

ANALYSIS & PDE

Volume 18

No. 7

2025

AIDAN BACKUS, JAMES LENG AND ZHONGKAI TAO

**THE FRACTAL UNCERTAINTY PRINCIPLE
VIA DOLGOPYAT'S METHOD IN HIGHER DIMENSIONS**



THE FRACTAL UNCERTAINTY PRINCIPLE VIA DOLGOPYAT’S METHOD IN HIGHER DIMENSIONS

AIDAN BACKUS, JAMES LENG AND ZHONGKAI TAO

We prove a fractal uncertainty principle with exponent $\frac{1}{2}d - \delta + \varepsilon$, $\varepsilon > 0$, for Ahlfors–David regular subsets of \mathbb{R}^d with dimension δ which satisfy a suitable “nonorthogonality condition”. This generalizes the application of Dolgopyat’s method by Dyatlov and Jin (2018) to higher dimensions. As a corollary, we get a quantitative essential spectral gap for the Laplacian on convex cocompact hyperbolic manifolds of arbitrary dimension with Zariski-dense fundamental groups.

1. Introduction

The *fractal uncertainty principle*, informally, is the assertion that a function cannot be microlocalized to a neighborhood of a fractal set in phase space. Such assertions have applications in spectral theory, where one can apply microlocal methods to show that fractal uncertainty principles imply the existence of essential spectral gaps [Dyatlov and Zahl 2016]. In particular, one can obtain $L^2 \rightarrow L^2$ bounds on the scattering resolvents of the Laplacian on convex cocompact hyperbolic manifolds, as well as improvements on the size of the maximal region in which certain zeta functions admit analytic continuation [Bourgain and Dyatlov 2018].

To make the fractal uncertainty principle more precise, we introduce the *semiclassical Fourier transform*

$$\mathcal{F}_h f(\xi) := (2\pi h)^{-d/2} \int_{\mathbb{R}^d} e^{-ix \cdot \xi / h} f(x) dx,$$

where $h > 0$ is a small parameter. If we have sets X, Y and we write X_h, Y_h for the sumsets $X_h := X + B_h$, $Y_h := Y + B_h$, $B_h := B(0, h)$, then the fractal uncertainty principle for X, Y asserts bounds of the form

$$\|1_{X_h} \mathcal{F}_h 1_{Y_h}\|_{L^2 \rightarrow L^2} \lesssim h^\beta \tag{1-1}$$

for some $\beta > 0$ in the limit $h \rightarrow 0$. We are interested in the case that X, Y are Ahlfors–David regular sets.

Definition 1.1. A compactly supported finite Borel measure μ on \mathbb{R}^d is *Ahlfors–David regular* of dimension $\delta \in [0, d]$ on scales $[\alpha, \beta]$ with regularity constant $C_R \geq 1$ if, for every closed square box I with side length $r \in [\alpha, \beta]$ or closed ball I with radius $r \in [\alpha, \beta]$,

$$\mu(I) \leq C_R r^\delta$$

MSC2020: primary 28A80, 35B34; secondary 81Q50.

Keywords: fractal uncertainty principle, resonances.

and if, in addition, I is centered on a point in $X := \text{supp } \mu$,

$$C_R^{-1} r^\delta \leq \mu(I).$$

In short we say that (X, μ) is δ -regular.

Applying Plancherel’s theorem and Hölder’s inequality, one can easily check that if X is δ -regular and Y is δ' -regular on scales $[h, 1]$ then

$$\|1_{X_h} \mathcal{F}_h 1_{Y_h}\|_{L^2 \rightarrow L^2} \lesssim h^{\max(0, \frac{1}{2}(d-\delta-\delta'))}, \tag{1-2}$$

this estimate is a straightforward modification of [Dyatlov 2019, (2.7)]. In fact, (1-2) is sharp if δ or δ' are either 0 or d , or if X, Y are orthogonal line segments in \mathbb{R}^2 .

Thus we say that X, Y satisfy the *fractal uncertainty principle* if (1-1) holds for some

$$\beta > \max(0, \frac{1}{2}(d - \delta - \delta')).$$

We note that the two ranges $\delta + \delta' \geq d$ and $\delta + \delta' \leq d$ are very different and the corresponding fractal uncertainty principles usually hold for different reasons. There are several cases in which the fractal uncertainty principle is known:

- (1) If $d = 1$ and $0 < \delta, \delta' < 1$, then the fractal uncertainty principle holds [Bourgain and Dyatlov 2018; Dyatlov and Jin 2018; Dyatlov and Zahl 2016].
- (2) If $d < \delta + \delta' < 2d$, then the fractal uncertainty principle holds under the additional assumption that either Y can be decomposed as a product of Ahlfors–David fractals in \mathbb{R} [Han and Schlag 2020] or Y is line-porous [Cohen 2023].
- (3) If d is odd and δ, δ' are very close to $\frac{1}{2}d$, then the fractal uncertainty principle holds [Cladek and Tao 2021].
- (4) If X, Y are arithmetic Cantor sets,¹ then the fractal uncertainty principle holds for $d = 1$ [Dyatlov and Jin 2017] and $d = 2, \delta + \delta' \geq 1$ under the condition that X does not contain any line [Cohen 2025].

1.1. The main theorem. In this paper we establish the fractal uncertainty principle for $0 < \delta + \delta' \leq d$ under the following additional hypothesis, which rules out the possibility that X, Y are orthogonal line segments. For $\Phi(x, y) := -x \cdot y$, it is a quantitative form of the statement that “ X and Y do not lie in submanifolds which have orthogonal tangent spaces”.

