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DAMPED STRICHARTZ ESTIMATES AND THE INCOMPRESSIBLE EULER–MAXWELL SYSTEM

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Euler–Maxwell systems describe the dynamics of inviscid plasmas. We consider an incompressible two-dimensional version of such a system and prove the existence and uniqueness of global weak solutions, uniformly with respect to the speed of light $c \in (c_0, \infty)$, for some threshold value $c_0 > 0$ depending only on the initial data. In particular, the condition $c > c_0$ ensures that the velocity of the plasma nowhere exceeds the speed of light and allows us to analyze the singular regime $c \to \infty$.

The functional setting for the fluid velocity lies in the framework of Yudovich's solutions of the two-dimensional Euler equations, whereas the analysis of the electromagnetic field hinges upon the refined interactions between the damping and dispersive phenomena in Maxwell's equations in the whole space. This analysis is enabled by the new development of a robust abstract method allowing us to incorporate the damping effect into a variety of existing estimates. The use of this method is illustrated by the derivation of damped Strichartz estimates (including endpoint cases) for several dispersive systems (including the wave and Schrödinger equations), as well as damped maximal regularity estimates for the heat equation. The ensuing damped Strichartz estimates supersede previously existing results on the same systems.

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1. Introduction

We are concerned with the existence and uniqueness of solutions to the incompressible Euler–Maxwell system

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + j \times B, & \operatorname{div} u = 0 \quad \text{(Euler's equation)}, \\ \frac{1}{c}\partial_t E - \nabla \times B = -j, & \operatorname{div} E = 0 \quad \text{(Ampère's equation)}, \\ \frac{1}{c}\partial_t B + \nabla \times E = 0, & \operatorname{div} B = 0 \quad \text{(Faraday's equation)}, \\ j = \sigma(cE + P(u \times B)), & \operatorname{div} j = 0 \quad \text{(Ohm's law)} \end{cases} \tag{1-1}$$

for some initial data $(u, E, B)|_{t=0} = (u_0, E_0, B_0)$, with the two-dimensional normal structure on the vector fields

$$u(t,x) = \begin{pmatrix} u_1(t,x) \\ u_2(t,x) \\ 0 \end{pmatrix}, \quad E(t,x) = \begin{pmatrix} E_1(t,x) \\ E_2(t,x) \\ 0 \end{pmatrix} \quad \text{and} \quad B(t,x) = \begin{pmatrix} 0 \\ 0 \\ b(t,x) \end{pmatrix}, \tag{1-2}$$

where $(t,x) \in [0,\infty) \times \mathbb{R}^2$ and $P = \mathrm{Id} - \Delta^{-1}\nabla \operatorname{div}$ denotes Leray's projector onto divergence-free vector fields. We will later see that the normal structure (1-2) is propagated by the flow and is therefore persistent.

Taking the divergence of Maxwell's system, which is made up of Ampère and Faraday's equations, notice that the divergence-free conditions $\operatorname{div} E = 0$ and $\operatorname{div} B = 0$ are also propagated by the evolution of the system, provided they hold initially. (In fact, notice that the condition $\operatorname{div} B = 0$ is a trivial consequence of the normal structure (1-2). Nevertheless, it is physically relevant, since magnetic fields are solenoidal.)

This model describes the evolution of a plasma, i.e., a charged gas or an electrically conducting fluid, subject to the self-induced electromagnetic Lorentz force $j \times B$. Here, the field u denotes the velocity of the fluid, E and B are the electric and magnetic fields, respectively, whereas j denotes the electric current. Moreover, the positive constants c and σ represent the speed of light and the electrical conductivity, respectively. We refer to [Biskamp 1993; Davidson 2001] for details about the physical principles behind the modeling of plasmas.

It is readily seen that any smooth solution $(u, E, B) \in C_c^1([0,\infty) \times \mathbb{R}^2)$ of (1-1) satisfies the energy inequality

$$\|u(t)\|_{L^2}^2 + \|E(t)\|_{L^2}^2 + \|B(t)\|_{L^2}^2 + \frac{2}{\sigma}\int_0^t \|j(\tau)\|_{L^2}^2\, d\tau \le \mathcal{E}_0^2 \tag{1-3}$$

for all $t \ge 0$, where we define

$$\mathcal{E}_0 := \|(u_0, E_0, B_0)\|_{L^2}.$$

(Observe that (1-3) actually holds with an equality sign for smooth functions, but this will not be used.) This is the only known global a priori estimate for solutions of (1-1), and the ensuing natural bound

$$(u, E, B) \in L^\infty([0,\infty); L^2(\mathbb{R}^2))$$

is insufficient to guarantee the existence of global weak solutions to (1-1). At least, no known method has so far been able to build such solutions, and the same holds for the classical two-dimensional incompressible Euler system

$$\partial_t u + u \cdot \nabla u = -\nabla p, \quad \operatorname{div} u = 0, \tag{1-4}$$

which corresponds to the case $(E, B) = 0$. This is due to the fact that the nonlinear terms in (1-1) and (1-4) are, in general, not stable under weak convergence of solutions.

1.1. *Main results.* Our main result on the Euler–Maxwell system (1-1) establishes the global existence and uniqueness of weak solutions for any initial data in suitable spaces, provided the speed of light c is sufficiently large. Note that this is seemingly the only known global existence result for incompressible Euler–Maxwell systems. It reads as follows.

Theorem 1.1. *Let p and s be any real numbers in $(2,\infty)$ and $\left(\frac{7}{4}, 2\right)$, respectively. For any initial data*

$$(u_0, E_0, B_0) \in ((H^1 \cap \dot{W}^{1,p}) \times H^s \times H^s)(\mathbb{R}^2),$$

with $\operatorname{div} u_0 = \operatorname{div} E_0 = \operatorname{div} B_0$ and the two-dimensional normal structure (1-2), there is a constant $c_0 > 0$ such that, for any speed of light $c \in (c_0, \infty)$, there is a global weak solution (u, E, B) to the two-dimensional Euler–Maxwell system (1-1), with the normal structure (1-2), satisfying the energy inequality (1-3) and enjoying the additional regularity

$$\begin{gathered} u \in L^\infty(\mathbb{R}^+; H^1 \cap \dot{W}^{1,p}), \quad (E, B) \in L^\infty(\mathbb{R}^+; H^s), \\ (cE, B) \in L^2(\mathbb{R}^+; \dot{H}^1), \quad cE \in L^2(\mathbb{R}^+; \dot{H}^s), \quad (E, B) \in L^2(\mathbb{R}^+; \dot{W}^{1,\infty}). \end{gathered} \tag{1-5}$$

It is to be emphasized that the bounds in (1-5) are uniform in $c \in (c_0, \infty)$ for any given initial data.

If, furthermore, the initial vorticity $\omega_0 := \nabla \times u_0$ belongs to $L^\infty(\mathbb{R}^2)$, then the solution enjoys the global bound

$$\omega := \nabla \times u \in L^\infty(\mathbb{R}^+; L^\infty),$$

and it is unique in the space of all solutions $(\bar{u}, \bar{E}, \bar{B})$ to the Euler–Maxwell system (1-1) satisfying the bounds, locally in time,

$$(\bar{u}, \bar{E}, \bar{B}) \in L^\infty_t L^2_x, \quad \bar{u} \in L^2_t L^\infty_x, \quad \bar{j} \in L^2_{t,x},$$

and having the same initial data.

Theorem 1.1 is a simple and more accessible reformulation of the results from Section 3, which are stated therein in full detail in the setting of Besov and Chemin–Lerner spaces (see Appendix A for a precise definition of these spaces). Indeed, it is readily seen that Theorem 1.1 follows directly from the combination of Theorems 3.1, 3.2, 3.3 and Corollary 3.13 with straightforward embeddings of functional spaces. The respective proofs of these results are also provided in complete detail in Section 3.

One should note that the constant c_0 in the above statement depends on norms of the initial data. Thus, for any given $c > 0$, the condition $c_0 < c$ can be interpreted, in a fully equivalent way, as a smallness condition on the initial data. In fact, a careful inspection of (3-4) in the statement of Theorem 3.3 readily provides an explicit expression for c_0 in terms of the norms of (u_0, E_0, B_0) in $(H^1 \cap \dot{W}^{1,p}) \times H^s \times H^s$. More specifically, for any given initial data, one could set, for example,

$$c_0 = \max\{1, (\|u_0\|_{H^1 \cap \dot{W}^{1,p}} + \|(E_0, B_0)\|_{H^s}) C e^{C\mathcal{E}_0^5}\}$$

for some suitable large constant $C > 0$ which only depends on p and s, and is independent of the initial data. Then, with this definition of c_0, it is straightforward to show that the condition $c > c_0$ implies the validity of (3-4). In particular, for a given speed of light c, we observe that the existence of solutions is a consequence of the smallness of the initial data. Finally, we also note that it is not difficult to provide sharper formulas for c_0, with increasing complexity.

A detailed scaling analysis of solutions to the Euler–Maxwell system (1-1) is conducted in Section 3.1, which further clarifies the significance of the initial conditions of our main results and their dependence on the physical constants c and σ.

We have already emphasized that the bounds (1-5) on the solutions of the incompressible Euler–Maxwell system (1-1) are uniform with respect to the speed of light $c > c_0$. This crucial feature allows us to deduce a simple but powerful convergence result in the asymptotic regime $c \to \infty$, which is of particular interest. We refer to [Arsénio et al. 2015] for a thorough discussion of this regime in the context of incompressible Navier–Stokes–Maxwell systems.

Generally speaking, the physical relevance of the regime $c \to \infty$ in Euler–Maxwell systems stems from the fact that the limiting magnetohydrodynamic systems are suitable to describe the behavior of flows which are influenced by self-induced magnetic fields. This is the case, for instance, of the terrestrial magnetic field, which is sustained by the earth's core through the dynamo effect, or the solar magnetic field, which is responsible for sunspots, or the galactic magnetic field, which plays a role in the formation of stars. We refer to [Davidson 2001] for more details on the physical background of magnetohydrodynamic systems.

The next result follows directly from Theorem 1.1 and establishes a magnetohydrodynamic system by taking the limit of the Euler–Maxwell system (1-1) in the singular regime $c \to \infty$. Observe that it recovers the classical Yudovich theorem for the incompressible Euler system (1-4) by setting $B \equiv 0$.

Corollary 1.2. *For any given initial data (u_0, E_0, B_0) as in Theorem 1.1 $\big($for some $p \in (2, \infty)$ and $s \in \big(\frac{7}{4}, 2\big)\big)$, consider the global solution (u^c, E^c, B^c) constructed therein for each $c \in (c_0, \infty)$. Then, the set of solutions $\{(u^c, E^c, B^c)\}_{c > c_0}$ is relatively compact in $L^2_{t,x,\mathrm{loc}}$. In particular, for any sequence $\{(u^{c_n}, E^{c_n}, B^{c_n})\}_{n \in \mathbb{N}}$, with $c_n \to \infty$, there is a convergent subsequence (which we do not distinguish, for simplicity)*

$$(u^{c_n}, E^{c_n}, B^{c_n}) \xrightarrow{n \to \infty} (u, 0, B) \quad \text{in } L^2_{t,x,\mathrm{loc}}, \tag{1-6}$$

where $(u, B) = ((u_1, u_2, 0), (0, 0, b))$ has the normal structure (1-2) and is a global weak solution of the system

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p, & \operatorname{div} u = 0, \\ \partial_t b - \frac{1}{\sigma} \Delta b + u \cdot \nabla b = 0, \end{cases} \tag{1-7}$$

with the bounds

$$u \in L^\infty(\mathbb{R}^+; H^1 \cap \dot{W}^{1,p}), \quad b \in L^\infty(\mathbb{R}^+; H^1) \cap L^2(\mathbb{R}^+; \dot{H}^1 \cap \dot{H}^2 \cap \dot{W}^{1,\infty}).$$

If, furthermore, the initial vorticity ω_0 *belongs to* $L^\infty(\mathbb{R}^2)$, *then the solution* (u, b) *to* (1-7) *satisfies the additional bound* $\omega \in L^\infty(\mathbb{R}^+; L^\infty)$ *and is unique in the space of all solutions* $(\bar{u}, \bar{b})$ *satisfying the bounds, locally in time,*

$$\bar{u} \in L^\infty_t L^2_x \cap L^2_t L^\infty_x, \quad \bar{b} \in L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x,$$

and having the same initial data. Moreover, one has the convergence

$$(u^c, E^c, B^c) \xrightarrow{c \to \infty} (u, 0, B) \quad \text{in } L^2_{t,x,\mathrm{loc}}, \tag{1-8}$$

without extraction of subsequences.

Remark. Note that (1-7) is a simple form of a magnetohydrodynamic system. Indeed, the equations for u and b are not genuinely coupled, for the incompressible Euler equation does not contain an external magnetic force. This can be interpreted as a consequence of the two-dimensional normal structure (1-2). More specifically, whenever the electric current is given by $j = \nabla \times B$, a straightforward calculation exploiting (1-2) shows that the Lorentz force satisfies

$$j \times B = (\nabla \times B) \times B = -\tfrac{1}{2} \nabla(b^2),$$

which can be absorbed in the pressure gradient. In particular, since u is independent of b in this regime, there can be no Alfvén waves (see [Davidson 2001] for an introduction to Alfvén waves). Therefore, in this case, the limiting magnetohydrodynamic system loses the feature of some important physical effects (such as Alfvén waves). This suggests that extending the results of the present article beyond the two-dimensional normal structure (1-2) is of particular interest and significance.

Proof. We begin by showing the relative compactness of the set of solutions $\{(u^c, E^c, B^c)\}_{c>c_0}$ in $L^2_{t,x}(K)$ for any compact set $K \subset \mathbb{R}^+ \times \mathbb{R}^2$. To that end, note that the energy inequality (1-3) and the global bounds (1-5) on the solutions hold uniformly in c. In particular, it is readily seen that $E^c \to 0$ in $L^2_{t,x,\mathrm{loc}}$ as $c \to \infty$. Therefore, we only need to focus on $\{(u^c, B^c)\}_{c>c_0}$.

Now, one can show from (1-5) that $u^c \in L^\infty_{t,x}$ and $B^c \in L^2_t L^\infty_x$ (for instance, using the Gagliardo–Nirenberg convexity inequality (3-16), which is recalled later on). It therefore follows directly from (1-9) that $\partial_t u^c = P(j^c \times B^c) - P(u^c \cdot \nabla u^c)$ is uniformly bounded in $L^1_{t,\mathrm{loc}} L^2_x$. Similarly, it is readily seen from Faraday's equation $\partial_t B^c = -c \nabla \times E^c$ that $\partial_t B^c$ is uniformly bounded in $L^2_{t,x}$. Then, further combining these controls of $\partial_t u^c$ and $\partial_t B^c$ with the uniform bound $(u^{c_n}, B^{c_n}) \in L^\infty_t H^1_x$ and the compactness of the embedding $H^1_{\mathrm{loc}} \subset L^2_{\mathrm{loc}}$, we deduce that $\{(u^c, B^c)\}_{c>c_0}$ is relatively compact in the topology of $L^2_{t,x,\mathrm{loc}}$ by a classical compactness result by Aubin and Lions. (See [Simon 1987] for a thorough discussion of such compactness results and, in particular, Section 9 therein, for convenient results which are easily applicable to our setting.)

Next, for any convergent subsequence (1-6), employing Ohm's law to substitute $c_n E^{c_n}$ into Faraday's equation in (1-1), observe that we only have to pass to the limit in the system

$$\begin{cases} \partial_t u^{c_n} + u^{c_n} \cdot \nabla u^{c_n} = -\nabla p^{c_n} + j^{c_n} \times B^{c_n}, & \operatorname{div} u^{c_n} = 0, \\ \frac{1}{c_n} \partial_t E^{c_n} - \nabla \times B^{c_n} = -j^{c_n}, & \operatorname{div} B^{c_n} = 0, \\ \partial_t B^{c_n} + \frac{1}{\sigma} \nabla \times j^{c_n} + u^{c_n} \cdot \nabla B^{c_n} = 0. \end{cases} \tag{1-9}$$

Moreover, up to further extraction of subsequences, it is also possible to assume that one has the weak convergence

$$j^{c_n} \rightharpoonup j \quad \text{in } L^2_{t,x}.$$

All in all, passing to the limit $n \to \infty$ in (1-9) in the sense of distributions and exploiting the strong convergence (1-6), we find that

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + j \times B, & \operatorname{div} u = 0, \\ \nabla \times B = j, & \operatorname{div} B = 0, \\ \partial_t B + \frac{1}{\sigma} \nabla \times j + u \cdot \nabla B = 0. \end{cases}$$

Then, recalling the vector identity $\nabla \times (\nabla \times B) = \nabla(\operatorname{div} B) - \Delta B$ and noticing that $(\nabla \times B) \times B = -\frac{1}{2}\nabla(b^2)$, we conclude that (u, b) is a solution of (1-7).

Finally, if we further assume the pointwise boundedness of the initial vorticity ω_0, then $\omega^c = \nabla \times u^c$ remains uniformly bounded in $L^\infty_{t,x}$, thereby yielding a similar bound for the limiting system (1-7). These bounds then fall in the framework of Yudovich's uniqueness theorem (see [Majda and Bertozzi 2002, Section 8.2.4], for instance), which guarantees the uniqueness of the solution u to the incompressible two-dimensional Euler system. Alternatively, one can also deduce the uniqueness of u by reproducing the arguments from Section 3.9 below, by setting $(E, B) = 0$. As for the uniqueness of b, it easily follows from classical energy estimates on the heat equation.

At last, the uniqueness of the limit point $(u, 0, B)$ allows us to deduce the validity of (1-8), which completes the proof of the corollary. □

1.2. ***Other models of incompressible plasmas.*** The Euler–Maxwell system (1-1) can be seen as the inviscid version of the Navier–Stokes–Maxwell system given by

$$\begin{cases} \partial_t u + u \cdot \nabla u - \nu \Delta u = -\nabla p + j \times B, & \operatorname{div} u = 0, \\ \frac{1}{c} \partial_t E - \nabla \times B = -j, & \operatorname{div} E = 0, \\ \frac{1}{c} \partial_t B + \nabla \times E = 0, & \operatorname{div} B = 0, \\ j = \sigma(cE + P(u \times B)), & \operatorname{div} j = 0, \end{cases} \tag{1-10}$$

where $\nu > 0$ denotes the viscosity of the fluid.

The derivation of (1-10) has been established rigorously in [Arsénio and Saint-Raymond 2019] through the analysis of the viscous incompressible hydrodynamic regimes of Vlasov–Maxwell–Boltzmann systems. In particular, it follows from the results therein that (1-10) can be obtained by letting $\delta \to 0$, with $\delta > 0$, in the more complete system

$$\begin{cases} \partial_t u + u \cdot \nabla u - \nu \Delta u = -\nabla p + \delta c n E + j \times B, & \operatorname{div} u = 0, \\ \frac{1}{c} \partial_t E - \nabla \times B = -j, & \operatorname{div} E = \delta n, \\ \frac{1}{c} \partial_t B + \nabla \times E = 0, & \operatorname{div} B = 0, \\ j - \delta n u = \sigma\left(-\frac{c}{\delta} \nabla n + cE + u \times B\right), \end{cases} \tag{1-11}$$

which takes the Coulomb force nE into account, where n is the electric charge density.

The work performed in [Arsénio and Saint-Raymond 2019] addresses the viscous incompressible regimes of Vlasov–Maxwell–Boltzmann systems only. However, inviscid incompressible regimes can also be achieved as an asymptotic limit of collisional kinetic equations. For instance, the incompressible Euler limit of the Boltzmann equation has been established in [Saint-Raymond 2003; 2009b]. (A general discussion of hydrodynamic regimes of the Boltzmann equation can also be found in [Saint-Raymond 2009a].) Similarly, in the vein of the results from [Arsénio and Saint-Raymond 2019], it is possible to derive (1-1) by considering the incompressible Euler regime of Vlasov–Maxwell–Boltzmann systems, at least formally. However, this remains to be done rigorously.

The well-posedness theory established in this article only concerns (1-1) and does not encompass the inviscid version of (1-11) (i.e., the corresponding Euler–Maxwell system obtained by setting $\nu = 0$ in (1-11)). However, we are hopeful that some adaptation of our results can be implemented to show the existence and uniqueness of solutions to (1-11), with $\nu = 0$. Nevertheless, for the sake of simplicity, we are going to stick to (1-1).

It turns out that there is yet another version of incompressible Navier–Stokes–Maxwell systems which is commonly found in the literature. It reads

$$\begin{cases} \partial_t u + u \cdot \nabla u - \nu \Delta u = -\nabla p + j \times B, & \operatorname{div} u = 0, \\ \frac{1}{c} \partial_t E - \nabla \times B = -j, & \operatorname{div} B = 0, \\ \frac{1}{c} \partial_t B + \nabla \times E = 0, \\ j = \sigma(cE + u \times B), \end{cases} \tag{1-12}$$

and a corresponding incompressible Euler–Maxwell system is given by setting $\nu = 0$. We refer to [Arsénio and Gallagher 2020; Germain et al. 2014; Masmoudi 2010] for details on the construction of global solutions to (1-12), with $\nu > 0$.

Unlike (1-10) and (1-11), it is to be emphasized that this model is not obtained as an asymptotic regime of Vlasov–Maxwell–Boltzmann systems, as shown in [Arsénio and Saint-Raymond 2019]. Furthermore, when compared to (1-10) and (1-11), it has the major drawback of not providing a strong control of $\operatorname{div} E$. For this reason, we do not make any claim concerning the extension of our work to the above model. It would, however, be interesting to clarify the well-posedness of the nonviscous version of (1-12).

Finally, we observe that there is also a rich family of compressible Euler–Maxwell systems which are commonly used to model the behavior of plasmas. The study of such systems is challenging, and corresponding results tend to focus on the stability of smooth solutions near specific equilibrium states. We refer to [Germain and Masmoudi 2014; Guo et al. 2016] for foundational results on three-dimensional compressible Euler–Maxwell systems. We note that the results therein do not require any specific vector structure, such as the normal structure (1-2). However, they are sensitive to the speed of light c and, therefore, may not provide uniform bounds as c tends to infinity.

1.3. ***Strategy of proof.*** We lay out now the strategy and the key ideas leading to the proof of Theorem 1.1, which will be implemented in Section 3 to establish the more precise Theorems 3.1, 3.2 and 3.3.

Observe first that, even if we add a dissipation term $-\Delta u$ to the first equation of (1-1), thereby yielding the incompressible Navier–Stokes–Maxwell system (1-10), it is still unknown whether or not global weak solutions do exist when the initial data are only square-integrable. This is due to the lack of strong compactness (or regularity) in electromagnetic fields (E, B), combined with the lack of stability of the source term $j \times B$ in weak topologies (see [Arsénio and Gallagher 2020] for further details). The same difficulty persists in the inviscid version of the same system, which stems from the propagation of singularities in Maxwell's system, as a result of its hyperbolic nature. The construction of solutions in L^2 to (1-1) is thus highly challenging — all the more so than in the viscous case.

One should therefore treat this system in some higher-regularity spaces. To this end, inspired by known results on the well-posedness of the two-dimensional Euler system (1-4), we shall look at the equivalent vorticity formulation of (1-1), which reads as

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = -j \cdot \nabla B, & \operatorname{div} u = 0, \\ \frac{1}{c}\partial_t E - \nabla \times B + \sigma c E = -\sigma P(u \times B), & \operatorname{div} E = 0, \\ \frac{1}{c}\partial_t B + \nabla \times E = 0, & \operatorname{div} B = 0, \\ j = \sigma(cE + P(u \times B)), & \operatorname{div} j = 0, \end{cases} \tag{1-13}$$

where $\omega := \nabla \times u$ and u can be reconstructed from ω through the Biot–Savart law

$$u = -\Delta^{-1} \nabla \times \omega. \tag{1-14}$$

Observe that the normal structure (1-2) has been used in (1-13) to write $\nabla \times (j \times B) = -j \cdot \nabla B$. This is crucial.

Much of our analysis of (1-13) will hinge on the dispersive properties of the damped Maxwell system

$$\begin{cases} \frac{1}{c}\partial_t E - \nabla \times B + \sigma c E = -\sigma P(u \times B), \\ \frac{1}{c}\partial_t B + \nabla \times E = 0, \\ \operatorname{div} u = \operatorname{div} E = \operatorname{div} B = 0. \end{cases} \tag{1-15}$$

This will require us to interpret the role of the velocity field u in (1-15), in the spatial variable x, as that of a coefficient in the algebra $L^\infty_x \cap \dot{H}^1_x$ (or some weaker variant), thereby allowing us to view (1-15) as a linear system in (E, B) and produce closed estimates on the electromagnetic field.

To be precise, the treatment of the source term $-\sigma P(u \times B)$ in (1-15) will necessitate the control of the velocity field u in a suitable algebra acting on $\dot{H}^s_x$ for appropriate values of s. In particular, according to the classical paradifferential product law

$$\|fg\|_{\dot{H}^s(\mathbb{R}^d)} \lesssim \|f\|_{L^\infty \cap \dot{B}^{d/2}_{2,\infty}(\mathbb{R}^d)} \|g\|_{\dot{H}^s(\mathbb{R}^d)},$$

which holds for any $s \in \left(-\frac{1}{2}d, \frac{1}{2}d\right)$ and $d \geq 1$, it will be natural to seek the control of u (in the space variable) in the weaker algebra $L^\infty \cap \dot{B}^1_{2,\infty}(\mathbb{R}^2)$; see Appendix A for a definition of Besov spaces.

In the context of two-dimensional viscous flows, such a control is expected in view of the strong bounds provided by the energy dissipation inequality. For example, in [Arsénio and Gallagher 2020, Theorem 1.2], the existence of weak solutions to a two-dimensional incompressible Navier–Stokes–Maxwell system was established by proving a uniform control of the velocity field in the algebra $L^\infty_x \cap \dot{H}^1_x(\mathbb{R}^2)$. More precisely, by building upon the methods from [Masmoudi 2010], it was shown therein (see [Arsénio and Gallagher 2020, Proposition 2.1]) that the control of the velocity field in the space $L^2_t(L^\infty_x \cap \dot{H}^1_x)$ was sufficient to propagate some $\dot{H}^s$-regularity, with $-1 < s < 1$, in Maxwell's equations (1-15), uniformly as $c \to \infty$.

In the setting of two-dimensional incompressible electrically conducting ideal fluids (i.e., plasmas), which is the focus of our work, global energy estimates are nowhere near as good as their viscous counterpart and, thus, fail to yield the control of u in a useful algebra. Instead, we need to take Yudovich's approach of propagating the $L^2_x \cap L^p_x$-norm of the vorticity ω, for some given $p > 2$, by exploiting the transport equation

$$\partial_t \omega + u \cdot \nabla \omega = -j \cdot \nabla B, \tag{1-16}$$

thereby providing a bound on u in the algebra $L^\infty_t(L^\infty_x \cap \dot{H}^1_x)$, by classical Sobolev embeddings combined with standard estimates on the Biot–Savart law (1-14). We refer to [Bahouri et al. 2011, Section 7.2] for a modern treatment of global existence results for two-dimensional perfect incompressible fluids and the Yudovich theorem.

In particular, elementary estimates on transport equations, which are performed in detail in Section 3.4, show that the control of ω in $L^\infty_t L^p_x$ follows from the control of the initial vorticity ω_0 and the nonlinear source term $j \cdot \nabla B$ in L^p_x and $L^1_t L^p_x$, respectively. Since j is naturally bounded in $L^2_{t,x}$, by virtue of the energy inequality (1-3), we conclude that ∇B should be controlled in $L^2_t L^\infty_x$.

Now, experience shows that such a Lipschitz bound on B cannot easily follow from energy estimates on the wave system (1-15). Indeed, energy estimates on hyperbolic systems are typically performed in L^2_x. Therefore, in order to control ∇B in L^∞_x, an energy estimate on (1-15) would lead us, in view of classical Sobolev embeddings, to seek a bound of B in $H^{2+\delta}_x$, with a small parameter $\delta > 0$. To that end, the source term $-\sigma P(u \times B)$ in (1-15) would also need to be controlled in $H^{2+\delta}_x$. However, employing paradifferential calculus to control $u \times B$ would require that ∇u be bounded in $L^\infty_x \cap \dot{H}^1_x$ at least. Unfortunately, such uniform bounds on perfect incompressible two-dimensional flows are largely out of reach in our context. This is where the damped dispersive properties of (1-15), on the whole Euclidean plane $\mathbb{R}^2$, come into play.

Maxwell's system (1-15) can be rewritten as a system of wave equations (more on this later on, see (1-17)). Thus, heuristically, one expects to be able to employ Strichartz estimates for the wave equation to control the electromagnetic field (E, B). In particular, by paying close attention to the admissibility criteria of functional spaces in Strichartz estimates (see [Bahouri et al. 2011, Section 8.3] or [Keel and Tao 1998]), one observes that it is possible to control the Lipschitz norm of a solution to a two-dimensional wave equation, provided one can bound $\frac{7}{4}$ derivatives of the initial data and the source term in some appropriate functional spaces (in some Besov spaces, for instance) of L^2 space-integrability. (For simplicity, we have omitted here the consideration of time integrability in Strichartz estimates and focused solely on space regularity and integrability.) Loosely speaking, such an estimate is better than a Sobolev embedding, which would require the control of over two derivatives in $L^2(\mathbb{R}^2)$ in order to bound a Lipschitz norm. This should give the reader some intuition concerning the special role played by the regularity parameter $s = \frac{7}{4}$ in Theorem 1.1.

Thus, so far, our strategy seems to yield some promising closed estimates. Indeed, on the one hand, the transport equation (1-16) gives us a bound on the $L^\infty_t(L^2_x \cap L^p_x)$-norm of the vorticity ω provided ∇B is controlled in $L^2_t L^\infty_x$, while, on the other hand, a control of ∇B in $L^2_t L^\infty_x$ can be achieved through dispersive estimates on the wave system (1-15) if the velocity field u is sufficiently smooth (at least $L^\infty_t(L^\infty_x \cap \dot{H}^1_x)$, say).

However, such a roadmap may not lead to global estimates in time. To see this, we need to take a closer look at the temporal norms associated with our strategy. Specifically, it is important to note that the classical Strichartz estimates for the two-dimensional wave equation do not actually give a global control of ∇B in $L^2_t L^\infty_x$. Instead, they only allow us to control ∇B in $L^4_t L^\infty_x$ globally, which then leads to a control in $L^2_t L^\infty_x$ locally in time. This difficulty is solved by complementing our strategy with a careful study of the damping phenomenon in (1-15) produced by the term $\sigma c E$. To that end, we provide, in Section 2, a robust analysis of the damping effect on general semigroup flows, which is formulated in precise terms in Lemma 2.1 (the damping lemma). We also give applications of the damping lemma to parabolic and dispersive equations in Sections 2.2 and 2.3, respectively.

Concerning Maxwell's system (1-15), the ensuing time decay of the electromagnetic field is encapsulated in Corollary 2.12. It is shown therein that (1-15) forms a damped hyperbolic system which is best understood by decomposing the frequencies of the solutions relative to the magnitude of the speed of light $c > 0$.

Indeed, by appropriately combining Ampère's equation and Faraday's equation from (1-15) and using that $\nabla \times (\nabla \times B) = -\Delta B$, observe that B solves the damped wave equation

$$\frac{1}{c^2}\partial_t^2 B + \sigma \partial_t B - \Delta B = -\sigma \nabla \times (u \times B), \tag{1-17}$$

where the damping term $\sigma \partial_t B$ comes from the term $\sigma c E$ in (1-15).

Heuristically, since waves described by (1-17) typically propagate with a characteristic speed c, it is then natural to expect a consistent hyperbolic behavior of the solutions of (1-15) on the range of frequencies larger than a suitable multiple of the speed of light c. In particular, Corollary 2.12 will confirm that solutions to (1-17) enjoy dispersive properties for those high frequencies, which are analogous to the nondamped case (obtained by setting $\sigma = 0$ in (1-17)) with drastically improved long-time integrability.

On the remaining range of frequencies, i.e., on frequencies slower than c, the same result will establish that the behavior of solutions to (1-15) is largely dictated by the heat equation

$$\sigma \partial_t B - \Delta B = -\sigma \nabla \times (u \times B),$$

which is formally achieved in the asymptotic regime $c \to \infty$ from (1-17).

All in all, the application of the sharp damped dispersive estimates from Section 2 to Maxwell's equations (1-15) will allow us to obtain closed estimates on the incompressible Euler–Maxwell system (1-1) which hold globally and lead to Theorem 1.1. The precise nonlinear analysis of (1-1) is detailed in Section 3 with complete proofs of our main theorems.

It is difficult to pinpoint the exact source of the breakdown of our proofs for small values of light velocity c. However, one can argue that the degeneracy of Maxwell's system in the limit $c \to 0$ results in a loss of the damped dispersive properties which are central to our nonlinear analysis. We believe that this provides some evidence that our method cannot be extended to the whole range of $c > 0$. Nevertheless, we are hopeful that other techniques may be used to construct solutions in the remaining range of light velocities.

1.4. ***Notation.*** Allow us to clarify some notation which will be used repeatedly throughout this article.

First of all, for clarity and convenience, note that all relevant functional spaces of Besov and Chemin–Lerner types are introduced in precise detail in Appendix A.

Next, Leray's projector

$$P : L^2(\mathbb{R}^3; \mathbb{R}^3) \to L^2(\mathbb{R}^3; \mathbb{R}^3)$$

onto divergence-free vector fields, which is used in (1-1), and the corresponding orthogonal projector $P^\perp = \mathrm{Id} - P$ onto conservative fields are given by

$$P = \mathrm{Id} - \Delta^{-1} \nabla \operatorname{div}, \qquad P^\perp = \Delta^{-1} \nabla \operatorname{div}.$$

Finally, when necessary, we will employ the letter C to denote a generic constant, which is allowed to differ from one estimate to another, and we will resort to the use of indices to distinguish specific constants. We will also often write $A \lesssim B$ to denote $A \leq CB$ for some positive constant C which only depends on fixed parameters, and $A \sim B$ whenever $A \lesssim B$ and $B \lesssim A$ are simultaneously true.

2. The effect of damping on semigroup flows

Here, we analyze the effect of damping on evolution flows, which are generally described by semigroups. More specifically, in Section 2.1, we begin by establishing a robust and general result — called the damping lemma — showing how damping terms act on integral operators. Then, in Sections 2.2 and 2.3, this result is applied to the context of damped parabolic and Strichartz estimates, which will be crucial to our analysis of Maxwell's system in Section 3. In particular, in Section 2.3, we give complete and sharp formulations of Strichartz estimates for the damped Schrödinger, half-wave, wave and Maxwell equations in Euclidean spaces.

2.1. ***The damping lemma.*** The result below provides a general and robust principle allowing us to take into account the influence of a damping term $e^{-\alpha t}$, with $\alpha > 0$, on an integral operator.

Lemma 2.1 (the damping lemma). *Let X and Y be Banach spaces and, for each $s, t \in [0, T)$, with $T > 0$, let $K(t, s) : X \to Y$ be an operator-valued kernel from X to Y such that*

$$K(t, s) \in L^1([0, T) \times [0, T); \mathcal{L}(X, Y)),$$

where $\mathcal{L}(X,Y)$ denotes the Banach space of bounded linear operators from X to Y. Further suppose that there are $0 < p_0 \le q_0 \le \infty$, with $q_0 \ge 1$, and a constant $A > 0$ such that the estimate

$$\left\| \int_0^T \chi(t,s) K(t,s) f(s)\, ds \right\|_{L^{q_0}([0,T);Y)} \le A \|f\|_{L^{p_0}([0,T);X)} \tag{2-1}$$

holds for all $f \in L^{p_0}([0,T);X)$ and any $\chi(t,s) \in L^\infty([0,T)^2;\mathbb{R})$, with $\|\chi\|_{L^\infty} \le 1$.

Then, for any $\alpha \ge 0$ and $p_0 \le p \le q \le q_0$, with $q \ge 1$, one has the damped estimate

$$\left\| \int_0^T e^{-\alpha|t-s|} \chi(t,s) K(t,s) f(s)\, ds \right\|_{L^q([0,T);Y)} \le C_\beta A \Big(\frac{T}{1+\alpha T} \Big)^\beta \|f\|_{L^p([0,T);X)}$$

for all $f \in L^p([0,T);X)$ and any $\chi(t,s) \in L^\infty([0,T)^2;\mathbb{R})$, with $\|\chi\|_{L^\infty} \le 1$, where $\beta \ge 0$ is defined by

$$\beta = \frac{1}{q} - \frac{1}{q_0} + \frac{1}{p_0} - \frac{1}{p}$$

and $C_\beta > 0$ only depends on β.

Proof. For $\alpha = 0$, the result follows straightforwardly from Hölder's inequality on the domain $[0,T)$ for all integrability parameters merely satisfying $0 < q \le q_0 \le \infty$ and $0 < p_0 \le p \le \infty$. We assume now that $\alpha > 0$ and $0 < p_0 \le p \le q \le q_0 \le \infty$, with $q \ge 1$.

For convenience of notation, we extend the definition of the kernel K and the functions χ and f to all real values of t and s by setting them equal to zero whenever t or s fall outside of the interval $[0,T)$.

We begin with the use of a partition

$$\mathbb{1}_{\{t \ne s\}} = \sum_{j \in \mathbb{Z}} \mathbb{1}_{\{2^j \le |t-s| < 2^{j+1}\}}$$

to deduce that

$$\left\| \int_0^T e^{-\alpha|t-s|} \chi(t,s) K(t,s) f(s)\, ds \right\|_{L^q(\mathbb{R};Y)} \le \sum_{\substack{j \in \mathbb{Z} \\ 2^j < T}} e^{-\alpha 2^j} \left\| \int_0^T \chi_j(t,s) K(t,s) f(s)\, ds \right\|_{L^q(\mathbb{R};Y)}, \tag{2-2}$$

where we have defined

$$\chi_j(t,s) = \mathbb{1}_{\{2^j \le |t-s| < 2^{j+1}\}} e^{-\alpha(|t-s| - 2^j)} \chi(t,s).$$

Observe that $\|\chi_j\|_{L^\infty} \le 1$.

Then, we further decompose the domain of t into the disjoint union

$$\bigcup_{k \in \mathbb{Z}} \{2^j k \le t < 2^j(k+1)\}$$

to write

$$\begin{aligned} \left\| \int_0^T \chi_j(t,s) K(t,s) f(s)\, ds \right\|_{L^q(\mathbb{R};Y)} &= \left\| \left\| \int_0^T \chi_j K f(s)\, ds \right\|_{L^q([2^j k, 2^j(k+1));Y)} \right\|_{\ell^q(k \in \mathbb{Z})} \\ &\le 2^{j(\frac{1}{q} - \frac{1}{q_0})} \left\| \left\| \int_0^T \chi_j K f(s)\, ds \right\|_{L^{q_0}([2^j k, 2^j(k+1));Y)} \right\|_{\ell^q(k \in \mathbb{Z})}, \end{aligned} \tag{2-3}$$

where we employed Hölder's inequality.

Now, notice that

$$2^j(k-2) < t-|t-s| \le s \le t+|t-s| < 2^j(k+3)$$

whenever $2^j \le |t-s| < 2^{j+1}$ and $2^jk \le t < 2^j(k+1)$. In particular, using (2-1), it follows that

$$\begin{aligned}\left\|\int_0^T \chi_j Kf(s)\,ds\right\|_{L^{q_0}([2^jk,2^j(k+1));Y)} &\le \left\|\int_0^T \chi_j Kf(s)\mathbb{1}_{\{2^j(k-2)<s<2^j(k+3)\}}\,ds\right\|_{L^{q_0}(\mathbb{R};Y)}\\ &\le A\sum_{n=-2}^{2}\|f\|_{L^{p_0}([2^j(k+n),2^j(k+1+n));X)}\\ &\le A2^{j(\frac{1}{p_0}-\frac{1}{p})}\sum_{n=-2}^{2}\|f\|_{L^{p}([2^j(k+n),2^j(k+1+n));X)},\end{aligned}\tag{2-4}$$

where we applied Hölder's inequality again.

All in all, combining (2-2), (2-3) with (2-4), and recalling that $\ell^p\subset\ell^q$ because $p\le q$, we infer that

$$\begin{aligned}&\left\|\int_0^T e^{-\alpha|t-s|}\chi(t,s)K(t,s)f(s)\,ds\right\|_{L^q(\mathbb{R};Y)}\\ &\qquad\le 5A\sum_{\substack{j\in\mathbb{Z}\\ 2^j<T}} e^{-\alpha 2^j}2^{j(\frac{1}{q}-\frac{1}{q_0}+\frac{1}{p_0}-\frac{1}{p})}\big\|\|f\|_{L^p([2^jk,2^j(k+1));X)}\big\|_{\ell^q(k\in\mathbb{Z})}\\ &\qquad\le 5A\|f\|_{L^p(\mathbb{R};X)}\sum_{\substack{j\in\mathbb{Z}\\ 2^j<T}} e^{-\alpha 2^j}2^{j(\frac{1}{q}-\frac{1}{q_0}+\frac{1}{p_0}-\frac{1}{p})}.\end{aligned}\tag{2-5}$$

It only remains to evaluate the constant resulting from the above sum in $j\in\mathbb{Z}$. If $p=p_0$ and $q=q_0$, the lemma trivially holds and there is nothing to prove. Thus, we may assume that $\beta>0$, thereby ensuring that the sum converges.

Now, observing that the function $e^{-x}(1+x)^{1+\beta}$ reaches its maximum on $[0,\infty)$ at $x=\beta$, we obtain

$$\begin{aligned}e^\beta(1+\beta)^{-(1+\beta)}\sum_{\substack{j\in\mathbb{Z}\\ 2^j<T}} e^{-\alpha 2^j}2^{j\beta} &\le \sum_{\substack{j\in\mathbb{Z}\\ 2^j<T}}\frac{2^{j\beta}}{(1+\alpha 2^j)^{1+\beta}} \le \sum_{\substack{j\in\mathbb{Z}\\ 2^j<T}}\int_{j-1}^{j}\frac{2^{(1+u)\beta}}{(1+\alpha 2^u)^{1+\beta}}\,du\\ &\le \frac{2^\beta}{\log 2}\int_0^T\frac{x^{\beta-1}}{(1+\alpha x)^{1+\beta}}\,dx = \frac{2^\beta}{\beta\log 2}\Big(\frac{T}{1+\alpha T}\Big)^\beta.\end{aligned}\tag{2-6}$$

Therefore, incorporating (2-6) into the estimate (2-5) concludes the proof of the lemma. $\square$

2.2. ***Damped parabolic estimates.*** Let us consider the general solution $w(t,x)$ of a damped heat equation on the Euclidean space $\mathbb{R}^d$ for any dimension $d\ge 1$

$$\begin{cases}\partial_t w+\alpha w-\Delta w = f,\\ w|_{t=0}=w_0,\end{cases}\tag{2-7}$$

where $(t,x)\in[0,T)\times\mathbb{R}^d$, with $T>0$ ($T=\infty$ may also be considered), the damping constant satisfies $\alpha\ge 0$, the right-hand side $f(t,x)$ is a source term and $w_0(x)$ is an initial datum.

Such equations naturally appear in dissipative physical systems. For instance, the heat equation (2-7) provides the linear structure of the damped incompressible Navier–Stokes equations, which arise from hydrodynamic regimes of inelastic particle systems.

Using standard semigroup notation, the solution $w(t, x)$ can be represented as

$$w(t) = e^{-t(\alpha-\Delta)} w_0 + \int_0^t e^{-(t-s)(\alpha-\Delta)} f(s)\, ds. \tag{2-8}$$

We are now going explore the jungle of parabolic smoothing estimates in Besov spaces for (2-8) by first reviewing the available results for the case $\alpha = 0$ and, then, extending these results to the setting $\alpha > 0$.

When $f \equiv 0$ and $\alpha = 0$, direct parabolic estimates on the semigroup $e^{t\Delta}$ yield the following result.

Proposition 2.2. *Let $\sigma \in \mathbb{R}$, $p \in [1, \infty]$ and $q \in [1, \infty]$. If $\alpha = 0$, w_0 belongs to $\dot{B}^{\sigma}_{p,q}$ and $f \equiv 0$, then the solution of the heat equation (2-7) satisfies*

$$\|e^{t\Delta} w_0\|_{L^\infty([0,\infty); \dot{B}^{\sigma}_{p,q})} \lesssim \|w_0\|_{\dot{B}^{\sigma}_{p,q}}.$$

Furthermore, if $q < \infty$, one also has the estimate

$$\|e^{t\Delta} w_0\|_{L^q([0,\infty); \dot{B}^{\sigma+2/q}_{p,1})} \lesssim \|w_0\|_{\dot{B}^{\sigma}_{p,q}}.$$

Remark. The above result somewhat reinforces the estimate

$$\|e^{t\Delta} w_0\|_{L^q([0,\infty); L^p)} \lesssim \|w_0\|_{\dot{B}^{-2/q}_{p,q}}$$

for any $1 \le p, q \le \infty$, which is commonly found in the literature; see [Bahouri et al. 2011, Theorem 2.34], for instance.

Remark. Note that taking $p = q = 2$ in the above proposition yields the estimate

$$\|e^{t\Delta} w_0\|_{L^2([0,\infty); \dot{B}^{\sigma+1}_{2,1})} \lesssim \|w_0\|_{\dot{H}^{\sigma}},$$

where we used that $\dot{H}^{\sigma} = \dot{B}^{\sigma}_{2,2}$ (see Appendix A for a precise definition of all relevant homogeneous spaces).

Remark. Throughout this section, we will routinely use the basic estimate

$$\|e^{t\Delta} \Delta_k u\|_{L^p} \le C e^{-C_* t 2^{2k}} \|\Delta_k u\|_{L^p} \tag{2-9}$$

for any $t > 0$, $p \in [1, \infty]$ and any dyadic block Δ_k, with $k \in \mathbb{Z}$, where C and C_* are positive independent constants. We refer to [Bahouri et al. 2011, Lemma 2.4] for a justification of (2-9).

Proof. The first part of the statement is a straightforward consequence of the definition of the homogeneous Besov norm. More precisely, using (2-9), we obtain

$$\|e^{t\Delta} w_0\|_{\dot{B}^{\sigma}_{p,q}} = \left(\sum_{k\in\mathbb{Z}} (2^{k\sigma} \|e^{t\Delta} \Delta_k w_0\|_{L^p})^q \right)^{\frac{1}{q}} \lesssim \left(\sum_{k\in\mathbb{Z}} e^{-C_* t 2^{2k}} (2^{k\sigma} \|\Delta_k w_0\|_{L^p})^q \right)^{\frac{1}{q}} \lesssim \|w_0\|_{\dot{B}^{\sigma}_{p,q}},$$

which, upon taking the supremum in $t > 0$, concludes the justification of the first estimate.

The second part of the statement is more subtle. Indeed, assuming now that $q<\infty$ and using (2-9), we find that

$$\|e^{t\Delta}w_0\|_{\dot B^{\sigma+2/q}_{p,1}}=\sum_{k\in\mathbb{Z}}2^{k(\sigma+\frac{2}{q})}\|e^{t\Delta}\Delta_k w_0\|_{L^p}\lesssim\sum_{k\in\mathbb{Z}}e^{-C_*t2^{2k}}2^{k(\sigma+\frac{2}{q})}\|\Delta_k w_0\|_{L^p}.$$

Next, further employing Hölder's inequality and taking a fixed positive value $\lambda>0$ such that $(q-1)\lambda<1$, we infer that

$$\|e^{t\Delta}w_0\|_{\dot B^{\sigma+2/q}_{p,1}} \lesssim\Big(\sum_{k\in\mathbb{Z}}(t2^{2k})^\lambda e^{-C_*t2^{2k}}\Big)^{\frac{q-1}{q}}\Big(\sum_{k\in\mathbb{Z}}\frac{1}{t}e^{-C_*t2^{2k}}(t2^{2k})^{1-\lambda(q-1)}(2^{k\sigma}\|\Delta_k w_0\|_{L^p})^q\Big)^{\frac{1}{q}}. \tag{2-10}$$

Now, for any positive t, considering the unique $j\in\mathbb{Z}$ such that $2^{2j}\le t<2^{2(j+1)}$, we find, since $\lambda>0$, that

$$\sup_{t>0}\sum_{k\in\mathbb{Z}}(t2^{2k})^\lambda e^{-C_*t2^{2k}}\le 2^{2\lambda}\sup_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}(2^{2(j+k)})^\lambda e^{-C_*2^{2(j+k)}}=2^{2\lambda}\sum_{k\in\mathbb{Z}}(2^{2k})^\lambda e^{-C_*2^{2k}}<\infty, \tag{2-11}$$

whereas, since $\lambda(q-1)<1$, we evaluate

$$\int_0^\infty e^{-C_*t2^{2k}}(t2^{2k})^{1-\lambda(q-1)}\frac{dt}{t}=\int_0^\infty e^{-C_*t}t^{-\lambda(q-1)}\,dt<\infty.$$

Therefore, integrating (2-10) in time, we finally arrive at the estimate

$$\begin{aligned}\|e^{t\Delta}w_0\|_{L^q_t\dot B^{\sigma+2/q}_{p,1}} &\lesssim\sup_{t>0}\Big(\sum_{k\in\mathbb{Z}}(t2^{2k})^\lambda e^{-C_*t2^{2k}}\Big)^{\frac{q-1}{q}}\Big(\sum_{k\in\mathbb{Z}}\int_0^\infty e^{-C_*t2^{2k}}(t2^{2k})^{1-\lambda(q-1)}\frac{dt}{t}(2^{k\sigma}\|\Delta_k w_0\|_{L^p})^q\Big)^{\frac{1}{q}}\\ &\lesssim\|w_0\|_{\dot B^\sigma_{p,q}},\end{aligned}$$

which concludes the proof of the proposition. □

In view of the preceding result, the effect of the damping term $e^{-\alpha t}$ on the initial data can be taken into account through a straightforward application of Hölder's inequality, thereby providing the following corollary.

Corollary 2.3. *Let $\sigma\in\mathbb{R}$, $p\in[1,\infty]$ and $q\in[1,\infty]$. If $\alpha\ge0$, w_0 belongs to $\dot B^\sigma_{p,q}$ and $f\equiv0$, then the solution of the heat equation (2-7) satisfies*

$$\|e^{-t(\alpha-\Delta)}w_0\|_{L^m([0,T);\dot B^\sigma_{p,q})}\lesssim\Big(\frac{T}{1+\alpha T}\Big)^{\frac{1}{m}}\|w_0\|_{\dot B^\sigma_{p,q}}$$

for every $0<m\le\infty$. Furthermore, if $0<m\le q<\infty$, one also has the estimate

$$\|e^{-t(\alpha-\Delta)}w_0\|_{L^m([0,T);\dot B^{\sigma+2/q}_{p,1})}\lesssim\Big(\frac{T}{1+\alpha T}\Big)^{\frac{1}{m}-\frac{1}{q}}\|w_0\|_{\dot B^\sigma_{p,q}}.$$

Proof. A direct use of Hölder's inequality followed by an application of Proposition 2.2 yields

$$\|e^{-t(\alpha-\Delta)}w_0\|_{L^m([0,T);\dot{B}^{\sigma}_{p,q})} \le \|e^{-t\alpha}\|_{L^m([0,T))}\|e^{t\Delta}w_0\|_{L^\infty([0,T);\dot{B}^{\sigma}_{p,q})} \lesssim \left(\frac{T}{1+\alpha T}\right)^{\frac{1}{m}}\|w_0\|_{\dot{B}^{\sigma}_{p,q}}$$

for all $0 < m \le \infty$ and

$$\begin{aligned}\|e^{-t(\alpha-\Delta)}w_0\|_{L^m([0,T);\dot{B}^{\sigma+2/q}_{p,1})} &\le \|e^{-t\alpha}\|_{L^{(1/m-1/q)^{-1}}([0,T))}\|e^{t\Delta}w_0\|_{L^q([0,T);\dot{B}^{\sigma+2/q}_{p,1})}\\ &\lesssim \left(\frac{T}{1+\alpha T}\right)^{\frac{1}{m}-\frac{1}{q}}\|w_0\|_{\dot{B}^{\sigma}_{p,q}}\end{aligned}$$

for all $0 < m \le q < \infty$, which completes the proof. □

Parabolic estimates are more involved when one includes a nonzero source term f. The coming results contain a wide range of smoothing estimates for the inhomogeneous heat equation. In preparation of these results, in order to reach a broader range of applicability, we are now going to introduce symbols

$$a(t,s,\xi) \in L^\infty([0,T)\times[0,T)\times\mathbb{R}^d),$$

which act as multipliers on the Fourier variable $\xi \in \mathbb{R}^d$ and are dependent on the time variables $t, s \in [0,T)$, thereby leading to time-dependent Fourier multipliers $a(t,s,D)$.

Definition. For a given $1 \le p \le \infty$, we say that $a(t,s,D)$ is *bounded* if there is a constant $C_a > 0$, independent of t and s, such that

$$\|a(t,s,D)f\|_{\dot{B}^0_{p,\infty}} \le C_a\|f\|_{\dot{B}^0_{p,\infty}} \tag{2-12}$$

for every $f \in \dot{B}^0_{p,\infty}(\mathbb{R}^d)$ and almost every $(t,s) \in [0,T)^2$. That is, the multiplier $a(t,s,D)$ is bounded if it is bounded over the Besov space $\dot{B}^0_{p,\infty}(\mathbb{R}^d)$, uniformly in t and s. The *norm* of $a(t,s,D)$, which we denote by

$$\|a(t,s,D)\|_{M_p},$$

is defined as the smallest possible constant $C_a > 0$ that fits in (2-12).

Remark. Equivalently, it is readily seen that (2-12) holds if and only if there is a constant $C_a' > 0$, independent of t and s, such that

$$\|a(t,s,D)\Delta_k f\|_{L^p} \le C_a'\|f\|_{L^p} \tag{2-13}$$

for every $k \in \mathbb{Z}$, $f \in L^p(\mathbb{R}^d)$ and almost every $(t,s) \in [0,T)^2$.

Remark. Observe that (2-12) and (2-13) hold if and only if one has

$$\|a(t,s,D)f\|_{\dot{B}^{\sigma}_{p,q}} \le C_{a,\sigma,q}\|f\|_{\dot{B}^{\sigma}_{p,q}},$$

with $C_{a,\sigma,q} > 0$, for all $\sigma \in \mathbb{R}$, $q \in [1,\infty]$ and every $f \in \dot{B}^{\sigma}_{p,q}(\mathbb{R}^d)$.

Since the space of Fourier multipliers over $L^2(\mathbb{R}^d)$ is isomorphic to $L^\infty(\mathbb{R}^d)$, it is readily seen, when $p=2$, that proving (2-12) and (2-13) is equivalent to establishing a bound

$$a(t,s,\xi)\in L^\infty([0,T)\times[0,T)\times\mathbb{R}^d).$$

More generally, when $p\neq 2$, in order to ensure that (2-12) or (2-13) hold, it is sufficient to require that

$$\mathcal{F}^{-1}[a(t,s,\xi)\varphi(2^{-k}\xi)]\in L^1(\mathbb{R}^d),$$

uniformly in t, s and k, where $\varphi(2^{-k}\xi)$ is a smooth compactly supported cutoff function used to define a Littlewood–Paley dyadic decomposition (see Appendix A). Therefore, in view of the straightforward classical estimate

$$\begin{aligned}\|\mathcal{F}^{-1}[a(t,s,\xi)\varphi(2^{-k}\xi)](x)\|_{L^1_x} &\lesssim 2^{-k\frac{d}{2}}\|(1+2^k|x|)^N\mathcal{F}^{-1}[a(t,s,\xi)\varphi(2^{-k}\xi)](x)\|_{L^2_x}\\ &\lesssim 2^{-k\frac{d}{2}}\sum_{\substack{\alpha\in\mathbb{N}^d\\|\alpha|\leq N}}\|2^{k|\alpha|}\partial_\xi^\alpha[a(t,s,\xi)\varphi(2^{-k}\xi)]\|_{L^2_\xi}\\ &\lesssim 2^{-k\frac{d}{2}}\sum_{\substack{\alpha,\beta\in\mathbb{N}^d\\|\alpha|+|\beta|\leq N}}\|2^{k|\alpha|}\partial_\xi^\alpha a(t,s,\xi)(\partial^\beta\varphi)(2^{-k}\xi)\|_{L^2_\xi},\end{aligned}$$

where we have used Plancherel's theorem and N is any integer larger than $\frac{1}{2}d$, we see that (2-12) and (2-13) both hold as soon as $a(t,s,\xi)$ is sufficiently differentiable in ξ (except possibly at the origin $\xi=0$) and satisfies the estimate

$$\||\xi|^{|\alpha|}\partial_\xi^\alpha a(t,s,\xi)\|_{L^\infty_{t,s,\xi}}<\infty \tag{2-14}$$

for every multi-index $\alpha\in\mathbb{N}^d$ with $|\alpha|\leq\left[\frac{1}{2}d\right]+1$. Observe that the above criterion establishes the boundedness of $a(t,s,D)$ over $\dot B^0_{p,\infty}(\mathbb{R}^d)$, uniformly in t and s, for all values of $1\leq p\leq\infty$, including the endpoints. Later on, we will be making use of (2-14) to show the boundedness of multipliers.

We return now to the smoothing estimates for the heat equation with a nontrivial source term f. The next result provides a large array of such estimates in the classical case $\alpha=0$.

Proposition 2.4. *Let $\sigma\in\mathbb{R}$, $1<r<m<\infty$ and $p\in[1,\infty]$. If f belongs to $L^r([0,T);\dot B^{\sigma+2/r}_{p,\infty})$ and $w_0=0$, then the solution of the heat equation (2-7), with $\alpha=0$, satisfies*

$$\left\|\int_0^t e^{(t-s)\Delta}a(t,s,D)f(s)\,ds\right\|_{L^m([0,T),\dot B^{\sigma+2+2/m}_{p,1})}\lesssim\|a\|_{M_p}\|f\|_{L^r([0,T),\dot B^{\sigma+2/r}_{p,\infty})}$$

for any Fourier multiplier $a(t,s,D)$.

Remark. We refer to [Arsénio 2019, Lemma 2] for a complete justification of the preceding proposition in the case $a(t,s,D)=\mathrm{Id}$. A straightforward adaptation of this proof readily extends the result to nontrivial multipliers $a(t,s,D)$.

Remark. The endpoint case $r=m$ above corresponds formally to a maximal gain of two derivatives on the solution of the heat equation. However, the method of proof of this result relies on the Hardy–Littlewood–Sobolev inequality, which typically falls short for endpoint settings. It is therefore not possible to extend the proof of [Arsénio 2019, Lemma 2] to the case $r=m$.

The next result generalizes Proposition 2.4 to incorporate the action of a damping term.

Proposition 2.5. *Let* $\sigma \in \mathbb{R}$, $p \in [1, \infty]$ *and*

$$1 \le r \le m \le \infty, \quad 0 < \theta < 1 + \frac{1}{m} - \frac{1}{r} \le 1,$$

or

$$1 < r < m < \infty, \quad 0 < \theta = 1 + \frac{1}{m} - \frac{1}{r} < 1.$$

Then, for any $\alpha \ge 0$, *one has the estimate*

$$\left\| \int_0^t e^{-(t-s)(\alpha-\Delta)} a(t,s,D) f(s)\, ds \right\|_{L^m([0,T), \dot{B}^{\sigma+2\theta}_{p,1})} \lesssim \left(\frac{T}{1+\alpha T} \right)^{1+\frac{1}{m}-\frac{1}{r}-\theta} \|a\|_{M_p} \|f\|_{L^r([0,T), \dot{B}^{\sigma}_{p,\infty})} \tag{2-15}$$

for any f *in* $L^r([0,T); \dot{B}^\sigma_{p,\infty})$ *and any Fourier multiplier* $a(t,s,D)$.

Remark. We emphasize that any implicit constant involved in the estimate of Proposition 2.5 is independent of T and α. Moreover, it is permitted to set $T = \infty$ and $\alpha > 0$ therein, in order to deduce a global estimate.

Remark. Observe that, choosing any $1 \le r \le m \le \infty$, $1 \le p, q \le \infty$ and $\sigma \in \mathbb{R}$, one has the simple estimate

$$\begin{aligned} \left\| \int_0^t e^{-(t-s)(\alpha-\Delta)} a(t,s,D) f(s)\, ds \right\|_{L^m([0,T), \dot{B}^{\sigma}_{p,q})} &\lesssim \|a\|_{M_p} \left\| \int_0^t e^{-\alpha(t-s)} \|f(s)\|_{\dot{B}^\sigma_{p,q}}\, ds \right\|_{L^m([0,T))} \\ &\lesssim \left(\frac{T}{1+\alpha T} \right)^{1+\frac{1}{m}-\frac{1}{r}} \|a\|_{M_p} \|f\|_{L^r([0,T), \dot{B}^\sigma_{p,q})} \end{aligned}$$

for all $\alpha \ge 0$, which corresponds to the case $\theta = 0$ in the previous proposition.

Proof in the case $1 < r \le m < \infty$. First of all, notice that the case

$$1 < r < m < \infty, \quad 0 < \theta = 1 + \frac{1}{m} - \frac{1}{r} < 1$$

follows from a direct application of Proposition 2.4 by absorbing the damping term $e^{-\alpha(t-s)}$ into the multiplier $a(t,s,D)$.

In order to treat the remaining case

$$1 < r \le m < \infty, \quad 0 < \theta < 1 + \frac{1}{m} - \frac{1}{r} \le 1,$$

we introduce auxiliary parameters

$$1 < r_0 < r \le m < m_0 < \infty$$

such that

$$\theta = 1 + \frac{1}{m_0} - \frac{1}{r_0}.$$

In particular, in view of Proposition 2.4, we have

$$\left\| \int_0^t e^{(t-s)\Delta} a(t,s,D) f(s)\, ds \right\|_{L^{m_0}([0,T),\dot{B}^{\sigma+2\theta}_{p,1})} \lesssim \|a\|_{M_p} \|f\|_{L^{r_0}([0,T),\dot{B}^{\sigma}_{p,\infty})}.$$

Then, an application of the damping lemma (Lemma 2.1) implies, for any $\alpha \geq 0$, that (2-15) holds, thereby concluding the proof. □

For the sake of completeness, since the preceding proof fails to treat the cases $r = 1$ and $m = \infty$, we provide now an alternative justification of Proposition 2.5, based on the proof of Lemma 2 from [Arsénio 2019], which works in full generality.

General proof. Following [Arsénio 2019], we begin by using (2-9) and (2-13) to deduce the existence of an independent constant $C_* > 0$ such that

$$\left\| \Delta_k \int_0^t e^{-(t-s)(\alpha-\Delta)} a(t,s,D) f(s)\, ds \right\|_{L^p} \lesssim \int_0^t e^{-(t-s)(\alpha+C_* 2^{2k})} \|\Delta_k f(s)\|_{L^p}\, ds.$$

For simplicity, we omit the norm $\|a\|_{M_p}$, which we absorb in the implicit constants. It then follows that

$$\begin{aligned} \left\| \int_0^t e^{-(t-s)(\alpha-\Delta)} a(t,s,D) f(s)\, ds \right\|_{\dot{B}^{\sigma+2\theta}_{p,1}} &\lesssim \int_0^t \sum_{k\in\mathbb{Z}} e^{-(t-s)(\alpha+C_* 2^{2k})} 2^{k(\sigma+2\theta)} \|\Delta_k f(s)\|_{L^p}\, ds \\ &\lesssim \int_0^T |t-s|^{-\theta} e^{-\alpha(t-s)} \|f(s)\|_{\dot{B}^{\sigma}_{p,\infty}}\, ds, \end{aligned}$$

where we have used (2-11), with the assumption that $\theta > 0$, to deduce that

$$\sum_{k\in\mathbb{Z}} 2^{2k\theta} e^{-C_*(t-s)2^{2k}} \lesssim |t-s|^{-\theta}.$$

Next, if $\theta = 1 + 1/m - 1/r$, by virtue of the Hardy–Littlewood–Sobolev inequality, which holds because $0 < \theta < 1$ and $1 < m, r < \infty$, we infer that

$$\left\| \int_0^t e^{-(t-s)(\alpha-\Delta)} a(t,s,D) f(s)\, ds \right\|_{L^m \dot{B}^{\sigma+2\theta}_{p,1}} \lesssim \left\| \int_0^T |t-s|^{-\theta} \|f(s)\|_{\dot{B}^{\sigma}_{p,\infty}}\, ds \right\|_{L^m} \lesssim \|f\|_{L^r \dot{B}^{\sigma}_{p,\infty}}.$$

Similarly, if $0 < \theta < 1 + 1/m - 1/r \leq 1$, we deduce from Young's convolution inequality that

$$\begin{aligned} \left\| \int_0^t e^{-(t-s)(\alpha-\Delta)} a(t,s,D) f(s)\, ds \right\|_{L^m \dot{B}^{\sigma+2\theta}_{p,1}} &\lesssim \left\| \int_0^T |t-s|^{-\theta} e^{-\alpha(t-s)} \|f(s)\|_{\dot{B}^{\sigma}_{p,\infty}}\, ds \right\|_{L^m} \\ &\lesssim \left(\int_0^T (t^{-\theta} e^{-\alpha t})^{(1+\frac{1}{m}-\frac{1}{r})^{-1}}\, dt \right)^{1+\frac{1}{m}-\frac{1}{r}} \|f\|_{L^r \dot{B}^{\sigma}_{p,\infty}} \\ &\lesssim \left(\frac{T}{1+\alpha T} \right)^{1+\frac{1}{m}-\frac{1}{r}-\theta} \|f\|_{L^r \dot{B}^{\sigma}_{p,\infty}}, \end{aligned}$$

which concludes the proof of the proposition. □

The shortcomings of Propositions 2.4 and 2.5 in the case $r = m$, with $\theta = 1$, naturally bring the question of the maximal regularity of the Laplacian in Banach spaces.

For a given Banach space X such that the Laplacian operator Δ is defined on a dense subspace of X, we say that the Laplacian (or another elliptic operator) has maximal L^p-regularity on $[0, T)$ for some $1 < p < \infty$ if the solution (2-8) of the heat equation (without damping, i.e., $\alpha = 0$) for a null initial data, i.e., $w_0 = 0$, is differentiable almost everywhere in t, takes values almost everywhere in the domain of Δ and satisfies the estimate

$$\|\partial_t w\|_{L^p([0,T);X)} + \|\Delta w\|_{L^p([0,T);X)} \leq C_p \|f\|_{L^p([0,T);X)}$$

for any source term $f \in L^p([0, T); X)$. We refer to [Kunstmann and Weis 2004] for an introduction to the theory of maximal L^p-regularity for parabolic equations.

The next important result, extracted from [Arsénio and Gallagher 2020], establishes the maximal regularity of the Laplacian in all homogeneous Besov spaces $\dot{B}^{\sigma}_{p,q}$. In particular, this result provides the basis which will allow us (in Section 3.6, for instance) to obtain stronger estimates, with sharp gains of parabolic regularity, by avoiding the use of Chemin–Lerner spaces.

Proposition 2.6 [Arsénio and Gallagher 2020]. *Let $\sigma \in \mathbb{R}$, $p, q \in [1, \infty]$ and $r \in (1, \infty)$. If f belongs to $L^r([0, T); \dot{B}^{\sigma}_{p,q})$ and $w_0 = 0$, then the solution of the heat equation (2-7), with $\alpha \geq 0$, satisfies*

$$\left\| \int_0^t e^{-(t-s)(\alpha-\Delta)} a(t, s, D) f(s)\, ds \right\|_{L^r([0,T),\dot{B}^{\sigma+2}_{p,q})} \lesssim \|a\|_{M_p} \|f\|_{L^r([0,T),\dot{B}^{\sigma}_{p,q})}$$

for any Fourier multiplier $a(t, s, D)$. The result remains valid if $r = q = 1$ or $r = q = \infty$.

Remark. Again, it is to be emphasized that any implicit constant involved in the above estimate is independent of T and α. In particular, one can set $T = \infty$ therein.

Remark. We refer to [Arsénio and Gallagher 2020, Proposition 3.1] or [Arsénio 2019, Lemma 3] for a proof of Proposition 2.6 in the case $a(t, s, D) = \mathrm{Id}$ and $\alpha = 0$. The original proof from [Arsénio and Gallagher 2020] deals first with the case $q = 1$ and then relies on an interpolation argument. The proof from [Arsénio 2019], however, offers a self-contained approach which avoids interpolation altogether.

Remark. In fact, the original statements of Proposition 3.1 in [Arsénio and Gallagher 2020] and Lemma 3 in [Arsénio 2019] only cover the range of parameters $1 \leq q \leq r < \infty$. Nevertheless, it is readily seen that the corresponding proofs can be used mutatis mutandis to show identical bounds on the adjoint operator, which is defined by

$$\int_t^T e^{-(s-t)(\alpha-\Delta)} \overline{a(s, t, D)} g(s)\, ds.$$

It then follows from a standard duality argument that these results hold for values $1 < r \leq q \leq \infty$ as well.

For the sake of completeness and clarity, we provide now a full justification of Proposition 2.6, based on a straightforward adaptation of the proof of [Arsénio 2019, Lemma 3].

Proof. First, noticing that the damping term $e^{-\alpha(t-s)}$ can be absorbed into the bounded multiplier $a(t,s,D)$, we assume, without loss of generality, that $\alpha=0$. Then, we follow the proof of Lemma 3 from [Arsénio 2019].

We start by considering the case $1\le q\le r<\infty$. By duality, it is enough to prove that, if g is a nonnegative function in $L^{b'}([0,T))$ with $b=r/q\ge 1$ and $1/b+1/b'=1$, then

$$\int_0^T g(t)\left\|\int_0^t e^{(t-s)\Delta}a(t,s,D)f(s)\,ds\right\|^q_{\dot B^{\sigma+2}_{p,q}}dt\lesssim \|a\|^q_{M_p}\|f\|^q_{L^r([0,T),\dot B^{\sigma}_{p,q})}\|g\|_{L^{b'}([0,T))}.$$

To this end, using (2-9) and (2-13), we deduce the existence of a constant $C_*>0$ such that

$$\begin{aligned}\int_0^T g(t)&\left\|\int_0^t e^{(t-s)\Delta}a(t,s,D)f(s)\,ds\right\|^q_{\dot B^{\sigma+2}_{p,q}}dt\\&=\sum_{k\in\mathbb Z}\int_0^T g(t)\left\|\int_0^t e^{(t-s)\Delta}a(t,s,D)\Delta_k f(s)\,ds\right\|^q_{L^p}2^{k(\sigma+2)q}\,dt\\&\lesssim\|a\|^q_{M_p}\sum_{k\in\mathbb Z}\int_0^T g(t)\left(\int_0^t e^{-C_*(t-s)2^{2k}}\|\Delta_k f(s)\|_{L^p}\,ds\right)^q2^{k(\sigma+2)q}\,dt\\&\lesssim\|a\|^q_{M_p}\sum_{k\in\mathbb Z}\int_0^T g(t)\int_0^t e^{-C_*(t-s)2^{2k}}\|\Delta_k f(s)\|^q_{L^p}\,ds\,2^{k(\sigma q+2)}\,dt.\end{aligned}$$

For simplicity, we omit the norm $\|a\|_{M_p}$ in the remaining estimates.

Next, we define a maximal operator by

$$Mg(s)=\sup_{\rho>0}\rho\int_0^T e^{-\rho|s-t|}|g(t)|\,dt.$$

Classical results from harmonic analysis (see [Grafakos 2014, Theorems 2.1.6 and 2.1.10]) establish that M is bounded over $L^c([0,T))$ for any $1<c\le\infty$. One can then write

$$\begin{aligned}\int_0^T g(t)&\left\|\int_0^t e^{(t-s)\Delta}a(t,s,D)f(s)\,ds\right\|^q_{\dot B^{\sigma+2}_{p,q}}dt\\&\lesssim\sum_{k\in\mathbb Z}\int_0^T\left[2^{2k}\int_s^T g(t)e^{-C_*(t-s)2^{2k}}\,dt\right]\|\Delta_k f(s)\|^q_{L^p}2^{k\sigma q}\,ds\\&\lesssim\sum_{k\in\mathbb Z}\int_0^T Mg(s)\|\Delta_k f(s)\|^q_{L^p}2^{k\sigma q}\,ds=\int_0^T Mg(s)\|f(s)\|^q_{\dot B^\sigma_{p,q}}\,ds.\end{aligned}$$

Therefore, by the boundedness properties of Mg and Hölder's inequality, we conclude that

$$\int_0^T g(t)\left\|\int_0^t e^{(t-s)\Delta}a(t,s,D)f(s)\,ds\right\|^q_{\dot B^{\sigma+2}_{p,q}}dt\lesssim\|Mg\|_{L^{b'}}\|f\|^q_{L^r\dot B^\sigma_{p,q}}\lesssim\|g\|_{L^{b'}}\|f\|^q_{L^r\dot B^\sigma_{p,q}},$$

which completes the proof of the proposition in the case $1\le q\le r<\infty$.

Now, observe that the exact same proof applies to the adjoint operator

$$\int_t^T e^{(s-t)\Delta}\overline{a(s,t,D)}\, f(s)\, ds,$$

thereby leading to the estimate

$$\left\| \int_t^T e^{(s-t)\Delta}\overline{a(s,t,D)}\, f(s)\, ds \right\|_{L^r([0,T),\dot{B}^{\sigma+2}_{p,q})} \lesssim \|f\|_{L^r([0,T),\dot{B}^{\sigma}_{p,q})}$$

whenever $1 \le q \le r < \infty$. Then, a standard duality argument establishes that

$$\left\| \int_0^t e^{(t-s)\Delta} a(t,s,D)\, f(s)\, ds \right\|_{L^{r'}([0,T),\dot{B}^{-\sigma}_{p',q'})} \lesssim \|f\|_{L^{r'}([0,T),\dot{B}^{-(\sigma+2)}_{p',q'})}$$

for parameter values in the range $1 < r' \le q' \le \infty$. Therefore, replacing p, q, r and σ by p', q', r' and $-(\sigma+2)$, respectively, shows the proposition in the case $1 < r \le q \le \infty$, which concludes the proof. □

2.3. ***Damped Strichartz estimates.*** We focus now on the interaction between damping and dispersion. More precisely, we are going to explore how the damping lemma (Lemma 2.1) applies to Strichartz estimates. To that end, we first recall the general result on Strichartz estimates for abstract semigroups from [Keel and Tao 1998]. We also refer to [Bahouri et al. 2011, Chapter 8] for a comprehensive exposition of Strichartz estimates.

Proposition 2.7 [Keel and Tao 1998]. *Let H be a Hilbert space and (X, dx) be a measure space. For each $t \in [0,T)$, with $T > 0$, let $U(t) : H \to L^2(X)$ be an operator such that*

$$U(t) \in L^\infty([0,T); \mathcal{L}(H, L^2(X)))$$

and, for some $\sigma > 0$,

$$\|U(t)U(s)^* g\|_{L^\infty(X)} \lesssim \frac{1}{|t-s|^\sigma} \|g\|_{L^1(X)}$$

for all $t, s \in [0,T)$, with $t \neq s$, and all $g \in L^1(X) \cap L^2(X)$.

Then, the estimate

$$\|U(t) f\|_{L^q_t L^r_x} \lesssim \|f\|_H$$

and its dual version

$$\left\| \int_0^T U(t)^* g(t)\, dt \right\|_H \lesssim \|g\|_{L^{q'}_t L^{r'}_x}$$

hold for any exponent pair $(q,r) \in [2,\infty]^2$, which is admissible in the sense that

$$\frac{1}{q} + \frac{\sigma}{r} = \frac{\sigma}{2} \quad \text{and} \quad (q,r,\sigma) \neq (2,\infty,1).$$

Furthermore, if $(\tilde{q},\tilde{r}) \in [2,\infty]^2$ is also an admissible exponent pair, then the estimate

$$\left\| \int_0^T \chi(t,s) U(t) U(s)^* g(s)\, ds \right\|_{L^q_t L^r_x} \lesssim \|\chi\|_{L^\infty} \|g\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x}$$

holds for any $\chi(t,s) \in L^\infty([0,T)^2; \mathbb{R})$.

Remark. In fact, the statement of the result from [Keel and Tao 1998] only considers the function $\chi(t,s)=\mathbb{1}_{\{s\le t\}}$. However, a straightforward alteration of the proof from [Keel and Tao 1998] easily shows that the result actually holds for all $\chi(t,s)\in L^\infty([0,T)^2;\mathbb{R})$. A detailed proof valid for all $\chi(t,s)$ can also be found in Section 8.2 of [Bahouri et al. 2011].

By combining the damping lemma with the preceding proposition, we obtain the damped Strichartz estimates, which are stated in precise terms in the next result.

Proposition 2.8 (damped Strichartz estimates). *Let H be a Hilbert space and (X,dx) be a measure space. For each $t\in[0,T)$, with $T>0$, let $U(t):H\to L^2(X)$ be an operator such that*

$$U(t)\in L^\infty([0,T);\mathcal{L}(H,L^2(X)))$$

and, for some $\sigma>0$,

$$\|U(t)U(s)^*g\|_{L^\infty(X)}\lesssim\frac{1}{|t-s|^\sigma}\|g\|_{L^1(X)}$$

for all $t,s\in[0,T)$, with $t\neq s$, and all $g\in L^1(X)\cap L^2(X)$.

Then, for any $\alpha\ge 0$, the estimate

$$\|e^{-\alpha t}U(t)f\|_{L^q_tL^r_x}\lesssim\Big(\frac{T}{1+\alpha T}\Big)^{\frac{1}{q}+\frac{\sigma}{r}-\frac{\sigma}{2}}\|f\|_H$$

and its dual version

$$\Big\|\int_0^T e^{-\alpha t}U(t)^*g(t)\,dt\Big\|_H\lesssim\Big(\frac{T}{1+\alpha T}\Big)^{\frac{1}{q}+\frac{\sigma}{r}-\frac{\sigma}{2}}\|g\|_{L^{q'}_tL^{r'}_x}$$

hold for any exponent pair $(q,r)\in[1,\infty]\times[2,\infty]$, which is admissible in the sense that

$$\frac{1}{q}+\frac{\sigma}{r}\ge\frac{\sigma}{2},\quad \frac{1}{2}+\frac{\sigma}{r}\ge\frac{\sigma}{2}\quad\text{and}\quad(r,\sigma)\neq(\infty,1).$$

Furthermore, if $(\tilde q,\tilde r)\in[1,\infty]\times[2,\infty]$ is also an admissible exponent pair such that

$$\frac{1}{q}+\frac{1}{\tilde q}\le 1,$$

then the estimate

$$\Big\|\int_0^T e^{-\alpha|t-s|}\chi(t,s)U(t)U(s)^*g(s)\,ds\Big\|_{L^q_tL^r_x}\lesssim\Big(\frac{T}{1+\alpha T}\Big)^{\frac{1}{q}+\frac{1}{\tilde q}+\sigma\left(\frac{1}{r}+\frac{1}{\tilde r}\right)-\sigma}\|\chi\|_{L^\infty}\|g\|_{L^{\tilde q'}_tL^{\tilde r'}_x}$$

holds for any $\chi(t,s)\in L^\infty([0,T)^2;\mathbb{R})$.

Proof. First of all, observe that all hypotheses of Proposition 2.7 are satisfied by $U(t)$. Then, introducing the parameter $q_0\in[2,\infty]$ by setting

$$\frac{1}{q_0}=\sigma\Big(\frac{1}{2}-\frac{1}{r}\Big),\tag{2-16}$$

we see that the exponent pair $(q_0,r)\in[2,\infty]^2$ is admissible for Proposition 2.7. Therefore, it follows from Proposition 2.7, with an application of Hölder's inequality, that

$$\|e^{-\alpha t}U(t)f\|_{L^q_tL^r_x}\le\|e^{-\alpha t}\|_{L^{(1/q-1/q_0)^{-1}}([0,T))}\|U(t)f\|_{L^{q_0}_tL^r_x}\lesssim\Big(\frac{T}{1+\alpha T}\Big)^{\frac{1}{q}-\frac{1}{q_0}}\|f\|_H,$$

which establishes the first estimate of the proposition. The second estimate then ensues from a dual reformulation of the first estimate.

It only remains to justify the validity of the third estimate. To that end, employing (2-16), we introduce auxiliary parameters $q_0, \tilde{q}_0 \in [2,\infty]$, so that the exponent pairs $(q_0, r), (\tilde{q}_0, \tilde{r}) \in [2,\infty]^2$ are admissible for Proposition 2.7. In particular, it follows that

$$\left\| \int_0^T \chi(t,s) U(t) U(s)^* g(s)\, ds \right\|_{L_t^{q_0} L_x^r} \lesssim \|\chi\|_{L^\infty} \|g\|_{L_t^{\tilde{q}_0'} L_x^{\tilde{r}'}}$$

for any $\chi(t,s) \in L^\infty([0,T)^2;\mathbb{R})$. Therefore, noticing that $\tilde{q}_0' \le \tilde{q}' \le q \le q_0$, we conclude from an application of Lemma 2.1 that

$$\left\| \int_0^T e^{-\alpha|t-s|} \chi(t,s) U(t) U(s)^* g(s)\, ds \right\|_{L_t^q L_x^r} \lesssim \left(\frac{T}{1+\alpha T}\right)^{\frac{1}{q}-\frac{1}{q_0}+\frac{1}{\tilde{q}_0'}-\frac{1}{\tilde{q}'}} \|\chi\|_{L^\infty} \|g\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}},$$

which completes the proof. □

Remark. We do not make any claim of optimality of Proposition 2.8. It would be interesting, though, to test the sharpness of the admissibility criteria for the exponent pairs (q,r) and $(\tilde{q},\tilde{r})$ in connection with the sensitivity in T and α of the estimates.

We proceed now to specific formulations of the damped Strichartz estimates for the Schrödinger and wave equations, as well as for Maxwell's system.

Corollary 2.9 (damped Schrödinger equation). *Let $d \ge 1$ and consider a solution $u(t,x)$ of the damped Schrödinger equation*

$$\begin{cases} (\partial_t + \alpha - i\Delta) u(t,x) = F(t,x), \\ u(0,x) = f(x), \end{cases}$$

with $\alpha \ge 0$, $t \in [0,T)$ and $x \in \mathbb{R}^d$.

For any exponent pairs $(q,r), (\tilde{q},\tilde{r}) \in [1,\infty] \times [2,\infty]$ which are admissible in the sense that

$$\frac{2}{q} + \frac{d}{r} \ge \frac{d}{2}, \qquad 1 + \frac{d}{r} \ge \frac{d}{2} \quad \text{and} \quad (r,d) \neq (\infty,2),$$

and similarly for $(\tilde{q},\tilde{r})$, and such that

$$\frac{1}{q} + \frac{1}{\tilde{q}} \le 1,$$

one has the estimate

$$\|u\|_{L_t^q L_x^r} \lesssim \left(\frac{T}{1+\alpha T}\right)^{\frac{1}{q}+\frac{d}{2}\left(\frac{1}{r}-\frac{1}{2}\right)} \|f\|_{L_x^2} + \left(\frac{T}{1+\alpha T}\right)^{\frac{1}{q}+\frac{1}{\tilde{q}}+\frac{d}{2}\left(\frac{1}{r}+\frac{1}{\tilde{r}}-1\right)} \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}.$$

Proof. The solution $u(t,x)$ can be expressed by Duhamel's representation formula as

$$u(t) = e^{-\alpha t} U(t) f + \int_0^t e^{-\alpha(t-s)} U(t) U(s)^* F(s)\, ds = e^{-\alpha t} U(t) f + \int_0^t e^{-\alpha(t-s)} U(t-s) F(s)\, ds,$$

where

$$U(t) = e^{it\Delta} \quad \text{and} \quad U(t)^* = e^{-it\Delta}.$$

In particular, one has the explicit formula (see Section 8.1.2 in [Bahouri et al. 2011])

$$U(t)f(x)=\frac{1}{(4\pi it)^{d/2}}\int_{\mathbb{R}^d}e^{-\frac{|x-y|^2}{4it}}f(y)\,dy,$$

which readily implies that $U(t)$ satisfies all hypotheses of Proposition 2.8 with $H=L^2(\mathbb{R}^d)$, $X=\mathbb{R}^d$ and $\sigma=\frac{1}{2}d$. Therefore, we conclude that the corollary follows from a direct application of the damped Strichartz estimates of Proposition 2.8. □

Corollary 2.10 (damped half-wave equation). *Let $d\geq 2$, and consider a solution $u(t,x)$ of the damped half-wave equation*

$$\begin{cases}(\partial_t+\alpha\mp i|D|)u(t,x)=F(t,x),\\ u(0,x)=f(x),\end{cases}$$

with $\alpha\geq 0$, $t\in[0,T)$ and $x\in\mathbb{R}^d$.

For any exponent pairs $(q,r),(\tilde q,\tilde r)\in[1,\infty]\times[2,\infty]$ which are admissible in the sense that

$$\frac{2}{q}+\frac{d-1}{r}\geq\frac{d-1}{2},\quad 1+\frac{d-1}{r}\geq\frac{d-1}{2}\quad\text{and}\quad (r,d)\neq(\infty,3),$$

and similarly for $(\tilde q,\tilde r)$, and such that

$$\frac{1}{q}+\frac{1}{\tilde q}\leq 1,$$

one has the estimate

$$\begin{aligned}&2^{-j\frac{d+1}{2}(\frac{1}{2}-\frac{1}{r})}\|\Delta_j u\|_{L^q_tL^r_x}\\&\quad\lesssim\Big(\frac{T}{1+\alpha T}\Big)^{\frac{1}{q}+\frac{d-1}{2}(\frac{1}{r}-\frac{1}{2})}\|\Delta_j f\|_{L^2_x}+\Big(\frac{T}{1+\alpha T}\Big)^{\frac{1}{q}+\frac{1}{\tilde q}+\frac{d-1}{2}(\frac{1}{r}+\frac{1}{\tilde r}-1)}2^{j\frac{d+1}{2}(\frac{1}{2}-\frac{1}{\tilde r})}\|\Delta_j F\|_{L^{\tilde q'}_tL^{\tilde r'}_x}\end{aligned}$$

for all $j\in\mathbb{Z}$.

Remark. If $\tilde q'\leq p\leq q$, then further multiplying the preceding estimate by $2^{j\sigma}$ for some $\sigma\in\mathbb{R}$ and summing over $j\in\mathbb{Z}$ in the ℓ^p-norm leads to

$$\begin{aligned}&\|u\|_{L^q_t\dot B^{\sigma-((d+1)/2)(1/2-1/r)}_{r,p,x}}\\&\quad\lesssim\Big(\frac{T}{1+\alpha T}\Big)^{\frac{1}{q}+\frac{d-1}{2}(\frac{1}{r}-\frac{1}{2})}\|f\|_{\dot B^{\sigma}_{2,p,x}}+\Big(\frac{T}{1+\alpha T}\Big)^{\frac{1}{q}+\frac{1}{\tilde q}+\frac{d-1}{2}(\frac{1}{r}+\frac{1}{\tilde r}-1)}\|F\|_{L^{\tilde q'}_t\dot B^{\sigma+((d+1)/2)(1/2-1/\tilde r)}_{\tilde r',p,x}}.\end{aligned}$$

Proof. The solution $u(t,x)$ can be expressed by Duhamel's representation formula as

$$u(t)=e^{-\alpha t}e^{\pm it|D|}f+\int_0^t e^{-\alpha(t-s)}e^{\pm i(t-s)|D|}F(s)\,ds.$$

For each $j\in\mathbb{Z}$, we introduce now the flow

$$U_j(t)f(x):=e^{\pm it|D|}\psi\Big(\frac{D}{2^j}\Big)f(x),$$

where $\psi(\xi)$ is a smooth compactly supported function such that $0\notin\operatorname{supp}\psi$ and $\psi\equiv 1$ on $\{\frac{1}{2}\leq|\xi|\leq 2\}$. In particular, if Δ_j is the Littlewood–Paley frequency cutoff operator, defined in Appendix A, which localizes frequencies to $\{2^{j-1} \le |\xi| \le 2^{j+1}\}$, one has the representation

$$\begin{aligned}\Delta_j u(t) &= e^{-\alpha t} U_j(t)\Delta_j f + \int_0^t e^{-\alpha(t-s)} U_j(t-s)\Delta_j F(s)\,ds \\ &= e^{-\alpha t} U_j(t)\Delta_j f + \int_0^t e^{-\alpha(t-s)} U_j(t)U_j(s)^*\Delta_j F(s)\,ds.\end{aligned}$$

We are now going to apply Proposition 2.8 to the operator $U_0(t)$ in order to control $\Delta_0 u(t)$.

Classical results, based on the stationary phase method, establish that

$$\|U_0(t)U_0(s)^* f\|_{L^\infty(\mathbb{R}^d)} \lesssim \frac{1}{|t-s|^{(d-1)/2}} \|f\|_{L^1(\mathbb{R}^d)}$$

for all $t \neq s$ and $f \in L^1(\mathbb{R}^d)$. (Proposition 8.15 from [Bahouri et al. 2011] contains a precise justification of the preceding dispersive estimate, and we further refer to Section 8.1.3 from the same work for more details on the stationary phase method.) It therefore follows that $U_0(t)$ satisfies all hypotheses of Proposition 2.8 with $H = L^2(\mathbb{R}^d)$, $X = \mathbb{R}^d$ and $\sigma = \frac{1}{2}(d-1)$. Hence, we conclude that

$$\|\Delta_0 u\|_{L^q_t L^r_x} \lesssim \Big(\frac{T}{1+\alpha T}\Big)^{\frac{1}{q}+\frac{d-1}{2}(\frac{1}{r}-\frac{1}{2})} \|\Delta_0 f\|_{L^2_x} + \Big(\frac{T}{1+\alpha T}\Big)^{\frac{1}{q}+\frac{1}{\tilde q}+\frac{d-1}{2}(\frac{1}{r}+\frac{1}{\tilde r}-1)} \|\Delta_0 F\|_{L^{\tilde q'}_t L^{\tilde r'}_x} \tag{2-17}$$

for all admissible exponent pairs.

In order to recover an estimate for all components $\Delta_j u$, where $j \in \mathbb{Z}$, we conduct a simple scaling argument by introducing

$$u_j(t,x) := u\Big(\frac{t}{2^j}, \frac{x}{2^j}\Big), \quad F_j(t,x) := \frac{1}{2^j} F\Big(\frac{t}{2^j}, \frac{x}{2^j}\Big), \quad f_j(x) := f\Big(\frac{x}{2^j}\Big).$$

Noticing that

$$\Delta_0 u_j(t,x) = (\Delta_j u)\Big(\frac{t}{2^j}, \frac{x}{2^j}\Big), \quad \Delta_0 F_j(t,x) = \frac{1}{2^j}(\Delta_j F)\Big(\frac{t}{2^j}, \frac{x}{2^j}\Big), \quad \Delta_0 f_j(x) = (\Delta_j f)\Big(\frac{x}{2^j}\Big)$$

and that $u_j(t,x)$ solves

$$\begin{cases}(\partial_t + 2^{-j}\alpha \mp i|D|)u_j(t,x) = F_j(t,x),\\ u_j(0,x) = f_j(x)\end{cases}$$

on $[0, 2^j T)$, we obtain, applying (2-17) to u_j,

$$\begin{aligned}&2^{j(\frac{1}{q}+\frac{d}{r})}\|\Delta_j u\|_{L^q_t L^r_x}\\ &= \|\Delta_0 u_j\|_{L^q_t L^r_x}\\ &\lesssim \Big(\frac{2^j T}{1+\alpha T}\Big)^{\frac{1}{q}+\frac{d-1}{2}(\frac{1}{r}-\frac{1}{2})} \|\Delta_0 f_j\|_{L^2_x} + \Big(\frac{2^j T}{1+\alpha T}\Big)^{\frac{1}{q}+\frac{1}{\tilde q}+\frac{d-1}{2}(\frac{1}{r}+\frac{1}{\tilde r}-1)} \|\Delta_0 F_j\|_{L^{\tilde q'}_t L^{\tilde r'}_x}\\ &= \Big(\frac{2^j T}{1+\alpha T}\Big)^{\frac{1}{q}+\frac{d-1}{2}(\frac{1}{r}-\frac{1}{2})} 2^{j\frac{d}{2}} \|\Delta_j f\|_{L^2_x} + \Big(\frac{2^j T}{1+\alpha T}\Big)^{\frac{1}{q}+\frac{1}{\tilde q}+\frac{d-1}{2}(\frac{1}{r}+\frac{1}{\tilde r}-1)} 2^{j(-\frac{1}{\tilde q}+d(1-\frac{1}{\tilde r}))} \|\Delta_j F\|_{L^{\tilde q'}_t L^{\tilde r'}_x}.\end{aligned}$$

Finally, reorganizing the terms above, we deduce that

$$\|\Delta_j u\|_{L^q_t L^r_x} \lesssim \Big(\frac{T}{1+\alpha T}\Big)^{\frac{1}{q}+\frac{d-1}{2}(\frac{1}{r}-\frac{1}{2})} 2^{j\frac{d+1}{2}(\frac{1}{2}-\frac{1}{r})}\|\Delta_j f\|_{L^2_x} + \Big(\frac{T}{1+\alpha T}\Big)^{\frac{1}{q}+\frac{1}{\tilde q}+\frac{d-1}{2}(\frac{1}{r}+\frac{1}{\tilde r}-1)} 2^{j\frac{d+1}{2}(1-\frac{1}{r}-\frac{1}{\tilde r})}\|\Delta_j F\|_{L^{\tilde q'}_t L^{\tilde r'}_x},$$

which concludes the proof. □

Corollary 2.11 (damped wave equation). *Let $d \geq 2$, and consider a solution $u(t,x)$ of the damped wave equation*

$$\begin{cases} (\partial_t^2 + \alpha\partial_t - \Delta)u(t,x) = F(t,x), \\ u(0,x) = f(x), \\ \partial_t u(0,x) = g(x), \end{cases}$$

with $\alpha \geq 0$, $t \in [0,T)$ and $x \in \mathbb{R}^d$.

For any exponent pairs $(q,r), (\tilde q, \tilde r) \in [1,\infty]\times[2,\infty]$ which are admissible in the sense that

$$\frac{2}{q}+\frac{d-1}{r} \geq \frac{d-1}{2}, \quad 1+\frac{d-1}{r} \geq \frac{d-1}{2} \quad \textit{and} \quad (r,d) \neq (\infty,3),$$

and similarly for $(\tilde q,\tilde r)$, and such that

$$\frac{1}{q}+\frac{1}{\tilde q} \leq 1,$$

one has the high-frequency estimate

$$2^{-j\frac{d+1}{2}(\frac{1}{2}-\frac{1}{r})}\|\Delta_j(\partial_t u, \nabla u)\|_{L^q_t L^r_x} \lesssim \Big(\frac{T}{1+\alpha T}\Big)^{\frac{1}{q}+\frac{d-1}{2}(\frac{1}{r}-\frac{1}{2})}\|\Delta_j(g,\nabla f)\|_{L^2_x} + \Big(\frac{T}{1+\alpha T}\Big)^{\frac{1}{q}+\frac{1}{\tilde q}+\frac{d-1}{2}(\frac{1}{r}+\frac{1}{\tilde r}-1)} 2^{j\frac{d+1}{2}(\frac{1}{2}-\frac{1}{\tilde r})}\|\Delta_j F\|_{L^{\tilde q'}_t L^{\tilde r'}_x}$$

for all $j \in \mathbb{Z}$ with $2^j \geq \alpha$, and the low-frequency estimates

$$2^{-jd(\frac{1}{2}-\frac{1}{r})}\|\Delta_j \partial_t u\|_{L^q_t L^r_x} \lesssim \Big(\frac{T}{1+\alpha T}\Big)^{\frac{1}{q}}\|\Delta_j g\|_{L^2_x} + \frac{1}{\alpha}\Big(\frac{\alpha 2^{2j}T}{\alpha+2^{2j}T}\Big)^{\frac{1}{q}} 2^{j(1-\frac{2}{q})}\|\Delta_j \nabla f\|_{L^2_x} + \Big(\frac{T}{1+\alpha T}\Big)^{\frac{1}{q}+\frac{1}{\tilde q}} 2^{jd(\frac{1}{2}-\frac{1}{\tilde r})}\|\Delta_j F\|_{L^{\tilde q'}_t L^{\tilde r'}_x}$$

and

$$2^{-j(d(\frac{1}{2}-\frac{1}{r})-\frac{2}{q})}\|\Delta_j \nabla u\|_{L^q_t L^r_x} \lesssim \frac{1}{\alpha}\Big(\frac{\alpha 2^{2j}T}{\alpha+2^{2j}T}\Big)^{\frac{1}{q}} 2^j\|\Delta_j g\|_{L^2_x} + \Big(\frac{\alpha 2^{2j}T}{\alpha+2^{2j}T}\Big)^{\frac{1}{q}}\|\Delta_j \nabla f\|_{L^2_x} + \frac{1}{\alpha}\Big(\frac{\alpha 2^{2j}T}{\alpha+2^{2j}T}\Big)^{\frac{1}{q}+\frac{1}{\tilde q}} 2^{j(1+d(\frac{1}{2}-\frac{1}{\tilde r})-\frac{2}{\tilde q})}\|\Delta_j F\|_{L^{\tilde q'}_t L^{\tilde r'}_x}$$

for all $j \in \mathbb{Z}$ with $2^j \leq \alpha$.

Remark. Summing the preceding inequalities in j easily leads to damped Strichartz estimates in Besov spaces with summability in ℓ^p for any $\tilde q' \leq p \leq q$. However, due to the dichotomy of the statement of Corollary 2.11 into high and low frequencies, the resulting estimates cannot be stated with homogeneous Besov spaces in a unified format, which is natural because the damped wave equation does not enjoy any scaling invariance. Observe, though, that the high- and low-frequency estimates match in the borderline case $\alpha = 2^j$, with $r = \tilde{r} = 2$.

Remark. Corollary 2.11 is optimal in the loose sense that it provides a similar result as Corollary 2.10 for high frequencies. Moreover, it recovers the optimal Strichartz estimates for the classical wave equation in the limit $\alpha \to 0$. Corollary 2.11 also displays optimality in its control of low frequencies. Indeed, let us consider a solution $u_c(t,x)$ for each $c > 0$ of the damped wave equation

$$\begin{cases} (c^{-2}\partial_t^2 + \alpha\partial_t - \Delta)u_c(t,x) = F(t,x), \\ u_c(0,x) = f(x), \\ \partial_t u_c(0,x) = g(x) \end{cases}$$

on $t \in [0,T)$. In particular, it follows that $\tilde{u}_c(t,x) := u_c(c^{-1}t, x)$ solves

$$\begin{cases} (\partial_t^2 + c\alpha\partial_t - \Delta)\tilde{u}_c(t,x) = F(c^{-1}t,x), \\ \tilde{u}_c(0,x) = f(x), \\ \partial_t \tilde{u}_c(0,x) = c^{-1}g(x) \end{cases}$$

on $t \in [0,cT)$. Therefore, applying Corollary 2.11 to $\tilde{u}_c$, with $r = \tilde{r} = 2$, yields the low-frequency estimates

$$\|\Delta_j \partial_t u_c\|_{L^q_t L^2_x} \lesssim \frac{1}{c^2}\Big(\frac{T}{1+c^2\alpha T}\Big)^{\frac{1}{q}} \|\Delta_j g\|_{L^2_x} + \frac{1}{\alpha}\Big(\frac{\alpha 2^{2j}T}{\alpha+2^{2j}T}\Big)^{\frac{1}{q}} 2^{j(1-\frac{2}{q})}\|\Delta_j \nabla f\|_{L^2_x} + c^2\Big(\frac{T}{1+c^2\alpha T}\Big)^{\frac{1}{q}+\frac{1}{\tilde{q}}}\|\Delta_j F\|_{L^{\tilde{q}'}_t L^2_x}$$

and

$$2^{j\frac{2}{q}}\|\Delta_j \nabla u_c\|_{L^q_t L^2_x} \lesssim \frac{1}{c^2\alpha}\Big(\frac{\alpha 2^{2j}T}{\alpha+2^{2j}T}\Big)^{\frac{1}{q}} 2^j\|\Delta_j g\|_{L^2_x} + \Big(\frac{\alpha 2^{2j}T}{\alpha+2^{2j}T}\Big)^{\frac{1}{q}}\|\Delta_j \nabla f\|_{L^2_x} + \frac{1}{\alpha}\Big(\frac{\alpha 2^{2j}T}{\alpha+2^{2j}T}\Big)^{\frac{1}{q}+\frac{1}{\tilde{q}}} 2^{j(1-\frac{2}{\tilde{q}})}\|\Delta_j F\|_{L^{\tilde{q}'}_t L^2_x}$$

for all $j \in \mathbb{Z}$ with $2^j \leq c\alpha$. Finally, letting c tend to infinity and denoting the limit of u_c (in the sense of distributions) by u, we obtain the estimates

$$\|\Delta_j \partial_t u\|_{L^q_t L^2_x} \lesssim \frac{1}{\alpha}\Big(\frac{\alpha 2^{2j}T}{\alpha+2^{2j}T}\Big)^{\frac{1}{q}} 2^{j2(1-\frac{1}{q})}\|\Delta_j f\|_{L^2_x} + \frac{1}{\alpha}\|\Delta_j F\|_{L^q_t L^2_x}$$

and

$$2^{j\frac{2}{q}}\|\Delta_j u\|_{L^q_t L^2_x} \lesssim \Big(\frac{\alpha 2^{2j}T}{\alpha+2^{2j}T}\Big)^{\frac{1}{q}}\|\Delta_j f\|_{L^2_x} + \frac{1}{\alpha}\Big(\frac{\alpha 2^{2j}T}{\alpha+2^{2j}T}\Big)^{\frac{1}{q}+\frac{1}{\tilde{q}}} 2^{-j\frac{2}{\tilde{q}}}\|\Delta_j F\|_{L^{\tilde{q}'}_t L^2_x}$$

for all $j \in \mathbb{Z}$ and every $q, \tilde{q} \in [1,\infty]$ such that $1/q + 1/\tilde{q} \leq 1$, which are optimal parabolic estimates for the heat equation

$$(\alpha\partial_t - \Delta)u = F$$

with initial data $u(0,x) = f(x)$.

Remark. Other attempts at establishing Strichartz estimates for the damped wave equation can be found in [Inui 2019; Inui and Wakasugi 2021]. However, the results obtained therein are suboptimal. Indeed, Corollary 2.11 supersedes the results from those two works in both the breadth of the range of integrability parameters that it handles and the sharpness of regularity gains that it produces.

Proof. We begin by introducing

$$e(t,x) := \partial_t u(t,x) \quad \text{and} \quad b(t,x) := i|D|u(t,x).$$

It then follows that (e,b) is a solution of the system

$$\begin{aligned} \partial_t e - i|D|b + \alpha e &= F, \\ \partial_t b - i|D|e &= 0, \end{aligned}$$

with initial data $(e(0,x), b(0,x)) = (g(x), i|D|f(x))$. This system is reminiscent of Maxwell's equations, which are studied in Section 3, and it can be recast as

$$\partial_t \begin{pmatrix} e \\ b \end{pmatrix} = \mathcal{L} \begin{pmatrix} e \\ b \end{pmatrix} + \begin{pmatrix} F \\ 0 \end{pmatrix},$$

where

$$\mathcal{L} = \mathcal{L}(D) := \begin{pmatrix} -\alpha & i|D| \\ i|D| & 0 \end{pmatrix}.$$

Now, a straightforward computation shows that $\mathcal{L}(\xi)$, where $\xi \in \mathbb{R}^d$, has the eigenvalues

$$\lambda_\pm(\xi) = -\tfrac{1}{2}\alpha \pm \sqrt{\tfrac{1}{4}\alpha^2 - |\xi|^2} \tag{2-18}$$

in the complex field $\mathbb{C}$. Moreover, exploiting the trivial identities $\lambda_+ + \lambda_- = -\alpha$ and $\lambda_+\lambda_- = |\xi|^2$, one can readily verify that

$$\begin{pmatrix} e \\ b \end{pmatrix} = P_+ \begin{pmatrix} e \\ b \end{pmatrix} + P_- \begin{pmatrix} e \\ b \end{pmatrix}$$

provides an eigenvector decomposition, where

$$P_+ \begin{pmatrix} e \\ b \end{pmatrix} = \frac{1}{\lambda_+ - \lambda_-} \begin{pmatrix} \lambda_+ e + i|D|b \\ i|D|e - \lambda_- b \end{pmatrix} \quad \text{and} \quad P_- \begin{pmatrix} e \\ b \end{pmatrix} = \frac{1}{\lambda_- - \lambda_+} \begin{pmatrix} \lambda_- e + i|D|b \\ i|D|e - \lambda_+ b \end{pmatrix}$$

are the projections onto the eigenspaces associated with $\lambda_+(D)$ and $\lambda_-(D)$ $\big($when these eigenvalues are distinct, i.e., when $|\xi| \neq \frac{1}{2}\alpha\big)$, respectively. In particular, this decomposition allows us to deduce the representation formula

$$\begin{aligned} \begin{pmatrix} e \\ b \end{pmatrix}(t) &= P_+ \begin{pmatrix} e \\ b \end{pmatrix}(t) + P_- \begin{pmatrix} e \\ b \end{pmatrix}(t) \\ &= \begin{pmatrix} \frac{e^{t\lambda_+}\lambda_+ - e^{t\lambda_-}\lambda_-}{\lambda_+ - \lambda_-} g - \frac{e^{t\lambda_+} - e^{t\lambda_-}}{\lambda_+ - \lambda_-} |D|^2 f \\ \frac{e^{t\lambda_+} - e^{t\lambda_-}}{\lambda_+ - \lambda_-} i|D|g - \frac{e^{t\lambda_+}\lambda_- - e^{t\lambda_-}\lambda_+}{\lambda_+ - \lambda_-} i|D|f \end{pmatrix} + \int_0^t \begin{pmatrix} \frac{e^{(t-s)\lambda_+}\lambda_+ - e^{(t-s)\lambda_-}\lambda_-}{\lambda_+ - \lambda_-} F \\ \frac{e^{(t-s)\lambda_+} - e^{(t-s)\lambda_-}}{\lambda_+ - \lambda_-} i|D|F \end{pmatrix} ds. \end{aligned} \tag{2-19}$$

We want now to use (2-19) to deduce an estimate on $\Delta_0 e$ and $\Delta_0 b$, which, by a scaling argument in the spirit of the proof of Corollary 2.10, will then result in a control on each dyadic component $\Delta_j e$ and $\Delta_j b$, with $j \in \mathbb{Z}$. Note, however, that the eigenvalues $\lambda_\pm(\xi)$ are of a fundamentally different nature depending on the relative size of the frequencies $\{\frac{1}{2} \le |\xi| \le 2\}$ with respect to $\alpha \ge 0$, which leads us to consider several cases.

More specifically, on the one hand, when $\lambda_\pm \in \mathbb{R}$ $\big($i.e., when $|\xi| \le \frac{1}{2}\alpha\big)$, we are going to employ the elementary controls

$$\begin{aligned} -\alpha \le \lambda_- \le -\frac{\alpha}{2} \le -\frac{2|\xi|^2}{\alpha} \le \lambda_+ \le -\frac{|\xi|^2}{\alpha}, \\ 0 \le \frac{e^{t\lambda_+} - e^{t\lambda_-}}{\lambda_+ - \lambda_-} = \frac{\int_{\lambda_-}^{\lambda_+} t e^{ts}\, ds}{\lambda_+ - \lambda_-} \le t e^{t\lambda_+} \le t e^{-t\frac{|\xi|^2}{\alpha}}, \end{aligned} \tag{2-20}$$

while, on the other hand, when $\lambda_\pm \in \mathbb{C} \setminus \mathbb{R}$ $\big($i.e., when $|\xi| \ge \frac{1}{2}\alpha\big)$, we are going to use the properties

$$\begin{aligned} |\lambda_\pm| = |\xi|, \quad |e^{-t\lambda_\pm}| = e^{-\frac{\alpha}{2}t}, \\ \left| \frac{e^{t\lambda_+} - e^{t\lambda_-}}{\lambda_+ - \lambda_-} \right| = e^{-\frac{\alpha}{2}t} \frac{|\sin(t\sqrt{|\xi|^2 - \frac{1}{4}\alpha^2})|}{\sqrt{|\xi|^2 - \frac{1}{4}\alpha^2}} \le t e^{-\frac{\alpha}{2}t}. \end{aligned} \tag{2-21}$$

Considering that the dyadic operator Δ_0 localizes frequencies to $\{\frac{1}{2} \le |\xi| \le 2\}$, we will then distinguish three cases:

- The complex case, where $0 \le \alpha \le \frac{1}{2}$, so that $\lambda_\pm$ and $(\lambda_+ - \lambda_-)^{-1}$ are complex and smooth on $\{\frac{1}{2} \le |\xi| \le 2\}$.
- The degenerate case, where $\frac{1}{2} < \alpha < 5$ and the eigenvalues may be equal.
- The real case, where $\alpha \ge 5$, which implies that the eigenvalues are real and the damping phenomenon dominates the behavior of solutions on $\{\frac{1}{2} \le |\xi| \le 2\}$.

The complex case. We begin by considering the range $0 \le \alpha \le \frac{1}{2}$. In this setting, it is readily seen that the functions $\lambda_+(\xi)$, $\lambda_-(\xi)$ and $(\lambda_+(\xi) - \lambda_-(\xi))^{-1}$, as well as any number of their derivatives, are uniformly bounded on $\{\frac{1}{3} < |\xi| < 3\}$, uniformly in $\alpha \in [0, \frac{1}{2}]$. In particular, by virtue of the criterion (2-14) for the boundedness of multipliers, further introducing a smooth cutoff function $\psi(\xi)$ compactly supported inside $\{\frac{1}{3} < |\xi| < 3\}$ and such that $\psi \equiv 1$ on $\{\frac{1}{2} \le |\xi| \le 2\}$, it follows that $\lambda_+(\xi)\psi(\xi)$, $\lambda_-(\xi)\psi(\xi)$ and $(\lambda_+(\xi) - \lambda_-(\xi))^{-1}\psi(\xi)$ are the symbols of bounded Fourier multipliers over any homogeneous Besov space. Therefore, we deduce from (2-19) that

$$\begin{aligned} &\|\Delta_0(e,b)\|_{L^q_t L^r_x} \\ &\qquad \lesssim \sum_\pm \|e^{t\lambda_\pm}\Delta_0(f,g)\|_{L^q_t L^r_x} + \left\| \int_0^t e^{(t-s)\lambda_\pm}\Delta_0 F\, ds \right\|_{L^q_t L^r_x} \\ &\qquad = \sum_\pm \|e^{-\frac{\alpha}{2}t} e^{\pm it\delta(D)}\Delta_0(f,g)\|_{L^q_t L^r_x} + \left\| \int_0^t e^{-\frac{\alpha}{2}(t-s)} e^{\pm i(t-s)\delta(D)}\Delta_0 F\, ds \right\|_{L^q_t L^r_x}, \end{aligned} \tag{2-22}$$

where we introduced the notation

$$\delta(\xi) := \sqrt{|\xi|^2 - \tfrac{1}{4}\alpha^2}$$

for convenience.

Now, classically, the stationary phase method can be used (see [Bahouri et al. 2011, Proposition 8.15], for instance) to show that

$$\left| \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{\pm it|\xi|} \psi(\xi)\, d\xi \right| \le \frac{C_\psi}{t^{(d-1)/2}}$$

for all $t > 0$, where the constant $C_\psi > 0$ is independent of t and x, and ψ is any smooth compactly supported function whose support does not contain the origin. A similar estimate holds, uniformly in $\alpha \in \left[0, \tfrac{1}{2}\right]$, if one replaces $|\xi|$ by $\delta(\xi)$ and if the support of ψ is disjoint from the closed ball $\left\{|\xi| \le \tfrac{1}{2}\alpha\right\}$. More precisely, we claim that

$$\left| \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{\pm it\delta(\xi)} \psi(\xi)\, d\xi \right| \le \frac{C_\psi}{t^{(d-1)/2}} \tag{2-23}$$

whenever $\operatorname{supp}\psi \subset \left\{|\xi| > \tfrac{1}{4} \ge \tfrac{1}{2}\alpha\right\}$, where $C_\psi > 0$ is independent of $t > 0$, $x \in \mathbb{R}^d$ and $\alpha \in \left[0, \tfrac{1}{2}\right]$. For the sake of completeness, we provide a justification of (2-23) in Appendix B.

Therefore, introducing the flow

$$U(t) f(x) := e^{\pm it\delta(D)} \psi(D) f(x)$$

for some fixed compactly supported cutoff $\psi(\xi)$ such that $\operatorname{supp}\psi \subset \left\{|\xi| > \tfrac{1}{4} \ge \tfrac{\alpha}{2}\right\}$ and $\psi \equiv 1$ on $\left\{\tfrac{1}{2} \le \xi \le 2\right\}$, we see, in view of (2-23), that $U(t)$ satisfies all hypotheses of Proposition 2.8 with

$$H = L^2(\mathbb{R}^d), \quad X = \mathbb{R}^d \quad \text{and} \quad \sigma = \tfrac{1}{2}(d-1).$$

Hence, we conclude from (2-22) that

$$\begin{aligned} &\|\Delta_0(e,b)\|_{L^q_t L^r_x} \\ &\quad\lesssim \left(\frac{T}{1+\alpha T}\right)^{\frac{1}{q}+\frac{d-1}{2}\left(\frac{1}{r}-\frac{1}{2}\right)} \|\Delta_0(f,g)\|_{L^2_x} + \left(\frac{T}{1+\alpha T}\right)^{\frac{1}{q}+\frac{1}{\tilde q}+\frac{d-1}{2}\left(\frac{1}{r}+\frac{1}{\tilde r}-1\right)} \|\Delta_0 F\|_{L^{\tilde q'}_t L^{\tilde r'}_x} \end{aligned} \tag{2-24}$$

for all admissible exponent pairs, when $0 \le \alpha \le \tfrac{1}{2}$.

The degenerate case. We are now looking at the range $\tfrac{1}{2} < \alpha < 5$. This case is easily settled by the use of (2-20) and (2-21), which allows us to deduce, whenever $\tfrac{1}{2} \le |\xi| \le 2$, that

$$\begin{aligned} \left\| \frac{e^{t\lambda_+} - e^{t\lambda_-}}{\lambda_+ - \lambda_-} \right\|_{L^c_t} &\le \|te^{-\frac{t}{20}}\|_{L^c_t} \lesssim \left(\frac{T^2}{1+T^2}\right)^{\frac{1}{c}} \le 2^{\frac{1}{c}} \left(\frac{T}{1+T}\right)^{\frac{2}{c}} \lesssim \left(\frac{T}{1+T}\right)^{\frac{1}{c}}, \\ \left\| \frac{e^{t\lambda_+}\lambda_+ - e^{t\lambda_-}\lambda_-}{\lambda_+ - \lambda_-} \right\|_{L^c_t} &\le \|e^{t\lambda_+}\|_{L^c_t} + \left\| \lambda_- \frac{e^{t\lambda_+} - e^{t\lambda_-}}{\lambda_+ - \lambda_-} \right\|_{L^c_t} \lesssim \left(\frac{T}{1+T}\right)^{\frac{1}{c}}, \\ \left\| \frac{e^{t\lambda_+}\lambda_- e^{t\lambda_-}\lambda_+}{\lambda_+ - \lambda_-} \right\|_{L^c_t} &\le \|e^{t\lambda_+}\|_{L^c_t} + \left\| \lambda_+ \frac{e^{t\lambda_+} - e^{t\lambda_-}}{\lambda_+ - \lambda_-} \right\|_{L^c_t} \lesssim \left(\frac{T}{1+T}\right)^{\frac{1}{c}} \end{aligned}$$

for any $1 \le c \le \infty$. Indeed, incorporating these controls into (2-19) and recalling that the space of Fourier multipliers on $L^2(\mathbb{R}^d)$ is isomorphic to $L^\infty(\mathbb{R}^d)$ leads to

$$\begin{aligned}
&\|\Delta_0(e,b)\|_{L^q_t L^r_x} \\
&\qquad \lesssim \|\Delta_0(e,b)\|_{L^q_t L^2_x} \\
&\qquad \lesssim \left(\frac{T}{1+T}\right)^{\frac{1}{q}} \|\Delta_0(f,g)\|_{L^2_x} + \left(\frac{T}{1+T}\right)^{\frac{1}{q}+\frac{1}{\tilde{q}}} \|\Delta_0 F\|_{L^{\tilde{q}'}_t L^2_x} \\
&\qquad \lesssim \left(\frac{T}{1+T}\right)^{\frac{1}{q}+\frac{d-1}{2}(\frac{1}{r}-\frac{1}{2})} \|\Delta_0(f,g)\|_{L^2_x} + \left(\frac{T}{1+T}\right)^{\frac{1}{q}+\frac{1}{\tilde{q}}+\frac{d-1}{2}(\frac{1}{r}+\frac{1}{\tilde{r}}-1)} \|\Delta_0 F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x}
\end{aligned} \tag{2-25}$$

for all admissible exponent pairs, when $\frac{1}{2} < \alpha < 5$.

The real case. In the remaining case, we assume that $\alpha \ge 5$. In particular, when $\frac{1}{2} \le |\xi| \le 2$,

$$\lambda_+ - \lambda_- = \sqrt{\alpha^2 - 4|\xi|^2} \ge \sqrt{\alpha^2 - 16} \ge \tfrac{3}{5}\alpha.$$

Furthermore, employing (2-20), one finds that

$$\|e^{t\lambda_+}\|_{L^c_t} \le \|e^{-\frac{t}{4\alpha}}\|_{L^c_t} \lesssim \left(\frac{\alpha T}{\alpha+T}\right)^{\frac{1}{c}} \quad \text{and} \quad \|e^{t\lambda_-}\|_{L^c_t} \le \|e^{-\frac{\alpha t}{2}}\|_{L^c_t} \lesssim \left(\frac{T}{1+\alpha T}\right)^{\frac{1}{c}}$$

for any $1 \le c \le \infty$. Therefore, we deduce from (2-19) and (2-20) that

$$\begin{aligned}
\|\Delta_0 e\|_{L^q_t L^r_x} &\lesssim \|\Delta_0 e\|_{L^q_t L^2_x} \\
&\lesssim \left(\frac{\alpha^{1-2q}T}{\alpha+T} + \frac{T}{1+\alpha T}\right)^{\frac{1}{q}} \|\Delta_0 g\|_{L^2_x} + \left(\frac{\alpha^{1-q}T}{\alpha+T} + \frac{\alpha^{-q}T}{1+\alpha T}\right)^{\frac{1}{q}} \|\Delta_0 f\|_{L^2_x} \\
&\qquad + \left(\frac{\alpha^{1-2(1/q+1/\tilde{q})^{-1}}T}{\alpha+T} + \frac{T}{1+\alpha T}\right)^{\frac{1}{q}+\frac{1}{\tilde{q}}} \|\Delta_0 F\|_{L^{\tilde{q}'}_t L^2_x} \\
&\lesssim \left(\frac{T}{1+\alpha T}\right)^{\frac{1}{q}} \|\Delta_0 g\|_{L^2_x} + \left(\frac{\alpha^{1-q}T}{\alpha+T}\right)^{\frac{1}{q}} \|\Delta_0 f\|_{L^2_x} + \left(\frac{T}{1+\alpha T}\right)^{\frac{1}{q}+\frac{1}{\tilde{q}}} \|\Delta_0 F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x}
\end{aligned} \tag{2-26}$$

and

$$\begin{aligned}
&\|\Delta_0 b\|_{L^q_t L^r_x} \\
&\quad \lesssim \|\Delta_0 b\|_{L^q_t L^2_x} \\
&\quad \lesssim \left(\frac{\alpha^{1-q}T}{\alpha+T} + \frac{\alpha^{-q}T}{1+\alpha T}\right)^{\frac{1}{q}} \|\Delta_0 g\|_{L^2_x} + \left(\frac{\alpha T}{\alpha+T} + \frac{\alpha^{-2q}T}{1+\alpha T}\right)^{\frac{1}{q}} \|\Delta_0 f\|_{L^2_x} \\
&\qquad\qquad + \left(\frac{\alpha^{1-(1/q+1/\tilde{q})^{-1}}T}{\alpha+T} + \frac{\alpha^{-(1/q+1/\tilde{q})^{-1}}T}{1+\alpha T}\right)^{\frac{1}{q}+\frac{1}{\tilde{q}}} \|\Delta_0 F\|_{L^{\tilde{q}'}_t L^2_x} \\
&\quad \lesssim \left(\frac{\alpha^{1-q}T}{\alpha+T}\right)^{\frac{1}{q}} \|\Delta_0 g\|_{L^2_x} + \left(\frac{\alpha T}{\alpha+T}\right)^{\frac{1}{q}} \|\Delta_0 f\|_{L^2_x} + \left(\frac{\alpha^{1-(1/q+1/\tilde{q})^{-1}}T}{\alpha+T}\right)^{\frac{1}{q}+\frac{1}{\tilde{q}}} \|\Delta_0 F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x}
\end{aligned} \tag{2-27}$$

for all admissible exponent pairs, whenever $\alpha \ge 5$.

Scaling argument and conclusion of proof. We are now in a position to conclude the justification of the corollary. In order to deduce an estimate on $\Delta_j(e,b)$ for all $j\in\mathbb{Z}$ from (2-24)–(2-27), we conduct now a scaling analysis in the spirit of the proof of Corollary 2.10. To that end, we introduce

$$(e_j,b_j)(t,x) := (e,b)\Big(\frac{t}{2^j},\frac{x}{2^j}\Big),\quad F_j(t,x) := \frac{1}{2^j}F\Big(\frac{t}{2^j},\frac{x}{2^j}\Big),$$
$$f_j(x) := 2^j f\Big(\frac{x}{2^j}\Big),\qquad g_j(x) := g\Big(\frac{x}{2^j}\Big)$$

and observe that (e_j,b_j) solves

$$\partial_t e_j - i|D|b_j + 2^{-j}\alpha e_j = F_j,$$
$$\partial_t b_j - i|D|e_j = 0$$

on $[0,2^jT)$, with initial data $(e_j(0,x),b_j(0,x)) = (g_j(x), i|D|f_j(x))$.

Then, noticing that

$$\|\Delta_j(e,b)\|_{L^q_tL^r_x} = 2^{-j(\frac1q+\frac dr)}\|\Delta_0(e_j,b_j)\|_{L^q_tL^r_x},$$
$$\|\Delta_j g\|_{L^2_x} = 2^{-j\frac d2}\|\Delta_0 g_j\|_{L^2_x},$$
$$\|\Delta_j f\|_{L^2_x} = 2^{-j(1+\frac d2)}\|\Delta_0 f_j\|_{L^2_x},$$
$$\|\Delta_j F\|_{L^{\tilde q'}_tL^{\tilde r'}_x} = 2^{j(\frac1{\tilde q}-d(1-\frac1{\tilde r}))}\|\Delta_0 F_j\|_{L^{\tilde q'}_tL^{\tilde r'}_x}$$

and applying (2-24) and (2-25) to (e_j,b_j) yields the estimate

$$\begin{aligned}\|\Delta_j(e,b)\|_{L^q_tL^r_x} \lesssim{}& \Big(\frac{T}{1+\alpha T}\Big)^{\frac1q+\frac{d-1}{2}(\frac1r-\frac12)}2^{j\frac{d+1}{2}(\frac12-\frac1r)}\|\Delta_j(g,\nabla f)\|_{L^2_x}\\ &+\Big(\frac{T}{1+\alpha T}\Big)^{\frac1q+\frac1{\tilde q}+\frac{d-1}{2}(\frac1r+\frac1{\tilde r}-1)}2^{j\frac{d+1}{2}(1-\frac1r-\frac1{\tilde r})}\|\Delta_jF\|_{L^{\tilde q'}_tL^{\tilde r'}_x}\end{aligned}$$

whenever $2^j\ge\alpha$.

Similarly, if $2^j\le\alpha$, then, applying (2-25), (2-26) and (2-27) to (e_j,b_j) leads to the controls

$$\begin{aligned}\|\Delta_j e\|_{L^q_tL^r_x} \lesssim{}& \Big(\frac{T}{1+\alpha T}\Big)^{\frac1q}2^{jd(\frac12-\frac1r)}\|\Delta_j g\|_{L^2_x}+\frac1\alpha\Big(\frac{\alpha T}{\alpha+2^{2j}T}\Big)^{\frac1q}2^{j(1+d(\frac12-\frac1r))}\|\Delta_j\nabla f\|_{L^2_x}\\ &+\Big(\frac{T}{1+\alpha T}\Big)^{\frac1q+\frac1{\tilde q}}2^{jd(1-\frac1r-\frac1{\tilde r})}\|\Delta_jF\|_{L^{\tilde q'}_tL^{\tilde r'}_x}\end{aligned}$$

and

$$\begin{aligned}\|\Delta_j b\|_{L^q_tL^r_x} \lesssim{}& \frac1\alpha\Big(\frac{\alpha T}{\alpha+2^{2j}T}\Big)^{\frac1q}2^{j(1+d(\frac12-\frac1r))}\|\Delta_j g\|_{L^2_x}+\Big(\frac{\alpha T}{\alpha+2^{2j}T}\Big)^{\frac1q}2^{jd(\frac12-\frac1r)}\|\Delta_j\nabla f\|_{L^2_x}\\ &+\frac1\alpha\Big(\frac{\alpha T}{\alpha+2^{2j}T}\Big)^{\frac1q+\frac1{\tilde q}}2^{j(1+d(1-\frac1r-\frac1{\tilde r}))}\|\Delta_jF\|_{L^{\tilde q'}_tL^{\tilde r'}_x},\end{aligned}$$

which concludes the proof of the corollary. □

Corollary 2.12 (damped Maxwell equations). *Let $d=2$ or $d=3$ and consider a solution $(E,B)(t,x):[0,T)\times\mathbb{R}^d\to\mathbb{R}^6$ of the damped Maxwell system*

$$\begin{cases}\frac{1}{c}\partial_t E-\nabla\times B+\sigma cE=G,\\ \frac{1}{c}\partial_t B+\nabla\times E=0,\\ \operatorname{div}B=0\end{cases}$$

for some initial data $(E,B)(0,x)=(E_0,B_0)(x)$, where $\sigma\geq 0$ and $c>0$.

For any exponent pairs $(q,r),(\tilde q,\tilde r)\in[1,\infty]\times[2,\infty]$ which are admissible in the sense that

$$\frac{2}{q}+\frac{d-1}{r}\geq\frac{d-1}{2},\qquad 1+\frac{d-1}{r}\geq\frac{d-1}{2}\quad\text{and}\quad (r,d)\neq(\infty,3),$$

and similarly for $(\tilde q,\tilde r)$, and such that

$$\frac{1}{q}+\frac{1}{\tilde q}\leq 1,$$

one has the high-frequency estimate

$$\begin{aligned}2^{-j\frac{d+1}{2}(\frac12-\frac1r)}&\|\Delta_j(PE,B)\|_{L^q_t([0,T);L^r_x)}\\ &\lesssim c^{\frac{d-1}{2}(\frac12-\frac1r)-\frac2q}\Big(\frac{c^2T}{1+\sigma c^2T}\Big)^{\frac1q+\frac{d-1}{2}(\frac1r-\frac12)}\|\Delta_j(PE_0,B_0)\|_{L^2_x}\\ &\quad+c^{1+\frac{d-1}{2}(1-\frac1r-\frac1{\tilde r})-\frac2q-\frac2{\tilde q}}\Big(\frac{c^2T}{1+\sigma c^2T}\Big)^{\frac1q+\frac1{\tilde q}+\frac{d-1}{2}(\frac1r+\frac1{\tilde r}-1)}2^{j\frac{d+1}{2}(\frac12-\frac1{\tilde r})}\|\Delta_jPG\|_{L^{\tilde q'}_t([0,T);L^{\tilde r'}_x)}\end{aligned}$$

for all $j\in\mathbb{Z}$ with $2^j\geq\sigma c$, and the low-frequency estimates

$$\begin{aligned}2^{-jd(\frac12-\frac1r)}\|\Delta_jPE\|_{L^q_t([0,T);L^r_x)}\lesssim\Big(\frac{T}{1+\sigma c^2T}\Big)^{\frac1q}\|\Delta_jPE_0\|_{L^2_x}+\frac{1}{\sigma c}\Big(\frac{\sigma 2^{2j}T}{\sigma+2^{2j}T}\Big)^{\frac1q}2^{j(1-\frac2q)}\|\Delta_jB_0\|_{L^2_x}\\ +c\Big(\frac{T}{1+\sigma c^2T}\Big)^{\frac1q+\frac1{\tilde q}}2^{jd(\frac12-\frac1{\tilde r})}\|\Delta_jPG\|_{L^{\tilde q'}_t([0,T);L^{\tilde r'}_x)}\end{aligned}$$

and

$$\begin{aligned}2^{-j(d(\frac12-\frac1r)-\frac2q)}\|\Delta_jB\|_{L^q_t([0,T);L^r_x)}\lesssim\frac{1}{\sigma c}\Big(\frac{\sigma 2^{2j}T}{\sigma+2^{2j}T}\Big)^{\frac1q}2^j\|\Delta_jPE_0\|_{L^2_x}+\Big(\frac{\sigma 2^{2j}T}{\sigma+2^{2j}T}\Big)^{\frac1q}\|\Delta_jB_0\|_{L^2_x}\\ +\frac{1}{\sigma}\Big(\frac{\sigma 2^{2j}T}{\sigma+2^{2j}T}\Big)^{\frac1q+\frac1{\tilde q}}2^{j(1+d(\frac12-\frac1{\tilde r})-\frac2{\tilde q})}\|\Delta_jPG\|_{L^{\tilde q'}_t([0,T);L^{\tilde r'}_x)}\end{aligned}$$

for all $j\in\mathbb{Z}$ with $2^j\leq\sigma c$.

Remark. Corollary 2.12 only provides estimates of the magnetic field B and the divergence-free part of the electric field PE. Notice, though, that the divergent component $P^\perp E$ can also be estimated directly from Maxwell's system. Indeed, applying the projector $P^\perp$ to Ampère's equation yields

$$\partial_tP^\perp E+\sigma c^2P^\perp E=cP^\perp G,$$

which leads to the representation formula

$$P^\perp E(t)=e^{-\sigma c^2t}P^\perp E_0+c\int_0^te^{-\sigma c^2(t-s)}P^\perp G(s)\,ds.$$

A direct estimate then easily gives

$$\begin{aligned}
2^{-jd\left(\frac{1}{2}-\frac{1}{r}\right)}\|\Delta_j P^\perp E\|_{L^q_t L^r_x}
&\lesssim \|\Delta_j P^\perp E\|_{L^q_t L^2_x}\\
&\lesssim \left(\frac{T}{1+\sigma c^2 T}\right)^{\frac{1}{q}}\|\Delta_j P^\perp E_0\|_{L^2_x} + c\left(\frac{T}{1+\sigma c^2 T}\right)^{\frac{1}{q}+\frac{1}{\tilde q}}\|\Delta_j P^\perp G\|_{L^{\tilde q'}_t L^2_x}\\
&\lesssim \left(\frac{T}{1+\sigma c^2 T}\right)^{\frac{1}{q}}\|\Delta_j P^\perp E_0\|_{L^2_x} + c\left(\frac{T}{1+\sigma c^2 T}\right)^{\frac{1}{q}+\frac{1}{\tilde q}} 2^{jd\left(\frac{1}{2}-\frac{1}{\tilde r}\right)}\|\Delta_j P^\perp G\|_{L^{\tilde q'}_t L^{\tilde r'}_x}
\end{aligned}$$

for any $q,\tilde q \in [1,\infty]$ and $r,\tilde r \in [2,\infty]$, with $1/q + 1/\tilde q \le 1$.

Proof. Since $B(t,x)$ is a solenoidal field, we begin by introducing a vector potential $A(t,x)$, with $t \in [0,cT)$ and $x \in \mathbb{R}^d$, such that

$$B(t,x) = \nabla \times A(ct,x). \tag{2-28}$$

Faraday's equation $c^{-1}\partial_t B + \nabla \times E = 0$ then implies that $(\partial_t A)(ct,x) + E(t,x)$ must be curl-free, whereby there exists a scalar potential $\varphi(t,x)$, with $t \in [0,cT)$ and $x \in \mathbb{R}^d$, such that

$$E(t,x) = \nabla\varphi(ct,x) - (\partial_t A)(ct,x). \tag{2-29}$$

Observe that A and φ are not uniquely determined. Indeed, for any scalar-valued potential $\psi(t,x)$, it is possible to apply the transformations

$$\begin{aligned} A(t,x) &\mapsto A(t,x) + \nabla\psi(t,x),\\ \varphi(t,x) &\mapsto \varphi(t,x) + \partial_t\psi(t,x)\end{aligned} \tag{2-30}$$

to produce new potentials representing the same electromagnetic field (E,B). Any particular choice of A and φ is called a gauge.

