

ANALYSIS & PDE

Volume 18

No. 7

2025

ALESSANDRO CARLOTTO, MARIO B. SCHULZ AND DAVID WIYGUL

**SPECTRAL ESTIMATES FOR FREE BOUNDARY MINIMAL
SURFACES
VIA MONTIEL-ROS PARTITIONING METHODS**



SPECTRAL ESTIMATES FOR FREE BOUNDARY MINIMAL SURFACES VIA MONTIEL–ROS PARTITIONING METHODS

ALESSANDRO CARLOTTO, MARIO B. SCHULZ AND DAVID WIYGUL

We adapt and extend the Montiel–Ros methodology to compact manifolds with boundary, allowing for mixed (including oblique) boundary conditions and also accounting for the action of a finite group G together with an additional twisting homomorphism $\sigma : G \rightarrow O(1)$. We then apply this machinery in order to obtain quantitative lower and upper bounds on the growth rate of the Morse index of free boundary minimal surfaces with respect to the topological data (i.e., the genus and the number of boundary components) of the surfaces in question. In particular, we compute the exact values of the equivariant Morse index and nullity for two infinite families of examples, with respect to their maximal symmetry groups, and thereby derive explicit two-sided linear bounds when the equivariance constraint is lifted.

1. Introduction	1715
2. Notation and standing assumptions	1718
3. Fundamental tools	1726
4. Free boundary minimal surfaces in the ball: a first application	1735
5. Effective index estimates for two sequences of examples	1738
Acknowledgements	1766
References	1767

1. Introduction

Despite a profusion of constructions of free boundary minimal surfaces in the Euclidean unit ball \mathbb{B}^3 over the course of the past decade — [Fraser and Schoen 2011; 2016; Girouard and Lagacé 2021; Karpukhin et al. 2014] via optimization of the first Steklov eigenvalue, [Carlotto et al. 2022a; Ketover 2016a; 2016b] via min-max methods for the area functional, and [Carlotto et al. 2022b; Folha et al. 2017; Kapouleas and Li 2021; Kapouleas and McGrath 2023; Kapouleas and Wiygul 2023; Kapouleas and Zou 2021] via gluing methods — many basic questions about the space of such surfaces remain open. The reader is referred to [Franz 2022; Fraser 2020; Li 2020] for recent overviews of the field. In particular, so far it is only for the rotationally symmetric examples, planar discs through the origin and critical catenoids, that the exact value of the Morse index is actually known; see [Devyver 2019; Smith and Zhou 2019; Tran 2020]. The present manuscript is the first in a series of works aimed at shedding new light on this fundamental invariant, which (also due to its variational content, and thus to its natural connection with min-max theory, see [Marques and Neves 2016; 2018; 2020]) has acquired great importance within geometric analysis.

MSC2020: primary 53A10; secondary 49Q05, 58C40.

Keywords: minimal surfaces, Morse index, equivariant spectrum.

Partly motivated by the corresponding conjectures concerning closed minimal hypersurfaces in manifolds of positive Ricci curvature (see [Ambrozio et al. 2018a; Neves 2014]), five years ago the first author proved with Ambrozio and Sharp a universal lower bound for the index of any free boundary minimal surface in any mean-convex subdomain Ω of \mathbb{R}^3 in terms of the topological data of the surface under consideration. Specifically, it was shown in [Ambrozio et al. 2018b] that the following estimate holds:

$$\text{index}(\Sigma) \geq \frac{1}{3}(2g + b - 1), \quad (1-1)$$

where Σ is any free boundary minimal surface in Ω and g and b denote its genus and the number of its boundary components, respectively. This result was then partly complemented by [Lima 2022, Theorem 4], that is, an affine upper bound with a very large, yet in principle computable, numerical constant. In this article we shall develop a general methodology, building upon the fundamental work [Montiel and Ros 1991], which allows us, among other things, to significantly refine such universal estimates bringing the geometry and symmetry group of the surfaces under consideration into play. This approach, while motivated by our goal to better understand the behavior of certain infinite families of free boundary minimal surfaces in \mathbb{B}^3 (aiming for two-sided bounds in terms of explicit, affine functions of the topological data), turns out to be of independent interest and much wider applicability.

In more abstract terms, we shall be concerned here with proving effective estimates for (part of) the spectrum of Schrödinger-type operators on bounded Lipschitz domains of Riemannian manifolds, combined with mixed boundary conditions, which will be — on disjoint portions of the boundary in question — of Dirichlet or Robin (oblique) type. Summarizing and oversimplifying things to the extreme, the number of eigenvalues of any such operator *below a given threshold* can be estimated by suitably partitioning the domain into finitely many subdomains, provided one adjoins Dirichlet boundary conditions in the interior boundaries when aiming for lower bounds, and Neumann boundary conditions in the interior boundaries for upper bounds instead. We refer the reader to Section 2 for the setup of our problem together with our standing assumptions, and to the first part of Section 3 (specifically to Proposition 3.1, and Corollary 3.2) for precise statements.

Often times (yet not always) the partitions mentioned above naturally relate to the underlying symmetries of the problem in question, which is in particular the case for some of the classes of free boundary minimal surfaces in \mathbb{B}^3 that have so far been constructed. With this remark in mind, a peculiar (and, a posteriori, fundamental) feature of our work is the development of the Montiel–Ros methodology in the presence of the action of a group G together with an additional twisting homomorphism $\sigma : G \rightarrow \text{O}(1)$, in the terms explained in Section 2.4. This allows us, for instance, to explicitly and transparently study how the Morse index of a given free boundary minimal surface depends on the symmetries one imposes, namely to look at the “functor” $(G, \sigma) \rightarrow \text{ind}_G^\sigma(T)$, where T denotes the index (Jacobi) form of the surface in question. As apparent even from the simplest examples we shall discuss, this perspective turns out to be very natural and effective in tackling the geometric problems we are interested in.

With this approach, *lower* bounds are sometimes relatively cheap to obtain. One way they can be derived is from ambient Killing vector fields, once it is shown that the associated (scalar-valued) Jacobi field on the surface under consideration vanishes along the (interior) boundary of any domain of the

chosen partition, which in practice amounts to suitably *designing the partition and picking the Killing field* given the geometry of the problem. We present one such simple yet paradigmatic result in Proposition 4.2, which concerns free boundary minimal surfaces with pyramidal or prismatic symmetry in \mathbb{B}^3 . Instead, *upper* bounds are often a lot harder to obtain and shall typically rely on finer information than the sole symmetries of the scene one deals with. Said otherwise, one needs to know *how* (i.e., by which method) the surface under study has been obtained.

We will develop here a detailed analysis of the Morse index of the two families of free boundary minimal surfaces we constructed in our recent, previous work [Carlotto et al. 2022b]. Very briefly, using gluing methods of essentially PDE-theoretic character, we obtained there a sequence $\Sigma_m^{-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0}$ of surfaces having genus m , three boundary components and antiprismatic symmetry group \mathbb{A}_{m+1} , and a sequence $\Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0}$ of surfaces having genus zero, $n + 2$ boundary components and prismatic symmetry group \mathbb{P}_n . As we described at length in Section 7 therein, with data (see Tables 2 and 3 in [Carlotto et al. 2022b]) and heuristics, numerical simulations for the Morse index of the surfaces in the former sequence display a seemingly “erratic” behavior, as such values do not align on the graph of any affine function, nor seem to exhibit any obvious periodic pattern. This is a rather unexpected behavior (by comparison, e.g., with other families of examples, say in the round three-dimensional sphere, see [Kapouleas and Wiygul 2020]), which obviously calls for a careful study that we carry through in Section 5 of the present article. In particular, we establish the following statement.

Theorem 1.1 (index estimates for $\Sigma_m^{-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0}$ and $\Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0}$). *There exist $m_0, n_0 > 0$ such that, for all integers $m > m_0$ and $n > n_0$, the Morse index and nullity of the free boundary minimal surfaces $\Sigma_m^{-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0}, \Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0} \subset \mathbb{B}^3$ satisfy the bounds*

$$\begin{aligned} 2m + 1 &\leq \text{ind}(\Sigma_m^{-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0}), & \text{ind}(\Sigma_m^{-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0}) + \text{nul}(\Sigma_m^{-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0}) &\leq 12m + 12, \\ 2n + 2 &\leq \text{ind}(\Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0}), & \text{ind}(\Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0}) + \text{nul}(\Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0}) &\leq 8n. \end{aligned}$$

In fact, the upper bound in this “absolute estimate” follows quite easily by combining the “relative estimate” associated to the equivariant Morse index of these surfaces (with respect to their respective *maximal* symmetry groups) with the aforementioned Proposition 3.1. The next statement thus pertains to such equivariant bounds for which we do obtain equality, thus settling part of Conjectures 7.7 (iv) and 7.9 (iv) of [Carlotto et al. 2022b]. We stress that neither family is constructed variationally, and thus there is actually no cheap index bound one can extract from the design methodology itself; on the contrary, this statement indicates a posteriori that the families of surfaces in question may in principle be constructed (even in a nonasymptotic regime) by means of min-max schemes generated by 2-parameter sweepouts, modulo the well-known problem of fully controlling the topology in the process (see [Carlotto et al. 2022a]).

Theorem 1.2 (equivariant index and nullity of $\Sigma_m^{-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0}$ and $\Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0}$). *There exist $m_0, n_0 > 0$ such that, for all integers $m > m_0$ and $n > n_0$, the equivariant Morse index and nullity of the free boundary minimal surfaces $\Sigma_m^{-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0}, \Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0} \subset \mathbb{B}^3$ satisfy*

$$\begin{aligned} \text{ind}_{\mathbb{A}_{m+1}}(\Sigma_m^{-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0}) &= 2, & \text{nul}_{\mathbb{A}_{m+1}}(\Sigma_m^{-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0}) &= 0, \\ \text{ind}_{\mathbb{P}_n}(\Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0}) &= 2, & \text{nul}_{\mathbb{P}_n}(\Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0}) &= 0. \end{aligned}$$

The main idea behind the proof of these results, or — more precisely — for the upper bounds, can only be explained by recalling, in a few words, how the surfaces in question have been constructed. Following the general methodology of [Kapouleas 1997], one first considers a singular configuration, that is a formal union of minimal surfaces in \mathbb{B}^3 (not necessarily free boundary), then its regularization — which needs the use of (wrapped) periodic minimal surfaces in \mathbb{R}^3 , to desingularize near the divisors, and controlled interpolation processes between the building blocks in play — and, thirdly and finally, the perturbation of such configurations to exact minimality (at least for *some* values of the parameters), while also ensuring proper embeddedness and accommodating the free boundary condition. Here we first get a complete understanding of the index and nullities of the building blocks for the concrete cases under consideration in Section 5. In somewhat more detail, the analysis of the Karcher–Scherk towers (the periodic building blocks employed in either construction) exploits, in a substantial fashion, the use of the Gauss map, which allows one to rephrase the initial geometric question into one for the spectrum of simple elliptic operators of the form $\Delta_g \mathbb{S}^2 + 2$ on suitable (typically singular, i.e., spherical triangles, wedges or lunes) subdomains of round \mathbb{S}^2 , with mixed boundary conditions, and possibly subject to additional symmetry requirements. The analysis of the other building blocks — disks and asymmetric catenoidal annuli — is more direct, although, in the latter case, trickier than it may first look (see, e.g., Lemma 5.8).

Once that preliminary analysis is done, we then prove that, corresponding to the (local) geometric convergence results (that are implied by the very gluing methodology), there are robust spectral convergence results that serve our scopes. However, a general challenge in the process is that gluing constructions typically have *transition regions* where different scales interact with each another: in our constructions of the sequences

$$\Sigma_m^{-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0} \quad \text{and} \quad \Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0},$$

such regions occur between the catenoidal annuli \mathbb{K}_0 (as well as the disk \mathbb{B}^2 in the former case) and the wrapped Karcher–Scherk towers, roughly at distances between m^{-1} and $m^{-1/2}$ (respectively n^{-1} and $n^{-1/2}$) from the equatorial \mathbb{S}^1 . As a result, we need to deal with delicate scale-picking arguments, an ad hoc study of the geometry of such regions (see Lemma 5.21) and — most importantly — prove the corresponding uniform bounds for eigenvalues and eigenfunctions (collected in Lemma 5.25), which allow us to rule out pathologic concentration phenomena, thereby leading to the desired conclusions.

2. Notation and standing assumptions

2.1. Boundary value problems for Schrödinger operators on Lipschitz domains. Let Ω be a Lipschitz domain of a smooth, compact d -dimensional manifold M with (possibly empty) boundary ∂M , by which we mean here a nonempty, open subset of M whose boundary is everywhere locally representable as the graph of a Lipschitz function. We do not require — at least in general — Ω to be connected, and we admit the case $\bar{\Omega} = M$ (where $\bar{\Omega}$ denotes the closure of Ω in M), when of course $\partial\Omega = \partial M$, the boundary of the ambient manifold in question. Throughout this article we will in fact assume $d \geq 2$.

We are going to study the spectrum of a given Schrödinger operator on Ω subject to boundary conditions and, sometimes, symmetry constraints. Such symmetry constraints will be encoded in terms of equivariance with respect to a certain group action, which we shall specify in due time.

The Schrödinger operator

$$\Delta_g + q$$

is determined by the data of a given smooth Riemannian metric g on $\bar{\Omega}$ and a given smooth (i.e., C^∞) function $q : \bar{\Omega} \rightarrow \mathbb{R}$. To avoid ambiguities, we remark here that a function (or tensor field) on $\bar{\Omega}$ is smooth if it is the restriction of a smooth tensor field on M or — equivalently — on a relatively open set containing $\bar{\Omega}$.

The boundary conditions are specified by another smooth function $r : \bar{\Omega} \rightarrow \mathbb{R}$ and a decomposition

$$\partial\Omega = \overline{\partial_D\Omega} \cup \overline{\partial_N\Omega} \cup \overline{\partial_R\Omega}, \quad (2-1)$$

where the sets on the right-hand side are the closures of pairwise disjoint open subsets $\partial_D\Omega$, $\partial_N\Omega$, and $\partial_R\Omega$ of $\partial\Omega$.

Somewhat more specifically, we will consider the spectrum of the operator $\Delta_g + q$ subject to the Dirichlet, Neumann, and Robin conditions

$$\begin{cases} u = 0 & \text{on } \partial_D\Omega, \\ du(\eta_g^\Omega) = 0 & \text{on } \partial_N\Omega, \\ du(\eta_g^\Omega) = ru & \text{on } \partial_R\Omega, \end{cases} \quad (2-2)$$

where η_g^Ω is the almost-everywhere defined outward unit normal induced by g on $\partial\Omega$.

It is obviously the case that the Neumann boundary conditions can be regarded as a special case of their inhomogeneous counterpart, however it is convenient — somewhat artificially — to distinguish them in view of the later applications we have in mind — the study of the Morse index of free boundary minimal surfaces.

2.2. Sobolev spaces and traces. To pose the problem precisely, we introduce the Sobolev space $H^1(\Omega, g)$ consisting of all real-valued functions in $L^2(\Omega, g)$ which have a weak g -gradient whose pointwise g -norm is also in $L^2(\Omega, g)$; then $H^1(\Omega, g)$ is a Hilbert space equipped with the inner product

$$\langle u, v \rangle_{H^1(\Omega, g)} := \int_{\Omega} (uv + g(\nabla_g u, \nabla_g v)) d\mathcal{H}^d(g),$$

integrating with respect to the d -dimensional Hausdorff measure induced by g . (We say a function $u \in L^1_{\text{loc}}(\Omega, g)$ has a weak g -gradient $\nabla_g u$ if $\nabla_g u$ is a measurable vector field on Ω with pointwise g -norm in $L^1_{\text{loc}}(\Omega, g)$ and $\int_{\Omega} g(X, \nabla_g u) d\mathcal{H}^d(g) = -\int_{\Omega} u \operatorname{div}_g X d\mathcal{H}^d(g)$ for every smooth vector field X on Ω of relatively compact support, where $\operatorname{div}_g X$ is the g divergence of X ; $\nabla_g u$ is uniquely defined whenever it exists, modulo vector fields vanishing almost everywhere.)

Under our assumptions on $\partial\Omega$, we have a bounded trace map $H^1(\Omega, g) \rightarrow L^2(\partial\Omega, g)$ extending the restriction map $C^1(\bar{\Omega}) \rightarrow C^0(\partial\Omega)$. (The Hilbert space $L^2(\partial\Omega, g)$ is defined using either the $(d-1)$ -dimensional Hausdorff measure $\mathcal{H}^{d-1}(g)$ induced by g or, equivalently, the almost-everywhere defined volume density induced by g on $\partial\Omega$.) In fact, we have not only boundedness of this map but also the stronger inequality

$$\|u|_{\partial\Omega}\|_{L^2(\partial\Omega, g)} \leq C(\Omega, g)(\epsilon\|u\|_{H^1(\Omega, g)} + C(\epsilon)\|u\|_{L^2(\Omega, g)}) \quad (2-3)$$

for all $u \in H^1(\Omega, g)$, all $\epsilon > 0$, some $C(\Omega, g)$ independent of u and ϵ , and some $C(\epsilon)$ independent of u and (Ω, g) . (This can be deduced, for example, by inspecting the proof of Theorem 4.6 in [Evans and Gariepy 2015]: specifically, we can apply the Cauchy–Schwarz inequality (weighting with ϵ , as is standard) to the inequality immediately above the line labeled $(\star\star\star)$ on page 158 of the preceding reference, whose treatment of Lipschitz domains in Euclidean space is readily adapted to our setting.)

For each $C \in \{D, N, R\}$, indicating one of the boundary conditions we wish to impose, by composing the preceding trace map with the restriction $L^2(\partial\Omega, g) \rightarrow L^2(\partial_C\Omega, g)$, since $\partial_C\Omega$ is open in $\partial\Omega$, we also get a trace map $\cdot|_{\partial_C} : H^1(\Omega, g) \rightarrow L^2(\partial_C\Omega, g)$. In practice we will consider traces on just $\partial_D\Omega$ and $\partial_R\Omega$. Considering the condition on $\partial_D\Omega$, we will then define

$$H^1_{\partial_D\Omega}(\Omega, g) := \{u \in H^1(\Omega, g) : u|_{\partial_D\Omega} = 0\},$$

which is obviously to be understood in the sense of traces, in the terms we just described, and we remark that (2-3) also clearly holds with $\partial\Omega$ on the left-hand side replaced by $\partial_R\Omega$ (or by $\partial_D\Omega$ or $\partial_N\Omega$, but we have no need of the inequality in these cases).

2.3. Bilinear forms and their eigenvalues and eigenspaces. Corresponding to the above data we define the bilinear form $T = T[\Omega, g, q, r, \partial_D\Omega, \partial_N\Omega, \partial_R\Omega]$ by

$$T : H^1_{\partial_D\Omega}(\Omega, g) \times H^1_{\partial_D\Omega}(\Omega, g) \rightarrow \mathbb{R},$$

$$(u, v) \mapsto \int_{\Omega} (g(\nabla_g u, \nabla_g v) - quv) d\mathcal{H}^d(g) - \int_{\partial_R\Omega} ruv d\mathcal{H}^{d-1}(g). \quad (2-4)$$

Then T is symmetric, bounded, and coercive as encoded in the following three equations, respectively:

$$\text{for all } u, v \in H^1_{\partial_D\Omega}(\Omega, g), \quad T(u, v) = T(v, u),$$

$$\text{for all } u \in H^1_{\partial_D\Omega}(\Omega, g), \quad T(u, u) \leq (1 + C(\Omega, g, q, r))\|u\|_{H^1(\Omega, g)}^2, \quad (2-5)$$

$$\text{for all } u \in H^1_{\partial_D\Omega}(\Omega, g), \quad T(u, u) \geq \frac{1}{2}\|u\|_{H^1(\Omega, g)}^2 - C(\Omega, g, q, r)\|u\|_{L^2(\Omega, g)}^2, \quad (2-6)$$

where, for (2-5) and (2-6), one can take $C(\Omega, g, q, r) = \|q\|_{C^0(\bar{\Omega})} + C(\Omega, g)\|r\|_{C^0(\bar{\partial_R\Omega})}$ thanks to the trace inequality (2-3). From these three properties and the Riesz representation theorem for Hilbert spaces, it follows that, for some constant $\Lambda = \Lambda(\Omega, g, q, r) > 0$, there exists a linear map $R : L^2(\Omega, g) \rightarrow H^1_{\partial_D\Omega}(\Omega, g)$ such that

$$T(Rf, v) + \Lambda \langle \iota Rf, \iota v \rangle_{L^2(\Omega, g)} = \langle f, \iota v \rangle_{L^2(\Omega, g)}$$

for all functions $f \in L^2(\Omega, g)$ and $v \in H^1_{\partial_D\Omega}(\Omega, g)$, where we have introduced the inclusion map $\iota : H^1_{\partial_D\Omega}(\Omega, g) \rightarrow L^2(\Omega, g)$.

(Of course, if f is smooth then standard elliptic *interior* regularity results ensure that u is as well smooth on Ω and there satisfies the equation $-(\Delta_g + q - \Lambda)u = f$ in a classical pointwise sense.) Since the inclusion $H^1(\Omega, g) \hookrightarrow L^2(\Omega, g)$ is compact (see for example Section 7 of Chapter 4 of [Taylor 1996]) and of course the inclusion of the closed subspace $H^1_{\partial_D\Omega}(\Omega, g) \hookrightarrow H^1(\Omega, g)$ is bounded, the aforementioned maps $\iota : H^1_{\partial_D\Omega}(\Omega, g) \rightarrow L^2(\Omega, g)$ and the composite $\iota R : L^2(\Omega, g) \rightarrow L^2(\Omega, g)$ are also

both compact operators. Furthermore, to confirm that ιR is symmetric, we simply note that (by appealing to the equation defining the operator R , with Rf_1 and Rf_2 in place of v)

$$\begin{aligned} \langle f_2, \iota Rf_1 \rangle_{L^2(\Omega, g)} &= T(Rf_2, Rf_1) + \Lambda \langle \iota Rf_2, \iota Rf_1 \rangle_{L^2(\Omega, g)} \\ &= T(Rf_1, Rf_2) + \Lambda \langle \iota Rf_1, \iota Rf_2 \rangle_{L^2(\Omega, g)} = \langle f_1, \iota Rf_2 \rangle_{L^2(\Omega, g)} \end{aligned}$$

for all $f_1, f_2 \in L^2(\Omega, g)$. That being clarified, to improve readability we will from now on refrain from explicitly indicating the inclusion map ι in our equations.

With slight abuse of language, in the setting above we call $\lambda \in \mathbb{R}$ an *eigenvalue* of T if there exists a nonzero $u \in H_{\partial_D}^1(\Omega, g)$ such that,

$$\text{for all } v \in H_{\partial_D}^1(\Omega, g), \quad T(u, v) = \lambda \langle u, v \rangle_{L^2(\Omega, g)}, \quad (2-7)$$

and we call any such u an *eigenfunction* of T with eigenvalue λ . (We caution that the notions of eigenfunctions and eigenvalues depend not only on T but also on the underlying metric g ; for the sake of convenience we choose to suppress the latter dependence from our notation.)

Hence, as a consequence of the key facts we presented before this definition, one can prove by well-known arguments the existence of a discrete spectrum for the “shifted” elliptic operator $(\Delta_g + q) - \Lambda$ subject to the very same boundary conditions (2-2). As a straightforward corollary, by accounting for the shift, we obtain the following conclusions for T :

- The set of eigenvalues of T is discrete in \mathbb{R} and bounded below.
- For each eigenvalue of T , the corresponding eigenspace has finite dimension.
- There exists a Hilbertian basis $\{e_j\}_{j=1}^\infty$ for $L^2(\Omega, g)$ consisting of eigenfunctions of T .
- $\{e_j\}_{j=1}^\infty$ has dense span in $H_{\partial_D}^1(\Omega, g)$.

(To avoid ambiguities, we remark that the phrase *Hilbertian basis* refers to a countable, complete orthonormal system for the Hilbert space in question.) For each integer $i \geq 1$, we write $\lambda_i(T)$ for the i -th eigenvalue of T (listed with repetitions in nondecreasing order, in the usual fashion). We have the usual min-max characterization

$$\lambda_i(T) = \min \left\{ \max \left\{ \frac{T(w, w)}{\|w\|_{L^2(\Omega, g)}^2} : 0 \neq w \in W \right\} : W \subset_{\text{subspace}} H_{\partial_D}^1(\Omega, g), \dim W = i \right\}. \quad (2-8)$$

Next, for any $t \in \mathbb{R}$, we let $E^{=t}(T)$ denote the (possibly trivial) linear span, in $H_{\partial_D}^1(\Omega, g)$, of the eigenfunctions of T with eigenvalue t , and, more generally, for any $t \in \mathbb{R}$ and any binary relation \sim on \mathbb{R} (in practice $<$, \leq , $>$, \geq , or $=$), we set

$$E^{\sim t}(T) := \text{Closure}_{L^2(\Omega, g)} \left(\text{Span} \left(\bigcup_{s \sim t} E^{=s}(T) \right) \right),$$

and we denote the corresponding orthogonal projection by

$$\pi_T^{\sim t} : L^2(\Omega, g) \rightarrow E^{\sim t}(T).$$

That is, the space $E^{\sim t}(T)$ has been defined to be the closure in $L^2(\Omega, g)$ of the span of all eigenfunctions of T having eigenvalue λ such that $\lambda \sim t$. Of course $E^{\sim t}(T)$ is a subspace of $H^1_{\partial\Omega}(\Omega, g)$ — in particular — whenever the former has finite dimension. Taking \sim to be equality clearly reproduces the originally defined space $E^{=t}(T)$.

For future use, observe that the above spectral theorem for T implies

$$(E^{\sim t}(T))^{\perp_{L^2(\Omega, g)}} = E^{\not\sim t}(T), \quad E^{<t}(T) \underset{\text{subspace}}{\subset} E^{\leq t}(T) \underset{\text{subspace}}{\subset} H^1_{\partial\Omega}(\Omega, g), \tag{2-9}$$

for all $u \in E^{\sim t}(T) \cap H^1_{\partial\Omega}(\Omega, g)$, $T(u, u) \sim t\|u\|^2_{L^2(\Omega, g)}$ for \sim any one of $<, \leq, >, \geq$,

and,

$$\text{for all } u \in H^1_{\partial\Omega}(\Omega, g) \cap (E^{\leq t}(T) \cup E^{\geq t}(T)), \quad T(u, u) = t\|u\|^2_{L^2(\Omega, g)} \implies u \in E^{=t}(T),$$

throughout which t is any real number (not necessarily an eigenvalue of T) and where in the first equality of (2-9) \sim is any relation on \mathbb{R} and $\not\sim$ its negation (so that $\{s \not\sim t\} = \mathbb{R} \setminus \{s \sim t\}$ for any $t \in \mathbb{R}$).

Index and nullity. In the setting above and under the corresponding standing assumption, we shall define the nonnegative integers

$$\text{ind}(T) := \dim E^{<0}(T) \quad \text{and} \quad \text{nul}(T) := \dim E^{=0}(T)$$

called, respectively, the *index* and *nullity* of T . Such invariants will be of primary interest in our applications.

2.4. Group actions. Let G be a finite group of smooth diffeomorphisms of M , each restricting to an isometry of $(\bar{\Omega}, g)$. Then, as for any group of diffeomorphism of Ω , we have the standard (left) action of G on functions on Ω via pullback:

$$(\phi, u) \mapsto u \circ \phi^{-1} = \phi^{-1*}u \quad \text{for all } \phi \in G, \quad u : \Omega \rightarrow \mathbb{R}.$$

We say that a function u is G -invariant if it is invariant under this action: equivalently $u \circ \phi = u$ for all $\phi \in G$.

We can also twist this action by orthogonal transformations on the fiber \mathbb{R} : given in addition to G a group homomorphism $\sigma : G \rightarrow \text{O}(1) = \{-1, 1\}$, we define the action

$$(\phi, u) \mapsto \sigma(\phi)(u \circ \phi^{-1}) = \sigma(\phi)\phi^{-1*}u \quad \text{for all } \phi \in G, \quad u : \Omega \rightarrow \mathbb{R},$$

and we call a function (G, σ) -invariant if it is invariant under this action. Obviously the above standard action $(\phi, u) \mapsto u \circ \phi^{-1}$ is recovered by taking the trivial homomorphism $\sigma \equiv 1$. We also comment that one could of course replace \mathbb{R} by \mathbb{C} and correspondingly $\text{O}(1)$ by $\text{U}(1)$ (and in the preceding sections instead work with Sobolev spaces over \mathbb{C}), though we restrict our attention to real-valued functions in this article.

Since, by virtue of our initial requirement, G is a group of isometries of $(\bar{\Omega}, g)$, the above twisted action yields a unitary representation of G in $L^2(\Omega, g)$, i.e., a group homomorphism

$$\hat{\sigma} : G \rightarrow \text{O}(L^2(\Omega, g)), \quad \phi \mapsto \sigma(\phi)\phi^{-1*}, \tag{2-10}$$

whose targets are the global isometries of $L^2(\Omega, g)$; we note that the same conclusions hold with $H^1(\Omega, g)$ in place of $L^2(\Omega, g)$. The corresponding subspaces of (G, σ) -invariant functions, in $L^2(\Omega, g)$ or $H^1(\Omega, g)$, are readily checked to be closed and thus Hilbert spaces themselves. That said, we define the orthogonal projection

$$\pi_{G,\sigma} : L^2(\Omega, g) \rightarrow L^2(\Omega, g), \quad u \mapsto \frac{1}{|G|} \sum_{\phi \in G} \hat{\sigma}(\phi)u. \quad (2-11)$$

Here $|G|$ is the order of G , which—we recall—is assumed throughout to be finite. The image of $L^2(\Omega, g)$ under $\pi_{G,\sigma}$ thus consists of (G, σ) -invariant functions.

Remark 2.1. One could lift the finiteness assumption, say by allowing G to be a compact Lie group, requiring σ to be continuous, and replacing the finite average in (2-11) with the average over G with respect to its Haar measure (which reduces to the former for finite G). However, with a view towards our later applications, in this article we content ourselves with the finiteness assumption, which allows for a lighter exposition.

Henceforth we make the additional assumptions that G globally (i.e., as sets) preserves each of $\partial_D \Omega$, $\partial_N \Omega$, and $\partial_R \Omega$, and that q and r are both G -invariant. Each element of $\hat{\sigma}(G)$ then preserves also $H_{\partial_D \Omega}^1(\Omega, g)$ and the bilinear form T , and the projection $\pi_{G,\sigma}$ commutes with the projection π_T^t for any $t \in \mathbb{R}$ and binary relation \sim on \mathbb{R} (as above). In particular $\pi_{G,\sigma}$ preserves each eigenspace $E^t(T)$ of T , and more generally the space

$$E_{G,\sigma}^{\sim t}(T) := \pi_{G,\sigma}(E^{\sim t}(T)) \quad (2-12)$$

is a subspace of $E^{\sim t}(T)$.

For each integer $i \geq 1$, we can then define $\lambda_i^{G,\sigma}(T)$, the i -th (G, σ) -eigenvalue of T , to be the i -th eigenvalue of T having a (G, σ) -invariant eigenfunction (by definition nonzero), counting with multiplicity as before; equivalently one can work with spaces of (G, σ) -invariant functions and derive the analogous conclusions as in Section 2.3 directly in that setting.

