\sharp$ extends the embedding $H_1 \subset H_2$.*

Proof. Define $P_0 : H_2 \to H_1^\sharp$ by

$$P_0(y)(x) = \langle y, x\rangle \quad \text{for all } y \in H_2 \text{ and } x \in H_1. \tag{3-1}$$

It is a bounded module map (by viewing $H_1^\sharp$ as a Hilbert module instead of the dual to avoid the conjugation) with $\|P_0\| = 1$. Note that $P_0|_{H_1} = \mathrm{id}_{H_1}$.

Let $\tau \in H_1^\sharp$. Since A is monotone complete, by Theorem 3.8 of [Lin 1992], there is $\tilde{\tau} \in H_2^\sharp = H_2$ such that $\tilde{\tau}|_{H_1} = \tau$ and $\|\tilde{\tau}\| = \|\tau\|$. This implies that P_0 is surjective.

Define $j : H_1^\sharp \to H_2^\sharp = H_2$ by

$$j(x)(y) = \langle x, P_0(y)\rangle \quad \text{for all } x \in H_1^\sharp \text{ and } y \in H_2. \tag{3-2}$$

Then j extends the embedding $H_1 \hookrightarrow H_2$. Now, for $x \in H_1^\sharp$ and $y \in H_2$, by (3-1) and (3-2),

$$P_0 \circ j(x)(y) = P_0(j(x))(y) = \langle j(x), y\rangle = \langle x, P_0(y)\rangle = \langle P_0(y), x\rangle^* = (P_0(y)(x))^* = \langle y, x\rangle^* = \langle x, y\rangle.$$

It follows that $P_0 \circ j = \mathrm{id}|_{H_1^\sharp}$, and thus $j : H_1^\sharp \to H_2$ is an embedding. With the identification of $H_1^\sharp$ and $j(H_1^\sharp)$, $P_0|_{H_1^\sharp} = \mathrm{id}|_{H_1^\sharp}$. It follows that P_0 is a projection and $H_1^\sharp$ is an orthogonal summand of H_2. □

Applying Propositions 3.1 and 2.6, we obtain the following characterization of $H^\sim$.

Proposition 3.2. *Let A be a C^*-algebra and H be a Hilbert A-module. Then $H^\sim$ is the smallest self-dual Hilbert A^{**}-module containing H as a Hilbert A-submodule.*

Proof. Let H_1 be a self-dual Hilbert A^{**}-module containing H as a Hilbert A-submodule. Then, by Proposition 2.6,

$$H \subset H \bullet A^{**} \subset H_1.$$

Applying Proposition 3.1, since H_1 is self-dual,

$$H^\sim = (H \bullet A^{**})^\sharp \subset H_1^\sharp = H_1.$$

The proposition follows. □

3.3. In the next proposition, let A be a C^*-algebra, and let $H_1 \subset H$ be Hilbert A-modules. Then, by Proposition 2.6, $H_1 \bullet A^{**} \subset H \bullet A^{**}$. Since A^{**} is monotone complete and $(H \bullet A^{**})^\sharp = H^\sim$ and $(H_1 \bullet A^{**})^\sharp = H_1^\sim$, by Proposition 3.1, we may write $H^\sim = H_1^\sim \oplus (H_1^\sim)^\perp$. Denote by $P : H^\sim \to H_1^\sim$ the projection. Note that $P \in L(H^\sim)$. By Lemma 3.2 of [Lin 1992], $K(H_1)$ is a hereditary C^*-subalgebra of $K(H)$. Let $\Psi_H : K(H)^{**} \to B(H^\sim)$ and $\Psi_1 : K(H_1)^{**} \to B(H_1^\sim)$ be the homomorphisms given by Proposition 2.14, respectively.

Proposition 3.4. *Using the notation above, we have that*

$$\Psi_1 = \Psi_H|_{K(H_1)^{**}} = P\Psi_H P|_{K(H_1)^{**}};$$

in particular, $\Psi_1(T) = \Psi_H(T)|_{H_1^\sim} = P\Psi_H(T)P|_{H_1^\sim}$ *for* $T \in K(H_1)^{**}$. *Moreover*,

$$P\Psi_H(L)P|_{K(H_1)^{**}} \subset \Psi_1(K(H_1)^{**}) \quad \text{for all } L \in K(H)^{**}.$$

Furthermore, $\Psi(Q) = P$, *where* Q *is the open projection in* $K(H)^{**}$ *corresponding to the hereditary* C^*-*subalgebra* $K(H_1)$.

Proof. Denote by $\Psi_{K(H),0}$ the injective homomorphism from $K(H)$ into $K(H \bullet A^{**})$ and by $\Psi_{K(H_1),0}$ the injective homomorphism from $K(H_1)$ into $K(H_1 \bullet A^{**})$ described in Definition 2.7, respectively.

Fix $S \in K(H_1)$. For each $x \in H_1$ and $b \in A^{**}$, by Proposition 2.8,

$$\Psi_{K(H),0}(S)(x \bullet b) = S(x) \bullet b,$$

$$P\Psi_{K(H),0}(S)P(x \bullet b) = P(S(x \bullet b)) = S(x) \bullet b = \Psi_{K(H_1),0}(S)(x \bullet b).$$

It follows that

$$\Psi_{K(H),0}(S)|_{H_1 \bullet A^{**}} = P\Psi_{K(H),0}(S)P|_{H_1 \bullet A^{**}} = \Psi_{K(H_1),0}(S).$$

Since the extensions of $\Psi_{K(H),0}(S)|_{H_1 \bullet A^{**}}$ and $\Psi_{K(H_1),0}(S)$ to bounded module maps on $H_1^\sim$ are unique, and $\Psi(S)|_{H_1^\sim}$ and $\Psi(S)$ are corresponding extensions, by Corollary 3.7 of [Paschke 1973], we conclude that $\Psi(S)|_{H_1^\sim} = P\Psi_H(S)P|_{H_1^\sim} = \Psi_1(S)$.

Let $T \in K(H_1)^{**}$ and $\{T_\lambda\} \subset K(H_1)$ be a net such that $T_\lambda \to T$ in the weak* topology. By Proposition 2.14, for any $g \in A^*$,

$$\lim_\lambda |g(\langle \Psi_H(T_\lambda)(x), y\rangle) - g(\langle \Psi_H(T)(x), y\rangle)| = 0, \tag{3-3}$$

$$\lim_\lambda |g(\langle \Psi_1(T_\lambda)(x), y\rangle) - g(\langle \Psi_1(T)(x), y\rangle)| = 0 \tag{3-4}$$

for all $x, y \in H_1^\sim$. Since we have shown that $\Psi_H(T_\lambda)|_{H_1^\sim} = P\Psi_H(T_\lambda)P|_{H_1^\sim} = \Psi_1(T_\lambda)$, we conclude that

$$\Psi_H(T)|_{H_1^\sim} = P\Psi_H(T)P|_{H_1^\sim} = \Psi_1(T). \tag{3-5}$$

Hence

$$\Psi_1 = P\Psi_H P|_{K(H_1)^{**}} = \Psi_H|_{K(H_1)^{**}}.$$

Let $\{q_\lambda\}$ be an approximate identity for $K(H_1)$. Then $q_\lambda \nearrow \mathrm{id}_{H_1} \in K(H_1)^{**}$. It follows from Proposition 2.14 that

$$\lim_\lambda f(\langle \Psi_1(q_\lambda(y)), z\rangle) = f(\langle y, z\rangle) \quad \text{for all } y, z \in H_1^\sim \text{ and } f \in A^*.$$

On the other hand, we also have that $q_\lambda \nearrow Q$ in $K(H)^{**}$. By Proposition 3.1, $H^\sim = H_1^\sim \oplus (H_1^\sim)^\perp$. Note that $q_\lambda(x) \in H_1$ for all $x \in H$. Then, for $x \in H$, $b \in A^{**}$ and $g \in (H_1^\sim)^\perp$, by Proposition 2.8,

$$\langle \Psi_H(q_\lambda)(x \bullet b), g\rangle = g(q_\lambda(x \bullet b))^* = g(q_\lambda(x) \bullet b)^* = 0.$$

It follows that, for any $y \in H \bullet A^{**}$ and $g \in (H_1^\sim)^\perp$,

$$\langle \Psi_H(q_\lambda)(y), g\rangle = 0.$$

Hence, for $g \in (H_1^{\sim})^{\perp}$,

$$\langle y, \Psi_H(q_\lambda)(g)\rangle = 0 \quad \text{for all } y \in H \bullet A^{**}.$$

It follows that $\Psi_H(q_\lambda)(g) = 0$ and

$$\langle \Psi_H(q_\lambda)(z), g\rangle = \langle z, \Psi_H(q_\lambda)(g)\rangle = 0 \quad \text{for all } z \in H^{\sim}.$$

In other words, $\Psi_H(q_\lambda)(z) \in H_1^{\sim}$ for all $z \in H$ and λ. Therefore

$$P\Psi_H(q_\lambda) = \Psi_H(q_\lambda) = \Psi_H(q_\lambda)P.$$

Note that $Pz \in H_1^{\sim}$ for any $z \in H^{\sim}$. Thus, by (3-5) and (3-4),

$$\begin{aligned}\lim_\lambda f(\langle \Psi(q_\lambda)(y), z\rangle) &= \lim_\lambda f(\langle \Psi(q_\lambda)(P(y)), P(z)\rangle) = \lim_\lambda f(\langle \Psi_1(q_\lambda)(P(y)), P(z)\rangle)\\ &= f(\langle P(y), P(z)\rangle) = f(\langle P(y), z\rangle).\end{aligned}$$

By (3-3) and (3-5), $\lim_\lambda f(\langle \Psi(q_\lambda)(y), z\rangle) = f(\langle \Psi(Q)(y), z\rangle)$. Therefore

$$\Psi(Q) = P.$$

This proves the "furthermore" part. In what follows we will identify Q with P as well as $\Psi(Q)$ and $\Psi(P)$.

Now let $L \in K(H)^{**}$ and $\{L_\lambda\} \subset K(H)$ be a net such that $L_\lambda \to L$ in the weak* topology. By Proposition 2.14, for any $g \in A^*$, $x, y \in H_1^{\sim}$,

$$\begin{aligned}\lim_\lambda |g(\langle \Psi_H(T_\lambda)(x), y\rangle) - g(\langle \Psi_H(T)(x), y\rangle)| &= 0,\\ \lim_\lambda |g(\langle \Psi_1(T_\lambda)(x), y\rangle) - g(\langle \Psi_1(T)(x), y\rangle)| &= 0\end{aligned}$$

(note that $\Psi_1(T_\lambda) = P\Psi_1(T_\lambda)P$). We also have, for any $x, y \in H_1^{\sim}$,

$$\begin{aligned}\langle \Psi_H(PT_\lambda P)(x), y\rangle &= \langle \Psi_H(T_\lambda)(x), y\rangle,\\ \langle P\Psi_H(T)P(x), y\rangle &= \langle \Psi_H(T)(x), y\rangle.\end{aligned}$$

Since $PT_\lambda P \in K(H_1)^{**}$, by the first part of the lemma, $\Psi_H(PT_\lambda P)(x) = \Psi_1(PT_\lambda P)(x)$ for $x \in H_1^{\sim}$. It follows that $P\Psi_H(T)P(x) = \Psi_1(PTP)(x)$ for all $x \in H_1^{\sim}$. Then

$$P\Psi_H(T)P = \Psi_1(PTP) \in \Psi_1(K(H_1)^{**}). \qquad \square$$

3.5. Let A be a C^*-algebra and let, for $n \in \mathbb{N}$,

$$H_n = A^{(n)} = \{(a_1, a_2, \dots, a_n) : a_j \in A,\ 1 \le j \le n\},$$

the direct sum of n copies of A, where $\langle a, b\rangle = \sum_{j=1}^n a_j^* b_j$ if $a = (a_1, a_2, \dots, a_n)$ and $b = (b_1, b_2, \dots, b_n)$. Let

$$H_A = \left\{ \{a_n\} : a_n \in A \text{ and } \sum_{i=1}^{n} a_k^* a_k \text{ converges in norm} \right\}$$

be the standard countably generated Hilbert (right) A-module. Note that

$$\langle \{a_n\}, \{b_n\} \rangle = \sum_{n=1}^{\infty} a_n^* b_n.$$

We note that H_A is the closure of $\bigcup_n A^{(n)}$. We may also view $H_n = A^{(n)}$ as an orthogonal summand of H_A. Then

$$H_A^\sharp = \left\{ \{a_n\} : \left\{ \left\| \sum_{k=1}^{n} a_k^* a_k \right\| \right\} \text{ is bounded} \right\}.$$

If $g = \{a_n\} \in H^\sharp$, then

$$g(x) = \sum_{n=1}^{\infty} a_n^* b_n \quad \text{for all } x = \{b_n\} \in H_A,$$

where the sum converges in norm. Moreover $\|g\| = \lim_{n\to\infty} \left\| \sum_{k=1}^{n} a_k^* a_k \right\|$.