Different choices of gauge lead to different insights into Maxwell's equations. It is therefore important to carefully select the properties fixing the gauge. A standard example of gauge fixing is the Coulomb gauge, which merely requires that A be solenoidal, i.e., $\operatorname{div} A = 0$. The Lorenz gauge, which imposes the condition

$$\operatorname{div} A(t,x) = \partial_t\varphi(t,x)$$

is another classical example with the property that it produces decoupled wave equations on A and φ when there is no damping, i.e., $\sigma = 0$.

Here, we introduce a damped Lorenz gauge by selecting potentials A and φ solving

$$\operatorname{div} A(t,x) = \partial_t\varphi(t,x) + \sigma c\varphi(t,x). \tag{2-31}$$

Observe that it is always possible to find a damped Lorenz gauge. Indeed, starting from any other gauge (A,φ), one can apply the transformations (2-30) with any solution $\psi(t,x)$ of the damped wave equation

$$\partial_t^2\psi + \sigma c\partial_t\psi - \Delta\psi = \operatorname{div} A - \partial_t\varphi - \sigma c\varphi,$$

thereby producing new potentials satisfying (2-31).

Now, by inserting (2-28) and (2-29) into Ampère's equation and then employing (2-31), a straightforward calculation shows that the damped Lorenz gauge is a solution of the damped wave system

$$(\partial_t^2+\sigma c\partial_t-\Delta)A(t,x)=-G(c^{-1}t,x) \tag{2-32}$$

on $t\in[0,cT)$.

Therefore, applying the Strichartz estimates for damped wave equations from Corollary 2.11 to this system, we find, concerning high frequencies, that

$$\begin{aligned}
&c^{\frac1q}2^{-j\frac{d+1}{2}(\frac12-\frac1r)}\|\Delta_j(PE,B)\|_{L^q_t([0,T);L^r_x)}\\
&\quad\lesssim 2^{-j\frac{d+1}{2}(\frac12-\frac1r)}\|\Delta_j(\partial_tPA,\nabla PA)\|_{L^q_t([0,cT);L^r_x)}\\
&\quad\lesssim\Big(\frac{cT}{1+\sigma c^2T}\Big)^{\frac1q+\frac{d-1}{2}(\frac1r-\frac12)}\|\Delta_j(PE_0,B_0)\|_{L^2_x}\\
&\qquad\qquad+c^{1-\frac1{\tilde q}}\Big(\frac{cT}{1+\sigma c^2T}\Big)^{\frac1q+\frac1{\tilde q}+\frac{d-1}{2}(\frac1r+\frac1{\tilde r}-1)}2^{j\frac{d+1}{2}(\frac12-\frac1{\tilde r})}\|\Delta_jPG\|_{L^{\tilde q'}_t([0,T);L^{\tilde r'}_x)}
\end{aligned}$$

for all $j\in\mathbb{Z}$ with $2^j\ge\sigma c$.

As for low frequencies, i.e., when $j\in\mathbb{Z}$ with $2^j\le\sigma c$, we obtain similarly from Corollary 2.11 that

$$\begin{aligned}
&c^{\frac1q}2^{-jd(\frac12-\frac1r)}\|\Delta_jPE\|_{L^q_t([0,T);L^r_x)}\\
&\quad=2^{-jd(\frac12-\frac1r)}\|\Delta_j\partial_tPA\|_{L^q_t([0,cT);L^r_x)}\\
&\quad\lesssim\Big(\frac{cT}{1+\sigma c^2T}\Big)^{\frac1q}\|\Delta_jPE_0\|_{L^2_x}+\frac{1}{\sigma c}\Big(\frac{\sigma c2^{2j}T}{\sigma+2^{2j}T}\Big)^{\frac1q}2^{j(1-\frac2q)}\|\Delta_jB_0\|_{L^2_x}\\
&\qquad\qquad+c^{1-\frac1{\tilde q}}\Big(\frac{cT}{1+\sigma c^2T}\Big)^{\frac1q+\frac1{\tilde q}}2^{jd(\frac12-\frac1r)}\|\Delta_jPG\|_{L^{\tilde q'}_t([0,T);L^{\tilde r'}_x)}
\end{aligned}$$

and

$$\begin{aligned}
&c^{\frac1q}2^{-j(d(\frac12-\frac1r)-\frac2q)}\|\Delta_jB\|_{L^q_t([0,T);L^r_x)}\\
&\quad\lesssim 2^{-j(d(\frac12-\frac1r)-\frac2q)}\|\Delta_j\nabla PA\|_{L^q_t([0,cT);L^r_x)}\\
&\quad\lesssim\frac{1}{\sigma c}\Big(\frac{\sigma c2^{2j}T}{\sigma+2^{2j}T}\Big)^{\frac1q}2^j\|\Delta_jPE_0\|_{L^2_x}+\Big(\frac{\sigma c2^{2j}T}{\sigma+2^{2j}T}\Big)^{\frac1q}\|\Delta_jB_0\|_{L^2_x}\\
&\qquad\qquad+c^{1-\frac1{\tilde q}}\frac{1}{\sigma c}\Big(\frac{\sigma c2^{2j}T}{\sigma+2^{2j}T}\Big)^{\frac1q+\frac1{\tilde q}}2^{j(1+d(\frac12-\frac1{\tilde r})-\frac2{\tilde q})}\|\Delta_jPG\|_{L^{\tilde q'}_t([0,T);L^{\tilde r'}_x)},
\end{aligned}$$

which concludes the proof of the corollary. □

The global low-frequency estimates from Corollaries 2.11 and 2.12 can be refined by considering the maximal regularity of the heat equation (without damping) discussed in Section 2.2. The next two results provide such low-frequency parabolic estimates for the wave equation and Maxwell's system, respectively.

Proposition 2.13. *Let $d \geq 2$, and consider a solution $u(t,x)$ of the damped wave equation*

$$\begin{aligned}(\partial_t^2 + \alpha\partial_t - \Delta)u(t,x) &= F(t,x),\\ u(0,x) &= f(x),\\ \partial_t u(0,x) &= g(x),\end{aligned}$$

with $\alpha > 0,\ t \in [0,T)$ and $x \in \mathbb{R}^d$.

For any $\chi \in C_c^\infty(\mathbb{R}^d)$ and $\sigma \in \mathbb{R}$, one has the low-frequency estimates

$$\|\chi(\alpha^{-1}D)\partial_t u\|_{L_t^m([0,T);\dot B^{\sigma+2/m}_{2,q})} \lesssim \alpha^{-\frac{1}{m}}\|g\|_{\dot B^{\sigma+2/m}_{2,q}} + \alpha^{\frac{1}{m}-1}\|f\|_{\dot B^{\sigma+2}_{2,m}} + \alpha^{-(1+\frac{1}{m}-\frac{1}{r})}\|F\|_{L_t^r([0,T);\dot B^{\sigma+2/m}_{2,q})}$$

for any $1 < r \leq m < \infty$ and $1 \leq q \leq \infty$, as well as

$$\|\chi(\alpha^{-1}D)\nabla u\|_{L_t^m([0,T);\dot B^{\sigma+2/m}_{2,1})} \lesssim \alpha^{\frac{1}{m}-1}\|g\|_{\dot B^{\sigma+1}_{2,m}} + \alpha^{\frac{1}{m}}\|f\|_{\dot B^{\sigma+1}_{2,m}} + \alpha^{\frac{1}{m}-\frac{1}{r}}\|F\|_{L_t^r([0,T);\dot B^{\sigma-1+2/r}_{2,\infty})}$$

for any $1 < r < m < \infty$, and

$$\|\chi(\alpha^{-1}D)\nabla u\|_{L_t^m([0,T);\dot B^{\sigma+2/m}_{2,q})} \lesssim \alpha^{\frac{1}{m}-1}\|g\|_{\dot B^{\sigma+1}_{2,m}} + \alpha^{\frac{1}{m}}\|f\|_{\dot B^{\sigma+1}_{2,m}} + \|F\|_{L_t^m([0,T);\dot B^{\sigma-1+2/m}_{2,q})}$$

for any $1 < m < \infty$ and $1 \leq q \leq \infty$.

Proof. Following the proof of Corollary 2.11, we consider

$$e(t,x) := \partial_t u(t,x) \quad \text{and} \quad b(t,x) := i|D|u(t,x).$$

In particular, one has the representation formula (2-19), which, for any given choice of $0 < A < \frac{1}{2}$, can be recast as

$$\begin{aligned}e(t) &= e^{\frac{A}{\alpha}t\Delta}m_2(t,D)g - e^{\frac{A}{\alpha}t\Delta}m_1(t,D)|D|^2 f + \int_0^t e^{\frac{A}{\alpha}(t-s)\Delta}m_2(t-s,D)F\,ds\\ &= (e^{\frac{A}{\alpha}t\Delta}|D|^2 m_2^+(t,D) - e^{-At\alpha}m_2^-(t,D))g - e^{\frac{A}{\alpha}t\Delta}m_1(t,D)|D|^2 f\\ &\qquad + \int_0^t (e^{\frac{A}{\alpha}(t-s)\Delta}|D|^2 m_2^+(t-s,D) - e^{-A(t-s)\alpha}m_2^-(t-s,D))F\,ds,\\ b(t) &= e^{\frac{A}{\alpha}t\Delta}m_1(t,D)i|D|g - e^{\frac{A}{\alpha}t\Delta}m_3(t,D)i|D|f + \int_0^t e^{\frac{A}{\alpha}(t-s)\Delta}m_1(t-s,D)i|D|F\,ds,\end{aligned} \tag{2-33}$$

with the time-dependent Fourier multipliers

$$\begin{aligned} m_1(t,\xi) &= \frac{e^{t\lambda_+} - e^{t\lambda_-}}{\lambda_+ - \lambda_-}e^{At\frac{|\xi|^2}{\alpha}}, & m_2(t,\xi) &= \frac{e^{t\lambda_+}\lambda_+ - e^{t\lambda_-}\lambda_-}{\lambda_+ - \lambda_-}e^{At\frac{|\xi|^2}{\alpha}},\\ m_2^+(t,\xi) &= \frac{e^{t\lambda_+}\lambda_+}{|\xi|^2(\lambda_+ - \lambda_-)}e^{At\frac{|\xi|^2}{\alpha}}, & m_2^-(t,\xi) &= \frac{e^{t\lambda_-}\lambda_-}{\lambda_+ - \lambda_-}e^{At\alpha},\\ m_3(t,\xi) &= \frac{e^{t\lambda_+}\lambda_- e^{t\lambda_-}\lambda_+}{\lambda_+ - \lambda_-}e^{At\frac{|\xi|^2}{\alpha}}, \end{aligned} \tag{2-34}$$

where the eigenvalues $\lambda_+(\xi)$ and $\lambda_-(\xi)$ are defined in (2-18).

Then, making use of the elementary controls (2-20), one can show that

$$\|m_1 \mathbb{1}_{\{|\xi|\le\frac14\alpha\}}\|_{L^\infty_{t,\xi}} \lesssim \alpha^{-1}, \quad \|m_2^+ \mathbb{1}_{\{|\xi|\le\frac14\alpha\}}\|_{L^\infty_{t,\xi}} \lesssim \alpha^{-2},$$
$$\|m_2^- \mathbb{1}_{\{|\xi|\le\frac14\alpha\}}\|_{L^\infty_{t,\xi}} \lesssim 1, \qquad \|m_3 \mathbb{1}_{\{|\xi|\le\alpha R\}}\|_{L^\infty_{t,\xi}} \lesssim 1.$$

In particular, since the space of Fourier multipliers over $L^2(\mathbb{R}^d)$ is isomorphic to $L^\infty(\mathbb{R}^d)$, we conclude that

$$\mathbb{1}_{\{|D|\le\frac14\alpha\}} m_1, \quad \mathbb{1}_{\{|D|\le\frac14\alpha\}} m_2^\pm \quad \text{and} \quad \mathbb{1}_{\{|D|\le\frac14\alpha\}} m_3$$

are bounded in the sense that they satisfy (2-12) for $p = 2$.

Therefore, applying Propositions 2.2 and 2.4 to the representation formulas (2-33) by suitably scaling time by A/α, we obtain the estimates

$$\begin{aligned}
\|\mathbb{1}_{\{|D|\le\frac14\alpha\}} e\|_{L^m_t \dot B^{\sigma+2/m}_{2,q}} &\lesssim \alpha^{\frac1m-2}\|\mathbb{1}_{\{|D|\le\frac14\alpha\}} g\|_{\dot B^{\sigma+2}_{2,m}} + \alpha^{-\frac1m}\|g\|_{\dot B^{\sigma+2/m}_{2,q}} + \alpha^{\frac1m-1}\|f\|_{\dot B^{\sigma+2}_{2,m}} \\
&\quad + \alpha^{\frac1m-\frac1r-1}\|\mathbb{1}_{\{|D|\le\frac14\alpha\}} F\|_{L^r_t \dot B^{\sigma+2/r}_{2,\infty}} + \alpha^{-(1+\frac1m-\frac1r)}\|F\|_{L^r_t \dot B^{\sigma+2/m}_{2,q}} \\
&\lesssim \alpha^{-\frac1m}\|g\|_{\dot B^{\sigma+2/m}_{2,q}} + \alpha^{\frac1m-1}\|f\|_{\dot B^{\sigma+2}_{2,m}} + \alpha^{-(1+\frac1m-\frac1r)}\|F\|_{L^r_t \dot B^{\sigma+2/m}_{2,q}}
\end{aligned} \tag{2-35}$$

and

$$\|\mathbb{1}_{\{|D|\le\frac14\alpha\}} b\|_{L^m_t \dot B^{\sigma+2/m}_{2,1}} \lesssim \alpha^{\frac1m-1}\|g\|_{\dot B^{\sigma+1}_{2,m}} + \alpha^{\frac1m}\|f\|_{\dot B^{\sigma+1}_{2,m}} + \alpha^{\frac1m-\frac1r}\|F\|_{L^r_t \dot B^{\sigma-1+2/r}_{2,\infty}} \tag{2-36}$$

for any $1 < r < m < \infty$ and $1 \le q \le \infty$.

If, instead of Proposition 2.4, one uses Proposition 2.6, then one arrives at the estimates

$$\|\mathbb{1}_{\{|D|\le\frac14\alpha\}} e\|_{L^m_t \dot B^{\sigma+2/m}_{2,q}} \lesssim \alpha^{-\frac1m}\|g\|_{\dot B^{\sigma+2/m}_{2,q}} + \alpha^{\frac1m-1}\|f\|_{\dot B^{\sigma+2}_{2,m}} + \alpha^{-1}\|F\|_{L^m_t \dot B^{\sigma+2/m}_{2,q}} \tag{2-37}$$

and

$$\|\mathbb{1}_{\{|D|\le\frac14\alpha\}} b\|_{L^m_t \dot B^{\sigma+2/m}_{2,q}} \lesssim \alpha^{\frac1m-1}\|g\|_{\dot B^{\sigma+1}_{2,m}} + \alpha^{\frac1m}\|f\|_{\dot B^{\sigma+1}_{2,m}} + \|F\|_{L^m_t \dot B^{\sigma-1+2/m}_{2,q}} \tag{2-38}$$

for any $1 < m < \infty$ and $1 \le q \le \infty$.

In order to handle frequencies lying in the range $\{\frac14\alpha < |\xi| \le \alpha R\}$ for any choice of parameter $R > 1$ with $2AR^2 < 1$, we employ (2-20) and (2-21) to deduce that the multipliers in (2-33) satisfy

$$\begin{aligned}
&\|m_1 \mathbb{1}_{\{\frac14\alpha<|\xi|\le\alpha R\}}\|_{L^\infty_{t,\xi}} \lesssim \alpha^{-1}, \\
&\|m_2 \mathbb{1}_{\{\frac14\alpha<|\xi|\le\alpha R\}}\|_{L^\infty_{t,\xi}} \le \|\lambda_- m_1 \mathbb{1}_{\{\frac14\alpha<|\xi|\le\alpha R\}}\|_{L^\infty_{t,\xi}} + \|e^{t(\lambda_+ + A\frac{|\xi|^2}{\alpha})}\mathbb{1}_{\{\frac14\alpha<|\xi|\le\alpha R\}}\|_{L^\infty_{t,\xi}} \lesssim 1, \\
&\|m_3 \mathbb{1}_{\{\frac14\alpha<|\xi|\le\alpha R\}}\|_{L^\infty_{t,\xi}} \le \|\lambda_+ m_1 \mathbb{1}_{\{\frac14\alpha<|\xi|\le\alpha R\}}\|_{L^\infty_{t,\xi}} + \|e^{t(\lambda_+ + A\frac{|\xi|^2}{\alpha})}\mathbb{1}_{\{\frac14\alpha<|\xi|\le\alpha R\}}\|_{L^\infty_{t,\xi}} \lesssim 1.
\end{aligned}$$

Therefore, as previously, by the boundedness of multipliers and the fact that $\|e^{-at}\|_{L^p_t([0,\infty))} = a^{-1/p}$ for any $a > 0$ and $1 \le p \le \infty$ (no need to use Propositions 2.2, 2.4 or 2.6, here), we conclude from (2-33) that

$$\begin{aligned}
&\|\mathbb{1}_{\{\frac14\alpha<|D|\le\alpha R\}} e\|_{L^m_t \dot B^{\sigma+2/m}_{2,q}} \\
&\qquad \lesssim \alpha^{-\frac1m}\|g\|_{\dot B^{\sigma+2/m}_{2,q}} + \alpha^{-\frac1m-1}\|\mathbb{1}_{\{|D|\le\alpha R\}} f\|_{\dot B^{\sigma+2+2/m}_{2,q}} + \alpha^{-(1+\frac1m-\frac1r)}\|F\|_{L^r_t \dot B^{\sigma+2/m}_{2,q}} \\
&\qquad \lesssim \alpha^{-\frac1m}\|g\|_{\dot B^{\sigma+2/m}_{2,q}} + \alpha^{\frac1m-1}\|f\|_{\dot B^{\sigma+2}_{2,m}} + \alpha^{-(1+\frac1m-\frac1r)}\|F\|_{L^r_t \dot B^{\sigma+2/m}_{2,q}}
\end{aligned} \tag{2-39}$$

and

$$
\begin{aligned}
&\|\mathbb{1}_{\{\frac14\alpha<|D|\le\alpha R\}}b\|_{L^m_t\dot B^{\sigma+2/m}_{2,q}} \\
&\quad\lesssim \alpha^{-\frac1m-1}\|\mathbb{1}_{\{|D|\le\alpha R\}}g\|_{\dot B^{\sigma+1+2/m}_{2,q}}+\alpha^{-\frac1m}\|\mathbb{1}_{\{|D|\le\alpha R\}}f\|_{\dot B^{\sigma+1+2/m}_{2,q}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad+\alpha^{-2-\frac1m+\frac1r}\|\mathbb{1}_{\{|D|\le\alpha R\}}F\|_{L^r_t\dot B^{\sigma+1+2/m}_{2,q}} \\
&\quad\lesssim \alpha^{\frac1m-1}\|g\|_{\dot B^{\sigma+1}_{2,m}}+\alpha^{\frac1m}\|f\|_{\dot B^{\sigma+1}_{2,m}}+\alpha^{\frac1m-\frac1r}\|F\|_{L^r_t\dot B^{\sigma-1+2/r}_{2,\infty}}
\end{aligned}
\tag{2-40}
$$

for any $1\le r\le m<\infty$ and $1\le q\le\infty$.

All in all, combining (2-35), (2-37) and (2-39), we obtain

$$
\|\mathbb{1}_{\{|D|\le\alpha R\}}e\|_{L^m_t\dot B^{\sigma+2/m}_{2,q}}\lesssim \alpha^{-\frac1m}\|g\|_{\dot B^{\sigma+2/m}_{2,q}}+\alpha^{\frac1m-1}\|f\|_{\dot B^{\sigma+2}_{2,m}}+\alpha^{-(1+\frac1m-\frac1r)}\|F\|_{L^r_t\dot B^{\sigma+2/m}_{2,q}}
$$

for any $1<r\le m<\infty$ and $1\le q\le\infty$. Similarly, combining (2-36), (2-38) and (2-40), we deduce that

$$
\|\mathbb{1}_{\{|D|\le\alpha R\}}b\|_{L^m_t\dot B^{\sigma+2/m}_{2,1}}\lesssim \alpha^{\frac1m-1}\|g\|_{\dot B^{\sigma+1}_{2,m}}+\alpha^{\frac1m}\|f\|_{\dot B^{\sigma+1}_{2,m}}+\alpha^{\frac1m-\frac1r}\|F\|_{L^r_t\dot B^{\sigma-1+2/r}_{2,\infty}}
$$

for any $1<r<m<\infty$ and

$$
\|\mathbb{1}_{\{|D|\le\alpha R\}}b\|_{L^m_t\dot B^{\sigma+2/m}_{2,q}}\lesssim \alpha^{\frac1m-1}\|g\|_{\dot B^{\sigma+1}_{2,m}}+\alpha^{\frac1m}\|f\|_{\dot B^{\sigma+1}_{2,m}}+\|F\|_{L^m_t\dot B^{\sigma-1+2/m}_{2,q}}
$$

for any $1<m<\infty$ and $1\le q\le\infty$.

Finally, selecting R large enough that $\operatorname{supp}\chi\subset\{|\xi|\le R\}$ yields the desired estimates on $\chi(\alpha^{-1}D)(e,b)$, thereby concluding the proof. □

Remark. The preceding proof raises a question — is it possible to extend the statement of Proposition 2.13 from the L^2-setting (in space integrability) to a general L^p-setting, with $p\neq 2$? Such an extension would require dealing with the boundedness of the multipliers defined in (2-34) over L^p. This is related to the boundedness of the Bochner–Riesz multiplier $(1-|\xi|^2)^{1/2}_+$, which is notoriously challenging and remains unsettled in general dimensions. We will therefore not be going into further detail on this subject.

Corollary 2.14. *Let $d=2$ or $d=3$, and consider a solution $(E,B)(t,x):[0,T)\times\mathbb{R}^d\to\mathbb{R}^6$ of the damped Maxwell system*

$$
\begin{cases}
\frac1c\partial_tE-\nabla\times B+\sigma cE=G,\\
\frac1c\partial_tB+\nabla\times E=0,\\
\operatorname{div}B=0
\end{cases}
$$

for some initial data $(E,B)(0,x)=(E_0,B_0)(x)$, where $\sigma>0$ and $c>0$.

For any $\chi\in C^\infty_c(\mathbb{R}^d)$ and $s\in\mathbb{R}$, one has the low-frequency estimates

$$
\begin{aligned}
&\|\chi(c^{-1}D)PE\|_{L^m_t([0,T);\dot B^{s+2/m}_{2,q})} \\
&\qquad\qquad\lesssim c^{-\frac2m}\|PE_0\|_{\dot B^{s+2/m}_{2,q}}+c^{-1}\|B_0\|_{\dot B^{s+1}_{2,m}}+c^{-1+\frac2r-\frac2m}\|PG\|_{L^r_t([0,T);\dot B^{s+2/m}_{2,q})}
\end{aligned}
$$

for any $1<r\le m<\infty$ and $1\le q\le\infty$, as well as

$$
\|\chi(c^{-1}D)B\|_{L^m_t([0,T);\dot B^{s+2/m}_{2,1})}\lesssim c^{-1}\|PE_0\|_{\dot B^{s+1}_{2,m}}+\|B_0\|_{\dot B^{s}_{2,m}}+\|PG\|_{L^r_t([0,T);\dot B^{s-1+2/r}_{2,\infty})}
$$

for any $1 < r < m < \infty$, *and*

$$\|\chi(c^{-1}D)B\|_{L^m_t([0,T);\dot B^{s+2/m}_{2,q})} \lesssim c^{-1}\|PE_0\|_{\dot B^{s+1}_{2,m}} + \|B_0\|_{\dot B^{s}_{2,m}} + \|PG\|_{L^m_t([0,T);\dot B^{s-1+2/m}_{2,q})}$$

for any $1 < m < \infty$ *and* $1 \le q \le \infty$.

Proof. We follow the steps of the proof of Corollary 2.12, which involves first fixing a damped Lorenz gauge satisfying (2-28), (2-29) and (2-31). Then, applying Proposition 2.13 instead of Corollary 2.11 to the damped wave system (2-32), we find, for any $\chi \in C^\infty_c(\mathbb{R}^d)$ and $s \in \mathbb{R}$, that

$$\begin{aligned}\|\chi(c^{-1}D)PE\|_{L^m_t([0,T);\dot B^{s+2/m}_{2,q})} &= c^{-\frac{1}{m}}\|\chi(c^{-1}D)\partial_t PA\|_{L^m_t([0,cT);\dot B^{s+2/m}_{2,q})}\\ &\lesssim c^{-\frac{2}{m}}\|PE_0\|_{\dot B^{s+2/m}_{2,q}} + c^{-1}\|B_0\|_{\dot B^{s+1}_{2,m}} + c^{-1+\frac{2}{r}-\frac{2}{m}}\|PG\|_{L^r_t([0,T);\dot B^{s+2/m}_{2,q})},\end{aligned}$$

for any $1 < r \le m < \infty$ and $1 \le q \le \infty$. We also obtain

$$\begin{aligned}\|\chi(c^{-1}D)B\|_{L^m_t([0,T);\dot B^{s+2/m}_{2,1})} &\lesssim c^{-\frac{1}{m}}\|\chi(c^{-1}D)\nabla PA\|_{L^m_t([0,cT);\dot B^{s+2/m}_{2,1})}\\ &\lesssim c^{-1}\|PE_0\|_{\dot B^{s+1}_{2,m}} + \|B_0\|_{\dot B^{s}_{2,m}} + \|PG\|_{L^r_t([0,T);\dot B^{s-1+2/r}_{2,\infty})},\end{aligned}$$

for any $1 < r < m < \infty$, whereas the limiting case $1 < r = m < \infty$ yields

$$\begin{aligned}\|\chi(c^{-1}D)B\|_{L^m_t([0,T);\dot B^{s+2/m}_{2,q})} &\lesssim c^{-\frac{1}{m}}\|\chi(c^{-1}D)\nabla PA\|_{L^m_t([0,cT);\dot B^{s+2/m}_{2,q})}\\ &\lesssim c^{-1}\|PE_0\|_{\dot B^{s+1}_{2,m}} + \|B_0\|_{\dot B^{s}_{2,m}} + \|PG\|_{L^m_t([0,T);\dot B^{s-1+2/m}_{2,q})}\end{aligned}$$

for any $1 \le q \le \infty$, which concludes the proof. □

3. Perfect incompressible two-dimensional plasmas

We are now going to apply the damped Strichartz estimates for Maxwell's system, established in the preceding section, to the analysis of the two-dimensional incompressible Euler–Maxwell system (1-1). The main goal of this section is to establish Theorems 3.1, 3.2 and 3.3 below.

In order to conveniently state the results, recall first that we denote the initial energy by

$$\mathcal{E}_0 := \|(u_0, E_0, B_0)\|_{L^2}.$$

For ease of notation, we also introduce a natural frequency decomposition of Besov and Chemin–Lerner spaces with respect to the speed of light $c > 0$. More precisely, we define the Besov seminorms

$$\|f\|_{\dot B^s_{p,q,<}} := \Bigg(\sum_{\substack{k\in\mathbb{Z}\\ 2^k<\sigma c}} 2^{ksq}\|\Delta_k f\|^q_{L^p}\Bigg)^{\frac{1}{q}} \quad\text{and}\quad \|f\|_{\dot B^s_{p,q,>}} := \Bigg(\sum_{\substack{k\in\mathbb{Z}\\ 2^k\ge\sigma c}} 2^{ksq}\|\Delta_k f\|^q_{L^p}\Bigg)^{\frac{1}{q}},$$

as well as the Chemin–Lerner seminorms

$$\|f\|_{\widetilde L^r_t\dot B^s_{p,q,<}} := \Bigg(\sum_{\substack{k\in\mathbb{Z}\\ 2^k<\sigma c}} 2^{ksq}\|\Delta_k f\|^q_{L^r_tL^p_x}\Bigg)^{\frac{1}{q}} \quad\text{and}\quad \|f\|_{\widetilde L^r_t\dot B^s_{p,q,>}} := \Bigg(\sum_{\substack{k\in\mathbb{Z}\\ 2^k\ge\sigma c}} 2^{ksq}\|\Delta_k f\|^q_{L^r_tL^p_x}\Bigg)^{\frac{1}{q}}$$

for any $s \in \mathbb{R}$ and $0 < p, q, r \leq \infty$ (with obvious modifications if q is infinite), where the constant $\sigma > 0$ is the electrical conductivity used in the original Euler–Maxwell system (1-1).

Theorem 3.1. *Let p and ε be any real numbers in $(2, \infty)$ and $(0, 1)$, respectively. There is a constant $C_* > 0$ such that, if the initial data (u_0, E_0, B_0), with $\operatorname{div} u_0 = \operatorname{div} E_0 = \operatorname{div} B_0$, has the two-dimensional normal structure (1-2) and belongs to $((H^1 \cap \dot{W}^{1,p}) \times (H^1 \cap \dot{B}^{7/4}_{2,1})^2)(\mathbb{R}^2)$ with*

$$(\mathcal{E}_0 + \|u_0\|_{\dot{H}^1 \cap \dot{W}^{1,p}} + \|(E_0, B_0)\|_{\dot{H}^1} + c^{-\frac{3}{4}} \|(E_0, B_0)\|_{\dot{B}^{7/4}_{2,1}}) C_* e^{C_* \mathcal{E}_0^{4+\varepsilon}} < c, \tag{3-1}$$

where $c > 0$ is the speed of light, then there is a global weak solution $(u, E, B) \in L^\infty(\mathbb{R}^+; L^2)$ to the two-dimensional Euler–Maxwell system (1-1), with the normal structure (1-2), satisfying the energy inequality (1-3) and enjoying the additional regularity

$$\begin{aligned} u \in L^\infty(\mathbb{R}^+; \dot{H}^1 \cap \dot{W}^{1,p}), \quad (E, B) \in L^\infty(\mathbb{R}^+; \dot{H}^1), \quad c^{-\frac{3}{4}}(E, B) \in \widetilde{L}^\infty(\mathbb{R}^+; \dot{B}^{7/4}_{2,1}), \\ (cE, B) \in L^2(\mathbb{R}^+; \dot{H}^1), \quad B \in L^2(\mathbb{R}^+; \dot{B}^2_{2,1,<}), \\ (E, B) \in \widetilde{L}^2(\mathbb{R}^+; \dot{B}^1_{\infty,1,>}), \quad c^{\frac{1}{4}} E \in \widetilde{L}^2(\mathbb{R}^+; \dot{B}^{7/4}_{2,1}), \quad c^{\frac{1}{4}} B \in \widetilde{L}^2(\mathbb{R}^+; \dot{B}^{7/4}_{2,1,>}). \end{aligned} \tag{3-2}$$

It is to be emphasized that the bounds in (3-2) are uniform in any set of initial data such that the left-hand side of (3-1) remains bounded.

Remark. For any fixed initial data (u_0, E_0, B_0) satisfying the requirements of Theorem 3.1, it is possible to improve the uniform controls (3-2) by showing that the bound

$$(E, B) \in \widetilde{L}^\infty(\mathbb{R}^+; \dot{B}^{7/4}_{2,1})$$

holds uniformly as $c \to \infty$. This is clarified in Section 3.11 below.

Remark. Note that we do not make any claim concerning the uniqueness of solutions produced by Theorem 3.1. However, Theorem 3.2 below strengthens the statement of Theorem 3.1 by achieving such uniqueness, provided the initial vorticity is bounded pointwise.

Remark. Employing the bounds (3-2), one can easily show that $(\nabla E, \nabla B) \in L^2(\mathbb{R}^+; L^\infty)$. Indeed, making use of straightforward embeddings in Besov and Chemin–Lerner spaces, we deduce that

$$\begin{aligned} \|\nabla E\|_{L^2(\mathbb{R}^+; L^\infty)} \leq \|\nabla E\|_{\widetilde{L}^2(\mathbb{R}^+; \dot{B}^0_{\infty,1})} &\lesssim \|E\|_{\widetilde{L}^2(\mathbb{R}^+; \dot{B}^1_{\infty,1,<})} + \|E\|_{\widetilde{L}^2(\mathbb{R}^+; \dot{B}^1_{\infty,1,>})} \\ &\lesssim \|cE\|_{\widetilde{L}^2(\mathbb{R}^+; \dot{B}^0_{\infty,\infty,<})} + \|E\|_{\widetilde{L}^2(\mathbb{R}^+; \dot{B}^1_{\infty,1,>})} \\ &\lesssim \|cE\|_{\widetilde{L}^2(\mathbb{R}^+; \dot{B}^1_{2,\infty,<})} + \|E\|_{\widetilde{L}^2(\mathbb{R}^+; \dot{B}^1_{\infty,1,>})} \end{aligned}$$

and

$$\begin{aligned} \|\nabla B\|_{L^2(\mathbb{R}^+; L^\infty)} \leq \|\nabla B\|_{L^2(\mathbb{R}^+; \dot{B}^0_{\infty,1})} &\lesssim \|B\|_{L^2(\mathbb{R}^+; \dot{B}^1_{\infty,1,<})} + \|B\|_{\widetilde{L}^2(\mathbb{R}^+; \dot{B}^1_{\infty,1,>})} \\ &\lesssim \|B\|_{L^2(\mathbb{R}^+; \dot{B}^2_{2,1,<})} + \|B\|_{\widetilde{L}^2(\mathbb{R}^+; \dot{B}^1_{\infty,1,>})}. \end{aligned} \tag{3-3}$$

Remark. The initial condition (3-1) can be interpreted as a mere strengthening of the property that the velocity of the fluid cannot exceed the speed of light, i.e.,

$$\|u\|_{L^\infty_{t,x}} \le c.$$

On purely physical grounds, this condition seems therefore quite reasonable and is not very restrictive.

The previous result only covers the case $\omega_0 \in L^2 \cap L^p$ in the range of parameters $p \in (2,\infty)$. The next result strengthens Theorem 3.1 by assuming that $\omega_0 \in L^2 \cap L^\infty$.

Theorem 3.2. *If, in addition to all hypotheses of Theorem 3.1, for some given $p \in (2,\infty)$, one also assumes that $\omega_0 \in L^\infty$ (but not necessarily $\nabla u_0 \in L^\infty$), then the solution produced by Theorem 3.1 satisfies the additional bound $\omega \in L^\infty(\mathbb{R}^+; L^\infty)$ and is unique in the space of all solutions $(\bar u, \bar E, \bar B)$ to the Euler–Maxwell system (1-1) satisfying the bounds, locally in time,*

$$(\bar u, \bar E, \bar B) \in L^\infty_t L^2_x, \quad \bar u \in L^2_t L^\infty_x, \quad \bar j \in L^2_{t,x},$$

and having the same initial data.

We address the propagation of regularity in the Euler–Maxwell system (1-1) with the following result.

Theorem 3.3. *Consider parameters $p \in (2,\infty)$, $\varepsilon \in (0,1)$, $s \in \left(\frac{7}{4}, 2\right)$ and $n \in [1,\infty]$. There is a constant $C_{**} > 0$ such that, if the initial data (u_0, E_0, B_0), with $\operatorname{div} u_0 = \operatorname{div} E_0 = \operatorname{div} B_0$, has the two-dimensional normal structure (1-2) and belongs to $((H^1 \cap \dot W^{1,p}) \times (H^1 \cap \dot B^s_{2,n})^2)(\mathbb{R}^2)$ with*

$$\left(\mathcal{E}_0 + \|u_0\|_{\dot H^1 \cap \dot W^{1,p}} + \|(E_0,B_0)\|_{\dot H^1} + c^{1-s}\|(E_0,B_0)\|_{\dot B^s_{2,n}}\right) C_{**} e^{C_{**}\mathcal{E}_0^{4+\varepsilon}} < c, \tag{3-4}$$

where $c > 0$ is the speed of light, then there is a global weak solution $(u, E, B) \in L^\infty(\mathbb{R}^+; L^2)$ to the two-dimensional Euler–Maxwell system (1-1), with the normal structure (1-2), satisfying the energy inequality (1-3) and enjoying the additional regularity

$$\begin{gathered} u \in L^\infty(\mathbb{R}^+; \dot H^1 \cap \dot W^{1,p}), \quad (E,B) \in L^\infty(\mathbb{R}^+; \dot H^1), \quad c^{1-s}(E,B) \in \widetilde L^\infty(\mathbb{R}^+; \dot B^s_{2,n}), \\ (cE, B) \in L^2(\mathbb{R}^+; \dot H^1), \quad B \in L^2(\mathbb{R}^+; \dot B^2_{2,1,<}), \\ c^{\frac{7}{4}-s}(E,B) \in \widetilde L^2(\mathbb{R}^+; \dot B^{s-3/4}_{\infty,n,>}), \quad c^{2-s} E \in \widetilde L^2(\mathbb{R}^+; \dot B^s_{2,n}), \quad c^{2-s} B \in \widetilde L^2(\mathbb{R}^+; \dot B^s_{2,n,>}). \end{gathered} \tag{3-5}$$

It is to be emphasized that the bounds in (3-5) are uniform in any set of initial data such that the left-hand side of (3-4) remains bounded.

Remark. As in the case of Theorem 3.1, for any fixed initial data (u_0, E_0, B_0) satisfying the requirements of Theorem 3.3, it is possible to improve the uniform controls (3-5) by showing that the bound

$$(E,B) \in \widetilde L^\infty(\mathbb{R}^+; \dot B^s_{2,n})$$

holds uniformly as $c \to \infty$. This is clarified in Section 3.11 below.

The remainder of this section builds up to the proofs of Theorems 3.1, 3.2 and 3.3 by implementing the strategy discussed in Section 1.3. The proofs of the theorems per se are given in Sections 3.8, 3.9 and 3.10, respectively.

3.1. ***Dimensional analysis.*** Prior to discussing specific elements of the proofs of the above theorems, we provide here a dimensional analysis of the Euler–Maxwell system (1-1), which, we hope, will shed light on the initial conditions (3-1) and (3-4).

Specifically, assuming that (u, E, B) is a solution of (1-1) for some fixed light velocity $c > 0$ and initial data (u_0, E_0, B_0), we observe, defining

$$u^\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x), \quad E^\lambda(t,x) = \lambda E(\lambda^2 t, \lambda x), \quad B^\lambda(t,x) = \lambda B(\lambda^2 t, \lambda x), \tag{3-6}$$

for any $\lambda > 0$, that $(u^\lambda, E^\lambda, B^\lambda)$ also solves (1-1) with a rescaled speed of light $c_\lambda = \lambda c$ (the electrical conductivity σ remains unchanged) and for the initial data

$$u_0^\lambda(x) = \lambda u(\lambda x), \quad E_0^\lambda(x) = \lambda E(\lambda x), \quad B_0^\lambda(x) = \lambda B(\lambda x).$$

In particular, we readily compute

$$\begin{aligned}\|(u_0^\lambda, E_0^\lambda, B_0^\lambda)\|_{L^2} + \|u_0^\lambda\|_{\dot W^{1,p}} + \|(u_0^\lambda, E_0^\lambda, B_0^\lambda)\|_{\dot H^1} + c_\lambda^{-3/4}\|(E_0^\lambda, B_0^\lambda)\|_{\dot B^{7/4}_{2,1}} \\ = \lambda(\lambda^{-1}\|(u_0, E_0, B_0)\|_{L^2} + \lambda^{1-\frac{2}{p}}\|u_0\|_{\dot W^{1,p}} + \|(u_0, E_0, B_0)\|_{\dot H^1} + c^{-\frac{3}{4}}\|(E_0, B_0)\|_{\dot B^{7/4}_{2,1}}),\end{aligned}$$

which, by an optimization procedure in λ, implies that Theorem 3.1 still holds if one replaces assumption (3-1) with the weaker inequality

$$(\mathcal{E}_0^{(p-2)/(2p-2)}\|u_0\|_{\dot W^{1,p}}^{p/(2p-2)} + \|(u_0, E_0, B_0)\|_{\dot H^1} + c^{-\frac{3}{4}}\|(E_0, B_0)\|_{\dot B^{7/4}_{2,1}})C_* e^{C_*\mathcal{E}_0^{4+\varepsilon}} < c, \tag{3-7}$$

where the independent constant $C_* > 0$ may take a different value. Note that this inequality is now invariant with respect to the parabolic scaling (3-6). Similarly, the same procedure can be used to optimize (3-4) and replace it with a scaling invariant assumption.

It turns out that the parabolic scaling (3-6) is the only available invariant dilation which leaves the electrical conductivity σ unchanged. However, if one allows σ to be redefined according to the dilation, then other scalings become available. For example, introducing the hyperbolic scaling

$$u^\lambda(t,x) = u(\lambda t, \lambda x), \quad E^\lambda(t,x) = E(\lambda t, \lambda x), \quad B^\lambda(t,x) = B(\lambda t, \lambda x) \tag{3-8}$$

for any $\lambda > 0$, we see that $(u^\lambda, E^\lambda, B^\lambda)$ now solves (1-1) with a rescaled electrical conductivity $\sigma_\lambda = \lambda\sigma$ (the speed of light c remains unchanged) and for the initial data

$$u_0^\lambda(x) = u(\lambda x), \quad E_0^\lambda(x) = E(\lambda x), \quad B_0^\lambda(x) = B(\lambda x).$$

In particular, by setting $\lambda = \sigma^{-1}$, it is now possible to deduce how the constants C_* and C_{**} in (3-1) and (3-4), respectively, depend on σ. More precisely, this process allows us to show that Theorem 3.1 holds if one further replaces (3-7) by the assumption

$$(\mathcal{E}_0^{(p-2)/(2p-2)}\|u_0\|_{\dot W^{1,p}}^{p/(2p-2)} + \|(u_0, E_0, B_0)\|_{\dot H^1} + (\sigma c)^{-\frac{3}{4}}\|(E_0, B_0)\|_{\dot B^{7/4}_{2,1}})C_* e^{C_*\sigma^{4+\varepsilon}\mathcal{E}_0^{4+\varepsilon}} < c \tag{3-9}$$

for some constant $C_* > 0$ which is now independent of the electrical conductivity σ. Observe that (3-9) is now invariant with respect to both the parabolic scaling (3-6) and the hyperbolic scaling (3-8). As previously noted, the same process can be applied to improve (3-4).

3.2. ***Approximation procedure and stability.*** The proofs of Theorems 3.1, 3.2 and 3.3 proceed by compactness arguments. More specifically, they follow the standard procedure of first considering smooth solutions to a regularized approximation of the original system (1-1), where all formal estimates can be conducted with full rigor, and then showing the stability of the approximation as it converges towards the original system.

Such approximation procedures are absolutely classical in the field of fluid dynamics. We are therefore only going to outline an example of approximation which can be used here to conveniently establish our results. Specifically, for any integer $n \geq 1$, we consider the unique solution (u_n, E_n, B_n) to the approximate Navier–Stokes–Maxwell system

$$\begin{cases} \partial_t u_n + (S_n u_n) \cdot \nabla u_n - \frac{1}{n}\Delta u_n = -\nabla p_n + (S_n j_n) \times B_n, & \operatorname{div} u_n = 0, \\ \frac{1}{c}\partial_t E_n - \nabla \times B_n = -j_n, & \operatorname{div} E_n = 0, \\ \frac{1}{c}\partial_t B_n + \nabla \times E_n = 0, & \operatorname{div} B_n = 0, \\ j_n = \sigma(cE_n + S_n P(u_n \times B_n)), & \operatorname{div} j_n = 0, \end{cases} \tag{3-10}$$

for the initial data $(u_n, E_n, B_n)|_{t=0} = S_n(u_0, E_0, B_0)$, with the two-dimensional normal structure (1-2), where S_n denotes the Fourier multiplier operator defined in Appendix A which restricts frequencies to the domain $\{|\xi| \leq 2^n\}$. The construction of the solution (u_n, E_n, B_n) is a standard procedure. One can, for instance, follow and adapt the steps detailed in [Lemarié-Rieusset 2016, Section 12.2].

Note that other approximation schemes can be employed. In particular, the dissipation term $-(1/n)\Delta u_n$ is not essential. However, as a matter of convenience, the use of this term allows us to comfortably construct the approximate solution (u_n, E_n, B_n) by relying on methods from the analysis of the incompressible Navier–Stokes system.

Observe that the corresponding energy inequality

$$\frac{1}{2}(\|u_n(t)\|_{L^2}^2 + \|E_n(t)\|_{L^2}^2 + \|B_n(t)\|_{L^2}^2) + \int_0^t \left(\frac{1}{n}\|\nabla u_n(\tau)\|_{L^2}^2 + \frac{1}{\sigma}\|j_n(\tau)\|_{L^2}^2\right) d\tau \\ \leq \frac{1}{2}\|S_n(u_0, E_0, B_0)\|_{L^2}^2 \leq \frac{1}{2}\|(u_0, E_0, B_0)\|_{L^2}^2$$

for all $t \geq 0$ is now fully justified and, since the initial data is smooth, it is possible to show that (u_n, E_n, B_n) remains smooth for all times, albeit not uniformly in n.

The above energy inequality only allows us to deduce the uniform bounds

$$(u_n, E_n, B_n) \in L_t^\infty L_x^2 \quad \text{and} \quad j_n \in L_{t,x}^2,$$

which are insufficient to establish the stability of the nonlinear terms

$$(S_n u_n) \cdot \nabla u_n, \quad (S_n j_n) \times B_n \quad \text{and} \quad S_n P(u_n \times B_n)$$

in the limit $n \to \infty$. The general strategy is therefore to show that the bounds and properties stated in Theorems 3.1, 3.2 and 3.3 can be fully justified on the smooth system (3-10) uniformly in n. Such uniform bounds are then sufficient to show the strong relative compactness of $\{(u_n, E_n, B_n)\}_{n=1}^\infty$ in $L^2_{t,x,\mathrm{loc}}$, which then allows us to take the limit $n \to \infty$ (up to extraction of subsequences) and establish the asymptotic stability of (3-10), thereby yielding suitable solutions of the original Euler–Maxwell system (1-1).

In what follows, for the sake of simplicity, keeping in mind that all computations can be fully justified on the approximate system (3-10), we shall perform all estimates formally on the original system (1-1). In particular, we emphasize that, even though, strictly speaking, the dissipation operator $-(1/n)\Delta$ cannot be ignored, it is self-adjoint and therefore will not impact the energy estimates which are performed in the proofs below.

3.3. *Paradifferential calculus and the normal structure.* The use of Besov and Chemin–Lerner spaces in the analysis of nonlinear systems often requires a careful use of product estimates. Such results from paradifferential calculus are rather standard, but their applicability is limited. In the following paradifferential lemma, we show how the normal structure (1-2) can be exploited to extend the range of applicability of classical product estimates. This plays a central role in our analysis of (1-1).

Lemma 3.4. *Let $F, G : \mathbb{R}_t \times \mathbb{R}^2_x \to \mathbb{R}^3$ be solenoidal vector fields with the normal structure*

$$F(t,x_1,x_2) = \begin{pmatrix} F_1(t,x_1,x_2) \\ F_2(t,x_1,x_2) \\ 0 \end{pmatrix} \quad \text{and} \quad G(t,x_1,x_2) = \begin{pmatrix} 0 \\ 0 \\ G_3(t,x_1,x_2) \end{pmatrix}. \tag{3-11}$$

Further consider integrability parameters in $[1,\infty]$ such that

$$\frac{1}{a} = \frac{1}{a_1} + \frac{1}{a_2} \quad \text{and} \quad \frac{1}{c} = \frac{1}{c_1} + \frac{1}{c_2}.$$

Then, recalling that $P = (-\Delta)^{-1} \operatorname{curl}\operatorname{curl}$ denotes Leray's projector onto solenoidal vector fields, one has the product estimate

$$\|P(F \times G)\|_{\widetilde{L}^a_t \dot{B}^{s+t-1}_{2,c}(\mathbb{R}^2)} \lesssim \|F\|_{\widetilde{L}^{a_1}_t \dot{B}^s_{2,c_1}(\mathbb{R}^2)} \|G\|_{\widetilde{L}^{a_2}_t \dot{B}^t_{2,c_2}(\mathbb{R}^2)} \tag{3-12}$$

for any $s \in (-\infty, 1)$ and $t \in (-\infty, 2)$, with $s + t > 0$. Furthermore, in the endpoint case $s = 1$, one has

$$\|P(F \times G)\|_{\widetilde{L}^a_t \dot{B}^t_{2,c}(\mathbb{R}^2)} \lesssim \|F\|_{L^{a_1}_t L^\infty_x(\mathbb{R}^2) \cap \widetilde{L}^{a_1}_t \dot{B}^1_{2,\infty}(\mathbb{R}^2)} \|G\|_{\widetilde{L}^{a_2}_t \dot{B}^t_{2,c}(\mathbb{R}^2)} \tag{3-13}$$

for any $t \in (-1, 2)$.

Remark. The significance of the preceding lemma lies in the fact that it allows us to consider parameters in the range $t \in [1, 2)$. Without the normal structure (3-11), we would be restricted to values $t < 1$.

Remark. A straightforward simplification of the proof below yields a corresponding result in Besov spaces for vector fields independent of the time variable t.

Proof. We are going to use the paradifferential decomposition

$$F \times G = T_F G - T_G F + R(F, G),$$

where the paraproducts are defined by

$$T_F G = \sum_{j \in \mathbb{Z}} S_{j-2} F \times \Delta_j G,$$

$$T_G F = \sum_{j \in \mathbb{Z}} S_{j-2} G \times \Delta_j F = -\sum_{j \in \mathbb{Z}} \Delta_j F \times S_{j-2} G$$

and the remainder is given by

$$R(F,G)=\sum_{\substack{j,k\in\mathbb{Z}\\|j-k|\le 2}}\Delta_j F\times\Delta_k G.$$

In particular, by virtue of the solenoidal and normal structures of F and G, one has

$$\nabla\times(F\times G)=F\times(\nabla\times G),$$

which leads to the identity

$$\nabla\times(F\times G)=\nabla\times T_F G-T_{\nabla\times G}F+\nabla\times R(F,G).$$

We therefore conclude, by standard embeddings of Besov spaces, that

$$\begin{aligned}\|P(F\times G)\|_{\tilde L^a_t\dot B^{s+t-1}_{2,c}}&\lesssim\|\nabla\times(F\times G)\|_{\tilde L^a_t\dot B^{s+t-2}_{2,c}}\\&\lesssim\|T_F G\|_{\tilde L^a_t\dot B^{s+t-1}_{2,c}}+\|T_{\nabla\times G}F\|_{\tilde L^a_t\dot B^{s+t-2}_{2,c}}+\|R(F,G)\|_{\tilde L^a_t\dot B^{s+t}_{1,c}}\end{aligned}$$

for every $s,t\in\mathbb{R}$.

Next, employing the classical paradifferential estimates (A-6) and (A-7) presented in the appendix and further exploiting standard embeddings of Besov spaces, we find that

$$\begin{aligned}&\|P(F\times G)\|_{\tilde L^a_t\dot B^{s+t-1}_{2,c}}\\&\qquad\lesssim\|F\|_{\tilde L^{a_1}_t\dot B^{s-1}_{\infty,c_1}}\|G\|_{\tilde L^{a_2}_t\dot B^{t}_{2,c_2}}+\|F\|_{\tilde L^{a_1}_t\dot B^{s}_{2,c_1}}\|\nabla\times G\|_{\tilde L^{a_2}_t\dot B^{t-2}_{\infty,c_2}}+\|F\|_{\tilde L^{a_1}_t\dot B^{s}_{2,c_1}}\|G\|_{\tilde L^{a_2}_t\dot B^{t}_{2,c_2}}\\&\qquad\lesssim\|F\|_{\tilde L^{a_1}_t\dot B^{s}_{2,c_1}}\|G\|_{\tilde L^{a_2}_t\dot B^{t}_{2,c_2}}\end{aligned}$$

for any $s<1$ and $t<2$, such that $s+t>0$. This establishes (3-12).

As for the endpoint case $s=1$, if, in addition to (A-6), one also uses (A-8), then similar bounds lead to

$$\begin{aligned}&\|P(F\times G)\|_{\tilde L^a_t\dot B^{t}_{2,c}}\\&\qquad\lesssim\|F\|_{L^{a_1}_tL^\infty_x}\|G\|_{\tilde L^{a_2}_t\dot B^{t}_{2,c}}+\|F\|_{\tilde L^{a_1}_t\dot B^{1}_{2,\infty}}\|\nabla\times G\|_{\tilde L^{a_2}_t\dot B^{t-2}_{\infty,c}}+\|F\|_{\tilde L^{a_1}_t\dot B^{1}_{2,\infty}}\|G\|_{\tilde L^{a_2}_t\dot B^{t}_{2,c}}\\&\qquad\lesssim\|F\|_{L^{a_1}_tL^\infty_x\cap\tilde L^{a_1}_t\dot B^{1}_{2,\infty}}\|G\|_{\tilde L^{a_2}_t\dot B^{t}_{2,c}}\end{aligned}$$

for any $-1<t<2$, thereby establishing (3-13) and concluding the proof of the lemma. □

The following ad hoc variant of a paraproduct estimate will be useful when handling vorticities which are bounded pointwise.

Lemma 3.5. *Let $F,G:\mathbb{R}^2_x\to\mathbb{R}^3$ be solenoidal vector fields with the normal structure (3-11). Then, one has the product estimate*

$$\|P(F\times G)\|_{\dot B^1_{2,1}(\mathbb{R}^2)}\lesssim\|F\|_{L^2(\mathbb{R}^2)}\|G\|_{\dot B^1_{\infty,1}(\mathbb{R}^2)}+\|F\|_{\dot H^1(\mathbb{R}^2)}\|G\|_{\dot H^1(\mathbb{R}^2)}.\tag{3-14}$$

Remark. The above statement is phrased in terms of mere Besov spaces. As usual, a straightforward extension of the same result to Chemin–Lerner spaces also exists.

Proof. We employ the method of proof of Lemma 3.4. In particular, we obtain

$$\|P(F\times G)\|_{\dot B^1_{2,1}} \lesssim \|T_F G\|_{\dot B^1_{2,1}} + \|T_{\nabla\times G}F\|_{\dot B^0_{2,1}} + \|R(F,G)\|_{\dot B^1_{2,1}}.$$

Then, by virtue of the classical paradifferential estimates (A-6), (A-7) and (A-8) , we arrive at

$$\begin{aligned}\|P(F\times G)\|_{\dot B^1_{2,1}} &\lesssim \|F\|_{L^2}\|G\|_{\dot B^1_{\infty,1}} + \|\nabla\times G\|_{\dot B^{-1}_{\infty,2}}\|F\|_{\dot B^1_{2,2}} + \|F\|_{\dot B^0_{2,\infty}}\|G\|_{\dot B^1_{\infty,1}}\\ &\lesssim \|F\|_{L^2}\|G\|_{\dot B^1_{\infty,1}} + \|G\|_{\dot B^0_{\infty,2}}\|F\|_{\dot B^1_{2,2}}.\end{aligned}$$

Finally, an application of the two-dimensional embedding $\dot B^1_{2,2}\subset \dot B^0_{\infty,2}$ concludes the proof. □

3.4. *Controlling the vorticity.* In order to carry out our strategy, previously laid out in Section 1.3, we need to control the vorticity ω in L^p_x, with $p\geq 2$, by exploiting Yudovich's approach of the two-dimensional incompressible Euler equations (1-4). The following basic lemma provides us with a simple tool to do so.

Lemma 3.6. *Let $(u,E,B)\in C^1([0,T)\times\mathbb{R}^2)\cap L^\infty([0,T);H^1(\mathbb{R}^2))$ be a smooth solution to (1-1) for some $T>0$, with the two-dimensional normal structure (1-2). Then, for all $t\in(0,T)$,*

$$\begin{aligned}\|\omega(t)\|_{L^2_x} &\leq \|\omega(0)\|_{L^2_x} + \|j\|_{L^2([0,t);L^2_x)}\|\nabla B\|_{L^2([0,t);L^\infty_x)},\\ \|\omega(t)\|_{L^p_x} &\lesssim \|\omega(0)\|_{L^p_x} + \|j\|^{2/p}_{L^2([0,t);\dot B^0_{2,\infty})}\|j\|^{1-2/p}_{L^2([0,t);\dot B^1_{2,\infty})}\|\nabla B\|_{L^2([0,t);L^\infty_x)},\\ \|\omega(t)\|_{L^\infty_x} &\lesssim \|\omega(0)\|_{L^\infty_x} + \|j\|_{L^2([0,t);L^\infty_x)}\|\nabla B\|_{L^2([0,t);L^\infty_x)}\\ &\lesssim \|\omega(0)\|_{L^\infty_x} + \|j\|_{L^2([0,t);\dot B^1_{2,1})}\|\nabla B\|_{L^2([0,t);L^\infty_x)}\end{aligned}$$

for any $2<p<\infty$.

Proof. By slight abuse of language, we assume here that the vorticity ω is defined as the scalar function $\omega=\partial_1u_2-\partial_2u_1$ and that

$$u=\begin{pmatrix}u_1\\u_2\end{pmatrix}\quad\text{and}\quad j=\begin{pmatrix}j_1\\j_2\end{pmatrix}.$$

In particular, a straightforward computation shows that the transport equation (1-16) can then be recast as

$$\partial_t\omega + u\cdot\nabla\omega = -j\cdot\nabla b, \tag{3-15}$$

where $b(t,x)$ is the third component of the magnetic field $B(t,x)$ as defined in (1-2).

Next, supposing, for simplicity, that p is finite and introducing the test function $\varphi(x)=e^{-|x|^2}$, we multiply the above vorticity transport equation by $p\omega|\omega|^{p-2}\varphi(\varepsilon x)$, where $0<\varepsilon<1$, and then integrate in space. This yields, since u is divergence-free,

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^2}|\omega|^p\varphi(\varepsilon x)\,dx &\leq p\int_{\mathbb{R}^2}|j||\nabla b||\omega|^{p-1}\varphi(\varepsilon x)\,dx + \varepsilon\int_{\mathbb{R}^2}|\omega|^p u\cdot\nabla\varphi(\varepsilon x)\,dx\\ &\leq p\|j\|_{L^p_x}\|\nabla b\|_{L^\infty_x}\Big(\int_{\mathbb{R}^2}|\omega|^p\varphi(\varepsilon x)\,dx\Big)^{\frac{p-1}{p}} + \varepsilon\|\omega\|^{p-1}_{L^\infty_x}\|\omega\|_{L^2_x}\|u\|_{L^2_x}\|\nabla\varphi\|_{L^\infty_x},\end{aligned}$$

whereby, if ω is nontrivial,

$$\frac{d}{dt}\left(\int_{\mathbb{R}^2}|\omega|^p\varphi(\varepsilon x)\,dx\right)^{\frac{1}{p}} \le \|j\|_{L^p_x}\|\nabla b\|_{L^\infty_x} + \frac{\varepsilon\|\omega\|_{L^\infty_x}^{p-1}\|\omega\|_{L^2_x}\|u\|_{L^2_x}\|\nabla\varphi\|_{L^\infty_x}}{p\left(\int_{\mathbb{R}^2}|\omega|^p\varphi(\varepsilon x)\,dx\right)^{(p-1)/p}}.$$

Finally, further integrating in time and letting ε tend to zero, we conclude that

$$\|\omega(t)\|_{L^p_x} \le \|\omega(0)\|_{L^p_x} + \|j\|_{L^2([0,t);L^p_x)}\|\nabla b\|_{L^2([0,t);L^\infty_x)}$$

for any $t \in (0,T)$. A classical modification of this argument gives the same result for an infinite value of the parameter p.

Now, if $p = 2$, the proof is finished. If $2 < p < \infty$, we further employ the following convexity inequality (see [Bergh and Löfström 1976] for details on interpolation theory)

$$\|j\|_{L^2([0,t);\dot{H}^{1-2/p})} \lesssim \|j\|_{L^2([0,t);\dot{B}^0_{2,\infty})}^{2/p}\|j\|_{L^2([0,t);\dot{B}^1_{2,\infty})}^{1-2/p},$$

in combination with the classical two-dimensional Sobolev embedding $\dot{H}^{1-2/p} \subset L^p$, to conclude that

$$\|\omega(t)\|_{L^p_x} \lesssim \|\omega(0)\|_{L^p_x} + \|j\|_{L^2([0,t);\dot{B}^0_{2,\infty})}^{2/p}\|j\|_{L^2([0,t);\dot{B}^1_{2,\infty})}^{1-2/p}\|\nabla b\|_{L^2([0,t);L^\infty_x)}.$$

Finally, the case $p = \infty$ is settled with an application of the continuous embeddings $\dot{B}^1_{2,1} \subset \dot{B}^0_{\infty,1} \subset L^\infty$, valid in two dimensions, thereby completing the proof of the lemma. □

Remark. Recall that the approximation procedure presented in Section 3.2 relies on a viscous approximation of the Euler system. It is therefore important to emphasize here that the method of proof of the preceding lemma also applies to viscous approximations of the transport equation (3-15). Indeed, observe that multiplying the transport-diffusion equation

$$\partial_t\omega + u\cdot\nabla\omega - \Delta\omega = -j\cdot\nabla b$$

by $\beta'(\omega)$, for some convex nonnegative renormalization $\beta \in C^2(\mathbb{R})$ with $\beta(0) = 0$, and then integrating in space leads, at least formally, to the estimate

$$\frac{d}{dt}\int_{\mathbb{R}^2}\beta(\omega)\,dx \le \frac{d}{dt}\int_{\mathbb{R}^2}\beta(\omega)\,dx + \int_{\mathbb{R}^2}|\nabla\omega|^2\beta''(\omega)\,dx = \int_{\mathbb{R}^2}(j\cdot\nabla b)\beta'(\omega)\,dx.$$

This observation can be exploited to derive the estimates stated in Lemma 3.6 in spite of the diffusion term. Thus, we conclude that the approximated system (3-10) is well-suited for an application of the estimates from Lemma 3.6.

Recall now that the space $\dot{H}^1(\mathbb{R}^2)$ barely fails to embed into $L^\infty(\mathbb{R}^2)$. Thus, the bounds produced in the preceding lemma will be very useful to recover, by interpolation, an L^∞-bound on the velocity field u, as explained in the following simple result.

Lemma 3.7. *Let (u, E, B) be a smooth solution to (1-1), with the two-dimensional normal structure (1-2). Then, for all $T > 0$,*

$$\|u(T)\|_{L^\infty_x} \lesssim \|u(T)\|_{L^2_x}^{(p-2)/(2(p-1))}\left(\|u(0)\|_{\dot{W}^{1,p}_x} + \|j\|_{L^2_t\dot{B}^0_{2,\infty}}^{2/p}\|j\|_{L^2_t\dot{B}^1_{2,\infty}}^{1-2/p}\|\nabla B\|_{L^2_tL^\infty_x}\right)^{\frac{p}{2(p-1)}}$$

for any $2 < p < \infty$, where all time-norms are taken over the interval $[0,T)$.

Proof. This result follows from the classical Gagliardo–Nirenberg interpolation inequality in two space-dimensions. For the sake of convenience, we provide a brief justification of the precise inequality which is employed here.

Specifically, for any $k \in \mathbb{Z}$ and $2^k \leq R < 2^{k+1}$, we estimate that

$$\|u\|_{L^\infty} \leq \|S_k u\|_{L^\infty} + \sum_{i=k}^{\infty} \|\Delta_i u\|_{L^\infty} \lesssim 2^k \|u\|_{L^2} + \sum_{i=k}^{\infty} 2^{i(\frac{2}{p}-1)} \|\nabla u\|_{L^p} \lesssim R\|u\|_{L^2} + R^{-(1-\frac{2}{p})} \|\nabla u\|_{L^p},$$

which yields, upon optimization of the interpolation parameter value $R > 0$, the Gagliardo–Nirenberg inequality

$$\|u\|_{L^\infty(\mathbb{R}^2)} \lesssim \|u\|_{L^2(\mathbb{R}^2)}^{(p-2)/(2(p-1))} \|\nabla u\|_{L^p(\mathbb{R}^2)}^{p/(2(p-1))} \tag{3-16}$$

for any $2 < p \leq \infty$.

Then, combining this convexity inequality with the estimates from Lemma 3.6 and recalling the equivalence $\|\nabla u\|_{L^p} \sim \|\omega\|_{L^p}$, because u is divergence-free, concludes the proof of the lemma. □

Remark. Note that the proof of (3-16) can be adapted to establish, for any divergence-free field u, that

$$\|u\|_{L^\infty(\mathbb{R}^2)} \lesssim \|u\|_{L^2(\mathbb{R}^2)}^{(p-2)/(2(p-1))} \|\nabla \times u\|_{L^p(\mathbb{R}^2)}^{p/(2(p-1))} \tag{3-17}$$

for all values $2 < p \leq \infty$. Indeed, this follows from the observation that

$$\|\Delta_i u\|_{L^\infty} \lesssim 2^{-i} \|\Delta_i \nabla \times u\|_{L^\infty} \lesssim 2^{i(\frac{2}{p}-1)} \|\nabla \times u\|_{L^p},$$

provided $\operatorname{div} u = 0$.

3.5. ***Control of high-frequency damped electromagnetic waves.*** The following result follows from a simple but careful combination of the damped Strichartz estimates for high electromagnetic frequencies, established in Section 2.3, with the paradifferential product estimates from Lemma 3.4.

Lemma 3.8. *Let $d = 2$. Assume that (E, B) is a smooth solution to (1-15) for some initial data (E_0, B_0) and some divergence-free vector field u, with the normal structure (1-2).*

Then, for any exponents $1 \leq p \leq q \leq \infty$, $2 \leq r \leq \infty$ and $1 \leq n \leq \infty$ which are admissible in the sense that

$$\frac{2}{q} + \frac{1}{r} \geq \frac{1}{2}, \tag{3-18}$$

one has the high-frequency estimate over any time interval $[0, T)$ for any $0 < \alpha < 1$ and $s < 2$, with $\alpha + s > 0$,

$$\begin{aligned}\|(E, B)\|_{\widetilde{L}^q_t \dot{B}^{s+\alpha-7/4+3/(2r)}_{r,n,>}} &\lesssim c^{\frac{1}{2}(\frac{1}{2}-\frac{1}{r})-\frac{2}{q}} \|(E_0, B_0)\|_{\dot{B}^{s+\alpha-1}_{2,n,>}} + c^{\frac{1}{2}(\frac{1}{2}-\frac{1}{r})+2(\frac{1}{p}-\frac{1}{q})-1} \|u\|^{1-\alpha}_{L^\infty_t \dot{B}^0_{2,\infty}} \|u\|^{\alpha}_{L^\infty_t \dot{B}^1_{2,\infty}} \|B\|_{\widetilde{L}^p_t \dot{B}^s_{2,\infty}}.\end{aligned}$$

In the endpoint cases $\alpha = 1$ and $\alpha = 0$, one also has the respective estimates

$$\begin{aligned}\|(E, B)\|_{\widetilde{L}^q_t \dot{B}^{s-(3/2)(1/2-1/r)}_{r,n,>}} &\lesssim c^{\frac{1}{2}(\frac{1}{2}-\frac{1}{r})-\frac{2}{q}} \|(E_0, B_0)\|_{\dot{B}^s_{2,n,>}} + c^{\frac{1}{2}(\frac{1}{2}-\frac{1}{r})+2(\frac{1}{p}-\frac{1}{q})-1} \|u\|_{L^\infty_{t,x} \cap L^\infty_t \dot{B}^1_{2,\infty}} \|B\|_{\widetilde{L}^p_t \dot{B}^s_{2,n}}\end{aligned} \tag{3-19}$$

for any $-1 < s < 2$ *and*

$$\|(E,B)\|_{\widetilde{L}^q_t\dot{B}^{s-7/4+3/(2r)}_{r,n,>}} \lesssim c^{\frac{1}{2}(\frac{1}{2}-\frac{1}{r})-\frac{2}{q}}\|(E_0,B_0)\|_{\dot{B}^{s-1}_{2,n,>}} + c^{\frac{1}{2}(\frac{1}{2}-\frac{1}{r})+2(\frac{1}{p}-\frac{1}{q})-1}\|u\|_{L^\infty_t\dot{B}^0_{2,\infty}}\|B\|_{\widetilde{L}^p_t\dot{B}^s_{2,n}}$$

for any $0 < s < 2$.

Proof. Considering Maxwell's system (1-15) and applying Corollary 2.12, with $\tilde{r} = 2$ and $\tilde{q} = p'$, yields the high-frequency estimate

$$2^{-j\frac{3}{2}(\frac{1}{2}-\frac{1}{r})}\|\Delta_j(E,B)\|_{L^q_tL^r_x} \lesssim c^{\frac{1}{2}(\frac{1}{2}-\frac{1}{r})-\frac{2}{q}}\big(\|\Delta_j(E_0,B_0)\|_{L^2_x} + c^{\frac{2}{p}-1}\|\Delta_j P(u\times B)\|_{L^p_tL^2_x}\big)$$

for all $j \in \mathbb{Z}$ with $2^j \geq \sigma c$, where $1 \leq p \leq q \leq \infty$ and $2 \leq r \leq \infty$ must satisfy (3-18). It is to be emphasized that, thanks to the damping phenomenon in (1-15), all estimates here hold uniformly over any time interval $[0,T)$, where $T = \infty$ is allowed.

Next, summing the preceding estimate in j and utilizing the paradifferential product law (3-13), we deduce that

$$\begin{aligned}\Big\|\mathbb{1}_{\{2^j\geq\sigma c\}}2^{j(s-\frac{3}{2}(\frac{1}{2}-\frac{1}{r}))}\|\Delta_j(E,B)\|_{L^q_tL^r_x}\Big\|_{\ell^n}&\\ \lesssim c^{\frac{1}{2}(\frac{1}{2}-\frac{1}{r})-\frac{2}{q}}&\|(E_0,B_0)\|_{\dot{B}^s_{2,n,>}} + c^{\frac{1}{2}(\frac{1}{2}-\frac{1}{r})+2(\frac{1}{p}-\frac{1}{q})-1}\|P(u\times B)\|_{\widetilde{L}^p_t\dot{B}^s_{2,n}}\\ \lesssim c^{\frac{1}{2}(\frac{1}{2}-\frac{1}{r})-\frac{2}{q}}&\|(E_0,B_0)\|_{\dot{B}^s_{2,n,>}} + c^{\frac{1}{2}(\frac{1}{2}-\frac{1}{r})+2(\frac{1}{p}-\frac{1}{q})-1}\|u\|_{L^\infty_{t,x}\cap\widetilde{L}^\infty_t\dot{B}^1_{2,\infty}}\|B\|_{\widetilde{L}^p_t\dot{B}^s_{2,n}}\end{aligned}$$

for any $-1 < s < 2$. If, instead of using (3-13), one employs the paradifferential product law (3-12), then one arrives at the estimate

$$\begin{aligned}\Big\|\mathbb{1}_{\{2^j\geq\sigma c\}}2^{j(s+\alpha-\frac{7}{4}+\frac{3}{2r})}&\|\Delta_j(E,B)\|_{L^q_tL^r_x}\Big\|_{\ell^n}\\ &\lesssim c^{\frac{1}{2}(\frac{1}{2}-\frac{1}{r})-\frac{2}{q}}\|(E_0,B_0)\|_{\dot{B}^{s+\alpha-1}_{2,n,>}} + c^{\frac{1}{2}(\frac{1}{2}-\frac{1}{r})+2(\frac{1}{p}-\frac{1}{q})-1}\|P(u\times B)\|_{\widetilde{L}^p_t\dot{B}^{s+\alpha-1}_{2,n}}\\ &\lesssim c^{\frac{1}{2}(\frac{1}{2}-\frac{1}{r})-\frac{2}{q}}\|(E_0,B_0)\|_{\dot{B}^{s+\alpha-1}_{2,n,>}} + c^{\frac{1}{2}(\frac{1}{2}-\frac{1}{r})+2(\frac{1}{p}-\frac{1}{q})-1}\|u\|_{\widetilde{L}^\infty_t\dot{B}^\alpha_{2,1}}\|B\|_{\widetilde{L}^p_t\dot{B}^s_{2,\infty}}\\ &\lesssim c^{\frac{1}{2}(\frac{1}{2}-\frac{1}{r})-\frac{2}{q}}\|(E_0,B_0)\|_{\dot{B}^{s+\alpha-1}_{2,n,>}} + c^{\frac{1}{2}(\frac{1}{2}-\frac{1}{r})+2(\frac{1}{p}-\frac{1}{q})-1}\|u\|^{1-\alpha}_{\widetilde{L}^\infty_t\dot{B}^0_{2,\infty}}\|u\|^\alpha_{\widetilde{L}^\infty_t\dot{B}^1_{2,\infty}}\|B\|_{\widetilde{L}^p_t\dot{B}^s_{2,\infty}},\end{aligned}$$

which is valid for any $0 < \alpha < 1$ and $s < 2$, with $\alpha + s > 0$, where we exploited the fact that $\widetilde{L}^\infty_t\dot{B}^\alpha_{2,1}$ is an interpolation space between $\widetilde{L}^\infty_t\dot{B}^0_{2,\infty}$ and $\widetilde{L}^\infty_t\dot{B}^1_{2,\infty}$ (see [Bergh and Löfström 1976] for details on interpolation theory).

Similarly, the case $\alpha = 0$ yields

$$\begin{aligned}\Big\|\mathbb{1}_{\{2^j\geq\sigma c\}}2^{j(s-\frac{7}{4}+\frac{3}{2r})}\|\Delta_j(E,B)\|_{L^q_tL^r_x}\Big\|_{\ell^n}&\\ \lesssim c^{\frac{1}{2}(\frac{1}{2}-\frac{1}{r})-\frac{2}{q}}&\|(E_0,B_0)\|_{\dot{B}^{s-1}_{2,n,>}} + c^{\frac{1}{2}(\frac{1}{2}-\frac{1}{r})+2(\frac{1}{p}-\frac{1}{q})-1}\|P(u\times B)\|_{\widetilde{L}^p_t\dot{B}^{s-1}_{2,n}}\\ \lesssim c^{\frac{1}{2}(\frac{1}{2}-\frac{1}{r})-\frac{2}{q}}&\|(E_0,B_0)\|_{\dot{B}^{s-1}_{2,n,>}} + c^{\frac{1}{2}(\frac{1}{2}-\frac{1}{r})+2(\frac{1}{p}-\frac{1}{q})-1}\|u\|_{\widetilde{L}^\infty_t\dot{B}^0_{2,\infty}}\|B\|_{\widetilde{L}^p_t\dot{B}^s_{2,n}}\end{aligned}$$

for any $0 < s < 2$. This completes the justification of the high-frequency estimates. □

3.6. ***Control of low-frequency damped electromagnetic waves.*** The statement of Corollary 2.12 has clearly emphasized how solutions to the damped Maxwell system enjoy intrinsically different properties on the distinct ranges of low and high frequencies.

In particular, the combination of Corollary 2.12 with the paradifferential Lemma 3.4 resulted in the nonlinear high-frequency estimates of Lemma 3.8. A similar strategy based on the same corollary could now be employed to deduce suitable nonlinear low-frequency estimates.

However, in the next lemma, we are instead going to exploit the refined estimates established in Corollary 2.14, which are a consequence of the maximal parabolic regularity studied in Section 2.2, to obtain a sharper control of low frequencies. This will lead to stronger statements of our main theorems.

Lemma 3.9. *Let $d = 2$. Assume that (E, B) is a smooth solution to (1-15) for some initial data (E_0, B_0) and some divergence-free vector field u, with the normal structure (1-2).*

Then, for any exponents $1 < p \le q < \infty$ and $1 \le n \le \infty$, one has the following low-frequency estimates over any time interval $[0, T)$. For any $0 < \alpha < 1$ and $s < 2$, with $\alpha + s > 0$,

$$\begin{aligned}\|E\|_{L^q_t \dot B^{s+\alpha-1}_{2,n,<}} &\lesssim c^{-\frac{2}{q}} \|E_0\|_{\dot B^{s+\alpha-1}_{2,n,<}} + c^{-1}\|B_0\|_{\dot B^{s+\alpha-2/q}_{2,q,<}} \\ &\qquad + c^{2(\frac1p-\frac1q)-1}\|u\|^{1-\alpha}_{L^\infty_t \dot B^0_{2,\infty}}\|u\|^{\alpha}_{L^\infty_t \dot B^1_{2,\infty}}\|B\|_{L^p_t \dot B^s_{2,\infty}}, \\ \|B\|_{L^q_t \dot B^{s+\alpha+2/q-2/p}_{2,1,<}} &\lesssim c^{-1}\|E_0\|_{\dot B^{s+\alpha+1-2/p}_{2,q,<}} + \|B_0\|_{\dot B^{s+\alpha-2/p}_{2,q,<}} \\ &\qquad + \|u\|^{1-\alpha}_{L^\infty_t \dot B^0_{2,\infty}}\|u\|^{\alpha}_{L^\infty_t \dot B^1_{2,\infty}}\|B\|_{L^p_t \dot B^s_{2,\infty}}.\end{aligned} \tag{3-20}$$

In the endpoint case $\alpha = 1$, with $-1 < s < 2$, one also has the estimates

$$\|E\|_{L^q_t \dot B^s_{2,n,<}} \lesssim c^{-\frac{2}{q}}\|E_0\|_{\dot B^s_{2,n,<}} + c^{-1}\|B_0\|_{\dot B^{s+1-2/q}_{2,q,<}} + c^{2(\frac1p-\frac1q)-1}\|u\|_{L^\infty_{t,x}\cap L^\infty_t \dot B^1_{2,\infty}}\|B\|_{L^p_t \dot B^s_{2,n}}, \tag{3-21}$$

as well as, if $p < q$,

$$\|B\|_{L^q_t \dot B^{s+1+2/q-2/p}_{2,1,<}} \lesssim c^{-1}\|E_0\|_{\dot B^{s+2-2/p}_{2,q,<}} + \|B_0\|_{\dot B^{s+1-2/p}_{2,q,<}} + \|u\|_{L^\infty_{t,x}\cap L^\infty_t \dot B^1_{2,\infty}}\|B\|_{L^p_t \dot B^s_{2,\infty}}$$

and, if $p = q$,

$$\|B\|_{L^q_t \dot B^{s+1}_{2,n,<}} \lesssim c^{-1}\|E_0\|_{\dot B^{s+2-2/q}_{2,q,<}} + \|B_0\|_{\dot B^{s+1-2/q}_{2,q,<}} + \|u\|_{L^\infty_{t,x}\cap L^\infty_t \dot B^1_{2,\infty}}\|B\|_{L^q_t \dot B^s_{2,n}}. \tag{3-22}$$

Finally, in the remaining endpoint case $\alpha = 0$, with $0 < s < 2$,

$$\|E\|_{L^q_t \dot B^{s-1}_{2,n,<}} \lesssim c^{-\frac{2}{q}}\|E_0\|_{\dot B^{s-1}_{2,n,<}} + c^{-1}\|B_0\|_{\dot B^{s-2/q}_{2,q,<}} + c^{2(\frac1p-\frac1q)-1}\|u\|_{L^\infty_t \dot B^0_{2,\infty}}\|B\|_{L^p_t \dot B^s_{2,n}},$$

as well as, if $p < q$,

$$\|B\|_{L^q_t \dot B^{s+2/q-2/p}_{2,1,<}} \lesssim c^{-1}\|E_0\|_{\dot B^{s+1-2/p}_{2,q,<}} + \|B_0\|_{\dot B^{s-2/p}_{2,q,<}} + \|u\|_{L^\infty_t \dot B^0_{2,\infty}}\|B\|_{L^p_t \dot B^s_{2,\infty}}$$

and, if $p = q$,

$$\|B\|_{L^q_t \dot B^s_{2,n,<}} \lesssim c^{-1}\|E_0\|_{\dot B^{s+1-2/q}_{2,q,<}} + \|B_0\|_{\dot B^{s-2/q}_{2,q,<}} + \|u\|_{L^\infty_t \dot B^0_{2,\infty}}\|B\|_{L^q_t \dot B^s_{2,n}}. \tag{3-23}$$

Remark. The devil is in the details. In the statement of Lemma 3.9 and its corresponding proof below, we have paid painstaking attention to the summability index of Besov spaces, occasionally referred to as the third index. Thus, in the above statement, sometimes the third index is q, other times it is n, 1 or ∞. Either way, we believe that these defining values of Besov spaces are optimal, which is a crucial step to reach sharp statements of our main results.

Proof. Let us consider some fixed regularity parameters $s \in \mathbb{R}$ and $0 \leq \alpha \leq 1$.

On the one hand, an application of Corollary 2.12 with $r = \tilde{r} = 2$ (note that this is the best possible choice for r and $\tilde{r}$, since all other estimates for values $2 \leq r, \tilde{r} \leq \infty$ follow from the case $r = \tilde{r} = 2$ by Sobolev embeddings) and $\tilde{q} = p'$ gives

$$2^{j(s+\alpha-1)}\|\Delta_j E\|_{L^q_t L^2_x} \lesssim c^{-\frac{2}{q}}\|E_0\|_{\dot{B}^{s+\alpha-1}_{2,\infty,<}} + c^{-1}\|B_0\|_{\dot{B}^{s+\alpha-2/q}_{2,\infty,<}} + c^{2(\frac{1}{p}-\frac{1}{q})-1}\|P(u\times B)\|_{L^p_t \dot{B}^{s+\alpha-1}_{2,\infty}}$$

and

$$2^{j(s+\alpha+\frac{2}{q}-\frac{2}{p})}\|\Delta_j B\|_{L^q_t L^2_x} \lesssim c^{-1}\|E_0\|_{\dot{B}^{s+\alpha+1-2/p}_{2,\infty,<}} + \|B_0\|_{\dot{B}^{s+\alpha-2/p}_{2,\infty,<}} + \|P(u\times B)\|_{L^p_t \dot{B}^{s+\alpha-1}_{2,\infty}}$$

for all $j \in \mathbb{Z}$ with $2^j < \sigma c$, where $1 \leq p \leq q \leq \infty$.

On the other hand, employing Corollary 2.14 yields

$$\begin{aligned}\Big\|\mathbb{1}_{\{2^j<\frac{1}{2}\sigma c\}}&2^{j(s+\alpha-1)}\|\Delta_j E\|_{L^2_x}\Big\|_{L^q_t \ell^n_j} \\ &\lesssim c^{-\frac{2}{q}}\|E_0\|_{\dot{B}^{s+\alpha-1}_{2,n,<}} + c^{-1}\|B_0\|_{\dot{B}^{s+\alpha-2/q}_{2,q,<}} + c^{2(\frac{1}{p}-\frac{1}{q})-1}\|P(u\times B)\|_{L^p_t \dot{B}^{s+\alpha-1}_{2,n}}\end{aligned}$$

for any $1 < p \leq q < \infty$ and $1 \leq n \leq \infty$, as well as

$$\begin{aligned}\Big\|\mathbb{1}_{\{2^j<\frac{1}{2}\sigma c\}}&2^{j(s+\alpha+\frac{2}{q}-\frac{2}{p})}\|\Delta_j B\|_{L^2_x}\Big\|_{L^q_t \ell^1_j} \\ &\lesssim c^{-1}\|E_0\|_{\dot{B}^{s+\alpha+1-2/p}_{2,q,<}} + \|B_0\|_{\dot{B}^{s+\alpha-2/p}_{2,q,<}} + \|P(u\times B)\|_{L^p_t \dot{B}^{s+\alpha-1}_{2,\infty}}\end{aligned}$$

for any $1 < p < q < \infty$, and

$$\Big\|\mathbb{1}_{\{2^j<\frac{1}{2}\sigma c\}}2^{j(s+\alpha)}\|\Delta_j B\|_{L^2_x}\Big\|_{L^q_t \ell^n_j} \lesssim c^{-1}\|E_0\|_{\dot{B}^{s+\alpha+1-2/q}_{2,q,<}} + \|B_0\|_{\dot{B}^{s+\alpha-2/q}_{2,q,<}} + \|P(u\times B)\|_{L^q_t \dot{B}^{s+\alpha-1}_{2,n}}$$

for any $1 < q < \infty$ and $1 \leq n \leq \infty$.

On the whole, combining the above estimates, we arrive at the conclusion that

$$\|E\|_{L^q_t \dot{B}^{s+\alpha-1}_{2,n,<}} \lesssim c^{-\frac{2}{q}}\|E_0\|_{\dot{B}^{s+\alpha-1}_{2,n,<}} + c^{-1}\|B_0\|_{\dot{B}^{s+\alpha-2/q}_{2,q,<}} + c^{2(\frac{1}{p}-\frac{1}{q})-1}\|P(u\times B)\|_{L^p_t \dot{B}^{s+\alpha-1}_{2,n}}$$

for any $1 < p \leq q < \infty$ and $1 \leq n \leq \infty$, as well as

$$\|B\|_{L^q_t \dot{B}^{s+\alpha+2/q-2/p}_{2,1,<}} \lesssim c^{-1}\|E_0\|_{\dot{B}^{s+\alpha+1-2/p}_{2,q,<}} + \|B_0\|_{\dot{B}^{s+\alpha-2/p}_{2,q,<}} + \|P(u\times B)\|_{L^p_t \dot{B}^{s+\alpha-1}_{2,\infty}}$$

for any $1 < p < q < \infty$, and

$$\|B\|_{L^q_t \dot{B}^{s+\alpha}_{2,n,<}} \lesssim c^{-1}\|E_0\|_{\dot{B}^{s+\alpha+1-2/q}_{2,q,<}} + \|B_0\|_{\dot{B}^{s+\alpha-2/q}_{2,q,<}} + \|P(u\times B)\|_{L^q_t \dot{B}^{s+\alpha-1}_{2,n}}$$

for any $1 < q < \infty$ and $1 \leq n \leq \infty$.

We are now going to apply the paradifferential product inequalities from Lemma 3.4 to the three preceding controls. More precisely, setting $\alpha = 1$, restricting the range of s to $(-1, 2)$ and utilizing the product law (3-13) to handle the nonlinear term $u \times B$, we obtain

$$\|E\|_{L^q_t \dot B^s_{2,n,<}} \lesssim c^{-\frac{2}{q}} \|E_0\|_{\dot B^s_{2,n,<}} + c^{-1} \|B_0\|_{\dot B^{s+1-2/q}_{2,q,<}} + c^{2(\frac{1}{p}-\frac{1}{q})-1} \|u\|_{L^\infty_{t,x} \cap L^\infty_t \dot B^1_{2,\infty}} \|B\|_{L^p_t \dot B^s_{2,n}}$$

for any $1 < p \le q < \infty$ and $1 \le n \le \infty$, as well as

$$\|B\|_{L^q_t \dot B^{s+1+2/q-2/p}_{2,1,<}} \lesssim c^{-1} \|E_0\|_{\dot B^{s+2-2/p}_{2,q,<}} + \|B_0\|_{\dot B^{s+1-2/p}_{2,q,<}} + \|u\|_{L^\infty_{t,x} \cap L^\infty_t \dot B^1_{2,\infty}} \|B\|_{L^p_t \dot B^s_{2,\infty}}$$

for any $1 < p < q < \infty$, and

$$\|B\|_{L^q_t \dot B^{s+1}_{2,n,<}} \lesssim c^{-1} \|E_0\|_{\dot B^{s+2-2/q}_{2,q,<}} + \|B_0\|_{\dot B^{s+1-2/q}_{2,q,<}} + \|u\|_{L^\infty_{t,x} \cap L^\infty_t \dot B^1_{2,\infty}} \|B\|_{L^q_t \dot B^s_{2,n}}$$

for any $1 < q < \infty$ and $1 \le n \le \infty$.

Similarly, choosing parameters $0 < \alpha < 1$ and $s < 2$, with $\alpha + s > 0$, utilizing (3-12) instead of (3-13), and exploiting again the fact that $L^\infty_t \dot B^\alpha_{2,1}$ is an interpolation space between $L^\infty_t \dot B^0_{2,\infty}$ and $L^\infty_t \dot B^1_{2,\infty}$, we find that

$$\begin{aligned}\|E\|_{L^q_t \dot B^{s+\alpha-1}_{2,n,<}} \lesssim c^{-\frac{2}{q}} \|E_0\|_{\dot B^{s+\alpha-1}_{2,n,<}} &+ c^{-1} \|B_0\|_{\dot B^{s+\alpha-2/q}_{2,q,<}} \\ &+ c^{2(\frac{1}{p}-\frac{1}{q})-1} \|u\|^{1-\alpha}_{L^\infty_t \dot B^0_{2,\infty}} \|u\|^\alpha_{L^\infty_t \dot B^1_{2,\infty}} \|B\|_{L^p_t \dot B^s_{2,\infty}}\end{aligned}$$

for any $1 < p \le q < \infty$ and $1 \le n \le \infty$, as well as

$$\|B\|_{L^q_t \dot B^{s+\alpha+2/q-2/p}_{2,1,<}} \lesssim c^{-1} \|E_0\|_{\dot B^{s+\alpha+1-2/p}_{2,q,<}} + \|B_0\|_{\dot B^{s+\alpha-2/p}_{2,q,<}} + \|u\|^{1-\alpha}_{L^\infty_t \dot B^0_{2,\infty}} \|u\|^\alpha_{L^\infty_t \dot B^1_{2,\infty}} \|B\|_{L^p_t \dot B^s_{2,\infty}}$$

for any $1 < p \le q < \infty$.

Finally, the case $\alpha = 0$, with $0 < s < 2$, is handled with the same product estimate (3-12) and results in

$$\|E\|_{L^q_t \dot B^{s-1}_{2,n,<}} \lesssim c^{-\frac{2}{q}} \|E_0\|_{\dot B^{s-1}_{2,n,<}} + c^{-1} \|B_0\|_{\dot B^{s-2/q}_{2,q,<}} + c^{2(\frac{1}{p}-\frac{1}{q})-1} \|u\|_{L^\infty_t \dot B^0_{2,\infty}} \|B\|_{L^p_t \dot B^s_{2,n}}$$

for any $1 < p \le q < \infty$ and $1 \le n \le \infty$, as well as

$$\|B\|_{L^q_t \dot B^{s+2/q-2/p}_{2,1,<}} \lesssim c^{-1} \|E_0\|_{\dot B^{s+1-2/p}_{2,q,<}} + \|B_0\|_{\dot B^{s-2/p}_{2,q,<}} + \|u\|_{L^\infty_t \dot B^0_{2,\infty}} \|B\|_{L^p_t \dot B^s_{2,\infty}}$$

for any $1 < p < q < \infty$, and

$$\|B\|_{L^q_t \dot B^s_{2,n,<}} \lesssim c^{-1} \|E_0\|_{\dot B^{s+1-2/q}_{2,q,<}} + \|B_0\|_{\dot B^{s-2/q}_{2,q,<}} + \|u\|_{L^\infty_t \dot B^0_{2,\infty}} \|B\|_{L^q_t \dot B^s_{2,n}}$$

for any $1 < q < \infty$ and $1 \le n \le \infty$, thereby completing the justifications of all low-frequency estimates. □

Note that the estimates from Lemma 3.9 do not include the value $q = \infty$. Rather than providing a technical extension of the preceding proof to incorporate the value $q = \infty$, we show in the next result a simple energy estimate on Maxwell's system (1-15), which corresponds to the case $q = \infty$ in Lemma 3.9. This energy estimate allows us to propagate the $\dot H^1_x$-norm of electromagnetic fields, which will come in handy in the proof of Theorem 3.1 below.

Lemma 3.10. *Let $d = 2$. Assume that (E, B) is a smooth solution to (1-15) for some initial data (E_0, B_0) and some divergence-free vector field u, with the normal structure (1-2).*

Then, one has the estimates

$$\begin{aligned}
&\|(E,B)\|_{L^\infty_t \dot H^1_x} + c\|E\|_{L^2_t \dot H^1_x} \lesssim \|(E_0,B_0)\|_{\dot H^1_x} + \|u\|_{L^\infty_t L^2_x}\|\nabla B\|_{L^2_t L^\infty_x},\\
&\|(E,B)\|_{L^\infty_t \dot H^1_x} + c\|E\|_{L^2_t \dot H^1_x} \lesssim \|(E_0,B_0)\|_{\dot H^1_x} + \|u\|^{1-\alpha}_{L^\infty_t \dot B^0_{2,\infty}}\|u\|^{\alpha}_{L^\infty_t \dot B^1_{2,\infty}}\|B\|_{L^2_t \dot B^{2-\alpha}_{2,\infty}},\\
&\|(E,B)\|_{L^\infty_t \dot H^1_x} + c\|E\|_{L^2_t \dot H^1_x} \lesssim \|(E_0,B_0)\|_{\dot H^1_x} + \|u\|_{L^\infty_{t,x}\cap L^\infty_t \dot B^1_{2,\infty}}\|B\|_{L^2_t \dot H^1_x}
\end{aligned}\tag{3-24}$$

over any time interval $[0,T)$ for any $0<\alpha<1$.

Proof. We perform a classical energy estimate on (1-15). More precisely, defining

$$(\widetilde E, \widetilde B) := (\nabla\times E, \nabla\times B),$$

we observe from (1-15) that $(\widetilde E, \widetilde B)$ solves the system

$$\begin{cases} \frac{1}{c}\partial_t \widetilde E - \nabla\times\widetilde B + \sigma c\widetilde E = -\sigma\nabla\times(u\times B) = \sigma(u\cdot\nabla)B - \sigma(B\cdot\nabla)u,\\ \frac{1}{c}\partial_t\widetilde B + \nabla\times\widetilde E = 0,\end{cases}$$

where we employed the fact that u and B are divergence-free fields. In fact, the preceding step holds in any dimension $d=2$ or $d=3$.

However, restricting ourselves to the two-dimensional setting and assuming that the field (u, E, B) satisfies the structure (1-2) allows us to discard the term $(B\cdot\nabla)u$. To be precise, we now have that $(\widetilde E, \widetilde B)$ solves

$$\begin{cases} \frac{1}{c}\partial_t \widetilde E - \nabla\times\widetilde B + \sigma c\widetilde E = -\sigma u\times\widetilde B,\\ \frac{1}{c}\partial_t\widetilde B + \nabla\times\widetilde E = 0.\end{cases}$$

Thus, multiplying the first equation by $\widetilde E$, the second by $\widetilde B$ and integrating in time and space, we deduce that

$$\begin{aligned}
\frac{1}{2c}\|(\widetilde E,\widetilde B)(T)\|^2_{L^2_x} + \sigma c\|\widetilde E\|^2_{L^2_t([0,T);L^2_x)}
&\le \frac{1}{2c}\|(\widetilde E,\widetilde B)(0)\|^2_{L^2_x} + \sigma\|u\times\widetilde B\|_{L^2_t([0,T);L^2_x)}\|\widetilde E\|_{L^2_t([0,T);L^2_x)}\\
&\le \frac{1}{2c}\|(\widetilde E,\widetilde B)(0)\|^2_{L^2_x} + \frac{\sigma}{2c}\|u\times\widetilde B\|^2_{L^2_t([0,T);L^2_x)} + \frac{\sigma c}{2}\|\widetilde E\|^2_{L^2_t([0,T);L^2_x)}.
\end{aligned}$$

Further employing the fact that

$$\|(E,B)\|_{\dot H^1_x} = \|(\nabla E,\nabla B)\|_{L^2_x} = \|(\widetilde E,\widetilde B)\|_{L^2_x},$$

we arrive at the estimate

$$\|(E,B)(T)\|^2_{\dot H^1_x} + \sigma c^2\|E\|^2_{L^2_t([0,T);\dot H^1_x)} \le \|(E_0,B_0)\|^2_{\dot H^1_x} + \sigma\|u\times\widetilde B\|^2_{L^2_t([0,T);L^2_x)}.$$

Therefore, the proof will be completed upon controlling the nonlinear term $\|u\times\widetilde B\|^2_{L^2_t([0,T);L^2_x)}$.

To this end, an elementary application of Hölder's inequality first leads to

$$\|u\times\widetilde{B}\|_{L^2_{t,x}}\le\|u\|_{L^\infty_tL^2_x}\|\nabla B\|_{L^2_tL^\infty_x},$$

thereby completing the justification of the first estimate of the lemma.

Alternatively, following the proof of Lemma 3.9, one can use the paraproduct estimates from Lemma 3.4 again. More specifically, utilizing the product law (3-13), we obtain

$$\|u\times\widetilde{B}\|_{L^2_{t,x}}=\|P(u\times B)\|_{L^2_t\dot{H}^1_x}\lesssim\|u\|_{L^\infty_{t,x}\cap L^\infty_t\dot{B}^1_{2,\infty}}\|B\|_{L^2_t\dot{H}^1_x}.$$

Similarly, choosing a parameter $0<\alpha<1$, utilizing (3-12) instead of (3-13), and exploiting the fact that $L^\infty_t\dot{B}^\alpha_{2,2}$ is an interpolation space between $L^\infty_t\dot{B}^0_{2,\infty}$ and $L^\infty_t\dot{B}^1_{2,\infty}$, we finally infer that

$$\|u\times\widetilde{B}\|_{L^2_{t,x}}=\|P(u\times B)\|_{L^2_t\dot{H}^1_x}\lesssim\|u\|_{L^\infty_t\dot{B}^\alpha_{2,2}}\|B\|_{L^2_t\dot{B}^{2-\alpha}_{2,\infty}}\lesssim\|u\|^{1-\alpha}_{L^\infty_t\dot{B}^0_{2,\infty}}\|u\|^{\alpha}_{L^\infty_t\dot{B}^1_{2,\infty}}\|B\|_{L^2_t\dot{B}^{2-\alpha}_{2,\infty}},$$

which concludes the proof. □

3.7. *Almost-parabolic estimates on the magnetic field.* In the singular limit $c\to\infty$, Maxwell's equations (1-15) formally converge towards the parabolic system

$$\partial_tB+(u\cdot\nabla)B-\frac{1}{\sigma}\Delta B=(B\cdot\nabla)u,$$

where we employed the fact that u and B are divergence-free. This holds in both dimensions $d=2$ and $d=3$. Further assuming the two-dimensional normal structure (1-2), the preceding system reduces to the simple transport-diffusion equation

$$\partial_tb+u\cdot\nabla b-\frac{1}{\sigma}\Delta b=0,$$

which satisfies the energy inequality

$$\frac{1}{2}\|b(T)\|^2_{L^2_x}+\frac{1}{\sigma}\|\nabla b\|^2_{L^2_x}\le\frac{1}{2}\|b(0)\|^2_{L^2_x}\tag{3-25}$$

for all $T>0$, at least formally.

The estimates provided by Lemma 3.9 fail to fully capture this asymptotic parabolic behavior of Maxwell's equations, because they always contain a control of the nonlinear term $u\cdot\nabla b$, whereas this term does not contribute to the energy dissipation inequality (3-25).

The next result establishes a singular almost-parabolic energy estimate for Maxwell's system (1-15) which recovers the classical a priori estimate (3-25) (up to multiplicative constants) for the heat equation in the limit $c\to\infty$. This is crucial to our work, as it will allow us to construct solutions to the Euler–Maxwell system (1-1) for arbitrarily large initial data as the speed of light c tends to infinity.

Lemma 3.11. *Let $d=2$. Assume that (E,B) is a smooth solution to (1-15) for some initial data (E_0,B_0) and some divergence-free vector field u, with the normal structure (1-2).*

Then, one has the estimate

$$\|B\|_{L^\infty_tL^2_x}+\|\nabla B\|_{L^2_tL^2_x}\lesssim\|B_0\|_{L^2_x}+c^{-1}\|(E_0,B_0)\|_{\dot{H}^1_x}+c^{-1}\|u\|_{L^\infty_tL^2_x}\|\nabla B\|_{L^2_tL^\infty_x}\tag{3-26}$$

over any time interval $[0,T)$.

Proof. First, we observe that taking the curl of Ampère's equation in (1-15) and then employing Faraday's equation to substitute $\nabla \times E = -(1/c)\partial_t B$, when necessary, leads to the system

$$\partial_t B + (u\cdot\nabla)B - \frac{1}{\sigma}\Delta B = (B\cdot\nabla)u + \frac{1}{\sigma c}\partial_t(\nabla\times E),$$

where we have also used that u and B are divergence-free. This holds in any dimension $d = 2$ or $d = 3$. Then, further assuming that (u, E, B) has the two-dimensional normal structure (1-2) yields

$$\partial_t b + u\cdot\nabla b - \frac{1}{\sigma}\Delta b = \frac{1}{\sigma c}\partial_t(\operatorname{curl} E).$$

Now, the elementary observation that

$$\int_{\mathbb{R}^2}(u\cdot\nabla b)b\,dx = \frac{1}{2}\int_{\mathbb{R}^2} u\cdot\nabla(b^2)\,dx = -\frac{1}{2}\int_{\mathbb{R}^2}(\operatorname{div} u)b^2\,dx = 0,$$

which is a consequence of the incompressibility of u, allows us to deduce the parabolic energy estimate

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} b^2\,dx + \frac{1}{\sigma}\int_{\mathbb{R}^2}|\nabla b|^2\,dx = \frac{1}{\sigma c}\frac{d}{dt}\int_{\mathbb{R}^2}(\operatorname{curl} E)b\,dx + \frac{1}{\sigma}\int_{\mathbb{R}^2}(\operatorname{curl} E)^2\,dx, \tag{3-27}$$

where we used Faraday's equation again to substitute $\partial_t b = -c\operatorname{curl} E$.

Then, integrating in time, we infer that

$$\begin{aligned}
&\frac{1}{2}\|b(T)\|_{L^2_x}^2 + \frac{1}{\sigma}\|\nabla b\|_{L^2_t([0,T);L^2_x)}^2 \\
&\quad\le \frac{1}{2}\|b(0)\|_{L^2_x}^2 + \frac{1}{\sigma c}\int_{\mathbb{R}^2}\operatorname{curl} E(T)b(T)\,dx - \frac{1}{\sigma c}\int_{\mathbb{R}^2}\operatorname{curl} E(0)b(0)\,dx + \frac{1}{\sigma}\|\operatorname{curl} E\|_{L^2_t([0,T);L^2_x)}^2 \\
&\quad\le \frac{1}{2}\|b(0)\|_{L^2_x}^2 + \frac{1}{4}(\|b(0)\|_{L^2_x}^2 + \|b(T)\|_{L^2_x}^2) + \frac{1}{\sigma^2 c^2}(\|E(0)\|_{\dot H^1_x}^2 + \|E(T)\|_{\dot H^1_x}^2) + \frac{1}{\sigma}\|E\|_{L^2_t([0,T);\dot H^1_x)}^2,
\end{aligned}$$

which leads to

$$\begin{aligned}
&\frac{1}{4}\|B(T)\|_{L^2_x}^2 + \frac{1}{\sigma}\|\nabla B\|_{L^2_t([0,T);L^2_x)}^2 \\
&\qquad\le \frac{3}{4}\|B_0\|_{L^2_x}^2 + \frac{1}{\sigma^2c^2}\|E_0\|_{\dot H^1_x}^2 + \frac{1}{\sigma^2c^2}(\|E(T)\|_{\dot H^1_x}^2 + \sigma\|cE\|_{L^2_t([0,T);\dot H^1_x)}^2).
\end{aligned} \tag{3-28}$$

Finally, combining (3-28) with the energy estimates (3-24) shows that (3-26) holds, thereby reaching the conclusion of the proof. □

Remark. The identity (3-27) contains the crucial calculation which enables us to extract a uniform bound on B in $L^2_t\dot H^1_x$, with remainder terms of order c^{-1}. This estimate will play an important role in the control of the low frequencies of nonlinear source terms in the Euler–Maxwell system.

3.8. *Proof of Theorem 3.1.* We proceed to the proof of our first main result — Theorem 3.1. Recall that we are taking the liberty of assuming, for simplicity, to be dealing with a smooth solution (u, E, B) of (1-1) for some smooth initial data (u_0, E_0, B_0), and that the justification of the theorem is only fully completed by relying on the approximation procedure laid out in Section 3.2.

Our proof hinges upon the preliminary lemmas established in Sections 3.4–3.7. Accordingly, we begin by carefully gathering all the relevant estimates on an arbitrary time interval $[0,T)$. We will then move on to construct the energy functional which will produce the uniform bounds we are seeking.

Control of velocity field. The control of u is obtained from Lemmas 3.6 and 3.7. Recalling the equivalence $\|\nabla u\|_{L^a_x} \sim \|\omega\|_{L^a_x}$ for any value $1<a<\infty$, these lemmas provide the estimates

$$\begin{aligned}
\|u\|_{L^\infty_t \dot H^1_x} &\lesssim \|u_0\|_{\dot H^1_x} + \|j\|_{L^2_{t,x}} \|\nabla B\|_{L^2_t L^\infty_x},\\
\|u\|_{L^\infty_t \dot W^{1,p}_x} &\lesssim \|u_0\|_{\dot W^{1,p}_x} + \|j\|^{2/p}_{L^2_{t,x}} \|j\|^{1-2/p}_{L^2_t \dot H^1_x} \|\nabla B\|_{L^2_t L^\infty_x},\\
\|u\|_{L^\infty_{t,x}} &\lesssim \|u\|^{(p-2)/(2(p-1))}_{L^\infty_t L^2_x} \|u\|^{p/(2(p-1))}_{L^\infty_t \dot W^{1,p}_x}\\
&\lesssim \|u\|^{p-2/(2(p-1))}_{L^\infty_t L^2_x} \big(\|u_0\|_{\dot W^{1,p}_x} + \|j\|^{2/p}_{L^2_{t,x}} \|j\|^{1-2/p}_{L^2_t \dot H^1_x} \|\nabla B\|_{L^2_t L^\infty_x}\big)^{p/(2(p-1))},
\end{aligned} \tag{3-29}$$

where $2<p<\infty$ is a fixed value.

Control of high electromagnetic frequencies. The control of high frequencies of (E,B) is obtained from Lemma 3.8. Specifically, setting $s=\frac74$ in (3-19) allows us to deduce that

$$\begin{aligned}
c^{-\frac34}\|(E,B)\|_{\widetilde L^\infty_t \dot B^{7/4}_{2,1,>}} + c^{\frac14}\|(E,B)\|_{\widetilde L^2_t \dot B^{7/4}_{2,1,>}} + \|(E,B)\|_{\widetilde L^2_t \dot B^1_{\infty,1,>}}&\\
\lesssim c^{-\frac34}\|(E_0,B_0)\|_{\dot B^{7/4}_{2,1,>}} + c^{-\frac34}\|u\|_{L^\infty_{t,x}\cap L^\infty_t \dot H^1_x} \|B\|_{\widetilde L^2_t \dot B^{7/4}_{2,1}}&.
\end{aligned}$$

Then, further decomposing high and low frequencies in the last term above, we obtain

$$\begin{aligned}
\|B\|_{\widetilde L^2_t \dot B^{7/4}_{2,1}} \lesssim \|B\|_{\widetilde L^2_t \dot B^{7/4}_{2,1,<}} + \|B\|_{\widetilde L^2_t \dot B^{7/4}_{2,1,>}} &\lesssim \|B\|^{1/4}_{\widetilde L^2_t \dot B^1_{2,\infty,<}} \|B\|^{3/4}_{\widetilde L^2_t \dot B^2_{2,\infty,<}} + \|B\|_{\widetilde L^2_t \dot B^{7/4}_{2,1,>}}\\
&\lesssim \|B\|^{1/4}_{L^2_t \dot H^1_x} \|B\|^{3/4}_{L^2_t \dot B^2_{2,1,<}} + \|B\|_{\widetilde L^2_t \dot B^{7/4}_{2,1,>}},
\end{aligned}$$

which yields the estimate

$$\begin{aligned}
c^{-\frac34}\|(E,B)\|_{\widetilde L^\infty_t \dot B^{7/4}_{2,1,>}} + c^{\frac14}\|(E,B)\|_{\widetilde L^2_t \dot B^{7/4}_{2,1,>}} + \|(E,B)\|_{\widetilde L^2_t \dot B^1_{\infty,1,>}}&\\
\lesssim c^{-\frac34}\|(E_0,B_0)\|_{\dot B^{7/4}_{2,1,>}} + c^{-\frac34}\|u\|_{L^\infty_{t,x}\cap L^\infty_t \dot H^1_x} \big(\|B\|^{1/4}_{L^2_t \dot H^1_x} \|B\|^{3/4}_{L^2_t \dot B^2_{2,1,<}} + \|B\|_{\widetilde L^2_t \dot B^{7/4}_{2,1,>}}\big)&.
\end{aligned} \tag{3-30}$$

Observe that the choice of regularity parameter $s=\frac74$ is dictated by the need to control ∇B in $L^2_t L^\infty_x$, as anticipated in the strategy presented in Section 1.3. Further notice that it is therefore crucial to be able to set a regularity parameter s with a value greater than one in (3-19). This flexibility comes from the use of the normal structure (1-2), which we exploited in the product estimates established in Section 3.3.

Control of low electromagnetic frequencies. The control of low frequencies of (E,B) is deduced from Lemmas 3.9 and 3.10. Here also, the choice of parameters is dictated by the need to control ∇B in $L^2_t L^\infty_x$. Thus, since $\dot B^2_{2,1}(\mathbb R^2)$ is contained in $L^\infty(\mathbb R^2)$, one could, for instance, set $s=2$ in (3-23), which gives

$$\|B\|_{L^2_t \dot B^2_{2,1,<}} \lesssim c^{-1}\|E_0\|_{\dot B^2_{2,2,<}} + \|B_0\|_{\dot B^1_{2,2,<}} + \|u\|_{L^\infty_t L^2_x} \|B\|_{L^2_t \dot B^2_{2,1}}.$$

But, due to the coefficient $\|u\|_{L^\infty_t L^2_x}$, such an estimate would eventually lead to a smallness condition, uniform in c, on the initial energy $\mathcal E_0$, which is not desirable. Therefore, instead, we employ (3-20) and (3-22). More precisely, by setting $s=1$ in (3-22) and $s=2-\alpha$ in (3-20), for any choice $0<\alpha<1$, we obtain the respective estimates

$$\|B\|_{L^2_t\dot B^2_{2,1,<}} \lesssim c^{-1}\|E_0\|_{\dot B^2_{2,2,<}} + \|B_0\|_{\dot B^1_{2,2,<}} + \|u\|_{L^\infty_{t,x}\cap L^\infty_t\dot H^1_x}\|B\|_{L^2_t\dot B^1_{2,1}},$$
$$\|B\|_{L^2_t\dot B^2_{2,1,<}} \lesssim c^{-1}\|E_0\|_{\dot B^2_{2,2,<}} + \|B_0\|_{\dot B^1_{2,2,<}} + \|u\|^{1-\alpha}_{L^\infty_t L^2_x}\|u\|^{\alpha}_{L^\infty_t\dot H^1_x}\|B\|^{\alpha}_{L^2_t\dot B^1_{2,\infty}}\|B\|^{1-\alpha}_{L^2_t\dot B^2_{2,\infty}}.$$

In fact, it is possible to straightforwardly adapt the proofs of the above estimates to derive the more useful combined control, where we split the high and low frequencies of B in the right-hand side,

$$\begin{aligned}\|B\|_{L^2_t\dot B^2_{2,1,<}} &\lesssim c^{-1}\|E_0\|_{\dot B^2_{2,2,<}} + \|B_0\|_{\dot B^1_{2,2,<}} + \|u\|_{L^\infty_{t,x}\cap L^\infty_t\dot H^1_x}\|\mathbb{1}_{\{|D|\geq\frac12\sigma c\}}B\|_{L^2_t\dot B^1_{2,1}}\\ &\qquad + \|u\|^{1-\alpha}_{L^\infty_t L^2_x}\|u\|^{\alpha}_{L^\infty_t\dot H^1_x}\|\mathbb{1}_{\{|D|<\frac12\sigma c\}}B\|^{\alpha}_{L^2_t\dot B^1_{2,\infty}}\|\mathbb{1}_{\{|D|<\frac12\sigma c\}}B\|^{1-\alpha}_{L^2_t\dot B^2_{2,\infty}}\\ &\lesssim \|(E_0,B_0)\|_{\dot H^1_x} + \|u\|_{L^\infty_{t,x}\cap L^\infty_t\dot H^1_x}\big(c^{-1}\|B\|_{L^2_t\dot B^2_{2,1,<}} + c^{-\frac34}\|B\|_{\tilde L^2_t\dot B^{7/4}_{2,1,>}}\big)\\ &\qquad + \|u\|^{1-\alpha}_{L^\infty_t L^2_x}\|u\|^{\alpha}_{L^\infty_t\dot H^1_x}\|B\|^{\alpha}_{L^2_t\dot H^1_x}\|B\|^{1-\alpha}_{L^2_t\dot B^2_{2,1,<}}.\end{aligned}\tag{3-31}$$

Further combining the preceding estimate with the energy estimate (3-24) from Lemma 3.10, we obtain

$$\begin{aligned}&\|(E,B)\|_{L^\infty_t\dot H^1_x} + c\|E\|_{L^2_t\dot H^1_x} + \|B\|_{L^2_t\dot B^2_{2,1,<}}\\ &\quad\lesssim \|(E_0,B_0)\|_{\dot H^1_x} + \|u\|_{L^\infty_{t,x}\cap L^\infty_t\dot H^1_x}\big(c^{-1}\|B\|_{L^2_t\dot B^2_{2,1,<}} + c^{-\frac34}\|B\|_{\tilde L^2_t\dot B^{7/4}_{2,1,>}}\big)\\ &\qquad\qquad + \|u\|^{1-\alpha}_{L^\infty_t L^2_x}\|u\|^{\alpha}_{L^\infty_t\dot H^1_x}\|B\|^{\alpha}_{L^2_t\dot H^1_x}\|B\|^{1-\alpha}_{L^2_t\dot B^2_{2,1,<}}.\end{aligned}$$

Finally, by a classical use of Young's inequality $ab\leq\alpha a^{1/\alpha}+(1-\alpha)b^{1/(1-\alpha)}$, with $a,b\geq 0$, aimed at absorbing the term $\|B\|^{1-\alpha}_{L^2_t\dot B^2_{2,1,<}}$ with the above left-hand side, we conclude that

$$\begin{aligned}&\|(E,B)\|_{L^\infty_t\dot H^1_x} + c\|E\|_{L^2_t\dot H^1_x} + \|B\|_{L^2_t\dot B^2_{2,1,<}}\\ &\quad\lesssim \|(E_0,B_0)\|_{\dot H^1_x} + \|u\|_{L^\infty_{t,x}\cap L^\infty_t\dot H^1_x}\big(c^{-1}\|B\|_{L^2_t\dot B^2_{2,1,<}} + c^{-\frac34}\|B\|_{\tilde L^2_t\dot B^{7/4}_{2,1,>}}\big)\\ &\qquad\qquad + \|u\|^{1/\alpha-1}_{L^\infty_t L^2_x}\|u\|_{L^\infty_t\dot H^1_x}\|B\|_{L^2_t\dot H^1_x}.\end{aligned}\tag{3-32}$$

Parabolic stability of magnetic field. The parabolic stability of the magnetic field comes as a result of the almost-parabolic estimates established in Lemma 3.11, which we conveniently reproduce here:

$$\|B\|_{L^2_t\dot H^1_x} \lesssim \|B_0\|_{L^2_x} + c^{-1}\|(E_0,B_0)\|_{\dot H^1_x} + c^{-1}\|u\|_{L^\infty_t L^2_x}\|\nabla B\|_{L^2_t L^\infty_x}.\tag{3-33}$$

This estimate will serve to control the term $\|B\|_{L^2_t\dot H^1_x}$ in (3-32).

Ohm's law estimate. Finally, we need to employ Ohm's law from (1-1) to control the electric current j. More precisely, by Ohm's law and the normal structure (1-2), we have

$$\|j\|_{L^2_t\dot H^1_x} \lesssim c\|E\|_{L^2_t\dot H^1_x} + \|P(u\times B)\|_{L^2_t\dot H^1_x} \lesssim c\|E\|_{L^2_t\dot H^1_x} + \|u\|_{L^\infty_t L^2_x}\|\nabla B\|_{L^2_t L^\infty_x},\tag{3-34}$$

which will be used to control $\|j\|_{L^2_t\dot H^1_x}$ in (3-29).

Nonlinear energy estimate. We are now in a position to derive the global nonlinear energy estimate which will yield a uniform bound on solutions to (1-1). Thus, inspired by the above set of estimates, we introduce the energy $\mathcal{H}(t_1,t_2)$, with $0\le t_1\le t_2$, by setting

$$\begin{aligned}\mathcal{H}(t_1,t_2):=\|u\|_{L^\infty_t\dot H^1_x}+\mathcal{E}_0^{(p-2)/(2(p-1))}\|u\|^{p/(2(p-1))}_{L^\infty_t\dot W^{1,p}_x}+c^{-\frac34}\|(E,B)\|_{\widetilde L^\infty_t\dot B^{7/4}_{2,1,>}}+c^{\frac14}\|(E,B)\|_{\widetilde L^2_t\dot B^{7/4}_{2,1,>}}\\+\|(E,B)\|_{\widetilde L^2_t\dot B^1_{\infty,1,>}}+\|(E,B)\|_{L^\infty_t\dot H^1_x}+c\|E\|_{L^2_t\dot H^1_x}+\|B\|_{L^2_t\dot B^2_{2,1,<}},\end{aligned}$$

where all time-norms are taken over the interval $[t_1,t_2)$. Since (u,E,B) is assumed, without any loss of generality, to be a smooth solution of (1-1), observe that $\mathcal{H}(t_1,t_2)$ is continuous on $\{0\le t_1\le t_2\}$. In particular, we can further define the continuous function

$$\mathcal{H}(t):=\mathcal{H}(t,t)=\|u(t)\|_{\dot H^1_x}+\mathcal{E}_0^{(p-2)/(2(p-1))}\|u(t)\|^{p/(2(p-1))}_{\dot W^{1,p}_x}+c^{-\frac34}\|(E,B)(t)\|_{\dot B^{7/4}_{2,1,>}}+\|(E,B)(t)\|_{\dot H^1_x}$$

for every $t\ge 0$. Note that

$$\mathcal{H}(t)\le\mathcal{H}(t_1,t_2)$$

for all $t\in[t_1,t_2]$.

It will also be useful to assign the notation

$$\mathcal{J}(t_1,t_2):=\|j\|_{L^2_{t,x}}$$

to the dissipation produced by the electric current in (1-3), where the L^2-norm is taken over the time-interval $[t_1,t_2)$, as well. In particular, one has the uniform bound

$$\mathcal{J}(t_1,t_2)\le\left(\frac{\sigma}{2}\right)^{\frac12}\mathcal{E}_0 \tag{3-35}$$

by virtue of the energy inequality (1-3).

Observe now that all the above estimates, which were stated on the time-interval $[0,T)$, could equally well be written over any other finite time-interval $[t_1,t_2)$, provided one replaces the initial data (u_0,E_0,B_0) by the data at time t_1. Thus, employing the energy inequality (1-3), the energies $\mathcal{H}(t_1,t_2)$ and $\mathcal{H}(t)$, and the embedding

$$\|\nabla B\|_{L^2_tL^\infty_x}\le\|B\|_{L^2_t\dot B^1_{\infty,1}}\lesssim\|B\|_{L^2_t\dot B^2_{2,1,<}}+\|B\|_{\widetilde L^2_t\dot B^1_{\infty,1,>}}\le\mathcal{H}(t_1,t_2),$$

one can write the parabolic stability estimate (3-33) as

$$\|B\|_{L^2_t\dot H^1_x}\lesssim\mathcal{E}_0+c^{-1}\mathcal{H}(t_1)+c^{-1}\mathcal{E}_0\mathcal{H}(t_1,t_2) \tag{3-36}$$

and the Ohm's law estimate (3-34) as

$$\|j\|_{L^2_t\dot H^1_x}\lesssim(1+\mathcal{E}_0)\mathcal{H}(t_1,t_2), \tag{3-37}$$

which are linear in $\mathcal{H}(t_1,t_2)$.

Then, incorporating the linear estimate (3-37) into the velocity control (3-29), we obtain

$$\begin{aligned}\|u\|_{L^\infty_t\dot H^1_x}&\lesssim\mathcal{H}(t_1)+\mathcal{J}(t_1,t_2)\mathcal{H}(t_1,t_2),\\ \|u\|_{L^\infty_{t,x}}\lesssim\mathcal{E}_0^{\frac{p-2}{2(p-1)}}\|u\|^{\frac{p}{2(p-1)}}_{L^\infty_t\dot W^{1,p}_x}&\lesssim\mathcal{H}(t_1)+\mathcal{E}_0^{\frac{p-2}{2(p-1)}}(1+\mathcal{E}_0)^{\frac{p-2}{2(p-1)}}\mathcal{J}(t_1,t_2)^{\frac{1}{p-1}}\mathcal{H}(t_1,t_2).\end{aligned} \tag{3-38}$$

Furthermore, the use of (3-36) and (3-38) in the high-frequency control (3-30) leads to

$$\begin{aligned}
&c^{-\frac{3}{4}}\|(E,B)\|_{\widetilde{L}^\infty_t\dot{B}^{7/4}_{2,1,>}}+c^{\frac{1}{4}}\|(E,B)\|_{\widetilde{L}^2_t\dot{B}^{7/4}_{2,1,>}}+\|(E,B)\|_{\widetilde{L}^2_t\dot{B}^{1}_{\infty,1,>}}\\
&\qquad\lesssim \mathcal{H}(t_1)+c^{-\frac{3}{4}}\mathcal{H}(t_1,t_2)(\mathcal{E}_0+c^{-1}\mathcal{H}(t_1)+c^{-1}\mathcal{E}_0\mathcal{H}(t_1,t_2))^{\frac{1}{4}}\mathcal{H}(t_1,t_2)^{\frac{3}{4}}+c^{-1}\mathcal{H}(t_1,t_2)^2\\
&\qquad\lesssim \mathcal{H}(t_1)+c^{-\frac{3}{4}}\mathcal{E}_0^{1/4}\mathcal{H}(t_1,t_2)^{\frac{7}{4}}+c^{-1}(1+\mathcal{E}_0^{1/4})\mathcal{H}(t_1,t_2)^2\\
&\qquad\lesssim \mathcal{H}(t_1)+\lambda\mathcal{H}(t_1,t_2)+\lambda^{-\frac{1}{3}}c^{-1}\mathcal{E}_0^{1/3}\mathcal{H}(t_1,t_2)^2+c^{-1}(1+\mathcal{E}_0^{1/4})\mathcal{H}(t_1,t_2)^2 \qquad (3\text{-}39)
\end{aligned}$$

for any $\lambda>0$, whereas a similar procedure applied to the low-frequency estimate (3-32) yields

$$\begin{aligned}
&\|(E,B)\|_{L^\infty_t\dot{H}^1_x}+c\|E\|_{L^2_t\dot{H}^1_x}+\|B\|_{L^2_t\dot{B}^2_{2,1,<}}\\
&\lesssim \mathcal{H}(t_1)+c^{-1}\mathcal{H}(t_1,t_2)^2+\mathcal{E}_0^{1/\alpha-1}(\mathcal{H}(t_1)+\mathcal{J}(t_1,t_2)\mathcal{H}(t_1,t_2))(\mathcal{E}_0+c^{-1}\mathcal{H}(t_1)+c^{-1}\mathcal{E}_0\mathcal{H}(t_1,t_2))\\
&\lesssim (1+\mathcal{E}_0^{1/\alpha})\mathcal{H}(t_1)+\mathcal{E}_0^{1/\alpha}\mathcal{J}(t_1,t_2)\mathcal{H}(t_1,t_2)+(1+\mathcal{E}_0^{1/\alpha+1})c^{-1}\mathcal{H}(t_1,t_2)^2. \qquad (3\text{-}40)
\end{aligned}$$

All in all, summing estimates (3-38), (3-39) and (3-40) together and setting λ in (3-39) small enough that the term $\lambda\mathcal{H}(t_1,t_2)$ can be absorbed by the resulting left-hand side, we finally arrive at the crucial nonlinear energy estimate

$$\begin{aligned}
\mathcal{H}(t_1,t_2)\lesssim{}&(1+\mathcal{E}_0^{1/\alpha})\mathcal{H}(t_1)+(1+\mathcal{E}_0^{1/\alpha})\mathcal{J}(t_1,t_2)\mathcal{H}(t_1,t_2)\\
&+\mathcal{E}_0^{(p-2)/(2(p-1))}(1+\mathcal{E}_0^{(p-2)/(2(p-1))})\mathcal{J}(t_1,t_2)^{\frac{1}{p-1}}\mathcal{H}(t_1,t_2)+(1+\mathcal{E}_0^{1/\alpha+1})c^{-1}\mathcal{H}(t_1,t_2)^2
\end{aligned}$$

for any $0\le t_1\le t_2$, which, using (3-35), can be slightly simplified to

$$\begin{aligned}
\mathcal{H}(t_1,t_2)\le{}& C_*(1+\mathcal{E}_0^{1/\alpha})\mathcal{H}(t_1)+C_*(1+\mathcal{E}_0^{1/\alpha+(p-2)/(p-1)})\mathcal{J}(t_1,t_2)^{\frac{1}{p-1}}\mathcal{H}(t_1,t_2)\\
&+C_*(1+\mathcal{E}_0^{1/\alpha+1})c^{-1}\mathcal{H}(t_1,t_2)^2, \qquad (3\text{-}41)
\end{aligned}$$

where $C_*>0$ only depends on fixed parameters (in particular, it is independent of time, the speed of light c and the initial data). We are now going to show that (3-41) leads to a global bound on $\mathcal{H}(t_1,t_2)$.

Conclusion of proof. Let us consider a partition of time

$$0=t_0<t_1<\cdots<t_n<t_{n+1}=\infty$$

such that, for every $i=0,1,\ldots,n-1$,

$$C_*(1+\mathcal{E}_0^{1/\alpha+(p-2)/(p-1)})\mathcal{J}(t_i,t_{i+1})^{\frac{1}{p-1}}=\tfrac{1}{2}\quad\text{and}\quad C_*(1+\mathcal{E}_0^{1/\alpha+(p-2)/(p-1)})\mathcal{J}(t_n,t_{n+1})^{\frac{1}{p-1}}\le\tfrac{1}{2}.$$

In particular, by virtue of (3-35), one has, for any $t\in[t_i,t_{i+1}]$ with $i=0,1,\ldots,n$, that

$$\frac{i}{(2C_*(1+\mathcal{E}_0^{1/\alpha+(p-2)/(p-1)}))^{2(p-1)}}=\sum_{k=0}^{i-1}\mathcal{J}(t_k,t_{k+1})^2\le\mathcal{J}(t_0,t)^2\le\frac{\sigma}{2}\mathcal{E}_0^2. \qquad (3\text{-}42)$$

It then follows from (3-41) that

$$\mathcal{H}(t_i,t)\le 2C_*(1+\mathcal{E}_0^{1/\alpha})\mathcal{H}(t_i)+2C_*(1+\mathcal{E}_0^{1/\alpha+1})c^{-1}\mathcal{H}(t_i,t)^2 \qquad (3\text{-}43)$$

for all $i=0,\ldots,n$ and $t\in[t_i,t_{i+1}]$.

Next, for fixed i, we introduce the quadratic polynomial

$$p(X) = 2C_*(1+\mathcal{E}_0^{1/\alpha+1})c^{-1}X^2 - X + 2C_*(1+\mathcal{E}_0^{1/\alpha})\mathcal{H}(t_i),$$

whose roots

$$\lambda_\pm = \frac{1 \pm \sqrt{1-16C_*^2(1+\mathcal{E}_0^{1/\alpha+1})(1+\mathcal{E}_0^{1/\alpha})c^{-1}\mathcal{H}(t_i)}}{4C_*(1+\mathcal{E}_0^{1/\alpha+1})c^{-1}}$$

are real and distinct, provided

$$\mathcal{H}(t_i) < \frac{c}{16C_*^2(1+\mathcal{E}_0^{1/\alpha+1})(1+\mathcal{E}_0^{1/\alpha})}. \tag{3-44}$$

Observe that (3-43) can be rewritten as

$$p(\mathcal{H}(t_i,t)) \geq 0$$

for all $t \in [t_i, t_{i+1}]$. Therefore, by continuity of $\mathcal{H}(t_i,t)$ and assuming that (3-44) is satisfied, we deduce that $\mathcal{H}(t_i,t) \leq \lambda_-$ for all $t \in [t_i, t_{i+1}]$ if it is true for $t = t_i$, i.e., $\mathcal{H}(t_i) \leq \lambda_-$.

Now, it is readily seen that (3-44) implies $\mathcal{H}(t_i) \leq \lambda_-$ if we assume, without any loss of generality, that $2C_*(1+\mathcal{E}_0^{1/\alpha}) \geq 1$. Thus, by continuity of $t \mapsto \mathcal{H}(t_i,t)$, we conclude that (3-44) is sufficient to deduce the bound

$$\mathcal{H}(t) \leq \mathcal{H}(t_i,t) \leq \lambda_- \leq 4C_*(1+\mathcal{E}_0^{1/\alpha})\mathcal{H}(t_i)$$

for all $t \in [t_i, t_{i+1}]$, where we used the elementary inequality $1-\sqrt{1-z} \leq z$ for all $z \in [0,1]$ in the last step. Then, a straightforward iterative process leads us to the estimate

$$\mathcal{H}(t_i,t) \leq 4C_*(1+\mathcal{E}_0^{1/\alpha})\mathcal{H}(t_i) \leq [4C_*(1+\mathcal{E}_0^{1/\alpha})]^{i+1}\mathcal{H}(t_0)$$

for each $i = 0, 1, \ldots, n$ and all $t \in [t_i, t_{i+1}]$, if the initial data satisfies

$$\mathcal{H}(t_0) < \frac{c}{4C_*(1+\mathcal{E}_0^{1/\alpha+1})[4C_*(1+\mathcal{E}_0^{1/\alpha})]^{i+1}}.$$

Further noticing, for any $t \in [t_i, t_{i+1}]$, that

$$\mathcal{H}(t_0,t) \leq \mathcal{H}(t_i,t) + \sum_{k=0}^{i-1}\mathcal{H}(t_k, t_{k+1}),$$

we conclude, in view of (3-42), that one has the global bound

$$\begin{aligned}\mathcal{H}(0,t) &\leq \sum_{k=0}^{i}[4C_*(1+\mathcal{E}_0^{1/\alpha})]^{k+1}\mathcal{H}(0) = 4C_*(1+\mathcal{E}_0^{1/\alpha})\frac{[4C_*(1+\mathcal{E}_0^{1/\alpha})]^{i+1}-1}{4C_*(1+\mathcal{E}_0^{1/\alpha})-1}\mathcal{H}(0)\\ &\leq 2[4C_*(1+\mathcal{E}_0^{1/\alpha})]^{i+1}\mathcal{H}(0) \leq 2[4C_*(1+\mathcal{E}_0^{1/\alpha})]^{1+[2C_*(1+\mathcal{E}_0^{1/\alpha+(p-2)/(p-1)})]^{2(p-1)}\mathcal{J}(0,t)^2}\mathcal{H}(0),\end{aligned}$$

provided

$$\begin{aligned}\mathcal{H}(0) &< \frac{c}{4C_*(1+\mathcal{E}_0^{1/\alpha+1})[4C_*(1+\mathcal{E}_0^{1/\alpha})]^{1+[2C_*(1+\mathcal{E}_0^{1/\alpha+(p-2)/(p-1)})]^{2(p-1)}(\sigma/2)\mathcal{E}_0^2}}\\ &\leq \frac{c}{4C_*(1+\mathcal{E}_0^{1/\alpha+1})[4C_*(1+\mathcal{E}_0^{1/\alpha})]^{1+n}}\end{aligned}$$

holds initially.

Summarizing the preceding developments, we have now established the existence of an independent constant $C_* > 0$ such that, if the smooth solution (u, E, B) has initial data satisfying

$$4C_*(1+\mathcal{E}_0^{1/\alpha+1})[4C_*(1+\mathcal{E}_0^{1/\alpha})]^{1+[2C_*(1+\mathcal{E}_0^{1/\alpha+(p-2)/(p-1)})]^{2(p-1)}(\sigma/2)\mathcal{E}_0^2}\mathcal{H}(0) < c, \tag{3-45}$$

then the bound

$$\mathcal{H}(0,t) \le 2[4C_*(1+\mathcal{E}_0^{1/\alpha})]^{1+[2C_*(1+\mathcal{E}_0^{1/\alpha+(p-2)/(p-1)})]^{2(p-1)}\mathcal{J}(0,t)^2}\mathcal{H}(0) \tag{3-46}$$

holds globally for any $t \in [0,\infty)$. In view of the approximation procedure laid out in Section 3.2, this uniform control allows us to complete the construction of solutions claimed in the statement of Theorem 3.1.

Moreover, observe that all global bounds on (u, E, B) stated in (3-2) are a direct consequence of (3-36) and (3-46).