Remark 2.2. We explicitly note, for the sake of completeness, that under no additional assumptions on the group G and the homomorphism σ it is possible that the space of (G, σ) -invariant functions be finite dimensional (possibly even of dimension zero). This type of phenomenon happens, for instance, when every point of the manifold M is a fixed point of an isometry on which σ takes the value -1 . In this case, all conclusions listed above still hold but need to be understood with a bit of care: the corresponding sequence of eigenvalues $\lambda_1^{G,\sigma}(T) \leq \lambda_2^{G,\sigma}(T) \leq \dots$ will in fact just be a finite sequence, consisting say of $I(G, \sigma)$ elements, counted with multiplicity as usual; we shall use the convention that $\lambda_i^{G,\sigma}(T) = +\infty$ for $i > I(G, \sigma)$. That being said, we also remark that this phenomenon patently does not occur for the Jacobi form of the two sequences of free boundary minimal surfaces we examine in Sections 4 and 5.

In this equivariant framework we still have the corresponding min-max characterization

$$\lambda_i^{G,\sigma}(T) = \min \left\{ \max \left\{ \frac{T(w, w)}{\|w\|_{L^2(\Omega, g)}^2} : 0 \neq w \in W \right\} : W \underset{\text{subspace}}{\subset} \pi_{G,\sigma}(H_{\partial_D \Omega}^1(\Omega, g)), \dim W = i \right\}. \quad (2-13)$$

We also define the (G, σ) -index and (G, σ) -nullity

$$\text{ind}_G^\sigma(T) := \dim E_{G,\sigma}^{<0}(T) \quad \text{and} \quad \text{nul}_G^\sigma(T) := \dim E_{G,\sigma}^{=0}(T)$$

of T . Obviously we can recover $E^{\sim t}(T)$, $\lambda_i(T)$, and the standard index and nullity by taking G to be the trivial group. As mentioned in the introduction, we reiterate that it is one of the goals of the present article to study, for fixed g and T , how these numbers (index $\text{ind}_G^\sigma(T)$ and nullity $\text{nul}_G^\sigma(T)$) depend on G and σ .

Terminology. For the sake of brevity, we shall employ the phrase *admissible data* to denote any tuple $(\Omega, g, q, r, \partial_D\Omega, \partial_N\Omega, \partial_R\Omega, G, \sigma)$ satisfying all the standing assumptions presented up to now. We digress briefly to highlight two important special cases, which warrant additional notation.

Example 2.3 (actions of order-2 groups). When $|G| = 2$, there are precisely two homomorphisms $G \rightarrow \text{O}(1)$. Considering such homomorphisms and the corresponding (G, σ) -invariant functions, we may define G -even or G -odd functions. Hence, we may call ind_G^+ and ind_G^- the G -even and G -odd index, and likewise for the nullity. Clearly, we always have

$$\begin{cases} \text{ind}(T) = \text{ind}_G^+(T) + \text{ind}_G^-(T), \\ \text{nul}(T) = \text{nul}_G^+(T) + \text{nul}_G^-(T). \end{cases} \tag{2-14}$$

Example 2.4 (actions of self-congruences of two-sided hypersurfaces). Suppose, momentarily, that (M, g) is isometrically embedded (as a codimension-one submanifold) in a Riemannian manifold (N, h) , that the set Ω is connected, and assume further that the normal bundle of M over Ω is trivial. Then we can pick a unit normal ν on Ω and thereby identify — as usual — sections of the normal bundle of $M|_\Omega$ with functions on Ω . With this interpretation of functions on Ω in mind and G now a finite group of diffeomorphisms of N that map Ω onto itself (as a set) and everywhere on Ω preserve the ambient metric h meaning that $\phi^*h = h$ for any $\phi \in G$, we have a natural action given by

$$(\phi, u) \mapsto \text{sgn}_\nu(\phi)(u \circ \phi^{-1}) \quad \text{for all } \phi \in G, \quad u : \Omega \rightarrow \mathbb{R},$$

where $\text{sgn}_\nu(\phi) := h(\phi_*\nu, \nu)$ is a constant in $\text{O}(1) = \{1, -1\}$. We shall further assume that the action of G on Ω is faithful, meaning that only the identity element fixes Ω pointly; this assumption is always satisfied in our applications.

In this context we continue to say that a function $u : \Omega \rightarrow \mathbb{R}$ is G -invariant if $u = u \circ \phi$ for all $\phi \in G$, and we say rather that u is G -equivariant if $u = \text{sgn}_\nu(\phi)u \circ \phi$ for all $\phi \in G$ (that is, noting the identity $\text{sgn}_\nu(\phi) = \text{sgn}_\nu(\phi^{-1})$, provided u is invariant under the sgn_ν -twisted G action).

Similarly, in this context, we set

$$\text{ind}_G(T) := \text{ind}_G^{\text{sgn}_\nu}(T) \quad \text{and} \quad \text{nul}_G T := \text{nul}_G^{\text{sgn}_\nu}(T), \tag{2-15}$$

which we may refer to as simply the G -equivariant index and G -equivariant nullity of T . We point out that we are abusing notation in the above definitions in that, on the right-hand side of each, in place of G we mean really the group, isomorphic to G by virtue of the faithfulness assumption, obtained by restricting each element of G to Ω , and in place of sgn_ν we mean really the corresponding homomorphism, well-defined by the faithfulness assumption, on this last group of isometries of Ω .

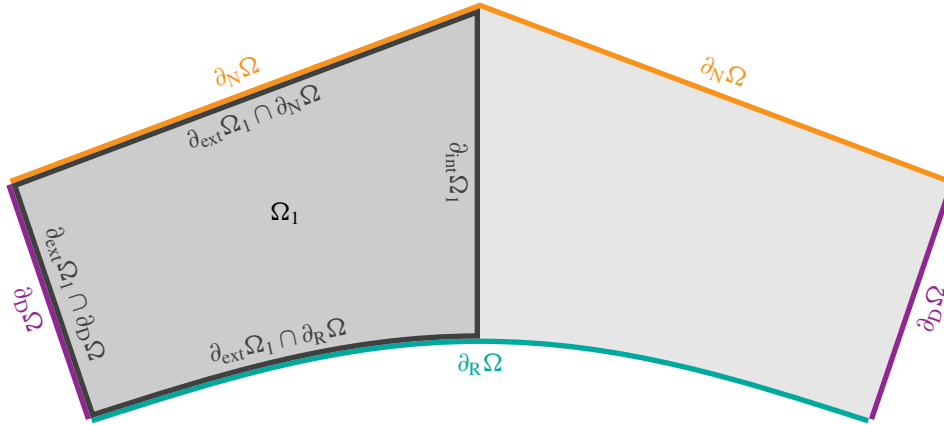


Figure 1. Example of a Lipschitz domain Ω with subdomain Ω_1 .

We now return to the more general assumptions on G preceding this paragraph.

2.5. Subdomains. Suppose that $\Omega_1 \subset \Omega$ is another Lipschitz domain of M (see Figure 1). We shall define

$$\begin{aligned}
 \partial_{\text{int}}\Omega_1 &:= \partial\Omega_1 \cap \Omega, & \partial_{\text{ext}}\Omega_1 &:= \partial\Omega_1 \setminus \overline{\partial_{\text{int}}\Omega_1}, \\
 \partial_{\text{D}}^{\text{Dint}}\Omega_1 &:= (\partial_{\text{ext}}\Omega_1 \cap \partial_{\text{D}}\Omega) \cup \partial_{\text{int}}\Omega_1, & \partial_{\text{D}}^{\text{Nint}}\Omega_1 &:= \partial_{\text{ext}}\Omega_1 \cap \partial_{\text{D}}\Omega, \\
 \partial_{\text{N}}^{\text{Dint}}\Omega_1 &:= \partial_{\text{ext}}\Omega_1 \cap \partial_{\text{N}}\Omega, & \partial_{\text{N}}^{\text{Nint}}\Omega_1 &:= (\partial_{\text{ext}}\Omega_1 \cap \partial_{\text{N}}\Omega) \cup \partial_{\text{int}}\Omega_1, \\
 \partial_{\text{R}}^{\text{Dint}}\Omega_1 &:= \partial_{\text{ext}}\Omega_1 \cap \partial_{\text{R}}\Omega, & \partial_{\text{R}}^{\text{Nint}}\Omega_1 &:= \partial_{\text{ext}}\Omega_1 \cap \partial_{\text{R}}\Omega.
 \end{aligned}
 \tag{2-16}$$

In this way we prepare to pose two different sets of boundary conditions on Ω_1 , whereby, roughly speaking, in both cases $\partial\Omega_1$ inherits whatever boundary condition is in effect on $\partial\Omega$ wherever the two meet (corresponding to $\partial_{\text{ext}}\Omega_1$) and the two sets of conditions are distinguished by placing either the Dirichlet or the Neumann condition on the remainder of the boundary (corresponding to $\partial_{\text{int}}\Omega_1$). Naturally associated to these two sets of conditions are the bilinear forms

$$\begin{aligned}
 T_{\Omega_1}^{\text{Dint}} &:= T[\Omega_1, g, q, r, \partial_{\text{D}}^{\text{Dint}}\Omega_1, \partial_{\text{N}}^{\text{Dint}}\Omega_1, \partial_{\text{R}}^{\text{Dint}}\Omega_1], \\
 T_{\Omega_1}^{\text{Nint}} &:= T[\Omega_1, g, q, r, \partial_{\text{D}}^{\text{Nint}}\Omega_1, \partial_{\text{N}}^{\text{Nint}}\Omega_1, \partial_{\text{R}}^{\text{Nint}}\Omega_1],
 \end{aligned}
 \tag{2-17}$$

defined, respectively, on the Sobolev spaces

$$H_{\partial_{\text{D}}^{\text{Dint}}\Omega_1}^1(\Omega_1, g) \quad \text{and} \quad H_{\partial_{\text{D}}^{\text{Nint}}\Omega_1}^1(\Omega_1, g).$$

Recalling (G, σ) from above, with the tacit understanding that $(\Omega, g, q, r, \partial_{\text{D}}\Omega, \partial_{\text{N}}\Omega, \partial_{\text{R}}\Omega, G, \sigma)$ is admissible, we further assume that each element of G maps Ω_1 onto itself; since G preserves Ω and respects the decomposition (2-1), it follows that it also respects the decompositions (2-16). Somewhat abusively, we shall write $\hat{\sigma}$ and $\pi_{G,\sigma}$ not only for the maps (2-10) and (2-11) but also for their counterparts with Ω replaced by Ω_1 , which are well-defined under our assumptions. The spaces $E_{G,\sigma}^{\sim t}(T_{\Omega_1}^{\text{Dint}})$ and $E_{G,\sigma}^{\sim t}(T_{\Omega_1}^{\text{Nint}})$ as in (2-12) are then also well-defined.

3. Fundamental tools

3.1. Index and nullity bounds in the style of Montiel and Ros. Recalling the notation and assumptions of Section 2, suppose now that we have not only $\Omega_1 \subset \Omega$ as above, but also (open) Lipschitz subdomains $\Omega_1, \dots, \Omega_n \subset \Omega$ which are pairwise disjoint, each of which satisfies the same assumptions as Ω_1 in Section 2.5 and whose closures cover $\bar{\Omega}$. In particular, we assume that each element of the group G maps each subdomain Ω_i onto itself. We assume further that G acts transitively on the connected components of Ω and note that this last condition is always satisfied in the important special case that Ω is connected.

Proposition 3.1 (Montiel–Ros bounds on the number of eigenvalues below a threshold). *With assumptions as in the preceding paragraph and notation as in Section 2, the following inequalities hold for any $t \in \mathbb{R}$:*

- (i) $\dim E_{G,\sigma}^{<t}(T) \geq \dim E_{G,\sigma}^{<t}(T_{\Omega_1}^{\text{Dint}}) + \sum_{i=2}^n \dim E_{G,\sigma}^{<t}(T_{\Omega_i}^{\text{Dint}}),$
- (ii) $\dim E_{G,\sigma}^{\leq t}(T) \leq \dim E_{G,\sigma}^{\leq t}(T_{\Omega_1}^{\text{Nint}}) + \sum_{i=2}^n \dim E_{G,\sigma}^{\leq t}(T_{\Omega_i}^{\text{Nint}}).$

The statement and proof of Proposition 3.1 are adapted from Lemmas 12 and 13 of [Montiel and Ros 1991], which concern the spectrum of the Laplacian on branched coverings of the round sphere and rely on standard, fundamental facts about eigenvalues and eigenfunctions of Schrödinger operators, much as in the proof of the classical Courant nodal domain theorem. These arguments are readily applied to more general Schrödinger operators on more general domains, as observed for instance in [Kapouleas and Wiygul 2020], where such bounds in the style of Montiel and Ros played a major role in the computation of the index and nullity of the $\xi_{g,1}$ Lawson surfaces. Here, instead, we present an extended version allowing for the imposition of mixed (Robin and Dirichlet) boundary conditions and invariance under a group action; as mentioned in the introduction, this level of generality is motivated by the goal of bounding (from above and below) the G -equivariant Morse index of free boundary minimal surfaces. (Our treatment of course includes the fundamental case when G is the trivial group.)

Proof. Throughout the proof we will make free use of the consequences (2-9) of the spectral theorem for the various bilinear forms appearing in the statement. Fix $t \in \mathbb{R}$. For (i) we will verify injectivity of the map

$$\iota^{\text{Dint}} : E_{G,\sigma}^{<t}(T_{\Omega_1}^{\text{Dint}}) \oplus \bigoplus_{i=2}^n E_{G,\sigma}^{\leq t}(T_{\Omega_i}^{\text{Dint}}) \rightarrow E_{G,\sigma}^{<t}(T), \quad (u_1, u_2, \dots, u_n) \mapsto \pi_T^{<t} \left(\sum_{i=1}^n U_i \right),$$

where each U_i is the extension to Ω of u_i such that U_i vanishes on $\Omega \setminus \Omega_i$. Clearly, each such extension lies in the image of $\pi_{G,\sigma}$, which, as observed above, commutes with $\pi_T^{<t}$, so that the map is indeed well-defined with its asserted target. Now suppose that (u_1, \dots, u_n) belongs to the domain of ι^{Dint} , and set $v := \sum_{i=1}^n U_i$. Then $v \in H_{\partial\Omega}^1(\Omega, g)$ and

$$T(v, v) = \sum_{i=1}^n T_{\Omega_i}^{\text{Dint}}(u_i, u_i) \leq t \|v\|_{L^2(\Omega, g)}^2,$$

with equality possible only when $u_1 = 0$. To check injectivity suppose next that $\iota^{\text{Dint}}(u_1, \dots, u_n) = 0$. By definition of ι^{Dint} this assumption means that v is $L^2(\Omega, g)$ -orthogonal to $E_{G,\sigma}^{<t}(T)$, and so in view of the preceding inequality and (2-9) we have $v \in E_{G,\sigma}^{\leq t}(T)$. Thus, v satisfies the elliptic equation

$(\Delta_g + q + t)u = 0$; moreover, we must also have $v|_{\Omega_1} = u_1 = 0$, but now the unique continuation principle [Aronszajn 1957] implies that $v = 0$, whence $(u_1, \dots, u_n) = 0$, completing the proof of (i).

For (ii) we verify injectivity of

$$\begin{aligned} \iota^{\text{Nint}} : E_{G,\sigma}^{\leq t}(T) &\rightarrow E_{G,\sigma}^{\leq t}(T_{\Omega_1}^{\text{Nint}}) \oplus \bigoplus_{i=2}^n E_{G,\sigma}^{\leq t}(T_{\Omega_i}^{\text{Nint}}), \\ u &\mapsto (\pi_{T_{\Omega_1}^{\text{Nint}}}^{\leq t} u|_{\Omega_1}, \pi_{T_{\Omega_2}^{\text{Nint}}}^{\leq t} u|_{\Omega_2}, \dots, \pi_{T_{\Omega_n}^{\text{Nint}}}^{\leq t} u|_{\Omega_n}) \end{aligned}$$

instead. Note that

$$u|_{\Omega_i} \in \pi_{G,\sigma}(L^2(\Omega_i, g)) \cap H_{\partial\Omega_i}^1(\Omega_i, g) \quad (3-1)$$

for each i ; in particular, the left inclusion and the commutativity of $\pi_{G,\sigma}$ with each of the spectral projections appearing in the definition of ι^{Nint} ensure that the latter really is well-defined. Suppose then that u belongs to the domain of ι^{Nint} and $\iota^{\text{Nint}}u = (0, \dots, 0)$. The second assumption (making use of the right inclusion in (3-1) in addition to (2-9)) implies

$$T(u, u) = \sum_{i=1}^n T_{\Omega_i}^{\text{Nint}}(u|_{\Omega_i}, u|_{\Omega_i}) \geq t \|u\|_{L^2(\Omega, g)}^2,$$

with equality possible only when $u|_{\Omega_1} = 0$. Recalling that, by assumption, $u \in E_{G,\sigma}^{\leq t}(T)$, we therefore conclude, appealing to (2-9), that $u \in E_{G,\sigma}^{\leq t}(T)$ and indeed this equality case holds. In particular, u satisfies the elliptic equation $(\Delta_g + q + t)u = 0$, but then the condition $u|_{\Omega_1} = 0$ and the unique continuation principle imply $u = 0$, ending the proof. \square

In particular, in our applications we will repeatedly (yet not always) appeal to the special case when $t = 0$ and Ω (most often equal to the whole ambient manifold itself M) is partitioned into a finite collection of pairwise isometric domains.

Corollary 3.2 (Montiel–Ros index and nullity bounds from isometric pieces). *In the setting of the previous proposition, let us suppose the domains $\Omega_1, \dots, \Omega_n$ to be pairwise isometric via isometries of Ω . Then*

- (i) $\text{ind}_G^\sigma(T) \geq n \text{ind}_G^\sigma(T_{\Omega_1}^{\text{Dint}}) + (n-1) \text{nul}_G^\sigma(T_{\Omega_1}^{\text{Dint}})$,
- (ii) $\text{ind}_G^\sigma(T) + \text{nul}_G^\sigma(T) \leq n \text{ind}_G^\sigma(T_{\Omega_1}^{\text{Nint}}) + \text{nul}_G^\sigma(T_{\Omega_1}^{\text{Nint}})$.

Remark 3.3. We further explicitly note how the two inequalities given in the previous corollary jointly imply the “compatibility condition” that

$$(n-1) \text{nul}_G^\sigma(T_{\Omega_1}^{\text{Dint}}) - \text{nul}_G^\sigma(T_{\Omega_1}^{\text{Nint}}) \leq n(\text{ind}_G^\sigma(T_{\Omega_1}^{\text{Nint}}) - \text{ind}_G^\sigma(T_{\Omega_1}^{\text{Dint}})), \quad (3-2)$$

which in general has nontrivial content.

Remark 3.4. The requirement that the domains in question be G -invariant implies, in certain examples, that some of them may in fact have to be taken disconnected. We will however discuss, in the next subsection, how this nuisance may actually be avoided in the totality of our later applications.

3.2. Reduction and extension of domain under symmetries. With our standing assumptions on (Ω, g) , T and (G, σ) in place, encoded in the requirement that they determine admissible data, we again assume that $\Omega_1, \dots, \Omega_n \subset \Omega$ are pairwise disjoint Lipschitz domains whose closures cover Ω . However, for the specific purposes of this section, we assume Ω is connected and, rather than assuming G -invariance of each Ω_i , we instead suppose that G preserves the collection $\{\Omega_i\}_{i=1}^n$ (while — as per our general postulate — also respecting the decomposition (2-1), which dictates the boundary conditions (2-2)) and acts transitively on its elements (so in particular the Ω_i are pairwise isometric). The (possibly trivial) subgroup of G which preserves Ω_1 we call H . Note that H preserves $\partial_{\text{int}}\Omega_1$ in particular.

For each $p \in \partial_{\text{int}}\Omega_1$, we define

$$G_p := \{ \phi \in G : \exists U \underset{\text{open}}{\subseteq} \partial_{\text{int}}\Omega_1, p \in U \text{ and } \phi|_U = \text{id} \}.$$

Then G_p is a subgroup of G having order at most 2, as we now explain. Let $\phi_1, \phi_2 \in G_p$. Then we have open neighborhoods U_1, U_2 of p in $\partial_{\text{int}}\Omega_1$ with U_i fixed pointwise by ϕ_i . By the Lipschitz assumption, there exists $q \in U_1 \cap U_2$ at which $\partial_{\text{int}}\Omega_1$ has a well-defined outward unit conormal η_q . Then, for each i , we have $(d_q \phi_i)(\eta_q) = \epsilon_i \eta_q$ for some $\epsilon_i = \pm 1$. If an $\epsilon_i = +1$, then, since ϕ_i fixes U_i pointwise and Ω is connected, ϕ_i must be the identity on Ω (which comes essentially by arguing, e.g., as in Lemma 4.5 of [Carlotto and Li 2024]). If $\epsilon_1 = \epsilon_2 = -1$, then similarly $\phi_1 \circ \phi_2^{-1}$ is the identity on Ω , establishing our claim. Note also that the set $\{p : |G_p| = 2\}$ is open in $\partial_{\text{int}}\Omega_1$ and that, for each $\chi \in H$, the map $\phi \mapsto \chi \circ \phi \circ \chi^{-1}$ defines an isomorphism from G_p to $G_{\chi(p)}$ which commutes with σ .

For each p , we next set

$$\sigma_p := \begin{cases} 0 & \text{if } |G_p| = 1, \\ 1 & \text{if } |G_p| = 2 \text{ but } \sigma(G_p) = \{+1\}, \\ -1 & \text{if } |G_p| = 2 \text{ and } \sigma(G_p) = \{+1, -1\}, \end{cases}$$

and we in turn define the subsets $\partial_+\Omega_1, \partial_-\Omega_1 \subseteq \partial_{\text{int}}\Omega_1$ by letting (respectively)

$$\partial_{\pm}\Omega_1 := \sigma_p^{-1}(\pm 1).$$

With the aid of the foregoing observations, we see that $\partial_+\Omega_1$ and $\partial_-\Omega_1$ are open and disjoint, and each is preserved by H . We now impose the additional assumption that their closures cover $\partial_{\text{int}}\Omega_1$, and finally we set $T_{\Omega_1} := T[\Omega_1, g, q, r, \partial_{\text{D}}\Omega_1, \partial_{\text{N}}\Omega_1, \partial_{\text{R}}\Omega_1]$, where

$$\begin{aligned} \partial_{\text{D}}\Omega_1 &:= \partial_-\Omega_1 \cup (\partial_{\text{ext}}\Omega_1 \cap \partial_{\text{D}}\Omega), \\ \partial_{\text{N}}\Omega_1 &:= \partial_+\Omega_1 \cup (\partial_{\text{ext}}\Omega_1 \cap \partial_{\text{N}}\Omega), \\ \partial_{\text{R}}\Omega_1 &:= \partial_{\text{ext}}\Omega_1 \cap \partial_{\text{R}}\Omega. \end{aligned}$$

Lemma 3.5 (reduction and extension of domain under symmetries). *Under the above assumptions, for every integer $i \geq 1$,*

$$\lambda_i^{G,\sigma}(T) = \lambda_i^{H,\sigma}(T_{\Omega_1}),$$

and the (H, σ) -invariant eigenfunctions of T_{Ω_1} are the restrictions to Ω_1 of the (G, σ) -invariant eigenfunctions of T .

Proof. First observe that

$$v \in \pi_{G,\sigma} H_{\partial\Omega}^1(\Omega, g) \implies v|_{\Omega_1} \in \pi_{H,\sigma} H_{\partial\Omega_1}^1(\Omega_1, g),$$

using in particular the fact that any (G, σ) -invariant function in $H^1(\Omega, g)$ must have vanishing trace along $\partial_-\Omega_1$. Next observe that our assumptions guarantee that each (H, σ) -invariant function u on Ω_1 has a unique (G, σ) -invariant extension \bar{u} to Ω . This is also true of vector fields, the action being $(\phi, X) \mapsto \sigma(\phi)\phi_*X$. Now suppose $u \in \pi_{H,\sigma} H_{\partial\Omega_1}^1(\Omega_1, g)$. Obviously $\bar{u} \in L^2(\Omega, g)$, and we next check that in fact $\bar{u} \in H^1(\Omega, g)$ with $\nabla_g \bar{u} = \overline{\nabla_g u}$.

For this let X be a smooth vector field with support contained in Ω . Let $Y := \pi_{G,\sigma} X$ (meaning we average as in (2-11) but with the appropriate action for vector fields, as above). Then (writing, with slight abuse of notation, $L^2(\Omega, g)$ and $L^2(\Omega_1, g)$ also for the Hilbert spaces of L^2 vector fields on Ω and Ω_1 , respectively, in metric g)

$$\begin{aligned} \langle X, \overline{\nabla_g u} \rangle_{L^2(\Omega, g)} &= \langle Y, \overline{\nabla_g u} \rangle_{L^2(\Omega, g)} = n \langle Y|_{\Omega_1}, \nabla_g u \rangle_{L^2(\Omega_1, g)} \\ &= n \langle 1, \operatorname{div}(uY)|_{\Omega_1} \rangle_{L^2(\Omega_1, g)} - n \langle u, \operatorname{div} Y|_{\Omega_1} \rangle_{L^2(\Omega_1, g)} \\ &= n \langle u|_{\partial\Omega_1}, g(\eta_g^{\Omega_1}, Y|_{\partial\Omega_1}) \rangle_{L^2(\partial\Omega_1, g)} - \langle \bar{u}, \operatorname{div} Y \rangle_{L^2(\Omega, g)} \\ &= 0 - \langle \operatorname{div} X, \bar{u} \rangle_{L^2(\Omega, g)}; \end{aligned}$$

in the third line we have used the divergence theorem (see for example Theorem 4.6 of [Evans and Gariepy 2015] for a statement serving our assumptions) with $u|_{\partial\Omega_1}$ of course the trace of u and $\eta_g^{\Omega_1}$ the almost everywhere defined outward unit conormal, and in the fourth line we have used the fact that the (G, σ) -invariance of Y forces it to be (almost everywhere) orthogonal to this last conormal on $\partial_+\Omega_1$, while on the other hand, as already noted above, $u|_{\partial\Omega_1}$ vanishes on $\partial_-\Omega_1$. Thus every element of $\pi_{H,\sigma} H_{\partial\Omega_1}^1(\Omega_1, g)$ extends uniquely to an element of $\pi_{G,\sigma} H_{\partial\Omega}^1(\Omega, g)$. It is now straightforward to verify that, for all $t \in \mathbb{R}$, restriction to Ω_1 furnishes a bijection $E_{G,\sigma}^{-t}(T) \rightarrow E_{H,\sigma}^{-t}(T_{\Omega_1})$, which implies the claims. \square

For the purposes of our later geometric applications, it is convenient to focus on two special cases, which correspond to the examples we presented in Section 2.4.

Example 3.6 (actions of order-2 groups). With respect to our general setup, let $\Omega = M$ and consider $G = \langle \phi \rangle$, where ϕ is a (nontrivial) isometric involution of M . Suppose further (which is not true in general) that the set of fixed points of the action divides M into two open regions, which we shall label Ω_1 and Ω_2 . Then note that, arguing as above, one must have $\phi(\Omega_1) = \Omega_2$ (as well as $\phi(\Omega_2) = \Omega_1$). In particular H is the trivial subgroup, just consisting of the identity element. That said, there are two cases depending on the choice of twisting homomorphism $\sigma : G \rightarrow \{-1, 1\}$ we consider:

- (1) If we let $\sigma(\phi) = +1$, then $\partial_+\Omega_1 = \partial_{\text{int}}\Omega_1$ and $\partial_-\Omega_1 = \emptyset$, so we are considering the (nonequivariant) spectrum of T_{Ω_1} adding a Neumann boundary condition along $\partial_{\text{int}}\Omega_1$.
- (2) If we let $\sigma(\phi) = -1$, then $\partial_+\Omega_1 = \emptyset$ and $\partial_-\Omega_1 = \partial_{\text{int}}\Omega_1$, so we are considering the (nonequivariant) spectrum of T_{Ω_1} adding a Dirichlet boundary condition along $\partial_{\text{int}}\Omega_1$.

Example 3.7 (actions of self-congruences of two-sided hypersurfaces). Here we follow up on the discussion of Example 2.4 but specified to $\Omega = M$ for $N = \mathbb{B}^3$ and $G = \mathbb{P}_n$ (i.e., we postulate the ambient manifold to be the Euclidean ball, and the surface M to have prismatic symmetry). We refer the reader to the first part of Section 4 for basic recollections about this and related group action. We let Ω_1 be an open fundamental domain for this action (so that M is covered by the closures of exactly $4n$ pairwise isometric domains); it follows that again H is the trivial subgroup. Considering the sign homomorphism $\sigma : \mathbb{P}_n \rightarrow \{-1, +1\}$ defined in Example 2.4, it is readily checked that $\partial_+ \Omega_1 = \partial_{\text{int}} \Omega_1$ and $\partial_- \Omega_1 = \emptyset$, and so — when applied to this case — Lemma 3.5 compares (and proves equality of) the (fully-)equivariant spectrum of the problem with the spectrum of a fundamental domain, with Neumann boundary conditions added on each interior side.

3.3. Spectral stability. As it has been anticipated in the introduction, in our applications we will analyze the spectrum of free boundary minimal surfaces obtained by gluing certain constituting blocks. In that respect, we will need to derive from “geometric convergence” results some corresponding “spectral convergence” results. Suppose we have a sequence $\{(\Omega_n, g_n, q_n, r_n, \partial_D \Omega_n, \partial_N \Omega_n, \partial_R \Omega_n, G_n, \sigma_n)\}$ of admissible data, as well as “limit data” $(\Omega, g, q, r, \partial_D \Omega, \partial_N \Omega, \partial_R \Omega, G_\infty, \sigma_\infty)$, satisfying all our assumptions on admissible data except that G_∞ is possibly allowed to have infinite order. For instance, in our later applications G_∞ is the compact Lie group $O(2)$. Although we originally introduced the notation $\lambda_i^{G_\infty, \sigma_\infty}(T)$, with T the bilinear form associated to the foregoing data, for G_∞ finite, the notion remains well-defined for infinite G_∞ . The quantities $\text{ind}_{G_\infty}^{\sigma_\infty}(T)$ and $\text{nul}_{G_\infty}^{\sigma_\infty}(T)$ are likewise defined in this setting; as a special case, we can in turn define $\text{ind}_{G_\infty}(T)$ and $\text{nul}_{G_\infty}(T)$ for G_∞ a suitable infinite-order symmetry group of a hypersurface (as per Example 2.4). That being said, alongside $T[\Omega, g, q, r, \partial_D \Omega, \partial_N \Omega, \partial_R \Omega]$, we then have the corresponding sequence $\{T_n\}$ with

$$T_n := T[\Omega_n, g_n, q_n, r_n, \partial_D \Omega_n, \partial_N \Omega_n, \partial_R \Omega_n].$$

We will present some conditions on the data that ensure

$$\lim_{n \rightarrow \infty} \lambda_i^{G_n, \sigma_n}(T_n) = \lambda_i^{G_\infty, \sigma_\infty}(T) \quad \text{for all } i. \tag{3-3}$$

As we are especially interested in index and nullity, we immediately point out that (3-3) implies

$$\begin{aligned} \text{ind}_{G_\infty}^{\sigma_\infty}(T) &\leq \liminf_{n \rightarrow \infty} \text{ind}_{G_n}^{\sigma_n}(T_n), \\ \limsup_{n \rightarrow \infty} \text{nul}_{G_n}^{\sigma_n}(T_n) &\leq \text{nul}_{G_\infty}^{\sigma_\infty}(T), \\ \limsup_{n \rightarrow \infty} (\text{ind}_{G_n}^{\sigma_n}(T_n) + \text{nul}_{G_n}^{\sigma_n}(T_n)) &\leq \text{ind}_{G_\infty}^{\sigma_\infty}(T) + \text{nul}_{G_\infty}^{\sigma_\infty}(T). \end{aligned} \tag{3-4}$$

Proposition 3.8. *Let $(\Omega, g, q, r, \partial_D \Omega, \partial_N \Omega, \partial_R \Omega, G_\infty, \sigma_\infty)$ satisfy all our assumptions on admissible data except that we allow G_∞ to have infinite order; let T be the bilinear form determined by the data. Let $\{(\Omega, g_n, q_n, r_n, \partial_D \Omega, \partial_N \Omega, \partial_R \Omega, G_n, \sigma_n)\}$ be a sequence of admissible data, with corresponding sequence $\{T_n\}$ of bilinear forms. Assume*

$$\sup_n \sup_\Omega (|g_n|_g + |g_n^{-1}|_g + |q_n| + |r_n|) < \infty \quad \text{and} \quad (g_n, q_n, r_n) \xrightarrow[n \rightarrow \infty]{\text{a.e. on } \Omega} (g, q, r).$$

Assume further that

- (1) $G_n \leq G_\infty$ for all n , and $\sigma_n(\phi_n) = \sigma(\phi_n)$ for all n and all $\phi_n \in G_n$;
- (2) for each $\phi \in G_\infty$, there exists a sequence $\{\phi_n\}$ such that
 - (a) $\phi_n \in G_n$ for all n ,
 - (b) $\phi_n^* \xrightarrow{n \rightarrow \infty} \phi^*$ strongly as linear endomorphisms of $L^2(\Omega, g)$,
 - (c) $\sigma_n(\phi_n) = \sigma(\phi)$ for all n .