Definition 1.2. Let $X, Y \subseteq \mathbb{R}^d$, and let $\Phi \in C^2(\mathbb{R}^d \times \mathbb{R}^d)$. We say that (X, Y) is Φ -nonorthogonal with constant $0 < c_N \leq 1$ from scales (α_0^X, α_0^Y) to (α_1^X, α_1^Y) if, for any $x_0 \in X, y_0 \in Y$ and $r_X \in (\alpha_0^X, \alpha_1^X)$ and $r_Y \in (\alpha_0^Y, \alpha_1^Y)$, there exists $x_1, x_2 \in X \cap B(x_0, r_X), y_1, y_2 \in Y \cap B(y_0, r_Y)$ such that

$$|\Phi(x_1, y_1) - \Phi(x_2, y_1) - \Phi(x_1, y_2) + \Phi(x_2, y_2)| \geq c_N r_X r_Y. \tag{1-3}$$

¹We define these fundamental examples in Section 1.2.1, but for now the reader may view them as Cantor sets where the removed boxes have rational vertices.

The motivation for this definition is as follows: we want nonorthogonality to be visible on virtually all scales; after all, orthogonality of fractals is a local property, so we want nonorthogonal examples on most balls centered on a point in X and Y . The Ahlfors–David regularity condition guarantees that each such ball contributes roughly the same amount of fractal mass and is hence the reason why we upgrade “most” to “all”. At the same time, we don’t want nonorthogonal points to lie too close to each other. This is why we take the right-hand side to be $r_X r_Y$ instead of $|x_1 - x_0| \cdot |y_1 - y_0|$. One can verify that this definition of nonorthogonality generalizes the nonorthogonality hypothesis of [Dyatlov 2019, Proposition 6.5].

The nonorthogonality condition (1-3) is based on the *local nonintegrability condition* (LNI) of [Naud 2005; Stoyanov 2011], which itself can be traced back to the *uniform nonintegrability condition* of [Chernov 1998; Dolgopyat 1998]. In such papers one is concerned with the nonintegrability of the stable and unstable foliations of an Axiom A (or perhaps even Anosov) flow. Roughly speaking, given fractals X, Y , one may define two laminations (in the sense of [Thurston 1979, Chapter 8]) in $\mathbb{R}_x^d \times \mathbb{R}_y^d$, the *vertical lamination* $\{x \in X\}$ and *horizontal lamination* $\{\xi = \partial_x \Phi(x, y) : y \in Y\}$, and then (1-3) essentially asserts that the vertical and horizontal laminations satisfy LNI.

In order to state our result, we need one more condition which involves how the measure of a cube I varies when we double it.

Definition 1.3. A measure μ is *doubling* on scales $[h, 1]$ if there exists $C_D > 0$ such that, for every $r \in [h, \frac{1}{2}]$ and every cube I of side length r centered at $x \in \text{supp } \mu$, we have $\mu(I \cdot 2) \leq C_D \mu(I)$, where $I \cdot 2$ is the cube with the same center as I and side length $2r$.

Clearly every regular measure is doubling; we highlight that our main theorem only needs to assume the measure is doubling rather than regular. It is essential that we only consider cubes centered at $x \in \text{supp } \mu$ in the definition. One can compare this doubling property with the Federer property in [Dolgopyat 1998, §7], in which case the Gibbs measure is supported everywhere.

What follows is our main theorem.

Theorem 1.4. Let μ_X, μ_Y be doubling probability measures on scales $[h, 1]$ with compact supports $X \subset I_0, Y \subset J_0$, where $I_0, J_0 \subset \mathbb{R}^d$ are rectangular boxes with unit length. Let \mathcal{B}_h be the semiclassical Fourier integral operator

$$\mathcal{B}_h f(x) = \int_Y \exp\left(\frac{i\Phi(x, y)}{h}\right) p(x, y) f(y) \, d\mu_Y(y), \tag{1-4}$$

where the phase Φ belongs to $C^3(I_0 \times J_0)$, X, Y are Φ -nonorthogonal from scales h to 1, and the symbol p belongs to $C^1(I_0 \times J_0)$. Then there exists $\varepsilon_0 > 0$ such that

$$\|\mathcal{B}_h\|_{L^2(\mu_Y) \rightarrow L^2(\mu_X)} \lesssim h^{\varepsilon_0}.$$

We use the notation \lesssim in the statements of our theorems to record the existence of a hidden constant. The constant could depend on the dimension d , the nonorthogonality constant c_N , the doubling constant C_D , the diameters of X, Y , $\|\Phi\|_{C^3}$, and $\|p\|_{C^1}$, but is *independent* of h .

If one additionally assumes $d = 1$ and that μ_X, μ_Y are regular with dimension $\in (0, 1)$, then Theorem 1.4 was proven in [Dyatlov and Jin 2018], extending the method of [Dolgopyat 1998] which had already been

applied to construct spectral gaps. Using the construction of dyadic cubes in [Christ 1990], it might be possible that Theorem 1.4 can be generalized to doubling metric spaces. Since there is no immediate application for metric spaces, we have not attempted to write down the more general version.

Following the methods of [Dyatlov and Jin 2018], Theorem 1.4 implies the following fractal uncertainty principle which is interesting in the range $\frac{1}{2}(d - \delta - \delta') + \varepsilon_0 > 0$.