Finally, in order to deduce the simpler initial condition (3-1) from (3-45), there only remains to notice, for any given $\varepsilon > 0$, by taking $0 < \alpha < 1$ and $2 < p < \infty$ sufficiently close to the values 1 and 2, respectively, that

$$4C_*(1+\mathcal{E}_0^{1/\alpha+1})[4C_*(1+\mathcal{E}_0^{1/\alpha})]^{1+[2C_*(1+\mathcal{E}_0^{1/\alpha+(p-2)/(p-1)})]^{2(p-1)}(\sigma/2)\mathcal{E}_0^2} \le C_{**}e^{C_{**}\mathcal{E}_0^{4+\varepsilon}}$$

for some large independent constant $C_{**} > 0$. Then, since the global bound (3-46) holds for that particular choice of p close to 2, one can use the ensuing uniform controls on the solution (u, E, B) in combination with (3-29) to propagate the L^p_x-norm of the vorticity ω for higher values $2 < p < \infty$, which completes the proof of Theorem 3.1. □

3.9. *Proof of Theorem 3.2.* The proof of Theorem 3.2 is a continuation of that of Theorem 3.1. Thus, assuming that the solution (u, E, B) produced by Theorem 3.1 is already constructed, we observe that Lemma 3.6 provides us with the additional bound

$$\begin{aligned}\|\omega(t)\|_{L^\infty_x} &\lesssim \|\omega(0)\|_{L^\infty_x} + \|j\|_{L^2([0,t);L^\infty_x)}\|\nabla B\|_{L^2([0,t);L^\infty_x)}\\ &\lesssim \|\omega(0)\|_{L^\infty_x} + (\|j\|_{L^2([0,t);\dot B^0_{\infty,1,<})} + \|j\|_{L^2([0,t);\dot B^0_{\infty,1,>})})\|\nabla B\|_{L^2([0,t);L^\infty_x)}\end{aligned}$$

for any $t \ge 0$, which, when combined with Ohm's law, the two-dimensional embedding $\dot B^1_{2,1} \subset \dot B^0_{\infty,1}$ and the paradifferential product law (3-14), further yields

$$\begin{aligned}\|\omega(t)\|_{L^\infty_x} &\lesssim \|\omega(0)\|_{L^\infty_x} + (\|j\|_{L^2_t\dot B^1_{2,1,<}} + c\|E\|_{L^2_t\dot B^0_{\infty,1,>}} + \|P(u\times B)\|_{L^2_t\dot B^1_{2,1}})\|\nabla B\|_{L^2_tL^\infty_x}\\ &\lesssim \|\omega(0)\|_{L^\infty_x} + \big(c\|j\|_{L^2_t\dot B^0_{2,\infty,<}} + \|E\|_{L^2_t\dot B^1_{\infty,1,>}}\\ &\qquad + \|u\|_{L^\infty_tL^2_x}\|B\|_{L^2_t\dot B^1_{\infty,1}} + \|u\|_{L^\infty_t\dot H^1_x}\|B\|_{L^2_t\dot H^1_x}\big)\|\nabla B\|_{L^2_tL^\infty_x}.\end{aligned}$$

Now, recall from (3-3) that ∇B belongs to $L^2(\mathbb{R}^+;\dot B^0_{\infty,1}) \subset L^2(\mathbb{R}^+;L^\infty_x)$. Therefore, by virtue of the energy inequality (1-3), the global bounds (3-2) and the assumption that the initial vorticity belongs to L^∞_x, we conclude the pointwise boundedness of $\omega(t)$ for all times.

Observe, though, that the ensuing bound $\omega \in L^\infty_{t,x}$ is global in time, but it is not uniform in c. Nevertheless, it is possible to derive another global bound on the vorticity in $L^\infty_{t,x}$, uniformly in c, by employing Ohm's law to control j in $L^2_t \dot{B}^1_{2,1,<}$ and requiring the additional initial assumption that $E_0 \in \dot{B}^1_{2,1}$.

Specifically, the use of Ohm's law to expand the low frequencies of j leads to the necessity of controlling cE uniformly in the space $L^2_t \dot{B}^1_{2,1,<}$, which can be achieved by relying on the low-frequency estimates from Lemma 3.9. More precisely, by combining (3-21), with $s = 1$, and (3-20), with $s = 2-\alpha$ and $0 < \alpha < 1$, it is possible to establish, by repeating the steps leading to (3-31), that

$$c\|E\|_{L^2_t \dot{B}^1_{2,1,<}} \lesssim \|E_0\|_{\dot{B}^1_{2,1}} + \|B_0\|_{\dot{H}^1_x} + \|u\|_{L^\infty_{t,x} \cap L^\infty_t \dot{H}^1_x} \big(c^{-1}\|B\|_{L^2_t \dot{B}^2_{2,1,<}} + c^{-\frac{3}{4}}\|B\|_{\tilde{L}^2_t \dot{B}^{7/4}_{2,1,>}}\big) \\ + \|u\|^{1-\alpha}_{L^\infty_t L^2_x} \|u\|^\alpha_{L^\infty_t \dot{H}^1_x} \|B\|^\alpha_{L^2_t \dot{H}^1_x} \|B\|^{1-\alpha}_{L^2_t \dot{B}^2_{2,1,<}}.$$

All terms in the right-hand side of this estimate are now uniformly controlled (in c) by the bounds (3-2), provided one further assumes that the initial data E_0 belongs to $\dot{B}^1_{2,1}$. This concludes the justification of the bound $\omega \in L^\infty_{t,x}$, uniformly in c.

We turn now to the uniqueness of solutions to (1-1), which rests upon Yudovich's fundamental ideas. To that end, suppose that

$$(u_i, E_i, B_i) \in L^\infty([0,T); L^2_x),$$

with $i = 1, 2$, are two weak solutions to the two-dimensional incompressible Euler–Maxwell system (1-1) for the same initial data and for some existence time $T > 0$. We are going to establish a weak-strong uniqueness principle by requiring a control on the solution (u_2, E_2, B_2) which is stronger than the one on (u_1, E_1, B_1).

In a natural way, we denote the vorticities and electric currents associated to each solution by ω_i and j_i, respectively. Furthermore, we assume that each solution satisfies its corresponding energy inequality (1-3) and that

$$u_1 \in L^2([0,T); L^\infty_x), \quad \omega_2 \in \bigcap_{2 \le q \le \infty} L^{q'}([0,T); L^q_x), \quad j_2 \in L^1([0,T); L^\infty_x).$$

By virtue of the Gagliardo–Nirenberg inequality (3-17), recall that the above bounds are sufficient to imply that

$$u_2 \in L^2([0,T); L^\infty_x)$$

as well. Note that we are not requiring here that the solutions have the normal structure (1-2).

Next, a straightforward duality argument on (1-1), similar to the computation which gives the energy inequality (1-3), leads to

$$\frac{d}{dt}\int_{\mathbb{R}^2} (u_1 \cdot u_2 + E_1 \cdot E_2 + B_1 \cdot B_2)\,dx + \frac{2}{\sigma}\int_{\mathbb{R}^2} j_1 \cdot j_2\,dx \\ = -\int_{\mathbb{R}^2} (u_2 \otimes (u_1 - u_2)) : \nabla(u_1 - u_2)\,dx \\ + \int_{\mathbb{R}^2} (((j_1 - j_2) \times (B_1 - B_2)) \cdot u_2 - (j_2 \times (B_1 - B_2)) \cdot (u_1 - u_2))\,dx.$$

It is to be emphasized that the solutions considered here have sufficient regularity and integrability to justify the above computation rigorously.

We introduce now the modulated energy

$$F_\varepsilon(t) := \tfrac{1}{2}(\|\tilde{u}(t)\|_{L^2}^2 + \|\widetilde{E}(t)\|_{L^2}^2 + \|\widetilde{B}(t)\|_{L^2}^2) + \varepsilon,$$

where

$$\tilde{u} := u_1 - u_2, \quad \widetilde{E} := E_1 - E_2, \quad \widetilde{B} := B_1 - B_2,$$

and $\varepsilon > 0$ merely ensures the positivity of F_ε.

Then, integrating the preceding identity in time and combining the result with the energy inequality (1-3) for each solution, we deduce the estimate

$$\begin{aligned}
&F_\varepsilon(t) + \frac{1}{\sigma}\int_0^t \|\tilde{j}(\tau)\|_{L_x^2}^2\,d\tau \\
&\quad\le \varepsilon - \int_0^t\int_{\mathbb{R}^2} [((\tilde{u}\cdot\nabla)u_2)\cdot\tilde{u} + (\tilde{j}\times\widetilde{B})\cdot u_2 - (j_2\times\widetilde{B})\cdot\tilde{u}](\tau)\,dx\,d\tau \\
&\quad\le \varepsilon + \int_0^t \|\nabla u_2\|_{L_x^q}\|\tilde{u}\|_{L_x^\infty}^{2/q}\|\tilde{u}\|_{L_x^2}^{2/q'}\,d\tau + \int_0^t \|\tilde{j}\|_{L_x^2}[\|\widetilde{B}\|_{L_x^2}\|u_2\|_{L_x^\infty} + \|j_2\|_{L_x^\infty}\|\widetilde{B}\|_{L_x^2}\|\tilde{u}\|_{L_x^2}]\,d\tau \\
&\quad\le \varepsilon + \int_0^t \|\nabla u_2\|_{L_x^q}\|\tilde{u}\|_{L_x^\infty}^{2/q}\|\tilde{u}\|_{L_x^2}^{2/q'}\,d\tau + \frac{1}{2\sigma}\int_0^t \|\tilde{j}(\tau)\|_{L_x^2}^2\,d\tau \\
&\qquad\qquad + \int_0^t \Big[\frac{\sigma}{2}\|\widetilde{B}\|_{L_x^2}^2\|u_2\|_{L_x^\infty}^2 + \|j_2\|_{L_x^\infty}\|\widetilde{B}\|_{L_x^2}\|\tilde{u}\|_{L_x^2}\Big]\,d\tau
\end{aligned}$$

for all $t \in [0,T)$ and any $2 \le q < \infty$, where we have defined $\tilde{j} := j_1 - j_2$.

The next step relies on a classical sharp estimate on the Biot–Savart law (1-14). More precisely, by exploiting that the map $\omega \mapsto \nabla(-\Delta^{-1}\nabla\times\omega) = \nabla u$ produces a Calderón–Zygmund singular integral operator, it is possible to show that

$$\|\nabla u\|_{L^a(\mathbb{R}^2)} \le C_{\mathrm{BS}}\frac{a^2}{a-1}\|\nabla\times u\|_{L^a(\mathbb{R}^2)}$$

for all $1 < a < \infty$ and any divergence-free vector field u, where $C_{\mathrm{BS}} > 0$ is independent of a. We refer to [Bahouri et al. 2011, Section 7.1.1] for more details concerning the Biot–Savart law and to [Grafakos 2014, Section 6.2.3] for a Fourier multiplier theorem which can be used to obtain the correct dependence of the above Biot–Savart estimate in the parameter a.

Thus, we deduce from the previous bound that

$$F_\varepsilon(t) \le \varepsilon + \int_0^t 2C_{\mathrm{BS}}q\|\omega_2\|_{L_x^q}\|\tilde{u}\|_{L_x^\infty}^{2/q}F_\varepsilon(\tau)^{\frac{1}{q'}} + [\sigma\|u_2\|_{L_x^\infty}^2 + \|j_2\|_{L_x^\infty}]F_\varepsilon(\tau)\,d\tau.$$

Moreover, denoting the right-hand side of the above inequality by $G_\varepsilon(t)$ and observing that it is absolutely continuous on $[0,T)$, we obtain

$$\frac{d}{dt}G_\varepsilon(t) \le 2C_{\mathrm{BS}}q\|\omega_2\|_{L_x^q}\|\tilde{u}\|_{L_x^\infty}^{2/q}G_\varepsilon(t)^{\frac{1}{q'}} + [\sigma\|u_2\|_{L_x^\infty}^2 + \|j_2\|_{L_x^\infty}]G_\varepsilon(t)$$

for almost every $t \in [0,T)$. Therefore, further introducing the absolutely continuous functional

$$\Theta_\varepsilon(t) := G_\varepsilon(t)e^{-\int_0^t[\sigma\|u_2\|_{L_x^\infty}^2 + \|j_2\|_{L_x^\infty}](\tau)\,d\tau},$$

we see that

$$\frac{d}{dt}\Theta_\varepsilon(t) \le 2C_{\mathrm{BS}}q\|\omega_2\|_{L^q_x}\|\tilde u\|^{2/q}_{L^\infty_x}\Theta_\varepsilon(t)^{\frac{1}{q'}}e^{-\frac{1}{q}\int_0^t[\sigma\|u_2\|^2_{L^\infty_x}+\|j_2\|_{L^\infty_x}](\tau)\,d\tau},$$

whence

$$\frac{d}{dt}\Theta_\varepsilon(t)^{\frac{1}{q}} \le 2C_{\mathrm{BS}}\|\omega_2\|_{L^q_x}\|\tilde u\|^{2/q}_{L^\infty_x}e^{-\frac{1}{q}\int_0^t[\sigma\|u_2\|^2_{L^\infty_x}+\|j_2\|_{L^\infty_x}](\tau)\,d\tau}.$$

Observe that all technical difficulties incurred when dividing by $\Theta_\varepsilon(t)^{1/q'}$ in the last step are removed by the use of $\varepsilon > 0$.

Now, integrating in time leads to

$$F_\varepsilon(t)^{\frac{1}{q}} \le G_\varepsilon(t)^{\frac{1}{q}} \le \varepsilon e^{\frac{1}{q}\int_0^t[\sigma\|u_2\|^2_{L^\infty_x}+\|j_2\|_{L^\infty_x}](\tau)\,d\tau} + 2C_{\mathrm{BS}}\int_0^t \|\omega_2(\tau)\|_{L^q_x}\|\tilde u(\tau)\|^{2/q}_{L^\infty_x}e^{\frac{1}{q}\int_\tau^t[\sigma\|u_2\|^2_{L^\infty_x}+\|j_2\|_{L^\infty_x}](s)\,ds}\,d\tau,$$

whereby, letting $\varepsilon \to 0$, we end up with

$$\begin{aligned}
&\frac{1}{2}(\|\tilde u(t)\|^2_{L^2}+\|\widetilde E(t)\|^2_{L^2}+\|\widetilde B(t)\|^2_{L^2})\\
&\le\left(2C_{\mathrm{BS}}\int_0^t \|\omega_2(\tau)\|_{L^q_x}\|\tilde u(\tau)\|^{2/q}_{L^\infty_x}\,d\tau\right)^q e^{\int_0^t[\sigma\|u_2\|^2_{L^\infty_x}+\|j_2\|_{L^\infty_x}](s)\,ds}\\
&\le(2C_{\mathrm{BS}}\|\omega_2\|_{L^{q'}([0,t);L^q_x)})^q\|\tilde u\|^2_{L^2([0,t);L^\infty_x)}e^{\sigma\|u_2\|^2_{L^2([0,t);L^\infty_x)}+\|j_2\|_{L^1([0,t);L^\infty_x)}}\\
&\le(2C_{\mathrm{BS}}\|\omega_2\|_{L^1([0,t);L^\infty_x)})^{q-2}(2C_{\mathrm{BS}}\|\omega_2\|_{L^2([0,t);L^2_x)})^2\|\tilde u\|^2_{L^2([0,t);L^\infty_x)}e^{\sigma\|u_2\|^2_{L^2([0,t);L^\infty_x)}+\|j_2\|_{L^1([0,t);L^\infty_x)}}.
\end{aligned}$$

Finally, considering values $t \in [0, T)$ such that

$$\|\omega_2\|_{L^1([0,t);L^\infty_x)} < \frac{1}{2C_{\mathrm{BS}}}$$

and then letting q tend to infinity, we conclude that $(\tilde u, \widetilde E, \widetilde B)(t) = 0$, thereby establishing the uniqueness of solutions on a time interval $[0, t)$ for some $t \in (0, T)$.

Now, observe that this argument can be reproduced on any time interval $[t_0, T)$, with $t_0 > 0$ and such that $(\tilde u, \widetilde E, \widetilde B)(t_0) = 0$, to prove the uniqueness of solutions on $[t_0, t)$ for some $t \in (t_0, T)$. In other words, we have shown that the set

$$S = \left\{t \in [0, T) : \int_0^t \|(\tilde u, \widetilde E, \widetilde B)(s)\|_{L^2_x}\,ds = 0\right\}$$

is open. Since the function $t \mapsto \int_0^t \|(\tilde u, \widetilde E, \widetilde B)(s)\|_{L^2_x}\,ds$ is continuous, the set S is actually both open and closed. Furthermore, it is nonempty and $[0, T)$ is connected. We conclude that $S = [0, T)$ and, therefore, that both solutions (u_1, E_1, B_1) and (u_2, E_2, B_2) match on the whole interval of existence $[0, T)$. This completes the proof of the weak-strong uniqueness principle and concludes the proof of the theorem. □

Remark. In view of the estimates on the electric current j established in the preceding proof, it seems also possible to propagate the boundedness of the $\dot B^s_{p,1}$-norms of the vorticity, with $p \in (1, \infty)$ and $s = 2/p$, by making use of the methods developed in [Vishik 1998] to prove the global well-posedness of the two-dimensional incompressible Euler system in critical spaces. We also refer to [Abidi et al. 2010; Hassainia 2022] for well-posedness results of similar models in critical spaces.

Remark. It is possible to propagate the boundedness of L^p-norms of the vorticity for values $1 \le p < 2$. Indeed, a variation of the proof of Lemma 3.6 gives

$$\begin{aligned}\|\omega(t)\|_{L^p_x} &\le \|\omega(0)\|_{L^p_x} + \|j\|_{L^2([0,t);L^2_x)} \|\nabla B\|_{L^2([0,t);L_x^{2p/(2-p)})} \\ &\le \|\omega(0)\|_{L^p_x} + \|j\|_{L^2([0,t);L^2_x)} \|\nabla B\|^{2/p-1}_{L^2([0,t);L^2_x)} \|\nabla B\|^{2-2/p}_{L^2([0,t);L^\infty_x)}\end{aligned}$$

for any $t > 0$. It is then readily seen that the terms involving ∇B are controlled by the bounds (3-2) and (3-3), whereas the electric current j remains bounded by virtue of the energy inequality (1-3).

3.10. ***Proof of Theorem 3.3.*** The proof of Theorem 3.3 builds upon the estimates established in Theorem 3.1. Thus, following the proof of that theorem, we assume that we have a smooth solution (u, E, B) of (1-1) for some smooth initial data (u_0, E_0, B_0), and we only derive the bounds relevant to our argument through formal estimates on (u, E, B), keeping in mind that the full justification of the result is then completed by carrying out the approximation strategy laid out in Section 3.2.

Now, the proof of Theorem 3.1 establishes that the bound (3-45) on the initial data (u_0, E_0, B_0) implies the global uniform bound (3-46) on the solution (u, E, B). In particular, combining the two inequalities (3-45) and (3-46), we see that, for any $0 < A < 1$, if the initial data satisfies (3-45) with its right-hand side replaced by Ac, then the solution (u, E, B) satisfies the estimate

$$\mathcal{H}(0, T) < \frac{Ac}{2C_*(1 + \mathcal{E}_0^{1/\alpha+1})} \tag{3-47}$$

for all $T \in [0, \infty)$. This global bound will come in handy below, with some small but fixed value for the constant A.

Next, in order to derive a higher-regularity estimate on the field (E, B), we extend the control of high electromagnetic frequencies (3-30) to higher smoothness parameters. Specifically, a direct application of estimate (3-19) from Lemma 3.8 yields

$$\begin{aligned}c^{1-s}\|(E,B)\|_{\tilde{L}^\infty_t \dot{B}^s_{2,n,>}} + c^{2-s}\|(E,B)\|_{\tilde{L}^2_t \dot{B}^s_{2,n,>}} + c^{\frac{7}{4}-s}\|(E,B)\|_{\tilde{L}^2_t \dot{B}^{s-3/4}_{\infty,n,>}} \\ \lesssim c^{1-s}\|(E_0,B_0)\|_{\dot{B}^s_{2,n,>}} + c^{1-s}\|u\|_{L^\infty_{t,x}\cap L^\infty_t \dot{H}^1_x}\|B\|_{\tilde{L}^2_t \dot{B}^s_{2,n}}\end{aligned}$$

on any time interval $[0, T)$ for any $\frac{7}{4} < s < 2$ and $1 \le n \le \infty$. Then, recalling that the energy $\mathcal{H}(0, T)$ controls the velocity u in $L^\infty_{t,x} \cap L^\infty_t \dot{H}^1_x$ (thanks to the Gagliardo–Nirenberg interpolation inequality (3-16)) and the magnetic field B in $L^2_t \dot{B}^2_{2,1,<}$, we infer that the last term above can be bounded by

$$\begin{aligned}&c^{1-s}\mathcal{H}(0,T)\big(\|B\|^{2-s}_{\tilde{L}^2_t \dot{B}^1_{2,\infty,<}}\|B\|^{s-1}_{\tilde{L}^2_t \dot{B}^2_{2,\infty,<}} + \|B\|_{\tilde{L}^2_t \dot{B}^s_{2,n,>}}\big) \\ &\quad\lesssim c^{1-s}\mathcal{H}(0,T)\big(\|B\|^{2-s}_{L^2_t \dot{H}^1_x}\|B\|^{s-1}_{L^2_t \dot{B}^2_{2,1,<}} + \|B\|_{\tilde{L}^2_t \dot{B}^s_{2,n,>}}\big) \\ &\quad\lesssim c^{1-s}\mathcal{H}(0,T)\big((\mathcal{E}_0 + c^{-1}\mathcal{H}(0) + c^{-1}\mathcal{E}_0\mathcal{H}(0,T))^{2-s}\mathcal{H}(0,T)^{s-1} + \|B\|_{\tilde{L}^2_t \dot{B}^s_{2,n,>}}\big),\end{aligned}$$

where we also employed (3-36) in the last step to control B in $L^2_t \dot{H}^1_x$.

All in all, combining the preceding inequalities and making use of (3-47), we arrive at

$$c^{1-s}\|(E,B)\|_{\tilde L^\infty_t \dot B^s_{2,n,>}} + c^{2-s}\|(E,B)\|_{\tilde L^2_t \dot B^s_{2,n,>}} + c^{\frac74 - s}\|(E,B)\|_{\tilde L^2_t \dot B^{s-3/4}_{\infty,n,>}} \\ \lesssim c^{1-s}\|(E_0,B_0)\|_{\dot B^s_{2,n,>}} + c^{1-s}\mathcal{E}_0^{2-s}\mathcal{H}(0,T)^s + c^{-1}(1+\mathcal{E}_0^{2-s})\mathcal{H}(0,T)^2 + Ac^{2-s}\|B\|_{\tilde L^2_t \dot B^s_{2,n,>}}.$$

Thus, setting the value of the constant A small enough (with respect to fixed parameters only) that the last term above can be absorbed by the left-hand side, the previous estimate can be recast as

$$c^{1-s}\|(E,B)\|_{\tilde L^\infty_t \dot B^s_{2,n,>}} + c^{2-s}\|(E,B)\|_{\tilde L^2_t \dot B^s_{2,n,>}} + c^{\frac74 - s}\|(E,B)\|_{\tilde L^2_t \dot B^{s-3/4}_{\infty,n,>}} \\ \lesssim c^{1-s}\|(E_0,B_0)\|_{\dot B^s_{2,n,>}} + c^{1-s}\mathcal{E}_0^{2-s}\mathcal{H}(0,T)^s + c^{-1}(1+\mathcal{E}_0^{2-s})\mathcal{H}(0,T)^2,$$

which provides the pursued uniform control of the solution (u,E,B) in higher-regularity spaces.

Finally, it only remains to notice that

$$c^{-\frac34}\|(E_0,B_0)\|_{\dot B^{7/4}_{2,1}} \le c^{-\frac34}\|(E_0,B_0)\|_{\dot B^{7/4}_{2,1,<}} + c^{-\frac34}\|(E_0,B_0)\|_{\dot B^{7/4}_{2,1,>}} \\ \lesssim \|(E_0,B_0)\|_{\dot H^1} + c^{1-s}\|(E_0,B_0)\|_{\dot B^s_{2,n}},$$

which allows us to deduce that a suitable choice of independent constant $C_{**}>0$ in the initial assumption (3-4) implies the corresponding initial condition (3-1) in Theorem 3.1, with its right-hand side replaced by Ac (this is necessary to guarantee the validity of (3-47)), thus completing the proof of the theorem. □

3.11. ***Uniform bounds for fixed initial data.*** As previously mentioned, the controls (3-2) and (3-5), from Theorems 3.1 and 3.3, hold for any families of initial data such that the left-hand sides of (3-1) and (3-4) remain respectively bounded. In particular, within such families, the corresponding collection of global solutions only satisfies the respective uniform bounds

$$c^{-\frac34}(E,B) \in \tilde L^\infty(\mathbb{R}^+;\dot B^{7/4}_{2,1}), \quad c^{1-s}(E,B)\in \tilde L^\infty(\mathbb{R}^+;\dot B^s_{2,n}).$$

Thus, there is, in general, no bound on the size of the electromagnetic field (E,B) in $\tilde L^\infty(\mathbb{R}^+;\dot B^{7/4}_{2,1})$ and $\tilde L^\infty(\mathbb{R}^+;\dot B^s_{2,n})$, uniformly in c, if the corresponding family of initial data (E_0,B_0) only satisfies a uniform control

$$c^{-\frac34}(E_0,B_0)\in \dot B^{7/4}_{2,1} \quad\text{and}\quad c^{1-s}(E_0,B_0)\in \dot B^s_{2,n},$$

respectively.

For example, such sets of initial electromagnetic fields occur naturally when considering mollifications

$$(u_0^c,E_0^c,B_0^c) = \varphi_c * (u_0,E_0,B_0),$$

where $\varphi_c(x) = c^2\varphi(cx)$ is a classical approximate identity and (u_0,E_0,B_0) is a given initial data satisfying

$$(\mathcal{E}_0 + \|u_0\|_{\dot H^1\cap \dot W^{1,p}} + \|(E_0,B_0)\|_{\dot H^1})C_{***}e^{C_{***}\mathcal{E}_0^{4+\varepsilon}} < c, \tag{3-48}$$

with $C_{***}>0$. Indeed, it is readily seen that (u_0^c,E_0^c,B_0^c) satisfies the bounds (3-1) and (3-4), provided (3-48) holds for some suitable constant C_{***}.

We are now going to show that the solutions constructed in Theorems 3.1 and 3.3 have, in fact, an electromagnetic field which remains uniformly bounded in

$$\widetilde{L}^\infty(\mathbb{R}^+;\dot{B}^{7/4}_{2,1}) \quad \text{and} \quad \widetilde{L}^\infty(\mathbb{R}^+;\dot{B}^{s}_{2,n}),$$

provided that their corresponding initial values are selected within a bounded family of $\dot{B}^{7/4}_{2,1}$ and $\dot{B}^{s}_{2,n}$, respectively. In particular, such uniform bounds hold whenever one considers fixed initial data independent of c. This is of special significance in the study of the limiting regime $c\to\infty$ in order to derive sharp asymptotic bounds.

The next result provides a suitable energy estimate on the damped Maxwell system (1-15), and the ensuing corollary clarifies the uniform boundedness properties of the solutions built in Theorems 3.1 and 3.3.

Lemma 3.12. *Let $d=2$. Assume that (E,B) is a smooth solution to (1-15) for some initial data (E_0,B_0) and some divergence-free vector field u, with the normal structure (1-2).*

Then, one has the estimate

$$\begin{aligned}\|(E,B)\|_{\widetilde{L}^\infty_t\dot{B}^s_{2,n}}&+c\|E\|_{\widetilde{L}^2_t\dot{B}^s_{2,n}}\\ &\lesssim \|(E_0,B_0)\|_{\dot{B}^s_{2,n}}+\|u\|_{L^\infty_{t,x}\cap L^\infty_t\dot{H}^1_x}\big(\|B\|_{L^2_t\dot{H}^1_x}+\|B\|_{L^2_t\dot{B}^2_{2,1,<}}+\|B\|_{\widetilde{L}^2_t\dot{B}^s_{2,n,>}}\big) \qquad (3\text{-}49)\end{aligned}$$

over any time interval $[0,T)$ for any $1<s<2$ and $1\le n\le\infty$.

Proof. This result follows from a direct energy estimate on the damped Maxwell system (1-15) and is an extension of Lemma 3.10.

In order to show (3-49), we first localize (1-15) in frequencies by applying Δ_j, for $j\in\mathbb{Z}$, and then perform a classical energy estimate on each dyadic frequency component (Δ_jE,Δ_jB). This procedure yields the control

$$\begin{aligned}\frac{1}{2c}\|(\Delta_jE,\Delta_jB)(T)\|^2_{L^2_x}&+\sigma c\|\Delta_jE\|^2_{L^2_t([0,T);L^2_x)}\\ &\le \frac{1}{2c}\|(\Delta_jE,\Delta_jB)(0)\|^2_{L^2_x}+\sigma\|\Delta_jP(u\times B)\|_{L^2_t([0,T);L^2_x)}\|\Delta_jE\|_{L^2_t([0,T);L^2_x)}\\ &\le \frac{1}{2c}\|(\Delta_jE,\Delta_jB)(0)\|^2_{L^2_x}+\frac{\sigma}{2c}\|\Delta_jP(u\times B)\|^2_{L^2_t([0,T);L^2_x)}+\frac{\sigma c}{2}\|\Delta_jE\|^2_{L^2_t([0,T);L^2_x)}.\end{aligned}$$

Then, summing over j in ℓ^n, with $1\le n\le\infty$, we deduce, for any $s\in\mathbb{R}$, that

$$\|(E,B)\|_{\widetilde{L}^\infty_t([0,T);\dot{B}^s_{2,n})}+c\|E\|_{\widetilde{L}^2_t([0,T);\dot{B}^s_{2,n})}\lesssim\|(E_0,B_0)\|_{\dot{B}^s_{2,n}}+\|P(u\times B)\|_{\widetilde{L}^2_t([0,T);\dot{B}^s_{2,n})}.$$

Next, by the paradifferential product law (3-13), we see that

$$\|P(u\times B)\|_{\widetilde{L}^2_t\dot{B}^s_{2,n}}\lesssim\|u\|_{L^\infty_{t,x}\cap L^\infty_t\dot{B}^1_{2,\infty}}\|B\|_{\widetilde{L}^2_t\dot{B}^s_{2,n}}$$

for any $-1<s<2$ and $1\le n\le\infty$. Therefore, combining the preceding inequalities, we conclude that

$$\begin{aligned}\|(E,B)\|_{\widetilde{L}^\infty_t\dot{B}^s_{2,n}}&+c\|E\|_{\widetilde{L}^2_t\dot{B}^s_{2,n}}\\ &\lesssim\|(E_0,B_0)\|_{\dot{B}^s_{2,n}}+\|u\|_{L^\infty_{t,x}\cap L^\infty_t\dot{B}^1_{2,\infty}}\big(\|B\|_{\widetilde{L}^2_t\dot{B}^s_{2,n,<}}+\|B\|_{\widetilde{L}^2_t\dot{B}^s_{2,n,>}}\big)\\ &\lesssim\|(E_0,B_0)\|_{\dot{B}^s_{2,n}}+\|u\|_{L^\infty_{t,x}\cap L^\infty_t\dot{B}^1_{2,\infty}}\big(\|B\|^{2-s}_{\widetilde{L}^2_t\dot{B}^1_{2,\infty,<}}\|B\|^{s-1}_{\widetilde{L}^2_t\dot{B}^2_{2,\infty,<}}+\|B\|_{\widetilde{L}^2_t\dot{B}^s_{2,n,>}}\big)\end{aligned}$$

for all $1<s<2$ and $1\le n\le\infty$.

Finally, combining the previous estimate with straightforward embeddings of Besov spaces proves (3-49), thereby concluding the proof of the lemma. □

Corollary 3.13. *Consider parameters $p \in (2,\infty)$, $\frac{7}{4} \le s < 2$ and $1 \le n \le \infty$ such that $n = 1$ if $s = \frac{7}{4}$. For any fixed initial data*

$$(u_0, E_0, B_0) \in ((H^1 \cap \dot W^{1,p}) \times (H^1 \cap \dot B^s_{2,n}) \times (H^1 \cap \dot B^s_{2,n}))(\mathbb{R}^2)$$

satisfying the assumptions of Theorem 3.1 $\big($if $s = \frac{7}{4}$ and $n = 1\big)$ or Theorem 3.3 $\big($if $s > \frac{7}{4}\big)$, the corresponding global solutions (u^c, E^c, B^c) constructed therein satisfy the bounds

$$u^c \in L^\infty(\mathbb{R}^+; H^1 \cap \dot W^{1,p}), \quad (E^c, B^c) \in L^\infty(\mathbb{R}^+; H^1), \quad (E^c, B^c) \in \widetilde L^\infty(\mathbb{R}^+; \dot B^s_{2,n}),$$
$$(cE^c, B^c) \in L^2(\mathbb{R}^+; \dot H^1), \quad B^c \in L^2(\mathbb{R}^+; \dot B^2_{2,1,<}),$$
$$c^{\frac{7}{4}-s}(E^c, B^c) \in \widetilde L^2(\mathbb{R}^+; \dot B^{s-3/4}_{\infty,n,>}), \quad cE^c \in \widetilde L^2(\mathbb{R}^+; \dot B^s_{2,n}), \quad c^{\frac{1}{4}} B^c \in \widetilde L^2(\mathbb{R}^+; \dot B^s_{2,n,>}),$$

uniformly in c.

Proof. This result follows from a straightforward combination of Theorems 3.1 and 3.3 with Lemma 3.12. Indeed, it is readily seen that the uniform bounds (3-2) and (3-5) are sufficient to control the right-hand side of (3-49) for appropriate values of (s,n), thereby showing the improved uniform boundedness of the solutions (u^c, E^c, B^c). □

Appendix A: Littlewood–Paley decompositions and Besov spaces

We recall here the fundamentals of Littlewood–Paley decompositions and introduce a precise definition of Besov spaces.

A.1. *Littlewood–Paley decompositions.*

We are going to use the Fourier transform

$$\mathcal{F} f(\xi) = \hat f(\xi) := \int_{\mathbb{R}^d} e^{-i\xi\cdot x} f(x)\,dx$$

and its inverse

$$\mathcal{F}^{-1} g(x) = \tilde g(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} g(\xi)\,d\xi$$

in any dimension $d \ge 1$.

Now, consider smooth cutoff functions $\psi(\xi), \varphi(\xi) \in C_c^\infty(\mathbb{R}^d)$ satisfying

$$\psi, \varphi \ge 0 \text{ are radial}, \quad \operatorname{supp}\psi \subset \{|\xi| \le 1\}, \quad \operatorname{supp}\varphi \subset \big\{\tfrac{1}{2} \le |\xi| \le 2\big\}$$

and

$$1 = \psi(\xi) + \sum_{k=0}^{\infty} \varphi(2^{-k}\xi) \quad \text{for all } \xi \in \mathbb{R}^d.$$

Defining the scaled cutoffs

$$\psi_k(\xi) := \psi(2^{-k}\xi), \quad \varphi_k(\xi) := \varphi(2^{-k}\xi)$$

for each $k \in \mathbb{Z}$, so that

$$\operatorname{supp}\psi_k \subset \{|\xi| \le 2^k\}, \quad \operatorname{supp}\varphi_k \subset \{2^{k-1} \le |\xi| \le 2^{k+1}\},$$

it is readily seen that

$$1 \equiv \psi + \sum_{k=0}^{\infty} \varphi_k$$

provides us with a dyadic partition of unity of $\mathbb{R}^d$. Notice also that

$$1 \equiv \psi_j + \sum_{k=j}^{\infty} \varphi_k$$

for any $j \in \mathbb{Z}$ and

$$1 \equiv \sum_{k=-\infty}^{\infty} \varphi_k$$

away from the origin $\xi = 0$.

Next, denoting the space of tempered distributions by $\mathcal{S}'$, we introduce the Fourier multiplier operators

$$S_k, \Delta_k : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d),$$

with $k \in \mathbb{Z}$, defined by

$$S_k f := \mathcal{F}^{-1}\psi_k \mathcal{F} f = (\mathcal{F}^{-1}\psi_k) * f \quad \text{and} \quad \Delta_k f := \mathcal{F}^{-1}\varphi_k \mathcal{F} f = (\mathcal{F}^{-1}\varphi_k) * f. \tag{A-1}$$

The Littlewood–Paley decomposition of $f \in \mathcal{S}'$ is then given by

$$S_0 f + \sum_{k=0}^{\infty} \Delta_k f = f,$$

where the series is convergent in $\mathcal{S}'$.

Similarly, one can verify that the homogeneous Littlewood–Paley decomposition

$$\sum_{k=-\infty}^{\infty} \Delta_k f = f$$

holds in $\mathcal{S}'$ if $f \in \mathcal{S}'$ satisfies

$$\lim_{k \to -\infty} \|S_k f\|_{L^\infty} = 0. \tag{A-2}$$

Observe that (A-2) holds if $\hat{f}$ is locally integrable around the origin, or whenever $S_0 f$ belongs to $L^p(\mathbb{R}^d)$ for some $1 \le p < \infty$. In particular, note that (A-2) excludes all nonzero polynomials.

A.2. ***Besov spaces.*** For any $s \in \mathbb{R}$ and $1 \le p, q \le \infty$, we define now the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R}^d)$ as the subspace of tempered distributions satisfying (A-2) endowed with the norm

$$\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^d)} = \left(\sum_{k \in \mathbb{Z}} 2^{ksq} \|\Delta_k f\|^q_{L^p(\mathbb{R}^d)} \right)^{\frac{1}{q}}$$

if $q < \infty$, and

$$\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^d)} = \sup_{k \in \mathbb{Z}} (2^{ks} \|\Delta_k f\|_{L^p(\mathbb{R}^d)})$$

if $q = \infty$. One can show that $\dot{B}^s_{p,q}$ is a Banach space if $s < d/p$, or if $s = d/p$ and $q = 1$; see [Bahouri et al. 2011, Theorem 2.25].

We also consider here the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^d)$ for any real value $s \in \mathbb{R}$, which is defined as the subspace of tempered distributions whose Fourier transform is locally integrable equipped with the norm

$$\|f\|_{\dot{H}^s} = \left(\int_{\mathbb{R}^d} |\xi|^{2s} |\hat{f}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}.$$

One verifies that $\dot{H}^s$ is a Hilbert space if and only if $s < \frac{1}{2}d$; see [Bahouri et al. 2011, Proposition 1.34]. Moreover, it is possible to show that $\dot{H}^s = \dot{B}^s_{2,2}$ whenever $s < \frac{1}{2}d$.

A.3. *Chemin–Lerner spaces.* For any time $T > 0$ and any choice of parameters $s \in \mathbb{R}$ and $1 \le p, q, r \le \infty$, with $s < d/p$ (or $s = d/p$ and $q = 1$), the spaces

$$L^r([0,T); \dot{B}^s_{p,q}(\mathbb{R}^d))$$

are naturally defined as L^r-spaces with values in the Banach spaces $\dot{B}^s_{p,q}$. In addition to these vector-valued Lebesgue spaces, we define the spaces

$$\widetilde{L}^r([0,T); \dot{B}^s_{p,q}(\mathbb{R}^d))$$

as the subspaces of tempered distributions such that

$$\lim_{k \to -\infty} \|S_k f\|_{L^r([0,T);L^p(\mathbb{R}^d))} = 0,$$

endowed with the norm

$$\|f\|_{\widetilde{L}^r([0,T);B^s_{p,q}(\mathbb{R}^d))} = \left(\sum_{k=-\infty}^{\infty} 2^{ksq} \|\Delta_k f\|^q_{L^r([0,T);L^p(\mathbb{R}^d))} \right)^{\frac{1}{q}}$$

if $q < \infty$, and with the obvious modifications in the case $q = \infty$.

One can easily check that, if $q \ge r$, then

$$L^r([0,T); \dot{B}^s_{p,q}(\mathbb{R}^d)) \subset \widetilde{L}^r([0,T); \dot{B}^s_{p,q}(\mathbb{R}^d))$$

and that, if $q \le r$, then

$$\widetilde{L}^r([0,T); \dot{B}^s_{p,q}(\mathbb{R}^d)) \subset L^r([0,T); \dot{B}^s_{p,q}(\mathbb{R}^d)).$$

We refer the reader to [Bahouri et al. 2011, Section 2.6.3] for more details on Chemin–Lerner spaces.

A.4. *Embeddings.* We present now a few embeddings and inequalities in Besov spaces which are routinely used throughout this work.

First, a direct application of Young's convolution inequality to (A-1) yields

$$\|\Delta_k f\|_{L^r(\mathbb{R}^d)} \lesssim 2^{kd\left(\frac{1}{p}-\frac{1}{r}\right)} \|\Delta_k f\|_{L^p(\mathbb{R}^d)} \tag{A-3}$$

for any $1 \le p \le r \le \infty$. A suitable use of (A-3) then leads to the embedding

$$\|f\|_{\dot{B}^s_{r,q}(\mathbb{R}^d)} \lesssim \|f\|_{\dot{B}^{s+d(1/p-1/r)}_{p,q}(\mathbb{R}^d)} \tag{A-4}$$

for any $1 \le p \le r \le \infty$, $1 \le q \le \infty$ and $s \in \mathbb{R}$, which can be interpreted as a Sobolev embedding in the framework of Besov spaces.

Moreover, recalling that $\ell^q \subset \ell^r$, for all $1 \le q \le r \le \infty$, one has

$$\dot{B}^s_{p,q}(\mathbb{R}^d) \subset \dot{B}^s_{p,r}(\mathbb{R}^d)$$

for all $s \in \mathbb{R}$, $1 \le p \le \infty$ and $1 \le q \le r \le \infty$.

Next, observe that

$$\|f\|_{L^p(\mathbb{R}^d)} = \Big\|\sum_{k\in\mathbb{Z}} \Delta_k f\Big\|_{L^p(\mathbb{R}^d)} \le \sum_{k\in\mathbb{Z}} \|\Delta_k f\|_{L^p(\mathbb{R}^d)} = \|\Delta_k f\|_{\dot{B}^0_{p,1}(\mathbb{R}^d)} \tag{A-5}$$

for every $1 \le p \le \infty$. Therefore, by combining (A-4) and (A-5), we obtain

$$\|f\|_{L^\infty(\mathbb{R}^d)} \lesssim \|f\|_{\dot{B}^{d/2}_{2,1}(\mathbb{R}^d)}.$$

This estimate is particularly useful in view of the failure of the embedding of the Sobolev space $\dot{H}^{d/2}(\mathbb{R}^d)$ into $L^\infty(\mathbb{R}^d)$.

Further considering any cutoff function $\chi \in C_c^\infty(\mathbb{R}^d)$ such that $\mathbb{1}_{\{|\xi|\le 1\}} \le \chi(\xi) \le \mathbb{1}_{\{|\xi|\le 2\}}$, one can show, for any $c > 0$, $\alpha > 0$ and $1 \le p \le \infty$, that

$$\Big\|\chi\Big(\frac{D}{c}\Big) f\Big\|_{\dot{B}^{s+\alpha}_{p,1}(\mathbb{R}^d)} \lesssim c^\alpha \|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^d)}$$

and

$$c^\alpha \Big\|(1-\chi)\Big(\frac{D}{c}\Big) f\Big\|_{\dot{B}^s_{p,1}(\mathbb{R}^d)} \lesssim \|f\|_{\dot{B}^{s+\alpha}_{p,\infty}(\mathbb{R}^d)},$$

where the operator $m(D)$ denotes the Fourier multiplier associated with the symbol $m(\xi)$ for any $m \in C_c^\infty(\mathbb{R}^d)$.

Finally, we mention another essential inequality in Besov spaces which is related to their interpolation properties. Specifically, one has the interpolation, or convexity, inequality

$$\|f\|_{\dot{B}^s_{p,1}} \lesssim \|f\|^{1-\theta}_{\dot{B}^{s_0}_{p,\infty}} \|f\|^{\theta}_{\dot{B}^{s_1}_{p,\infty}}$$

for any $p \in [1,\infty]$, $s, s_0, s_1 \in \mathbb{R}$ and $\theta \in (0,1)$ such that $s = (1-\theta)s_0 + \theta s_1$ and $s_0 \ne s_1$.

Note that the preceding estimates and embeddings can be adapted to the setting of Chemin–Lerner spaces in a straightforward way.

A.5. ***Paradifferential product estimates.*** Here, we recall the basic principles of paraproduct decompositions and some essential paradifferential product estimates that follow from it.

For any two suitable tempered distributions f and g, introducing the paraproduct

$$T_f g = \sum_{j\in\mathbb{Z}} S_{j-2} f \Delta_j g$$

readily leads to the decomposition

$$fg = T_f g + T_g f + R(f,g),$$

where

$$R(f,g)=\sum_{\substack{j,k\in\mathbb{Z}\\ |j-k|\le 2}}\Delta_j f\Delta_k g$$

is the remainder. For any choice of integrability parameters in $[1,\infty]$ such that

$$\frac{1}{a}=\frac{1}{a_1}+\frac{1}{a_2},\quad \frac{1}{b}=\frac{1}{b_1}+\frac{1}{b_2},\quad \frac{1}{c}=\frac{1}{c_1}+\frac{1}{c_2},$$

it can be shown, in the context of Chemin–Lerner spaces, that

$$\|T_f g\|_{\tilde L^a_t \dot B^{\alpha+\beta}_{b,c}}\lesssim \|f\|_{\tilde L^{a_1}_t \dot B^{\alpha}_{b_1,c_1}}\|g\|_{\tilde L^{a_2}_t \dot B^{\beta}_{b_2,c_2}} \tag{A-6}$$

for any $\alpha<0$ and $\beta\in\mathbb{R}$, and that

$$\|R(f,g)\|_{\tilde L^a_t \dot B^{\alpha+\beta}_{b,c}}\lesssim \|f\|_{\tilde L^{a_1}_t \dot B^{\alpha}_{b_1,c_1}}\|g\|_{\tilde L^{a_2}_t \dot B^{\beta}_{b_2,c_2}} \tag{A-7}$$

for any $\alpha,\beta\in\mathbb{R}$ with $\alpha+\beta>0$. Moreover, one has the endpoint estimates

$$\|T_f g\|_{\tilde L^a_t \dot B^{\beta}_{b,c}}\lesssim \|f\|_{L^{a_1}_t L^{b_1}_x}\|g\|_{\tilde L^{a_2}_t \dot B^{\beta}_{b_2,c}} \tag{A-8}$$

for all $\beta\in\mathbb{R}$ and

$$\|R(f,g)\|_{L^a_t L^b_x}\lesssim \|f\|_{\tilde L^{a_1}_t \dot B^{\alpha}_{b_1,c_1}}\|g\|_{\tilde L^{a_2}_t \dot B^{-\alpha}_{b_2,c_2}}$$

for all $\alpha\in\mathbb{R}$, provided $1/c_1+1/c_2=1$. Similar bounds hold for Besov spaces, and we refer to [Bahouri et al. 2011, Section 2.6] for more details on such paradifferential estimates.

We finally recall two important product rules of paradifferential calculus in the context of Besov spaces, which are direct consequences of the preceding bounds. First, exploiting (A-6) and (A-7) (for Besov spaces), we have

$$\|fg\|_{\dot B^{s+t-d/2}_{2,1}}\lesssim \|f\|_{\dot H^s}\|g\|_{\dot H^t}$$

for any $-\frac12 d<s,t<\frac12 d$, with $s+t>0$.

Second, we find that

$$\|fg\|_{\dot H^s}\lesssim \|f\|_{L^\infty\cap \dot B^{d/2}_{2,\infty}}\|g\|_{\dot H^s}$$

for all $-\frac12 d<s<\frac12 d$, which follows from a suitable combination of (A-6), (A-7) and (A-8) (for Besov spaces, as well).

Appendix B: Oscillatory integrals and dispersion

We give here a justification of the dispersive estimate (2-23) employing the stationary phase method. This method is classical and we will rely on [Bahouri et al. 2011], when needed, to refer the reader to complete details on the technical results pertaining to the method.

Generally speaking, we are considering here oscillatory integrals of the form

$$I_\psi(t) = \int_{\mathbb{R}^d} e^{it\phi(\xi)} \psi(\xi)\, d\xi$$

for smooth test functions $\psi \in C_c^\infty(\mathbb{R}^d)$ and $t \in \mathbb{R}$, where the smooth phase $\phi(\xi)$ only needs to be defined on the support of ψ. It is readily seen that

$$|I_\psi(t)| \le \|\psi\|_{L^1(\mathbb{R}^d)}.$$

We seek now to understand the asymptotic behavior of $I_\psi(t)$ when $|t|$ is large, which requires us to exploit the cancellations in the integral $I_\psi(t)$ due to the oscillatory term $e^{it\phi(\xi)}$. There are two cases to consider: the stationary phase and the nonstationary phase.

The stationary phase. This case analyzes the asymptotic behavior of $I_\psi(t)$ near critical points of the phase, i.e., near points in the integration domain where $\nabla\phi(\xi) = 0$. More precisely, we suppose here that

$$|\nabla\phi(\xi)| \le \varepsilon_0$$

for some $\varepsilon_0 \in (0, 1]$ and for all $\xi \in \operatorname{supp}\psi$. Under such assumptions, Theorem 8.9 from [Bahouri et al. 2011] establishes that, for any positive integer N, there is a constant $C_{N,\phi,\psi} > 0$ such that

$$|I_\psi(t)| \le C_{N,\phi,\psi} \int_{\operatorname{supp}\psi} \frac{d\xi}{(1 + \varepsilon_0 t |\nabla\phi(\xi)|^2)^N} \tag{B-1}$$

for all $t > 0$.

The nonstationary phase. The decay of $I_\psi(t)$ is better when $\nabla\phi$ does not vanish on the support of ψ. More precisely, assuming now that

$$|\nabla\phi(\xi)| \ge \varepsilon_0$$

for some $\varepsilon_0 \in (0, 1]$ and for all $\xi \in \operatorname{supp}\psi$, Theorem 8.8 from [Bahouri et al. 2011] shows that, for any positive integer N, there is a constant $C_{N,\phi,\psi} > 0$ such that

$$|I_\psi(t)| \le \frac{C_{N,\phi,\psi}}{(\varepsilon_0 t)^N} \tag{B-2}$$

for all $t > 0$.

The asymptotic estimate (B-2) always offers a faster decay than (B-1) and, therefore, the oscillatory integral $I_\psi(t)$ can be treated as a remainder term wherever the phase $\nabla\phi$ does not vanish. In conclusion, the overall asymptotic behavior of $I_\psi(t)$ is, in general, determined by the critical points of the phase.

All in all, as explained in Theorem 8.12 from [Bahouri et al. 2011], it is possible to combine the preceding estimates to show, for all $\psi \in C_c^\infty(\mathbb{R}^d)$, $\varepsilon_0 \in (0, 1]$ and any positive numbers N and N', that there are positive constants C_N and $C_{N'}$ such that

$$|I_\psi(t)| \le \frac{C_N}{(\varepsilon_0 t)^N} + C_{N'} \int_{A_\phi} \frac{d\xi}{(1 + \varepsilon_0 t |\nabla\phi(\xi)|^2)^{N'}} \tag{B-3}$$

for all $t > 0$, where the set A_ϕ is defined as

$$A_\phi := \{\xi \in \operatorname{supp}\psi : |\nabla\phi(\xi)| \le \varepsilon_0\}.$$

We are now in a position to prove (2-23). To be precise, we are going to establish the equivalent estimate (up to a scaling of the variable x)

$$\left|\int_{\mathbb{R}^d} e^{it\phi(x,\xi)}\psi(\xi)\,d\xi\right| \le \frac{C_\psi}{t^{(d-1)/2}} \tag{B-4}$$

for all $t > 0$ and $x \in \mathbb{R}^d$, where the phase is defined by

$$\phi(x,\xi) = x\cdot\xi \pm \delta(\xi), \quad \text{with } \delta(\xi) = \sqrt{|\xi|^2 - \tfrac{1}{4}\alpha^2} \text{ and } 0 \le \alpha \le \tfrac{1}{2},$$

and the test function $\psi \in C_c^\infty(\mathbb{R}^d)$ satisfies $\operatorname{supp}\psi \subset \left\{\frac{1}{4} < |\xi| < R\right\}$ for some $R > \frac{1}{4}$, while the constant $C_\psi > 0$ is independent of t, x and α.

To that end, noting that $\phi(x,\xi)$ is smooth on the support of ψ and setting $\varepsilon_0 = \frac{1}{2}$, $N = \frac{1}{2}(d-1)$ and $N' = d$ in (B-3), we find that

$$\left|\int_{\mathbb{R}^d} e^{it\phi(x,\xi)}\psi(\xi)\,d\xi\right| \lesssim \frac{1}{t^{(d-1)/2}} + \int_A \frac{d\xi}{\left(1 + t\left|x \pm \frac{\xi}{\delta(\xi)}\right|^2\right)^d},$$

where

$$A := \left\{\frac{1}{4} < |\xi| < R,\ \left|x \pm \frac{\xi}{\delta(\xi)}\right| \le \frac{1}{2}\right\}.$$

Now, notice that $x \neq 0$ if A is nonempty. In particular, for any $\xi \in A$, we can write the decomposition

$$\xi = \zeta_1 + \zeta', \quad \text{with } \zeta_1 := \left(\frac{\xi\cdot x}{|x|^2}\right)x \text{ and } \zeta' := \xi - \zeta_1,$$

whence, since $\zeta'\cdot x = 0$,

$$\left|x \pm \frac{\xi}{\delta(\xi)}\right|^2 = \left|x \pm \frac{\zeta_1}{\delta(\xi)}\right|^2 + \left|\frac{\zeta'}{\delta(\xi)}\right|^2 \ge \frac{|\zeta'|^2}{\delta(\xi)^2} \ge \frac{|\zeta'|^2}{R^2}.$$

We therefore conclude that

$$\left|\int_{\mathbb{R}^d} e^{it\phi(x,\xi)}\psi(\xi)\,d\xi\right| \lesssim \frac{1}{t^{(d-1)/2}} + \int_{\{|\zeta'|<R\}\subset\mathbb{R}^{d-1}} \frac{d\zeta'}{(1 + t|\zeta'|^2)^d} \lesssim \frac{1}{t^{(d-1)/2}},$$

which completes the justification of (B-4). □

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EQUIVARIANT PROPERTY GAMMA AND THE TRACIAL LOCAL-TO-GLOBAL PRINCIPLE FOR C*-DYNAMICS

GÁBOR SZABÓ AND LISE WOUTERS

We consider the notion of equivariant uniform property Gamma for actions of countable discrete groups on C*-algebras that admit traces. In case the group is amenable and the C*-algebra has a compact tracial state space, we prove that this property implies a kind of tracial local-to-global principle for the C*-dynamical system, generalizing a recent such principle for C*-algebras exhibited in work of Castillejos et al. For actions on simple nuclear $\mathcal{Z}$-stable C*-algebras, we use this to prove that equivariant uniform property Gamma is equivalent to equivariant $\mathcal{Z}$-stability, generalizing a result of Gardella, Hirshberg, and Vaccaro.

Introduction

This article aims to extend the fine structure theory for actions of amenable groups on finite simple C*-algebras, in particular those covered by the Elliott program. The classification of such C*-algebras, which mirrors the celebrated Connes–Haagerup classification of injective factors [Connes 1976; Haagerup 1987], has been nearly completed as a culmination of numerous articles by many researchers over the past decade, such as [Elliott et al. 2024; 2020; Gong and Lin 2020; 2022; Gong et al. 2020a; 2020b; Schafhauser 2020; Tikuisis et al. 2017]. Furthermore, Carrion, Gabe, Schafhauser, Tikuisis, and White have announced an eagerly awaited new conceptual proof of the classification theorem [Carrión et al. 2023b], which does not directly rely on the prior works related to tracial approximation. By now it has been recognized that the next natural step is to understand the underlying symmetries of classifiable C*-algebras, which can be interpreted as the goal to classify group actions on them. This mirrors the work

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of Connes, Jones, Ocneanu and others [Connes 1977; Jones 1980; Katayama et al. 1998; Kawahigashi et al. 1992; Masuda 2007; 2013; Ocneanu 1985; Sutherland and Takesaki 1989]. When it comes to classifying group actions on C*-algebras, a number of researchers have introduced many sophisticated methods over the years to classify specific kinds of group actions utilizing certain Rokhlin-type properties [Evans and Kishimoto 1997; Izumi 2004a; 2004b; Izumi and Matui 2010; 2021a; 2021b; Katsura and Matui 2008; Kishimoto 1995; 1998a; 1998b; Matui 2008; 2010; 2011; Nakamura 2000; Sato 2010; Szabó 2021a]. In direct comparison to the generality achieved for actions on von Neumann algebras, the implementation of the involved methods (in particular the Evans–Kishimoto intertwining argument) for actions on C*-algebras remained challenging beyond some specific classes of actions or acting groups.

To combat these methodological obstacles, the first author introduced a categorical framework in [Szabó 2021c] to open up the classification of C*-dynamics up to cocycle conjugacy to methodology directly inspired by [Elliott 2010]. For actions on classifiable C*-algebras without traces, the so-called Kirchberg algebras [1995], this idea led to the recent breakthrough in [Gabe and Szabó 2022; 2024]. The main result of said work implies that, given any countable amenable group G, any outer G-action on a Kirchberg algebra is uniquely determined by its KK^G-class up to cocycle conjugacy.

Although one might be tempted to guess that similar breakthrough results ought to be in reach for actions on finite classifiable C*-algebras, one still has a long way to go before such a goal can be achieved. In analogy to the original obstacles to classify all simple nuclear C*-algebras [Rørdam 2003; Toms 2008; Villadsen 1999], there are basic structural questions to be settled before a classification theory such as in [Gabe and Szabó 2024] can be attempted on finite C*-algebras. When concerned with just the underlying C*-algebras, this is already a serious challenge. On the one hand, there is the question whether the C*-algebras under consideration automatically satisfy certain properties predicted by classification. For the purpose of this article we highlight the property of Jiang–Su stability. If $\mathcal{Z}$ is the so-called Jiang–Su algebra from [Jiang and Su 1999], then a C*-algebra A is called Jiang–Su stable or $\mathcal{Z}$-stable when $A \cong A \otimes \mathcal{Z}$. Although this might seem like a technical property at first glance, it becomes natural with more context: Firstly, $\mathcal{Z}$ behaves (as a C*-algebra) very much like an infinite-dimensional version of the complex numbers $\mathbb{C}$, for instance at the level of K-theory and traces. Secondly, there is by now a pile of evidence that $\mathcal{Z}$-stability holds automatically for many C*-algebras arising from various constructions like the crossed product [Kerr 2020; Kerr and Naryshkin 2021; Kerr and Szabó 2020; Toms and Winter 2013]. The discovery that $\mathcal{Z}$-stability does in fact *not* hold automatically for all simple nuclear C*-algebras has, among other things, led to the nearly proven Toms–Winter conjecture, which asserts that $\mathcal{Z}$-stability can only hold or fail in conjunction with some other, a priori different, regularity conditions.

On the other hand, there is the question about precisely what additional structural consequences (not necessarily equivalent characterizations) are shared by Jiang–Su stable C*-algebras, a good example of which is the recent breakthrough work [Castillejos et al. 2021b] (which was in turn continuing work from [Bosa et al. 2019; Matui and Sato 2014a; Sato et al. 2015]). The most novel technical achievement therein can be identified as the *tracial local-to-global principle* for C*-algebras satisfying the so-called uniform property Gamma, which is a weaker assumption than Jiang–Su stability. Said principle concerns the behavior of elements in a given C*-algebra A with respect to the 2-seminorm $\|\cdot\|_{2,\tau}$ induced by individual tracial states τ on A on the one hand, and the behavior with respect to the uniform tracial 2-norm $\|\cdot\|_u = \sup_\tau \|\cdot\|_{2,\tau}$ on the other hand. While the latter is often of interest in the deeper structure and classification theory for C*-algebras, the former can be understood by studying the tracial von Neumann algebra $\pi_\tau(A)''$ arising as the weak closure of A under the GNS representation associated to an individual trace τ. In a nutshell, the tracial local-to-global principle asserts that any suitable behavior that can be observed one trace at a time can also be observed uniformly, which often allows one to *transfer*, so to speak, phenomena from von Neumann algebras to the C*-algebraic context. This became in turn the main technical driving force behind the main results in [Castillejos et al. 2021b; 2022], which can be summarized by saying that the Toms–Winter conjecture holds for all simple nuclear C*-algebras having the uniform property Gamma. Whether the latter property automatically holds for simple nuclear nonelementary C*-algebras is presently unknown but is of high interest as it currently represents the main obstacle towards a full solution to the Toms–Winter conjecture.

When we turn our attention to C*-dynamics instead of C*-algebras, we can (and should) study analogous well-behavedness properties as for C*-algebras, one important example of which is equivariant Jiang–Su stability. An action $\alpha : G \curvearrowright A$ on a C*-algebra is called (equivariantly) Jiang–Su stable or $\mathcal{Z}$-stable, if α is cocycle conjugate to $\alpha \otimes \mathrm{id}_{\mathcal{Z}} : G \curvearrowright A \otimes \mathcal{Z}$. Assuming G is amenable, it is presently open if this happens automatically when A is simple nuclear and $\mathcal{Z}$-stable; see [Szabó 2021b, Conjecture A]. We note that the analogous question for nonamenable groups is known to have a negative answer [Gardella and Lupini 2021; Jones 1983], although recent insights as in [Gardella et al. 2024; Suzuki 2021] leave some hope for the class of amenable actions of nonamenable groups, which we shall not investigate in this article. If one stresses the point again that $\mathcal{Z}$ essentially looks like an infinite-dimensional version of $\mathbb{C}$, it should not come as a surprise that we can only expect a classification of G-actions on classifiable C*-algebras by reasonable invariants if they are equivariantly $\mathcal{Z}$-stable. The existing work in this direction seems to indicate that equivariant Jiang–Su stability may indeed hold automatically whenever one can reasonably expect it [Gardella et al. 2022; Matui and Sato 2012; 2014b; Sato 2019; Szabó 2018a; Wouters 2023]. The other possible line of investigation, namely further structural consequences of equivariant Jiang–Su stability for group actions, was initiated in [Gardella et al. 2022] as a direct adaption of techniques in [Castillejos et al. 2021b], albeit under rather restrictive assumptions on the actions. We recall one of the key concepts from both said paper and the present work but restrict ourselves in this introduction to the case of unital simple C*-algebras for convenience, despite actually investigating the concept in broader generality.[1]

Definition A. Let G be a countable discrete group. Let A be a separable unital simple C*-algebra such that all 2-quasitraces on A are traces and $T(A) \neq \varnothing$. Given a free ultrafilter ω on $\mathbb{N}$, form the uniform tracial ultrapower A^ω of A. An action $\alpha : G \curvearrowright A$ is said to have equivariant uniform property Gamma if, for any $k \geq 2$, there exist pairwise orthogonal projections $p_1, \dots, p_k \in (A^\omega \cap A')^{\alpha^\omega}$ such that

$$\tau(ap_j) = \frac{1}{k}\tau(a) \quad \text{for all } j = 1, \dots, k,\ a \in A,\ \tau \in T_\omega(A).$$

[1] The reader might consult Definitions 1.7, 1.8, and 2.1 and compare with [Gardella et al. 2022, Definition 3.1].

One can observe easily (Remark 2.2) that equivariant uniform property Gamma is always implied by equivariant Jiang–Su stability. This is important to note because it means that all nontrivial consequences of this property will also hold for Jiang–Su stable actions, and may in fact turn out to be useful for subsequent applications. The most important technical consequence relevant to the present work is the tracial local-to-global principle for C*-dynamical systems over amenable groups. An ad hoc version of this principle appeared in [Gardella et al. 2022, Lemma 4.5] but is only applicable (see [Gardella et al. 2022, Remark 2.2]) for actions that induce an action on tracial states with finite orbits of uniformly bounded size. This assumption appeared not only as a prerequisite for the theory in [Gardella et al. 2022] but is implicitly crucial for the usefulness of the conclusion of this ad hoc principle, which only involves tracial states that are fixed by the action.[2] In this article we aim to remove any assumptions about how actions $G \curvearrowright A$ are allowed to act on the traces of A, as well as strengthen the conclusion of the tracial local-to-global principle compared to [Gardella et al. 2022], in such a way as to directly generalize and strengthen the known principle for C*-algebras [Castillejos et al. 2021b, Lemma 4.1]. In addition to formulating our result in the language of $*$-polynomials as all prior papers did, we would also like to promote the following (formally equivalent) formulation of the tracial local-to-global principle for C*-dynamics, which becomes our main technical result. We not only restrict ourselves for the moment to actions on unital simple nuclear C*-algebras (similarly to before) for convenience but give a slightly weaker version here that is easier to state. We treat a stronger version of the statement in broader generality in the main body of the paper; see Theorems 4.2 and 4.6.

Theorem B. *Let G be a countable amenable group and A a separable unital simple nuclear* C**-algebra. Let $\alpha : G \curvearrowright A$ be an action with equivariant uniform property Gamma. Let $\delta : G \curvearrowright D$ be an action on a separable* C**-algebra and $B \subseteq D$ a δ-invariant* C**-subalgebra. Suppose that $\varphi : (B, \delta) \to (A^\omega, \alpha^\omega)$ is an equivariant $*$-homomorphism. Then φ extends to an equivariant $*$-homomorphism $\bar{\varphi} : (D, \delta) \to (A^\omega, \alpha^\omega)$ if and only if, for every trace $\tau \in \overline{T_\omega(A)}^{w^*}$, there exists a $*$-homomorphism $\varphi^\tau : (D, \delta) \to (\pi_\tau^{\alpha^\omega}(A^\omega)'', \alpha^\omega)$ with $\varphi^\tau|_B = \pi_\tau^{\alpha^\omega} \circ \varphi$.[3] Here π_τ denotes the GNS representation associated to the trace τ, and $\pi_\tau^{\alpha^\omega}$ denotes the direct sum representation $\bigoplus_{g \in G} \pi_\tau \circ \alpha^\omega_{g^{-1}}$.*

Not unlike previous approaches, the proof of this main result factors through a kind of dynamical version of CPoU (the existence of so-called *complemented partitions of unity*) that we establish along the way; see Lemma 3.2. There are two things to note about this, however. Firstly, the present version of dynamical CPoU does not generally match the property suggested for this purpose in [Gardella et al. 2022], and based on our work we are in fact uncertain whether that property can be expected to hold even under the validity of the above theorem. Secondly, we would like to propose a slight perspective shift by viewing the above local-to-global principle as the primary conceptual property to be studied and exploited instead of the dynamical CPoU, which we feel — especially compared to previous iterations — to be rather unwieldy by itself due to its elaborate technical nature.

[2]This uses that if $\alpha : G \curvearrowright A$ is assumed to induce an action $G \curvearrowright T(A)$ with finite orbits of uniformly bounded cardinality $M > 0$, then one has $\|\cdot\|_{2,u} \leq M \|\cdot\|_{2,T(A)^\alpha}$ as norms on A.

[3]In actuality one may even allow φ^τ to have range in the tracial ultrapower of this von Neumann algebra, but this requires more cumbersome notation to state rigorously.

In the main body of the paper, we actually prove a stronger version of Theorem B for a class of actions on much more general C*-algebras. We would like to comment that the starting point of our theory presumes that the underlying C*-algebra satisfies a kind of *weak CPoU*, namely the one shown to hold for nuclear C*-algebras in [Castillejos et al. 2021b, Lemma 3.6]. Fortunately, a result in the recent preprint [Carrión et al. 2023a] implies that this kind of weak CPoU in fact holds automatically for all C*-algebras with compact tracial state space, which we can use to our advantage.

As for the rest of the paper, we apply Theorem B (or rather Theorems 4.2 and 4.6) to gain insight on equivariant Jiang–Su stability. A famous argument due to Matui and Sato [2012] and the main result of [Szabó 2021b] allows us to argue (as explained in Section 5) that an action α as above is equivariantly Jiang–Su stable if and only if $A \cong A \otimes \mathcal{Z}$ and α is uniformly McDuff, i.e., there exist unital $*$-homomorphisms

$$M_n \to (A^\omega \cap A')^{\alpha^\omega} \quad \text{for all } n \in \mathbb{N}.$$

Once we note that the latter property is known to hold one trace at a time as a consequence of Ocneanu's theorem [1985] (in the generality we need it, this is imported from [Szabó and Wouters 2024]), the above result can be applied to the α-equivariant inclusion

$$1_n \otimes \mathrm{id}_A : A \to M_n(A)$$

to deduce the following consequence. As before, we note that we prove this result in greater generality than stated here; see Theorem 5.7.

Corollary C (cf. [Gardella et al. 2022, Theorem 7.6]). *Let $\alpha : G \curvearrowright A$ be an action of a countable amenable group on a separable unital simple nuclear $\mathcal{Z}$-stable* C*-*algebra. Then α has equivariant uniform property Gamma if and only if α is equivariantly Jiang–Su stable.*

We expect the main result of this article to have an impact on subsequent applications of equivariant uniform property Gamma or equivariant Jiang–Su stability, in particular in the context of classifying actions on tracial C*-algebras.

As far as potential further research is concerned, let us point out that, for group actions $\alpha : G \curvearrowright A$ that are assumed to be "sufficiently free",[4] the theory pursued in this article can be seen as an instance where one studies uniform property Gamma for the inclusion of C*-algebras $A \subseteq A \rtimes_{\alpha,r} G$ in such a way as to strengthen uniform property Gamma for A. It is a tantalizing issue to determine a common framework encompassing all applications of interest regarding uniform property Gamma for more general inclusions of C*-algebras. For instance, it has been hypothesized in past work [Kerr and Szabó 2020, Remark 9.6] that, for a free minimal action $G \curvearrowright X$ of an amenable group on a compact metric space, some desirable dynamical properties ought to follow from a different kind of uniform property Gamma for the inclusion $\mathcal{C}(X) \subseteq \mathcal{C}(X) \rtimes G$, namely the one that strengthens uniform property Gamma for the crossed product; see also [Liao and Tikuisis 2022].

[4] A priori, this may have several different interpretations.

1. Preliminaries

Notation 1.1. Throughout this paper, we will use the following notation and conventions unless specified otherwise:

- By default, ω denotes some free ultrafilter on $\mathbb{N}$. At times it can make it easier to state a claim using two free ultrafilters, in which case we denote a second one by κ.
- If F is a finite subset inside another set M, we often denote this by $F \Subset M$.
- $\mathbb{K}$ denotes the compact operators on the Hilbert space $\ell^2(\mathbb{N})$.
- Let A be a C*-algebra. We denote its positive elements by A_+ and its minimal unitization by $\tilde{A}$. We will also make use of its Pedersen ideal, denoted by $\mathcal{P}(A)$. We assume the reader is familiar with the basic properties of this object. Given a positive element $a \in A$ and $\varepsilon > 0$, we denote by $(a-\varepsilon)_+$ the positive part of the self-adjoint element $a - \varepsilon 1_{\tilde{A}}$.
- The topological cone of lower semicontinuous traces on A_+ will be denoted by $\widetilde{T}(A)$; cf. [Elliott et al. 2011]. We call such a trace τ on A *trivial* if it is $\{0,\infty\}$-valued. It is well known that trivial traces are in one-to-one correspondence with the ideal lattice of A by mapping a trivial trace τ to the linear span of $\tau^{-1}(0)$. The set of nontrivial lower semicontinuous traces on A_+ will be denoted by $T^+(A)$ and the set of tracial states will be denoted by $T(A)$. In this paper, we say that a compact subset $K \subset T^+(A)$ is a *compact generator* for $T^+(A)$ if $\mathbb{R}^{>0}K = T^+(A)$.[5]
- In addition, we denote by $Q\widetilde{T}_2(A)$ the set of lower semicontinuous 2-quasitraces (see [Blanchard and Kirchberg 2004, Definition 2.22]) on A, which contains $\widetilde{T}(A)$. We usually only mention them to assume in appropriate contexts that there are no genuine quasitraces, i.e., $Q\widetilde{T}_2(A) = \widetilde{T}(A)$.

We recall the following existence theorem for traces. This follows from a combination of the work of Blackadar and Cuntz [1982, Theorem 1.5] and Haagerup [2014]; see also [Blanchard and Kirchberg 2004, Remark 2.29 (i)].

Theorem 1.2. *Let A be a simple, exact* C**-algebra such that $A \otimes \mathbb{K}$ contains no infinite projections. Then $Q\widetilde{T}_2(A) = \widetilde{T}(A)$ and $T^+(A) \neq \varnothing$.*

In particular, this implies that each stably finite, simple, separable, nuclear C^*-algebra admits a nontrivial trace.

Definition 1.3 [Kirchberg 2006, Definition 1.1; Kirchberg and Rørdam 2014, Definition 4.3]. Let A be a C*-algebra with an action $\alpha : G \curvearrowright A$ of a discrete group.

(1) The *ultrapower* of A is defined as

$$A_\omega := \ell^\infty(A) / \big\{ (a_n)_{n\in\mathbb{N}} \in \ell^\infty(A) : \lim_{n\to\omega} \|a_n\| = 0 \big\}.$$

(2) Pointwise application of α on representing sequences induces an action on the ultrapower, which we will denote by $\alpha_\omega : G \curvearrowright A_\omega$.

[5] In case A is simple, an example of such a compact generator is given by $\{\tau \in T^+(A) \mid \tau(a) = 1\}$ for some $a \in \mathcal{P}(A)_+ \setminus \{0\}$.

(3) There is a natural inclusion $A \subset A_\omega$ by identifying an element of A with its constant sequence. Define

$$A_\omega \cap A' := \{x \in A_\omega \mid [x, A] = 0\} \quad \text{and} \quad A_\omega \cap A^\perp := \{x \in A_\omega \mid xA = Ax = 0\}.$$

The quotient

$$F_\omega(A) := (A_\omega \cap A')/(A_\omega \cap A^\perp)$$

is called the (*corrected*) *central sequence algebra*. If A is σ-unital, then $F_\omega(A)$ is unital, where the unit is represented by a sequential approximate unit $(e_n)_{n\in\mathbb{N}}$.

(4) Since A is α_ω-invariant, so are $A_\omega \cap A'$ and $A_\omega \cap A^\perp$. Thus, α_ω induces an action on $F_\omega(A)$, which we will denote by $\tilde{\alpha}_\omega : G \curvearrowright F_\omega(A)$.

Definition 1.4. Let A be a C*-algebra. A sequence of tracial states $(\tau_n)_{n\in\mathbb{N}}$ on A defines a trace on A_ω via

$$[(a_n)_{n\in\mathbb{N}}] \mapsto \lim_{n\to\omega} \tau_n(a_n).$$

A trace of this form is called a *limit trace*. The set of all limit traces on A_ω will be denoted by $T_\omega(A)$. More generally, following [Szabó 2021b, Definition 2.1], a sequence $(\tau_n)_{n\in\mathbb{N}}$ in $\widetilde{T}(A)$ defines a lower semicontinuous trace $\tau : \ell^\infty(A)_+ \to [0, \infty]$ by

$$\tau((a_n)_{n\in\mathbb{N}}) = \sup_{\varepsilon>0} \lim_{n\to\omega} \tau_n((a_n - \varepsilon)_+).$$

This trace is the lower semicontinuous regularization of the trace given by $\lim_{n\to\omega} \tau_n(a_n)$; see [Elliott et al. 2011, Lemma 3.1]. This regularization ensures that $\tau((a_n)_{n\in\mathbb{N}}) = 0$ if $\lim_{n\to\omega} \|a_n\| = 0$, so τ also induces a lower semicontinuous trace on A_ω. A trace of this form on A_ω is called a *generalized limit trace*. The set of all generalized limit traces is denoted by $\widetilde{T}_\omega(A)$.

For the next part, assume A is separable. For any $a \in A_+$ and $\tau \in \widetilde{T}_\omega(A)$, we can define a trace

$$\tau_a : (A_\omega \cap A')_+ \to [0, \infty], \quad x \mapsto \tau(ax).$$

We have that $\tau_a(x) \le \|x\|\tau(a)$, so this trace is bounded whenever $\tau(a) < \infty$. Note that this trace also induces a trace on $F_\omega(A)$, which by abuse of notation will also be denoted by τ_a. Clearly this yields a tracial state under the assumption $\tau(a) = 1$. Let us say that a given tracial state τ on $F_\omega(A)$ is a *canonical trace* if it belongs to the weak-$*$-closed convex hull of $\{\tau_a \mid \tau \in \widetilde{T}_\omega(A), a \in A_+, \tau(a) = 1\}$.

Remark 1.5. We point out that it is not necessary to consider generalized limit traces in an important subcase that often occurs in the literature. Namely, assume A is a separable simple C*-algebra with $\varnothing \neq T^+(A) = \mathbb{R}^{>0}T(A)$ such that $T(A)$ is compact.[6] Then it follows by [Castillejos and Evington 2021, Proposition 2.3] that every generalized limit trace $\tau \in \widetilde{T}_\omega(A)$ that is finite on some nonzero positive element of A is a multiple of an ordinary limit trace.

Next we recall how various versions of tracial ultrapowers are defined.

[6]For instance, this is automatic when A is separable, simple, nuclear, unital, and stably finite.

Definition 1.6 [Ando and Haagerup 2014, Propositions 3.1 and 3.2]. Suppose $\mathcal{M}$ is a finite von Neumann algebra with faithful normal tracial state τ. Then the *tracial von Neumann algebra ultrapower* is defined as

$$\mathcal{M}^{\omega} := \ell^{\infty}(\mathcal{M})/\big\{(x_n)_{n\in\mathbb{N}} \in \ell^{\infty}(\mathcal{M}) \mid \lim_{n\to\infty} \|x_n\|_{2,\tau} = 0\big\}. \tag{1-1}$$

This is again a von Neumann algebra with a faithful normal tracial state τ^{ω} that is defined on representative sequences by $\tau^{\omega}((x_n)_{n\in\mathbb{N}}) = \lim_{n\to\omega} \tau(x_n)$.

The notation used for the tracial von Neumann algebra ultrapower is the same as for the uniform tracial ultrapower of a suitable C^*-algebra as defined below. It will be clear from context which of the two notions we use. In the special case that A is a C^*-algebra with unique tracial state τ and no unbounded traces, the uniform tracial ultrapower A^{ω} is naturally isomorphic, by Kaplansky's density theorem, to the von Neumann tracial ultrapower $(\pi_\tau(A)'')^{\omega}$, where π_τ denotes the GNS representation associated to τ. Note that on a tracial von Neumann algebra $(\mathcal{M}, \tau)$, the topology induced by the $\|\cdot\|_{2,\tau}$-norm agrees with the $*$-strong operator topology on bounded subsets. So equivalently, in (1-1) one can quotient out by the sequences that converge to 0 in the $*$-strong operator topology, which makes the construction equivalent to the Ocneanu ultrapower; cf. [Ando and Haagerup 2014].

Definition 1.7. Let A be a C*-algebra. Given a constant $p \geq 1$ and $\tau \in T(A)$, we define a seminorm $\|\cdot\|_{p,\tau}$ on A by

$$\|a\|_{p,\tau} = \tau(|a|^p)^{1/p}, \quad a \in A.$$

We will in particular appeal to the cases $p = 1$ or $p = 2$ subsequently. For a nonempty set $X \subset T(A)$, we define a seminorm $\|\cdot\|_{2,X}$ on A by

$$\|a\|_{2,X} := \sup_{\tau\in X} \|a\|_{2,\tau}$$

for all $a \in A$. The seminorm $\|\cdot\|_{2,T(A)}$ is also denoted by $\|\cdot\|_{2,u}$. This is a norm if and only if, for all nonzero $a \in A$, there exists some $\tau \in T(A)$ such that $\tau(a^*a) > 0$, which is in particular the case when A is simple with $T(A)$ nonempty.

We note that in the construction below, we deviate from other sources by making a very explicit distinction in terminology between C*-algebras that do or do not admit nontrivial unbounded traces.

Definition 1.8 (cf. [Castillejos et al. 2021b, Section 1.3][7]). Let A be a C*-algebra with $T(A) \neq \varnothing$. Then the *trace-kernel ideal* (*with respect to bounded traces*) inside A_ω is defined by

$$J_A^{\mathrm{b}} := \big\{[(a_n)_{n\in\mathbb{N}}] \in A_\omega \mid \lim_{n\to\omega} \|a_n\|_{2,T(A)} = 0\big\}.$$

The *uniform bounded tracial ultrapower* is defined as the quotient

$$A^{\omega,\mathrm{b}} := A_\omega / J_A^{\mathrm{b}}.$$

Whenever $\|\cdot\|_{2,T(A)}$ defines a norm on A, there also exists a canonical embedding of A into $A^{\omega,\mathrm{b}}$. Then $A^{\omega,\mathrm{b}} \cap A'$ is called the *uniform bounded tracial central sequence algebra*. Whenever we have an action

[7]The cited source assumes separability, but we generalize the definition beyond that case.

$\alpha : G \curvearrowright A$ of a discrete group, the ideal J_A^{b} is α_ω-invariant. Hence, there is an induced action on the uniform bounded tracial ultrapower, which we will denote by $\alpha^\omega : G \curvearrowright A^{\omega,\mathrm{b}}$.

Clearly, every limit trace vanishes on J_A^{b} and hence also induces a tracial state on $A^{\omega,\mathrm{b}}$. We will also use $T_\omega(A)$ to denote the collection of limit traces on $A^{\omega,\mathrm{b}}$. Note that

$$J_A^{\mathrm{b}} = \{x \in A_\omega : \|x\|_{2,T_\omega(A)} = 0\},$$

so in particular $\|\cdot\|_{2,T_\omega(A)}$ defines a norm on $A^{\omega,\mathrm{b}}$.

Finally, if we assume A is a simple C*-algebra such that $Q\widetilde{T}_2(A) = \widetilde{T}(A)$ and $\varnothing \neq T^+(A) = \mathbb{R}^{>0} \cdot T(A)$ with $T(A)$ compact, then we simply call $J_A = J_A^{\mathrm{b}}$ the trace-kernel ideal, $A^\omega = A^{\omega,\mathrm{b}}$ the uniform tracial ultrapower, and $A^\omega \cap A'$ the uniform tracial central sequence algebra.

Remark 1.9. Our choice to add the extra "bounded" in the terminology above and the extra letter "b" in the notation, which is usually not included in other sources such as the ones we cite, is deliberate and has the purpose to not overuse the word "uniform", in particular in cases where it becomes rather misleading. This is most apparent for nonsimple C*-algebras; if B is any unital simple C*-algebra with $T(B) \neq \varnothing$, then the above construction applied to $A = B \oplus \mathbb{K}$ yields $A^{\omega,\mathrm{b}} = B^\omega$ by virtue of the fact that the canonical trace on $\mathbb{K}$ is unbounded. Since one of the two tracial direct summands is entirely forgotten in this construction, this object seems unfit to be called "uniform tracial". However, even the case of simple C*-algebras is enough to illustrate why one should not equate $A^{\omega,\mathrm{b}}$ with the object capturing all "uniform tracial" data. Namely, the range result [Gong and Lin 2022] combined with a little playing around with invariants allows one to see that, given any metrizable Choquet simplex S with $\partial_e S$ admitting some isolated point, there exists a (nonunital) classifiable C*-algebra A such that $T^+(A)$ has a Choquet base affinely homeomorphic to S, yet A has a unique tracial state τ. In this scenario, we have $A^{\omega,\mathrm{b}} \cong (\pi_\tau(A)'')^\omega \cong \mathcal{R}^\omega$ as a consequence of Connes' theorem. So despite A having a rich tracial structure, the only trace captured by this construction is τ, which compels us to not apply the word "uniform" or the notation "A^ω" to such an example.

Note that the phenomenon discussed here is also what motivated us to subsequently revise the definition of (equivariant) uniform property Gamma in the spirit of [Castillejos and Evington 2021], as well as introduce an auxiliary version of it that explicitly only takes into account tracial states, even when the surrounding C*-algebra may have other unbounded traces.

Remark 1.10. Let A be a σ-unital C*-algebra with $T(A) \neq \varnothing$. By [Castillejos et al. 2021b, Proposition 1.11] the uniform bounded tracial ultrapower $A^{\omega,\mathrm{b}}$ is unital if and only if $T(A)$ is compact.[8] Moreover, [Castillejos et al. 2021b, Lemma 1.10] shows that, in that case, the natural map $A_\omega \cap A' \to A^{\omega,\mathrm{b}} \cap A'$ factors through $F_\omega(A)$. In case A is separable, this natural map is surjective by a combination of Propositions 4.5 (iii) and 4.6 in [Kirchberg and Rørdam 2014] (the unitality hypothesis in the second cited proposition is not needed, as it suffices to take a unit in the minimal unitization for the proof).

As we have argued above, there are some issues if one is trying to define the object A^ω for a C*-algebra A that may possess many unbounded traces. In fact, trying to find a viable general definition that has the

[8]The cited statement assumes separability of A, but a closer look at the proof shows that σ-unitality is sufficient.

same level of utility as in the case of unital C*-algebras has eluded a number of researchers for years. By introducing the next few definitions and observations, however, we wish to promote the viewpoint that there is a rather natural way to define the object $A^\omega \cap A'$ for any separable C*-algebra A, even if we do not know at present how to properly define the object A^ω itself. Since we are unsure of the viability of this definition when A admits genuine quasitraces, we wish to be cautious and shall only define the concepts below under the assumption that A does not admit them.[9]

Definition 1.11. Let A be a separable C*-algebra with $Q\widetilde{T}_2(A) = \widetilde{T}(A)$ and $T^+(A) \neq \varnothing$. The *trace-kernel ideal* $\mathcal{J}_A$ inside $F_\omega(A)$ is defined as the set of elements $x \in F_\omega(A)$ such that, for every generalized limit trace $\tau \in \widetilde{T}_\omega(A)$ and $a \in A_+$ with $0 < \tau(a) < \infty$, we have $\tau_a(x^*x) = 0$. With some abuse of notation, we denote the quotient by

$$A^\omega \cap A' = F_\omega(A)/\mathcal{J}_A. \tag{1-2}$$

It is clear from construction that a canonical trace on $F_\omega(A)$ vanishes on $\mathcal{J}_A$, so it descends to a tracial state on $A^\omega \cap A'$. As before, we call a given tracial state on $A^\omega \cap A'$ a *canonical trace* if it is induced by a canonical trace on $F_\omega(A)$, or equivalently if it belongs to the weak-$*$-closed convex hull of the tracial states τ_a on $A^\omega \cap A'$, where $\tau \in \widetilde{T}_\omega(A)$ and $a \in A_+$ with $\tau(a) = 1$.

If $\alpha : G \curvearrowright A$ is an action of a discrete group with induced action $\tilde{\alpha}_\omega : G \curvearrowright F_\omega(A)$, then clearly $\mathcal{J}_A$ is $\tilde{\alpha}_\omega$-invariant, so we obtain an induced action $\alpha^\omega : G \curvearrowright A^\omega \cap A'$.

Remark 1.12. In the case that A is simple, and $Q\widetilde{T}_2(A) = \widetilde{T}(A)$ and $\varnothing \neq T^+(A) = \mathbb{R}^{>0}T(A)$ with $T(A)$ compact, it follows from Remarks 1.5 and 1.10 that $F_\omega(A)/\mathcal{J}_A = A^{\omega,\mathrm{b}} \cap A'$, so the notation $A^\omega \cap A'$ is consistent with the last part of Definition 1.8.

Remark 1.13 (see remark after [Kirchberg and Rørdam 2014, Definition 4.3]). Let $p \geq 1$ be any constant. Given any element x in a C*-algebra B with a tracial state θ, one has the inequalities

$$\|x\|_{1,\theta} \leq \|x\|_{p,\theta} \leq \|x\|_{1,\theta}^{1/p} \|x\|^{1-1/p}.$$

This implies that an element in either Definition 1.8 or 1.11 belongs to the trace-kernel ideal if and only if its tracial p-norms vanish with respect to the appropriately chosen (limit) traces. We will frequently use this without further mention for $p = 1$.

Remark 1.14. One of Kirchberg's initial observations about $F_\omega(A)$, which attests to the naturality of its construction, is that it is a stable invariant. We are about to argue that the same applies to the construction $A \mapsto A^\omega \cap A'$. For this purpose, let $\{e_{k,\ell} \mid k, \ell \geq 1\}$ be a set of matrix units generating $\mathbb{K}$, and let $1_n \in \mathbb{K}$ be the increasing approximate unit given by $1_n = \sum_{j=1}^n e_{j,j}$. We recall (see [Kirchberg 2006, Proposition 1.9, Corollary 1.10]) that there is a canonical isomorphism $\theta : F_\omega(A) \to F_\omega(A \otimes \mathbb{K})$ defined as follows: given an element $x \in F_\omega(A)$ represented by a central sequence $(x_n)_{n\in\mathbb{N}}$ in A, it is sent to the element $\theta(x)$ represented by the central sequence $(x_n \otimes 1_n)_{n\in\mathbb{N}}$.

[9]At the same time, we note that the concepts make sense formally anyway, and none of the subsequent arguments hinge on the assumption that A does not admit genuine quasitraces.

Proposition 1.15. *Let A be a separable* C*-*algebra with $Q\widetilde{T}_2(A) = \widetilde{T}(A)$ and $T^+(A) \neq \varnothing$. Then the canonical isomorphism $F_\omega(A) \cong F_\omega(A \otimes \mathbb{K})$ preserves the canonical traces on both sides. Consequently, it descends to a canonical isomorphism*

$$A^\omega \cap A' \cong (A \otimes \mathbb{K})^\omega \cap (A \otimes \mathbb{K})'.$$

Proof. As we set up before the proposition, we denote the canonical isomorphism by θ. It is clear that it induces an affine homeomorphism between all tracial states on $F_\omega(A)$ and on $F_\omega(A \otimes \mathbb{K})$ via $\tau \mapsto \tau \circ \theta^{-1}$. The claim amounts to showing that the image of the canonical traces on the left is equal to the canonical traces on the right.

Let Tr be the unique lower semicontinuous trace on $\mathbb{K}$ with $\mathrm{Tr}(e_{1,1}) = 1$. We keep in mind that the assignment $\widetilde{T}(A) \to \widetilde{T}(A \otimes \mathbb{K})$ given by $\tau \mapsto \tau \otimes \mathrm{Tr}$ is an affine homeomorphism. Given a generalized limit trace $\tau \in \widetilde{T}_\omega(A)$ induced by a sequence $(\tau_n)_{n \in \mathbb{N}}$ in $\widetilde{T}(A)$, let us denote by $\tau^s \in \widetilde{T}_\omega(A \otimes \mathbb{K})$ the generalized limit trace induced by the sequence $(\tau_n \otimes \mathrm{Tr})_{n \in \mathbb{N}}$ in $\widetilde{T}(A \otimes \mathbb{K})$. Clearly the assignment $\tau \mapsto \tau^s$ is also a bijection between generalized limit traces. Let such a generalized limit trace τ be given on A_ω. Given $a \in (A \otimes \mathbb{K})_+$, we can write $a = \sum_{k,\ell=1}^\infty a_{k,\ell} \otimes e_{k,\ell}$ for uniquely determined elements $a_{k,\ell} \in A$. It then follows from [Castillejos and Evington 2021, Proposition 2.9] that we have a norm-convergent sum expression

$$\tau_a^s \circ \theta = \sum_{\ell=1}^\infty \tau_{a_{\ell,\ell}}. \tag{1-3}$$

Applied to $a = b \otimes e_{1,1}$ for some $b \in A_+$ with $\tau(b) = 1$, this gives $\tau_b \circ \theta^{-1} = \tau_a^s$. From this we can infer that canonical traces are mapped to canonical traces. The general expression (1-3) applied to $a \in (A \otimes \mathbb{K})_+$ with $\tau^s(a) = 1$ shows that we have a bijection. □

We give two more technical lemmas that will be useful later on.

Lemma 1.16. *Let A be a* C*-*algebra with positive element $a \in A_+$. Let $(\varepsilon_n)_{n \in \mathbb{N}}$ be a sequence of positive constants such that $\lim_{n \to \infty} \varepsilon_n = 0$, and let $(b_n)_{n \in \mathbb{N}}$ be a sequence of positive elements such that $\|b_n - a\| < \varepsilon_n$. For each $\tau \in \widetilde{T}_\omega(A)$ and $c \in F_\omega(A)$, one has*

$$\lim_{n \to \infty} \tau_{(b_n - \varepsilon_n)_+}(c) = \tau_a(c).$$

Proof. For each $n \in \mathbb{N}$, there exists a contraction $d_n \in A$ such that $(b_n - \varepsilon_n)_+ = d_n a d_n^*$ by [Kirchberg and Rørdam 2002, Lemma 2.2]. This implies that $\tau_{(b_n - \varepsilon_n)_+}(c) \leq \tau_a(c)$. Since the sequence $(b_n - \varepsilon_n)_+$ converges to a and τ is lower semicontinuous, this leads to the desired result. □

Lemma 1.17. *Let A be a simple* C*-*algebra with $a \in \mathcal{P}(A)_+ \setminus \{0\}$, and let K be a compact generator for $T^+(A) \neq \varnothing$. Take a generalized limit trace $\tau \in \widetilde{T}_\omega(A)$ such that $0 < \tau(a) < \infty$. Then there exists a sequence $(\theta_n)_{n \in \mathbb{N}}$ in K such that the associated generalized limit trace θ on A_ω is a scalar multiple of τ and such that, for each sequence $(b_n)_{n \in \mathbb{N}}$ representing an element of A_ω, we have that*

$$\theta((ab_n)_{n \in \mathbb{N}}) = \lim_{n \to \omega} \theta_n(ab_n). \tag{1-4}$$

Proof. The fact that there exists a sequence $(\theta_n)_{n\in\mathbb{N}}$ in K such that the associated generalized limit trace is a multiple of τ follows directly from [Szabó 2021b, Lemma 2.10 and Remark 2.11]. Let $B := \overline{a^{1/2}Aa^{1/2}}$ denote the hereditary subalgebra generated by $a^{1/2}$. Then $B \subseteq \mathcal{P}(A)$; see for example [Pedersen 1979, Proposition 5.6.2]. Since K is compact, we claim that $\sup_{\sigma\in K}\|\sigma|_B\| < \infty$. Suppose that this would not be the case; then, for all $n\in\mathbb{N}$, we could find a $\sigma_n\in K$ and a positive contraction $d_n\in B$ such that $\sigma_n(d_n)\geq n2^n$. Consider $d := \sum_{n=1}^\infty 2^{-n}d_n \in B$ and $\sigma = \lim_{n\to\omega}\sigma_n \in K$ (using the compactness of K). Then, for each $n\in\mathbb{N}$, we would get

$$\sigma(d) = \lim_{n\to\omega}\sigma_n(d) \geq \lim_{n\to\omega}\sigma_n(2^{-n}d_n) = \infty,$$

but this is a contradiction, since d belongs to the Pedersen ideal. As a consequence we get that, when restricted to the hereditary subalgebra $\overline{a^{1/2}\ell^\infty(A)a^{1/2}} \subseteq \ell^\infty(B)$, the trace formed by $\lim_{n\to\omega}\theta_n$ is already bounded and hence continuous, so formula (1-4) holds. □

The following proposition is a useful lifting property in various contexts. The proof relies on the concept of G-σ-ideals; see [Szabó 2018c, Definition 4.1]. Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be actions on C*-algebras. As in [Kirchberg 2006], we call an equivariant surjective $*$-homomorphism $\pi : (A,\alpha)\to(B,\beta)$ strongly locally semisplit, if for every separable β-invariant C*-subalgebra $D\subseteq B$, there exists an equivariant c.p.c. order-zero map $\phi : (D,\beta)\to(A,\alpha)$ such that $\pi\circ\phi = \mathrm{id}_D$.

Proposition 1.18. *Let A be a separable simple* C**-algebra with $Q\widetilde{T}_2(A) = \widetilde{T}(A)$ and $T^+(A)\neq\varnothing$. Let $\alpha : G\curvearrowright A$ be an action of a countable discrete group. Then the quotient map*

$$(F_\omega(A),\tilde{\alpha}_\omega) \to (A^\omega\cap A',\alpha^\omega)$$

is strongly locally semisplit.

Proof. By [Szabó 2018c, Proposition 4.5 (ii)], it suffices to prove that $\mathcal{J}_A \subset F_\omega(A)$ is a G-σ-ideal. Fix an element $0\neq a\in\mathcal{P}(A)_+$ and a compact generator $K\subset T^+(A)$.[10] By [Szabó 2021b, Proposition 2.4] and Lemma 1.16, we can conclude that $\mathcal{J}_A$ coincides with the ideal of those elements $x\in F_\omega(A)$ such that $\tau_a(x^*x) = 0$ for all $\tau\in\widetilde{T}_\omega(A)$ induced by any sequence $\tau_n\in K$. Since a belongs to the Pedersen ideal and K is compact, this further implies that an element $x\in F_\omega(A)$ represented by a sequence $(x_n)_{n\in\mathbb{N}}$ in A belongs to $\mathcal{J}_A$ precisely when $\lim_{n\to\omega}\max_{\tau\in K}\tau(a^{1/2}x_n^*x_na^{1/2}) = 0$.

We proceed to show that $\mathcal{J}_A$ is a G-σ-ideal. Let $D\subset F_\omega(A)$ be a separable $\tilde{\alpha}_\omega$-invariant C*-subalgebra. Let $(d_{k,n})_{n,k\in\mathbb{N}}$ and $(c_{k,n})_{n,k\in\mathbb{N}}$ be two bounded double sequences in A such that, for each $k\in\mathbb{N}$, the sequences $(d_{k,n})_{n\in\mathbb{N}}$ and $(c_{k,n})_{n\in\mathbb{N}}$ are approximately central, the set $\{d^{(k)} = [(d_{k,n})_{n\in\mathbb{N}}] \mid k\in\mathbb{N}\}$ defines a dense subset in the unit ball of D, and the set $\{c^{(k)} = [(c_{k,n})_{n\in\mathbb{N}}] \mid k\in\mathbb{N}\}$ defines a dense subset in the unit ball of $D\cap\mathcal{J}_A$. By Kasparov's lemma [1988, Lemma 1.4], we can find, for any $\varepsilon>0$, $F\Subset G$, and $m\in\mathbb{N}$, a positive element $e\in\mathcal{J}_A$ such that

$$\max_{k\leq m}\|[e,d^{(k)}]\|\leq\varepsilon,\quad \max_{k\leq m}\|(1-e)c^{(k)}\|\leq\varepsilon,\quad\text{and}\quad \max_{g\in F}\|e-\tilde{\alpha}_\omega(e)\|\leq\varepsilon.$$

[10] As pointed out in the footnote after defining compact generators in Notation 1.1, this always exists as a consequences of simplicity.

Let $b \in A$ be a strictly positive contraction. If we represent e by an approximately central sequence $(e_n)_{n\in\mathbb{N}}$ of positive contractions in A, then it follows that

$$\max_{k\le m} \lim_{n\to\omega} \|[e_n, d_{k,n}]b\| \le \varepsilon, \quad \max_{k\le m} \lim_{n\to\omega} \|(1-e_n)c_{k,n}b\| \le \varepsilon,$$

and

$$\max_{g\in F} \lim_{n\to\omega} \|(e_n - \alpha_g(e_n))b\| \le \varepsilon, \quad \lim_{n\to\omega} \max_{\tau\in K} \tau(a^{1/2}e_n a^{1/2}) = 0.$$

Appealing to Kirchberg's ε-test [Kirchberg and Rørdam 2014, Lemma 3.1], we can find another approximately central sequence $(e_n)_{n\in\mathbb{N}}$ of positive contractions in A satisfying the stronger property

$$\lim_{n\to\omega} (\|[e_n, d_{k,n}]b\| + \|(1-e_n)c_{k,n}b\| + \|(e_n - \alpha_g(e_n))b\|) = 0 \quad \text{and} \quad \lim_{n\to\omega} \max_{\tau\in K} \tau(a^{1/2}e_n a^{1/2}) = 0$$

for all $k \in \mathbb{N}$ and $g \in G$. This means that this sequence represents a positive contraction $e \in (\mathcal{J}_A \cap D')^{\tilde{\alpha}_\omega}$ such that $ec = c$ for all $c \in \mathcal{J}_A \cap D$. This finishes the proof. □

To end this preliminary section, we prove the following tracial inequality.

Lemma 1.19. *Let B be a* C*-*algebra with $a, b \in B_+$ and $\tau \in T(B)$. Then*

$$\|a-b\|_{2,\tau}^2 \le \|a^2 - b^2\|_{1,\tau}.$$

Proof. If we replace B by its weak closure of the GNS representation $\pi_\tau(B)''$, it is enough to show this in the case that B is a von Neumann algebra with faithful normal tracial state τ.

Historically, this was proved by Powers and Størmer [1970, Lemma 4.1] in the case $B = M_n(\mathbb{C})$ for some $n \in \mathbb{N}$. When B is a von Neumann algebra with faithful normal tracial state τ, this follows from applying [Haagerup 1975, Lemma 2.10], which is formulated for the space $L^2(B,\tau)$, to elements in $B \subset L^2(B,\tau)$. For the reader's convenience we give here a more direct proof using an idea from [Anantharaman and Popa 2014, Theorem 7.3.7].

Given $a, b \in B_+$, let p and q denote the spectral projections of $a-b$ corresponding to $[0,+\infty)$ and $(-\infty, 0)$, respectively. This means that $a-b = (p-q)|a-b|$ and $p \perp q$. First of all, we have

$$\tau((a^2-b^2)p) - \tau((a-b)^2 p) = \tau(b(a-b)p) + \tau((a-b)bp) = 2\tau(b^{1/2}(a-b)pb^{1/2}) \ge 0$$

since $(a-b)p \ge 0$. So we get

$$\tau((a-b)^2 p) \le \tau((a^2-b^2)p), \tag{1-5}$$

and in a similar way we can obtain that

$$\tau((b-a)^2 q) \le \tau((b^2-a^2)q). \tag{1-6}$$

Combining (1-5) and (1-6) gives

$$\tau((a-b)^2) = \tau((a-b)^2(p+q)) \le \tau((a^2-b^2)(p-q)).$$

Also,

$$\tau((a^2-b^2)(p-q)) = \tau(p(a^2-b^2)p) + \tau(q(b^2-a^2)q) \le \tau(|a^2-b^2|(p+q)) \le \|a^2-b^2\|_{1,\tau}$$

since $\|p+q\| \le 1$. This implies the result. □

2. Equivariant uniform property Gamma

The notion of uniform property Gamma was introduced in [Castillejos et al. 2021b] and further studied in [Castillejos et al. 2022], where it served as a uniform C*-algebraic version of property Gamma introduced by Murray and von Neumann for II_1 factors [1943]. Recently, a dynamical version of this property was introduced in the separable unital setting in [Gardella et al. 2022] called the *equivariant uniform property Gamma*. Here we revise the definition to account for separable C*-algebras with possibly unbounded traces, generalizing the concept called "stabilised property Gamma" by Castillejos and Evington [2021, Definition 2.5]. We choose not to adopt that name because one can argue that uniform property Gamma ought to be a stable property in the first place, just like property Gamma is for von Neumann algebras. In light of recent work by Lin [2023] who proposed a more general framework for C*-algebras that admit genuine quasitraces, we shall state the definition only in the absence of such.

For separable unital simple exact C*-algebras, the definition below corresponds to the earlier definition given in [Gardella et al. 2022] (see Proposition 2.4 below) but not in the nonsimple case, as demonstrated by C*-algebras that arise as extensions of unital classifiable C*-algebras by the compacts (see the type of example mentioned in Remark 1.9, for instance).

Definition 2.1. Let A be a separable C*-algebra with $Q\widetilde{T}_2(A)=\widetilde{T}(A)$ and $T^+(A)\neq\varnothing$, and let $\alpha : G \curvearrowright A$ be an action by a countable discrete group. We say that α has *equivariant uniform property Gamma* (or *equivariant property Gamma* for short) if, for all $n\in\mathbb{N}$, there exist pairwise orthogonal projections $p_1,\dots,p_n\in(A^\omega\cap A')^{\alpha^\omega}$ such that, for all $a\in A_+$ and $\tau\in\widetilde{T}_\omega(A)$ with $\tau(a)<\infty$,

$$\tau_a(p_i)=\frac{1}{n}\tau(a).$$

Remark 2.2. We can notice immediately from the naturality of the isomorphism in Proposition 1.15 that equivariant uniform property Gamma is preserved under stable cocycle conjugacy. That is, if A and B are C*-algebras as above and we have actions $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ such that $\alpha\otimes\mathrm{id}_{\mathbb{K}}$ is cocycle conjugate to $\beta\otimes\mathrm{id}_{\mathbb{K}}$, then α^ω is conjugate to β^ω via a map preserving the canonical traces. In particular, equivariant uniform property Gamma holds for α if and only if it holds for β.

Next, we observe (cf. [Castillejos et al. 2021b, Proposition 2.3]) that, whenever $\alpha : G \curvearrowright A$ is an equivariantly $\mathcal{Z}$-stable action on a separable C*-algebra with $Q\widetilde{T}_2(A)=\widetilde{T}(A)$ and $T^+(A)\neq\varnothing$, it automatically has equivariant property Gamma. Indeed, a cocycle conjugacy between α and $\alpha\otimes\mathrm{id}_{\mathcal{Z}}$ is easily seen to give rise to a unital $*$-homomorphism

$$\mathcal{Z}^\omega\cap\mathcal{Z}'\to(A^\omega\cap A')^{\alpha^\omega}.$$

For this purpose one chooses an approximate unit $e_n\in A$ and considers a sequence of maps $\mathcal{Z}\to A\otimes\mathcal{Z}$, $x\mapsto e_n\otimes x$, composed with such a cocycle conjugacy, which is seen to induce such a homomorphism. It is well known that $\mathcal{Z}^\omega\cap\mathcal{Z}'$ admits unital embeddings of matrix algebras of arbitrary size $n\geq 2$. So if we fix n and define $p_1,\dots,p_n\in(A^\omega\cap A')^{\alpha^\omega}$ as the image of the canonical rank-one projections inside a matrix algebra under the aforementioned $*$-homomorphism, then they satisfy the necessary requirements for equivariant property Gamma by uniqueness of the trace on the $n\times n$ matrices.

The following is a version of equivariant property Gamma for possibly nonseparable C*-algebras that exclusively takes into account the bounded traces. This agrees with [Gardella et al. 2022, Definition 3.1] for separable unital C*-algebras but not with the general definition of equivariant property Gamma given above.

Definition 2.3. Let A be a σ-unital C*-algebra with $T(A)$ nonempty and compact, and let $\alpha : G \curvearrowright A$ be an action of a countable discrete group. We say that α has *local equivariant property Gamma with respect to bounded traces* if, for all $n \in \mathbb{N}$ and $\|\cdot\|_{2,T_\omega(A)}$-separable subsets $S \subset A^{\omega,\mathrm{b}}$, there exist pairwise orthogonal projections $p_1, \dots, p_n \in (A^{\omega,\mathrm{b}})^{\alpha^\omega} \cap S'$ such that $\tau(ap_i) = \tau(a)/n$ for all $a \in S$ and $\tau \in T_\omega(A)$.

In the unital separable simple setting, Definitions 2.1 and 2.3 are equivalent. We prove this fact in a slightly more general setting in the proposition below.

Proposition 2.4. *Let A be a simple separable* C*-*algebra with $Q\widetilde{T}_2(A) = \widetilde{T}(A)$, and such that $T(A) \neq \varnothing$ is compact and $T^+(A) = \mathbb{R}^{>0}T(A)$.*[11] *Then an action $\alpha : G \curvearrowright A$ of a countable discrete group has equivariant property Gamma if and only if α has local equivariant property Gamma with respect to bounded traces.*

Proof. The assumptions on A imply that every generalized limit trace on A that is finite on some nonzero positive element of A is a multiple of an ordinary limit trace (see Remark 1.5) and that $A^\omega \cap A' = A^{\omega,\mathrm{b}} \cap A'$ (see Remark 1.12). Therefore, it suffices to show that the existence of pairwise orthogonal projections $p_1, \dots, p_n \in (A^\omega \cap A')^{\alpha^\omega}$ such that $\tau(ap_i) = \tau(a)/n$ for all $a \in A$ and $\tau \in T_\omega(A)$ implies, for any $\|\cdot\|_{2,T_\omega(A)}$-separable $S \subset A^\omega$, the existence of pairwise orthogonal projections $p'_1, \dots, p'_n \in (A^\omega \cap S')^{\alpha^\omega}$ such that $\tau(ap'_i) = \tau(a)/n$ for all $a \in S$ and $\tau \in T_\omega(A)$. This follows by a standard reindexation argument, which we omit. □

The next part of this section is devoted to proving an equivalence between equivariant property Gamma for an action $\alpha : G \curvearrowright A$ and local equivariant property Gamma with respect to bounded traces for its induced action $\alpha^\omega : G \curvearrowright A^\omega \cap A'$, at least in the setting when A is simple nuclear and has stable rank one.[12] Recall that A is said to have stable rank one if the invertibles of $\tilde{A}$ are dense in $\tilde{A}$. We start by observing the following description of the tracial state space of $A^\omega \cap A'$.

Proposition 2.5. *Let A be a separable, simple, nuclear* C*-*algebra with uniform property Gamma and stable rank one. Then every tracial state on $A^\omega \cap A'$ is a canonical trace, i.e., one has*

$$T(A^\omega \cap A') = \overline{\mathrm{conv}}^{w^*}\{\tau_a \mid \tau \in \widetilde{T}_\omega(A),\, a \in A_+,\, \tau(a) = 1\}.$$

Proof. Using exactly the same argument as in the proof of [Castillejos and Evington 2021, Lemma 3.3] and modifying it as hinted in the remark stated before [Castillejos and Evington 2021, Theorem 3.4], we may appeal to [Antoine et al. 2022, Theorem 7.13] (since we assume stable rank one) and pick a nonzero hereditary C*-subalgebra $B \subset A \otimes \mathbb{K}$ with $T^+(B) = \mathbb{R}^{>0}T(B)$ and for which $T(B)$ is nonempty and

[11] We note that this is automatic if one assumes, e.g., that A has continuous scale (see [Lin 1991, Definition 2.5]), which is a rather common assumption in the context of classification.

[12] Although we use it in the proof, it is likely that stable rank one is not so important for the claim to hold, although we take no guess as to pinning down the correct general assumptions. We note, however, that simple finite $\mathcal{Z}$-stable C*-algebras have stable rank one; see [Fu et al. 2022; Rørdam 2004].

compact. By Brown's theorem, it follows that A and B are stably isomorphic. Proposition 1.15 implies that we have an isomorphism $A^\omega \cap A' \cong B^\omega \cap B'$ that induces a bijection between the canonical traces on the left and the right. Hence the claim holds for A if and only if it holds for B.

Now B has uniform property Gamma (see Remark 2.2), so [Castillejos et al. 2021b, Lemma 3.7] implies that B has CPoU. By the "no silly trace" theorem [Castillejos et al. 2021a, Proposition 2.5],[13] one has that $T(B^\omega)$ is the weak-$*$-closed convex hull of the limit traces. If B is unital, then the claim follows directly from [Castillejos et al. 2021b, Proposition 4.6]. If B is nonunital, we can extend the inclusion map $B \subset B^\omega$ to a unital inclusion $B^\dagger \subset B^\omega$. From this point of view, we have a trivial equality of algebras

$$B^\omega \cap B' = B^\omega \cap (B^\dagger)' \cap \{1_{B^\omega} - 1_{B^\dagger}\}^\perp.$$

In this case it follows from [Castillejos and Evington 2020, Proposition 5.7] that $T(B^\omega \cap B')$ is the closed convex hull of traces of the form τ_a, where $\tau \in T_\omega(B)$ is a limit trace and $a \in B^\dagger$ is a positive element with $\tau(a) = 1$. If $(e_n)_{n\in\mathbb{N}}$ is an increasing approximate unit in B, then $b_n = e_n a e_n \in B$ converges to a strictly, and hence $\|b_n - a\|_{2,\tau} \to 0$. This implies the convergence of tracial states $\tau(b_n)^{-1}\tau_{b_n} \to \tau_a$ in the norm topology, so we observe the equality

$$T(B^\omega \cap B') = \overline{\mathrm{conv}}^{w^*}\{\tau_b \mid \tau \in T_\omega(B),\ b \in B_+,\ \tau(b) = 1\}. \qquad \square$$

Theorem 2.6. *Let A be a separable, simple, nuclear* C*-*algebra with stable rank one. Then $\alpha : G \curvearrowright A$ has equivariant uniform property Gamma if and only if $\alpha^\omega : G \curvearrowright A^\omega \cap A'$ has local equivariant uniform property Gamma with respect to bounded traces.*

Proof. In order to increase readability in this proof, let us specify another free ultrafilter κ on $\mathbb{N}$ (which may or may not be equal to ω).

We shall show the "if" part first, which actually holds for arbitrary separable simple C*-algebras with $Q\widetilde{T}_2(A) = \widetilde{T}(A)$ and $T^+(A) \neq \varnothing$. Let $k \geq 2$. Assuming α^ω has local equivariant property Gamma with respect to bounded traces, we can find pairwise orthogonal projections $p_1, \dots, p_k \in ((A^\omega \cap A')^{\kappa,\mathrm{b}})^{(\alpha^\omega)^\kappa}$ such that

$$\tau(ap_j) = \frac{1}{k}\tau(a) \quad \text{for } j = 1, \dots, k,\ a \in A,\ \tau \in T_\kappa(A^\omega \cap A').$$

For each $j = 1, \dots, k$, let p_j be represented by a sequence of positive contractions $(p_{j,n})_{n\in\mathbb{N}}$ in $A^\omega \cap A'$. Let in turn each element $p_{j,n}$ be represented by a central sequence $(x_{j,n,\ell})_{\ell\in\mathbb{N}}$ of positive contractions in A. Traces in $T_\kappa(A^\omega \cap A')$ in particular include limit traces associated to sequences of canonical traces. Let $C \subset \mathcal{P}(A)_+ \setminus \{0\}$ be a countable dense subset. Let $K \subset T^+(A)$ be a compact generator. By the conclusion of Lemma 1.17, it follows, for all $a \in C$ and all sequences $(\theta_\ell)_{\ell\in\mathbb{N}}$ in K, that, if τ is the limit trace on A_ω induced by $(\theta_\ell)_{\ell\in\mathbb{N}}$ and τ_a is the induced bounded trace on $A^\omega \cap A'$ that we view in a trivial way as a multiple of a (constant) limit trace on $(A^\omega \cap A')^{\kappa,\mathrm{b}}$, then

$$0 = \lim_{n\to\kappa} \|p_{j,n} - p_{j,n}^2\|_{1,\tau_a} = \lim_{n\to\kappa} \tau(a|p_{j,n} - p_{j,n}^2|) \overset{\text{Lemma 1.17}}{=} \lim_{n\to\kappa}\lim_{\ell\to\omega} \theta_\ell(a|x_{j,n,\ell} - x_{j,n,\ell}^2|).$$

[13] Strictly speaking the conclusion is about the reduced tracial product B^∞ in the reference, but this makes no difference to the argument there.

Since the sequence $(\theta_\ell)_{\ell\in\mathbb{N}}$ in K was arbitrary, we may rewrite this as

$$0=\lim_{n\to\kappa}\lim_{\ell\to\omega}\max_{\theta\in K}\theta(a|x_{j,n,\ell}-x_{j,n,\ell}^2|).$$

We may argue in a completely analogous fashion to see that

$$0=\lim_{n\to\kappa}\lim_{\ell\to\omega}\max_{\theta\in K}\theta(a|x_{j,n,\ell}-\alpha_g(x_{j,n,\ell})|),\quad g\in G,$$

as well as

$$0=\lim_{n\to\kappa}\lim_{\ell\to\omega}\max_{\theta\in K}\Big|\theta(ax_{j,n,\ell})-\frac{1}{k}\theta(a)\Big|=\lim_{n\to\kappa}\lim_{\ell\to\omega}\max_{\theta\in K}\theta(ax_{j,n,\ell}x_{i,n,\ell})$$

for all $i, j=1,\dots,k$ with $i\neq j$. Lastly, we have by definition that $(x_{j,n,\ell})_{\ell\in\mathbb{N}}$ is a central sequence as $\ell\to\omega$. Appealing to Kirchberg's ε-test, we can find central sequences of positive contractions $e_\ell^{(j)}$ in A for $j=1,\dots,k$ satisfying, for all $a\in C$, the properties

$$0=\lim_{\ell\to\omega}\max_{\theta\in K}\Big|\theta(ae_\ell^{(j)})-\frac{1}{k}\theta(a)\Big|=\lim_{\ell\to\omega}\max_{\theta\in K}\theta(a|e_\ell^{(j)}-e_\ell^{(j)2}|)$$

and

$$0=\lim_{\ell\to\omega}\max_{\theta\in K}\theta(ae_\ell^{(j)}e_\ell^{(i)})=\lim_{\ell\to\omega}\max_{\theta\in K}\theta(a|e_\ell^{(j)}-\alpha_g(e_\ell^{(j)})|),\quad g\in G\text{ and } i\neq j.$$

We consider the resulting elements $e_j\in A^\omega\cap A'$ represented by $(e_\ell^{(j)})_{\ell\in\mathbb{N}}$. Given that C was dense in A_+, we may conclude that they are pairwise orthogonal projections belonging to $(A^\omega\cap A')^{\alpha^\omega}$ satisfying $\tau_a(e_j)=\tau(a)/k$ for all $\tau\in\widetilde{T}_\omega(A)$ and $a\in C$ with $\tau(a)<\infty$. In conclusion, this shows that α has equivariant uniform property Gamma.

For the "only if" part, suppose that α has equivariant property Gamma. Given $k\geq 2$, there exist pairwise orthogonal projections $p_1,\dots,p_k\in(A^\omega\cap A')^{\alpha^\omega}$ such that, for all $a\in A_+$ and $\tilde\tau\in\widetilde{T}_\omega(A)$ with $\tilde\tau(a)<\infty$,

$$\tilde\tau_a(p_j)=\frac{1}{k}\tilde\tau(a)\quad\text{for } j=1,\dots,k.$$

As above, choose a compact generator $K\subset T^+(A)$. If we represent each element p_j by a central sequence of positive contractions $(p_{j,n})_{n\in\mathbb{N}}$ in A, then we can argue as before and see that, for all $a\in\mathcal{P}(A)_+\setminus\{0\}$, $g\in G$, and $i\neq j$, one has the limit properties

$$0=\lim_{n\to\omega}\max_{\theta\in K}\Big|\theta(ap_{j,n})-\frac{1}{k}\theta(a)\Big|=\lim_{n\to\omega}\max_{\theta\in K}\theta(a|p_{j,n}-\alpha_g(p_{j,n})|)$$

and

$$0=\lim_{n\to\omega}\max_{\theta\in K}\theta(ap_{j,n}p_{i,n})=\lim_{n\to\omega}\max_{\theta\in K}\theta(a|p_{j,n}-p_{j,n}^2|).$$

Now take a countable subset $S\subset(A^\omega\cap A')^{\kappa,\mathrm{b}}$ whose closure would represent a separable subset as in Definition 2.3. Without loss of generality, let us assume S consists of positive elements. Choose a countable subset $S_0\subset(A^\omega\cap A')_+$ such that every element of S is represented by a bounded S_0-valued sequence. Next, choose an increasing sequence of finite sets $F_n\subset\mathcal{P}(A)_+\setminus\{0\}$ such that their union is dense in A_+ and every element in S_0 has a representing sequence in $\prod_{n\in\mathbb{N}}F_n$. Appealing to the above stated properties of the sequences $(p_{j,n})_{n\in\mathbb{N}}$ for $j=1,\dots,n$, we may find an increasing sequence of natural numbers $\ell\mapsto n_\ell$ such that the resulting subsequences satisfy

$$0=\lim_{\ell\to\infty}\max_{a\in F_\ell}\|[a,p_{j,n_\ell}]\|=\lim_{\ell\to\infty}\max_{a\in F_\ell}\max_{\theta\in K}\theta(a|p_{j,n_\ell}-\alpha_g(p_{j,n_\ell})|), \tag{2-1}$$

$$0=\lim_{\ell\to\infty}\max_{a\in F_\ell}\max_{\theta\in K}\theta(ap_{j,n_\ell}p_{i,n_\ell})=\lim_{\ell\to\infty}\max_{a\in F_\ell}\max_{\theta\in K}\theta(a|p_{j,n_\ell}-p_{j,n_\ell}^2|), \tag{2-2}$$

and

$$0=\lim_{\ell\to\infty}\max_{a,b\in F_\ell}\max_{\theta\in K}\left|\theta(abp_{j,n_\ell})-\frac{1}{k}\theta(ab)\right| \tag{2-3}$$

for all $i,j=1,\dots,k$ with $i\neq j$. By the choice of the sets F_ℓ, we can see that $(p_{j,n_\ell})_{\ell\in\mathbb{N}}$ defines a central sequence in A, and its induced element $e_j\in A^\omega\cap A'$ commutes with elements in S_0. We keep in mind the conclusion of Lemma 1.17. Then conditions (2-1) and (2-2) imply that $e_1,\dots,e_k$ are pairwise orthogonal projections in $(A^\omega\cap A')^{\alpha^\omega}$. Condition (2-3) implies that, for all $j=1,\dots,k$, $\tau\in\widetilde{T}_\omega(A)$, every $a\in A_+$ with $\tau(a)=1$, and every $b\in S_0$, we have

$$\tau_a(be_j)=\tau(abe_j)=\frac{1}{k}\tau(ab)=\frac{1}{k}\tau_a(b).$$

By Proposition 2.5, the weak-$*$-closed convex hull of such tracial states τ_a yields the whole tracial state space of $A^\omega\cap A'$. In other words, we may conclude

$$\tau(be_j)=\frac{1}{k}\tau(b)\quad\text{for all } j=1,\dots,k,\ b\in S_0 \text{ and } \tau\in T(A^\omega\cap A').$$

We may view e_j as constant elements inside $(A^\omega\cap A')^{\kappa,\mathrm{b}}$. Since every element in S was represented by a sequence in S_0, we may conclude that the elements $e_1,\dots,e_k$ satisfy the required property from Definition 2.3 applied to the action

$$\alpha^\omega: G\curvearrowright A^\omega\cap A'.$$

We conclude that α^ω has local equivariant property Gamma with respect to bounded traces. □

3. Dynamical complemented partitions of unity

This section contains the most involved technical arguments of the article, namely the proof that local equivariant property Gamma implies the existence of a dynamical version of complemented partitions of unity [Castillejos et al. 2021b, Definition 3.1], or dynamical CPoU for short. In the case where the induced action on the tracial state space has the property that all orbits are finite with uniformly bounded cardinality, a different iteration of dynamical CPoU was proved in [Gardella et al. 2022, Theorem 4.3]. However, we note that the general statement we prove is a weaker and more intricate version compared to earlier versions but will nevertheless be sufficient to deduce the tracial local-to-global principle.

The starting point for the approach in this section is the following weaker version of CPoU shown in [Castillejos et al. 2021b, Lemma 3.6] for nuclear C*-algebras, which turns out to hold automatically with the aid of the theory of tracially complete C*-algebras [Carrión et al. 2023a].

Proposition 3.1. *Let A be a σ-unital* C*-*algebra with $T(A)$ nonempty and compact. Then, for every $\|\cdot\|_{2,T_\omega(A)}$-separable subset $S \subset A^{\omega,\mathrm{b}}$, every $k \in \mathbb{N}$, every family $a_1, \dots, a_k \in A_+$, and every*

$$\delta > \sup_{\tau \in T(A)} \min_{i=1,\dots,k} \tau(a_i),$$

there exist $e_1, \dots, e_k \in (A^{\omega,\mathrm{b}} \cap S')^1_+$ such that, for all $\tau \in T_\omega(A)$,

- $\tau\big(\sum_{i=1}^k e_i\big) = 1,$
- $\tau(a_i e_i) \leq \delta\tau(e_i)$ *for $i = 1, \dots, k$.*

Proof. Let $S,\ k,\ a_1, \dots, a_k$, and δ be chosen as in the assumption. Set

$$\delta_0 := \sup_{\tau \in T(A)} \min_{i=1,\dots,k} \tau(a_i) < \delta.$$

Since S is $\|\cdot\|_{2,T_\omega(A)}$-separable, it is first of all clear that one may find a nondegenerate separable C*-subalgebra $A_0 \subseteq A$ containing the tuple $a_1, \dots, a_k$ such that every element of S can be represented by a bounded sequence in A_0. As every tracial state on A restricts to one on A_0, the tracial state space of A_0 is still nonempty and compact, and furthermore

$$\sup_{\tau \in T(A_0)} \min_{i=1,\dots,k} \tau(a_i) \geq \delta_0.$$

Let $\eta > 0$. We claim that there exists a finite set $F_\eta \Subset A$ and $\varepsilon_\eta > 0$ such that, if ρ is any state on A with

$$\max_{x \in F_\eta} |\rho(x^*x) - \rho(xx^*)| < \varepsilon_\eta,$$

then $\min_{i=1,\dots,k} \rho(a_i) < \delta_0 + \eta$. If we suppose for a moment that this were false, then it follows that, for every finite set $F \Subset A$ and every $\varepsilon > 0$, there exists a state $\rho_{(F,\varepsilon)}$ on A with

$$\max_{x \in F} |\rho_{(F,\varepsilon)}(x^*x) - \rho_{(F,\varepsilon)}(xx^*)| < \varepsilon \quad \text{and} \quad \min_{i=1,\dots,k} \rho_{(F,\varepsilon)}(a_i) \geq \delta_0 + \eta.$$

We can view $\rho_{(F,\varepsilon)}$ as a net of states by equipping the set of pairs (F, ε) with the obvious order. By the Banach–Anaoglu theorem, there exists a subset $(\rho_\lambda)_{\lambda \in \Lambda}$ that weak-$*$-converges to a positive functional ρ' with norm at most one on A. By the properties of the net $\rho_{(F,\varepsilon)}$, it is clear that ρ' is tracial. Hence $\min_{i=1,\dots,k} \rho'(a_i) \leq \delta_0$, while at the same time

$$\min_{i=1,\dots,k} \rho'(a_i) = \lim_{(F,\varepsilon)} \min_{i=1,\dots,k} \rho_{(F,\varepsilon)}(a_i) \geq \delta_0 + \eta,$$

which is a contradiction.

Using this intermediate claim, we choose for each $n \geq 1$ a finite set $F_n \Subset A$ and $\varepsilon_n > 0$ satisfying the above conclusion for $\eta = 1/n$. Let $A_1 \subseteq A$ be the C*-algebra generated by A_0 and all the finite sets F_n, which is clearly still separable. Since A_1 contains all the finite sets F_n, it follows that every tracial state τ on A_1 must satisfy

$$\min_{i=1,\dots,k} \tau(a_i) \leq \delta_0 + \frac{1}{n}, \quad n \geq 1,$$

which leads to

$$\sup_{\tau\in T(A_1)}\min_{i=1,\dots,k}\tau(a_i)=\delta_0<\delta.$$

By all the properties arranged for the subalgebra $A_1\subseteq A$ so far, it is clear for proving our main claim that we may swap A for the subalgebra A_1. In other words, we may assume without loss of generality that A is separable.

By [Carrión et al. 2023a, Definition 3.19, Proposition 3.23], the tracial completion $\overline{A}^{T(A)}$ of A yields a factorial tracially complete C*-algebra. Note that as per the ultraproduct construction of tracially complete C*-algebras in [Carrión et al. 2023a], the object $(\overline{A}^{T(A)})^\omega$ in that sense becomes canonically isomorphic to the C*-algebra $A^{\omega,\mathrm{b}}$ as considered in Definition 1.8. Because A is separable, $\overline{A}^{T(A)}$ is $\|\cdot\|_{2,T(A)}$-separable. Thus we may directly apply [Carrión et al. 2023a, Theorem 6.15] (inserting the unit in place of the projection q appearing there) and find the elements $e_1,\dots,e_k\in(A^{\omega,\mathrm{b}}\cap S')_+^1$ with the desired properties. □

The main achievement of this section is the following technical lemma.

Lemma 3.2. *Given $\varepsilon>0$ and $t\in(0,1)$, there exists a universal constant $\eta=\eta(\varepsilon,t)>0$ such that the following holds: Let A be a σ-unital* C*-*algebra with $T(A)$ nonempty and compact. Let G be a countable discrete group, and let $\alpha:G\curvearrowright A$ be an action with local equivariant property Gamma with respect to bounded traces. Suppose that $F,H\Subset G$ are finite subsets such that*

$$|gH\Delta H|<\eta|H|\quad \textit{for all } g\in F.$$

Then, for every $\|\cdot\|_{2,T_\omega(A)}$-separable subset $S\subset A^{\omega,\mathrm{b}}$, every family $a_1,\dots,a_k\in(A^{\omega,\mathrm{b}})_+$, and every constant $\delta>0$ with

$$\frac{\delta}{|H|}>\sup_{\tau\in T_\omega(A)}\min_{i=1,\dots,k}\tau(a_i),\tag{3-1}$$

there exist pairwise orthogonal projections $p_1,\dots,p_k\in A^{\omega,\mathrm{b}}\cap S'$ such that, for all $\tau\in T_\omega(A)$, one has

$$\tau(p_1+\cdots+p_k)>t,\tag{3-2}$$

$$\tau(a_ip_i)\le\delta\tau(p_i)\quad \textit{for } i=1,\dots,k,\tag{3-3}$$

$$\max_{g\in F}\sum_{i=1}^k\|\alpha_g^\omega(p_i)-p_i\|_{2,\tau}^2<\varepsilon.\tag{3-4}$$

Remark 3.3. A standard argument shows that the statement in Lemma 3.2 is equivalent to the existence of a universal constant $\eta(\varepsilon,t)>0$ satisfying the following statement (using approximations instead of the uniform bounded tracial ultrapower):

If $\alpha:G\curvearrowright A$ is an action and $F,H\Subset G$ are all given as in Lemma 3.2, then, for every finite subset $S\Subset A$, every $\xi>0$, every family $a_1,\dots,a_k\in A_+$, and every $\delta>0$ with

$$\frac{\delta}{|H|}>\sup_{\tau\in T(A)}\min_{i=1,\dots,k}\tau(a_i),\tag{3-5}$$

there exist pairwise orthogonal contractions $e_1, \dots, e_k \in A_+$ *such that*

$$\begin{aligned} \|[e_i, x]\|_{2,u} &< \xi \quad \textit{for } x \in S,\ i = 1, \dots, k, \\ \|e_i - e_i^2\|_{2,u} &< \xi \quad \textit{for } i = 1, \dots, k, \\ \tau(e_1 + \dots + e_k) &> t - \xi \quad \textit{for } \tau \in T(A), \\ \tau(a_i e_i) &< \delta\tau(e_i) + \xi \quad \textit{for } \tau \in T(A),\ i = 1, \dots, k, \\ \max_{g \in F} \sum_{i=1}^{k} \|\alpha_g(e_i) - e_i\|_{2,\tau} &< \varepsilon + \xi \quad \textit{for } \tau \in T(A). \end{aligned}$$

In particular, this means that it suffices to prove Lemma 3.2 for positive elements $a_1, \dots, a_k$ taken in A instead of $A^{\omega,\mathrm{b}}$. In this case, (3-1) and (3-5) are equivalent.

The proof of Lemma 3.2 is an adapted version of the proof in the nondynamical setting (cf. [Castillejos et al. 2021b, Section 3]) but also incorporates new ideas related to the dynamical structure. Before we delve into the details, we shall give an overview of the strategy. The construction of the pairwise orthogonal projections $p_1, \dots, p_k$ in the statement of Lemma 3.2 is done in three steps:

(1) Instead of producing pairwise orthogonal projections $p_1, \dots, p_k$, we start by producing (not yet pairwise orthogonal) positive contractions $e_1, \dots, e_k \in A^{\omega,\mathrm{b}} \cap S'$ that satisfy

$$\tau(e_1 + \dots + e_k) = 1, \quad \tau(a_i e_i) \le \delta\tau(e_i) \quad \text{for } i = 1, \dots, k,\ \tau \in T_\omega(A),$$

and that are approximately invariant under α^ω in the right sense. This is done in Lemma 3.4 and is the only part of the proof that makes use of the approximate Følner property that appears in the assumption of the lemma.

(2) Next, we use equivariant property Gamma to turn these contractions into orthogonal projections $p_1', \dots, p_k' \in A^{\omega,\mathrm{b}} \cap S'$. As a consequence of this procedure we get that

$$\tau(p_1' + \dots + p_k') = \frac{1}{k} \quad \text{for } \tau \in T_\omega(A),$$

but they still satisfy (3-3) and are still approximately invariant in the right sense. This is done in Lemma 3.6.

(3) In order to enlarge the trace of the sum of the projections, we repeat the above steps underneath the projection $1_{A^\omega} - \sum_{i=1}^{k} p_i'$.[14] We continue this procedure inductively until we end up with orthogonal projections $p_1, \dots, p_k$ whose sum exceeds t in trace and that still satisfy (3-3). If everything is done carefully from the start and $\eta > 0$ is chosen correctly, we can control the error in the invariance of the projections and make sure they satisfy (3-4) in the end. (We note, informally, that this error grows with the number of times this procedure is repeated, which is the ultimate reason why we cannot simply work with $t = 1$ in the statement.)

We shall now implement the above strategy. Combining the contractions arising from Proposition 3.1 with an averaging argument over suitable Følner sets allows us to carry out the first step.

[14] For this one actually needs a somewhat stronger version of the second step; see Lemma 3.7.