If A is a W^*-algebra, as mentioned earlier, $H_A^\sharp$ becomes a Hilbert A-module in a natural way (see Theorem 3.2 of [Paschke 1973]). In fact, we may define

$$\langle x, y \rangle = \sum_{n=1}^{\infty} a_n^* b_n \quad \text{for all } x = \{a_n\},\ y = \{b_n\} \in H_A^\sharp. \tag{3-6}$$

To see the right side converges in the weak* topology, we first let $f \in A^*$. Note that, if $\{a_n\} \in H_A^\sharp$,

$$\left| \sum_{k=1}^{N} f(a_k^* a_k) \right| = \left| f\left(\sum_{k=1}^{N} a_k^* a_k \right) \right| \le \|f\| \left\| \sum_{k=1}^{N} a_k^* a_k \right\|$$

for any integer N. Hence $\left\{ \sum_{k=1}^{n} f(a_k^* a_k) \right\}$ is bounded, is increasing and converges for any positive linear functional f. Hence, for any $m > n$,

$$\sum_{k=n}^{m} f(a_k^* a_k) \to 0 \quad \text{as } n \to \infty \quad \text{for all } f \in A^*. \tag{3-7}$$

For any positive linear functional f of A and for any $m > n$ in $\mathbb{N}$,

$$\begin{aligned}
\left| f\left(\sum_{k=n}^{m} a_k^* b_k \right) \right| &= \left| \sum_{k=n}^{m} f(a_k^* b_k) \right| \le \sum_{k=n}^{m} |f(a_k^* b_k)| \\
&\le \sum_{k=n}^{m} |f(a_k^* a_k)|^{1/2} |f(b_k^* b_k)|^{1/2} \\
&\le \left(\left(\sum_{k=n}^{m} |f(a_k^* a_k)| \right) \left(\sum_{k=n}^{m} |f(b_k^* b_k)| \right) \right)^{1/2} \\
&\le \|f\|^{1/2} \|\{b_k\}\| \left(\sum_{k=n}^{m} |f(a_k^* a_k)| \right)^{1/2} \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

It follows that $f\left(\sum_{k=1}^{n} a_k^* b_k\right)$ converges for all $f \in A^*$ as $n \to \infty$. Let us write the limit as $f\left(\sum_{k=1}^{\infty} a_k^* b_k\right)$. Then, by the above inequalities (with $n = 1$), we also have

$$\left| f\left(\sum_{k=1}^{\infty} a_k^* b_k\right)\right| \le \|f\| M_b M_a,$$

where

$$M_a = \sup\left\{\left\|\sum_{k=1}^{n} a_k^* a_k\right\|\right\}^{1/2} \quad \text{and} \quad M_b = \sup\left\{\left\|\sum_{k=1}^{n} b_k^* b_k\right\|\right\}^{1/2}.$$

Thus $\sum_{k=1}^{\infty} a_k^* b_k$ defines a bounded linear functional on A^*. Its restriction on A_* gives an element in A (recall that A is assumed to be a W^*-algebra). This shows the infinite series in the right side of (3-6) converges in the weak* topology. It is then standard to verify that (3-6) defines an inner product which extends the inner product on H_A.

Let A act on a Hilbert space X (as a W^*-algebra). Consider $l^2(X)$, the Hilbert space direct sum of countably many copies of X. Suppose that $b = \{b_n\} \in H_A^\sharp$. Then the infinite matrix $\bar{b} = (b_{i,j})$, with $b_{i,1} = b_i$, $i \in \mathbb{N}$ and $b_{i,j} = 0$ if $j \ge 2$, defines a bounded linear operator on $l^2(X)$, by $\bar{b}(v) = (b_1(v_1), b_2(v_1), \dots, b_n(v_1), \dots)$, where $v = (v_1, v_2, \dots, v_n, \dots) \in l^2(X)$. Moreover

$$\|\bar{b}\|^2 = \|\bar{b}^*\bar{b}\| = \left\|\sum_{i=1}^{\infty} b_i^* b_i\right\| = \sup\left\{\left\|\sum_{i=1}^{n} b_i^* b_i\right\| : n \in \mathbb{N}\right\} \tag{3-8}$$

(some of these details in this subsection may be found in [Lin 1991b]).

Proposition 3.6. *Let C be a unital C^*-algebra and $A \subset C$ be a C^*-subalgebra such that $1_{\tilde{A}} = 1_C$. Denote by $R = \overline{AC}$ the closed right ideal of C generated by A. Then*:

(1) $H_A \bullet C = \{\{b_n\} \in H_C : b_n \in R\}$.

(2) *If C is a W^*-algebra and $e_\alpha \nearrow 1_C$, where $\{e_\alpha\}$ is an approximate identity for A, then*

$$(H_A \bullet C)^\sharp = H_C^\sharp.$$

Proof. To see (1), we first note that $A \bullet C = R$ as Hilbert C-modules. Hence $A^{(n)} \bullet C = R^{(n)}$. Clearly, $H_A \bullet C \subset H_C$. We note that $\{\{r_n\} \in H_C : r_n \in R\}$ is closed in H_C. Since both $\bigcup_n A^{(n)} \bullet C$ and $\bigcup_{n=1}^{\infty} R^{(n)}$ are dense in $\{\{r_n\} \in H_C : r_n \in R\}$, and $\bigcup_n A^{(n)} \bullet C$ is dense $H_A \bullet C$, we obtain

$$\{\{r_n\} \in H_C : r_n \in R\} = H_A \bullet C.$$

This proves (1).

For (2) we may assume that $A \subset C \subset B(X)$, where X is a Hilbert space, $1_C = \mathrm{id}_X$, and the range $C(X)$ equals X. Otherwise, we replace X by $1_C(X)$.

Claim 1: $\overline{R(X)} = C(X) = X$. Since $e_\alpha \nearrow 1_C = \mathrm{id}_X$, for any $v \in X$, $e_\alpha(v) \to v$. This proves the claim.

Claim 2: $R^\sharp = C$, where $R^\sharp$ is the dual of the Hilbert C-module R (as we assume that C is a W^*-algebra).

Let $f \in R^\sharp$. Then $f(e_\alpha)r = f(e_\alpha r) \to f(r)$ for all $r \in R$ in norm as $e_\alpha r \to r$ in norm. Hence $f(e_\alpha)r(v) \to f(r)(v)$ for all $r \in R$ and $v \in X$. Define T on $R(X)$ by $T(r(v)) = \lim_\alpha f(e_\alpha)r(v)$ for all $v \in X$ and $r \in R$. One checks that T is a well-defined linear map on $R(X)$. Moreover, we have

$\|T\| \le \sup\{\|f(e_\alpha)\| : \alpha\} \le \|f\|$. Since, by Claim 1, $\overline{R(X)} = X$, we have that T extends uniquely to a bounded linear operator (denote by T again) on X. Moreover, $f(e_\alpha)$ converges to T on X. Since C is closed in the weak operator topology, $T \in C$. Moreover, $Tr(v) = f(r)(v)$ for all $v \in X$. It follows that $Tr = f(r)$ for all $r \in R$.

For each $c \in C$, define $f_c \in R^\sharp$ by

$$f_c(r) = c^* r \quad \text{for all } r \in R.$$

For the above T, we note that $f_{T^*}(r) = Tr$ for all $r \in R$. Hence the map $c \to f_c$ is surjective. To see it is injective, suppose that $c^* r = 0$ for all $r \in R$. Then

$$c^* e_\alpha c = 0 \quad \text{for all } \alpha.$$

Since $c^* e_\alpha c \nearrow c^* c$, this implies that $c^* c = 0$. Thus the map $c \mapsto f_c$ is injective, which extends the identity map on R. It follows that $R^\sharp = C$, and Claim 2 is proved.

By Claim 2, we obtain that $((A^{(n)}) \bullet C)^\sharp = C^{(n)}$. By (1), $(A^{(n)}) \bullet C$ is a direct summand of $H_A \bullet C$. Hence we may write $((A^{(n)}) \bullet C)^\sharp \subset (H_A \bullet C)^\sharp$. Together with (1), we obtain that

$$H_A \bullet C \subset H_C \subset (H_A \bullet C)^\sharp.$$

Note H_C is a Hilbert C-submodule of the self-dual Hilbert C module $(H_A \bullet C)^\sharp$. It follows from Proposition 3.1 that

$$(H_A \bullet C)^\sharp \subset H_C^\sharp \subset (H_A \bullet C)^\sharp.$$

Consequently, $H_C^\sharp = (H_A \bullet C)^\sharp$. □

3.7. Note that, if A is unital, $H_A \bullet C = H_C$.

From the above discussion, we obtain the following result.

Lemma 3.8. *Let A be a C^*-algebra, $H_n = (A^{**})^{(n)}$ and $P_n : H_{A^{**}}^\sharp \to H_n$ be the projection.*

(1) *Let $S \subset H_{A^{**}}^\sharp$ be a bounded subset. Then, for any $f \in A^*$ and $x \in H_{A^{**}}^\sharp$,*

$$\lim_{n\to\infty} \sup\{|f(\langle P_n(x), y\rangle) - f(\langle x, y\rangle)| : y \in S\} = 0,$$
$$\lim_{n\to\infty} \sup\{|f(\langle y, P_n(x)\rangle) - f(\langle y, x\rangle)| : y \in S\} = 0.$$

(2) *Moreover,*

$$\lim_{n\to\infty} |f(\langle P_n(x), P_n(x)\rangle) - f(\langle x, x\rangle)| = 0 \quad \textit{for all } x \in H_{A^{**}}^\sharp \textit{ and } f \in A^*.$$

Proof. Set $M = \sup\{\|y\| : y \in S\} + 1$. Let f be a positive linear functional in A^* and $x = \{a_n\} \in H_{A^{**}}^\sharp$. For each $y = \{b_n\} \in S$,

$$|f(\langle P_n(x), y\rangle) - f(\langle x, y\rangle)| = \left|\sum_{k=n+1}^\infty f(a_k^* b_k)\right| \le \left(\sum_{k=n+1}^\infty f(a_k^* a_k)\right)^{1/2} \left(\sum_{k=n+1}^\infty f(b_k^* b_k)\right)^{1/2}$$
$$\le \|f\|\|y\| \left(\sum_{k=n+1}^\infty f(a_k^* a_k)\right)^{1/2} \le M\|f\| \left(\sum_{k=n+1}^\infty f(a_k^* a_k)\right)^{1/2}.$$

By what has been discussed in Section 3.5,

$$\lim_{n\to\infty}\left(\sum_{k=n+1}^{\infty} f(a_k^*a_n)\right)^{1/2}=0.$$

Thus, for this f and x, we have that $|f(\langle P_n(x), y\rangle) - f(\langle x, y\rangle)|$ converges uniformly on S. Almost identical estimates show that $|f(\langle y, P_n(x)\rangle) - f(\langle y, x\rangle)|$ converges uniformly on S.

Since any $f \in A^*$ can be written as a linear combination of four positive linear functionals in A^*, the first part of the statement holds.

For the second part, we note that, for any $f \in A^*$ and $x \in H^{\sharp}_{A^{**}}$, by the first part of the lemma (since $\|P_n(x)\| \le \|x\|$),

$$\lim_{n\to\infty} |f(\langle P_n(x), P_n(x)\rangle) - f(\langle x, P_n(x)\rangle)| = 0.$$

We also have

$$\lim_{n\to\infty} |f(\langle P_n(x), x\rangle) - f(\langle x, x\rangle)| = 0.$$

Hence the second part of the lemma also follows. □

The following are two easy facts which we present here for convenience.

Lemma 3.9. *Let A be a C^*-algebra.*

(1) *Let H be a Hilbert A-module and $\{E_\lambda\}$ be an approximate identity for $K(H)$. Suppose $T \in K(H)^{**}$ is a nonzero positive element. Then there is λ_0 such that*

$$E_\lambda T E_\lambda \neq 0 \quad \textit{for all } \lambda \ge \lambda_0.$$

(2) *Let $T \in K(H_A)^{**}$ be a nonzero positive element and $P_n : H_A \to H_n = A^{(n)}$ be the projection ($n \in \mathbb{N}$). Then, there exists $n_0 \in \mathbb{N}$ such that*

$$P_n T P_n \neq 0 \quad \textit{for all } n \ge n_0.$$

Proof. Let $f \in K(H)^*$ be a positive linear functional. Then

$$|f(T^{1/2}(1-E_\lambda))|^2 \le f(T) f((1-E_\lambda)^2) \le f(T) f(1-E_\lambda) \to 0.$$

It follows that $f(T^{1/2}E_\lambda) \to f(T^{1/2})$ for all positive linear functionals in $K(H)^*$, whence for all $f \in K(H)^*$. Since $T^{1/2} \neq 0$ for some λ_0, we have that $T^{1/2}E_\lambda \neq 0$ for all $\lambda \ge \lambda_0$. It follows that

$$E_\lambda T E_\lambda \neq 0$$

for all $\lambda \ge \lambda_0$. This proves (1).

There are several easy proofs for (2). Let us use part (1). Choose an approximate identity $\{e_\alpha\}$ for A. Let $\lambda = (\alpha, n)$ and $\lambda_1 = (\beta_1, n) \le \lambda_2 = (\beta_2, m)$ if $\beta_1 \le \beta_2$ and $n \le m$. Define

$$E_{\beta,n} = \mathrm{diag}(\overbrace{e_\beta, e_\beta, \ldots, e_\beta}^{n}, 0, \ldots).$$

Then $\{E_{\beta,n}\}$ forms an approximate identity for $K(H_A) \cong A \otimes \mathcal{K}$. Let $T \in K(H_A)_+^{**}$ be a nonzero positive element. By (1), there is β_0 and $n_0 \in \mathbb{N}$ such that

$$E_{\beta,n} T E_{\beta,n} \neq 0 \quad \text{for all } (\beta, n) \geq (\beta_0, n_0).$$

Hence $\|T^{1/2} E_{\beta,n}^2 T^{1/2}\| = \|E_{\beta,n} T E_{\beta,n}\| \neq 0$ for all $(\beta, n) \geq (\beta_0, n_0)$. Since

$$T^{1/2} P_n T^{1/2} \geq T^{1/2} E_{\beta,n}^2 T^{1/2} \neq 0,$$

we have $T^{1/2} P_n T^{1/2} \neq 0$. It follows that

$$P_n T P_n \neq 0 \quad \text{for all } n \geq n_0. \qquad \square$$

Lemma 3.10. *Let A be a C^*-algebra and H be a countably generated Hilbert A-module. Then the homomorphism Ψ from $K(H)^{**}$ into $B(H^\sim)$ (given by Proposition 2.16) is injective.*

Proof. Let $H_n = A^{(n)} = \{(a_1, a_2, \ldots, a_n) : a_j \in A\}$ be the Hilbert A-module whose inner product is defined by $\langle x, y\rangle = \sum_{j=1}^n a_j^* b_j$, where $x = \{a_j\}_{1\leq j\leq n}$ and $y = \{b_j\}_{1\leq j\leq n}$. One identifies $K(H_n)$ with $M_n(A)$.