Then

$$\lim_{n \rightarrow \infty} \lambda_i^{G_n, \sigma_n}(T_n) = \lambda_i^{G_\infty, \sigma_\infty}(T) \quad \text{for all } i.$$

Proof. For expository convenience, we will first focus on the case when $G_n = G_\infty$ and $\sigma_n = \sigma_\infty$ for all n , thereby implicitly assuming $(\Omega, g, q, r, \partial_D \Omega, \partial_N \Omega, \partial_R \Omega, G_\infty, \sigma_\infty)$ to be admissible data (in our standard sense); we shall simply denote by G the group in question, and by σ the associated homomorphism.

Fix the index $i \geq 1$. We will start by showing that

$$\limsup_{n \rightarrow \infty} \lambda_i^{G, \sigma}(T_n) \leq \lambda_i^{G, \sigma}(T). \quad (3-5)$$

For this we start with an $L^2(\Omega, g)$ -orthonormal set $\{u_j\}_{j=1}^i$ such that u_j is a (G, σ) -invariant eigenfunction of T with eigenvalue $\lambda_j^{G, \sigma}(T)$. Then our assumptions on the coefficients together with the dominated convergence theorem imply that, for all $1 \leq j, k \leq i$,

$$\begin{aligned} \lim_{n \rightarrow \infty} \langle u_j, u_k \rangle_{L^2(\Omega, g_n)} &= \langle u_j, u_k \rangle_{L^2(\Omega, g)}, \\ \lim_{n \rightarrow \infty} \langle u_j, q_n u_k \rangle_{L^2(\Omega, g_n)} &= \langle u_j, q u_k \rangle_{L^2(\Omega, g)}, \\ \lim_{n \rightarrow \infty} \int_{\Omega} g_n (\nabla_{g_n} u_j, \nabla_{g_n} u_k) d\mathcal{H}^d(g_n) &= \int_{\Omega} g (\nabla_g u_j, \nabla_g u_k) d\mathcal{H}^d(g), \\ \lim_{n \rightarrow \infty} \int_{\partial_R \Omega} r_n u_j u_k d\mathcal{H}^{d-1}(g_n) &= \int_{\partial_R \Omega} r u_j u_k d\mathcal{H}^{d-1}(g). \end{aligned}$$

In conjunction with the min-max characterization (2-13) this proves (3-5). To conclude it thus suffices to prove the complementary inequality

$$\liminf_{n \rightarrow \infty} \lambda_i^{G, \sigma}(T_n) \geq \lambda_i^{G, \sigma}(T). \quad (3-6)$$

By (3-5) the sequence $\lambda_i^{G, \sigma}(T_n)$ is bounded from above uniformly in n , and by the min-max characterization (2-13) of eigenvalues along with the assumed uniform bounds on q_n and r_n and the trace inequality (2-3) it is also bounded from below. Therefore the left-hand side of (3-6) is a real number, and, by passing to a subsequence of the data if necessary (without renaming), we in fact assume without loss of generality that

$$\{\lambda_j^{G, \sigma}(T_n)\} \text{ converges to } \lambda_j^\infty \in \mathbb{R} \quad \text{for each } j \leq i, \quad (3-7)$$

with λ_j^∞ the \liminf of the j -th (G, σ) -eigenvalue of the original sequence.

For each $j \leq i$ and each n , let $v_j^{(n)}$ be a (G, σ) -invariant eigenfunction of T_n with eigenvalue $\lambda_j^{G, \sigma}(T_n)$ such that, for each n , the set $\{v_j^{(n)}\}_{j=1}^i$ is $L^2(\Omega, g_n)$ -orthonormal. It follows from the assumed unit $L^2(\Omega, g_n)$ bounds on the $v_j^{(n)}$, the definitions of eigenvalues and eigenfunctions, the eigenvalue bound following from (3-7), and the assumed bounds on q_n and r_n as well as g_n and g_n^{-1} that the sequence $\|v_j^{(n)}\|_{H^1(\Omega, g_n)}$ is bounded uniformly in n . (The assumptions on the metrics are needed there to ensure that the constants in the trace inequality (2-3), as applied here, can be chosen independently of n .) It then follows, in turn, using again the assumed bounds on g_n and g_n^{-1} , that $\|v_j^{(n)}\|_{H^1(\Omega, g)}$ is likewise bounded. Consequently, passing to a further subsequence if needed, for each $j \leq i$, there exists $v_j \in H^1(\Omega, g)$ which is simultaneously a limit in $L^2(\Omega, g)$ and a weak limit in $H^1(\Omega, g)$ of $v_j^{(n)}$ as $n \rightarrow \infty$. Note in particular that each v_j is (G, σ) -invariant.

The dominated convergence theorem, our assumptions on the metrics, and the $L^2(\Omega, g)$ -convergence for each j of $\{v_j^{(n)}\}$ to v_j imply that $\{v_j\}_{j=1}^i$ is $L^2(\Omega, g)$ -orthonormal, so in particular this finite family is linearly independent. In the same fashion, but also appealing to the assumptions on the q_n , we get, for all $1 \leq j \leq i$ and all $w \in L^2(\Omega, g)$,

$$\lim_{n \rightarrow \infty} \langle v_j^{(n)}, w \rangle_{L^2(\Omega, g_n)} = \langle v_j, w \rangle_{L^2(\Omega, g)}, \quad \lim_{n \rightarrow \infty} \langle q_n v_j^{(n)}, w \rangle_{L^2(\Omega, g_n)} = \langle q v_j, w \rangle_{L^2(\Omega, g)}.$$

Thanks to the weak convergence in $H^1(\Omega, g)$ of $\{v_j^{(n)}\}$ to v_j for each j (and again using the dominated convergence theorem, the assumptions on the metrics, and the L^2 convergence of each $\{v_j^{(n)}\}$), we further conclude that, for all $1 \leq j \leq i$ and all $w \in H^1(\Omega, g)$,

$$\lim_{n \rightarrow \infty} \int_{\Omega} g_n(\nabla_{g_n} v_j^{(n)}, \nabla_{g_n} w) d\mathcal{H}^d(g_n) = \int_{\Omega} g(\nabla_g v_j, \nabla_g w) d\mathcal{H}^d(g).$$

We use the trace inequality (2-3) in conjunction with boundedness in $H^1(\Omega, g)$ of $\{v_j^{(n)}\} \cup \{v_j\}$ and the convergence in $L^2(\Omega, g)$ for each j of $v_j^{(n)}$ to v_j to deduce that we also have $L^2(\partial\Omega, g)$ -convergence of the traces. As one consequence we see that each v_j in fact belongs to $H_{\partial\Omega}^1(\Omega, g)$. As another, by virtue of the assumptions on the r_n and once again the dominated convergence theorem, we obtain, for all $1 \leq j \leq i$ and $w \in H^1(\Omega, g)$,

$$\lim_{n \rightarrow \infty} \int_{\partial\mathbb{R}\Omega} r_n v_j^{(n)} w d\mathcal{H}^{d-1}(g_n) = \int_{\partial\mathbb{R}\Omega} r v_j w d\mathcal{H}^{d-1}(g_n).$$

From the definition of the $v_j^{(n)}$, the assumption (3-7), and the above three displayed equations, we conclude that, for all $1 \leq j \leq i$ and $w \in H_{\partial\Omega}^1(\Omega, g)$, we eventually have

$$T(v_j, w) = \lim_{n \rightarrow \infty} T_n(v_j^{(n)}, w) = \lim_{n \rightarrow \infty} \lambda_j^{G, \sigma}(T_n) \langle v_j^{(n)}, w \rangle_{L^2(\Omega, g_n)} = \lambda_j^\infty \langle v_j, w \rangle_{L^2(\Omega, g)}.$$

Specifically, for the second equality above we have used the fact that $v_j^{(n)}$ is an eigenfunction of T_n ; together, the inequalities then show that v_j is an eigenfunction of T . Since $\{v_j\}_{j=1}^i$ is a linearly independent subset of $\pi_{G, \sigma} H_{\partial\Omega}^1(\Omega, g)$, it follows that

$$\lambda_i^\infty \geq \lambda_i^{G, \sigma}(T),$$

completing the proof in the case of a “fixed symmetry group”.

However, it is actually straightforward to generalize the above argument to capture also continuity in the symmetries. The proof above goes through with mostly superficial modification, and we address the only two salient points. First, in proving (3-5) but with (G, σ) replaced on the left by (G_n, σ_n) and on the right by $(G_\infty, \sigma_\infty)$, note that each u_j , now assumed $(G_\infty, \sigma_\infty)$ -invariant, is by our hypotheses also (G_n, σ_n) -invariant for each n . Second, in proving the corresponding analogue of (3-6), note that each v_j is, as the $L^2(\Omega, g)$ limit of a sequence whose n -th term is (G_n, σ_n) -invariant, by our hypotheses, itself $(G_\infty, \sigma_\infty)$ -invariant. \square

We now turn our attention to the related yet different problem of handling controlled changes in the domain. We switch to slightly different notation, which is again tailor-made to best fit our later applications.

Proposition 3.9. *Let $(\Omega, g, q, r, \partial_D \Omega, \partial_N \Omega, \partial_R \Omega, G, \sigma)$ be admissible data, with corresponding bilinear form T . Suppose that, for any $\delta > 0$ less than the injectivity radius of (M, g) , say δ_0 , we are given a Lipschitz domain $\Omega_\delta \subset \Omega$ such that $(\Omega_\delta, g, q, r, \partial_D \Omega_\delta, \partial_N \Omega_\delta, \partial_R \Omega_\delta, G, \sigma)$ are also admissible data (with suitable restrictions of tensors and functions tacitly understood) and whose complement $K_\delta := \Omega \setminus \Omega_\delta$ satisfies*

$$\bigcup_{p \in S} \overline{B_{f_1(\delta)}(p)} \subset K_\delta \subset \bigcup_{p \in S} \overline{B_{f_2(\delta)}(p)} \quad (3-8)$$

for some finite set of points $S \subset \overline{\Omega}$ and monotone functions $f_1, f_2: [0, \delta_0[\rightarrow \mathbb{R}_{\geq 0}$ such that $\lim_{\delta \rightarrow 0} f_2(\delta) = 0$. Consider the sets as in (2-16) with Ω_δ in lieu of Ω_1 as well as the associated bilinear form

$$T_{\Omega_\delta}^{\text{Dint}} := T[\Omega_\delta, g, q, r, \partial_D^{\text{Dint}} \Omega_\delta, \partial_N^{\text{Dint}} \Omega_\delta, \partial_R^{\text{Dint}} \Omega_\delta].$$

Then, for each integer $i \geq 1$,

$$\lambda_i^{G, \sigma}(T_{\Omega_\delta}^{\text{Dint}}) \geq \lambda_i^{G, \sigma}(T), \quad (3-9)$$

and we have

$$\lim_{\delta \rightarrow 0} \lambda_i^{G, \sigma}(T_{\Omega_\delta}^{\text{Dint}}) = \lambda_i^{G, \sigma}(T). \quad (3-10)$$

The conclusion simply relies on the fact that points have null $W^{1,s}$ -capacity in \mathbb{R}^n for $1 \leq s \leq n$ and so, in particular, have null $W^{1,2}$ -capacity in \mathbb{R}^n for any $n \geq 2$; for the sake of completeness, we provide a self-contained argument focusing on the case of surfaces ($d = 2$), where a logarithmic cutoff trick is required and omit the simpler modifications for $d \geq 3$.

Proof. Given any $u_\delta, v_\delta \in H_{\partial_D^{\text{Dint}} \Omega_\delta}^1(\Omega_\delta)$, postulated to be (G, σ) -invariant, it is standard to note that their extensions by 0, say $\bar{u}_\delta, \bar{v}_\delta$ respectively, belong to $H_{\partial_D \Omega}^1(\Omega)$, that such functions are themselves (G, σ) -invariant, and, for any $\delta \in (0, \delta_0)$ we have $\langle u_\delta, v_\delta \rangle_{L^2(\Omega_\delta, g)} = \langle \bar{u}_\delta, \bar{v}_\delta \rangle_{L^2(\Omega, g)}$ and $T_{\Omega_\delta}^{\text{Dint}}(u_\delta, u_\delta) = T(\bar{u}_\delta, \bar{u}_\delta)$. Hence, it follows at once from the variational characterization of eigenvalues, (2-13), that, for each integer $i \geq 1$, we have indeed $\lambda_i^{G, \sigma}(T_{\Omega_\delta}^{\text{Dint}}) \geq \lambda_i^{G, \sigma}(T)$, which is the first claim. Appealing again to the domain monotonicity, it actually suffices to check (3-10) in the case when K_δ is in fact a union of metric balls, namely when we have equality in (3-8), for $f_1 = f_2$. To simplify the notation we can (without loss of generality, up to reparametrization) assume in fact $f_2(\delta) = \delta$ for any δ in the assumed domain. That said,

given any $\bar{u}, \bar{v} \in H^1_{\partial_D \Omega}(\Omega)$, (G, σ) -invariant, and $\delta > 0$ (small as in the statement), one can simply define $u_\delta = \bar{u}\varphi_\delta$ and $v_\delta = \bar{v}\varphi_\delta$, where (for $r := d_g(p, q)$ and $p \in S$) we set

$$\varphi_\delta(q) = \begin{cases} 0 & \text{if } r \leq \delta^{3/4}, \\ 3 - 4 \log r / \log \delta & \text{if } \delta^{3/4} \leq r \leq \delta^{1/2}, \\ 1 & \text{otherwise.} \end{cases}$$

It is then clear that $u_\delta, v_\delta \in H^1_{\partial^D \text{int} \Omega_\delta}(\Omega_\delta)$, that such functions are (G, σ) -invariant, and, in addition,

$$\lim_{\delta \rightarrow 0} T_{\Omega_\delta}^{\text{Dint}}(u_\delta, u_\delta) = T(\bar{u}, \bar{u}), \quad \lim_{\delta \rightarrow 0} \langle u_\delta, v_\delta \rangle_{L^2(\Omega_\delta, g)} = \langle \bar{u}, \bar{v} \rangle_{L^2(\Omega, g)}.$$

Hence, again appealing to (2-13), we must conclude

$$\limsup_{\delta \rightarrow 0} \lambda_i^{G, \sigma}(T_{\Omega_\delta}^{\text{Dint}}) \leq \lambda_i^{G, \sigma}(T), \tag{3-11}$$

whence, combining this inequality with the one above, the conclusion follows. □

Corollary 3.10. *Given the setting and the assumptions of Proposition 3.9, we have*

$$\lim_{\delta \rightarrow 0} \text{ind}_G^\sigma(T_{\Omega_\delta}^{\text{Dint}}) = \text{ind}_G^\sigma(T).$$

3.4. Conformal change in dimension 2. In this section we suppose, in addition to the assumptions above, that $d = \dim M = 2$ and that we are given a smooth, strictly positive, G -invariant function ρ on $\bar{\Omega}$. Note that the above bilinear form T of (2-4) is invariant under scaling, namely under the simultaneous transformations $g \mapsto \rho^2 g$, $q \mapsto \rho^{-2} q$, and $r \mapsto \rho^{-1} r$:

$$T[\Omega, \rho^2 g, \rho^{-2} q, \rho^{-1} r, \partial_D \Omega, \partial_N \Omega, \partial_R \Omega] = T[\Omega, g, q, r, \partial_D \Omega, \partial_N \Omega, \partial_R \Omega],$$

with the corresponding domains $H^1_{\partial_D \Omega}(\Omega, \rho^2 g)$ and $H^1_{\partial_D \Omega}(\Omega, g)$ agreeing as sets of functions and having equivalent norms. This claim needs a clarification: the standard H^1 norms of $H^1_{\partial_D \Omega}(\Omega, \rho^2 g)$ and $H^1_{\partial_D \Omega}(\Omega, g)$ are only equivalent up to constants that depend on the extremal (inf and sup) values of the conformal factor ρ .

In general, the eigenvalues (as defined in Section 2.3) will be affected by the conformal scaling, and yet the index and nullity are nonetheless invariant when this operation is performed.

Proposition 3.11 (invariance of index and nullity under conformal change in dimension 2). *With assumptions as in the preceding paragraph,*

$$\text{ind}_G^\sigma(T, \rho^2 g) = \text{ind}_G^\sigma(T, g) \quad \text{and} \quad \text{nul}_G^\sigma(T, \rho^2 g) = \text{nul}_G^\sigma(T, g).$$

Proof. By definition, we have that $u \in E_{G, \sigma}^{\leq 0}(T, g)$ if and only if u is (G, σ) -invariant and $T(u, v) = 0$ for all $v \in H^1_{\partial_D \Omega}(\Omega, g)$ (and likewise if each g is replaced by $\rho^2 g$), so the nullity equality is clear. For the index, because we can reverse the roles of g and $\rho^2 g$ by replacing ρ with ρ^{-1} , it suffices to check that the claim holds with \geq in place of $=$. This follows at once from the min-max characterization (2-13) applied to the (G, σ) -eigenvalues of $(T, \rho^2 g)$, by considering the “competitor” subspace $E_{G, \sigma}^{\leq 0}(T, g)$ in the minimization problem therein, for $i = \text{ind}_G^\sigma(T, g)$. □

4. Free boundary minimal surfaces in the ball: a first application

From now on, we specialize our study to the case when $\bar{\Omega} = M$ is a properly embedded free boundary minimal surface, henceforth denoted by Σ , of the closed unit ball $\mathbb{B}^3 := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$ in Euclidean space $(\mathbb{R}^3, g^{\mathbb{R}^3})$. Observe that, by the maximum principle, every embedded free boundary minimal surface is properly embedded.

As anticipated in the introduction, our task here will be to obtain quantitative estimates on the Morse index of free boundary minimal surfaces; hence our Schrödinger operator is the Jacobi (or stability) operator on Σ acting on functions subject to the Robin condition

$$du(\eta_{g^{\mathbb{R}^3}}^\Sigma) = u \quad \text{on } \partial\Sigma, \quad (4-1)$$

namely: $q = |A^\Sigma|^2$, the squared norm of the second fundamental form of Σ , and $\partial_D \Sigma = \partial_N \Sigma = \emptyset$, $\partial_R \Sigma = \partial \Sigma$, $r = 1$. Correspondingly, as our bilinear form T we will consider the index (or stability or Jacobi) form of Σ , which we will denote by Q^Σ . We define the index and nullity of Σ in the usual way, setting

$$\text{ind}(\Sigma) := \text{ind}(Q^\Sigma) \quad \text{and} \quad \text{nul}(\Sigma) := \text{nul}(Q^\Sigma),$$

and we likewise define the G -equivariant index and nullity of Σ , $\text{ind}_G(\Sigma)$ and $\text{nul}_G(\Sigma)$, in the sense of (2-15), when given a group $G < O(3)$ of symmetries of Σ one considers the associated sign homomorphism. More generally, we will also study the (G, σ) -index and (G, σ) -nullity of Σ , $\text{ind}_G^\sigma(\Sigma)$ and $\text{nul}_G^\sigma(\Sigma)$, when given a group G and, further, a homomorphism $\sigma : G \rightarrow O(1)$ (thus, in either case, these expressions are to be understood by replacing Σ by Q^Σ).

It has already been mentioned above how general lower bounds for the index, linear in the topological data (genus and number of boundary components), have been obtained in [Ambrozio et al. 2018b] and by [Sargent 2017] in the special case when the ambient manifold is a convex body in Euclidean \mathbb{R}^3 . We begin this section by presenting an alternative lower bound (Proposition 4.2 below) in terms of symmetries, which, though much less general in nature, nevertheless yields sharper lower bounds for many of the known examples (in terms of the coefficients describing the linear growth rate as a function of the topological data). Before proceeding, we pause to explain some notation we will find convenient.

Cylindrical coordinates and wedges. We shall describe points in Euclidean \mathbb{R}^3 , endowed with standard Cartesian coordinates (x, y, z) and also in terms of cylindrical coordinates (r, θ, z) , so that the point with cylindrical coordinates (r_0, θ_0, z_0) has Cartesian coordinates $(x, y, z) = (r_0 \cos \theta_0, r_0 \sin \theta_0, z_0)$. However we wish to stress that, for our purposes, it will be convenient to allow arbitrary real values for both r and θ ; thus the triples (r, θ, z) and $(-r, \theta + \pi, z)$ describe the same point in Euclidean space. Given real numbers $\alpha \leq \beta$, we also define the closed wedge

$$W_\alpha^\beta := \{(r \cos \theta, r \sin \theta, z) : r \geq 0, \theta \in [\alpha, \beta], z \in \mathbb{R}\}, \quad (4-2)$$

with the half-plane W_α^α accommodated as a degenerate wedge. In particular, our convention implies

$$\{\theta = \alpha\} = W_\alpha^\alpha \cup W_{\alpha+\pi}^{\alpha+\pi}.$$

Notation for symmetries. Given a plane $\Pi \subset \mathbb{R}^3$ through the origin, we write $\underline{R}_\Pi \in O(3)$ for reflection through Π . Similarly, given a directed line $\xi \subset \mathbb{R}^3$ through the origin and an angle $\theta \in \mathbb{R}$, we write \underline{R}_ξ^θ for rotation about ξ through angle α in the usual right-handed sense. Typically we will be interested not exclusively in such a rotation \underline{R}_ξ^θ but rather in the cyclic subgroup it generates, with the result that it will never really be important to associate a direction to ξ . Given symmetries $T_1, \dots, T_n \in O(3)$, we write $\langle T_1, \dots, T_n \rangle$ for the subgroup they generate.

The order-2 groups generated by reflections through planes will figure repeatedly in the sequel (beginning with the following proposition), so for succinctness of notation, given a plane $\Pi \subset \mathbb{R}^3$ through the origin, we agree to set $\mathbf{\Pi} := \langle \underline{R}_\Pi \rangle$. In such a context, consistent with the general convention we defined above, we will employ the apex $+$ (resp. $-$) to denote functions that are even (resp. odd) with respect to the reflection through Π . Similarly (but less frequently), if ξ is a line through the origin in \mathbb{R}^3 , we will write $\mathbf{\xi}$ for the order-2 group generated by reflection \underline{R}_ξ through ξ (equivalently rotation through angle π in either sense about ξ).

We also pause to name the following three subgroups of $O(3)$, which will be realized as subgroups of the symmetry groups of the examples we study below and which partly pertain to the statement of the next proposition: for each integer $k \geq 1$, we set

$$\begin{aligned} \mathbb{Y}_k &:= \langle \underline{R}_{\{\theta=-\pi/(2k)\}}, \underline{R}_{\{\theta=\pi/(2k)\}} \rangle && \text{(pyramidal group of order } 2k), \\ \mathbb{P}_k &:= \langle \underline{R}_{\{\theta=-\pi/(2k)\}}, \underline{R}_{\{\theta=\pi/(2k)\}}, \underline{R}_{\{z=0\}} \rangle && \text{(prismatic group of order } 4k), \\ \mathbb{A}_k &:= \langle \underline{R}_{\{\theta=\pi/(2k)\}}, \underline{R}_{\{y=z=0\}}^\pi \rangle && \text{(antiprismatic group of order } 4k). \end{aligned} \tag{4-3}$$

Note in particular that we have $\mathbb{Y}_k = \mathbb{P}_k \cap \mathbb{A}_k$.

Remark 4.1. The above three groups are so named because they are the (maximal) symmetry groups of, respectively, a right pyramid, prism, or antiprism over a regular k -gon. See, e.g., Section 2 of [Carlotto et al. 2022b] for pictures and further details, but we caution that the above definition of the subgroup \mathbb{P}_k differs slightly from that given in [Carlotto et al. 2022b]: the two subgroups are conjugate to one another via rotation through angle $\pi/(2k)$ about the z -axis.

With this terminology and notation in place, we proceed with the aforementioned lower index bound, which illustrates the Montiel–Ros methodology as developed in Section 3 and is interesting in its own right.

Proposition 4.2 (index lower bounds under pyramidal and prismatic symmetry; see [Choe 1990; Kapouleas and Wiygul 2020]). *Let Σ be a connected, embedded free boundary minimal surface in \mathbb{B}^3 . Assume that Σ is not a disc or critical catenoid, that Σ is invariant under reflection through a plane Π_1 , and that Σ is also invariant under rotation through an angle $\alpha \in]0, 2\pi[$ about a line $\xi \subset \Pi_1$. Then α is a rational multiple of 2π , there is a largest integer $k \geq 2$ such that rotation about ξ through angle $2\pi/k$ is also a symmetry of Σ , and*

- (a) $\text{ind}(\Sigma) \geq 2k - 1$,
- (b) $\text{ind}_{\Pi_1}^-(\Sigma) \geq k - 1$, and
- (c) if Σ is additionally invariant under reflection through a plane Π_\perp orthogonal to ξ , then in fact $\text{ind}_{\Pi_\perp}^+(\Sigma) \geq 2k - 1$.

Note that the symmetries assumed in the preamble of Proposition 4.2 generate, up to conjugacy in $O(3)$, the group \mathbb{V}_k from (4-3), while one instead obtains (again up to conjugacy) the group \mathbb{P}_k by adjoining the additional symmetry assumed in item (c).

The proof below is an abstraction and transplantation to the free boundary setting of some index lower bounds obtained in the course of [Kapouleas and Wiygul 2020] and drawing on ideas from [Montiel and Ros 1991]. The estimates ultimately depend on a lower bound on the number of nodal domains of a suitable Jacobi field, which was also the basis for earlier index estimates (of complete minimal surfaces in \mathbb{R}^3 and closed minimal surfaces in \mathbb{S}^3) established in [Choe 1990].

Proof. By excluding the discs and critical catenoids we ensure that Σ is not \mathbb{S}^1 -invariant about ξ , implying the claim on α and the existence of the rotational symmetry about ξ through angle of the form $2\pi/k$, as follows. First, if the cyclic subgroup generated by rotation about ξ through angle α were not finite, then it would be dense in the $SO(2)$ subgroup of rotations about ξ , but the symmetry group of Σ is closed in $O(3)$; yet, as already observed, our assumptions ensure that Σ has no $SO(2)$ symmetry subgroup. Thus α must be a rational multiple of 2π , as claimed. Now let β be the least angle in $]0, 2\pi[$ through which rotation about ξ is generated by the assumed rotational symmetry through angle α , and let k be the least positive integer such that $k\beta \geq 2\pi$. Then rotation through angle $k\beta - 2\pi$, which lies in $[0, \beta[$, is also generated by the assumed rotational symmetry. The presumed minimality of β then forces $\beta = 2\pi/k$.

By composing the assumed symmetries, it follows that Σ is also invariant under reflection through each of the $k - 1$ planes Π_2, \dots, Π_k containing ξ and there meeting Π_1 at angle an integer multiple of π/k . Now suppose $\Pi \in \{\Pi_i\}_{i=1}^k$. We necessarily have $\Pi \cap \Sigma \neq \emptyset$ (for example since Π separates \mathbb{B}^3 into two components and is a plane of symmetry for Σ , which is assumed to be connected). Because Π is a plane of symmetry and Σ is embedded, these two surfaces must intersect either orthogonally or tangentially, but in the latter case Σ must be a disc, which possibility we have excluded by assumption; consequently, the intersection is orthogonal. Moreover, by the symmetries each of the $2k$ components W_1, \dots, W_{2k} of $\mathbb{B}^3 \setminus \bigcup_{i=1}^k \Pi_i$ then has nontrivial intersection $\Omega_i := \Sigma \cap W_i$ with Σ . Without loss of generality, let us agree to label the domains under consideration in counterclockwise order such that $\Omega_1, \dots, \Omega_k$ all lie on the same side of Π_1 .

Note that the members of the family $\{\Omega_i\}_{i=1}^{2k}$ are pairwise isometric and each is connected. (Indeed, Σ is itself connected, so any two points in any single Ω_i can be joined by some path in Σ , but this path can leave Ω_i only through the latter's intersection with planes of symmetry, so we can always produce a path connecting the two points that is entirely contained in Ω_i by repeated reflection and replacement, if necessary.) Furthermore, each Ω_i has Lipschitz boundary contained in $\mathbb{S}^2 \cup \bigcup_{i=1}^k \Pi_i$ because the intersection of Σ with either \mathbb{S}^2 and any of the planes Π_1, \dots, Π_k is orthogonal (thus transverse) and exactly k of the Ω_i lie on each side of Π_1 .

Next, letting κ_ξ be a choice of (scalar-valued) Jacobi field on Σ induced by the rotations about ξ and again using the fact that Σ is not rotationally symmetric (and so, in particular, not planar either), we conclude that κ_ξ vanishes on $\Sigma \cap \bigcup_{i=1}^k \Pi_i$ (because of the aforementioned orthogonality) but does not vanish identically on any Ω_i . As a result, imposing, for each i , the Robin condition (4-1) on $\mathbb{S}^2 \cap \partial\Omega_i$ and the Dirichlet condition on $\partial\Omega_i \cap \bigcup_{i=1}^k \Pi_i$, the corresponding nullity of Ω_i is at least 1. An appeal to

item (i) of Corollary 3.2 (for claims (a) and (c)) and to item (i) of Proposition 3.1 (for (b)) now completes the proof. Specifically:

- for (a), we consider the partition of Σ into the $2k$ domains $\Omega_1, \dots, \Omega_{2k}$ and take G to be the trivial group;
- for (b), we take $G = \langle \underline{R}_{\Pi_1} \rangle$ to be the group with two elements (as in Example 2.3), the homomorphism determined by $\sigma(\underline{R}_{\Pi_1}) = -1$ (thereby imposing *odd* symmetry) and, correspondingly, we consider the partition of Σ into k domains obtained by equivariant pairing, i.e., by taking $\Omega_{i+1} \cup \Omega_{2k-i}$ for $i = 0, \dots, k - 1$;
- for (c), we consider the partition of Σ into the $2k$ domains $\Omega_1, \dots, \Omega_{2k}$, take $G = \langle \underline{R}_{\Pi_\perp} \rangle$ to be the group with two elements, and the homomorphism determined by $\sigma(\underline{R}_{\Pi_\perp}) = +1$ (thereby imposing *even* symmetry).