Lemma 3.4. *Let A be a σ-unital C^*-algebra with $T(A)$ nonempty and compact. Let $\alpha : G \curvearrowright A$ be an action by a countable discrete group. Let $\varepsilon > 0$, and let finite subsets $F \Subset G$ and $H \Subset G$ be given such that $|gH \Delta H| < \varepsilon |H|$ for all $g \in F$. Then, for all $\|\cdot\|_{2,T_\omega(A)}$-separable subsets $S \subset A^{\omega,\mathrm{b}}$, all $\delta > 0$, and all $a_1, \dots, a_k \in A_+$ with*

$$\frac{\delta}{|H|} > \sup_{\tau \in T(A)} \min_{i=1,\dots,k} \tau(a_i),$$

there exist $e_1, \dots, e_k \in (A^{\omega,\mathrm{b}} \cap S')^1_+$ such that, for $\tau \in T_\omega(A)$,

- $\tau\big(\sum_{i=1}^k e_i\big) = 1$,
- $\tau(a_i e_i) \le \delta \tau(e_i)$ *for $i = 1, \dots, k$, and*
- $\max_{g \in F} \sum_{i=1}^k \|\alpha_g^\omega(e_i) - e_i\|_{1,\tau} < \varepsilon$.

Proof. Given $S \subset A^{\omega,\mathrm{b}}$, $\delta > 0$, and $a_1, \dots, a_k \in A_+$ as above, we define

$$a_i' := \frac{1}{|H|} \sum_{g \in H} \alpha_{g^{-1}}(a_i), \quad i = 1, \dots, k.$$

Note that, for each $\tau \in T(A)$, the trace $\frac{1}{|H|} \sum_{g \in H} \tau \circ \alpha_{g^{-1}}$ is again an element of $T(A)$, so we see that

$$\frac{\delta}{|H|} > \sup_{\tau \in T(A)} \min_{i=1,\dots,k} \tau(a_i').$$

By Proposition 3.1, we know that there exist

$$e_1', \dots, e_k' \in \bigg(A^\omega \cap \Big(\bigcup_{g \in G} \alpha_g^\omega(S)\Big)'\bigg)^1_+$$

such that, for $\tau \in T_\omega(A)$,

$$\tau\bigg(\sum_{i=1}^k e_i'\bigg) = 1, \tag{3-6}$$

$$\tau(a_i' e_i') \le \frac{\delta}{|H|} \tau(e_i') \quad \text{for } i = 1, \dots, k.$$

In particular, this last equation implies that

$$\tau(\alpha_{g^{-1}}(a_i) e_i') \le \delta \tau(e_i'), \quad g \in H, \quad \tau \in T_\omega(A). \tag{3-7}$$

Now, for $i = 1, \dots, k$, define

$$e_i := |H|^{-1} \sum_{g \in H} \alpha_g^\omega(e_i').$$

Clearly this still is a positive contraction in $A^\omega \cap S'$. Notice that

$$\tau\bigg(\sum_{i=1}^k e_i\bigg) = |H|^{-1} \sum_{g \in H} (\tau \circ \alpha_g^\omega)\bigg(\sum_{i=1}^k e_i'\bigg) \overset{(3\text{-}6)}{=} 1 \quad \text{for } \tau \in T_\omega(A).$$

For $\tau \in T_\omega(A)$ and $i = 1, \dots, k$, we have

$$\tau(a_i e_i) = |H|^{-1}\tau\Big(a_i \sum_{g\in H} \alpha_g^\omega(e_i')\Big) = |H|^{-1} \sum_{g\in H} (\tau \circ \alpha_g^\omega)(\alpha_{g^{-1}}(a_i) e_i')$$
$$\overset{(3\text{-}7)}{\le} |H|^{-1} \sum_{g\in H} \delta(\tau\circ\alpha_g^\omega)(e_i') = \delta\tau(e_i).$$

Lastly, we see that, for $g \in F$ and $\tau \in T_\omega(A)$, we have

$$\sum_{i=1}^k \|\alpha_g^\omega(e_i) - e_i\|_{1,\tau} \le |H|^{-1} \sum_{i=1}^k \sum_{h\in gH\Delta H} \|\alpha_h^\omega(e_i')\|_{1,\tau} = |H|^{-1} \sum_{h\in gH\Delta H} \sum_{i=1}^k \tau(\alpha_h^\omega(e_i'))$$
$$\overset{(3\text{-}6)}{=} |H|^{-1}|gH\Delta H| < \varepsilon. \qquad \square$$

Analogously as in the nondynamical setting (cf. [Castillejos et al. 2021b, Lemma 2.4]), (local) equivariant property Gamma allows one to replace positive contractions by projections without changing the tracial values in $A^{\omega,\mathrm{b}}$. A different generalization of this lemma was proved in [Gardella et al. 2022, Proposition 3.4], but for the purposes of this paper we need a way to control the (tracially) approximate fixedness of the elements for the action.

Lemma 3.5. *Let A be a σ-unital* C**-algebra with $T(A)$ nonempty and compact, and let $\alpha : G \curvearrowright A$ be an action of a countable discrete group. Assume that α has local equivariant property Gamma with respect to bounded traces. Let $S \subset A^{\omega,\mathrm{b}}$ be a $\|\cdot\|_{2,T_\omega(A)}$-separable subset, and let $b \in A^{\omega,\mathrm{b}} \cap S'$ be a positive contraction. Then there exists a projection $p \in A^{\omega,\mathrm{b}} \cap S'$ such that*

$$\tau(ap) = \tau(ab) \quad \text{for } a \in S,\ \tau \in T_\omega(A), \tag{3-8}$$

and such that, for all $g \in G$ and $\tau \in T_\omega(A)$, one has

$$\|\alpha_g^\omega(p) - p\|_{2,\tau}^2 \le \|\alpha_g^\omega(b)^{1/2} - b^{1/2}\|_{2,\tau} \|\alpha_g^\omega(b)^{1/2} + b^{1/2}\|_{2,\tau}. \tag{3-9}$$

Proof. Fix $n \in \mathbb{N}$. By a common reindexation trick, it suffices to find a positive contraction $p \in A^{\omega,\mathrm{b}} \cap S'$ satisfying (3-8), (3-9), and $\|p - p^2\|_{2,T_\omega(A)}^2 \le 1/n$. An element p satisfying all the necessary properties except (3-9) is constructed in the proof of [Castillejos et al. 2021b, Lemma 2.4] with the use of uniform property Gamma. We show that when the construction is carried out using (local) equivariant property Gamma instead, the resulting projection also satisfies the extra condition (3-9).

Just as in [Castillejos et al. 2021b], we define functions $f_1, \dots, f_n \in C([0,1])$ such that $f_i|_{[0,(i-1)/n]} = 0$, $f_i|_{[i/n,1]} = 1$, and f_i is linear on $[(i-1)/n, i/n]$. Note that not only $(1/n)\sum_{i=1}^n f_i = \mathrm{id}_{[0,1]}$, but the monotonicity of each f_i also implies

$$\frac{1}{n}\sum_{i=1}^n |f_i(t_1) - f_i(t_2)| = |t_1 - t_2|, \qquad t_1, t_2 \in [0,1]. \tag{3-10}$$

By local equivariant property Gamma with respect to bounded traces, we can find pairwise orthogonal projections $p_1, \dots, p_n \in (A^{\omega,\mathrm{b}})^{\alpha^\omega} \cap S' \cap \{b\}'$ such that $\tau(p_i x) = \tau(x)/n$ for $i = 1, \dots, n$, $\tau \in T_\omega(A)$,

$x \in C^*(S \cup \{b\})$. Define

$$p := \sum_{i=1}^{n} p_i f_i(b) \in A^{\omega,\mathrm{b}} \cap S'.$$

By repeating the arguments in the proof of [Castillejos et al. 2021b, Lemma 2.4] verbatim, we may conclude that $\|p - p^2\|_{2,T_\omega(A)}^2 \le 1/n$ and $\tau(ap) = \tau(ab)$ for all $a \in S$ and $\tau \in T_\omega(A)$. We need to show that (3-9) holds as well. Fix $\tau \in T_\omega(A)$ and $g \in G$. By [Connes 1976, I.1], there exists a positive Radon measure ν on $[0, 1]^2$ such that, for every pair of functions $h_1, h_2 \in C_0((0, 1])$, the functions $(s, t) \mapsto h_1(s)$ and $(s, t) \mapsto h_2(t)$ are square integrable and

$$\|h_1(\alpha_g^\omega(b)) - h_2(b)\|_{2,\tau}^2 = \int_{[0,1]^2} |h_1(s) - h_2(t)|^2 \,\mathrm{d}\nu(s, t).$$

Then we get

$$\begin{aligned}
\|\alpha_g^\omega(p) - p\|_{2,\tau}^2 &= \Bigl\| \sum_{i=1}^{n} p_i (f_i(\alpha_g^\omega(b)) - f_i(b)) \Bigr\|_{2,\tau}^2 \\
&= \frac{1}{n} \sum_{i=1}^{n} \| f_i(\alpha_g^\omega(b)) - f_i(b)\|_{2,\tau}^2 \\
&= \frac{1}{n} \sum_{i=1}^{n} \int_{[0,1]^2} |f_i(s) - f_i(t)|^2 \,\mathrm{d}\nu(s, t) \\
&\le \frac{1}{n} \sum_{i=1}^{n} \int_{[0,1]^2} |f_i(s) - f_i(t)| \,\mathrm{d}\nu(s, t) \\
&\overset{\text{(3-10)}}{=} \int_{[0,1]^2} |s - t| \,\mathrm{d}\nu(s, t) \\
&\le \sqrt{\int_{[0,1]^2} |s^{1/2} - t^{1/2}|^2 \,\mathrm{d}\nu(s, t)} \sqrt{\int_{[0,1]^2} |s^{1/2} + t^{1/2}|^2 \,\mathrm{d}\nu(s, t)} \\
&= \|\alpha_g^\omega(b)^{1/2} - b^{1/2}\|_{2,\tau} \|\alpha_g^\omega(b)^{1/2} + b^{1/2}\|_{2,\tau}.
\end{aligned}$$

□

Using Lemma 3.5, we can construct orthogonal projections that play a similar role to the positive elements in Lemma 3.4.

Lemma 3.6. *Let A be a σ-unital C^*-algebra with $T(A)$ nonempty and compact. Let $\alpha : G \curvearrowright A$ be an action by a countable discrete group and assume it has local equivariant property Gamma with respect to bounded traces. Let $\varepsilon > 0$, $F \Subset G$, and $H \Subset G$ be such that $|gH \Delta H| < \varepsilon |H|$ for all $g \in F$. Then, for every $\|\cdot\|_{2,T_\omega(A)}$-separable subset $S \subset A^{\omega,\mathrm{b}}$, all $\delta > 0$, and all $a_1, \dots, a_k \in A_+$ with*

$$\frac{\delta}{|H|} > \sup_{\tau \in T(A)} \min_{i=1,\dots,k} \tau(a_i),$$

there exist pairwise orthogonal projections $p_1, \dots, p_k \in A^{\omega,\mathrm{b}} \cap S'$ such that, for all $\tau \in T_\omega(A)$,

- $\tau\bigl(\sum_{i=1}^{k} p_i\bigr) = 1/k$,
- $\tau(a_i p_i) \le \delta \tau(p_i)$ *for* $i = 1, \dots, k$, *and*
- $\max_{g \in F} \sum_{i=1}^{k} \|\alpha_g^\omega(p_i) - p_i\|_{2,\tau}^2 < 2\sqrt{\varepsilon}/k$.

Proof. By Lemma 3.4, we can find $e_1, \dots, e_k \in (A^{\omega,\mathrm{b}} \cap S')^1_+$ such that, for all $\tau \in T_\omega(A)$,

$$\tau\Big(\sum_{i=1}^k e_i\Big) = 1, \tag{3-11}$$

$$\tau(a_i e_i) \le \delta\tau(e_i) \quad \text{for } i = 1, \dots, k, \tag{3-12}$$

$$\max_{g\in F} \sum_{i=1}^k \|\alpha_g^\omega(e_i) - e_i\|_{1,\tau} < \varepsilon. \tag{3-13}$$

Let $S_0 = S \cup \{1_{A^{\omega,\mathrm{b}}}, a_1, \dots, a_k\}$. Apply Lemma 3.5 for each $i \in \{1, \dots, k\}$ and find a projection $p_i \in A^{\omega,\mathrm{b}} \cap S_0'$ such that, for all $a \in S_0$, $\tau \in T_\omega(A)$, and $g \in G$, we have

$$\tau(ap_i) = \tau(ae_i), \tag{3-14}$$

$$\|\alpha_g^\omega(p_i) - p_i\|_{2,\tau}^2 \le \|\alpha_g^\omega(e_i)^{1/2} - e_i^{1/2}\|_{2,\tau}\|\alpha_g^\omega(e_i)^{1/2} + e_i^{1/2}\|_{2,\tau}. \tag{3-15}$$

This already implies the two following facts:

$$\tau\Big(\sum_{i=1}^k p_i\Big) \overset{(3\text{-}14)}{=} \tau\Big(\sum_{i=1}^k e_i\Big) \overset{(3\text{-}11)}{=} 1 \quad \text{for } \tau \in T_\omega(A), \tag{3-16}$$

$$\tau(a_i p_i) \overset{(3\text{-}14)}{=} \tau(a_i e_i) \overset{(3\text{-}12)}{\le} \delta\tau(e_i) \overset{(3\text{-}14)}{=} \delta\tau(p_i) \quad \text{for } i = 1, \dots, k,\ \tau \in T_\omega(A). \tag{3-17}$$

Furthermore, we get, for $g \in F$ and $\tau \in T_\omega(A)$, that

$$\begin{aligned}
\sum_{i=1}^k \|\alpha_g^\omega(p_i) - p_i\|_{2,\tau}^2 &\overset{(3\text{-}15)}{\le} \sum_{i=1}^k \|\alpha_g^\omega(e_i)^{1/2} - e_i^{1/2}\|_{2,\tau}\|\alpha_g^\omega(e_i)^{1/2} + e_i^{1/2}\|_{2,\tau} \\
&\le \sum_{i=1}^k \|\alpha_g^\omega(e_i)^{1/2} - e_i^{1/2}\|_{2,\tau}(\|\alpha_g^\omega(e_i)^{1/2}\|_{2,\tau} + \|e_i^{1/2}\|_{2,\tau}) \\
&\overset{\text{Lemma 1.19}}{\le} \sum_{i=1}^k \|\alpha_g^\omega(e_i) - e_i\|_{1,\tau}^{1/2}(\|\alpha_g^\omega(e_i)^{1/2}\|_{2,\tau} + \|e_i^{1/2}\|_{2,\tau}) \\
&= \sum_{i=1}^k \|\alpha_g^\omega(e_i) - e_i\|_{1,\tau}^{1/2}\tau(\alpha_g^\omega(e_i))^{1/2} + \sum_{i=1}^k \|\alpha_g^\omega(e_i) - e_i\|_{1,\tau}^{1/2}\tau(e_i)^{1/2} \\
&\le \sqrt{\sum_{i=1}^k \|\alpha_g^\omega(e_i) - e_i\|_{1,\tau}\sum_{i=1}^k (\tau\circ\alpha_g^\omega)(e_i)} + \sqrt{\sum_{i=1}^k \|\alpha_g^\omega(e_i) - e_i\|_{1,\tau}\sum_{i=1}^k \tau(e_i)} \\
&\overset{(3\text{-}11),(3\text{-}13)}{<} 2\sqrt{\varepsilon}.
\end{aligned} \tag{3-18}$$

Set

$$S_1 = S \cup \mathrm{C}^*(\{a_1, \dots, a_k\} \cup \{\alpha_g^\omega(p_j) \mid 1 \le j \le k,\ g \in G\}) \subset A^{\omega,\mathrm{b}}.$$

Since A has local equivariant property Gamma with respect to bounded traces, we can find pairwise orthogonal projections $r_1, \dots, r_k \in (A^{\omega,\mathrm{b}})^{\alpha^\omega} \cap S_1'$ such that

$$\tau(r_i a) = \frac{1}{k}\tau(a) \quad \text{for } \tau \in T_\omega(A),\ a \in S_1,\ i = 1, \dots, k. \tag{3-19}$$

Set $p_i' := r_i p_i$. Then clearly $p_1', \dots, p_k' \in A^{\omega,\mathrm{b}} \cap S'$ are pairwise orthogonal projections. We get, for each $\tau \in T_\omega(A)$, that

$$\tau\Bigl(\sum_{i=1}^k p_i'\Bigr) = \tau\Bigl(\sum_{i=1}^k r_i p_i\Bigr) \overset{\text{(3-19)}}{=} \frac{1}{k}\tau\Bigl(\sum_{i=1}^k p_i\Bigr) \overset{\text{(3-16)}}{=} \frac{1}{k}.$$

For $i = 1, \dots, k$ and $\tau \in T_\omega(A)$, we get

$$\tau(a_i p_i') = \tau(a_i r_i p_i) \overset{\text{(3-19)}}{=} \frac{1}{k}\tau(a_i p_i) \overset{\text{(3-17)}}{\leq} \frac{\delta}{k}\tau(p_i) \overset{\text{(3-19)}}{=} \delta\tau(p_i').$$

Lastly, for $g \in F$ and $\tau \in T_\omega(A)$, we get

$$\sum_{i=1}^k \|\alpha_g^\omega(p_i') - p_i'\|_{2,\tau}^2 \overset{\text{(3-19)}}{=} \frac{1}{k}\sum_{i=1}^k \|\alpha_g^\omega(p_i) - p_i\|_{2,\tau}^2 \overset{\text{(3-18)}}{<} \frac{2\sqrt{\varepsilon}}{k}.$$

Hence the projections p_i' satisfy all the required properties. This finishes the proof. □

In order to carry out the inductive argument to enlarge the trace of the sum of the constructed orthogonal projections, we need a stronger version of the previous lemma that allows us to find the orthogonal projections under an arbitrary tracially constant projection q.

Lemma 3.7. *Let A be a σ-unital C^*-algebra with $T(A)$ nonempty and compact. Let $\alpha : G \curvearrowright A$ be an action by a countable discrete group and assume it has local equivariant property Gamma with respect to bounded traces. Let $\varepsilon > 0$, $F \Subset G$, and $H \Subset G$ be such that $|gH \Delta H| < \varepsilon|H|$ for all $g \in F$. Choose $\delta > 0$ and $a_1, \dots, a_k \in A_+$ such that*

$$\frac{\delta}{|H|} > \sup_{\tau \in T(A)} \min_{i=1,\dots,k} \tau(a_i).$$

For every $\mu \in (0, 1]$ and $\|\cdot\|_{2,T_\omega(A)}$-separable subset $S_0 \subset A^{\omega,\mathrm{b}}$, there exists a $\|\cdot\|_{2,T_\omega(A)}$-separable subset $S \subset A^{\omega,\mathrm{b}}$ such that, if $q \in A^{\omega,\mathrm{b}} \cap S'$ is a projection with $\tau(q) = \mu$ for all $\tau \in T_\omega(A)$, there exist pairwise orthogonal projections $p_1, \dots, p_k \in A^{\omega,\mathrm{b}} \cap S_0' \cap \{\alpha_g^\omega(q) \mid g \in G\}'$ such that, for all $\tau \in T_\omega(A)$,

- $\sum_{i=1}^k \tau(p_i q) = \mu/k$,
- $\tau(a_i p_i q) \leq \delta\tau(p_i q)$ *for $i = 1, \dots, k$,*
- $\max_{g \in F} \sum_{i=1}^k \|q(\alpha_g^\omega(p_i) - p_i)\|_{2,\tau}^2 \leq 2\mu\sqrt{\varepsilon}/k$, *and*
- $\sum_{i=1}^k \|\alpha_g^\omega(p_i)(\alpha_g^\omega(q) - q)\|_{2,\tau}^2 \leq (1/k)\|\alpha_g^\omega(q) - q\|_{2,T_\omega(A)}^2$ *for all $g \in G$.*

Proof. Let $\varepsilon > 0$, $F, H \Subset G$, $\delta > 0$, and $a_1, \dots, a_k \in A_+$ be as in the statement of the lemma. In order to prove the claim, it suffices to prove the following local statement:

For every $\mu \in (0, 1]$, $\zeta > 0$, $T \Subset A$, and $E \Subset G$, there exist $S \Subset A$ and $\xi > 0$ such that, if $q \in A_+^1$ satisfies

$$\|q - q^2\|_{2,u} < \xi, \qquad \sup_{\tau \in T(A)} |\tau(q) - \mu| < \xi, \qquad \text{and} \qquad \|[q, s]\|_{2,u} < \xi \ \text{ for } s \in S,$$

then there exist pairwise orthogonal projections $p_1, \ldots, p_k \in A_+^1$ such that

$$\|p_i - p_i^2\|_{2,u} < \zeta,$$
$$\|[p_i, t]\|_{2,u} < \zeta \quad \text{for } t \in T \cup \{\alpha_g(q) \mid g \in E\},$$
$$\sup_{\tau \in T(A)} \left| \sum_{i=1}^{k} \tau(p_i q) - \frac{\mu}{k} \right| < \zeta,$$
$$\tau(a_i p_i q) < \delta \tau(p_i q) + \zeta \quad \text{for } \tau \in T(A),$$
$$\sup_{\tau \in T(A)} \max_{g \in F} \sum_{i=1}^{k} \|q(\alpha_g(p_i) - p_i\|_{2,\tau}^2 < \frac{2\mu\sqrt{\varepsilon}}{k} + \zeta,$$
$$\sup_{\tau \in T(A)} \max_{g \in E} \sum_{i=1}^{k} \|\alpha_g(p_i)^{1/2}(\alpha_g(q) - q)\|_{2,\tau}^2 < \frac{1}{k} \|\alpha_g(q) - q\|_{2,u}^2 + \zeta.$$

We prove this local statement by contradiction. Suppose there exist $\mu \in (0, 1]$, $\zeta > 0$, $T \Subset A$, and $E \Subset G$ for which the statement does not hold. In other words, this means that, for every $\varnothing \neq S \Subset A$, we can find a $q_S \in A_+^1$ such that

$$\|q_S - q_S^2\|_{2,u} < 1/|S|, \qquad \sup_{\tau \in T(A)} |\tau(q_S) - \mu| < 1/|S|, \qquad \text{and} \qquad \|[q_S, s]\|_{2,u} < 1/|S| \ \text{ for } s \in S,$$

but there exist no pairwise orthogonal projections $p_1, \ldots, p_k \in A_+^1$ satisfying

$$\|p_i - p_i^2\|_{2,u} < \zeta, \tag{3-20}$$
$$\|[p_i, t]\|_{2,u} < \zeta \quad \text{for } t \in T \cup \{\alpha_g(q_S) \mid g \in E\}, \tag{3-21}$$
$$\sup_{\tau \in T(A)} \left| \sum_{i=1}^{k} \tau(p_i q_S) - \frac{\mu}{k} \right| < \zeta, \tag{3-22}$$
$$\tau(a_i p_i q_S) < \delta \tau(p_i q_S) + \zeta \quad \text{for } \tau \in T(A), \tag{3-23}$$
$$\sup_{\tau \in T(A)} \max_{g \in F} \sum_{i=1}^{k} \|q_S(\alpha_g(p_i) - p_i\|_{2,\tau}^2 < \frac{2\mu\sqrt{\varepsilon}}{k} + \zeta, \tag{3-24}$$
$$\sup_{\tau \in T(A)} \max_{g \in E} \sum_{i=1}^{k} \|\alpha_g(p_i)^{1/2}(\alpha_g(q_S) - q_S)\|_{2,\tau}^2 < \frac{1}{k} \|\alpha_g(q_S) - q_S\|_{2,u}^2 + \zeta. \tag{3-25}$$

In this way we get a net $(q_S)_S$ indexed by the finite subsets of A equipped with inclusion as the natural partial order. We can take a free ultrafilter $\tilde{\omega}$ on this index set of finite subsets of A as follows. For each $I \Subset A$ consider the set

$$P_I = \{J \Subset A \mid I \subseteq J\}.$$

As the collection of P_I is closed under finite intersections, they form a filter basis and hence there is a minimal filter on the set of finite subsets of A containing all the sets P_I for $I \Subset A$. This filter will be free and can be extended to a free ultrafilter $\tilde{\omega}$ by Zorn's lemma.

Similarly as in Definitions 1.3 and 1.8, we can define the norm and bounded tracial ultrapower of A over the ultrafilter $\tilde{\omega}$. We also get a set of limit traces $T_{\tilde{\omega}}(A)$ over $\tilde{\omega}$. Then the net $(q_S)_S$ defines a projection $q \in A^{\tilde{\omega},\mathrm{b}} \cap A'$ with value μ on all limit traces on $A^{\tilde{\omega},\mathrm{b}}$. By Lemma 3.6, we can find pairwise orthogonal projections $p_1', \dots, p_k' \in A_+^1$ such that

$$\|p_i' - p_i'^2\|_{2,u} < \zeta \quad \text{for } i = 1, \dots, k, \tag{3-26}$$

$$\|[p_i', t]\|_{2,u} < \zeta \quad \text{for } t \in T, \tag{3-27}$$

$$\sup_{\tau \in T(A)} \left| \tau\left(\sum_{i=1}^k p_i'\right) - \frac{1}{k}\right| < \frac{1}{4}\zeta, \tag{3-28}$$

$$\tau(a_i p_i') < \delta\tau(p_i') + \zeta \quad \text{for } i = 1, \dots, k,\ \tau \in T(A), \tag{3-29}$$

$$\max_{g \in F} \sum_{i=1}^k \|\alpha_g(p_i') - p_i'\|_{2,\tau}^2 < \frac{2\sqrt{\varepsilon}}{k} \quad \text{for } \tau \in T(A). \tag{3-30}$$

Since $q \in A^{\tilde{\omega},\mathrm{b}} \cap A'$, it follows, for each $\tau \in T_{\tilde{\omega}}(A)$ and $g \in G$, that the assignment

$$A \to \mathbb{C} : a \mapsto \tau(a\alpha_g^{\tilde{\omega}}(q))/\tau(\alpha_g^{\tilde{\omega}}(q)) = \tau(a\alpha_g^{\tilde{\omega}}(q))/\mu$$

defines a tracial state on A. In particular, we get that, for $\tau \in T_{\tilde{\omega}}(A)$, the following hold:

$$\left| \tau\left(\sum_{i=1}^k p_i'\alpha_g^{\tilde{\omega}}(q)\right) - \frac{\mu}{k}\right| \overset{(3\text{-}28)}{<} \frac{1}{4}\mu\zeta \le \frac{1}{4}\zeta \quad \text{for } g \in G, \tag{3-31}$$

$$\tau(a_i p_i' q) \overset{(3\text{-}29)}{<} \delta\tau(p_i' q) + \mu\zeta \le \delta\tau(p_i' q) + \zeta, \tag{3-32}$$

$$\max_{g \in F} \sum_{i=1}^k \|q(\alpha_g(p_i') - p_i')\|_{2,\tau}^2 \overset{(3\text{-}30)}{<} \frac{2\mu\sqrt{\varepsilon}}{k}. \tag{3-33}$$

Next we show that, for all $\tau \in T_{\tilde{\omega}}(A)$ and $g \in G$,

$$\sum_{i=1}^k \|\alpha_g(p_i')^{1/2}(\alpha_g^{\tilde{\omega}}(q) - q)\|_{2,\tau}^2 < \frac{1}{k}\|\alpha_g^{\tilde{\omega}}(q) - q\|_{2,u}^2 + \zeta. \tag{3-34}$$

Fix $\tau \in T_{\tilde{\omega}}(A)$ and $g \in G$. Assume $\tau(q\alpha_g^{\tilde{\omega}}(q)) > 0$. Then the map

$$A \to \mathbb{C}, \quad a \mapsto \tau(aq\alpha_g^{\tilde{\omega}}(q))/\tau(q\alpha_g^{\tilde{\omega}}(q))$$

defines a tracial state on A. This means that

$$\left| \tau\left(\sum_{i=1}^k \alpha_g(p_i')q\alpha_g^{\tilde{\omega}}(q)\right) - \frac{1}{k}\tau(q\alpha_g^{\tilde{\omega}}(q))\right| \overset{(3\text{-}28)}{<} \frac{1}{4}\zeta\tau(q\alpha_g^{\tilde{\omega}}(q)) \le \frac{1}{4}\zeta. \tag{3-35}$$

Note that, if $\tau(q\alpha_g^{\tilde{\omega}}(q)) = 0$, then

$$\tau\left(\sum_{i=1}^{k} \alpha_g(p_i')q\alpha_g^{\tilde{\omega}}(q)\right) = 0 = \frac{1}{k}\tau(q\alpha_g^{\tilde{\omega}}(q)),$$

and hence (3-35) also holds in this case. Now

$$\begin{aligned}\tau\left(\sum_{i=1}^{k} \alpha_g(p_i')(\alpha_g^{\tilde{\omega}}(q)-q)^2\right) &= \tau\left(\sum_{i=1}^{k} \alpha_g(p_i')(\alpha_g^{\tilde{\omega}}(q)+q-\alpha_g^{\tilde{\omega}}(q)q-q\alpha_g^{\tilde{\omega}}(q))\right)\\ &\overset{(3\text{-}31),(3\text{-}35)}{<} \frac{1}{k}\tau(\alpha_g^{\tilde{\omega}}(q)+q-\alpha_g^{\tilde{\omega}}(q)q-q\alpha_g^{\tilde{\omega}}(q))+\zeta\\ &= \frac{1}{k}\tau((\alpha_g^{\tilde{\omega}}(q)-q)^2)+\zeta.\end{aligned}$$

This proves (3-34). If we combine this with (3-26), (3-27), and (3-31)–(3-33), we can conclude that, for some q_S in the net, (3-20)–(3-25) must hold. This gives the desired contradiction. $\square$

Proof of Lemma 3.2. Given $\varepsilon > 0$ and $t \in (0, 1)$, choose $\eta > 0$ small enough that

$$4\left\lceil \frac{t}{1-t}\right\rceil \sqrt{\eta} < \varepsilon. \tag{3-36}$$

We show that such a constant η satisfies the required properties. Let $\alpha : G \curvearrowright A$, finite sets $F, H \Subset G$, and $S \subset A^{\omega,\mathrm{b}}$ be given as in the statement of the lemma. By Remark 3.3, it suffices to consider $\delta > 0$ and $a_1, \dots, a_k \in A_+$ such that

$$\frac{\delta}{|H|} > \sup_{\tau\in T(A)} \min_{i=1,\dots,k} \tau(a_i).$$

We construct the pairwise orthogonal projections $p_1, \dots, p_k$ in $N := k\lceil t/(1-t)\rceil$ steps. Define a sequence (s_n) in $[0, 1)$ inductively by setting $s_0 = 0$ and setting $s_{i+1} = s_i + (1-s_i)/k$. Note that, when $s < t$, we have $s + (1-s)/k > s + (1-t)/k$. If we assumed for a moment that $s_N < t$, then this sequence is less than t for all of the first N steps, leading to $s_N > N(1-t)/k \geq t$, which is a contradiction; hence we must have $s_N > t$. Next, we construct separable subsets $S_0, \dots, S_N$ as follows: Set $S_N := S$. Given $i \in \{1, \dots, N\}$ such that S_i is defined, let S_{i-1} be the union of S_i and the set determined by Lemma 3.7 with $1 - s_{i-1}$ in place of μ and S_i in place of S_0.

In the initial step, we set $p_1^{(0)} = \cdots = p_k^{(0)} = 0$. Now suppose that, for some $n \in \mathbb{N}$, we have pairwise orthogonal projections $p_1^{(n)}, \dots, p_k^{(n)} \in A^{\omega,\mathrm{b}} \cap S_n'$ such that, for all $\tau \in T_\omega(A)$,

$$\tau(p_1^{(n)} + \cdots + p_k^{(n)}) = s_n, \tag{3-37}$$

$$\tau(a_i p_i^{(n)}) \leq \delta\tau(p_i^{(n)}) \quad \text{for } i = 1, \dots, k, \tag{3-38}$$

$$\max_{g\in F} \sum_{i=1}^{k} \|\alpha_g^{\omega}(p_i^{(n)}) - p_i^{(n)}\|_{2,\tau}^2 \leq \frac{4n\sqrt{\eta}}{k}, \tag{3-39}$$

$$\max_{g\in F} \left\| \sum_{i=1}^{k} \alpha_g^{\omega}(p_i^{(n)}) - \sum_{i=1}^{k} p_i^{(n)} \right\|_{2,T_\omega(A)}^2 \leq 2s_n\sqrt{\eta}. \tag{3-40}$$

Note that the $p_i^{(0)}$ trivially satisfy (3-37)–(3-40). We show that we can construct pairwise orthogonal projections $p_1^{(n+1)}, \dots, p_k^{(n+1)} \in A^{\omega,\mathrm{b}} \cap S_{n+1}'$ such that, for all $\tau \in T_\omega(A)$,

$$\tau(p_1^{(n+1)} + \cdots + p_k^{(n+1)}) = s_{n+1}, \tag{3-41}$$

$$\tau(a_i p_i^{(n+1)}) \leq \delta\tau(p_i^{(n+1)}) \quad \text{for } i = 1, \dots, k, \tag{3-42}$$

$$\max_{g \in F} \sum_{i=1}^k \|\alpha_g^\omega(p_i^{(n+1)}) - p_i^{(n+1)}\|_{2,\tau}^2 \leq \frac{4(n+1)\sqrt{\eta}}{k}, \tag{3-43}$$

$$\max_{g \in F} \Big\| \sum_{i=1}^k \alpha_g^\omega(p_i^{(n+1)}) - \sum_{i=1}^k p_i^{(n+1)} \Big\|_{2,T_\omega(A)}^2 \leq 2s_{n+1}\sqrt{\eta}. \tag{3-44}$$

Define

$$q := 1_{A^{\omega,\mathrm{b}}} - \sum_{i=1}^k p_i^{(n)}.$$

Note that q is a projection in $A^{\omega,\mathrm{b}} \cap S_n'$ with $\tau(q) = 1 - s_n$ for all $\tau \in T_\omega(A)$. By Lemma 3.7 and our choice of S_n, we can find pairwise orthogonal projections

$$r_1, \dots, r_k \in A^{\omega,\mathrm{b}} \cap S_{n+1}' \cap \{\alpha_g^\omega(q) \mid g \in G\}'$$

such that, for all $\tau \in T_\omega(A)$,

$$\tau\Big(\sum_{i=1}^k r_i q\Big) = \frac{1 - s_n}{k}, \tag{3-45}$$

$$\tau(a_i r_i q) \leq \delta\tau(r_i q) \quad \text{for } i = 1, \dots, k, \tag{3-46}$$

$$\max_{g \in F} \sum_{i=1}^k \|q(\alpha_g^\omega(r_i) - r_i)\|_{2,\tau}^2 \leq \frac{2(1 - s_n)\sqrt{\eta}}{k}, \tag{3-47}$$

$$\sum_{i=1}^k \|\alpha_g^\omega(r_i)(\alpha_g^\omega(q) - q)\|_{2,\tau}^2 \leq \frac{1}{k}\|\alpha_g^\omega(q) - q\|_{2,T_\omega(A)}^2 \quad \text{for } g \in G. \tag{3-48}$$

Define

$$p_i^{(n+1)} := p_i^{(n)} + q r_i.$$

By construction (recall that $S_{n+1} \subset S_n$), the elements $p_1^{(n+1)}, \dots, p_k^{(n+1)}$ are pairwise orthogonal projections in $A^{\omega,\mathrm{b}} \cap S_{n+1}'$. We show that they satisfy (3-41)–(3-44). Fix $\tau \in T_\omega(A)$. Note first of all that

$$\tau\Big(\sum_{i=1}^k p_i^{(n+1)}\Big) = \tau\Big(\sum_{i=1}^k p_i^{(n)}\Big) + \tau\Big(\sum_{i=1}^k q r_i\Big) \overset{(3\text{-}37),(3\text{-}45)}{=} s_n + \frac{1}{k}(1 - s_n) = s_{n+1}.$$

Moreover, for $i = 1, \dots, k$, we have

$$\tau(a_i p_i^{(n+1)}) = \tau(a_i p_i^{(n)}) + \tau(a_i q r_i) \overset{(3\text{-}38),(3\text{-}46)}{\leq} \delta\tau(p_i^{(n)}) + \delta\tau(q r_i) = \delta\tau(p_i^{(n+1)}).$$

This shows already that the projections satisfy (3-41) and (3-42). Next we prove that they satisfy (3-43). Note that $p_i^{(n)}$ is orthogonal to q for $i=1,\dots,k$. Hence, we get, for each $g\in G$ and $\tau\in T_\omega(A)$, that

$$\begin{aligned}\sum_{i=1}^{k}\|\alpha_g^\omega(p_i^{(n+1)})-p_i^{(n+1)}\|_{2,\tau}&\\ &=\sum_{i=1}^{k}\tau((\alpha_g^\omega(p_i^{(n+1)})-p_i^{(n+1)})^2)\\ &=\sum_{i=1}^{k}\bigl(\tau((\alpha_g^\omega(p_i^{(n)})-p_i^{(n)})^2)+\tau((\alpha_g^\omega(qr_i)-qr_i)^2)-2\tau(\alpha_g^\omega(p_i^{(n)})qr_i)-2\tau(p_i^{(n)}\alpha_g^\omega(qr_i))\bigr)\\ &\le\sum_{i=1}^{k}\|\alpha_g^\omega(p_i^{(n)})-p_i^{(n)}\|_{2,\tau}^2+\sum_{i=1}^{k}\|\alpha_g^\omega(qr_i)-qr_i\|_{2,\tau}^2.\end{aligned}\tag{3-49}$$

For $i=1,\dots,k$, we have

$$\begin{aligned}\|\alpha_g^\omega(qr_i)-qr_i\|_{2,\tau}^2&\le(\|(\alpha_g^\omega(q)-q)\alpha_g^\omega(r_i)\|_{2,\tau}+\|q(\alpha_g^\omega(r_i)-r_i)\|_{2,\tau})^2\\ &\le 2(\|(\alpha_g^\omega(q)-q)\alpha_g^\omega(r_i)\|_{2,\tau}^2+\|q(\alpha_g^\omega(r_i)-r_i)\|_{2,\tau}^2).\end{aligned}$$

Combining this with (3-49), we find that, for $g\in F$,

$$\begin{aligned}\sum_{i=1}^{k}\|\alpha_g^\omega(p_i^{(n+1)})-p_i^{(n+1)}\|_{2,\tau}^2&\\ &\le\sum_{i=1}^{k}\|\alpha_g^\omega(p_i^{(n)})-p_i^{(n)}\|_{2,\tau}^2+2\sum_{i=1}^{k}(\|(\alpha_g^\omega(q)-q)\alpha_g^\omega(r_i)\|_{2,\tau}^2+\|q(\alpha_g^\omega(r_i)-r_i)\|_{2,\tau}^2)\\ &\overset{(3\text{-}39),(3\text{-}48)}{\le}\frac{4n\sqrt{\eta}}{k}+\frac{2}{k}\Bigl\|\sum_{i=1}^{k}\alpha_g^\omega(p_i^{(n)})-\sum_{i=1}^{k}p_i^{(n)}\Bigr\|_{2,T_\omega(A)}^2+2\sum_{i=1}^{k}\|q(\alpha_g^\omega(r_i)-r_i)\|_{2,\tau}^2\\ &\overset{(3\text{-}40),(3\text{-}47)}{\le}\frac{4n\sqrt{\eta}}{k}+\frac{4s_n\sqrt{\eta}}{k}+\frac{4(1-s_n)\sqrt{\eta}}{k}=\frac{4(n+1)\sqrt{\eta}}{k}.\end{aligned}$$

Lastly, we show that the elements $p_i^{(n+1)}$ satisfy (3-44). We get, for each $g\in G$, that

$$\begin{aligned}\sum_{i=1}^{k}(\alpha_g^\omega(p_i^{(n+1)})-p_i^{(n+1)})&=\sum_{i=1}^{k}(\alpha_g^\omega(p_i^{(n)})-p_i^{(n)})+\alpha_g^\omega\Bigl(q\Bigl(\sum_{j=1}^{k}r_j\Bigr)\Bigr)-q\Bigl(\sum_{j=1}^{k}r_j\Bigr)\\ &=q-\alpha_g^\omega(q)+\alpha_g^\omega\Bigl(q\Bigl(\sum_{j=1}^{k}r_j\Bigr)\Bigr)-q\Bigl(\sum_{j=1}^{k}r_j\Bigr)\\ &=q\Bigl(1-\sum_{j=1}^{k}r_j\Bigr)-\alpha_g^\omega\Bigl(q\Bigl(1-\sum_{j=1}^{k}r_j\Bigr)\Bigr)\\ &=q\Bigl(\sum_{j=1}^{k}(\alpha_g^\omega(r_j)-r_j)\Bigr)+q\Bigl(1-\sum_{j=1}^{k}\alpha_g^\omega(r_j)\Bigr)-\alpha_g^\omega\Bigl(q\Bigl(1-\sum_{j=1}^{k}r_j\Bigr)\Bigr)\\ &=q\Bigl(\sum_{j=1}^{k}(\alpha_g^\omega(r_j)-r_j)\Bigr)+(q-\alpha_g^\omega(q))\Bigl(1-\sum_{j=1}^{k}\alpha_g^\omega(r_j)\Bigr).\end{aligned}$$

Keeping in mind that q is a projection and the elements r_j are pairwise orthogonal projections, we have, for all $g \in F$ and $\tau \in T_\omega(A)$, that

$$\begin{aligned}
&\Big\| \sum_{i=1}^k \alpha_g^\omega(p_i^{(n+1)}) - \sum_{i=1}^k p_i^{(n+1)} \Big\|_{2,\tau}^2 \\
&\quad = \tau\biggl(\Bigl(\sum_{i=1}^k \alpha_g^\omega(p_i^{(n+1)}) - \sum_{i=1}^k p_i^{(n+1)}\Bigr)^2\biggr) = \tau\biggl(\Bigl(\alpha_g^\omega(p_i^{(n)} + qr_i) - \sum_{i=1}^k p_i^{(n)} + qr_i\Bigr)^2\biggr) \\
&\quad = \tau\biggl(\Bigl(\sum_{i=1}^k q(\alpha_g^\omega(r_i) - r_i) + (q - \alpha_g^\omega(q))(1 - \alpha_g^\omega(r_i))\Bigr)^2\biggr) \\
&\quad = \tau\biggl(q\Bigl(\sum_{i,j=1}^k \alpha_g^\omega(r_i r_j) + r_i r_j - r_i\alpha_g^\omega(r_j) - \alpha_g^\omega(r_i) r_j\Bigr) + (q - \alpha_g^\omega(q))^2\Bigl(1 - \sum_{j=1}^k \alpha_g^\omega(r_j)\Bigr)\biggr) \\
&\qquad\qquad - 2\tau\biggl(\Bigl(\sum_{j=1}^k r_j\Bigr) q(1 - \alpha_g^\omega(q))\Bigl(1 - \sum_{j=1}^k \alpha_g^\omega(r_j)\Bigr)\biggr) \\
&\quad \le \tau\biggl(q\Bigl(\sum_{i,j=1}^k \alpha_g^\omega(r_i r_j) + r_i r_j - r_i\alpha_g^\omega(r_j) - \alpha_g^\omega(r_i) r_j\Bigr) + (q - \alpha_g^\omega(q))^2\biggr) \\
&\quad \le \tau\biggl(q\Bigl(\sum_{i=1}^k \alpha_g^\omega(r_i) + r_i - r_i\alpha_g^\omega(r_i) - \alpha_g^\omega(r_i) r_i\Bigr) + (q - \alpha_g^\omega(q))^2\biggr) \\
&\quad = \tau\biggl(q\Bigl(\sum_{i=1}^k (\alpha_g^\omega(r_i) - r_i)^2\Bigr) + (q - \alpha_g^\omega(q))^2\biggr) = \sum_{i=1}^k \|q(r_i - \alpha_g^\omega(r_i))\|_{2,\tau}^2 + \|q - \alpha_g^\omega(q)\|_{2,\tau}^2 \\
&\overset{\text{(3-40),(3-47)}}{\le} 2\Bigl(s_n + \frac{1}{k}(1 - s_n)\Bigr)\sqrt{\eta} = 2s_{n+1}\sqrt{\eta}.
\end{aligned}$$

In the first inequality in the above computation, we used the fact that q commutes with all the elements r_i, thus the term $\bigl(\sum_{j=1}^k r_j\bigr) q(1 - \alpha_g^\omega(q))\bigl(1 - \sum_{j=1}^k \alpha_g^\omega(r_j)\bigr)$ is a product of two positive elements, whose trace value must be nonnegative. In the second inequality, we used the pairwise orthogonality of the projections r_i, so the mixed terms in the double sum appearing above contribute the trace value of $-(r_i\alpha_g^\omega(r_j) + \alpha_g^\omega(r_i) r_j)$, which likewise is nonpositive.

As explained before, one has $s_N > t$. So if we start the inductive procedure with the projections $p_1^{(0)} = \cdots = p_k^{(0)} = 0$, then after N steps we obtain projections $p_1^{(N)}, \dots, p_k^{(N)}$ satisfying

$$\sum_{i=1}^k \tau(p_i^{(N)}) > t \quad \text{for } \tau \in T_\omega(A).$$

Moreover, at that point we have

$$\tau(a_i p_i^{(N)}) \overset{\text{(3-42)}}{\le} \delta\tau(p_i^{(N)}) \quad \text{for } i = 1, \dots, k,\ \tau \in T_\omega(A),$$

$$\max_{g \in F} \sum_{i=1}^k \|\alpha_g^\omega(p_i^{(N)}) - p_i^{(N)}\|_{2,\tau}^2 \overset{\text{(3-43)}}{\le} \frac{4N\sqrt{\eta}}{k} \overset{\text{(3-36)}}{<} \varepsilon.$$

We conclude that $p_1^{(N)}, \dots, p_k^{(N)}$ satisfy all the required properties. $\square$

4. Dynamical tracial local-to-global principle

Here we prove our main technical result, namely that equivariant property Gamma implies a *tracial local-to-global principle* for actions of amenable groups. Roughly, this means that whenever a $*$-polynomial identity has (local) approximate solutions one tracial presentation at a time, then it has (global) approximate solutions in the uniform tracial 2-norm. We begin by precisely defining these polynomial identities.

Definition 4.1 (cf. [Gardella et al. 2022, Definition 4.4]). Let G be a discrete group, and let X be a countable set of noncommutative variables. A *noncommutative G-$*$-polynomial* in the variables X is a noncommutative $*$-polynomial in the variables $\{g \cdot x \mid g \in G, x \in X\}$.

Let A be a C*-algebra with action $\alpha : G \curvearrowright A$. Suppose that $h(x_1, \dots, x_r)$ is a G-$*$-polynomial in r noncommuting variables. Given a tuple $(a_1, \dots, a_r) \in A^r$, the evaluation $h(a_1, \dots, a_r)$ is computed by interpreting $g \cdot x_i$ as $\alpha_g(a_i)$ for $g \in G$ and $i = 1, \dots, r$.

The main theorem that we prove in this section is the following.

Theorem 4.2. *Let A be a σ-unital* C*-*algebra with $T(A)$ nonempty and compact*, *and with weak* CPoU. *Let $\alpha : G \curvearrowright A$ be an action by an amenable countable discrete group and assume it has local equivariant property Gamma with respect to bounded traces. For each $m \in \mathbb{N}$, let*

$$h_m(x_1, \dots, x_{r_m}, z_1, \dots, z_{s_m})$$

be a G-$*$-polynomial in $r_m + s_m$ noncommuting variables. Let $(a_i)_{i\in\mathbb{N}}$ be a sequence in $A^{\omega,\mathrm{b}}$. Suppose, for every $\varepsilon > 0$, $\ell \in \mathbb{N}$, and $\tau \in \overline{T_\omega(A)}^{w^*}$, there exist contractions $(y_i^\tau)_{i\in\mathbb{N}}$ in $A^{\omega,\mathrm{b}}$ such that

$$\|h_m(a_1, \dots, a_{r_m}, y_1^\tau, \dots, y_{s_m}^\tau)\|_{2,\tau} < \varepsilon \quad \textit{for } m = 1, \dots, \ell.$$

Then there exist contractions $(y_i)_{i\in\mathbb{N}}$ in $A^{\omega,\mathrm{b}}$ such that

$$h_m(a_1, \dots, a_{r_m}, y_1, \dots, y_{s_m}) = 0 \quad \textit{for all } m \in \mathbb{N}. \tag{4-1}$$

We reduce the complexity of the polynomials involved in the proof of this result with the following.

Lemma 4.3. *Let G be a countable discrete group. Consider sets of variables $X = \{x_i \mid i \in \mathbb{N}\}$ and $Z = \{z_i \mid i \in \mathbb{N}\}$. Assume*

$$\mathcal{P} = \{h_m(x_1, \dots, x_{r_m}, z_1, \dots, z_{s_m}) \mid m \in \mathbb{N}\}$$

is a countable set of noncommutative G-$*$-polynomials in the variables $X \cup Z$. Then there exists another set of variables $Z' = \{z'_i \mid i \in \mathbb{N}\}$ and another countable set of noncommutative G-$*$-polynomials

$$\mathcal{P}' = \{h'_m(x_1, \dots, x_{r'_m}, z'_1, \dots, z'_{s'_m}) \mid m \in \mathbb{N}\}$$

in the variables $X \cup Z'$ such that every G-$*$-polynomial h'_m satisfies one of the following properties:

1. $h'_m(x_1, \dots, x_{r'_m}, 0, \dots, 0) = 0$ (*i.e.*, *no terms in the polynomial with variables only in X*) *and* $h'_m(1, \dots, 1, z'_1, \dots, z'_{s'_m})$ *is an ordinary $*$-polynomial in the variables Z'*;
2. $h'_m(x_1, \dots, x_{r'_m}, z'_1, \dots, z'_{s'_m}) = \|h'_m(x_1, \dots, x_{r'_m}, 0, \dots, 0)\| z'_{s'_m} - h'_m(x_1, \dots, x_{r'_m}, 0, \dots, 0)$;
3. $h'_m(x_1, \dots, x_{r'_m}, z'_1, \dots, z'_{s'_m}) = g \cdot z'_i - z'_j$ *for some* $1 \le i, j \le s'_m$ *and some* $g \in G$.

Moreover we have that, for every action $\beta : G \curvearrowright B$ on any C*-*algebra, any sequence $(b_i)_{i\in\mathbb{N}}$ in B, and subset $T \subset T(B)$ the following two statements hold*:

(a) *There exist contractions $(y_i)_{i\in\mathbb{N}}$ in B such that*

$$\|h_m(b_1, \dots, b_{r_m}, y_1, \dots, y_{s_m})\|_{2,T} = 0 \quad \textit{for all } m \in \mathbb{N}$$

if and only if there exist contractions $(y'_i)_{i\in\mathbb{N}}$ in B such that

$$\|h'_m(b_1, \dots, b_{r'_m}, y'_1, \dots, y'_{s'_m})\|_{2,T} = 0 \quad \textit{for all } m \in \mathbb{N}.$$

(b) *For each $\varepsilon > 0$ and each $\ell \in \mathbb{N}$, there exist contractions $(y_i)_{i\in\mathbb{N}}$ in B such that*

$$\|h_m(b_1, \dots, b_{r_m}, y_1, \dots, y_{s_m})\|_{2,T} < \varepsilon \quad \textit{for } m = 1, \dots, \ell$$

if and only if, for each $\varepsilon' > 0$ and each $\ell' \in \mathbb{N}$, there exist contractions $(y'_i)_{i\in\mathbb{N}}$ in B such that

$$\|h'_m(b_1, \dots, b_{r'_m}, y'_1, \dots, y'_{s'_m})\|_{2,T} < \varepsilon' \quad \textit{for } m = 1, \dots, \ell'.$$

Proof. Define

$$Z' = \{z_i \mid i \in \mathbb{N}\} \cup \{z_{i,g} \mid i \in \mathbb{N},\ g \in G \setminus \{e\}\} \cup \{w_m \mid m \in \mathbb{N}\}.$$

For each $m \in \mathbb{N}$, we can take the G-$*$-polynomial h_m in the variables $X \cup Z$ and define h''_m in the variables $X \cup Z'$ by replacing every instance of a variable $g \cdot z_i$ for some $i \in \mathbb{N}$ and $g \in G \setminus \{e\}$ by $z_{i,g}$, e.g., the polynomial $g \cdot z_1 - z_2$ would be transformed into $z_{1,g} - z_2$. Next, we define a new G-$*$-polynomial h'''_m for each $m \in \mathbb{N}$ by setting

$$h'''_m(X \cup Z') = h''_m(X \cup Z') + \|h_m(x_1, \dots, x_{r_m}, 0, \dots, 0)\| w_m - h_m(x_1, \dots, x_{r_m}, 0, \dots, 0). \tag{4-2}$$

Set $\mathcal{P}'_1 := \{h'''_m(X \cup Z') \mid m \in \mathbb{N}\}$. All the polynomials in this set are of type (1) mentioned in the statement of this lemma. Next, define the sets of G-$*$-polynomials

$$\mathcal{P}'_2 := \{\|h_m(x_1, \dots, x_{r_m}, 0, \dots, 0)\| w_m - h_m(x_1, \dots, x_{r_m}, 0, \dots, 0) \mid m \in \mathbb{N}\}$$

and

$$\mathcal{P}'_3 := \{g \cdot z_i - z_{i,g} \mid i \in \mathbb{N},\ g \in G \setminus \{e\}\}.$$

These sets consist of polynomials of type (2) and (3), respectively.

Consider $\mathcal{P}' = \mathcal{P}'_1 \cup \mathcal{P}'_2 \cup \mathcal{P}'_3$. The G-$*$-polynomials in this set are all of the right form, and we claim that this does the job. It suffices to show part (a). This is because, for a given C*-algebra B, action $\beta : G \curvearrowright B$, sequence $(b_i)_{i\in\mathbb{N}} \in B$, and subset $T \subset T(B)$, statement (b) immediately follows from statement (a) when applied to the C*-algebra B_ω, with action β_ω, sequence $(b_i)_{i\in\mathbb{N}}$ in $B \subset B_\omega$, and the set of limit traces on B_ω arising from sequences of traces in T.

To show part (a), fix a C*-algebras B, an action $\beta : G \curvearrowright B$, a sequence $(b_i)_{i\in\mathbb{N}}$, and subset $T \subset T(B)$. Let $(y_i)_{i\in\mathbb{N}}$ be any sequence of contractions in B. For notational brevity, we denote these sequences by $\bar{b} = (b_i)_i$ and $\bar{y} = (y_i)_i$. Furthermore we shall also write (exclusively in this proof), for two elements $x, y \in B$, the expression "$x =_T y$" as shorthand for $\|x - y\|_{2,T} = 0$.

We set $y_{i,g} = \beta_g(y_i)$ for $i \in \mathbb{N}$ and $g \in G \setminus \{e\}$ and

$$w_m = \begin{cases} 0, & h_m(b_1, \dots, b_{r_m}, 0, \dots, 0) = 0, \\ \dfrac{h_m(b_1, \dots, b_{r_m}, 0, \dots, 0)}{\|h_m(b_1, \dots, b_{r_m}, 0, \dots, 0)\|}, & h_m(b_1, \dots, b_{r_m}, 0, \dots, 0) \neq 0. \end{cases}$$

Then the tuple

$$\bar{z} := (y_i)_{i\in\mathbb{N}} \times (y_{i,g})_{i\in\mathbb{N}, g\in G\setminus\{e\}} \times (w_m)_{m\in\mathbb{N}}$$

represents a choice for the free variables of Z' inside B. By definition we have $p(\bar{b}, \bar{z}) = 0$ for all $p \in \mathcal{P}_2' \cup \mathcal{P}_3'$. By definition of the polynomials h_m''', we have

$$h_m'''(b_1, \dots, b_{r_m}, \bar{z}) = h_m''(b_1, \dots, b_{r_m}, \bar{z}) + q(\bar{b}, \bar{z}) = h_m''(b_1, \dots, b_{r_m}, \bar{z})$$

for some $*$-polynomial $q \in \mathcal{P}_2'$. Due to the vanishing of all the $*$-polynomials of $\mathcal{P}_3'$ in $\bar{z}$ and given how the polynomial h_m'' arises from the polynomial h_m via substitution of variables, we may finally observe

$$h_m(b_1, \dots, b_{r_m}, y_1, \dots, y_{s_m}) = h_m''(b_1, \dots, b_{r_m}, \bar{z}).$$

This shows immediately that if the sequence $(y_i)_i$ satisfies

$$\|h_m(b_1, \dots, b_{r_m}, y_1, \dots, y_{s_m})\|_{2,T} = 0 \quad \text{for all } m \in \mathbb{N},$$

then we also have $\|p(\bar{b}, \bar{z})\|_{2,T} = 0$ for all $p \in \mathcal{P}'$. In particular, we get the "only if" part in (a).

Conversely, suppose that

$$\bar{z} := (y_i)_{i\in\mathbb{N}} \times (y_{i,g})_{i\in\mathbb{N}, g\in G\setminus\{e\}} \times (w_m)_{m\in\mathbb{N}}$$

is an arbitrary tuple with values in the unit ball of B representing a choice for the free variables in Z' such that $p(\bar{b}, \bar{z}) = 0$ for all $p \in \mathcal{P}'$. By doing the above computations in reverse, we can see that $p(\bar{z}) =_T 0$ for $p \in \mathcal{P}_3'$ forces $y_{i,g} =_T \beta_g(y_i)$ for all $g \in G \setminus \{e\}$. Moreover, the vanishing $p(\bar{b}, \bar{z}) =_T 0$ for $p \in \mathcal{P}_2'$ forces the equation

$$w_m = \frac{h_m(b_1, \dots, b_{r_m}, 0, \dots, 0)}{\|h_m(b_1, \dots, b_{r_m}, 0, \dots, 0)\|}$$

when $h_m(b_1, \dots, b_{r_m}, 0, \dots, 0) \neq 0$. Similar to how we argued above, this implies, for all $m \geq 1$, that

$$h_m'''(b_1, \dots, b_{r_m}, \bar{z}) = h_m''(b_1, \dots, b_{r_m}, \bar{z}) + q(\bar{b}, \bar{z}) =_T h_m''(b_1, \dots, b_{r_m}, \bar{z})$$

for some $*$-polynomial $q \in \mathcal{P}_2'$. Given how the polynomial h_m'' arises from the polynomial h_m via substituion of variables, we may finally observe

$$h_m(b_1, \dots, b_{r_m}, y_1, \dots, y_{s_m}) =_T h_m''(b_1, \dots, b_{r_m}, \bar{z}), \quad m \geq 1.$$

This shows the "if" part of (a) and finishes the proof. □

Proof of Theorem 4.2. Equation (4-1) is equivalent to

$$\|h_m(a_1, \dots, a_{r_m}, y_1, \dots, y_{s_m})\|_{2,T_\omega(A)} = 0 \quad \text{for all } m \in \mathbb{N}.$$

By the previous lemma, we may assume that the $*$-polynomials h_m are all of one of the following types:

(1) $h_m(a_1,\dots,a_{r_m},0,\dots,0)=0$ and $h_m(1_{A^{\omega,\mathrm{b}}},\dots,1_{A^{\omega,\mathrm{b}}},z_1,\dots,z_{s_m})$ is an ordinary $*$-polynomial.

(2) h_m is of the form $\|h_m(x_1,\dots,x_{r_m},0,\dots,0)\|z_i-h_m(x_1,\dots,x_{r_m},0,\dots,0)$ for some $i\in\mathbb{N}$. Equivalently, since this doesn't change the solutions, we may assume that $h_m(a_1,\dots,a_{r_m},z_1,\dots,z_{s_m})$ is of the form z_i-a for some $i\in\mathbb{N}$ and $a\in \mathrm{C}^*(\{\alpha_g^\omega(a_i)\mid i\in\mathbb{N},\,g\in G\})$ with $\|a\|=1$.

(3) h_m is of the form $g\cdot z_i-z'_i$ for some $i,i'\in\mathbb{N}$ and $g\in G$.

By Kirchberg's ε-test, it suffices to find, for each $\varepsilon>0$ and $\ell\in\mathbb{N}$, contractions $(y_i)_{i\in\mathbb{N}}$ in $A^{\omega,\mathrm{b}}$ such that

$$\|h_m(a_1,\dots,a_{r_m},y_1,\dots,y_{s_m})\|_{2,\tau}\le\varepsilon\quad\text{for } m=1,\dots,\ell \text{ and } \tau\in T_\omega(A).$$

Choose $\varepsilon>0$ and $\ell\in\mathbb{N}$ arbitrarily. Denote by $F\Subset G$ the set of $g\in G$ appearing in h_m for some $m=1,\dots,\ell$. Set

$$\delta=\tfrac{1}{18}\varepsilon^2. \tag{4-3}$$

Choose $t\in(0,1)$ such that

$$1-t<\tfrac{1}{2}\varepsilon^2. \tag{4-4}$$

Let $\eta>0$ be the universal constant from Lemma 3.2 corresponding to the tuple (δ^2,t). Since G is amenable, we can find $H\Subset G$ such that $|gH\Delta H|<\eta|H|$ for each $g\in F$. By assumption, for each $\tau\in\overline{T_\omega(A)}^{w^*}$, we can find contractions $(y_i^\tau)_{i\in\mathbb{N}}\in A^{\omega,\mathrm{b}}$ such that

$$\|h_m(a_1,\dots,a_{r_m},y_1^\tau,\dots,y_{s_m}^\tau)\|_{2,\tau}^2<\frac{\varepsilon^2}{2\ell|H|}\quad\text{for } m=1,\dots,\ell.$$

Define

$$b^\tau:=\sum_{m=1}^{\ell}|h_m(a_1,\dots,a_{r_m},y_1^\tau,\dots,y_{s_m}^\tau)|^2\in A^{\omega,\mathrm{b}}. \tag{4-5}$$

Then we get

$$\tau(b^\tau)=\sum_{m=1}^{\ell}\|h_m(a_1,\dots,a_{r_m},y_1^\tau,\dots,y_{s_m}^\tau)\|_{2,\tau}^2<\frac{\varepsilon^2}{2|H|}.$$

By continuity and compactness of $\overline{T_\omega(A)}^{w^*}$, we can find finitely many tracial states $\tau_1,\dots,\tau_k\in\overline{T_\omega(A)}^{w^*}$ such that

$$\frac{\varepsilon^2}{2|H|}>\sup_{\tau\in T_\omega(A)}\min_{i=1,\dots,k}\tau(b^{\tau_i}).$$

By Lemma 3.2, it follows that we can find pairwise orthogonal projections

$$p_1,\dots,p_k\in A^{\omega,\mathrm{b}}\cap\Big(\bigcup_{g\in G}\alpha_g^\omega(\{y_i^{\tau_1},\dots,y_i^{\tau_n},a_i\mid i\in\mathbb{N}\})\Big)'$$

such that, for $\tau\in T_\omega(A)$,

$$\tau\Big(\sum_{j=1}^{k}p_j\Big)>t, \tag{4-6}$$

$$\tau(b^{\tau_j}p_j)\le\tfrac{1}{2}\varepsilon^2\tau(p_j)\quad\text{for } j=1,\dots,k, \tag{4-7}$$

$$\max_{g\in F}\sum_{j=1}^{k}\|\alpha_g(p_j)-p_j\|_{2,\tau}^2<\delta^2. \tag{4-8}$$

Define $y_i = \sum_{j=1}^k p_j y_i^{\tau_j}$ for $i \in \mathbb{N}$. We show that, for $\tau \in T_\omega(A)$ and $m = 1, \dots, \ell$,

$$\left|\tau(|h_m(a_1, \dots, a_{r_m}, y_1, \dots, y_{s_m})|^2) - \tau\left(\sum_{j=1}^k p_j |h_m(a_1, \dots, a_{r_m}, y_1^{\tau_j}, \dots, y_{s_m}^{\tau_j})|^2\right)\right| < \tfrac{1}{2}\varepsilon^2. \tag{4-9}$$

We distinguish three cases. First, assume that $h_m(a_1, \dots, a_{r_m}, 0, \dots, 0) = 0$ and that

$$h_m(1_{A^{\omega,\mathrm{b}}}, \dots, 1_{A^{\omega,\mathrm{b}}}, z_1, \dots, z_{s_m})$$

is an ordinary $*$-polynomial. In this case

$$|h_m(a_1, \dots, a_{r_m}, y_1, \dots, y_{s_m})|^2 = \sum_{j=1}^k p_j |h_m(a_1, \dots, a_{r_m}, y_1^{\tau_j}, \dots, y_{s_m}^{\tau_j})|^2$$

since $p_1, \dots, p_k$ are pairwise orthogonal projections. Second, assume h_m is of the form $z_i - a$ for some $i \in \mathbb{N}$ and some element $a \in \mathrm{C}^*(\{\alpha_g^\omega(a_i) \mid i \in \mathbb{N}, g \in G\})$ with $\|a\| = 1$. In this case we have

$$\begin{aligned}
&\left|\tau(|h_m(a_1, \dots, a_{r_m}, y_1, \dots, y_{s_m})|^2) - \tau\left(\sum_{j=1}^k p_j |h_m(a_1, \dots, a_{r_m}, y_1^{\tau_j}, \dots, y_{s_m}^{\tau_j})|^2\right)\right| \\
&\qquad = \left|\tau\left(\left|\sum_{j=1}^k p_j y_i^{\tau_j} - a\right|^2\right) - \tau\left(\sum_{j=1}^k p_j |y_i^{\tau_j} - a|^2\right)\right| = \left|\tau(|a|^2) - \tau\left(\sum_{j=1}^k p_j |a|^2\right)\right| \\
&\qquad \le \tau\left(1 - \sum_{j=1}^k p_j\right) \overset{(4\text{-}6)}{<} 1 - t \overset{(4\text{-}4)}{<} \tfrac{1}{2}\varepsilon^2.
\end{aligned}$$

Third, assume h_m is of the form $g \cdot z_i - z_{i'}$ for some $i, i' \in \mathbb{N}$ and $g \in F$. Then we have

$$\begin{aligned}
&\left|\tau(|h_m(a_1, \dots, a_{r_m}, y_1, \dots, y_{s_m})|^2) - \tau\left(\sum_{j=1}^k p_j |h_m(a_1, \dots, a_{r_m}, y_1^{\tau_j}, \dots, y_{s_m}^{\tau_j})|^2\right)\right| \\
&\quad = \left|\tau\left(\left|\alpha_g^\omega\left(\sum_{j=1}^k p_j y_i^{\tau_j}\right) - \sum_{j'=1}^k p_{j'} y_{i'}^{\tau_{j'}}\right|^2\right) - \tau\left(\sum_{j=1}^k p_j |\alpha_g^\omega(y_i^{\tau_j}) - y_{i'}^{\tau_j}|^2\right)\right| \\
&\quad = \left|\tau\left(\sum_{j=1}^k (\alpha_g^\omega(p_j) - p_j)\alpha_g^\omega(|y_i^{\tau_j}|^2)\right) - \tau\left(\sum_{j=1}^k (\alpha_g^\omega(p_j) - p_j)\alpha_g^\omega(y_i^{\tau_j})^* \sum_{j'=1}^k p_{j'} y_{i'}^{\tau_{j'}}\right)\right. \\
&\qquad\qquad \left. - \tau\left(\left(\sum_{j'=1}^k p_{j'} y_{i'}^{\tau_{j'}}\right)^* \sum_{j=1}^k (\alpha_g^\omega(p_j) - p_j)\alpha_g^\omega(y_i^{\tau_j})\right)\right| \\
&\quad \le \left\|\sum_{j=1}^k (\alpha_g^\omega(p_j) - p_j)\alpha_g^\omega(|y_i^{\tau_j}|^2)\right\|_{2,\tau} + 2\left\|\sum_{j=1}^k (\alpha_g^\omega(p_j) - p_j)\alpha_g^\omega(y_i^{\tau_j})\right\|_{2,\tau} \left\|\sum_{j=1}^k p_j y_{i'}^{\tau_j}\right\|_{2,\tau} \\
&\quad \le \left\|\sum_{j=1}^k (\alpha_g^\omega(p_j) - p_j)\alpha_g^\omega(|y_i^{\tau_j}|^2)\right\|_{2,\tau} + 2\left\|\sum_{j=1}^k (\alpha_g^\omega(p_j) - p_j)\alpha_g^\omega(y_i^{\tau_j})\right\|_{2,\tau}.
\end{aligned} \tag{4-10}$$

Note that, for any $g\in G$ and any positive contractions $c_1,\dots,c_k\in A^{\omega,\mathrm{b}}$ commuting with the p_j and $\alpha_g^\omega(p_j)$, one has that

$$\begin{aligned}\Big\|\sum_{i=1}^k(\alpha_g^\omega(p_i)-p_i)c_i\Big\|_{2,\tau}^2&=\sum_{i=1}^k\tau((\alpha_g^\omega(p_i)-p_i)^2c_i^2)+\sum_{\substack{i,j=1\\ i\neq j}}^k\tau(c_i(\alpha_g^\omega(p_i)-p_i)(\alpha_g^\omega(p_j)-p_j)c_j)\\&\le\sum_{i=1}^k\Big(\tau((\alpha_g^\omega(p_i)-p_i)^2)-\sum_{\substack{j=1,\dots,k\\ j\neq i}}\tau(c_i\alpha_g^\omega(p_i)p_jc_j)-\sum_{\substack{j=1,\dots,k\\ j\neq i}}\tau(c_ip_i\alpha_g^\omega(p_j)c_j)\Big)\\&\le\sum_{i=1}^k\|\alpha_g^\omega(p_i)-p_i\|_{2,\tau}^2,\end{aligned}$$

where in the last inequality we used the tracial property and the fact that the c_i commute with the p_j and $\alpha_g^\omega(p_j)$ to show that the last two terms can be rewritten as the negative of the trace of positive elements.

In particular we have, for $g\in F$, that

$$\Big\|\sum_{j=1}^k(\alpha_g^\omega(p_j)-p_j)\alpha_g^\omega(|y_i^{\tau_j}|^2)\Big\|_{2,\tau}\le\sqrt{\sum_{i=1}^k\|\alpha_g^\omega(p_i)-p_i\|_{2,\tau}^2}\overset{\text{(4-8)}}{<}\delta\overset{\text{(4-3)}}{=}\tfrac{1}{18}\varepsilon^2. \tag{4-11}$$

For $j=1,\dots,k$, we have that $\alpha_g^\omega(y_i^{\tau_j})$ is a contraction that can be written as a linear combination of positive contractions commuting with the p_i and $\alpha_g^\omega(p_i)$. An application of the triangle inequality yields

$$\Big\|\sum_{i=1}^k(\alpha_g^\omega(p_i)-p_i)\alpha_g^\omega(y_i^{\tau_j})\Big\|_{2,\tau}\le4\sqrt{\sum_{i=1}^k\|\alpha_g^\omega(p_i)-p_i\|_{2,\tau}^2}\overset{\text{(4-8)}}{<}4\delta\overset{\text{(4-3)}}{=}\tfrac{2}{9}\varepsilon^2. \tag{4-12}$$

Combining (4-10) with (4-11) and (4-12), we get

$$\Big|\tau(|h_m(a_1,\dots,a_{r_m},y_1,\dots,y_{s_m})|^2)-\tau\Big(\sum_{j=1}^kp_j|h_m(a_1,\dots,a_{r_m},y_1^{\tau_j},\dots,y_{s_m}^{\tau_j})|^2\Big)\Big|<\tfrac{9}{18}\varepsilon^2=\tfrac12\varepsilon^2.$$

Thus, we have indeed shown that (4-9) holds for all $\tau\in T_\omega(A)$ and $m=1,\dots,\ell$. From (4-5) we see that

$$\sum_{j=1}^kp_j|h_m(a_1,\dots,a_{r_m},y_1^{\tau_j},\dots,y_{s_m}^{\tau_j})|^2\le\sum_{j=1}^kp_jb^{\tau_j}\quad\text{for } m=1,\dots,\ell. \tag{4-13}$$

As a consequence, for $\tau\in T_\omega(A)$ and $m=1,\dots,\ell$, we get

$$\begin{aligned}\|h_m(a_1,\dots,a_{r_m},y_1,\dots,y_{s_m})\|_{2,\tau}^2&=\tau(|h_m(a_1,\dots,a_{r_m},y_1,\dots,y_{s_m})|^2)\\&\overset{\text{(4-9)}}{<}\tau\Big(\sum_{j=1}^kp_j|h_m(a_1,\dots,a_{r_m},y_1^{\tau_j},\dots,y_{s_m}^{\tau_j})|^2\Big)+\tfrac12\varepsilon^2\\&\overset{\text{(4-13)}}{\le}\sum_{j=1}^k\tau(p_jb^{\tau_j})+\tfrac12\varepsilon^2\overset{\text{(4-7)}}{\le}\sum_{j=1}^k\tfrac12\varepsilon^2\tau(p_j)+\tfrac12\varepsilon^2\le\tfrac12\varepsilon^2+\tfrac12\varepsilon^2=\varepsilon^2.\qquad\square\end{aligned}$$

The next theorem gives an alternative formulation of the tracial local-to-global principle that is convenient to use in certain applications. Before we state it, we introduce some notation.

Notation 4.4. Let A be a C*-algebra with an action $\alpha : G \curvearrowright A$ of a countable discrete group. Given a tracial state $\tau \in T(A)$, denote by $\pi_\tau : A \to \mathcal{B}(H_\tau)$ the corresponding GNS representation. Then we define the representation

$$\pi_\tau^\alpha : A \to \mathcal{B}(\ell^2(G, H_\tau)), \quad \pi_\tau^\alpha(x)(\xi)(h) = \pi_\tau(\alpha_h^{-1}(x))\xi(h).$$

The left-regular representation $\lambda : G \to \mathcal{U}(\ell^2(G, H_\tau))$, defined by $(g \cdot \xi)(h) = \xi(g^{-1}h)$ for $\xi \in \ell^2(G, H_\tau)$ and $g, h \in G$, implements the action α on $\pi_\tau^\alpha(A)$, so we get a continuous extension of the action $\alpha : G \curvearrowright \pi_\tau^\alpha(A)''$ on the weak closure.

Notice that $\pi_\tau^\alpha(A)'' \subseteq \prod_{g\in G} \pi_\tau(A)''$. The trace τ on A extends to a faithful normal trace on $\pi_\tau(A)''$ and, by composition with the natural quotient map $q_g : \prod_{g\in G} \pi_\tau(A)'' \to \pi_\tau(A)''$ onto the summand with index $g \in G$, also to a normal trace on $\prod_{g\in G} \pi_\tau(A)''$, which we will denote by $\tilde{\tau}_g$. Notice that $\tilde{\tau}_g \circ \pi_\tau^\alpha = \tau \circ \alpha_g^{-1}$. Let $(c_g)_{g\in G}$ be a sequence in $(0, 1)$ such that $\sum_{g\in G} c_g = 1$. Then $\tilde{\tau} := \sum_{g\in G} c_g \tilde{\tau}_g$ defines a faithful normal tracial state on $\prod_{g\in G} \pi_\tau(A)''$ and hence also on the subalgebra $\pi_\tau^\alpha(A)''$. In this way we can form the tracial von Neumann algebra ultrapower $(\pi_\tau^\alpha(A)'')^\omega$. Note that, on bounded subsets of $\prod_{g\in G} \pi_\tau(A)''$, the strong operator topology is induced by the norm $\|\cdot\|_{2,\tilde{\tau}}$, or equivalently by the seminorms $\{\|\cdot\|_{2,\tilde{\tau}_g} \mid g \in G\}$. Since $\pi_\tau^\alpha(A)''$ is a von Neumann subalgebra, it follows that on bounded subsets its strong operator topology is also induced by (the restrictions of) these (semi)norms.

Remark 4.5. With the above notation and terminology, the condition in Theorem 4.2 that requires, for every $\varepsilon > 0$, $\ell \in \mathbb{N}$, and $\tau \in \overline{T_\omega(A)}^{w^*}$, the existence of contractions $(y_i)_{i\in\mathbb{N}}$ in $A^{\omega,\mathrm{b}}$ such that

$$\|h_m(a_1, \dots, a_{r_m}, y_1, \dots, y_{s_m})\|_{2,\tau} < \varepsilon \quad \text{for } m = 1, \dots, \ell$$

is equivalent to the following statement (by Kaplansky's density theorem): for every $\varepsilon > 0$, $\ell \in \mathbb{N}$, and $\tau \in \overline{T_\omega(A)}^{w^*}$, there exist contractions $(y_i')_{i\in\mathbb{N}}$ in $\pi_\tau^{\alpha^\omega}(A^{\omega,\mathrm{b}})''$ such that

$$\|h_m(a_1, \dots, a_{r_m}, y_1', \dots, y_{s_m}')\|_{2,\tilde{\tau}} < \varepsilon \quad \text{for } m = 1, \dots, \ell.$$

Making use of the tracial von Neumann algebra ultrapowers, this is also equivalent to the following statement: for every $\tau \in \overline{T_\omega(A)}^{w^*}$, there are contractions $(y_i'')_{i\in\mathbb{N}}$ in $(\pi_\tau^{\alpha^\omega}(A^{\omega,\mathrm{b}})'')^\kappa$ such that $h_m(a_1, \dots, a_{r_m}, y_1'', \dots, y_s'') = 0$ for every $m \in \mathbb{N}$.

Theorem 4.6. *Let A be a σ-unital* C*-*algebra with $T(A)$ nonempty and compact. Let $\alpha : G \curvearrowright A$ be an action by an amenable countable discrete group G and assume it has local equivariant property Gamma with respect to bounded traces. Let ω and κ be two free ultrafilters on $\mathbb{N}$. Let $\delta : G \curvearrowright D$ be an action on a separable* C*-*algebra and let $B \subset D$ be a separable, δ-invariant* C*-*subalgebra. Suppose $\varphi : (B, \delta) \to (A^{\omega,\mathrm{b}}, \alpha^\omega)$ is an equivariant $*$-homomorphism. Then the following are equivalent*:

(1) *For every $\tau \in \overline{T_\omega(A)}^{w^*}$, there exists an equivariant $*$-homomorphism*

$$\varphi^\tau : (D, \delta) \to ((\pi_\tau^{\alpha^\omega}(A^{\omega,\mathrm{b}})'')^\kappa, (\alpha^\omega)^\kappa)$$

such that $\varphi^\tau|_B = \pi_\tau^{\alpha^\omega} \circ \varphi$.

(2) *There is an equivariant $*$-homomorphism $\bar{\varphi} : (D, \delta) \to (A^{\omega,\mathrm{b}}, \alpha^\omega)$ with $\bar{\varphi}|_B = \varphi$.*

Proof. It is clear that (2) implies (1). To prove the other implication, take a countable dense $\mathbb{Q}[i]$-$*$-subalgebra $C \subset D$ such that it is δ-invariant and such that $C \cap B$ is also dense in B. By inductively enlarging C we may in addition assume that, for each contraction $x \in C$, one has $1 - \sqrt{1 - x^*x} \in C$. Let $\mathcal{P}$ denote the countable family of G-$*$-polynomials with coefficients in $A^{\omega,\mathrm{b}}$ in the variables $\{X_c\}_{c\in C}$ encoding all relations in C:

- $g \cdot X_c - X_{\delta_g(c)}$ for all $c \in C$ and $g \in G$,
- $\lambda X_c + X_{c'} - X_{\lambda c + c'}$ for all $c, c' \in C$ and $\lambda \in \mathbb{Q}[i]$,
- $X_c X_{c'} - X_{cc'}$ for $c, c' \in C$,
- $X_c^* - X_{c^*}$ for $c \in C$,
- $\varphi(b) - X_b$ for $b \in B \cap C$.

It follows from (1) that, for every $\tau \in \overline{T_\omega(A)}^{w^*}$, the equations in $\mathcal{P}$ have exact solutions in $(\pi_\tau^{\alpha^\omega}(A^{\omega,\mathrm{b}})'')^\kappa$. By Remark 4.5 this means precisely that all conditions to apply Theorem 4.2 are fulfilled, and we can find exact solutions to all equations in $\mathcal{P}$ in $A^{\omega,\mathrm{b}}$. This is equivalent to the existence of a $\mathbb{Q}[i]$-linear, $*$-preserving, multiplicative, equivariant map $\bar{\varphi} : C \to A^{\omega,\mathrm{b}}$ with $\bar{\varphi}|_{B\cap C} = \varphi|_{B\cap C}$. We observe that $\bar{\varphi}$ is contractive. Indeed, if $x \in C$ is a contraction, then $y = 1 - \sqrt{1 - x^*x}$ is a self-adjoint element also belonging to C, which satisfies

$$x^*x + y^2 - 2y = x^*x + (y-1)^2 - 1 = 0.$$

Hence

$$\bar{\varphi}(x)^*\bar{\varphi}(x) + \bar{\varphi}(y)^2 - 2\bar{\varphi}(y) = 0,$$

or equivalently,

$$\bar{\varphi}(x)^*\bar{\varphi}(x) + (1 - \bar{\varphi}(y))^2 = 1.$$

We see that $\bar{\varphi}(x)^*\bar{\varphi}(x)$ is a contraction, and hence $\bar{\varphi}(x)$ is as well. In conclusion, φ extends to an equivariant $*$-homomorphism $\bar{\varphi} : (D, \delta) \to (A^{\omega,\mathrm{b}}, \alpha^\omega)$ with $\bar{\varphi}_B = \varphi$. □

In many cases of interest we get the following corollary from Proposition 2.4, which directly generalizes and recovers the technical machinery related to uniform property Gamma from the nondynamical setting; see [Castillejos et al. 2021b, Lemma 4.1]. We note that, upon close inspection of our proof so far, this particular corollary can be obtained based on [Castillejos et al. 2021b, Lemma 3.6] without relying on the preprint [Carrión et al. 2023a].

Corollary 4.7. *Let A be a separable, simple, nuclear* C**-algebra with $T(A)$ nonempty and compact, and such that $T^+(A) = \mathbb{R}^{>0}T(A)$. Let $\alpha : G \curvearrowright A$ be an action by a countable amenable discrete group that has equivariant property Gamma. Then α satisfies the conclusion of Theorems 4.2 and 4.6.*

Remark 4.8. For potential subsequent applications of the theory in this article, let us reflect on how we ended up with the main result of this section. It is worthwhile to note that the amenability of the group G is used (in the proof of Theorem 4.2) through the Følner condition exclusively for the purpose of having access to a finite set $H \Subset G$ that satisfies the conclusion of Lemma 3.4. At no other point in the whole chain of argument is it necessary to know that H is actually a set that is almost invariant with respect to F, or anything else about H for that matter. This culminates in the following more explicit observation, which we suspect may be, at some point, interesting to consider for certain actions of nonamenable groups.

Let A be a σ-unital C*-algebra with $T(A)$ nonempty and compact. Let $\alpha : G \curvearrowright A$ be an action of a countable discrete group. Suppose that, for all $\varepsilon > 0$ and $F \Subset G$, there exists a finite subset $H \Subset G$ that satisfies the same conclusion as in Lemma 3.4. If α has local equivariant property Gamma with respect to bounded traces, then α also satisfies the conclusion of Theorems 4.2 and 4.6.

5. Equivariant Jiang–Su stability

In this section we use the dynamical tracial local-to-global principle derived in the previous section combined with von Neumann algebraic results to conclude that, for actions of countable amenable groups on separable, simple, nuclear, finite, $\mathcal{Z}$-stable C*-algebras, equivariant property Gamma implies equivariant $\mathcal{Z}$-stability. Although one can get by with known variations of Ocneanu's theorem [1985], for many applications treated in this section, our most general results here need a more general McDuff-type theorem for actions of amenable groups on von Neumann algebras, which we import from our recent work [Szabó and Wouters 2024].

We begin by reducing the problem of equivariant $\mathcal{Z}$-stability to the existence of so-called tracially large c.p.c. order-zero maps $M_n \to F_\omega(A)^{\tilde{\alpha}_\omega}$ for $n \geq 2$. The argument is well known to experts and traces back to the work of Matui and Sato [2012]. It makes use of an equivariant version of their property (SI), for which the general framework needed here was developed in [Szabó 2021b].

Definition 5.1 [Szabó 2021b, Definition 2.5; Castillejos et al. 2023, Definition 1.3]. Let A be a separable, simple C*-algebra with $T^+(A) \neq \varnothing$.

(1) We say that a positive contraction $f \in F_\omega(A)$ is *tracially supported at* 1 if the following holds: for every nonzero positive element $a \in \mathcal{P}(A)$, there exists a constant $\kappa = \kappa(f, a) > 0$ such that, for every $\tau \in \widetilde{T}_\omega(A)$ with $0 < \tau(a) < \infty$, one has $\inf_{k\in\mathbb{N}} \tau_a(f^k) \geq \kappa\tau(a)$.

(2) A positive element $e \in F_\omega(A)$ is called *tracially null* if $e \in \mathcal{J}_A$ in the sense of Definition 1.11.

(3) Let B be a unital C*-algebra. A c.p.c. order-zero map $\phi : B \to F_\omega(A)$ is called *tracially large* if $\tau_a \circ \phi(1) = \tau(a)$ for all nonzero positive elements $a \in \mathcal{P}(A)$ and $\tau \in \widetilde{T}_\omega(A)$ with $\tau(a) < \infty$.

Remark 5.2. It follows from [Szabó 2021b, Proposition 2.4] that any of the conditions above hold for all nonzero positive elements $a \in \mathcal{P}(A)$ if and only if they hold for just a single such element, so in practice it suffices to check them for a single $a \in \mathcal{P}(A)_+ \setminus \{0\}$.

Definition 5.3 [Szabó 2021b, Definition 2.7]. Let A be a separable, simple C*-algebra with $T^+(A) \neq \varnothing$ and an action $\alpha : G \curvearrowright A$ of a countable discrete group. We say that α has *equivariant property* (*SI*) if the following holds:

Whenever $e, f \in F_\omega(A)^{\tilde{\alpha}_\omega}$ are two positive contractions such that f is tracially supported at 1 and e is tracially null, there exists a contraction $s \in F_\omega(A)^{\tilde{\alpha}_\omega}$ such that $fs = s$ and $s^*s = e$.

It follows from [Szabó 2021b, Corollary 4.3] that all actions of amenable groups on nonelementary, separable, simple, nuclear C*-algebras with strict comparison have property (SI). Combined with the following theorem, it gives a powerful sufficient criterion for equivariant Jiang–Su stability. This is not new to the experts but has never been formally stated in this generality before, so we shall give the proof for the reader's convenience.

Theorem 5.4. *Let A be a separable, simple* C**-algebra with $T^+(A) \neq \varnothing$, and let $\alpha : G \curvearrowright A$ be an action of a countable discrete group with equivariant property* (*SI*). *Then α is equivariantly $\mathcal{Z}$-stable if and only if, for every $n \in \mathbb{N}$, there exists a unital $*$-homomorphism $M_n \to (A^\omega \cap A')^{\alpha^\omega}$.*

Proof. Since the "only if" part can be obtained with the standard argument sketched in Remark 2.2, we prove the "if" part. Given $n \in \mathbb{N}$, let $\phi' : M_n \to (A^\omega \cap A')^{\alpha^\omega}$ be a unital $*$-homomorphism. By Proposition 1.18, we can find a tracially large c.p.c. order-zero map $\phi : M_n \to F_\omega(A)^{\tilde{\alpha}_\omega}$ that lifts ϕ'. Set $e := 1_{F_\omega(A)} - \phi(1)$, and set $f := \phi(e_{1,1})$. Both are positive contractions in $F_\omega(A)^{\tilde{\alpha}_\omega}$. Since ϕ is tracially large, we can conclude immediately that e is tracially null. Since $e_{1,1}$ is a projection, it follows that $\phi(e_{1,1}) - \phi(e_{1,1})^m$ is tracially null for any $m \geq 1$. Moreover, for every $\tau \in \widetilde{T}_\omega(A)$ and $a \in \mathcal{P}(A)_+ \setminus \{0\}$ such that $\tau(a) < \infty$, the functional $\tau_a \circ \phi$ is a bounded trace and therefore a multiple of the unique tracial state on M_n. So, for every $k \in \mathbb{N}$, we have

$$\tau_a(f^k) = \tau_a(\phi(e_{1,1})^k) = \tau_a(\phi(e_{1,1})) = \frac{1}{n}\tau(a).$$

This proves that f is tracially supported at 1.

Since α has equivariant property (SI), we can find a contraction $s \in F_\omega(A)^{\tilde{\alpha}_\omega}$ such that $fs = s$ and $s^*s = e$. By [Rørdam and Winter 2010, Theorem 5.1], this implies the existence of a unital $*$-homomorphism from the dimension drop algebra $Z_{n,n+1}$ into $F_\omega(A)^{\tilde{\alpha}_\omega}$. As $\mathcal{Z}$ is an inductive limit of those algebras, we find a unital $*$-homomorphism $\mathcal{Z} \to F_\omega(A)^{\tilde{\alpha}_\omega}$.[15] This implies equivariant $\mathcal{Z}$-stability by [Szabó 2018b, Corollary 3.8]. □

We shall now prove that, for actions of amenable groups on simple nuclear $\mathcal{Z}$-stable C*-algebras, equivariant uniform property Gamma is equivalent to equivariant $\mathcal{Z}$-stability. We end up giving two separate arguments to prove this result in two cases. Firstly, we prove this result for actions on C*-algebras that have a compact nonempty tracial state space and no unbounded traces, for which it is sufficient to appeal to Corollary 4.7. Secondly, we prove the result in full generality, but this requires the full power of our theory based on the results from [Carrión et al. 2023a].

Let us proceed in the first case.

Theorem 5.5. *Let A be a separable, nuclear, simple $\mathcal{Z}$-stable* C**-algebra with $T(A)$ nonempty and compact, and such that $T^+(A) = \mathbb{R}^{>0}T(A)$. Let $\alpha : G \curvearrowright A$ be an action of a countable discrete amenable group. If α has equivariant property Gamma, then α is equivariantly $\mathcal{Z}$-stable.*

Proof. By Theorem 5.4, it suffices to construct a unital $*$-homomorphism $M_n \to (A^\omega \cap A')^{\alpha^\omega}$ for $n \geq 2$. We appeal to Corollary 4.7 and hence know that α obeys the conclusion of Theorem 4.2.

[15] This is a standard reindexation trick. Alternatively one can deduce this for example by a combination of Corollary 3.9 and Lemma 4.2 in [Barlak and Szabó 2016].

Let $(a_k)_{k\in\mathbb{N}}$ be a dense sequence in A. Then the existence of such a desired $*$-homomorphism is equivalent to the existence of elements $e_{1,1}, e_{2,1}, \dots, e_{n,1} \in A^\omega$ satisfying the equations

$$e_{i,1}e_{j,1} = 0, \quad e_{i,1}^* e_{j,1} = \delta_{ij} e_{1,1}, \quad e_{1,1}^2 = e_{1,1}, \quad \alpha_g^\omega(e_{i,1}) = e_{i,1}, \quad \text{and} \quad a_k e_{i,1} - e_{i,1} a_k = 0$$

for all $g \in G$, $i, j = 2, \dots, n$, and $k \in \mathbb{N}$. By Theorem 4.2, it suffices to show that, for each $\varepsilon > 0$, finite subset $F \Subset G$, $m \in \mathbb{N}$, and every tracial state $\tau \in T(A)$, there exists contractions $f_{1,1}, f_{2,1}, \dots, f_{n,1} \in A$ satisfying

$$\|f_{i,1}f_{j,1}\|_{2,\tau} < \varepsilon, \quad \|f_{i,1}^* f_{j,1} - \delta_{ij} f_{1,1}\|_{2,\tau} < \varepsilon, \quad \|f_{1,1}^2 - f_{1,1}\|_{2,\tau} < \varepsilon,$$
$$\|\alpha_g(f_{i,1}) - f_{i,1}\|_{2,\tau} < \varepsilon, \quad \text{and} \quad \|a_k f_{i,1} - f_{i,1} a_k\|_{2,\tau} < \varepsilon$$

for all $g \in F$, $i, j = 2, \dots, n$, and $1 \le k \le m$. This is the case, however, if and only if, for every $\tau \in T(A)$, there exists a unital equivariant $*$-homomorphism $M_n \to ((\pi_\tau^\alpha(A)'')^\omega \cap A')^{\alpha^\omega}$. Since A is nuclear, the tracial von Neumann algebra $N_\tau := \pi_\tau^\alpha(A)''$ is injective; see for example [Blackadar 2006, Theorem IV.2.2.13]. Since it does not have any direct summand of type I, it follows from Connes' theorem [1976] that $N_\tau \mathbin{\overline{\otimes}} \mathcal{R} \cong N_\tau$. Hence the claim follows directly from [Szabó and Wouters 2024, Theorem A]. $\square$

Next we proceed in the second and more general case.

Lemma 5.6. *Let A be a σ-unital* C**-algebra with $T(A)$ nonempty and compact. Let $\alpha : G \curvearrowright A$ be an action by a countable discrete amenable group G and assume it has local equivariant property Gamma with respect to bounded traces. Then the following are equivalent*:

(1) *For all $\tau \in T(A)$, there exists a unital $*$-homomorphisms $\varphi^\tau : M_n \to ((\pi_\tau^\alpha(A)'')^\omega)^{\alpha^\omega}$.*

(2) *There exists a unital $*$-homomorphism $\varphi : M_n \to (A^{\omega,\mathrm{b}})^{\alpha^\omega}$.*

Proof. For each $\tau \in T(A)$, the map π_τ^α induces a unital $*$-homomorphism

$$(A^{\omega,\mathrm{b}})^{\alpha^\omega} \to ((\pi_\tau^\alpha(A)'')^\omega)^{\alpha^\omega},$$

so (2) implies (1). We use the fact that α has local equivariant property Gamma to prove the other implication. By Theorem 4.6 the following are equivalent:

(a) For all $\tau \in \overline{T_\omega(A)}^{w^*}$, there exists a unital equivariant $*$-homomorphism

$$\varphi^\tau : (M_n, \mathrm{id}_{M_n}) \to ((\pi_\tau^{\alpha^\omega}(A^{\omega,\mathrm{b}})'')^\kappa, (\alpha^\omega)^\kappa).$$

(b) There exists a unital equivariant $*$-homomorphism $\varphi : (M_n, \mathrm{id}_{M_n}) \to (A^{\omega,\mathrm{b}}, \alpha^\omega)$.

Statement (b) is equivalent to statement (2) above. Hence, in order to prove the implication it suffices to prove that statement (1) implies (a). Take $\tau \in \overline{T_\omega(A)}^{w^*}$ and denote its restriction to A by σ. The canonical map $A \to A^{\omega,\mathrm{b}}$ induces an equivariant $*$-homomorphism $(\pi_\sigma^\alpha(A), \alpha) \to (\pi_\tau^{\alpha^\omega}(A^{\omega,\mathrm{b}})'', \alpha^\omega)$ that is continuous on the unit ball with respect to the strong operator topology. Hence, it can be extended to a unital equivariant $*$-homomorphism $\pi_\sigma^\alpha(A)'' \to \pi_\tau^{\alpha^\omega}(A^{\omega,\mathrm{b}})''$. Combining this with (1), this means we can find a unital $*$-homomorphism $M_n \to ((\pi_\tau^{\alpha^\omega}(A^{\omega,\mathrm{b}})'')^\kappa)^{(\alpha^\omega)^\kappa}$. This ends the proof. $\square$

Theorem 5.7. *Let A be a separable, simple, nuclear $\mathcal{Z}$-stable* C*-*algebra such that $T^+(A) \neq \varnothing$. Let $\alpha : G \curvearrowright A$ be an action of a countable discrete amenable group. If α has equivariant property Gamma, then α is equivariantly $\mathcal{Z}$-stable.*

Proof. Combining [Szabó 2021b, Corollary 4.3] with Theorem 5.4, we see that, given $n \geq 2$, it suffices to construct a unital $*$-homomorphism $M_n \to (A^\omega \cap A')^{\alpha^\omega}$. For convenience, let us specify a (possibly different) free ultrafilter κ on $\mathbb{N}$. It suffices to show that we can construct a unital $*$-homomorphism $M_n \to ((A^\omega \cap A')^{\kappa,\mathrm{b}})^{(\alpha^\omega)^\kappa}$, as a reindexation trick will then yield the required unital $*$-homomorphism $M_n \to (A^\omega \cap A')^{\alpha^\omega}$. By Theorem 2.6, we conclude that $\alpha^\omega = G \curvearrowright A^\omega \cap A'$ has local equivariant property Gamma with respect to bounded traces. Thus, by Lemma 5.6, it suffices to prove that, for all $\tau \in T(A^\omega \cap A')$, there exists a unital $*$-homomorphism $\varphi^\tau : M_n \to ((\pi_\tau^{\alpha^\omega}(A^\omega \cap A')'')^\kappa)^{(\alpha^\omega)^\kappa}$. We note that the C*-algebra A has uniform property Gamma. Using the same trick as in the proof of Proposition 2.5, we conclude that $A^\omega \cap A' \cong B^\omega \cap B'$ for a hereditary subalgebra $B \subset A \otimes \mathbb{K}$ such that $T^+(B) = \mathbb{R}^{>0} T(B)$ and $T(B)$ is compact. Using [Castillejos et al. 2022, Theorem 4.6], we can hence conclude that there exists a unital $*$-homomorphism $M_2 \to A^\omega \cap A'$. By a standard reindexation trick, we can argue that such a $*$-homomorphism can be chosen to additionally commute with any specified separable subset of $A^\omega \cap A'$.

Fix $\tau \in T(A^\omega \cap A')$. We show first that $N_\tau := \pi_\tau^{\alpha^\omega}(A^\omega \cap A')''$ contains a $\|\cdot\|_{2,\tau}$-separable, α^ω-invariant von Neumann subalgebra that tensorially absorbs the hyperfinite II_1-factor. By the aforementioned property of A, we can find a unital embedding $\phi_1 : M_2 \to \pi_\tau^{\alpha^\omega}(A^\omega \cap A') \subseteq N_\tau$. Set

$$B_1 := C^*\bigg(\bigcup_{g \in G} \alpha_g^\omega(\phi_1(M_2))\bigg).$$

Using that B_1 is a separable subquotient of $A^\omega \cap A'$, we again use the aforementioned property of A and find a unital embedding

$$\phi_2 : M_2 \to \pi_\tau^{\alpha^\omega}(A_\omega \cap A') \cap B_1' \subseteq N_\tau \cap B_1'.$$

Set

$$B_2 := C^*\bigg(B_1 \cup \bigcup_{g \in G} \alpha_g^\omega(\phi_2(M_2))\bigg).$$

Carry on with this procedure inductively, i.e., given the C*-algebra $B_i \subset \pi_\tau^{\alpha^\omega}(A^\omega \cap A')$, find a unital $*$-homomorphism $\phi_{i+1} : M_2 \to \pi_\tau^{\alpha^\omega}(A^\omega \cap A') \cap B_i'$ and set

$$B_{i+1} := C^*\bigg(B_i \cup \bigcup_{g \in G} \alpha_g^\omega(\phi_{i+1}(M_2))\bigg).$$

Define $\mathcal{B} := \overline{\bigcup_{i \in \mathbb{N}} B_i}^{\|\cdot\|_{2,\tau}} \subset N_\tau$. Then $\mathcal{B}$ is a $\|\cdot\|_{2,\tau}$-separable, α^ω-invariant von Neumann subalgebra of N_τ such that additionally $\mathcal{B} \cong \mathcal{B} \mathbin{\overline{\otimes}} \mathcal{R}$ by [Szabó and Wouters 2024, Corollary 3.8] because it satisfies the McDuff-type criterion (existence of a unital $*$-homomorphism $\mathcal{R} \to \mathcal{B}_\omega$) by construction. Denote the restriction of α^ω to $\mathcal{B}$ by β. By [Szabó and Wouters 2024, Theorem A], it follows that β is cocycle conjugate to $\beta \otimes \mathrm{id}_{\mathcal{R}}$. In particular, we can find a unital $*$-homomorphism

$$M_n \to (\mathcal{B}^\kappa)^{\beta^\kappa} \subset ((\pi_\tau^{\alpha^\omega}(A^\omega \cap A')'')^\kappa)^{(\alpha^\omega)^\kappa}. \qquad \square$$

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MULTIJET BUNDLES AND APPLICATION TO THE FINITENESS OF MOMENTS FOR ZEROS OF GAUSSIAN FIELDS

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We define a notion of multijet for functions on $\mathbb{R}^n$, which extends the classical notion of jets in the sense that the multijet of a function is defined by contact conditions at several points. For all $p \geqslant 1$ we build a vector bundle of p-multijets, defined over a well-chosen compactification of the configuration space of p distinct points in $\mathbb{R}^n$. As an application, we prove that the linear statistics associated with the zero set of a centered Gaussian field on a Riemannian manifold have a finite p-th moment as soon as the field is of class $\mathcal{C}^p$ and its $(p-1)$-jet is nowhere degenerate. We prove a similar result for the linear statistics associated with the critical points of a Gaussian field and those associated with the vanishing locus of a holomorphic Gaussian field.

1. Introduction

This paper is concerned with two different but related problems. The first one is to define a natural notion of multijet for a $\mathcal{C}^k$ function on $\mathbb{R}^n$, generalizing the usual notion of k-jet. By multijet we mean that we want to consider a collection of jets at different points in $\mathbb{R}^n$ and patch them together in a relevant way. The second one is to find natural conditions on a Gaussian field $f : \mathbb{R}^n \to \mathbb{R}^r$ ensuring that the $(n-r)$-dimensional volume of $f^{-1}(0) \cap \mathbb{B}$ admits finite higher moments, where $\mathbb{B}$ stands for the unit ball in $\mathbb{R}^n$. One way to tackle this second problem is by considering the multijet of the random field f. In the following, we give more details about our contributions concerning the previous two problems, as well as some variations on these questions.

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1.1. ***Multijet bundles.*** Let us start by recalling some standard facts about jets. See [Saunders 1989] for background on this matter. Let $x \in \mathbb{R}^n$ and $k \geqslant 0$. Two smooth functions f and g on $\mathbb{R}^n$ are said to have the same k-jet at x if $f - g$ vanishes at x, as well as all its partial derivatives up to order k. Having the same k-jet at x is an equivalence relation on $\mathcal{C}^\infty(\mathbb{R}^n)$, and the space $\mathcal{J}_k(\mathbb{R}^n)_x$ of k-jets at x is the set of equivalence classes for this relation. We denote by $\mathrm{j}_k(f, x)$ the k-jet of f at x, that is, its class in $\mathcal{J}_k(\mathbb{R}^n)_x$. The map $\mathrm{j}_k(\,\cdot\,, x)$ is a linear surjection from $\mathcal{C}^\infty(\mathbb{R}^n)$ onto the finite-dimensional vector space $\mathcal{J}_k(\mathbb{R}^n)_x$. Of course, $\mathrm{j}_k(f, x)$ makes sense even if f is only $\mathcal{C}^k$ and defined on some neighborhood of x.

Considering the family of k-jet spaces for all $x \in \mathbb{R}^n$, the set $\mathcal{J}_k(\mathbb{R}^n) = \bigsqcup_{x \in \mathbb{R}^n} \mathcal{J}_k(\mathbb{R}^n)_x$ is equipped with a natural vector bundle structure over $\mathbb{R}^n$. Then, if $\Omega \subset \mathbb{R}^n$ is open and $f : \Omega \to \mathbb{R}$ is $\mathcal{C}^k$, the map $\mathrm{j}_k(f, \cdot)$ is a local section of $\mathcal{J}_k(\mathbb{R}^n) \to \mathbb{R}^n$ over Ω. These definitions are well-behaved with respect to smooth changes of coordinates, so one can define similarly the vector bundle of k-jets of functions on a manifold M. More generally, if $E \to M$ is a vector bundle over M, there is a corresponding vector bundle $\mathcal{J}_k(M, E) \to M$ of k-jets of sections of $E \to M$.

In this paper, we are interested in defining similarly a notion of multijet and the associated vector bundles. That is, we want to consider smooth functions on $\mathbb{R}^n$ up to an equivalence relation defined by the vanishing of some derivatives at several points.

Let us make this more precise. Let $p \geqslant 1$ and $\Delta_p = \{(x_1, \dots, x_p) \in (\mathbb{R}^n)^p \mid \exists i \neq j \text{ such that } x_i = x_j\}$ denote the diagonal in $(\mathbb{R}^n)^p$. Given $\underline{x} = (x_1, \dots, x_p) \notin \Delta_p$, we say that f and g have the same multijet at $\underline{x}$ if $f(x_i) = g(x_i)$ for all $i \in [\![1, p]\!]$ (here we use the notation $[\![a, b]\!] = [a, b] \cap \mathbb{N}$). This is an equivalence relation on functions, defined by the vanishing of $f - g$ on the set $\{x_1, \dots, x_p\} \subset \mathbb{R}^n$, that is, by p independent linear conditions. Thus the corresponding set of classes is a vector space of dimension p, which we denote by $\mathcal{MJ}_p(\mathbb{R}^n)_{\underline{x}}$. We also denote by $\mathrm{mj}_p(f, \underline{x})$ the class of f in this space, that is, its multijet at $\underline{x}$.

As will be explained later, this defines a vector bundle $\mathcal{MJ}_p(\mathbb{R}^n)$ of rank p over $(\mathbb{R}^n)^p \setminus \Delta_p$. Moreover, for all $\underline{x} \notin \Delta_p$ the linear map $\mathrm{mj}_p(\,\cdot\,, \underline{x}) : \mathcal{C}^\infty(\mathbb{R}^n) \to \mathcal{MJ}_p(\mathbb{R}^n)_{\underline{x}}$ is surjective, and, for all smooth f, its multijet $\mathrm{mj}_p(f, \cdot)$ is smooth. We would like to extend this picture over the whole of $(\mathbb{R}^n)^p$. Note that the surjectivity conditions rule out defining $\mathcal{MJ}_p(\mathbb{R}^n)$ as $\mathcal{J}_0(\mathbb{R}^n)^p$ with $\mathrm{mj}_p(f, \underline{x}) = (\mathrm{j}_0(f, x_i))_{1 \leqslant i \leqslant p}$. When $\underline{x} \notin \Delta_p$, the previous notion of multijet is defined by p independent linear conditions: vanishing at each of the x_i. The main issue is that, when $\underline{x} \in \Delta_p$, these conditions are no longer independent and we need to replace them by another p-tuple of independent conditions.

A first natural idea is to look at vanishing with multiplicities. In dimension $n = 1$ this works very well. Let $\underline{x} \in \mathbb{R}^p$ be a permutation of $(y_1, \dots, y_1, \dots, y_m, \dots, y_m)$, where $(y_j)_{1 \leqslant j \leqslant m} \in \mathbb{R}^m \setminus \Delta_m$ and y_j appears exactly $k_j + 1$ times. We say that f and g have the same multijet at $\underline{x}$ if $(f - g)^{(k)}(y_j) = 0$ for all $j \in [\![1, m]\!]$ and $k \in [\![0, k_j]\!]$. In this sense, having the same multijet is equivalent to having the same Hermite interpolating polynomials at $\underline{x}$. Thus, we can define $\mathcal{MJ}_p(\mathbb{R})$ as the trivial bundle $\mathbb{R}_{p-1}[X] \times \mathbb{R}^p \to \mathbb{R}^p$ and $\mathrm{mj}_p(f, \underline{x})$ as the Hermite interpolating polynomial of f at $\underline{x}$.

If $n > 1$, the previous approach fails already for $p = 2$. Let us consider $x \in \mathbb{R}^n$ and the corresponding $\underline{x} = (x, x) \in \Delta_2$. Asking for the vanishing of $f - g$ and its differential at x gives us $n + 1$ independent conditions, which define the 1-jet space $\mathcal{J}_1(\mathbb{R}^n)_x$. This space has dimension $n + 1 > 2$; hence it is too large to be the $\mathcal{MJ}_2(\mathbb{R}^n)_{\underline{x}}$ we are looking for. The next natural idea is to ask only for the vanishing at x of $f-g$ and one of its directional derivatives. But whatever choice of directional derivative we make will lead to $\mathrm{mj}_2(f,\cdot)$ not being continuous at x for most $f \in \mathcal{C}^\infty(\mathbb{R}^n)$. Actually, one cannot extend $\mathcal{MJ}_2(\mathbb{R}^n)$ nicely over $(\mathbb{R}^n)^2$ if $n>1$. However, we can extend it nicely over a larger space: the blow-up $\mathrm{Bl}_{\Delta_2}((\mathbb{R}^n)^2)$ of $(\mathbb{R}^n)^2$ along Δ_2. The key idea is that $\mathrm{Bl}_{\Delta_2}((\mathbb{R}^n)^2)$ contains $(\mathbb{R}^n)^2 \setminus \Delta_2$ as a dense open subset and that points in the complement of $(\mathbb{R}^n)^2 \setminus \Delta_2$ can be described by the following data: a base point $x \in \mathbb{R}^n$ and a direction $u \in \mathbb{RP}^{n-1}$. This data tells us exactly which directional derivative to consider at the corresponding point in the exceptional locus of $\mathrm{Bl}_{\Delta_2}((\mathbb{R}^n)^2)$. We will come back to this example later; see Example 5.9.

This long discussion shows that there is a natural way to define a multijet bundle $\mathcal{MJ}_p(\mathbb{R}^n)$ over the configuration space $(\mathbb{R}^n)^p \setminus \Delta_p$, but that it does not extend nicely over $(\mathbb{R}^n)^p$ in general. The case $p=2$ hints that it might however be possible to define a natural multijet bundle over a slightly larger space, containing a copy of $(\mathbb{R}^n)^p \setminus \Delta_p$ as a dense open subset. Our first main contribution is to define such an object. Its main properties are gathered in the following statement, where $\mathcal{C}^k(\mathbb{R}^n, V)$ denotes the space of $\mathcal{C}^k$ functions from $\mathbb{R}^n$ to V.

Theorem 1.1 (existence of multijet bundles). *Let $n \geqslant 1$ and $p \geqslant 1$ and let V be a real vector space of finite dimension $r \geqslant 1$. There exist a smooth manifold $C_p[\mathbb{R}^n]$ of dimension np without boundary and a smooth vector bundle $\mathcal{MJ}_p(\mathbb{R}^n, V) \to C_p[\mathbb{R}^n]$ of rank rp with the following properties*:

(1) *There exists a smooth proper surjection $\pi : C_p[\mathbb{R}^n] \to (\mathbb{R}^n)^p$ such that $\pi^{-1}((\mathbb{R}^n)^p \setminus \Delta_p)$ is a dense open subset of $C_p[\mathbb{R}^n]$, and π restricted to $\pi^{-1}((\mathbb{R}^n)^p \setminus \Delta_p)$ is a $\mathcal{C}^\infty$-diffeomorphism onto $(\mathbb{R}^n)^p \setminus \Delta_p$.*

(2) *There exists a map $\mathrm{mj}_p : \mathcal{C}^{p-1}(\mathbb{R}^n, V) \times C_p[\mathbb{R}^n] \to \mathcal{MJ}_p(\mathbb{R}^n, V)$ such that*

- *for all $z \in C_p[\mathbb{R}^n]$, the linear map $\mathrm{mj}_p(\cdot, z) : \mathcal{C}^{p-1}(\mathbb{R}^n, V) \to \mathcal{MJ}_p(\mathbb{R}^n, V)_z$ is surjective*;
- *for all $f \in \mathcal{C}^{l+p-1}(\mathbb{R}^n, V)$, the section $\mathrm{mj}_p(f, \cdot)$ of $\mathcal{MJ}_p(\mathbb{R}^n, V) \to C_p[\mathbb{R}^n]$ is $\mathcal{C}^l$.*

(3) *Let $z \in C_p[\mathbb{R}^n]$ be such that $\pi(z) = (x_1, \dots, x_p) \notin \Delta_p$. Then for all $f \in \mathcal{C}^{p-1}(\mathbb{R}^n, V)$ we have*

$$\mathrm{mj}_p(f, z) = 0 \quad \iff \quad \forall i \in [\![1, p]\!], \quad f(x_i) = 0.$$

(4) *Let $z \in C_p[\mathbb{R}^n]$ be such that $\pi(z)$ is obtained as a permutation of $(y_1, \dots, y_1, \dots, y_m, \dots, y_m)$, where y_j appears exactly $k_j + 1$ times and $y_1, \dots, y_m$ are pairwise distinct vectors in $\mathbb{R}^n$. Then, there exists a linear surjection $\Theta_z : \prod_{i=1}^m \mathcal{J}_{k_j}(\mathbb{R}^n, V)_{y_j} \to \mathcal{MJ}_p(\mathbb{R}^n, V)_z$ such that*

$$\forall f \in \mathcal{C}^{p-1}(\mathbb{R}^n, V), \quad \mathrm{mj}_p(f, z) = \Theta_z(\mathrm{j}_{k_1}(f, y_1), \dots, \mathrm{j}_{k_m}(f, y_m)).$$

Remark 1.2. In Theorem 1.1, the manifold $C_p[\mathbb{R}^n]$ does not depend on V. Part (1) means that we can consider $(\mathbb{R}^n)^p \setminus \Delta_p$ as a dense open subset in $C_p[\mathbb{R}^n]$. Part (2) consists of properties that we expect any reasonable notion of multijet to satisfy. Part (3) means that, as in the previous discussions, if $\pi(z) \notin \Delta_p$ then $\mathcal{MJ}_p(\mathbb{R}^n)_z = \mathcal{C}^{p-1}(\mathbb{R}^n, V)/\sim$, where $f \sim g$ if and only if $f(x_i) = g(x_i)$ for all $i \in [\![1, p]\!]$. Part (4) means that, more generally, $\mathrm{mj}_p(f, z)$ only depends on the collection of jets $(\mathrm{j}_{k_j}(f, y_j))_{1 \leqslant j \leqslant m}$. In particular, $\mathrm{mj}_p(f, z)$ still makes sense if f is only $\mathcal{C}^{k_j}$ on some neighborhood of y_j. This last condition also means that we can think of $\mathrm{mj}_p(f, z)$ intuitively as a family of p independent linear combinations of partial derivatives of f, up to order k_j at y_j. However this family is neither explicit nor unique in general.

Let us now introduce some definitions and notation.

Definition 1.3 (multijets). Let $\Omega \subset \mathbb{R}^n$ be open. We let $C_p[\Omega] = \pi^{-1}(\Omega^p)$ and denote by $\mathcal{MJ}_p(\Omega, V) \to C_p[\Omega]$ the restriction of $\mathcal{MJ}_p(\mathbb{R}^n, V)$ to $C_p[\Omega]$. We call $\mathcal{MJ}_p(\Omega, V) \to C_p[\Omega]$ the *bundle of p-multijets* of functions from Ω to V. Its fiber $\mathcal{MJ}_p(\mathbb{R}^n, V)_z$ above $z \in C_p[\Omega]$ is the *space of p-multijets at z*. If $V = \mathbb{R}$, we drop it from the notation and write $\mathcal{MJ}_p(\Omega) \to C_p[\Omega]$. Let $f : \Omega \to V$ be of class $\mathcal{C}^{p-1}$, we call the section $\mathrm{mj}_p(f, \cdot)$ of $\mathcal{MJ}_p(\Omega, V)$ the *p-multijet of f* and its value at $z \in C_p[\Omega]$ the *p-multijet of f at z*.

The manifold $C_p[\mathbb{R}^n]$ is what is called in the literature a "compactification" of the configuration space $(\mathbb{R}^n)^p \setminus \Delta_p$. We will use this terminology, even though it is ill-chosen in our case since $C_p[\mathbb{R}^n]$ is not compact. However $C_p[\mathbb{R}^n]$ contains a diffeomorphic copy of $(\mathbb{R}^n)^p \setminus \Delta_p$ as a dense open subset and it is equipped with a proper surjection onto $(\mathbb{R}^n)^p$ so that, in a sense, it is locally a compactification of $(\mathbb{R}^n)^p \setminus \Delta_p$.

Compactifications of configuration spaces are built to understand how a configuration (ordered or not) of p distinct points can degenerate as these points converge toward one another. They are usually obtained by blowing up various pieces of the diagonal. Points in the exceptional locus then correspond to singular configurations, with some extra data encoding along which paths regular configurations are allowed to degenerate in order to reach this singular configuration. The hope is that the extra data attached to singular configurations is enough to lift the singularities of the problem under consideration. The simplest example of this kind is the blow-up $\mathrm{Bl}_{\Delta_2}((\mathbb{R}^n)^2)$ discussed above. More evolved examples are the space defined by Le Barz [1988], the compactification of Fulton and MacPherson [1994] (see also [Axelrod and Singer 1994; Sinha 2004]), Olver's multispace [2001], the polydiagonal compactification of Ulyanov [2002], the construction of Evain [2005] using Hilbert schemes, and many others.

In dimension $n = 1$, most of the compactifications of configuration spaces that we found in the literature coincide and can be used to define multijets; see for example [Ancona 2021], where Olver's multispace is used. In higher dimensions they are different and none of them exactly suited our needs. Thus to the best of our knowledge, the manifold $C_p[\mathbb{R}^n]$ in Theorem 1.1 is a new addition to the previous list. We define it by resolving the singularities of some real-algebraic variety, using Hironaka's theorem [1964a; 1964b]. In particular, $C_p[\mathbb{R}^n]$ is obtained by a sequence of blow-ups along Δ_p. Note that this sequence of blow-ups is neither explicit nor unique. Actually, $C_p[\mathbb{R}^n]$ itself is not uniquely defined, but this is not an issue for the applications we have in mind.

1.2. ***Finiteness of moments for zeros of Gaussian fields.*** Let us now describe our contributions concerning zeros of Gaussian fields. Let $n \geqslant 1$ and let $r \in [\![1, n]\!]$. In the following n will always denote the dimension of the ambient space and r the codimension of the random objects we are interested in.

Let $\Omega \subset \mathbb{R}^n$ be open and let $f : \Omega \to \mathbb{R}^r$ be a centered Gaussian field of class $\mathcal{C}^1$. We will always assume that f is nondegenerate, in the sense that $\det \mathrm{Var}(f(x)) > 0$ for all $x \in \Omega$. Under this hypothesis the zero set $Z = f^{-1}(0)$ is almost surely $(n-r)$-rectifiable; see [Armentano et al. 2023b]. As such, it admits a well-defined $(n-r)$-dimensional volume measure dVol_Z induced by the Euclidean metric on $\mathbb{R}^n$.

We denote by ν the random Radon measure on Ω defined by

$$\forall \phi \in \mathcal{C}_c^0(\Omega), \quad \langle \nu, \phi \rangle = \int_Z \phi(x) \, \mathrm{dVol}_Z(x), \tag{1-1}$$

where $\mathcal{C}_c^0(\Omega)$ denotes the space of continuous functions on Ω with compact support.

Actually, $\langle \nu, \phi \rangle$ makes sense as an almost surely defined random variable as soon as $\phi \in L^\infty(\Omega)$ and has compact support. This kind of test-function includes $\mathcal{C}_c^0(\Omega)$ and indicator functions of bounded Borel subsets, which are the examples we are most interested in. Random variables of the type $\langle \nu, \phi \rangle$ are called the linear statistics of ν (or of f). Understanding the distribution of these linear statistics is one way to understand the distribution of the random measure ν, or equivalently of the random set Z. For example, if $B \subset \Omega$ is a bounded Borel set and $\mathbf{1}_B$ denotes its indicator function, then $\langle \nu, \mathbf{1}_B \rangle$ is the $(n-r)$-dimensional volume of $Z \cap B$.

In this setting, a classical question is to determine conditions on the field f ensuring that its linear statistics admit finite moments. If $n = r = 1$, such conditions were first obtained in [Belyaev 1966]. More generally see [Azaïs and Wschebor 2009, Theorem 3.6], which holds even if f is not Gaussian. If $n \geqslant r = 1$, a similar result is proved in [Armentano et al. 2023a]; see also [Armentano et al. 2019, Theorem 4.4]. For a survey of previous results for hypersurfaces (i.e., $r = 1$), we refer to [Azaïs and Wschebor 2009, Chapter 3, Section 2.7] in dimension $n = 1$ and to the introduction of [Armentano et al. 2023a] in dimension $n \geqslant 1$. Note that [Priya 2020, Theorem 1.2] implies the finiteness of all moments of the nodal length for some Gaussian fields in $\mathbb{R}^2$. This problem was also studied in [Malevich and Volodina 1993] for points in $\mathbb{R}^2$.

Our second main result gives simple conditions on the field f ensuring the finiteness of the p-th moments of its linear statistics in any dimension and codimension. These conditions are of two kinds: we require the field to be regular enough, and to be nondegenerate in the following sense.

Definition 1.4 (p-nondegeneracy). Let $p \geqslant 1$ and let $f : \Omega \to \mathbb{R}^r$ be a $\mathcal{C}^p$ centered Gaussian field. We say that the field f is *p-nondegenerate* if for all $x \in \Omega$ the centered Gaussian vector

$$(f(x), D_x f, \dots, D_x^p f) \in \bigoplus_{k=0}^{p} \mathrm{Sym}^k(\mathbb{R}^n) \otimes \mathbb{R}^r$$

is nondegenerate, where $\mathrm{Sym}^k(\mathbb{R}^n)$ denotes the space of symmetric k-linear forms on $\mathbb{R}^n$ and $D_x^k f \in \mathrm{Sym}^k(\mathbb{R}^n) \otimes \mathbb{R}^r$ stands the k-th differential of f at x.

Remark 1.5. If $f = (f_1, \dots, f_r)$, the p-nondegeneracy condition means more concretely that for all $x \in \Omega$ the Gaussian vector $(\partial^\alpha f_i(x))_{1 \leqslant i \leqslant r; |\alpha| \leqslant p}$ is nondegenerate, where we used multi-index notation (see Section 2.2). More abstractly, this condition means that the p-jet $\mathrm{j}_p(f, x)$ of f is nondegenerate for all $x \in \Omega$.

Theorem 1.6 (finiteness of moments). *Let $n \geqslant 1$, let $r \in [\![1, n]\!]$ and let $p \geqslant 1$. Let $\Omega \subset \mathbb{R}^n$ be open, let $f : \Omega \to \mathbb{R}^r$ be a centered Gaussian field and let ν be defined as in (1-1). If f is $\mathcal{C}^p$ and $(p-1)$-nondegenerate then $\mathbb{E}[|\langle \nu, \phi \rangle|^p] < +\infty$ for all $\phi \in L^\infty(\Omega)$ with compact support.*

Example 1.7. Let us give some examples of fields satisfying the assumptions of Theorem 1.6.

- The Bargmann–Fock field, i.e., the smooth stationary Gaussian field on $\mathbb{R}^n$ whose covariance function is $x \mapsto e^{-\|x\|^2/2}$, satisfies the hypotheses of Theorem 1.6.
- Let $f : \mathbb{R}^n \to \mathbb{R}$ be a stationary $\mathcal{C}^p$ centered Gaussian field. If the support of its spectral measure has nonempty interior then f is $(p-1)$-nondegenerate.
- In codimension r, if $(f_i)_{1\leqslant i\leqslant r}$ are r independent $(p-1)$-nondegenerate $\mathcal{C}^p$ Gaussian fields then so is $f = (f_1, \dots, f_r)$.
- The Berry field, i.e., the smooth stationary Gaussian field f on $\mathbb{R}^n$ whose spectral measure is the uniform measure on $\mathbb{S}^{n-1}$, is 1-nondegenerate but not 2-nondegenerate. Indeed it almost surely satisfies $\Delta f + f = 0$, so that $(f(x), D_x f, D_x^2 f)$ is degenerate for all $x \in \mathbb{R}^n$.

We can consider the same question in a more geometric setting. Let (M, g) be a Riemannian manifold of dimension $n \geqslant 1$ without boundary and let $E \to M$ be a smooth vector bundle of rank $r \in [\![1, n]\!]$ over M. Let s be a centered Gaussian field on M with values in E, in the sense that s is a random section of $E \to M$ such that for all $m \geqslant 1$ and all $x_1, \dots, x_m \in M$ the random vector $(s(x_1), \dots, s(x_m))$ is a centered Gaussian. We assume that s is almost surely $\mathcal{C}^1$ and that $\det \mathrm{Var}(s(x)) > 0$ for all $x \in M$.

As in the Euclidean setting, $Z = s^{-1}(0)$ is almost surely $(n-r)$-rectifiable. As before, we denote by ν the random Radon measure on M defined by integrating over Z with respect to the $(n-r)$-dimensional volume measure dVol_Z induced by g. For all $\phi \in L^\infty(M)$ with compact support, we define the linear statistic $\langle \nu, \phi \rangle$ as in (1-1). In this context Definition 1.4 adapts as follows.

Definition 1.8 (p-nondegeneracy for Gaussian sections). Let $p \geqslant 1$ and let s be a $\mathcal{C}^p$ centered Gaussian field on M with values in E. We say that s is *p-nondegenerate* if, for all $x \in M$, the centered Gaussian vector $\mathrm{j}_p(s, x) \in \mathcal{J}_p(M, E)_x$ is nondegenerate.

Theorem 1.9 (finiteness of moments for zeros of Gaussian sections). *Let $p \geqslant 1$, let s be a centered Gaussian field on M with values in E and let ν be defined as in (1-1). If s is $\mathcal{C}^p$ and $(p-1)$-nondegenerate then $\mathbb{E}[|\langle \nu, \phi \rangle|^p] < +\infty$ for all $\phi \in L^\infty(M)$ with compact support.*

We are aware of the very recent paper [Gass and Stecconi 2024], in which the authors prove a result similar to Theorem 1.6, as well as its analogue for zeros of Gaussian fields on a Riemannian manifold. Their work and ours are independent, and the proofs are different. Their idea is to compare the Kac–Rice densities (see Section 6.3) of the field f with those of a well-chosen Gaussian polynomial P. Then they deduce the result for f from the result for P, which is a consequence of Bézout's theorem. Our proof follows a different path, as it relies on the multijet bundle that we defined in Theorem 1.1. Our idea is to observe that the zero set of $F : (x_1, \dots, x_p) \mapsto (f(x_1), \dots, f(x_p))$ in the configuration space $\Omega^p \setminus \Delta_p$ is exactly the vanishing locus of the multijet $\mathrm{mj}_p(f, \cdot)$ restricted to $\Omega^p \setminus \Delta_p \subset C_p[\Omega]$. Instead of working with F, which degenerates along Δ_p, we work with the field $\mathrm{mj}_p(f, \cdot)$ that we built to be nondegenerate everywhere. Then, we deduce Theorem 1.6 from the Kac–Rice formula for the expectation (see Proposition 6.17) applied to the p-multijet of f and a compactness argument.

1.3. ***Higher-order multijets and holomorphic multijets.*** Let us now discuss two important variations on our main results, Theorems 1.1 and 1.9. In Section 1.1, we said that two functions f and g on $\mathbb{R}^n$ have the same p-multijet at a point $\underline{x} = (x_1, \dots, x_p) \in (\mathbb{R}^n)^p \setminus \Delta_p$ if and only if f and g have the same value, i.e., the same 0-jet, at x_i for all $i \in [\![1, p]\!]$. In a sense, the p-multijet of f at $\underline{x}$ is obtained by patching together the 0-jets of f at each of the x_i in a relevant way. A natural generalization is to define a higher-order multijet of f at $\underline{x}$ by patching together the k-jets of f at each of the x_i. We define such a higher-order multijet in Section 7. More generally, we define a multijet bundle adapted to a differential operator $\mathcal{D}$. The case of higher-order multijets corresponds to $\mathcal{D} = \mathrm{j}_k$. The analogue of Theorem 1.1 in this framework is Theorem 7.4 below. We use it to prove an analogue of Theorem 1.9 adapted to $\mathcal{D}$; see Theorem 7.8 for a general statement. In the special case where $\mathcal{D} = D$ is the standard differential, the statement is the following.

Theorem 1.10 (finiteness of moments for critical points). *Let M be a smooth manifold without boundary. Let $f : M \to \mathbb{R}$ be a centered Gaussian field and let ν_D denote the counting measure of its critical locus. Let $p \geqslant 1$, we assume that f is $\mathcal{C}^{2p}$ and $(2p-1)$-nondegenerate. Then, for all $\phi \in L^\infty_c(M)$, we have $\mathbb{E}[|\langle \nu_D, \phi\rangle|^p] < +\infty$.*

Another variation on Theorem 1.1 is to define holomorphic multijets for holomorphic maps. This is done in Section 8, and more precisely in Theorem 8.2. This is used to prove a holomorphic version of Theorem 1.9. The general statement is given in Theorem 8.13. For a holomorphic Gaussian field on an open subset of $\mathbb{C}^n$, it takes the following form.

Theorem 1.11 (finiteness of moments for zeros of holomorphic Gaussian fields). *Let $\Omega \subset \mathbb{C}^n$ be open and let $f : \Omega \to \mathbb{C}^r$ be a centered holomorphic Gaussian field, where $r \in [\![1, n]\!]$. Let ν be as in Definition 6.11. Let $p \geqslant 1$, we assume that, for all $x \in \Omega$, the complex Gaussian vector*

$$(f(x), D_x f, \dots, D_x^{p-1} f) \in \bigoplus_{k=0}^{p-1} \mathrm{Sym}^k(\mathbb{C}^n) \otimes \mathbb{C}^r$$

is nondegenerate. Then, for all $\phi \in L^\infty_c(\Omega)$, we have $\mathbb{E}[|\langle \nu, \phi\rangle|^p] < +\infty$.

Note that Theorems 1.10 and 1.11 are not consequences of Theorem 1.9. Indeed, if $f : M \to \mathbb{R}$ is a smooth Gaussian field then Df cannot be 1-nondegenerate because $D^2 f$ is symmetric. Similarly, if M is a complex manifold and s is holomorphic, then s is never 1-nondegenerate because it satisfies the Cauchy–Riemann equations.

Gass and Stecconi [2024] proved, independently and by a different method, results analogous to Theorems 1.10 and 1.11. Actually, they prove Theorem 1.10 under the weaker and optimal hypotheses that f is $\mathcal{C}^{p+1}$ and p-nondegenerate. The finiteness of the third moment for the number of critical points of a stationary Gaussian field on $\mathbb{R}^d$ was proved in [Beliaev et al. 2024, Theorem 1.6]. For holomorphic Gaussian fields in dimension $n = 1$, see [Nazarov and Sodin 2012].

1.4. ***Organization of the paper.*** In Section 2 we gather useful notation that appears in several parts of the paper. In Section 3 we discuss Kergin interpolation, which is a multivariate polynomial interpolation appearing in the definition of multijets. Section 4 is dedicated to evaluations maps on spaces of polynomials, and more precisely the properties of their kernels. We define our multijet bundles and prove Theorem 1.1 in Section 5. Section 6 is concerned with the application of multijets to the finiteness of moments for the zeros of Gaussian fields and the proofs of Theorems 1.6 and 1.9. Multijets adapted to a differential operator are discussed in Section 7, where we also prove the analogue of Theorem 1.9 for critical points. Finally, holomorphic multijets are defined in Section 8, where we prove the analogue of Theorem 1.9 for holomorphic Gaussian fields.

2. Notation: partitions and function spaces

The goal of this section is to quickly introduce definitions and notation that will appear in different parts of the paper. We gather them here for the reader's convenience.

2.1. ***Sets, partitions and diagonals.*** In this paper, we denote by $\mathbb{N}$ the set of nonnegative integers. Let a and $b \in \mathbb{N}$, we use the following notation for integer intervals $[\![a, b]\!] = [a, b] \cap \mathbb{N}$.

Let A be a nonempty finite set. For simplicity, in all the notation introduced in this section, if $A = [\![1, p]\!]$ we allow ourselves to replace A by p in the indices and exponents. We denote by $|A|$ the cardinality of A. Let M be any set. We denote by M^A the Cartesian product of $|A|$ copies of M indexed by the elements of A. A generic element of M^A is usually denoted by $\underline{x} = (x_a)_{a \in A}$. If $\varnothing \neq B \subset A$, we denote by $\underline{x}_B = (x_a)_{a \in B}$.

Definition 2.1 (large diagonal). We denote by Δ_A the large diagonal in M^A, that is,

$$\Delta_A = \{(x_a)_{a \in A} \in M^A \mid \exists a, b \in A \text{ such that } a \neq b \text{ and } x_a = x_b\}.$$

Definition 2.2 (partitions). Let A be a nonempty and finite set, a *partition* of A is a family $\mathcal{I} = \{I_1, \dots, I_m\}$ of nonempty disjoint subsets of A such that $\bigsqcup_{i=1}^m I_i = A$. The subsets $I_1, \dots, I_m$ are called the *cells* of $\mathcal{I}$. Given $a \in A$, we denote by $[a]_\mathcal{I}$ the only cell of $\mathcal{I}$ that contains a. Finally, we denote by $\mathcal{P}_A$ the set of partitions of A.

Definition 2.3 (clustering partition). Let $\underline{x} = (x_a)_{a \in A} \in M^A$. We denote by $\mathcal{I}(\underline{x}) \in \mathcal{P}_A$ the only partition such that for all a and $b \in A$ we have $x_a = x_b$ if and only if $[a]_{\mathcal{I}(\underline{x})} = [b]_{\mathcal{I}(\underline{x})}$.

Example 2.4. If $\underline{x} = (x, \dots, x)$ then $\mathcal{I}(\underline{x}) = \{A\}$. If $\underline{x} \in M^A \setminus \Delta_A$ then $\mathcal{I}(\underline{x}) = \{\{a\} \mid a \in A\} = \mathcal{I}_0$.