Theorem 1.5. *Let X and Y be Ahlfors–David regular sets in \mathbb{R}^d which are nonorthogonal with respect to the dot product on $\mathbb{R}^d \times \mathbb{R}^d$. Assume that X is δ -regular, Y is δ' -regular, $0 < \delta, \delta' < d$. Then there exists $\varepsilon_0 > 0$ such that*

$$\|1_{X_h} \mathcal{F}_h 1_{Y_h}\|_{L^2 \rightarrow L^2} \lesssim h^{\frac{1}{2}(d - \delta - \delta') + \varepsilon_0}.$$

1.1.1. Lower bounds on the uncertainty exponent. If we let

$$L := \frac{10^{14} d^3}{c_N^3} \max(1, \|\partial_{xy}^2 \Phi\|_{C^1}^3), \tag{1-5}$$

then we can take in Theorem 1.4

$$\frac{1}{\varepsilon_0} \leq 6 \cdot 10^9 c_N^{-2} d^2 (C_D(X) C_D(Y))^{4 \lceil \log_2(20L^{5/3}) \rceil} L^{2/3} \log L. \tag{1-6}$$

In the model case where $X = Y$ is regular, $d = 1$, and $\Phi(x, y) = -xy$, we can always take $c_N = C_R^{-4/\delta}$ and $C_D = 2^\delta C_R^2$, which gives a subexponential bound of the form $1/\varepsilon_0 \lesssim e^{C(\delta) \log_2 C_R}$. This is because of the rather poor dependence of ε_0 on the doubling constant; if one modified our proof to use the Ahlfors–David regularity directly, they would obtain a bound of the form $1/\varepsilon_0 \lesssim C_R^{O(1+1/\delta)}$, which is comparable with the bound $1/\varepsilon_0 \lesssim C_R^{160/(\delta(1-\delta))}$ of [Dyatlov and Jin 2018].

In any case, it does not seem that one can use Dolgopyat’s method to obtain sharp fractal uncertainty principles, which therefore remains an interesting and challenging open problem. To drive this point home, we recall that in the case $d = 1$, $\delta = \frac{1}{2}$, an unpublished manuscript of Murphy claims that $1/\varepsilon_0 \lesssim \log C_R \log \log C_R$ [Cladek and Tao 2021, §1].

1.1.2. Applications to spectral gaps. Suppose $M = \Gamma \backslash \mathbb{H}^{d+1}$ is a (noncompact) convex cocompact hyperbolic manifold and $\Lambda(\Gamma)$ is the limit set (see Section 5.2 for the definition). The Patterson–Sullivan measure μ on $\Lambda(\Gamma)$ is Ahlfors–David regular of dimension $\delta_\Gamma \in [0, d)$ [Sullivan 1979, Theorem 7]. Under the condition that Γ is Zariski dense in $G = \text{SO}(d + 1, 1)_0$, we have that $(\Lambda(\Gamma), \mu)$ satisfies the nonorthogonality condition (1-3) for very general $\Phi(x, y)$ (see Corollary 5.4). So we have the fractal uncertainty principle for $\Lambda(\Gamma)$ with very general phase functions.

Dyatlov and Zahl [2016] showed that fractal uncertainty principles can be used to prove essential spectral gaps. Let Δ be the Laplace–Beltrami operator on M . Then the resolvent

$$R(\lambda) := \left(-\Delta - \frac{1}{4}d^2 - \lambda^2\right)^{-1} : L^2_{\text{comp}}(M) \rightarrow H^2_{\text{loc}}(M)$$

is well defined for $\text{Im}(\lambda) \gg 1$ with a meromorphic continuation to $\lambda \in \mathbb{C}$; see [Guillarmou 2005; Mazzeo and Melrose 1987] for (even) asymptotically hyperbolic manifolds and [Guillopé and Zworski 1995] for manifolds with constant negative curvature near infinity. Vasy [2013a; 2013b] gave a new construction of the meromorphic continuation, which is the one used in [Dyatlov and Zahl 2016].

The standard Patterson–Sullivan gap [Patterson 1976; Sullivan 1979] says

$$R(\lambda) \text{ has only finitely many poles in } \{\text{Im}(\lambda) \geq -\max(0, \frac{1}{2}d - \delta_\Gamma)\}. \tag{1-7}$$

Moreover, there is no pole in $\{\text{Im}(\lambda) > \delta_\Gamma - \frac{1}{2}d\}$, and there are conditions on δ_Γ such that $\lambda = i(\delta_\Gamma - \frac{1}{2}d)$ is the first pole (see [Sullivan 1979; Patterson 1988]). Using methods of [Dyatlov and Zahl 2016], we can improve the essential spectral gap when $\delta_\Gamma \leq \frac{1}{2}d$.

Theorem 1.6. *Let M be a noncompact convex cocompact hyperbolic $(d+1)$ -dimensional manifold such that $\Gamma = \pi_1(M)$ is Zariski dense in $\text{SO}(d + 1, 1)_0$. Let $\delta_\Gamma \in (0, d)$ be the Hausdorff dimension of the limit set $\Lambda(\Gamma)$. Then there exists $\varepsilon_0 > 0$ such that, for any $\varepsilon > 0$, $R(\lambda)$ has only finitely many poles λ with $\text{Im} \lambda > \delta_\Gamma - \frac{1}{2}d - \varepsilon_0 + \varepsilon$. Moreover, for any $\chi \in C_0^\infty(M)$, there exists $C_0 = C_0(\varepsilon) > 0$ and $C = C(\varepsilon, \chi) > 0$ such that*

$$\|\chi R(\lambda)\chi\|_{L^2 \rightarrow L^2} \leq C|\lambda|^{-1-2\min(0, \text{Im}\lambda)+\varepsilon}, \quad |\lambda| > C_0, \quad \text{Im} \lambda \geq \delta_\Gamma - \frac{1}{2}d - \varepsilon_0 + \varepsilon. \tag{1-8}$$

Dyatlov and Jin [2018, Theorem 2] showed Theorem 1.6 with $d = 1$ by proving Theorem 1.4 for $d = 1$ and X, Y δ -regular and applying [Dyatlov and Zahl 2016, Theorem 3]; our result is the natural higher-dimensional generalization of this theorem. Note the statement of the theorem holds for $\delta_\Gamma \in (0, d)$ in the whole range, but when $\delta_\Gamma > \frac{1}{2}d + \varepsilon_0$, our theorem says nothing more than the Lax–Phillips gap coming from unitarity. On the other hand, it improves the Lax–Phillips gap when $\delta_\Gamma < \frac{1}{2}d + \varepsilon_0$, which slightly passes the threshold $\frac{1}{2}d$.