Claim: The map $\Psi : K(H_n)^{**} \to B(H_n^\sim)$ is a W^*-isomorphism.

Since we identify $K(H_n)$ with $M_n(A)$, we have $K(H_n)^{**} = M_n(A^{**})$.

By Proposition 3.6 (2) (and Claim 2 of the proof), $(H_n \bullet A^{**})^\sharp = (A^{**})^{(n)}$. So $H_n^\sim = (A^{**})^{(n)}$. Note that $B(H_n^\sim) = M_n(A^{**})$. One then easily checks that $\Psi : K(H_n)^{**} \to B(H_n^\sim)$ is a W^*-isomorphism. This proves the claim.

Let us consider the homomorphism $\Psi_{H_A} : K(H_A)^{**} \to B(H_A^\sim)$ given by Proposition 2.14. Put $T \in K(H_A)_+^{**} \setminus \{0\}$.

By Lemma 3.9 (2), there exists $n_0 \in \mathbb{N}$ such that $P_n T P_n \neq 0$ for all $n \geq n_0$. Recall that H_n is a direct summand of H_A. Hence by the claim and applying Proposition 3.4, we conclude that $\Psi_{H_A}(P_n T P_n) \neq 0$ for all $n \geq n_0$. There must be an element $x \in H_n$ such that

$$\langle \Psi_{H_A}(P_n T P_n)(x), x\rangle \neq 0.$$

It follows that $\langle \Psi_{H_A}(T)x, x\rangle \neq 0$. Hence $\Psi_{H_A}(T) \neq 0$. This implies that $\ker \Psi_{H_A} = \{0\}$.

In general, since H is countably generated, by Kasparov's absorbing theorem [1980, Theorem 2], we may write $H_A = H \oplus H^\perp$. To show Ψ is injective, let $T \in B(H)^{**}$ be a nonzero element. Then $K(H)^{**} = P K(H_A)^{**} P$, where $P : H_A \to H$ is the projection. Hence $PTP = T$ in $K(H_A)^{**}$. We have shown that $\Psi_{H_A}(PTP) \neq 0$. By Proposition 3.4, we have that $\Psi(T) = P \Psi_{H_A}(T) P|_{H^\sim} \neq 0$. This implies that Ψ is injective. $\square$

Lemma 3.11. *Let A be a C^*-algebra and H be a countably generated Hilbert A-module. Then there is an isomorphism Ψ from $K(H)^{**}$ onto $B(H^\sim)$ as W^*-algebras.*

Proof. By Lemma 3.10 (and by Proposition 2.14), it suffices to show that Ψ is surjective. Let us first consider the case $H = H_A$ (even though H_A is not countably generated when A is not σ-unital). By the end of Definition 2.4 (see also Remark 3.9 (and Proposition 3.10) of [Paschke 1973]), to show that

$T \in B(H^{\sim}) = B(H^{\sharp}_{A^{**}})$ is in $\Psi(K(H_A)^{**})$, it suffices to show that, for any $\epsilon > 0$, any finite subsets $X \subset H^{\sharp}_{A^{**}}$ and a finite subset $\mathcal{F} \subset A^*$, there exists $S \in K(H)^{**}$ such that

$$|f(\langle \Psi(S)(x), y\rangle) - f(\langle T(x), y\rangle)| < \epsilon \quad \text{for all } x, y \in X \text{ and } f \in \mathcal{F}.$$

We now fix ϵ, X and $\mathcal{F}$.

For any $T \in B(H^{\sim}_A) = B(H^{\sharp}_A)$,

$$\begin{aligned}|f(\langle P_n T P_n(x), y\rangle - f(\langle T(x), y\rangle)| \\ \le |f(T P_n(x), P_n(y)\rangle) - f(\langle T(x), P_n(y)\rangle)| + |f(\langle T(x), P_n(y)\rangle) - f(\langle T(x), y\rangle)|\end{aligned} \tag{3-9}$$

for any $x, y \in H^{\sharp}_{A^{**}}$ and $f \in A^*$. However, $\|P_n(y)\| \le \|y\|$ for all $n \in \mathbb{N}$. By Lemma 3.8 (1),

$$|f(T P_n(x), P_n(y)\rangle) - f(\langle T(x), P_n(y)\rangle)| \to 0,$$

and by Lemma 3.8 (2),

$$|f(\langle T(x), P_n(y)\rangle) - f(\langle T(x), y\rangle)| \to 0.$$

It follows that (by (3-9))

$$\lim_{n\to\infty} |f(\langle P_n T P_n(x), y\rangle - f(\langle T(x), y\rangle)| = 0$$

for all $x, y \in H^{\sharp}_A$ and $f \in A^*$.

We then choose $n_0 \in \mathbb{N}$ such that, for all $n \ge n_0$ (recall P_n is a projection),

$$|f(\langle P_n T P_n(x), P_n(y)\rangle) - f(T(x), y\rangle)| < \epsilon \tag{3-10}$$

for all $x, y \in X$ and $f \in \mathcal{F}$.

Now fix $n \ge n_0$. Then we have $P_n(x), P_n(y) \in (H_n)^{\sim}$ for all $x, y \in X$, and $P_n T P_n \subset B(H^{\sim}_n)$. By the claim for H_n in the proof of Lemma 3.10, we obtain an element $S \in K(H_n)^{**}$ such that $\Psi_n(S) = (P_n T P_n)|_{H^{\sim}_n}$, where

$$\Psi_n : K(H_n)^{**} \cong M_n(A^{**}) \to B(H^{\sim}_n) = M_n(A^{**})$$

is the isomorphism given by the claim. Note, by Proposition 3.4, that $\Psi(S) = P_n\Psi(S) = \Psi(S)P_n = \Psi_n(S)$. Hence it follows that, for all $x, y \in X$ and $f \in \mathcal{F}$ (and $n \ge n_0$), applying (3-10),

$$\begin{aligned}&|f(\langle \Psi(S)(x), y\rangle) - f(\langle T(x), y\rangle)| \\ &\quad = |f(\langle P_n\Psi(S)P_n(x), y\rangle) - f(T(x), y\rangle)| \\ &\quad = |(f(\langle P_n\Psi(S)P_n(x), P_n(y)\rangle) - f(\langle P_n T P_n(x), P_n(y)\rangle)| + |f(\langle P_n T P_n(x), P_n(y)\rangle) - f(T(x), y\rangle)| \\ &\quad < 0 + \epsilon = \epsilon.\end{aligned}$$

As mentioned above, this implies that Ψ is surjective.

For a general countably generated Hilbert A-module H, by Kasparov's absorbing theorem [1980, Theorem 2], we may write $H_A = H \oplus H^{\perp}$. By Proposition 3.4, $H^{\sim}_A = H^{\sim} \oplus (H^{\perp})^{\sim}$. Let $S \in B(H^{\sim}) \setminus \{0\}$. Define $T \in B(H^{\sim}_A)$ by $T|_{H^{\sim}} = S$ and $S|_{(H^{\perp})^{\sim}} = \{0\}$. We have shown that there is $L \in B(H_A)^{**}$ such that $\Psi_{H_A}(L) = S$. Then $PSP = S$, and, by Proposition 3.4, $\Psi(L) = P\Psi_{H_A}(L)P|_{H^{\sim}} = T$. Hence Ψ is surjective. □

Theorem 3.12. *Let A be a C^*-algebra and H be a Hilbert A-module. Then there is an isomorphism Ψ (given by Proposition 2.14) from $K(H)^{**}$ onto $B(H^\sim)$ as W^*-algebras. Moreover,*

$$\Psi|_{B(H)} = \widetilde{\Psi}_0.$$

Proof. By Proposition 2.14, it suffices to show that Ψ is bijective. If $K(H)$ is unital, by Proposition 2.8 of [Brown and Lin 2025], H is finitely generated. The theorem then follows from Lemma 3.11. So we will assume that $K(H)$ is not unital.

Let $\{E_\lambda\}$ be an approximate identity for $K(H)$ and $H_\lambda = \overline{E_\lambda(H)}$. Then $K(H_\lambda) = \overline{E_\lambda K(H) E_\lambda}$ is σ-unital. By Proposition 3.2 of [Brown and Lin 2025], H_λ is countably generated.

Denote by $P_\lambda : H^\sim \to H_\lambda^\sim$ the projection given by Proposition 3.1 and let $\Psi_\lambda : K(H_\lambda)^{**} \to B(H_\lambda^\sim)$ be the map given by Proposition 2.14.

To see Ψ is injective, let $T \in K(H)_+^{**} \setminus \{0\}$. It follows from Lemma 3.9 that $E_\lambda T E_\lambda \neq 0$ for all $\lambda \geq \lambda_0$ and some λ_0. Since H_λ is countably generated, by Lemma 3.11, $\Psi_\lambda(E_\lambda T E_\lambda) \neq 0$ (for $\lambda \geq \lambda_0$). By Proposition 3.4,

$$\Psi(E_\lambda T E_\lambda)|_{H_\lambda^\sim} = \Psi_\lambda(E_\lambda T E_\lambda).$$

It follows that $\Psi(E_\lambda T E_\lambda)|_{H_\lambda^\sim} \neq 0$ for all $\lambda \geq \lambda_0$. For $\lambda \geq \lambda_0$, there are $x, y \in H_\lambda$ such that

$$\langle \Psi(T)(E_\lambda(x)), E_\lambda(y)\rangle = \langle \Psi(E_\lambda T E_\lambda)(x), y\rangle \neq 0.$$

Hence $\Psi(T) \neq 0$. This shows that Ψ is injective.

To see that Ψ is surjective, let $L \in B(H^\sim)$. Since, by Proposition 2.14, $\Psi(K(H)^{**})$ is weak*-closed in the W^*-algebra $B(H^\sim)$, it suffices to show the following: for any $\epsilon > 0$, any finite subsets $X, Y \subset H^\sim$ and finite subset $\mathcal{F} \subset A^*$, there exists $T \in K(H)^{**}$ such that

$$|f(\langle \Psi(T)(x), y\rangle) - f(\langle L(x), y\rangle)| < \epsilon \quad \text{for all } x \in X, \quad y \in Y, \quad f \in \mathcal{F} \tag{3-11}$$

(see the last part of Definition 2.4). We now fix ϵ, X, Y and $\mathcal{F}$. By Proposition 2.14 (since $E_\lambda \nearrow 1_{K(H)^{**}}$),

$$\lim_\lambda f(\langle x, \Psi(E_\lambda)(y)\rangle) = \lim_\lambda f(\langle \Psi(E_\lambda)(x), y\rangle) = f(\langle x, y\rangle)$$

for all $x, y \in H^\sim$ and $f \in A^*$. It follows that there is λ_0 such that, for all $\lambda \geq \lambda_0$,

$$\begin{aligned} &|f(\langle \Psi(E_\lambda)(x), L^*(y)\rangle) - f(\langle x, L^*(y)\rangle)| < \tfrac{1}{2}\epsilon \\ \text{or} \quad &|f(\langle L\Psi(E_\lambda)(x), y\rangle) - f(\langle L(x), y\rangle)| < \tfrac{1}{2}\epsilon \end{aligned}$$

for all $x \in X$, $y \in Y$ and $f \in \mathcal{F}$. We note that the proof would be shorter if we knew

$$\lim_\lambda f(\langle \Psi(E_\lambda) L \Psi(E_\lambda)(x), y\rangle) = f(\langle L(x), y\rangle).$$

However, we may also assume that, for fixed λ_0, there is $\lambda_1 \geq \lambda_0$ such that

$$\begin{aligned} &|f(\langle L\Psi(E_{\lambda_0})(x), \Psi(E_\lambda)(y)\rangle) - f(\langle L\Psi(E_{\lambda_0})(x), y\rangle)| < \tfrac{1}{2}\epsilon \\ \text{or} \quad &|f(\langle \Psi(E_\lambda) L\Psi(E_{\lambda_0})(x), y\rangle) - f(\langle L\Psi(E_{\lambda_0})(x), y\rangle)| < \tfrac{1}{2}\epsilon \end{aligned}$$

for all $x \in X,\ y \in Y$ and $f \in \mathcal{F}$, and $\lambda \geq \lambda_1$. It follows that, for all $x \in X,\ y \in Y$ and $f \in \mathcal{F}$, if $\lambda \geq \lambda_1$,

$$\begin{aligned}
&|f(\langle L(x), y\rangle) - f(\langle \Psi(E_\lambda)L\Psi(E_{\lambda_0})(x), y\rangle)| \\
&\leq |f(\langle L(x), y\rangle) - f(\langle L\Psi(E_{\lambda_0})(x), y\rangle)| + |f(\langle L\Psi(E_{\lambda_0})(x), y\rangle) - f(\langle \Psi(E_\lambda)L\Psi(E_{\lambda_0})(x), y\rangle)| \\
&< \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon.
\end{aligned} \tag{3-12}$$