Thereby the proof is complete. □

5. Effective index estimates for two sequences of examples

5.1. Review of the construction and lower index bounds. As we have already alluded to in the introduction, in [Carlotto et al. 2022b] two families of embedded free boundary minimal surfaces in \mathbb{B}^3 were constructed by desingularizing (in the spirit of [Kapouleas 1997]) the configurations $-\mathbb{K}_0 \cup \mathbb{K}_0$ and $-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0$, where \mathbb{K}_0 is the intersection with \mathbb{B}^3 of a certain catenoid having axis of symmetry $\{x = y = 0\}$ and meeting $\partial\mathbb{B}^3$ (not orthogonally) along the equator $\partial\mathbb{B}^2$ and orthogonally along one additional circle of latitude at height $h > 0$.

Proposition 5.1 (existence and basic properties of \mathbb{K}_0). *There exists a minimal annulus \mathbb{K}_0 which is properly embedded in \mathbb{B}^3 and intersects the unit sphere $\partial\mathbb{B}^3$ exactly along the equator $\partial_0\mathbb{K}_0 := \partial\mathbb{B}^3 \cap \{z = 0\}$ and orthogonally along a circle of latitude at height $z = h \approx 0.87028$, which we denote by*

$$\partial_\perp\mathbb{K}_0 := \partial\mathbb{K}_0 \setminus \partial_0\mathbb{K}_0.$$

Moreover, \mathbb{K}_0 coincides with the surface of revolution of the graph of $r : [0, h] \rightarrow]0, 1[$ given by $r(\zeta) = (1/a) \cosh(a\zeta - s)$ for suitable $a \approx 2.3328$ and $s \approx 1.4907$.

Proof. The existence of \mathbb{K}_0 is proven in [Carlotto et al. 2022b, Lemma 3.3]. For the numerical values of a , h , and s , we refer to [loc. cit., Remark 3.9]. □

That being said, these are (somewhat simplified) versions of the main existence results we proved in [loc. cit.].

Theorem 5.2 (desingularizations of $-\mathbb{K}_0 \cup \mathbb{K}_0$ [Carlotto et al. 2022b]). *For each sufficiently large integer n , there exists in \mathbb{B}^3 a properly embedded free boundary minimal surface $\Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0}$ that has genus 0, exactly $n + 2$ boundary components, and is invariant under the prismatic group \mathbb{P}_n from (4-3). Moreover, $\Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0}$ converges to $-\mathbb{K}_0 \cup \mathbb{K}_0$ in the sense of varifolds, with unit multiplicity, and smoothly away from the equator, as $n \rightarrow \infty$.*

Theorem 5.3 (desingularizations of $-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0$ [Carlotto et al. 2022b]). *For each sufficiently large integer m , there exists in \mathbb{B}^3 a properly embedded free boundary minimal surface $\Sigma_m^{-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0}$ that has genus m , exactly three boundary components and is invariant under the antiprismatic group \mathbb{A}_{m+1} from (4-3). Moreover, $\Sigma_m^{-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0}$ converges to $-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0$ in the sense of varifolds, with unit multiplicity, and smoothly away from the equator, as $m \rightarrow \infty$.*

Proposition 5.4 (lower bounds by symmetry on the index of the examples of [Carlotto et al. 2022b]). *There exist $n_0, m_0 > 0$ such that we have the following index estimates for all integers $n > n_0$ and $m > m_0$:*

$$\text{ind}_{\{z=0\}}^+(\Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0}) \geq 2n - 1 \quad \text{and} \quad \text{ind}(\Sigma_m^{-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0}) \geq 2m + 1.$$

Proof. As stated in Theorem 5.2, $\Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0}$ is invariant under the action of the prismatic group \mathbb{P}_n , which is generated by the reflections through the vertical planes $\{\theta = -\pi/(2n)\}$ and $\{\theta = \pi/(2n)\}$ and through the horizontal plane $\{z = 0\}$. As a composition of the first two reflections, \mathbb{P}_n also contains the rotation by angle $2\pi/n$ about the vertical axis $\xi_0 = \{r = 0\}$. Applying Proposition 4.2 (c) with $k = n$, $\xi = \xi_0$, $\Pi_1 = \{\theta = \pi/(2n)\}$, and $\Pi_\perp = \{z = 0\}$, we obtain

$$\text{ind}_{\{z=0\}}^+(\Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0}) \geq 2n - 1.$$

Similarly, Theorem 5.3 states that $\Sigma_m^{-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0}$ is invariant under the action of the antiprismatic group \mathbb{A}_{m+1} , which contains the reflection through the vertical plane $\{\theta = \pi/(2(m+1))\}$ and also the rotation by angle $2\pi/(m+1)$ about the vertical axis ξ_0 . Applying Proposition 4.2 (a) then yields

$$\text{ind}(\Sigma_m^{-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0}) \geq 2m + 1. \quad \square$$

In terms of topological data, the previous proposition (compared to [Ambrozio et al. 2018b]) provides a coefficient 2 for the growth rate of the Morse index of $\Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0}$ (resp. $\Sigma_m^{-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0}$) with respect to the number of boundary components (resp. of the genus), modulo an additive term. In fact, the lower bound on the Morse index of $\Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0}$ can be further improved via the following observation, which pertains to the *odd* contributions to the index instead (again with respect to reflections across the $\{z = 0\}$ plane in \mathbb{R}^3); incidentally this is also an example of an application of Proposition 3.1 to a collection of domains that are *not* pairwise isometric.

Proposition 5.5. *There exists $n_0 > 0$ such that we have the following index estimates for all integers $n > n_0$:*

$$\text{ind}_{\{z=0\}}^-(\Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0}) \geq 3.$$

Proof. Let Π_1 denote a vertical plane of symmetry, passing through the origin, of the surface $\Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0}$ (which, we recall, has prismatic symmetry \mathbb{P}_n), let ξ be the line obtained as the intersection of such a plane with $\{z = 0\}$, and let finally $\Pi_2 = \xi^\perp$ be the vertical plane, again passing through the origin, that is orthogonal to Π_1 . Consider on $\Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0}$ the function $\kappa_\xi = K_\xi \cdot \nu$, where K_ξ is the Killing vector field associated to rotations around ξ (oriented either way) and ν is a choice of the unit normal to the surface in question. Clearly, the flow of K_ξ generates a curve of free boundary minimal surfaces around $\Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0}$, hence the function κ_ξ lies in the kernel of the Jacobi operator of $\Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0}$ and satisfies the

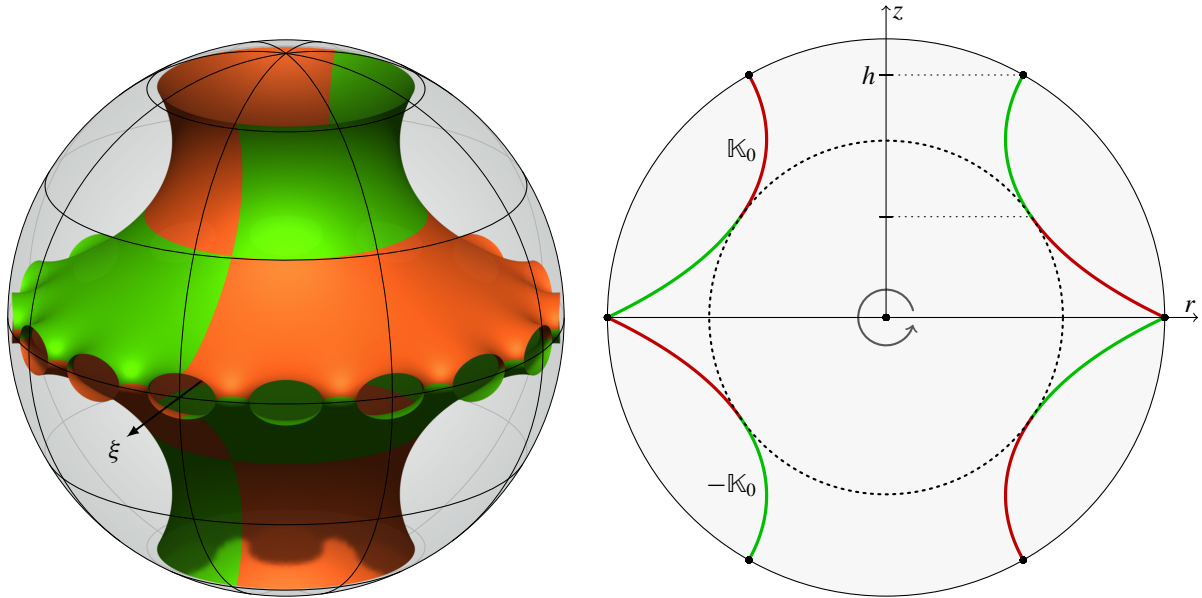


Figure 2. Nodal domains of the function induced by rotations around the symmetry axis ξ .

natural Robin boundary condition along the free boundary. Furthermore, concerning its nodal set, we first note it contains the curves

$$\Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0} \cap \{z = 0\} \quad \text{and} \quad \Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0} \cap \Pi_1.$$

We also claim that, for any sufficiently large n , the function κ_ξ changes sign along the connected arc

$$\Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0} \cap \Pi_2^+ \cap \{z \geq z_0\}, \tag{5-1}$$

where Π_2^+ denotes either of the half-planes determined by Π_1 on Π_2 and $z_0 > 0$ is any sufficiently small value (as we are about to describe, stressing that we can choose it independently of n). Since one has smooth convergence of $\Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0}$ to $-\mathbb{K}_0 \cup \mathbb{K}_0$ as $n \rightarrow \infty$ away from the equator, it suffices to verify an analogous claim for \mathbb{K}_0 . In fact, it then follows from an explicit calculation that the function induced by rotations around the symmetry axis ξ (the analogue of κ_ξ on \mathbb{K}_0) has opposite signs on the two endpoints of the arc $\mathbb{K}_0 \cap \Pi_2^+$ (see Figure 2, right image), and so — assuming without loss of generality it is negative on the equatorial point — by continuity there exists $\bar{z}_0 > 0$ such that the same function is also strictly negative at all points of $\mathbb{K}_0 \cap \Pi_2^+$ at height $z_0 \in [0, \bar{z}_0]$. In particular, we can indeed choose one such value $z_0 \in (0, \bar{z}_0)$ once and for all.

Hence, appealing to the aforementioned smooth convergence, by the intermediate value theorem for any sufficiently large n , there must be a point along the arc (5-1) where κ_ξ vanishes. Now, standard results about the structure of the nodal sets of eigenfunctions of Schrödinger operators ensure that such a zero is not isolated, but is either a regular point of a smooth curve or a branch point out of which finitely many smooth arcs emanate. In either case, combining all facts above we must conclude that on $\Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0} \cap \{z \geq 0\}$ the function κ_ξ has at least four nodal domains, and thus an application of Proposition 3.1 with $t = 0$, $G = \langle \underline{R}_\Pi \rangle$ for $\Pi = \{z = 0\}$, and $\sigma(\underline{R}_\Pi) = -1$ ensures the conclusion. \square

Remark 5.6. Note that the very same argument would lead, when applied with no equivariance constraint at all (i.e., when G is the trivial group) to the conclusion that, for any sufficiently large n , the index of $\Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0}$ is bounded from below by 7, which however is a lot worse than the bound provided by combining Proposition 5.4 with Proposition 5.5. Furthermore, we note that one can show that the function κ_ξ has *exactly* 8 nodal domains and not more, as visualized in Figure 2.

Remark 5.7. Concerning the sharpness of the estimate given in Proposition 5.5, we note that numerical simulations of \mathbb{K}_0 with fixed lower boundary $\partial_0 \mathbb{K}_0$ and upper boundary $\partial_\perp \mathbb{K}_0$ constrained to the unit sphere indicate that it has in fact index equal to 3. Roughly speaking, one negative direction comes from “pinching” the catenoidal neck and the other two negative directions correspond to “translations” of $\partial_\perp \mathbb{K}_0$ on the northern hemisphere.

The rest of this section is aimed at obtaining *upper* bounds on the Morse index of our examples, which is a more delicate task and one that relies crucially not only on the symmetries of the surfaces in question but also on the way they were actually constructed (which we encode in suitable convergence results).

5.2. Equivariant index and nullity of the models. For upper bounds we will exploit the regionwise convergence of the two families to the models glued together in their construction. Therefore we first study the index and nullity on these models.

Equivariant index and nullity of \mathbb{K}_0 . We begin with a summary of the properties of the minimal annulus \mathbb{K}_0 we will need. Let $\partial_0 \mathbb{K}_0 = \partial \mathbb{K}_0 \cap \{z = 0\}$ and $\partial_\perp \mathbb{K}_0 = \partial \mathbb{K}_0 \setminus \partial_0 \mathbb{K}_0$ be as in Proposition 5.1, so that $\partial_\perp \mathbb{K}_0$ is the boundary component along which \mathbb{K}_0 meets the sphere $\partial \mathbb{B}^3$ orthogonally. Referring to (2-4), we define

$$Q_N^{\mathbb{K}_0} := T[\mathbb{K}_0, g^{\mathbb{K}_0}, q := |A^{\mathbb{K}_0}|^2, r := 1, \partial_D \mathbb{K}_0 := \emptyset, \partial_N \mathbb{K}_0 := \partial_0 \mathbb{K}_0, \partial_R \mathbb{K}_0 := \partial_\perp \mathbb{K}_0]$$

(where we abuse notation in that by \mathbb{K}_0 we really mean its topological interior) to be the Jacobi form of \mathbb{K}_0 subject to the natural geometric Robin condition (4-1) on $\partial_\perp \mathbb{K}_0$ and to the Neumann condition on $\partial_0 \mathbb{K}_0$. Clearly, for each $k \geq 1$, the pyramidal group \mathbb{Y}_k from (4-3) preserves \mathbb{K}_0 and each of its boundary components individually.

Lemma 5.8 (\mathbb{Y}_k -equivariant index and nullity of \mathbb{K}_0). *With notation as above, for each sufficiently large integer k ,*

$$\text{ind}_{\mathbb{Y}_k}(Q_N^{\mathbb{K}_0}) = 1 \quad \text{and} \quad \text{nul}_{\mathbb{Y}_k}(Q_N^{\mathbb{K}_0}) = 0.$$

Proof. We shall start by recalling [Carlotto et al. 2022b, Lemma 4.4], which states that when imposing the Dirichlet condition on $\partial_0 \mathbb{K}_0$ and the Robin condition on $\partial_\perp \mathbb{K}_0$, then the Jacobi operator acting on \mathbb{Y}_k -equivariant functions on \mathbb{K}_0 is invertible provided that k is sufficiently large, which means that the equivariant nullity vanishes in this case. Considering that the coordinate function $u = z$ on \mathbb{K}_0 , which is harmonic, satisfies the Dirichlet condition on $\partial_0 \mathbb{K}_0$ and the Robin condition on $\partial_\perp \mathbb{K}_0$, it is also evident that the equivariant index is at least 1 in this case (see [loc. cit., Lemma 7.2]). This implies that when instead the Neumann condition is imposed on $\partial_0 \mathbb{K}_0$, the equivariant index is again at least 1. Below we prove that it is exactly 1 and the equivariant nullity is exactly 0 in the Neumann case by showing that the second eigenvalue is strictly positive. (We note here, incidentally, that this information also proves that a posteriori the equivariant index is also exactly 1 in the case that a Dirichlet condition is imposed on $\partial_0 \mathbb{K}_0$.)

Let $a, h, s > 0$ and $r(\zeta) = (1/a) \cosh(a\zeta - s)$ be as in Proposition 5.1. In particular, we have $(r')^2 + 1 = \cosh^2(a\zeta - s)$. Thus, when \mathbb{K}_0 is parametrized as a surface of revolution in terms of the coordinates (θ, ζ) with profile function $r(\zeta)$, the metric $g_{\mathbb{K}_0}$ and the squared norm of the second fundamental form $A_{\mathbb{K}_0}$ on \mathbb{K}_0 are given by

$$g_{\mathbb{K}_0} = ((r')^2 + 1) d\zeta^2 + r^2 d\theta^2, \quad |A_{\mathbb{K}_0}|^2 = \frac{(-r'')^2}{((r')^2 + 1)^3} + \frac{1}{((r')^2 + 1)^2 r^2} = \frac{a^2 + a^{-2}}{\cosh^4(a\zeta - s)}.$$

The outward unit conormal along $\partial_{\perp} \mathbb{K}_0 = \mathbb{K}_0 \cap \{\zeta = h\}$ is given by

$$\eta_{\mathbb{K}_0} = \frac{1}{\sqrt{(r')^2(h) + 1}} \partial_{\zeta} = \frac{1}{\cosh(ah - s)} \partial_{\zeta} = \frac{1}{ar(h)} \partial_{\zeta}.$$

Assume, for the sake of a contradiction, that $\lambda_2 = \lambda_2^{\mathbb{Y}_k, \text{sgn}} \leq 0$, where we are considering the spectrum of the Jacobi operator of \mathbb{K}_0 acting on \mathbb{Y}_k -equivariant functions (see Example 2.4), and subject to the boundary conditions described above. Then, by first invoking the Courant nodal domain theorem as in the proof of [Carlotto et al. 2022b, Lemma 4.4] we may assume that the associated eigenfunction u_2 is rotationally symmetric provided that k is sufficiently large, i.e., u_2 only depends on ζ and not on θ .

That said, let u be a function on \mathbb{K}_0 which is rotationally symmetric, i.e., constant in θ . Then

$$\Delta_{\mathbb{K}_0} u = \frac{1}{\cosh^2(a\zeta - s)} \frac{\partial^2 u}{\partial \zeta^2},$$

and we shall consider the Jacobi operator $J = \Delta_{\mathbb{K}_0} + |A_{\mathbb{K}_0}|^2$ and the eigenvalue problem

$$\begin{cases} Ju = -\lambda u, \\ u'(0, \cdot) = 0 & \text{(Neumann condition on } \partial_0 \mathbb{K}_0), \\ u'(h, \cdot) = \cosh(ah - s)u(h, \cdot) & \text{(Robin condition on } \partial_{\perp} \mathbb{K}_0). \end{cases}$$

Since u_2 must change sign, there exists $z_0 \in]0, h[$ such that $u_2(z_0) = 0$. Multiplying the eigenvalue equation

$$\frac{\partial^2 u_2}{\partial \zeta^2} + \frac{a^2 + a^{-2}}{\cosh^2(a\zeta - s)} u_2 = -\lambda_2 u_2 \cosh^2(a\zeta - s) \tag{5-2}$$

with u_2 and integrating from $\zeta = 0$ to $\zeta = z_0$, we obtain

$$\int_0^{z_0} -\lambda_2 u_2^2 \cosh^2(a\zeta - s) d\zeta = -\int_0^{z_0} |u_2'|^2 d\zeta + \int_0^{z_0} \frac{a^2 + a^{-2}}{\cosh^2(a\zeta - s)} u_2^2 d\zeta.$$

Since $u(z_0) = 0$, we can obtain the Poincaré-type inequality

$$\int_0^{z_0} |u_2(\zeta)|^2 d\zeta = \int_0^{z_0} \left| \int_{z_0}^{\zeta} u_2'(t) dt \right|^2 d\zeta \leq \int_0^{z_0} (z_0 - \zeta) \int_{\zeta}^{z_0} |u_2'(t)|^2 dt d\zeta \leq \frac{z_0^2}{2} \int_0^{z_0} |u_2'(\zeta)|^2 d\zeta.$$

Hence,

$$\int_0^{z_0} -\lambda_2 u_2^2 \cosh^2(a\zeta - s) d\zeta \leq \int_0^{z_0} \left(\frac{a^2 + a^{-2}}{\cosh^2(a\zeta - s)} - \frac{2}{z_0^2} \right) u_2^2 d\zeta.$$

The right-hand side is negative if

$$z_0 < \sqrt{\frac{2}{a^2 + a^{-2}}} \approx 0.5962,$$

and so, in this case, we conclude $\lambda_2 > 0$, a contradiction.

Integrating the eigenvalue equation (5-2) instead from $\zeta = z_0$ to $\zeta = h$ and recalling the Robin condition $u'(h) = \cosh(ah - s)u(h)$ along $\partial_{\perp} \mathbb{K}_0$, we obtain the alternative estimate

$$\begin{aligned} \int_{z_0}^h -\lambda_2 u_2^2 \cosh^2(a\zeta - s) d\zeta &= |u_2(h)|^2 \cosh(ah - s) - \int_{z_0}^h |u_2'|^2 d\zeta + \int_{z_0}^h \frac{a^2 + a^{-2}}{\cosh^2(a\zeta - s)} u_2^2 d\zeta \\ &\leq ((h - z_0) \cosh(ah - s) - 1) \int_{z_0}^h |u_2'|^2 d\zeta + \int_{z_0}^h \frac{a^2 + a^{-2}}{\cosh^2(a\zeta - s)} u_2^2 d\zeta \\ &\leq \left(a^2 + a^{-2} + \frac{2}{(h - z_0)^2} ((h - z_0) \cosh(ah - s) - 1) \right) \int_{z_0}^h u_2^2 d\zeta, \end{aligned}$$

provided that $(h - z_0) \cosh(ah - s) - 1 < 0$. Now the right-hand side is negative if $z_0 > 0.4443$.

Since the intervals $[0, 0.5962]$ and $[0.4443, h]$ intersect, we anyway obtain a contradiction. Thus, we confirm the claim $\lambda_2 > 0$, as desired. \square

Observing (as we have already done in the previous proof) that any eigenfunction “generating” the index in Lemma 5.8 is rotationally invariant, we have the following obvious corollary (which in fact can conversely be used to prove the lemma, with the aid of Proposition 3.8). In the statement, $G^{\mathbb{K}_0}$ denotes the subgroup of $O(3)$ preserving \mathbb{K}_0 . Note that $G^{\mathbb{K}_0}$ consists of rotations about the z -axis and reflections through planes containing the z -axis. In particular $G^{\mathbb{K}_0}$ is isomorphic to $O(2)$, and each element of $G^{\mathbb{K}_0}$ preserves either choice of unit normal of \mathbb{K}_0 .

Corollary 5.9 (fully equivariant index and nullity of \mathbb{K}_0). *With notation as above and recalling the comments immediately preceding Proposition 3.8, we have*

$$\text{ind}_{G^{\mathbb{K}_0}}(Q_N^{\mathbb{K}_0}) = 1 \quad \text{and} \quad \text{nul}_{G^{\mathbb{K}_0}}(Q_N^{\mathbb{K}_0}) = 0.$$

Equivariant index and nullity of \mathbb{B}^2 . The analysis for the flat disc \mathbb{B}^2 (featured in the construction of just one of the families) is trivial, and the conclusions are as follows; in the statement we write $Q_N^{\mathbb{B}^2}$ for the index form of \mathbb{B}^2 as a minimal surface with boundary in $(\mathbb{R}^3, g^{\mathbb{R}^3})$ subject to the Neumann boundary condition, namely

$$Q_N^{\mathbb{B}^2} := T[\mathbb{B}^2, g^{\mathbb{B}^2}, 0, 0, \emptyset, \partial_N \mathbb{B}^2 := \partial \mathbb{B}^2, \emptyset].$$

Lemma 5.10 ($(\mathbb{A}_{m+1}$ -equivariant) index and nullity of \mathbb{B}^2). *With notation as above,*

$$\text{ind}(Q_N^{\mathbb{B}^2}) = 0 \quad \text{and} \quad \text{nul}(Q_N^{\mathbb{B}^2}) = 1.$$

Moreover, for each integer $m \geq 0$, the antiprismatic group \mathbb{A}_{m+1} preserves \mathbb{B}^2 and

$$\text{ind}_{\mathbb{A}_{m+1}}(Q_N^{\mathbb{B}^2}) = \text{nul}_{\mathbb{A}_{m+1}}(Q_N^{\mathbb{B}^2}) = 0.$$

Proof. The first line of equalities is clear, since the Jacobi operator on \mathbb{B}^2 is simply the standard Laplacian, whose Neumann kernel is spanned by the constants (to rule out index one can for instance just appeal to the Hopf boundary point lemma). The invariance of \mathbb{B}^2 under each \mathbb{A}_{m+1} is obvious, and the proof is then completed by the observation that the constants are not \mathbb{A}_{m+1} -equivariant (for any $m \geq 0$). \square

From Lemma 5.10 we immediately obtain, analogously to Corollary 5.9 from Lemma 5.8, the following corollary. In the statement $O(2)$ refers to the group of intrinsic isometries of \mathbb{B}^2 (extended to isometries of \mathbb{R}^2), rather than to some subgroup of $O(3)$, and we write 1 and \det for the trivial and determinant homomorphisms $O(2) \rightarrow O(1)$, respectively. The $(O(2), 1)$ -invariant functions on \mathbb{B}^2 are thus the radial functions, while the space of $(O(2), \det)$ -invariant functions is trivial.

Corollary 5.11 (indices and nullities of \mathbb{B}^2 under $O(2)$ actions). *With notation as above, we have*

$$\text{ind}_{O(2)}^1(Q_N^{\mathbb{B}^2}) = 0, \quad \text{nul}_{O(2)}^1(Q_N^{\mathbb{B}^2}) = 1, \quad \text{ind}_{O(2)}^{\det}(Q_N^{\mathbb{B}^2}) = \text{nul}_{O(2)}^{\det}(Q_N^{\mathbb{B}^2}) = 0.$$

Equivariant index and nullity of \mathbb{M}^Ξ and \mathbb{M}^Σ . We recall how, away from the equator \mathbb{S}^1 , the surfaces $\Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0}$ and $\Sigma_m^{-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0}$ are constructed as graphs over (subsets of) $-\mathbb{K}_0 \cup \mathbb{K}_0$ and $-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0$. In the vicinity of \mathbb{S}^1 , the surfaces are instead modeled on certain singly periodic minimal surfaces that belong to a family discovered by Karcher [1988] and generalize the classical singly periodic minimal surfaces of Scherk [1835]. We now summarize the key properties of such models to the extent we will need later.

Proposition 5.12 (desingularizing models). *There exist in \mathbb{R}^3 complete, connected, properly embedded minimal surfaces \mathbb{M}^Ξ and \mathbb{M}^Σ having the following properties, which uniquely determine the surfaces up to congruence:*

- (i) \mathbb{M}^Ξ and \mathbb{M}^Σ are periodic in the y direction with period 2π , and the corresponding quotient surfaces have genus zero.
- (ii) \mathbb{M}^Ξ and \mathbb{M}^Σ are invariant under $\underline{R}_{\{x=0\}}$, $\underline{R}_{\{y=\pi/2\}}$, and $\underline{R}_{\{y=-\pi/2\}}$.
- (iii) \mathbb{M}^Ξ is invariant under $\underline{R}_{\{z=0\}}$ and \mathbb{M}^Σ under $\underline{R}_{\{y=z=0\}}$.
- (iv) \mathbb{M}^Ξ has four ends and \mathbb{M}^Σ has six ends, all asymptotically planar.
- (v) Each of \mathbb{M}^Ξ and \mathbb{M}^Σ has an end contained in $\{x \leq 0\} \cap \{z \geq 0\}$ whose asymptotic plane intersects $\{z = 0\}$ at the same angle $\omega_0 > 0$ at which \mathbb{K}_0 intersects \mathbb{B}^2 , and \mathbb{M}^Σ has additionally $\{z = 0\}$ as an asymptotic plane.
- (vi) $\mathbb{M}_{\text{fb}}^\Xi := \mathbb{M}^\Xi \cap \{x \leq 0\} \cap \{|y| \leq \pi/2\}$ and $\mathbb{M}_{\text{fb}}^\Sigma := \mathbb{M}^\Sigma \cap \{x \leq 0\} \cap \{|y| \leq \pi/2\}$ are connected free boundary minimal surfaces in the half-slab $\{x \leq 0\} \cap \{|y| \leq \pi/2\}$, with $\mathbb{M}_{\text{fb}}^\Xi$ invariant under $\underline{R}_{\{z=0\}}$ and $\mathbb{M}_{\text{fb}}^\Sigma$ invariant under $\underline{R}_{\{y=z=0\}}$ (see Figure 3).
- (vii) Each of $\mathbb{M}_{\text{fb}}^\Xi \setminus \{z = 0\}$ and $\mathbb{M}_{\text{fb}}^\Sigma \setminus \{y = z = 0\}$ has exactly two connected components.
- (viii) \mathbb{M}^Ξ has no umbilics, while the set of umbilic points of \mathbb{M}^Σ is $\{(0, n\pi, 0) : n \in \mathbb{Z}\}$.
- (ix) The Gauss map v^Ξ of \mathbb{M}^Ξ restricted to the closure of either component of $\mathbb{M}_{\text{fb}}^\Xi \setminus \{z = 0\}$ is a bijection onto a solid spherical triangle with all sides geodesic segments of length $\pi/2$ (in other words, a quarter hemisphere), less a point in the interior of one side.
- (x) The Gauss map v^Σ of \mathbb{M}^Σ restricted to the closure of either component of $\mathbb{M}_{\text{fb}}^\Sigma \setminus \{y = z = 0\}$ is a bijection onto a spherical lune of dihedral angle $\pi/2$ (in other words, a half-hemisphere), less one vertex and a point in the interior of one side.

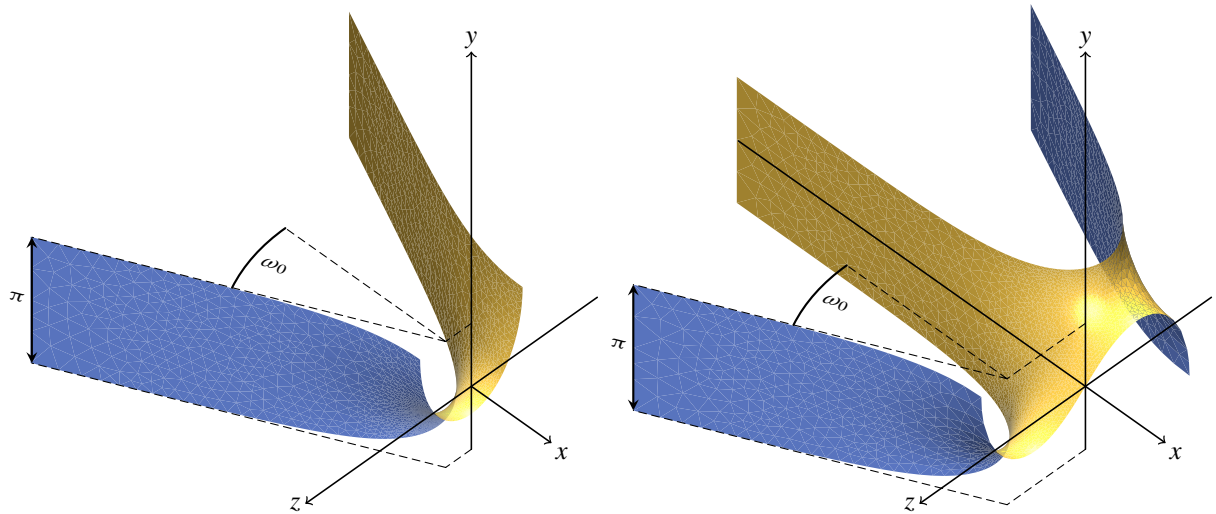


Figure 3. The minimal surfaces M_{fb}^E (left) and M_{fb}^Σ (right) as defined in Proposition 5.12 (vi).