Definition 2.5 (strata of the diagonal). For all $\mathcal{I} \in \mathcal{P}_A$, we set $\Delta_{A,\mathcal{I}} = \{\underline{x} \in M^A \mid \mathcal{I}(\underline{x}) = \mathcal{I}\}$, so that $\Delta_{A,\mathcal{I}_0} = M^A \setminus \Delta_A$ and $\Delta_A = \bigsqcup_{\mathcal{I} \neq \mathcal{I}_0} \Delta_{A,\mathcal{I}}$.

Definition 2.6 (diagonal inclusions). Let $\mathcal{I} \in \mathcal{P}_A$. We denote by $\iota_\mathcal{I} : M^\mathcal{I} \setminus \Delta_\mathcal{I} \to \Delta_{A,\mathcal{I}}$ the bijection defined by $\iota_\mathcal{I}((y_I)_{I \in \mathcal{I}}) = (y_{[a]_\mathcal{I}})_{a \in A}$.

2.2. ***Spaces of functions, sections and jets.*** We use the following multi-index notation. Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$. We denote its length by $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Let ∂_i denote the i-th partial derivative in some product space we denote by $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$. Finally, if $X = (X_1, \dots, X_n)$, we let $X^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$.

Definition 2.7 (polynomials). We denote by $\mathbb{R}_d[X]$ the space of real polynomials in n variables of degree at most d, where $d \in \mathbb{N}$ and $X = (X_1, \dots, X_n)$ is multivariate.

Definition 2.8 (symmetric forms and differentials). Let $k \in \mathbb{N}$. We denote by $\mathrm{Sym}^k(\mathbb{R}^n)$ the space of symmetric k-linear forms on $\mathbb{R}^n$. Let V be a finite-dimensional real vector space. Then $\mathrm{Sym}^k(\mathbb{R}^n) \otimes V$ is the space of symmetric k-linear maps from $\mathbb{R}^n$ to V. Given a $\mathcal{C}^k$ map $f : \mathbb{R}^n \to V$, we denote by $D^k_x f \in \mathrm{Sym}^k(\mathbb{R}^n) \otimes V$ its k-th differential at $x \in \mathbb{R}^n$.

Let M and N be two manifolds without boundary. For all $k \in \mathbb{N} \cup \{\infty\}$, we denote by $\mathcal{C}^k(M, N)$ the space of $\mathcal{C}^k$ maps from M to N. If $N = \mathbb{R}$, we drop it from the notation and we simply write $\mathcal{C}^k(M)$. We denote by $L^1_{\mathrm{loc}}(M)$ the space of locally integrable functions on M. We denote by $\mathcal{C}^0_c(M)$ (resp. $L^\infty_c(M)$) the space of continuous (resp. L^∞) functions on M with compact support. Finally, for any Borel subset $B \subset M$, we denote by $\mathbf{1}_B : M \to \mathbb{R}$ its indicator function.

Let $E \to M$ be a vector bundle of finite rank over M, we denote by E_x the fiber above $x \in M$. For all $k \in \mathbb{N} \cup \{\infty\}$, we denote by $\Gamma^k(M, E)$ the space of $\mathcal{C}^k$ sections of $E \to M$.

Definition 2.9 (jets). Let $k \in \mathbb{N}$, we denote by $\mathcal{J}_k(M, E) \to M$ the vector bundle of k-jets of sections of $E \to M$. If $E = V \times M$ is trivial with fiber V, we denote its k-jet bundle by $\mathcal{J}_k(M, V) \to M$. If $V = \mathbb{R}$, we simply write $\mathcal{J}_k(M) \to M$. Given $s \in \Gamma^k(M, E)$, we denote by $\mathrm{j}_k(s, x) \in \mathcal{J}_k(M, E)_x$ its k-jet at $x \in M$.

3. Divided differences and Kergin interpolation

An important step in our construction of a multijet for $\mathcal{C}^k$ functions is to reduce the problem to that of defining a multijet for polynomials. This is done by polynomial interpolation. In several variables, polynomial interpolation is rather ill-behaved, at least compared with the one-variable case. However, a multivariate polynomial interpolation suiting our needs was defined by Kergin [1980]. A constructive version of his proof was then given in [Micchelli and Milman 1980], using a multivariate version of the so-called divided differences. In this section, we give the definitions of these objects and recall their relevant properties. We refer to the survey [Lorentz 2000] for more background on polynomial interpolation in $\mathbb{R}^n$.

3.1. ***Divided differences.*** In this section, we recall the definition of multivariate divided differences; see [Micchelli and Milman 1980]. Let $k \in \mathbb{N}$. We denote by σ_k the standard simplex of dimension k, that is,

$$\sigma_k = \left\{ \underline{t} = (t_0, \dots, t_k) \in [0,1]^{k+1} \;\middle|\; \sum_{i=0}^{k} t_i = 1 \right\} \subset \mathbb{R}^{k+1}. \tag{3-1}$$

The simplex σ_k is a subset of $\{\underline{t} \in \mathbb{R}^{k+1} \mid \sum t_i = 1\}$, and we denote by ν_k the (k-dimensional) Lebesgue measure on this hyperplane, normalized so that $\nu_k(\sigma_k) = 1/k!$. One can check that its restriction to σ_k satisfies

$$\int_{\sigma_k} \phi(\underline{t})\, \mathrm{d}\nu_k(\underline{t}) = \int_{\substack{t_1,\dots,t_k \geqslant 0 \\ \sum_{i=1}^k t_i \leqslant 1}} \phi\left(1 - \sum_{i=1}^{k} t_i, t_1, \dots, t_k\right) \mathrm{d}t_1 \cdots \mathrm{d}t_k, \tag{3-2}$$

where $\mathrm{d}t_1 \cdots \mathrm{d}t_k$ is the Lebesgue measure on $\mathbb{R}^k$. For any $\underline{x} = (x_0, \dots, x_k) \in (\mathbb{R}^n)^{k+1}$, we denote by $\sigma(\underline{x})$ the convex hull of the x_i and we define $\upsilon_{\underline{x}} : \underline{t} \mapsto \sum_{i=0}^k t_i x_i$ from σ_k onto $\sigma(\underline{x})$. Recalling Definition 2.8, we have the following.

Definition 3.1 (divided differences). Let $\underline{x} = (x_i)_{0\leqslant i\leqslant k} \in (\mathbb{R}^n)^{k+1}$ and let f be a $\mathcal{C}^k$ function defined on some open neighborhood of $\sigma(\underline{x})$ in $\mathbb{R}^n$. We define the *divided difference* of f at $\underline{x}$ by

$$f[x_0, \dots, x_k] = \int_{\sigma_k} D^k_{\upsilon_{\underline{x}}(\underline{t})} f \, \mathrm{d}\nu_k(\underline{t}) \in \mathrm{Sym}^k(\mathbb{R}^n),$$

that is, as the average of $D^k f$ over $\sigma(\underline{x})$ with respect to the pushed-forward measure $(\upsilon_{\underline{x}})_*(\nu_k)$.

Remark 3.2. • If $\underline{x} = (x, \dots, x)$ for some $x \in \mathbb{R}^n$ then $f[x, \dots, x] = (1/k!) D^k_x f$.

• Definition 3.1 is invariant under permutation of $(x_0, \dots, x_k)$.

• When $n = 1$, Definition 3.1 coincides with the classical definition of divided differences, under the canonical isomorphism $\mathrm{Sym}^k(\mathbb{R}) \simeq \mathbb{R}$. This is known as the Hermite–Genocchi formula [Micchelli and Milman 1980].

Lemma 3.3 (regularity of divided differences). *For all $\underline{x} \in (\mathbb{R}^n)^{k+1}$, the map $f \mapsto f[x_0, \dots, x_k]$ is linear. Moreover, if f is of class $\mathcal{C}^{k+l}$ then $\underline{x} \mapsto f[x_0, \dots, x_k]$ is of class $\mathcal{C}^l$.*

Proof. The linearity with respect to f is clear. The regularity with respect to $\underline{x}$ is obtained by derivation under the integral, using Definition 3.1 and (3-2). □

3.2. *Kergin interpolation.* This section is dedicated to Kergin interpolation. In the following, we recall the construction of Kergin interpolation in [Micchelli and Milman 1980], which relies on the divided differences introduced in Definition 3.1. We will use the notation introduced in Definition 2.7.

Proposition 3.4 (Kergin interpolation). *Let $\underline{x} \in (\mathbb{R}^n)^p$ and let f be a function of class $\mathcal{C}^{p-1}$ defined on some neighborhood of $\sigma(\underline{x})$ in $\mathbb{R}^n$. There exists a unique polynomial $K(f, \underline{x}) \in \mathbb{R}_{p-1}[X]$ such that, for all nonempty $I \subset [\![1, p]\!]$, we have $f[\underline{x}_I] = (K(f, \underline{x}))[\underline{x}_I]$. Moreover,*

$$K(f, \underline{x}) = \sum_{k=1}^{p} f[x_1, \dots, x_k](X - x_1, \dots, X - x_{k-1}). \tag{3-3}$$

Proof. This is the content of [Bojanov et al. 1993, Theorem 12.5] for $m = 0$. See also [Micchelli and Milman 1980]. □

Remark 3.5. In particular, Proposition 3.4 implies the following:

• The restriction of $K(\cdot, \underline{x})$ to $\mathbb{R}_{p-1}[X]$ is the identity.

• If x appears with multiplicity at least $k + 1$ in $\underline{x}$, then

$$D^k_x f = k!\, f[\underbrace{x, \dots, x}_{k+1 \text{ times}}] = k!\, (K(f, \underline{x}))[\underbrace{x, \dots, x}_{k+1 \text{ times}}] = D^k_x(K(f, \underline{x})).$$

• The map $P \mapsto (P[x_1, \dots, x_j])_{1\leqslant j\leqslant p}$ is an isomorphism from $\mathbb{R}_{p-1}[X]$ to $\bigoplus_{j=0}^{p-1} \mathrm{Sym}^j(\mathbb{R}^n)$ whose inverse map is given by $(S_j)_{0\leqslant j\leqslant p-1} \mapsto \sum_{j=0}^{p-1} S_j(X - x_1, \dots, X - x_j)$.

Definition 3.6 (Kergin polynomial). The polynomial $K(f, \underline{x})$ from Proposition 3.4 is called the *Kergin interpolating polynomial* of f at $\underline{x}$.

Example 3.7. If $n = 1$, then $K(f, \underline{x})$ is the Hermite interpolating polynomial of f at $\underline{x} \in \mathbb{R}^p$. If $\underline{x} = (x, \dots, x)$, then $K(f, \underline{x})$ is the Taylor polynomial of order $p-1$ of f at $x \in \mathbb{R}^n$.

Lemma 3.8 (regularity of the Kergin polynomial). *For all $\underline{x} \in (\mathbb{R}^n)^p$, the map $K(\,\cdot\,, \underline{x})$ is linear. Moreover, if f is $\mathcal{C}^{l+p-1}$ then $K(f, \cdot\,)$ is of class $\mathcal{C}^l$.*

Proof. This is a consequence of Lemma 3.3 and (3-3). □

We need to prove a form of compatibility in Kergin interpolation, when the set of interpolation points is refined. We will use this fact to prove that the multijet bundle we define below satisfies (4) in Theorem 1.1. The following lemma is stated using the clustering partition $\mathcal{I}(\underline{x})$ from Definition 2.3.

Lemma 3.9 (compatibility in Kergin interpolation). *For all $\underline{x} \in (\mathbb{R}^n)^p$ the linear map from $\mathbb{R}_{p-1}[X]$ to $\prod_{I \in \mathcal{I}(\underline{x})} \mathbb{R}_{|I|-1}[X]$ defined by $(K(\,\cdot\,, \underline{x}_I))_{I \in \mathcal{I}(\underline{x})} : P \mapsto (K(P, \underline{x}_I))_{I \in \mathcal{I}(\underline{x})}$ is surjective.*

Proof. Let $\underline{x} \in (\mathbb{R}^n)^p$ and let us write $\mathcal{I} = \mathcal{I}(\underline{x})$ for simplicity. As explained at the end of Section 2.1, there exists a unique $\underline{y} = (y_I)_{I \in \mathcal{I}} \in (\mathbb{R}^n)^{\mathcal{I}} \setminus \Delta_{\mathcal{I}}$ such that $\underline{x} = \iota_{\mathcal{I}}(\underline{y})$. Let $(\chi_I)_{I \in \mathcal{I}}$ be smooth functions on $\mathbb{R}^n$ with pairwise disjoint compact supports and such that χ_I is equal to 1 in a neighborhood of y_I.

Let $(P_I)_{I \in \mathcal{I}} \in \prod_{I \in \mathcal{I}} \mathbb{R}_{|I|-1}[X]$. We consider the function $f = \sum_{I \in \mathcal{I}} \chi_I P_I \in \mathcal{C}^\infty(\mathbb{R}^n)$. Let $P = K(f, \underline{x})$ and let us prove that $K(P, \underline{x}_I) = P_I$ for all $I \in \mathcal{I}$. For all $k \leqslant |I| - 1$ we have $D^k_{y_I} P = D^k_{y_I} f = D^k_{y_I} P_I$. Indeed y_I appears with multiplicity $|I|$ in $\underline{x}$ (see Remark 3.5) and f is equal to P_I in a neighborhood of y_I. Recalling Example 3.7, we know that $K(P, \underline{x}_I)$ is the Taylor polynomial of order $|I| - 1$ at y_I of P, and hence of P_I. Since $P_I \in \mathbb{R}_{|I|-1}[X]$, we get $K(P, \underline{x}_I) = P_I$. □

4. Evaluation maps and their kernels

The goal of this section is to study evaluation maps on spaces of polynomials and their kernels. Defining multijets is closely related to these objects. Indeed, let $n \geqslant 1$ and $p \geqslant 1$ and recall that Δ_p stands for the large diagonal in $(\mathbb{R}^n)^p$; see Definition 2.1. As explained in the Introduction, when $\underline{x} \notin \Delta_p$ we want the multijet of a $\mathcal{C}^{p-1}$ function f at $\underline{x}$ to be the class of f in $\mathcal{C}^{p-1}(\mathbb{R}^n)/\sim$, where $f \sim g$ if and only if $(f(x_i))_{1 \leqslant i \leqslant p} = (g(x_i))_{1 \leqslant i \leqslant p}$. The Kergin interpolation of Section 3.2 shows that any such class can be represented by a polynomial. Hence, the space of p-multijets at $\underline{x}$ is canonically isomorphic to $\mathbb{R}_{p-1}[X]/\ker \mathrm{ev}_{\underline{x}}$, where $\mathrm{ev}_{\underline{x}} : P \mapsto (P(x_1), \dots, P(x_p))$.

Definition 4.1 (Grassmannian). Let V be a vector space of finite dimension N and $k \in [\![0, N]\!]$. We denote by $\mathrm{Gr}_k(V)$ the *Grassmannian* of vector subspaces of V of *codimension* k.

Remark 4.2. Beware that this notation is slightly unusual, since in most textbooks $\mathrm{Gr}_k(V)$ stands for the Grassmannian of subspaces of dimension k.

Let us denote by $\mathcal{L}_{\mathrm{reg}}(V, \mathbb{R}^k) \subset V^* \otimes \mathbb{R}^k$ the open dense subset of linear surjective maps from V to $\mathbb{R}^k$. The group $\mathrm{GL}_k(\mathbb{R})$ acts on $\mathcal{L}_{\mathrm{reg}}(V, \mathbb{R}^k)$ by multiplication on the left. On the other hand, $L \mapsto \ker(L)$ defines a surjective map from $\mathcal{L}_{\mathrm{reg}}(V, \mathbb{R}^k)$ to $\mathrm{Gr}_k(V)$, and $\ker(L_1) = \ker(L_2)$ if and only if there exists $M \in \mathrm{GL}_k(\mathbb{R})$ such that $L_2 = ML_1$. Thus, one can identify $\mathrm{Gr}_k(V)$ with the orbit space $\mathcal{L}_{\mathrm{reg}}(V, \mathbb{R}^k)/\mathrm{GL}_k(\mathbb{R})$ of the previous action. This is one of the many ways to describe $\mathrm{Gr}_k(V)$ as a smooth real-algebraic manifold.

Definition 4.3 (evaluation map). Let $\underline{x} \in (\mathbb{R}^n)^p$. We set $\mathrm{ev}_{\underline{x}} : f \mapsto (f(x_1), \dots, f(x_p))$ from any space of functions defined at the x_i to $\mathbb{R}^p$. The source space will always be clear from the context.

Lemma 4.4 (nondegeneracy of $\mathrm{ev}_{\underline{x}}$). *Let $\underline{x} \notin \Delta_p$. Then $\mathrm{ev}_{\underline{x}} : \mathbb{R}_{p-1}[X] \to \mathbb{R}^p$ is surjective.*

Proof. Since $\underline{x} \notin \Delta_p$, we have $\mathcal{I}(\underline{x}) = \{\{1\}, \dots, \{p\}\}$ and $\mathrm{ev}_{\underline{x}} = (K(\cdot, x_i))_{1 \leqslant i \leqslant p}$ under the canonical identification $\mathbb{R}_0[X] \simeq \mathbb{R}$. Hence this is just a special case of Lemma 3.9. Alternatively, in the right basis, one can extract a Vandermonde matrix from that of $\mathrm{ev}_{\underline{x}}$. □

Lemma 4.4 shows that the following map is well-defined from $(\mathbb{R}^n)^p \setminus \Delta_p$ to $\mathrm{Gr}_p(\mathbb{R}_{p-1}[X])$:

$$\mathcal{G} : \underline{x} \longmapsto \ker \mathrm{ev}_{\underline{x}} . \tag{4-1}$$

Lemma 4.5 (algebraicity of $\mathcal{G}$). *The map $\mathcal{G} : (\mathbb{R}^n)^p \setminus \Delta_p \to \mathrm{Gr}_p(\mathbb{R}_{p-1}[X])$ is algebraic.*

Proof. Recalling the previous discussion, we have $\mathcal{L}_{\mathrm{reg}}(\mathbb{R}_{p-1}[X], \mathbb{R}^p)/\mathrm{GL}_p(\mathbb{R}) \simeq \mathrm{Gr}_p(\mathbb{R}_{p-1}[X])$, where the isomorphism is obtained as the quotient map of $\ker : L \mapsto \ker(L)$. In particular,

$$\ker : \mathcal{L}_{\mathrm{reg}}(\mathbb{R}_{p-1}[X], \mathbb{R}^p) \longrightarrow \mathrm{Gr}_p(\mathbb{R}_{p-1}[X]) \simeq \mathcal{L}_{\mathrm{reg}}(\mathbb{R}_{p-1}[X], \mathbb{R}^p)/\mathrm{GL}_p(\mathbb{R})$$

is just the canonical projection, which is algebraic.

Writing $\mathrm{ev} : \underline{x} \mapsto \mathrm{ev}_{\underline{x}}$, we have $\mathcal{G} = \ker \circ \mathrm{ev}$. Thus it is enough to prove that ev is algebraic from $(\mathbb{R}^n)^p \setminus \Delta_p$ to $\mathcal{L}_{\mathrm{reg}}(\mathbb{R}_{p-1}[X], \mathbb{R}^p)$. In the basis of $\mathbb{R}_{p-1}[X]$ formed by the monomials $(X^\alpha)_{|\alpha|<p}$, the matrix of $\mathrm{ev}_{\underline{x}}$ is $(x_i^\alpha)_{1 \leqslant i \leqslant p; |\alpha|<p}$, which depends algebraically on $\underline{x}$. □

Let $\underline{x} \in (\mathbb{R}^n)^p \setminus \Delta_p$, we defined $\mathcal{G}(\underline{x}) \in \mathrm{Gr}_p(\mathbb{R}_{p-1}[X])$ by (4-1). For any nonempty $I \subset [\![1, p]\!]$, we define similarly

$$\mathcal{G}_I(\underline{x}) = \ker \mathrm{ev}_{\underline{x}_I} \in \mathrm{Gr}_{|I|}(\mathbb{R}_{|I|-1}[X]) \quad \text{and} \quad \widetilde{\mathcal{G}}_I(\underline{x}) = \ker \mathrm{ev}_{\underline{x}_I} \in \mathrm{Gr}_{|I|}(\mathbb{R}_{p-1}[X]). \tag{4-2}$$

Because of the interpolation properties of the Kergin polynomials (see Remark 3.5), we have that $\mathrm{ev}_{\underline{x}_I} = (\mathrm{ev}_{\underline{x}_I})_{|\mathbb{R}_{|I|-1}[X]} \circ K(\cdot, \underline{x}_I)$ on $\mathbb{R}_{p-1}[X]$. Hence $\widetilde{\mathcal{G}}_I(\underline{x}) = K(\cdot, \underline{x}_I)^{-1}(\mathcal{G}_I(\underline{x}))$. Since $K(\cdot, \underline{x}_I)$ is surjective from $\mathbb{R}_{p-1}[X]$ to $\mathbb{R}_{|I|-1}[X]$, this shows that $\widetilde{\mathcal{G}}_I(\underline{x})$ has indeed codimension $|I|$, like $\mathcal{G}_I(\underline{x})$.

This collection of subspaces satisfies some incidence relations that will be useful in the following. For all nonempty $I \subset [\![1, p]\!]$, we have $\mathcal{G}(\underline{x}) \subset \widetilde{\mathcal{G}}_I(\underline{x})$. Actually, we can be more precise: for any $\mathcal{I} \in \mathcal{P}_p$, we have $\mathcal{G}(\underline{x}) = \bigcap_{I \in \mathcal{I}} \widetilde{\mathcal{G}}_I(\underline{x})$, and this intersection is transverse by a codimension argument.

Remark 4.6. The map $\mathcal{G} : (\mathbb{R}^n)^p \setminus \Delta_p \to \mathrm{Gr}_p(\mathbb{R}_{p-1}[X])$ does not admit an extension as a regular map from $(\mathbb{R}^n)^p$ to $\mathrm{Gr}_p(\mathbb{R}_{p-1}[X])$, except if $n = 1$ or $p = 1$, that is, if $\mathrm{Gr}_p(\mathbb{R}_{p-1}[X])$ is a point.

For example, when $n = 2 = p$, the Grassmannian $\mathrm{Gr}_p(\mathbb{R}_{p-1}[X])$ is the set of lines in $\mathbb{R}_1[X_1, X_2]$. Taking $x = R(\cos\theta, \sin\theta) \in \mathbb{R}^2 \setminus \{0\}$, the reader can check that $\mathcal{G}(0, x) = \mathrm{Span}(X_1 \sin\theta - X_2 \cos\theta)$, which does not converge as $R \to 0$. However, in this case, $\mathcal{G}(0, \cdot)$ extends to the blow-up $\mathrm{Bl}_0(\mathbb{R}^2)$ of $\mathbb{R}^2$ at 0 and similarly $\mathcal{G}$ extends smoothly to $\mathrm{Bl}_{\Delta_2}((\mathbb{R}^2)^2)$. This suggests that, even though $\mathcal{G}$ does not extend smoothly to $(\mathbb{R}^n)^p$, it might extend to a larger space.

5. Definition of the multijet bundles

In this section we define the vector bundle $\mathcal{MJ}_p(\mathbb{R}^n, V) \to C_p[\mathbb{R}^n]$ of p-multijets for functions from $\mathbb{R}^n$ to some finite-dimensional vector space V and prove Theorem 1.1. The singularity of $\mathcal{G}$ along Δ_p makes it impossible to define such a bundle over $(\mathbb{R}^n)^p$, which is why we define it over a compactification $C_p[\mathbb{R}^n]$ of the configuration space $(\mathbb{R}^n)^p \setminus \Delta_p$.

The manifold $C_p[\mathbb{R}^n]$ does not depend on V. It is defined in Section 5.1. In the next two sections, we work in the case $V = \mathbb{R}$. All important ideas appear in this case but the notation is slightly simpler. In Section 5.2, we define the bundle $\mathcal{MJ}_p(\mathbb{R}^n)$. In Section 5.3, we prove that p-multijets are local, in the sense of (4) in Theorem 1.1. Finally, we define the bundle $\mathcal{MJ}_p(\mathbb{R}^n, V)$ of multijets for vector-valued maps and prove Theorem 1.1 in Section 5.4.

5.1. ***Definition of the basis $C_p[\mathbb{R}^n]$.*** In this section, we define the basis $C_p[\mathbb{R}^n]$ over which our p-multijet bundles are defined. This is a smooth manifold, obtained a compactification of the configuration space $(\mathbb{R}^n)^p \setminus \Delta_p$ such that $(\mathcal{G}_I)_{I\subset[\![1,p]\!]}$ extends smoothly to $C_p[\mathbb{R}^n]$. Let us first introduce some notation. We denote by Π_0 the projection from the product space

$$(\mathbb{R}^n)^p \times \prod_{\varnothing\neq I\subset[\![1,p]\!]} \mathrm{Gr}_{|I|}(\mathbb{R}_{|I|-1}[X])$$

onto the factor $(\mathbb{R}^n)^p$. Similarly, we denote by Π_I the projection onto $\mathrm{Gr}_{|I|}(\mathbb{R}_{|I|-1}[X])$. Then, let

$$\Sigma = \{(\underline{x}, (\mathcal{G}_I(\underline{x}))_{I\subset[\![1,p]\!]}) \mid \underline{x} \in (\mathbb{R}^n)^p \setminus \Delta_p\} \subset (\mathbb{R}^n)^p \times \prod_{\varnothing\neq I\subset[\![1,p]\!]} \mathrm{Gr}_{|I|}(\mathbb{R}_{|I|-1}[X]) \tag{5-1}$$

denote the graph of the map $(\mathcal{G}_I)_{I\subset[\![1,p]\!]}$. We denote by $\overline{\Sigma}$ the closure of Σ in the product space on the right-hand side of (5-1).

Lemma 5.1 (surjectivity of $(\Pi_0)_{|\overline{\Sigma}}$). *Let $\underline{x} \in (\mathbb{R}^n)^p$. Then there exists $z \in \overline{\Sigma}$ such that $\Pi_0(z) = \underline{x}$.*

Proof. Let $(\underline{x}_n)_{n\in\mathbb{N}}$ be a sequence of points in $(\mathbb{R}^n)^p \setminus \Delta_p$ converging to $\underline{x}$. Since Grassmannians are compact manifolds, up to extracting subsequences finitely many times, we can assume that for all nonempty $I \subset [\![1, p]\!]$ there exists $G_I \in \mathrm{Gr}_{|I|}(\mathbb{R}_{|I|-1}[X])$ such that $\mathcal{G}_I(\underline{x}_n) \xrightarrow[n\to+\infty]{} G_I$. Then

$$(\underline{x}_n, (\mathcal{G}_I(\underline{x}_n))_{I\subset[\![1,p]\!]}) \xrightarrow[n\to+\infty]{} (\underline{x}, (G_I)_{I\subset[\![1,p]\!]}) = z \in \overline{\Sigma}. \qquad \square$$

Lemma 5.2 (location of the new points). *We have $\overline{\Sigma} \setminus \Sigma \subset \Pi_0^{-1}(\Delta_p)$.*

Proof. Since Σ is the graph of a continuous function on $(\mathbb{R}^n)^p \setminus \Delta_p$, it is closed in the open subset $\Pi_0^{-1}((\mathbb{R}^n)^p \setminus \Delta_p)$. Hence $\overline{\Sigma} \cap \Pi_0^{-1}((\mathbb{R}^n)^p \setminus \Delta_p) = \Sigma$ and $\overline{\Sigma} \setminus \Sigma \subset \Pi_0^{-1}(\Delta_p)$. $\square$

Lemma 5.3 (algebraicity of Σ and $\overline{\Sigma}$). *The graph Σ is a smooth real-algebraic manifold and $(\Pi_0)_{|\Sigma} : \Sigma \to (\mathbb{R}^n)^p \setminus \Delta_p$ is an isomorphism. Moreover, $\overline{\Sigma}$ is a real-algebraic variety whose singular locus is contained in $\overline{\Sigma} \setminus \Sigma$.*

Proof. By Lemma 4.5, the set Σ is the graph of an algebraic map, hence a smooth real-algebraic manifold. Additionally, Π_0 is algebraic and its restriction to Σ is the inverse of $\underline{x} \mapsto (\underline{x}, (\mathcal{G}_I(\underline{x}))_{I\subset[\![1,p]\!]})$. Thus

$(\Pi_0)_{|\Sigma}$ is an algebraic isomorphism from Σ onto $(\mathbb{R}^n)^p \setminus \Delta_p$. Since Σ is real-algebraic, so is its closure $\overline{\Sigma}$. By Lemma 5.2, we know that $\overline{\Sigma} \cap \Pi_0^{-1}((\mathbb{R}^n)^p \setminus \Delta_p) = \Sigma$ is smooth. Hence, the singular locus of $\overline{\Sigma}$ is contained in $\overline{\Sigma} \cap \Pi_0^{-1}(\Delta_p) = \overline{\Sigma} \setminus \Sigma$. □

Example 5.4. In simple cases, we understand very well what $\overline{\Sigma}$ is.

- If $p = 1$ and $n \geqslant 1$, then $\Delta_p = \varnothing$ and $\mathrm{Gr}_p(\mathbb{R}_{p-1}[X]) = \{\{0\}\}$, so that $\overline{\Sigma} = \Sigma = \mathbb{R}^n$.
- If $n = 1$ and $p \geqslant 1$, then $\mathrm{Gr}_{|I|}(\mathbb{R}_{|I|-1}[X]) = \{\{0\}\}$ for all $I \subset [\![1, p]\!]$ and $\overline{\Sigma} = \mathbb{R}^p$.
- If $p = 2$ and $n \geqslant 2$, then for $x \neq y$ in $\mathbb{R}^n$ we know that $\mathcal{G}(x, y) \subset \mathbb{R}_1[X]$ is the subspace of affine forms on $\mathbb{R}^n$ vanishing at x and y, i.e., on the affine line through x and y. Thus $\mathcal{G}(x, y)$ encodes this line. As $y \to x$, the accumulation points of $\mathcal{G}(x, y)$ correspond to all the affine lines passing through x, and they encode "the direction from which y converges to x". In this case, one can check that $\overline{\Sigma} = \mathrm{Bl}_{\Delta_2}((\mathbb{R}^n)^2)$.

In the previous examples the variety $\overline{\Sigma}$ is smooth, hence the following natural question.

Question. *Is $\overline{\Sigma}$ smooth for all $n \geqslant 1$ and $p \geqslant 1$?*

Lacking a positive answer to this question, since we want $C_p[\mathbb{R}^n]$ to be a smooth manifold, we will define it by resolving the singularities of $\overline{\Sigma}$. The existence of a resolution of singularities is given by Hironaka's theorem [1964a; 1964b]. Our references on this matter are [Kollár 2007; Włodarczyk 2005]. See also [Hauser 2003] for a softer introduction to this theory.

Proposition 5.5 (resolution of singularities). *There exists a smooth manifold $C_p[\mathbb{R}^n]$ without boundary of dimension np and a smooth proper $\Pi : C_p[\mathbb{R}^n] \to (\mathbb{R}^n)^p \times \prod_{\varnothing \neq I \subset [\![1,p]\!]} \mathrm{Gr}_{|I|}(\mathbb{R}_{|I|-1}[X])$ such that*

(1) $\Pi(C_p[\mathbb{R}^n]) = \overline{\Sigma}$;

(2) $\Pi^{-1}(\Sigma)$ *is an open dense subset of* $C_p[\mathbb{R}^n]$;

(3) $\Pi_{|\Pi^{-1}(\Sigma)}$ *is $\mathcal{C}^\infty$-diffeomorphism from* $\Pi^{-1}(\Sigma)$ *onto* Σ.

Proof. We apply Hironaka's theorem [Kollár 2007, Theorem 3.27] to resolve the singularities of $\overline{\Sigma}$. Since $\overline{\Sigma}$ is algebraic by Lemma 5.3, there exists a smooth real-algebraic manifold $C_p[\mathbb{R}^n]$ and a projective morphism $\Pi : C_p[\mathbb{R}^n] \to \overline{\Sigma}$ such that Π is an isomorphism over the smooth locus of $\overline{\Sigma}$.

In particular $C_p[\mathbb{R}^n]$ is smooth, the map $\Pi : C_p[\mathbb{R}^n] \to (\mathbb{R}^n)^p \times \prod_{\varnothing \neq I \subset [\![1,p]\!]} \mathrm{Gr}_{|I|}(\mathbb{R}_{|I|-1}[X])$ is smooth and proper, and $\Pi(C_p[\mathbb{R}^n]) \subset \overline{\Sigma}$. Since Σ is contained in the smooth locus of $\overline{\Sigma}$, the restriction of Π to $\Pi^{-1}(\Sigma)$ is an isomorphism; in particular (3) is satisfied.

According to [Włodarczyk 2005, Theorem 1.0.2], the manifold $C_p[\mathbb{R}^n]$ and the projection Π are obtained by a sequence of blow-ups along smooth submanifolds that do not intersect the regular locus of $\overline{\Sigma}$, and hence Σ. This ensures that conditions (1) and (2) are satisfied. □

The following corollary proves the existence of the manifold $C_p[\mathbb{R}^n]$ and the proper surjection $\pi : C_p[\mathbb{R}^n] \to (\mathbb{R}^n)^p$ satisfying (1) in Theorem 1.1.

Corollary 5.6 (existence of the basis $C_p[\mathbb{R}^n]$). *There exists a smooth manifold $C_p[\mathbb{R}^n]$ without boundary of dimension np and a smooth proper surjection $\pi : C_p[\mathbb{R}^n] \to (\mathbb{R}^n)^p$ such that*

(1) *the open subset* $\pi^{-1}((\mathbb{R}^n)^p \setminus \Delta_p)$ *is dense in* $C_p[\mathbb{R}^n]$ *and* π *induces a* $\mathcal{C}^\infty$*-diffeomorphism from this set onto* $(\mathbb{R}^n)^p \setminus \Delta_p$;

(2) *for any nonempty* $I \subset [\![1, p]\!]$, *the map* $\mathcal{G}_I \circ \pi$ *admits a unique smooth extension to* $C_p[\mathbb{R}^n]$.

Proof. We consider $\Pi : C_p[\mathbb{R}^n] \to (\mathbb{R}^n)^p \times \prod_{\emptyset \neq I \subset [\![1,p]\!]} \mathrm{Gr}_{|I|}(\mathbb{R}_{|I|-1}[X])$ given by Proposition 5.5 and we let $\pi = \Pi_0 \circ \Pi$. Since Grassmannians are compact, Π_0 is proper. Hence π is smooth and proper because Π and Π_0 are. The surjectivity of π is given by Lemma 5.1 and (1) in Proposition 5.5.

Item (1) in Corollary 5.6 is a consequence of Lemmas 5.2 and 5.3 and of conditions (2) and (3) in Proposition 5.5. Let $I \subset [\![1, p]\!]$ be nonempty. On the dense open subset $\pi^{-1}((\mathbb{R}^n)^p \setminus \Delta_p)$ we have $\mathcal{G}_I \circ \pi = \Pi_I \circ \Pi$ by definition. In the last equality, the right-hand side is well-defined and smooth on $C_p[\mathbb{R}^n]$, which yields the unique extension we are looking for. □

Since it is defined using Hironaka's theorem, the manifold $C_p[\mathbb{R}^n]$ is not unique. However, the value of the smooth extension of $\mathcal{G}_I \circ \pi = \Pi_I \circ \Pi$ at $z \in C_p[\mathbb{R}^n]$ only depends on $\Pi(z) \in \overline{\Sigma}$. So this extension does not really depend on the choice of a resolution of singularities. In the following we choose once and for all a realization of $\pi : C_p[\mathbb{R}^n] \to (\mathbb{R}^n)^p$ as in Corollary 5.6. Thanks to (1), we can identify the configuration space $(\mathbb{R}^n)^p \setminus \Delta_p$ with its open dense preimage by π. Under this identification, (2) states that the maps $(\mathcal{G}_I)_{I \subset [\![1,p]\!]}$ extend smoothly to $C_p[\mathbb{R}^n]$. So, from now on, we consider $\mathcal{G}_I$ as a smooth map from $C_p[\mathbb{R}^n]$ to $\mathrm{Gr}_{|I|}(\mathbb{R}_{|I|-1}[X])$.

5.2. ***Definition of the bundle*** $\mathcal{MJ}_p(\mathbb{R}^n)$**.** Now that we have defined the base space $C_p[\mathbb{R}^n]$ of our multijet bundle, we can define the bundle itself. The purpose of this section is to construct the vector bundle $\mathcal{MJ}_p(\mathbb{R}^n) \to C_p[\mathbb{R}^n]$ of multijets for functions from $\mathbb{R}^n$ to $\mathbb{R}$, and the associated multijet map. The construction for vector-valued maps, explained in Section 5.4, is basically a fiberwise direct sum of this simpler case.

Recall that we defined the projections

$$C_p[\mathbb{R}^n] \xrightarrow{\Pi} \overline{\Sigma} \xrightarrow{\Pi_0} (\mathbb{R}^n)^p$$

and that $\pi = \Pi_0 \circ \Pi$. Thanks to Corollary 5.6, and under the identification discussed above, the map $\mathcal{G} = \mathcal{G}_{[\![1,p]\!]}$ defined by (4-1) extends as a smooth map from $C_p[\mathbb{R}^n]$ to $\mathrm{Gr}_p(\mathbb{R}_{p-1}[X])$. Seen as a collection of subspaces of $\mathbb{R}_{p-1}[X]$ indexed by $C_p[\mathbb{R}^n]$, this means that $\mathcal{G}$ defines a smooth vector sub-bundle of corank p in the trivial bundle $\mathbb{R}_{p-1}[X] \times C_p[\mathbb{R}^n] \to C_p[\mathbb{R}^n]$. We define our multijet bundle as the quotient of this trivial bundle by $\mathcal{G}$.

Definition 5.7 (vector bundle of multijets). Let $n \geqslant 1$ and $p \geqslant 1$. The *vector bundle of multijets of order* p *on* $\mathbb{R}^n$ is the smooth vector bundle of rank p over $C_p[\mathbb{R}^n]$ defined by

$$\mathcal{MJ}_p(\mathbb{R}^n) = (\mathbb{R}_{p-1}[X] \times C_p[\mathbb{R}^n])/\mathcal{G}.$$

In particular, for any $z \in C_p[\mathbb{R}^n]$, the fiber $\mathcal{MJ}_p(\mathbb{R}^n)_z = \mathbb{R}_{p-1}[X]/\mathcal{G}(z)$ only depends on $\Pi(z) \in \overline{\Sigma}$.

Recalling the definition of Kergin polynomials given in Section 3.2, we can now define the p-multijet of a $\mathcal{C}^{p-1}$ function on $\mathbb{R}^n$.

Definition 5.8 (multijet of a function). Let $f \in \mathcal{C}^{p-1}(\mathbb{R}^n)$ and $z \in C_p[\mathbb{R}^n]$. The *multijet of f at z* is the element of $\mathcal{MJ}_p(\mathbb{R}^n)_z$ defined as

$$\mathrm{mj}_p(f, z) = K(f, \pi(z)) \bmod \mathcal{G}(z).$$

In particular, as an element of $\mathbb{R}_{p-1}[X]/\mathcal{G}(z)$, the multijet $\mathrm{mj}_p(f, z)$ only depends on $\Pi(z) \in \overline{\Sigma}$.

Example 5.9. In Example 5.4 we saw that in simple cases $\overline{\Sigma}$ is smooth. In these cases we set $C_p[\mathbb{R}^n] = \overline{\Sigma}$ and we can describe the bundle $\mathcal{MJ}_p(\mathbb{R}^n) \to C_p[\mathbb{R}^n]$ and the map mj_p.

- If $p = 1$, then $C_1[\mathbb{R}^n] = \mathbb{R}^n$ and $\mathcal{G} : x \mapsto \{0\} \subset \mathbb{R}_0[X] \simeq \mathbb{R}$. Thus $\mathcal{MJ}_1(\mathbb{R}^n)$ is the trivial bundle $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$. Moreover, if $f \in \mathcal{C}^0(\mathbb{R}^n)$ then $K(f, x) = f(x) \in \mathbb{R}_0[X] \simeq \mathbb{R}$ and $\mathrm{mj}_1(f, x) = f(x)$ for all $x \in \mathbb{R}^n$.

- If $n = 1$, then $C_p[\mathbb{R}] = \mathbb{R}^p$ and $\mathcal{G} : \underline{x} \mapsto \{0\} \subset \mathbb{R}_{p-1}[X]$. Thus $\mathcal{MJ}_p(\mathbb{R})$ is the trivial bundle $\mathbb{R}_{p-1}[X] \times \mathbb{R}^p \to \mathbb{R}^p$. If $f \in \mathcal{C}^{p-1}(\mathbb{R})$ then $\mathrm{mj}_p(f, \underline{x}) = K(f, \underline{x})$ is the Hermite interpolating polynomial of f at $\underline{x}$; see Example 3.7.

Given $\underline{x} = (x_1, \dots, x_p) \notin \Delta_p$, Lemma 4.4 shows that $\mathrm{ev}_{\underline{x}} : \mathbb{R}_{p-1}[X] \to \mathbb{R}^p$ is an isomorphism. We can then consider the Lagrange basis $(L_i(\underline{x}))_{1 \leqslant i \leqslant p}$ of $\mathbb{R}_{p-1}[X]$ which is the preimage by $\mathrm{ev}_{\underline{x}}$ of the canonical basis of $\mathbb{R}^p$. We then have $\mathrm{mj}_p(f, \underline{x}) = K(f, \underline{x}) = \sum_{i=1}^p f(x_i) L_i(\underline{x})$. Geometrically, this means that the map $(P, \underline{x}) \mapsto (\mathrm{ev}_{\underline{x}}(P), \underline{x})$ defines a local trivialization of $\mathcal{MJ}_p(\mathbb{R}) \to C_p[\mathbb{R}]$ over $\mathbb{R}^p \setminus \Delta_p$ and that $\underline{x} \mapsto (L_i(\underline{x}))_{1 \leqslant i \leqslant p}$ is the corresponding frame. Moreover, it is tautological that $\mathrm{mj}_p(f, \underline{x})$ reads as $(f(x_1), \dots, f(x_p))$ in this trivialization.

In this example, one can also define a global trivialization of $\mathcal{MJ}_p(\mathbb{R})$ by considering the global frame of Newton polynomials $\underline{x} \mapsto (N_k(\underline{x}))_{1 \leqslant k \leqslant p}$, where $N_k(\underline{x}) = \prod_{1 \leqslant i < k} (X - x_i)$. By (3-3) we have $K(f, \underline{x}) = \sum_{k=1}^p f[x_1, \dots, x_k] N_k(\underline{x})$, so that $\mathrm{mj}_p(f, \underline{x})$ reads as $(f[x_1, \dots, x_k])_{1 \leqslant k \leqslant p}$ in this trivialization, where the divided differences are the classical ones in dimension 1. In this setting, we used in [Ancona and Letendre 2021] a strategy that can be roughly summarized as replacing $(f(x_i))_{1 \leqslant i \leqslant p}$ by $(f[x_1, \dots, x_k])_{1 \leqslant k \leqslant p}$. Our present point of view shows that we were actually considering $\mathrm{mj}_p(f, \underline{x})$ all along, but read in different trivializations.

- If $p = 2$, we saw that $C_2[\mathbb{R}^n] = \mathrm{Bl}_{\Delta_2}((\mathbb{R}^n)^2)$. Given $z \in C_2[\mathbb{R}^n]$, if $\pi(z) = (x_1, x_2) \notin \Delta_2$, we know that $\mathcal{G}(z) \subset \mathbb{R}_1[X]$ is the subspace of affine forms vanishing on the line $L_z \subset \mathbb{R}^n$ through x_1 and x_2. It is then natural to think of the class of P modulo $\mathcal{G}(z)$ as its restriction to L_z. Parametrizing L_z by

$$t \mapsto x_1 + \frac{x_2 - x_1}{\|x_2 - x_1\|} t,$$

one can check that

$$P \mapsto P\left(x_1 + \frac{x_2 - x_1}{\|x_2 - x_1\|} T\right)$$

induces an isomorphism $\mathcal{MJ}_2(\mathbb{R}^n)_z \simeq \mathbb{R}_1[T] \simeq \mathbb{R}^2$, where T is univariate and the second isomorphism is obtained by reading coordinates in the canonical basis $(1, T)$ of $\mathbb{R}_1[T]$.

Recalling Definition 3.1, we have

$$P[x_1, x_2] \cdot \frac{x_2 - x_1}{\|x_2 - x_1\|} T = \left(\int_0^1 D_{x_1 + t(x_2 - x_1)} P \cdot (x_2 - x_1) \,\mathrm{d}t\right) \frac{T}{\|x_2 - x_1\|} = \frac{P(x_2) - P(x_1)}{\|x_2 - x_1\|} T,$$

and $P = K(P, x_1, x_2)$ is given by (3-3). Letting $\widetilde{P}(z) = P(x_2) - P(x_1)/\|x_2 - x_1\|$, we have

$$P\left(x_1 + \frac{x_2 - x_1}{\|x_2 - x_1\|}T\right) = K(P, x_1, x_2)\left(x_1 + \frac{x_2 - x_1}{\|x_2 - x_1\|}T\right) = P(x_1) + \widetilde{P}(z)T.$$

Thus, the previous isomorphism $\mathcal{MJ}_2(\mathbb{R}^n)_z \to \mathbb{R}^2$ is $(P \mod \mathcal{G}(z)) \mapsto (P(\pi(z)_1), \widetilde{P}(z))$.

Let us now consider $z \in \pi^{-1}(\Delta_2)$. This exceptional divisor is the projectivized normal bundle of Δ_2 in $(\mathbb{R}^n)^2$. So we can think of z as a point in the diagonal, say $(x, x) \in \Delta_2$, and a line in $(\mathbb{R}^n)^2$ which is orthogonal to Δ_2, say spanned by $(u, -u)$ with $u \in \mathbb{S}^{n-1}$. Then $z = \lim_{\varepsilon\to 0}(x + \varepsilon u, x - \varepsilon u)$ in $C_2[\mathbb{R}^n]$. By continuity, $\mathcal{G}(z)$ is the space of affine forms vanishing on the line $L_z \subset \mathbb{R}^n$ parametrized by $t \mapsto x + tu$. As above, mapping P to the coefficients of $P(x + Tu) = P(x) + (D_x P \cdot u)T \in \mathbb{R}_1[T]$ induces an isomorphism $\mathcal{MJ}_2(\mathbb{R}^n)_z \to \mathbb{R}^2$. Letting $\widetilde{P}(z) = D_x P \cdot u$, this isomorphism is again $(P \mod \mathcal{G}(z)) \mapsto (P(\pi(z)_1), \widetilde{P}(z))$.

Actually, one can check that everything depends smoothly on the base point $z \in C_2[\mathbb{R}^n]$, so that the bundle map $(P \mod \mathcal{G}(z), z) \mapsto (P(\pi(z)_1), \widetilde{P}(z), z)$ defines a global trivialization $\mathcal{MJ}_2(\mathbb{R}^n) \to \mathbb{R}^2 \times C_2[\mathbb{R}^n]$. If $f \in \mathcal{C}^1(\mathbb{R}^n)$, with the same notation as above, $\mathrm{mj}_2(f, z)$ reads in this trivialization as

$$\left(f(x_1), \frac{f(x_2) - f(x_1)}{\|x_2 - x_1\|}\right)$$

if $z \notin \pi^{-1}(\Delta_2)$ and as $(f(x), D_x f \cdot u)$ otherwise.

In these examples, the multijet bundle $\mathcal{MJ}_p(\mathbb{R}^n) \to C_p[\mathbb{R}^n]$ is trivial. This raises the following.

Question. *Is $\mathcal{MJ}_p(\mathbb{R}^n) \to C_p[\mathbb{R}^n]$ trivial for all $n \geqslant 1$ and $p \geqslant 1$?*

The following two lemmas prove that the bundle map $\mathrm{mj}_p : \mathcal{C}^{p-1}(\mathbb{R}^n) \times C_p[\mathbb{R}^n] \to \mathcal{MJ}_p(\mathbb{R}^n)$ satisfies (2) and (3) in Theorem 1.1.

Lemma 5.10 (regularity of multijets). *The map $\mathrm{mj}_p(\,\cdot\,, z) : \mathcal{C}^{p-1}(\mathbb{R}^n) \to \mathcal{MJ}_p(\mathbb{R}^n)_z$ is a linear surjection for all $z \in C_p[\mathbb{R}^n]$. Additionally, let $l \geqslant 0$ and let $f \in \mathcal{C}^{l+p-1}(\mathbb{R}^n)$. Then $\mathrm{mj}_p(f, \cdot\,)$ is a section of class $\mathcal{C}^l$ of $\mathcal{MJ}_p(\mathbb{R}^n) \to C_p[\mathbb{R}^n]$.*

Proof. Let $z \in C_p[\mathbb{R}^n]$. The map $K(\,\cdot\,, \pi(z)) : \mathcal{C}^{p-1}(\mathbb{R}^n) \to \mathbb{R}_{p-1}[X]$ is linear by Lemma 3.8. It is also surjective since its restriction to $\mathbb{R}_{p-1}[X]$ is the identity. Since $\mathrm{mj}_p(\,\cdot\,, z)$ is the composition of $K(\,\cdot\,, \pi(z))$ with the canonical projection from $\mathbb{R}_{p-1}[X]$ onto $\mathcal{MJ}_p(\mathbb{R}^n)_z$, it is a linear surjection.

Let $l \geqslant 0$ and let $f \in \mathcal{C}^{l+p-1}(\mathbb{R}^n)$. By Lemma 3.8, we have $K(f, \cdot\,) \in \mathcal{C}^l((\mathbb{R}^n)^p, \mathbb{R}_{p-1}[X])$. Since π is smooth, we get $K(f, \cdot\,) \circ \pi \in \mathcal{C}^l(C_p[\mathbb{R}^n], \mathbb{R}_{p-1}[X])$. In other words, $K(f, \cdot\,) \circ \pi$ defines a section of class $\mathcal{C}^l$ of the trivial bundle $\mathbb{R}_{p-1}[X] \times C_p[\mathbb{R}^n] \to C_p[\mathbb{R}^n]$. Since $\mathcal{G}$ is a smooth sub-bundle of $\mathbb{R}_{p-1}[X] \times C_p[\mathbb{R}^n]$, projecting onto the quotient bundle $\mathcal{MJ}_p(\mathbb{R}^n)$ does not decrease the regularity. $\square$

Lemma 5.11 (multijets and evaluation). *Let $z \in C_p[\mathbb{R}^n]$ be such that $\pi(z) = (x_1, \ldots, x_p) \notin \Delta_p$. Then for all $f \in \mathcal{C}^{p-1}(\mathbb{R}^n)$ we have $\mathrm{mj}_p(f, z) = 0$ if and only if, for all $i \in [\![1, p]\!]$, $f(x_i) = 0$.*

Proof. Let us denote by $\underline{x} = (x_1, \ldots, x_p) = \pi(z) \notin \Delta_p$. For all $f \in \mathcal{C}^{p-1}(\mathbb{R}^n)$, we have

$$\mathrm{mj}_p(f, z) = 0 \iff K(f, \underline{x}) \in \mathcal{G}(\underline{x}) \iff \mathrm{ev}_{\underline{x}}(K(f, \underline{x})) = 0 \iff \mathrm{ev}_{\underline{x}}(f) = 0,$$

since $K(f, \underline{x})$ interpolates the values of f on at $x_1, \ldots, x_p$ (see Remark 3.5). $\square$

Actually, we can describe more precisely the relation between multijets and evaluation outside of the diagonal. This will appear in the proof of Theorem 6.26 below. Thanks to Lemma 4.4, the smooth bundle map $\mathrm{ev} : (P, \underline{x}) \mapsto (\mathrm{ev}_{\underline{x}}(P), \underline{x})$ from $\mathbb{R}_{p-1}[X] \times ((\mathbb{R}^n)^p \setminus \Delta_p)$ to $\mathbb{R}^p \times ((\mathbb{R}^n)^p \setminus \Delta_p)$ is surjective and its kernel is exactly the sub-bundle $\ker \mathrm{ev} = \mathcal{G}$. Thus it induces a smooth bundle map $\tau : \mathcal{MJ}_p(\mathbb{R}^n)_{|(\mathbb{R}^n)^p \setminus \Delta_p} \to \mathbb{R}^p \times ((\mathbb{R}^n)^p \setminus \Delta_p)$ defined by $\tau(P \mod \mathcal{G}(\underline{x})) = (\mathrm{ev}_{\underline{x}}(P), \underline{x})$, which is bijective. Thus τ defines a smooth local trivialization of $\mathcal{MJ}_p(\mathbb{R}^n)$ over $(\mathbb{R}^n)^p \setminus \Delta_p$. Moreover, for all $f \in \mathcal{C}^{p-1}(\mathbb{R}^n)$ and $z \in C_p[\mathbb{R}^n]$ such that $\underline{x} = \pi(z) \notin \Delta_p$ we have

$$\tau(\mathrm{mj}_p(f, z)) = \tau(K(f, \underline{x}) \mod \mathcal{G}(\underline{x})) = (\mathrm{ev}_{\underline{x}}(K(f, \underline{x})), \underline{x}) = (\mathrm{ev}_{\underline{x}}(f), \underline{x}).$$

Hence $\mathrm{mj}_p(f, z)$ simply reads as $(f(x_1), \dots, f(x_p))$ in this trivialization.

5.3. ***Localness of multijets.*** The goal of this section is to prove that the multijet bundle $\mathcal{MJ}_p(\mathbb{R}^n) \to C_p[\mathbb{R}^n]$ defined in the previous section satisfies (4) in Theorem 1.1. Let $z \in C_p[\mathbb{R}^n]$, let $\underline{x} = \pi(z)$ and let $\mathcal{I} = \mathcal{I}(\underline{x})$ be as in Definition 2.3. As explained in Section 2.1, there is a unique $\underline{y} = (y_I)_{I \in \mathcal{I}} \in \mathbb{R}^{\mathcal{I}} \setminus \Delta_{\mathcal{I}}$ such that $\underline{x} = \iota_{\mathcal{I}}(\underline{y})$. Recalling that we dropped $V = \mathbb{R}$ from the notation in the present case, we can restate (4) in Theorem 1.1 as: there exists $\Theta_z : \prod_{I \in \mathcal{I}} \mathcal{J}_{|I|-1}(\mathbb{R}^n)_{y_I} \to \mathcal{MJ}_p(\mathbb{R}^n)_z$ a linear surjection such that $\mathrm{mj}_p(f, z) = \Theta_z((\mathrm{j}_{|I|-1}(f, y_I))_{I \in \mathcal{I}})$ for all $f \in \mathcal{C}^{p-1}(\mathbb{R}^n)$.

This property is fundamental. First, it shows that $\mathrm{mj}_p(f, z)$ is obtained by patching together (part of) the jets of order $|I| - 1$ of f at y_I, which justifies the name multijet. Second, it shows that $\mathrm{mj}_p(f, z)$ only depends on the values of f in arbitrarily small neighborhoods of the y_I. This is not obvious at all since the definition of $\mathrm{mj}_p(f, z)$ involves divided differences of f, which are obtained by integrating on the whole convex hull $\sigma(\underline{x})$ of the x_i (see Definition 3.1). In particular, it shows that $\mathrm{mj}_p(f, z)$ makes sense even if f is only $\mathcal{C}^{|I|-1}$ in some neighborhood of y_I for all $I \in \mathcal{I}$. Hence Definition 1.3 makes sense even if Ω is not convex.

In the following, we consider what we think of as the I-th part of a multijet, where $I \subset [\![1, p]\!]$. This is just a variation on what we did in Definitions 5.7 and 5.8 and it is defined as follows.

Definition 5.12 (I-multijets). Let $n \geqslant 1$ and $p \geqslant 1$ and let $I \subset [\![1, p]\!]$ be nonempty, we define the bundle of *I-multijets* as the following smooth bundle of rank $|I|$ over $C_p[\mathbb{R}^n]$:

$$\mathcal{MJ}_I(\mathbb{R}^n) = (\mathbb{R}_{|I|-1}[X] \times C_p[\mathbb{R}^n])/\mathcal{G}_I.$$

Let $f \in \mathcal{C}^{|I|-1}(\mathbb{R}^n)$ and $z \in C_p[\mathbb{R}^n]$. We define by $\mathrm{mj}_I(f, z) = K(f, \pi(z)_I) \mod \mathcal{G}_I(z) \in \mathcal{MJ}_I(\mathbb{R}^n)_z$ the *I-multijet* of f at z.

As explained in Section 5.1, for all $\varnothing \neq I \subset [\![1, p]\!]$ we have a map $\mathcal{G}_I : C_p[\mathbb{R}^n] \to \mathrm{Gr}_{|I|}(\mathbb{R}_{|I|-1}[X])$ extending the one on $(\mathbb{R}^n)^p \setminus \Delta_p$. Let $z \in C_p[\mathbb{R}^n]$ and $\underline{x} = \pi(z)$. As in Section 4 we define $\widetilde{\mathcal{G}}_I(z) = K(\cdot, \underline{x}_I)^{-1}(\mathcal{G}_I(z)) \in \mathrm{Gr}_{|I|}(\mathbb{R}_{p-1}[X])$, where $K(\cdot, \underline{x}_I) : \mathbb{R}_{p-1}[X] \to \mathbb{R}_{|I|-1}[X]$. Note that $\widetilde{\mathcal{G}}_I(z)$ has the same codimension as $\mathcal{G}_I(z)$ since $K(\cdot, \underline{x}_I)$ is surjective.

Lemma 5.13 (compatibility of the $\mathcal{G}_I$). *For all $I \subset [\![1, p]\!]$ and $z \in C_p[\mathbb{R}^n]$ we have $\mathcal{G}(z) \subset \widetilde{\mathcal{G}}_I(z)$.*

Proof. Recall from Section 4 that $\mathcal{G}(z) \subset \widetilde{\mathcal{G}}_I(z)$ for any $z \in (\mathbb{R}^n)^p \setminus \Delta_p \subset C_p[\mathbb{R}^n]$, that is, $K(\cdot, \pi(z)_I)(\mathcal{G}(z)) \subset \mathcal{G}_I(z)$. This incidence relation is a closed condition. By construction, the subset $(\mathbb{R}^n)^p \setminus \Delta_p$ is dense in $C_p[\mathbb{R}^n]$ and both terms in the previous inclusion are continuous with respect to z; see Lemma 3.8 and Corollary 5.6. Hence the inclusion actually holds for any $z \in C_p[\mathbb{R}^n]$. Thus $\mathcal{G}(z) \subset \widetilde{\mathcal{G}}_I(z)$ for all $z \in C_p[\mathbb{R}^n]$. □

Let $\varnothing \neq I \subset [\![1, p]\!]$, let $z \in C_p[\mathbb{R}^n]$ and $\underline{x} = \pi(z)$. We consider $\mathrm{mj}_I(\,\cdot\,, z) : \mathbb{R}_{p-1}[X] \to \mathcal{MJ}_I(\mathbb{R}^n)_z$ from Definition 5.12. This linear map is surjective as the composition of $K(\,\cdot\,, \underline{x}_I)$ and the projection modulo $\mathcal{G}_I(z)$. Moreover, $\ker(\mathrm{mj}_I(\,\cdot\,, z)) = \widetilde{\mathcal{G}}_I(z)$ contains $\mathcal{G}(z)$ by Lemma 5.13. Hence, $\mathrm{mj}_I(\,\cdot\,, z)$ induces a surjective linear map from $\mathcal{MJ}_p(\mathbb{R}^n)_z = \mathbb{R}_{p-1}[X]/\mathcal{G}(z)$ onto $\mathcal{MJ}_I(\mathbb{R}^n)_z$ that we still denote by $\mathrm{mj}_I(\,\cdot\,, z)$. This is summarized in the following commutative diagram, where the vertical arrows are the canonical projections and all arrows are surjective:

$$\begin{tikzcd} \mathbb{R}_{p-1}[X] \arrow[r, "K(\cdot\,,\underline{x}_I)"] \arrow[d] \arrow[dr, "\mathrm{mj}_I(\cdot\,,z)"] & \mathbb{R}_{|I|-1}[X] \arrow[d] \\ \mathcal{MJ}_p(\mathbb{R}^n)_z \arrow[r, "\mathrm{mj}_I(\cdot\,,z)"] & \mathcal{MJ}_I(\mathbb{R}^n)_z \end{tikzcd} \tag{5-2}$$

Note that $(P, z) \mapsto (K(P, \pi(z)_I), z)$ is a smooth bundle map over $C_p[\mathbb{R}^n]$ from $\mathbb{R}_{p-1}[X] \times C_p[\mathbb{R}^n]$ to $\mathbb{R}_{|I|-1}[X] \times C_p[\mathbb{R}^n]$. Hence, the previous diagram (5-2) defines a smooth surjective bundle map $\mathrm{mj}_I : (P \mod \mathcal{G}(z)) \mapsto (P \mod \mathcal{G}_I(z))$ from $\mathcal{MJ}_p(\mathbb{R}^n)$ to $\mathcal{MJ}_I(\mathbb{R}^n)$ over $C_p[\mathbb{R}^n]$.

Definition 5.14 (partitioned multijet). For all $\mathcal{I} \in \mathcal{P}_p$ and $z \in C_p[\mathbb{R}^n]$, we define a linear map from $\mathcal{MJ}_p(\mathbb{R}^n)_z$ to $\prod_{I\in\mathcal{I}} \mathcal{MJ}_I(\mathbb{R}^n)_z$ by $\mathrm{mj}_{\mathcal{I}}(\,\cdot\,, z) : \alpha \mapsto (\mathrm{mj}_I(\alpha, z))_{I\in\mathcal{I}}$.

As above, $\mathrm{mj}_{\mathcal{I}} : (\alpha, z) \mapsto \mathrm{mj}_{\mathcal{I}}(\alpha, z)$ defines a smooth bundle map over $C_p[\mathbb{R}^n]$ from $\mathcal{MJ}_p(\mathbb{R}^n)$ to $\bigoplus_{I\in\mathcal{I}} \mathcal{MJ}_I(\mathbb{R}^n)$, which is obtained as the quotient of $(P, z) \mapsto ((K(P, \pi(z)_I))_{I\in\mathcal{I}}, z)$. However, $\mathrm{mj}_{\mathcal{I}}(\,\cdot\,, z)$ is not surjective in general. The following lemma proves its surjectivity in some cases.

Lemma 5.15 (splitting of multijets). *Let $z \in C_p[\mathbb{R}^n]$, let $\underline{x} = \pi(z)$ and let $\mathcal{I}(\underline{x})$ be defined as in Definition 2.3. Then $\mathrm{mj}_{\mathcal{I}(\underline{x})}(\,\cdot\,, z) : \mathcal{MJ}_p(\mathbb{R}^n)_z \to \prod_{I\in\mathcal{I}(\underline{x})} \mathcal{MJ}_I(\mathbb{R}^n)_z$ is an isomorphism.*

Proof. The map $\mathrm{mj}_{\mathcal{I}(\underline{x})}(\,\cdot\,, z)$ is linear between two spaces of the same dimension $p = \sum_{I\in\mathcal{I}(\underline{x})} |I|$, so it is enough to prove its surjectivity. Let $(\alpha_I)_{I\in\mathcal{I}(\underline{x})} \in \prod_{I\in\mathcal{I}(\underline{x})} \mathcal{MJ}_I(\mathbb{R}^n)_z$. For each $I \in \mathcal{I}(\underline{x})$ there exists $P_I \in \mathbb{R}_{|I|-1}[X]$ such that $\alpha_I = P_I \mod \mathcal{G}_I(z)$. By Lemma 3.9, there exists $P \in \mathbb{R}_{p-1}[X]$ such that $K(P, \underline{x}_I) = P_I$ for all $I \in \mathcal{I}(\underline{x})$. Let $\alpha = P \mod \mathcal{G}(z) \in \mathcal{MJ}_p(\mathbb{R}^n)_z$. Then, for all $I \in \mathcal{I}(\underline{x})$, we have

$$\mathrm{mj}_I(\alpha, z) = \mathrm{mj}_I(P, z) = K(P, \underline{x}_I) \mod \mathcal{G}_I(z) = P_I \mod \mathcal{G}_I(z) = \alpha_I.$$

Hence $\mathrm{mj}_{\mathcal{I}(\underline{x})}(\alpha, z) = (\alpha_I)_{I\in\mathcal{I}(\underline{x})}$, and $\mathrm{mj}_{\mathcal{I}(\underline{x})}(\,\cdot\,, z)$ is indeed surjective. □

Let $k \in \mathbb{N}$ and let $x \in \mathbb{R}^n$. By definition, two maps f and $g \in \mathcal{C}^k(\mathbb{R}^n)$ have the same k-jet at x if and only if they have the same Taylor polynomial of order k at x. Let $\underline{x} = (x, \dots, x)$. Recalling Example 3.7, the linear map $K(\,\cdot\,, \underline{x}) : \mathcal{C}^k(\mathbb{R}^n) \to \mathbb{R}_k[X]$ is surjective, and it induces an isomorphism $\mathcal{J}_k(\mathbb{R}^n)_x \simeq \mathbb{R}_k[X]$.

Let $z \in C_p[\mathbb{R}^n]$, let $\underline{x} = \pi(z)$, let $\mathcal{I} = \mathcal{I}(\underline{x})$ and let $(y_I)_{I\in\mathcal{I}} = \iota_{\mathcal{I}}^{-1}(\underline{x})$; see Definitions 2.3 and 2.6. For all $I \in \mathcal{I}$, the canonical isomorphism $\mathcal{J}_{|I|-1}(\mathbb{R}^n)_{y_I} \simeq \mathbb{R}_{|I|-1}[X]$ allows us to see the projection from $\mathbb{R}_{|I|-1}[X]$ onto $\mathcal{MJ}_I(\mathbb{R}^n)_z$ as a canonical linear surjection $\varpi_{z,I} : \mathcal{J}_{|I|-1}(\mathbb{R}^n)_{y_I} \to \mathcal{MJ}_I(\mathbb{R}^n)_z$.

Definition 5.16 (gluing map). Let $z \in C_p[\mathbb{R}^n]$, let $\underline{x} = \pi(z)$ and let $(y_I)_{I\in\mathcal{I}} = \iota_{\mathcal{I}}^{-1}(\underline{x})$, where $\mathcal{I} = \mathcal{I}(\underline{x})$. We define $\varpi_z : (\alpha_I)_{I\in\mathcal{I}} \mapsto (\varpi_{z,I}(\alpha_I))_{I\in\mathcal{I}}$ from $\prod_{I\in\mathcal{I}} \mathcal{J}_{|I|-1}(\mathbb{R}^n)_{y_I}$ to $\prod_{I\in\mathcal{I}} \mathcal{MJ}_I(\mathbb{R}^n)_z$. We also define $\Theta_z = \mathrm{mj}_{\mathcal{I}}(\,\cdot\,, z)^{-1} \circ \varpi_z$.

We can now check that Θ_z satisfies (4) in Theorem 1.1.

Lemma 5.17 (localness of multijets). *For all $z \in C_p[\mathbb{R}^n]$, the map Θ_z is a linear surjection from $\prod_{I\in\mathcal{I}} \mathcal{J}_{|I|-1}(\mathbb{R}^n)_{y_I}$ to $\mathcal{MJ}_p(\mathbb{R}^n)_z$. Moreover, it is the only map such that*

$$\forall f \in \mathcal{C}^{p-1}(\mathbb{R}^n), \quad \Theta_z((\mathrm{j}_{|I|-1}(f, y_I))_{I\in\mathcal{I}}) = \mathrm{mj}_p(f, z).$$

Proof. With the same notation as in Definition 5.16, for all $I \in \mathcal{I}$ the map $\varpi_{z,I}$ is a linear surjection by definition. Hence so is ϖ_z. Since $\mathrm{mj}_{\mathcal{I}}(\,\cdot\,, z)$ is an isomorphism by Lemma 5.15, the map $\Theta_z = \mathrm{mj}_{\mathcal{I}}(\,\cdot\,, z)^{-1} \circ \varpi_z$ is also a linear surjection.

Let $f \in \mathcal{C}^{p-1}(\mathbb{R}^n)$. For all $I \in \mathcal{I}$, the image of $\mathrm{j}_{|I|-1}(f, y_I)$ under the canonical isomorphism $\mathcal{J}_{|I|-1}(\mathbb{R}^n)_{y_I} \simeq \mathbb{R}_{|I|-1}[X]$ is the Taylor polynomial $K(f, \underline{x}_I)$. Hence

$$\varpi_{z,I}(\mathrm{j}_{|I|-1}(f, y_I)) = K(f, \underline{x}_I) \bmod \mathcal{G}_I(z) = \mathrm{mj}_I(K(f, \underline{x}_I), z).$$

Thus, we have

$$\varpi_z((\mathrm{j}_{|I|-1}(f, y_I))_{I\in\mathcal{I}}) = (\mathrm{mj}_I(K(f, \underline{x}_I), z))_{I\in\mathcal{I}} = \mathrm{mj}_{\mathcal{I}}(\mathrm{mj}_p(f, z), z),$$

and finally

$$\Theta_z((\mathrm{j}_{|I|-1}(f, y_I))_{I\in\mathcal{I}}) = \mathrm{mj}_p(f, z).$$

Since the y_I are pairwise distinct, every element of $\prod_{I\in\mathcal{I}} \mathcal{J}_{|I|-1}(\mathbb{R}^n)_{y_I}$ can be realized as $(\mathrm{j}_{|I|-1}(f, y_I))_{I\in\mathcal{I}}$ for some $f \in \mathcal{C}^\infty(\mathbb{R}^n)$. Hence the previous relation completely defines Θ_z. ☐

5.4. *Multijets of vector-valued maps.* So far we have only defined multijets of real-valued functions. In this section, we extend the previous construction to maps from $\mathbb{R}^n$ to some vector space V of dimension $r \geqslant 1$. Let $\pi : C_p[\mathbb{R}^n] \to (\mathbb{R}^n)^p$ be given by Corollary 5.6 as before. We define $\mathcal{MJ}_p(\mathbb{R}^n, V) \to C_p[\mathbb{R}^n]$ as the tensor product of the bundle $\mathcal{MJ}_p(\mathbb{R}^n) \to C_p[\mathbb{R}^n]$ from Definition 5.7 with the trivial bundle $V \times C_p[\mathbb{R}^n] \to C_p[\mathbb{R}^n]$.

Definition 5.18 (multijet bundle of vector-valued maps). Let $n \geqslant 1$ and $p \geqslant 1$; let V be a real vector space of dimension $r \geqslant 1$. We define the *bundle of p-multijets of V-valued maps on* $\mathbb{R}^n$ as the following smooth bundle of rank pr over $C_p[\mathbb{R}^n]$:

$$\mathcal{MJ}_p(\mathbb{R}^n, V) = \mathcal{MJ}_p(\mathbb{R}^n) \otimes V.$$

Definition 5.19 (multijet of a map). Let $(v_1, \ldots, v_r)$ denote a basis of V. Let $z \in C_p[\mathbb{R}^n]$ and let $f = \sum_{i=1}^r f_i v_i \in \mathcal{C}^{p-1}(\mathbb{R}^n, V)$. We define by $\mathrm{mj}_p(f, z) = \sum_{i=1}^r \mathrm{mj}_p(f_i, z) \otimes v_i \in \mathcal{MJ}_p(\mathbb{R}^n, V)_z$ the p-multijet of f at z.

Lemma 5.20 (independence from the basis). *In Definition 5.19, the vector* $\mathrm{mj}_p(f, z)$ *does not depend on the choice of the basis* $(v_1, \ldots, v_r)$.

Proof. Let $(w_1, \dots, w_r)$ be another basis of V. There exists a matrix $(a_{ij})_{1\leqslant i,j\leqslant r} \in \mathrm{GL}_r(\mathbb{R})$ such that $v_i = \sum_{j=1}^r a_{ij} w_j$ for all $i \in [\![1, r]\!]$. Letting $g_j = \sum_{i=1}^r a_{ij} f_i$ for all $j \in [\![1, r]\!]$, we get

$$f = \sum_{i=1}^{r} f_i v_i = \sum_{1\leqslant i,j\leqslant r} a_{ij} f_i w_j = \sum_{j=1}^{r} g_j w_j.$$

Then, by linearity of the p-multijet for functions, we have

$$\sum_{j=1}^{r} \mathrm{mj}_p(g_j, z) \otimes w_j = \sum_{1\leqslant i,j\leqslant r} a_{ij}\, \mathrm{mj}_p(f_i, z) \otimes w_j = \sum_{i=1}^{r} \mathrm{mj}_p(f_i, z) \otimes v_i. \qquad \square$$

Example 5.21. If $V = \mathbb{R}^r$, then for all $z \in C_p[\mathbb{R}^n]$ and $f = (f_1, \dots, f_r) \in \mathcal{C}^{p-1}(\mathbb{R}^n, \mathbb{R}^r)$ we have

$$\mathcal{MJ}_p(\mathbb{R}^n, \mathbb{R}^r)_z = \mathcal{MJ}_p(\mathbb{R}^n)_z \otimes \mathbb{R}^r \simeq (\mathcal{MJ}_p(\mathbb{R}^n)_z)^r,$$

and under this canonical isomorphism $\mathrm{mj}_p(f, z) = (\mathrm{mj}_p(f_i, z))_{1\leqslant i\leqslant r}$, as one would expect.

Let $x \in \mathbb{R}^n$ and $k \in \mathbb{N}$. We have a canonical isomorphism $\mathcal{J}_k(\mathbb{R}^n, V)_x \simeq \mathcal{J}_k(\mathbb{R}^n)_x \otimes V$. If $(v_1, \dots, v_r)$ is a basis of V, this isomorphism is totally determined by the fact that the image of $\mathrm{j}_k(f, x)$ is $\sum_{i=1}^r \mathrm{j}_k(f_i, x) \otimes v_i$ for all $f = \sum_{i=1}^r f_i v_i \in \mathcal{C}^k(\mathbb{R}^n, V)$. As in the proof of Lemma 5.20, this does not depend on the choice of the basis $(v_1, \dots, v_r)$.

Definition 5.22 (gluing map for vector-valued multijets). Let $z \in C_p[\mathbb{R}^n]$, let $\underline{x} = \pi(z)$, let $\mathcal{I} = \mathcal{I}(\underline{x})$ and let $(y_I)_{I\in\mathcal{I}} = \iota_{\mathcal{I}}^{-1}(\underline{x})$. Using the previous canonical isomorphisms, we have

$$\prod_{I\in\mathcal{I}} \mathcal{J}_{|I|-1}(\mathbb{R}^n, V)_{y_I} = \prod_{I\in\mathcal{I}} \mathcal{J}_{|I|-1}(\mathbb{R}^n)_{y_I} \otimes V = \left(\prod_{I\in\mathcal{I}} \mathcal{J}_{|I|-1}(\mathbb{R}^n)_{y_I}\right) \otimes V.$$

Recalling Definition 5.16, we define a linear map

$$\Theta_z : \prod_{I\in\mathcal{I}} \mathcal{J}_{|I|-1}(\mathbb{R}^n, V)_{y_I} \to \mathcal{MJ}_p(\mathbb{R}^n, V)_z$$

by $\Theta_z(\alpha \otimes v) = \Theta_z(\alpha) \otimes v$ for all $\alpha \in \prod_{I\in\mathcal{I}} \mathcal{J}_{|I|-1}(\mathbb{R}^n)_{y_I}$ and all $v \in V$.

We now have everything we need to prove Theorem 1.1.

Proof of Theorem 1.1. The base space $C_p[\mathbb{R}^n]$ and the projection π are given by Corollary 5.6. In particular, they satisfy (1) in Theorem 1.1. Definition 5.19 and Lemma 5.10 show that mj_p satisfies (2). Similarly, (3) is satisfied thanks to Lemma 5.11 and the definition of mj_p for V-valued maps.

Let us check that the linear maps Θ_z from Definition 5.22 satisfy (4). Let $z \in C_p[\mathbb{R}^n]$, let $\underline{x} = \pi(z)$, let $\mathcal{I} = \mathcal{I}(\underline{x})$ and let $(y_I)_{I\in\mathcal{I}} = \iota_{\mathcal{I}}^{-1}(\underline{x})$. Let us also denote by $(v_1, \dots, v_r)$ a basis of V. Let $\alpha \in \mathcal{MJ}_p(\mathbb{R}^n, V)_z$. There exists $\alpha_1, \dots, \alpha_r \in \mathcal{MJ}_p(\mathbb{R}^n)_z$ such that $\alpha = \sum_{i=1}^r \alpha_i \otimes v_i$. By Lemma 5.17, for each $i \in [\![1, r]\!]$, there exists $\beta_i \in \prod_{I\in\mathcal{I}} \mathcal{J}_{|I|-1}(\mathbb{R}^n)_{y_I}$ such that $\alpha_i = \Theta_z(\beta_i)$. Hence,

$$\Theta_z\left(\sum_{i=1}^{r} \beta_i \otimes v_i\right) = \sum_{i=1}^{r} \Theta_z(\beta) \otimes v_i = \alpha,$$

and Θ_z is indeed surjective.

Finally, let us consider $f = \sum_{i=1}^{r} f_i v_i \in \mathcal{C}^{p-1}(\mathbb{R}^n, V)$. Then, by Lemma 5.17 once again,

$$\begin{aligned}\Theta_z((\mathrm{j}_{|I|-1}(f, y_I))_{I\in\mathcal{I}}) &= \Theta_z\left(\sum_{i=1}^{r}(\mathrm{j}_{|I|-1}(f_i, y_I))_{I\in\mathcal{I}} \otimes v_i\right) = \sum_{i=1}^{r}\Theta_z((\mathrm{j}_{|I|-1}(f_i, y_I))_{I\in\mathcal{I}}) \otimes v_i \\ &= \sum_{i=1}^{r}\mathrm{mj}_p(f_i, z) \otimes v_i = \mathrm{mj}_p(f, z).\end{aligned}$$ □

Remark 5.23. Another way to define $\mathcal{MJ}_p(\mathbb{R}^n, V)$ and mj_p is the following. If $f : \mathbb{R}^n \to V$ is regular enough, then the divided differences from Definition 3.1 still make sense, only this time $f[x_0, \dots, x_k] \in \mathrm{Sym}^k(\mathbb{R}^n) \otimes V$. Then, one can still define $K(f, \underline{x})$ as in Proposition 3.4, and it defines an element of $\mathbb{R}_{p-1}[X] \otimes V$ that interpolates the divided differences of f. Similarly, everything we did from Section 3.1 to Section 5.3 can be adapted to the case of V-valued maps, simply by tensoring each vector space by V, and each linear map by Id_V. One can check that we recover the same objects as in Definitions 5.18 and 5.19, up to canonical isomorphisms.

6. Application to zeros of Gaussian fields

This section is concerned with our application of multijet bundles to Gaussian fields. In Section 6.1, we describe the local model for the Gaussian fields with values in a vector bundle that we consider. In Section 6.2, we prove a Bulinskaya-type lemma and a Kac–Rice formula for the zeros of these fields. Section 6.3 is dedicated to the definition of the Kac–Rice densities of order larger than 2. We also relate the properties of these functions with the moments of the linear statistics associated with our field. Finally, we prove Theorems 1.6 and 1.9 in Section 6.4, using the multijet bundles defined in Theorem 1.1.

6.1. ***Gaussian vectors and Gaussian sections.*** In this section, we briefly recall some notation and conventions concerning Gaussian vectors. Then we describe the local model for Gaussian fields with values in a vector bundle. We will mostly consider centered random vectors in finite-dimensional vector spaces, so we restrict ourselves to this setting. In the following, V is a finite-dimensional real vector space.

Definition 6.1 (Gaussian vector). We say that a random vector X with values in V is a *centered Gaussian vector* if, for all $\eta \in V^*$, the real random variable $\eta(X)$ is a centered Gaussian in $\mathbb{R}$.

In particular, a centered Gaussian vector in V has finite moments up to any order. Let us assume that V is endowed with an inner product $\langle \cdot\,, \cdot \rangle$. Then for all $v \in V$, we define $v^* = \langle v, \cdot \rangle \in V^*$.

Definition 6.2 (variance operator). Let X be a centered Gaussian vector in $(V, \langle \cdot\,, \cdot \rangle)$. Then its *variance operator* is the nonnegative self-adjoint endomorphism $\mathrm{Var}(X) = \mathbb{E}[X \otimes X^*]$ of V. We say that X is *nondegenerate* if $\mathrm{Var}(X)$ is invertible.

Recall that a centered Gaussian vector in $(V, \langle \cdot\,, \cdot \rangle)$ is completely determined by its variance. In the following, we denote by $\mathcal{N}(0, \Lambda)$ the centered Gaussian distribution of variance Λ, and by $X \sim \mathcal{N}(0, \Lambda)$ the fact that X follows this distribution.

Definition 6.3 (Gaussian field). Let $E \to M$ be a vector bundle over some manifold M. We say that a random section s of $E \to M$ is a *centered Gaussian field* if for all $m \geqslant 1$ and all $x_1, \dots, x_m$ the random vector $(s(x_1), \dots, s(x_m))$ is a centered Gaussian. We say that this field is *nondegenerate* if $s(x)$ is nondegenerate for all $x \in M$.

If the centered Gaussian field s is $\mathcal{C}^p$, then its jet $\mathrm{j}_k(s, x)$ is a centered Gaussian for all $x \in M$. Thus, the definition of p-nondegeneracy of the field makes sense; see Definition 1.8. Note that 0-nondegenerate simply means nondegenerate.

Since this will appear in several places later on, let us describe the local model for Gaussian fields in this context. Let $x_0 \in M$. There exists a chart (U, φ) of M around x_0. That is $\varphi : U \to \Omega$ is a diffeomorphism between an open neighborhood U of x_0 and an open subset $\Omega \subset \mathbb{R}^n$. Up to reducing U, we can assume that E is trivial over U, i.e., there exists a trivialization $\tau : E_{|U} \to \mathbb{R}^r \times U$. Letting $\tau_\varphi = (\mathrm{Id}, \varphi) \circ \tau$, we have the following commutative diagram, where arrows on the top row are bundle maps covering the maps on the bottom row:

$$\begin{tikzcd}
E_{|U} \arrow[r, "\tau"] \arrow[d] \arrow[rr, bend left, "\tau_\varphi"] & \mathbb{R}^r \times U \arrow[r, "(\mathrm{Id},\varphi)"] \arrow[d] & \mathbb{R}^r \times \Omega \arrow[d] \\
U \arrow[r, "\mathrm{Id}"] & U \arrow[r, "\varphi"] & \Omega
\end{tikzcd} \tag{6-1}$$

Let s be a local section of $E_{|U}$. Then $\tau_\varphi \circ s \circ \varphi^{-1}$ is a section of the trivial bundle on the right-hand side of (6-1). Hence there exists a map $f : \Omega \to \mathbb{R}^r$ such that $\tau_\varphi \circ s \circ \varphi^{-1} = (f, \mathrm{Id})$. For all $x \in \Omega$, the vector $f(x)$ is the image of $s(\varphi^{-1}(x))$ by a linear bijection. Thus, if $s : M \to E$ is a centered Gaussian field, its restriction to U corresponds to a centered Gaussian field $f : \Omega \to \mathbb{R}^r$. Moreover, f has the same regularity as s.

The local trivializations on the diagram (6-1) induce a similar picture for jet bundles so that we have a local trivialization $\mathcal{J}_p(U, E_{|U}) \simeq \mathcal{J}_p(\Omega, \mathbb{R}^r)$, under which $\mathrm{j}_p(f, x)$ corresponds to $\mathrm{j}_p(s, \varphi^{-1}(x))$. Thus, the Gaussian section s is p-nondegenerate in the sense of Definition 1.8 if and only if f is p-nondegenerate in the sense of Definition 1.4. If this is the case, up to replacing Ω by a smaller Ω' such that $\overline{\Omega}' \subset \Omega$ is compact, we can assume that f is uniformly p-nondegenerate, in the sense that $\det \mathrm{Var}(\mathrm{j}_p(f, x))$ is bounded from below on Ω. This local picture is summarized in the following lemma.

Lemma 6.4 (local model for Gaussian fields). *Let $s : M \to E$ be a centered p-nondegenerate Gaussian field. For all $x_0 \in M$, there exist an open neighborhood U of x_0 and a local trivialization of the form (6-1) such that s reads in local coordinates as a centered Gaussian field $f : \Omega \to \mathbb{R}^r$ of the same regularity as s and which is uniformly p-nondegenerate on Ω.*

6.2. ***Bulinskaya lemma and Kac–Rice formula for the expectation.*** Let (M, g) be a Riemannian manifold of dimension $n \geqslant 1$ without boundary and let $E \to M$ be a vector bundle of rank $r \in [\![1, n]\!]$. We consider a nondegenerate centered Gaussian field $s : M \to E$, in the sense of Definition 6.3. The goal of this section is to state a Bulinskaya-type lemma and a Kac–Rice formula for the expectation of the linear statistics of s.

When M is an open subset of $\mathbb{R}^n$ and $E = \mathbb{R}^r \times M$ is trivial, these results are proved by Armentano, Azaïs and Leòn [Armentano et al. 2023b, Proposition 2.1 and Theorem 2.2]. They are extended to fields on submanifolds of $\mathbb{R}^N$ in [Armentano et al. 2023b, Section 9.1]. In the following we check that the results of that work can be adapted to the case of Gaussian sections.

Remark 6.5. Some readers are most interested in the zeros of Gaussian fields from $\mathbb{R}^n$ to $\mathbb{R}^r$ and the present geometric setting may seem overly complicated to them. Let us stress that, even in the simpler setting of fields from $\mathbb{R}^n$ to $\mathbb{R}^r$, our proof of Theorem 1.6 uses the Kac–Rice formula in the more general setting we are studying here.

In order to state the Bulinskaya lemma and the Kac–Rice formula, we need the following.

Definition 6.6 (Jacobian determinant). Let $L : V \to V'$ be a linear map between Euclidean spaces and let L^* denote its adjoint map. The *Jacobian* of L is defined as $\mathrm{Jac}(L) = \det(LL^*)^{1/2}$.

Remark 6.7. We have $\mathrm{Jac}(L) \geqslant 0$, and $\mathrm{Jac}(L) > 0$ if and only if L is surjective. In particular, the fact that $\mathrm{Jac}(L) = 0$ depends only on L and not on the Euclidean structures on V and V'. Thus the condition $\mathrm{Jac}(L) = 0$ makes sense even if no inner product is specified.

Proposition 6.8 (weak Bulinskaya lemma). *Let ∇ be a connection on $E \to M$. If the centered Gaussian field $s : M \to E$ is $\mathcal{C}^1$ and nondegenerate, the $(n-r)$-dimensional Hausdorff measure of*

$$\{x \in M \mid s(x) = 0 \text{ and } \mathrm{Jac}(\nabla_x s) = 0\}$$

is almost surely 0.

Remark 6.9. If $s(x) = 0$ then $\nabla_x s$ does not depend on ∇. Hence the random set we are interested in Proposition 6.8 does not depend on the choice ∇.

Proof of Proposition 6.8. We can cover M by countably many open trivialization domains of the type described in Lemma 6.4. Then it is enough to prove the result in each of these domains.

Let $U \subset M$ be as Lemma 6.4. In local coordinates, the restriction of s reads as a uniformly nondegenerate $\mathcal{C}^1$ centered Gaussian field $f : \Omega \to \mathbb{R}^r$, where $\Omega \subset \mathbb{R}^n$ is open. Moreover, for any $x \in \Omega$ such that $f(x) = 0$, the covariant derivative of s reads as $D_x f$, independently of the choice of ∇. Thus, we are left with proving that the $(n-r)$-dimensional Hausdorff measure of

$$\{x \in \Omega \mid f(x) = 0 \text{ and } \mathrm{Jac}(D_x f) = 0\}$$

is almost surely 0, which is given by [Armentano et al. 2023b, Proposition 2.1]. □

Let us assume from now on that the centered Gaussian field $s : M \to E$ is $\mathcal{C}^1$ and nondegenerate. We denote its zero set by $Z = s^{-1}(0)$. Let us define $Z_{\mathrm{sing}} = \{x \in Z \mid \mathrm{Jac}(\nabla_x s) = 0\}$ and $Z_{\mathrm{reg}} = Z \setminus Z_{\mathrm{sing}}$. By Proposition 6.8, the $(n-r)$-dimensional Hausdorff measure of the singular part Z_{sing} is almost surely 0. On the other hand, the regular part Z_{reg} is the set of points where s vanishes transversally. As such, it is a (possibly empty) $\mathcal{C}^1$ submanifold of M of codimension r without boundary. Thus, Z is almost surely the union of an open (in Z) regular part Z_{reg} of dimension $n - r$, and a negligible singular part Z_{sing} that we can think of as a set of lower dimension. Let us mention that, under additional assumptions on the field, the singular part is almost surely empty.

Proposition 6.10 (strong Bulinskaya lemma). *If the centered Gaussian field $s : M \to E$ is $\mathcal{C}^2$ and 1-nondegenerate, then $Z_{\mathrm{sing}} = \varnothing$ almost surely.*

Proof. As in the proof of Proposition 6.8, it is enough to prove the result in local coordinates given by Lemma 6.4. In these coordinates, s reads as a $\mathcal{C}^2$ centered Gaussian field $f : \Omega \to \mathbb{R}^r$ which is uniformly 1-nondegenerate, that is, $\det \mathrm{Var}(f(x), D_x f)$ is bounded from below on Ω. Then the result follows from [Azaïs and Wschebor 2009, Proposition 6.12]. □

The Riemannian metric g induces a metric on Z_{reg}, which in turn defines an $(n-r)$-dimensional Riemannian volume measure $\mathrm{d}\mathrm{Vol}_Z$. This measure coincides with the $(n-r)$-dimensional Hausdorff measure on Z. In the following, we consider this measure as a Radon measure on M defined as follows. Recall that the space of Radon measures is the topological dual of $\mathcal{C}^0_c(M)$, and that being a nonnegative Radon is equivalent to being a Borel measure which is finite on compact subsets.

Definition 6.11 (random measure associated with Z). We denote by ν the random nonnegative Radon measure on M defined by

$$\forall \phi \in \mathcal{C}^0_c(M), \qquad \langle \nu, \phi \rangle = \int_{Z_{\mathrm{reg}}} \phi(x)\, \mathrm{d}\mathrm{Vol}_Z(x).$$

We define $\langle \nu, \phi \rangle$ similarly if ϕ is nonnegative Borel function (in which case $\langle \nu, \phi \rangle \in [0, +\infty]$) or if ϕ is a Borel function such that $\langle \nu, |\phi| \rangle < +\infty$ almost surely.

Example 6.12. If $n = r$ then Z is almost surely locally finite. In this case $\nu = \sum_{x \in Z} \delta_x$ is the random counting measure of this point process.

Let us go back to the local model of Lemma 6.4. Around any $x_0 \in M$ there exists a chart (U, φ) and a local trivialization of the kind described by (6-1). Since s is $\mathcal{C}^1$ and nondegenerate, it corresponds in local coordinates to a $\mathcal{C}^1$ nondegenerate Gaussian field $f : \Omega \to \mathbb{R}^r$. We still denote by Z (resp. Z_{reg}) the image of Z (resp. Z_{reg}) by φ, which is the zero set of f (resp. its regular part). Similarly, we still denote by g (resp. $\mathrm{d}\mathrm{Vol}_Z$) the push-forward to Ω of the metric g (resp. of the measure $\mathrm{d}\mathrm{Vol}_Z$), and we identify test-functions on U with test-functions on Ω. Thus, if $\phi \in L^\infty_c(U)$ we have

$$\langle \nu, \phi \rangle = \int_{Z_{\mathrm{reg}}} \phi(x)\, \mathrm{d}\mathrm{Vol}_Z(x),$$

where we think of everything on the right-hand side as defined on Ω. Now, the measure $\mathrm{d}\mathrm{Vol}_Z$ is the $(n-r)$-dimensional Riemannian volume on $Z \subset \Omega \subset \mathbb{R}^n$ induced by g. In the following, we will need to understand how it compares with the Riemannian volume $\mathrm{d}\mathrm{Vol}^0_Z$ on Z induced by the Euclidean metric. This is the purpose of what comes next.

Definition 6.13 (Riemannian densities). Let $x \in \Omega$ and let G be a subspace of $\mathbb{R}^n$. We denote by $\det(g(x)_{|G})$ the determinant of the restriction to G of the inner product $g(x)$, in any basis of G which is orthonormal for the Euclidean inner product of $\mathbb{R}^n$.

For all $r \in [\![0, n]\!]$, we denote by $\gamma_r : \Omega \times \mathrm{Gr}_r(\mathbb{R}^n) \to (0, +\infty)$ the continuous map defined by $\gamma_r : (x, G) \mapsto \det(g(x)_{|G})^{1/2}$. We also write $\gamma : x \mapsto \gamma_0(x, \mathbb{R}^n)$ for simplicity.

Lemma 6.14 (comparing volumes). *Let Z be a submanifold of codimension r of Ω and let* $\mathrm{d}\mathrm{Vol}_Z$ *(resp.* $\mathrm{d}\mathrm{Vol}_Z^0$*) denote the* $(n-r)$*-dimensional Riemannian volume on Z induced by g (resp. the Euclidean metric). Then* $\mathrm{d}\mathrm{Vol}_Z$ *admits the density* $x \mapsto \gamma_r(x, T_x Z)$ *with respect to* $\mathrm{d}\mathrm{Vol}_Z^0$*. In particular* $\mathrm{d}\mathrm{Vol}_\Omega$ *admits the density γ with respect to the Lebesgue measure on Ω.*

Proof. This follows directly from the definition of the Riemannian volume measures; see [Lee 2018, Chapter 3] for example. □

We can now state and prove the Kac–Rice formula for the expectation of the linear statistics in our setting of Gaussian fields on M with values in a vector bundle E.

Definition 6.15 (Kac–Rice density). Let $\rho_1 : M \to [0, +\infty)$ be defined by

$$\rho_1 : x \longmapsto \frac{\mathbb{E}[\mathrm{Jac}(\nabla_x s) \mid s(x) = 0]}{\det(2\pi \operatorname{Var}(s(x)))^{1/2}},$$

where the numerator stands for the conditional expectation of $\mathrm{Jac}(\nabla_x s)$ given that $s(x) = 0$.

Remark 6.16. Since s is nondegenerate and $\mathcal{C}^1$, the function ρ_1 is well-defined and continuous. Moreover, it does not depend on the choice of ∇, nor on the choice of a metric on E.

Proposition 6.17 (Kac–Rice formula for the expectation). *Let s be a nondegenerate $\mathcal{C}^1$ centered Gaussian field. Then, for any Borel function $\phi : M \to \mathbb{R}$ which is nonnegative or such that $\phi\rho_1 \in L^1(M)$, we have*

$$\mathbb{E}[\langle \nu, \phi\rangle] = \int_M \phi(x)\rho_1(x)\,\mathrm{d}\mathrm{Vol}_M(x),$$

i.e., $\mathbb{E}[\nu]$ *is the Radon measure on M with density ρ_1 with respect to the Riemannian volume* $\mathrm{d}\mathrm{Vol}_M$*.*

Remark 6.18. For any Borel maps ϕ_1 and ϕ_2, we have

$$\mathbb{E}[|\langle \nu, \phi_1\rangle - \langle \nu, \phi_2\rangle|] \leqslant \mathbb{E}[\langle \nu, |\phi_1 - \phi_2|\rangle] = \int_M |\phi_1(x) - \phi_2(x)|\rho_1(x)\,\mathrm{d}\mathrm{Vol}_M(x).$$

Then, if $\phi_1 = \phi_2$ almost everywhere on M, we have $\langle \nu, \phi_1\rangle = \langle \nu, \phi_2\rangle$ almost surely. Thus, $\langle \nu, \phi\rangle$ makes sense as a random variable even if ϕ is only defined up to modification on a negligible set.

Proof of Proposition 6.17. By a partition of unity argument, it enough to prove the result if ϕ is compactly supported in an open domain U satisfying the same properties as in Lemma 6.4. In this case, in local coordinates, the field s corresponds to a nondegenerate $\mathcal{C}^1$ centered Gaussian field $f : \Omega \to \mathbb{R}^r$ with $\Omega \subset \mathbb{R}^n$ open. Thanks to Remark 6.16, we can assume that ∇ corresponds in this trivialization to the standard derivation for maps from Ω to $\mathbb{R}^r$ and that the metric on E corresponds to the canonical inner product on $\mathbb{R}^r$. Identifying Z_{reg}, the metric g, the measure $\mathrm{d}\mathrm{Vol}_Z$ and the test-function ϕ with their images in the trivialization, we have reduced our problem to proving the result for the vanishing locus of f with the volume measures induced by g.

By Lemma 6.14, we have

$$\mathbb{E}[\langle \nu, \phi\rangle] = \mathbb{E}\left[\int_{Z_{\mathrm{reg}}} \phi(x)\,\mathrm{d}\mathrm{Vol}_Z(x)\right] = \mathbb{E}\left[\int_{Z_{\mathrm{reg}}} \phi(x)\gamma_r(x, \ker D_x f)\,\mathrm{d}\mathrm{Vol}_Z^0(x)\right].$$

For all $x \in \Omega$ and $\lambda \in \mathcal{C}^0(\Omega, \mathcal{L}(\mathbb{R}^n, \mathbb{R}^r))$ we define $\Psi(x, \lambda) = \phi(x)\gamma_r(x, \ker \lambda(x))\mathbf{1}_O(\lambda(x))$, where $O = \{L \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^r) \mid \operatorname{Jac}(L) > 0\}$. Since O is open and the maps $\ker : O \to \operatorname{Gr}_r(\mathbb{R}^n)$ and γ_r are continuous, the map Ψ is lower semicontinuous with respect to each variable, where $\mathcal{C}^0(\Omega, \mathcal{L}(\mathbb{R}^n, \mathbb{R}^r))$ is equipped with the weak topology. Thus, we can apply the Euclidean Kac–Rice formula with weight from [Armentano et al. 2023b, Theorem 7.1 and Remark 8] to $\Psi(x, Df)$. This yields

$$\mathbb{E}[\langle \nu, \phi\rangle] = \int_\Omega \phi(x) \frac{\mathbb{E}[\gamma_r(x, \ker D_x f) \operatorname{Jac}^0(D_x f) \mid f(x) = 0]}{\det(2\pi \operatorname{Var}(f(x)))^{1/2}} \,\mathrm{d}x,$$

where Jac^0 means that we computed the Jacobian with respect to the Euclidean metric on $\mathbb{R}^n$.

To conclude, we need to compare Jac^0 with the Jacobian Jac with respect to g. This is the content of Lemma 6.19 below, which yields that $\gamma_r(x, \ker D_x f) \operatorname{Jac}^0(D_x f) = \gamma(x) \operatorname{Jac}(D_x f)$. Since $\gamma(x)$ is deterministic, by Lemma 6.14 we have

$$\mathbb{E}[\langle \nu, \phi\rangle] = \int_\Omega \phi(x) \frac{\mathbb{E}[\operatorname{Jac}(D_x f) \mid f(x) = 0]}{\det(2\pi \operatorname{Var}(f(x)))^{1/2}} \gamma(x) \,\mathrm{d}x = \int_\Omega \phi(x)\rho_1(x) \,\mathrm{d}\mathrm{Vol}_\Omega(x).$$

This proves that the result holds locally, that is, for a field $f : \Omega \to \mathbb{R}^r$, with the volume measures induced by any Riemannian metric on Ω, which concludes the proof. □

Lemma 6.19 (comparing Jacobians). *Let $x \in \Omega$ and let $L : \mathbb{R}^n \to \mathbb{R}^r$ be a surjective linear map. With the same notation as above, we have $\gamma_r(x, \ker L) \operatorname{Jac}^0(L) = \gamma(x) \operatorname{Jac}(L)$.*

Proof. We denote by L_g^* (resp. L_0^*) the adjoint of L with respect to the inner product $g(x)$ (resp. the Euclidean inner product). In a Euclidean orthonormal basis adapted to $\ker(L)^\perp \oplus \ker(L)$, the matrix of $g(x)$ is symmetric of the form $\left(\begin{smallmatrix} A & {}^tB \\ B & C \end{smallmatrix}\right)$ with A and C positive-definite, the matrix of L is $(F\ 0)$, that of L_0^* is $\left(\begin{smallmatrix} {}^tF \\ 0 \end{smallmatrix}\right)$ and that of L_g^* is $\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)$. We have $\left(\begin{smallmatrix} {}^tF \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} A & {}^tB \\ B & C \end{smallmatrix}\right)\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)$, which leads to ${}^tF = (A - {}^tBC^{-1}B)X$. Hence,

$$\det(LL_0^*) = \det(F\,{}^tF) = \det(A - {}^tBC^{-1}B)\det(FX) = \det(A - {}^tBC^{-1}B)\det(LL_g^*),$$

and $\gamma_r(x, G) \operatorname{Jac}^0(L) = \det(C)^{1/2} \det(A - {}^tBC^{-1}B)^{1/2} \operatorname{Jac}(L)$. Since $A - {}^tBC^{-1}B$ is the Schur complement of C in the matrix of $g(x)$, we have $\det(C)^{1/2}\det(A - {}^tBC^{-1}B)^{1/2} = \gamma(x)$. □

6.3. ***Factorial moment measures and Kac–Rice densities.*** As in the previous section, we consider a nondegenerate $\mathcal{C}^1$ centered Gaussian field $s : M \to E$ which is a random section of some vector bundle $E \to M$. Recall that $n = \dim(M)$ and $r \in [\![1, n]\!]$ is the rank of E. We are interested in the finiteness of the moments of the linear statistics $\langle \nu, \phi\rangle$ with $\phi \in L_c^\infty(M)$; see Definition 6.11. In this section, we introduce the factorial moment measures and Kac–Rice densities of the fields s, and we relate them to higher moments of the linear statistics.

In the following, p will always denote the order of the moment we are considering. Recall that, under our hypothesis on s, the random measure ν is almost surely a nonnegative Radon measure on M; see Remark 6.18.

Definition 6.20 (product measures). Let $p \geqslant 1$. We denote by $\nu^{\otimes p}$ the product measure of ν with itself p-times. We also denote by $\nu^{[p]}$ the restriction of $\nu^{\otimes p}$ to $M^p \setminus \Delta_p$. That is, for any test-function Φ,

$$\langle \nu^{\otimes p}, \Phi \rangle = \int_{Z_{\mathrm{reg}}^p} \Phi(\underline{x}) \, \mathrm{d}\mathrm{Vol}_Z^{\otimes p}(\underline{x}) \quad \text{and} \quad \langle \nu^{[p]}, \Phi \rangle = \int_{Z_{\mathrm{reg}}^p \setminus \Delta_p} \Phi(\underline{x}) \, \mathrm{d}\mathrm{Vol}_Z^{\otimes p}(\underline{x}).$$

Almost surely, these measures are Radon measures on M^p. More generally, if we want to consider product spaces indexed by a nonempty finite set A instead of $[\![1, p]\!]$, we denote by $\nu^{\otimes A}$ the product measure of ν with itself $|A|$ times on M^A and by $\nu^{[A]}$ its restriction to $M^A \setminus \Delta_A$.

The following lemma describes the relation between $\nu^{\otimes p}$ and $\nu^{[p]}$, using the notation introduced in Section 2.1.

Lemma 6.21 (relation between $\nu^{\otimes p}$ and $\nu^{[p]}$). *Let $p \geqslant 1$. If $r < n$ then $\nu^{\otimes p} = \nu^{[p]}$. If $r = n$ then $\nu^{\otimes p} = \sum_{\mathcal{I} \in \mathcal{P}_p} (\iota_{\mathcal{I}})_*(\nu^{[\mathcal{I}]})$.*

Proof. If $r < n$ then Z_{reg} is a $\mathcal{C}^1$ submanifold of positive dimension in M. In particular, the large diagonal in $(Z_{\mathrm{reg}})^p$ has positive codimension, and hence is negligible for $\mathrm{d}\mathrm{Vol}_Z^{\otimes p}$. Thus $\nu^{\otimes p} = \nu^{[p]}$ in this case.

If $r = n$ then Z_{reg} is a locally finite set and ν is its counting measure. Similarly, $\nu^{\otimes p}$ is the counting measure of the locally finite $(Z_{\mathrm{reg}})^p$, and Δ_p is no longer negligible for this measure. If $r = n = 1$, we proved in [Ancona and Letendre 2021, Lemma 2.7] that $\nu^{\otimes p} = \sum_{\mathcal{I} \in \mathcal{P}_p} (\iota_{\mathcal{I}})_*(\nu^{[\mathcal{I}]})$. The proof is purely combinatorics and it extends immediately to the case $r = n \geqslant 1$. □

Our interest in these measures is that, by the Fubini theorem, for all $\phi \in L_c^\infty(M)$ we have $\mathbb{E}[\langle \nu, \phi \rangle^p] = \mathbb{E}[\langle \nu^{\otimes p}, \phi^{\otimes p} \rangle] = \langle \mathbb{E}[\nu^{\otimes p}], \phi^{\otimes p} \rangle$, where $\phi^{\otimes p} : (x_1, \dots, x_p) \mapsto \phi(x_1) \cdots \phi(x_p)$. Thus, the measure $\mathbb{E}[\nu^{\otimes p}]$ is closely related with the computation of moments of linear statistics. For technical reasons, it is more convenient to consider $\mathbb{E}[\nu^{[p]}]$ instead.

Definition 6.22 (moment measures). Let $p \geqslant 1$. The measure $\mathbb{E}[\nu^{\otimes p}]$ is called the *p-th moment measure* of the field s and $\mathbb{E}[\nu^{[p]}]$ is called its *p-th factorial moment measure*.

Definition 6.23 (Kac–Rice density of order p). Let $p \geqslant 1$ and let us assume that the random vector $(s(x_1), \dots, s(x_p))$ is nondegenerate for all $(x_1, \dots, x_p) \in M^p \setminus \Delta_p$. Then we define

$$\rho_p : (x_1, \dots, x_p) \longmapsto \frac{\mathbb{E}\big[\prod_{i=1}^p \mathrm{Jac}(\nabla_{x_i} s) \mid \forall i \in [\![1, p]\!], s(x_i) = 0\big]}{\det(2\pi \operatorname{Var}(s(x_1), \dots, s(x_p)))^{1/2}}$$

from $M^p \setminus \Delta_p$ to $[0, +\infty)$, where the numerator is the conditional expectation of $\prod_{i=1}^p \mathrm{Jac}(\nabla_{x_i} s)$ given that $s(x_i) = 0$ for all $i \in [\![1, p]\!]$.

Once again, ρ_p is well-defined and continuous on $M^p \setminus \Delta_p$ thanks to our nondegeneracy hypothesis. However, its expression is singular along Δ_p. In particular, ρ_p is in general not bounded, which raises the question of its local integrability near Δ_p. For example, if $f : \mathbb{R}^n \to \mathbb{R}$ is a nondegenerate enough stationary Gaussian field and $p = 2$, one can check that as $y \to x$, the corresponding Kac–Rice density $\rho_2(x, y)$ behaves like $\|y - x\|$ if $n = 1$ and like $1/\|y - x\|$ if $n > 1$.

Proposition 6.24 (Kac–Rice formula for the p-th factorial moment). *Let s be a $\mathcal{C}^1$ centered Gaussian field such that $(s(x_1),\dots,s(x_p))$ is nondegenerate for all $(x_1,\dots,x_p)\in M^p\setminus\Delta_p$. Then, for any Borel function $\Phi: M^p\to\mathbb{R}$ which is nonnegative or such that $\Phi\rho_p\in L^1(M^p)$, we have*

$$\mathbb{E}[\langle\nu^{[p]},\Phi\rangle]=\int_{M^p}\Phi(\underline{x})\rho_p(\underline{x})\,\mathrm{d}\mathrm{Vol}_M^{\otimes p}(\underline{x}),$$

i.e., $\mathbb{E}[\nu^{[p]}]$ is the measure on M^p with density ρ_p with respect to the Riemannian volume $\mathrm{d}\mathrm{Vol}_M^{\otimes p}$.

Proof. Let us consider $S:(x_1,\dots,x_p)\mapsto(s(x_1),\dots,s(x_p))$ on $M^p\setminus\Delta_p$, which is a random section of the restriction over $M^p\setminus\Delta_p$ of the vector bundle $E^p\to M^p$. This is a nondegenerate $\mathcal{C}^1$ centered Gaussian field on $M^p\setminus\Delta_p$, and $\nu^{[p]}$ is the measure of integration over its zero set. Bearing in mind that Δ_p is negligible in M^p for $\mathrm{d}\mathrm{Vol}_M^{\otimes p}$, the result follows from Proposition 6.17 applied to S. □

The following proposition relates the properties of the Kac–Rice densities, the moment measures and the moments of linear statistics.

Proposition 6.25 (relation between moments, measures and densities). *Let $p\geqslant 1$ and let s be a $\mathcal{C}^1$ centered Gaussian field such that $(s(x_1),\dots,s(x_p))$ is nondegenerate for all $(x_1,\dots,x_p)\notin\Delta_p$. Then the following four properties are equivalent*:

(1) *For all $\phi\in L_c^\infty(M)$, we have $\mathbb{E}[|\langle\nu,\phi\rangle|^p]<+\infty$.*

(2) *For all $k\in[\![1,p]\!]$, the moment measure $\mathbb{E}[\nu^{\otimes k}]$ is Radon on M^k, i.e., finite on compact sets.*

(3) *For all $k\in[\![1,p]\!]$, the factorial moment measure $\mathbb{E}[\nu^{[k]}]$ is Radon on M^k.*

(4) *For all $k\in[\![1,p]\!]$, the Kac–Rice density satisfies $\rho_k\in L^1_{\mathrm{loc}}(M^k)$.*

Proof. Let us assume (1). Let $k\in[\![1,p]\!]$ and let $K_0\subset M^k$ be compact. There exists a compact set $K\subset M$ such that $K_0\subset K^k$. Then,

$$\langle\mathbb{E}[\nu^{\otimes k}],\mathbf{1}_{K_0}\rangle\leqslant\langle\mathbb{E}[\nu^{\otimes k}],\mathbf{1}_K^{\otimes k}\rangle=\mathbb{E}[\langle\nu,\mathbf{1}_K\rangle^k]=\mathbb{E}[|\langle\nu,\mathbf{1}_K\rangle|^k].$$

Since $\mathbf{1}_K\in L_c^\infty(M)$, the p-th absolute moment of $\langle\nu,\mathbf{1}_K\rangle$ is finite; hence so is its k-th absolute moment. Thus $\langle\mathbb{E}[\nu^{\otimes k}],\mathbf{1}_{K_0}\rangle<+\infty$ for all compact K_0 and (2) is satisfied.

If (2) is satisfied then so is (3). Indeed, for any $k\in[\![1,p]\!]$, the measure $\nu^{[k]}$ is the restriction of $\nu^{\otimes k}$ to $M^k\setminus\Delta_k$. Thus, for any compact $K\subset M^k$ we have

$$\langle\mathbb{E}[\nu^{[k]}],\mathbf{1}_K\rangle=\mathbb{E}[\langle\nu^{[k]},\mathbf{1}_K\rangle]\leqslant\mathbb{E}[\langle\nu^{\otimes k},\mathbf{1}_K\rangle]=\langle\mathbb{E}[\nu^{\otimes k}],\mathbf{1}_K\rangle<+\infty.$$

If (3) is satisfied, let $k\in[\![1,p]\!]$ and let $K\subset M^k$ be a compact. By Proposition 6.24 we have

$$\int_K\rho_k(\underline{x})\,\mathrm{d}\mathrm{Vol}_M^{\otimes p}(\underline{x})=\int_M\mathbf{1}_K(\underline{x})\rho_k(\underline{x})\,\mathrm{d}\mathrm{Vol}_M^{\otimes p}(\underline{x})=\mathbb{E}[\langle\nu^{[k]},\mathbf{1}_K\rangle]=\langle\mathbb{E}[\nu^{[k]}],\mathbf{1}_K\rangle<+\infty.$$

Thus ρ_k is integrable on any compact set, that is, $\rho_k\in L^1_{\mathrm{loc}}(M^k)$. This proves (4) in this case.

Finally, let us assume that (4) holds. Let $\phi\in L_c^\infty(M)$, and let us denote by $K\subset M$ its compact support. We have $\mathbb{E}[|\langle\nu,\phi\rangle|^p]\leqslant\mathbb{E}[\langle\nu,|\phi|\rangle^p]\leqslant\|\phi\|_\infty^p\mathbb{E}[\langle\nu,\mathbf{1}_K\rangle^p]$, so it is enough to prove that $\mathbb{E}[\langle\nu,\mathbf{1}_K\rangle^p]=$

$\mathbb{E}[\langle \nu^{\otimes p}, \mathbf{1}_{K^p}\rangle]$ is finite. By Lemma 6.21, whether $r = n$ or not, we have

$$\mathbb{E}[\langle \nu^{\otimes p}, \mathbf{1}_{K^p}\rangle] \leqslant \sum_{\mathcal{I}\in\mathcal{P}_p} \mathbb{E}[\langle \nu^{[\mathcal{I}]}, \mathbf{1}_{K^p} \circ \iota_{\mathcal{I}}\rangle] = \sum_{\mathcal{I}\in\mathcal{P}_p} \langle \mathbb{E}[\nu^{[\mathcal{I}]}], \mathbf{1}_{K^{\mathcal{I}}}\rangle.$$

Then, the Kac–Rice formula for moments and the local integrability of the $(\rho_k)_{1\leqslant k\leqslant p}$ yields

$$\mathbb{E}[\langle \nu, \mathbf{1}_K\rangle^p] \leqslant \sum_{\mathcal{I}\in\mathcal{P}_p} \langle \mathbb{E}[\nu^{[\mathcal{I}]}], \mathbf{1}_{K^{\mathcal{I}}}\rangle = \sum_{\mathcal{I}\in\mathcal{P}_p} \int_{K^{|\mathcal{I}|}} \rho_{|\mathcal{I}|}(\underline{x})\, \mathrm{d}\mathrm{Vol}_M^{\otimes|\mathcal{I}|}(\underline{x}) < +\infty,$$

which proves (1) and concludes the proof. □

6.4. *Proofs of Theorems 1.6 and 1.9: finiteness of moments.* The goal of this section is to prove Theorems 1.6 and 1.9, which give simple conditions for the finiteness of the moments of the linear statistics of a Gaussian field. We begin by proving a local version of Theorem 1.6, under a nondegeneracy hypothesis for the multijets of the field. This is Theorem 6.26 below. Then we deduce Theorem 1.9 from Theorem 6.26, in the case of Gaussian fields with value in a vector bundle. Finally, Theorem 1.6 is obtained as a special case of Theorem 1.9.

Let $\Omega \subset \mathbb{R}^n$ be open. Recall that $\mathcal{MJ}_p(\Omega, \mathbb{R}^r) \to C_p[\Omega]$ is defined in Definition 1.3 as the restriction over $C_p[\Omega] \subset C_p[\mathbb{R}^n]$ of the vector bundle $\mathcal{MJ}_p(\mathbb{R}^n, \mathbb{R}^r) \to C_p[\mathbb{R}^n]$ from Theorem 1.1.

Theorem 6.26 (finiteness of moments, local version). *Let $f : \Omega \to \mathbb{R}^r$ be a centered Gaussian field and ν be as in Definition 6.11. Let $p \geqslant 1$. If f is $\mathcal{C}^p$ and for all $k \in [\![1, p]\!]$ the Gaussian field $\mathrm{mj}_k(f, \cdot) : C_k[\Omega] \to \mathcal{MJ}_k(\Omega, \mathbb{R}^r)$ is nondegenerate, then the four equivalent statements in Proposition 6.25 hold.*

Proof. Let $f : \Omega \to \mathbb{R}^r$ be a $\mathcal{C}^p$ centered Gaussian field such that $\mathrm{mj}_k(f, \cdot) : C_k[\Omega] \to \mathcal{MJ}_k(\Omega, \mathbb{R}^r)$ is nondegenerate for all $k \in [\![1, p]\!]$.

Step 1: Gaussianity and nondegeneracy of the multijets. Since f is $\mathcal{C}^p$, for all $k \in [\![1, p]\!]$ we have $\mathrm{mj}_k(f, \cdot) \in \Gamma^1(C_k[\Omega], \mathcal{MJ}_k(\Omega, \mathbb{R}^r))$ because of (2) in Theorem 1.1. Since f is centered and Gaussian, so is any finite collection of jets of f. Then, for all $m \geqslant 1$ and all $z_1, \dots, z_m \in C_k[\Omega]$ we have that $(\mathrm{mj}_k(f, z_1), \dots, \mathrm{mj}_k(f, z_m))$ is a centered Gaussian. Indeed, by (4) in Theorem 1.1, this is the image of a centered Gaussian by a linear map. Thus, $\mathrm{mj}_k(f, \cdot)$ is a nondegenerate $\mathcal{C}^1$ centered Gaussian field on $C_k[\Omega]$ with values in $\mathcal{MJ}_k(\Omega, \mathbb{R}^r)$.

Let $z \notin \pi^{-1}(\Delta_p)$ and let $\underline{x} = (x_1, \dots, x_p) = \pi(z)$. By (4) in Theorem 1.1, the map Θ_z is a linear surjection. A dimension argument shows that it is actually a bijection. Thus

$$(f(x_1), \dots, f(x_p)) = (\mathrm{j}_0(f, x_1), \dots, \mathrm{j}_0(f, x_p)) = \Theta_z^{-1}(\mathrm{mj}_p(f, z)),$$

which proves that $(f(x_1), \dots, f(x_p))$ is nondegenerate. Thus, the hypotheses of Proposition 6.25 are satisfied, and the four statements appearing in this proposition are indeed equivalent.

Step 2: Comparing zeros of f and $\mathrm{mj}_k(f, \cdot)$. Let $k \in [\![1, p]\!]$. In the following we are going to prove that $\mathbb{E}[\nu^{[k]}]$ is a Radon measure on Ω^k, which is enough to conclude the proof. In the following, we say that a subset of $C_k[\Omega]$ (resp. Ω^k) is negligible if its $k(n-r)$-dimensional Hausdorff measure is 0.

Let us consider the Gaussian field $\mathrm{mj}_k(f,\cdot): C_k[\Omega] \to \mathcal{MJ}_k(\Omega,\mathbb{R}^r)$. We have checked above that it satisfies the hypotheses of Proposition 6.8. Let $X \subset C_k[\Omega]$ denote the zero set of $\mathrm{mj}_k(f,\cdot)$. As in Section 6.2, we define $X_{\mathrm{sing}} = \{z \in C_k[\Omega] \mid \mathrm{mj}_k(f,x) = 0 \text{ and } \mathrm{Jac}(\nabla_z \mathrm{mj}_k(f,\cdot)) = 0\}$ and $X_{\mathrm{reg}} = X \setminus X_{\mathrm{sing}}$. Recall that X_{reg} is a $\mathcal{C}^1$ submanifold of codimension kr and that X_{sing} is almost surely negligible by Proposition 6.8. Let $Y = X \cap \pi^{-1}(\Omega^k \setminus \Delta_k)$. We also let $Y_{\mathrm{sing}} = Y \cap X_{\mathrm{sing}}$ and $Y_{\mathrm{reg}} = Y \cap X_{\mathrm{reg}}$.

Recalling that $Z = f^{-1}(0) \subset \Omega$, for all $z \in C_k[\Omega]$ we have $z \in Y$ if and only if $\pi(z) \in Z^k \setminus \Delta_k$; see (3) in Theorem 1.1. By (1) in the same theorem, the restriction of π to $\pi^{-1}(\Omega^k \setminus \Delta_k)$ is a diffeomorphism. Hence $\pi(Y) = Z^k \setminus \Delta_k$, the set $\pi(Y_{\mathrm{reg}})$ is a $\mathcal{C}^1$ submanifold of $\Omega^k \setminus \Delta_k$, and $\pi(Y_{\mathrm{sing}})$ is almost surely negligible. Since $Z^k \setminus (Z_{\mathrm{reg}})^k$ is also almost surely negligible, the submanifolds $Z^k_{\mathrm{reg}} \setminus \Delta_k$ and $\pi(Y_{\mathrm{reg}})$ are almost surely the same, up to a negligible set (actually the reader can check that $\pi(Y_{\mathrm{reg}}) = Z^k_{\mathrm{reg}} \setminus \Delta_k$ using the trivialization τ introduced at the end of Section 5.2). Recalling Definition 6.20, this shows that $\nu^{[k]}$ is the same as the integral over $\pi(Y_{\mathrm{reg}})$ with respect to the Riemannian volume $\mathrm{dVol}_{\pi(Y)}$ induced by the Euclidean metric on $(\mathbb{R}^n)^k$. At this stage we know that, almost surely,

$$\forall \Phi \in L^\infty_c(\Omega^k), \quad \langle \nu^{[k]}, \Phi\rangle = \int_{\pi(Y_{\mathrm{reg}})} \Phi(\underline{x})\,\mathrm{dVol}_{\pi(Y)}(\underline{x}). \tag{6-2}$$

<u>Step 3</u>: Comparing volumes. Let us introduce a Riemannian metric g on $C_k[\Omega]$. It induces a volume measure dVol_X on X_{reg}, and hence on Y_{reg}. Additionally, let $z \in C_k[\Omega]$ and let $G \subset T_z(C_k[\Omega])$ be a vector subspace of codimension kr. We define $J(G) = \det((D_z\pi_{|G})^* D_z\pi_{|G})^{1/2}$, where the adjoint of $D_z\pi_{|G}$ is computed with respect to g on G and to the Euclidean metric on $(\mathbb{R}^n)^k$. Since π is smooth, this defines a smooth nonnegative function J on the total space of the Grassmannian bundle $\mathrm{Gr}_{kr}(T(C_k[\Omega])) \to C_k[\Omega]$ of subspaces of codimension kr in the tangent of $C_k[\Omega]$. Our interest in this map is that if $z \in Y_{\mathrm{reg}}$ and $G = T_z Y_{\mathrm{reg}}$ then $J(G)$ is the Jacobian determinant of $D_z(\pi_Y)$, where the $\mathcal{C}^\infty$-diffeomorphism $\pi_Y : Y_{\mathrm{reg}} \to \pi(Y_{\mathrm{reg}})$ is the restriction of π on both sides.

Let $K \subset \Omega^k$ be compact and let us apply (6-2) to $\mathbf{1}_K$. Using the previous notation, the change of variables π_Y yields

$$\langle \nu^{[k]}, \mathbf{1}_K\rangle = \int_{Y_{\mathrm{reg}}} \mathbf{1}_K(\pi(z)) J(T_z Y_{\mathrm{reg}})\,\mathrm{dVol}_X(z) \leqslant \int_{X_{\mathrm{reg}}} \mathbf{1}_{\pi^{-1}(K)}(z) J(T_z X_{\mathrm{reg}})\,\mathrm{dVol}_X(z).$$

By (1) in Theorem 1.1, the projection π is proper; hence $\widetilde{K} = \pi^{-1}(K)$ is compact. Since the bundle $\mathrm{Gr}_{kr}(T(C_k[\Omega])) \to C_k[\Omega]$ has compact fiber, its restriction over $\widetilde{K} \subset C_k[\Omega]$ is compact. By continuity, the function J is bounded on this compact set by some constant C_K. Finally, we have proved that, almost surely,

$$\langle \nu^{[k]}, \mathbf{1}_K\rangle \leqslant C_K \langle \tilde{\nu}, \mathbf{1}_{\widetilde{K}}\rangle,$$

where $\tilde{\nu}$ is defined by integrating over X_{reg} with respect to dVol_X. Taking expectation on both sides we get $\langle \mathbb{E}[\nu^{[k]}], \mathbf{1}_K\rangle \leqslant C_K \langle \mathbb{E}[\tilde{\nu}], \mathbf{1}_{\widetilde{K}}\rangle$.

<u>Step 4</u>: Applying the Kac–Rice formula to multijets. Now, X_{reg} is the regular part of the zero set X of the Gaussian field $\mathrm{mj}_k(f,\cdot)$. We have checked at the beginning of the proof that $\mathrm{mj}_k(f,\cdot)$ satisfies the hypotheses of Proposition 6.17. This proposition yields that $\mathbb{E}[\tilde{\nu}]$ is a Radon measure on $C_k[\Omega]$. Hence $\langle \mathbb{E}[\tilde{\nu}], \mathbf{1}_{\widetilde{K}}\rangle$ is finite, and so is $\langle \mathbb{E}[\nu^{[k]}], \mathbf{1}_K\rangle$. Thus $\mathbb{E}[\nu^{[k]}]$ is Radon on Ω^k, which concludes the proof. $\square$

We can now prove Theorem 1.9, which gives a criterion for the finiteness of the p-th moments of the linear statistics associated with a centered Gaussian field $s : M \to E$, where $E \to M$ is some vector bundle of rank r over a Riemannian manifold (M, g) without boundary of dimension $n \geqslant r$. The idea of the proof is to patch together the local results obtained by applying Theorem 6.26 in nice local trivializations.

Proof of Theorem 1.9. Let $p \geqslant 1$ and let $s \in \Gamma^p(M, E)$ be a centered Gaussian field which is $\mathcal{C}^p$ and $(p-1)$-nondegenerate.

Step 1: Existence of nice local trivializations. Let $x_0 \in M$. There exists an open neighborhood U of x_0 and a local trivialization of E over U given by Lemma 6.4. In this trivialization, the Gaussian section s corresponds to a centered Gaussian field $f : \Omega \to \mathbb{R}^r$ which is $\mathcal{C}^p$ and $(p-1)$-nondegenerate. We denote by $x \in \Omega$ the image of x_0 in local coordinates.

Let $k \in [\![1, p]\!]$ and let $\underline{x} = (x, \ldots, x) \in \Omega^k$. Since $\mathrm{j}_{p-1}(f, x)$ is nondegenerate so is $\mathrm{j}_{k-1}(f, x)$; see Definition 1.4. Then, by (4) in Theorem 1.1, for all $z \in \pi^{-1}(\{\underline{x}\}) \subset C_k[\Omega]$ the Gaussian vector $\mathrm{mj}_k(f, z) = \Theta_z(\mathrm{j}_{k-1}(f, x)) \in \mathcal{MJ}_k(\Omega, \mathbb{R}^r)_z$ is nondegenerate. By (1) in Theorem 1.1, the map π is proper; hence $\pi^{-1}(\{\underline{x}\})$ is compact. One the other hand, since f is $\mathcal{C}^p$, we know $\mathrm{mj}_k(f, \cdot)$ is at least $\mathcal{C}^1$. Thus $z \mapsto \det \mathrm{Var}(\mathrm{mj}_k(f, z))$ is continuous on $C_k[\Omega]$ and positive on the compact $\pi^{-1}(\{\underline{x}\})$, and hence on some neighborhood V_k of $\pi^{-1}(\{\underline{x}\})$ in $C_k[\Omega]$.

Up to reducing V_k we can assume that $V_k = \pi^{-1}(W_k)$, where W_k is an open neighborhood of $\underline{x}$ in Ω^k. Otherwise, there would exist a sequence $(z_n)_{n\in\mathbb{N}} \in C_k[\Omega] \setminus V_k$ such that $\pi(z_n) \xrightarrow[n\to+\infty]{} \underline{x}$. By properness of π, up to extracting a subsequence, we could assume that $z_n \xrightarrow[n\to+\infty]{} z$. By continuity $z \in \pi^{-1}(\{\underline{x}\})$, which would be absurd. Since W_k is a neighborhood of $\underline{x}$ in Ω^k, there exists an open neighborhood Υ_k of x in Ω such that $(\Upsilon_k)^k \subset W_k$.

Let us define $\Upsilon = \bigcap_{k=1}^p \Upsilon_k$, which is an open neighborhood of x. For all $k \in [\![1, p]\!]$, we have $C_k[\Upsilon] = \pi^{-1}(\Upsilon^k) \subset \pi^{-1}((\Upsilon_k)^k) \subset \pi^{-1}(W_k) = V_k$ and $\mathrm{mj}_k(f, \cdot)$ is nondegenerate on $C_k[\Upsilon]$. Thus, up to replacing Ω by the smaller neighborhood Υ of x in Ω and replacing U by the corresponding neighborhood of x_0 on M, we can assume that the local trivialization given by Lemma 6.4 is such that, for all $k \in [\![1, p]\!]$, the Gaussian field $\mathrm{mj}_k(f, \cdot) : C_k[\Omega] \to \mathcal{MJ}_k(\Omega, \mathbb{R}^r)$ is nondegenerate.

Step 2: Reduction to the local case. Let $\phi \in L^\infty_c(M)$ and let K denote its support. By compactness, there exists a finite family $(U_i)_{1=1}^m$ of open subsets such that $K \subset \bigcup_{i=1}^m U_i$ and each U_i is the domain of nice trivialization of the type we built in the previous paragraph. Letting $U_0 = M \setminus K$, there exists a smooth partition of unity $(\chi_i)_{i=0}^m$ subordinated to the open covering $(U_i)_{i=0}^m$ of M. Then $\phi = \sum_{i=1}^m \chi_i \phi$ by the definition of K.

Recall that ν is the measure from Definition 6.11. We have $|\langle \nu, \phi\rangle| \leqslant \langle \nu, |\phi|\rangle = \sum_{i=1}^m \langle \nu, \phi_i\rangle$, where $\phi_i = \chi_i |\phi|$ for all $i \in [\![1, m]\!]$. Let $p \geqslant 1$, by Hölder's inequality we get

$$\mathbb{E}[|\langle \nu, \phi\rangle|^p] \leqslant \mathbb{E}\left[\left(\sum_{i=1}^m \langle \nu, \phi_i\rangle\right)^p\right] = \sum_{1\leqslant i_1,\ldots,i_p\leqslant m} \mathbb{E}\left[\prod_{j=1}^p \langle \nu, \phi_{i_j}\rangle\right] \leqslant \sum_{1\leqslant i_1,\ldots,i_p\leqslant m} \prod_{j=1}^p \mathbb{E}[\langle \nu, \phi_{i_j}\rangle^p]^{1/p}$$
$$\leqslant m^p \max_{1\leqslant i\leqslant m} \mathbb{E}[\langle \nu, \phi_i\rangle^p].$$

Thus, in order to prove Theorem 1.9, it is enough to prove that $\mathbb{E}[\langle \nu, \phi\rangle^p] < +\infty$ for any nonnegative $\phi \in L^\infty_c(M)$ whose support is included in the domain of a nice trivialization.

Step 3: Local case. Let $U \subset M$ be an open subset over which we have a nice trivialization of E and s. That is, U is as in Lemma 6.4, the section s reads as $f : \Omega \to \mathbb{R}^r$ in local coordinates, and in addition we can assume that for all $k \in [\![1, p]\!]$ the field $\mathrm{mj}_k(f, \cdot)$ is nondegenerate on $C_k[\Omega]$. Identifying objects on U with their image in the local trivialization, we reduced our problem to proving that $\mathbb{E}[\langle \nu, \phi\rangle^p] < +\infty$ for all nonnegative $\phi \in L^\infty_c(\Omega)$. Note that ν is the measure of integration over Z_{reg} with respect to the Riemannian volume measure $\mathrm{d}\mathrm{Vol}_Z$ induced by the metric g. In order to apply Theorem 6.26, we need to compare ν with $\tilde{\nu}$, which is the measure of integration over Z_{reg} with respect to the Euclidean volume measure $\mathrm{d}\mathrm{Vol}^0_Z$.

Let $\phi \in L^\infty_c(\Omega)$ be nonnegative and let K denote its compact support. Recalling Definition 6.13, Lemma 6.14 shows that

$$\langle \nu, \phi\rangle = \int_{Z_{\mathrm{reg}}} \phi(x)\, \mathrm{d}\mathrm{Vol}_Z(x) = \int_{Z_{\mathrm{reg}}\cap K} \phi(x)\gamma_r(x, \ker D_x f)\, \mathrm{d}\mathrm{Vol}^0_Z(x).$$

Since γ_r is continuous and $K \times \mathrm{Gr}_r(\mathbb{R}^n)$ is compact, the nonnegative function γ_r is bounded by some $C_K > 0$ on this set. Thus $\langle \nu, \phi\rangle \leqslant C_K\langle \tilde{\nu}, \phi\rangle$. Since $f : \Omega \to \mathbb{R}^r$ satisfies the hypotheses of Theorem 6.26 and $\tilde{\nu}$ is the measure of integration over its zero set induced by the Euclidean metric, we have $\mathbb{E}[\langle \nu, \phi\rangle^p] \leqslant C_K^p \mathbb{E}[\langle \tilde{\nu}, \phi\rangle^p] < +\infty$. ☐

We conclude this section with the proof of Theorem 1.6, which is a corollary of Theorem 1.9.