Fix $\lambda \geq \lambda_1 \geq \lambda_0$. Then $H_\lambda = \overline{E_\lambda(H)} \supset H_{\lambda_0}$. Hence

$$P_\lambda \Psi(E_\lambda) = \Psi(E_\lambda) \quad \text{and} \quad \Psi(E_{\lambda_0})P_\lambda = \Psi(E_{\lambda_0}). \tag{3-13}$$

We also note that $\Psi(E_\lambda)L\Psi(E_{\lambda_0})|_{H_\lambda^\sim} \in B(H_\lambda)$. Since H_λ is countably generated, by Lemma 3.11, there is $T_\lambda \in K(H_\lambda)^{**}$ such that

$$\Psi_\lambda(T_\lambda) = \Psi(E_\lambda)L\Psi(E_{\lambda_0})|_{H_\lambda^\sim}. \tag{3-14}$$

However, by Proposition 3.4,

$$P_\lambda \Psi(T_\lambda)P_\lambda|_{H_\lambda^\sim} = \Psi(T_\lambda)|_{H_\lambda^\sim} = \Psi_\lambda(T_\lambda). \tag{3-15}$$

Fix $\lambda \geq \lambda_1 \geq \lambda_0$. Then, for any $x \in X,\ y \in Y$ and $f \in A^{**}$, by (3-15), (3-14), (3-13) and (3-12),

$$\begin{aligned}
|f(\langle \Psi(T_\lambda)(x), (y)\rangle) - f(\langle L(x), y\rangle)| &= |f(\langle \Psi(T_\lambda)P_\lambda(x), P_\lambda(y)\rangle) - f(\langle L(x), y\rangle)| \\
&= |f(\langle \Psi(E_\lambda)L\Psi(E_{\lambda_0})P_\lambda(x), P_\lambda(y)\rangle) - f(\langle L(x), y\rangle)| \\
&= |f(\langle P_\lambda\Psi(E_\lambda)L\Psi(E_{\lambda_0})P_\lambda(x), y\rangle) - f(\langle L(x), y\rangle)| \\
&= |f(\langle \Psi(E_\lambda)L\Psi(E_{\lambda_0})(x), y\rangle) - f(\langle L(x), y\rangle)| < \epsilon.
\end{aligned}$$

As mentioned above, this implies that Ψ is surjective. □

Corollary 3.13. *Let A be a W^*-algebra and H be a Hilbert A-module. Then $F : K(H)^{**} \to B(H^\sharp)$, the map given by Proposition 2.16, is a surjective map.*

Proof. Consider the pair A and A^{**} and $H^\sim = (H \bullet A^{**})^\sharp$. By Corollary 4.3 of [Paschke 1973], $H^\sim = B(H, A^{**})$, the A^{**}-module of all bounded A^{**}-valued A-module maps from H into A^{**}. It follows that $H^\sharp \subset H^\sim$ as an A-submodule. It then follows from Proposition 2.6 that $H^\sharp \bullet A^{**} \subset H^\sim$ as Hilbert A^{**}-modules. Then, by applying Proposition 3.1,

$$(H^\sharp \bullet A^{**})^\sharp \subset H^\sim.$$

However, $H \bullet A^{**} \subset H^\sharp \bullet A^{**}$. By applying Proposition 3.1 again, we obtain

$$H^\sim = (H \bullet A^{**})^\sharp \subset (H^\sharp \bullet A^{**})^\sharp \subset H^\sim.$$

Hence $(H^\sharp \bullet A^{**})^\sharp = H^\sim$. Denote by $\widetilde{\Psi} : K(H)^{**} \to B(H^\sim)$ the isomorphism given by Theorem 3.12 and by $\widetilde{\Psi}_{H^\sharp} : B(H^\sharp) \to B((H^\sharp \bullet A^{**})^\sharp) = B(H^\sim)$ the map given by Theorem 3.12.

Now let $T \in B(H^\sharp)$. Then, by applying Theorem 3.12, we obtain $a \in K(H)^{**}$ such that $\widetilde{\Psi}(a) = \widetilde{\Psi}_{H^\sharp}(T)$. It follows that (viewing $H^\sharp \subset H^\sim$)

$$\widetilde{\Psi}(a)|_{H^\sharp} = T.$$

Since $a \in K(H)^{**}$, there exists a net $\{a_\alpha\}$ in $K(H)$ such that $a_\alpha \to a$ in the weak* topology. Therefore, by Proposition 2.14, for any $f \in A^*$ and any $\xi, \zeta \in H^{\sim}$,

$$\lim_\alpha f(\langle(\widetilde{\Psi}(a) - \widetilde{\Psi}(a_\alpha))(\xi), \zeta\rangle) = 0.$$

Note, by Theorem 3.12, $\widetilde{\Psi}(a_\alpha) = \widetilde{\Psi}_0(a_\alpha)$. On the other hand, by Proposition 2.16, for any $g \in A_*$ and any $x, y \in H$,

$$\lim_\alpha g(\langle(F(a) - a_\alpha)(x), y\rangle) = 0.$$

Hence (since $\widetilde{\Psi}_0(a_\alpha)x = a_\alpha(x)$ for all $x \in H$, see Definition 2.12)

$$g(\langle(F(a) - \widetilde{\Psi}(a))(x), y\rangle) = 0 \quad \text{for all } x, y \in H \text{ and } g \in A_*.$$

Since $\widetilde{\Psi}(a)|_{H^\sharp} = T$, we actually have

$$g(\langle(F(a) - T)(x), y\rangle) = 0 \quad \text{for all } x, y \in H \text{ and } g \in A_*. \tag{3-16}$$

Note that $F(a), T \in B(H^\sharp)$. So $F(a)(x), T(x) \in H^\sharp$ for all $x \in H$. It follows that

$$\langle(F(a) - T)(x), y\rangle \in A \quad \text{for all } x, y \in H.$$

Then, by (3-16),

$$\langle(F(a) - T)(x), y\rangle = 0 \quad \text{for all } x, y \in H.$$

Hence $F(a) = T$. In other words, F is surjective. □

4. A Kaplansky density theorem in Hilbert modules

As mentioned in the introduction, in this section we study the density of H in $H \bullet A^{**}$.

Definition 4.1. Let X be a Hilbert space and $A \subset B(X)$ be a C^*-subalgebra of $B(X)$. Let $M = \bar{A}^{\mathrm{SOT}}$, the strong operator closure of A, and let H be a Hilbert A-module. Recall, by Proposition 2.6, $H \bullet M$ is the smallest Hilbert M-module containing H as a Hilbert A-module. We consider the question of how large H is in $H \bullet M$ as a submodule.

Let $\epsilon > 0$ and V be a finite subset of X. For each $\xi \in H \bullet M$, define

$$N_{\xi,\epsilon,V} = \{z \in H \bullet M : \|\langle \xi - z, \xi - z\rangle(v)\| < \epsilon,\ v \in V\}.$$

Let $\mathcal{T}_s$ be the topology generated by $N_{\xi,\epsilon,V}$ for all $\xi \in H \bullet M$, $\epsilon \in \mathbb{R}_+ \setminus \{0\}$, and any finite subset $V \subset X$. In other words, in $\mathcal{T}_s$, a net $\{z_\alpha\}$ converges to ξ in $H \bullet M$ if and only if

$$\lim_\alpha \|\langle \xi - z_\alpha, \xi - z_\alpha\rangle(v)\| = 0 \quad \text{for all } v \in X.$$

In the special case that $X = H_U$ is the Hilbert space corresponding to the universal representation π_U of A and $M = A^{**}$, we use $\mathcal{T}_{su}$ for the topology generated by $N_{\xi,\epsilon,V}$ for all $\xi \in H \bullet A^{**}$, $\epsilon \in \mathbb{R}_+ \setminus \{0\}$, and any finite subset $V \subset H_U$.

We note that H is dense in $H \bullet M$ in the topology $\mathcal{T}_s$, but to be more useful, we will show in Theorem 4.4 that the unit ball of H is dense in the unit ball of $H \bullet M$ in $\mathcal{T}_s$, a Kaplansky-style density theorem.

Lemma 4.2. *Suppose that $x \in H \bullet M$ and $\{x_\alpha\} \subset H \bullet M$ is a bounded net. Then $x_\alpha \to x$ in $\mathcal{T}_s$ if and only if, for any $v \in X$,*

$$\lim_\alpha \sup\{\|\langle y, x_\alpha - x\rangle(v)\| : y \in H \bullet M,\ \|y\| \le 1\} = 0.$$

Moreover, if $x_\alpha \to x$ in $\mathcal{T}_s$, then, for any $f \in M_*$,

$$\lim_\alpha \sup\{|f(\langle y, x_\alpha - x\rangle)| : y \in H \bullet M,\ \|y\| \le 1\} = 0.$$

Proof. Suppose that $x_\alpha \to x$ in $\mathcal{T}_s$. We have (see Proposition 2.3 (ii) of [Paschke 1973]), for any $y \in H \bullet M$,

$$\langle x_\alpha - x, y\rangle \cdot \langle y, x_\alpha - x\rangle \le \|y\|^2 \langle x_\alpha - x, x_\alpha - x\rangle.$$

Then, for any $v \in X$ and any $y \in H \bullet M$ with $\|y\| \le 1$,

$$\begin{aligned}\|\langle y, x_\alpha - x\rangle(v)\|^2 &= \langle \langle x_\alpha - x, y\rangle \cdot \langle y, x_\alpha - x\rangle v, v\rangle_X \\ &\le \|y\|^2 \langle \langle x_\alpha - x, x_\alpha - x\rangle v, v\rangle_X \le \|\langle x_\alpha - x, x_\alpha - x\rangle v\| \|v\| \to 0\end{aligned}$$

(where $\langle\,\cdot\,,\,\cdot\,\rangle_X$ is the inner product in the Hilbert space X). Conversely, let $K = \sup_\alpha\{\|x_\alpha\| + \|x\|\} + 1$. Then

$$\|\langle x_\alpha - x, x_\alpha - x\rangle(v)\| \le K \sup\{\|\langle y, x_\alpha - x\rangle(v)\| : y \in H \bullet M,\ \|y\| \le 1\} \to 0$$

For the "moreover" part of the lemma, suppose that $\langle x_\alpha - x, x_\alpha - x\rangle \to 0$ in the strong operator topology. Then it converges in the weak operator topology. However, since $\{\langle x_\alpha - x, x_\alpha - x\rangle\}$ is bounded, this also implies that it converges to zero in the σ-weak topology and in the weak* topology. Hence

$$\lim_\alpha f(\langle x_\alpha - x, x_\alpha - x\rangle) = 0 \quad \text{for all } f \in M_*.$$

Let $f \in M_*$ be a positive normal functional. Then, $f(\langle\,\cdot\,,\,\cdot\,\rangle)$ defines a pseudo inner product on $H \bullet M$. Hence, for any $y \in H \bullet M$, we have, by the Cauchy–Bunyakovsky–Schwarz inequality,

$$|f(\langle y, x_\alpha - x\rangle)|^2 \le f(\langle y, y\rangle) f(\langle x_\alpha - x, x_\alpha - x\rangle) \le \|f\|^2 \|y\|^2 f(\langle x_\alpha - x, x_\alpha - x\rangle).$$

Thus

$$\lim_\alpha \sup\{|f(\langle y, x_\alpha - x\rangle)| : y \in H \bullet M,\ \|y\| \le 1\} = 0. \qquad \square$$

Lemma 4.3. *Let X be a Hilbert space, $A \subset B(X)$ be a C^*-subalgebra and $M = \bar{A}^{\mathrm{SOT}}$ such that $\mathrm{id}_X \in M$. Then the unit ball of H_A is dense in the unit ball of $H_A \bullet M$ in $\mathcal{T}_s$.*

Proof. Let $\xi \in H_A \bullet M$ with $\|\xi\| \le 1$. We will show that there is a net $\{x_\alpha\} \in H$ such that $\|x_\alpha\| \le \|\xi\|$ and $\lim_\alpha \|\langle x_\alpha - \xi, x_\alpha - \xi\rangle(v)\| = 0$ for all $v \in X$. From the inequality

$$\|\langle x_\alpha - \xi, x_\alpha - \xi\rangle(v)\| \le \|\langle x_\alpha - \xi, x_\alpha - \xi\rangle^{1/2}\| \|\langle x_\alpha - \xi, x_\alpha - \xi\rangle^{1/2}(v)\| \le 2\|\langle x_\alpha - \xi, x_\alpha - \xi\rangle^{1/2}(v)\|,$$

we conclude that it is enough to show that there is a net $\{x_\alpha\} \in H$ such that

$$\|x_\alpha\| \le \|\xi\| \quad \text{and} \quad \lim_\alpha \|\langle x_\alpha - \xi, x_\alpha - \xi\rangle^{1/2} v\| = 0$$

for all $v \in X$. Therefore it suffices to show that, for any $\epsilon > 0$ and any finite subset $V \subset X$, there exists $z \in H$ with $\|z\| \le 1$ such that

$$\|(\langle \xi - z, \xi - z\rangle)^{1/2}(v)\| < \epsilon \quad \text{for all } v \in V.$$

To simplify notation, we may also assume that $\|v\| \le 1$ for all $v \in V$.