We refer the reader to Section 3 and Appendix A of [Carlotto et al. 2022b] for further details and a fine analysis of the properties of both surfaces in question. The free boundary minimal surfaces M_{fb}^E and M_{fb}^Σ are visualized in Figure 3.

Now we examine the index and nullity of M_{fb}^E and M_{fb}^Σ as free boundary minimal surfaces in the half-slab $\{x \leq 0\} \cap \{|y| \leq \pi/2\}$. Because the boundary of such a domain is piecewise planar, the corresponding Robin condition associated with the index forms of these surfaces is in fact homogeneous (Neumann).

Let us prove an ancillary result. We will observe (in the proof of Lemma 5.16, to follow shortly) that, by virtue of the behavior of the Gauss maps described in Proposition 5.12, the analysis of the index and nullity of M_{fb}^E and M_{fb}^Σ reduces to the following index and nullity computations for boundary value problems on suitable Lipschitz domains of S^2 .

Lemma 5.13 (index and nullity of $\Delta_{g_{S^2}} + 2$ on images of Gauss maps of M_{fb}^E and M_{fb}^Σ). *Set*

$$\begin{aligned} \Omega_{S^2}^E &:= S^2 \cap \{x > 0\} \cap \{y > 0\} \cap \{z > 0\}, \\ \Omega_{S^2}^\Sigma &:= S^2 \cap \{x > 0\} \cap \{y > 0\}. \end{aligned}$$

Then we have the following indices and nullities, where the final row holds for any $\zeta \in]-1, 1[$ and, throughout, T is the bilinear form (2-4) with Ω as indicated, $g = g^{S^2}$ is the round metric, $q = 2$ (and so is associated to the Schrödinger operator $\Delta_{g^{S^2}} + 2$), $\partial_R \Omega = \emptyset$, $\partial_D \Omega$ is as indicated, and $\partial_N \Omega = \partial \Omega \setminus \overline{\partial_D \Omega}$:

Ω	$\partial_D \Omega$	$\text{ind}(T)$	$\text{nul}(T)$
$\Omega_{S^2}^E$	\emptyset	1	0
	$\{z = 0\}$	0	1
$\Omega_{S^2}^\Sigma$	\emptyset	1	1
	$\{x = 0\}$	0	1
	$\{x = 0\} \cap \{z > \zeta\}$	1	0

(5-3)

Proof. By Lemma 3.5, we can fill in the first four rows by identifying the index and nullity of $\Delta_g \mathbb{S}^2 + 2$ on the entire sphere subject to appropriate symmetries, the relevant spherical harmonics being simply the restrictions of affine functions on \mathbb{R}^3 . Lemma 3.5 is not directly applicable to the final row, but by the min-max characterization (2-13) of eigenvalues, the i -th eigenvalue for the bilinear form specified in that row must lie between the i -th eigenvalues of the forms specified in the two preceding rows (\geq that of the third row and \leq that of the fourth); moreover, the unique continuation principle implies that both inequalities must be strict ($>$ and $<$). The entries of the final row now follow, concluding the proof. \square

We shall fix components of

$$\mathbb{M}_{\text{fb}}^{\Xi} \setminus \{z = 0\} \quad \text{and} \quad \mathbb{M}_{\text{fb}}^{\Sigma} \setminus \{y = z = 0\}$$

once and for all and write Ω^{Ξ} and Ω^{Σ} for their respective interiors: it follows from Proposition 5.12 that $\nu^{\Xi}|_{\Omega^{\Xi}}$ and $\nu^{\Sigma}|_{\Omega^{\Sigma}}$ are diffeomorphisms onto their images, which we can and will identify with the triangle $\Omega_{\mathbb{S}^2}^{\Xi}$ and lune $\Omega_{\mathbb{S}^2}^{\Sigma}$ of Lemma 5.13, respectively, and in particular

$$\begin{aligned} \{x = 0\} \cap \partial\Omega_{\mathbb{S}^2}^{\Xi} &= \nu^{\Xi}(\{x = 0\} \cap \partial\Omega^{\Xi}), \\ \{y = 0\} \cap \partial\Omega_{\mathbb{S}^2}^{\Xi} &= \overline{\nu^{\Xi}(\{y = \pm\pi/2\} \cap \partial\Omega^{\Xi})}, \\ \{z = 0\} \cap \partial\Omega_{\mathbb{S}^2}^{\Xi} &= \nu^{\Xi}(\{z = 0\} \cap \partial\Omega^{\Xi}), \end{aligned}$$

and

$$\begin{aligned} \{x = 0\} \cap \partial\Omega_{\mathbb{S}^2}^{\Sigma} &= \overline{\nu^{\Sigma}(\{x = 0\} \cup \{y = z = 0\} \cap \partial\Omega^{\Sigma})}, \\ \{y = 0\} \cap \partial\Omega_{\mathbb{S}^2}^{\Sigma} &= \overline{\nu^{\Sigma}(\{y = \pm\pi/2\} \cap \partial\Omega^{\Sigma})}. \end{aligned}$$

In what follows, recalling, e.g., that the index of a minimal surface, when finite, can be computed by exhaustion (see [Fischer-Colbrie 1985]), we conveniently introduce this notation, which pertains to certain truncations of \mathbb{M}^{Ξ} , \mathbb{M}^{Σ} , $\mathbb{M}_{\text{fb}}^{\Xi}$, and $\mathbb{M}_{\text{fb}}^{\Sigma}$. To do so, we first fix $R_1 > 0$ large enough that $\mathbb{M}^{\Xi} \setminus \{x^2 + z^2 = R_1^2\}$ consists of five connected components: one component C in $\{x^2 + z^2 < R_1^2\}$ and four components W_1, W_2, W_3, W_4 in the complement, each of which is a graph over (a subset of) an asymptotic half-plane (see Figure 4). For each W_i let $\tau^{(i)}$ be a unit vector parallel to the asymptotic half-plane of W_i , perpendicular to the y -axis (the axis of periodicity), and directed away from ∂W_i toward the corresponding end, namely (up to relabeling)

$$\begin{aligned} \tau^{(1)} &= (\cos \omega_0, 0, \sin \omega_0) = -\tau^{(3)}, \\ \tau^{(2)} &= (-\cos \omega_0, 0, \sin \omega_0) = -\tau^{(4)}, \end{aligned}$$

where we recall that $\omega_0 > 0$ is the angle at which \mathbb{K}_0 intersects \mathbb{B}^2 . Now, given $s > R_1$, we define the truncations

$$\begin{aligned} W_i(s) &:= W_i \cap \{\tau^{(i)} \cdot (x, y, z) \leq s\}, \\ \mathbb{M}^{\Xi}(s) &:= \bar{C} \cup \bigcup_{i=1}^4 W_i(s), & \mathbb{M}^{\Sigma}(s) &\text{ analogously (for six ends),} \\ \mathbb{M}_-^{\Xi}(s) &:= \mathbb{M}^{\Xi}(s) \cap \{x \leq 0\}, & \mathbb{M}_-^{\Sigma}(s) &:= \mathbb{M}^{\Sigma}(s) \cap \{x \leq 0\}, \\ \mathbb{M}_{\text{fb}}^{\Xi}(s) &:= \mathbb{M}^{\Xi}(s) \cap \mathbb{M}_{\text{fb}}^{\Xi}, & \mathbb{M}_{\text{fb}}^{\Sigma}(s) &:= \mathbb{M}^{\Sigma}(s) \cap \mathbb{M}_{\text{fb}}^{\Sigma}. \end{aligned} \tag{5-4}$$

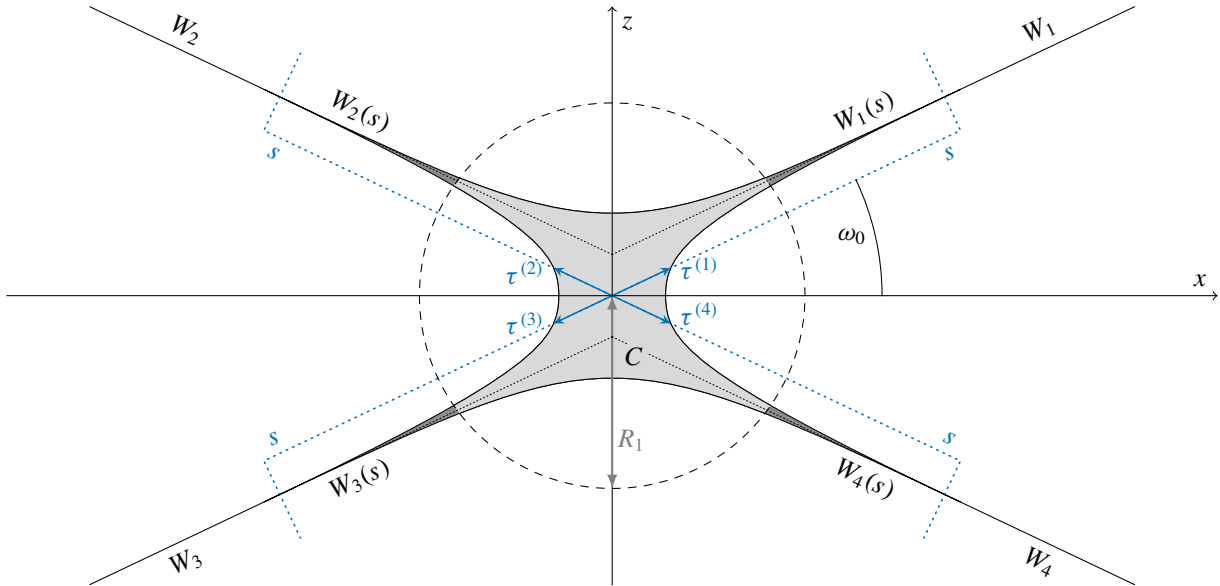


Figure 4. A view of $M^{\Xi}(s)$.

For each $\epsilon, \epsilon' > 0$, we then set similarly $M_{fb}^{\Sigma}(\epsilon^{-1}, \epsilon') := M_{fb}^{\Sigma}(\epsilon^{-1}) \cap \{x^2 + y^2 + z^2 > \epsilon'\}$ and

$$\begin{aligned} \Omega^{\Xi}(\epsilon) &:= \Omega^{\Xi} \cap M_{fb}^{\Xi}(\epsilon^{-1}), \\ \Omega^{\Sigma}(\epsilon, \epsilon') &:= \Omega^{\Sigma} \cap M_{fb}^{\Sigma}(\epsilon^{-1}, \epsilon'), \end{aligned}$$

truncating Ω^{Ξ} and Ω^{Σ} at (affine) distance ϵ^{-1} and excising from Ω^{Σ} a disc with radius $\sqrt{\epsilon'}$ and center at the umbilic $(0, 0, 0)$. We then in turn define $\Omega_{\mathbb{S}^2}^{\Xi}(\epsilon) := \nu^{\Xi}(\Omega^{\Xi}(\epsilon)) \subset \Omega_{\mathbb{S}^2}^{\Xi}$ as well as $\Omega_{\mathbb{S}^2}^{\Sigma}(\epsilon, \epsilon') := \nu^{\Sigma}(\Omega^{\Sigma}(\epsilon, \epsilon')) \subset \Omega_{\mathbb{S}^2}^{\Sigma}$. As a direct consequence of Lemma 5.13 and Proposition 3.9 we get what follows.

Corollary 5.14. *In the setting above, consider for any $\epsilon, \epsilon' > 0$ the Schrödinger operator $\Delta_{g^{\mathbb{S}^2}} + 2$ on the domains given by $\Omega_{\mathbb{S}^2}^{\Xi}(\epsilon)$ and $\Omega_{\mathbb{S}^2}^{\Sigma}(\epsilon, \epsilon')$, respectively, and subject to any of the boundary conditions specified in the table (5-3), where the boundary is contained in $\partial\Omega_{\mathbb{S}^2}^{\Sigma}$ and $\partial\Omega_{\mathbb{S}^2}^{\Xi}$, respectively, and subject to Dirichlet conditions elsewhere. In other words, let T^{Ξ} be either bilinear form corresponding to the top two rows of (5-3), let T^{Σ} be any bilinear form corresponding to the bottom three rows of (5-3), and consider also the bilinear forms*

$$\begin{aligned} T_{\epsilon}^{\Xi} &:= (T^{\Xi})_{\Omega_{\mathbb{S}^2}^{\Xi}(\epsilon)}^{\text{Dir}} = T[\Omega_{\mathbb{S}^2}^{\Xi}(\epsilon), g^{\mathbb{S}^2}, 2, 0, \partial_D\Omega_{\mathbb{S}^2}^{\Xi} \cup (\partial\Omega_{\mathbb{S}^2}^{\Xi}(\epsilon) \setminus \partial\Omega_{\mathbb{S}^2}^{\Xi}), \partial_N\Omega_{\mathbb{S}^2}^{\Xi}, \emptyset], \\ T_{\epsilon, \epsilon'}^{\Sigma} &:= (T^{\Sigma})_{\Omega_{\mathbb{S}^2}^{\Sigma}(\epsilon, \epsilon')}^{\text{Dir}} = T[\Omega_{\mathbb{S}^2}^{\Sigma}(\epsilon, \epsilon'), g^{\mathbb{S}^2}, 2, 0, \partial_D\Omega_{\mathbb{S}^2}^{\Sigma} \cup (\partial\Omega_{\mathbb{S}^2}^{\Sigma}(\epsilon, \epsilon') \setminus \partial\Omega_{\mathbb{S}^2}^{\Sigma}), \partial_N\Omega_{\mathbb{S}^2}^{\Sigma}, \emptyset] \end{aligned}$$

using the notation (2-17). Then there exists $\epsilon_0 > 0$ such that, for all $0 < \epsilon, \epsilon' < \epsilon_0$,

$$\text{ind}(T_{\epsilon}^{\Xi}) = \text{ind}(T^{\Xi}) \quad \text{and} \quad \text{ind}(T_{\epsilon, \epsilon'}^{\Sigma}) = \text{ind}(T^{\Sigma}).$$

In particular, we can derive the following geometric conclusions.

Corollary 5.15 (index of $\mathbb{M}_{\text{fb}}^{\Xi}$ and $\mathbb{M}_{\text{fb}}^{\Sigma}$). *We have the following even and odd indices for $\mathbb{M}_{\text{fb}}^{\Xi}$ and $\mathbb{M}_{\text{fb}}^{\Sigma}$:*

S	G	$\text{ind}_G^+(S)$	$\text{ind}_G^-(S)$
$\mathbb{M}_{\text{fb}}^{\Xi}$	$\{z = 0\}$	1	0
$\mathbb{M}_{\text{fb}}^{\Sigma}$	$\{y = z = 0\}$	1	1

Proof. We will verify (as a sample) the even index asserted in the second row of the table; the other claims are checked in the same fashion. The Gauss map of a minimal surface in \mathbb{R}^3 is (anti)conformal away from its umbilics, with conformal factor (one half of) the pointwise square of the norm of its second fundamental form, so by Proposition 3.11, for each $\epsilon, \epsilon' > 0$, the index of $\Omega^{\Sigma}_{\mathbb{S}^2}(\epsilon, \epsilon')$ with the foregoing boundary conditions (as in Corollary 5.14, according to the third row of the table in Lemma 5.13) agrees also with the index of $\Omega^{\Sigma}(\epsilon, \epsilon')$ subject to the corresponding boundary conditions. By Lemma 3.5, this last index agrees with the $\{y = z = 0\}$ -even index of $\mathbb{M}_{\text{fb}}^{\Sigma}(\epsilon^{-1}, \epsilon')$ subject to the Dirichlet condition along the excisions and the Neumann condition everywhere else. Hence, thanks to Corollary 5.14, such a value of the index is equal to 1 for any sufficiently small ϵ, ϵ' . We now conclude, first letting $\epsilon' \rightarrow 0$ and appealing to Proposition 3.9 to control the effect of the excision near $(0, 0, 0)$, and then appealing to the aforementioned characterization of the Morse index via exhaustions, that $\mathbb{M}_{\text{fb}}^{\Sigma}$ indeed has $\{y = z = 0\}$ -index 1. \square

For use in the following subsection, we fix a smooth cutoff function $\Psi : [0, \infty[\rightarrow [0, 1]$ that is constantly 1 on $\{x \leq 1\}$ and constantly 0 on $\{x \geq 2\}$, and we define on \mathbb{M}^{Ξ} and \mathbb{M}^{Σ} the functions and metrics

$$\begin{aligned} \psi^{\Xi} &:= (\Psi \circ |x|)|_{\mathbb{M}^{\Xi}}, & \rho^{\Xi} &:= \sqrt{\psi^{\Xi} + \frac{1}{2}|A^{\mathbb{M}^{\Xi}}|^2(1 - \psi^{\Xi})}, & h^{\Xi} &:= (\rho^{\Xi})^2 g^{\mathbb{M}^{\Xi}}, \\ \psi^{\Sigma} &:= (\Psi \circ |x|)|_{\mathbb{M}^{\Sigma}}, & \rho^{\Sigma} &:= \sqrt{\psi^{\Sigma} + \frac{1}{2}|A^{\mathbb{M}^{\Sigma}}|^2(1 - \psi^{\Sigma})}, & h^{\Sigma} &:= (\rho^{\Sigma})^2 g^{\mathbb{M}^{\Sigma}}. \end{aligned} \tag{5-5}$$

Note that ρ^{Ξ} is invariant under $\mathbb{R}_{\{z=0\}}$, ρ^{Σ} under $\mathbb{R}_{\{y=z=0\}}$, and both are invariant under $\mathbb{R}_{\{x=0\}}$, $\mathbb{R}_{\{y=-\pi/2\}}$, and $\mathbb{R}_{\{y=\pi/2\}}$. It is natural to associate to $\mathbb{M}_{\text{fb}}^{\Xi}$, regarded as a free boundary minimal surface in the slab $\{x \leq 0\} \cap \{|y| \leq \pi/2\}$, the stability form $Q^{\mathbb{M}_{\text{fb}}^{\Xi}}$, defined at least on smooth functions of compact support by

$$Q^{\mathbb{M}_{\text{fb}}^{\Xi}}(u, v) := \int_{\mathbb{M}_{\text{fb}}^{\Xi}} g^{\mathbb{M}^{\Xi}}(\nabla_{g^{\mathbb{M}^{\Xi}}} u, \nabla_{g^{\mathbb{M}^{\Xi}}} v) d\mathcal{H}^2(g^{\mathbb{M}^{\Xi}}) - \int_{\mathbb{M}_{\text{fb}}^{\Xi}} |A^{\mathbb{M}^{\Xi}}|_{g^{\mathbb{M}^{\Xi}}}^2 uv d\mathcal{H}^2(g^{\mathbb{M}^{\Xi}}).$$

From the identity

$$Q^{\mathbb{M}_{\text{fb}}^{\Xi}}(u, v) = \int_{\mathbb{M}_{\text{fb}}^{\Xi}} h^{\Xi}(\nabla_{h^{\Xi}} u, \nabla_{h^{\Xi}} v) d\mathcal{H}^2(h^{\Xi}) - \int_{\mathbb{M}_{\text{fb}}^{\Xi}} |A^{\mathbb{M}^{\Xi}}|_{h^{\Xi}}^2 uv d\mathcal{H}^2(h^{\Xi})$$

and the manifest boundedness of $|A^{\mathbb{M}^{\Xi}}|_{h^{\Xi}}^2 = (\rho^{\Xi})^{-2}|A^{\mathbb{M}^{\Xi}}|_{g^{\mathbb{M}^{\Xi}}}^2$, we see that $Q^{\mathbb{M}_{\text{fb}}^{\Xi}}$ is in fact well-defined on $H^1(\mathbb{M}_{\text{fb}}^{\Xi}, h^{\Xi})$. Likewise, the analogously defined $Q^{\mathbb{M}_{\text{fb}}^{\Sigma}}$ is well-defined on $H^1(\mathbb{M}_{\text{fb}}^{\Sigma}, h^{\Sigma})$.

We now point out that we can identify the interiors of $\mathbb{M}_{\text{fb}}^{\Xi}$ and $\mathbb{M}_{\text{fb}}^{\Sigma}$ under the metrics h^{Ξ} and h^{Σ} , respectively, as Lipschitz domains in the setting of Section 2. Concretely, we first consider the Riemannian quotients $\tilde{\mathbb{M}}^{\Xi}$ and $\tilde{\mathbb{M}}^{\Sigma}$ of $(\mathbb{M}^{\Xi}, h^{\Xi})$ and $(\mathbb{M}^{\Sigma}, h^{\Sigma})$ under a fundamental period. Then $\tilde{\mathbb{M}}^{\Xi}$ is diffeomorphic to \mathbb{S}^2 with four points removed and $\tilde{\mathbb{M}}^{\Sigma}$ is diffeomorphic to \mathbb{S}^2 with six points removed. By virtue of (5-5) and the behavior of the Gauss maps outlined in Proposition 5.12, we can in fact choose the last

two diffeomorphisms so that they are isometries on neighborhoods of the punctures. In this way we obtain smooth Riemannian compactifications. By composing the defining projection of each tower onto its quotient by a fundamental period with the corresponding embedding into the compactification, we identify (via isometric embedding) the interior of $\mathbb{M}_{\text{fb}}^{\Xi}$ under h^{Ξ} and the interior of $\mathbb{M}_{\text{fb}}^{\Sigma}$ under h^{Σ} with Lipschitz domains $\widehat{\mathbb{M}}_{\text{fb}}^{\Xi}$ and $\widehat{\mathbb{M}}_{\text{fb}}^{\Sigma}$ in the two respective compactifications, and we likewise identify $\partial\mathbb{M}_{\text{fb}}^{\Xi}$ and $\partial\mathbb{M}_{\text{fb}}^{\Sigma}$ with subsets of $\partial\widehat{\mathbb{M}}_{\text{fb}}^{\Xi}$ and $\partial\widehat{\mathbb{M}}_{\text{fb}}^{\Sigma}$, respectively. Of course, the role of the ‘‘ambient manifold’’ for such Lipschitz domains is played by the Riemannian manifolds (\mathbb{S}^2, h^{Ξ}) and $(\mathbb{S}^2, h^{\Sigma})$, respectively; here, with slight abuse of notation, we have tacitly extended the metrics in question across the four and six punctures respectively.

Next, recalling the definition of T from (2-4), we define the bilinear form

$$Q^{\widehat{\mathbb{M}}_{\text{fb}}^{\Xi}} := T[\widehat{\mathbb{M}}_{\text{fb}}^{\Xi}, h^{\Xi}, q = (\rho^{\Xi})^{-2}|A^{\mathbb{M}^{\Xi}}|_{g^{\mathbb{M}^{\Xi}}}^2, r = 0, \partial_{\text{D}}\widehat{\mathbb{M}}_{\text{fb}}^{\Xi} = \emptyset, \partial_{\text{N}}\widehat{\mathbb{M}}_{\text{fb}}^{\Xi} = \partial\widehat{\mathbb{M}}_{\text{fb}}^{\Xi}, \partial_{\text{R}}\widehat{\mathbb{M}}_{\text{fb}}^{\Xi} = \emptyset],$$

where (as we shall do generally in the sequel for functions defined on \mathbb{M}^{Ξ} or \mathbb{M}^{Σ} , without further comment) for the potential we tacitly interpret the right-hand side as a function on $\widehat{\mathbb{M}}_{\text{fb}}^{\Xi}$; we define $Q^{\widehat{\mathbb{M}}_{\text{fb}}^{\Sigma}}$ in analogous fashion. We then have (see Section 3.4) the equalities

$$Q^{\widehat{\mathbb{M}}_{\text{fb}}^{\Xi}} = Q^{\mathbb{M}_{\text{fb}}^{\Xi}} \text{ on } H^1(\mathbb{M}_{\text{fb}}^{\Xi}, h^{\Xi}) \quad \text{and} \quad Q^{\widehat{\mathbb{M}}_{\text{fb}}^{\Sigma}} = Q^{\mathbb{M}_{\text{fb}}^{\Sigma}} \text{ on } H^1(\mathbb{M}_{\text{fb}}^{\Sigma}, h^{\Sigma}). \tag{5-6}$$

Lemma 5.16 (index and nullity of $Q^{\widehat{\mathbb{M}}_{\text{fb}}^{\Xi}}$ and $Q^{\widehat{\mathbb{M}}_{\text{fb}}^{\Sigma}}$). *With definitions as in the preceding paragraph, we have the following indices and nullities:*

S	G	$\text{ind}_G^+(Q^S)$	$\text{nul}_G^+(Q^S)$	$\text{ind}_G^-(Q^S)$	$\text{nul}_G^-(Q^S)$
$\widehat{\mathbb{M}}_{\text{fb}}^{\Xi}$	$\{\mathbf{z} = \mathbf{0}\}$	1	0	0	1
$\widehat{\mathbb{M}}_{\text{fb}}^{\Sigma}$	$\{\mathbf{y} = \mathbf{z} = \mathbf{0}\}$	1	1	1	0

Proof. The first row follows from a direct application of Proposition 3.11 in conjunction with the first two rows of the table in Lemma 5.13. Indeed, in this case there are no umbilic points in play (for, recall, \mathbb{M}^{Ξ} has no umbilic points) and the Gauss map furnishes an (anti)conformal map from the compactified quotient onto \mathbb{S}^2 . For \mathbb{M}^{Σ} , however, the corresponding conformal factor degenerates at the umbilic at $(0, 0, 0)$, as all of its translates. Nevertheless, aided by Lemma 3.5 and Corollary 3.10, we can verify the indices in the second row in much the same fashion, applying Proposition 3.11 on suitable subdomains (obtained by removing smaller and smaller neighborhoods of the origin).

For the nullities, however, we employ an ad hoc argument since one cannot expect an analogue of the aforementioned Corollary 3.10 to hold true in general. That said, we observe first that the translations in the z direction induce a nontrivial, smooth, bounded, $(\{\mathbf{y} = \mathbf{z} = \mathbf{0}\}, +)$ -invariant (scalar-valued) Jacobi field on \mathbb{M}^{Σ} which readily implies that it defines an element of $H^1(\mathbb{M}_{\text{fb}}^{\Sigma}, h^{\Sigma})$. This shows, in view of (5-6), that the nullities in question are at least the values indicated in the table. On the other hand (appealing to Lemma 3.5 for the regularity), each element, say $u : \widehat{\mathbb{M}}_{\text{fb}}^{\Sigma} \rightarrow \mathbb{R}$, of the eigenspace with eigenvalue zero corresponding to the nullities in question is smooth and bounded. If we restrict it to $\Omega^{\Sigma} \subset \widehat{\mathbb{M}}_{\text{fb}}^{\Sigma}$ and consider the precomposition with the inverse of the Gauss map (which, let us recall, yields an (anti)conformal diffeomorphism $\nu^{\mathbb{M}^{\Sigma}} : \Omega^{\Sigma} \rightarrow \Omega_{\mathbb{S}^2}^{\Sigma}$), then the resulting function $u_0 := u \circ (\nu^{\mathbb{M}^{\Sigma}})^{-1}$

satisfies $(\Delta_{g^{S^2}} + 2)u_0 = 0$, and so we get an element contributing to $\text{nul}(T)$, where T is as encoded in the third (resp. fifth) row of the table (5-3) when starting from the $(\{\mathbf{y} = \mathbf{z} = \mathbf{0}\}, +)$ -invariant (resp. $(\{\mathbf{y} = \mathbf{z} = \mathbf{0}\}, -)$ -invariant) problem on $\widehat{\mathbb{M}}_{\text{fb}}^\Sigma$. It is clear that one thereby gets injective maps of vector spaces, and so from Lemma 5.13

$$\text{nul}_G^+(Q^{\widehat{\mathbb{M}}_{\text{fb}}^\Sigma}) \leq 1, \quad \text{nul}_G^-(Q^{\widehat{\mathbb{M}}_{\text{fb}}^\Sigma}) \leq 0,$$

which in particular implies that such maps are, a posteriori, linear isomorphisms, and thus completes the proof. \square

When we wish to consider the sets $\mathbb{M}_{\text{fb}}^\Xi(s)$ and $\mathbb{M}_{\text{fb}}^\Sigma(s)$ endowed with the metrics h^Ξ and h^Σ , respectively, we shall denote them by $\widehat{\mathbb{M}}_{\text{fb}}^\Xi(s)$ and $\widehat{\mathbb{M}}_{\text{fb}}^\Sigma(s)$. Recalling the notation of Section 2.5, we further define

$$Q_{\text{D}}^{\widehat{\mathbb{M}}_{\text{fb}}^\Xi(s)} := (Q^{\widehat{\mathbb{M}}_{\text{fb}}^\Xi(s)})_{\widehat{\mathbb{M}}_{\text{fb}}^\Xi(s)}^{\text{D}_{\text{int}}} \quad \text{and} \quad Q_{\text{N}}^{\widehat{\mathbb{M}}_{\text{fb}}^\Xi(s)} := (Q^{\widehat{\mathbb{M}}_{\text{fb}}^\Xi(s)})_{\widehat{\mathbb{M}}_{\text{fb}}^\Xi(s)}^{\text{N}_{\text{int}}}. \tag{5-7}$$

In short, we are adjoining respectively Dirichlet or Neumann boundary conditions along the cuts.

Lemma 5.17 (spectra of $Q^{\widehat{\mathbb{M}}_{\text{fb}}^\Xi(s)}$ and $Q^{\widehat{\mathbb{M}}_{\text{fb}}^\Sigma(s)}$). *For each integer $i \geq 1$,*

$$\begin{aligned} \lim_{s \rightarrow \infty} \lambda_i^{\{\mathbf{z}=\mathbf{0}\}, \pm}(Q_{\text{D}}^{\widehat{\mathbb{M}}_{\text{fb}}^\Xi(s)}) &= \lim_{s \rightarrow \infty} \lambda_i^{\{\mathbf{z}=\mathbf{0}\}, \pm}(Q_{\text{N}}^{\widehat{\mathbb{M}}_{\text{fb}}^\Xi(s)}) = \lambda_i^{\{\mathbf{z}=\mathbf{0}\}, \pm}(Q^{\widehat{\mathbb{M}}_{\text{fb}}^\Xi}), \\ \lim_{s \rightarrow \infty} \lambda_i^{\{\mathbf{y}=\mathbf{z}=\mathbf{0}\}, \pm}(Q_{\text{D}}^{\widehat{\mathbb{M}}_{\text{fb}}^\Sigma(s)}) &= \lim_{s \rightarrow \infty} \lambda_i^{\{\mathbf{y}=\mathbf{z}=\mathbf{0}\}, \pm}(Q_{\text{N}}^{\widehat{\mathbb{M}}_{\text{fb}}^\Sigma(s)}) = \lambda_i^{\{\mathbf{y}=\mathbf{z}=\mathbf{0}\}, \pm}(Q^{\widehat{\mathbb{M}}_{\text{fb}}^\Sigma}) \end{aligned}$$

for any consistent choice of $+$ or $-$ on both sides of each equality.