The spectral gap in Theorem 1.6 was first proved [Naud 2005] for surfaces and generalized [Stoyanov 2011] to higher dimensions. The size of their gap is implicit but our method gives an explicit constant ε_0 as in (1-6) depending on the fractal dimension δ_Γ , the regularity constant and the nonorthogonality constant of the limit set $\Lambda(\Gamma)$. We give a method for computing nonorthogonality constants from the generators of a classical Schottky group $\Gamma \subset \text{SL}(2, \mathbb{C})$ in the Appendix.

Another advantage of the method of [Dyatlov and Zahl 2016] is that we also get the resolvent estimate (1-8), which is hard to obtain using transfer operator techniques and in particular is not included in [Naud 2005; Stoyanov 2011]. The resolvent bound is useful in applications; see, e.g., [Vacossin 2023].

Corollary 1.7. *Let M be convex cocompact with Γ Zariski-dense. Let ζ_M be the Selberg zeta function*

$$\zeta_M(s) = \prod_{l \in \mathcal{L}_M} \prod_{k=0}^{\infty} (1 - e^{-(s+k)l}), \quad s = \frac{1}{2}d - i\lambda,$$

where \mathcal{L}_M consists of the lengths of all primitive closed geodesics on M (with multiplicity). Then $\zeta_M(s)$ has only finitely many singularities (i.e., zeroes or poles) in the half-plane $\{\text{Re } s > \delta_\Gamma - \varepsilon_0 + \varepsilon\}$ for any $\varepsilon > 0$.

Proof. This follows from Theorem 1.6 and [Bunke and Olbrich 1999; Patterson and Perry 2001]. □

The spectral gap is closely related to asymptotics of closed geodesics and exponential decay of correlations, which are important and well-studied questions in dynamical systems. We list a few references.

- Chernov [1998] gave the first dynamical proof showing subexponential decay of correlations for 3-dimensional contact Anosov flows. The groundbreaking work of Dolgopyat [1998] showed exponential decay of correlations for transitive Anosov flows with jointly nonintegrable C^1 stable/unstable foliations.
- Naud [2005] applied Dolgopyat’s method to establish a spectral gap for convex cocompact hyperbolic surfaces.
- Stoyanov [2008; 2011] showed exponential mixing for a general class of Axiom A flows satisfying his local nonintegrability condition.
- Sarkar and Winter [2021] used Dolgopyat’s method to prove exponential mixing of the frame flow for convex cocompact hyperbolic manifolds. Chow and Sarkar [2022] extended it to locally symmetric spaces.
- It is interesting to ask what happens on hyperbolic manifolds with cusps. We direct the readers to [Li and Pan 2023; Li et al. 2023] for more details.

All the above works require certain *nonintegrability conditions* which should be thought of as the analogue of our nonorthogonality condition (1-3).

We would like to mention some other related works on the spectral gap for convex cocompact hyperbolic manifolds.

- Dyatlov and Zahl [2016], Dyatlov and Jin [2018], Bourgain and Dyatlov [2018], and Jin and Zhang [2020] proved the fractal uncertainty principle for $d = 1$ and hence gave explicit essential spectral gaps.
- Bourgain and Dyatlov [2017] used Fourier decay of the Patterson–Sullivan measure to get an essential spectral gap that only depends on δ_Γ when $d = 1$, $\delta_\Gamma \leq \frac{1}{2}$. This is generalized to Kleinian Schottky groups when $d = 2$ by Li, Naud and Pan [Li et al. 2021], but in this case the essential spectral gap will depend on δ_Γ and another quantity related to our nonorthogonality constant c_N (see [Li et al. 2021, Lemma 4.4]). See also [Khalil 2023; 2024] for a method using additive combinatorics.
- Oh and Winter [2016] showed a uniform spectral gap for a large family of congruence arithmetic surfaces, which was then generalized to arbitrary dimensions by Sarkar [2022].

1.2. Idea of the proof.

1.2.1. Model problem: Arithmetic Cantor sets. We first describe the problem in the model case that X, Y are arithmetic Cantor sets. Let $M \geq 3$ be an integer and $A, B \subseteq \{0, 1, \dots, M-1\}^d$ be sets with

$$\delta_A := \frac{\log |A|}{\log(M)} \leq \frac{1}{2}d, \quad \delta_B := \frac{\log |B|}{\log(M)} \leq \frac{1}{2}d.$$

We let $N := M^k$ and define the *arithmetic Cantor sets*

$$C_{k,A} := \{a_0 + a_1M + \dots + a_{k-1}M^{k-1} : a_i \in A\} \subset (\mathbb{Z}/N\mathbb{Z})^d,$$

$$C_{k,B} := \{b_0 + b_1M + \dots + b_{k-1}M^{k-1} : b_i \in B\} \subset (\mathbb{Z}/N\mathbb{Z})^d.$$

We introduce the *discrete Fourier transform*

$$\mathcal{F}_N f(j) := N^{-d/2} \sum_{\ell \in \{0,1,\dots,N-1\}^d} \exp\left(2\pi i j \cdot \frac{\ell}{N}\right) f(\ell), \quad j \in (\mathbb{Z}/N\mathbb{Z})^d.$$