Denote by $R = \overline{AM}$, the closed right ideal of M generated by A. Note, by Proposition 3.6,

$$H_A \bullet M = \{\{b_n\} \in H_B : b_n \in R\}.$$

We write $\xi = \{b_n\} \in H_A \bullet M$. There exists $n_0 \in \mathbb{N}$ such that, for all $n \ge n_0$,

$$\left\| \sum_{k=n}^{\infty} b_n^* b_n \right\| < \tfrac{1}{2}\epsilon.$$

Fix an integer $n \ge n_0$. Let $P_n : H_A \bullet M \to R^{(n)} = \{(c_1, c_2, \ldots, c_n) : c_i \in R\}$ be the projection. Put

$$S = \begin{pmatrix} b_1 & 0 & 0 & \cdots & 0 \\ b_2 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ b_n & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

For any $v \in V$, put

$$u_v = \begin{pmatrix} v \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

By the Kaplansky density theorem, there is $L \in M_n(A)$ such that

$$\|L\| \le \|S\| \quad \text{and} \quad \|L(u_v) - S(u_v)\| < \tfrac{1}{2}\epsilon \tag{4-1}$$

for all $v \in V$. Hence, denoting by $\langle\,\cdot\,,\cdot\,\rangle_X$ the inner product in X,

$$\langle (L-S)^*(L-S)u_v, u_v\rangle_X < \tfrac{1}{2}\epsilon \quad \text{for all } v \in V. \tag{4-2}$$

Define $q = \mathrm{diag}(1, 0, \ldots, 0) \in M_n(M)$. Then $S = Sq$. Replacing L by Lq, we may write

$$L = \begin{pmatrix} a_1 & 0 & 0 & \cdots & 0 \\ a_2 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_n & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where $a_i \in A,\ i = 1, 2, \ldots, n$. Then

$$\left\| \sum_{i=1}^{n} a_i^* a_i \right\| = \|L^*L\| = \|L\|^2 \le \|S\|^2 = \left\| \sum_{i=1}^{n} b_i^* b_i \right\| \le \|\xi\|^2. \tag{4-3}$$

It follows from (4-2) that

$$\left\langle \sum_{i=1}^{n} (b_i - a_i)^*(b_i - a_i)(v), v \right\rangle_X < \tfrac{1}{2}\epsilon.$$

Put $x = (a_1, a_2, \dots, a_n, 0, 0, \dots) \in H_A$. Then, by (4-3), we have $\|x\| \le \|\xi\|$ and

$$\langle\langle \xi - x, \xi - x\rangle(v), v\rangle_X = \left\langle \sum_{i=1}^{n} (b_i - a_i)^*(b_i - a_i)(v), v\right\rangle_X + \left\langle \sum_{i=n+1}^{\infty} b_i^* b_i(v), v\right\rangle_X$$
$$< \tfrac{1}{2}\epsilon + \left\| \sum_{i=n+1}^{\infty} b_i^* b_i \right\| < \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon.$$

In other words, for any $v \in V$,

$$\|(\langle \xi - x_\alpha, \xi - x_\alpha\rangle^{1/2}(v)\| = \langle\langle \xi - x, \xi - x\rangle(v), v\rangle_X < \epsilon.$$

The lemma then follows. □

Theorem 4.4. *Let X be a Hilbert space, $A \subset B(X)$ be a C^*-subalgebra and $M = \bar{A}^{\mathrm{SOT}}$, with $\mathrm{id}_X \in M$. Let H be a Hilbert A-module. Then the unit ball of H is dense in the unit ball of $H \bullet M$ in $\mathcal{T}_s$.*

Proof. Let $\xi \in H \bullet M$, with $\|\xi\| \le 1$.

Let us first assume that H is a countably generated A-module. By Lemma 4.2, it suffices to show that, for any $\epsilon > 0$ and any finite subset $V \subset X$, there exists $z \in H$ with $\|z\| \le 1$ such that

$$\|\langle y, \xi - z\rangle)(v)\| < \epsilon \quad \text{for all } y \in H,\ \|y\| \le 1 \ \text{ and } \ v \in V.$$

To simplify notation, we may also assume that $\|v\| \le 1$ for all $v \in V$.

By Kasparov's absorbing theorem [1980, Theorem 2], we may write $H_A = H \oplus H^\perp$. It follows that

$$H_A \bullet M = H \bullet M \oplus H^\perp \bullet M.$$

Define $Q : H_A \to H$ to be the projection. Then $Q \in L(H_A) = M(K(H_A))$. We identify Q with $\Psi_0(Q)$ in the sense that $Q \in L(H_M)$ which extends $Q|_{H_A}$. In particular, $H \bullet M = Q(H_A \bullet M)$.

By applying Lemmas 4.3 and 4.2, we obtain $z \in H_A$ with $\|z\| \le \|\xi\|$ such that

$$\|\langle y, \xi - z\rangle(v)\| < \epsilon \quad \text{for all } y \in H \bullet M,\ \|y\| \le 1, \ \text{ and } \ v \in V.$$

Note $Q(\xi) = \xi$ and $Q(y) = y$ for all $y \in H$. Put $x = Q(z) \in H$. We have

$$\|\langle y, \xi - x\rangle(v)\| = \|\langle y, Q(\xi) - Q(z))\rangle(v)\| = \|\langle Q(y), \xi - z\rangle(v)\| = \|\langle y, \xi - z\rangle(v)\| < \epsilon.$$

This proves the case that H is countably generated.

Next we let H be a general Hilbert A-module. We will show that, for any $\epsilon > 0$ and any finite subset $V \subset X$, there exists $z \in H$ with $\|z\| \le 1$ such that

$$\|\langle \xi - z, \xi - z\rangle)(v)\| < \epsilon \quad \text{for all } v \in V.$$

Again, we may also assume that $\|v\| \le 1$ for all $v \in V$.

Let $\{E_\lambda\}$ be an approximate identity for $K(H)$. Then, as in the proof of Theorem 3.12, $H_\lambda = \overline{E_\lambda(H)}$ is countably generated for each λ. It follows from Lemma 2.9 that there is λ such that

$$\|\Psi_0(E_\lambda)(\xi) - \xi\| < \tfrac{1}{4}\epsilon. \tag{4-4}$$

Fix such a λ. Note that, by Proposition 2.8, $\Psi_0(E_\lambda)(\xi) \in H_\lambda \bullet M \subset H \bullet M$. Since H_λ is countably generated, by the first part of the proof, we obtain $x \in H_\lambda$ with $\|x\| \le \|\Psi_0(E_\lambda)(\xi)\| \le \|\xi\|$ such that

$$\sup\{\|\langle y, \Psi_0(E_\lambda)(\xi) - x\rangle(v)\| : y \in H \bullet M,\ \|y\| \le 1\} < \tfrac{1}{4}\epsilon. \tag{4-5}$$

Then, applying (4-4) and then (4-5), for any $v \in V$,

$$\begin{aligned}\|\langle \xi - x, \xi - x\rangle(v)\| &\le \|\langle \xi - x, \xi - \Psi_0(E_\lambda)(\xi)\rangle(v)\| + \|\langle \xi - x, \Psi_0(E_\lambda)(\xi) - x\rangle(v)\| \\ &< 2\|\xi - \Psi_0(E_\lambda)(\xi)\| + 2\big\|\big\langle \tfrac{1}{2}(\xi - x), \Psi_0(E_\lambda)(\xi) - x\big\rangle(v)\big\| < \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon. \qquad \square\end{aligned}$$

We then obtain the following corollary as a Kaplansky density theorem.

Theorem 4.5. *Let A be a C^*-algebra and H be a Hilbert A-module. Then the unit ball of H is dense in the unit ball of $H \bullet A^{**}$ in $\mathcal{T}_{su}$.*

5. Closeness of *H*

Let H be a Hilbert A-module., Then, by Theorem 6.1 of [Brown and Lin 2025], the unit ball of H is A-weakly dense (see Definition 3.3 of [Brown and Lin 2025]) in the unit ball of $H^\sharp$, i.e., for any $f \in H^\sharp$, there is a net $\{x_\alpha\}$ in H with $\|x_\alpha\| \le \|f\|$ such that $\lim_\alpha \|\langle f - x_\alpha, y\rangle\| = 0$ for all $y \in H$. In the case that A is a W^*-algebra, $H^\sharp$ is a Hilbert A-module. One may ask: can one find the net $\{x_\alpha\} \in H$ with $\|x_\alpha\| \le \|f\|$ such that $\lim_\alpha \|\langle f - x_\alpha, \xi\rangle\| = 0$ for all $\xi \in H^\sharp$?

We begin with the following example.

Example 5.1. Let M be a W^*-algebra which contains a self-adjoint element a with infinite spectrum. Then, by the spectral theory, one obtains a sequence of mutually orthogonal nonzero projections $p_1, p_2, \ldots, p_n, \ldots$. Let $H = H_M$, and let $\xi = \{p_n\} \in H_M^\sharp$. Note that $\|\xi\| = \big\|\sum_{n=1}^\infty p_n\big\| = 1$ (the convergence is in the strong operator topology and weak* topology of M). We claim that there is *no* net $\{x_\alpha\}$ in H_M such that

$$\lim_\alpha \|\langle \xi - x_\alpha, \xi\rangle\| = 0.$$

Otherwise, there would be $x \in H_M$ such that

$$\|\langle \xi - x, \xi\rangle\| < \tfrac{1}{4}. \tag{5-1}$$

Since $x = \{a_n\} \in H_M$, there is $N \in \mathbb{N}$ such that

$$\Big\|\sum_{N+1} a_n^* a_n\Big\| < \big(\tfrac{1}{16}(1 + \|x\|)\big)^2.$$

Choose $q = \sum_{n=N+1}^\infty p_n \in M$. Define $P_N : H_M^\sharp \to M^{(N)} = \{(b_1, b_2, \ldots, b_N) : b_i \in M\}$ to be the projection. Then

$$\begin{aligned}\|\langle \xi - P_N(x), \xi\rangle\| &\le \|\langle \xi - x, \xi\rangle\| + \|\langle (1 - P_N)(x), \xi\rangle\| \\ &< \tfrac{1}{4} + \|(1 - P_N)(x)\|\|\xi\| < \tfrac{1}{4} + \tfrac{1}{16} = \tfrac{5}{16}.\end{aligned}$$

On the other hand,

$$\tfrac{5}{16} \geq \|\langle \xi - P_N(x), \xi\rangle\| \geq \|\langle \xi - P_N(x), \xi\rangle q\| = \left\| \left(\sum_{N+1}^{\infty} p_n - \sum_{i=1}^{N} (p_i - a_i)^* p_i\right) q \right\| = \left\| \sum_{N+1}^{\infty} p_n q \right\| = 1.$$

A contradiction. In other words, the question at the beginning of this section is negative. This also follows from Corollary 5.7 below. However, we think that the example above might also be helpful.

Lemma 5.2. *Let A be a C^*-algebra. Suppose that $\xi \in H_A^\sharp$ and $\{x_\alpha\}$ is a bounded net in H_A such that*

$$\lim_\alpha \|\xi(x) - x_\alpha(x)\| = 0 \quad \textit{for all } x \in H_A$$

and $\xi(x_\alpha) := \langle \xi, x_\alpha\rangle$ converges in norm. Then $\xi \in H_A$ and $\langle \xi, \xi\rangle = \lim_\alpha \langle \xi, x_\alpha\rangle$.

Proof. Write $\xi = \{b_n\}$ and $x_\alpha = \{a_{\alpha,n}\}$, where $\{b_n\} \in H_A^\sharp$, $a_{\alpha,n} \in A$ and, for each α, $\{a_{\alpha,n}\} \in H_A$.

Put

$$M = 1 + \sup\{\|x_\alpha\| : \alpha\} + \|\xi\| < \infty \quad \text{and} \quad a = \lim_\alpha \langle x_\alpha, \xi\rangle.$$

Note $\xi(x_\alpha) = \langle \xi, x_\alpha\rangle \in A$ for all α. Hence $a \in A$.

Let $P_n : H_A^\sharp \to H_n := A^{(n)}$ be the projection to the first n copies of A, $n \in \mathbb{N}$. Then $P_n\xi \in H_n \subset H_A$. It follows that, for each $n \in \mathbb{N}$,

$$\lim_\alpha \langle x_\alpha, P_n(\xi)\rangle = \langle \xi, P_n(\xi)\rangle = \sum_{j=1}^{n} b_j^* b_j. \tag{5-2}$$

Fix $f \in A^*$. Let $\epsilon > 0$. By Lemma 3.8, since $\{x_\alpha\}$ is bounded, there is an integer $N \in \mathbb{N}$ such that, for all $n \geq N$,

$$|f(\langle x_\alpha, \xi\rangle) - f(\langle x_\alpha, P_n(\xi)\rangle)| < \tfrac{1}{3}\epsilon \quad \text{for all } \alpha. \tag{5-3}$$

Fix any $n \geq N$. By (5-2), choose α_0 such that, for all $\alpha \geq \alpha_0$,

$$\left\| \langle x_\alpha, P_n(\xi)\rangle - \sum_{j=1}^{n} b_j^* b_j \right\| < \tfrac{1}{3}\epsilon(1 + \|f\|), \tag{5-4}$$

$$\|\langle x_\alpha, \xi\rangle - a\| < \tfrac{1}{3}\epsilon(1 + \|f\|). \tag{5-5}$$

It follows that, for all $n \geq N$, by (5-5), (5-3) and (5-4),

$$\begin{aligned} &\left| f(a) - f\left(\sum_{j=1}^{n} b_j^* b_j\right) \right| \\ &\quad \leq |f(a - \langle x_{\alpha_0}, \xi\rangle)| + |f(\langle x_{\alpha_0}, \xi\rangle) - f(x_{\alpha_0}, P_n(\xi)\rangle)| + \|f\| \left\| \langle x_{\alpha_0}, P_n(\xi)\rangle - \sum_{j=1}^{n} b_j^* b_j \right\| \\ &\quad < \tfrac{1}{3}\epsilon + \tfrac{1}{3}\epsilon + \tfrac{1}{3}\epsilon = \epsilon. \end{aligned}$$