Proof of Theorem 1.6. Let $f : \Omega \to \mathbb{R}^r$ be a centered Gaussian field which is $\mathcal{C}^p$ and $(p-1)$-nondegenerate in the sense of Definition 1.4. Then $s = (f, \mathrm{Id})$ is a random section of the trivial bundle $\mathbb{R}^r \times \Omega \to \Omega$. This s is also a $\mathcal{C}^p$ and $(p-1)$-nondegenerate centered Gaussian field. Its vanishing locus (as a section) is the same as the vanishing locus of f. Hence, the result follows from applying Theorem 1.9 to s. ☐

7. Multijets adapted to a differential operator

In Theorem 1.1 we defined multijets such that, over the configuration space $(\mathbb{R}^n)^p \setminus \Delta_p \subset C_p[\mathbb{R}^n]$, the p-multijet $\mathrm{mj}_p(f, \underline{x})$ reads as $(f(x_1), \dots, f(x_p))$ in the natural trivialization τ (see the end of Section 5.2). Thus $\mathrm{mj}_p(f, \underline{x})$ is a way to patch together the 0-jets of f at x_i into a smooth object that does not degenerate along Δ_p. In this section, we explain how a similar construction allows us to build a multijet that patches together the k-jets of f at x_i, and more generally the values at x_i of $\mathcal{D}f$, where $\mathcal{D}$ is a differential operator. In Section 7.1 we recall the definition of a differential operator. Then we define a multijet adapted to a given differential operator in Section 7.2. Finally, in Section 7.3, we prove Theorem 1.10.

7.1. ***Differential operator.*** In this section, we recall a few fact about differential operators. In the following, we use the multi-index notation introduced in Section 2.2.

Definition 7.1 (differential operator). Let $\Omega \subset \mathbb{R}^n$ be open, let $q, r \geqslant 1$ and let $d \geqslant 0$. We say a *differential operator of order at most d* is a linear map $\mathcal{D} : \mathcal{C}^d(\Omega, \mathbb{R}^q) \to \mathcal{C}^0(\Omega, \mathbb{R}^r)$ such that there exist continuous functions $(a_{ij\alpha})_{1\leqslant i\leqslant r;1\leqslant j\leqslant q;|\alpha|\leqslant d}$ on Ω such that, for all $f=(f_1,\dots,f_q)\in\mathcal{C}^d(\Omega,\mathbb{R}^q)$,

$$\mathcal{D}(f):x\longmapsto\left(\sum_{j=1}^{q}\sum_{|\alpha|\leqslant d}a_{ij\alpha}(x)\partial^\alpha f_j(x)\right)_{1\leqslant i\leqslant r}. \tag{7-1}$$

More generally, let M be a manifold of dimension n and let $E\to M$ and $F\to M$ be two vector bundles of ranks q and r respectively. We say that $\mathcal{D}:\Gamma^d(M,E)\to\Gamma^0(M,F)$ is a *differential operator of order at most d* if around any point $x\in M$ there exist a chart and local trivializations of E and F such that $\mathcal{D}$ is of the form (7-1) in the corresponding local coordinates. We say that $\mathcal{D}$ is of *order $d\in\mathbb{N}$* if it is of order at most d and not of order at most $d-1$. If $s\in\Gamma^d(M,E)$ and $x\in M$, we write $\mathcal{D}s=\mathcal{D}(s)$ and $\mathcal{D}_xs=\mathcal{D}(s)(x)$ for simplicity.

Remark 7.2. Let us make some important comments.

- If $\mathcal{D}:\Gamma^d(M,E)\to\Gamma^0(M,F)$ is a differential operator of order at most d, then it is of the form (7-1) in any set of local coordinates on M, E and F.
- An equivalent definition of a differential operator of order at most d is that it factors linearly through the bundle of d-jets. That is, there exists $L\in\Gamma^0(M,\mathcal{J}_d(M,E)^*\otimes F)$ such that $\mathcal{D}_xs=L(x)\,\mathrm{j}_d(s,x)\in F_x$ for all $s\in\Gamma^d(M,E)$ and $x\in M$.

In the following we always assume that M, E, F and L are smooth. In particular, the functions $(a_{ij\alpha})$ appearing in the local expression (7-1) of $\mathcal{D}$ are smooth. This implies that if $s\in\Gamma^{d+l}(M,E)$ then $\mathcal{D}s\in\Gamma^l(M,F)$.

Example 7.3. The main examples we have in mind are the following.

- The differential $D:\mathcal{C}^1(M)\to\Gamma^0(M,T^*M)$ is a differential operator of order 1.
- For all $k\in\mathbb{N}$, the jet map $\mathrm{j}_k:\Gamma^k(M,E)\to\Gamma^0(M,\mathcal{J}_k(M,E))$ is a differential operator of order k corresponding to $L(x)$ being the identity of $\mathcal{J}_k(M,E)_x$ for all $x\in M$.
- If M is equipped with a Riemannian metric, the Laplace–Beltrami operator Δ acting on $\mathcal{C}^2(M)$ is a differential operator of order 2.
- If ∇ is a connection on $E\to M$ then $\nabla:\Gamma^1(M,E)\to\Gamma^0(M,T^*M\otimes E)$ is a differential operator of order 1. Indeed, in a local frame $(e_1,\dots,e_q)$ of E and local coordinates $(x_1,\dots,x_n)$ on M the covariant derivative of $s=\sum_{j=1}^q f_je_j\in\Gamma^1(M,E)$ at x is given by

$$\nabla_xs=\sum_{i=1}^{n}\sum_{j=1}^{q}\left(\partial_if_j(x)+\sum_{k=1}^{q}\mu_{ijk}(x)f_k(x)\right)dx_i\otimes e_j(x),$$

where the (μ_{ijk}) are defined by the relations $\nabla e_k=\sum_{i=1}^n\sum_{j=1}^q\mu_{ijk}dx_i\otimes e_j$ for all $k\in[\![1,q]\!]$.

7.2. *Multijets adapted to $\mathcal{D}$.* The purpose of this section is to explain how to modify the construction of Section 5 in order to define a multijet bundle adapted to a given differential operator.

Let n, q and $r\geqslant 1$. We consider a differential operator $\mathcal{D}:\mathcal{C}^d(\mathbb{R}^n,\mathbb{R}^q)\to\mathcal{C}^0(\mathbb{R}^n,\mathbb{R}^r)$ of order d. As in Remark 7.2, there exists a section L of $\mathcal{J}_d(\mathbb{R}^n,\mathbb{R}^q)^*\otimes\mathbb{R}^r$ such that for any $f\in\mathcal{C}^d(\mathbb{R}^n,\mathbb{R}^q)$ and $x\in\mathbb{R}^n$

we have $\mathcal{D}_x f = L(x)\,\mathrm{j}_d(f,x)$. We assume that for all $x \in \mathbb{R}^n$ the linear map $\mathcal{D}_x : \mathcal{C}^d(\mathbb{R}^n,\mathbb{R}^q) \to \mathbb{R}^r$ is surjective, which is equivalent to $L(x) : \mathcal{J}_d(\mathbb{R}^n,\mathbb{R}^q)_x \to \mathbb{R}^r$ being surjective. Moreover, we assume that L is smooth. In this context, we have the following analogue of Theorem 1.1. It holds in particular if $\mathcal{D} = \mathrm{j}_k$ or $\mathcal{D} = D$ is the differential.

Theorem 7.4 (existence of multijets adapted to $\mathcal{D}$). *Let $\mathcal{D} : \mathcal{C}^d(\mathbb{R}^n,\mathbb{R}^q) \to \mathcal{C}^0(\mathbb{R}^n,\mathbb{R}^r)$ be a differential operator of order d as above. Let $p \geqslant 1$. There exist a smooth manifold $C_p^{\mathcal{D}}[\mathbb{R}^n]$ of dimension np without boundary and a smooth vector bundle $\mathcal{MJ}_p^{\mathcal{D}}(\mathbb{R}^n) \to C_p^{\mathcal{D}}[\mathbb{R}^n]$ of rank rp with the following properties*:

(1) *There exists a smooth proper surjection $\pi : C_p^{\mathcal{D}}[\mathbb{R}^n] \to (\mathbb{R}^n)^p$ such that $\pi^{-1}((\mathbb{R}^n)^p \setminus \Delta_p)$ is a dense open subset of $C_p^{\mathcal{D}}[\mathbb{R}^n]$, and π restricted to $\pi^{-1}((\mathbb{R}^n)^p \setminus \Delta_p)$ is a $\mathcal{C}^\infty$-diffeomorphism onto $(\mathbb{R}^n)^p \setminus \Delta_p$.*

(2) *There exists a map $\mathrm{mj}_p^{\mathcal{D}} : \mathcal{C}^{(d+1)p-1}(\mathbb{R}^n,\mathbb{R}^r) \times C_p^{\mathcal{D}}[\mathbb{R}^n] \to \mathcal{MJ}_p^{\mathcal{D}}(\mathbb{R}^n)$ such that*

- *for all $z \in C_p^{\mathcal{D}}[\mathbb{R}^n]$, the map $\mathrm{mj}_p^{\mathcal{D}}(\,\cdot\,,z) : \mathcal{C}^{(d+1)p-1}(\mathbb{R}^n,\mathbb{R}^r) \to \mathcal{MJ}_p^{\mathcal{D}}(\mathbb{R}^n)_z$ is surjective*;
- *for all $f \in \mathcal{C}^{l+(d+1)p-1}(\mathbb{R}^n,\mathbb{R}^r)$, the section $\mathrm{mj}_p^{\mathcal{D}}(f,\cdot)$ of $\mathcal{MJ}_p^{\mathcal{D}}(\mathbb{R}^n) \to C_p^{\mathcal{D}}[\mathbb{R}^n]$ is $\mathcal{C}^l$.*

(3) *Let $z \in C_p^{\mathcal{D}}[\mathbb{R}^n]$ be such that $\pi(z) = (x_1,\dots,x_p) \notin \Delta_p$. Then for all $f \in \mathcal{C}^{(d+1)p-1}(\mathbb{R}^n,\mathbb{R}^q)$*

$$\mathrm{mj}_p^{\mathcal{D}}(f,z) = 0 \quad\iff\quad \forall i \in [\![1,p]\!],\ \ \mathcal{D}_{x_i} f = 0.$$

(4) *Let $z \in C_p^{\mathcal{D}}[\mathbb{R}^n]$, let $\mathcal{I} = \mathcal{I}(\pi(z))$ and let $(y_I)_{I\in\mathcal{I}} = \iota_{\mathcal{I}}^{-1}(\pi(z)) \in (\mathbb{R}^n)^{\mathcal{I}} \setminus \Delta_{\mathcal{I}}$. There exists a linear surjection $\Theta_z^{\mathcal{D}} : \prod_{I\in\mathcal{I}} \mathcal{J}_{(d+1)|I|-1}(\mathbb{R}^n,\mathbb{R}^q)_{y_I} \to \mathcal{MJ}_p^{\mathcal{D}}(\mathbb{R}^n)_z$ such that*

$$\forall f \in \mathcal{C}^{(d+1)p-1}(\mathbb{R}^n,\mathbb{R}^r),\quad \mathrm{mj}_p^{\mathcal{D}}(f,z) = \Theta_z^{\mathcal{D}}((\mathrm{j}_{(d+1)|I|-1}(f,y_I))_{I\in\mathcal{I}}).$$

Proof. The proof follows the same strategy as what we did in Sections 4 and 5 in order to prove Theorem 1.1. Let us sketch its main steps.

Let $\underline{x} = (x_1,\dots,x_p) \in (\mathbb{R}^n)^p$ and let $\hat{\underline{x}} = (\underline{x},\dots,\underline{x}) \in (\mathbb{R}^n)^{(d+1)p}$. Let $f \in \mathcal{C}^{(d+1)p-1}(\mathbb{R}^n,\mathbb{R}^q)$. The polynomial map $K(f,\hat{\underline{x}}) \in \mathbb{R}_{(d+1)p-1}[X] \otimes \mathbb{R}^q$ is defined as in Definition 3.6. For all $i \in [\![1,p]\!]$, since x_i appears with multiplicity $d+1$ in $\hat{\underline{x}}$, the map $K(f,\hat{\underline{x}})$ has the same d-jet as f at x_i. Hence $\mathcal{D}_{x_i} f = L(x_i)\,\mathrm{j}_d(f,x_i) = L(x_i)\,\mathrm{j}_d(K(f,\hat{\underline{x}}),x_i) = \mathcal{D}_{x_i}(K(f,\hat{\underline{x}}))$.

If $\underline{x} \notin \Delta_p$, let us define $\mathrm{ev}_{\underline{x}}^{\mathcal{D}} : P \mapsto (\mathcal{D}_{x_i} P)_{1\leqslant i\leqslant p}$ from $\mathbb{R}_{(d+1)p-1}[X] \otimes \mathbb{R}^q$ to $(\mathbb{R}^r)^p$. Since we assumed that $L(x_i)$ is surjective for all $i \in [\![1,p]\!]$, the previous interpolation result proves that $\mathrm{ev}_{\underline{x}}^{\mathcal{D}}$ is surjective. Then, as in (4-2), for all nonempty $I \subset [\![1,p]\!]$ we define

$$\mathcal{G}_I^{\mathcal{D}}(\underline{x}) = \ker \mathrm{ev}_{\underline{x}_I}^{\mathcal{D}} \in \mathrm{Gr}_{r|I|}(\mathbb{R}_{(d+1)|I|-1}[X] \otimes \mathbb{R}^q).$$

We also define $\mathcal{G}^{\mathcal{D}}(\underline{x}) = \mathcal{G}_{[\![1,p]\!]}^{\mathcal{D}}(\underline{x})$.

Following the same strategy as in Section 5, we denote by $\Sigma_{\mathcal{D}}$ the graph of $(\mathcal{G}_I^{\mathcal{D}})_{I\subset[\![1,p]\!]}$ defined on $(\mathbb{R}^n)^p \setminus \Delta_p$. We define $C_p^{\mathcal{D}}[\mathbb{R}^n]$ as a resolution of the singularities of the algebraic variety

$$\overline{\Sigma}_{\mathcal{D}} \subset (\mathbb{R}^n)^p \times \prod_{\varnothing\neq I\subset[\![1,p]\!]} \mathrm{Gr}_{r|I|}(\mathbb{R}_{(d+1)|I|-1}[X] \otimes \mathbb{R}^q).$$

The manifold $C_p^{\mathcal{D}}[\mathbb{R}^n]$ satisfies the analogue of Corollary 5.6. In particular the maps $\mathcal{G}_I^{\mathcal{D}}$ with $I \subset [\![1, p]\!]$ extend smoothly to $C_p^{\mathcal{D}}[\mathbb{R}^n]$. Then we define the p-multijet bundle adapted to $\mathcal{D}$ as

$$\mathcal{MJ}_p^{\mathcal{D}}(\mathbb{R}^n) = \big((\mathbb{R}_{(d+1)p-1}[X] \otimes \mathbb{R}^q) \times C_p^{\mathcal{D}}[\mathbb{R}^n]\big)/\mathcal{G}^{\mathcal{D}}$$

over $C_p^{\mathcal{D}}[\mathbb{R}^n]$, similarly to Definition 5.7. Given a function $f \in \mathcal{C}^{(d+1)p-1}(\mathbb{R}^n, \mathbb{R}^q)$, we define its p-multijet adapted to $\mathcal{D}$ at $z \in C_p^{\mathcal{D}}[\mathbb{R}^n]$ as

$$\mathrm{mj}_p^{\mathcal{D}}(f, z) = K(f, \widehat{\pi(z)}) \bmod \mathcal{G}^{\mathcal{D}}(z).$$

Then, following the same steps as in Section 5, one can check that the objects we just defined satisfy the conditions in Theorem 7.4. □

As before, thanks to the localness condition in Theorem 7.4(4), the multijet $\mathrm{mj}_p^{\mathcal{D}}(f, z)$ makes sense even if f is only defined and $\mathcal{C}^{(d+1)|I|-1}$ near y_I for all $I \in \mathcal{I}(\pi(z))$. Hence, the following definition makes sense.

Definition 7.5 (multijets adapted to $\mathcal{D}$). Let $\Omega \subset \mathbb{R}^n$ be open. We define $C_p^{\mathcal{D}}[\Omega] = \pi^{-1}(\Omega^p)$ and denote by $\mathcal{MJ}_p^{\mathcal{D}}(\Omega)$ the restriction of $\mathcal{MJ}_p^{\mathcal{D}}(\mathbb{R}^n)$ to $C_p^{\mathcal{D}}[\Omega]$. We call $\mathcal{MJ}_p^{\mathcal{D}}(\Omega) \to C_p^{\mathcal{D}}[\Omega]$ the *bundle of p-multijets adapted to $\mathcal{D}$*. Let $f : \Omega \to \mathbb{R}^q$ be of class $\mathcal{C}^{(d+1)p-1}$. We call the section $\mathrm{mj}_p^{\mathcal{D}}(f, \cdot)$ of $\mathcal{MJ}_p^{\mathcal{D}}(\Omega)$ the *p-multijet of f adapted to $\mathcal{D}$*.

7.3. *Finiteness of moments for critical points.* The purpose of this section is to prove Theorem 1.10. More generally we prove an analogous result for the zero set of $\mathcal{D}s$, where s is a section of a vector bundle $E \to M$ and $\mathcal{D}$ is a differential operator; see Theorem 7.8. This is done by adapting what we did in Section 6 to this framework.

Let (M, g) be Riemannian manifold of dimension $n \geqslant 1$ without boundary. Let $E \to M$ (resp. $F \to M$) be a smooth vector bundle of rank $q \geqslant 1$ (resp. $r \in [\![1, n]\!]$). We consider a differential operator $\mathcal{D} : \Gamma^d(M, E) \to \Gamma^0(M, F)$ of order $d \geqslant 0$, corresponding to a smooth section $L \in \Gamma^\infty(M, \mathcal{J}_d(M, E)^* \otimes F)$; see Remark 7.2. Thanks to this smoothness assumption we have $\mathcal{D} : \Gamma^{d+l}(M, E) \to \Gamma^l(M, F)$ for all $l \geqslant 0$. Finally we assume that $L(x) : \mathcal{J}_d(M, E)_x \to F_x$ (or equivalently $\mathcal{D}_x : \Gamma^d(M, E) \to F_x$) is surjective for all $x \in M$.

Let $s : M \to E$ be a centered Gaussian field on M with values in E in the sense of Definition 6.3. We assume that s is $\mathcal{C}^{d+1}$ and d-nondegenerate, so that $\mathrm{j}_d(s, \cdot)$ is a centered Gaussian field with values in $\mathcal{J}_d(M, E)$ which is $\mathcal{C}^1$ and nondegenerate. Then $\mathcal{D}s \in \Gamma^1(M, F)$ is a centered Gaussian field with values in F which is nondegenerate because of the surjectivity assumption on L. Everything we did in Sections 6.2 and 6.3 applies to $\mathcal{D}s$. In particular, $\mathcal{D}s$ satisfies the weak Bulinskaya lemma (see Proposition 6.8). Hence its vanishing locus is almost surely the union of a codimension-r submanifold of M and a negligible singular set. We denote by $\nu_{\mathcal{D}}$ the random Radon measure on M defined by integrating over the zero set of $\mathcal{D}s$. The formal definition is similar to Definition 6.11.

Example 7.6. Let us assume that $E = \mathbb{R} \times M$ is trivial. Then we can identify $\Gamma^1(M, E)$ with $\mathcal{C}^1(M)$ and consider the differential $D : \mathcal{C}^1(M) \to \Gamma^0(M, T^*M)$, which is a differential operator of order 1. Let $f : M \to \mathbb{R}$ be a $\mathcal{C}^2$ and 1-nondegenerate centered Gaussian field. Then Df is a nondegenerate $\mathcal{C}^1$

centered Gaussian field on M with values in T^*M, the vanishing locus of Df is the set of critical points of f, and ν_D is the counting measure this random set.

We can now state the analogue of Theorem 6.26 in this context, bearing in mind that Proposition 6.25 applies to $\nu_{\mathcal{D}}$.

Theorem 7.7 (finiteness of moments for $\nu_{\mathcal{D}}$, local version). *Let $\Omega \subset \mathbb{R}^n$ be open and $f : \Omega \to \mathbb{R}^q$ be a centered Gaussian field. Let $r \in [\![1, n]\!]$. Let $\mathcal{D} : \mathcal{C}^d(\Omega, \mathbb{R}^q) \to \mathcal{C}^0(\Omega, \mathbb{R}^r)$ be a differential operator of order d satisfying the previous hypotheses and $\nu_{\mathcal{D}}$ denote the measure of integration over the zero set of $\mathcal{D}f$. Let $p \geqslant 1$. If f is $\mathcal{C}^{(d+1)p}$ and the Gaussian field $\mathrm{mj}_k^{\mathcal{D}}(f, \cdot) : C_k^{\mathcal{D}}[\Omega] \to \mathcal{MJ}_k^{\mathcal{D}}(\Omega)$ is nondegenerate for all $k \in [\![1, p]\!]$, then the four equivalent statements in Proposition 6.25 hold for $\nu_{\mathcal{D}}$.*

Proof. Under these hypotheses, for all $k \in [\![1, p]\!]$ the Gaussian field $\mathrm{mj}_k^{\mathcal{D}}(f, \cdot)$ is at least $\mathcal{C}^1$. Then the proof is the same as that of Theorem 6.26. □

Theorem 7.8 (finiteness of moments for zeros of $\mathcal{D}s$). *In the setting introduced at the beginning of this section, let $s : M \to E$ be a centered Gaussian field and $\nu_{\mathcal{D}}$ denote the measure of integration over the zero set of $\mathcal{D}s$. Let $p \geqslant 1$. If s is $\mathcal{C}^{(d+1)p}$ and $((d+1)p-1)$-nondegenerate then $\mathbb{E}[|\langle \nu_{\mathcal{D}}, \phi\rangle|^p] < +\infty$ for all $\phi \in L_c^\infty(M)$.*

Proof. We deduce Theorem 7.8 from Theorem 7.7 in the same way that we deduced Theorem 1.9 from Theorem 6.26; see Section 6.4. □

8. Multijets of holomorphic maps

The purpose of this section is to explain how to adapt what we did in Sections 3 to 6 to the case of holomorphic maps. Theorem 1.6 asks for the $(p-1)$-nondegeneracy of the field f, that is, $\mathrm{j}_{p-1}(f, x)$ needs to be nondegenerate for all x. If $f : \mathbb{C}^n \to \mathbb{C}$ is a centered holomorphic Gaussian field, then $(f(x), D_x f)$ is always degenerate. Indeed, identifying canonically $\mathbb{C}$ with $\mathbb{R}^2$, the differential $D_x f$ takes values in the subspace of $\mathcal{L}(\mathbb{R}^{2n}, \mathbb{R}^2)$ consisting of $\mathbb{R}$-linear maps that are actually $\mathbb{C}$-linear. Thus, if we see the holomorphic field f as a smooth field from $\mathbb{R}^{2n}$ to $\mathbb{R}^2$, we cannot apply Theorem 1.6. From the point of view of multijets, the multijet $\mathrm{mj}_p(f, \cdot)$ of a holomorphic function $f : \mathbb{C}^n \to \mathbb{C}$ takes values in a strict sub-bundle of $\mathcal{MJ}_p(\mathbb{R}^{2n}, \mathbb{R}^2)$, which is similar to what happens for jet bundles. Thus, the field $\mathrm{mj}_p(f, \cdot)$ associated with a holomorphic Gaussian field f is necessarily degenerated and Theorem 6.26 does not apply. To remedy this situation, we define in Section 8.1 a multijet bundle adapted to holomorphic maps. Then, in Section 8.2, we use this holomorphic multijet to prove Theorem 1.11.

8.1. ***Definition of the holomorphic multijet bundles.*** In this section, we define a multijet bundle for holomorphic maps. Our main result is an equivalent of Theorem 1.1 in this context. Let us first introduce some notation.

Definition 8.1 (spaces of holomorphic maps). We define the following spaces.

- We denote by $\mathbb{C}_d[X]$ the space of complex polynomials of degree at most d in n variables, where $X = (X_1, \dots, X_n)$ is multivariate.

• If M and N are two complex manifolds, we denote by $\mathcal{O}(M,N)$ the space of holomorphic maps from M to N. If $N=\mathbb{C}$, we simply write $\mathcal{O}(M)$.

• If $E\to M$ is a holomorphic vector bundle, we denote by $\mathcal{J}_k^{\mathbb{C}}(M,E)\to M$ the holomorphic bundle of k-jets of holomorphic sections of E. If $E=V\times M$ is trivial with fiber V, we denote its holomorphic k-jet bundle by $\mathcal{J}_k^{\mathbb{C}}(M,V)\to M$. If $V=\mathbb{C}$, we simply write $\mathcal{J}_k^{\mathbb{C}}(M)\to M$. Given a holomorphic section s of E, we denote by $\mathrm{j}_k^{\mathbb{C}}(s,x)$ its holomorphic k-jet at $x\in M$.

Theorem 8.2 (existence of holomorphic multijet bundles). *Let $n\geqslant 1$ and $p\geqslant 1$ and let V be a complex vector space of dimension $r\geqslant 1$. There exist a complex manifold $C_p^{\mathbb{C}}[\mathbb{C}^n]$ of dimension np and a holomorphic vector bundle $\mathcal{MJ}_p^{\mathbb{C}}(\mathbb{C}^n,V)\to C_p^{\mathbb{C}}[\mathbb{C}^n]$ of rank rp with the following properties*:

(1) *There exists a holomorphic proper surjection $\pi: C_p^{\mathbb{C}}[\mathbb{C}^n]\to(\mathbb{C}^n)^p$ such that $\pi^{-1}((\mathbb{C}^n)^p\setminus\Delta_p)$ is a dense open subset of $C_p^{\mathbb{C}}[\mathbb{C}^n]$, and π restricted to $\pi^{-1}((\mathbb{C}^n)^p\setminus\Delta_p)$ is a biholomorphism onto $(\mathbb{C}^n)^p\setminus\Delta_p$.*

(2) *There exists a map $\mathrm{mj}_p^{\mathbb{C}}:\mathcal{O}(\mathbb{C}^n,V)\times C_p^{\mathbb{C}}[\mathbb{C}^n]\to\mathcal{MJ}_p^{\mathbb{C}}(\mathbb{C}^n,V)$ such that*

- *for all $z\in C_p^{\mathbb{C}}[\mathbb{C}^n]$, the linear map $\mathrm{mj}_p^{\mathbb{C}}(\,\cdot\,,z):\mathcal{O}(\mathbb{C}^n,V)\to\mathcal{MJ}_p^{\mathbb{C}}(\mathbb{C}^n,V)_z$ is surjective*;
- *for all $f\in\mathcal{O}(\mathbb{C}^n,V)$, the section $\mathrm{mj}_p^{\mathbb{C}}(f,\cdot)$ of $\mathcal{MJ}_p^{\mathbb{C}}(\mathbb{C}^n,V)\to C_p^{\mathbb{C}}[\mathbb{C}^n]$ is holomorphic.*

(3) *Let $z\in C_p^{\mathbb{C}}[\mathbb{C}^n]$ be such that $\pi(z)=(x_1,\dots,x_p)\notin\Delta_p$. Then for all $f\in\mathcal{O}(\mathbb{C}^n,V)$ we have*

$$\mathrm{mj}_p^{\mathbb{C}}(f,z)=0\quad\iff\quad\forall i\in[\![1,p]\!],\quad f(x_i)=0.$$

(4) *Let $z\in C_p^{\mathbb{C}}[\mathbb{C}^n]$, let $\mathcal{I}=\mathcal{I}(\pi(z))$ and let $(y_I)_{I\in\mathcal{I}}=\iota_{\mathcal{I}}^{-1}(\pi(z))\in(\mathbb{C}^n)^{\mathcal{I}}\setminus\Delta_{\mathcal{I}}$. There exists a linear surjection $\Theta_z^{\mathbb{C}}:\prod_{I\in\mathcal{I}}\mathcal{J}_{|I|-1}^{\mathbb{C}}(\mathbb{C}^n,V)_{y_I}\to\mathcal{MJ}_p^{\mathbb{C}}(\mathbb{C}^n,V)_z$ such that*

$$\forall f\in\mathcal{O}(\mathbb{C}^n,V),\quad \mathrm{mj}_p^{\mathbb{C}}(f,z)=\Theta_z^{\mathbb{C}}((\mathrm{j}_{|I|-1}^{\mathbb{C}}(f,y_I))_{I\in\mathcal{I}}).$$

Proof. The proof follows the same steps as what we did in Sections 3, 4 and 5 to prove Theorem 1.1. In the following, we sketch how the proof of Theorem 1.1 adapts to the holomorphic case.

Step 1: Divided differences and Kergin interpolation. Let us consider $f\in\mathcal{O}(\mathbb{C}^n)$ and $\underline{x}=(x_0,\dots,x_k)\in(\mathbb{C}^n)^{k+1}$. The divided difference $f[x_0,\dots,x_k]$ from Definition 3.1 still makes sense. Since f is holomorphic, it is now a symmetric $\mathbb{C}$-multilinear form on $\mathbb{C}^n$ that depends linearly on f and is holomorphic with respect to $\underline{x}$. As explained in [Kergin 1980, Proposition 5.1], the Kergin interpolating polynomial is well-behaved with respect to holomorphic maps. Given $f\in\mathcal{O}(\mathbb{C}^n)$ and $\underline{x}\in(\mathbb{C}^n)^p$, (3-3) defines $K(f,\underline{x})\in\mathbb{C}_{p-1}[X]$ that interpolates the values of $f[\underline{x}_I]$ for all nonempty $I\subset[\![1,p]\!]$. The equivalent of Lemma 3.8 is true, in the sense that $K(\,\cdot\,,\underline{x})$ is $\mathbb{C}$-linear and $K(f,\cdot)$ is holomorphic from $(\mathbb{C}^n)^p$ to $\mathbb{C}_{p-1}[X]$.

In this complex framework, the equivalent of Lemma 3.9 holds, that is: for all $\underline{x}\in(\mathbb{C}^n)^p$ the map $P\mapsto(K(P,\underline{x}_I))_{I\in\mathcal{I}(\underline{x})}$ is surjective from $\mathbb{C}_{p-1}[X]$ to $\prod_{I\in\mathcal{I}(\underline{x})}\mathbb{C}_{|I|-1}[X]$. Note however that the proof we gave of Lemma 3.9 does not adapt to the holomorphic setting since it uses bump functions. Here, we deduce the surjectivity of $(K(\,\cdot\,,\underline{x}_I))_{I\in\mathcal{I}(\underline{x})}$ from a general amplitude result in algebraic geometry. Let $\mathcal{I}=\mathcal{I}(\underline{x})$ and $(y_I)_{I\in\mathcal{I}}=\iota_{\mathcal{I}}^{-1}(\underline{x})$. For all $I\in\mathcal{I}$, let $P_I\in\mathbb{C}_{|I|-1}[X]$. Multiplying each monomial in P_I by the right power of X_0 yields a homogeneous polynomial $\widetilde{P}_I\in\mathbb{C}_{p-1}^{\mathrm{hom}}[X_0,\dots,X_n]$, that is, a global holomorphic section of the line bundle $\mathcal{O}(p-1)\to\mathbb{CP}^n$. Recall that $\mathcal{O}(p-1)$ is the $(p-1)$-th tensor power of the hyperplane line bundle $\mathcal{O}(1) \to \mathbb{CP}^n$. Since $\mathcal{O}(1)$ is very ample, the bundle $\mathcal{O}(p-1)$ is $(p-1)$-jet ample; see [Beltrametti and Sommese 1993, Corollary 2.1]. This means that there exists $\widetilde{P} \in \mathbb{C}^{\mathrm{hom}}_{p-1}[X_0, \dots, X_n]$ with the same $(|I|-1)$-jet as $\widetilde{P}_I$ at y_I for all $I \in \mathcal{I}$, where we see $\mathbb{C}^n$ as a standard affine chart in $\mathbb{CP}^n$. Then, for all $I \in \mathcal{I}$, the polynomial $P = \widetilde{P}(1, X_1, \dots, X_n) \in \mathbb{C}_{p-1}[X]$ has the same $(|I|-1)$-jet (i.e., the same Taylor polynomial of order $|I|-1$) as P_I at y_I. Thus $(K(P, \underline{x}_I))_{I \in \mathcal{I}} = (P_I)_{I \in \mathcal{I}}$.

Step 2: Kernel of the evaluation and resolution of singularities. As in Definition 4.3, we define a complex evaluation map by $\mathrm{ev}^{\mathbb{C}}_{\underline{x}} : f \mapsto (f(x_1), \dots, f(x_p))$, where $\underline{x} \in (\mathbb{C}^n)^p$. If $\underline{x} \notin \Delta_p$, this map is surjective from $\mathbb{C}_{p-1}[X]$ to $\mathbb{C}^p$. Hence we can define $\mathcal{G}^{\mathbb{C}}_I(\underline{x}) = \ker \mathrm{ev}^{\mathbb{C}}_{\underline{x}_I} \in \mathrm{Gr}_{|I|}(\mathbb{C}_{|I|-1}[X])$ for all nonempty $I \subset [\![1, p]\!]$, where the Grassmannian is now the Grassmannian of complex subspaces of codimension $|I|$. Then, everything we did in Sections 4 and 5 works in the holomorphic setting after replacing $\mathbb{R}$-linear objects by $\mathbb{C}$-linear ones.

We define $\Sigma_{\mathbb{C}}$ as the graph of $(\mathcal{G}^{\mathbb{C}}_I)_{I \subset [\![1,p]\!]}$ from $(\mathbb{C}^n)^p \setminus \Delta_p$ to $\prod_{\varnothing \neq I \subset [\![1,p]\!]} \mathrm{Gr}_{|I|}(\mathbb{C}_{|I|-1}[X])$ and $C^{\mathbb{C}}_p[\mathbb{C}^n]$ as a resolution of the singularities of its closure $\overline{\Sigma}_{\mathbb{C}}$ in $(\mathbb{C}^n)^p \times \prod_{\varnothing \neq I \subset [\![1,p]\!]} \mathrm{Gr}_{|I|}(\mathbb{C}_{|I|-1}[X])$. The resolution of singularities is a result from algebraic geometry which holds over fields of characteristic 0. In particular, in Proposition 5.5 and Corollary 5.6, "smooth" can be replaced by "algebraic" everywhere. The same results hold over $\mathbb{C}$, in which case algebraic implies holomorphic. Thus, $C^{\mathbb{C}}_p[\mathbb{C}^n]$ is a complex manifold of dimension np, which satisfies the equivalent of Corollary 5.6 with "smooth" replaced by "holomorphic".

Step 3: Definition of the holomorphic multijet bundles. Everything we did in Sections 5.2, 5.3 and 5.4 adapts to the holomorphic setting. It is enough to replace $\mathcal{C}^k$ functions by holomorphic ones and to write the linear arguments over $\mathbb{C}$ instead of $\mathbb{R}$. We can then define the holomorphic vector bundle of p-multijets of holomorphic functions on $\mathbb{C}^n$ by

$$\mathcal{MJ}^{\mathbb{C}}_p(\mathbb{C}^n) = (\mathbb{C}_{p-1}[X] \times C^{\mathbb{C}}_p[\mathbb{C}^n])/\mathcal{G}^{\mathbb{C}} \tag{8-1}$$

and the p-multijet of $f \in \mathcal{O}(\mathbb{C}^n)$ by $\mathrm{mj}^{\mathbb{C}}_p(f, z) = K(f, \pi(z)) \bmod \mathcal{G}^{\mathbb{C}}(z)$ for all $z \in C^{\mathbb{C}}_p[\mathbb{C}^n]$. If V is a complex vector space of finite dimension, we define as in Definition 5.18

$$\mathcal{MJ}^{\mathbb{C}}_p(\mathbb{C}^n, V) = \mathcal{MJ}^{\mathbb{C}}_p(\mathbb{C}^n) \otimes V. \tag{8-2}$$

Then we define the p-multijet of $f \in \mathcal{O}(\mathbb{C}^n, V)$ as in Definition 5.19. If $(v_1, \dots, v_r)$ is a basis of V and $f = \sum_{i=1}^r f_i v_i$ is holomorphic, then $\mathrm{mj}^{\mathbb{C}}_p(f, z) = \sum_{i=1}^r \mathrm{mj}^{\mathbb{C}}_p(f_i, z) \otimes v_i$ for all $z \in C^{\mathbb{C}}_p[\mathbb{C}^n]$. As in Lemma 5.20, this definition does not depend on a choice of basis. This defines the holomorphic p-multijet that we are looking for. ☐

As in the real case, thanks to (4) in Theorem 8.2, the holomorphic multijet $\mathrm{mj}^{\mathbb{C}}_p(f, z)$ of f at $z \in C^{\mathbb{C}}_p[\mathbb{C}^n]$ only depends on the germ of f near the x_i, where $(x_i)_{1 \leqslant i \leqslant p} = \pi(z)$. Thus, we can define a holomorphic multijet bundle over any open subset of $\mathbb{C}^n$.

Definition 8.3 (holomorphic multijets). Let $\Omega \subset \mathbb{C}^n$ be open. We denote by $C^{\mathbb{C}}_p[\Omega] = \pi^{-1}(\Omega^p)$ and by $\mathcal{MJ}^{\mathbb{C}}_p(\Omega, V) \to C^{\mathbb{C}}_p[\Omega]$ the restriction of $\mathcal{MJ}^{\mathbb{C}}_p(\mathbb{C}^n, V)$ to $C_p[\Omega]$. If $V = \mathbb{C}$, we drop it from the notation and write $\mathcal{MJ}^{\mathbb{C}}_p(\Omega) \to C^{\mathbb{C}}_p[\Omega]$. Let $f \in \mathcal{O}(\Omega, V)$, we call the section $\mathrm{mj}^{\mathbb{C}}_p(f, \cdot)$ of $\mathcal{MJ}^{\mathbb{C}}_p(\Omega, V)$ the *holomorphic p-multijet of f*.

8.2. ***Application to the zeros of holomorphic Gaussian fields.*** In this section, we explain how the holomorphic multijets defined in Section 8.1 allow us to prove Theorem 1.11, and the analogue of Theorem 6.26 for holomorphic Gaussian fields. We start by recalling a few facts about complex Gaussian vectors; see [Andersen et al. 1995, Chapter 2].

A random variable $X \in \mathbb{C}$ is called a *centered complex Gaussian* if its real and imaginary parts are independent real centered Gaussian variables of the same variance, i.e., there exists $\lambda \geqslant 0$ such that $X = X_{\Re} + iX_{\Im}$, with $(X_{\Re}, X_{\Im}) \sim \mathcal{N}(0, \lambda\,\mathrm{Id})$ in $\mathbb{R}^2$.

Definition 8.4 (complex Gaussian vector). We say that a random vector X in a finite-dimensional complex vector space V is a *centered Gaussian* if for all $\eta \in V^*$ the complex variable $\eta(X)$ is a centered complex Gaussian.

If V is equipped with a Hermitian inner product $\langle\cdot\,,\cdot\rangle$ and we define $v^* = \langle v, \cdot\rangle$ then the *variance* of X is the nonnegative Hermitian operator $\mathrm{Var}_{\mathbb{C}}(X) = \mathbb{E}[X \otimes X^*]$. We say that X is *nondegenerate* if $\mathrm{Var}_{\mathbb{C}}(X)$ is positive-definite.

Remark 8.5. As in the real case, the Gaussianity and nondegeneracy of X do not depend on $\langle\cdot\,,\cdot\rangle$, but the variance operator does.

A centered complex Gaussian vector X in $(V, \langle\cdot\,,\cdot\rangle)$ is completely determined by its variance. For example, if $\mathrm{Var}_{\mathbb{C}}(X) = \Lambda$ is positive-definite, then X admits the density $v \mapsto e^{-\langle v, \Lambda^{-1} v\rangle}/\det(\pi\Lambda)$ with respect to the Lebesgue measure on V. We denote by $\mathcal{N}_{\mathbb{C}}(0, \Lambda)$ the centered complex Gaussian distribution of variance Λ. Then $X \sim \mathcal{N}_{\mathbb{C}}(0, \Lambda)$ in $\mathbb{C}^n$ if and only if its real and imaginary part satisfy

$$(X_{\Re}, X_{\Im}) \sim \mathcal{N}\left(0, \frac{1}{2}\begin{pmatrix} \Re(\Lambda) & \Im(\Lambda) \\ -\Im(\Lambda) & \Re(\Lambda) \end{pmatrix}\right)$$

in $\mathbb{R}^{2n}$.

Let $E \to M$ be a holomorphic vector bundle over a complex manifold M. We denote by $H^0(M, E)$ the vector space of global holomorphic sections of $E \to M$.

Definition 8.6 (holomorphic Gaussian field). We say that a random section $s \in H^0(M, E)$ is a *centered holomorphic Gaussian field* if for all $m \geqslant 1$ and all $x_1, \dots, x_m$ the random vector $(s(x_1), \dots, s(x_m))$ is a centered complex Gaussian. We say that this field is *nondegenerate* if $s(x)$ is nondegenerate for all $x \in M$.

Note that if $s \in H^0(M, E)$ is a centered holomorphic Gaussian field then, for all $k \in \mathbb{N}$, its holomorphic k-jet $\mathrm{j}^{\mathbb{C}}_k(s, \cdot)$ defines a centered holomorphic Gaussian field with values in $\mathcal{J}^{\mathbb{C}}_k(M, E)$.

Definition 8.7 (p-nondegeneracy for holomorphic fields). Let $p \geqslant 1$. We say that the centered holomorphic Gaussian field $s \in H^0(M, E)$ is *p-nondegenerate* if the centered complex Gaussian $\mathrm{j}^{\mathbb{C}}_p(s, x) \in \mathcal{J}^{\mathbb{C}}_p(M, E)_x$ is nondegenerate for all $x \in M$.

As in the real framework, we need the following definition.

Definition 8.8 (complex Jacobian). Let $L : V \to V'$ be a $\mathbb{C}$-linear map between Hermitian spaces and let L^* denote its adjoint map. The *complex Jacobian* of L is defined as $\mathrm{Jac}_{\mathbb{C}}(L) = \det(LL^*)$.

Remark 8.9. If we see V, V' and L as $\mathbb{R}$-linear objects and we equip V and V' with the Euclidean structures induced by their Hermitian inner products, then the real and complex Jacobians are related by $\mathrm{Jac}_{\mathbb{C}}(L) = \mathrm{Jac}(L)^2$; see [Andersen et al. 1995, Theorem A.5].

Let us consider a complex manifold M of complex dimension n equipped with a Riemannian metric g and a holomorphic vector bundle $E \to M$ of complex rank $r \in [\![1, n]\!]$. In the following, we denote by ∇ a connection on E which is compatible with the complex structure. As in the real case, the choice of this connection will not matter. Let $s \in H^0(M, E)$ be a centered holomorphic Gaussian field on M with values in E and let $Z = s^{-1}(0)$ denote its vanishing locus. We will always assume that s is nondegenerate. In this setting, the random section s satisfies a strong Bulinskaya-type lemma.

Proposition 8.10 (holomorphic Bulinskaya lemma). *Almost surely the following set is empty*:

$$\{x \in M \mid s(x) = 0 \textit{ and } \mathrm{Jac}_{\mathbb{C}}(\nabla_x s) = 0\}.$$

In particular, the zero set Z is almost surely a (possibly empty) complex submanifold of complex codimension r in M.

Proof. It is enough to check the result locally. On an open subset $\Omega \subset M$ over which E is trivial, we can consider s as a nondegenerate smooth centered Gaussian field from Ω to $\mathbb{C}^r \simeq \mathbb{R}^{2r}$. Then, the local result follows from [Lerario and Stecconi 2019, Theorem 7]. $\square$

Let us consider Z as random submanifold of real codimension $2r$ in the Riemannian manifold (M, g) of real dimension $2n$. The metric g induces a $(2n-2r)$-dimensional Riemannian volume dVol_Z on Z and we can define ν as in Definition 6.11, bearing in mind that $Z = Z_{\mathrm{reg}}$. Then Propositions 6.17, 6.24 and 6.25 hold for the holomorphic field s and the associated linear statistics $\langle \nu, \phi\rangle$ with $\phi \in L^\infty_c(M)$. More generally, everything we did in Sections 6.2, 6.3 and 6.4 adapts to the holomorphic setting.

Remark 8.11. In Definition 6.23, the function ρ_p is defined in terms of real Jacobians and the variance of $(s(x_1), \dots, s(x_p))$ seen as a real Gaussian vector. One can check that another expression of $\rho_p(x_1, \dots, x_p)$ is the following, which is more natural in our holomorphic framework:

$$\forall (x_1, \dots, x_p) \notin \Delta_p, \quad \rho_p(x_1, \dots, x_p) = \frac{\mathbb{E}\big[\prod_{i=1}^p \mathrm{Jac}_{\mathbb{C}}(\nabla_{x_i} s) \mid \forall i \in [\![1, p]\!], s(x_i) = 0\big]}{\det(\pi \,\mathrm{Var}_{\mathbb{C}}(s(x_1), \dots, s(x_p)))}.$$

We can now state the equivalent of Theorem 6.26 for holomorphic Gaussian fields. Let $\Omega \subset \mathbb{C}^n$ be open. Recall that $\mathcal{MJ}^{\mathbb{C}}_p(\Omega, \mathbb{C}^r) \to C^{\mathbb{C}}_p[\Omega]$ is defined in Definition 8.3 as the restriction over $C^{\mathbb{C}}_p[\Omega] \subset C^{\mathbb{C}}_p[\mathbb{C}^n]$ of the vector bundle $\mathcal{MJ}^{\mathbb{C}}_p(\mathbb{C}^n, \mathbb{C}^r) \to C^{\mathbb{C}}_p[\mathbb{C}^n]$ from Theorem 8.2.

Theorem 8.12 (finiteness of moments for holomorphic fields, local version). *Let $f : \Omega \to \mathbb{C}^r$ be a centered holomorphic Gaussian field and ν be as in Definition 6.11. Let $p \geqslant 1$. If for all $k \in [\![1, p]\!]$ the holomorphic Gaussian field $\mathrm{mj}^{\mathbb{C}}_k(f, \cdot) \in H^0(C^{\mathbb{C}}_k[\Omega], \mathcal{MJ}^{\mathbb{C}}_k(\Omega, \mathbb{C}^r))$ is nondegenerate, then the four equivalent statements in Proposition 6.25 hold.*

Proof. The proof of Theorem 6.26 relies mostly on two facts that are valid for all $k \in [\![1, p]\!]$. First, on the open dense subset $\Omega^k \setminus \Delta_k \subset C_k[\Omega]$, the zero set of $\mathrm{mj}_k(f, \cdot)$ is the same as that of $(x_1, \dots, x_k) \mapsto$

$(f(x_1), \dots, f(x_k))$. And second, the field $\mathrm{mj}_k(f, \cdot)$ is nondegenerate on $C_k[\Omega]$, so that we can apply the Kac–Rice formula (Proposition 6.17) to the k-multijet.

These two facts are still true in the present holomorphic setting. Hence, the same proof as that of Theorem 6.26 yields the result. □

We deduce from this result the equivalent of Theorem 1.9 for a centered holomorphic Gaussian field s on a complex manifold M of dimension n with values in a holomorphic vector bundle E of rank $r \in [\![1, n]\!]$. Theorem 1.11 is a special case of the following.

Theorem 8.13 (finiteness of moments for zeros of holomorphic Gaussian sections). *Let $p \geqslant 1$, let $s \in H^0(M, E)$ be a centered holomorphic Gaussian field and let ν be as in Definition 6.11. If s is $(p-1)$-nondegenerate then $\mathbb{E}[|\langle \nu, \phi \rangle|^p] < +\infty$ for all $\phi \in L^\infty_c(M)$.*

Proof. We deduce Theorem 8.13 from Theorem 8.12 in the same way that we deduced Theorem 1.9 from Theorem 6.26; see Section 6.4. □

Remark 8.14. In particular, Theorem 8.13 proves the local integrability of the p-points correlation functions studied in [Bleher et al. 2000] and their scaling limit.

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EXISTENCE OF SOLUTIONS TO A FRACTIONAL SEMILINEAR HEAT EQUATION IN UNIFORMLY LOCAL WEAK ZYGMUND-TYPE SPACES

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We introduce uniformly local weak Zygmund-type spaces and obtain an optimal sufficient condition for the existence of solutions to the critical fractional semilinear heat equation.

1. Introduction

Consider the Cauchy problem for the fractional semilinear heat equation

$$\begin{cases} \partial_t u + (-\Delta)^{\frac{\theta}{2}} u = |u|^{p-1}u, & x \in \mathbb{R}^n, \ t > 0, \\ u(x,0) = \varphi(x), & x \in \mathbb{R}^n, \end{cases} \tag{P}$$

where $n \geq 1$, $\partial_t := \partial/\partial t$, $\theta \in (0,2]$, $p > 1$, and φ is a locally integrable function in $\mathbb{R}^n$. Here $(-\Delta)^{\theta/2}$ denotes the fractional power of the Laplace operator $-\Delta$ in $\mathbb{R}^n$. In this paper we establish the local-in-time existence of solutions to problem (P) in the critical case

$$p = p_\theta := 1 + \frac{\theta}{n}$$

in the framework of uniformly local weak Zygmund-type spaces.

The solvability of the Cauchy problem for semilinear heat equations has fascinated many mathematicians since the pioneering work by Fujita [1966]. The literature is very large, and we refer the reader to the comprehensive monograph [Quittner and Souplet 2007] and the papers [Andreucci and DiBenedetto 1991; Baras and Pierre 1985; Brezis and Cazenave 1996; Fujishima et al. 2023; 2024; Fujishima and Ioku 2021; 2022; Giraudon and Miyamoto 2022; Hisa and Ishige 2018; Hisa et al. 2023; Ishige et al. 2014; 2020; 2022; Kozono and Yamazaki 1994; Laister and Sierżęga 2020; 2021; Laister et al. 2016; Miyamoto 2021; Robinson and Sierżęga 2013; Sugitani 1975; Weissler 1981; Zhanpeisov 2023], some of which are closely related to this paper, while the others include recent developments in this subject. The study of the solvability of problem (P) is divided into the following three cases:

$$1 < p < p_\theta \ \text{(subcritical case)}, \qquad p > p_\theta \ \text{(supercritical case)}, \qquad p = p_\theta \ \text{(critical case)}.$$

We collect some known results on necessary conditions and sufficient conditions for the existence of solutions to problem (P).

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Keywords: solvability, fractional semilinear heat equation, Zygmund-type spaces.

(1) Subcritical case ($1 < p < p_\theta$).

(a) Necessity: There exists $C_1 = C_1(n, \theta, p) > 0$ such that if problem (P) possesses a nonnegative solution in $\mathbb{R}^n \times (0, T)$ for some $T > 0$, then

$$\sup_{x\in\mathbb{R}^n} \int_{B(x,T^{1/\theta})} \varphi(y)\,dy \le C_1 T^{\frac{n}{\theta}-\frac{1}{p-1}}.$$

See [Andreucci and DiBenedetto 1991; Baras and Pierre 1985] for $\theta = 2$ and [Hisa and Ishige 2018] for $\theta \in (0, 2]$.

(b) Sufficiency: There exists $\epsilon_1 = \epsilon_1(n, \theta, p) > 0$ such that if

$$\sup_{x\in\mathbb{R}^n} \int_{B(x,T^{1/\theta})} |\varphi(y)|\,dy \le \epsilon_1 T^{\frac{n}{\theta}-\frac{1}{p-1}}$$

for some $T \in (0, \infty)$, then problem (P) possesses a solution in $\mathbb{R}^n \times (0, T)$. See, e.g., [Andreucci and DiBenedetto 1991; Hisa and Ishige 2018; Weissler 1981].

The results in (a) and (b) (see also (1-4)) imply that, for any nonnegative measurable initial function φ in $\mathbb{R}^n$, problem (P) possesses a local-in-time nonnegative solution if and only if

$$\sup_{x\in\mathbb{R}^n} \int_{B(x,1)} \varphi(y)\,dy < \infty.$$

(2) Supercritical case ($p > p_\theta$).

(a) Necessity: There exists $C_2 = C_2(n, \theta, p) > 0$ such that if problem (P) possesses a nonnegative solution in $\mathbb{R}^n \times (0, T)$ for some $T > 0$, then

$$\sup_{x\in\mathbb{R}^n} \sup_{\sigma\in(0,T^{1/\theta})} |B(x,\sigma)|^{\frac{\theta}{n(p-1)}-1} \int_{B(x,\sigma)} \varphi(y)\,dy \le C_2.$$

See [Andreucci and DiBenedetto 1991; Baras and Pierre 1985] for $\theta = 2$ and [Hisa and Ishige 2018] for $\theta \in (0, 2]$.

(b) Sufficiency: For any $r \in (1, \infty)$, there exists $\epsilon_2 = \epsilon_2(n, \theta, p, r) > 0$ such that if

$$\sup_{x\in\mathbb{R}^n} \sup_{\sigma\in(0,T^{1/\theta})} |B(x,\sigma)|^{\frac{\theta}{n(p-1)}-\frac{1}{r}} \left[\int_{B(x,\sigma)} |\varphi(y)|^r\,dy\right]^{\frac{1}{r}} \le \epsilon_2$$

for some $T \in (0, \infty]$, then problem (P) possesses a solution in $\mathbb{R}^n \times (0, T)$. See [Kozono and Yamazaki 1994; Robinson and Sierżęga 2013] for $\theta = 2$ and [Hisa and Ishige 2018; Ishige et al. 2020; 2022; Zhanpeisov 2023] for $\theta \in (0, 2]$. See, e.g., [Andreucci and DiBenedetto 1991; Ishige et al. 2014; Weissler 1981] for related results.

(3) Critical case ($p = p_\theta$).

(a) Necessity: There exists $C_3 = C_3(n, \theta) > 0$ such that if problem (P) possesses a nonnegative solution in $\mathbb{R}^n \times (0, T)$ for some $T > 0$, then

$$\sup_{x\in\mathbb{R}^n} \int_{B(x,\sigma)} \varphi(y)\,dy \le C_3 \left[\log\left(e + \frac{T^{1/\theta}}{\sigma}\right)\right]^{-\frac{n}{\theta}}, \quad \sigma \in (0, T^{\frac{1}{\theta}}).$$

See [Baras and Pierre 1985] for $\theta = 2$ and [Hisa and Ishige 2018] for $\theta \in (0, 2]$.

(b) Sufficiency: For any $\alpha > 0$, there exists $\epsilon_3 = \epsilon_3(n, \theta, \alpha) > 0$ such that if

$$\sup_{x\in\mathbb{R}^n} \Psi_\alpha^{-1}\left[\frac{1}{|B(x,\sigma)|}\int_{B(x,\sigma)} \Psi_\alpha(T^{\frac{1}{p-1}}|\varphi(y)|)\,dy\right] \le \epsilon_3\rho(\sigma T^{-\frac{1}{\theta}}), \quad \sigma \in (0, T^{\frac{1}{\theta}}),$$

for some $T > 0$, then problem (P) possesses a solution in $\mathbb{R}^n \times (0, T)$, where

$$\Psi_\alpha(s) := s[\log(e+s)]^\alpha, \quad \rho(s) := s^{-n}\left[\log\left(e+\frac{1}{s}\right)\right]^{-\frac{n}{\theta}}.$$

See [Hisa and Ishige 2018; Ishige et al. 2020; 2022].

Furthermore, the results in (2) and (3) imply the following results.

(4) Let $p \ge p_\theta$, and set

$$\varphi_c(x) := \begin{cases} |x|^{-n}\left[\log\left(e+\frac{1}{|x|}\right)\right]^{-\frac{n}{\theta}-1} & \text{if } p = p_\theta, \\ |x|^{-\frac{\theta}{p-1}} & \text{if } p > p_\theta \end{cases} \quad \text{for } x \in \mathbb{R}^n. \tag{1-1}$$

(a) There exists $C_4 = C_4(n, \theta, p) > 0$ such that if

$$\varphi(x) \ge C_4\varphi_c(x)$$

for almost all x in a neighborhood of the origin, then problem (P) possesses no local-in-time nonnegative solutions.

(b) There exists $\epsilon_4 = \epsilon_4(n, \theta, p) > 0$ such that if

$$|\varphi(x)| \le \epsilon_4\varphi_c(x) + K, \quad \text{a.a. } x \in \mathbb{R}^n,$$

for some $K \ge 0$, then problem (P) possesses a local-in-time solution.

The results in (4) show that the "strength" of the singularity at the origin of the function φ_c is the critical threshold for the local-in-time solvability of problem (P). The function φ_c is quite useful for identifying optimal function spaces to which initial functions belong from the view of the solvability of problem (P). We remark that assertion (2b) with $r = 1$ and assertion (3b) with $\alpha = 0$ do not hold. (See [Takahashi 2016, Theorem 1 and Proposition 1], which treat only the case of $\theta = 2$ but which is also applicable to the case of $\theta \in (0, 2)$. See also [Kan and Takahashi 2017, Section 4].)

There are (at least) two useful strategies for the proof of the existence of solutions to problem (P). One is the supersolution method (SSM) and the other is the contraction mapping theorem (CMT). SSM depends on the following principle: if there exists a nonnegative supersolution v to problem (P) in $\mathbb{R}^n \times (0, T)$ for some $T > 0$, then problem (P) possesses a nonnegative solution u in $\mathbb{R}^n \times (0, T)$ such that $u \le v$ in $\mathbb{R}^n \times (0, T)$. In our problem (P) with nonnegative initial function φ, the following functions have been used as supersolutions in $\mathbb{R}^n \times (0, T)$ for some $T > 0$:

$$2S_\theta(t)\varphi \ (1 < p < p_\theta), \quad 2(S_\theta(t)\varphi^r)^{\frac{1}{r}} \ (p > p_\theta), \quad 2\Psi_\alpha^{-1}(S_\theta(t)\Psi_\alpha(\varphi)) \ (p = p_\theta),$$

where $S_\theta(t)\varphi$ is a solution to the fractional heat equation (see (1-5)), $r > 1$, and Ψ_α is as in assertion (3b). (See, e.g., [Weissler 1981] for $1 < p < p_\theta$; [Hisa and Ishige 2018; Robinson and Sierżęga 2013] for $p > p_\theta$; [Hisa and Ishige 2018] for $p = p_\theta$.) Furthermore, thanks to the arguments in [Tayachi and Weissler 2014], SSM is also applicable to the study of the existence of sign-changing solutions to problem (P) (see [Ishige et al. 2020; 2022]), however we require additional arguments in which sense the solution converges to the initial function. On the other hand, CMT is widely used in the proof of the existence of solutions in various evolution equations, and the choice of function spaces is crucial. For our problem (P) with $p > p_\theta$, the existence of solutions has been proved by CMT in the framework of weak Lebesgue spaces (see [Fujishima and Ioku 2021; Ishige et al. 2014]) and Morrey spaces (see [Kozono and Yamazaki 1994; Zhanpeisov 2023]). The results in [Fujishima and Ioku 2021; Ishige et al. 2014; Kozono and Yamazaki 1994; Zhanpeisov 2023] cover the result in (4b) with $p > p_\theta$. However, in the critical case $p = p_\theta$, the arguments in [Fujishima and Ioku 2021; Ishige et al. 2014; Kozono and Yamazaki 1994; Zhanpeisov 2023] are not applicable to the proof of assertion (4b) by the logarithmic singularity of φ_c.

	supersolution method (SSM)	weak spaces (CMT)	Morrey spaces (CMT)
$p > p_\theta$	[Hisa and Ishige 2018] [Robinson and Sierżęga 2013]	[Fujishima and Ioku 2021] [Ishige et al. 2014]	[Kozono and Yamazaki 1994] [Zhanpeisov 2023]
$p = p_\theta$	[Hisa and Ishige 2018]	open	not applicable (see [Takahashi 2016])

The aim of this paper is to establish a sharp sufficient condition on the existence of solutions to problem (P) in the critical case $p = p_\theta$ in the framework of Banach spaces. For the critical case $p = p_\theta$, the weak Zygmund space $L^{1,\infty}(\log L)^{1+n/\theta}$ seems a reasonable Banach space since $\varphi_c \in L^{1,\infty}(\log L)^{1+n/\theta}$. (See Remark 4.4 (i) for the definition of the weak Zygmund spaces $L^{q,\infty}(\log L)^\alpha$, where $1 \le q < \infty$ and $\alpha \ge 0$.) Then we require sharp decay estimates of solutions to the fractional heat equation in the weak Zygmund spaces $L^{q,\infty}(\log L)^\alpha$; however, by the peculiarity of $L^{1,\infty}(\log L)^\alpha$, it seems difficult to obtain our desired sharp decay estimates. (See Remark 4.4 (ii) for further details.)

In this paper we introduce new weak Zygmund-type spaces $\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha$ and uniformly local weak Zygmund-type spaces $\mathfrak{L}^{q,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^\alpha$. Then we establish sharp decay estimates of solutions to the fractional heat equation in the spaces $\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha$ and $\mathfrak{L}^{q,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^\alpha$, and obtain a sufficient condition on the existence of solutions to problem (P) with $p = p_\theta$ in the framework of the space $\mathfrak{L}^{q,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^\alpha$. Our sufficient condition is simpler than that of assertion (3b) and covers assertion (4b) with $p = p_\theta$.

We introduce some notation and define the weak Zygmund-type spaces $\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha$ and the uniformly local weak Zygmund-type spaces $\mathfrak{L}^{q,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^\alpha$. We also formulate the definition of solutions to problem (P). Let $\mathcal{M}$ be the set of Lebesgue measurable sets in $\mathbb{R}^n$. For any $E \in \mathcal{M}$, we denote by $|E|$ and χ_E the n-dimensional Lebesgue measure of E and the characteristic function of E, respectively. Let L^1_{loc} be the set of locally integrable functions in $\mathbb{R}^n$. For any $q \in [1, \infty]$, we denote by L^q and $\|\cdot\|_{L^q}$ the usual L^q-space on $\mathbb{R}^n$ and its norm, respectively.

Let $q \in [1, \infty]$ and $\alpha \in [0, \infty)$. We define the weak Zygmund-type space $\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha$ by

$$\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha := \{ f \in L^1_{\mathrm{loc}} : \|f\|_{\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha} < \infty \},$$

where

$$\|f\|_{\mathfrak{L}^{q,\infty}(\log\mathfrak{L})^\alpha} := \begin{cases} \sup_{s>0}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^\alpha \sup_{|E|=s} \int_E |f(x)|^q\,dx\right\}^{\frac{1}{q}} & \text{if } q\in[1,\infty),\\ \|f\|_{L^\infty} & \text{if } q=\infty. \end{cases} \tag{1-2}$$

Then $\mathfrak{L}^{q,\infty}(\log\mathfrak{L})^\alpha$ is a Banach space equipped with the norm $\|\cdot\|_{\mathfrak{L}^{q,\infty}(\log\mathfrak{L})^\alpha}$ (see Lemma 2.1). See (2-9) for different expressions of the norm $\|\cdot\|_{\mathfrak{L}^{q,\infty}(\log\mathfrak{L})^\alpha}$. We remark that

$$L^q = \mathfrak{L}^{q,\infty}(\log\mathfrak{L})^0 \supset \mathfrak{L}^{q,\infty}(\log\mathfrak{L})^\alpha \quad \text{for } \alpha\geq 0. \tag{1-3}$$

Next, we introduce the uniformly local weak Zygmund-type space $\mathfrak{L}^{q,\infty}_{\mathrm{ul}}(\log\mathfrak{L})^\alpha$ by

$$\mathfrak{L}^{q,\infty}_{\mathrm{ul}}(\log\mathfrak{L})^\alpha := \{f\in L^1_{\mathrm{loc}} : \|f\|_{\mathfrak{L}^{q,\infty}_{\mathrm{ul}}(\log\mathfrak{L})^\alpha} < \infty\},$$

where

$$\|f\|_{\mathfrak{L}^{q,\infty}_{\mathrm{ul}}(\log\mathfrak{L})^\alpha} := \sup_{z\in\mathbb{R}^n} \|f\chi_{B(z,1)}\|_{\mathfrak{L}^{q,\infty}(\log\mathfrak{L})^\alpha}.$$

Then $\mathfrak{L}^{q,\infty}_{\mathrm{ul}}(\log\mathfrak{L})^\alpha$ is also a Banach space equipped with the norm $\|\cdot\|_{\mathfrak{L}^{q,\infty}_{\mathrm{ul}}(\log\mathfrak{L})^\alpha}$. We often write, for any $f\in\mathfrak{L}^{q,\infty}_{\mathrm{ul}}(\log\mathfrak{L})^\alpha$ and $\rho>0$,

$$|||f|||_{q,\alpha;\rho} := \sup_{z\in\mathbb{R}^n} \|f\chi_{B(z,\rho)}\|_{\mathfrak{L}^{q,\infty}(\log\mathfrak{L})^\alpha}$$

for simplicity. We remark that $\mathfrak{L}^{\infty,\infty}_{\mathrm{ul}}(\log\mathfrak{L})^\alpha = L^\infty$ and $|||\cdot|||_{\infty,\alpha;\rho} = \|\cdot\|_{L^\infty}$ for all $\alpha\in[0,\infty)$. Notice that, for any $k\geq 1$, there exists $C=C(n,k)>0$ such that

$$|||f|||_{q,\alpha;k\rho} \leq C\,|||f|||_{q,\alpha;\rho} \tag{1-4}$$

for $f\in\mathfrak{L}^{q,\infty}_{\mathrm{ul}}(\log\mathfrak{L})^\alpha$ and $\rho>0$.

We formulate the definition of solutions to problem (P). Let $\theta\in(0,2]$. Let G_θ be the fundamental solution to the fractional heat equation

$$\partial_t v + (-\Delta)^{\frac{\theta}{2}} v = 0 \quad \text{in } \mathbb{R}^n\times(0,\infty).$$

For any φ in L^1_{loc}, we write

$$(S_\theta(t)\varphi)(x) := \int_{\mathbb{R}^n} G_\theta(x-y,t)\varphi(y)\,dy, \quad (x,t)\in\mathbb{R}^n\times(0,\infty), \tag{1-5}$$

for simplicity.

Definition 1.1. Let $\theta\in(0,2]$, $p>1$, and $T>0$. Set $F_p(s) := |s|^{p-1}s$ for $s\in\mathbb{R}$. Let u be a measurable and finite almost everywhere function in $\mathbb{R}^n\times(0,T)$. We say that u is a solution to problem (P) in $\mathbb{R}^n\times(0,T)$ if, for almost all $(x,t)\in\mathbb{R}^n\times(0,T)$,

- $G_\theta(x-y,t)\varphi(y)$ is integrable in $\mathbb{R}^n$ with respect to $y\in\mathbb{R}^n$,
- $G_\theta(x-y,t-s)F_p(u(y,s))$ is integrable in $\mathbb{R}^n\times(0,t)$ with respect to $(y,s)\in\mathbb{R}^n\times(0,t)$,
- u satisfies

$$u(x,t) = [S_\theta(t)\varphi](x) + \int_0^t [S_\theta(t-s)F_p(u(s))](x)\,ds.$$

We are ready to state our main results.

Theorem 1.2. *Let $\theta \in (0,2]$, $p = p_\theta = 1+\theta/n$, and $T_* \in (0,\infty)$. Then there exists $\epsilon > 0$ such that if $\varphi \in \mathfrak{L}^{1,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^{n/\theta}$ satisfies*

$$|||\varphi|||_{1,\frac{n}{\theta};T^{1/\theta}} \le \epsilon \quad \text{for some } T \in (0, T_*],$$

then problem (P) possesses a solution $u \in C((0,T) : \mathfrak{L}^{1,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^{n/\theta}) \cap L^\infty_{\mathrm{loc}}((0,T) : L^\infty)$ in $\mathbb{R}^n \times (0,T)$, with u satisfying

$$\sup_{t\in(0,T)} |||u(t)|||_{1,\frac{n}{\theta};T^{1/\theta}} + \sup_{t\in(0,T)} t^{\frac{n}{\theta}} \left[\log\left(e+\frac{1}{t}\right)\right]^{\frac{n}{\theta}} \|u(t)\|_{L^\infty} < \infty. \tag{1-6}$$

Furthermore, the solution u satisfies

$$\begin{aligned} \lim_{t\to+0} \|u(t) - S_\theta(t)\varphi\|_{\mathfrak{L}^{1,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^\gamma} &= 0 \quad \text{for any } \gamma \in [0, n/\theta), \\ \lim_{t\to+0} u(t) &= \varphi \quad \text{in the sense of distributions.} \end{aligned} \tag{1-7}$$

We remark that Theorem 1.2 with $T_* = \infty$ does not hold since problem (P) possesses no global-in-time positive solutions (see [Sugitani 1975]). As a direct consequence of Theorem 1.2, we obtain assertion (4b).

Corollary 1.3. *Let $\theta \in (0,2]$ and $p = p_\theta$. Let φ_c be as in (1-1). Then there exists $\epsilon > 0$ such that if*

$$|\varphi(x)| \le \epsilon \varphi_c(x) + K, \quad \text{a.a. } x \in \mathbb{R}^n,$$

for some $K \ge 0$, then problem (P) possesses a local-in-time solution.

Furthermore, as a consequence of Theorem 1.2, we have the following.

Theorem 1.4. *Let $\theta \in (0,2]$ and $p = p_\theta$. If $\varphi \in \mathfrak{L}^{1,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^\alpha$ for some $\alpha > n/\theta$, then problem (P) possesses a solution u in $\mathbb{R}^n \times (0,T)$ for some $T > 0$, with u satisfying (1-6) and (1-7).*

The rest of this paper is organized as follows. In Section 2 we collect some properties of nonincreasing rearrangements of measurable functions and prove some lemmas in $\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha$ and $\mathfrak{L}^{q,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^\alpha$. Furthermore, we recall Hardy's inequalities and some properties of $S_\theta(t)\varphi$. In Section 3 we establish decay estimates of $S_\theta(t)\varphi$ in weak Zygmund-type spaces (see Proposition 3.1). Furthermore, we obtain decay estimates of $S_\theta(t)\varphi$ in $\mathfrak{L}^{q,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^\alpha$ using Besicovitch's covering lemma. In Section 4 we apply the contraction mapping theorem in $\mathfrak{L}^{q,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^\alpha$ to prove Theorems 1.2 and 1.4. In the Appendix we give two propositions on relations among the weak Zygmund-type spaces $\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha$, the Zygmund spaces $L^q(\log L)^\alpha$, and the weak Zygmund spaces $L^{q,\infty}(\log L)^\alpha$.

2. Preliminaries

In this section we introduce some notation and give some lemmas on our weak Zygmund-type spaces. Furthermore, we recall some lemmas on Hardy's inequalities. In all that follows, we will use C to denote generic positive constants and point out that C may take different values within a calculation. For any positive functions f_1 and f_2 in $(0,\infty)$, we write

$$f_1 \asymp f_2 \ \text{ for } s > 0 \quad \text{if } C^{-1} f_2(s) \le f_1(s) \le C f_2(s) \ \text{ for } s > 0.$$

2.1. ***Weak Zygmund-type spaces.*** For any (Lebesgue) measurable function f in $\mathbb{R}^n$, we denote by μ_f the distribution function of f, that is,

$$\mu_f(\lambda) := |\{x : |f(x)| > \lambda\}| \quad \text{for } \lambda > 0.$$

We define the nonincreasing rearrangement f^* of f by

$$f^*(s) := \inf\{\lambda > 0 : \mu_f(\lambda) \le s\} \quad \text{for } s \in [0, \infty).$$

Here we adopt the convention $\inf \varnothing = \infty$. Then f^* is nonincreasing and right-continuous in $[0, \infty)$, and it has the following properties (see [Grafakos 2008, Proposition 1.4.5]):

$$(kf)^* = |k| f^*, \quad (|f|^q)^* = (f^*)^q, \quad \int_{\mathbb{R}^n} |f(x)|^q \, dx = \int_0^\infty f^*(s)^q \, ds, \quad f^*(0) = \|f\|_{L^\infty}, \tag{2-1}$$

where $q \in (0, \infty)$ and $k \in \mathbb{R}$. We remark that if $E \in \mathcal{M}$ with $|E| < \infty$, then

$$(\chi_E)^*(s) = \chi_{[0,|E|)}(s) \quad \text{for } s \ge 0. \tag{2-2}$$

Define

$$f^{**}(s) := \frac{1}{s} \int_0^s f^*(\tau) \, d\tau \quad \text{for } s \in (0, \infty). \tag{2-3}$$

Here we collect properties of f^* and f^{**} used in the paper.

(a) Since f^* is nonincreasing in $(0, \infty)$, it follows that

$$f^{**}(s) \ge f^*(s) \quad \text{for } s \in (0, \infty). \tag{2-4}$$

(b) For any $q \in [1, \infty)$, Jensen's inequality together with (2-1) yields

$$(f^{**}(s))^q \le \frac{1}{s} \int_0^s f^*(\tau)^q \, d\tau = \frac{1}{s} \int_0^s (|f|^q)^*(\tau) \, d\tau = (|f|^q)^{**}(s) \quad \text{for } s \in (0, \infty). \tag{2-5}$$

(c) It follows from [Bennett and Sharpley 1988, Chapter 2, Proposition 3.3] that

$$f^{**}(s) = \frac{1}{s} \int_0^s f^*(\tau) \, d\tau = \frac{1}{s} \sup_{|E|=s} \int_E |f(x)| \, dx \quad \text{for } s \in (0, \infty). \tag{2-6}$$

(d) (O'Neil's inequality) For any $f, g \in L^1$, it follows from [O'Neil 1963, Lemma 1.6] that

$$(f * g)^{**}(s) \le \int_s^\infty f^{**}(\tau) g^{**}(\tau) \, d\tau \quad \text{for } s \in (0, \infty), \tag{2-7}$$

where

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x - y) g(y) \, dy.$$

(e) For any $f_1, f_2 \in L^1_{\mathrm{loc}}$, it follows from [O'Neil 1963, Theorem 3.3] that

$$(f_1 f_2)^{**}(s) \le \frac{1}{s} \int_0^s f_1^*(\tau) f_2^*(\tau) \, d\tau \quad \text{for } s \in (0, \infty). \tag{2-8}$$

Let $q \in [1,\infty)$ and $\alpha \geq 0$. For any L^1_{loc}-function f, by (1-2), (2-1), and (2-6), we have

$$\begin{aligned} \|f\|_{\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha} &= \sup_{s>0}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^\alpha s(|f|^q)^{**}(s)\right\}^{\frac{1}{q}} = \sup_{s>0}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^\alpha \int_0^s (|f|^q)^*(\tau)\,d\tau\right\}^{\frac{1}{q}} \\ &= \sup_{s>0}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^\alpha \int_0^s f^*(\tau)^q\,d\tau\right\}^{\frac{1}{q}}. \end{aligned} \tag{2-9}$$

Furthermore, for any $E \in \mathcal{M}$ with $|E| < \infty$, it follows from (2-2) and (2-8) that

$$(f\chi_E)^{**}(s) = (f\chi_E\chi_E)^{**}(s) \leq \frac{1}{s}\int_0^s (f\chi_E)^*(\tau)(\chi_E)^*(\tau)\,d\tau = \frac{1}{s}\int_0^{\min\{s,|E|\}} (f\chi_E)^*(\tau)\,d\tau. \tag{2-10}$$

For any $\beta \in [\alpha,\infty)$, since

$$\text{the function } (0,\infty) \ni \tau \mapsto \left[\log\left(e+\frac{1}{\tau}\right)\right]^{\alpha-\beta} \in \mathbb{R} \text{ is nondecreasing,} \tag{2-11}$$

by (2-1), (2-9), and (2-10), we have

$$\begin{aligned} \|f\chi_E\|_{\mathfrak{L}^{1,\infty}(\log \mathfrak{L})^\alpha} &= \sup_{s>0}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^\alpha s(f\chi_E)^{**}(s)\right\} \\ &\leq \sup_{s>0}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^\alpha \int_0^{\min\{s,|E|\}} (f\chi_E)^*(\tau)\,d\tau\right\} \\ &= \sup_{0<s\leq|E|}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^{\beta+\alpha-\beta} \int_0^{\min\{s,|E|\}} (f\chi_E)^*(\tau)\,d\tau\right\} \\ &\leq \left[\log\left(e+\frac{1}{|E|}\right)\right]^{\alpha-\beta} \sup_{0<s\leq|E|}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^{\beta} \int_0^{\min\{s,|E|\}} (f\chi_E)^*(\tau)\,d\tau\right\} \\ &\leq \left[\log\left(e+\frac{1}{|E|}\right)\right]^{\alpha-\beta} \sup_{s>0}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^{\beta} \int_0^{s} (f\chi_E)^*(\tau)\,d\tau\right\} \\ &= \left[\log\left(e+\frac{1}{|E|}\right)\right]^{\alpha-\beta} \|f\chi_E\|_{\mathfrak{L}^{1,\infty}(\log \mathfrak{L})^\beta}. \end{aligned}$$

In particular,

$$|||f|||_{1,\alpha;\rho} \leq C\left[\log\left(e+\frac{1}{\rho}\right)\right]^{\alpha-\beta} |||f|||_{1,\beta;\rho} \tag{2-12}$$

for $f \in \mathfrak{L}^{1,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^\beta$, $0 \leq \alpha \leq \beta$, and $\rho > 0$. Here we show that $\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha$ and $\mathfrak{L}^{q,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^\alpha$ are Banach spaces.

Lemma 2.1. *For any $1 \leq q < \infty$ and $\alpha \geq 0$, the weak Zygmund-type space $\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha$ and the uniformly local weak Zygmund-type space $\mathfrak{L}^{q,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^\alpha$ are Banach spaces.*

Proof. Let $1 \leq q < \infty$ and $\alpha \geq 0$. It suffices to prove that $\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha$ (resp. $\mathfrak{L}^{q,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^\alpha$) is a complete metric space with the norm $\|\cdot\|_{\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha}$ (resp. $\|\cdot\|_{\mathfrak{L}^{q,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^\alpha}$). Let $\{f_n\}$ be a Cauchy sequence in $\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha$. It follows from (1-3) that $\{f_n\}$ is a Cauchy sequence in L^q, and hence there exists $f \in L^q$

such that $f_n \to f$ as $n \to \infty$ in L^q. Since the Cauchy sequence $\{f_n\}$ is bounded in $\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha$, we observe from (1-2) that $f \in \mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha$. It remains to prove that $f_n \to f$ as $n \to \infty$ in $\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha$. For this aim, we take a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}\}$ converges almost everywhere to f. Then Fatou's lemma gives us that

$$\begin{aligned}\|f - f_{n_j}\|_{\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha} &\le \sup_{s>0}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^\alpha \sup_{|E|=s}\liminf_{k\to\infty}\int_E |f_{n_k}(x) - f_{n_j}(x)|^q\,dx\right\}^{\frac{1}{q}}\\ &\le \liminf_{k\to\infty}\|f_{n_k} - f_{n_j}\|_{\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha}.\end{aligned}$$

This implies that f_{n_j} converges to f in $\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha$. Thus $\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha$ is a complete metric space. Similarly, we see that $\mathfrak{L}^{q,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^\alpha$ is a complete metric space. □

Next, we prove two lemmas on our weak Zygmund-type spaces and uniformly local weak Zygmund-type spaces.

Lemma 2.2. *Let $q_1, q_2 \in [1,\infty]$ and $\alpha_1, \alpha_2 \ge 0$ be such that*

$$1 = \frac{1}{q_1} + \frac{1}{q_2}, \qquad \alpha = \frac{\alpha_1}{q_1} + \frac{\alpha_2}{q_2}. \tag{2-13}$$

Then

$$\|f_1 f_2\|_{\mathfrak{L}^{1,\infty}(\log \mathfrak{L})^\alpha} \le \|f_1\|_{\mathfrak{L}^{q_1,\infty}(\log \mathfrak{L})^{\alpha_1}} \|f_2\|_{\mathfrak{L}^{q_2,\infty}(\log \mathfrak{L})^{\alpha_2}} \tag{2-14}$$

for $f_1 \in \mathfrak{L}^{q_1,\infty}(\log \mathfrak{L})^{\alpha_1}$ and $f_2 \in \mathfrak{L}^{q_2,\infty}(\log \mathfrak{L})^{\alpha_2}$. Furthermore,

$$|||\tilde{f}_1 \tilde{f}_2|||_{1,\alpha;\rho} \le |||\tilde{f}_1|||_{q_1,\alpha_1;\rho} |||\tilde{f}_2|||_{q_2,\alpha_2;\rho} \tag{2-15}$$

for $\tilde{f}_1 \in \mathfrak{L}^{q_1,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^{\alpha_1}$, $\tilde{f}_2 \in \mathfrak{L}^{q_2,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^{\alpha_2}$, and $\rho > 0$.

Proof. Let $q_1, q_2 \in [1,\infty)$ and $\alpha_1, \alpha_2 \ge 0$ satisfy (2-13). Let

$$f_1 \in \mathfrak{L}^{q_1,\infty}(\log \mathfrak{L})^{\alpha_1} \quad \text{and} \quad f_2 \in \mathfrak{L}^{q_2,\infty}(\log \mathfrak{L})^{\alpha_2}.$$

It follows from Hölder's inequality, (2-8), and (2-9) that

$$\begin{aligned}\|f_1 f_2\|_{\mathfrak{L}^{1,\infty}(\log \mathfrak{L})^\alpha} &= \sup_{s>0}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^\alpha s (f_1 f_2)^{**}(s)\right\}\\ &\le \sup_{s>0}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^\alpha \int_0^s f_1^*(\tau) f_2^*(\tau)\,d\tau\right\}\\ &\le \sup_{s>0}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^\alpha \left(\int_0^s f_1^*(\tau)^{q_1}\,d\tau\right)^{\frac{1}{q_1}} \left(\int_0^s f_2^*(\tau)^{q_2}\,d\tau\right)^{\frac{1}{q_2}}\right\}\\ &\le \sup_{s>0}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^{\alpha_1} \int_0^s f_1^*(\tau)^{q_1}\,d\tau\right\}^{\frac{1}{q_1}} \sup_{s>0}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^{\alpha_2} \int_0^s f_2^*(\tau)^{q_2}\,d\tau\right\}^{\frac{1}{q_2}}\\ &= \|f_1\|_{\mathfrak{L}^{q_1,\infty}(\log \mathfrak{L})^{\alpha_1}} \|f_2\|_{\mathfrak{L}^{q_2,\infty}(\log \mathfrak{L})^{\alpha_2}}.\end{aligned}$$

Thus (2-14) holds. Furthermore, for any $\tilde{f}_1 \in \mathfrak{L}^{q_1,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^{\alpha_1}$, $\tilde{f}_2 \in \mathfrak{L}^{q_2,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^{\alpha_2}$, and $\rho > 0$, by (2-14) we have

$$
\begin{aligned}
|||\tilde{f}_1 \tilde{f}_2|||_{1,\alpha;\rho} &= \sup_{x\in\mathbb{R}^n} \|\tilde{f}_1 \tilde{f}_2 \chi_{B(x,\rho)}\|_{\mathfrak{L}^{1,\infty}(\log \mathfrak{L})^\alpha} \\
&\le \sup_{x\in\mathbb{R}^n} \{\|\tilde{f}_1 \chi_{B(x,\rho)}\|_{\mathfrak{L}^{q_1,\infty}(\log \mathfrak{L})^{\alpha_1}} \|\tilde{f}_2 \chi_{B(x,\rho)}\|_{\mathfrak{L}^{q_2,\infty}(\log \mathfrak{L})^{\alpha_2}}\} \\
&\le \sup_{x\in\mathbb{R}^n} \|\tilde{f}_1 \chi_{B(x,\rho)}\|_{\mathfrak{L}^{q_1,\infty}(\log \mathfrak{L})^{\alpha_1}} \cdot \sup_{x\in\mathbb{R}^n} \|\tilde{f}_2 \chi_{B(x,\rho)}\|_{\mathfrak{L}^{q_2,\infty}(\log \mathfrak{L})^{\alpha_2}} \\
&= |||\tilde{f}_1|||_{q_1,\alpha_1;\rho} |||\tilde{f}_2|||_{q_2,\alpha_2;\rho}.
\end{aligned}
$$

Thus (2-15) holds, and Lemma 2.2 follows for $q_1, q_2 \in [1,\infty)$. If $q_1 = \infty$ or $q_2 = \infty$, the conclusion follows from (1-2). □

Lemma 2.3. *Let $q \in [1,\infty)$ and $\alpha \ge 0$. Then, for any $r > 0$ with $rq \ge 1$,*

$$
\begin{aligned}
\| |f|^r \|_{\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha} &= \|f\|^r_{\mathfrak{L}^{rq,\infty}(\log \mathfrak{L})^\alpha} \quad && \textit{for } f \in \mathfrak{L}^{rq,\infty}(\log \mathfrak{L})^\alpha, \\
||| |\tilde{f}|^r |||_{q,\alpha;\rho} &= |||\tilde{f}|||^r_{rq,\alpha;\rho} && \textit{for } \tilde{f} \in \mathfrak{L}^{q,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^\alpha \textit{ and } \rho > 0.
\end{aligned}
$$

Proof. It follows from (2-9) that

$$
\begin{aligned}
\| |f|^r \|_{\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha} &= \sup_{s>0} \left\{ \left[\log\left(e + \frac{1}{s}\right)\right]^\alpha \int_0^s ((|f|^r)^q)^*(\tau)\, d\tau \right\}^{\frac{1}{q}} \\
&= \sup_{s>0} \left\{ \left[\log\left(e + \frac{1}{s}\right)\right]^\alpha \int_0^s (|f|^{rq})^*(\tau)\, d\tau \right\}^{\frac{r}{rq}} \\
&= \|f\|^r_{\mathfrak{L}^{rq,\infty}(\log \mathfrak{L})^\alpha}
\end{aligned}
$$

for $f \in \mathfrak{L}^{rq,\infty}(\log \mathfrak{L})^\alpha$. Then

$$
\begin{aligned}
||| |\tilde{f}|^r |||_{q,\alpha;\rho} &= \sup_{x\in\mathbb{R}^n} \| |\tilde{f}|^r \chi_{B(x,\rho)} \|_{\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha} \\
&= \sup_{x\in\mathbb{R}^n} \| |\tilde{f}| \chi_{B(x,\rho)} \|^r_{\mathfrak{L}^{rq,\infty}(\log \mathfrak{L})^\alpha} \\
&= |||\tilde{f}|||^r_{rq,\alpha;\rho}
\end{aligned}
$$

for $\tilde{f} \in \mathfrak{L}^{rq,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^\alpha$ and $\rho > 0$. Thus Lemma 2.3 follows. □

2.2. *Hardy's inequalities.* We recall the following two lemmas on Hardy's inequality. (See [Muckenhoupt 1972, Theorems 1 and 2].) Throughout this paper, for any $q \in [1,\infty]$, we denote by q' the Hölder conjugate of q, that is, $q' = q/(q-1)$ if $q \in (1,\infty)$, $q' = \infty$ if $q = 1$, and $q' = 1$ if $q = \infty$.

Lemma 2.4. *Let $q \in [1,\infty]$. Let U and V be locally integrable functions in $[0,\infty)$. Then there exists $C > 0$ such that*

$$
\|U\tilde{F}\|_{L^q((0,\infty))} \le C\|Vf\|_{L^q((0,\infty))}, \quad \textit{with } \tilde{F}(s) := \int_0^s f(\tau)\, d\tau,
$$

holds for all locally integrable functions f in $[0,\infty)$ if and only if

$$
\sup_{s>0}\{\|U\|_{L^q((s,\infty))}\|V^{-1}\|_{L^{q'}((0,s))}\} < \infty.
$$

Lemma 2.5. *Let $q \in [1,\infty]$. Let U and V be locally integrable functions in $[0,\infty)$. Then there exists $C > 0$ such that*

$$\|U\hat{F}\|_{L^q((0,\infty))} \le C\|Vf\|_{L^q((0,\infty))}, \quad \textit{with } \hat{F}(s) := \int_s^\infty f(\tau)\,d\tau,$$

holds for all locally integrable functions f in $(0,\infty)$, with $f \in L^1((1,\infty))$, if and only if

$$\sup_{s>0}\{\|U\|_{L^q((0,s))}\|V^{-1}\|_{L^{q'}((s,\infty))}\} < \infty.$$

2.3. *Fundamental solutions.* Let $\theta \in (0,2]$. Let G_θ be the fundamental solution to the fractional heat equation

$$\partial_t v + (-\Delta)^{\frac{\theta}{2}} v = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty).$$

The function G_θ is positive and smooth in $\mathbb{R}^n \times (0,\infty)$, and it satisfies

$$\begin{aligned} &G_\theta(x,t) = (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{|x|^2}{4t}\right) \le C h_{\theta,t}(x) && \text{if } \theta = 2, \\ &G_\theta(x,t) \asymp h_{\theta,t}(x) && \text{if } 0 < \theta < 2 \end{aligned} \tag{2-16}$$

for $(x,t) \in \mathbb{R}^n \times (0,\infty)$, where

$$h_{\theta,t}(x) := t^{-\frac{n}{\theta}}(1 + t^{-\frac{1}{\theta}}|x|)^{-n-\theta}. \tag{2-17}$$

Furthermore,

- $G_\theta(x,t) = t^{-\frac{n}{\theta}} G_\theta(t^{-\frac{1}{\theta}}x, 1)$, $\int_{\mathbb{R}^n} G_\theta(x,t)\,dx = 1$,
- $G_\theta(\cdot,1)$ is radially symmetric and $G_\theta(x,1) \le G_\theta(y,1)$ if $|x| \ge |y|$,
- $G_\theta(x,t) = \int_{\mathbb{R}^n} G_\theta(x-y,t-s)G_\theta(y,s)\,dy$

for x, $y \in \mathbb{R}^n$ and $0 < s < t$ (see, e.g., [Bogdan and Jakubowski 2007; Brandolese and Karch 2008; Sugitani 1975]), and

$$\lim_{t\to+0} \|S_\theta(t)\eta - \eta\|_{L^\infty} = 0 \quad \text{for } \eta \in C_0(\mathbb{R}^n). \tag{2-18}$$

In addition, it follows from Young's inequality that

$$\|S_\theta(t)\eta\|_{L^q} \le C t^{-\frac{n}{\theta}\left(\frac{1}{r}-\frac{1}{q}\right)} \|\eta\|_{L^r} \tag{2-19}$$

for $\eta \in L^r$, $1 \le r \le q \le \infty$, and $t > 0$.

3. Decay estimates of $S_\theta(t)\varphi$

In this section we obtain decay estimates of $S_\theta(t)\varphi$ in our weak Zygmund-type spaces and uniformly local weak Zygmund-type spaces. For simplicity we write $g_t := G_\theta(\cdot,t)$ and $h_t := h_{\theta,t}$.

Proposition 3.1. *Let $\theta \in (0,2]$, $1 \le r \le q \le \infty$, and α, $\beta \ge 0$. Assume that $\alpha \le \beta$ if $r = q$. Then there exists $C > 0$ such that*

$$\|S_\theta(t)\varphi\|_{\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\beta} \le C t^{-\frac{n}{\theta}\left(\frac{1}{r}-\frac{1}{q}\right)} \left[\log\left(e + \frac{1}{t}\right)\right]^{-\frac{\alpha}{r}+\frac{\beta}{q}} \|\varphi\|_{\mathfrak{L}^{r,\infty}(\log \mathfrak{L})^\alpha}$$

for $\varphi \in \mathfrak{L}^{r,\infty}(\log \mathfrak{L})^\alpha$ and $t > 0$.

Before starting the proof, we recall the following relations on logarithmic functions: for any fixed $L > 1$ and $k > 0$,

$$\log\left(e+\frac{1}{s}\right) \asymp \log\left(L+\frac{1}{s}\right) \asymp \log\left(e+\frac{k}{s}\right) \asymp \log\left(e+\frac{1}{s^k}\right) \quad \text{for } s > 0. \tag{3-1}$$

Furthermore, we have the following results.

Lemma 3.2. (1) *Let $q > -1$ and $\alpha \in \mathbb{R}$. Then there exists $C_1 > 0$ such that*

$$\int_0^s \tau^q \left[\log\left(e+\frac{1}{\tau}\right)\right]^\alpha d\tau \le C_1 s^{q+1}\left[\log\left(e+\frac{1}{s}\right)\right]^\alpha \quad \textit{for } s > 0.$$

(2) *Let $S > 0$ and $\alpha < -1$. Then there exists $C_2 > 0$ such that*

$$\int_0^s \tau^{-1} \left[\log\left(e+\frac{1}{\tau}\right)\right]^\alpha d\tau \le C_2 \left[\log\left(e+\frac{1}{s}\right)\right]^{\alpha+1} \quad \textit{for } s \in (0, S).$$

(3) *Let $q < -1$ and $\alpha \in \mathbb{R}$. Then there exists $C_3 > 0$ such that*

$$\int_s^\infty \tau^q \left[\log\left(e+\frac{1}{\tau}\right)\right]^\alpha d\tau \le C_3 s^{q+1}\left[\log\left(e+\frac{1}{s}\right)\right]^\alpha \quad \textit{for } s > 0.$$

Proof. We prove assertion (1). Let $\delta > 0$ be such that $q - \delta > -1$. Then there exists $L \in [e, \infty)$ such that

$$\text{the function } (0,\infty) \ni \tau \mapsto \tau^\delta \left[\log\left(L+\frac{1}{\tau}\right)\right]^\alpha \text{ is nondecreasing.} \tag{3-2}$$

This together with (3-1) implies that

$$\begin{aligned}\int_0^s \tau^q \left[\log\left(e+\frac{1}{\tau}\right)\right]^\alpha d\tau &\le C\int_0^s \tau^{q-\delta}\cdot\tau^\delta\left[\log\left(L+\frac{1}{\tau}\right)\right]^\alpha d\tau \\ &\le Cs^\delta\left[\log\left(L+\frac{1}{s}\right)\right]^\alpha \int_0^s \tau^{q-\delta}\,d\tau \le Cs^{q+1}\left[\log\left(e+\frac{1}{s}\right)\right]^\alpha\end{aligned}$$

for $s > 0$. Thus assertion (1) follows.

We prove assertion (2). Let $S > 0$. It follows that

$$\int_0^s \tau^{-1}\left[\log\left(e+\frac{1}{\tau}\right)\right]^\alpha d\tau \le C\int_0^s \tau^{-1}|\log\tau|^\alpha\,d\tau \le C|\log s|^{\alpha+1} \le C\left[\log\left(e+\frac{1}{s}\right)\right]^{\alpha+1}$$

for $s \in \left(0, \frac{1}{2}\right)$. If $S \ge \frac{1}{2}$, then

$$\int_0^s \tau^{-1}\left[\log\left(e+\frac{1}{\tau}\right)\right]^\alpha d\tau \le \int_{\frac{1}{4}}^S \tau^{-1}\left[\log\left(e+\frac{1}{\tau}\right)\right]^\alpha d\tau + C \le C \le C\left[\log\left(e+\frac{1}{s}\right)\right]^{\alpha+1}$$

for $s \in \left[\frac{1}{2}, S\right)$. Thus assertion (2) follows.

It remains to prove assertion (3). Let $\delta' > 0$ be such that $q + \delta' < -1$. Then there exists $L' \in [e, \infty)$ such that

$$\text{the function } (0,\infty) \ni \tau \mapsto \tau^{-\delta'}\left[\log\left(L'+\frac{1}{\tau}\right)\right]^\alpha \text{ is nonincreasing.}$$

This together with (3-1) implies that

$$\int_s^\infty \tau^q \left[\log\left(e+\frac{1}{\tau}\right)\right]^\alpha d\tau \le C\int_s^\infty \tau^{q+\delta'}\cdot\tau^{-\delta'}\left[\log\left(L'+\frac{1}{\tau}\right)\right]^\alpha d\tau$$
$$\le Cs^{-\delta'}\left[\log\left(L'+\frac{1}{s}\right)\right]^\alpha \int_s^\infty \tau^{q+\delta'}\,d\tau \le Cs^{q+1}\left[\log\left(e+\frac{1}{s}\right)\right]^\alpha$$

for $s>0$. Thus assertion (3) follows. □

Next, we prepare the following lemma on h_t^*, where $h_t = h_{\theta,t}$ is as in (2-17).

Lemma 3.3. *Let $1\le r\le q<\infty$ and $\gamma\in\mathbb{R}$. Assume that $\gamma\ge 0$ if $r=q$. Then there exists $C>0$ such that*

$$\int_0^\infty \tau^{q(1-\frac{1}{r})}\left[\log\left(e+\frac{1}{\tau}\right)\right]^\gamma (h_t^*(\tau))^q\,d\tau \le Ct^{-\frac{nq}{\theta}(\frac{1}{r}-\frac{1}{q})}\left[\log\left(e+\frac{1}{t}\right)\right]^\gamma \tag{3-3}$$

for $t>0$.

Proof. It follows from (2-17) that

$$(h_t)^*(s) = h_t((\omega_n^{-1}s)^{\frac{1}{n}}e_1) \le Ct^{-\frac{n}{\theta}}(1+t^{-\frac{1}{\theta}}s^{\frac{1}{n}})^{-n-\theta}$$

for $s\in[0,\infty)$ and $t\in(0,\infty)$, where ω_n is the volume of the n-dimensional unit ball $B(0,1)$ and $e_1 := (1,0,\dots,0)\in\mathbb{R}^n$. Then

$$\begin{aligned} I &:= \int_0^\infty \tau^{q(1-\frac{1}{r})}\left[\log\left(e+\frac{1}{\tau}\right)\right]^\gamma (h_t^*(\tau))^q\,d\tau \\ &\le Ct^{-\frac{nq}{\theta}}\int_0^\infty \tau^{q(1-\frac{1}{r})}\left[\log\left(e+\frac{1}{\tau}\right)\right]^\gamma (1+t^{-\frac{1}{\theta}}\tau^{\frac{1}{n}})^{-q(n+\theta)}\,d\tau \\ &\le Ct^{-\frac{nq}{\theta}(\frac{1}{r}-\frac{1}{q})}\int_0^\infty \xi^{nq(1-\frac{1}{r})+n-1}(1+\xi)^{-q(n+\theta)}\left[\log\left(e+\frac{1}{(t^{1/\theta}\xi)^n}\right)\right]^\gamma d\xi \end{aligned} \tag{3-4}$$

for $t>0$.

We first consider the case of $\gamma\ge 0$. It follows from (3-1) that

$$\begin{aligned} \left[\log\left(e+\frac{1}{(t^{1/\theta}\xi)^n}\right)\right]^\gamma &\le C\left[\log\left(e+\frac{1}{t^{1/\theta}\xi}\right)\right]^\gamma \\ &\le C\left[\log\left(e+\frac{1}{t^{1/\theta}}\right)+\log\left(e+\frac{1}{\xi}\right)\right]^\gamma \\ &\le C\left[\log\left(e+\frac{1}{t}\right)\right]^\gamma + C\left[\log\left(e+\frac{1}{\xi}\right)\right]^\gamma \end{aligned} \tag{3-5}$$

for $t>0$ and $\xi\in\left(0,\frac{1}{2}\right)$. Similarly, by (3-1), we have

$$\left[\log\left(e+\frac{1}{(t^{1/\theta}\xi)^n}\right)\right]^\gamma \le \left[\log\left(e+\frac{2^n}{(t^{1/\theta})^n}\right)\right]^\gamma \le C\left[\log\left(e+\frac{1}{t}\right)\right]^\gamma \tag{3-6}$$

for $t > 0$ and $\xi \in \left[\frac{1}{2}, \infty\right)$. Since

$$nq\left(1-\frac{1}{r}\right)+n-1-q(n+\theta) = -\frac{nq}{r}+n-1-q\theta = -nq\left(\frac{1}{r}-\frac{1}{q}\right)-1-q\theta < -1, \tag{3-7}$$

by Lemma 3.2, (3-4), (3-5), and (3-6), we obtain

$$\begin{aligned} I &\le Ct^{-\frac{nq}{\theta}\left(\frac{1}{r}-\frac{1}{q}\right)} \int_0^{\frac{1}{2}} \xi^{nq\left(1-\frac{1}{r}\right)+n-1} \left(\left[\log\left(e+\frac{1}{t}\right)\right]^\gamma + \left[\log\left(e+\frac{1}{\xi}\right)\right]^\gamma\right) d\xi \\ &\qquad + Ct^{-\frac{nq}{\theta}\left(\frac{1}{r}-\frac{1}{q}\right)} \int_{\frac{1}{2}}^\infty \xi^{nq\left(1-\frac{1}{r}\right)+n-1}(1+\xi)^{-q(n+\theta)} \left[\log\left(e+\frac{1}{t}\right)\right]^\gamma d\xi \\ &\le Ct^{-\frac{nq}{\theta}\left(\frac{1}{r}-\frac{1}{q}\right)} \left(1+\left[\log\left(e+\frac{1}{t}\right)\right]^\gamma\right) \\ &\le Ct^{-\frac{nq}{\theta}\left(\frac{1}{r}-\frac{1}{q}\right)} \left[\log\left(e+\frac{1}{t}\right)\right]^\gamma \end{aligned}$$

for $t > 0$. This implies (3-3) in the case of $\gamma \ge 0$.

Consider the case of $\gamma < 0$. Then, by (3-1), we have

$$\left[\log\left(e+\frac{1}{(t^{1/\theta}\xi)^n}\right)\right]^\gamma \le \left[\log\left(e+\frac{2^n}{(t^{1/\theta})^n}\right)\right]^\gamma \le C\left[\log\left(e+\frac{1}{t}\right)\right]^\gamma \tag{3-8}$$

for $t > 0$ and $\xi \in \left(0, \frac{1}{2}\right)$. Let $0 < \delta < \theta q/|\gamma|$. We find $L \in [e, \infty)$ such that the function f in $(0, \infty)$ defined by

$$f(z) := z^\delta \log\left(L+\frac{1}{z^n}\right)$$

is nondecreasing in $(0, \infty)$. Since $\gamma < 0$, by (3-1), we obtain

$$\begin{aligned} \left[\log\left(e+\frac{1}{(t^{1/\theta}\xi)^n}\right)\right]^\gamma &\le C\left[\log\left(L+\frac{1}{(t^{1/\theta}\xi)^n}\right)\right]^\gamma = C[z^{-\delta\gamma} f(z)^\gamma]|_{z=t^{1/\theta}\xi} \\ &\le C(t^{\frac{1}{\theta}}\xi)^{-\delta\gamma} f(z)^\gamma\big|_{z=\frac{t^{1/\theta}}{2}} \le C\xi^{-\delta\gamma}\left[\log\left(e+\frac{2^n}{t^{n/\theta}}\right)\right]^\gamma \\ &\le C\xi^{-\delta\gamma}\left[\log\left(e+\frac{1}{t}\right)\right]^\gamma \end{aligned}$$

for $t > 0$ and $\xi \in \left[\frac{1}{2}, \infty\right)$. This together with (3-7) and (3-8) implies that

$$\begin{aligned} I &\le Ct^{-\frac{nq}{\theta}\left(\frac{1}{r}-\frac{1}{q}\right)} \int_0^{\frac{1}{2}} \xi^{nq\left(1-\frac{1}{r}\right)+n-1} \left[\log\left(e+\frac{1}{t}\right)\right]^\gamma d\xi \\ &\qquad + Ct^{-\frac{nq}{\theta}\left(\frac{1}{r}-\frac{1}{q}\right)} \int_{\frac{1}{2}}^\infty \xi^{nq\left(1-\frac{1}{r}\right)+n-1-\delta\gamma}(1+\xi)^{-q(n+\theta)} \left[\log\left(e+\frac{1}{t}\right)\right]^\gamma d\xi \\ &\le Ct^{-\frac{nq}{\theta}\left(\frac{1}{r}-\frac{1}{q}\right)} \left[\log\left(e+\frac{1}{t}\right)\right]^\gamma \end{aligned}$$

for $t > 0$. This implies (3-3) in the case of $\gamma < 0$. Thus Lemma 3.3 follows. □

Proof of Proposition 3.1. The proof is divided into the following three cases:

$$1 \le r < q < \infty, \quad 1 \le r = q < \infty, \quad 1 \le r \le q = \infty.$$

Step 1. Consider the case of $1 \le r < q < \infty$. It follows from (2-4), (2-7), (2-9), and (2-16) that

$$\begin{aligned}
\|S_\theta(t)\varphi\|^q_{\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\beta} &= \sup_{s>0}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^\beta \int_0^s ((S_\theta(t)\varphi)^*(\tau))^q \, d\tau\right\} \\
&\le \sup_{s>0}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^\beta \int_0^s ((S_\theta(t)\varphi)^{**}(\tau))^q \, d\tau\right\} \\
&\le \sup_{s>0}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^\beta \int_0^s \left(\int_\tau^\infty g_t^{**}(\eta)\varphi^{**}(\eta)\, d\eta\right)^q d\tau\right\} \\
&\le C \sup_{s>0}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^\beta \int_0^s \left(\int_\tau^\infty h_t^{**}(\eta)\varphi^{**}(\eta)\, d\eta\right)^q d\tau\right\}
\end{aligned}$$

for $t > 0$. Furthermore, thanks to (2-11), we have

$$\|S_\theta(t)\varphi\|^q_{\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\beta} \le C \int_0^\infty \left(\left[\log\left(e+\frac{1}{\tau}\right)\right]^{\frac{\beta}{q}} \int_\tau^\infty h_t^{**}(\eta)\varphi^{**}(\eta)\, d\eta\right)^q d\tau \tag{3-9}$$

for $t > 0$. On the other hand, set

$$U(\tau) := \left[\log\left(e+\frac{1}{\tau}\right)\right]^{\frac{\beta}{q}}, \quad V(\tau) := \tau\left[\log\left(e+\frac{1}{\tau}\right)\right]^{\frac{\beta}{q}}$$

for $\tau > 0$. It follows from Lemma 3.2 (1) and (3) that

$$\begin{aligned}
&\sup_{s>0}\left(\int_0^s |U(\tau)|^q \, d\tau\right)^{\frac{1}{q}} \left(\int_s^\infty |V(\tau)|^{-q'} \, d\tau\right)^{\frac{1}{q'}} \\
&\qquad\qquad \le \sup_{s>0}\left\{C s^{\frac{1}{q}}\left[\log\left(e+\frac{1}{s}\right)\right]^{\frac{\beta}{q}} \cdot C s^{-1+\frac{1}{q'}}\left[\log\left(e+\frac{1}{s}\right)\right]^{-\frac{\beta}{q}}\right\} < \infty.
\end{aligned}$$

Then, by Lemma 2.5, (2-3), and (3-9), we have

$$\begin{aligned}
\|S_\theta(t)\varphi\|^q_{\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\beta} &\le C \int_0^\infty \left(\tau\left[\log\left(e+\frac{1}{\tau}\right)\right]^{\frac{\beta}{q}} h_t^{**}(\tau)\varphi^{**}(\tau)\right)^q d\tau \\
&\le C \sup_{s>0}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^\alpha s(\varphi^{**}(s))^r\right\}^{\frac{q}{r}} \int_0^\infty \left(\tau^{1-\frac{1}{r}}\left[\log\left(e+\frac{1}{\tau}\right)\right]^{-\frac{\alpha}{r}+\frac{\beta}{q}} h_t^{**}(\tau)\right)^q d\tau
\end{aligned}$$

for $t > 0$. This together with (2-5) and (2-9) implies that

$$\|S_\theta(t)\varphi\|^q_{\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\beta} \le C\|\varphi\|^q_{\mathfrak{L}^{r,\infty}(\log \mathfrak{L})^\alpha} \int_0^\infty \left(\tau^{-\frac{1}{r}}\left[\log\left(e+\frac{1}{\tau}\right)\right]^\gamma \int_0^\tau h_t^*(\xi)\, d\xi\right)^q d\tau \tag{3-10}$$

for $t > 0$, where

$$\gamma := -\frac{\alpha}{r} + \frac{\beta}{q}.$$

Set

$$\widetilde{U}(\tau) = \tau^{-\frac{1}{r}}\left[\log\left(e + \frac{1}{\tau}\right)\right]^{\gamma}, \quad \widetilde{V}(\tau) = \tau^{1-\frac{1}{r}}\left[\log\left(e + \frac{1}{\tau}\right)\right]^{\gamma}.$$

Since $q > r$ and $q' < r'$, by Lemma 3.2 (1) and (3), we have

$$\begin{aligned}
\sup_{s>0}\left(\int_s^\infty |\widetilde{U}(\tau)|^q \, d\tau\right)^{\frac{1}{q}} & \left(\int_0^s |\widetilde{V}(\tau)|^{-q'} \, d\tau\right)^{\frac{1}{q'}} \\
&= \sup_{s>0}\left(\int_s^\infty \tau^{-\frac{q}{r}}\left[\log\left(e + \frac{1}{\tau}\right)\right]^{q\gamma} d\tau\right)^{\frac{1}{q}} \left(\int_0^s \tau^{-\frac{q'}{r'}}\left[\log\left(e + \frac{1}{\tau}\right)\right]^{-q'\gamma} d\tau\right)^{\frac{1}{q'}} \\
&\le \sup_{s>0}\left\{C s^{\frac{1}{q}-\frac{1}{r}}\left[\log\left(e + \frac{1}{s}\right)\right]^{\gamma} \cdot C s^{\frac{1}{q'}-\frac{1}{r'}}\left[\log\left(e + \frac{1}{s}\right)\right]^{-\gamma}\right\} < \infty. \qquad (3\text{-}11)
\end{aligned}$$

Applying Lemma 2.4 to (3-10), by (3-11), we obtain

$$\|S_\theta(t)\varphi\|^q_{\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\beta} \le C\|\varphi\|^q_{\mathfrak{L}^{r,\infty}(\log \mathfrak{L})^\alpha} \int_0^\infty \left(\tau^{1-\frac{1}{r}}\left[\log\left(e + \frac{1}{\tau}\right)\right]^{\gamma} h_t^*(\tau)\right)^q d\tau$$

for $t > 0$. This together with Lemma 3.3 implies that

$$\|S_\theta(t)\varphi\|^q_{\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\beta} \le C t^{-\frac{nq}{\theta}\left(\frac{1}{r}-\frac{1}{q}\right)}\left[\log\left(e + \frac{1}{t}\right)\right]^{q\gamma} \|\varphi\|^q_{\mathfrak{L}^{r,\infty}(\log \mathfrak{L})^\alpha}$$

for $t > 0$. Thus Proposition 3.1 follows in the case of $1 \le r < q < \infty$.

Step 2. Consider the case of $1 \le r = q < \infty$. It follows from Hölder's inequality and (2-16) that

$$\begin{aligned}
|[S_\theta(t)\varphi](x)|^r &\le C\left(\int_{\mathbb{R}^n} |h_t(x-y)||\varphi(y)| \, dy\right)^r \\
&\le C\left(\int_{\mathbb{R}^n} |h_t(x-y)| \, dy\right)^{r-1} \int_{\mathbb{R}^n} |h_t(x-y)||\varphi(y)|^r \, dy \\
&\le C\int_{\mathbb{R}^n} |h_t(x-y)||\varphi(y)|^r \, dy.
\end{aligned}$$

Then it follows from (2-7) and (2-9) that

$$\begin{aligned}
\|S_\theta(t)\varphi\|^r_{\mathfrak{L}^{r,\infty}(\log \mathfrak{L})^\beta} &= \sup_{s>0}\left\{\left[\log\left(e + \frac{1}{s}\right)\right]^{\beta} s(|S_\theta(t)\varphi|^r)^{**}(s)\right\} \\
&\le C\sup_{s>0}\left\{\left[\log\left(e + \frac{1}{s}\right)\right]^{\beta} s\int_s^\infty (h_t)^{**}(\tau)(|\varphi|^r)^{**}(\tau) \, d\tau\right\} \qquad (3\text{-}12)
\end{aligned}$$

for $t > 0$. Set

$$\widehat{U}(r) = r\left[\log\left(e + \frac{1}{r}\right)\right]^{\beta}, \quad \widehat{V}(r) = r^2\left[\log\left(e + \frac{1}{r}\right)\right]^{\beta}.$$

Similarly to (3-2), we find $L \in [e, \infty)$ such that

$$\text{the function } (0,\infty) \ni r \mapsto r\left[\log\left(L + \frac{1}{r}\right)\right]^{\beta} \text{ is nondecreasing.}$$

Then, by (3-1), we have

$$\begin{aligned}\|\widehat{U}\|_{L^\infty(0,s)} &\le C \sup_{r\in(0,s)}\left\{r\left[\log\left(L + \frac{1}{r}\right)\right]^{\beta}\right\}\\ &\le Cs\left[\log\left(L + \frac{1}{s}\right)\right]^{\beta}\\ &\le Cs\left[\log\left(e + \frac{1}{s}\right)\right]^{\beta}\end{aligned}$$

for $s > 0$. This together with Lemma 3.2 (3) implies that

$$\sup_{s>0}\left\{\|\widehat{U}\|_{L^\infty((0,s))}\int_s^\infty |\widehat{V}(\tau)|^{-1}\,d\tau\right\} \le \sup_{s>0}\left\{Cs\left[\log\left(e+\frac{1}{s}\right)\right]^{\beta}\cdot Cs^{-1}\left[\log\left(e+\frac{1}{s}\right)\right]^{-\beta}\right\} < \infty. \tag{3-13}$$

Applying Lemma 2.5 with $q = \infty$, by (2-9), (3-12) and (3-13), we obtain

$$\begin{aligned}\|S_\theta(t)\varphi\|^r_{\mathfrak{L}^{r,\infty}(\log\mathfrak{L})^\beta} &\le C\sup_{s>0}\left\{s^2\left[\log\left(e+\frac{1}{s}\right)\right]^{\beta}(h_t)^{**}(s)(|\varphi|^r)^{**}(s)\right\}\\ &\le C\sup_{s>0}\left\{s\left[\log\left(e+\frac{1}{s}\right)\right]^{\beta-\alpha}(h_t)^{**}(s)\right\}\cdot\sup_{s>0}\left\{s\left[\log\left(e+\frac{1}{s}\right)\right]^{\alpha}(|\varphi|^r)^{**}(s)\right\}\\ &= C\|h_t\|_{\mathfrak{L}^{1,\infty}(\log\mathfrak{L})^{\beta-\alpha}}\|\varphi\|^r_{\mathfrak{L}^{r,\infty}(\log\mathfrak{L})^\alpha}\end{aligned} \tag{3-14}$$

for $t > 0$. Furthermore, since $\alpha \le \beta$, by Lemma 3.3, (2-9), and (2-11), we have

$$\begin{aligned}\|h_t\|_{\mathfrak{L}^{1,\infty}(\log\mathfrak{L})^{\beta-\alpha}} &= \sup_{s>0}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^{\beta-\alpha}\int_0^s (h_t)^*(\tau)\,d\tau\right\}\\ &\le \int_0^\infty\left[\log\left(e+\frac{1}{\tau}\right)\right]^{\beta-\alpha}(h_t)^*(\tau)\,d\tau\\ &\le C\left[\log\left(e+\frac{1}{t}\right)\right]^{\beta-\alpha}\end{aligned}$$

for $t > 0$. This together with (3-14) implies that

$$\|S_\theta(t)\varphi\|^r_{\mathfrak{L}^{r,\infty}(\log\mathfrak{L})^\beta} \le C\left[\log\left(e+\frac{1}{t}\right)\right]^{\beta-\alpha}\|\varphi\|^r_{\mathfrak{L}^{r,\infty}(\log\mathfrak{L})^\alpha} \quad \text{for } t > 0.$$

Thus Proposition 3.1 follows in the case of $1 \le r = q < \infty$.

Step 3. It remains to consider the case of $1 \le r \le q = \infty$. If $r = q = \infty$, then it follows from (2-16) that

$$\|S_\theta(t)\varphi\|_{L^\infty} \le C\|\varphi\|_{L^\infty} \int_{\mathbb{R}^n} h_t(y)\,dy \le C\|\varphi\|_{L^\infty}$$

for $t > 0$, and Proposition 3.1 follows. On the other hand, in the case of $1 \le r < q = \infty$, let $\tilde{q} \in (r, \infty)$. Then, by Proposition 3.1 with $q = \tilde{q} > r$, (1-3), (2-19), and (3-1), we have

$$\begin{aligned}\|S_\theta(t)\varphi\|_{L^\infty} &= \Big\|S_\theta\Big(\frac{t}{2}\Big)S_\theta\Big(\frac{t}{2}\Big)\varphi\Big\|_{L^\infty} \le Ct^{-\frac{n}{\theta\tilde{q}}}\Big\|S_\theta\Big(\frac{t}{2}\Big)\varphi\Big\|_{L^{\tilde{q}}} = C^{-\frac{n}{\theta\tilde{q}}}\Big\|S_\theta\Big(\frac{t}{2}\Big)\varphi\Big\|_{\mathfrak{L}^{\tilde{q},\infty}(\log\mathfrak{L})^0} \\ &\le Ct^{-\frac{n}{\theta\tilde{q}}}\cdot Ct^{-\frac{n}{\theta}(\frac{1}{r}-\frac{1}{\tilde{q}})}\Big[\log\Big(e+\frac{2}{t}\Big)\Big]^{-\frac{\alpha}{r}}\|\varphi\|_{\mathfrak{L}^{r,\infty}(\log\mathfrak{L})^\alpha} \\ &= Ct^{-\frac{n}{\theta r}}\Big[\log\Big(e+\frac{1}{t}\Big)\Big]^{-\frac{\alpha}{r}}\|\varphi\|_{\mathfrak{L}^{r,\infty}(\log\mathfrak{L})^\alpha}\end{aligned}$$

for $t > 0$. Thus Proposition 3.1 follows in the case of $1 \le r < q = \infty$. The proof of Proposition 3.1 is complete. □

Furthermore, by Proposition 3.1, we employ the arguments in the proof of [Hisa and Ishige 2018, Lemma 2.1] to obtain decay estimates of $S_\theta(t)\varphi$ in uniformly local weak Zygmund-type spaces.

Proposition 3.4. *Let $\theta \in (0, 2]$, $1 \le r \le q \le \infty$, and $\alpha, \beta \ge 0$. Assume that $\alpha \le \beta$ if $r = q$. There exists $C > 0$ such that, for any $T > 0$,*

$$|||S_\theta(t)\varphi|||_{q,\beta;T^{1/\theta}} \le Ct^{-\frac{n}{\theta}(\frac{1}{r}-\frac{1}{q})}\Big[\log\Big(e+\frac{1}{t}\Big)\Big]^{-\frac{\alpha}{r}+\frac{\beta}{q}}|||\varphi|||_{r,\alpha;T^{1/\theta}} \tag{3-15}$$

for $\varphi \in \mathfrak{L}^{r,\infty}_{\mathrm{ul}}(\log\mathfrak{L})^\alpha$ and $t \in (0, T]$.

Proof. We first consider the case of $\theta \in (0, 2)$. It suffices to prove

$$t^{\frac{n}{\theta}(\frac{1}{r}-\frac{1}{q})}\Big[\log\Big(e+\frac{1}{t}\Big)\Big]^{\frac{\alpha}{r}-\frac{\beta}{q}}\|\chi_{B(z,T^{1/\theta})}S_\theta(t)\varphi\|_{\mathfrak{L}^{q,\infty}(\log\mathfrak{L})^\beta} \le C|||\varphi|||_{r,\alpha;T^{1/\theta}} \tag{3-16}$$

for $z \in \mathbb{R}^n$ and $0 < t \le T$. For the proof, by translating if necessary, we have only to consider the case of $z = 0$.

By Besicovitch's covering lemma, we can find an integer m depending only on n and a set

$$\{x_{k,i}\}_{k=1,\dots,m,\,i\in\mathbb{N}} \subset \mathbb{R}^n \setminus B(0, 10T^{\frac{1}{\theta}})$$

such that

$$B_{k,i} \cap B_{k,j} = \varnothing \ \text{ if } i \ne j \quad\text{and}\quad \mathbb{R}^n \setminus B(0, 10T^{\frac{1}{\theta}}) \subset \bigcup_{k=1}^m\bigcup_{i=1}^\infty B_{k,i}, \tag{3-17}$$

where $B_{k,i} := \overline{B(x_{k,i}, T^{1/\theta})}$. Then

$$|[S_\theta(t)\varphi](x)| \le |u_0(x,t)| + \sum_{k=1}^m\sum_{i=1}^\infty |u_{k,i}(x,t)|, \quad (x,t) \in \mathbb{R}^n \times (0,T), \tag{3-18}$$

where

$$u_0(x,t) := [S_\theta(t)(\varphi\chi_{B(0,10T^{1/\theta})})](x), \quad u_{k,i}(x,t) := [S_\theta(t)(\varphi\chi_{B_{k,i}})](x).$$

By Proposition 3.1 and (1-4), we have

$$\begin{aligned}\|u_0(t)\chi_{B(0,T^{1/\theta})}\|_{\mathfrak{L}^{q,\infty}(\log\mathfrak{L})^\beta} &\le \|u_0(t)\|_{\mathfrak{L}^{q,\infty}(\log\mathfrak{L})^\beta}\\ &\le Ct^{-\frac{n}{\theta}(\frac{1}{r}-\frac{1}{q})}\left[\log\left(e+\frac{1}{t}\right)\right]^{-\frac{\alpha}{r}+\frac{\beta}{q}}\|\varphi\chi_{B(0,10T^{1/\theta})}\|_{\mathfrak{L}^{r,\infty}(\log\mathfrak{L})^\alpha}\\ &\le Ct^{-\frac{n}{\theta}(\frac{1}{r}-\frac{1}{q})}\left[\log\left(e+\frac{1}{t}\right)\right]^{-\frac{\alpha}{r}+\frac{\beta}{q}}|||\varphi|||_{r,\alpha;10T^{1/\theta}}\\ &\le Ct^{-\frac{n}{\theta}(\frac{1}{r}-\frac{1}{q})}\left[\log\left(e+\frac{1}{t}\right)\right]^{-\frac{\alpha}{r}+\frac{\beta}{q}}|||\varphi|||_{r,\alpha;T^{1/\theta}}\end{aligned}\tag{3-19}$$

for $t\in(0,T]$.

Let $k=1,\dots,m$ and $i\in\mathbb{N}$. By (2-16), we have

$$|u_{k,i}(x,t)|\le C\int_{B(x_{k,i},T^{1/\theta})}h_t(x-y)|\varphi(y)|\,dy = C\int_{\mathbb{R}^n}h_t(x-z-x_{k,i})\varphi_{k,i}(z)\,dz\tag{3-20}$$

for $(x,t)\in\mathbb{R}^n\times(0,\infty)$, where $\varphi_{k,i}(x)=|\varphi(x+x_{k,i})|\chi_{B(0,T^{1/\theta})}$. Since $|x_{k,i}|\ge 10T^{1/\theta}$, it follows that

$$\begin{aligned}(1+T^{-\frac{1}{\theta}}|x_{k,i}|)(1+t^{-\frac{1}{\theta}}|x-z|) &= 1+T^{-\frac{1}{\theta}}|x_{k,i}|+t^{-\frac{1}{\theta}}|x-z|+t^{-\frac{1}{\theta}}T^{-\frac{1}{\theta}}|x_{k,i}||x-z|\\ &\le 1+3t^{-\frac{1}{\theta}}|x_{k,i}|+t^{-\frac{1}{\theta}}|x-z|\\ &= 1+4t^{-\frac{1}{\theta}}(|x_{k,i}|-|x-z|)-t^{-\frac{1}{\theta}}|x_{k,i}|+5t^{-\frac{1}{\theta}}|x-z|\\ &\le 4(1+t^{-\frac{1}{\theta}}(|x_{k,i}|-|x-z|))\le 4(1+t^{-\frac{1}{\theta}}|x-z-x_{k,i}|)\end{aligned}$$

for $x,z\in B(0,T^{1/\theta})$ and $t\in(0,T)$. This together with (2-16) implies that

$$\begin{aligned}h_t(x-z-x_{k,i}) &\le Ct^{-\frac{n}{\theta}}(1+T^{-\frac{1}{\theta}}|x_{k,i}|)^{-n-\theta}(1+t^{-\frac{1}{\theta}}|x-z|)^{-n-\theta}\\ &\le C(1+T^{-\frac{1}{\theta}}|x_{k,i}|)^{-n-\theta}g_t(x-z)\end{aligned}\tag{3-21}$$

for $x,z\in B(0,T^{1/\theta})$ and $t\in(0,T)$. We observe from (3-20) and (3-21) that

$$|u_{k,i}(x,t)|\le C(1+T^{-\frac{1}{\theta}}|x_{k,i}|)^{-n-\theta}[S_\theta(t)\varphi_{k,i}](x)$$

for $x\in B(0,T^{1/\theta})$ and $t\in(0,T)$. Then, by Proposition 3.1, we obtain

$$\begin{aligned}&\|u_{k,i}(t)\chi_{B(0,T^{1/\theta})}\|_{\mathfrak{L}^{q,\infty}(\log\mathfrak{L})^\beta}\\ &\quad\le C(1+T^{-\frac{1}{\theta}}|x_{k,i}|)^{-n-\theta}\|S_\theta(t)\varphi_{k,i}\|_{\mathfrak{L}^{q,\infty}(\log\mathfrak{L})^\beta}\\ &\quad\le C(1+T^{-\frac{1}{\theta}}|x_{k,i}|)^{-n-\theta}t^{-\frac{n}{\theta}(\frac{1}{r}-\frac{1}{q})}\left[\log\left(e+\frac{1}{t}\right)\right]^{-\frac{\alpha}{r}+\frac{\beta}{q}}\|\varphi_{k,i}\|_{\mathfrak{L}^{r,\infty}(\log\mathfrak{L})^\alpha}\\ &\quad= C(1+T^{-\frac{1}{\theta}}|x_{k,i}|)^{-n-\theta}t^{-\frac{n}{\theta}(\frac{1}{r}-\frac{1}{q})}\left[\log\left(e+\frac{1}{t}\right)\right]^{-\frac{\alpha}{r}+\frac{\beta}{q}}\|\varphi\chi_{B(x_{k,i},T^{1/\theta})}\|_{\mathfrak{L}^{r,\infty}(\log\mathfrak{L})^\alpha}\\ &\quad\le C(1+T^{-\frac{1}{\theta}}|x_{k,i}|)^{-n-\theta}t^{-\frac{n}{\theta}(\frac{1}{r}-\frac{1}{q})}\left[\log\left(e+\frac{1}{t}\right)\right]^{-\frac{\alpha}{r}+\frac{\beta}{q}}|||\varphi|||_{r,\alpha;T^{1/\theta}}\end{aligned}\tag{3-22}$$

for $t\in(0,T)$.

On the other hand, since

$$\tfrac{1}{2}|y| \le \tfrac{1}{2}(|x_{k,i}| + T^{\frac{1}{\theta}}) \le |x_{k,i}| \quad \text{for } y \in B_{k,i},$$

we have

$$\frac{1}{|B_{k,i}|}\int_{B_{k,i}} \big(1+\tfrac{1}{2}T^{-\frac{1}{\theta}}|y|\big)^{-n-\theta}\,dy \ge (1+T^{-\frac{1}{\theta}}|x_{k,i}|)^{-n-\theta}.$$

Then, by (3-17), we see that

$$\begin{aligned}\sum_{i=1}^{\infty}(1+T^{-\frac{1}{\theta}}|x_{k,i}|)^{-n-\theta} &\le CT^{-\frac{n}{\theta}}\sum_{i=1}^{\infty}\int_{B_{k,i}}\big(1+\tfrac{1}{2}T^{-\frac{1}{\theta}}|y|\big)^{-n-\theta}\,dy\\ &\le CT^{-\frac{n}{\theta}}\int_{\mathbb{R}^n}\big(1+\tfrac{1}{2}T^{-\frac{1}{\theta}}|y|\big)^{-n-\theta}\,dy \le C \qquad (3\text{-}23)\end{aligned}$$

for $T > 0$. Combining (3-18), (3-19), (3-22), and (3-23), we obtain

$$\begin{aligned}&t^{\frac{n}{\theta}(\frac{1}{r}-\frac{1}{q})}\left[\log\left(e+\frac{1}{t}\right)\right]^{\frac{\alpha}{r}-\frac{\beta}{q}}\|\chi_{B(0,T^{1/\theta})}S_\theta(t)\varphi\|_{\mathfrak{L}^{q,\infty}(\log\mathfrak{L})^\beta}\\ &\qquad\le C\,|||\varphi|||_{r,\alpha;T^{1/\theta}} + C\,|||\varphi|||_{r,\alpha;T^{1/\theta}}\sum_{k=1}^{m}\sum_{i=1}^{\infty}(1+T^{-\frac{1}{\theta}}|x_{k,i}|)^{-n-\theta}\\ &\qquad\le C\,|||\varphi|||_{r,\alpha;T^{1/\theta}}\end{aligned}$$

for $t \in (0,T)$. This implies (3-16) with $z = 0$; that is, (3-15) holds. Thus Proposition 3.4 follows in the case of $0 < \theta < 2$.

Consider the case of $\theta = 2$; that is, $S_\theta(t) = e^{t\Delta}$. Let $\tau = t^{1/2}$. It follows from (2-16) that

$$\begin{aligned}|[e^{t\Delta}\varphi](x)| &\le C\int_{\mathbb{R}^n} t^{-\frac{n}{2}}(1+t^{-\frac{1}{2}}|x-y|)^{-n-1}|\varphi(y)|\,dy\\ &\le C[S_1(\tau)|\varphi|](x)\end{aligned}$$

for $(x,t) \in \mathbb{R}^n \times (0,\infty)$. This together with Proposition 3.4 in the case of $\theta = 1$ implies that

$$\begin{aligned}|||e^{t\Delta}\varphi|||_{q,\beta;T^{1/2}} &\le C\,|||S_1(\tau)|\varphi|\,|||_{q,\beta;T^{1/2}}\\ &\le C\tau^{-n(\frac{1}{r}-\frac{1}{q})}\left[\log\left(e+\frac{1}{\tau}\right)\right]^{-\frac{\alpha}{r}+\frac{\beta}{q}}|||\varphi|||_{r,\alpha;T^{1/2}}\\ &\le Ct^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\left[\log\left(e+\frac{1}{t}\right)\right]^{-\frac{\alpha}{r}+\frac{\beta}{q}}|||\varphi|||_{r,\alpha;T^{1/2}} \quad \text{for } t \in (0,T).\end{aligned}$$

Thus Proposition 3.4 follows in the case of $\theta = 2$. The proof of Proposition 3.4 is complete. □

4. Proof of Theorems 1.2 and 1.4

We apply the contraction mapping theorem to problem (P) in uniformly local weak Zygmund-type spaces and prove Theorems 1.2 and 1.4. We also prove Corollary 1.3. We first prove the following proposition.

Proposition 4.1. *Let $p = p_\theta$, $T_* \in (0,\infty)$, and $\gamma \in [0, n/\theta)$. Then there exists $\epsilon > 0$ such that if $\varphi \in \mathfrak{L}^{1,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^{n/\theta}$ satisfies*

$$|||\varphi|||_{1,\frac{n}{\theta};T^{1/\theta}} \le \epsilon \quad \textit{for some } T \in (0, T_*], \tag{4-1}$$

then problem (P) possesses a solution

$$u \in C((0,T) : \mathfrak{L}^{1,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^{n/\theta}) \cap L^\infty_{\mathrm{loc}}(0,T : L^\infty) \quad \textit{in } \mathbb{R}^n \times (0,T),$$

with u satisfying

$$\begin{aligned} |||u(t)|||_{1,\frac{n}{\theta};T^{1/\theta}} &\le C|||\varphi|||_{1,\frac{n}{\theta};T^{1/\theta}}, \\ |||u(t)|||_{p,\gamma;T^{1/\theta}} &\le C t^{-\frac{n}{\theta}(1-\frac{1}{p})}\left[\log\left(e+\frac{1}{t}\right)\right]^{-\frac{n}{\theta}+\frac{\gamma}{p}} |||\varphi|||_{1,\frac{n}{\theta};T^{1/\theta}}, \\ \|u(t)\|_{L^\infty} &\le C t^{-\frac{n}{\theta}}\left[\log\left(e+\frac{1}{t}\right)\right]^{-\frac{n}{\theta}} |||\varphi|||_{1,\frac{n}{\theta};T^{1/\theta}} \end{aligned} \tag{4-2}$$

for $t \in (0,T)$. Here C is a positive constant depending only on T_*, n, θ, and γ.