The fractal uncertainty principle states that there exists some $\varepsilon_0 > 0$ such that

$$\|1_{C_{k,A}} \mathcal{F}_N 1_{C_{k,B}}\|_{\ell^2 \rightarrow \ell^2} \lesssim N^{-\beta - \varepsilon_0}, \tag{1-9}$$

where $\beta := \frac{1}{2}(d - \delta_A - \delta_B)$ [Dyatlov and Jin 2017, §3]. Analyzing the Hilbert–Schmidt norm, we have

$$\|1_{C_{k,A}} \mathcal{F}_N 1_{C_{k,B}}\|_{\ell^2 \rightarrow \ell^2} \leq \|1_{C_{k,A}} \mathcal{F}_N 1_{C_{k,B}}\|_{HS} = \sqrt{\frac{|A|^k |B|^k}{N^d}} = N^{-\beta}. \tag{1-10}$$

Thus, our goal is to obtain additional gain beyond β . To prove this, one can show as in [Dyatlov 2019, Lemma 6.4] that if we let

$$r_k := \|1_{C_{k,A}} \mathcal{F}_N 1_{C_{k,B}}\|_{\ell^2 \rightarrow \ell^2}$$

then $r_{k_1+k_2} \leq r_{k_1} r_{k_2}$. This can be used to show that if we can get any gain at all at some scale k then we get a gain on all further levels, so we suppose for the sake of contradiction that we cannot obtain any gain at any scale, or that the inequality present in (1-10) is an equality. Then, since the Hilbert–Schmidt norm measures the square root of the sum of squares of the singular values and the operator norm measures the largest singular value, it follows that the operator $N^{d/2} 1_{C_{k,A}} \mathcal{F}_N 1_{C_{k,B}}$ must be rank 1. A simple computation then shows that the operator $N^{d/2} 1_{C_{k,A}} \mathcal{F}_N 1_{C_{k,B}}$ is the matrix $(\exp(2\pi i j \cdot \ell / N))_{j \in C_{k,A}, \ell \in C_{k,B}}$ (and is zero in the unspecified entries). Computing the determinant of 2×2 minors, we see that

$$\left| \det \begin{pmatrix} \exp(2\pi i j \cdot \ell / N) & \exp(2\pi i j' \cdot \ell / N) \\ \exp(2\pi i j \cdot \ell' / N) & \exp(2\pi i j' \cdot \ell' / N) \end{pmatrix} \right| = \left| \exp\left(2\pi i \frac{\langle j - j', \ell - \ell' \rangle}{N}\right) - 1 \right| = 0$$

for all $j, j' \in C_{k,A}$ and $\ell, \ell' \in C_{k,B}$. Thus, (1-9) holds as long as a *nonorthogonality* condition

$$\langle j - j', \ell - \ell' \rangle \neq 0$$

holds for some choice of $j, j' \in A, \ell, \ell' \in B$. If nonorthogonality is violated at all scales, then (1-9) cannot hold; see Example 1.9.

1.2.2. Nonorthogonality and Dolgopyat’s method. Our proof and the proof of [Dyatlov and Jin 2018] lies in the continuous setting where the fractal is not necessarily self-similar. Thus, we must construct a tree of tiles that discretizes the doubling measure μ and which is regular enough so that each tile has two children which are spaced far enough apart. While very nice submultiplicativity does not hold as it does in the discrete case, we can still, via an induction on scales argument, propagate gain on one scale to gain on all scales. The key tool allowing us to obtain gain on all scales is nonorthogonality, which we formulated in (1-3); it asserts that we can find many points in the intersections of the vertical and horizontal laminations where the phase is “oscillating faster than the function \mathcal{B}_h is being tested against” at every scale, and so we must obtain a gain at every scale. This technique, called *Dolgopyat’s method*, has been used to obtain fractal uncertainty principles, spectral gaps, or exponential mixing in previous works, including [Dolgopyat 1998; Dyatlov and Jin 2018; Liverani 2004; Naud 2005; Stoyanov 2008; 2011; Tsujii and Zhang 2023].

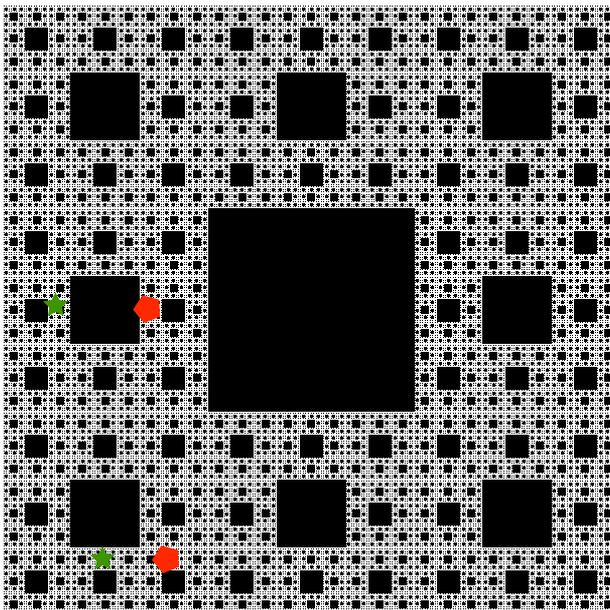


Figure 1. Nonorthogonality of the Sierpiński carpet X (the white region) to itself at scale $\frac{1}{3}$ (where $\text{diam } X = \sqrt{2}$). Given any two points $x_1, y_1 \in X$ (green stars), we can find two points $x_2, y_2 \in X$ (red pentagons) such that $|x_1 - x_2|$ and $|y_1 - y_2|$ are both ≈ 0.15 , and $|\sin \angle(x_2 - x_1, y_2 - y_1)| \ll 1$, so (X, X) is nonorthogonal with constant $(3 \cdot 0.14)^2 \approx 0.42$. Image adapted from [Rössel 2008].