Proof. We will write down the proof of the two equalities in the first line for the $+$ choice, as the remaining cases can be proved in the same way. First note that Proposition 3.9 gives us

$$\lim_{s \rightarrow \infty} \lambda_i^{\{\mathbf{z}=\mathbf{0}\}, +}(Q_{\text{D}}^{\widehat{\mathbb{M}}_{\text{fb}}^\Xi(s)}) = \lambda_i^{\{\mathbf{z}=\mathbf{0}\}, +}(Q^{\widehat{\mathbb{M}}_{\text{fb}}^\Xi}).$$

Using the min-max characterization (2-13) of eigenvalues, we then also get

$$\limsup_{s \rightarrow \infty} \lambda_i^{\{\mathbf{z}=\mathbf{0}\}, +}(Q_{\text{N}}^{\widehat{\mathbb{M}}_{\text{fb}}^\Xi(s)}) \leq \limsup_{s \rightarrow \infty} \lambda_i^{\{\mathbf{z}=\mathbf{0}\}, +}(Q_{\text{D}}^{\widehat{\mathbb{M}}_{\text{fb}}^\Xi(s)}) = \lambda_i^{\{\mathbf{z}=\mathbf{0}\}, +}(Q^{\widehat{\mathbb{M}}_{\text{fb}}^\Xi}).$$

The key step now toward the goal of establishing

$$\liminf_{s \rightarrow \infty} \lambda_i^{\{\mathbf{z}=\mathbf{0}\}, +}(Q_{\text{N}}^{\widehat{\mathbb{M}}_{\text{fb}}^\Xi(s)}) \geq \lambda_i^{\{\mathbf{z}=\mathbf{0}\}, +}(Q^{\widehat{\mathbb{M}}_{\text{fb}}^\Xi})$$

(which completes the proof) is to construct a family of (appropriately symmetric) linear extension operators $E_s : H^1(\widehat{\mathbb{M}}_{\text{fb}}^\Xi(s)) \rightarrow H^1(\widehat{\mathbb{M}}_{\text{fb}}^\Xi)$ uniformly bounded in s , assuming $s \geq s_0$ for some universal $s_0 > 0$. With these extensions in hand it is straightforward, for example, to adapt the argument for (3-6) in the proof of Proposition 3.8.

We now outline the construction of the E_s extension operators. By the imposed symmetry (in the case under discussion even reflection through $\{z = 0\}$) and by taking s large enough, it suffices to specify the extension on a single end W , a graph over a subset of the corresponding asymptotic plane Π (with τ the corresponding defining vector, recalling the notation preceding (5-4)). Let $\varpi : W \rightarrow \Pi$ be the associated projection. By partitioning the given function using appropriately chosen smooth cutoff functions (fixed independently of s), it in fact suffices to consider the extension problem for a function

$v \in H^1(W \cap \mathbb{M}_{\text{fb}}^{\Xi}(s), h^{\Xi})$ such that the support of ϖ^*v is compactly contained in the rectangle (expressed in the notation of (5-4))

$$\{0 < \tau \cdot (x, y, z) \leq s\} \cap \{-\pi \leq 2y \leq \pi\}.$$

We can extend ϖ^*v via even reflection through the s side of the above rectangle, thereby obtaining an extension of v to an element of $H^1(W, h^{\Xi})$. The asymptotic convergence of W to Π , the monotonic decay of ρ^{Ξ} along W toward ∞ , and the conformal invariance (in the current two-dimensional setting) of the Dirichlet energy ensure that this extension has the desired properties. \square

5.3. Deconstruction of the surfaces and regionwise geometric convergence. We first take a moment to briefly review the constructions of the surfaces from [Carlotto et al. 2022b]. First (see [loc. cit., Section 3]), an approximate minimal surface in \mathbb{B}^3 , called the initial surface, whose boundary is contained in $\partial\mathbb{B}^3$ and which meets $\partial\mathbb{B}^3$ exactly orthogonally, is fashioned by hand, via suitable interpolations, from the models $(\mathbb{K}_0, \mathbb{M}^{\Xi}, \text{ or } \mathbb{M}^{\Sigma}, \text{ and for } \Sigma_m^{-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0} \text{ also } \mathbb{B}^2)$. Second (see [loc. cit., Section 5]), the final exact solution is identified as the normal graph of a small function over the approximate solution. For what pertains to this second step we wish only to highlight that the assignment of graph to function is made using not the usual Euclidean metric $g^{\mathbb{R}^3}$ but instead an $O(3)$ -invariant metric (fixed once and for all, independently of the data n or m) conformally Euclidean and called the auxiliary metric. On a neighborhood of the origin this metric agrees exactly with the Euclidean one, while on a neighborhood of $\partial\mathbb{B}^3 = \mathbb{S}^2$ it agrees exactly with the cylindrical metric on $\mathbb{S}^2 \times \mathbb{R}$; this last property and the orthogonality of the intersection of the initial surface with $\partial\mathbb{B}^3$ ensure that the boundary of the resulting graph is also in $\partial\mathbb{B}^3$. We will write $\widehat{\Sigma}_n^{-\mathbb{K}_0 \cup \mathbb{K}_0}$ and $\widehat{\Sigma}_m^{-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0}$ for the initial surfaces and $\varpi_n^{\Xi} : \mathfrak{E}_n^{-\mathbb{K}_0 \cup \mathbb{K}_0} \rightarrow \widehat{\Sigma}_n^{-\mathbb{K}_0 \cup \mathbb{K}_0}$ and $\varpi_m^{\Sigma} : \Sigma_m^{-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0} \rightarrow \widehat{\Sigma}_m^{-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0}$ for the nearest-point projections under the above auxiliary metric.

Turning to the first step, actually (because of the presence of a cokernel) one constructs for each given n or m not just a single initial surface but a (continuous) one-parameter family of them. In the construction this parameter is treated as an unknown and is determined only in the second step, simultaneously with the defining function for the final surface. Here, however, we can take the construction for granted and accordingly speak of a single initial surface, whose defining parameter value is some definite (though not explicit) function of n or m as appropriate. Nevertheless we must explain that this parameter enters the construction at the level of the building blocks, except for \mathbb{B}^2 , which is unaffected, as follows. First, the catenoidal annulus \mathbb{K}_0 is just one in a family \mathbb{K}_{ϵ} (see the beginning of Section 3.1 in [loc. cit.]) of such annuli, all rotationally symmetric about the z -axis, depending smoothly on ϵ . The details are not critical here, but each \mathbb{K}_{ϵ} is the intersection with \mathbb{B}^3 of a complete catenoid with axis the z -axis, and \mathbb{K}_{ϵ} meets \mathbb{S}^2 at two circles of latitude, the upper one a circle of orthogonal intersection and the lower one the circle at height $z = \epsilon$. Similarly, from \mathbb{M}^{Ξ} and \mathbb{M}^{Σ} we define, by explicit graphical deformation, families which here we will call $\mathbb{M}_{\delta}^{\Xi}$ and $\mathbb{M}_{\delta}^{\Sigma}$ (see the beginning of Section 3.2 of [loc. cit.]). These deformations are the identity on the “cores” of \mathbb{M}^{Ξ} and \mathbb{M}^{Σ} and smoothly transition to translations on the ends, in the z -direction, up or down depending on the end, and through a displacement determined by δ . Importantly, all the $\mathbb{M}_{\delta}^{\Xi}$ and $\mathbb{M}_{\delta}^{\Sigma}$ have the same symmetries as \mathbb{M}^{Ξ} and \mathbb{M}^{Σ} , respectively. Now the datum n determines building blocks $\mathbb{M}_{\delta^{\Xi}(n)}^{\Xi}$ and $\mathbb{K}_{\epsilon^{\Xi}(n)}$, while the datum m determines building blocks $\mathbb{M}_{\delta^{\Sigma}(m)}^{\Sigma}$, $\mathbb{K}_{\epsilon^{\Sigma}(m)}$, and \mathbb{B}^2 .

We next define maps Φ_n^Ξ and Φ_m^Σ [loc. cit., (3.37)] from neighborhoods of $(1/n)\mathbb{M}_{\delta^\Xi(n)}^\Xi \cap \{x \leq 0\}$ and $(1/(m+1))\mathbb{M}_{\delta^\Sigma(m)}^\Sigma \cap \{x \leq 0\}$, respectively, into \mathbb{B}^3 , so as to “wrap” the cores of these surfaces around the equator \mathbb{S}^1 approximately isometrically but to take their asymptotic half-planes (in $\{x \leq 0\}$) onto $\pm\mathbb{K}_\epsilon^\Xi(n)$ in the first case and onto $\pm\mathbb{K}_\epsilon^\Sigma(m)$ and \mathbb{B}^2 in the second. Thus, just referring to the family $\Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0}$ for the sake of brevity, we truncate the surface $\mathbb{M}_{\delta^\Xi(n)}^\Xi$ by intersecting with $\{x \geq -n^{3/4}\}$, and then apply Φ_n^Ξ to the truncated surface scaled-down by a factor of $1/n$. The image is embedded (for n large enough) and contained in the ball, and is in fact contained in a tubular neighborhood of \mathbb{S}^1 with radius of order $n^{-1/4}$.

Near the two truncation boundary components, the surface is a small graph over either $\pm\mathbb{K}_\epsilon^\Xi(n)$. We smoothly cut off the defining function in a $(1/n)$ -neighborhood of the boundary to make the surface exactly catenoidal there and then extend using these annuli on the other side of the truncation boundary all the way to $\partial\mathbb{B}^3$. The result is our initial surface $\widehat{\Xi}_n^{-\mathbb{K}_0 \cup \mathbb{K}_0}$. The initial surface $\widehat{\Sigma}_m^{-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0}$ is constructed analogously, now also smoothly transitioning from the middle truncation boundary to coincide with \mathbb{B}^2 on a neighborhood of the origin. In what follows we will distill those objects and ancillary results that are needed for the spectral convergence theorems we will prove in Section 5.4.

Decompositions. Recalling (5-4) for the definition of the below domains, our construction in [Carlotto et al. 2022b] provides, in particular, smooth maps

$$\begin{aligned} \varphi^{M_n^\Xi} : \mathbb{M}_-^\Xi(n^{5/8}) &\rightarrow \Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0}, \\ \varphi^{M_m^\Sigma} : \mathbb{M}_-^\Sigma((m+1)^{5/8}) &\rightarrow \Sigma_m^{-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0}, \end{aligned}$$

which are smooth coverings of their images. For all $0 < s \leq \sqrt{n}$, or, respectively, $0 < s \leq \sqrt{m+1}$, we in turn define

$$\begin{aligned} M_n^\Xi(s) &:= \varphi^{M_n^\Xi}(\mathbb{M}_-^\Xi(s)) \subset \Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0}, \\ M_m^\Sigma(s) &:= \varphi^{M_m^\Sigma}(\mathbb{M}_-^\Sigma(s)) \subset \Sigma_m^{-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0}. \end{aligned}$$

In practice, in addition to the upper bound required on s , we will be interested only in s greater than a universal constant set by \mathbb{M}^Ξ and \mathbb{M}^Σ : we want to truncate far enough out (in the domain) that near and beyond the truncation boundary the surface is already the graph of a small function over the asymptotic planes. In a typical application to follow we will take s large in absolute terms and then take n or m large with respect to s , so we will not always repeat either restriction. When they do hold, $\Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0} \setminus M_n^\Xi(s)$ consists of two connected components and $\Sigma_m^{-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0} \setminus M_m^\Sigma(s)$ consists of three, and we define

$$\begin{aligned} K_n^\Xi(s) &:= \text{the closure of the component of } \Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0} \setminus M_n^\Xi(s) \text{ on which } z \text{ is maximized,} \\ K_m^\Sigma(s) &:= \text{the closure of the component of } \Sigma_m^{-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0} \setminus M_m^\Sigma(s) \text{ on which } z \text{ is maximized,} \\ B_m^\Sigma(s) &:= \text{the closure of the component of } \Sigma_m^{-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0} \setminus M_m^\Sigma(s) \text{ that contains the origin.} \end{aligned}$$

Observe that each $M_n^\Xi(s)$ is invariant under $\underline{R}_{\{z=0\}}$, that the interiors of $M_n^\Xi(s)$, $K_n^\Xi(s)$, and $\underline{R}_{\{z=0\}}K_n^\Xi(s)$ are pairwise disjoint, and that the last three regions cover $\Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0}$. In particular, considering the interior of such sets, one thereby determines a candidate partition for the application of Proposition 3.1. Similarly, $M_m^\Sigma(s)$ and $B_m^\Sigma(s)$ are invariant under $\underline{R}_{\{y=z=0\}}$; the interiors of $M_m^\Sigma(s)$, $B_m^\Sigma(s)$, $K_m^\Sigma(s)$, and $\underline{R}_{\{y=z=0\}}K_m^\Sigma(s)$ are pairwise disjoint, and these four surfaces also cover $\Sigma_m^{-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0}$.

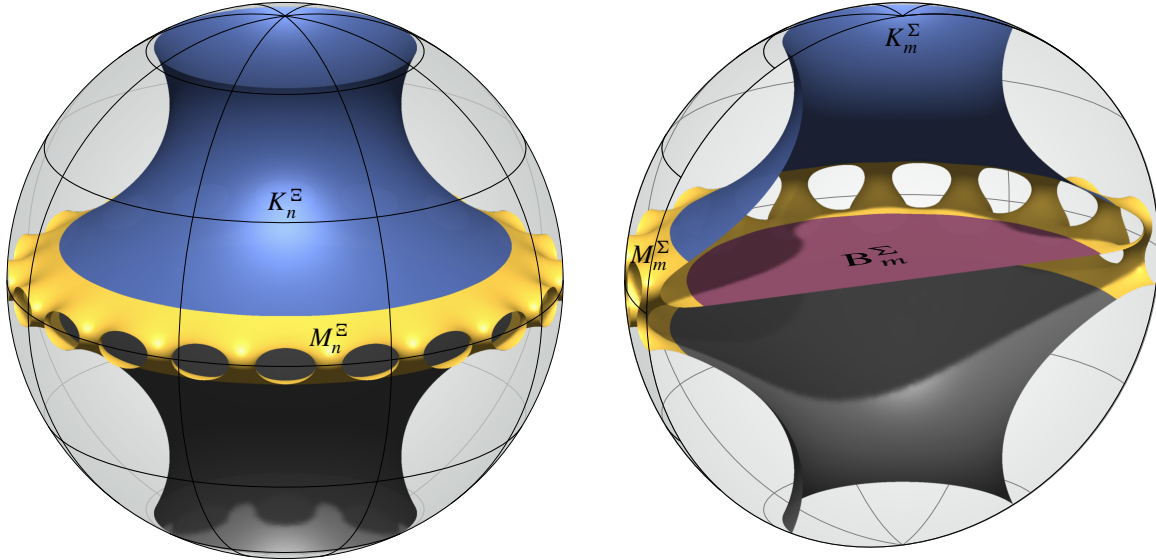


Figure 5. Decomposition of $\Xi_n^{-\mathbb{K}_0} \cup \mathbb{K}_0$ (left) and $\Sigma_m^{-\mathbb{K}_0} \cup \mathbb{B}^2 \cup \mathbb{K}_0$ (right, cutaway view).

We agree to distinguish the choices $s = \sqrt{n}$ and $s = \sqrt{m+1}$ by omission of the parameter value:

$$\begin{aligned} M_n^\Xi &:= M_n^\Xi(\sqrt{n}), & K_n^\Xi &:= K_n^\Xi(\sqrt{n}), \\ M_m^\Sigma &:= M_m^\Sigma(\sqrt{m+1}), & K_m^\Sigma &:= K_m^\Sigma(\sqrt{m+1}), & B_m^\Sigma &:= B_m^\Sigma(\sqrt{m+1}), \end{aligned}$$

as visualized in Figure 5. We also define the dilated truncations (see Figure 6)

$$\begin{aligned} M_{\text{fb},n}^\Xi(s) &:= n(M_n^\Xi(s) \cap W_{-\pi/(2n)}^{\pi/(2n)}) = n\varphi^{M_n^\Xi}(\mathbb{M}_{\text{fb}}^\Xi(s)), & M_{\text{fb},n}^\Xi &:= M_{\text{fb},n}^\Xi(\sqrt{n}), \\ M_{\text{fb},m}^\Sigma(s) &:= (m+1)(M_m^\Sigma(s) \cap W_{-\pi/(2(m+1))}^{\pi/(2(m+1))}) = (m+1)\varphi^{M_m^\Sigma}(\mathbb{M}_{\text{fb}}^\Sigma(s)), & M_{\text{fb},m}^\Sigma &:= M_{\text{fb},m}^\Sigma(\sqrt{m+1}), \end{aligned}$$

where the notation for wedges has been given in (4-2), and finally introduce the transition regions

$$\Lambda_n^\Xi(s) := \overline{M_{\text{fb},n}^\Xi} \setminus M_{\text{fb},n}^\Xi(s), \quad \Lambda_m^\Sigma(s) := \overline{M_{\text{fb},m}^\Sigma} \setminus M_{\text{fb},m}^\Sigma(s).$$

Geometric estimates. Before proceeding, we declare the following abbreviated notation for the metrics and second fundamental forms on $M_{\text{fb},n}^\Xi$ and $M_{\text{fb},m}^\Sigma$ (induced by their inclusions in $(\mathbb{R}^3, g^{\mathbb{R}^3})$):

$$g_n^\Xi := g^{M_{\text{fb},n}^\Xi}, \quad g_m^\Sigma := g^{M_{\text{fb},m}^\Sigma}, \quad A_n^\Xi := A^{M_{\text{fb},n}^\Xi}, \quad A_m^\Sigma := A^{M_{\text{fb},m}^\Sigma}.$$

In analogy with (5-5), we first write $\psi_n^\Xi, \psi_m^\Sigma$ for the unique functions on $M_{\text{fb},n}^\Xi, M_{\text{fb},m}^\Sigma$ such that

$$\psi^\Xi = (n \circ \varphi^{M_n^\Xi})^* \psi_n^\Xi, \quad \psi^\Sigma = ((m+1) \circ \varphi^{M_m^\Sigma})^* \psi_m^\Sigma,$$

and then in turn define

$$\begin{aligned} \rho_n^\Xi &:= \sqrt{\psi_n^\Xi + \frac{1}{2}|A_n^\Xi|_{g_n^\Xi}^2(1 - \psi_n^\Xi) + e^{-2n}}, & h_n^\Xi &:= (\rho_n^\Xi)^2 g_n^\Xi, \\ \rho_m^\Sigma &:= \sqrt{\psi_m^\Sigma + \frac{1}{2}|A_m^\Sigma|_{g_m^\Sigma}^2(1 - \psi_m^\Sigma) + e^{-2m}}, & h_m^\Sigma &:= (\rho_m^\Sigma)^2 g_m^\Sigma. \end{aligned} \tag{5-8}$$

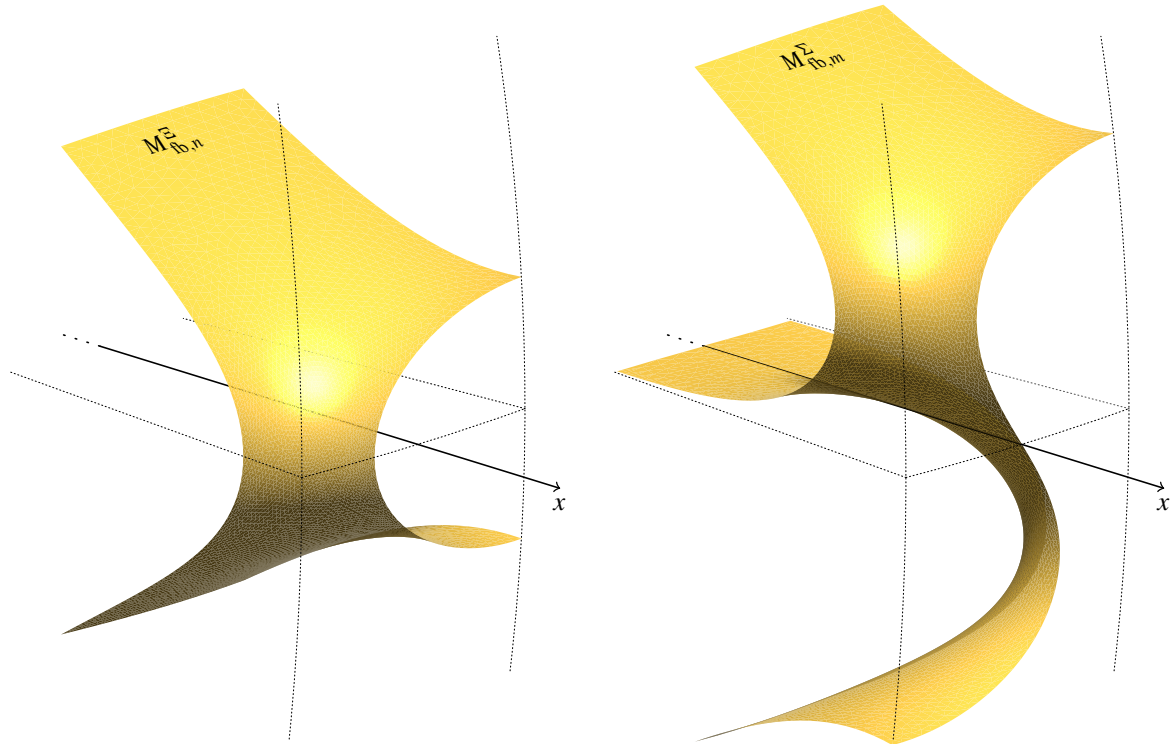


Figure 6. The dilated truncations $M_{fb,n}^{\Xi}$ (left) and $M_{fb,m}^{\Sigma}$ (right).

The terms e^{-2n} and e^{-2m} above are included to ensure the conformal factors vanish nowhere. For the sake of brevity, and consistent with the notation adopted in the previous subsections, we set

$\widehat{M}_{fb,n}^{\Xi} := (M_{fb,n}^{\Xi}, h_n^{\Xi})$, $\widehat{M}_{fb,m}^{\Sigma} := (M_{fb,m}^{\Sigma}, h_m^{\Sigma})$, $\widehat{M}_{fb,n}^{\Xi}(s) := (M_{fb,n}^{\Xi}(s), h_n^{\Xi})$, $\widehat{M}_{fb,m}^{\Sigma}(s) := (M_{fb,m}^{\Sigma}(s), h_m^{\Sigma})$, so that $\widehat{M}_{fb,n}^{\Xi}$ and $\widehat{M}_{fb,m}^{\Sigma}$ and their truncations $\widehat{M}_{fb,n}^{\Xi}(s) \subset M_{fb,n}^{\Xi}$ and $M_{fb,m}^{\Sigma}(s) \subset M_{fb,m}^{\Sigma}$ are always understood as being equipped with the conformal metrics h_n^{Ξ} and h_m^{Σ} , rather than g_n^{Ξ} and g_m^{Σ} .

Lemma 5.18 (convergence of $M_{fb,n}^{\Xi}(s)$ and $M_{fb,m}^{\Sigma}(s)$). *For every $s > 0$, there exists $m_s > 0$ such that, for every integer $m > m_s$,*

- (i) *the region $M_{fb,m}^{\Sigma}(s)$ is defined and is the diffeomorphic image under $(m + 1)\varphi^{M_m^{\Sigma}}$ of $\mathbb{M}_{fb}^{\Sigma}(s)$,*
- (ii) $(m + 1)\varphi^{M_m^{\Sigma}}(\mathbb{M}_{fb}^{\Sigma}(s) \cap \{x = 0\}) = M_{fb,m}^{\Sigma}(s) \cap (m + 1)\mathbb{S}^2$,
- (iii) $\varphi^{M_m^{\Sigma}}$ commutes with $\mathbb{R}_{\{z=0\}}$, and
- (iv) $M_m^{\Sigma}(s) = (m + 1)^{-1}\mathbb{A}_{m+1}M_{fb,m}^{\Sigma}(s)$ is a surface with smooth boundary.

Moreover, for every $s > 0$ and $\alpha \in]0, 1[$,

- (v) $((m + 1) \circ \varphi^{M_m^{\Sigma}})^* g_m^{\Sigma} \xrightarrow[m \rightarrow \infty]{C^{1,\alpha}(\mathbb{M}_{fb}^{\Sigma}(s), g^{M^{\Sigma}})} g^{M^{\Sigma}}$ and
- (vi) $((m + 1) \circ \varphi^{M_m^{\Sigma}})^* A_m^{\Sigma} \xrightarrow[m \rightarrow \infty]{C^{0,\alpha}(\mathbb{M}_{fb}^{\Sigma}(s), g^{M^{\Sigma}})} A^{M^{\Sigma}}$.

All the above statements have analogues for $\Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0}$ in place of $\Sigma_m^{-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0}$, mutatis mutandis.

The first four claims are immediate from the definitions, while the convergence assertions are ensured, in the case of $\Sigma_m^{-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0}$, by the following estimates from [Carlotto et al. 2022b], the case of $\Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0}$ being completely analogous. Namely, the estimate [loc. cit., (5.20)] provides $C^{2,\alpha}$ bounds for the defining function of $\Sigma_m^{-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0}$ as a graph over the corresponding initial surface, and so controls the projection map ϖ_m^Σ from $\Sigma_m^{-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0}$ to the initial surface. The same estimate [loc. cit., (5.20)] also bounds the parameter value for the initial surface from the one-parameter family that is selected to produce the final one. On the other hand, [loc. cit., Proposition 3.18] provides estimates on the initial surface in terms of the datum g as well as the value of the continuous parameter. (As an aid to extracting the required information, we point out that the map $\varpi_{M_{m,\varepsilon}}$ in [loc. cit., (3.43)] is essentially (that is, up to some quotienting and the exact extent of the domains) the inverse of the map $\varpi_{m-1}^\Sigma \circ \varphi^{M_{m-1}^\Sigma}$ of the present article.)

Let us consider the other portions of our surfaces. By construction $\varpi_n^\Xi(K_n^\Xi)$ and $\varpi_m^\Sigma(K_m^\Sigma)$ (subsets of the initial surfaces) are graphs (under the Euclidean metric $g^{\mathbb{R}^3}$) over subsets of $\mathbb{K}_{\varepsilon^\Xi(n)}$ and $\mathbb{K}_{\varepsilon^\Sigma(m)}$, and $\varpi_m^\Sigma(B_m^\Sigma)$ is a graph over \mathbb{B}^2 . Thus, by composition with a further projection, we obtain injective maps

$$\varpi_n^\Xi(K_n^\Xi) \rightarrow \mathbb{K}_{\varepsilon^\Xi(n)}, \quad \varpi_m^\Sigma(K_m^\Sigma) \rightarrow \mathbb{K}_{\varepsilon^\Sigma(m)}, \quad \text{and} \quad B_m^\Sigma \rightarrow \mathbb{B}^2.$$

Moreover, the image of each of these three maps is $O(2)$ invariant: the image of the third is a disc with radius tending to 1 as $m \rightarrow \infty$, the image of the second is a catenoidal annulus with upper boundary circle coinciding with that of $\mathbb{K}_{\varepsilon^\Sigma(m)}$ and lower boundary circle tending to that of $\mathbb{K}_{\varepsilon^\Sigma(m)}$ as $m \rightarrow \infty$; the image of the first admits an analogous description.

In particular, by composing further with dilations of scale factor tending to 1, we obtain diffeomorphisms

$$\varphi^{B_m^\Sigma} : \mathbb{B}^2 \rightarrow B_m^\Sigma;$$

similarly reparametrizing in the radial direction one also obtains diffeomorphisms

$$\varphi^{K_n^\Xi} : \mathbb{K}_0 \rightarrow K_n^\Xi, \quad \varphi^{K_m^\Sigma} : \mathbb{K}_0 \rightarrow K_m^\Sigma.$$

The inverses of these maps may be regarded as small perturbations (for n and m large) of the nearest-point projection onto $\mathbb{R}^2 \subset \mathbb{B}^2$ or onto the complete catenoid containing \mathbb{K}_0 , as appropriate. Somewhat more formally, by reference to [loc. cit.] (specifically Proposition 3.18 and estimate (5.20) therein), much as in the proof of Lemma 5.18, we confirm the following properties of K_n^Ξ , K_m^Σ , and B_m^Σ .

Lemma 5.19 (convergence of K_n^Ξ and K_m^Σ). *There exists $m_0 > 0$ such that, for each integer $m > m_0$,*

- (i) $\varphi^{K_m^\Sigma}$ is defined and a diffeomorphism from \mathbb{K}_0 onto K_m^Σ ,
- (ii) $\varphi^{K_m^\Sigma}$ commutes with each element of \mathbb{Y}_{m+1} , and
- (iii) $\varphi^{K_m^\Sigma}$ takes the upper boundary component of \mathbb{K}_0 to the upper boundary component of K_m^Σ .

Moreover, for every $\alpha \in]0, 1[$,

- (vi) $(\varphi^{K_m^\Sigma})^* g^{\Sigma_m^{-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0}}|_{K_m^\Sigma} \xrightarrow{m \rightarrow \infty} g^{\mathbb{K}_0}$ and
- (vii) $(\varphi^{K_m^\Sigma})^* A^{\Sigma_m^{-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0}}|_{K_m^\Sigma} \xrightarrow{m \rightarrow \infty} A^{\mathbb{K}_0}$.

All the above statements have analogues for $\Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0}$ in place of $\Sigma_m^{-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0}$, *mutatis mutandis*.

Lemma 5.20 (convergence of B_g^Σ). *There exists $m_0 > 0$ such that, for each integer $m > m_0$,*

- (i) $\varphi^{B_m^\Sigma}$ is defined and a diffeomorphism from \mathbb{B}^2 onto B_m^Σ and
- (ii) $\varphi^{B_m^\Sigma}$ commutes with each element of \mathbb{A}_{m+1} .

Moreover, for each $\alpha \in]0, 1[$,

- (iii) $(\varphi^{B_m^\Sigma})^* g^{\Sigma - \mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0} |_{B_m^\Sigma} \xrightarrow{m \rightarrow \infty} g^{\mathbb{B}^2}$ and
- (iv) $(\varphi^{B_m^\Sigma})^* A^{\Sigma - \mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0} |_{B_m^\Sigma} \xrightarrow{m \rightarrow \infty} 0$.

Last we focus on the transition regions. Let us agree to write t_n^Ξ and t_m^Σ for the distance functions on $n\mathbb{K}_{\epsilon^\Xi(n)}$ and $(m+1)\mathbb{K}_{\epsilon^\Sigma(m)}$ from their respective lower boundary circles. By construction (assuming s large enough in absolute terms) $n\varpi_n^\Xi(n^{-1}\Lambda_n^\Xi(s))$ has two connected components, one a graph over the catenoidal annular wedge

$$\{s \leq t_n^\Xi \leq \sqrt{n}\} \cap W_{-\pi/(2n)}^{\pi/(2n)} \subset n\mathbb{K}_{\epsilon^\Xi(n)}$$

and the other the reflection of this last one through $\{z = 0\}$, while $(m+1)\varpi_m^\Sigma((m+1)^{-1}\Lambda_m^\Sigma(s))$ has three connected components, one a graph over the planar annular wedge

$$\{s \leq (m+1) - r \leq \sqrt{m+1}\} \cap W_{-\pi/(2(m+1))}^{\pi/(2(m+1))} \cap (m+1)\mathbb{B}^2,$$

another a graph over the catenoidal annular wedge

$$\{s \leq t_m^\Sigma \leq \sqrt{m+1}\} \cap W_{-\pi/(2(m+1))}^{\pi/(2(m+1))} \subset (m+1)\mathbb{K}_{\epsilon^\Sigma(m)},$$

and the third the reflection of this last one through $\{y = z = 0\}$.