Hence, on the state space $S(A)$ of A,

$$\lim_{n\to\infty} f\left(\sum_{j=1}^{n} b_j^* b_j\right) = f(a). \tag{5-6}$$

On the compact space $S(A)$ (in the weak* topology), $\hat{a}(f) = f(a)$ is a continuous function for all $f \in S(A)$, and $\widehat{\sum_{j=1}^n b_j^* b_j}$ is increasing. By the Dini theorem, $\widehat{\sum_{j=1}^n b_j^* b_j}$ converges uniformly to $\hat{a}$ on $S(A)$. It follows that

$$\sum_{j=1}^n b_j^* b_j \to a$$

in norm. This implies that $\xi = \{b_n\} \in H_A$ and $\langle \xi, \xi \rangle = a = \lim_\alpha \langle \xi, x_\alpha \rangle$. □

Proposition 5.3. *Let A be a C^*-algebra and H be a Hilbert A-module. Then, for any $T \in K(H)$, one has $\Psi_0(T)(H^\sharp) \subset H$, where Ψ_0 is given in Definition 2.7.*

Proof. Suppose that $T \in F(H)$ and $T = \sum_{i=1}^m \theta_{x_i, y_i}$ for some $x_i, y_i \in H$, $i = 1, 2, \ldots, m$. Then, for any $\xi \in H^\sharp$,

$$\Psi_0(T)(\xi) = \sum_{i=1}^m x_i \langle y_i, \xi \rangle = \sum_{i=1}^m x_i (\xi(y_i)^*) \in H.$$

Since $F(H)$ is dense in $K(H)$, this implies that $\Psi_0(T)(H^\sharp) \subset H$. □

Lemma 5.4. *Let A be a C^*-algebra, H be a Hilbert A-module and $\{E_\lambda\}$ be an approximate identity for $K(H)$. Then, for any $\xi \in H^\sim$ and any $f \in A^*$,*

$$\lim_\alpha \sup\{f(\langle \xi - \Psi_0(E_\lambda)(\xi), y\rangle)| : y \in H^\sim, \|y\| \le 1\} = 0.$$

Proof. By Lemma 2.9, $\{\Psi_0(E_\lambda)\}$ is an approximate identity for $K(H \bullet A^{**})$. In the universal representation of $K(H \bullet A^{**})$, $1 - \Psi_0(E_\lambda)$ converges to zero in the strong operator topology. Note that $\|1 - \Psi_0(E_\lambda)\| \le 1$. Therefore $(1 - \Psi_0(E_\lambda))(1 - \Psi_0(E_\lambda))$ also converges to zero in the strong operator topology. Hence it converges to zero in the weak operator topology. Since $\{(1 - \Psi_0(E_\lambda))^2\}$ is bounded, it also converges to zero in the weak* topology of $K(H \bullet A^{**})$. Recall that $(H \bullet A^{**})^\sharp = H^\sim$. It follows from Proposition 2.16, for any $\xi \in H^\sim$, that

$$\begin{aligned}\lim_\alpha |f(\langle \xi - \Psi_0(E_\lambda)(\xi), \xi - \Psi_0(E_\lambda)(\xi)\rangle)| &= \lim_\alpha |f(\langle \xi - F \circ \Psi_0(E_\lambda)(\xi), \xi - F \circ \Psi_0(E_\lambda)(\xi)\rangle) \\ &= \lim_\alpha |f(\langle (1 - F \circ \Psi_0(E_\lambda))^2(\xi), \xi\rangle)| = 0,\end{aligned}$$

where $F : K(H)^{**} \to B(H^\sharp)$ is the homomorphism given by Proposition 2.16. Suppose that $y \in H^\sim$ and $\|y\| \le 1$. Then, for any positive linear functional $f \in A^*$,

$$\begin{aligned}f(\langle \xi - \Psi_0(E_\lambda)(\xi), y\rangle)^2 &\le f(\langle \xi - \Psi_0(E_\lambda)(\xi), \xi - \Psi_0(E_\lambda)(\xi)\rangle) f(\langle y, y\rangle) \\ &\le \|f\| f(\langle \xi - \Psi_0(E_\lambda)(\xi), \xi - \Psi_0(E_\lambda)(\xi)\rangle).\end{aligned}$$

It follows that, for any $f \in A^*$,

$$\lim_\alpha \sup\{f(\langle \xi - \Psi_0(E_\lambda)(\xi), y\rangle)| : y \in H^\sim, \|y\| \le 1\} = 0.$$ □

Theorem 5.5. *Let A be a C^*-algebra and H be a Hilbert A-module. Suppose that $\xi \in H^\sharp$ and there is a bounded net $\{x_\alpha\}$ in H such that*

$$\lim_\alpha \|\xi(x) - \langle x_\alpha, x\rangle\| = 0 \quad \textit{for all } x \in H$$

and $\xi(x_\alpha) := \langle \xi, x_\alpha \rangle$ converges in norm. Then $\xi \in H$ and $\langle \xi, \xi \rangle = \lim_\alpha \langle \xi, x_\alpha \rangle \in A$.

Proof. First let us assume H is countably generated. Then, by Kasparov's absorbing theorem [1980, Theorem 2], we may write $H_A = H \oplus H^{\perp}$. Then $\xi \in H^{\sharp} \subset H_A^{\sharp}$. By applying Lemma 5.2, we obtain that

$$\xi \in H_A \quad \text{and} \quad \langle \xi, \xi \rangle = \lim_{\alpha} \langle \xi, x_\alpha \rangle.$$

Since $\xi(x_\alpha) \in A$, we have $a = \langle \xi, \xi \rangle \in A$. Let $P : H_A \to H$ be the projection. Then $P \in L(H_A)$. Put $\eta = P(\xi) \in H$. Note that $\langle P(\xi) - \xi, x \rangle = 0$ for all $x \in H$. Hence $\xi = \eta$. Therefore this case follows.

In what follows we will work in $H^{\sim}$ and use the inner product in $H^{\sim}$ whenever it is convenient.

In general, let $a = \lim_\alpha \langle \xi, x_\alpha \rangle$. Since $\langle \xi, x_\alpha \rangle = \xi(x_\alpha) \in A$ for all α, we have $a \in A$.

Claim: $a = \langle \xi, \xi \rangle$ (in the inner product of $H^{\sim}$).

Let $\{E_\lambda\}$ be an approximate identity for $K(H)$. Let $\epsilon > 0$ and $f \in A^*$, with $\|f\| \le 1$. By applying Lemma 5.4, we have (since $\{\|\xi - x_\alpha\|\}$ is bounded)

$$\lim_{\lambda} \Big(\sup_{\alpha} \{ |f(\langle \xi - \Psi_0(E_\lambda)(\xi), \xi - x_\alpha \rangle)| \} \Big) = 0. \tag{5-7}$$

Thus, by applying Lemma 5.4 and (5-7), we obtain λ_0 such that, for all $\lambda \ge \lambda_0$,

$$\begin{aligned} |f(\langle \xi - \Psi_0(E_\lambda)(\xi), \xi \rangle)| &< \tfrac{1}{3}\epsilon, \\ |f(\langle \xi - \Psi_0(E_\lambda)(\xi), \xi - x_\alpha \rangle)| &< \tfrac{1}{3}\epsilon \quad \text{for all } \alpha. \end{aligned} \tag{5-8}$$

Recall that, by Proposition 5.3, $\Psi_0(E_\lambda)(\xi) \in H$. Fix any $\lambda \ge \lambda_0$. Choose α_0 such that, for any $\alpha \ge \alpha_0$,

$$\|\langle \xi, x_\alpha \rangle - a\| < \tfrac{1}{3}\epsilon \quad \text{and} \quad |f(\langle \Psi_0(E_\lambda)(\xi), \xi - x_\alpha \rangle)| < \tfrac{1}{3}\epsilon. \tag{5-9}$$

Now, by the first inequality of (5-9), (5-8) and then the second inequality of (5-9),

$$\begin{aligned} |f(\langle \xi, \xi \rangle - a)| &< |f(\langle \xi, \xi \rangle - \langle \xi, x_\alpha \rangle)| + \tfrac{1}{3}\epsilon = |f(\langle \xi, \xi - x_\alpha \rangle)| + \tfrac{1}{3}\epsilon \\ &\le |f(\langle \xi - \Psi_0(E_\lambda)(\xi), \xi - x_\alpha \rangle)| + |f(\langle \Psi_0(E_\lambda)(\xi), \xi - x_\alpha \rangle)| + \tfrac{1}{3}\epsilon < \tfrac{1}{3}\epsilon + \tfrac{1}{3}\epsilon + \tfrac{1}{3}\epsilon = \epsilon. \end{aligned}$$

Since this holds for any ϵ, we conclude that

$$f(\langle \xi, \xi \rangle) = f(a) \quad \text{for all } f \in A^*.$$

By the Hahn–Banach theorem, we obtain $\langle \xi, \xi \rangle = a$. The claim is proved.

There exists $x_1 \in \{x_\alpha\}$ and then $x_2 \in \{x_\alpha\}$ such that

$$\|\langle x_1, \xi \rangle - a\| < \tfrac{1}{2}, \quad \|\langle \xi - x_2, x_1 \rangle\| < \tfrac{1}{4} \quad \text{and} \quad \|\langle x_2, \xi \rangle - a\| < \tfrac{1}{4}.$$

Suppose that we have found $x_1, x_2, \ldots, x_n$ such that

$$\|\langle \xi - x_j, x_i \rangle\| < 1/2^j \quad \text{and} \quad \|\langle x_j, \xi \rangle - a\| < 1/2^j, \quad i = 1, 2, \ldots, j-1,$$

and $j = 1, 2, \ldots, n$. Then choose $x_{n+1} \in \{x_\alpha\}$ such that

$$\|\langle \xi - x_{n+1}, x_i \rangle\| < 1/2^{n+1} \quad \text{and} \quad \|\langle x_{n+1}, \xi \rangle - a\| < 1/2^{n+1}, \quad i = 1, 2, \ldots, n.$$

Thus, by induction, we obtain a subsequence $\{x_n\}$ in $\{x_\alpha\}$ such that

$$\lim_{n\to\infty} \|\langle x_n, \xi\rangle - a\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|\langle \xi - x_n, x_i\rangle\| = 0 \ \text{ for } i \in \mathbb{N}.$$

Denote by H_0 the Hilbert A-submodule generated by $\{x_1, x_2, \dots, x_n, \dots\}$. In particular, $x_n \in H_0$ and $n \in \mathbb{N}$. Let $\eta = \xi|_{H_0}$.

Now H_0 is countably generated and $x_n \in H_0$, so we have

$$\lim_{n\to\infty} \|\eta(x_n) - a\| = \lim_{n\to\infty} \|\xi(x_n) - a\| = 0.$$

Moreover, if $y = \sum_{i=1}^m x_i \cdot a_i$, where $a_i \in A$, then

$$\lim_{n\to\infty} \|\eta(y) - \langle x_n, y\rangle\| = 0.$$

Since $\{x_n\}$ is bounded (since $\{x_\alpha\}$ is bounded), this implies that

$$\lim_{n\to\infty} \|\eta(y) - \langle x_n, y\rangle\| = 0 \quad \text{for all } y \in H_0.$$

Applying what has been proved, we conclude that $\eta \in H_0$ and $\lim_{n\to\infty}\langle \eta, x_n\rangle = \langle \eta, \eta\rangle = a$.

We now consider Hilbert A^{**}-modules $H_0 \bullet A^{**} \subset H \bullet A^{**}$. By Proposition 3.1, we obtain a projection $P : H^\sim \to H_0^\sim$ such that $P|_{H_0 \bullet A^{**}} = \mathrm{id}_{H_0 \bullet A^{**}}$. Then $\eta = P(\xi)$. Hence, by the claim,

$$\begin{aligned}\|(1-P)\xi\|^2 &= \|\langle (1-P)(\xi), (1-P)(\xi)\rangle\| \le \|\langle (1-P)(\xi), \xi\rangle\| + \|\langle (1-P)(\xi), P(\xi)\rangle\| \\ &= \|\langle \xi, \xi\rangle - \langle P(\xi), \xi\rangle\| + 0 = \|a - \langle P(\xi), P(\xi)\rangle\| = \|a - \langle \eta, \eta\rangle\| = 0.\end{aligned}$$

In other words, $P(\xi) = \eta = \xi$. The theorem follows. □

Definition 5.6. Let A be a C^*-algebra and H be a Hilbert A-module. Then $H^\sharp \subset H^\sim$.

For each $\xi \in H^\sharp$, $\epsilon > 0$ and a finite subset $Y \subset H^\sharp$, define

$$O_{\xi,\epsilon,Y} = \{\zeta \in H^\sharp : \|\langle \xi - \zeta, y\rangle\| < \epsilon,\ y \in Y\},$$

where the inner product is taken from $H^\sharp$ if $H^\sharp$ is a Hilbert A-module, or from $H^\sim$ (with values in A^{**}).

Denote by $\mathcal{T}_{NW}$ the topology in $H^\sharp$ generated by $O_{\xi,\epsilon,Y}$ for all $\xi \in H^\sharp$, $\epsilon \in \mathbb{R}_+ \setminus \{0\}$ and finite subsets $Y \subset H^\sharp$. Note that a net $\{\zeta_\alpha\}$ converges to ξ in $H^\sharp$ in $\mathcal{T}_{NW}$ if and only if

$$\lim_\alpha \|\langle \xi - \zeta_\alpha, y\rangle\| = 0$$

for all $y \in H^\sharp$, where the inner product is the one defined above.