The improvement on each child is measured in the spaces $C_\theta(I)$ that were introduced in [Naud 2005, Lemma 5.4]. Informally speaking, localizations of \mathcal{B}_h to a tile I have roughly constant oscillation when normalized by $\theta \text{diam}(I)$ for some appropriate choice of θ [Dyatlov and Jin 2018, §2.2]. The $C_\theta(I)$ norms are meant to capture this fact and to measure cancellation on scale I , similar to how algebraic manipulations on M^k -dimensional vectors can be used to measure cancellation in the arithmetic Cantor case.

1.2.3. Improvements over Dyatlov–Jin. The method of [Dyatlov and Jin 2018] does not immediately generalize to $d \geq 2$ for two reasons. First, in order to ensure that each interval has at least two children that are sufficiently far apart, Dyatlov–Jin allows intervals of varying length to appear in the tree by merging together consecutive intervals that intersect the fractal. However, in higher dimensions this leads to long, narrow, winding tiles appearing in the tree; these do not satisfy suitable doubling estimates, as exemplified by the following example.

Example 1.8. Let X be a Sierpiński carpet, and consider the merged discretization for X (see Section 3 or [Dyatlov and Jin 2018, §2.1]). Since X is path-connected, every scale consists of a single tile, the only child of the single tile at the previous scale! It is impossible to prove that every tile has two children which enjoy phase cancellation.

However, our method must be able to handle the Sierpiński carpet, since it meets the hypotheses of Theorem 1.5 if it is embedded in \mathbb{R}^4 . Indeed, $2\delta_X \approx 3.8 < 4$. Moreover, X is nonorthogonal to itself at one scale (see Figure 1), so it is at every scale by self-similarity.

Secondly, as remarked above, one cannot obtain cancellation for arbitrary children I_1, I_2 , but only those which are “not orthogonal to each other”. Otherwise, even if we construct I_1, I_2 to be the appropriate distance apart to impose cancellation, it will not follow that the phases actually cancel each other.

Example 1.9. Let $X := [-5, 5] \times \{0\}$ and $Y := \{0\} \times [-5, 5]$. The Gaussian

$$f(x, y) := e^{-x^2/2 - y^2/(2h^2)}$$

is localized to X_{5h} and its Fourier transform is localized to Y_{5h} . So the fractal uncertainty principle is simply false for (X, Y) , even though

$$\delta_X + \delta_Y = 2 \leq 2,$$

and we must use the nonorthogonality hypothesis somehow. One can also see that if $X' \subset X$ and $Y' \subset Y$ are fractals then the fractal uncertainty principle does not hold for (X', Y') .

To overcome these difficulties, we improve on Dyatlov–Jin as follows:

- (1) We carefully construct the tree, so that tiles in the tree are very close to cubes and therefore satisfy good doubling estimates, but also so that each tile contains two children with a suitable distance from each other.
- (2) We prove that if X, Y are nonorthogonal then we may choose tangent vectors to X, Y so that the phases cannot decouple.

These goals are accomplished by [Proposition 3.3](#), which asserts that we can construct the so-called *perturbed standard discretization* of μ , and [Proposition 3.13](#), which asserts that many quadruples of tiles in the perturbed standard discretization have the properties above.

We found it convenient to use the language of probability theory to state [Proposition 3.13](#), as we then could interpret the various quantities appearing in the induction on scale ([Proposition 4.3](#)) as expected values or variances of certain averages of $\mathcal{B}_h f$ taken over random tiles. The necessary estimates needed to obtain a contradiction then follow from the *second moment method*—namely, the observation that, if [Proposition 4.3](#) is false, then the variance of such random variables is impossibly small given the large size of their tails. A similar approach was taken by [[Dyatlov and Jin 2018](#)], which used the strict convexity of balls in Hilbert spaces [[Dyatlov and Jin 2018](#), Lemma 2.7] to accomplish the same goals.

1.3. Outline of the paper. In [Section 2](#) we recall some preliminaries.

In [Section 3](#) we construct our discretization and show that it has good statistical properties, as made precise by [Proposition 3.13](#).

In [Section 4](#) we carry out our inductive argument. The main proposition is the iterative step, [Proposition 4.3](#); we then use this to prove [Theorem 1.4](#).

We then turn to the applications in [Section 5](#), where we reduce [Theorems 1.5](#) and [1.6](#) to [Theorem 1.4](#) by standard techniques.

In the [Appendix](#), we demonstrate how one can compute the nonorthogonality constant in a typical application: classical Schottky groups.

2. Preliminaries

2.1. Probability theory. We shall have probability spaces A, B , and will denote by a, a', a'' and b, b', b'' outcomes in those spaces (or equivalently random variables with values in A, B). The expected value of a random variable X is denoted $\mathbb{E} X$, while $\mathbb{E}(X | E)$ refers to the conditional expectation of X assuming an event E . The probability of the event E is denoted $\Pr(E)$, and the variance of a random variable is

$$\text{Var } X := \mathbb{E}(X^2) - (\mathbb{E} X)^2.$$

If X, Y are i.i.d., then

$$\mathbb{E} |X - Y|^2 = \mathbb{E} |X|^2 + \mathbb{E} |Y|^2 - 2 \mathbb{E}(XY) = 2(\mathbb{E} |X|^2 - (\mathbb{E} X)^2),$$

and so

$$\mathbb{E} |X - Y|^2 = 2 \text{Var } X. \tag{2-1}$$

We also record *Cantelli's inequality*, valid for any constant $\lambda > 0$ [Lugosi 2009, Theorem 1]:

$$\begin{aligned} \Pr(X \geq \mathbb{E} X + \lambda) &\leq \frac{\text{Var } X}{\lambda^2 + \text{Var } X}, \\ \Pr(X \leq \mathbb{E} X - \lambda) &\leq \frac{\text{Var } X}{\lambda^2 + \text{Var } X}. \end{aligned} \tag{2-2}$$

2.2. A geometric mean value theorem. We shall need an analogue of the mean value theorem for phase functions [Dyatlov and Jin 2018, Lemma 2.5]. To formulate it, we shall recall some differential geometry.