Projecting onto these rotationally invariant sets and parametrizing them by arc length t in the “radial” direction and $\vartheta := n\theta$, or, respectively, $\vartheta := (m+1)\theta$ in the angular direction (with θ restricted to the appropriate interval containing 0), we obtain injective maps

$$\varphi^{\Lambda_n^\Xi(s), \mathbb{K}} : [s, \sqrt{n}] \times \left[\frac{\pi}{2}, \frac{\pi}{2} \right] \rightarrow \Lambda_n^\Xi, \quad \varphi^{\Lambda_m^\Sigma(s), \mathbb{K}}, \varphi^{\Lambda_m^\Sigma(s), \mathbb{B}^2} : [s, \sqrt{m+1}] \times \left[-\frac{\pi}{2}, \frac{\pi}{2} \right] \rightarrow \Lambda_m^\Sigma,$$

whose images are components of $\Lambda_n^\Xi(s)$ and $\Lambda_m^\Sigma(s)$ that generate the latter regions under $\{z = \mathbf{0}\}$ and $\{y = z = \mathbf{0}\}$, respectively.

Lemma 5.21 (estimates on $\Lambda_n^\Xi(s)$ and $\Lambda_m^\Sigma(s)$). *Let $\alpha \in]0, 1[$. There exists $s_0 > 0$ such that, for each $s > s_0$, there exists $m_s > 0$ such that, for every integer $m > m_s$,*

- (i) $(\varphi^{\Lambda_m^\Sigma(s), \mathbb{K}})^* |A_m^\Sigma|_{g_m^\Sigma}^2(t, \vartheta) = a_1(t)m^{-2} + a_2(t, \vartheta)e^{-t/4}$ for some smooth functions a_1 and a_2 having $C^{0,\alpha}(dt^2 + d\vartheta^2)$ norm bounded independently of m and s ,
- (ii) $(\varphi^{\Lambda_m^\Sigma(s), \mathbb{B}^2})^* |A_m^\Sigma|_{g_m^\Sigma}^2(t, \vartheta) = a_3(t, \vartheta)e^{-t/4}$ for some smooth function a_3 having $C^{0,\alpha}(dt^2 + d\vartheta^2)$ norm bounded independently of m and s ,
- (iii) $(\varphi^{\Lambda_m^\Sigma(s), \mathbb{K}})^* g_m^\Sigma = dt^2 + (1 + m^{-1}tf^1(t))d\vartheta^2 + f_{uv}^1(t, \vartheta)e^{-t/4}dudv$ for some smooth functions f^1, f_{uv}^1 having $C^{1,\alpha}(dt^2 + d\vartheta^2)$ norm bounded independently of m and s ,
- (iv) $(\varphi^{\Lambda_m^\Sigma(s), \mathbb{B}^2})^* g_m^\Sigma = dt^2 + (1 + m^{-1}tf^2(t))d\vartheta^2 + f_{uv}^2(t, \vartheta)e^{-t/4}dudv$ for some smooth functions f^2 and f_{uv}^2 having $C^{1,\alpha}(dt^2 + d\vartheta^2)$ norm bounded independently of m and s ,

- (v) $\Delta_{(\varphi^{\Lambda_m^\Sigma(s), \mathbb{K}})^* g_m^\Sigma} = \partial_t^2 + m^{-1} c_1^t(t) \partial_t + (1 + m^{-1/2} b_1^{\vartheta}(t)) \partial_\vartheta^2 + e^{-t/4} (b_2^{uv}(t, \vartheta) \partial_u \partial_v + c_2^u(t, \vartheta) \partial_u)$ for some smooth functions $b_1^{\vartheta}, b_2^{uv}, c_1^t, c_2^u$ having $C^{0,\alpha}(dt^2 + d\vartheta^2)$ norm bounded independently of m and s , and
- (vi) $\Delta_{(\varphi^{\Lambda_m^\Sigma(s), \mathbb{B}^2})^* g_m^\Sigma} = \partial_t^2 + m^{-1} c_3^t(t) \partial_t + (1 + m^{-1/2} b_3^{\vartheta}(t)) \partial_\vartheta^2 + e^{-t/4} (b_4^{uv}(t, \vartheta) \partial_u \partial_v + c_4^u(t, \vartheta) \partial_u)$ for some smooth functions $b_3^{\vartheta}, b_4^{uv}, c_3^t, c_4^u$ having $C^{0,\alpha}(dt^2 + d\vartheta^2)$ norm bounded independently of m and s .

It is understood that, in items (iii), (iv), (v), (vi), one sums over $u, v \in \{t, \vartheta\}$.

Furthermore,

$$(vii) \lim_{s \rightarrow \infty} \lim_{m \rightarrow \infty} \mathcal{H}^2(h_m^\Sigma)(\Lambda_m^\Sigma(s)) = 0.$$

The same claims hold for $\Lambda_n^\Xi(s)$, *mutatis mutandis*.

Proof. Again the estimates are ultimately justified by reference to the construction in [Carlotto et al. 2022b], most specifically (5.20) and Proposition 3.18 therein. That said, we also note how (v) follows easily from (iii), as does (vi) from (iv); furthermore, it is clear that the justification of (ii) is analogous to (in fact simpler than) (i), and (iv) is analogous to (iii). As a result, we briefly explain the ideas behind the elementary computations required for the proof, in the case of $\Lambda_m^\Sigma(s)$, with regard to items (i) and (iii).

The projection of this region onto the blown-up initial surface $(m+1)\widehat{\Sigma}_m^{-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0}$ is itself constructed as a graph over $(m+1)\mathbb{K}_\varepsilon^\Sigma(m)$ or \mathbb{B}^2 . Estimate [loc. cit., (5.20)] ensures that $m\varepsilon^\Sigma(m)$ is bounded uniformly in m . The defining function of the above graph is obtained by “transferring” the defining functions of the corresponding ends of \mathbb{M}^Σ over their asymptotic planes. These defining functions decay exponentially in the distance along the planes. In turn $\Lambda_m^\Sigma(s)$ is a graph over this portion of the initial surface with defining function that is also guaranteed (by [loc. cit., (5.20)]) to decay exponentially, though a priori at a slower rate; we have chosen $\frac{1}{4}$ somewhat arbitrarily. This accounts for all exponential factors appearing in the estimates.

The m -dependent terms in the estimates for the metric (and Laplacian) arise simply from the choice of (t, ϑ) coordinates on disc and catenoidal models. The m^{-2} term in the first item arises from scaling the second fundamental form of the “asymptotic” catenoid to this component (while the corresponding term for the disc vanishes). With the estimates for the second fundamental form in place, the final item—the area estimate—follows (recalling the definitions (5-8)) from the bound

$$\int_{-\pi/2}^{\pi/2} \int_s^{\sqrt{m+1}} (a_1 m^{-2} + a_2 e^{-t/4}) dt d\vartheta \leq C(m^{-3/2} + e^{-s/4})$$

and the analogous estimate concerning the disk-type component instead. \square

5.4. Regionwise spectral convergence. For each region S among $M_n^\Xi, M_m^\Sigma, K_n^\Xi, K_n^\Sigma,$ and B_m^Σ (depicted in Figure 5), we write Q_N^S for the Jacobi form of S as a minimal surface in \mathbb{B}^3 with boundary, subject to the Robin condition (4-1) where ∂S meets $\partial \mathbb{B}^3$ and subject to the Neumann condition elsewhere: recalling (2-17), we set

$$Q_N^S := \begin{cases} (Q_{\Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0}})_S^{N_{\text{int}}} & \text{for } S \subset \Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0}, \\ (Q_{\Sigma_m^{-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0}})_S^{N_{\text{int}}} & \text{for } S \subset \Sigma_m^{-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0} \end{cases}$$

(where on the right-hand side we slightly abuse notation in that in place of S we really mean its interior). Similarly, for S either $M_{fb,n}^\Xi$ or $M_{fb,m}^\Sigma$, we write Q_N^S for the Jacobi form of S as a minimal surface in either $n\mathbb{B}^3$ or $(m+1)\mathbb{B}^3$, subject to the Robin condition either $du(\eta) = n^{-1}u$ or $du(\eta) = (m+1)^{-1}u$ where ∂S meets either $n\mathbb{S}^2$ or $(m+1)\mathbb{S}^2$, respectively, and subject to the Neumann condition elsewhere. Keeping in mind the statement of Proposition 3.1, we stress that the adjunction of Neumann conditions in the “interior” boundaries is motivated by our task of deriving *upper* bounds on the Morse index of our examples. Recalling the notation $\widehat{M}_{fb,n}^\Xi$ and $\widehat{M}_{fb,m}^\Sigma$, we remark that the bilinear forms Q_N^S and $Q_N^{\widehat{S}}$ agree by definition for each S as above, but whenever we refer to the eigenvalues, eigenfunctions, index, and nullity of the latter we shall always mean those defined with respect to the h_n^Ξ or h_m^Σ metric.

In the notation of (2-4), we have in particular (see Proposition 3.11)

$$\begin{aligned} Q_N^{M_{fb,n}^\Xi} &= T[M_{fb,n}^\Xi, g_n^\Xi, q_n^\Xi = |A_n^\Xi|_{g_n^\Xi}^2, r_n^\Xi = n^{-1}, \\ &\quad \partial_D M_{fb,n}^\Xi = \emptyset, \partial_N M_{fb,n}^\Xi = \partial M_{fb,n}^\Xi \setminus n\mathbb{S}^2, \partial_R M_{fb,n}^\Xi = \partial M_{fb,n}^\Xi \setminus \overline{\partial_N M_{fb,n}^\Xi}] \\ &= T[M_{fb,n}^\Xi, h_n^\Xi, (\rho_n^\Xi)^{-2}q_n^\Xi, (\rho_n^\Xi)^{-1}n^{-1}, \emptyset, \partial M_{fb,n}^\Xi \setminus n\mathbb{S}^2, \partial M_{fb,n}^\Xi \setminus \overline{\partial_N M_{fb,n}^\Xi}] = Q_N^{\widehat{M}_{fb,n}^\Xi} \end{aligned} \tag{5-9}$$

and similarly for $Q_N^{M_{fb,m}^\Sigma} = Q_N^{\widehat{M}_{fb,m}^\Sigma}$. Observe further (see Lemma 3.5 and Proposition 3.11) that

$$\begin{aligned} \text{ind}_{\mathbb{P}_n}(Q_N^{M_n^\Xi}) &= \text{ind}_{\{z=0\}}^+(Q_N^{\widehat{M}_{fb,n}^\Xi}), & \text{ind}_{\mathbb{Y}_n}(Q_N^{M_n^\Xi}) &= \text{ind}(Q_N^{\widehat{M}_{fb,n}^\Xi}), \\ \text{ind}_{\mathbb{A}_{m+1}}(Q_N^{M_m^\Sigma}) &= \text{ind}_{\{y=z=0\}}^-(Q_N^{\widehat{M}_{fb,m}^\Sigma}), & \text{ind}_{\mathbb{Y}_{m+1}}(Q_N^{M_m^\Sigma}) &= \text{ind}(Q_N^{\widehat{M}_{fb,m}^\Sigma}), \end{aligned}$$

and likewise for the corresponding nullities.

Lemma 5.22 (equivariant index and nullity on K_n^Ξ , K_m^Σ , and B_m^Σ). *There exist $n_0, m_0 > 0$ such that we have the following indices and nullities for all integers $n > n_0$ and $m > m_0$:*

S	G	$\text{ind}_G(Q_N^S)$	$\text{nul}_G(Q_N^S)$
K_n^Ξ	\mathbb{Y}_n	1	0
K_m^Σ	\mathbb{Y}_{m+1}	1	0
B_m^Σ	\mathbb{A}_{m+1}	0	0

Additionally, still assuming $m > m_0$, we have the upper bound

$$\text{ind}_{\mathbb{Y}_{m+1}}(Q_N^{B_m^\Sigma}) + \text{nul}_{\mathbb{Y}_{m+1}}(Q_N^{B_m^\Sigma}) \leq 1.$$

Proof. We use the convergence described in Lemmas 5.19 and 5.20 along with Proposition 3.8 to compare the low eigenvalues of the regions in question with those of their limiting models, as recorded in Lemmas 5.8 and 5.10. □

While we have cut the surfaces $\Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0}$ and $\Sigma_m^{-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0}$ in such a way that the resulting regions K_n^Ξ and K_m^Σ converge uniformly to \mathbb{K}_0 and likewise B_m^Σ to \mathbb{B}^2 , thereby securing the preceding lemma in a straightforward fashion, the cases of M_n^Ξ and M_m^Σ are more subtle. Our approach here (especially the proof of eigenfunction bounds in Lemma 5.25 and their application to Lemma 5.26) draws inspiration from the analysis Kapouleas makes of the invertibility of the Jacobi operator on “extended standard regions” in many gluing constructions; for a specific example, concerning Scherk towers glued to catenoids, we refer the reader to the proof of [Kapouleas 1997, Lemma 7.4].

To proceed, recalling (2-17), for each $s > 0$ and each integer n (sufficiently large in terms of s), we define

$$Q_D^{\widehat{M}_{fb,n}^\Xi(s)} := (Q_N^{\widehat{M}_{fb,n}^\Xi})_{\widehat{M}_{fb,n}^\Xi(s)}^{D_{int}} \quad \text{and} \quad Q_N^{\widehat{M}_{fb,n}^\Xi(s)} := (Q_N^{\widehat{M}_{fb,n}^\Xi})_{\widehat{M}_{fb,n}^\Xi(s)}^{N_{int}}$$

and analogously for $\widehat{M}_{fb,m}^\Sigma(s)$ in place of $\widehat{M}_{fb,n}^\Xi(s)$.

Lemma 5.23 (spectral convergence for $\widehat{M}_{fb,n}^\Xi(s)$ and $\widehat{M}_{fb,m}^\Sigma(s)$). *With the above notation, we have*

$$\lambda_i^{\{z=0\},\pm}(Q_{\widehat{M}_{fb}^\Xi}) = \lim_{s \rightarrow \infty} \lim_{n \rightarrow \infty} \lambda_i^{\{z=0\},\pm}(Q_D^{\widehat{M}_{fb,n}^\Xi(s)}) = \lim_{s \rightarrow \infty} \lim_{n \rightarrow \infty} \lambda_i^{\{z=0\},\pm}(Q_N^{\widehat{M}_{fb,n}^\Xi(s)})$$

for each integer $i \geq 1$ and each common choice of sign \pm on both sides of each equation. The analogous statements hold, *mutatis mutandis*, for $\widehat{M}_{fb,m}^\Sigma$ in place of $\widehat{M}_{fb,n}^\Xi$.

Proof. Fix i . By Lemma 5.18 and Proposition 3.8, for each $s > 0$, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \lambda_i^{\{z=0\},+}(Q_D^{\widehat{M}_{fb,n}^\Xi(s)}) &= \lambda_i^{\{z=0\},+}(Q_D^{\widehat{M}_{fb}^\Xi}), \\ \lim_{n \rightarrow \infty} \lambda_i^{\{z=0\},+}(Q_N^{\widehat{M}_{fb,n}^\Xi(s)}) &= \lambda_i^{\{z=0\},+}(Q_N^{\widehat{M}_{fb}^\Xi}). \end{aligned}$$

An application of Lemma 5.17 completes the proof in this case, and the proofs of the remaining three cases are structurally identical to this one. □

Lemma 5.24 (eigenvalue upper bounds on $\widehat{M}_{fb,n}^\Xi$ and $\widehat{M}_{fb,m}^\Sigma$). *With the above notation, we have*

$$\begin{aligned} \limsup_{n \rightarrow \infty} \lambda_i^{\{z=0\},\pm}(Q_N^{\widehat{M}_{fb,n}^\Xi}) &\leq \lambda_i^{\{z=0\},\pm}(Q_N^{\widehat{M}_{fb}^\Xi}), \\ \limsup_{m \rightarrow \infty} \lambda_i^{\{y=z=0\},\pm}(Q_N^{\widehat{M}_{fb,m}^\Sigma}) &\leq \lambda_i^{\{y=z=0\},\pm}(Q_N^{\widehat{M}_{fb}^\Sigma}) \end{aligned}$$

for each integer $i \geq 1$ and each common choice of sign \pm on both sides of each equation.

Proof. We give the proof for the $+$ choice on both sides of the top equation, the proofs for the remaining three cases being identical in structure to this one. Fix $i \geq 1$. By (2-13), considering extensions by zero of functions corresponding to the right-hand side below to obtain valid test functions corresponding to the left, we get at once the inequality

$$\lambda_i^{\{z=0\},+}(Q_N^{\widehat{M}_{fb,n}^\Xi}) \leq \lambda_i^{\{z=0\},+}(Q_D^{\widehat{M}_{fb,n}^\Xi(s)})$$

for all $s > 0$ and all n sufficiently large (in terms of s) such that $\widehat{M}_{fb,n}^\Xi(s)$ is defined. We then finish by applying Lemma 5.23. □

Lemma 5.25 (uniform bounds on eigenvalues and eigenfunctions of $Q_N^{\widehat{M}_{fb,n}^\Xi}$ and $Q_N^{\widehat{M}_{fb,m}^\Sigma}$). *For each integer $i \geq 1$, there exist $C_i, k_i > 0$ such that, for each integer $k > k_i$ and whenever $\lambda_i^{(k)}$ is the i -th eigenvalue of $Q_N^{\widehat{M}_{fb,k}^\Xi}$ or $Q_N^{\widehat{M}_{fb,k}^\Sigma}$ and $v_i^{(k)}$ is any corresponding eigenfunction of unit L^2 norm (under either h_k^Ξ or h_k^Σ as appropriate), we have the bounds*

$$\max\{|\lambda_i^{(k)}|, \|v_i^{(k)}\|_{H^1}, \|v_i^{(k)}\|_{C^0}\} \leq C_i$$

(where the H^1 norm is defined via either h_n^Ξ or h_m^Σ as applicable and we emphasize that C_i does not depend on k).

Proof. We will give the proof for $\widehat{M}_{fb,n}^{\Xi}$, that for $\widehat{M}_{fb,m}^{\Sigma}$ being identical in structure. Fix $i \geq 1$, and let $\lambda^{(n)}$ and $v^{(n)}$ be as in the statement for each integer n (suppressing the fixed index i); it is our task to show that, by assuming n large enough in terms of just i , we can ensure the asserted bounds on $\lambda^{(n)}$ and $v^{(n)}$. In particular our assumptions include the normalization $\|v^{(n)}\|_{L^2(M_{fb,n}^{\Xi}, h_n^{\Xi})} = 1$.

Lemma 5.24 provides an upper bound on $\lambda^{(n)}$ independent of n . We deduce a lower bound on $\lambda^{(n)}$ as follows. Keeping in mind the min-max characterization (2-13), we observe that in the ratio

$$\frac{\langle u, q_n u \rangle_{L^2(M_{fb,n}^{\Xi}, h_n^{\Xi})} + \langle u |_{\partial_R M_{fb,n}^{\Xi}}, r_n u |_{\partial_R M_{fb,n}^{\Xi}} \rangle_{L^2(\partial_R M_{fb,n}^{\Xi}, h_n^{\Xi})}}{\|u\|_{L^2(M_{fb,n}^{\Xi}, h_n^{\Xi})}^2}, \tag{5-10}$$

with

$$r_n := (\rho_n^{\Xi})^{-1} |_{\partial_R M_{fb,n}^{\Xi}} n^{-1} = (1 + e^{-2n})^{-1/2} n^{-1}, \quad q_n := (\rho_n^{\Xi})^{-2} |_{A_n^{\Xi} g_n^{\Xi}},$$

we have not only a uniform upper bound on r_n , but also, by inspecting (5-8) and bearing in mind the convergence described in Lemma 5.18 as well as the boundedness (with decay) of the second fundamental form of \mathbb{M}^{Ξ} ,

$$\sup_n \|q_n\|_{C^0(M_{fb,n}^{\Xi})} < \infty.$$

In addition, the convergence in Lemma 5.18 further ensures that the constants appearing in (2-3), with $(\Omega, g) = (M_{fb,n}^{\Xi}, h_n^{\Xi})$ and $\partial_R \Omega$ in place of $\partial \Omega$, can be chosen uniformly in n : thus, employing such a trace inequality and exploiting the foregoing uniform bounds we secure the promised uniform lower bound on $\lambda^{(n)}$.

In turn, from the definitions of eigenvalues and eigenfunctions and the normalization of $v^{(n)}$, we have

$$\|\nabla_{h_n^{\Xi}} v^{(n)}\|_{L^2(M_{fb,n}^{\Xi}, h_n^{\Xi})}^2 = \lambda^{(n)} + \langle v^{(n)}, q_n v^{(n)} \rangle_{L^2(M_{fb,n}^{\Xi}, h_n^{\Xi})} + \langle v^{(n)} |_{\partial_R M_{fb,n}^{\Xi}}, r_n v^{(n)} |_{\partial_R M_{fb,n}^{\Xi}} \rangle_{L^2(\partial_R M_{fb,n}^{\Xi}, h_n^{\Xi})}.$$

The uniform bound on $\|v^{(n)}\|_{H^1(M_{fb,n}^{\Xi}, h_n^{\Xi})}$ now follows, in view of the above equality, from the upper bound on $\lambda^{(n)}$ as well as again the above uniform bounds on q_n and r_n .

It remains to establish the uniform C^0 bound. To start, by Lemma 3.5 and standard elliptic regularity, $v^{(n)}$ is smooth up to the boundary: indeed, it satisfies

$$\begin{cases} (\Delta_{h_n^{\Xi}} + (\rho_n^{\Xi})^{-2} |_{A_n^{\Xi} g_n^{\Xi}} + \lambda^{(n)}) v^{(n)} = 0 & \text{in } M_{fb,n}^{\Xi}, \\ h_n^{\Xi}(\eta_n^{\Xi}, \nabla_{h_n^{\Xi}} v^{(n)}) = (1 + e^{-2n})^{-1/2} n^{-1} v^{(n)} & \text{on } \partial_R M_{fb,n}^{\Xi}, \\ h_n^{\Xi}(\eta_n^{\Xi}, \nabla_{h_n^{\Xi}} v^{(n)}) = 0 & \text{on } \partial_N M_{fb,n}^{\Xi}, \end{cases} \tag{5-11}$$

with η_n^{Ξ} the outward h_n^{Ξ} unit conormal to $M_{fb,n}^{\Xi}$. As established above, we have bounds independent of n on $|\lambda^{(n)}|$ and the q_n and r_n functions. By Lemma 5.18 (and the uniform geometry of \mathbb{M}^{Ξ}), we also have uniform control over the geometry of $(M_{fb,n}^{\Xi}(s), h_n^{\Xi})$ for each $s > 0$ and all n sufficiently large in terms of s .

Standard elliptic regularity therefore ensures that, for every $s > 0$, there exist $n_s > 0$ and $\gamma(s) > 0$ such that

$$\|v^{(n)} |_{M_{fb,n}^{\Xi}(s)}\|_{C^0(M_{fb,n}^{\Xi}(s), h_n^{\Xi})} \leq \gamma(s) \quad \text{for every integer } n > n_s. \tag{5-12}$$

Since we do not have uniform control on the geometry of $(M_{\text{fb},n}^{\Xi} = M_{\text{fb},n}^{\Xi}(\sqrt{n}), h_n^{\Xi})$, we do not obtain a global bound independent of n in the same fashion. Instead the proof will be completed by securing a C^0 bound for $v^{(n)}$, independent of n , on $\Lambda_n^{\Xi}(s)$ for some $s > 0$ to be determined. In the remainder of the proof, $\gamma(s)$ will continue to denote the above constant, depending on s , while C will denote a strictly positive constant whose value may change from instance to instance but can always be selected independently of s and n .

To proceed we multiply both sides of the PDE in (5-11) by $(\rho_n^{\Xi})^2$ to get

$$(\Delta_{g_n^{\Xi}} + |A_n^{\Xi}|_{g_n^{\Xi}}|^2 + \lambda^{(n)}(\rho_n^{\Xi})^2)v^{(n)} = 0, \tag{5-13}$$

and we aim to bound $v^{(n)}$ on $\Lambda_n^{\Xi}(s)$ on the basis of this equation, with unknown but controlled (as we explain momentarily) Dirichlet data on the portion of $\partial\Lambda_n^{\Xi}(s)$ contained in the interior of $M_{\text{fb},n}^{\Xi}$ and with homogeneous Neumann data on the rest of the boundary. By the symmetries it suffices to establish the estimate on just the component of $\Lambda_n^{\Xi}(s)$ that is a graph over a subset of $n\mathbb{K}_0$. (For $\Lambda_m^{\Sigma}(s)$ one must also consider the component which is a graph over a subset of $(m + 1)\mathbb{B}^2$, but this case does not differ in substance from the one we treat now.)

Recall the map

$$\varphi^{\Lambda_n^{\Xi}(s), \mathbb{K}} : [s, \sqrt{n}] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow \Lambda_n(s)$$

introduced above Lemma 5.21, and continue to write (t, ϑ) for the standard coordinates on its domain. For the remainder of this proof we abbreviate $\varphi^{\Lambda_n^{\Xi}(s), \mathbb{K}}$ to $\varphi_{n,s}$ and its domain to $R_{n,s}$. Setting $w^{(n)} := \varphi_{n,s}^* v^{(n)}$, we pull back (5-13) to get

$$\Delta_{\varphi_{n,s}^* g_n^{\Xi}} w^{(n)} = -w^{(n)} \varphi_{n,s}^* (|A_n^{\Xi}|_{g_n^{\Xi}}|^2 + \lambda^{(n)}(\rho_n^{\Xi})^2).$$

From the uniform bound on $\lambda^{(n)}$, the expression for the conformal factor in (5-8), and item (i) of Lemma 5.21, we in turn obtain

$$\Delta_{\varphi_{n,s}^* g_n^{\Xi}} w^{(n)} = (c_{n,s} e^{-t/4} + d_{n,s} n^{-2}) w^{(n)} \tag{5-14}$$

for some smooth functions $c_{n,s}, d_{n,s}$ having $C^{0,\alpha}(dt^2 + d\vartheta^2)$ norms uniformly bounded in n and s , with $\alpha \in]0, 1[$ now fixed for the rest of the proof. (Here and below when referring to items of Lemma 5.21 we have in mind of course the corresponding statements for $\Lambda_n^{\Xi}(s)$ in place of $\Lambda_m^{\Sigma}(s)$.)

Noting that we have (5-14) for all sufficiently large s , it now follows from the C^0 bound (5-12) and standard interior Schauder estimates (using also item (iii) of Lemma 5.21) that

$$\|w^{(n)}(s, \cdot)\|_{C^{2,\alpha}(d\vartheta^2)} \leq C\gamma(s + 1) \quad \text{for every integer } n > n_{s+1}. \tag{5-15}$$

Since $v^{(n)}$ satisfies the homogeneous Neumann condition along $\partial M_{\text{fb},n}^{\Xi}$, with the aid of item (iii) of Lemma 5.21, we have

$$(\partial_t w^{(n)})(\sqrt{n}, \vartheta) = e_{n,s} e^{-\sqrt{n}/4} (\partial_{\vartheta} w^{(n)})(\sqrt{n}, \vartheta), \tag{5-16}$$

$$(\partial_{\vartheta} w^{(n)})(\cdot, \pm\pi/2) = 0 \tag{5-17}$$

for some smooth function $e_{n,s}$ having $C^{1,\alpha}(dt^2 + d\vartheta^2)$ norm bounded independently of n and s . (For (5-17) we simply use the fact that $\varphi_{n,s}$ has been constructed by composing and restricting maps which commute with the symmetries of the construction, including the reflections through planes corresponding to $\vartheta = \pm\pi/2$.)

Appealing again to standard Schauder estimates, now also up to the boundary, we can conclude from (5-14)–(5-17) that

$$\|w^{(n)}\|_{C^{2,\alpha}(dt^2+d\vartheta^2)} \leq C(\gamma(s+1) + \|w^{(n)}\|_{C^0}) \tag{5-18}$$

for n and s sufficiently large in terms of the bounds assumed on the functions $c_{n,s}$, $d_{n,s}$, and $e_{n,s}$, as well as constants, which can be chosen uniformly, that appear in local Schauder estimates on $R_{n,s}$. If we exploit (5-18) in (5-16), we get

$$\|(\partial_t w^{(n)})(\sqrt{n}, \cdot)\|_{C^{1,\alpha}(d\vartheta^2)} \leq C e^{-\sqrt{n}/4} (\gamma(s+1) + \|w^{(n)}\|_{C^0}), \tag{5-19}$$

once again for n and s assumed large enough in terms of absolute constants.

We next decompose $w^{(n)}$ into

$$w_0^{(n)} := \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} w^{(n)}(\cdot, \vartheta) d\vartheta, \quad w_{\perp}^{(n)} := w^{(n)} - w_0^{(n)}.$$

From (5-14), (5-18), and item (v) of Lemma 5.21, we obtain

$$\partial_t^2 w_0^{(n)} = a_{n,s}^0 e^{-t/4} + b_{n,s}^0 n^{-2} + c_{n,s}^0 n^{-1} \partial_t w_0^{(n)}, \tag{5-20}$$

with

$$\frac{\|a_{n,s}^0\|_{C^0} + \|b_{n,s}^0\|_{C^0}}{\gamma(s+1) + \|w^{(n)}\|_{C^0}} + \|c_{n,s}^0\|_{C^0} \leq C$$

and

$$\|\Delta_{dt^2+d\vartheta^2} w_{\perp}^{(n)}\|_{C^0} \leq C(e^{-s/4} + n^{-1/2})(\gamma(s+1) + \|w^{(n)}\|_{C^0}). \tag{5-21}$$

For (5-20) we have in particular integrated (5-14) in ϑ , making use of the ϑ -invariance (see item (v) of Lemma 5.21) of the coefficients of the $n^{-1} \partial_t$ and $n^{-1/2} \partial_{\vartheta}^2$ terms and observing that the $n^{-1/2} \partial_{\vartheta}^2$ term integrates to zero because of (5-17); for (5-21) we have made use of the fact that

$$\|\Delta_{dt^2+d\vartheta^2} w_{\perp}^{(n)}\|_{C^0} \leq 2\|\Delta_{dt^2+d\vartheta^2} w^{(n)}\|_{C^0}$$

and then appealed to (5-14).