Throughout this section, we set

$$T_* \in (0,\infty), \quad T \in (0, T_*], \quad p := p_\theta = 1 + \frac{\theta}{n}, \quad \alpha := \frac{n}{\theta}, \quad 0 \le \gamma < \alpha, \quad \varphi \in \mathfrak{L}^{1,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^{\alpha}.$$

Let $\epsilon > 0$, and assume (4-1). By Proposition 3.4, we find $C_* > 0$ such that

$$\begin{aligned} \sup_{0<t<T} |||S_\theta(t)\varphi|||_{1,\alpha;T^{1/\theta}} &\le C_*|||\varphi|||_{1,\alpha;T^{1/\theta}} \le C_*\epsilon, \\ \sup_{0<t<T} t^{\frac{n}{\theta}(1-\frac{1}{p})}\left[\log\left(e+\frac{1}{t}\right)\right]^{-\frac{\gamma}{p}+\alpha} |||S_\theta(t)\varphi|||_{p,\gamma;T^{1/\theta}} &\le C_*|||\varphi|||_{1,\alpha;T^{1/\theta}} \le C_*\epsilon, \\ \sup_{0<t<T} t^{\frac{n}{\theta}}\left[\log\left(e+\frac{1}{t}\right)\right]^{\alpha} \|S_\theta(t)\varphi\|_{L^\infty} &\le C_*|||\varphi|||_{1,\alpha;T^{1/\theta}} \le C_*\epsilon. \end{aligned} \tag{4-3}$$

Define

$$X_T := C((0,T) : \mathfrak{L}^{1,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^{\alpha}) \cap L^\infty_{\mathrm{loc}}((0,T) : \mathfrak{L}^{p,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^{\gamma}) \cap L^\infty_{\mathrm{loc}}((0,T) : L^\infty).$$

Setting $C^* = 2C_*$, for any $u \in X_T$, we say that $u \in X_T(C^*\epsilon)$ if u satisfies

$$\sup_{0<t<T} |||u(t)|||_{1,\alpha;T^{1/\theta}} + \sup_{0<t<T} t^{\frac{n}{\theta}(1-\frac{1}{p})}\left[\log\left(e+\frac{1}{t}\right)\right]^{-\frac{\gamma}{p}+\alpha} |||u(t)|||_{p,\gamma;T^{1/\theta}} + \sup_{0<t<T} t^{\frac{n}{\theta}}\left[\log\left(e+\frac{1}{t}\right)\right]^{\alpha} \|u(t)\|_{L^\infty} \le C^*\epsilon. \tag{4-4}$$

For any $u, v \in X_T(C^*\epsilon)$, set

$$d_X(u,v) := d^1_X(u,v) + d^2_X(u,v) + d^3_X(u,v),$$

where

$$d^1_X(u,v) := \sup_{0<t<T} |||u(t)-v(t)|||_{1,\alpha;T^{1/\theta}},$$

$$d_X^2(u,v) := \sup_{0<t<T} t^{\frac{n}{\theta}(1-\frac{1}{p})}\left[\log\left(e+\frac{1}{t}\right)\right]^{-\frac{\gamma}{p}+\alpha} |||u(t)-v(t)|||_{p,\gamma;T^{1/\theta}},$$
$$d_X^3(u,v) := \sup_{0<t<T} t^{\frac{n}{\theta}}\left[\log\left(e+\frac{1}{t}\right)\right]^{\alpha} \|u(t)-v(t)\|_{L^\infty}.$$

Then (X_T, d_X) is a Banach space and $X_T(C^*\epsilon)$ is closed in (X_T, d_X). Define

$$\Phi(u) := S_\theta(t)\varphi + \int_0^t S_\theta(t-s)F_p(u(s))\,ds \quad \text{for } u \in X_T(C^*\epsilon),$$

where $F_p(s) = |s|^{p-1}s$ for $s \in \mathbb{R}$. For the proof of Proposition 4.1 we prepare the following two lemmas.

Lemma 4.2. *Let* $\epsilon > 0$, *and assume that* (4-1) *holds for some* $T \in (0, T_*]$. *Then there exists* $C = C(n,\theta,C_*,T_*) > 0$ *such that*

$$d_X^1(\Phi(u),\Phi(v)) + d_X^2(\Phi(u),\Phi(v)) \le C\epsilon^{p-1}\, d_X^2(u,v) \quad \textit{for } u,v \in X_T(C^*\epsilon).$$

Proof. Let $u, v \in X_T(C^*\epsilon)$. Let $0 < s < t < T$. It follows that

$$|F_p(u(x,s)) - F_p(v(x,s))| \le w(x,s)|u(x,s)-v(x,s)| \quad \text{for } x \in \mathbb{R}^n, \tag{4-5}$$

where $w(x,s) := p(|u(x,s)|^{p-1} + |v(x,s)|^{p-1})$. Then, by Lemmas 2.2 and 2.3, we have

$$\begin{aligned} |||F_p(u(s)) - F_p(v(s))|||_{1,\gamma;T^{1/\theta}} &\le |||w(s)|||_{\frac{p}{(p-1)},\gamma;T^{1/\theta}} |||u(s)-v(s)|||_{p,\gamma;T^{1/\theta}} \\ &\le p(|||u(s)|||^{p-1}_{p,\gamma;T^{1/\theta}} + |||v(s)|||^{p-1}_{p,\gamma;T^{1/\theta}})|||u(s)-v(s)|||_{p,\gamma;T^{1/\theta}}. \end{aligned} \tag{4-6}$$

Since $u, v \in X_T(C^*\epsilon)$, by (4-4), we obtain

$$\begin{aligned} &|||F_p(u(s)) - F_p(v(s))|||_{1,\gamma;T^{1/\theta}} \\ &\le C s^{-\frac{n(p-1)}{\theta}(1-\frac{1}{p})}\left[\log\left(e+\frac{1}{s}\right)\right]^{\frac{\gamma(p-1)}{p}-\alpha(p-1)} (C^*\epsilon)^{p-1} C s^{-\frac{n}{\theta}(1-\frac{1}{p})}\left[\log\left(e+\frac{1}{s}\right)\right]^{\frac{\gamma}{p}-\alpha} d_X^2(u,v) \\ &= C\epsilon^{p-1} s^{-\frac{n(p-1)}{\theta}}\left[\log\left(e+\frac{1}{s}\right)\right]^{\gamma-\alpha p} d_X^2(u,v). \end{aligned} \tag{4-7}$$

This together with Proposition 3.4 implies that

$$\begin{aligned} &\left|\left|\left|\int_0^t S_\theta(t-s)[F_p(u(s)) - F_p(v(s))]\,ds\right|\right|\right|_{q,\beta;T^{1/\theta}} \\ &\le \int_0^t |||S_\theta(t-s)[F_p(u(s)) - F_p(v(s))]|||_{q,\beta;T^{1/\theta}}\,ds \\ &\le C\int_0^t (t-s)^{-\frac{n}{\theta}(1-\frac{1}{q})}\left[\log\left(e+\frac{1}{t-s}\right)\right]^{-\gamma+\frac{\beta}{q}} |||F_p(u(s)) - F_p(v(s))|||_{1,\gamma;T^{1/\theta}}\,ds \\ &\le C\epsilon^{p-1} d_X^2(u,v)\int_0^t (t-s)^{-\frac{n}{\theta}(1-\frac{1}{q})}\left[\log\left(e+\frac{1}{t-s}\right)\right]^{-\gamma+\frac{\beta}{q}} s^{-\frac{n}{\theta}(p-1)}\left[\log\left(e+\frac{1}{s}\right)\right]^{\gamma-\alpha p} ds \end{aligned} \tag{4-8}$$

for $q \in [1, p]$ and $\beta \in [\gamma, \alpha]$.

On the other hand, since

$$\gamma-\alpha p=\gamma-\frac{n}{\theta}\Big(1+\frac{\theta}{n}\Big)=\gamma-\frac{n}{\theta}-1=\gamma-\alpha-1<-1, \tag{4-9}$$

by Lemma 3.2 (2) and (3-1), we have

$$\begin{aligned}
&\int_0^{\frac{t}{2}}(t-s)^{-\frac{n}{\theta}(1-\frac{1}{q})}\Big[\log\Big(e+\frac{1}{t-s}\Big)\Big]^{-\gamma+\frac{\beta}{q}}s^{-\frac{n}{\theta}(p-1)}\Big[\log\Big(e+\frac{1}{s}\Big)\Big]^{\gamma-\alpha p}\,ds\\
&\quad\le Ct^{-\frac{n}{\theta}(1-\frac{1}{q})}\Big[\log\Big(e+\frac{1}{t}\Big)\Big]^{-\gamma+\frac{\beta}{q}}\int_0^{\frac{t}{2}}s^{-1}\Big[\log\Big(e+\frac{1}{s}\Big)\Big]^{\gamma-\alpha p}\,ds\\
&\quad\le Ct^{-\frac{n}{\theta}(1-\frac{1}{q})}\Big[\log\Big(e+\frac{1}{t}\Big)\Big]^{-\gamma+\frac{\beta}{q}}\cdot C\Big[\log\Big(e+\frac{1}{t}\Big)\Big]^{\gamma-\frac{n}{\theta}}=Ct^{-\frac{n}{\theta}(1-\frac{1}{q})}\Big[\log\Big(e+\frac{1}{t}\Big)\Big]^{\frac{\beta}{q}-\alpha}
\end{aligned}\tag{4-10}$$

for $t\in(0,T)$. Similarly, since

$$-\frac{n}{\theta}\Big(1-\frac{1}{q}\Big)\ge-\frac{n(p-1)}{\theta p}=-\frac{1}{p}>-1,$$

by Lemma 3.2 (1) and (3-1), we obtain

$$\begin{aligned}
&\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{n}{\theta}(1-\frac{1}{q})}\Big[\log\Big(e+\frac{1}{t-s}\Big)\Big]^{-\gamma+\frac{\beta}{q}}s^{-\frac{n}{\theta}(p-1)}\Big[\log\Big(e+\frac{1}{s}\Big)\Big]^{\gamma-\alpha p}\,ds\\
&\qquad\le Ct^{-\frac{n}{\theta}(p-1)}\Big[\log\Big(e+\frac{1}{t}\Big)\Big]^{\gamma-\alpha p}\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{n}{\theta}(1-\frac{1}{q})}\Big[\log\Big(e+\frac{1}{t-s}\Big)\Big]^{-\gamma+\frac{\beta}{q}}\,ds\\
&\qquad\le Ct^{-1}\Big[\log\Big(e+\frac{1}{t}\Big)\Big]^{\gamma-\alpha p}\cdot Ct^{-\frac{n}{\theta}(1-\frac{1}{q})+1}\Big[\log\Big(e+\frac{1}{t}\Big)\Big]^{-\gamma+\frac{\beta}{q}}\\
&\qquad=Ct^{-\frac{n}{\theta}(1-\frac{1}{q})}\Big[\log\Big(e+\frac{1}{t}\Big)\Big]^{\frac{\beta}{q}-\alpha p}\le Ct^{-\frac{n}{\theta}(1-\frac{1}{q})}\Big[\log\Big(e+\frac{1}{t}\Big)\Big]^{\frac{\beta}{q}-\alpha}
\end{aligned}\tag{4-11}$$

for $t\in(0,T)$. Combining (4-8), (4-10), and (4-11) with $(q,\beta)=(1,\alpha)$ and (p,γ), we deduce that

$$\begin{aligned}
&d_X^1(\Phi(u),\Phi(v))+d_X^2(\Phi(u),\Phi(v))\\
&\quad=\sup_{0<t<T}\Big|\Big|\Big|\int_0^t S_\theta(t-s)[F_p(u(s))-F_p(v(s))]\,ds\Big|\Big|\Big|_{1,\alpha;T^{1/\theta}}\\
&\qquad\qquad+\sup_{0<t<T}t^{\frac{n}{\theta}(1-\frac{1}{p})}\Big[\log\Big(e+\frac{1}{t}\Big)\Big]^{-\frac{\gamma}{p}+\alpha}\Big|\Big|\Big|\int_0^t S_\theta(t-s)[F_p(u(s))-F_p(v(s))]\,ds\Big|\Big|\Big|_{p,\gamma;T^{1/\theta}}\\
&\quad\le C\epsilon^{p-1}d_X^2(u,v)
\end{aligned}$$

for $u,v\in X_T(C^*\epsilon)$. Thus Lemma 4.2 follows. □

Lemma 4.3. *Let $\epsilon>0$, and assume that (4-1) holds for some $T\in(0,T_*]$. Then there exists $C=C(n,\theta,C_*,T_*)>0$ such that*

$$d_X^3(\Phi(u),\Phi(v))\le C\epsilon^{p-1}(d_X^2(u,v)+d_X^3(u,v))\quad\textit{for } u,v\in X_T(C^*\epsilon).$$

Proof. Let $u, v \in X_T(C^*\epsilon)$. Let $0 < s < t < T$. Similarly to (4-6), we have

$$\begin{aligned}\|F_p(u(s)) - F_p(v(s))\|_{L^\infty} &\le \|w(s)\|_{L^\infty}\|u(s)-v(s)\|_{L^\infty} \\ &\le p(\|u(s)\|_{L^\infty}^{p-1} + \|v(s)\|_{L^\infty}^{p-1})\|u(s)-v(s)\|_{L^\infty}.\end{aligned}$$

Since $u, v \in X_T(C^*\epsilon)$, by (4-4), we obtain

$$\begin{aligned}\|F_p(u(s)) - F_p(v(s))\|_{L^\infty} &\le C s^{-\frac{n(p-1)}{\theta}}\left[\log\left(e+\frac{1}{s}\right)\right]^{-\alpha(p-1)} (C^*\epsilon)^{p-1}\cdot s^{-\frac{n}{\theta}}\left[\log\left(e+\frac{1}{s}\right)\right]^{-\alpha} d_X^3(u,v) \\ &= C\epsilon^{p-1}s^{-\frac{np}{\theta}}\left[\log\left(e+\frac{1}{s}\right)\right]^{-\alpha p} d_X^3(u,v).\end{aligned}$$

This together with Proposition 3.4 and (4-7) implies that

$$\begin{aligned}&\left\|\int_0^t S_\theta(t-s)[F_p(u(s)) - F_p(v(s))]\,ds\right\|_{L^\infty} \\ &\le \int_0^t \|S_\theta(t-s)[F_p(u(s)) - F_p(v(s))]\|_{L^\infty}\,ds \\ &\le C\int_0^{\frac{t}{2}} (t-s)^{-\frac{n}{\theta}}\left[\log\left(e+\frac{1}{t-s}\right)\right]^{-\gamma} |||F_p(u(s)) - F_p(v(s))|||_{1,\gamma;T^{1/\theta}}\,ds \\ &\qquad + C\int_{\frac{t}{2}}^t \|F_p(u(s)) - F_p(v(s))\|_{L^\infty}\,ds \\ &\le C\epsilon^{p-1}d_X^2(u,v)\int_0^{\frac{t}{2}} (t-s)^{-\frac{n}{\theta}}\left[\log\left(e+\frac{1}{t-s}\right)\right]^{-\gamma} s^{-\frac{n}{\theta}(p-1)}\left[\log\left(e+\frac{1}{s}\right)\right]^{\gamma-\alpha p} ds \\ &\qquad + C\epsilon^{p-1}d_X^3(u,v)\int_{\frac{t}{2}}^t s^{-\frac{np}{\theta}}\left[\log\left(e+\frac{1}{s}\right)\right]^{-\alpha p} ds \\ &\le C\epsilon^{p-1}t^{-\frac{n}{\theta}}\left[\log\left(e+\frac{1}{t}\right)\right]^{-\gamma} d_X^2(u,v)\int_0^{\frac{t}{2}} s^{-1}\left[\log\left(e+\frac{1}{s}\right)\right]^{\gamma-\alpha p} ds \\ &\qquad + C\epsilon^{p-1}t^{-\frac{np}{\theta}+1}\left[\log\left(e+\frac{1}{t}\right)\right]^{-\alpha p} d_X^3(u,v).\end{aligned}$$

Since $np = n+\theta$ and $\alpha p > \alpha$, we have

$$\begin{aligned}&\left\|\int_0^t S_\theta(t-s)[F_p(u(s)) - F_p(v(s))]\,ds\right\|_{L^\infty} \\ &\le C\epsilon^{p-1}t^{-\frac{n}{\theta}}\left[\log\left(e+\frac{1}{t}\right)\right]^{-\gamma} d_X^2(u,v)\int_0^{\frac{t}{2}} s^{-1}\left[\log\left(e+\frac{1}{s}\right)\right]^{\gamma-\alpha p} ds \\ &\qquad + C\epsilon^{p-1}t^{-\frac{n}{\theta}}\left[\log\left(e+\frac{1}{t}\right)\right]^{-\alpha} d_X^3(u,v). \end{aligned} \tag{4-12}$$

Furthermore, by Lemma 3.2 (2) and (4-9) we see that

$$\int_0^{\frac{t}{2}} s^{-1}\left[\log\left(e+\frac{1}{s}\right)\right]^{\gamma-\alpha p} ds \le C\left[\log\left(e+\frac{1}{t}\right)\right]^{\gamma-\alpha} \tag{4-13}$$

for $t \in (0, T)$. Combining (4-12) and (4-13), we deduce that

$$\begin{aligned} d_X^3(\Phi(u), \Phi(v)) &= \sup_{0<t<T} t^{\frac{n}{\theta}} \left[\log\left(e + \frac{1}{t}\right)\right]^{\alpha} \left\| \int_0^t S_\theta(t-s)[F_p(u(s)) - F_p(v(s))]\, ds \right\|_{L^\infty} \\ &\le C\epsilon^{p-1}(d_X^2(u,v) + d_X^3(u,v)) \end{aligned}$$

for $u, v \in X_T(C^*\epsilon)$. Thus Lemma 4.3 follows. □

Proof of Proposition 4.1. Let $T_* > 0$. Let $\epsilon > 0$ be small enough. Let $\varphi \in \mathfrak{L}^{1,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^{\alpha}$ be such that $|||\varphi|||_{1,\alpha;T^{1/\theta}} < \epsilon$ for some $T \in (0, T_*]$. By (4-3), (4-4), and Lemma 4.2, we have

$$\begin{aligned} &\sup_{t\in(0,T)} |||\Phi(u(t))|||_{1,\alpha;T^{1/\theta}} + \sup_{0<t<T} \left\{ t^{\frac{n}{\theta}(1-\frac{1}{p})} \left[\log\left(e+\frac{1}{t}\right)\right]^{-\frac{\gamma}{p}+\alpha} |||\Phi(u(t))|||_{p,\gamma;T^{1/\theta}} \right\} \\ &\quad \le |||S_\theta(t)\varphi|||_{1,\alpha,T^{1/\theta}} + \sup_{0<t<T} \left\{ t^{\frac{n}{\theta}(1-\frac{1}{p})} \left[\log\left(e+\frac{1}{t}\right)\right]^{-\frac{\gamma}{p}+\alpha} |||S_\theta(t)\varphi|||_{p,\gamma;T^{1/\theta}} \right\} \\ &\qquad\qquad + d_X^1(\Phi(u), \Phi(0)) + d_X^2(\Phi(u), \Phi(0)) \\ &\quad \le C_*\epsilon + C\epsilon^{p-1} d_X^2(u, 0) \\ &\quad \le C_*\epsilon + C\epsilon^{p-1} \cdot C^*\epsilon \\ &\quad \le C^*\epsilon \end{aligned} \tag{4-14}$$

for $u \in X_T(C^*\epsilon)$. Similarly, we observe from Lemma 4.3, (4-3), and (4-4) that

$$\begin{aligned} \sup_{0<t<T} \left\{ t^{\frac{n}{\theta}} \left[\log\left(e+\frac{1}{t}\right)\right]^{\alpha} \|\Phi(u(t))\|_{L^\infty} \right\} &\le \sup_{0<t<T} \left\{ t^{\frac{n}{\theta}} \left[\log\left(e+\frac{1}{t}\right)\right]^{\alpha} \|S_\theta(t)\varphi\|_{L^\infty} \right\} + d_X^3(\Phi(u), \Phi(0)) \\ &\le C_*\epsilon + C\epsilon^{p-1}(d_X^2(u,0) + d_X^3(u,0)) \\ &\le C_*\epsilon + C\epsilon^{p-1} \cdot 2C^*\epsilon \\ &\le C^*\epsilon \end{aligned} \tag{4-15}$$

for $u \in X_T(C^*\epsilon)$. By (4-14) and (4-15), we see that $\Phi(u) \in X_T(C^*\epsilon)$ for $u \in X_T(C^*\epsilon)$. Furthermore, taking small enough $\epsilon > 0$ if necessary, by Lemmas 4.2 and 4.3, we have

$$\begin{aligned} d_X(\Phi(u), \Phi(v)) &= d_X^1(\Phi(u), \Phi(v)) + d_X^2(\Phi(u), \Phi(v)) + d_X^3(\Phi(u), \Phi(v)) \\ &\le C\epsilon^{p-1}(d_X^2(u,v) + d_X^3(u,v)) \\ &\le \tfrac{1}{2} d_X(u,v) \end{aligned}$$

for $u, v \in X_T(C^*\epsilon)$. Then we apply the contraction mapping theorem to find a unique $u_* \in X_T(C^*\epsilon)$ such that $\Phi(u_*) = u_*$ in $X_T(C^*\epsilon)$. The function u_* is a solution to problem (P) in $\mathbb{R}^n \times (0, T)$, with u_* satisfying (4-2). Thus Proposition 4.1 follows. □

Proof of Theorem 1.2. Let $T > 0$. Let $\varphi \in \mathfrak{L}^{1,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^{\alpha}$ be such that $|||\varphi|||_{1,\alpha;T^{1/\theta}}$ is small enough. Then, by Proposition 4.1, we find a solution u to problem (P) in $\mathbb{R}^n \times (0, T)$, with u satisfying (4-2). Let

$\beta \in (\gamma, n/\theta)$. Then, by Proposition 3.4, Lemma 2.3, and (4-2), we obtain

$$\begin{aligned}
|||u(t)-S_\theta(t)\varphi|||_{1,\beta;T^{1/\theta}} &\le \int_0^t |||S_\theta(t-s)F_p(u(s))|||_{1,\beta;T^{1/\theta}}\,ds \\
&\le C\int_0^t \left[\log\left(e+\frac{1}{t-s}\right)\right]^{-\gamma+\beta} |||F_p(u(s))|||_{1,\gamma;T^{1/\theta}}\,ds \\
&= C\int_0^t \left[\log\left(e+\frac{1}{t-s}\right)\right]^{-\gamma+\beta} |||u(s)|||^p_{p,\gamma;T^{1/\theta}}\,ds \\
&\le C|||\varphi|||^p_{1,\alpha;T^{1/\theta}} \int_0^t \left[\log\left(e+\frac{1}{t-s}\right)\right]^{-\gamma+\beta} s^{-1}\left[\log\left(e+\frac{1}{s}\right)\right]^{\gamma-\alpha p} ds \qquad (4\text{-}16)
\end{aligned}$$

for $t \in (0, T)$. On the other hand, since $\beta < \theta/n$, by Lemma 3.2 (2) and (4-9), we have

$$\begin{aligned}
&\int_0^{\frac{t}{2}} \left[\log\left(e+\frac{1}{t-s}\right)\right]^{-\gamma+\beta} s^{-1}\left[\log\left(e+\frac{1}{s}\right)\right]^{\gamma-\alpha p} ds \\
&\qquad \le C\left[\log\left(e+\frac{1}{t}\right)\right]^{-\gamma+\beta} \int_0^{\frac{t}{2}} s^{-1}\left[\log\left(e+\frac{1}{s}\right)\right]^{\gamma-\alpha p} ds \\
&\qquad \le C\left[\log\left(e+\frac{1}{t}\right)\right]^{-\gamma+\beta} \cdot C\left[\log\left(e+\frac{1}{t}\right)\right]^{\gamma-\frac{n}{\theta}} \to 0 \qquad (4\text{-}17)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\frac{t}{2}}^{t} \left[\log\left(e+\frac{1}{t-s}\right)\right]^{-\gamma+\beta} s^{-1}\left[\log\left(e+\frac{1}{s}\right)\right]^{\gamma-\alpha p} ds \\
&\qquad \le Ct^{-1}\left[\log\left(e+\frac{1}{t}\right)\right]^{\gamma-\frac{n}{\theta}-1} \int_{\frac{t}{2}}^{t} \left[\log\left(e+\frac{1}{t-s}\right)\right]^{-\gamma+\beta} ds \\
&\qquad \le Ct^{-1}\left[\log\left(e+\frac{1}{t}\right)\right]^{\gamma-\frac{n}{\theta}-1} \cdot Ct\left[\log\left(e+\frac{1}{t}\right)\right]^{-\gamma+\beta} \to 0 \qquad (4\text{-}18)
\end{aligned}$$

as $t \to +0$. Combining (4-16), (4-17), and (4-18), we see that

$$\lim_{t\to+0} |||u(t)-S_\theta(t)\varphi|||_{1,\beta;T^{1/\theta}} = 0 \quad \text{for } \beta \in (\gamma, n/\theta).$$

This together with (2-12) implies that

$$\lim_{t\to+0} |||u(t)-S_\theta(t)\varphi|||_{1,\beta;T^{1/\theta}} = 0 \quad \text{for } \beta \in [0, n/\theta). \qquad (4\text{-}19)$$

It remains to prove that $u \to \varphi$ in the sense of distributions. Let $\eta \in C_0(\mathbb{R}^n)$. Let $R > 0$ be such that $\operatorname{supp}\eta \subset B(0, R)$. By (1-3), (1-4), and (4-19), we have

$$\begin{aligned}
\left|\int_{\mathbb{R}^n} (u(x,t)-[S_\theta(t)\varphi](x))\eta(x)\,dx\right| &\le C\|\eta\|_{L^\infty} \int_{B(0,R)} |u(x,t)-[S_\theta(t)\varphi](x)|\,dx \\
&\le C\|\eta\|_{L^\infty} |||u(t)-S_\theta(t)\varphi|||_{1,0;T^{1/\theta}} \to 0 \qquad (4\text{-}20)
\end{aligned}$$

as $t \to +0$. Set

$$\eta(x,t) := \int_{\mathbb{R}^n} G_\theta(x-y,t)\eta(y)\,dy \quad \text{for } (x,t) \in \mathbb{R}^n \times (0,\infty).$$

It follows from (2-18) that

$$\lim_{t\to+0} \|\eta(\cdot,t)-\eta\|_{L^\infty} = 0. \tag{4-21}$$

On the other hand, by (2-16), we have

$$|\eta(x,t)| \le Ct^{-\frac{n}{\theta}} \int_{B(0,R)} (1+t^{-\frac{1}{\theta}}|x-y|)^{-n-\theta}|\eta(y)|\,dy \le C\|\eta\|_{L^\infty}t^{-\frac{n}{\theta}}\cdot C(t^{-\frac{1}{\theta}}|x|)^{-n-\theta} \le T|x|^{-n-\theta}$$

for $x \in \mathbb{R}^n \setminus B(0,2R)$ and $t \in (0,T)$. Since $\|\eta(\cdot,t)\|_{L^\infty} \le \|\eta\|_{L^\infty}$ for $t>0$, we obtain

$$|\eta(x,t)| \le C(1+|x|)^{-n-\theta} \quad \text{for } (x,t) \in \mathbb{R}^n \times (0,T). \tag{4-22}$$

Furthermore, it follows from Proposition 3.4 with $q=\infty$ that

$$[S_\theta(1)|\varphi|](0) = \int_{\mathbb{R}^n} G_\theta(y,1)|\varphi(y)|\,dy < \infty.$$

This together with (2-16) implies that

$$\int_{\mathbb{R}^n} (1+|y|)^{-n-\theta}|\varphi(y)|\,dy < \infty. \tag{4-23}$$

Therefore, by (4-21), (4-22), and (4-23), we apply the Fubini theorem and the Lebesgue convergence theorem to obtain

$$\begin{aligned}\int_{\mathbb{R}^n} &[S_\theta(t)\varphi](x)\eta(x)\,dx \\ &= \int_{\mathbb{R}^n}\biggl(\int_{\mathbb{R}^n} G_\theta(x-y,t)\varphi(y)\,dy\biggr)\eta(x)\,dx = \int_{\mathbb{R}^n}\biggl(\int_{\mathbb{R}^n} G_\theta(x-y,t)\eta(x)\,dx\biggr)\varphi(y)\,dy \\ &= \int_{\mathbb{R}^n} \eta(y,t)\varphi(y)\,dy \to \int_{\mathbb{R}^n} \eta(y)\varphi(y)\,dy\end{aligned}$$

as $t \to +0$. Then we deduce from (4-20) that

$$\lim_{t\to+0}\int_{\mathbb{R}^n} u(x,t)\eta(x)\,dx = \int_{\mathbb{R}^n} \varphi(x)\eta(x)\,dx \quad \text{for } \eta \in C_0(\mathbb{R}^n);$$

that is, $u(t) \to \varphi$ in the sense of distributions. The proof of Theorem 1.2 is complete. ☐

Proof of Corollary 1.3. Let φ_c be as in (1-1) with $p = p_\theta$. It follows from the definition of the nonincreasing rearrangements that

$$(\varphi_c)^*(s) \le Cs^{-1}\biggl[\log\biggl(e+\frac{1}{s}\biggr)\biggr]^{-\frac{n}{\theta}-1} \quad \text{for } s \in (0,\infty). \tag{4-24}$$

Let $S>0$. Then, by Lemma 3.2 (2), (2-3), and (4-24), we see that

$$(\varphi_c)^{**}(s) \le Cs^{-1}\biggl[\log\biggl(e+\frac{1}{s}\biggr)\biggr]^{-\frac{n}{\theta}} \quad \text{for } s \in (0,S).$$

This implies that $\varphi_c \in \mathfrak{L}^{1,\infty}_{\mathrm{ul}}(\log \mathfrak{L})^{n/\theta}$. Then Corollary 1.3 follows from Theorem 1.2. ☐

Proof of Theorem 1.4. Since $\alpha > n/\theta$, it follows from (2-12) that

$$|||\varphi|||_{1,\frac{n}{\theta};T^{1/\theta}} \le C\left[\log\left(e+\frac{1}{T^{1/\theta}}\right)\right]^{\frac{n}{\theta}-\alpha} |||\varphi|||_{1,\alpha;T^{1/\theta}} \to 0 \quad \text{as } T \to +0.$$

Then, by Theorem 1.2, we find a solution u to problem (P) in $\mathbb{R}^n \times (0, T)$ for some small enough $T > 0$, with u satisfying (1-6) and (1-7). Thus Theorem 1.4 follows. □

At the end of this paper we recall the definitions of the usual Zygmund space and the usual weak Zygmund space, and explain the advantage of our weak Zygmund-type spaces.

Remark 4.4. (i) We recall the Zygmund space $L^q(\log L)^\alpha$ and the weak Zygmund space $L^{q,\infty}(\log L)^\alpha$. For any $q \in [1,\infty]$ and $\alpha \ge 0$, set

$$L^q(\log L)^\alpha := \{f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{L^q(\log L)^\alpha} < \infty\},$$
$$L^{q,\infty}(\log L)^\alpha := \{f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{L^{q,\infty}(\log L)^\alpha} < \infty\},$$

where

$$\|f\|_{L^q(\log L)^\alpha} := \left(\int_0^\infty \left[\log\left(e+\frac{1}{s}\right)\right]^\alpha f^*(s)^q\, ds\right)^{\frac{1}{q}}, \tag{4-25}$$

$$\|f\|_{L^{q,\infty}(\log L)^\alpha} := \sup_{s>0}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^\alpha s f^*(s)^q\right\}^{\frac{1}{q}}. \tag{4-26}$$

See, e.g., [Bennett and Sharpley 1988, Chapter 4, Section 6] and [Wadade 2014]. For the case $q > 1$, as in the Lorentz space (see, e.g., [Grafakos 2008, Chapter 1, Exercises 1.4.3]), applying Hardy's inequality (see Lemma 2.4) and Lemma 3.2 with (2-3), for any $f \in L^1_{\mathrm{loc}}$, we see that $f \in L^q(\log L)^\alpha$ if and only if

$$[f]_{L^q(\log L)^\alpha} := \left(\int_0^\infty \left[\log\left(e+\frac{1}{s}\right)\right]^\alpha f^{**}(s)^q\, ds\right)^{\frac{1}{q}} < \infty.$$

In contrast, the above relation does not hold for the case $q = 1$. In fact, applying integration by parts, we see that

$$\int_0^\infty \left[\log\left(e+\frac{1}{s}\right)\right]^\alpha f^*(s)\, ds = \alpha\int_0^\infty \left[\log\left(e+\frac{1}{s}\right)\right]^{\alpha-1} f^{**}(s)\frac{ds}{es+1} + \|f\|_{L^1}.$$

(ii) By O'Neil's inequality (2-7), we have the inequality

$$(G_\theta(\cdot,t)*\varphi)^{**}(s) \le \int_s^\infty (G_\theta(\cdot,t))^{**}(\tau)\varphi^{**}(\tau)\, d\tau, \quad s > 0,$$

which is crucial in the proof of our sharp decay estimates of $S_\theta(t)\varphi$. Our Zygmund-type spaces are defined by the average of the nonincreasing rearrangement, and they are effectively used in the proof of our sharp decay estimates of $S_\theta(t)\varphi$ (see the proof of Proposition 3.1). These sharp decay estimates of $S_\theta(t)\varphi$ in the spaces $\mathfrak{L}^{q,\infty}(\log\mathfrak{L})^\alpha$ enable us to obtain Theorem 1.2.

On the other hand, since the weak Zygmund space $L^{q,\infty}(\log L)^\alpha$ is defined by the nonincreasing rearrangement, the inequality

$$(G_\theta(\cdot,t)*\varphi)^*(s) \le (G_\theta(\cdot,t)*\varphi)^{**}(s) \le \int_s^\infty (G_\theta(\cdot,t))^{**}(\tau)\varphi^{**}(\tau)\,d\tau, \quad s>0, \tag{4-27}$$

seems useful for the study of decay estimates of $S_\theta(t)\varphi$ in the space $L^{q,\infty}(\log L)^\alpha$. The first inequality in (4-27) follows from inequality (2-4). However, in general, inequality (2-4) is not sharp in $L^{1,\infty}(\log L)^\alpha$, where $\alpha>1$. Indeed, let $f\in L^1_{\mathrm{loc}}$ be such that

$$f^*(s) = s^{-1}\left[\log\left(e+\frac{1}{s}\right)\right]^{-\alpha}, \quad s>0,$$

where $\alpha>1$. Then $f\in L^{1,\infty}(\log L)^\alpha$ and

$$f^{**}(s) \asymp s^{-1}\left[\log\left(e+\frac{1}{s}\right)\right]^{-\alpha+1}$$

for small enough $s>0$. Then $f^*(s)/f^{**}(s)\to 0$ as $s\to +0$, and we see that inequality (2-4) is not sharp. This suggests that it is difficult to obtain sharp decay estimates of $S_\theta(t)\varphi$ in the usual weak Zygmund spaces.

(iii) In order to overcome the disadvantage of the usual weak Zygmund spaces, one might consider the weak Zygmund-type spaces

$$\mathbb{L}^{q,\infty}(\log\mathbb{L})^\alpha := \{f\in L^1_{\mathrm{loc}} : \|f\|_{\mathbb{L}^{q,\infty}(\log\mathbb{L})^\alpha}<\infty\},$$

where $1\le q<\infty$, $\alpha\ge 0$, and

$$\|f\|_{\mathbb{L}^{q,\infty}(\log\mathbb{L})^\alpha} := \sup_{s>0}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^\alpha s f^{**}(s)^q\right\}^{\frac{1}{q}}.$$

Indeed, applying the arguments to those in the proof of Proposition 3.4, we can obtain similar sharp decay estimates of $S_\theta(t)\varphi$ in the weak Zygmund-type space $\mathbb{L}^{q,\infty}(\log\mathbb{L})^\alpha$ to those in Proposition 3.4.

On the other hand, in the proof of Theorem 1.2, we used the inequality

$$\||f|^p\|_{\mathfrak{L}^{1,\infty}(\log\mathfrak{L})^\alpha} \le C\|f\|^p_{\mathfrak{L}^{p,\infty}(\log\mathfrak{L})^\alpha} \quad \text{for } f\in\mathfrak{L}^{p,\infty}(\log\mathfrak{L})^\alpha \tag{4-28}$$

in order to estimate the nonlinear term $|u|^{p-1}u$, where $p>1$ and $\alpha\ge 0$. Actually, (4-28) holds with $C=1$ and "$\le$" replace by "$=$" (see Lemma 2.3). In the case of $\mathbb{L}^{q,\infty}(\log\mathbb{L})^\alpha$, it follows from (2-5) that

$$\begin{aligned}\||f|^p\|_{\mathbb{L}^{1,\infty}(\log\mathbb{L})^\alpha} &= \sup_{s>0}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^\alpha s(|f|^p)^{**}(s)\right\}\\ &\ge \sup_{s>0}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^\alpha s(f^{**}(s))^p\right\} = \|f\|^p_{\mathbb{L}^{p,\infty}(\log\mathbb{L})^\alpha}\end{aligned}$$

for $f\in\mathbb{L}^{p,\infty}(\log\mathbb{L})^\alpha$; that is, the reverse to the desired inequality holds. This suggests that it is difficult to obtain a similar result to that of Theorem 1.2 in the framework of weak Zygmund-type spaces $\mathbb{L}^{q,\infty}(\log\mathbb{L})^\alpha$.

Appendix

Here we prove two propositions on relations between $L^q(\log L)^\alpha$, $L^{q,\infty}(\log L)^\alpha$, and $\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha$. We remark that the following relations hold for $\alpha = 0$:

$$L^q = L^q(\log L)^0 = \mathfrak{L}^{q,\infty}(\log \mathfrak{L})^0 \subsetneq L^{q,\infty} = L^{q,\infty}(\log L)^0 \quad \text{if } q \in [1,\infty).$$

Proposition A.1. *Let* $1 \le q < \infty$ *and* $\alpha \ge 0$. *Then*

$$\|f\|_{\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha} \le \|f\|_{L^q(\log L)^\alpha} \qquad \textit{for } f \in L^q(\log L)^\alpha,$$
$$\|f\|_{L^{q,\infty}(\log L)^\alpha} \le \|f\|_{\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha} \quad \textit{for } f \in \mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha.$$

Furthermore,

$$L^q(\log L)^\alpha \subsetneq \mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha \subsetneq L^{q,\infty}(\log L)^\alpha, \quad \alpha > 0.$$

Proof. By (2-9), (2-11), and (4-25), we see that

$$\begin{aligned}\|f\|_{\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha} &= \sup_{s>0}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^\alpha \int_0^s (f^*(\tau))^q\,d\tau\right\}^{\frac{1}{q}}\\ &\le \sup_{s>0}\left(\int_0^s \left[\log\left(e+\frac{1}{\tau}\right)\right]^\alpha (f^*(\tau))^q\,d\tau\right)^{\frac{1}{q}} = \|f\|_{L^q(\log L)^\alpha}\end{aligned}$$

for $f \in L^q(\log L)^\alpha$. This implies that $L^q(\log L)^\alpha \subset \mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha$. Let g be a function in $\mathbb{R}^n$ such that

$$g^*(s) = \frac{d}{ds}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^{-\alpha}\right\}\chi_{(0,\delta)}(s) = \frac{\alpha}{es^2+s}\left[\log\left(e+\frac{1}{s}\right)\right]^{-\alpha-1}\chi_{(0,\delta)}(s),$$

where $\delta > 0$ is chosen so that g^* is decreasing. Set $f(x) := |g(x)|^{1/q}$. It follows from (2-1) that $f^*(s)^q = g^*(s)$. Furthermore,

$$\|f\|^q_{\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha} = \sup_{s>0}\left[\log\left(e+\frac{1}{s}\right)\right]^\alpha \int_0^s g^*(\eta)\,d\eta = 1$$

and

$$\|f\|^q_{L^q(\log L)^\alpha} = \int_0^\infty \left[\log\left(e+\frac{1}{\eta}\right)\right]^\alpha g^*(\eta)\,d\eta = \int_0^\delta \frac{\alpha}{e\eta^2+\eta}\left[\log\left(e+\frac{1}{\eta}\right)\right]^{-1} d\eta = \infty.$$

Thus $L^q(\log L)^\alpha \subsetneq \mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha$.

On the other hand, it follows from (2-1), (2-4), (2-9), and (4-26) that

$$\begin{aligned}\|f\|_{\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha} &= \sup_{s>0}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^\alpha s(|f|^q)^{**}(s)\right\}^{\frac{1}{q}}\\ &\ge \sup_{s>0}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^\alpha s(|f|^q)^{*}(s)\right\}^{\frac{1}{q}}\\ &= \sup_{s>0}\left\{\left[\log\left(e+\frac{1}{s}\right)\right]^\alpha s(f^*(s))^q\right\}^{\frac{1}{q}} = \|f\|_{L^{q,\infty}(\log L)^\alpha}\end{aligned}$$

for $f \in \mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha$, and hence $\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha \subset L^{q,\infty}(\log L)^\alpha$. We finally show that the inclusion is strict. Let f be a function such that

$$f^*(s) = s^{-\frac{1}{q}}\left[\log\left(e+\frac{1}{s}\right)\right]^{-\frac{\alpha}{q}} \chi_{(0,\delta)}(s),$$

where $\delta > 0$ is chosen so that f^* is decreasing. Then $\|f\|_{L^{q,\infty}(\log L)^\alpha} = 1$. On the other hand, for the case $\alpha \le 1$, we see that

$$s(|f|^q)^{**}(s) = \int_0^s \eta^{-1}\left[\log\left(e+\frac{1}{\eta}\right)\right]^{-\alpha} d\eta \ge \int_0^s \eta^{-1}\left[\log\left(e+\frac{1}{\eta}\right)\right]^{-1} d\eta = \infty$$

for $s \in (0,\delta)$. This implies that $f \notin \mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha$. Furthermore, for the case $\alpha > 1$, there exists $C > 0$ such that

$$\begin{aligned} s(|f|^q)^{**}(s) &= \int_0^s \eta^{-1}\left[\log\left(e+\frac{1}{\eta}\right)\right]^{-\alpha} d\eta \\ &\ge C\int_0^s (e\eta^2+\eta)^{-1}\left[\log\left(e+\frac{1}{\eta}\right)\right]^{-\alpha} d\eta = \frac{C}{\alpha-1}\left[\log\left(e+\frac{1}{s}\right)\right]^{1-\alpha} \end{aligned}$$

for $s \in (0,\delta)$. In conclusion, there exists $C > 0$ such that

$$\|f\|_{\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha} \ge C \sup_{0<s<\delta}\left[\log\left(e+\frac{1}{s}\right)\right]^{\frac{1}{q}} = \infty.$$

Thus $\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha \subsetneq L^{q,\infty}(\log L)^\alpha$. The proof of Proposition A.1 is complete. □

Let f be a locally integrable function in $\mathbb{R}^n$ such that

$$\left[\log\left(e+\frac{1}{s}\right)\right]^{\alpha} s(|f|^q)^{**}(s) = 1, \quad s > 0, \tag{A-1}$$

which is a typical function in $\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha$. By (A-1), we see

$$f^*(s)^q = \frac{d}{ds}(s(|f|^q)^{**}(s)) = \frac{\alpha}{es^2+s}\left[\log\left(e+\frac{1}{s}\right)\right]^{-\alpha-1}, \quad s > 0.$$

Since

$$f^*(s) \asymp s^{-\frac{1}{q}}\left[\log\left(e+\frac{1}{s}\right)\right]^{-\frac{\alpha+1}{q}} \quad \text{for small enough } s > 0,$$

we see that f also has a typical singularity of functions in $L^{q,\infty}(\log L)^{\alpha+1}$. These arguments suggest that $\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha$ is closely related to $L^{q,\infty}(\log L)^{\alpha+1}$.

Proposition A.2. *Let $1 \le q < \infty$ and $\alpha > 0$. Then there exists $C > 0$ such that*

$$\|f\|_{\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha} \le C\|f\|_{L^{q,\infty}(\log L)^{\alpha+1}} \tag{A-2}$$

for $f \in L^{q,\infty}(\log L)^{\alpha+1}$. Furthermore,

$$\inf\left\{\frac{\|f\|_{\mathfrak{L}^{q,\infty}(\log \mathfrak{L})^\alpha}}{\|f\|_{L^{q,\infty}(\log L)^{\alpha+1}}} : f \in L^{q,\infty}(\log L)^{\alpha+1}\right\} = 0. \tag{A-3}$$

Proof. By Lemma 2.3 and (2-1), it suffices to consider the case $q = 1$. We first prove (A-2) with $q = 1$. Let $f \in L^{1,\infty}(\log L)^{\alpha+1}$, where $\alpha > 0$. By Lemma 3.2 (2), for any $R > 0$, we have

$$sf^{**}(s) = \int_0^s f^*(\eta)\,d\eta \le \left(\int_0^s \eta^{-1}\left[\log\left(e+\frac{1}{\eta}\right)\right]^{-\alpha-1} d\eta\right)\left(\sup_{\eta>0}\left[\log\left(e+\frac{1}{\eta}\right)\right]^{\alpha+1}\eta f^*(\eta)\right)$$
$$\le C\left[\log\left(e+\frac{1}{s}\right)\right]^{-\alpha}\left(\sup_{\eta>0}\left[\log\left(e+\frac{1}{\eta}\right)\right]^{\alpha+1}\eta f^*(\eta)\right)$$

for $s \in (0, R)$. This together with (2-9) implies that

$$\|f\|_{\mathfrak{L}^{1,\infty}(\log \mathfrak{L})^\alpha} = \sup_{0<s<R}\left[\log\left(e+\frac{1}{s}\right)\right]^{\alpha} sf^{**}(s)$$
$$\le C\sup_{\eta>0}\left[\log\left(e+\frac{1}{\eta}\right)\right]^{\alpha+1}\eta f^*(\eta) = C\|f\|_{L^{1,\infty}(\log L)^{\alpha+1}}.$$

Thus (A-2) holds for $q = 1$, and the proof of (A-2) is complete.

Next, we prove (A-3) with $q = 1$. Let $\{f_n\}$ be a sequence in L^1_{loc} such that

$$f_n^*(s) = n[\log(e+n)]^{-\alpha-1}\chi_{(0,\frac{1}{n})}(s).$$

Since

$$f_n^{**}(s) = \begin{cases} n[\log(e+n)]^{-\alpha-1} & \text{for } s \in \left(0, \frac{1}{n}\right), \\ s^{-1}[\log(e+n)]^{-\alpha-1} & \text{for } s \in \left[\frac{1}{n}, \infty\right), \end{cases} \tag{A-4}$$

we have

$$\|f_n\|_{\mathfrak{L}^{1,\infty}(\log \mathfrak{L})^\alpha} = \sup_{s>0}\left[\log\left(e+\frac{1}{s}\right)\right]^{\alpha} sf_n^{**}(s) = \left[\log\left(e+\frac{1}{s}\right)\right]^{\alpha} sf_n^{**}(s)\bigg|_{s=\frac{1}{n}} = [\log(e+n)]^{-1}$$

for $n = 1, 2, \ldots$. On the other hand, similarly to (3-2), we find $L \in [e, \infty)$ such that

$$\text{the function } (0,\infty) \ni \tau \mapsto \tau\left[\log\left(L+\frac{1}{\tau}\right)\right]^{\alpha+1} \text{ is nondecreasing.}$$

Then, by (3-1) and (A-4), we have

$$\|f_n\|_{L^{1,\infty}(\log L)^{\alpha+1}} = \sup_{s>0}\left[\log\left(e+\frac{1}{s}\right)\right]^{\alpha+1} sf_n^*(s)$$
$$\ge C\sup_{s>0}\left[\log\left(L+\frac{1}{s}\right)\right]^{\alpha+1} sf_n^*(s) = C\left[\log\left(L+\frac{1}{s}\right)\right]^{\alpha+1} sf_n^*(s)\bigg|_{s=\frac{1}{n}} \ge C$$

for $n = 1, 2, \ldots$. These imply (A-3). Thus Proposition A.2 follows. □

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A MARCINKIEWICZ MULTIPLIER THEORY FOR SCHUR MULTIPLIERS

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We prove a Marcinkiewicz-type multiplier theory for the boundedness of Schur multipliers on the Schatten p-classes. This generalizes a previous result of J. Bourgain for Toeplitz-type Schur multipliers and complements a recent result by J. Conde-Alonso et al. (*Ann. of Math.* (2) **198**:3 (2023), 1229–1260). As a corollary, we obtain a new unconditional decomposition for the Schatten p-classes, $1 < p < \infty$. We extend our main result to the $\mathbb{Z}^d$ and $\mathbb{R}^d$ cases, and include an operator-valued version of it using Pisier's noncommutative $L^\infty(\ell_1)$-norm.

1. Introduction

Let $A \in B(H)$ be a bounded operator on a (separable) Hilbert space H. We can write A in its matrix representation

$$A = (a_{k,j})_{k,j\in\mathbb{Z}},$$

with $a_{k,j} = \langle Ae_k, e_j\rangle$ for a given orthonormal basis $\{e_k\}_{k\in\mathbb{Z}}$ of H. Given a bounded function m on $\mathbb{Z}\times\mathbb{Z}$, we call the map

$$M_m : (a_{k,j}) \mapsto (m(k,j)a_{k,j}) \tag{1-1}$$

a Schur multiplier with symbol m. The study of the boundedness of Schur multipliers with respect to the Schatten p-norms has a rich history [Bennett 1977; Arazy 1982; Bożejko and Fendler 1984; Berkson and Gillespie 1994; Pisier 1998; 2001; Harcharras 1999; Clément et al. 2000; Aleksandrov and Peller 2002; Doust and Gillespie 2005]. The recent study of noncommutative analysis on the approximation properties of operator functions and operator algebras [Haagerup et al. 2010; Neuwirth and Ricard 2011; Caspers and de la Salle 2015; Potapov et al. 2015; 2017; de Laat and de la Salle 2018; Caspers et al. 2019; Parcet et al. 2022; Mei et al. 2022; Conde-Alonso et al. 2023], especially the work [Lafforgue and de la Salle 2011] on the approximation property of higher-rank Lie groups, draws a lot of attention to the boundedness of Schur multipliers for the case where $1 < p \neq 2 < \infty$. Conde-Alonso, González–Pérez, Parcet and Tablate [Conde-Alonso et al. 2023] recently proved a Hörmander–Mikhlin-type Schur multiplier theory for S^p, $1 < p \neq 2 < \infty$, in their remarkable work.

In this article, we prove a Marcinkiewicz-type Schur multiplier theory. Hörmander–Mikhlin-type multipliers and Marcinkiewicz-type multipliers are rooted in classical Fourier analysis. Like their counterpart in Fourier analysis, Marcinkiewicz-type Schur multipliers are a larger class of multipliers

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and their p-boundedness is more subtle; it was shown in [Tao and Wright 2001] that the L^p-bounds of Marcinkiewicz Fourier multipliers are of order $p^{3/2}$ as $p \to \infty$.

When m is of Toeplitz-type, i.e., $m(k, j) = \dot{m}(k - j)$ for some function $\dot{m} : \mathbb{Z} \to \mathbb{C}$, one may apply a well-known transference method and obtain bounded Schur multipliers from the classical Fourier multiplier theory. J. Bourgain's work [1986, Theorem 4, Corollary 20] on scalar-valued Fourier multipliers acting on Schatten p-valued functions implies that the following Marcinkiewicz-type condition is sufficient for the boundedness of M_m on the Schatten p-classes for all $1 < p < \infty$:

$$\sum_{2^{n-1} \leq |k| < 2^n} |\dot{m}(k+1) - \dot{m}(k)| < C \tag{1-2}$$

for all $n \in \mathbb{N}$. Let $\dot{m}_\varepsilon(k) = \varepsilon_n$ for $2^{n-1} \leq |k| < 2^n$. Then the associated multiplier M_{m_ε} is bounded for any sequence $\varepsilon_n = \pm 1$.

To extend Bourgain's result to general non-Toeplitz-type Schur multipliers, one may ask whether the condition that

$$\sum_{2^{n-1} \leq |k| < 2^n} |m(k+j+1, j) - m(k+j, j)| < C \tag{1-3}$$

for all $n \in \mathbb{N}$, $j \in \mathbb{Z}$ implies the S^p boundedness of general Schur multipliers. The answer is yes if m is Toeplitz since condition (1-3) reduces to Bourgain's condition (1-2) in that case. The answer would be yes for general Schur multipliers as well if the family M_{m_ε}, defined after condition (1-2), is R-bounded for any family of sequences $\varepsilon = (\varepsilon_k)_k$ valued in $\{\pm 1\}$. This implication was proved in the works of Berkson and Gillespie [1994], Doust and Gillespie [2005] and Clément, de Pagter, Sukochev and Witvliet [Clément et al. 2000], in which they studied the connection between vector-valued Littlewood–Paley theory and Marcinkiewicz multiplier theory. We show in Section 5.1 that this is not true in general and the condition (1-3) is not sufficient for the S^p-boundedness of the associated Schur multiplier.[1]

The main result of this article is the following.

Theorem 1.1. *M_m defined in (1-1) extends to a bounded map on the Schatten p-classes S^p for all $1 < p < \infty$ with bounds $\lesssim (p^2/(p-1))^3$ if m is bounded and there exists a constant C such that*

$$\sum_{2^{n-1} \leq |k| < 2^n} |m(k+j+1, j) - m(k+j, j)| < C, \tag{1-4}$$

$$\sum_{2^{n-1} \leq |k| < 2^n} |m(j, k+j+1) - m(j, k+j)| < C \tag{1-5}$$

for all $n \in \mathbb{N}$, $j \in \mathbb{Z}$.

The writing of this article was motivated by the recent article [Conde-Alonso et al. 2023], although the third author had known Theorem 1.1 previously. The authors of [loc. cit.] further studied Schatten-p-classes indexed in d-dimensional Euclidean spaces, aiming for possible applications to the approximation properties of higher-rank Lie groups. Following this trend, we extend Theorem 1.1 to the higher-dimensional

[1]This also shows that the family of "Littlewood–Paley operators" M_{m_ε} mentioned above is *not* R-bounded.

cases as well. The proof of Theorem 1.1 relies on the crucial property that a Schur multiplier is an operator-valued Fourier multiplier multiplying from the left, and is simultaneously an operator-valued Fourier multiplier multiplying from the right. This property was already used by the authors of [Conde-Alonso et al. 2023] in proving a Hörmander–Mikhlin-type criterion for the boundedness of Schur multipliers.

Theorem 1.1 implies a new unconditional decomposition for Schatten p classes. For $(n,\ell)\in\mathbb{Z}\times\mathbb{Z}$, let $E_{0,\ell}=\{(\ell,\ell)\}\subset\mathbb{Z}\times\mathbb{Z}$. Let

$$E_{n,\ell}=\{(k,j)\in\mathbb{Z}\times\mathbb{Z}:2^{n-1}\le k-j<2^n,\ \ell 2^n\le k<(\ell+1)2^n\}$$

for $n>0$, and

$$E_{n,\ell}=\{(k,j)\in\mathbb{Z}\times\mathbb{Z}:-2^{|n|}<k-j\le -2^{|n|-1},\ \ell 2^{|n|}\le k<(\ell+1)2^{|n|}\}$$

for $n<0$. We then have the decomposition

$$\mathbb{Z}\times\mathbb{Z}=\bigcup_{(n,\ell)\in\mathbb{Z}\times\mathbb{Z}}E_{n,\ell}.$$

Let $P_{n,\ell}$ be the projection onto $\operatorname{span}\{e_{k,j},(k,j)\in E_{n,\ell}\}$. It is easy to see that $\sum_{n,\ell}\varepsilon_{n,\ell}P_{n,\ell}$ is a Schur multiplier satisfying the assumptions of Theorem 1.1 for any bounded sequence $\varepsilon_{n,\ell}$. We then obtain an unconditional decomposition of S^p.

Corollary 1.2. *$\sum_{n,\ell}\varepsilon_{n,\ell}P_{n,\ell}$ extends to a bounded map on S^p for all $1<p<\infty$ for any bounded sequence $\varepsilon_{n,\ell}$.*

We will prove Theorem 1.1 in Section 3. We will explain how to extend Theorem 1.1 to the higher-dimensional case in Section 4 and explain that the ball-type Schur multipliers remain bounded on $S^p(\ell^2(\mathbb{Z}^d))$ for $d>1$ (Example 4.4), contrary to the behavior of Fourier multipliers. We will show that the condition (1-4) alone is not sufficient in Section 5.1 and explain an operator-valued version of Theorem 1.1 in Section 5.2.

2. Preliminaries

Given $d\in\mathbb{N}$, denote by $\mathcal{B}(\ell^2(\mathbb{Z}^d))$ the set of bounded linear operators on $\ell^2(\mathbb{Z}^d)$. We represent the operator $A\in\mathcal{B}(\ell^2(\mathbb{Z}^d))$ as $A=(a_{i,j})_{(i,j)\in\mathbb{Z}^d\times\mathbb{Z}^d}$ with $a_{i,j}=\langle Ae_i,e_j\rangle$ for the canonical basis $\{e_i\}$ of $\ell^2(\mathbb{Z}^d)$. Given a bounded function m on $\mathbb{Z}^d\times\mathbb{Z}^d$, the associated Schur multiplier

$$M_m(A)=(m(i,j)a_{ij})$$

extends to a bounded operator on the Hilbert–Schmidt class $S^2(\ell^2(\mathbb{Z}^d))$. We call m the symbol of M_m. Recall that the Schatten p-class S^p, $1\le p<\infty$, is the collection of all compact operators A with a finite Schatten-p norm, which is defined as

$$\|A\|_p=(\operatorname{tr}[(A^*A)^{\frac{p}{2}}])^{\frac{1}{p}}=\Bigl(\sum_i s_i^p\Bigr)^{\frac{1}{p}} \tag{2-1}$$

for $1\le p<\infty$, where s_i is the i-th singular value of A. The Schatten p norm is unitary invariant and does not depend on the choice of the orthonormal basis. The Schatten-class S^p, $1\le p<\infty$, and

$B(\ell^2(\mathbb{Z}^d))$ share many similar properties with ℓ^p, $1 \le p \le \infty$. In particular, the dual space of S^1 (resp. S^p, $1 < p < \infty$) is isomorphic to $B(\ell^2)$ (resp. $S^{p/(p-1)}$). The family forms an interpolation scale

$$[B(\ell^2), S^1]_{\frac{1}{p}} = S^p$$

for $1 < p < \infty$. However, S^p does not admit an unconditional basis whenever $p \neq 2$. We will prove that, for m satisfying additional conditions (1-4) and (1-5), M_m extends to a bounded map on S^p for all $1 < p < \infty$, which immediately implies the unconditional decomposition for S^p as stated in Corollary 1.2.

Given $d \in \mathbb{N}$, we denote by $L^p(\mathbb{T}^d; S^p)$ the space of S^p-valued Bochner integrable functions f such that

$$\|f\|_{L^p} = \left(\operatorname{tr}\left[\int_{[0,1)^d} |f|^p(z)\, d\theta\right]\right)^{\frac{1}{p}} < \infty.$$

Here we let $z = e^{i2\pi\theta}$, with $\theta \in [0,1)^d$, and $|f|^p = (f^* f)^{p/2}$ is defined via the functional calculus.

For $f \in L^p(\mathbb{T}^d; S^p)$ with $1 < p < \infty$, we have the Fourier expansion

$$f(z) \sim \sum_{k \in \mathbb{Z}^d} \hat{f}(k) z^k,$$

with $\hat{f}(k) = \int_{[0,1]^d} f(z)\bar{z}^k\, d\theta \in S^p$. Given R a finite subset of $\mathbb{Z}^d$, denote by $S_R f$ the partial Fourier sum

$$S_R f(z) = \sum_{k \in R} \hat{f}(k) z^k. \tag{2-2}$$

Choose $\delta \in C^\infty(\mathbb{R})$ such that $0 \le \delta \le 1$, $\operatorname{supp}(\delta) \subset \left[-2\sqrt{d}, -\frac{1}{4}\right] \cup \left[\frac{1}{4}, 2\sqrt{d}\right]$ and $\delta(x) = 1$ when $\frac{1}{2} \le |x| \le \sqrt{d}$. For $j \ge 0$, define $\delta_j(x) := \delta(2^{-j}x)$. For $f \in L^2(\mathbb{T}^d; S^2)$, we define

$$S_j f(z) = \sum_{k \in \mathbb{Z}^d} \delta_j(|k|_2) \hat{f}(k) z^k.$$

We denote by $|k|_2$ the ℓ_2 norm of $k \in \mathbb{Z}^d$ in the formula above and will denote by $|k|_\infty$ the ℓ_∞ norm of k. Let $(E_j)_{j \ge 0}$ be the cubes with squared holes in $\mathbb{Z}^d$ given by

$$E_j = \begin{cases} \{k \in \mathbb{Z}^d : 2^{j-1} \le |k|_\infty < 2^j\}, & j > 0, \\ \{0\}, & j = 0. \end{cases} \tag{2-3}$$

Note our construction implies $S_{E_j} S_j = S_{E_j}$, which we will need later.

For a sequence (f_k) in $L^p(\mathbb{T}^d; S^p)$, we use the classical notation

$$\|(f_k)\|_{L^p(\ell_2^c)} = \left\| \left(\sum_k |f_k|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{T}^d; S^p)}, \quad \|(f_k)\|_{L^p(\ell_2^r)} = \left\| \left(\sum_k |f_k^*|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{T}^d; S^p)},$$

and

$$\|(f_k)\|_{L^p(\ell_2)} = \begin{cases} \max\{\|(f_k)\|_{L^p(\ell_2^c)}, \|(f_k^*)\|_{L^p(\ell_2^c)}\} & \text{if } 2 \le p \le \infty, \\ \inf_{y_k + z_k = f_k} \|(y_k)\|_{L^p(\ell_2^c)} + \|(z_k)\|_{L^p(\ell_2^r)} & \text{if } 0 < p < 2. \end{cases}$$

The above definition is justified by the following noncommutative Khintchine inequality:

Lemma 2.1 [Lust-Piquard 1986; Lust-Piquard and Pisier 1991]. *Let (ε_k) be a sequence of independent Rademacher random variables. Then, for $1 \le p < \infty$,*

$$\alpha_p^{-1} E_\varepsilon \Big\| \sum_k \varepsilon_k \otimes f_k \Big\|_{L^p(\mathbb{T}^d; S^p)} \le \|(f_k)\|_{L^p(\ell_2)} \le \beta_p E_\varepsilon \Big\| \sum_k \varepsilon_k \otimes f_k \Big\|_{L^p(\mathbb{T}^d; S^p)}. \tag{2-4}$$

The optimal constant β_p is no greater than $\sqrt{3}$ for $1 \le p \le 2$ and is 1 for $p \ge 2$ (see [Haagerup and Musat 2007]); α_p is 1 for $1 \le p \le 2$ and is of order $\sqrt{p}$ as $p \to \infty$. Inequality (2-4) was pushed further to the case where $0 < p < 1$ (see [Pisier and Ricard 2017]).

We will need the following noncommutative Littlewood–Paley theorem on $\mathbb{Z}^d$.

Lemma 2.2. *There is a constant $C_d > 0$ that depends only on d such that*

$$\|(S_j f)_{j \ge 0}\|_{L^p(\ell_2)} \le C_d \frac{p^2}{p-1} \|f\|_{L^p(\mathbb{T}^d; S^p)}, \tag{2-5}$$

$$\|f\|_{L^p(\mathbb{T}^d; S^p)} \le C_d \frac{p^2}{p-1} \|(S_{E_j} f)_{j \ge 0}\|_{L^p(\ell_2)} \tag{2-6}$$

for all $f \in L^p(\mathbb{T}^d; S^p)$ and $1 < p < \infty$.

Proof. This lemma is well known. We explain here that the dependence of the constants on p is in the order of $p^2/(p-1)$. Given $\varepsilon_j = \pm 1$, let $M_\varepsilon = \sum_{j \ge 0} \varepsilon_j S_j$. Our choice of S_j's makes M_ε a so-called Hörmander–Mikhlin multiplier, which in particular is a Calderón–Zygmund operator. So it is bounded from the classical Hardy space H^1 to L^1. Moreover, it is from H^1 to H^1 since it commutes with the classical Hilbert transform. By [Mei 2007, Theorem 6.4], it extends to a bounded operator on the semicommutative BMO space $\mathrm{BMO}_{\mathrm{cr}}(L^\infty(\mathbb{T}^d) \bar{\otimes} \mathcal{B}(\ell^2(\mathbb{Z}^d)))$. Inequality (2-5) then follows from the interpolation result [Mei 2007, Theorem 6.2] and the Khintchine inequality (2-4). Inequality (2-6) follows from (2-5) by duality because of the identity $\langle f, g \rangle = \sum_j \langle S_{E_j} f, S_{E_j} g \rangle = \sum_j \langle S_j f, S_{E_j} g \rangle$. □

Lemma 2.3. *Suppose R_j is a family of boxes with sides parallel to the axes in $\mathbb{R}^d$. Then there is a constant $C_d > 0$ that depends only on d such that, for all $1 < p < \infty$ and for all families of measurable functions f_j on $\mathbb{R}^d$, we have*

$$\Big\| \Big(\sum_j |S_{R_j}(f_j)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\mathbb{T}^d; S^p)} \le C_d \Big(\frac{p^2}{p-1} \Big)^d \Big\| \Big(\sum_j |f_j|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\mathbb{T}^d; S^p)}. \tag{2-7}$$

Proof. Assume $d = 1$ and $R_j = [a_j, b_j]$. Let T_a to be the operator that sends $f(\cdot)$ to $f(\cdot) e^{i 2\pi a_j (\cdot)}$. Let P_+ be the analytic projection. Then

$$S_{R_j} = T_{a_j} P_+ T_{-a_j} - T_{b_j} P_+ T_{-b_j}.$$

Note that $|T_{a_j} f_j|^2 = |f_j|^2$ and we obtain the inequality for $d = 1$ by the boundedness of P_+. The case $d > 1$ holds due to Fubini's theorem. □

Lemma 2.4. *For sequences $(a_n), (c_n) \in B(H)$, we have*

$$\Big| \sum_n a_n^* c_n \Big|^2 \le \Big\| \sum_n a_n^* a_n \Big\| \Big(\sum_n c_n^* c_n \Big). \tag{2-8}$$

Proof. Given any $v \in S^2$, we have by the Cauchy–Schwarz inequality that

$$\mathrm{tr}\left(v^*\left|\sum_n a_n^* c_n\right|^2 v\right) = \mathrm{tr}\left(\left|\sum_n a_n^* c_n v\right|^2\right) \le \left\|\sum_n a_n^* a_n\right\| \mathrm{tr}\left[v^*\left(\sum_n c_n^* c_n\right)v\right].$$

Since v is arbitrary, we obtain (2-8). □

3. Proof of Theorem 1.1

For $z \in \mathbb{T}^d$ given, let Π_z be the *-homomorphism on $\mathcal{B}(\ell^2(\mathbb{Z}^d))$ defined as

$$\Pi_z(A) = U_z A U_z^*,$$

with U_z the unitary sending e_k to $z^k e_k$. It is easy to see that Π_z has the presentation

$$\Pi_z : A = (a_{k,j}) \longmapsto (a_{k,j} z^{k-j}), \tag{3-1}$$

with $k, j \in \mathbb{Z}^d$. Π_z defines an isometric isomorphism on $\mathcal{B}(\ell^2(\mathbb{Z}^d))$ and $S^p(\ell^2(\mathbb{Z}^d))$ for all $1 \le p < \infty$ because all these norms are unitary invariant. Considering z as a variable on $\mathbb{T}^d$, define $\Pi : \mathcal{B}(\ell^2(\mathbb{Z}^d)) \to L^\infty(\mathbb{T}^d) \otimes \mathcal{B}(\ell^2(\mathbb{Z}^d))$ as

$$\Pi(A)(z) = \Pi_z(A). \tag{3-2}$$

Then Π is an isometric isomorphism from $\mathcal{B}(\ell^2(\mathbb{Z}^d))$ to $L^\infty(\mathbb{T}^d) \otimes \mathcal{B}(\ell^2(\mathbb{Z}^d))$ and from $S^p(\ell^2(\mathbb{Z}^d))$ to $L^p(\mathbb{T}^d; S^p(\ell^2(\mathbb{Z}^d)))$ for all $1 \le p < \infty$.

Given a symbol $m = (m(i,j))_{i,j \in \mathbb{Z}^d}$ and $A = (a_{i,j})_{i,j \in \mathbb{Z}^d} \in S^p(\ell^2(\mathbb{Z}^d))$, set

$$\begin{gathered} M_l(n) = \sum_{s \in \mathbb{Z}^d} m(s, s-n)\, e_{s,s}, \quad M_r(n) = \sum_{s \in \mathbb{Z}^d} m(s+n, s)\, e_{s,s}, \\ A(n) = \sum_{s \in \mathbb{Z}^d} a_{s,s-n} e_{s,s-n} = \sum_{s \in \mathbb{Z}^d} a_{s+n,s} e_{s+n,s} \end{gathered} \tag{3-3}$$

for $n \in \mathbb{Z}^d$. Here $e_{s,t}$ denotes the operator on $\ell^2(\mathbb{Z}^d)$ sending e_t to e_s. Then $M_l(n), M_r(n) \in \mathcal{B}(\ell^2(\mathbb{Z}^d))$ with norm bounded by C, $A(n) \in S^p(\ell^2(\mathbb{Z}^d))$ for all $n \in \mathbb{Z}^d$, and

$$\Pi(A)(z) = \sum_n A(n) z^n, \quad M_m(A) = \sum_n M_l(n) A(n) = \sum_n A(n) M_r(n).$$

Here $M_l(n)A(n)$ and $A(n)M_r(n)$ denote the products of operators in $\mathcal{B}(\ell^2(\mathbb{Z}^d))$. Let $f = \Pi(A)$, i.e.,

$$f(z) = \sum_{n \in \mathbb{Z}^d} A(n) z^n. \tag{3-4}$$

Denote by $\Pi(S^p)$ the image of Π, i.e., the subspace of $L_p(\mathbb{T}^d; S^p)$ consisting of all f in the form of (3-4). Define the operator-valued Fourier multiplier T_M on $\Pi(S^p)$ as

$$T_M f(z) = \sum_{n \in \mathbb{Z}^d} M_l(n) A(n) z^n. \tag{3-5}$$

Note that $M_l(n)A(n) = A(n)M_r(n)$ for all $n \in \mathbb{Z}^d$; we can represent T_M as a multiplier from the right:

$$T_M f(z) = \sum_{n\in\mathbb{Z}^d} A(n)M_r(n)z^n. \tag{3-6}$$

T_M is defined so that the following identity holds:

$$T_M f = \Pi(M_m(A)).$$

Since Π is a trace preserving $*$-homomorphism, we have

$$\|A\|_{S^p} = \|f\|_{L^p(\mathbb{T}^d;S^p)}, \quad \|M_m A\|_{S^p} = \|T_M(f)\|_{L^p(\mathbb{T}^d;S^p)}. \tag{3-7}$$

In order to prove M_m's boundedness on S^p, we only need to prove that T_M is bounded on $\Pi(S^p)$, i.e., the subspace of $L_p(\mathbb{T}^d; S^p)$ consisting all f in the form of (3-4). By Lemma 2.2 and the transference relation (3-7), it is sufficient to show the inequality

$$\Big\|\Big(\sum_{j\geq 0}|S_{E_j}(T_M f)|^2\Big)^{\frac12}\Big\|_{L_p(\mathbb{T}^d;S^p)} \leq C_d\Big(\frac{p^2}{p-1}\Big)^d\Big\|\Big(\sum_{j\geq 0}|S_j f|^2\Big)^{\frac12}\Big\|_{L_p(\mathbb{T}^d;S^p)} \tag{3-8}$$

and its adjoint form

$$\Big\|\Big(\sum_{j\geq 0}|(S_{E_j}(T_M f))^*|^2\Big)^{\frac12}\Big\|_{L_p(\mathbb{T}^d;S^p)} \leq C_d\Big(\frac{p^2}{p-1}\Big)^d\Big\|\Big(\sum_{j\geq 0}|(S_j f)^*|^2\Big)^{\frac12}\Big\|_{L_p(\mathbb{T}^d;S^p)} \tag{3-9}$$

for $p \geq 2$. By duality, we will obtain M_m's boundedness on S^p for $1 < p < 2$ as well.

We will use (3-5) as the presentation of T_M to prove (3-8) and will use the presentation (3-6) to prove (3-9). Note that E_j is symmetric so $(S_{E_j}(T_M f))^* = S_{E_j}(T_M f)^*$ and

$$(T_M f)^* = \Big(\sum_{n\in\mathbb{Z}^d} A(n)M_r(n)z^n\Big)^* = \sum_{n\in\mathbb{Z}^d}(M_r(-n))^*(A(-n))^* z^n.$$

So, both $S_{E_j}(T_M f)$ and $(S_{E_j}(T_m f))^*$ have the multiplier symbols on the left. This allows us to write the corresponding squares in the forms with M_r or M_l sitting in the middle for both $S_{E_j}(T_M f)$ and $(S_{E_j}(T_m f))^*$ and avoid the usual trouble caused by the noncommutativity of the operator products. After noting these facts, the argument for the case $d = 1$ is rather standard, which we record below.

Proof of Theorem 1.1. We now set $d = 1$. By the notation in (3-3), the conditions (1-4) and (1-5) are equivalent to

$$\sup_{j\geq 0}\Big\|\sum_{n\in E_j}|M_l(n+1)-M_l(n)|\Big\|_\infty < C, \quad \sup_{j\geq 0}\Big\|\sum_{n\in E_j}|M_r(n+1)-M_r(n)|\Big\|_\infty < C, \tag{3-10}$$

where E_j is defined as in (2-3). Following the definition (2-2), we define $S_{(a,b)}$ as

$$S_{(a,b)}g(z) = \sum_{a<n<b}\hat{g}(n)z^n \tag{3-11}$$

for $g(z) = \sum_{n\in\mathbb{Z}}\hat{g}(n)z^n \in L^p(\mathbb{T}; S^p)$. We will deliberately extend the use of this notation and set

$$S_{(a,b)}g = -S_{(b,a)}g$$

when $a > b$. For $j \in \mathbb{N}$, write $E_{j,1} = (-2^j, -2^{j-1}]$, $E_{j,2} = [2^{j-1}, 2^j)$. Let $2_1^{(j)} = -2^j$, $2_2^{(j)} = 2^j$ and

$$\Delta M_l(n) = \begin{cases} M_l(n) - M_l(n-1), & n < 0, \\ M_l(n+1) - M_l(n), & n > 0. \end{cases} \tag{3-12}$$

By applying summation by parts and the presentation of $T_M f$ in (3-5) and (3-6), we obtain

$$S_{E_j}(T_M f) = \sum_{i=1,2} S_{E_{j,i}}(T_M f) = \sum_{i=1,2} \Big(M_l(2_i^{(j-1)})(S_{E_{j,i}} f) + \sum_{n \in E_{j,i}} \Delta M_l(n)(S_{(n,2_i^{(j)})} f) \Big) \tag{3-13}$$

$$= \sum_{i=1,2} \Big((S_{E_{j,i}} f) M_r(2_i^{(j-1)}) + \sum_{n \in E_{j,i}} (S_{(n,2_i^{(j)})} f) \Delta M_r(n) \Big), \tag{3-14}$$

with $\Delta M_r(n)$ defined similarly. We will use the presentation (3-13) to prove (3-8) and will use (3-14) to prove (3-9). The arguments are similar. So we will only give the argument for (3-8). We will ignore the term $j = 0$ in (3-8) because $\|S_{E_0}(T_M f)\|_{L^p(\mathbb{T};S^p)} \le C \|f\|_{L^p(\mathbb{T};S^p)}$.

Note $\Delta M_l(n)$ is a diagonal operator; we can write $\Delta M_l(n) = a_n^* b_n$, with a_n, b_n diagonal operators and $|a_n|^2 = |b_n|^2 = |\Delta M_l(n)|$. Then by Lemma 2.4, we have, for $i = 1, 2$,

$$\Big| \sum_{n \in E_{j,i}} \Delta M_\ell(n) S_{(n,2_i^{(j)})} f \Big|^2 \le \Big\| \sum_{n \in E_{j,i}} |\Delta M_l(n)| \Big\|_\infty \Big(\sum_{n \in E_{j,i}} |b_n S_{(n,2_i^{(j)})} f|^2 \Big)$$

$$\le C \Big(\sum_{n \in E_{j,i}} |b_n S_{(n,2_i^{(j)})} f|^2 \Big) \tag{3-15}$$

$$= C \Big(\sum_{n \in E_{j,i}} |S_{(n,2_i^{(j)})}(b_n S_j f)|^2 \Big). \tag{3-16}$$

Thus,

$$\Big\| \Big(\sum_{j \in \mathbb{N}} \Big| \sum_{n \in E_{j,i}} \Delta M_\ell(n) S_{(n,2_i^{(j)})} f \Big|^2 \Big)^{\frac12} \Big\|_{L^p(\mathbb{T};S^p)} \le C^{\frac12} \Big\| \Big(\sum_{j \in \mathbb{N}} \sum_{n \in E_{j,i}} |S_{(n,2_i^{(j)})}(b_n S_j f)|^2 \Big)^{\frac12} \Big\|_{L^p(\mathbb{T};S^p)}.$$

By Lemma 2.3, we get

$$\Big\| \Big(\sum_{j \in \mathbb{N}} \Big| \sum_{n \in E_{j,i}} \Delta M_\ell(n) S_{(n,2_i^{(j)})} f \Big|^2 \Big)^{\frac12} \Big\|_{L^p(\mathbb{T};S^p)}$$

$$\le C_2 \frac{p^2}{p-1} C^{\frac12} \Big\| \Big(\sum_{j \in \mathbb{N}} \sum_{n \in E_{j,i}} |(b_n S_j f)|^2 \Big)^{\frac12} \Big\|_{L^p(\mathbb{T};S^p)}$$

$$= C_2 \frac{p^2}{p-1} C^{\frac12} \Big\| \Big(\sum_{j \in \mathbb{N}} (S_j f)^* \Big(\sum_{n \in E_{j,i}} |b_n|^2 \Big) S_j f \Big)^{\frac12} \Big\|_{L^p(\mathbb{T};S^p)}$$

$$= C_2 \frac{p^2}{p-1} C^{\frac12} \Big\| \Big(\sum_{j \in \mathbb{N}} (S_j f)^* \Big(\sum_{n \in E_{j,i}} |\Delta M_l(n)| \Big) S_j f \Big)^{\frac12} \Big\|_{L^p(\mathbb{T};S^p)}$$

$$\le C_2 \frac{p^2}{p-1} C \Big\| \Big(\sum_{j \in \mathbb{N}} |S_j f|^2 \Big)^{\frac12} \Big\|_{L^p(\mathbb{T};S^p)}. \tag{3-17}$$

Hence, by (3-13), (3-17), and Lemma 2.3

$$
\begin{aligned}
&\Big\|\Big(\sum_{j\in\mathbb{N}}|S_{E_{j,i}}(T_M f)|^2\Big)^{\frac12}\Big\|_{L^p(\mathbb{T};S^p)}\\
&\qquad\le \Big\|\Big(\sum_{j\in\mathbb{N}}|M_l(2_i^{(j-1)})(S_{E_{j,i}}f)|^2\Big)^{\frac12}\Big\|_{L^p(\mathbb{T};S^p)}+C\frac{p^2}{p-1}C_2\Big\|\Big(\sum_{j\in\mathbb{N}}|S_j f|^2\Big)^{\frac12}\Big\|_{L^p(\mathbb{T};S^p)}\\
&\qquad\le \Big\|C\Big(\sum_{j\in\mathbb{N}}|(S_{E_{j,i}}f)|^2\Big)^{\frac12}\Big\|_{L^p(\mathbb{T};S^p)}+C\frac{p^2}{p-1}C_2\Big\|\Big(\sum_{j\in\mathbb{N}}|S_j f|^2\Big)^{\frac12}\Big\|_{L^p(\mathbb{T};S^p)}\\
&\qquad= \Big\|C\Big(\sum_{j\in\mathbb{N}}|(S_{E_{j,i}}S_j f)|^2\Big)^{\frac12}\Big\|_{L^p(\mathbb{T};S^p)}+C\frac{p^2}{p-1}C_2\Big\|\Big(\sum_{j\in\mathbb{N}}|S_j f|^2\Big)^{\frac12}\Big\|_{L^p(\mathbb{T};S^p)}\\
&\qquad\le C\frac{p^2}{p-1}\Big\|\Big(\sum_{j\in\mathbb{N}}|S_j f|^2\Big)^{\frac12}\Big\|_{L^p(\mathbb{T};S^p)}
\end{aligned}
$$

for $i=1,2$. Therefore we finish the proof of (3-8). The arguments for the adjoint version of (3-9) are similar. We then complete the proof of Theorem 1.1. □

Corollary 3.1 [Conde-Alonso et al. 2023, Corollary 3.5]. *The following Mikhlin conditions imply the boundedness of M_m on S^p for all* $1<p<\infty$:

$$|m(s,s+k)-m(s,s+k+1)|\le\frac{C}{|k|},\tag{3-18}$$

$$|m(s+k,s)-m(s+k+1,s)|\le\frac{C}{|k|}.\tag{3-19}$$

Proof. It is clear that the Mikhlin conditions (3-18), (3-19) imply the Marcinkiewicz-type conditions (1-4), (1-5). □

4. The case $d>1$

In this part, we generalize Theorem 1.1 to the d-dimensional case. Before we proceed to the main statement of the theorem, we need to borrow some notation from the calculus of finite differences.

Definition 4.1. Let $\sigma:\mathbb{Z}^d\to\mathbb{C}$ and $j=(j_1,\dots,j_d)\in\mathbb{Z}^d$. Let $\{e_j\}_{j=1}^d$ be standard basis of $\mathbb{Z}^d$, i.e., the j-th entry of e_j is 1 and all other entries are 0 for $j=1,\dots,d$. We define the forward partial difference operators Δ_{t_j} by

$$\Delta_{t_j}\sigma(t):=\sigma(t+e_j)-\sigma(t),\tag{4-1}$$

and for $\alpha\in\{0,1\}^d$, define

$$\Delta_t^\alpha:=\Delta_{t_1}^{\alpha_1}\cdots\Delta_{t_d}^{\alpha_d}.$$

For $\alpha=(1,\dots,1)\in\{0,1\}^d$, we simplify the notation Δ_t^α as Δ_t. Readers can find more information on the calculus of finite differences in Chapter 3 of [Ruzhansky and Turunen 2010].

4.1. ***The case $d=2$.*** Recall that we have the partition $\mathbb{Z}^2=\bigcup_{j\ge0}E_j$ with E_j defined as

$$E_j=\begin{cases}\{(0,0)\}, & j=0,\\ \{(n_1,n_2)\in\mathbb{Z}^2:2^{j-1}\le|(n_1,n_2)|_\infty<2^j\}, & j\ge1.\end{cases}\tag{4-2}$$

Theorem 4.2. *Given $m=(m_{s,t})_{s,t\in\mathbb{Z}^2}\in\mathcal{B}(\ell^2(\mathbb{Z}^2))$, suppose m satisfies*:

(i) $\sup_{s,t\in\mathbb{Z}^2}|m_{s,t}|<C_1$.

(ii) *For any $k\in\mathbb{N}$, $s\in\mathbb{Z}^2$, there are constants C_2, C_3 such that*

$$\Bigg(\sum_{t=(t_1,\pm 2^{k-1})\in E_k}|\Delta_{t_1}m_{s,s+t}|+\sum_{t=(\pm 2^{k-1},t_2)\in E_k}|\Delta_{t_2}m_{s,s+t}|\Bigg)<C_2, \tag{4-3}$$

$$\sum_{t=(t_1,t_2)\in E_k}|\Delta_t m_{s,s+t}|<C_3, \tag{4-4}$$

and

$$\Bigg(\sum_{t=(t_1,\pm 2^{k-1})\in E_k}|\Delta_{t_1}m_{s+t,s}|+\sum_{t=(\pm 2^{k-1},t_2)\in E_k}|\Delta_{t_2}m_{s+t,s}|\Bigg)<C_2, \tag{4-5}$$

$$\sum_{t=(t_1,t_2)\in E_k}|\Delta_t m_{s+t,s}|<C_3. \tag{4-6}$$

Then M_m is a bounded Schur multiplier on $S^p(\ell^2(\mathbb{Z}^2))$ for $p\in(1,\infty)$ with an upper bound $\lesssim(p^2/(p-1))^4$. Here C_1, C_2 and C_3 are positive absolute constants.

Now we come to the proof of Theorem 4.2. As explained at the beginning of Section 3, we only need to prove (3-8) and its adjoint version. Recall that S_{E_j} is the partial sum projection on $L^p(\mathbb{T}^2;S^p(\mathbb{Z}^2))$ given by $S_{E_j}f(z)=\sum_{n\in E_j}\hat{f}(n)z^n$, where $z\in\mathbb{T}^2$.

Applying the definition of M_l (3-3), we see that (4-3) and (4-4) imply

$$\Bigg\|\sum_{n=(n_1,\pm 2^{j-1})\in E_j}|\Delta_{n_1}M_l(n)|\Bigg\|_\infty+\Bigg\|\sum_{n=(\pm 2^{j-1},n_2)\in E_j}|\Delta_{n_2}M_l(n)|\Bigg\|_\infty<C_2, \tag{4-7}$$

$$\Bigg\|\sum_{n=(n_1,n_2)\in E_j}|\Delta_n M_l(n)|\Bigg\|_\infty<C_3. \tag{4-8}$$

To prove (3-8), we will cut E_j into four rectangles $E_{j,k}$, $k=1,\dots,4$, for $j\geq 1$. Let $I_j=[2^{j-1},2^j)\cap\mathbb{Z}$, $J_j=[-2^{j-1},2^j)\cap\mathbb{Z}$, and set

$$E_{j,1}=J_j\times I_j,\qquad E_{j,2}=(-I_j)\times J_j,$$
$$E_{j,3}=I_j\times(-J_j),\quad E_{j,4}=(-J_j)\times(-I_j).$$

Thus, we have

$$S_{E_j}T_Mf=\sum_{i=1}^4 S_{E_{j,i}}T_Mf. \tag{4-9}$$

To prove (3-8), it is sufficient to prove

$$\Bigg\|\Bigg(\sum_{j=0}^\infty|S_{E_{j,i}}T_Mf|^2\Bigg)^{\frac12}\Bigg\|_{L^p(\mathbb{T}^2;S^p)}\leq C'\Big(\frac{p^2}{p-1}\Big)^2\Bigg\|\Bigg(\sum_{j=0}^\infty|S_jf|^2\Bigg)^{\frac12}\Bigg\|_{L^p(\mathbb{T}^2;S^p)} \tag{4-10}$$

for $p\geq 2$, $i=1,2,3,4$. The arguments for $i=1,2,3,4$ are similar. We will give the argument for $i=1$ only. By the fundamental theorem of calculus,

$$
\begin{aligned}
S_{E_{j,1}} T_M f = M_l(-2^{j-1}, 2^{j-1}) S_{E_{j,1}} f + \sum_{n_1 \in J_j} \Delta_{n_1} M_l(n_1, 2^{j-1}) S_{(n_1, 2^j) \times I_j} f \\
+ \sum_{n_2 \in E_j} \Delta_{n_2} M_l(-2^{j-1}, n_2) S_{J_j \times (n_2, 2^j)} f + \sum_{n=(n_1,n_2) \in E_{j,1}} \Delta_n M_l(n_1, n_2) S_{(n_1,2^j) \times (n_2, 2^j)} f \\
=: P_j^1 + P_j^2 + P_j^3 + P_j^4. \qquad (4\text{-}11)
\end{aligned}
$$

By the operator inequality $\left|\sum_{k=1}^n a_k\right|^2 \le n \sum_{k=1}^n |a_k|^2$, we have

$$
|S_{E_{j,1}} T_M f|^2 = |P_j^1 + P_j^2 + P_j^3 + P_j^4|^2 \le 4(|P_j^1|^2 + |P_j^2|^2 + |P_j^3|^2 + |P_j^4|^2). \qquad (4\text{-}12)
$$

For part P_j^1, by assumption (i) of Theorem 4.2, we have

$$
|P_j^1|^2 = |M_l(-2^{j-1}, 2^{j-1})\ S_{E_{j,1}} f|^2 \le C_1^2 |S_{E_{j,1}} f|^2 = C_1^2 |S_{E_{j,1}} S_j f|^2. \qquad (4\text{-}13)
$$

By Lemma 2.3,

$$
\left\| \Big(\sum_{j \ge 0} |P_j^1|^2 \Big)^{\frac{1}{2}} \right\|_{L^p(\mathbb{T}^2; S^p(\mathbb{Z}^2))} \le C \Big(\frac{p^2}{p-1} \Big)^2 \left\| \Big(\sum_{j \ge 0} |S_j f|^2 \Big)^{\frac{1}{2}} \right\|_{L^p(\mathbb{T}^2; S^p(\mathbb{Z}^2))}. \qquad (4\text{-}14)
$$

For part P_j^2, we follow the arguments similar to (3-16) and (3-17) in the one-dimensional case and write $\Delta_{n_1} M_l(n_1, 2^{j-1}) = a_n^* b_n$, with $|a_n|^2 = |b_n|^2 = |\Delta_{n_1} M_l(n_1, 2^{j-1})|$. Letting $R_{n_1,j} = (n_1, 2^j) \times I_j$, we have

$$
|P_j^2|^2 = \left| \sum_{n_1 \in J_j} \Delta_{n_1} M_l(n_1, 2^{j-1}) S_{(n_1, 2^j) \times I_j} f \right|^2 \le \left\| \sum_{n_1 \in J_j} |\Delta_{n_1} M_l(n_1, 2^{j-1})| \right\|_\infty \Big(\sum_{n_1 \in J_j} |S_{R_{n_1,j}}(b_n S_j f)|^2 \Big). \quad (4\text{-}15)
$$

Thus, by (4-15), (4-7) and Lemma 2.3 and following the arguments similar to the case $d = 1$, we get

$$
\left\| \Big(\sum_{j \ge 0} |P_j^2|^2 \Big)^{\frac{1}{2}} \right\|_{L^p(\mathbb{T}^2; S^p(\mathbb{Z}^2))} \le C_2 \Big(\frac{p^2}{p-1} \Big)^2 \left\| \Big(\sum_{j \ge 0} |S_j f|^2 \Big)^{\frac{1}{2}} \right\|_{L^p(\mathbb{T}^2; S^p)}. \qquad (4\text{-}16)
$$

Similarly, we have

$$
\left\| \Big(\sum_{j \ge 0} |P_j^3|^2 \Big)^{\frac{1}{2}} \right\|_{L^p(\mathbb{T}^2; S^p)} \le C_2 \Big(\frac{p^2}{p-1} \Big)^2 \left\| \Big(\sum_{j \ge 0} |S_j f|^2 \Big)^{\frac{1}{2}} \right\|_{L^p(\mathbb{T}^2; S^p)}. \qquad (4\text{-}17)
$$

Now we come to the estimate of part P_j^4. Define $R_{n,j} = (n_1, 2^j) \times (n_2, 2^j)$. Similarly,

$$
\begin{aligned}
\left\| \Big(\sum_{j \ge 0} |P_j^4|^2 \Big)^{\frac{1}{2}} \right\|_{L^p(\mathbb{T}^2; S^p)} &\le C_3^{\frac{1}{2}} \left\| \Big(\sum_{j \ge 0} \sum_{n=(n_1,n_2) \in E_{j,1}} |S_{R_{n,j}} S_j |\Delta_n M_l(n)|^{\frac{1}{2}} f|^2 \Big)^{\frac{1}{2}} \right\|_{L^p(\mathbb{T}^2; S^p)} \\
&\le C_3^{\frac{1}{2}} \Big(\frac{p^2}{p-1} \Big)^2 \left\| \Big(\sum_{j \ge 0} \sum_{n=(n_1,n_2) \in E_{j,1}} |S_j |\Delta_n M_l(n)|^{\frac{1}{2}} f|^2 \Big)^{\frac{1}{2}} \right\|_{L^p(\mathbb{T}^2; S^p)} \\
&\le C_3 \Big(\frac{p^2}{p-1} \Big)^2 \left\| \Big(\sum_{j \ge 0} |S_j f|^2 \Big)^{\frac{1}{2}} \right\|_{L^p(\mathbb{T}^2; S^p)}. \qquad (4\text{-}18)
\end{aligned}
$$

Therefore, by (4-9), (4-11) and (4-15)–(4-18), we have

$$\Big\| \Big(\sum_{j=0}^{\infty} |S_{E_{j,1}} T_M f|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\mathbb{T}^2; S^p)} \le C' \Big(\frac{p^2}{p-1} \Big)^2 \Big\| \Big(\sum_{j=0}^{\infty} |S_j f|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\mathbb{T}^2; S^p)}. \tag{4-19}$$

Thus, (4-10) is proved. Hence, we finish the proof of Theorem 4.2.

4.2. ***Higher-dimensional case.*** We need some additional notation to deal with the case $d > 2$. Borrowing the notation from [Hytönen et al. 2016], we denote by $\mathbb{Z}^\alpha$ the space

$$\mathbb{Z}^\alpha := \{(n_i)_{i:\alpha_i=1} : n_i \in \mathbb{Z}\}$$

for $\alpha \in \{0,1\}^d$. For any $n \in \mathbb{Z}^d$ and $E = I_1 \times \cdots \times I_d \subseteq \mathbb{Z}^d$, let

$$n_\alpha := (n_i)_{i:\alpha_i=1} \in \mathbb{Z}^\alpha, \qquad E_\alpha := \prod_{i:\alpha_i=1} I_i \subseteq \mathbb{Z}^\alpha$$

be their natural projections onto $\mathbb{Z}^\alpha$. In particular, we will use the splittings $n = (n_\alpha, n_{1-\alpha}) \in \mathbb{Z}^\alpha \times \mathbb{Z}^{\mathbf{1}-\alpha}$ and $E = E_\alpha \times E_{\mathbf{1}-\alpha}$, where $\mathbf{1} = (1, \dots, 1)$. Suppose $s, t \in \mathbb{Z}^d$ and we abbreviate the interval notation $[s,t) \cap \mathbb{Z}^d$ as $[s,t)$.

Similarly, denote by $\mathcal{J}^d$ the partition $\mathcal{J}^d := \{E_j : j \ge 0\}$ of $\mathbb{Z}^d$, where

$$E_j = \begin{cases} \{(0,\dots,0)\}, & j = 0, \\ \{(n_1,\dots,n_d) \in \mathbb{Z}^d : 2^{j-1} \le |(n_1,\dots,n_d)|_\infty < 2^j\}, & j \ge 1. \end{cases} \tag{4-20}$$

Each E_j can be further decomposed into $2^d(2^d-1)$ subsets and each of the subsets can be obtained by translation of the cube $F_j = [2^{j-1}, 2^j) \times \cdots \times [2^{j-1}, 2^j)$. Following similar procedures to those in the two-dimensional case and using the discrete fundamental theorem formula,

$$\begin{aligned} \chi_{[s,t)}(n) m(n) &= \chi_{[s,t)} \sum_{\alpha \in \{0,1\}^d} \sum_{k_\alpha \in [s,n)_\alpha} \Delta^\alpha m(s_{\mathbf{1}-\alpha}, k_\alpha) \\ &= \sum_{\alpha \in \{0,1\}^d} \sum_{k_\alpha \in [s,t)_\alpha} \chi_{[k,t)_\alpha \times [s,t)_{\mathbf{1}-\alpha}}(n) \Delta^\alpha m(s_{\mathbf{1}-\alpha}, k_\alpha), \end{aligned} \tag{4-21}$$

we can obtain the following theorem. The details are left to the interested reader.

Theorem 4.3. *Given $m = (m_{s,t})_{s,t \in \mathbb{Z}^d} \in \mathcal{B}(\ell^2(\mathbb{Z}^d))$. Suppose m satisfies that, for some $C > 0$:*

(i) $\sup_{s,t \in \mathbb{Z}^d} |m_{s,t}| < C$.

(ii) *For any $n \in \mathbb{N}$, $s \in \mathbb{Z}^d$, $\alpha \in \{0,1\}^d$, $\alpha \ne 0$, and any $r^{(n)} \in \mathbb{Z}^d$ satisfying $|r_i^{(n)}| = 2^{n-1}$ for all $i = 1, \dots, d$,*

$$\sum_{t = (t_\alpha, r^{(n)}_{\mathbf{1}-\alpha}) \in E_n} |\Delta_t^\alpha m_{s,s+t}| < C, \qquad \sum_{t = (t_\alpha, r^{(n)}_{\mathbf{1}-\alpha}) \in E_n} |\Delta_t^\alpha m_{s+t,s}| < C. \tag{4-22}$$

Then M_m extends to a bounded Schur multiplier on $S^p(\ell^2(\mathbb{Z}^d))$ for $p \in (1,\infty)$ with an upper bound $C_d(p^2/(p-1))^{d+2}$. Here C_d is a constant dependent only on the dimension d.

Note that we cannot hope for an analogue of Theorem 4.3 with E_n defined by the ℓ^2-metric instead of the ℓ^∞ metric because the ball-type Fourier multipliers are not uniformly bounded on $L^p(\mathbb{T}^d)$ for any $d > 1$. Doust and Gillespie [2005, Theorem 6.2] gave an example of ball-type Schur multipliers for the case $d = 1$. Their argument does not seem to extend to the case $d > 1$.

Example 4.4 (ball Schur multipliers). Let $X_0 = \{(0, 0)\} \subset \mathbb{Z}^d \times \mathbb{Z}^d$. For $i \in \mathbb{N}$, let

$$X_i = \{(k, j) \in \mathbb{Z}^d \times \mathbb{Z}^d : 2^{i-1} \le |(k, j)|_2 < 2^i\}.$$

Let $m_X = \sum_i \varepsilon_i \mathbb{1}_{X_i}$, with $|\varepsilon_i| \le 1$. Then $|\Delta_t^\alpha m| \le 2^{|\alpha|}$. Note that

$$|(k, j)|_2 \simeq |k|_2 + |j|_2 \simeq |k - j|_2 + |j|_2 \simeq |k - j|_\infty + |j|_\infty \simeq |k - j|_\infty + |k|_\infty.$$

We can find a constant K_d which only depends on d such that the set

$$\{(s_0, s_0 + t) : t \in E_n\} \cup \{(s_0 + t, s_0) : t \in E_n\}$$

intersects with at most K_d many X_i's for any fixed s_0. Since $\bigcup_{0\le i\le n} X_i$ is convex for all n, we conclude that there are at most $2^d K_d$ many nonzero terms in the two summations in (4-22), and the summations are bounded by $2^d K_d$. So (4-22) is satisfied and $M_m = \sum_i \varepsilon_i P_{X_i}$ is bounded on S^p for any $1 < p < \infty$.

4.3. ***The case of continuous indices.*** We explain in this section that Theorems 1.1 and 4.3 extend to the continuous case by approximation. Let $S^p(\mathbb{R}^d)$ be the space of Schatten p-class operators acting on the Hilbert space $L^2(\mathbb{R}^d)$. We identify $S^2(\mathbb{R}^d)$ as $L^2(\mathbb{R}^d \times \mathbb{R}^d)$, so for $A \in S^2(\mathbb{R}^d)$ we can talk about its pointwise value $a_{s,t}$. For $m \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, we consider the Schur-multiplier-type map

$$M_m(A) = (m(s, t)a_{s,t})_{s,t\in\mathbb{R}^d}.$$

Motivated by the work of [Lafforgue and de la Salle 2011; Conde-Alonso et al. 2023], we wish to find sufficient conditions on m so that M_m extends to a bounded map with respect to the S^p-norm for $1 < p < \infty$.

Theorem 4.5. *For $p \in (1, \infty)$, consider the Schur multiplier M_m on $S^p(\mathbb{R})$ with symbol $m(\cdot, \cdot)$ in $L^\infty(\mathbb{R}^2)$ whose partial derivatives are continuous on $(-2^{j+1}, -2^j) \cup (2^j, 2^{j+1})$ for all $j \in \mathbb{Z}$. Suppose there exists an absolute constant C such that, for all $j \in \mathbb{Z}$ and $x, y \in \mathbb{R}$,*

$$\int_{-2^{j+1}}^{-2^j} |\partial_1 m(y + t, y)|\, dt + \int_{2^j}^{2^{j+1}} |\partial_1 m(y + t, y)|\, dt \le C, \tag{4-23}$$

$$\int_{-2^{j+1}}^{-2^j} |\partial_2 m(x, x + t)|\, dt + \int_{2^j}^{2^{j+1}} |\partial_2 m(x, x + t)|\, dt \le C. \tag{4-24}$$

Then, the Schur multiplier M_m extends to a bounded map on $S^p(\mathbb{R})$ with $\|M_m\| \le C \max\{p^3, 1/(p - 1)^3\}$.

Proof. Let $\mathcal{D}_k$ be the σ-algebra generated by dyadic cubes

$$Q_{k,s,t} = \left(\frac{s}{2^k}, \frac{s + 1}{2^k}\right] \times \left(\frac{t}{2^k}, \frac{t + 1}{2^k}\right], \quad s, t \in \mathbb{Z}.$$

Then $(\mathcal{D}_k)_{k=1}^{\infty}$ is the usual dyadic filtration for $\mathbb{R}^2$. Given $m \in L^\infty(\mathbb{R}^2)$, let $m_k = \mathbb{E}_k(m)$ be the conditional expectation of m with respect to the σ-algebra $\mathcal{D}_k$. That is to say

$$m_k(x) = \sum_{Q\in\mathcal{D}_k} \frac{1}{|Q|}\left[\int_Q m(y)\,dy\right]\chi_Q(x) \quad \text{for all } x \in \mathbb{R}^2.$$

Let $L^2(\mathbb{R}, \mathcal{D}_k)$ be the L^2 space of all $\mathcal{D}_k$-measurable functions. Let $\tilde{m}_k(s,t) = m(s/2^k, t/2^k)$ for $s, t \in \mathbb{Z}$. Note that $S^p(L^2(\mathbb{R}, \mathcal{D}_k))$ is isometrically isomorphic to $S^p(\ell^2(\mathbb{Z}))$. We see that $M_{\tilde{m}_k}$ extends to a bounded Schur multiplier on $S^p(\ell^2(\mathbb{Z}))$ with the same norm if M_{m_k} extends to a bounded Schur multiplier on $S^p(L^2(\mathbb{R}, \mathcal{D}_k))$ and vice versa. By Lemma 1.11 of [Lafforgue and de la Salle 2011],

$$\|M_m\| = \overline{\lim}_{k\to\infty}\|M_{m_k}\| = \overline{\lim}_{k\to\infty}\|M_{\widetilde{m}_k}\|.$$

So, we need to show that $M_{\tilde{m}_k}$ satisfies conditions (1-4), (1-5). First, we verify condition (1-5). For each $j \in \mathbb{N}$ and $s \in \mathbb{Z}$,

$$\begin{aligned}
&\sum_{\ell=0}^{2^j-2} |m_k(s, s+2^j+\ell+1) - m_k(s, s+2^j+\ell)| \\
&\quad = 2^{2k}\sum_{\ell=0}^{2^j-2}\left|\int_{\frac{s}{2^k}}^{\frac{s+1}{2^k}}\int_{\frac{2^j+\ell}{2^k}}^{\frac{2^j+\ell+1}{2^k}} M\left(y, y+x+\frac{1}{2^k}\right)dx\,dy - \int_{\frac{s}{2^k}}^{\frac{s+1}{2^k}}\int_{\frac{2^j+\ell}{2^k}}^{\frac{2^j+\ell+1}{2^k}} M(y, y+x)\,dx\,dy\right| \\
&\quad = 2^{2k}\sum_{\ell=0}^{2^j-2}\left|\int_{\frac{s}{2^k}}^{\frac{s+1}{2^k}}\int_{\frac{2^j+\ell}{2^k}}^{\frac{2^j+\ell+1}{2^k}}\int_{y+x}^{y+x+\frac{1}{2^k}} \partial_2 M(y,t)\,dt\,dx\,dy\right| \\
&\quad \le 2^{2k}\sum_{\ell=0}^{2^j-2}\int_{\frac{s}{2^k}}^{\frac{s+1}{2^k}}\int_{\frac{2^j+\ell}{2^k}}^{\frac{2^j+\ell+1}{2^k}}\int_{y+x}^{y+x+\frac{1}{2^k}} |\partial_2 M(y,t)|\,dt\,dx\,dy \\
&\quad = 2^{2k}\int_{\frac{s}{2^k}}^{\frac{s+1}{2^k}}\int_{\frac{2^j}{2^k}}^{\frac{2^{j+1}-1}{2^k}}\int_{y+x}^{y+x+\frac{1}{2^k}} |[\partial_2(M)](y,t)|\,dt\,dx\,dy \\
&\quad = 2^{2k}\int_{\frac{s}{2^k}}^{\frac{s+1}{2^k}}\int_{\frac{2^j}{2^k}}^{\frac{2^{j+1}-1}{2^k}}\int_{y+\frac{2^j}{2^k}}^{y+\frac{2^{j+1}}{2^k}} \chi_{(x,x+\frac{1}{2^k})}(t)|[\partial_2(M)](y,t)|\,dt\,dx\,dy \\
&\quad = 2^{2k}\int_{\frac{s}{2^k}}^{\frac{s+1}{2^k}}\int_{y+\frac{2^j}{2^k}}^{y+\frac{2^{j+1}}{2^k}}\int_{\frac{2^j}{2^k}}^{\frac{2^{j+1}-1}{2^k}} \chi_{(t-\frac{1}{2^k},t)}(x)\,dx|[\partial_2(M)](y,t)|\,dt\,dy \\
&\quad \le 2^k\int_{\frac{s}{2^k}}^{\frac{s+1}{2^k}}\int_{y+\frac{2^j}{2^k}}^{y+\frac{2^{j+1}}{2^k}} |\partial_2 M(y,t)|\,dt\,dy \le \sup_{y\in\mathbb{R}}\int_{\frac{2^j}{2^k}}^{\frac{2^{j+1}}{2^k}} |[\partial_2(M)](y,y+t)|\,dt \le A.
\end{aligned} \tag{4-25}$$

The last inequality follows from the assumption in (4-24). So, condition (1-5) is verified. Applying the same argument, we utilize the assumption in (4-23) to prove condition (1-4). Therefore, by Theorem 1.1, $\|M_m\| = \overline{\lim}_{k\to\infty}\|M_{\widetilde{m}_k}\| \le C\max\{p^3, 1/(p-1)^3\}$. □

Similarly, Theorem 4.3 and Example 4.4 have analogues in the continuous case as well.

Theorem 4.6. *Define $E_j := \{t \in \mathbb{R}^d : 2^{j-1} \le |t|_\infty < 2^j\}$ for $j \in \mathbb{Z}$. For $p \in (1, \infty)$, consider the Schur multiplier $m \in L^\infty(\mathbb{R}^{2d})$ whose partial derivatives are continuous up to the boundary of E_k for all $k \in \mathbb{Z}$. Assume there exists a constant C such that*

$$\int_{(t_\alpha, r^{(j)}_{1-\alpha}) \in E_j} |\partial^\alpha m(s, s+t)|\, dt_\alpha \le C, \tag{4-26}$$

$$\int_{(t_\alpha, r^{(j)}_{1-\alpha}) \in E_j} |\partial^\alpha m(s+t, s)|\, dt_\alpha \le C \tag{4-27}$$

for any $j \in \mathbb{Z}$, $s \in \mathbb{R}^d$ and any $r^{(j)} \in \mathbb{R}^d$ with $|r^{(j)}_i| = 2^{j-1}$ for all $1 \le i \le d$. Then, the Schur multiplier M_m extends to a bounded operator on $S^p(\mathbb{R}^d)$ for all $1 < p < \infty$ with $\|M_m\| \le C_d \max\{p^{d+2}, 1/(p-1)^{d+2}\}$. Here t_α is defined as in Section 4.2.

Remark 4.7. The Schur multipliers in all theorems of this article are also completely bounded on S^p for $1 < p < \infty$; the arguments are exactly the same.

5. Discussions

5.1. *Counterexamples.*

(1) We show in the following that (1-4) alone is not sufficient for the boundedness of M_m.

Choose a large $K \in \mathbb{N}$. Let $m(s, t) = \exp(i2\pi kj/K)$ if $s = 2^k$, $t = 2^j$ for some $j, k \in \mathbb{N}$ satisfying $1 \le j < k \le K$, and $m(s, t) = 0$ for other $s, t \in \mathbb{N}$. Let $\tilde{m}(k, j) = \exp(i2\pi kj/K)$ if $1 \le j < k \le K$, and $\tilde{m}(k, j) = 0$ for other $k, j \in \mathbb{N}$. Let U be the partial isometry on $\ell_2(\mathbb{N})$ sending e_k to e_{2^k}. Then, we have $M_{\tilde{m}}(A) = U^* M_m(UAU^*)U$ for any $A \in S^p(\ell_2(\mathbb{N}))$ and $\|M_{\tilde{m}}\| \le \|M_m\|$.

Note that, for any N, j given, there exists at most one k (actually $k = N$) satisfying $k > j$ and

$$2^{N-1} - 1 \le |2^k - 2^j| < 2^N.$$

Using the fact that $|m(s, t)| \le 1$, for any N, t, we get

$$\sum_{2^{N-1} \le |r| < 2^N} |m(t+r+1, t) - m(t+r, t)| \le 2,$$

because there are at most two nonzero terms in the sum above. This means $(m(s, t))_{s,t}$ satisfies the row condition (1-4). On the other hand, if $s = 2^N$, then $|m(s, t) - m(s, t+1)|$ does not vanish if t or $t+1$ has the form of 2^j, $j = 1, \dots, N-1$, by the definition of $m(s, t)$. Hence we have

$$\sum_{2^{N-1} \le |t-s| < 2^N} |m(s, t+1) - m(s, t)| = \sum_{2^{N-1} \le s-t < 2^N} |m(s, t+1) - m(s, t)| = 2(N-1),$$

which shows that $(m(s, t))_{s,t}$ fails the column condition (1-5).

Let A be the $K \times K$ matrix $(\exp(-i2\pi kj/K))_{1 \le k, j \le K}$. Then A has S^p norm $K^{1/2+1/p}$. $M_{\tilde{m}}(A)$ is the lower triangular matrix with all nonzero coefficients being 1 which has S^p norm $\simeq K$ for any given p, $1 < p < \infty$. This shows that $K^{1/2-1/p} \lesssim \|M_{\tilde{m}}\| \le \|M_m\|$. We then conclude that (1-4) alone is not sufficient for the boundedness of M_m. By symmetry, (1-5) alone is not sufficient for the boundedness of M_m either.

(2) A smooth version of the example above implies that neither the assumption (3-18) nor the assumption (3-19) is removable in Corollary 3.1. Indeed, fix a large $K > 0$, let

$$m_1(s,t) = \exp\left(\frac{i2\pi \log_2 s \log_2 t}{K}\right)$$

for $1 \le s \le t \le 2^K$, $s, t \in \mathbb{N}$,

$$m_1(s,t) = \frac{2^{K+1} - t}{2^K} \exp(i2\pi \log_2 s)$$

for $1 \le s \le t$, $2^K < t \le 2^{K+1}$, $s, t \in \mathbb{N}$ and $m_1(s,t) = 0$ otherwise. Then m_1 satisfies (3-18) because

$$\left|\frac{\partial}{\partial t} \exp\left(\frac{i2\pi \log_2 s \log_2 t}{K}\right)\right| \lesssim \frac{1}{t} \le \frac{1}{t-s}$$

whenever $s < t \le 2^K$. Assuming the sufficiency of (3-18) would imply the uniform boundedness of M_{m_1} for all $1 < p < \infty$, which is wrong because $M_m(A) = M_{m_1}(VAV)$ for $A \in S^p(\ell^2(\mathbb{N}))$ and m, V defined above. We conclude that neither the assumption (3-18) nor the assumption (3-19) is removable.

(3) Let

$$F_{N,t} = \{(s,t) \in \mathbb{N} \times \mathbb{N} : 2^{N-1} \le |s-t| < 2^N\}$$

for $N, t \in \mathbb{N}$. Let $Q_{N,t}$ be the projection from $S^2(\ell_2(\mathbb{N}))$ onto the span of $\{e_{s,t} : (s,t) \in F_{N,t}\}$. One may wonder whether Corollary 1.2 can be improved so that the Schur multiplier $S_\varepsilon = \sum_{N,t\in\mathbb{N}} \varepsilon(N,t) Q_{N,t}$ is bounded for any sequence $|\varepsilon(N,t)| \le 1$. This is impossible as well.[2] To see this, let $\varepsilon(N,t) = \exp(i2\pi Nj/K)$ if $t = 2^j$ for some $j \in \mathbb{N}$ and $j < N \le K$. Let $\varepsilon(N,t) = 0$ otherwise. Let V be the projection on $\ell^2(\mathbb{N})$ such that $V(e_i) = e_i$ if $i = 2^k$ for some $k \in \mathbb{N}$, and $V(e_i) = 0$ otherwise. Then, for M_m defined in the first example and $A \in S^p(\ell_2(\mathbb{N}))$, we have

$$M_m(A) = S_\varepsilon(VAV).$$

Therefore, $K^{1/2-1/p} \lesssim \|S_\varepsilon\|$.

5.2. *Operator-valued symbol.* The Schatten p class has a natural operator space structure inherited from the operator space complex interpolation $S^p = (S^\infty, S^1)_{1/p}$, $1 < p < \infty$. Pisier [1998, Lemma 1.7] proved that, with respect to S^p's natural operator space structure, a map M on $S^p(\ell^2)$ is completely bounded if and only if $M \otimes \mathrm{id}_{S^p(H)}$ is bounded on $S^p(S^p) = S^p(\ell^2 \otimes H)$ for any separable Hilbert space H. We will explain an operator-valued version of Theorem 1.1 which particularly implies the complete boundedness of the Schur multipliers considered in Theorem 1.1. We will assume the readers are familiar with the terminology of operator spaces in this subsection.

We will consider $A \in S^2(\ell^2(\mathbb{Z}) \otimes H)$ with H a separable Hilbert space. We present A in its matrix form $(a_{i,j})_{i,j\in\mathbb{Z}}$ with $a_{i,j} \in S^2(H)$. More precisely, denote by e_i the canonical basis of ℓ_2, let $e_{j,i}$ be the rank-1 operator on ℓ^2 sending e_i to e_j. Denote by tr (resp. τ) the canonical trace on $B(\ell^2(\mathbb{Z}))$ (resp. $B(H)$). We set

$$a_{i,j} = (\mathrm{tr} \otimes \mathrm{id})(A(e_{j,i} \otimes \mathrm{id}_H)).$$

[2] We can get the same conclusion for sequences $\varepsilon_k = \pm 1$ by choosing Hadamard orthogonal matrices instead of the matrices $(\exp((-i2\pi kj)/K))_{1\le k,j\le K}$.

Let $\mathcal{M}$ be a finite von Neumann algebra with a normal faithful tracial state τ. Given an $\mathcal{M}$-valued bounded function m on $\mathbb{Z}\times\mathbb{Z}$ and $A \in S^2(\ell^2(\mathbb{Z})\otimes H)$ in its matrix form $(a_{i,j})_{i,j\in\mathbb{Z}}$, we define $M_m(A)$ as the matrix

$$M_m(A) = (m_{i,j}\otimes a_{i,j})_{i,j}. \tag{5-1}$$

We will show that an analogue of Theorem 1.1 holds, that is, there exists $C_p \simeq (p^2/(p-1))^3$ for $1<p<\infty$ such that

$$\|M_m(A)\|_{L^p(\mathcal{M}\otimes B(\ell^2(\mathbb{Z})\otimes H))} \le C_p\|A\|_{S^p(\ell^2\otimes H)}$$

for all $A\in S^2\cap S^p$. By the density of $S^2\cap S^p$, M_m extends to a bounded operator from $S^p(\ell^2(\mathbb{Z})\otimes H)$ to $L^p(\mathcal{M}\otimes B(\ell^2(\mathbb{Z})\otimes H))$ when m satisfies Marcinkiewicz-type conditions. When $\mathcal{M}=\mathbb{C}$, this implies the complete boundedness of M_m in Theorem 1.1 by Pisier's result. We will need Pisier's $L_\infty(\ell_1)$ norm to express this Marcinkiewicz-type condition.

Definition 5.1 (Pisier's $L^\infty(\ell_1)$ norm). Given N-tuples $(x_1,\dots,x_N)$ in $\mathcal{M}$, set

$$\|x\|_{L^\infty(\mathcal{M};\ell_1)} = \inf\left\{\left\|\left(\sum a_ja_j^*\right)^{\frac12}\right\|\cdot\left\|\left(\sum b_j^*b_j\right)^{\frac12}\right\|\right\}, \tag{5-2}$$

where the infimum runs over all possible factorizations $x_j=a_jb_j$, with $a_j,b_j\in\mathcal{M}$.

When $x_k\ge 0$, we have $\|x\|_{L^\infty(\mathcal{M};\ell_1)} = \left\|\sum_k |x_k|\right\|$ but the two quantities are not comparable in general. Pisier showed that $\|x\|_{L^\infty(\mathcal{M};\ell_1)}<\infty$ if and only if there is a decomposition $x_k = x_{k,1}-x_{k,2}+ix_{k,3}-ix_{k,4}$ such that $x_{k,\ell}\ge 0$ and $\|(x_{k,\ell})_k\|_{L^\infty(\mathcal{M};\ell_1)}<\infty$ for all $\ell=1,2,3,4$.

Given M_m defined as in (5-1), let

$$\Delta_s m(s,t) = m(s+1,t)-m(s,t),\qquad \Delta_t m(s,t) = m(s,t+1)-m(s,t)$$

for $s,t\in\mathbb{Z}$.

Theorem 5.2. *M_m defined as in (5-1) extends to a bounded map from Schatten p-classes $S^p(\ell^2\otimes H)$ to $L^p(\mathcal{M}\otimes B(\ell^2(\mathbb{Z})\otimes H))$ for all $1<p<\infty$ with bounds $\lesssim (p^2/(p-1))^3$ if m is bounded in $\mathcal{M}$ and there is a constant C such that,*

(i) *for any $n\in\mathbb{N}$, $t\in\mathbb{Z}$,*

$$\|(\Delta_s m(s+t,t))_{2^{n-1}\le|s|<2^n}\|_{L^\infty(\mathcal{M};\ell_1)} < C, \tag{5-3}$$

(ii) *for any $n\in\mathbb{N}$, $s\in\mathbb{Z}$,*

$$\|(\Delta_t m(s,s+t))_{2^{n-1}\le|t|<2^n}\|_{L^\infty(\mathcal{M};\ell_1)} < C. \tag{5-4}$$

Sketch of proof. Define $\tilde m(s,t) = m(s,t)\otimes 1_{\ell_2\otimes H}$ and $\tilde a_{s,t} = 1_{\mathcal{M}}\otimes a_{s,t}$. Then $\tilde m(s,t)$ commutes with $a_{s',t'}$ for any $s,t,s',t'\in\mathbb{Z}$. Let

$$\begin{aligned}\widetilde M_l(j) = \sum_{s\in\mathbb{Z}}\tilde m(s,s-j)\otimes e_{s,s},\qquad \widetilde M_r(j) = \sum_{s\in\mathbb{Z}}\tilde m(s+j,s)\otimes e_{s,s},\\ \tilde A(j) = \sum_{s\in\mathbb{Z}}\tilde a_{s,s-j}\otimes e_{s,s-j} = \sum_{s\in\mathbb{Z}}\tilde a_{s+j,s}\otimes e_{s+j,s},\end{aligned} \tag{5-5}$$

with $e_{s,t}$ the canonical basis of $S^2(\ell_2(\mathbb{Z}))$. Let $f(z)=\sum_{j\in\mathbb{Z}}\tilde{A}(j)z^j$ and

$$T_{\widetilde{M}}f(z)=\sum_{j\in\mathbb{Z}}\widetilde{M}_l(j)\tilde{A}(j)z^j. \tag{5-6}$$

We still have

$$T_{\widetilde{M}}f(z)=\sum_{j\in\mathbb{Z}}\tilde{A}(j)\widetilde{M}_r(j)z^j \tag{5-7}$$

and the identities

$$\|f\|_{L^p(\mathcal{M}\otimes B(\ell^2(\mathbb{Z})\otimes H))}=\|A\|_{S^p(\ell_2\otimes H)},$$
$$|T_{\widetilde{M}}f\|_{L^p(\mathcal{M}\otimes B(\ell^2(\mathbb{Z})\otimes H))}=\|M_m(A)\|_{S^p(\ell_2\otimes H)}.$$

Moreover, the conditions (5-3) and (5-4) imply that

$$\|\Delta_l\widetilde{M}(j)_{2^{n-1}<|j|\le 2^n}\|_{L^\infty(\mathcal{M}\otimes B(\ell^2(\mathbb{Z})),\ell_1)},\ \|\Delta_r\widetilde{M}(j)_{2^{n-1}<|j|\le 2^n}\|_{L^\infty(M\otimes B(\ell^2(\mathbb{Z})),\ell_1)}<C$$

for

$$\Delta_l\widetilde{M}(j)=\widetilde{M}_l(j+1)-\widetilde{M}_l(j),\quad \Delta_r\widetilde{M}(j)=\widetilde{M}_r(j+1)-\widetilde{M}_r(j).$$

After these, it is not hard to check that the arguments for the proof of Theorem 1.1 work as well for the tensor case. □

Corollary 5.3. *The Schur multipliers considered in Theorem 1.1 are completely bounded on the Schatten classes S^p, $1<p<\infty$, with bounds $\lesssim (p^2/(p-1))^3$ with respect to their natural operator space structure.*

Remark 5.4. The optimal constant for the L^p bounds of the classical Marcinkiewicz Fourier multipliers is $p^{3/2}$ as $p\to\infty$ [Tao and Wright 2001]. It is unclear what is the optimal asymptotic order for the S^p-bounds of the Schur multipliers in Theorem 1.1.

Open Question. Assume m is a bounded map on $\mathbb{Z}\times\mathbb{Z}$ such that

$$\sum_s|m(k,j_s)-m(k,j_{s+1})|^2<C$$

for all possible increasing sequences $j_s\in\mathbb{Z}$. Does M_m extend to a bounded map on S^p for all $1<p<\infty$?

Remark 5.5. The authors heard this question from Potapov and Sukochev. They told the authors that it stems from the work of Birman and Solomyak on double operator integrals. The third author noticed Theorem 1.1 during his effort of attacking this question.

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DOUBLE DUALS AND HILBERT MODULES

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Let A be a C^*-algebra, H be a Hilbert A-module and $K(H)$ be the closure of the set of finite-rank module maps. We show that the W^*-algebra of all bounded A^{**}-module maps on the smallest self-dual Hilbert A^{**}-module containing H is isomorphic to $K(H)^{**}$ as W^*-algebras. We also show that the unit ball of H is closed in $H^\sharp$, the dual of H in an A-weak topology of $H^\sharp$, and the unit ball of H is also dense in the unit ball of $H^\sharp$ in a weak* topology. Some versions of the Kaplansky density theorem for Hilbert C^*-modules are also presented.

1. Introduction

Hilbert C^*-modules as a generalization of Hilbert spaces were first introduced by I. Kaplansky [1953] in special cases and later by W. Paschke [1973] for general C^*-algebras. Hilbert C^*-modules are crucial to Kasparov's formulation of KK-theory [1980]. Early applications also include C^*-algebraic quantum group theory; see [Baaj and Skandalis 1993]. Later, in the study of Cuntz semigroups in connection with the classification of amenable C^*-algebras, Hilbert C^*-modules play an important role; see, for example, [Brown and Ciuperca 2009; Brown and Lin 2025; Coward et al. 2008; Ortega et al. 2011].

Let A be a C^*-algebra. Unlike Hilbert spaces, bounded module maps on a Hilbert A-module H may not have adjoints and the dual module $H^\sharp$, i.e., the Banach A-module of all bounded module maps from H to A, may not be identified as elements in H. Moreover, the C^*-algebra $L(H)$ of all bounded module maps with adjoints may not be a W^*-algebra. If $H_0 \subset H$ is a Hilbert A-submodule, a bounded module map $\varphi : H_0 \to A$ may not be extended to a bounded module map from H to A. In general, one should not expect that H can be decomposed into an orthogonal direct sum of H_0 and its orthogonal complement. In fact, H_0 may not even have an orthogonal complement. Study of these phenomena may be found, for example, in [Lin 1991a; 1992] and more recently in [Brown and Lin 2025].

However, Paschke [1973] found that, if A is a W^*-algebra, then the dual module $H^\sharp$ of a Hilbert A-module H can be made into a Hilbert A-module in a natural way which extends H, and $H^\sharp$ is a self-dual Hilbert A-module. Even if A is not a W^*-algebra, one can extend H into an A^{**}-module $H \bullet A^{**}$ naturally. Then its dual $H^\sim := (H \bullet A^{**})^\sharp$ becomes a self-dual Hilbert A^{**}-module containing H. In fact, $H^\sim$ is the smallest self-dual Hilbert A^{**}-module containing H as a Hilbert A-submodule; see Proposition 3.2. Paschke showed that the Banach algebra of all bounded module maps on $H^\sim$ becomes a W^*-algebra.

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For a Hilbert A-module H, the rank-1 module maps are the module maps T of the form $T(h) = x\langle y, h\rangle$ for all $h \in H$ (and fixed $x, y \in H$, where $\langle\cdot,\cdot\rangle$ is the A-valued inner product). Denote by $F(H)$ the linear span of rank-1 module maps and denote by $K(H)$ the norm closure of $F(H)$. $K(H)$ is a C^*-algebra and an important algebra related to the Hilbert module H. It was proved by Kasparov [1980, Theorem 1] that the C^*-algebra $L(H)$ may be identified with $M(K(H))$, the multiplier algebra of $K(H)$, and it was proved in [Lin 1991a] that the Banach algebra of all bounded module maps on H is identified with the left multipliers of $K(H)$. (All Hilbert A-modules considered in this paper are right A-modules.) Over the decades, we eventually realized that it is rather convenient to work in $B(H^\sim)$ in many occasions as we study module maps on a Hilbert module H. It is not difficult to establish a natural normal homomorphism $\Psi : K(H)^{**} \to B(H^\sim)$ which extends beyond $M(K(H))$ and $LM(K(H))$. It remained unknown for many years whether Ψ is an isomorphism. The original motivation of this paper is to show that indeed Ψ is an isomorphism between W^*-algebras $K(H)^{**}$ and $B(H^\sim)$.

As we study the relation among Hilbert modules H, $H \bullet A^{**}$ and $H^\sim$, naturally we ask: how dense is H in $H \bullet A^{**}$ and in $H^\sim$? Since $H^\sim = (H \bullet A^{**})^\sharp$, the dual of $H \bullet A^{**}$, one may also ask about the density of H in $H^\sharp$ in general.

We first note that it was shown (Theorem 6.1 of [Brown and Lin 2025]) that H is dense in $H^\sharp$ in an A-weak topology. More precisely, for any $\xi \in H^\sharp$, there is a net $\{x_\alpha\}$ in H with $\|x_\alpha\| \le \|\xi\|$ for all α such that $\lim_\alpha \|\xi(x) - \langle x_\alpha, x\rangle\| = 0$ for all $x \in H$. However, we show here that the unit ball of H is closed in $H^\sharp$ in the topology where $x_\alpha \to \xi$ if and only if $\lim_\alpha \|\langle \xi - x_\alpha, \zeta\rangle\| = 0$ for all $\zeta \in H^\sharp$, and where the inner product is extended to $H^\sim$.

On the other hand, it is easy to see that, for any $\xi \in H \bullet A^{**}$, there is a net $\{x_\lambda\}$ in H such that $\lim_\lambda \pi_U(\langle x_\lambda, y\rangle)(v) = \pi_U(\langle \xi, y\rangle)(v)$ for all $y \in H \bullet A^{**}$ and $v \in H_U$, where H_U is the Hilbert space corresponding to the universal representation π_U of A. To be a more useful approximation, one may ask whether the net can be chosen to be bounded (by $\|\xi\|$). We will present a Kaplansky-style density theorem. Perhaps a more interesting question is: how dense is H in $H^\sim = (H \bullet A^{**})^\sharp$? Since $H^\sim$ is the dual of $H \bullet A^{**}$, it is relatively easy to show that, for any $\zeta \in H^\sim$, there is a net $\{z_\alpha\}$ in H such that

$$\lim_\lambda f(\langle z_\alpha, y\rangle) = f(\langle \zeta, y\rangle) \quad \text{for all } y \in H \bullet A^{**} \text{ and } f \in A^*.$$

It is more challenging to show that y can be replaced by any element in $H^\sim = (H \bullet A^{**})^\sharp$. We show that the unit ball of H is actually dense in the unit ball of $H^\sim$ in the weak* topology (as $H^\sim$ is a conjugate space), another Kaplansky-style density theorem. In fact, we show a stronger density theorem that, for any $\xi \in H^\sim$, there is a net $\{x_\alpha\}$ in H with $\|x_\alpha\| \le \|\xi\|$ such that

$$\lim_\alpha f(\langle \xi - x_\alpha, \xi - x_\alpha\rangle) = 0 \quad \text{for all } f \in A^*.$$

2. Self-duals

Definition 2.1. Let A be a C^*-algebra. Denote by $\tilde{A}$ the minimum unitization of A. We use the following convention: if A is a C^*-subalgebra of a unital C^*-algebra B, we write $1_{\tilde{A}} = 1_B$ if either A is unital and $1_A = 1_{\tilde{A}} = 1_B$, or $A^\perp = \{b \in B : ba = ab = 0\} = \{0\}$, and we unitize A by adjoining 1_B to form $\tilde{A} \subset B$.

Definition 2.2. Let X be a Hilbert space and $B(X)$ be the C^*-algebra of all bounded linear operators on X. Suppose that $A \subset B(X)$. Then $\bar{A}^{\mathrm{SOT}}$ is the closure of A in the strong operator topology. Note that if $\{e_\alpha\}$ is an approximate identity for A, then $e_\alpha \nearrow 1_M$, i.e., e_α increasingly converges to the identity of $M = \bar{A}^{\mathrm{SOT}}$ in the strong operator topology as well as in the weak* topology (of M). In particular, we may write $1_{\tilde{A}} = 1_M$.

This works particularly for the pair A and A^{**} (where X is H_u, the Hilbert space corresponding to the universal representation of A).

In general, if M is a W^*-algebra, we denote by M_* the predual of M.

Definition 2.3. Let A be a C^*-algebra. In this paper, we use the formal definition of Hilbert modules in [Paschke 1973] and consider only right A-modules. Recall that a linear space H is a pre-Hilbert module if it is also a right A-module with an inner product $H \times H \to A$ satisfying the following properties: for any $x, y, z \in H$, $a \in A$ and $\lambda \in \mathbb{C}$,

(1) $\langle x, \lambda y + z\rangle = \langle x, y\rangle + \lambda\langle x, y\rangle$,

(2) $\langle x, ya\rangle = \langle x, y\rangle a$,

(3) $\langle x, y\rangle^* = \langle y, x\rangle$,

(4) $\langle x, x\rangle \geq 0$; if $\langle x, x\rangle = 0$, then $x = 0$.

Define $\|x\| = \|\langle x, x\rangle\|^{1/2}$ for $x \in H$. Then H becomes a normed space. H is a Hilbert A-module if H is complete with this norm.

Denote by $H^\sharp$ the Banach space of all bounded module maps from H into A. A Hilbert A-module is said to be self-dual if, for every $f \in H^\sharp$, there is $x \in H$ such that

$$f(y) = \langle x, y\rangle \quad \text{for all } y \in H.$$

Denote by $B(H)$ the Banach algebra of all bounded module maps from H into itself, and by $L(H)$ the C^*-algebra of all those bounded module maps T with an adjoint T^* in $L(H)$ defined by

$$\langle T(x), y\rangle = \langle x, T^*(y)\rangle \quad \text{for all } x, y \in H.$$

Let $F(H)$ be the algebra of all finite-rank module maps, i.e., the linear span of all bounded module maps of the form $\theta_{x,y} : H \to H$ defined by

$$\theta_{x,y}(\xi) = x\langle y, \xi\rangle$$

for all $\xi \in H$ and $x, y \in H$. Denote by $K(H)$ the norm closure of $F(H)$, which is a C^*-algebra.

By Theorem 1 of [Kasparov 1980], we identify $L(H)$ with $M(K(H))$, the multiplier algebra of $K(H)$ and, by Theorem 1.5 of [Lin 1991a], $B(H)$ with $LM(K(H))$, the Banach algebra of left multipliers of $K(H)$ (in $K(H)^{**}$). If H is self-dual, then $B(H) = L(H)$.

We refer to [Kasparov 1980; Lin 1991a; 1992; Paschke 1973] for common terminologies related to Hilbert C^*-modules.

Definition 2.4. Let A be a C^*-algebra and H a Hilbert A-module. Let us give the definition of a self-dual Hilbert A^{**}-module $H^{\sim}$; see Definition 1.3 of [Lin 1991a].

We may view H as a Hilbert $\tilde{A}$-module. Let B be a unital C^*-algebra containing A and $1_{\tilde{A}} = 1_B$ (see the convention in Definition 2.1). The algebraical tensor product $H \otimes B$ becomes a right B-module if we set $(h \otimes a) \cdot b = h \otimes ab$ for any $h \in H$ and $a, b \in B$. Define $\langle -, - \rangle : H \otimes B \times H \otimes B \to B$ by

$$\left\langle \sum_i h_i \otimes a_i, \sum_j x_j \otimes b_j \right\rangle = \sum_{i,j} a_i^* \langle h_i, x_j \rangle b_j$$

and $N = \{z \in H \otimes A^{**} : \langle z, z \rangle = 0\}$. Then $(H \otimes B)/N$ becomes a pre-Hilbert B-module (see Section 4 of [Paschke 1973], but exchange B with A). Denote by $H \bullet B := ((H \otimes B)/N)^{-}$ (the completion of) the Hilbert B-module.

We are particularly interested in the case that $B = A^{**}$. We view $\tilde{A}$ as a C^*-subalgebra of A^{**}. Then $H^{\sim} := (H \bullet A^{**})^{\sharp}$ is a self-dual Hilbert A^{**}-module.