If R is a nondegenerate 2-dimensional rectangle in $\mathbb{R}_x^d \times \mathbb{R}_y^d$ and v, w are unit vectors tangent to the edges of R , then we write $\gamma_R := v \otimes w$ for the *unit bitangent* to R and dA_R for the area element on R .² We will consider the case when $v \in \mathbb{R}_x^d$ and $w \in \mathbb{R}_y^d$. In that case, γ_R and the off-diagonal Hessian $\partial_{xy}^2 \Phi$ both lie in $\mathbb{R}_x^d \otimes \mathbb{R}_y^d$, so we can consider their contraction

$$\langle \partial_{xy}^2 \Phi, \gamma_R \rangle = \partial_v \partial_w \Phi.$$

Lemma 2.1. *Let $\Phi \in C^2(\mathbb{R}^d \times \mathbb{R}^d)$. Let $x_0, x_1, y_0, y_1 \in \mathbb{R}^d$, and let R be the rectangle with vertices (x_i, y_j) , $i, j \in \{0, 1\}$. Then*

$$\int_R \langle \partial_{xy}^2 \Phi, \gamma_R \rangle dA_R = \Phi(x_0, y_0) - \Phi(x_0, y_1) - \Phi(x_1, y_0) + \Phi(x_1, y_1). \tag{2-3}$$

Proof. Both sides of (2-3) are preserved by orientation-preserving isometries which preserve the product structure on $\mathbb{R}^d \times \mathbb{R}^d$. In particular, we may take $x_0, y_0 = 0$, $x_1 = (\xi^*, 0, \dots, 0)$, and $y_1 = (\eta^*, 0, \dots, 0)$ for some $\xi^*, \eta^* \in \mathbb{R}$. We then set

$$\varphi(\xi, \eta) := \Phi((\xi, 0, \dots, 0), (\eta, 0, \dots, 0)).$$

²Strictly speaking, the unit bitangent should be defined using the exterior algebra, but since R is assumed nondegenerate this adds more complication for no gain.

Then by Fubini's theorem,

$$\begin{aligned} \int_R \langle \partial_{xy}^2 \Phi, \gamma_R \rangle dA_R &= \int_0^{\xi^*} \int_0^{\eta^*} \partial_\xi \partial_\eta \varphi(\xi, \eta) d\eta d\xi = \int_0^{\xi^*} \partial_\xi \varphi(\xi, \eta^*) - \partial_\xi \varphi(\xi, 0) d\xi \\ &= \varphi(\xi^*, \eta^*) - \varphi(\xi^*, 0) - (\varphi(0, \eta^*) - \varphi(0, 0)) \\ &= \Phi(x_0, y_0) - \Phi(x_0, y_1) - \Phi(x_1, y_0) + \Phi(x_1, y_1). \end{aligned} \quad \square$$

We now estimate the difference between (2-3) evaluated over two different rectangles R, R' by differentiating Φ along a homotopy between R, R' . This estimate will be useful when applying the nonorthogonality hypothesis.

Lemma 2.2. *Let $\Phi \in C^3(\mathbb{R}^d \times \mathbb{R}^d)$, and let $R_t = [x_0(t), x_1(t)] \times [y_0(t), y_1(t)]$, where $t = 0, 1$ and $x_i(t), y_i(t) \in \mathbb{R}^d$. Let $\gamma_t := \gamma_{R_t}$ be the unit bitangent to R_t . Assume that, for some $0 \leq \varepsilon_x, \varepsilon_y, c_x, c_y \leq 1$,*

- (1) *for every $i \in \{0, 1\}$, we have $|x_i(1) - x_i(0)| \leq \varepsilon_x$ and $|y_i(1) - y_i(0)| \leq \varepsilon_y$,*
- (2) *for every $t \in [0, 1]$, we have $|x_1(t) - x_0(t)| \leq c_x$ and $|y_1(t) - y_0(t)| \leq c_y$.*

Then

$$\left| \int_{R_1} \langle \partial_{xy}^2 \Phi, \gamma_1 \rangle dA_{R_1} - \int_{R_0} \langle \partial_{xy}^2 \Phi, \gamma_0 \rangle dA_{R_0} \right| \leq 7 \|\partial_{xy}^2 \Phi\|_{C^1} (\varepsilon_x c_y + \varepsilon_y c_x). \quad (2-4)$$

Proof. By taking convex combinations, we define $x_i(t)$ and $y_i(t)$ for any $t \in [0, 1]$ and hence also R_t and γ_t . Now introduce the parametrization

$$\Psi_t(\xi, \eta) := \begin{bmatrix} \xi x_1(t) + (1 - \xi)x_0(t) \\ \eta y_1(t) + (1 - \eta)y_0(t) \end{bmatrix} \in \mathbb{R}^d \times \mathbb{R}^d$$

which maps $[0, 1]^2$ to R_t . Also let $v_t := x_1(t) - x_0(t)$ and $w_t := y_1(t) - y_0(t)$, so $|v_t| |w_t|$ is the (unoriented) Jacobian of the map Ψ_t . We record for later that $|v_t| \leq c_x$ and $|w_t| \leq c_y$.