To complete the analysis we will need some basic estimates for

$$\Delta_{dt^2+d\vartheta^2} = \partial_t^2 + \partial_{\vartheta}^2$$

on $R_{n,s}$. For any bounded (real-valued) function f on $R_{n,s}$ and for each nonnegative integer κ , let us define on $[s, \sqrt{n}]$ the Fourier coefficients f_{κ} by

$$f_{\kappa}(t) := \begin{cases} \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} f(t, \vartheta) d\vartheta & \text{for } \kappa = 0, \\ \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} f(t, \vartheta) \cos \kappa(\vartheta - \pi/2) d\vartheta & \text{for } \kappa > 0. \end{cases}$$

Then the Fourier coefficients of any $u \in C^2(R_{n,s}, dt^2 + d\vartheta^2)$ satisfying $(\partial_\vartheta u) = 0$ at $\vartheta = \pm\pi/2$ admit the representations

$$\begin{aligned} u_0(t) &= u_0(s) + (\partial_t u_0)(\sqrt{n}) \cdot (t - s) + \int_s^t \int_{\sqrt{n}}^\tau \partial_t^2 u_0(\sigma) d\sigma d\tau \\ &= u_0(s) + (\partial_t u_0)(\sqrt{n}) \cdot (t - s) + \int_s^t \int_{\sqrt{n}}^\tau (\Delta_{dt^2+d\vartheta^2} u)_0(\sigma) d\sigma d\tau, \end{aligned} \quad (5-22)$$

$$\begin{aligned} u_{\kappa \neq 0}(t) &= \frac{u_\kappa(s)}{\cosh \kappa(\sqrt{n} - s)} \cosh \kappa(t - \sqrt{n}) + \frac{(\partial_t u_\kappa)(\sqrt{n})}{\kappa \cosh \kappa(\sqrt{n} - s)} \sinh \kappa(t - s) \\ &\quad - \frac{\cosh \kappa(t - \sqrt{n})}{\kappa \cosh \kappa(\sqrt{n} - s)} \int_s^t (\Delta_{dt^2+d\vartheta^2} u)_\kappa(\tau) \sinh \kappa(\tau - s) d\tau \\ &\quad - \frac{\sinh \kappa(t - s)}{\kappa \cosh \kappa(\sqrt{n} - s)} \int_t^{\sqrt{n}} (\Delta_{dt^2+d\vartheta^2} u)_\kappa(\tau) \cosh \kappa(\tau - \sqrt{n}) d\tau. \end{aligned} \quad (5-23)$$

In particular (5-23) implies, for any $\kappa \geq 1$, the inequality

$$|u_\kappa(t)| \leq |u_\kappa(s)| + \frac{1}{\kappa} |(\partial_t u_\kappa)(\sqrt{n})| + \frac{1}{\kappa^2} \|(\Delta_{dt^2+d\vartheta^2} u)_\kappa\|_{C^0}. \quad (5-24)$$

Since u is C^2 , the Fourier series $\sum_{\kappa=0}^\infty u_\kappa(t) \cos \kappa(\vartheta - \pi/2)$ converges (at least) pointwise to $u(t, \vartheta)$; furthermore (again appealing to the C^2 assumption in order to control the first two terms of (5-24)) we obtain the implication

$$\int_{-\pi/2}^{\pi/2} u(\cdot, \vartheta) d\vartheta = 0 \implies \|u\|_{C^0} \leq C(\|u(s, \cdot)\|_{C^2(d\vartheta^2)} + \|(\partial_t u)(\sqrt{n}, \cdot)\|_{C^1(d\vartheta^2)} + \|\Delta_{dt^2+d\vartheta^2} u\|_{C^0}).$$

This last estimate in conjunction with (5-21), (5-15), and (5-19) yields

$$\|w_\perp^{(n)}\|_{C^0} \leq C(\gamma(s+1) + (e^{-\sqrt{n}/4} + e^{-s/4} + n^{-1/2})\|w^{(n)}\|_{C^0}). \quad (5-25)$$

On the other hand, differentiating (5-22) with respect to t and applying (5-20) and (5-19), we find

$$\|\partial_t w_0^{(n)}\|_{C^0} \leq C(\gamma(s+1) + \|w^{(n)}\|_{C^0})(e^{-\sqrt{n}/4} + e^{-s/4} + n^{-3/2}) + Cn^{-1/2}\|\partial_t w_0^{(n)}\|_{C^0} \quad (5-26)$$

and therefore, by absorption,

$$\|\partial_t w_0^{(n)}\|_{C^0} \leq C(\gamma(s+1) + \|w^{(n)}\|_{C^0})(e^{-s/4} + n^{-3/2}) \quad (5-27)$$

for n sufficiently large in terms of s and the constants appearing in the above estimate. Feeding (5-27) into (5-20) and applying the result, along with (5-15) and (5-19), in (5-22), we get

$$\|w_0^{(n)}\|_{C^0} \leq C(\gamma(s+1) + (\sqrt{n}e^{-\sqrt{n}/4} + e^{-s/4} + n^{-1})\|w^{(n)}\|_{C^0}). \quad (5-28)$$

Finally, since $\|w^{(n)}\|_{C^0} \leq \|w_0^{(n)}\|_{C^0} + \|w_\perp^{(n)}\|_{C^0}$, estimates (5-28) and (5-25) jointly imply the desired bound on the C^0 norm of $w^{(n)}$ provided we first choose s and then, in turn, n sufficiently large, in terms of the absolute constants appearing in the two estimates, to be able to absorb the $\|w^{(n)}\|_{C^0}$ terms appearing on their right-hand sides. This ends the proof. \square

Lemma 5.26 (eigenvalue lower bounds on $\widehat{M}_{\text{fb},n}^{\Xi}$ and $\widehat{M}_{\text{fb},m}^{\Sigma}$). *For each integer $i \geq 1$,*

$$\begin{aligned} \liminf_{n \rightarrow \infty} \lambda_i^{\{z=0\}, \pm}(Q_N^{\widehat{M}_{\text{fb},n}^{\Xi}}) &\geq \lambda_i^{\{z=0\}, \pm}(Q_N^{\widehat{M}_{\text{fb}}^{\Xi}}), \\ \liminf_{m \rightarrow \infty} \lambda_i^{\{y=z=0\}, \pm}(Q_N^{\widehat{M}_{\text{fb},m}^{\Sigma}}) &\geq \lambda_i^{\{y=z=0\}, \pm}(Q_N^{\widehat{M}_{\text{fb}}^{\Sigma}}) \end{aligned}$$

for each common choice of sign \pm on both sides of each equation.

Proof. We give the proof for the $+$ choice on both sides of the top equation, the argument for the remaining three cases being identical in structure to this one. Fix $i \geq 1$, and, for each n , let $\{v_j^{(n)}\}_{j=1}^i$ be an $L^2(M_{\text{fb},n}^{\Xi}, h_n^{\Xi})$ orthonormal set such that each $v_j^{(n)}$ is a j -th $(\{z=0\}, +)$ -invariant eigenfunction of $Q_N^{\widehat{M}_{\text{fb},n}^{\Xi}}$. Fix $C > 0$, as afforded by Lemma 5.25, such that

$$\sup_n \sup_{1 \leq j \leq i} (\|v_j^{(n)}\|_{C^0} + \lambda_j^{\{z=0\}, +}(Q_N^{\widehat{M}_{\text{fb},n}^{\Xi}})) \leq C.$$

Given any $\epsilon > 0$ (fixed from now on) and taking $s > 0$ and correspondingly $n_s > 0$ large enough, as afforded by Lemmas 5.21 and 5.23, we have

$$\mathcal{H}^2(h_n^{\Xi})(\Lambda_n(s)) < \epsilon, \quad \lambda_i^{\{z=0\}, +}(Q_N^{\widehat{M}_{\text{fb}}^{\Xi}}) < \lambda_i^{\{z=0\}, +}(Q_N^{\widehat{M}_{\text{fb},n}^{\Xi}(s)}) + \epsilon. \tag{5-29}$$

Now, for $n > n_s$ and any v in the span of $\{v_j^{(n)}\}_{j=1}^i$, we estimate

$$\begin{aligned} \|v\|_{L^2(M_{\text{fb},n}^{\Xi}, h_n^{\Xi})}^2 - \|v|_{M_{\text{fb},n}^{\Xi}(s)}\|_{L^2(M_{\text{fb},n}^{\Xi}(s), h_n^{\Xi})}^2 &\leq C^2 i \epsilon \|v\|_{L^2(M_{\text{fb},n}^{\Xi}, h_n^{\Xi})}^2, \\ \|\nabla_{h_n^{\Xi}} v|_{M_{\text{fb},n}^{\Xi}(s)}\|_{L^2(M_{\text{fb},n}^{\Xi}(s), h_n^{\Xi})} &\leq \|\nabla_{h_n^{\Xi}} v\|_{L^2(M_{\text{fb},n}^{\Xi}, h_n^{\Xi})}, \\ \langle v|_{M_{\text{fb},n}^{\Xi}(s)}, (\rho_n^{\Xi})^{-2} |A_n^{\Xi}|_{g_n^{\Xi}}^2 v|_{M_{\text{fb},n}^{\Xi}(s)} \rangle_{L^2(M_{\text{fb},n}^{\Xi}(s), h_n^{\Xi})} &\geq \langle v, (\rho_n^{\Xi})^{-2} |A_n^{\Xi}|_{g_n^{\Xi}}^2 v \rangle_{L^2(M_{\text{fb},n}^{\Xi}, h_n^{\Xi})} - 2C^2 i \epsilon \|v\|_{L^2(M_{\text{fb},n}^{\Xi}, h_n^{\Xi})}^2, \end{aligned}$$

where for the last inequality we have used the fact that, on $\Lambda_n(s)$, the potential function appearing here is bounded above by 2, as is obvious from inspection of (5-8).

We conclude that, for all $n > n_s$, the set $\{v_j^{(n)}|_{M_{\text{fb},n}^{\Xi}(s)}\}_{j=1}^i$ is linearly independent, and for all v as above we have

$$\frac{Q_N^{\widehat{M}_{\text{fb},n}^{\Xi}(s)}(v|_{M_{\text{fb},n}^{\Xi}(s)}, v|_{M_{\text{fb},n}^{\Xi}(s)})}{\|v|_{M_{\text{fb},n}^{\Xi}(s)}\|_{L^2(M_{\text{fb},n}^{\Xi}(s), h_n^{\Xi})}^2} \leq \frac{\lambda_j^{\{z=0\}, +}(Q_N^{\widehat{M}_{\text{fb},n}^{\Xi}}) + 2C^2 i \epsilon}{1 - C^2 i \epsilon},$$

and so by virtue of the min-max characterization (2-13) for the eigenvalues, it follows that

$$\lambda_j^{\{z=0\}, +}(Q_N^{\widehat{M}_{\text{fb},n}^{\Xi}(s)}) \leq \frac{\lambda_j^{\{z=0\}, +}(Q_N^{\widehat{M}_{\text{fb},n}^{\Xi}}) + 2C^2 i \epsilon}{1 - C^2 i \epsilon}$$

for all $n > n_s$ and $1 \leq j \leq i$. Thus, using the second inequality in (5-29), we get in particular

$$\lambda_i^{\{z=0\}, +}(Q_N^{\widehat{M}_{\text{fb}}^{\Xi}}) \leq \frac{\lambda_i^{\{z=0\}, +}(Q_N^{\widehat{M}_{\text{fb},n}^{\Xi}}) + 2C^2 i \epsilon}{1 - C^2 i \epsilon} + \epsilon$$

for all $n > n_s$. The claim now follows since this inequality holds for all $\epsilon > 0$, with C independent of ϵ and n . □

By combining Lemmas 5.24 with 5.26, we immediately derive the following conclusion.

Corollary 5.27 (eigenvalues on $\widehat{M}_{fb,n}^{\Xi}$ and $\widehat{M}_{fb,m}^{\Sigma}$). For each integer $i \geq 1$,

$$\begin{aligned} \lim_{n \rightarrow \infty} \lambda_i^{\{z=0\}, \pm}(Q_N^{\widehat{M}_{fb,n}^{\Xi}}) &= \lambda_i^{\{z=0\}, \pm}(Q_{\widehat{M}_{fb}^{\Xi}}), \\ \lim_{m \rightarrow \infty} \lambda_i^{\{y=z=0\}, \pm}(Q_N^{\widehat{M}_{fb,m}^{\Sigma}}) &= \lambda_i^{\{y=z=0\}, \pm}(Q_{\widehat{M}_{fb}^{\Sigma}}) \end{aligned}$$

for each common choice of sign \pm on both sides of each equation.

Corollary 5.28 (equivariant index and nullity on M_n^{Ξ} and M_m^{Σ}). There exist $n_0, m_0 > 0$ such that we have the following indices and nullities for all integers $n > n_0$ and $m > m_0$:

S	G	$\text{ind}_G(Q_N^S)$	$\text{nul}_G(Q_N^S)$
M_n^{Ξ}	\mathbb{P}_n	1	0
M_m^{Σ}	\mathbb{A}_{m+1}	1	0

Additionally, still assuming $m > m_0$, we have the upper bound

$$\text{ind}_{\mathbb{Y}_{m+1}}(Q_N^{M_m^{\Sigma}}) + \text{nul}_{\mathbb{Y}_{m+1}}(Q_N^{M_m^{\Sigma}}) \leq 3.$$

Proof. All claims follow from the conjunction of Lemma 3.5 (to reduce to the appropriately even and odd indices and nullities on $n^{-1}M_{fb,n}^{\Xi}$ and $(m+1)^{-1}M_{fb,m}^{\Sigma}$ with Neumann boundary data), Proposition 3.11 (to dispense with the above scale factors $n, m+1$ and, more substantially, to pass from the natural metric to h_n^{Ξ} or h_m^{Σ}), Corollary 5.27 (to reduce to the appropriate indices and nullities of \widehat{M}_{fb}^{Ξ} and $\widehat{M}_{fb}^{\Sigma}$), and finally Lemma 5.16 (which provides these last quantities). \square

5.5. Proofs of Theorems 1.2 and 1.1. The following statement collects, from the broader analysis conducted in the previous section, those conclusions we shall need to prove the two main results stated in the introduction.

Corollary 5.29 (equivariant index and nullity upper bounds for $\Sigma_m^{-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0}$ and $\Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0}$). There exists $m_0, n_0 > 0$ such that, for all integers $m > m_0$ and $n > n_0$, we have the bounds

$$\begin{aligned} \text{ind}_{\mathbb{A}_{m+1}}(\Sigma_m^{-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0}) + \text{nul}_{\mathbb{A}_{m+1}}(\Sigma_m^{-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0}) &\leq 2, \\ \text{ind}_{\mathbb{Y}_{m+1}}(\Sigma_m^{-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0}) + \text{nul}_{\mathbb{Y}_{m+1}}(\Sigma_m^{-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0}) &\leq 6, \\ \text{ind}_{\mathbb{P}_n}(\Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0}) + \text{nul}_{\mathbb{P}_n}(\Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0}) &\leq 2. \end{aligned}$$

Proof. We apply item (ii) of Proposition 3.1 for the partition “into building blocks” defined in Section 5.3 (see Figure 5), in conjunction with Lemma 5.22 and Corollary 5.28 for the ancillary estimates for the index and nullity of the various blocks. We find that the three index-plus-nullity sums appearing in the statement are respectively bounded above by

$$\begin{aligned} \text{ind}_{\mathbb{Y}_{m+1}}(Q_N^{K_m^{\Sigma}}) + \text{ind}_{\mathbb{A}_{m+1}}(Q_N^{M_m^{\Sigma}}) + [\text{ind}_{\mathbb{A}_{m+1}}(Q_N^{B_m^{\Sigma}}) + \text{nul}_{\mathbb{A}_{m+1}}(Q_N^{B_m^{\Sigma}})] &\leq 1 + 1 + 0 = 2, \\ 2 \text{ind}_{\mathbb{Y}_{m+1}}(Q_N^{K_m^{\Sigma}}) + \text{ind}_{\mathbb{Y}_{m+1}}(Q_N^{M_m^{\Sigma}}) + [\text{ind}_{\mathbb{Y}_{m+1}}(Q_N^{B_m^{\Sigma}}) + \text{nul}_{\mathbb{Y}_{m+1}}(Q_N^{B_m^{\Sigma}})] &\leq 2 + 3 + 1 = 6, \\ \text{ind}_{\mathbb{P}_n}(Q_N^{K_m^{\Xi}}) + [\text{ind}_{\mathbb{P}_n}(Q_N^{M_m^{\Xi}}) + \text{nul}_{\mathbb{P}_n}(Q_N^{M_m^{\Xi}})] &\leq 1 + 1 = 2. \end{aligned}$$

The first term in the first line arises as an upper bound for the \mathbb{A}_{m+1} -equivariant index of $K_m^{\Sigma} \cup_{\mathbb{R}_{\{y=z=0\}}} K_m^{\Sigma}$ subject to the natural (free boundary) Robin condition on the portion of its boundary in \mathbb{S}^2 and subject to

the homogeneous Neumann boundary condition on the remainder of the boundary. To obtain this upper bound, we have used the fact that a function on $K_m^\Sigma \cup \underline{R}_{\{y=z=0\}} K_m^\Sigma$ (a disjoint union with each annulus disjoint from $\{z=0\}$) is \mathbb{A}_{m+1} -equivariant if and only if its restriction to K_m^Σ is \mathbb{Y}_{m+1} -equivariant and it is odd with respect to any one (so all) of the $m+1$ reflections through horizontal lines in \mathbb{A}_{m+1} . The first term of the final line is obtained in similar fashion. \square

So, we are in position to fully determine the (maximally) equivariant index and nullity for the two families of free boundary minimal surfaces we constructed in [Carlotto et al. 2022b].

Proof of Theorem 1.2. We combine the upper bounds of the preceding corollary with the lower bounds from our earlier paper [Carlotto et al. 2022b], specifically with the content of Proposition 7.1 (see Remark 7.5) therein for what pertains to the index. At that stage, the fact that both nullities are zero then follows from the first and third inequality in Corollary 5.29. \square

Finally, we can obtain the absolute estimates on the Morse index of the same families.

Proof of Theorem 1.1. The lower bounds have already been established: specifically, for $\Sigma_m^{-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0}$ this is just part of Proposition 5.4, while for $\Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0}$ it follows from just combining Proposition 5.4 with Proposition 5.5. For the upper bound we can apply the Montiel–Ros argument making use of the equivariant upper bounds above, as we are about to explain. In the case of $\Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0}$, the \mathbb{P}_n -equivariant upper bound on the Morse index (and nullity) is equivalent to an upper bound on the index and nullity on each domain $\Omega_i^n = \Xi_n^{-\mathbb{K}_0 \cup \mathbb{K}_0} \cap W_i$, where W_1, \dots, W_{4n} are the open domains defined, in \mathbb{B}^3 , by the horizontal plane $\{z=0\}$ together with the n vertical planes passing through the origin and having equations $\theta = \pi/(2n) + i\pi/n$, $i = 0, 1, \dots, n-1$, (in the cylindrical coordinates defined at the beginning of Section 4), subject to Neumann conditions in the interior boundary as prescribed by Lemma 3.5. Thus the conclusion comes immediately by appealing to Corollary 3.2 given the third displayed equation of Corollary 5.29. Similarly, for $\Sigma_m^{-\mathbb{K}_0 \cup \mathbb{B}^2 \cup \mathbb{K}_0}$ we can interpret the second inequality in the statement of Corollary 5.29 as a statement on the index and nullity of the portions of surfaces that are contained in any of the $2(m+1)$ sets obtained by cutting with the $m+1$ vertical planes passing through the origin and having equations $\theta = \pi/(2(m+1)) + i\pi/(m+1)$, $i = 0, 1, \dots, m$, again subject to Neumann conditions. This completes the proof. \square

Acknowledgements

The authors wish to express their sincere gratitude to Giada Franz for a number of conversations on themes related to the subject of the present manuscript. This project has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement no. 947923). The research of Schulz was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany’s Excellence Strategy EXC-2044-390685587, Mathematics Münster: Dynamics–Geometry–Structure, and the Collaborative Research Centre CRC 1442, Geometry: Deformations and Rigidity. Part of this article was finalized while Carlotto was visiting the ETH-FIM, whose support and excellent working conditions are gratefully acknowledged.

References

- [Ambrozio et al. 2018a] L. Ambrozio, A. Carlotto, and B. Sharp, “Comparing the Morse index and the first Betti number of minimal hypersurfaces”, *J. Differential Geom.* **108**:3 (2018), 379–410. MR Zbl
- [Ambrozio et al. 2018b] L. Ambrozio, A. Carlotto, and B. Sharp, “Index estimates for free boundary minimal hypersurfaces”, *Math. Ann.* **370**:3-4 (2018), 1063–1078. MR Zbl
- [Aronszajn 1957] N. Aronszajn, “A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order”, *J. Math. Pures Appl.* (9) **36** (1957), 235–249. MR Zbl
- [Carlotto and Li 2024] A. Carlotto and C. Li, “Constrained deformations of positive scalar curvature metrics”, *J. Differential Geom.* **126**:2 (2024), 475–554. MR Zbl
- [Carlotto et al. 2022a] A. Carlotto, G. Franz, and M. B. Schulz, “Free boundary minimal surfaces with connected boundary and arbitrary genus”, *Camb. J. Math.* **10**:4 (2022), 835–857. MR Zbl
- [Carlotto et al. 2022b] A. Carlotto, M. B. Schulz, and D. Wiygul, “Infinitely many pairs of free boundary minimal surfaces with the same topology and symmetry group”, 2022. To appear in Mem. Amer. Math. Soc. arXiv 2205.04861
- [Choe 1990] J. Choe, “Index, vision number and stability of complete minimal surfaces”, *Arch. Ration. Mech. Anal.* **109**:3 (1990), 195–212. MR Zbl
- [Devyver 2019] B. Devyver, “Index of the critical catenoid”, *Geom. Dedicata* **199** (2019), 355–371. MR Zbl
- [Evans and Gariepy 2015] L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions*, revised ed., CRC, Boca Raton, FL, 2015. MR Zbl
- [Fischer-Colbrie 1985] D. Fischer-Colbrie, “On complete minimal surfaces with finite Morse index in three-manifolds”, *Invent. Math.* **82**:1 (1985), 121–132. MR Zbl
- [Folha et al. 2017] A. Folha, F. Pacard, and T. Zolotareva, “Free boundary minimal surfaces in the unit 3-ball”, *Manuscripta Math.* **154**:3-4 (2017), 359–409. MR Zbl
- [Franz 2022] G. Franz, *Contributions to the theory of free boundary minimal surfaces*, Ph.D. thesis, ETH Zürich, 2022. arXiv 2208.12188
- [Fraser 2020] A. Fraser, “Extremal eigenvalue problems and free boundary minimal surfaces in the ball”, pp. 1–40 in *Geometric analysis*, Lecture Notes in Math. **2263**, Springer, 2020. MR Zbl
- [Fraser and Schoen 2011] A. Fraser and R. Schoen, “The first Steklov eigenvalue, conformal geometry, and minimal surfaces”, *Adv. Math.* **226**:5 (2011), 4011–4030. MR Zbl
- [Fraser and Schoen 2016] A. Fraser and R. Schoen, “Sharp eigenvalue bounds and minimal surfaces in the ball”, *Invent. Math.* **203**:3 (2016), 823–890. MR Zbl
- [Girouard and Lagacé 2021] A. Girouard and J. Lagacé, “Large Steklov eigenvalues via homogenisation on manifolds”, *Invent. Math.* **226**:3 (2021), 1011–1056. MR Zbl
- [Kapouleas 1997] N. Kapouleas, “Complete embedded minimal surfaces of finite total curvature”, *J. Differential Geom.* **47**:1 (1997), 95–169. MR Zbl
- [Kapouleas and Li 2021] N. Kapouleas and M. M.-c. Li, “Free boundary minimal surfaces in the unit three-ball via desingularization of the critical catenoid and the equatorial disc”, *J. Reine Angew. Math.* **776** (2021), 201–254. MR Zbl
- [Kapouleas and McGrath 2023] N. Kapouleas and P. McGrath, “Generalizing the linearized doubling approach, I: General theory and new minimal surfaces and self-shrinkers”, *Camb. J. Math.* **11**:2 (2023), 299–439. MR Zbl
- [Kapouleas and Wiygul 2020] N. Kapouleas and D. Wiygul, “The index and nullity of the Lawson surfaces $\xi_{g,1}$ ”, *Camb. J. Math.* **8**:2 (2020), 363–405. MR Zbl
- [Kapouleas and Wiygul 2023] N. Kapouleas and D. Wiygul, “Free boundary minimal surfaces with connected boundary in the 3-ball by tripling the equatorial disc”, *J. Differential Geom.* **123**:2 (2023), 311–362. MR Zbl
- [Kapouleas and Zou 2021] N. Kapouleas and J. Zou, “Free boundary minimal surfaces in the Euclidean three-ball close to the boundary”, preprint, 2021. arXiv 2111.11308
- [Karcher 1988] H. Karcher, “Embedded minimal surfaces derived from Scherk’s examples”, *Manuscripta Math.* **62**:1 (1988), 83–114. MR Zbl

- [Karpukhin et al. 2014] M. Karpukhin, G. Kokarev, and I. Polterovich, “Multiplicity bounds for Steklov eigenvalues on Riemannian surfaces”, *Ann. Inst. Fourier (Grenoble)* **64**:6 (2014), 2481–2502. MR Zbl
- [Ketover 2016a] D. Ketover, “Equivariant min-max theory”, preprint, 2016. arXiv 1612.08692
- [Ketover 2016b] D. Ketover, “Free boundary minimal surfaces of unbounded genus”, preprint, 2016. arXiv 1612.08691
- [Li 2020] M. M.-c. Li, “Free boundary minimal surfaces in the unit ball: recent advances and open questions”, pp. 401–435 in *Proceedings of the International Consortium of Chinese Mathematicians* (Guangzhou, China, 2017), edited by S. Y. Cheng et al., Int. Press, Boston, MA, 2020. MR Zbl
- [Lima 2022] V. Lima, “Bounds for the Morse index of free boundary minimal surfaces”, *Asian J. Math.* **26**:2 (2022), 227–252. MR Zbl
- [Marques and Neves 2016] F. C. Marques and A. Neves, “Morse index and multiplicity of min-max minimal hypersurfaces”, *Camb. J. Math.* **4**:4 (2016), 463–511. MR Zbl
- [Marques and Neves 2018] F. C. Marques and A. Neves, “The space of cycles, a Weyl law for minimal hypersurfaces and Morse index estimates”, pp. 319–329 in *Surveys in differential geometry* (Cambridge, MA, 2017), edited by H.-D. Cao et al., Surv. Differ. Geom. **22**, Int. Press, Somerville, MA, 2018. MR Zbl
- [Marques and Neves 2020] F. C. Marques and A. Neves, “Applications of min-max methods to geometry”, pp. 41–77 in *Geometric analysis*, Lecture Notes in Math. **2263**, Springer, 2020. MR Zbl
- [Montiel and Ros 1991] S. Montiel and A. Ros, “Schrödinger operators associated to a holomorphic map”, pp. 147–174 in *Global differential geometry and global analysis* (Berlin, 1990), Lecture Notes in Math. **1481**, Springer, 1991. MR Zbl
- [Neves 2014] A. Neves, “New applications of min-max theory”, pp. 939–957 in *Proceedings of the International Congress of Mathematicians, II*, edited by S. Y. Jang et al., Kyung Moon Sa, Seoul, 2014. MR Zbl
- [Sargent 2017] P. Sargent, “Index bounds for free boundary minimal surfaces of convex bodies”, *Proc. Amer. Math. Soc.* **145**:6 (2017), 2467–2480. MR Zbl
- [Scherk 1835] H. F. Scherk, “Bemerkungen über die kleinste Fläche innerhalb gegebener Grenzen”, *J. Reine Angew. Math.* **13** (1835), 185–208. MR Zbl
- [Smith and Zhou 2019] G. Smith and D. Zhou, “The Morse index of the critical catenoid”, *Geom. Dedicata* **201** (2019), 13–19. MR Zbl
- [Taylor 1996] M. E. Taylor, *Partial differential equations, I: Basic theory*, Appl. Math. Sci. **115**, Springer, 1996. MR Zbl
- [Tran 2020] H. Tran, “Index characterization for free boundary minimal surfaces”, *Comm. Anal. Geom.* **28**:1 (2020), 189–222. MR Zbl

Received 15 Jan 2023. Revised 3 May 2024. Accepted 20 Jul 2024.

ALESSANDRO CARLOTTO: alessandro.carlotto@unitn.it
Dipartimento di Matematica, Università di Trento, Povo, Italy

MARIO B. SCHULZ: mario.schulz@unitn.it
Dipartimento di Matematica, Università di Trento, Povo, Italy

DAVID WIYGUL: davidjames.wiygul@unitn.it
Dipartimento di Matematica, Università di Trento, Povo, Italy

Analysis & PDE

msp.org/apde

EDITOR-IN-CHIEF

Clément Mouhot Cambridge University, UK
c.mouhot@dpmms.cam.ac.uk

BOARD OF EDITORS

Massimiliano Berti	Scuola Intern. Sup. di Studi Avanzati, Italy berti@sissa.it	William Minicozzi II	Johns Hopkins University, USA minicozz@math.jhu.edu
Zbigniew Blocki	Uniwersytet Jagielloński, Poland zbigniew.blocki@uj.edu.pl	Werner Müller	Universität Bonn, Germany mueller@math.uni-bonn.de
Charles Fefferman	Princeton University, USA cf@math.princeton.edu	Igor Rodnianski	Princeton University, USA irod@math.princeton.edu
David Gérard-Varet	Université de Paris, France david.gerard-varet@imj-prg.fr	Yum-Tong Siu	Harvard University, USA siu@math.harvard.edu
Colin Guillarmou	Université Paris-Saclay, France colin.guillarmou@universite-paris-saclay.fr	Terence Tao	University of California, Los Angeles, USA tao@math.ucla.edu
Ursula Hamenstaedt	Universität Bonn, Germany ursula@math.uni-bonn.de	Michael E. Taylor	Univ. of North Carolina, Chapel Hill, USA met@math.unc.edu
Peter Hintz	ETH Zurich, Switzerland peter.hintz@math.ethz.ch	Gunther Uhlmann	University of Washington, USA gunther@math.washington.edu
Vadim Kaloshin	Institute of Science and Technology, Austria vadim.kaloshin@gmail.com	András Vasy	Stanford University, USA andras@math.stanford.edu
Izabella Laba	University of British Columbia, Canada ilaba@math.ubc.ca	Dan Virgil Voiculescu	University of California, Berkeley, USA dvv@math.berkeley.edu
Anna L. Mazzucato	Penn State University, USA alm24@psu.edu	Jim Wright	University of Edinburgh, UK j.r.wright@ed.ac.uk
Richard B. Melrose	Massachusetts Inst. of Tech., USA rbm@math.mit.edu	Maciej Zworski	University of California, Berkeley, USA zworski@math.berkeley.edu
Frank Merle	Université de Cergy-Pontoise, France merle@ihes.fr		

PRODUCTION

production@msp.org

Silvio Levy, Scientific Editor

Cover image: Eric J. Heller: “Linear Ramp”


See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2025 is US \$475/year for the electronic version, and \$735/year (+\$70, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscriber address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2025 Mathematical Sciences Publishers

ANALYSIS & PDE

Volume 18 No. 7 2025

Regularized Brascamp–Lieb inequalities NEAL BEZ and SHOHEI NAKAMURA	1567
Cosmic censorship near FLRW spacetimes with negative spatial curvature DAVID FAJMAN and LIAM URBAN	1615
Spectral estimates for free boundary minimal surfaces via Montiel–Ros partitioning methods ALESSANDRO CARLOTTO, MARIO B. SCHULZ and DAVID WIYGUL	1715
The fractal uncertainty principle via Dolgopyat’s method in higher dimensions AIDAN BACKUS, JAMES LENG and ZHONGKAI TAO	1769
Consequences of the random matrix solution to the Peterson–Thom conjecture BEN HAYES, DAVID JEKEL and SRIVATSAV KUNNAWALKAM ELAYAVALLI	1805