Corollary 5.7. *Let A be a C^*-algebra and H be a Hilbert A-module. Then, with $\mathcal{T}_{NW}$, the unit ball of H is closed in $H^\sharp$.*

Proof. Let $\xi \in H^\sharp$. Suppose that there is a net $\{x_\alpha\}$ in H with $\|x_\alpha\| \le 1$ such that

$$\lim_\alpha \|\langle \xi - x_\alpha, \eta\rangle\| = 0 \quad \text{for all } \eta \in H^\sharp,$$

where the inner product is in $H^\sim$. Then, for each $x \in H$, $\lim_\alpha \|\langle \xi - x_\alpha, x\rangle\| = 0$ and (by choosing $\eta = \xi$) $\{\xi(x_\alpha)\} = \{\langle \xi, x_\alpha\rangle\}$ converges in norm to $\langle \xi, \xi\rangle$. By Theorem 5.5, $\xi \in H$. □

Corollary 5.8. *Let A be a monotone complete C^*-algebra and H be a Hilbert A-module. Then the unit ball of H is closed in $H^\sharp$ in the topology $\mathcal{T}_{NW}$, where we view $H^\sharp$ as a self-dual Hilbert A-module.*

Lemma 5.9. *Let X be a Hilbert space, $A \subset B(X)$ be a C^*-subalgebra and $M = \bar{A}^{\mathrm{SOT}}$, with $\mathrm{id}_X \in M$. Let H be a Hilbert A-module. Suppose that $\xi \in H \bullet M$ and $\langle \xi, x\rangle \in A$ for all $x \in H$. Then $\xi \in H$.*

Proof. First let us consider the case that $H = H_A$. Then, by Proposition 3.6,

$$H_A \bullet M = \left\{ \{a_n\} : a_n \in \overline{AM} \text{ and } \sum_{k=1}^{n} a_k^* a_k \text{ converges in norm} \right\}.$$

Write $\xi = \{b_n\} \in H_A \bullet M$. The condition that $\langle \xi, x\rangle \in A$ for all $x \in H_A$ implies that $\xi \in H_A^\sharp$. It follows that $b_n \in A$. Hence $\xi \in H_A$.

Next, let us assume that H is countably generated. Let $\xi \in H \bullet M$ and $\langle \xi, x\rangle \in A$ for all $x \in H$. By Kasparov's absorbing theorem, we may write $H_A = H \oplus H^\perp$. It follows from what has been proved that $\xi \in H_A$. Let $P : H_A \to H$ be the projection. Then $P(\xi) \in H$. However, $\langle \xi - P(\xi), x\rangle = 0$ for all $x \in H$. For any $y \in H^\perp$, since $\xi \in H \bullet M$, we have $\langle \xi, y\rangle = 0$ for all $y \in H$. Hence $\xi = P(\xi) \in H$.

In general, since $\xi \in H \bullet M$, there are $x_{n,i} \in H$, $i = 1, 2, \ldots, k(n)$, $b_{n,i} \in M$, $i = 1, 2, \ldots, k(n)$, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} \left\| \xi - \sum_{i=1}^{k(n)} x_{n,i} \bullet b_{n,i} \right\| = 0.$$

Let H_0 be the Hilbert A-submodule generated by $\{x_{n,i} : 1 \le i \le k(n), n \in \mathbb{N}\}$. Then $\xi \in H_0 \bullet M$ and $\xi|_{H_0} \in H_0^\sharp$, as $\langle \xi, h\rangle \in A$ for all $h \in H_0 \subset H$. From what has just been proved, $\xi \in H_0 \subset H$. ☐

We end this section with the following result.

Theorem 5.10. *Let A be a C^*-algebra and H be a Hilbert A-module. Then the unit ball of H is closed in $H^\sim$ in the topology $\mathcal{T}_{NW}$ of $H^\sim = (H \bullet A^{**})^\sharp$.*

Proof. Let $\{x_\alpha\}$ be a net in the unit ball of H and $\xi \in H^\sim$ such that

$$\lim_\alpha \|\langle \xi - x_\alpha, \zeta\rangle\| = 0 \quad \text{for all } \zeta \in H^\sim.$$

Since $H^\sim = (H \bullet A^{**})^\sharp$ and $H \subset H \bullet A^{**}$, by applying Corollary 5.8, we conclude that $\xi \in H \bullet A^{**}$.

We also have, for all $y \in H$,

$$\lim_\alpha \|\langle \xi - x_\alpha, y\rangle\| = 0.$$

Since $\langle x_\alpha, y\rangle \in A$, it follows that $\langle \xi, y\rangle \in A$. By Lemma 5.9, $\xi \in H$. ☐

6. A Kaplansky-style density theorem in the self-dual Hilbert modules

In the last section, we show that H is closed in $H^\sharp$ and $H^\sim$ in the topology $\mathcal{T}_{NW}$ of $H^\sharp$ and that of $H^\sim$, respectively. In this section, however, we will show that H is dense in $H^\sim$ in a weaker topology. In fact, by Theorem 4.5, it is easy to show that H is dense in $H^\sharp$ in $\mathcal{T}_0$, the topology defined below (see Definition 6.1). A similar question is whether one can replace x in (6-1) by any element in $H^\sharp$.

Definition 6.1. Let A be a W^*-algebra and H be a Hilbert A-module.

Let $\epsilon > 0$, and let $Y \subset H$ and $\mathcal{F} \subset A_*$ be finite subsets. Let $\xi \in H^\sharp$. Define

$$O_{\xi,\epsilon,Y,\mathcal{F}} = \{\zeta \in H^\sharp : |f(\langle \xi - \zeta, x\rangle)| < \epsilon,\ x \in Y,\ f \in \mathcal{F}\} \subset H^\sharp. \tag{6-1}$$

Let $\mathcal{T}_0$ be the topology of $H^\sharp$ generated by the subsets $O_{\xi,\epsilon,Y,\mathcal{F}}$.

Let $\epsilon > 0$, and let $Z \subset H^\sharp$ and $\mathcal{F} \subset A_*$ be finite subsets. Let $\xi \in H^\sharp$. Define

$$O_{\xi,\epsilon,Z,\mathcal{F}} = \{\zeta \in H^\sharp : |f(\langle \xi - \zeta, x\rangle)| < \epsilon,\ x \in Z,\ f \in \mathcal{F}\} \subset H^\sharp.$$

Let $\mathcal{T}_w$ be the topology of $H^\sharp$ generated by the subsets $O_{\xi,\epsilon,Z,\mathcal{F}}$.

In fact, by [Paschke 1973, Proposition 3.8] and the definition before it, $\mathcal{T}_w$ is the weak* topology of $H^\sharp$ as a conjugate space. So a natural question is whether H is dense in $H^\sharp$ in $\mathcal{T}_w$. To be more useful (but perhaps not useful enough to be used twice on Sundays — see [Pedersen 1979, 2.3.4]), we will also prove a Kaplansky-style density theorem in Theorem 6.4.

Let us also consider another topology. Let $\epsilon > 0$, $\xi \in H^\sharp$, and let $\mathcal{F} \subset A_*$ be a finite subset. Define

$$O_{\epsilon,\xi,\mathcal{F}} = \{\zeta \in H^\sharp : |f(\langle \xi - \zeta, \xi - \zeta\rangle)| < \epsilon,\ f \in \mathcal{F}\}.$$

Let $\mathcal{T}_{ws}$ be the topology generated by $O_{\epsilon,\xi,\mathcal{F}}$ for all $\epsilon > 0$, $\xi \in H^\sharp$ and finite subsets $\mathcal{F} \subset A_*$. Note that $\mathcal{T}_{ws}$ is stronger than $\mathcal{T}_w$, which is stronger than $\mathcal{T}_0$.

Lemma 6.2. *Let X be a Hilbert space and $A \subset B(X)$ be a C^*-subalgebra. Suppose that $M = \bar{A}^{\mathrm{SOT}}$, with $\mathrm{id}_X \in M$ and $b = \{b_k\} \in H_M^\sharp$. There is a net $a_\alpha = \{(a_{1,\alpha}, a_{2,\alpha}, \ldots, a_{n,\alpha}, \ldots)\} \in H_A$ such that*

$$\left\| \sum_{j=1}^{\infty} a_{j,\alpha}^* a_{j,\alpha} \right\|^{1/2} \le \|b\|, \tag{6-2}$$

$$\lim_\alpha f\left(\sum_{j=1}^{\infty} (b_j - a_{j,\alpha})^* (b_j - a_{j,\alpha}) \right) = 0 \tag{6-3}$$

for all $f \in M_*$.

Proof. Let $Y = l^2(X)$, the Hilbert space direct sum of countably many copies of X. Let $\bar{b} = (c_{i,j}) \in B(Y)$, where $c_{i,1} = b_i$, $i \in \mathbb{N}$, and $c_{i,j} = 0$ if $j \ge 2$ (see (3-8)). Denote by $P_n : Y \to X^{(n)}$ the projection, where $X^{(n)}$ is the direct sum of (first) n copies of X. Let $\epsilon > 0$ and $V \in L^2(X)$ be a finite subset. Then there is $n_0 \in \mathbb{N}$ such that

$$\|(1 - P_{n_0})(v)\| < \tfrac{1}{2}\epsilon(1 + \|b\|) \quad \text{for all } v \in V.$$

There is $d \in M_{n_0}(A)$ such that

$$\|(\bar{b} - d)(P_{n_0}(v))\| < \tfrac{1}{4}\epsilon \quad \text{for all } v \in V.$$

We have

$$\begin{aligned}\|(\bar{b} - dP_{n_0})(v)\| &\le \|(\bar{b} - dP_{n_0})(1 - P_{n_0})(v)\| + \|(\bar{b} - d)P_{n_0}(v)\| \\ &= \|\bar{b}(1 - P_{n_0})(v)\| + \tfrac{1}{4}\epsilon < \epsilon \qquad \text{for all } v \in V.\end{aligned}$$

Let B be the self-adjoint algebra of those bounded operators on Y which can be expressed as infinite matrices with entries in A, where all are zero except finitely many of them. Then, by what has been proved, we conclude that, in the strong operator topology (of $B(Y)$), operator $\bar{b}$ is in the closure of operators in B in the strong operator topology.

Then, by the Kaplansky density theorem, there is a net $\{d_\alpha\} \in B$ with $\|d_\alpha\| \le \|\bar{b}\|$ such that

$$\lim_\alpha \|(\bar{b} - d_\alpha)v\| = 0 \quad \text{for all } v \in Y.$$

Since $\{\|\bar{b} - d_\alpha\|\}$ is bounded, we also have

$$\lim_\alpha \|(\bar{b} - d_\alpha)^*(\bar{b} - d_\alpha)v\| = 0 \quad \text{for all } v \in Y.$$

We further note that

$$\|\bar{b}\|^2 = \|(\bar{b})^*\bar{b}\| = \left\| \sum_{j=1}^{\infty} b_j^* b_j \right\| \le \|b\|.$$

Then

$$\lim_\alpha \|(\bar{b} - d_\alpha)^*(\bar{b} - d_\alpha)P_1 v\| = 0 \quad \text{for all } v \in Y. \tag{6-4}$$

Note $\bar{b}P_1 = \bar{b}$. Let $d'_\alpha = d_\alpha P_1 = (d_{i,j,\alpha})$, where $d_{i,j,\alpha} = 0$ if $j \ge 2$. Put $a_{j,\alpha} = d_{1,j,\alpha}$, $j \in \mathbb{N}$. Then, for all $n \in \mathbb{N}$,

$$\left\| \sum_{j=1}^{n} a_{j,\alpha}^* a_{j,\alpha} \right\| \le \|(d'_\alpha)^* d'_\alpha\| = \|d'_\alpha\|^2 \le \|d_\alpha\|^2 \le \|\bar{b}\|^2 \le \|b\|^2.$$

Put $a_\alpha = \{a_{j,\alpha}\}$. Since $d_\alpha \in B$, for each α, there are only finitely many $a_{j,\alpha}$ which are not zero. Hence $a_\alpha \in H_A$. Then $\|a_\alpha\| \le \|b\|$. Thus (6-2) holds. On the other hand, by (6-4),

$$\lim_\alpha \|(\bar{b} - d'_\alpha)^*(\bar{b} - d'_\alpha)P_1 v\| = 0. \tag{6-5}$$

Let $h \in X$. By (6-5),

$$\lim_\alpha \left\| \sum_{j=1}^{\infty} (b_j - a_{j,\alpha})^*(b_j - a_{j,\alpha})h \right\| = 0.$$

In other words, $\sum_{i=1}^{\infty}(b_j - a_{j,\alpha})^*(b_j - a_{j,\alpha}) = \langle b - a_\alpha, b - a_\alpha\rangle \to 0$ in the strong operator topology. However,

$$\left\| \sum_{j=1}^{n} (b_j - a_{j,\alpha})^*(b_j - a_{j,\alpha}) \right\| = \|(\bar{b} - d'_\alpha)\|^2 \le (\|\bar{b}\| + \|d_\alpha\|)^2 \le 4\|b\|^2.$$

Therefore $\sum_{i=1}^{n}(b_j - a_{j,\alpha})^*(b_j - a_{j,\alpha}) \to 0$ in the σ-weak operator topology and hence in the weak* topology (see, for example, 4.6.13 of [Pedersen 1989]). Therefore (6-3) holds. □

Lemma 6.3. *Let $A \subset B(X)$ be a C^*-subalgebra, and let $M = \bar{A}^{\mathrm{SOT}}$, with $1_X \in M$. Suppose that H is a countably generated Hilbert A-module. Then H is dense in $(H \bullet M)^\sharp$ in the following sense: for any $\xi \in (H \bullet M)^\sharp$, there is a net $x_\alpha \in H$ with $\|x_\alpha\| \le \|\xi\|$ such that*

$$\lim_\alpha \sup\{|f(\langle \xi - x_\alpha, \zeta\rangle)| : \zeta \in (H \bullet M)^\sharp,\ \|\zeta\| \le 1\} = 0 \quad \textit{for all } f \in M_*. \tag{6-6}$$

Proof. Let us first prove this for $H = H_A$, even though when A is not σ-unital, H_A is not countably generated. Lemma 6.2 provides a net $\{x_\alpha\}$ in H_A with $\|x_\alpha\| \le \|\xi\|$ such that

$$\lim_\alpha f(\langle \xi - x_\alpha, \xi - x_\alpha\rangle) = 0 \quad \text{for all } f \in M_*.$$

Recall that, for any positive linear functional f, the map $H_M^\sharp \times H_M^\sharp \to \mathbb{R}$ defined by $[x, y]_f = f(\langle x, y\rangle)$ (for all $x, y \in H_M^\sharp$) is a pseudo inner product. Therefore, by the Cauchy–Bunyakovsky–Schwarz inequality,

$$f(\langle x, y\rangle)^2 \le f(\langle x, x\rangle) f(\langle y, y\rangle) \quad \text{for all } x, y \in H_M^\sharp.$$

It follows that, for any positive normal linear functional f,

$$\begin{aligned}\sup\{|f(\langle \xi - x_\alpha, \zeta\rangle)| : \zeta \in H_M^\sharp,\ \|\zeta\| \le 1\}^2 &\le \sup\{f(\langle \zeta, \zeta\rangle) f(\langle \xi - x_\alpha, \xi - x_\alpha\rangle) : \zeta \in H_M^\sharp,\ \|\zeta\| \le 1\}\\ &= \|f\| f(\langle \xi - x_\alpha, \xi - x_\alpha\rangle) \to 0.\end{aligned}$$

Thus we proved (6-6) holds for $H = H_A$.