Note that $\tilde{A}$ is ultraweakly dense in A^{**} (since A is). By applying the result [Paschke 1973, Theorem 4.2] to the pair A^{**} (as A in that result) and $\tilde{A}$ (as B in that result, see also the remark right after the proof of that result), we obtain an isometric (surjective) isomorphism $\iota : H^{\sim} := (H \bullet A^{**})^{\sharp} \to B(H, A^{**})$, with $B(H, A^{**})$ the Banach space of all bounded A-module maps from H to A^{**} (written as $M(H, A^{**})$ in that same result).

Let $x \in H$ and $b \in B$. Then

$$\|(x \otimes b)/N\|^2 = \|b^* \langle x, x \rangle b\| \le \|x\|^2 \|b^* b\|.$$

Hence

$$\|(x \otimes b)/N\| \le \|x\| \|b\|.$$

In what follows, for $x \in H$ and $b \in B$, we write $x \bullet b := (x \otimes b)/N$.

In general, if E is a self-dual Hilbert module, then $B(E) = L(E)$; see [Paschke 1973, Corollary 3.5]. If in addition A is a W^*-algebra, $B(E)$ is also a W^* -algebra; see [Paschke 1973, Proposition 3.11].

Let us recall the description of the predual of $B(E)$ in this case. Denote by $E_{\sim}$ the linear space E with the "twisted" scalar multiplication (i.e., $\lambda x = \bar{\lambda} x$ for $x \in E$ and $\lambda \in \mathbb{C}$) and consider $E_{\sim} \otimes E \otimes A_*$ with the greatest cross-norm, where A_* is the usual predual of the W^*-algebra A. For each $T \in B(E)$, define a linear functional $\check{T}$ on $E_{\sim} \otimes E \otimes A_*$ by

$$\check{T}\left(\sum_{j=1}^n x_j \otimes y_j \otimes g_j \right) = \sum_{j=1}^n g_j(\langle T(x_j), y_j \rangle)$$

for $x_j, y_j \in E$ and $g_j \in A_*$, $1 \le j \le n$. The map $T \to \check{T}$ is a linear isometry of $B(E) = L(E)$ into $(E_{\sim} \otimes E \otimes A_*)^*$. It was shown [Paschke 1973, Proposition 3.10] that $\check{B(E)}$ is weak*-closed in $E_{\sim} \otimes E \otimes A_*$. A bounded net $\{T_\alpha\}$ in $B(E)$ converges to $T \in B(E)$ in the weak* topology if and only if

$$f(\langle T_\alpha(x), y \rangle) \to f(\langle T(x), y \rangle) \quad \text{for all } x, y \in E \text{ and } f \in A_*$$

[Paschke 1973, Remark 3.9 and Proposition 3.10]. In particular, $B(H^{\sim})$ is a W^*-algebra.

Definition 2.5. Keep the notation in Definition 2.4. Recall that H is a Hilbert A-module and $\tilde{A} \subset B$. Then $\iota : H \to H \bullet B$ defined by $x \to x \otimes 1$ is an injective map. Note that, for all $a \in A$,

$$\begin{aligned}\langle (x \cdot a) \otimes 1 - x \otimes a, (x \cdot a) \otimes 1 - x \otimes a \rangle &= \langle x \cdot a, x \cdot a \rangle - \langle x \cdot a, x \rangle a - a^* \langle x, x \cdot a \rangle + a^* \langle x, x \rangle a \\ &= a^* \langle x, x \rangle a - a^* \langle x, x \rangle a - a^* \langle x, x \rangle a + a^* \langle x, x \rangle a = 0.\end{aligned}$$

Hence $\iota(x \cdot a) = x \otimes a / N$ for all $a \in \tilde{A}$. In the case $B = A^{**}$, we then extend ι from $H^\sharp$ to $(H \bullet A^{**})^\sharp$ by

$$\iota(f)(x \bullet b) = f(x) b \quad \text{for all } x \in H \text{ and } b \in A^{**}$$

and $f \in H^\sharp$. Note that the map is a module map from $H^\sharp$ to $(H^\sim)^\sharp$, which is conjugate module isomorphic to $H^\sim$.

From now on, we may view H as a submodule of $H^\sim$ and, sometimes, omit the map ι.

The following result provides a convenient and easy fact that $H \bullet B$ is the smallest Hilbert B-module containing H as a Hilbert A-module.

Proposition 2.6. *Let A and B be a pair of C^*-algebras such that $A \subset B$, B is unital and $1_{\tilde{A}} = 1_B$. Suppose that H is a Hilbert A-module, H_1 is a Hilbert B-module and there is an embedding $\iota : H \to H_1$ as Hilbert modules, i.e., ι is a linear and A-module map such that*

$$\langle \iota(x), \iota(y) \rangle = \langle x, y \rangle \quad \textit{for all } x, y \in H.$$

Then there is a unique B-module embedding $\tilde{\iota} : H \bullet B \to H_1$ such that

$$\tilde{\iota}(x \bullet b) = \iota(x) b \quad \textit{for all } x \in H \textit{ and } b \in B, \qquad \langle \tilde{\iota}(\xi), \tilde{\iota}(\zeta) \rangle = \langle \xi, \zeta \rangle \quad \textit{for all } \xi, \zeta \in H \bullet B.$$

Proof. For any $\xi = \sum_{i=1}^n x_i \bullet a_i$, where $x_i \in H$ and $a_i \in B$ $(1 \le i \le n)$, define

$$\tilde{\iota}(\xi) = \sum_{i=1}^n \iota(x_i) a_i.$$

Then, for $\zeta = \sum_{i=1}^n y_i \bullet b_i$,

$$\langle \tilde{\iota}(\xi), \tilde{\iota}(\zeta) \rangle = \sum_{i,j}^n a_i^* \langle x_i, y_j \rangle b_j = \langle \xi, \zeta \rangle.$$

In particular,

$$\| \tilde{\iota}(\xi), \tilde{\iota}(\xi) \rangle \| = \left\| \sum_{i,j}^n a_i^* \langle x_i, x_j \rangle b_j \right\| = \| \xi \|^2.$$

Therefore $\|\tilde{\iota}\| \le 1$ on $(H \otimes B)/N$. So $\tilde{\iota}$ is uniquely extended to a contractive linear map from $H \bullet B$ into H_1. It is a B-module map. Since $(H \otimes B)/N$ is dense in $H \bullet B$,

$$\langle \tilde{\iota}(x), \tilde{\iota}(y) \rangle = \langle x, y \rangle \quad \text{for all } x, y \in H \bullet B.$$

To see this embedding is unique, let $\tilde{\iota}_1$ be another such embedding. Then $(\tilde{\iota} - \tilde{\iota}_1)|_H = 0$. For any $\xi = \sum_{i=1}^n x_i \bullet a_i$, where $x_i \in H$ and $a_i \in B$,

$$(\tilde{\iota} - \tilde{\iota}_1)(\xi) = \sum_{i=1}^n (\iota(x_i) - \iota(x_i)) \bullet a_i = 0.$$

In other words, $\tilde{\iota}_1 = \tilde{\iota}$. $\square$

Definition 2.7. Keep the notation in Definitions 2.3, 2.4 and 2.5. Recall that $F(H)$ is the algebra of all finite-rank module maps. Define $\Psi_0 : F(H) \to F(H \bullet B) \subset B(H \bullet B)$ by

$$\Psi_0(\theta_{x,y})(\zeta) = \iota(x)\langle \iota(y), \zeta\rangle$$

for all $\zeta \in H \bullet B,\ x, y \in H$. Ψ is a $*$-preserving homomorphism from the $*$-algebra $F(H)$ into $F(H \bullet B)$. Moreover, Ψ_0 is an isometry on $F(H)$. In particular, $\|\Psi_0\| = 1$. Therefore it extends uniquely to a C^*-algebra homomorphism from $K(H)$ to $K(H \bullet B)$, which preserves the norm. It has to be an isometry as $F(H)$ is dense in $K(H)$.

In the case that $B = A^{**}$, we may define $\widetilde{\Psi}_0 : F(H) \to F(H^{\sim}) \subset B(H^{\sim})$ by

$$\widetilde{\Psi}_0(\theta_{x,y})(\zeta) = \iota(x)\langle \iota(y), \zeta\rangle$$

for all $\zeta \in H^{\sim},\ x, y \in H$. Then $\widetilde{\Psi}_0$ is a $*$-preserving homomorphism from the $*$-algebra $F(H)$ into $F(H^{\sim})$ and it extends uniquely to a C^*-algebra homomorphism $\widetilde{\Psi}_0$ from $K(H)$ to $K(H^{\sim})$, which preserves the norm. Recall that $\iota(H^{\sharp}) \subset H^{\sim}$.

Proposition 2.8. *Let $A \subset B$ be a pair of C^*-algebras, where B is unital and $1_B = 1_{\tilde{A}}$. Let $T \in K(H)$. Then $\Psi_0(T)(x \bullet b) = T(x) \bullet b$ for all $x \in H$ and $b \in B$.*

Proof. From the definition, for any $S \in F(H)$, any $x \in H$ and any $b \in B$,

$$\Psi_0(S)(x \otimes b) = S(x) \otimes b \pmod{\mathbb{N}}.$$

Fix $T \in K(H)$, and let $\epsilon > 0$. There exists $S \in F(H)$ such that

$$\|T - S\| < \tfrac{1}{4}\epsilon(1 + \|x \bullet b\| + \|x\|\,\|b\|).$$

Then

$$\|\Psi_0(T) - \Psi_0(S)\| < \tfrac{1}{4}\epsilon(1 + \|x \otimes b\| + \|x\|\,\|b\|) \quad \text{and} \quad \|T(x) \bullet b - S(x) \bullet b\| \le \tfrac{1}{2}\epsilon.$$

Hence

$$\|\Psi_0(T)(x \bullet b) - T(x) \bullet b\| < \epsilon.$$

Since this holds for all $\epsilon > 0$, we conclude that

$$\Psi_0(x \bullet b) = T(x) \bullet b. \qquad \square$$

Lemma 2.9. *Let A and B be as in Proposition 2.8 and H be a Hilbert A-module. Suppose that $\{E_\lambda\}$ is an approximate identity for $K(H)$. Then $\{\Psi_0(E_\lambda)\}$ forms an approximate identity for $K(H \bullet B)$. Moreover*

$$\lim_\lambda \|\Psi_0(E_\lambda)(x) - x\| = 0 \quad \textit{for all } x \in H \bullet B.$$

Proof. By Lemma 3.1 of [Brown and Lin 2025],

$$\lim_\lambda \|E_\lambda(x) - x\| = 0 \quad \text{for all } x \in H. \tag{2-1}$$

Let $S = \sum_{i=1}^n \theta_{x_i,y_i}$, where $x_i, y_i \in (H \otimes B)/N,\ 1 \le i \le n$. Write $x_i = \sum_{j=1}^{k(i)} \xi_{j,i} \bullet b_{j,i}$, where $\xi_{j,i} \in H$ and $b_{j,i} \in B,\ j = 1, 2, \ldots, k(i),\ i = 1, 2, \ldots, n$. By Proposition 2.8,

$$\Psi_0(E_\lambda)(\xi_{j,i} \bullet b_{j,i}) = E_\lambda(\xi_{j,i}) \bullet b_{j,i}.$$

By (2-1),

$$\lim_{\lambda} \|\Psi_0(E_\lambda)(\xi_{j,i} \bullet b_{j,i}) - (\xi_{j,i} \bullet b_{j,i})\| = 0 \tag{2-2}$$

for $j = 1, 2, \ldots, k(i)$, $i = 1, 2, \ldots, n$. It follows that

$$\lim_{\lambda} \|\Psi_0(E_\lambda)(x_i) - x_i\| = 0, \quad i = 1, 2, \ldots, n.$$

For any $z \in H \bullet B$,

$$\Psi_0(E_\lambda)\theta_{x_i,y_i}(z) = (\Psi_0(E_\lambda)x_i)\langle y_i, z\rangle = E_\lambda(x_i)\langle y_i, z\rangle.$$

It follows that, for $1 \le i \le n$,

$$\lim_{\lambda} \|\Psi_0(E_\lambda)\theta_{x_i,y_i} - \theta_{x_i,y_i}\| = 0.$$

Hence

$$\lim_{\lambda} \|\Psi_0(E_\lambda)S - S\| = 0.$$

The set of those module maps with the form of S is norm-dense in $K(H \bullet B)$. Therefore we conclude that

$$\lim_{\lambda} \|\Psi_0(E_\lambda)S - S\| = 0 \quad \text{for all } S \in K(H \bullet B).$$

It follows that $\{\Psi_0(E_\lambda)\}$ forms an approximate identity for $K(H \bullet B)$. □

2.10. Let A be a C^*-algebra and H be a Hilbert A-module. Then $H^\sharp$ is a Banach A-module in general. Recall that, for each $T \in B(H)$, one may define a bounded conjugate module map $T^* : H \to H^\sharp$ as follows: for $x, y \in H$, define

$$T^*(x)(y) = \langle x, T(y)\rangle.$$

So, for a fixed x, we have that $T^*(x)$ gives an element in $H^\sharp$. Moreover, T^* is a bounded conjugate module map from H to $H^\sharp$ with $\|T^*\| = \|T\|$. However, if we view H as a submodule of $H^\sharp$, then T^* is a bounded module map. Note that, if $T \in L(H)$, then $T^* \in L(H)$ and $T^*(H) \subset H$.

If A is a W^*-algebra, by Theorem 3.2 of [Paschke 1973], $H^\sharp$ becomes a Hilbert A module in a natural way. For $T \in B(H)$ and $f \in H^\sharp$, define, for each $x \in H$,

$$\widetilde{T}(f)(x) = \langle f, T^*(x)\rangle, \tag{2-3}$$

where T^* is defined above. Thus $\widetilde{T}(f)$ is a bounded linear module map from H to A with $\|\widetilde{T}(f)\| \le \|T\|\|f\|$. Hence we extend T to a bounded (conjugate) module map from $H^\sharp$ to $H^\sharp$. As we view $H^\sharp$ as a Hilbert A-submodule in this case, T is in fact a bounded module map on $H^\sharp$ (we will take the conjugate as Hilbert space cases). By Corollary 3.7 of [Paschke 1973], such an extension is unique.

By Lemma 3.7 of [Lin 1992], one may ease the assumption that A is a W^*-algebra to the assumption that A is a monotone complete C^*-algebra.

Proposition 2.11. *Let A and B be as in Proposition 2.8, H be a Hilbert A-module and $\{E_\lambda\}$ an approximate identity for $K(H)$. Then*

$$\lim_{\lambda}(\sup\{\|\widetilde{\Psi}_0(E_\lambda)(f)(x) - f(x)\| : f \in (H \bullet B)^\sharp,\ \|f\| \le 1\}) = 0 \quad \textit{for all } x \in H \bullet B. \tag{2-4}$$

Moreover, suppose that $(H \bullet B)^\sharp$ extends $H \bullet B$ as a Hilbert B-module, then, for any $T \in B((H \bullet B)^\sharp)$,

$$\lim_{\lambda} \|\langle \widetilde{\Psi}_0(E_\lambda)T\widetilde{\Psi}_0(E_\lambda)(x), y\rangle - \langle T(x), y\rangle\| = 0 \quad \textit{for all } x, y \in H \bullet B.$$

Proof. Fix $f \in H^\sharp$. For any $x \in H \bullet B$, by Lemma 2.9,

$$\|\widetilde{\Psi}_0(E_\lambda)(f)(x) - f(x)\| = \|f(E_\lambda(x)) - f(x)\| \le \|f\|\|E_\lambda(x) - x\| \to 0.$$

Hence (2-4) holds.

To see the "moreover" part of the lemma, let $T \in B((H \bullet B)^\sharp)$. Then, for any $x, y \in H \bullet B$,

$$\begin{aligned}&\|\langle \widetilde{\Psi}_0(E_\lambda) T \widetilde{\Psi}_0(E_\lambda)(x), y\rangle - \langle T(x), y\rangle\| \\ &\qquad \le \|\langle T\widetilde{\Psi}_0(E_\lambda)(x), \widetilde{\Psi}_0(E_\lambda)(y)\rangle - \langle T(x), \widetilde{\Psi}_0(E_\lambda)(y)\rangle\| + \|\langle T(x), \widetilde{\Psi}_0(E_\lambda)(y)\rangle - \langle T(x), y\rangle\| \\ &\qquad \le \|y\|\|T\|\|\Psi_0(E_\lambda)(x) - x\| + \|T\|\|x\|\|\Psi_0(E_\lambda)(y) - y\|\end{aligned}$$

By applying Lemma 2.9 to the two terms of the last inequality above, we conclude that

$$\lim_\lambda \|\langle \widetilde{\Psi}_0(E_\lambda) T \widetilde{\Psi}_0(E_\lambda)(x), y\rangle - \langle T(x), y\rangle\| = 0 \quad \text{for all } x, y \in H \bullet B. \qquad \square$$

Definition 2.12. Let A be a C^*-algebra and H be a Hilbert A-module. Recall [Lin 1991a, Theorem 1.5] that we identify $B(H)$ with $LM(K(H))$, the Banach algebra of left multipliers of $K(H)$ (in $K(H)^{**}$).

By Lemma 2.9, Ψ_0 maps $K(H)$ into $K(H \bullet B)$ which maps approximate identities to approximate identities. We may then extend a homomorphism $\Psi_0 : B(H) = LM(K(H)) \to LM(K(H \bullet B)) = B(H \bullet B)$ by

$$\Psi_0(T) = \lim_\lambda \Psi_0(TE_\lambda),$$

where the convergence is in the left strict topology of $LM(K(H \bullet B))$. Since $\Psi_0|_{K(H)}$ is an isometry, so is Ψ_0.

We are mostly interested in the case that $B = A^{**}$. By Theorem 3.2 of [Paschke 1973], $(H \bullet A^{**})^\sharp$ is a self-dual Hilbert A^{**}-module. Therefore, by Section 2.10, for each $T \in B(H)$, the extension $\widetilde{\Psi}_0(T)$ is unique. Hence Ψ_0 may be extended to a Banach algebra isomorphism $\widetilde{\Psi}_0$ from $B(H)$ into $B(H^\sim)$ such that

$$\widetilde{\Psi}_0(T)|_{H \bullet A^{**}} = \Psi_0(T) \quad \text{for all } T \in B(H). \tag{2-5}$$

We will visualize the map Ψ_0 a bit more.

Proposition 2.13. *Let A and B be a pair of C^*-algebras as in Proposition 2.8 and H be a Hilbert A-module. Then, for any $T \in B(H)$,*

$$\lim_\lambda \|\Psi_0(T)\Psi_0(E_\lambda)(x) - \Psi_0(T)(x)\| = 0 \quad \textit{for all } x \in H \bullet B. \tag{2-6}$$

Moreover

$$\Psi_0(T)(x \bullet b) = T(x) \bullet b \quad \textit{for all } x \in H \textit{ and } b \in B.$$

Consequently, $\widetilde{\Psi}_0(\mathrm{id}_H) = \mathrm{id}_{H^\sim}$.

Proof. The identity (2-6) follows immediately from Lemma 2.9.

Since

$$\Psi_0(TE_\lambda)(x \bullet b) = TE_\lambda(x) \bullet b,$$

by (2-6) and by Lemma 3.1 of [Brown and Lin 2025],

$$\Psi_0(T)(x \bullet b) = T(x) \bullet b$$

for all $x \in H$ and $b \in B$.

For the last part of the proposition, we note that, by considering the pair A and A^{**}, and by the "moreover" part of the proposition, $\Psi_0(\mathrm{id}_H) = \mathrm{id}_{H\bullet A^{**}}$. Therefore, since the extension $\widetilde{\Psi}_0(\mathrm{id}_{H\bullet A^{**}})$ is unique (Corollary 3.7 of [Paschke 1973], see Section 2.10 for convenience), we must have that $\widetilde{\Psi}_0(\mathrm{id}_H) = \mathrm{id}_{H^\sim}$. □

The following is a slightly strengthened restatement of [Brown and Lin 2025, Proposition 2.3].

Proposition 2.14. *Let A be a C^*-algebra and H a Hilbert A-module. Then there is a homomorphism Ψ from $K(H)^{**}$ into $B(H^\sim)$ such that $\Psi|_{B(H)} = \widetilde{\Psi}_0$. Moreover, if $T \in K(H)^{**}$ and $T_\lambda \in K(H)^{**}$ such that $T_\lambda \to T$ in the weak* topology, then*

$$\lim_\lambda f(\langle \Psi(T_\lambda)(x), y\rangle) = f(\langle \Psi(T)(x), y\rangle) \quad \text{for all } x, y \in H^\sim \text{ and } f \in A^*.$$

Proof. By Definition 2.4, $B(H^\sim) = L(H^\sim)$ is a W^*-algebra; see [Paschke 1973, Proposition 3.11]. Let $\pi : B(H^\sim) \to B(H_\pi)$ be a faithful normal representation such that $\pi(B(H^\sim))$ is weakly closed in $B(H_\pi)$. Then, by, for example, [Pedersen 1979, Theorem 3.7.7] and [Conway 2000, Corollary 46.5], there is a normal homomorphism $\Phi : K(H)^{**} \to B(H_\pi)$ such that $\Phi|_{K(H)} = \pi \circ \widetilde{\Psi}_0|_{K(H)}$ and $\pi \circ \widetilde{\Psi}_0(K(H))$ is weakly dense in $\Phi(K(H)^{**})$. Since $\pi(B(H^\sim))$ is a von Neumann algebra, $\Phi(K(H)^{**}) \subset \pi(B(H^\sim))$. Since π is injective, we may define $\Psi = \pi^{-1} \circ \Phi$. Recall that π^{-1} is an isomorphism between W^*-algebras $\pi(B(H^\sim))$ and $B(H^\sim)$. It follows that Ψ is weak*-continuous. Then, $\Psi|_{K(H)} = \pi^{-1} \circ \pi \circ \widetilde{\Psi}_0|_{K(H)} = \widetilde{\Psi}_0|_{K(H)}$.

Let $V = B(H^\sim)_*$ be the predual (as Banach spaces). Then Ψ induces a map $\Psi^* : V \to K(H)^*$, the predual of $K(H)^{**}$, by $L(\Psi^*(v)) = \Psi(L)(v)$ for all $L \in (K(H)^*)^*$ and $v \in V$. Thus if $T_\lambda \in K(H)^{**}$ such that $T_\lambda \to T$ in the weak* topology in $K(H)^{**}$, then $\Psi(T_\lambda)(v) = T_\lambda(\Psi^*(v))$ converges to $T(\Psi^*(v)) = \Psi(T)(v)$ for all $v \in V$. In other words, $\Psi(T_\lambda) \to \Psi(T)$ in the weak* topology in $V^* = B(H^\sim)$. By Definition 2.4 (see Remark 3.9 and proof of Theorem 3.10 of [Paschke 1973]), this implies, in particular, for any $f \in A^*$, $x, y \in H^\sim$, that $f(\langle \Psi(T_\lambda)(x), y\rangle) \to f(\langle \Psi(T)(x), y\rangle)$.

By Theorem 1.5 of [Lin 1991a], $B(H) = LM(K(H))$. Let $\{E_\lambda\}$ be an approximate identity for $K(H)$. Then $TE_\lambda \in K(H)$ for all $T \in B(H)$. It follows from Proposition 2.13 that, for $T \in B(H)$,

$$\lim_\lambda \|\Psi(TE_\lambda)(f)(x) - \Psi(T)(f)(x)\| = \lim_\lambda \|\Psi(T)\Psi(E_\lambda)(f)(x) - \Psi(T)(f)(x)\| = 0$$

for all $x \in H \bullet A^{**}$ and $f \in (H \bullet A^{**})^\sharp$. On the other hand, by Lemma 2.9,

$$\lim_\lambda \|\Psi_0(TE_\lambda)(x) - \Psi_0(T)(x)\| = 0 \quad \text{for all } x \in H \bullet A^{**}.$$

However, we have shown that $\Psi(TE_\lambda)(y) = \widetilde{\Psi}_0(TE_\lambda)(y) = \Psi_0(TE_\lambda)(y)$ for all $y \in H \bullet A^{**}$ (see also Definition 2.12). Therefore, combining these three facts, for $x, y \in H \bullet A^{**}$, we obtain

$$\langle \Psi(T)(x), y\rangle = \langle \Psi_0(T)(x), y\rangle.$$

It follows that $\Psi(T)|_{H\bullet A^{**}} = \Psi_0(T)$. Since the extension of $\Psi_0(T)$ to a bounded module map on $(H\bullet A^{**})^\sharp$ is unique (see the end of Section 2.10 and [Lin 1992, Lemma 3.5]), we have $\Psi(T) = \widetilde{\Psi}_0(T)$ for all $T \in B(H)$. Hence

$$\Psi|_{B(H)} = \widetilde{\Psi}_0. \qquad \square$$

Definition 2.15. Let M be a W^*-algebra and H be a Hilbert M-module. Then, $H^\sharp$ is a self-dual Hilbert M-module by [Paschke 1973, Theorem 3.2]. Let $F_0 : F(H) \to F(H^\sharp)$ be the homomorphism defined by

$$F_0(\theta_{x,y})(z) = x\langle y, z\rangle \quad \text{for all } z \in H^\sharp \text{ and } x, y \in H.$$

Clearly F_0 is an isometry. It extends uniquely to a homomorphism $F_0 : K(H) \to K(H^\sharp)$. We further extend $F : \widetilde{K(H)} \to \widetilde{K(H^\sharp)}$ by $F(\mathrm{id}_H) = \mathrm{id}_{H^\sharp}$.

Proposition 2.16. *Let M be a W^*-algebra and H a Hilbert M-module. Then there exists a unital normal homomorphism $F : K(H)^{**} \to B(H^\sharp)$ such that $F|_{K(H)} = F_0$ and, if $T_\lambda \to T$ in the weak* topology of $K(H)^{**}$, then*

$$\lim_\lambda f(\langle F(T_\lambda)(x), y\rangle) = f(\langle F(T)(x), y\rangle)$$

for all $x, y \in H^\sharp$ and $f \in M_*$, the predual of M. Moreover, $F(T) = \widetilde{T}$ for all $T \in B(H)$ as defined by (2-3).

Proof. Recall that $B(H^\sharp)$ is a W^*-algebra. We may assume that $B(H^\sharp)$ acts on a Hilbert space X as a von Neumann algebra with $1_{B(H^\sharp)} = \mathrm{id}_X$. Then, by [Lin 2001, Theorem 1.8.2] (see also [Pedersen 1979, Theorem 3.7.7]), there is a unital normal homomorphism $F : K(H)^{**} \to \overline{F_0(K(H))}^{\mathrm{SOT}} \subset B(H^\sharp)$ such that $F|_{K(H)} = F_0$. So F is weak*-continuous (see, for example, [Conway 2000, Corollary 46.5]).

Suppose that $T_\lambda \to T$ in the weak* topology of $K(H)^{**}$. Then $F(T_\lambda) \to F(T)$ in the weak* topology of $B(H^\sharp)$. Therefore (see the later part of Definition 2.4, also, Remark 3.9 and the proof of Proposition 3.9 of [Paschke 1973]),

$$f(\langle F(T_\lambda)(x), y\rangle) \to f(\langle F(T)(x), y\rangle) \quad \text{for all } x, y \in H^\sharp \text{ and } f \in M_*.$$

Let $\{E_\lambda\}$ be an approximate identity for $K(H)$. Then, for any $T \in B(H)$, by Lemma 3.1 of [Brown and Lin 2025],

$$\lim_\lambda \|F(T)F(E_\lambda)(x) - F(T)(x)\| = \lim_\lambda \|F(T)E_\lambda(x) - F(T)(x)\| = 0 \quad \text{for all } x \in H.$$

On the other hand, since $F(T)F(E_\lambda)|_H = F(TE_\lambda)|_H = TE_\lambda$ and (by [Brown and Lin 2025, Lemma 3.1])

$$\lim_\lambda \|TE_\lambda(x) - T(x)\| = 0,$$

we conclude that

$$T(x) = F(T)(x) \quad \text{for all } x \in H.$$

Since the extension of T to $H^\sharp$ is unique (by Proposition 3.6 of [Paschke 1973], see also Lemma 3.5 of [Lin 1992]), $\widetilde{T} = F(T)$. □

3. Isomorphism of $B(H^\sim)$ and $K(H)^{**}$

Let A be a monotone complete C^*-algebra and H be a Hilbert A-module. Then, by Lemma 3.7 of [Lin 1992], $H^\sharp$ becomes a self-dual Hilbert A-module such that $\langle \tau, x\rangle = \tau(x)$ for all $x \in H$ and $\tau \in H^\sharp$. Note that, if E is self-dual, we conjugate map $E^\sharp$ onto E just as in the case of Hilbert spaces.

We will apply the following lemma several times.

Proposition 3.1. *Let A be a monotone complete C^*-algebra and $H_1 \subset H_2$ be Hilbert A-modules such that H_2 is self-dual. Then $H_1^\sharp$ is an orthogonal summand of $H_2^\sharp$ and the embedding $H_1^\sharp \to H_2^\sharp$ extends the embedding $H_1 \subset H_2$.*

Proof. Define $P_0 : H_2 \to H_1^\sharp$ by

$$P_0(y)(x) = \langle y, x\rangle \quad \text{for all } y \in H_2 \text{ and } x \in H_1. \tag{3-1}$$

It is a bounded module map (by viewing $H_1^\sharp$ as a Hilbert module instead of the dual to avoid the conjugation) with $\|P_0\| = 1$. Note that $P_0|_{H_1} = \mathrm{id}_{H_1}$.

Let $\tau \in H_1^\sharp$. Since A is monotone complete, by Theorem 3.8 of [Lin 1992], there is $\tilde\tau \in H_2^\sharp = H_2$ such that $\tilde\tau|_{H_1} = \tau$ and $\|\tilde\tau\| = \|\tau\|$. This implies that P_0 is surjective.

Define $j : H_1^\sharp \to H_2^\sharp = H_2$ by

$$j(x)(y) = \langle x, P_0(y)\rangle \quad \text{for all } x \in H_1^\sharp \text{ and } y \in H_2. \tag{3-2}$$

Then j extends the embedding $H_1 \hookrightarrow H_2$. Now, for $x \in H_1^\sharp$ and $y \in H_2$, by (3-1) and (3-2),

$$P_0 \circ j(x)(y) = P_0(j(x))(y) = \langle j(x), y\rangle = \langle x, P_0(y)\rangle = \langle P_0(y), x\rangle^* = (P_0(y)(x))^* = \langle y, x\rangle^* = \langle x, y\rangle.$$

It follows that $P_0 \circ j = \mathrm{id}|_{H_1^\sharp}$, and thus $j : H_1^\sharp \to H_2$ is an embedding. With the identification of $H_1^\sharp$ and $j(H_1^\sharp)$, $P_0|_{H_1^\sharp} = \mathrm{id}|_{H_1^\sharp}$. It follows that P_0 is a projection and $H_1^\sharp$ is an orthogonal summand of H_2. □

Applying Propositions 3.1 and 2.6, we obtain the following characterization of $H^\sim$.

Proposition 3.2. *Let A be a C^*-algebra and H be a Hilbert A-module. Then $H^\sim$ is the smallest self-dual Hilbert A^{**}-module containing H as a Hilbert A-submodule.*

Proof. Let H_1 be a self-dual Hilbert A^{**}-module containing H as a Hilbert A-submodule. Then, by Proposition 2.6,

$$H \subset H \bullet A^{**} \subset H_1.$$

Applying Proposition 3.1, since H_1 is self-dual,

$$H^\sim = (H \bullet A^{**})^\sharp \subset H_1^\sharp = H_1.$$

The proposition follows. □

3.3. In the next proposition, let A be a C^*-algebra, and let $H_1 \subset H$ be Hilbert A-modules. Then, by Proposition 2.6, $H_1 \bullet A^{**} \subset H \bullet A^{**}$. Since A^{**} is monotone complete and $(H \bullet A^{**})^\sharp = H^\sim$ and $(H_1 \bullet A^{**})^\sharp = H_1^\sim$, by Proposition 3.1, we may write $H^\sim = H_1^\sim \oplus (H_1^\sim)^\perp$. Denote by $P : H^\sim \to H_1^\sim$ the projection. Note that $P \in L(H^\sim)$. By Lemma 3.2 of [Lin 1992], $K(H_1)$ is a hereditary C^*-subalgebra of $K(H)$. Let $\Psi_H : K(H)^{**} \to B(H^\sim)$ and $\Psi_1 : K(H_1)^{**} \to B(H_1^\sim)$ be the homomorphisms given by Proposition 2.14, respectively.

Proposition 3.4. *Using the notation above, we have that*

$$\Psi_1 = \Psi_H|_{K(H_1)^{**}} = P\Psi_H P|_{K(H_1)^{**}};$$

in particular, $\Psi_1(T) = \Psi_H(T)|_{H_1^\sim} = P\Psi_H(T)P|_{H_1^\sim}$ *for* $T \in K(H_1)^{**}$. *Moreover,*

$$P\Psi_H(L)P|_{K(H_1)^{**}} \subset \Psi_1(K(H_1)^{**}) \quad \textit{for all } L \in K(H)^{**}.$$

Furthermore, $\Psi(Q) = P$, *where* Q *is the open projection in* $K(H)^{**}$ *corresponding to the hereditary* C^**-subalgebra* $K(H_1)$.

Proof. Denote by $\Psi_{K(H),0}$ the injective homomorphism from $K(H)$ into $K(H \bullet A^{**})$ and by $\Psi_{K(H_1),0}$ the injective homomorphism from $K(H_1)$ into $K(H_1 \bullet A^{**})$ described in Definition 2.7, respectively.

Fix $S \in K(H_1)$. For each $x \in H_1$ and $b \in A^{**}$, by Proposition 2.8,

$$\Psi_{K(H),0}(S)(x \bullet b) = S(x) \bullet b,$$
$$P\Psi_{K(H),0}(S)P(x \bullet b) = P(S(x \bullet b)) = S(x) \bullet b = \Psi_{K(H_1),0}(S)(x \bullet b).$$

It follows that

$$\Psi_{K(H),0}(S)|_{H_1 \bullet A^{**}} = P\Psi_{K(H),0}(S)P|_{H_1 \bullet A^{**}} = \Psi_{K(H_1),0}(S).$$

Since the extensions of $\Psi_{K(H),0}(S)|_{H_1 \bullet A^{**}}$ and $\Psi_{K(H_1),0}(S)$ to bounded module maps on $H_1^\sim$ are unique, and $\Psi(S)|_{H_1^\sim}$ and $\Psi(S)$ are corresponding extensions, by Corollary 3.7 of [Paschke 1973], we conclude that $\Psi(S)|_{H_1^\sim} = P\Psi_H(S)P|_{H_1^\sim} = \Psi_1(S)$.

Let $T \in K(H_1)^{**}$ and $\{T_\lambda\} \subset K(H_1)$ be a net such that $T_\lambda \to T$ in the weak* topology. By Proposition 2.14, for any $g \in A^*$,

$$\lim_\lambda |g(\langle \Psi_H(T_\lambda)(x), y\rangle) - g(\langle \Psi_H(T)(x), y\rangle)| = 0, \tag{3-3}$$
$$\lim_\lambda |g(\langle \Psi_1(T_\lambda)(x), y\rangle) - g(\langle \Psi_1(T)(x), y\rangle)| = 0 \tag{3-4}$$

for all $x, y \in H_1^\sim$. Since we have shown that $\Psi_H(T_\lambda)|_{H_1^\sim} = P\Psi_H(T_\lambda)P|_{H_1^\sim} = \Psi_1(T_\lambda)$, we conclude that

$$\Psi_H(T)|_{H_1^\sim} = P\Psi_H(T)P|_{H_1^\sim} = \Psi_1(T). \tag{3-5}$$

Hence

$$\Psi_1 = P\Psi_H P|_{K(H_1)^{**}} = \Psi_H|_{K(H_1)^{**}}.$$

Let $\{q_\lambda\}$ be an approximate identity for $K(H_1)$. Then $q_\lambda \nearrow \mathrm{id}_{H_1} \in K(H_1)^{**}$. It follows from Proposition 2.14 that

$$\lim_\lambda f(\langle \Psi_1(q_\lambda(y)), z\rangle) = f(\langle y, z\rangle) \quad \text{for all } y, z \in H_1^\sim \text{ and } f \in A^*.$$

On the other hand, we also have that $q_\lambda \nearrow Q$ in $K(H)^{**}$. By Proposition 3.1, $H^\sim = H_1^\sim \oplus (H_1^\sim)^\perp$. Note that $q_\lambda(x) \in H_1$ for all $x \in H$. Then, for $x \in H$, $b \in A^{**}$ and $g \in (H_1^\sim)^\perp$, by Proposition 2.8,

$$\langle \Psi_H(q_\lambda)(x \bullet b), g\rangle = g(q_\lambda(x \bullet b))^* = g(q_\lambda(x) \bullet b)^* = 0.$$

It follows that, for any $y \in H \bullet A^{**}$ and $g \in (H_1^\sim)^\perp$,

$$\langle \Psi_H(q_\lambda)(y), g\rangle = 0.$$

Hence, for $g \in (H_1^\sim)^\perp$,

$$\langle y, \Psi_H(q_\lambda)(g)\rangle = 0 \quad \text{for all } y \in H \bullet A^{**}.$$

It follows that $\Psi_H(q_\lambda)(g) = 0$ and

$$\langle \Psi_H(q_\lambda)(z), g\rangle = \langle z, \Psi_H(q_\lambda)(g)\rangle = 0 \quad \text{for all } z \in H^\sim.$$

In other words, $\Psi_H(q_\lambda)(z) \in H_1^\sim$ for all $z \in H$ and λ. Therefore

$$P\Psi_H(q_\lambda) = \Psi_H(q_\lambda) = \Psi_H(q_\lambda)P.$$

Note that $Pz \in H_1^\sim$ for any $z \in H^\sim$. Thus, by (3-5) and (3-4),

$$\begin{aligned}\lim_\lambda f(\langle\Psi(q_\lambda)(y), z\rangle) &= \lim_\lambda f(\langle\Psi(q_\lambda)(P(y)), P(z)\rangle) = \lim_\lambda f(\langle\Psi_1(q_\lambda)(P(y)), P(z)\rangle)\\ &= f(\langle P(y), P(z)\rangle) = f(\langle P(y), z\rangle).\end{aligned}$$

By (3-3) and (3-5), $\lim_\lambda f(\langle\Psi(q_\lambda)(y), z\rangle) = f(\langle\Psi(Q)(y), z\rangle)$. Therefore

$$\Psi(Q) = P.$$

This proves the "furthermore" part. In what follows we will identify Q with P as well as $\Psi(Q)$ and $\Psi(P)$.

Now let $L \in K(H)^{**}$ and $\{L_\lambda\} \subset K(H)$ be a net such that $L_\lambda \to L$ in the weak* topology. By Proposition 2.14, for any $g \in A^*$, $x, y \in H_1^\sim$,

$$\begin{aligned}\lim_\lambda |g(\langle\Psi_H(T_\lambda)(x), y\rangle) - g(\langle\Psi_H(T)(x), y\rangle)| &= 0,\\ \lim_\lambda |g(\langle\Psi_1(T_\lambda)(x), y\rangle) - g(\langle\Psi_1(T)(x), y\rangle)| &= 0\end{aligned}$$

(note that $\Psi_1(T_\lambda) = P\Psi_1(T_\lambda)P$). We also have, for any $x, y \in H_1^\sim$,

$$\begin{aligned}\langle\Psi_H(PT_\lambda P)(x), y\rangle &= \langle\Psi_H(T_\lambda)(x), y\rangle,\\ \langle P\Psi_H(T)P(x), y\rangle &= \langle\Psi_H(T)(x), y\rangle.\end{aligned}$$

Since $PT_\lambda P \in K(H_1)^{**}$, by the first part of the lemma, $\Psi_H(PT_\lambda P)(x) = \Psi_1(PT_\lambda P)(x)$ for $x \in H_1^\sim$. It follows that $P\Psi_H(T)P(x) = \Psi_1(PTP)(x)$ for all $x \in H_1^\sim$. Then

$$P\Psi_H(T)P = \Psi_1(PTP) \in \Psi_1(K(H_1)^{**}). \qquad \square$$

3.5. Let A be a C^*-algebra and let, for $n \in \mathbb{N}$,

$$H_n = A^{(n)} = \{(a_1, a_2, \ldots, a_n) : a_j \in A,\ 1 \le j \le n\},$$

the direct sum of n copies of A, where $\langle a, b\rangle = \sum_{j=1}^n a_j^* b_j$ if $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$. Let

$$H_A = \left\{\{a_n\} : a_n \in A \text{ and } \sum_{i=1}^n a_k^* a_k \text{ converges in norm}\right\}$$

be the standard countably generated Hilbert (right) A-module. Note that

$$\langle \{a_n\}, \{b_n\}\rangle = \sum_{n=1}^{\infty} a_n^* b_n.$$

We note that H_A is the closure of $\bigcup_n A^{(n)}$. We may also view $H_n = A^{(n)}$ as an orthogonal summand of H_A. Then

$$H_A^\sharp = \left\{\{a_n\} : \left\{\left\|\sum_{k=1}^{n} a_k^* a_k\right\|\right\} \text{ is bounded}\right\}.$$

If $g = \{a_n\} \in H^\sharp$, then

$$g(x) = \sum_{n=1}^{\infty} a_n^* b_n \quad \text{for all } x = \{b_n\} \in H_A,$$

where the sum converges in norm. Moreover $\|g\| = \lim_{n\to\infty}\left\|\sum_{k=1}^{n} a_k^* a_k\right\|$.

If A is a W^*-algebra, as mentioned earlier, $H_A^\sharp$ becomes a Hilbert A-module in a natural way (see Theorem 3.2 of [Paschke 1973]). In fact, we may define

$$\langle x, y\rangle = \sum_{n=1}^{\infty} a_n^* b_n \quad \text{for all } x = \{a_n\},\ y = \{b_n\} \in H_A^\sharp. \tag{3-6}$$

To see the right side converges in the weak* topology, we first let $f \in A^*$. Note that, if $\{a_n\} \in H_A^\sharp$,

$$\left|\sum_{k=1}^{N} f(a_k^* a_k)\right| = \left|f\left(\sum_{k=1}^{N} a_k^* a_k\right)\right| \le \|f\|\left\|\sum_{k=1}^{N} a_k^* a_k\right\|$$

for any integer N. Hence $\left\{\sum_{k=1}^{n} f(a_k^* a_k)\right\}$ is bounded, is increasing and converges for any positive linear functional f. Hence, for any $m > n$,

$$\sum_{k=n}^{m} f(a_k^* a_k) \to 0 \quad \text{as } n \to \infty \quad \text{for all } f \in A^*. \tag{3-7}$$

For any positive linear functional f of A and for any $m > n$ in $\mathbb{N}$,

$$\begin{aligned}\left|f\left(\sum_{k=n}^{m} a_k^* b_k\right)\right| &= \left|\sum_{k=n}^{m} f(a_k^* b_k)\right| \le \sum_{k=n}^{m} |f(a_k^* b_k)| \\ &\le \sum_{k=n}^{m} |f(a_k^* a_k)|^{1/2} |f(b_k^* b_k)|^{1/2} \\ &\le \left(\left(\sum_{k=n}^{m} |f(a_k^* a_k)|\right)\left(\sum_{k=n}^{m} |f(b_k^* b_k)|\right)\right)^{1/2} \\ &\le \|f\|^{1/2} \|\{b_k\}\| \left(\sum_{k=n}^{m} |f(a_k^* a_k)|\right)^{1/2} \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

It follows that $f\left(\sum_{k=1}^{n} a_k^* b_k\right)$ converges for all $f \in A^*$ as $n \to \infty$. Let us write the limit as $f\left(\sum_{k=1}^{\infty} a_k^* b_k\right)$. Then, by the above inequalities (with $n = 1$), we also have

$$\left| f\left(\sum_{k=1}^{\infty} a_k^* b_k\right) \right| \le \|f\| M_b M_a,$$

where

$$M_a = \sup\left\{ \left\| \sum_{k=1}^{n} a_k^* a_k \right\| \right\}^{1/2} \quad \text{and} \quad M_b = \sup\left\{ \left\| \sum_{k=1}^{n} b_k^* b_k \right\| \right\}^{1/2}.$$

Thus $\sum_{k=1}^{\infty} a_k^* b_k$ defines a bounded linear functional on A^*. Its restriction on A_* gives an element in A (recall that A is assumed to be a W^*-algebra). This shows the infinite series in the right side of (3-6) converges in the weak* topology. It is then standard to verify that (3-6) defines an inner product which extends the inner product on H_A.

Let A act on a Hilbert space X (as a W^*-algebra). Consider $l^2(X)$, the Hilbert space direct sum of countably many copies of X. Suppose that $b = \{b_n\} \in H_A^\sharp$. Then the infinite matrix $\bar{b} = (b_{i,j})$, with $b_{i,1} = b_i$, $i \in \mathbb{N}$ and $b_{i,j} = 0$ if $j \ge 2$, defines a bounded linear operator on $l^2(X)$, by $\bar{b}(v) = (b_1(v_1), b_2(v_1), \ldots, b_n(v_1), \ldots)$, where $v = (v_1, v_2, \ldots, v_n, \ldots) \in l^2(X)$. Moreover

$$\|\bar{b}\|^2 = \|\bar{b}^* \bar{b}\| = \left\| \sum_{i=1}^{\infty} b_i^* b_i \right\| = \sup\left\{ \left\| \sum_{i=1}^{n} b_i^* b_i \right\| : n \in \mathbb{N} \right\} \tag{3-8}$$

(some of these details in this subsection may be found in [Lin 1991b]).

Proposition 3.6. *Let C be a unital C^*-algebra and $A \subset C$ be a C^*-subalgebra such that $1_{\tilde{A}} = 1_C$. Denote by $R = \overline{AC}$ the closed right ideal of C generated by A. Then*:

(1) *$H_A \bullet C = \{\{b_n\} \in H_C : b_n \in R\}$.*

(2) *If C is a W^*-algebra and $e_\alpha \nearrow 1_C$, where $\{e_\alpha\}$ is an approximate identity for A, then*

$$(H_A \bullet C)^\sharp = H_C^\sharp.$$

Proof. To see (1), we first note that $A \bullet C = R$ as Hilbert C-modules. Hence $A^{(n)} \bullet C = R^{(n)}$. Clearly, $H_A \bullet C \subset H_C$. We note that $\{\{r_n\} \in H_C : r_n \in R\}$ is closed in H_C. Since both $\bigcup_n A^{(n)} \bullet C$ and $\bigcup_{n=1}^{\infty} R^{(n)}$ are dense in $\{\{r_n\} \in H_C : r_n \in R\}$, and $\bigcup_n A^{(n)} \bullet C$ is dense $H_A \bullet C$, we obtain

$$\{\{r_n\} \in H_C : r_n \in R\} = H_A \bullet C.$$

This proves (1).

For (2) we may assume that $A \subset C \subset B(X)$, where X is a Hilbert space, $1_C = \mathrm{id}_X$, and the range $C(X)$ equals X. Otherwise, we replace X by $1_C(X)$.

Claim 1: $\overline{R(X)} = C(X) = X$. Since $e_\alpha \nearrow 1_C = \mathrm{id}_X$, for any $v \in X$, $e_\alpha(v) \to v$. This proves the claim.

Claim 2: $R^\sharp = C$, where $R^\sharp$ is the dual of the Hilbert C-module R (as we assume that C is a W^*-algebra).

Let $f \in R^\sharp$. Then $f(e_\alpha) r = f(e_\alpha r) \to f(r)$ for all $r \in R$ in norm as $e_\alpha r \to r$ in norm. Hence $f(e_\alpha) r(v) \to f(r)(v)$ for all $r \in R$ and $v \in X$. Define T on $R(X)$ by $T(r(v)) = \lim_\alpha f(e_\alpha) r(v)$ for all $v \in X$ and $r \in R$. One checks that T is a well-defined linear map on $R(X)$. Moreover, we have

$\|T\| \leq \sup\{\|f(e_\alpha)\| : \alpha\} \leq \|f\|$. Since, by Claim 1, $\overline{R(X)} = X$, we have that T extends uniquely to a bounded linear operator (denote by T again) on X. Moreover, $f(e_\alpha)$ converges to T on X. Since C is closed in the weak operator topology, $T \in C$. Moreover, $Tr(v) = f(r)(v)$ for all $v \in X$. It follows that $Tr = f(r)$ for all $r \in R$.

For each $c \in C$, define $f_c \in R^\sharp$ by

$$f_c(r) = c^* r \quad \text{for all } r \in R.$$

For the above T, we note that $f_{T^*}(r) = Tr$ for all $r \in R$. Hence the map $c \to f_c$ is surjective. To see it is injective, suppose that $c^* r = 0$ for all $r \in R$. Then

$$c^* e_\alpha c = 0 \quad \text{for all } \alpha.$$

Since $c^* e_\alpha c \nearrow c^* c$, this implies that $c^* c = 0$. Thus the map $c \mapsto f_c$ is injective, which extends the identity map on R. It follows that $R^\sharp = C$, and Claim 2 is proved.

By Claim 2, we obtain that $((A^{(n)}) \bullet C)^\sharp = C^{(n)}$. By (1), $(A^{(n)}) \bullet C$ is a direct summand of $H_A \bullet C$. Hence we may write $((A^{(n)}) \bullet C)^\sharp \subset (H_A \bullet C)^\sharp$. Together with (1), we obtain that

$$H_A \bullet C \subset H_C \subset (H_A \bullet C)^\sharp.$$

Note H_C is a Hilbert C-submodule of the self-dual Hilbert C module $(H_A \bullet C)^\sharp$. It follows from Proposition 3.1 that

$$(H_A \bullet C)^\sharp \subset H_C^\sharp \subset (H_A \bullet C)^\sharp.$$

Consequently, $H_C^\sharp = (H_A \bullet C)^\sharp$. □

3.7. Note that, if A is unital, $H_A \bullet C = H_C$.

From the above discussion, we obtain the following result.

Lemma 3.8. *Let A be a C^*-algebra, $H_n = (A^{**})^{(n)}$ and $P_n : H_{A^{**}}^\sharp \to H_n$ be the projection.*

(1) *Let $S \subset H_{A^{**}}^\sharp$ be a bounded subset. Then, for any $f \in A^*$ and $x \in H_{A^{**}}^\sharp$,*

$$\lim_{n\to\infty} \sup\{|f(\langle P_n(x), y\rangle) - f(\langle x, y\rangle)| : y \in S\} = 0,$$
$$\lim_{n\to\infty} \sup\{|f(\langle y, P_n(x)\rangle) - f(\langle y, x\rangle)| : y \in S\} = 0.$$

(2) *Moreover,*

$$\lim_{n\to\infty} |f(\langle P_n(x), P_n(x)\rangle) - f(\langle x, x\rangle)| = 0 \quad \textit{for all } x \in H_{A^{**}}^\sharp \textit{ and } f \in A^*.$$

Proof. Set $M = \sup\{\|y\| : y \in S\} + 1$. Let f be a positive linear functional in A^* and $x = \{a_n\} \in H_{A^{**}}^\sharp$. For each $y = \{b_n\} \in S$,

$$|f(\langle P_n(x), y\rangle) - f(\langle x, y\rangle)| = \left|\sum_{k=n+1}^\infty f(a_k^* b_k)\right| \leq \left(\sum_{k=n+1}^\infty f(a_k^* a_k)\right)^{1/2} \left(\sum_{k=n+1}^\infty f(b_k^* b_k)\right)^{1/2}$$
$$\leq \|f\|\|y\| \left(\sum_{k=n+1}^\infty f(a_k^* a_k)\right)^{1/2} \leq M\|f\| \left(\sum_{k=n+1}^\infty f(a_k^* a_k)\right)^{1/2}.$$

By what has been discussed in Section 3.5,

$$\lim_{n\to\infty}\left(\sum_{k=n+1}^{\infty} f(a_k^* a_n)\right)^{1/2} = 0.$$

Thus, for this f and x, we have that $|f(\langle P_n(x), y\rangle) - f(\langle x, y\rangle)|$ converges uniformly on S. Almost identical estimates show that $|f(\langle y, P_n(x)\rangle) - f(\langle y, x\rangle)|$ converges uniformly on S.

Since any $f \in A^*$ can be written as a linear combination of four positive linear functionals in A^*, the first part of the statement holds.

For the second part, we note that, for any $f \in A^*$ and $x \in H^\sharp_{A^{**}}$, by the first part of the lemma (since $\|P_n(x)\| \le \|x\|$),

$$\lim_{n\to\infty} |f(\langle P_n(x), P_n(x)\rangle) - f(\langle x, P_n(x)\rangle)| = 0.$$

We also have

$$\lim_{n\to\infty} |f(\langle P_n(x), x\rangle) - f(\langle x, x\rangle)| = 0.$$

Hence the second part of the lemma also follows. □

The following are two easy facts which we present here for convenience.

Lemma 3.9. *Let A be a C^*-algebra.*

(1) *Let H be a Hilbert A-module and $\{E_\lambda\}$ be an approximate identity for $K(H)$. Suppose $T \in K(H)^{**}$ is a nonzero positive element. Then there is λ_0 such that*

$$E_\lambda T E_\lambda \neq 0 \quad \textit{for all } \lambda \ge \lambda_0.$$

(2) *Let $T \in K(H_A)^{**}$ be a nonzero positive element and $P_n : H_A \to H_n = A^{(n)}$ be the projection ($n \in \mathbb{N}$). Then, there exists $n_0 \in \mathbb{N}$ such that*

$$P_n T P_n \neq 0 \quad \textit{for all } n \ge n_0.$$

Proof. Let $f \in K(H)^*$ be a positive linear functional. Then

$$|f(T^{1/2}(1-E_\lambda))|^2 \le f(T) f((1-E_\lambda)^2) \le f(T) f(1-E_\lambda) \to 0.$$

It follows that $f(T^{1/2}E_\lambda) \to f(T^{1/2})$ for all positive linear functionals in $K(H)^*$, whence for all $f \in K(H)^*$. Since $T^{1/2} \neq 0$ for some λ_0, we have that $T^{1/2}E_\lambda \neq 0$ for all $\lambda \ge \lambda_0$. It follows that

$$E_\lambda T E_\lambda \neq 0$$

for all $\lambda \ge \lambda_0$. This proves (1).

There are several easy proofs for (2). Let us use part (1). Choose an approximate identity $\{e_\alpha\}$ for A. Let $\lambda = (\alpha, n)$ and $\lambda_1 = (\beta_1, n) \le \lambda_2 = (\beta_2, m)$ if $\beta_1 \le \beta_2$ and $n \le m$. Define

$$E_{\beta,n} = \operatorname{diag}(\overbrace{e_\beta, e_\beta, \ldots, e_\beta}^{n}, 0, \ldots).$$

Then $\{E_{\beta,n}\}$ forms an approximate identity for $K(H_A) \cong A \otimes \mathcal{K}$. Let $T \in K(H_A)^{**}_+$ be a nonzero positive element. By (1), there is β_0 and $n_0 \in \mathbb{N}$ such that

$$E_{\beta,n} T E_{\beta,n} \neq 0 \quad \text{for all } (\beta, n) \geq (\beta_0, n_0).$$

Hence $\|T^{1/2} E^2_{\beta,n} T^{1/2}\| = \|E_{\beta,n} T E_{\beta,n}\| \neq 0$ for all $(\beta, n) \geq (\beta_0, n_0)$. Since

$$T^{1/2} P_n T^{1/2} \geq T^{1/2} E^2_{\beta,n} T^{1/2} \neq 0,$$

we have $T^{1/2} P_n T^{1/2} \neq 0$. It follows that

$$P_n T P_n \neq 0 \quad \text{for all } n \geq n_0. \qquad \square$$

Lemma 3.10. *Let A be a C^*-algebra and H be a countably generated Hilbert A-module. Then the homomorphism Ψ from $K(H)^{**}$ into $B(H^\sim)$ (given by Proposition 2.16) is injective.*

Proof. Let $H_n = A^{(n)} = \{(a_1, a_2, \ldots, a_n) : a_j \in A\}$ be the Hilbert A-module whose inner product is defined by $\langle x, y\rangle = \sum_{j=1}^n a_j^* b_j$, where $x = \{a_j\}_{1\leq j\leq n}$ and $y = \{b_j\}_{1\leq j\leq n}$. One identifies $K(H_n)$ with $M_n(A)$.

Claim: The map $\Psi : K(H_n)^{**} \to B(H_n^\sim)$ is a W^*-isomorphism.

Since we identify $K(H_n)$ with $M_n(A)$, we have $K(H_n)^{**} = M_n(A^{**})$.

By Proposition 3.6 (2) (and Claim 2 of the proof), $(H_n \bullet A^{**})^\sharp = (A^{**})^{(n)}$. So $H_n^\sim = (A^{**})^{(n)}$. Note that $B(H_n^\sim) = M_n(A^{**})$. One then easily checks that $\Psi : K(H_n)^{**} \to B(H_n^\sim)$ is a W^*-isomorphism. This proves the claim.

Let us consider the homomorphism $\Psi_{H_A} : K(H_A)^{**} \to B(H_A^\sim)$ given by Proposition 2.14. Put $T \in K(H_A)^{**}_+ \setminus \{0\}$.

By Lemma 3.9 (2), there exists $n_0 \in \mathbb{N}$ such that $P_n T P_n \neq 0$ for all $n \geq n_0$. Recall that H_n is a direct summand of H_A. Hence by the claim and applying Proposition 3.4, we conclude that $\Psi_{H_A}(P_n T P_n) \neq 0$ for all $n \geq n_0$. There must be an element $x \in H_n$ such that

$$\langle \Psi_{H_A}(P_n T P_n)(x), x\rangle \neq 0.$$

It follows that $\langle \Psi_{H_A}(T)x, x\rangle \neq 0$. Hence $\Psi_{H_A}(T) \neq 0$. This implies that $\ker \Psi_{H_A} = \{0\}$.

In general, since H is countably generated, by Kasparov's absorbing theorem [1980, Theorem 2], we may write $H_A = H \oplus H^\perp$. To show Ψ is injective, let $T \in B(H)^{**}$ be a nonzero element. Then $K(H)^{**} = P K(H_A)^{**} P$, where $P : H_A \to H$ is the projection. Hence $PTP = T$ in $K(H_A)^{**}$. We have shown that $\Psi_{H_A}(PTP) \neq 0$. By Proposition 3.4, we have that $\Psi(T) = P\Psi_{H_A}(T)P|_{H^\sim} \neq 0$. This implies that Ψ is injective. $\square$

Lemma 3.11. *Let A be a C^*-algebra and H be a countably generated Hilbert A-module. Then there is an isomorphism Ψ from $K(H)^{**}$ onto $B(H^\sim)$ as W^*-algebras.*

Proof. By Lemma 3.10 (and by Proposition 2.14), it suffices to show that Ψ is surjective. Let us first consider the case $H = H_A$ (even though H_A is not countably generated when A is not σ-unital). By the end of Definition 2.4 (see also Remark 3.9 (and Proposition 3.10) of [Paschke 1973]), to show that

$T \in B(H^{\sim}) = B(H^{\sharp}_{A^{**}})$ is in $\Psi(K(H_A)^{**})$, it suffices to show that, for any $\epsilon > 0$, any finite subsets $X \subset H^{\sharp}_{A^{**}}$ and a finite subset $\mathcal{F} \subset A^*$, there exists $S \in K(H)^{**}$ such that

$$|f(\langle \Psi(S)(x), y\rangle) - f(\langle T(x), y\rangle)| < \epsilon \quad \text{for all } x, y \in X \text{ and } f \in \mathcal{F}.$$

We now fix ϵ, X and $\mathcal{F}$.

For any $T \in B(H^{\sim}_A) = B(H^{\sharp}_A)$,

$$\begin{aligned} &|f(\langle P_n T P_n(x), y\rangle - f(\langle T(x), y\rangle)| \\ &\qquad \le |f(T P_n(x), P_n(y)\rangle) - f(\langle T(x), P_n(y)\rangle)| + |f(\langle T(x), P_n(y)\rangle) - f(\langle T(x), y\rangle)| \end{aligned} \tag{3-9}$$

for any $x, y \in H^{\sharp}_{A^{**}}$ and $f \in A^*$. However, $\|P_n(y)\| \le \|y\|$ for all $n \in \mathbb{N}$. By Lemma 3.8 (1),

$$|f(T P_n(x), P_n(y)\rangle) - f(\langle T(x), P_n(y)\rangle)| \to 0,$$

and by Lemma 3.8 (2),

$$|f(\langle T(x), P_n(y)\rangle) - f(\langle T(x), y\rangle)| \to 0.$$

It follows that (by (3-9))

$$\lim_{n\to\infty} |f(\langle P_n T P_n(x), y\rangle - f(\langle T(x), y\rangle)| = 0$$

for all $x, y \in H^{\sharp}_A$ and $f \in A^*$.

We then choose $n_0 \in \mathbb{N}$ such that, for all $n \ge n_0$ (recall P_n is a projection),

$$|f(\langle P_n T P_n(x), P_n(y)\rangle) - f(T(x), y\rangle)| < \epsilon \tag{3-10}$$

for all $x, y \in X$ and $f \in \mathcal{F}$.

Now fix $n \ge n_0$. Then we have $P_n(x), P_n(y) \in (H_n)^{\sim}$ for all $x, y \in X$, and $P_n T P_n \subset B(H^{\sim}_n)$. By the claim for H_n in the proof of Lemma 3.10, we obtain an element $S \in K(H_n)^{**}$ such that $\Psi_n(S) = (P_n T P_n)|_{H^{\sim}_n}$, where

$$\Psi_n : K(H_n)^{**} \cong M_n(A^{**}) \to B(H^{\sim}_n) = M_n(A^{**})$$

is the isomorphism given by the claim. Note, by Proposition 3.4, that $\Psi(S) = P_n \Psi(S) = \Psi(S) P_n = \Psi_n(S)$. Hence it follows that, for all $x, y \in X$ and $f \in \mathcal{F}$ (and $n \ge n_0$), applying (3-10),

$$\begin{aligned} &|f(\langle \Psi(S)(x), y\rangle) - f(\langle T(x), y\rangle)| \\ &\qquad = |f(\langle P_n \Psi(S) P_n(x), y\rangle) - f(T(x), y\rangle)| \\ &\qquad = |(f(\langle P_n \Psi(S) P_n(x), P_n(y)\rangle) - f(\langle P_n T P_n(x), P_n(y))| + |f(\langle P_n T P_n(x), P_n(y)\rangle) - f(T(x), y\rangle)| \\ &\qquad < 0 + \epsilon = \epsilon. \end{aligned}$$

As mentioned above, this implies that Ψ is surjective.

For a general countably generated Hilbert A-module H, by Kasparov's absorbing theorem [1980, Theorem 2], we may write $H_A = H \oplus H^{\perp}$. By Proposition 3.4, $H^{\sim}_A = H^{\sim} \oplus (H^{\perp})^{\sim}$. Let $S \in B(H^{\sim}) \setminus \{0\}$. Define $T \in B(H^{\sim}_A)$ by $T|_{H^{\sim}} = S$ and $S|_{(H^{\perp})^{\sim}} = \{0\}$. We have shown that there is $L \in B(H_A)^{**}$ such that $\Psi_{H_A}(L) = S$. Then $PSP = S$, and, by Proposition 3.4, $\Psi(L) = P \Psi_{H_A}(L) P|_{H^{\sim}} = T$. Hence Ψ is surjective. $\square$

Theorem 3.12. *Let A be a C^*-algebra and H be a Hilbert A-module. Then there is an isomorphism Ψ (given by Proposition 2.14) from $K(H)^{**}$ onto $B(H^\sim)$ as W^*-algebras. Moreover,*

$$\Psi|_{B(H)} = \widetilde{\Psi}_0.$$

Proof. By Proposition 2.14, it suffices to show that Ψ is bijective. If $K(H)$ is unital, by Proposition 2.8 of [Brown and Lin 2025], H is finitely generated. The theorem then follows from Lemma 3.11. So we will assume that $K(H)$ is not unital.

Let $\{E_\lambda\}$ be an approximate identity for $K(H)$ and $H_\lambda = \overline{E_\lambda(H)}$. Then $K(H_\lambda) = \overline{E_\lambda K(H) E_\lambda}$ is σ-unital. By Proposition 3.2 of [Brown and Lin 2025], H_λ is countably generated.

Denote by $P_\lambda : H^\sim \to H_\lambda^\sim$ the projection given by Proposition 3.1 and let $\Psi_\lambda : K(H_\lambda)^{**} \to B(H_\lambda^\sim)$ be the map given by Proposition 2.14.

To see Ψ is injective, let $T \in K(H)^{**}_+ \setminus \{0\}$. It follows from Lemma 3.9 that $E_\lambda T E_\lambda \neq 0$ for all $\lambda \geq \lambda_0$ and some λ_0. Since H_λ is countably generated, by Lemma 3.11, $\Psi_\lambda(E_\lambda T E_\lambda) \neq 0$ (for $\lambda \geq \lambda_0$). By Proposition 3.4,

$$\Psi(E_\lambda T E_\lambda)|_{H_\lambda^\sim} = \Psi_\lambda(E_\lambda T E_\lambda).$$

It follows that $\Psi(E_\lambda T E_\lambda)|_{H_\lambda^\sim} \neq 0$ for all $\lambda \geq \lambda_0$. For $\lambda \geq \lambda_0$, there are $x, y \in H_\lambda$ such that

$$\langle \Psi(T)(E_\lambda(x)), E_\lambda(y)\rangle = \langle \Psi(E_\lambda T E_\lambda)(x), y\rangle \neq 0.$$

Hence $\Psi(T) \neq 0$. This shows that Ψ is injective.

To see that Ψ is surjective, let $L \in B(H^\sim)$. Since, by Proposition 2.14, $\Psi(K(H)^{**})$ is weak*-closed in the W^*-algebra $B(H^\sim)$, it suffices to show the following: for any $\epsilon > 0$, any finite subsets $X, Y \subset H^\sim$ and finite subset $\mathcal{F} \subset A^*$, there exists $T \in K(H)^{**}$ such that

$$|f(\langle \Psi(T)(x), y\rangle) - f(\langle L(x), y\rangle)| < \epsilon \quad \text{for all } x \in X, \quad y \in Y, \quad f \in \mathcal{F} \tag{3-11}$$

(see the last part of Definition 2.4). We now fix ϵ, X, Y and $\mathcal{F}$. By Proposition 2.14 (since $E_\lambda \nearrow 1_{K(H)^{**}}$),

$$\lim_\lambda f(\langle x, \Psi(E_\lambda)(y)\rangle) = \lim_\lambda f(\langle \Psi(E_\lambda)(x), y\rangle) = f(\langle x, y\rangle)$$

for all $x, y \in H^\sim$ and $f \in A^*$. It follows that there is λ_0 such that, for all $\lambda \geq \lambda_0$,

$$\begin{aligned} & |f(\langle \Psi(E_\lambda)(x), L^*(y)\rangle) - f(\langle x, L^*(y)\rangle)| < \tfrac{1}{2}\epsilon \\ \text{or} \quad & \\ & |f(\langle L\Psi(E_\lambda)(x), y\rangle) - f(\langle L(x), y\rangle)| < \tfrac{1}{2}\epsilon \end{aligned}$$

for all $x \in X$, $y \in Y$ and $f \in \mathcal{F}$. We note that the proof would be shorter if we knew

$$\lim_\lambda f(\langle \Psi(E_\lambda) L \Psi(E_\lambda)(x), y\rangle) = f(\langle L(x), y\rangle).$$

However, we may also assume that, for fixed λ_0, there is $\lambda_1 \geq \lambda_0$ such that

$$\begin{aligned} & |f(\langle L\Psi(E_{\lambda_0})(x), \Psi(E_\lambda)(y)\rangle) - f(\langle L\Psi(E_{\lambda_0})(x), y\rangle)| < \tfrac{1}{2}\epsilon \\ \text{or} \quad & \\ & |f(\langle \Psi(E_\lambda) L\Psi(E_{\lambda_0})(x), y\rangle) - f(\langle L\Psi(E_{\lambda_0})(x), y\rangle)| < \tfrac{1}{2}\epsilon \end{aligned}$$

for all $x \in X$, $y \in Y$ and $f \in \mathcal{F}$, and $\lambda \geq \lambda_1$. It follows that, for all $x \in X$, $y \in Y$ and $f \in \mathcal{F}$, if $\lambda \geq \lambda_1$,

$$\begin{aligned} &|f(\langle L(x), y\rangle) - f(\langle \Psi(E_\lambda) L \Psi(E_{\lambda_0})(x), y\rangle)| \\ &\leq |f(\langle L(x), y\rangle) - f(\langle L\Psi(E_{\lambda_0})(x), y\rangle)| + |f(\langle L\Psi(E_{\lambda_0})(x), y\rangle) - f(\langle \Psi(E_\lambda) L \Psi(E_{\lambda_0})(x), y\rangle)| \\ &< \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon. \end{aligned} \tag{3-12}$$

Fix $\lambda \geq \lambda_1 \geq \lambda_0$. Then $H_\lambda = \overline{E_\lambda(H)} \supset H_{\lambda_0}$. Hence

$$P_\lambda \Psi(E_\lambda) = \Psi(E_\lambda) \quad \text{and} \quad \Psi(E_{\lambda_0}) P_\lambda = \Psi(E_{\lambda_0}). \tag{3-13}$$

We also note that $\Psi(E_\lambda) L \Psi(E_{\lambda_0})|_{H_\lambda^\sim} \in B(H_\lambda)$. Since H_λ is countably generated, by Lemma 3.11, there is $T_\lambda \in K(H_\lambda)^{**}$ such that

$$\Psi_\lambda(T_\lambda) = \Psi(E_\lambda) L \Psi(E_{\lambda_0})|_{H_\lambda^\sim}. \tag{3-14}$$

However, by Proposition 3.4,

$$P_\lambda \Psi(T_\lambda) P_\lambda|_{H_\lambda^\sim} = \Psi(T_\lambda)|_{H_\lambda^\sim} = \Psi_\lambda(T_\lambda). \tag{3-15}$$

Fix $\lambda \geq \lambda_1 \geq \lambda_0$. Then, for any $x \in X$, $y \in Y$ and $f \in A^{**}$, by (3-15), (3-14), (3-13) and (3-12),

$$\begin{aligned} |f(\langle \Psi(T_\lambda)(x), (y)\rangle) - f(\langle L(x), y\rangle)| &= |f(\langle \Psi(T_\lambda) P_\lambda(x), P_\lambda(y)\rangle) - f(\langle L(x), y\rangle)| \\ &= |f(\langle \Psi(E_\lambda) L \Psi(E_{\lambda_0}) P_\lambda(x), P_\lambda(y)\rangle) - f(\langle L(x), y\rangle)| \\ &= |f(\langle P_\lambda \Psi(E_\lambda) L \Psi(E_{\lambda_0}) P_\lambda(x), y\rangle) - f(\langle L(x), y\rangle)| \\ &= |f(\langle \Psi(E_\lambda) L \Psi(E_{\lambda_0})(x), y\rangle) - f(\langle L(x), y\rangle)| < \epsilon. \end{aligned}$$

As mentioned above, this implies that Ψ is surjective. □

Corollary 3.13. *Let A be a W^*-algebra and H be a Hilbert A-module. Then $F : K(H)^{**} \to B(H^\sharp)$, the map given by Proposition 2.16, is a surjective map.*

Proof. Consider the pair A and A^{**} and $H^\sim = (H \bullet A^{**})^\sharp$. By Corollary 4.3 of [Paschke 1973], $H^\sim = B(H, A^{**})$, the A^{**}-module of all bounded A^{**}-valued A-module maps from H into A^{**}. It follows that $H^\sharp \subset H^\sim$ as an A-submodule. It then follows from Proposition 2.6 that $H^\sharp \bullet A^{**} \subset H^\sim$ as Hilbert A^{**}-modules. Then, by applying Proposition 3.1,

$$(H^\sharp \bullet A^{**})^\sharp \subset H^\sim.$$

However, $H \bullet A^{**} \subset H^\sharp \bullet A^{**}$. By applying Proposition 3.1 again, we obtain

$$H^\sim = (H \bullet A^{**})^\sharp \subset (H^\sharp \bullet A^{**})^\sharp \subset H^\sim.$$

Hence $(H^\sharp \bullet A^{**})^\sharp = H^\sim$. Denote by $\widetilde{\Psi} : K(H)^{**} \to B(H^\sim)$ the isomorphism given by Theorem 3.12 and by $\widetilde{\Psi}_{H^\sharp} : B(H^\sharp) \to B((H^\sharp \bullet A^{**})^\sharp) = B(H^\sim)$ the map given by Theorem 3.12.

Now let $T \in B(H^\sharp)$. Then, by applying Theorem 3.12, we obtain $a \in K(H)^{**}$ such that $\widetilde{\Psi}(a) = \widetilde{\Psi}_{H^\sharp}(T)$. It follows that (viewing $H^\sharp \subset H^\sim$)

$$\widetilde{\Psi}(a)|_{H^\sharp} = T.$$

Since $a \in K(H)^{**}$, there exists a net $\{a_\alpha\}$ in $K(H)$ such that $a_\alpha \to a$ in the weak* topology. Therefore, by Proposition 2.14, for any $f \in A^*$ and any $\xi, \zeta \in H^{\sim}$,

$$\lim_\alpha f(\langle(\widetilde{\Psi}(a) - \widetilde{\Psi}(a_\alpha))(\xi), \zeta\rangle) = 0.$$

Note, by Theorem 3.12, $\widetilde{\Psi}(a_\alpha) = \widetilde{\Psi}_0(a_\alpha)$. On the other hand, by Proposition 2.16, for any $g \in A_*$ and any $x, y \in H$,

$$\lim_\alpha g(\langle(F(a) - a_\alpha)(x), y\rangle) = 0.$$

Hence (since $\widetilde{\Psi}_0(a_\alpha)x = a_\alpha(x)$ for all $x \in H$, see Definition 2.12)

$$g(\langle(F(a) - \widetilde{\Psi}(a))(x), y\rangle) = 0 \quad \text{for all } x, y \in H \text{ and } g \in A_*.$$

Since $\widetilde{\Psi}(a)|_{H^\sharp} = T$, we actually have

$$g(\langle(F(a) - T)(x), y\rangle) = 0 \quad \text{for all } x, y \in H \text{ and } g \in A_*. \tag{3-16}$$

Note that $F(a), T \in B(H^\sharp)$. So $F(a)(x), T(x) \in H^\sharp$ for all $x \in H$. It follows that

$$\langle(F(a) - T)(x), y\rangle \in A \quad \text{for all } x, y \in H.$$

Then, by (3-16),

$$\langle(F(a) - T)(x), y\rangle = 0 \quad \text{for all } x, y \in H.$$

Hence $F(a) = T$. In other words, F is surjective. □

4. A Kaplansky density theorem in Hilbert modules

As mentioned in the introduction, in this section we study the density of H in $H \bullet A^{**}$.

Definition 4.1. Let X be a Hilbert space and $A \subset B(X)$ be a C^*-subalgebra of $B(X)$. Let $M = \bar{A}^{\mathrm{SOT}}$, the strong operator closure of A, and let H be a Hilbert A-module. Recall, by Proposition 2.6, $H \bullet M$ is the smallest Hilbert M-module containing H as a Hilbert A-module. We consider the question of how large H is in $H \bullet M$ as a submodule.

Let $\epsilon > 0$ and V be a finite subset of X. For each $\xi \in H \bullet M$, define

$$N_{\xi,\epsilon,V} = \{z \in H \bullet M : \|\langle \xi - z, \xi - z\rangle(v)\| < \epsilon,\ v \in V\}.$$

Let $\mathcal{T}_s$ be the topology generated by $N_{\xi,\epsilon,V}$ for all $\xi \in H \bullet M$, $\epsilon \in \mathbb{R}_+ \setminus \{0\}$, and any finite subset $V \subset X$. In other words, in $\mathcal{T}_s$, a net $\{z_\alpha\}$ converges to ξ in $H \bullet M$ if and only if

$$\lim_\alpha \|\langle \xi - z_\alpha, \xi - z_\alpha\rangle(v)\| = 0 \quad \text{for all } v \in X.$$

In the special case that $X = H_U$ is the Hilbert space corresponding to the universal representation π_U of A and $M = A^{**}$, we use $\mathcal{T}_{su}$ for the topology generated by $N_{\xi,\epsilon,V}$ for all $\xi \in H \bullet A^{**}$, $\epsilon \in \mathbb{R}_+ \setminus \{0\}$, and any finite subset $V \subset H_U$.

We note that H is dense in $H \bullet M$ in the topology $\mathcal{T}_s$, but to be more useful, we will show in Theorem 4.4 that the unit ball of H is dense in the unit ball of $H \bullet M$ in $\mathcal{T}_s$, a Kaplansky-style density theorem.

Lemma 4.2. *Suppose that $x \in H \bullet M$ and $\{x_\alpha\} \subset H \bullet M$ is a bounded net. Then $x_\alpha \to x$ in $\mathcal{T}_s$ if and only if, for any $v \in X$,*

$$\lim_\alpha \sup\{\|\langle y, x_\alpha - x\rangle(v)\| : y \in H \bullet M,\ \|y\| \le 1\} = 0.$$

Moreover, if $x_\alpha \to x$ in $\mathcal{T}_s$, then, for any $f \in M_*$,

$$\lim_\alpha \sup\{|f(\langle y, x_\alpha - x\rangle)| : y \in H \bullet M,\ \|y\| \le 1\} = 0.$$

Proof. Suppose that $x_\alpha \to x$ in $\mathcal{T}_s$. We have (see Proposition 2.3 (ii) of [Paschke 1973]), for any $y \in H \bullet M$,

$$\langle x_\alpha - x, y\rangle \cdot \langle y, x_\alpha - x\rangle \le \|y\|^2 \langle x_\alpha - x, x_\alpha - x\rangle.$$

Then, for any $v \in X$ and any $y \in H \bullet M$ with $\|y\| \le 1$,

$$\begin{aligned}\|\langle y, x_\alpha - x\rangle(v)\|^2 &= \langle \langle x_\alpha - x, y\rangle \cdot \langle y, x_\alpha - x\rangle v, v\rangle_X \\ &\le \|y\|^2 \langle \langle x_\alpha - x, x_\alpha - x\rangle v, v\rangle_X \le \|\langle x_\alpha - x, x_\alpha - x\rangle v\| \|v\| \to 0\end{aligned}$$

(where $\langle \cdot, \cdot \rangle_X$ is the inner product in the Hilbert space X). Conversely, let $K = \sup_\alpha\{\|x_\alpha\| + \|x\|\} + 1$. Then

$$\|\langle x_\alpha - x, x_\alpha - x\rangle(v)\| \le K \sup\{\|\langle y, x_\alpha - x\rangle(v)\| : y \in H \bullet M,\ \|y\| \le 1\} \to 0$$

For the "moreover" part of the lemma, suppose that $\langle x_\alpha - x, x_\alpha - x\rangle \to 0$ in the strong operator topology. Then it converges in the weak operator topology. However, since $\{\langle x_\alpha - x, x_\alpha - x\rangle\}$ is bounded, this also implies that it converges to zero in the σ-weak topology and in the weak* topology. Hence

$$\lim_\alpha f(\langle x_\alpha - x, x_\alpha - x\rangle) = 0 \quad \text{for all } f \in M_*.$$

Let $f \in M_*$ be a positive normal functional. Then, $f(\langle \cdot, \cdot \rangle)$ defines a pseudo inner product on $H \bullet M$. Hence, for any $y \in H \bullet M$, we have, by the Cauchy–Bunyakovsky–Schwarz inequality,

$$|f(\langle y, x_\alpha - x\rangle)|^2 \le f(\langle y, y\rangle) f(\langle x_\alpha - x, x_\alpha - x\rangle) \le \|f\|^2 \|y\|^2 f(\langle x_\alpha - x, x_\alpha - x\rangle).$$

Thus

$$\lim_\alpha \sup\{|f(\langle y, x_\alpha - x\rangle)| : y \in H \bullet M,\ \|y\| \le 1\} = 0. \qquad \square$$

Lemma 4.3. *Let X be a Hilbert space, $A \subset B(X)$ be a C^*-subalgebra and $M = \bar{A}^{\mathrm{SOT}}$ such that $\mathrm{id}_X \in M$. Then the unit ball of H_A is dense in the unit ball of $H_A \bullet M$ in $\mathcal{T}_s$.*

Proof. Let $\xi \in H_A \bullet M$ with $\|\xi\| \le 1$. We will show that there is a net $\{x_\alpha\} \in H$ such that $\|x_\alpha\| \le \|\xi\|$ and $\lim_\alpha \|\langle x_\alpha - \xi, x_\alpha - \xi\rangle(v)\| = 0$ for all $v \in X$. From the inequality

$$\|\langle x_\alpha - \xi, x_\alpha - \xi\rangle(v)\| \le \|\langle x_\alpha - \xi, x_\alpha - \xi\rangle^{1/2}\| \|\langle x_\alpha - \xi, x_\alpha - \xi\rangle^{1/2}(v)\| \le 2\|\langle x_\alpha - \xi, x_\alpha - \xi\rangle^{1/2}(v)\|,$$

we conclude that it is enough to show that there is a net $\{x_\alpha\} \in H$ such that

$$\|x_\alpha\| \le \|\xi\| \quad \text{and} \quad \lim_\alpha \|\langle x_\alpha - \xi, x_\alpha - \xi\rangle^{1/2} v\| = 0$$

for all $v \in X$. Therefore it suffices to show that, for any $\epsilon > 0$ and any finite subset $V \subset X$, there exists $z \in H$ with $\|z\| \le 1$ such that

$$\|(\langle \xi - z, \xi - z\rangle)^{1/2}(v)\| < \epsilon \quad \text{for all } v \in V.$$

To simplify notation, we may also assume that $\|v\| \le 1$ for all $v \in V$.

Denote by $R = \overline{AM}$, the closed right ideal of M generated by A. Note, by Proposition 3.6,

$$H_A \bullet M = \{\{b_n\} \in H_B : b_n \in R\}.$$

We write $\xi = \{b_n\} \in H_A \bullet M$. There exists $n_0 \in \mathbb{N}$ such that, for all $n \ge n_0$,

$$\left\|\sum_{k=n}^{\infty} b_n^* b_n\right\| < \tfrac{1}{2}\epsilon.$$

Fix an integer $n \ge n_0$. Let $P_n : H_A \bullet M \to R^{(n)} = \{(c_1, c_2, \dots, c_n) : c_i \in R\}$ be the projection. Put

$$S = \begin{pmatrix} b_1 & 0 & 0 & \cdots & 0 \\ b_2 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ b_n & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

For any $v \in V$, put

$$u_v = \begin{pmatrix} v \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

By the Kaplansky density theorem, there is $L \in M_n(A)$ such that

$$\|L\| \le \|S\| \quad \text{and} \quad \|L(u_v) - S(u_v)\| < \tfrac{1}{2}\epsilon \tag{4-1}$$

for all $v \in V$. Hence, denoting by $\langle\,\cdot\,,\cdot\,\rangle_X$ the inner product in X,

$$\langle (L-S)^*(L-S)u_v, u_v\rangle_X < \tfrac{1}{2}\epsilon \quad \text{for all } v \in V. \tag{4-2}$$

Define $q = \operatorname{diag}(1, 0, \dots, 0) \in M_n(M)$. Then $S = Sq$. Replacing L by Lq, we may write

$$L = \begin{pmatrix} a_1 & 0 & 0 & \cdots & 0 \\ a_2 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_n & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where $a_i \in A$, $i = 1, 2, \dots, n$. Then

$$\left\|\sum_{i=1}^{n} a_i^* a_i\right\| = \|L^*L\| = \|L\|^2 \le \|S\|^2 = \left\|\sum_{i=1}^{n} b_i^* b_i\right\| \le \|\xi\|^2. \tag{4-3}$$

It follows from (4-2) that

$$\left\langle \sum_{i=1}^{n} (b_i - a_i)^*(b_i - a_i)(v), v \right\rangle_X < \tfrac{1}{2}\epsilon.$$

Put $x = (a_1, a_2, \dots, a_n, 0, 0, \dots) \in H_A$. Then, by (4-3), we have $\|x\| \le \|\xi\|$ and

$$\langle\langle \xi - x, \xi - x\rangle(v), v\rangle_X = \left\langle \sum_{i=1}^{n} (b_i - a_i)^*(b_i - a_i)(v), v \right\rangle_X + \left\langle \sum_{i=n+1}^{\infty} b_i^* b_i(v), v \right\rangle_X$$
$$< \tfrac{1}{2}\epsilon + \left\| \sum_{i=n+1}^{\infty} b_i^* b_i \right\| < \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon.$$

In other words, for any $v \in V$,

$$\|(\langle \xi - x_\alpha, \xi - x_\alpha\rangle^{1/2}(v)\| = \langle\langle \xi - x, \xi - x\rangle(v), v\rangle_X < \epsilon.$$

The lemma then follows. ☐

Theorem 4.4. *Let X be a Hilbert space, $A \subset B(X)$ be a C^*-subalgebra and $M = \bar{A}^{\mathrm{SOT}}$, with $\mathrm{id}_X \in M$. Let H be a Hilbert A-module. Then the unit ball of H is dense in the unit ball of $H \bullet M$ in $\mathcal{T}_s$.*

Proof. Let $\xi \in H \bullet M$, with $\|\xi\| \le 1$.

Let us first assume that H is a countably generated A-module. By Lemma 4.2, it suffices to show that, for any $\epsilon > 0$ and any finite subset $V \subset X$, there exists $z \in H$ with $\|z\| \le 1$ such that

$$\|\langle y, \xi - z\rangle)(v)\| < \epsilon \quad \text{for all } y \in H,\ \|y\| \le 1 \text{ and } v \in V.$$

To simplify notation, we may also assume that $\|v\| \le 1$ for all $v \in V$.

By Kasparov's absorbing theorem [1980, Theorem 2], we may write $H_A = H \oplus H^\perp$. It follows that

$$H_A \bullet M = H \bullet M \oplus H^\perp \bullet M.$$

Define $Q : H_A \to H$ to be the projection. Then $Q \in L(H_A) = M(K(H_A))$. We identify Q with $\Psi_0(Q)$ in the sense that $Q \in L(H_M)$ which extends $Q|_{H_A}$. In particular, $H \bullet M = Q(H_A \bullet M)$.

By applying Lemmas 4.3 and 4.2, we obtain $z \in H_A$ with $\|z\| \le \|\xi\|$ such that

$$\|\langle y, \xi - z\rangle(v)\| < \epsilon \quad \text{for all } y \in H \bullet M,\ \|y\| \le 1, \text{ and } v \in V.$$

Note $Q(\xi) = \xi$ and $Q(y) = y$ for all $y \in H$. Put $x = Q(z) \in H$. We have

$$\|\langle y, \xi - x\rangle(v)\| = \|\langle y, Q(\xi) - Q(z))\rangle(v)\| = \|\langle Q(y), \xi - z\rangle(v)\| = \|\langle y, \xi - z\rangle(v)\| < \epsilon.$$

This proves the case that H is countably generated.

Next we let H be a general Hilbert A-module. We will show that, for any $\epsilon > 0$ and any finite subset $V \subset X$, there exists $z \in H$ with $\|z\| \le 1$ such that

$$\|\langle \xi - z, \xi - z\rangle)(v)\| < \epsilon \quad \text{for all } v \in V.$$

Again, we may also assume that $\|v\| \le 1$ for all $v \in V$.

Let $\{E_\lambda\}$ be an approximate identity for $K(H)$. Then, as in the proof of Theorem 3.12, $H_\lambda = \overline{E_\lambda(H)}$ is countably generated for each λ. It follows from Lemma 2.9 that there is λ such that

$$\|\Psi_0(E_\lambda)(\xi) - \xi\| < \tfrac{1}{4}\epsilon. \tag{4-4}$$

Fix such a λ. Note that, by Proposition 2.8, $\Psi_0(E_\lambda)(\xi) \in H_\lambda \bullet M \subset H \bullet M$. Since H_λ is countably generated, by the first part of the proof, we obtain $x \in H_\lambda$ with $\|x\| \le \|\Psi_0(E_\lambda)(\xi)\| \le \|\xi\|$ such that

$$\sup\{\|\langle y, \Psi_0(E_\lambda)(\xi) - x\rangle(v)\| : y \in H \bullet M, \ \|y\| \le 1\} < \tfrac{1}{4}\epsilon. \tag{4-5}$$

Then, applying (4-4) and then (4-5), for any $v \in V$,

$$\begin{aligned}\|\langle \xi - x, \xi - x\rangle(v)\| &\le \|\langle \xi - x, \xi - \Psi_0(E_\lambda)(\xi)\rangle(v)\| + \|\langle \xi - x, \Psi_0(E_\lambda)(\xi) - x\rangle(v)\| \\ &< 2\|\xi - \Psi_0(E_\lambda)(\xi)\| + 2\big\|\big\langle \tfrac{1}{2}(\xi - x), \Psi_0(E_\lambda)(\xi) - x\big\rangle(v)\big\| < \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon. \qquad \square\end{aligned}$$

We then obtain the following corollary as a Kaplansky density theorem.

Theorem 4.5. *Let A be a C^*-algebra and H be a Hilbert A-module. Then the unit ball of H is dense in the unit ball of $H \bullet A^{**}$ in $\mathcal{T}_{su}$.*

5. Closeness of H

Let H be a Hilbert A-module., Then, by Theorem 6.1 of [Brown and Lin 2025], the unit ball of H is A-weakly dense (see Definition 3.3 of [Brown and Lin 2025]) in the unit ball of $H^\sharp$, i.e., for any $f \in H^\sharp$, there is a net $\{x_\alpha\}$ in H with $\|x_\alpha\| \le \|f\|$ such that $\lim_\alpha \|\langle f - x_\alpha, y\rangle\| = 0$ for all $y \in H$. In the case that A is a W^*-algebra, $H^\sharp$ is a Hilbert A-module. One may ask: can one find the net $\{x_\alpha\} \in H$ with $\|x_\alpha\| \le \|f\|$ such that $\lim_\alpha \|\langle f - x_\alpha, \xi\rangle\| = 0$ for all $\xi \in H^\sharp$?

We begin with the following example.

Example 5.1. Let M be a W^*-algebra which contains a self-adjoint element a with infinite spectrum. Then, by the spectral theory, one obtains a sequence of mutually orthogonal nonzero projections $p_1, p_2, \dots, p_n, \dots$. Let $H = H_M$, and let $\xi = \{p_n\} \in H_M^\sharp$. Note that $\|\xi\| = \big\|\sum_{n=1}^\infty p_n\big\| = 1$ (the convergence is in the strong operator topology and weak* topology of M). We claim that there is *no* net $\{x_\alpha\}$ in H_M such that

$$\lim_\alpha \|\langle \xi - x_\alpha, \xi\rangle\| = 0.$$

Otherwise, there would be $x \in H_M$ such that

$$\|\langle \xi - x, \xi\rangle\| < \tfrac{1}{4}. \tag{5-1}$$

Since $x = \{a_n\} \in H_M$, there is $N \in \mathbb{N}$ such that

$$\Big\|\sum_{N+1} a_n^* a_n\Big\| < \big(\tfrac{1}{16}(1 + \|x\|)\big)^2.$$

Choose $q = \sum_{n=N+1}^\infty p_n \in M$. Define $P_N : H_M^\sharp \to M^{(N)} = \{(b_1, b_2, \dots, b_N) : b_i \in M\}$ to be the projection. Then

$$\begin{aligned}\|\langle \xi - P_N(x), \xi\rangle\| &\le \|\langle \xi - x, \xi\rangle\| + \|\langle (1 - P_N)(x), \xi\rangle\| \\ &< \tfrac{1}{4} + \|(1 - P_N)(x)\|\|\xi\| < \tfrac{1}{4} + \tfrac{1}{16} = \tfrac{5}{16}.\end{aligned}$$

On the other hand,

$$\tfrac{5}{16} \ge \|\langle \xi - P_N(x), \xi\rangle\| \ge \|\langle \xi - P_N(x), \xi\rangle q\| = \left\| \left(\sum_{N+1}^{\infty} p_n - \sum_{i=1}^{N} (p_i - a_i)^* p_i \right) q \right\| = \left\| \sum_{N+1}^{\infty} p_n q \right\| = 1.$$

A contradiction. In other words, the question at the beginning of this section is negative. This also follows from Corollary 5.7 below. However, we think that the example above might also be helpful.

Lemma 5.2. *Let A be a C^*-algebra. Suppose that $\xi \in H_A^\sharp$ and $\{x_\alpha\}$ is a bounded net in H_A such that*

$$\lim_\alpha \|\xi(x) - x_\alpha(x)\| = 0 \quad \textit{for all } x \in H_A$$

and $\xi(x_\alpha) := \langle \xi, x_\alpha\rangle$ converges in norm. Then $\xi \in H_A$ and $\langle \xi, \xi\rangle = \lim_\alpha \langle \xi, x_\alpha\rangle$.

Proof. Write $\xi = \{b_n\}$ and $x_\alpha = \{a_{\alpha,n}\}$, where $\{b_n\} \in H_A^\sharp$, $a_{\alpha,n} \in A$ and, for each α, $\{a_{\alpha,n}\} \in H_A$.

Put

$$M = 1 + \sup\{\|x_\alpha\| : \alpha\} + \|\xi\| < \infty \quad \text{and} \quad a = \lim_\alpha \langle x_\alpha, \xi\rangle.$$

Note $\xi(x_\alpha) = \langle \xi, x_\alpha\rangle \in A$ for all α. Hence $a \in A$.

Let $P_n : H_A^\sharp \to H_n := A^{(n)}$ be the projection to the first n copies of A, $n \in \mathbb{N}$. Then $P_n\xi \in H_n \subset H_A$. It follows that, for each $n \in \mathbb{N}$,

$$\lim_\alpha \langle x_\alpha, P_n(\xi)\rangle = \langle \xi, P_n(\xi)\rangle = \sum_{j=1}^{n} b_j^* b_j. \tag{5-2}$$

Fix $f \in A^*$. Let $\epsilon > 0$. By Lemma 3.8, since $\{x_\alpha\}$ is bounded, there is an integer $N \in \mathbb{N}$ such that, for all $n \ge N$,

$$|f(\langle x_\alpha, \xi\rangle) - f(\langle x_\alpha, P_n(\xi)\rangle)| < \tfrac{1}{3}\epsilon \quad \text{for all } \alpha. \tag{5-3}$$

Fix any $n \ge N$. By (5-2), choose α_0 such that, for all $\alpha \ge \alpha_0$,

$$\left\| \langle x_\alpha, P_n(\xi)\rangle - \sum_{j=1}^{n} b_j^* b_j \right\| < \tfrac{1}{3}\epsilon(1 + \|f\|), \tag{5-4}$$

$$\|\langle x_\alpha, \xi\rangle - a\| < \tfrac{1}{3}\epsilon(1 + \|f\|). \tag{5-5}$$

It follows that, for all $n \ge N$, by (5-5), (5-3) and (5-4),

$$\begin{aligned}
&\left| f(a) - f\left(\sum_{j=1}^{n} b_j^* b_j \right) \right| \\
&\qquad \le |f(a - \langle x_{\alpha_0}, \xi\rangle)| + |f(\langle x_{\alpha_0}, \xi\rangle) - f(x_{\alpha_0}, P_n(\xi)\rangle)| + \|f\| \left\| \langle x_{\alpha_0}, P_n(\xi)\rangle - \sum_{j=1}^{n} b_j^* b_j \right\| \\
&\qquad < \tfrac{1}{3}\epsilon + \tfrac{1}{3}\epsilon + \tfrac{1}{3}\epsilon = \epsilon.
\end{aligned}$$

Hence, on the state space $S(A)$ of A,

$$\lim_{n\to\infty} f\left(\sum_{j=1}^{n} b_j^* b_j \right) = f(a). \tag{5-6}$$

On the compact space $S(A)$ (in the weak* topology), $\hat{a}(f) = f(a)$ is a continuous function for all $f \in S(A)$, and $\widehat{\sum_{j=1}^{n} b_j^* b_j}$ is increasing. By the Dini theorem, $\widehat{\sum_{j=1}^{n} b_j^* b_j}$ converges uniformly to $\hat{a}$ on $S(A)$. It follows that

$$\sum_{j=1}^{n} b_j^* b_j \to a$$

in norm. This implies that $\xi = \{b_n\} \in H_A$ and $\langle \xi, \xi \rangle = a = \lim_\alpha \langle \xi, x_\alpha \rangle$. □

Proposition 5.3. *Let A be a C^*-algebra and H be a Hilbert A-module. Then, for any $T \in K(H)$, one has $\Psi_0(T)(H^\sharp) \subset H$, where Ψ_0 is given in Definition 2.7.*

Proof. Suppose that $T \in F(H)$ and $T = \sum_{i=1}^{m} \theta_{x_i, y_i}$ for some $x_i, y_i \in H$, $i = 1, 2, \ldots, m$. Then, for any $\xi \in H^\sharp$,

$$\Psi_0(T)(\xi) = \sum_{i=1}^{m} x_i \langle y_i, \xi \rangle = \sum_{i=1}^{m} x_i (\xi(y_i)^*) \in H.$$

Since $F(H)$ is dense in $K(H)$, this implies that $\Psi_0(T)(H^\sharp) \subset H$. □

Lemma 5.4. *Let A be a C^*-algebra, H be a Hilbert A-module and $\{E_\lambda\}$ be an approximate identity for $K(H)$. Then, for any $\xi \in H^\sim$ and any $f \in A^*$,*

$$\lim_\alpha \sup\{f(\langle \xi - \Psi_0(E_\lambda)(\xi), y \rangle)| : y \in H^\sim, \|y\| \le 1\} = 0.$$

Proof. By Lemma 2.9, $\{\Psi_0(E_\lambda)\}$ is an approximate identity for $K(H \bullet A^{**})$. In the universal representation of $K(H \bullet A^{**})$, $1 - \Psi_0(E_\lambda)$ converges to zero in the strong operator topology. Note that $\|1 - \Psi_0(E_\lambda)\| \le 1$. Therefore $(1 - \Psi_0(E_\lambda))(1 - \Psi_0(E_\lambda))$ also converges to zero in the strong operator topology. Hence it converges to zero in the weak operator topology. Since $\{(1 - \Psi_0(E_\lambda))^2\}$ is bounded, it also converges to zero in the weak* topology of $K(H \bullet A^{**})$. Recall that $(H \bullet A^{**})^\sharp = H^\sim$. It follows from Proposition 2.16, for any $\xi \in H^\sim$, that

$$\begin{aligned} \lim_\alpha |f(\langle \xi - \Psi_0(E_\lambda)(\xi), \xi - \Psi_0(E_\lambda)(\xi) \rangle)| &= \lim_\alpha |f(\langle \xi - F \circ \Psi_0(E_\lambda)(\xi), \xi - F \circ \Psi_0(E_\lambda)(\xi) \rangle) \\ &= \lim_\alpha |f(\langle (1 - F \circ \Psi_0(E_\lambda))^2(\xi), \xi \rangle)| = 0, \end{aligned}$$

where $F : K(H)^{**} \to B(H^\sharp)$ is the homomorphism given by Proposition 2.16. Suppose that $y \in H^\sim$ and $\|y\| \le 1$. Then, for any positive linear functional $f \in A^*$,

$$\begin{aligned} f(\langle \xi - \Psi_0(E_\lambda)(\xi), y \rangle)^2 &\le f(\langle \xi - \Psi_0(E_\lambda)(\xi), \xi - \Psi_0(E_\lambda)(\xi) \rangle) f(\langle y, y \rangle) \\ &\le \|f\| f(\langle \xi - \Psi_0(E_\lambda)(\xi), \xi - \Psi_0(E_\lambda)(\xi) \rangle). \end{aligned}$$

It follows that, for any $f \in A^*$,

$$\lim_\alpha \sup\{f(\langle \xi - \Psi_0(E_\lambda)(\xi), y \rangle)| : y \in H^\sim, \|y\| \le 1\} = 0.$$ □

Theorem 5.5. *Let A be a C^*-algebra and H be a Hilbert A-module. Suppose that $\xi \in H^\sharp$ and there is a bounded net $\{x_\alpha\}$ in H such that*

$$\lim_\alpha \|\xi(x) - \langle x_\alpha, x \rangle\| = 0 \quad \text{for all } x \in H$$

and $\xi(x_\alpha) := \langle \xi, x_\alpha \rangle$ converges in norm. Then $\xi \in H$ and $\langle \xi, \xi \rangle = \lim_\alpha \langle \xi, x_\alpha \rangle \in A$.

Proof. First let us assume H is countably generated. Then, by Kasparov's absorbing theorem [1980, Theorem 2], we may write $H_A = H \oplus H^{\perp}$. Then $\xi \in H^{\sharp} \subset H_A^{\sharp}$. By applying Lemma 5.2, we obtain that

$$\xi \in H_A \quad \text{and} \quad \langle \xi, \xi \rangle = \lim_{\alpha} \langle \xi, x_\alpha \rangle.$$

Since $\xi(x_\alpha) \in A$, we have $a = \langle \xi, \xi \rangle \in A$. Let $P : H_A \to H$ be the projection. Then $P \in L(H_A)$. Put $\eta = P(\xi) \in H$. Note that $\langle P(\xi) - \xi, x \rangle = 0$ for all $x \in H$. Hence $\xi = \eta$. Therefore this case follows.

In what follows we will work in $H^{\sim}$ and use the inner product in $H^{\sim}$ whenever it is convenient.

In general, let $a = \lim_\alpha \langle \xi, x_\alpha \rangle$. Since $\langle \xi, x_\alpha \rangle = \xi(x_\alpha) \in A$ for all α, we have $a \in A$.

Claim: $a = \langle \xi, \xi \rangle$ (in the inner product of $H^{\sim}$).

Let $\{E_\lambda\}$ be an approximate identity for $K(H)$. Let $\epsilon > 0$ and $f \in A^*$, with $\|f\| \le 1$. By applying Lemma 5.4, we have (since $\{\|\xi - x_\alpha\|\}$ is bounded)

$$\lim_{\lambda} \Big(\sup_{\alpha} \{ |f(\langle \xi - \Psi_0(E_\lambda)(\xi), \xi - x_\alpha \rangle)| \} \Big) = 0. \tag{5-7}$$

Thus, by applying Lemma 5.4 and (5-7), we obtain λ_0 such that, for all $\lambda \ge \lambda_0$,

$$\begin{aligned} |f(\langle \xi - \Psi_0(E_\lambda)(\xi), \xi \rangle)| &< \tfrac{1}{3}\epsilon, \\ |f(\langle \xi - \Psi_0(E_\lambda)(\xi), \xi - x_\alpha \rangle)| &< \tfrac{1}{3}\epsilon \quad \text{for all } \alpha. \end{aligned} \tag{5-8}$$

Recall that, by Proposition 5.3, $\Psi_0(E_\lambda)(\xi) \in H$. Fix any $\lambda \ge \lambda_0$. Choose α_0 such that, for any $\alpha \ge \alpha_0$,

$$\|\langle \xi, x_\alpha \rangle - a\| < \tfrac{1}{3}\epsilon \quad \text{and} \quad |f(\langle \Psi_0(E_\lambda)(\xi), \xi - x_\alpha \rangle)| < \tfrac{1}{3}\epsilon. \tag{5-9}$$

Now, by the first inequality of (5-9), (5-8) and then the second inequality of (5-9),

$$\begin{aligned} |f(\langle \xi, \xi \rangle - a)| &< |f(\langle \xi, \xi \rangle - \langle \xi, x_\alpha \rangle)| + \tfrac{1}{3}\epsilon = |f(\langle \xi, \xi - x_\alpha \rangle)| + \tfrac{1}{3}\epsilon \\ &\le |f(\langle \xi - \Psi_0(E_\lambda)(\xi), \xi - x_\alpha \rangle)| + |f(\langle \Psi_0(E_\lambda)(\xi), \xi - x_\alpha \rangle)| + \tfrac{1}{3}\epsilon < \tfrac{1}{3}\epsilon + \tfrac{1}{3}\epsilon + \tfrac{1}{3}\epsilon = \epsilon. \end{aligned}$$

Since this holds for any ϵ, we conclude that

$$f(\langle \xi, \xi \rangle) = f(a) \quad \text{for all } f \in A^*.$$

By the Hahn–Banach theorem, we obtain $\langle \xi, \xi \rangle = a$. The claim is proved.

There exists $x_1 \in \{x_\alpha\}$ and then $x_2 \in \{x_\alpha\}$ such that

$$\|\langle x_1, \xi \rangle - a\| < \tfrac{1}{2}, \quad \|\langle \xi - x_2, x_1 \rangle\| < \tfrac{1}{4} \quad \text{and} \quad \|\langle x_2, \xi \rangle - a\| < \tfrac{1}{4}.$$

Suppose that we have found $x_1, x_2, \dots, x_n$ such that

$$\|\langle \xi - x_j, x_i \rangle\| < 1/2^j \quad \text{and} \quad \|\langle x_j, \xi \rangle - a\| < 1/2^j, \quad i = 1, 2, \dots, j-1,$$

and $j = 1, 2, \dots, n$. Then choose $x_{n+1} \in \{x_\alpha\}$ such that

$$\|\langle \xi - x_{n+1}, x_i \rangle\| < 1/2^{n+1} \quad \text{and} \quad \|\langle x_{n+1}, \xi \rangle - a\| < 1/2^{n+1}, \quad i = 1, 2, \dots, n.$$

Thus, by induction, we obtain a subsequence $\{x_n\}$ in $\{x_\alpha\}$ such that

$$\lim_{n\to\infty} \|\langle x_n, \xi\rangle - a\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|\langle \xi - x_n, x_i\rangle\| = 0 \ \text{ for } i \in \mathbb{N}.$$

Denote by H_0 the Hilbert A-submodule generated by $\{x_1, x_2, \ldots, x_n, \ldots\}$. In particular, $x_n \in H_0$ and $n \in \mathbb{N}$. Let $\eta = \xi|_{H_0}$.

Now H_0 is countably generated and $x_n \in H_0$, so we have

$$\lim_{n\to\infty} \|\eta(x_n) - a\| = \lim_{n\to\infty} \|\xi(x_n) - a\| = 0.$$

Moreover, if $y = \sum_{i=1}^m x_i \cdot a_i$, where $a_i \in A$, then

$$\lim_{n\to\infty} \|\eta(y) - \langle x_n, y\rangle\| = 0.$$

Since $\{x_n\}$ is bounded (since $\{x_\alpha\}$ is bounded), this implies that

$$\lim_{n\to\infty} \|\eta(y) - \langle x_n, y\rangle\| = 0 \quad \text{for all } y \in H_0.$$

Applying what has been proved, we conclude that $\eta \in H_0$ and $\lim_{n\to\infty}\langle \eta, x_n\rangle = \langle \eta, \eta\rangle = a$.

We now consider Hilbert A^{**}-modules $H_0 \bullet A^{**} \subset H \bullet A^{**}$. By Proposition 3.1, we obtain a projection $P : H^\sim \to H_0^\sim$ such that $P|_{H_0 \bullet A^{**}} = \mathrm{id}_{H_0 \bullet A^{**}}$. Then $\eta = P(\xi)$. Hence, by the claim,

$$\begin{aligned}\|(1-P)\xi\|^2 &= \|\langle (1-P)(\xi), (1-P)(\xi)\rangle\| \le \|\langle (1-P)(\xi), \xi\rangle\| + \|\langle (1-P)(\xi), P(\xi)\rangle\| \\ &= \|\langle \xi, \xi\rangle - \langle P(\xi), \xi\rangle\| + 0 = \|a - \langle P(\xi), P(\xi)\rangle\| = \|a - \langle \eta, \eta\rangle\| = 0.\end{aligned}$$

In other words, $P(\xi) = \eta = \xi$. The theorem follows. $\square$

Definition 5.6. Let A be a C^*-algebra and H be a Hilbert A-module. Then $H^\sharp \subset H^\sim$.

For each $\xi \in H^\sharp$, $\epsilon > 0$ and a finite subset $Y \subset H^\sharp$, define

$$O_{\xi,\epsilon,Y} = \{\zeta \in H^\sharp : \|\langle \xi - \zeta, y\rangle\| < \epsilon,\ y \in Y\},$$

where the inner product is taken from $H^\sharp$ if $H^\sharp$ is a Hilbert A-module, or from $H^\sim$ (with values in A^{**}).

Denote by $\mathcal{T}_{NW}$ the topology in $H^\sharp$ generated by $O_{\xi,\epsilon,Y}$ for all $\xi \in H^\sharp$, $\epsilon \in \mathbb{R}_+ \setminus \{0\}$ and finite subsets $Y \subset H^\sharp$. Note that a net $\{\zeta_\alpha\}$ converges to ξ in $H^\sharp$ in $\mathcal{T}_{NW}$ if and only if

$$\lim_\alpha \|\langle \xi - \zeta_\alpha, y\rangle\| = 0$$

for all $y \in H^\sharp$, where the inner product is the one defined above.

Corollary 5.7. *Let A be a C^*-algebra and H be a Hilbert A-module. Then, with $\mathcal{T}_{NW}$, the unit ball of H is closed in $H^\sharp$.*

Proof. Let $\xi \in H^\sharp$. Suppose that there is a net $\{x_\alpha\}$ in H with $\|x_\alpha\| \le 1$ such that

$$\lim_\alpha \|\langle \xi - x_\alpha, \eta\rangle\| = 0 \quad \text{for all } \eta \in H^\sharp,$$

where the inner product is in $H^\sim$. Then, for each $x \in H$, $\lim_\alpha \|\langle \xi - x_\alpha, x\rangle\| = 0$ and (by choosing $\eta = \xi$) $\{\xi(x_\alpha)\} = \{\langle \xi, x_\alpha\rangle\}$ converges in norm to $\langle \xi, \xi\rangle$. By Theorem 5.5, $\xi \in H$. $\square$

Corollary 5.8. *Let A be a monotone complete C^*-algebra and H be a Hilbert A-module. Then the unit ball of H is closed in $H^\sharp$ in the topology $\mathcal{T}_{NW}$, where we view $H^\sharp$ as a self-dual Hilbert A-module.*

Lemma 5.9. *Let X be a Hilbert space, $A \subset B(X)$ be a C^*-subalgebra and $M = \bar{A}^{\mathrm{SOT}}$, with $\mathrm{id}_X \in M$. Let H be a Hilbert A-module. Suppose that $\xi \in H \bullet M$ and $\langle \xi, x\rangle \in A$ for all $x \in H$. Then $\xi \in H$.*

Proof. First let us consider the case that $H = H_A$. Then, by Proposition 3.6,

$$H_A \bullet M = \left\{ \{a_n\} : a_n \in \overline{AM} \text{ and } \sum_{k=1}^{n} a_k^* a_k \text{ converges in norm} \right\}.$$

Write $\xi = \{b_n\} \in H_A \bullet M$. The condition that $\langle \xi, x\rangle \in A$ for all $x \in H_A$ implies that $\xi \in H_A^\sharp$. It follows that $b_n \in A$. Hence $\xi \in H_A$.

Next, let us assume that H is countably generated. Let $\xi \in H \bullet M$ and $\langle \xi, x\rangle \in A$ for all $x \in H$. By Kasparov's absorbing theorem, we may write $H_A = H \oplus H^\perp$. It follows from what has been proved that $\xi \in H_A$. Let $P : H_A \to H$ be the projection. Then $P(\xi) \in H$. However, $\langle \xi - P(\xi), x\rangle = 0$ for all $x \in H$. For any $y \in H^\perp$, since $\xi \in H \bullet M$, we have $\langle \xi, y\rangle = 0$ for all $y \in H$. Hence $\xi = P(\xi) \in H$.

In general, since $\xi \in H \bullet M$, there are $x_{n,i} \in H,\ i = 1, 2, \ldots, k(n),\ b_{n,i} \in M,\ i = 1, 2, \ldots, k(n)$, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} \left\| \xi - \sum_{i=1}^{k(n)} x_{n,i} \bullet b_{n,i} \right\| = 0.$$

Let H_0 be the Hilbert A-submodule generated by $\{x_{n,i} : 1 \le i \le k(n),\ n \in \mathbb{N}\}$. Then $\xi \in H_0 \bullet M$ and $\xi|_{H_0} \in H_0^\sharp$, as $\langle \xi, h\rangle \in A$ for all $h \in H_0 \subset H$. From what has just been proved, $\xi \in H_0 \subset H$. □

We end this section with the following result.

Theorem 5.10. *Let A be a C^*-algebra and H be a Hilbert A-module. Then the unit ball of H is closed in $H^\sim$ in the topology $\mathcal{T}_{NW}$ of $H^\sim = (H \bullet A^{**})^\sharp$.*

Proof. Let $\{x_\alpha\}$ be a net in the unit ball of H and $\xi \in H^\sim$ such that

$$\lim_\alpha \|\langle \xi - x_\alpha, \zeta\rangle\| = 0 \quad \text{for all } \zeta \in H^\sim.$$

Since $H^\sim = (H \bullet A^{**})^\sharp$ and $H \subset H \bullet A^{**}$, by applying Corollary 5.8, we conclude that $\xi \in H \bullet A^{**}$.

We also have, for all $y \in H$,

$$\lim_\alpha \|\langle \xi - x_\alpha, y\rangle\| = 0.$$

Since $\langle x_\alpha, y\rangle \in A$, it follows that $\langle \xi, y\rangle \in A$. By Lemma 5.9, $\xi \in H$. □

6. A Kaplansky-style density theorem in the self-dual Hilbert modules

In the last section, we show that H is closed in $H^\sharp$ and $H^\sim$ in the topology $\mathcal{T}_{NW}$ of $H^\sharp$ and that of $H^\sim$, respectively. In this section, however, we will show that H is dense in $H^\sim$ in a weaker topology. In fact, by Theorem 4.5, it is easy to show that H is dense in $H^\sharp$ in $\mathcal{T}_0$, the topology defined below (see Definition 6.1). A similar question is whether one can replace x in (6-1) by any element in $H^\sharp$.

Definition 6.1. Let A be a W^*-algebra and H be a Hilbert A-module.

Let $\epsilon > 0$, and let $Y \subset H$ and $\mathcal{F} \subset A_*$ be finite subsets. Let $\xi \in H^\sharp$. Define

$$O_{\xi,\epsilon,Y,\mathcal{F}} = \{\zeta \in H^\sharp : |f(\langle \xi - \zeta, x\rangle)| < \epsilon,\ x \in Y,\ f \in \mathcal{F}\} \subset H^\sharp. \tag{6-1}$$

Let $\mathcal{T}_0$ be the topology of $H^\sharp$ generated by the subsets $O_{\xi,\epsilon,Y,\mathcal{F}}$.

Let $\epsilon > 0$, and let $Z \subset H^\sharp$ and $\mathcal{F} \subset A_*$ be finite subsets. Let $\xi \in H^\sharp$. Define

$$O_{\xi,\epsilon,Z,\mathcal{F}} = \{\zeta \in H^\sharp : |f(\langle \xi - \zeta, x\rangle)| < \epsilon,\ x \in Z,\ f \in \mathcal{F}\} \subset H^\sharp.$$

Let $\mathcal{T}_w$ be the topology of $H^\sharp$ generated by the subsets $O_{\xi,\epsilon,Z,\mathcal{F}}$.

In fact, by [Paschke 1973, Proposition 3.8] and the definition before it, $\mathcal{T}_w$ is the weak* topology of $H^\sharp$ as a conjugate space. So a natural question is whether H is dense in $H^\sharp$ in $\mathcal{T}_w$. To be more useful (but perhaps not useful enough to be used twice on Sundays — see [Pedersen 1979, 2.3.4]), we will also prove a Kaplansky-style density theorem in Theorem 6.4.

Let us also consider another topology. Let $\epsilon > 0$, $\xi \in H^\sharp$, and let $\mathcal{F} \subset A_*$ be a finite subset. Define

$$O_{\epsilon,\xi,\mathcal{F}} = \{\zeta \in H^\sharp : |f(\langle \xi - \zeta, \xi - \zeta\rangle)| < \epsilon,\ f \in \mathcal{F}\}.$$

Let $\mathcal{T}_{ws}$ be the topology generated by $O_{\epsilon,\xi,\mathcal{F}}$ for all $\epsilon > 0$, $\xi \in H^\sharp$ and finite subsets $\mathcal{F} \subset A_*$. Note that $\mathcal{T}_{ws}$ is stronger than $\mathcal{T}_w$, which is stronger than $\mathcal{T}_0$.

Lemma 6.2. *Let X be a Hilbert space and $A \subset B(X)$ be a C^*-subalgebra. Suppose that $M = \bar{A}^{\mathrm{SOT}}$, with $\mathrm{id}_X \in M$ and $b = \{b_k\} \in H_M^\sharp$. There is a net $a_\alpha = \{(a_{1,\alpha}, a_{2,\alpha}, \ldots, a_{n,\alpha}, \ldots)\} \in H_A$ such that*

$$\left\| \sum_{j=1}^\infty a_{j,\alpha}^* a_{j,\alpha} \right\|^{1/2} \le \|b\|, \tag{6-2}$$

$$\lim_\alpha f\left(\sum_{j=1}^\infty (b_j - a_{j,\alpha})^*(b_j - a_{j,\alpha}) \right) = 0 \tag{6-3}$$

for all $f \in M_*$.

Proof. Let $Y = l^2(X)$, the Hilbert space direct sum of countably many copies of X. Let $\bar{b} = (c_{i,j}) \in B(Y)$, where $c_{i,1} = b_i$, $i \in \mathbb{N}$, and $c_{i,j} = 0$ if $j \ge 2$ (see (3-8)). Denote by $P_n : Y \to X^{(n)}$ the projection, where $X^{(n)}$ is the direct sum of (first) n copies of X. Let $\epsilon > 0$ and $V \in L^2(X)$ be a finite subset. Then there is $n_0 \in \mathbb{N}$ such that

$$\|(1 - P_{n_0})(v)\| < \tfrac{1}{2}\epsilon(1 + \|b\|) \quad \text{for all } v \in V.$$

There is $d \in M_{n_0}(A)$ such that

$$\|(\bar{b} - d)(P_{n_0}(v))\| < \tfrac{1}{4}\epsilon \quad \text{for all } v \in V.$$

We have

$$\begin{aligned} \|(\bar{b} - dP_{n_0})(v)\| &\le \|(\bar{b} - dP_{n_0})(1 - P_{n_0})(v)\| + \|(\bar{b} - d)P_{n_0}(v)\| \\ &= \|\bar{b}(1 - P_{n_0})(v)\| + \tfrac{1}{4}\epsilon < \epsilon \qquad \text{for all } v \in V. \end{aligned}$$

Let B be the self-adjoint algebra of those bounded operators on Y which can be expressed as infinite matrices with entries in A, where all are zero except finitely many of them. Then, by what has been proved, we conclude that, in the strong operator topology (of $B(Y)$), operator $\bar{b}$ is in the closure of operators in B in the strong operator topology.

Then, by the Kaplansky density theorem, there is a net $\{d_\alpha\} \in B$ with $\|d_\alpha\| \le \|\bar{b}\|$ such that

$$\lim_\alpha \|(\bar{b} - d_\alpha)v\| = 0 \quad \text{for all } v \in Y.$$

Since $\{\|\bar{b} - d_\alpha\|\}$ is bounded, we also have

$$\lim_\alpha \|(\bar{b} - d_\alpha)^*(\bar{b} - d_\alpha)v\| = 0 \quad \text{for all } v \in Y.$$

We further note that

$$\|\bar{b}\|^2 = \|(\bar{b})^*\bar{b}\| = \left\|\sum_{j=1}^{\infty} b_j^* b_j\right\| \le \|b\|.$$

Then

$$\lim_\alpha \|(\bar{b} - d_\alpha)^*(\bar{b} - d_\alpha)P_1 v\| = 0 \quad \text{for all } v \in Y. \tag{6-4}$$

Note $\bar{b}P_1 = \bar{b}$. Let $d'_\alpha = d_\alpha P_1 = (d_{i,j,\alpha})$, where $d_{i,j,\alpha} = 0$ if $j \ge 2$. Put $a_{j,\alpha} = d_{1,j,\alpha}$, $j \in \mathbb{N}$. Then, for all $n \in \mathbb{N}$,

$$\left\|\sum_{j=1}^{n} a_{j,\alpha}^* a_{j,\alpha}\right\| \le \|(d'_\alpha)^* d'_\alpha\| = \|d'_\alpha\|^2 \le \|d_\alpha\|^2 \le \|\bar{b}\|^2 \le \|b\|^2.$$

Put $a_\alpha = \{a_{j,\alpha}\}$. Since $d_\alpha \in B$, for each α, there are only finitely many $a_{j,\alpha}$ which are not zero. Hence $a_\alpha \in H_A$. Then $\|a_\alpha\| \le \|b\|$. Thus (6-2) holds. On the other hand, by (6-4),

$$\lim_\alpha \|(\bar{b} - d'_\alpha)^*(\bar{b} - d'_\alpha)P_1 v\| = 0. \tag{6-5}$$

Let $h \in X$. By (6-5),

$$\lim_\alpha \left\|\sum_{j=1}^{\infty} (b_j - a_{j,\alpha})^*(b_j - a_{j,\alpha})h\right\| = 0.$$

In other words, $\sum_{i=1}^{\infty} (b_j - a_{j,\alpha})^*(b_j - a_{j,\alpha}) = \langle b - a_\alpha, b - a_\alpha\rangle \to 0$ in the strong operator topology. However,

$$\left\|\sum_{j=1}^{n} (b_j - a_{j,\alpha})^*(b_j - a_{j,\alpha})\right\| = \|(\bar{b} - d'_\alpha)\|^2 \le (\|\bar{b}\| + \|d_\alpha\|)^2 \le 4\|b\|^2.$$

Therefore $\sum_{i=1}^{n} (b_j - a_{j,\alpha})^*(b_j - a_{j,\alpha}) \to 0$ in the σ-weak operator topology and hence in the weak* topology (see, for example, 4.6.13 of [Pedersen 1989]). Therefore (6-3) holds. □

Lemma 6.3. *Let $A \subset B(X)$ be a C^*-subalgebra, and let $M = \bar{A}^{\mathrm{SOT}}$, with $1_X \in M$. Suppose that H is a countably generated Hilbert A-module. Then H is dense in $(H \bullet M)^\sharp$ in the following sense: for any $\xi \in (H \bullet M)^\sharp$, there is a net $x_\alpha \in H$ with $\|x_\alpha\| \le \|\xi\|$ such that*

$$\lim_\alpha \sup\{|f(\langle \xi - x_\alpha, \zeta\rangle)| : \zeta \in (H \bullet M)^\sharp,\ \|\zeta\| \le 1\} = 0 \quad \textit{for all } f \in M_*. \tag{6-6}$$

Proof. Let us first prove this for $H = H_A$, even though when A is not σ-unital, H_A is not countably generated. Lemma 6.2 provides a net $\{x_\alpha\}$ in H_A with $\|x_\alpha\| \le \|\xi\|$ such that

$$\lim_\alpha f(\langle \xi - x_\alpha, \xi - x_\alpha\rangle) = 0 \quad \text{for all } f \in M_*.$$

Recall that, for any positive linear functional f, the map $H_M^\sharp \times H_M^\sharp \to \mathbb{R}$ defined by $[x, y]_f = f(\langle x, y\rangle)$ (for all $x, y \in H_M^\sharp$) is a pseudo inner product. Therefore, by the Cauchy–Bunyakovsky–Schwarz inequality,

$$f(\langle x, y\rangle)^2 \le f(\langle x, x\rangle) f(\langle y, y\rangle) \quad \text{for all } x, y \in H_M^\sharp.$$

It follows that, for any positive normal linear functional f,

$$\begin{aligned}\sup\{|f(\langle \xi - x_\alpha, \zeta\rangle)| : \zeta \in H_M^\sharp, \|\zeta\| \le 1\}^2 &\le \sup\{f(\langle \zeta, \zeta\rangle) f(\langle \xi - x_\alpha, \xi - x_\alpha\rangle) : \zeta \in H_M^\sharp, \|\zeta\| \le 1\} \\ &= \|f\| f(\langle \xi - x_\alpha, \xi - x_\alpha\rangle) \to 0.\end{aligned}$$

Thus we proved (6-6) holds for $H = H_A$.

Now let H be a countably generated Hilbert A-module. Then, by Kasparov's absorbing theorem, we may write $H_A = H \oplus H^\perp$. Hence $H_A \bullet M = H \bullet M \oplus (H^\perp \bullet M)$. It follows that

$$H_M^\sharp = (H_A \bullet M)^\sharp = (H \bullet M)^\sharp \oplus (H^\perp \bullet M)^\sharp.$$

Let $P : H_M^\sharp \to (H \bullet M)^\sharp$ be the projection such that $P|_H = \mathrm{id}_H$. Then, by what has been proved for H_A, there is a net $y_\alpha \in H_A$ such that $\|y_\alpha\| \le \|\xi\|$ and, for any $f \in M_*$,

$$\lim_\alpha \sup\{f(\langle \xi - y_\alpha, \zeta\rangle) : \zeta \in H_M^\sharp, \|\zeta\| \le 1\} = 0.$$

Put $x_\alpha = P(y_\alpha) \in (H \bullet M)^\sharp$. Note that $P(\xi) = \xi$. Then, for any $f \in M_*$,

$$\begin{aligned}\lim_\alpha \sup\{f(\langle \xi - x_\alpha, \zeta\rangle) : \zeta \in (H \bullet M)^\sharp, \|\zeta\| \le 1\} &= \lim_\alpha \sup\{f(\langle \xi - x_\alpha, P(\zeta)\rangle) : \zeta \in (H \bullet M)^\sharp, \|\zeta\| \le 1\} \\ &= \lim_\alpha \sup\{f(\langle \xi - y_\alpha, \zeta\rangle) : \zeta \in (H \bullet M)^\sharp, \|\zeta\| \le 1\} \\ &\le \lim_\alpha \sup\{f(\langle \xi - y_\alpha, \zeta\rangle) : \zeta \in H_M^\sharp, \|\zeta\| \le 1\} = 0. \qquad \square\end{aligned}$$

Theorem 6.4. *Let X be a Hilbert space, $A \subset B(X)$ a C^*-subalgebra and $M = \bar{A}^{\mathrm{SOT}}$, with $1_M = \mathrm{id}_X$, and let H be a Hilbert A-module. Then the unit ball of H is dense in the unit ball of $(H \bullet M)^\sharp$ in $\mathcal{T}_{ws}$ (the topology on $(H \bullet M)^\sharp$).*

Proof. Let $\xi \in (H \bullet M)^\sharp$ with $\|\xi\| \le 1$. It suffices to show that, for any $\epsilon > 0$, any finite subset $Y \subset (H \bullet M)^\sharp$ and any finite subset $\mathcal{F} \subset M_*$, there is $x \in H$ such that

$$\|x\| \le \|\xi\| \quad \text{and} \quad |f(\langle \xi - x, y\rangle)| < \epsilon \quad \text{for all } y \in (H \bullet M)^\sharp,\ \|y\| \le 1, \ \text{ and } \ f \in \mathcal{F}.$$

Let us fix ϵ and $\mathcal{F}$. Choose an approximate identity $\{E_\lambda\}$ for $K(H)$. It follows that $E_\lambda \nearrow \mathrm{id}_H$. Note that $\mathrm{id}_H \in M(K(H))$. By the last part of Proposition 2.13, $\Psi_0(\mathrm{id}_H) = \mathrm{id}_{H \bullet M}$. By [Paschke 1973, Corollary 3.7], $F \circ \Psi_0(\mathrm{id}_H) = \mathrm{id}_{(H \bullet M)^\sharp}$, where $F : K(H \bullet M)^{**} \to B(H \bullet M)^\sharp$ is the map given by Proposition 2.16. Note also that, by Lemma 2.9, $\{\Psi_0(E_\lambda)\}$ is an approximate identity for $K(H \bullet M)$. In the universal representation of $K(H \bullet M)$, we have that $1 - \Psi_0(E_\lambda)$ converges to zero in the strong operator topology. Note that $\|1-\Psi_0(E_\lambda)\| \le 1$. Therefore $(1-\Psi_0(E_\lambda))^*(1-\Psi_0(E_\lambda))$ also converges to zero in the strong operator topology. Hence (since $\{\|(1-\Psi_0(E_\lambda))^*(1-\Psi_0(E_\lambda))\|\}$ is bounded), it converges to zero in the weak* topology. By Proposition 2.16, we have, for all $f \in M_*$,

$$\begin{aligned} f(\langle \xi - F\circ\Psi_0(E_\lambda)(\xi), \xi - F\circ\Psi_0(E_\lambda)(\xi)\rangle) \\ = f(\langle (1-F\circ\Psi_0(E_\lambda))^*(1-F\circ\Psi_0(E_\lambda))(\xi), \xi\rangle) \to 0. \end{aligned} \tag{6-7}$$

Next let g be a positive normal linear functional in M_*. Then, for any $y \in (H\bullet M)^\sharp$ with $\|y\| \le 1$,

$$\begin{aligned} |g(\langle \xi - F\circ\Psi_0(E_\lambda)(\xi), y\rangle)|^2 &\le g(\langle \xi - F\circ\Psi_0(E_\lambda)(\xi), \xi - F\circ\Psi_0(E_\lambda)(\xi)\rangle) g(\langle y, y\rangle) \\ &\le \|g\|\|y\|^2 g(\langle \xi - F\circ\Psi_0(E_\lambda)(\xi), \xi - F\circ\Psi_0(E_\lambda)(\xi)\rangle). \end{aligned}$$

Hence, by (6-7),

$$\lim_\alpha \big(\sup\{|g(\langle \xi - F\circ\Psi_0(E_\lambda)(\xi), y\rangle)| : y \in (H\bullet M)^\sharp,\ \|y\| \le 1\}\big) = 0.$$

It follows that, for any $f \in M_*$,

$$\lim_\alpha \big(\sup\{|f(\langle \xi - F\circ\Psi_0(E_\lambda)(\xi), y\rangle)| : y \in (H\bullet M)^\sharp,\ \|y\| \le 1\}\big) = 0.$$

Put $\Phi := F\circ\Psi_0$. We obtain λ_0 such that, for all $\lambda \ge \lambda_0$,

$$|f(\langle \xi - \Phi(E_\lambda)(\xi), y\rangle)| < \tfrac{1}{2}\epsilon \quad \text{for all } y \in (H\bullet M)^\sharp,\ \|y\| \le 1, \ \text{ and } \ f \in \mathcal{F}. \tag{6-8}$$

Let $H_\lambda = \overline{E_\lambda(H)}$. As in the proof of Theorem 3.12, we have that H_λ is countably generated. Moreover, by Proposition 3.1,

$$(H\bullet M)^\sharp = (H_\lambda\bullet M)^\sharp \oplus ((H_\lambda\bullet M)^\sharp)^\perp.$$

Let $P_\lambda : (H\bullet M)^\sharp \to (H_\lambda\bullet M)^\sharp$ be the projection. Note that

$$\Phi(E_\lambda)(\xi), \Phi(E_\lambda)(y) \in P_\lambda((H\bullet M)^\sharp) = (H_\lambda\bullet M)^\sharp$$

for all $y \in (H\bullet M)^\sharp$.

It follows from Lemma 6.3 that there is $x \in H_\lambda$ with $\|x\| \le \|\Phi(E_\lambda)(\xi)\| \le \|\xi\|$ such that

$$|f(\langle \Psi(E_\lambda)(\xi) - x, P_\lambda(y)\rangle)| < \tfrac{1}{2}\epsilon \quad \text{for all } y \in (H_\lambda\bullet M)^\sharp,\ \|y\| \le 1.$$

Since $P_\lambda\Phi(E_\lambda) = \Phi(E_\lambda)$ and $x \in H_\lambda$, we have, for all $y \in (H_\lambda\bullet M)^\sharp$, $\|y\| \le 1$,

$$|f(\langle \Phi(E_\lambda)(\xi) - x, y\rangle)| = |f(\langle P_\lambda\Phi(E_\lambda)(\xi) - P_\lambda(x), y\rangle)| = |f(\langle \Phi(E_\lambda)(\xi) - x, P_\lambda(y)\rangle)| < \tfrac{1}{2}\epsilon.$$

Thus (also applying (6-8)) for all $y \in (H\bullet M)^\sharp$ with $\|y\| \le 1$ and $f \in \mathcal{F}$,

$$|f(\langle \xi - x, y\rangle)| \le |f(\langle \xi - \Phi(E_\lambda)(\xi), y\rangle)| + |f(\langle \Phi(E_\lambda)(\xi) - x, y\rangle)| < \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon. \qquad \square$$

The next two statements are the main results of this section.

Corollary 6.5. *Let A be a W^*-algebra and H be a Hilbert A-module. Then the unit ball of H is dense in $H^\sharp$ in $\mathcal{T}_{ws}$.*

Proof. Let $M = A$ and then apply Theorem 6.4. $\square$

Theorem 6.6. *Let A be a C^*-algebra and H be a Hilbert A-module. Then the unit ball of H is dense in $H^\sim$ in $\mathcal{T}_{ws}$ (as $H^\sim = (H \bullet A^{**})^\sharp$).*

Proof. We choose the universal representation π_U and its strong operator closure $A'' = A^{**}$, then apply Theorem 6.4. □

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