Now let H be a countably generated Hilbert A-module. Then, by Kasparov's absorbing theorem, we may write $H_A = H \oplus H^\perp$. Hence $H_A \bullet M = H \bullet M \oplus (H^\perp \bullet M)$. It follows that

$$H_M^\sharp = (H_A \bullet M)^\sharp = (H \bullet M)^\sharp \oplus (H^\perp \bullet M)^\sharp.$$

Let $P : H_M^\sharp \to (H \bullet M)^\sharp$ be the projection such that $P|_H = \mathrm{id}_H$. Then, by what has been proved for H_A, there is a net $y_\alpha \in H_A$ such that $\|y_\alpha\| \le \|\xi\|$ and, for any $f \in M_*$,

$$\lim_\alpha \sup\{f(\langle \xi - y_\alpha, \zeta\rangle) : \zeta \in H_M^\sharp,\ \|\zeta\| \le 1\} = 0.$$

Put $x_\alpha = P(y_\alpha) \in (H \bullet M)^\sharp$. Note that $P(\xi) = \xi$. Then, for any $f \in M_*$,

$$\begin{aligned}\lim_\alpha \sup\{f(\langle \xi - x_\alpha, \zeta\rangle) : \zeta \in (H\bullet M)^\sharp,\ \|\zeta\| \le 1\} &= \lim_\alpha \sup\{f(\langle \xi - x_\alpha, P(\zeta)\rangle) : \zeta \in (H\bullet M)^\sharp,\ \|\zeta\| \le 1\}\\ &= \lim_\alpha \sup\{f(\langle \xi - y_\alpha, \zeta\rangle) : \zeta \in (H\bullet M)^\sharp,\ \|\zeta\| \le 1\}\\ &\le \lim_\alpha \sup\{f(\langle \xi - y_\alpha, \zeta\rangle) : \zeta \in H_M^\sharp,\ \|\zeta\| \le 1\} = 0. \quad \square\end{aligned}$$

Theorem 6.4. *Let X be a Hilbert space, $A \subset B(X)$ a C^*-subalgebra and $M = \bar{A}^{\mathrm{SOT}}$, with $1_M = \mathrm{id}_X$, and let H be a Hilbert A-module. Then the unit ball of H is dense in the unit ball of $(H \bullet M)^\sharp$ in $\mathcal{T}_{ws}$ (the topology on $(H \bullet M)^\sharp$).*

Proof. Let $\xi \in (H\bullet M)^\sharp$ with $\|\xi\| \le 1$. It suffices to show that, for any $\epsilon > 0$, any finite subset $Y \subset (H\bullet M)^\sharp$ and any finite subset $\mathcal{F} \subset M_*$, there is $x \in H$ such that

$$\|x\| \le \|\xi\| \quad \text{and} \quad |f(\langle \xi - x, y\rangle)| < \epsilon \quad \text{for all } y \in (H \bullet M)^\sharp,\ \|y\| \le 1, \text{ and } f \in \mathcal{F}.$$

Let us fix ϵ and $\mathcal{F}$. Choose an approximate identity $\{E_\lambda\}$ for $K(H)$. It follows that $E_\lambda \nearrow \mathrm{id}_H$. Note that $\mathrm{id}_H \in M(K(H))$. By the last part of Proposition 2.13, $\Psi_0(\mathrm{id}_H) = \mathrm{id}_{H\bullet M}$. By [Paschke 1973, Corollary 3.7], $F \circ \Psi_0(\mathrm{id}_H) = \mathrm{id}_{(H\bullet M)^\sharp}$, where $F : K(H \bullet M)^{**} \to B(H \bullet M)^\sharp$ is the map given by Proposition 2.16. Note also that, by Lemma 2.9, $\{\Psi_0(E_\lambda)\}$ is an approximate identity for $K(H \bullet M)$. In the universal representation of $K(H \bullet M)$, we have that $1 - \Psi_0(E_\lambda)$ converges to zero in the strong operator topology. Note that $\|1-\Psi_0(E_\lambda)\| \le 1$. Therefore $(1-\Psi_0(E_\lambda))^*(1-\Psi_0(E_\lambda))$ also converges to zero in the strong operator topology. Hence (since $\{\|(1-\Psi_0(E_\lambda))^*(1-\Psi_0(E_\lambda))\|\}$ is bounded), it converges to zero in the weak* topology. By Proposition 2.16, we have, for all $f \in M_*$,

$$f(\langle \xi - F\circ\Psi_0(E_\lambda)(\xi), \xi - F\circ\Psi_0(E_\lambda)(\xi)\rangle) = f(\langle (1-F\circ\Psi_0(E_\lambda))^*(1-F\circ\Psi_0(E_\lambda))(\xi), \xi\rangle) \to 0. \quad (6\text{-}7)$$

Next let g be a positive normal linear functional in M_*. Then, for any $y \in (H\bullet M)^\sharp$ with $\|y\| \le 1$,

$$\begin{aligned}|g(\langle \xi - F\circ\Psi_0(E_\lambda)(\xi), y\rangle)|^2 &\le g(\langle \xi - F\circ\Psi_0(E_\lambda)(\xi), \xi - F\circ\Psi_0(E_\lambda)(\xi)\rangle)g(\langle y, y\rangle)\\ &\le \|g\|\|y\|^2 g(\langle \xi - F\circ\Psi_0(E_\lambda)(\xi), \xi - F\circ\Psi_0(E_\lambda)(\xi)\rangle).\end{aligned}$$

Hence, by (6-7),

$$\lim_\alpha\bigl(\sup\{|g(\langle \xi - F\circ\Psi_0(E_\lambda)(\xi), y\rangle)| : y \in (H\bullet M)^\sharp,\ \|y\| \le 1\}\bigr) = 0.$$

It follows that, for any $f \in M_*$,

$$\lim_\alpha\bigl(\sup\{|f(\langle \xi - F\circ\Psi_0(E_\lambda)(\xi), y\rangle)| : y \in (H\bullet M)^\sharp,\ \|y\| \le 1\}\bigr) = 0.$$

Put $\Phi := F\circ\Psi_0$. We obtain λ_0 such that, for all $\lambda \ge \lambda_0$,

$$|f(\langle \xi - \Phi(E_\lambda)(\xi), y\rangle)| < \tfrac{1}{2}\epsilon \quad \text{for all } y \in (H\bullet M)^\sharp,\ \|y\| \le 1, \ \text{ and } \ f \in \mathcal{F}. \qquad (6\text{-}8)$$

Let $H_\lambda = \overline{E_\lambda(H)}$. As in the proof of Theorem 3.12, we have that H_λ is countably generated. Moreover, by Proposition 3.1,

$$(H\bullet M)^\sharp = (H_\lambda\bullet M)^\sharp \oplus ((H_\lambda\bullet M)^\sharp)^\perp.$$

Let $P_\lambda : (H\bullet M)^\sharp \to (H_\lambda\bullet M)^\sharp$ be the projection. Note that

$$\Phi(E_\lambda)(\xi), \Phi(E_\lambda)(y) \in P_\lambda((H\bullet M)^\sharp) = (H_\lambda\bullet M)^\sharp$$

for all $y \in (H\bullet M)^\sharp$.

It follows from Lemma 6.3 that there is $x \in H_\lambda$ with $\|x\| \le \|\Phi(E_\lambda)(\xi)\| \le \|\xi\|$ such that

$$|f(\langle \Psi(E_\lambda)(\xi) - x, P_\lambda(y)\rangle)| < \tfrac{1}{2}\epsilon \quad \text{for all } y \in (H_\lambda\bullet M)^\sharp,\ \|y\| \le 1.$$

Since $P_\lambda\Phi(E_\lambda) = \Phi(E_\lambda)$ and $x \in H_\lambda$, we have, for all $y \in (H_\lambda\bullet M)^\sharp$, $\|y\| \le 1$,

$$|f(\langle \Phi(E_\lambda)(\xi) - x, y\rangle)| = |f(\langle P_\lambda\Phi(E_\lambda)(\xi) - P_\lambda(x), y\rangle)| = |f(\langle \Phi(E_\lambda)(\xi) - x, P_\lambda(y)\rangle)| < \tfrac{1}{2}\epsilon.$$

Thus (also applying (6-8)) for all $y \in (H\bullet M)^\sharp$ with $\|y\| \le 1$ and $f \in \mathcal{F}$,

$$|f(\langle \xi - x, y\rangle)| \le |f(\langle \xi - \Phi(E_\lambda)(\xi), y\rangle)| + |f(\langle \Phi(E_\lambda)(\xi) - x, y\rangle)| < \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon. \qquad \square$$

The next two statements are the main results of this section.

Corollary 6.5. *Let A be a W^*-algebra and H be a Hilbert A-module. Then the unit ball of H is dense in $H^\sharp$ in $\mathcal{T}_{ws}$.*

Proof. Let $M = A$ and then apply Theorem 6.4. □

Theorem 6.6. *Let A be a C^*-algebra and H be a Hilbert A-module. Then the unit ball of H is dense in $H^{\sim}$ in $\mathcal{T}_{ws}$ (as $H^{\sim} = (H \bullet A^{**})^{\sharp}$).*

Proof. We choose the universal representation π_U and its strong operator closure $A'' = A^{**}$, then apply Theorem 6.4. □

References

[Baaj and Skandalis 1993] S. Baaj and G. Skandalis, "Unitaires multiplicatifs et dualité pour les produits croisés de C^*-algèbres", *Ann. Sci. École Norm. Sup.* (4) **26**:4 (1993), 425–488. MR Zbl

[Brown and Ciuperca 2009] N. P. Brown and A. Ciuperca, "Isomorphism of Hilbert modules over stably finite C^*-algebras", *J. Funct. Anal.* **257**:1 (2009), 332–339. MR Zbl

[Brown and Lin 2025] L. G. Brown and H. Lin, "Projective Hilbert modules and sequential approximations", *Sci. China Math.* **68**:1 (2025), 137–168. MR Zbl

[Conway 2000] J. B. Conway, *A course in operator theory*, Grad. Stud. Math. **21**, Amer. Math. Soc., Providence, RI, 2000. MR Zbl

[Coward et al. 2008] K. T. Coward, G. A. Elliott, and C. Ivanescu, "The Cuntz semigroup as an invariant for C^*-algebras", *J. Reine Angew. Math.* **623** (2008), 161–193. MR Zbl

[Kaplansky 1953] I. Kaplansky, "Modules over operator algebras", *Amer. J. Math.* **75** (1953), 839–858. MR Zbl

[Kasparov 1980] G. G. Kasparov, "Hilbert C^*-modules: theorems of Stinespring and Voiculescu", *J. Operator Theory* **4**:1 (1980), 133–150. MR Zbl

[Lin 1991a] H. X. Lin, "Bounded module maps and pure completely positive maps", *J. Operator Theory* **26**:1 (1991), 121–138. MR Zbl

[Lin 1991b] H. X. Lin, "Hilbert C^*-modules and their bounded module maps", *Sci. China Ser. A* **34**:12 (1991), 1409–1421. MR Zbl

[Lin 1992] H. X. Lin, "Injective Hilbert C^*-modules", *Pacific J. Math.* **154**:1 (1992), 131–164. MR Zbl

[Lin 2001] H. Lin, *An introduction to the classification of amenable C^*-algebras*, World Sci., River Edge, NJ, 2001. MR Zbl

[Ortega et al. 2011] E. Ortega, M. Rørdam, and H. Thiel, "The Cuntz semigroup and comparison of open projections", *J. Funct. Anal.* **260**:12 (2011), 3474–3493. MR Zbl

[Paschke 1973] W. L. Paschke, "Inner product modules over B^*-algebras", *Trans. Amer. Math. Soc.* **182** (1973), 443–468. MR Zbl

[Pedersen 1979] G. K. Pedersen, *C^*-algebras and their automorphism groups*, Lond. Math. Soc. Monogr. **14**, Academic Press, London, 1979. MR Zbl

[Pedersen 1989] G. K. Pedersen, *Analysis now*, Grad. Texts in Math. **118**, Springer, 1989. MR Zbl

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