We estimate

$$\begin{aligned} \left| \int_{R_1} \langle \partial_{xy}^2 \Phi, \gamma_1 \rangle dA_{R_1} - \int_{R_0} \langle \partial_{xy}^2 \Phi, \gamma_0 \rangle dA_{R_0} \right| &= \left| \int_0^1 \partial_t \int_{R_t} \langle \partial_{xy}^2 \Phi, \gamma_t \rangle dA_{R_t} dt \right| \\ &\leq \int_0^1 \int_0^1 \int_0^1 |\partial_t (\langle \partial_{xy}^2 \Phi \circ \Psi_t(\xi, \eta), \gamma_t \rangle \cdot |v_t| \cdot |w_t|)| d\xi d\eta dt. \end{aligned}$$

We next split up the above integrand:

$$\begin{aligned} &|\partial_t (\langle \partial_{xy}^2 \Phi \circ \Psi_t(\xi, \eta), \gamma_t \rangle |v_t| |w_t|)| \\ &\leq |\langle \partial_t (\partial_{xy}^2 \Phi \circ \Psi_t(\xi, \eta)), \gamma_t \rangle| \cdot |v_t| \cdot |w_t| + |\langle \partial_{xy}^2 \Phi \circ \Psi_t(\xi, \eta), \partial_t \gamma_t \rangle| \cdot |v_t| \cdot |w_t| \\ &\quad + |\langle \partial_{xy}^2 \Phi \circ \Psi_t(\xi, \eta), \gamma_t \rangle| \cdot |\partial_t |v_t|| \cdot |w_t| + |\langle \partial_{xy}^2 \Phi \circ \Psi_t(\xi, \eta), \gamma_t \rangle| \cdot |v_t| \cdot |\partial_t |w_t|| \\ &=: \text{I} + \text{II} + \text{III} + \text{IV}. \end{aligned}$$

To estimate I, we compute

$$\partial_t \Psi_t(\xi, \eta) = \begin{bmatrix} \xi(x_1(1) - x_1(0)) + (1 - \xi)(x_0(1) - x_0(0)) \\ \eta(y_1(1) - y_1(0)) + (1 - \eta)(y_0(1) - y_0(0)) \end{bmatrix}$$

and conclude that $\|\partial_t \Psi_t\|_{C^0} \leq \varepsilon_x + \varepsilon_y$. Therefore, by the chain rule,

$$I \leq \|\nabla \partial_{xy}^2 \Phi\|_{C^0} \|\partial_t \Psi_t\|_{C^0} |v_t| \cdot |w_t| \leq \|\partial_{xy}^2 \Phi\|_{C^1} c_x c_y (\varepsilon_x + \varepsilon_y) \leq \|\partial_{xy}^2 \Phi\|_{C^1} (c_x \varepsilon_y + c_y \varepsilon_x).$$

We furthermore estimate

$$|\partial_t v_t| = |x_1(1) - x_1(0) - x_0(1) + x_0(0)| \leq 2\varepsilon_x$$

and similarly for w_t . Now to estimate II, we recall

$$\gamma_t = \frac{v_t}{|v_t|} \otimes \frac{w_t}{|w_t|}.$$

By the product rule,

$$|\partial_t \gamma_t| \leq \frac{2}{|v_t|} |\partial_t v_t| + \frac{2}{|w_t|} |\partial_t w_t| \leq 4 \left[\frac{\varepsilon_x}{|v_t|} + \frac{\varepsilon_y}{|w_t|} \right].$$

So

$$II \leq 4 \|\partial_{xy}^2 \Phi\|_{C^0} (c_x \varepsilon_y + c_y \varepsilon_x) \leq 4 \|\partial_{xy}^2 \Phi\|_{C^1} (c_x \varepsilon_y + c_y \varepsilon_x).$$

To estimate III, we use Kato’s inequality $|\partial_t |v_t|| \leq |\partial_t v_t|$ to bound

$$III \leq 2 \|\partial_{xy}^2 \Phi\|_{C^0} c_y \varepsilon_x \leq 2 \|\partial_{xy}^2 \Phi\|_{C^1} c_y \varepsilon_x.$$

The estimate on IV is similar but with x and y swapped. Adding up these terms and integrating, we conclude the result. □

3. Discretization of sets and measures

3.1. A new discretization. As in previous works on the fractal uncertainty principle, such as [Bourgain and Dyatlov 2018; Dyatlov and Jin 2018], we will discretize fractals as trees.

Definition 3.1. Let $X \subseteq \mathbb{R}^d$ be a set. A *discretization* of X is a family $V(X) = (V_n(X))_{n \in \mathbb{Z}}$ of sets, where $V_n(X)$ is a set of nonempty subsets of \mathbb{R}^d such that

- $X = \bigcup \{I \cap X : I \in V_n(X)\}$ for each n and the union is disjoint;
- for any $I \in V_n(x)$, there exist $I_k \in V_{n+1}(X)$ such that $I = \bigcup_k I_k$.

Given $I \in \bigcup_n V_n(X)$, the *height* of I is defined as $H(I) = \sup\{n : I \in V_n(X)\}$.

Definition 3.2. For a compact set $X \subset \mathbb{R}^d$ and base $L \geq 2$, its *standard L -adic discretization* $V^0 = (V_n^0)_{n \in \mathbb{Z}}$ is defined by $I \in V_n^0(X)$ if and only if

$$I = I_n(q) := [q_1, L^{-n} + q_1) \times [q_2, L^{-n} + q_2) \times \cdots \times [q_d, L^{-n} + q_d)$$

for some $q \in L^{-n} \mathbb{Z}^d$ and $I \cap X \neq \emptyset$.

The standard discretization was used in [Bourgain and Dyatlov 2018] to prove the fractal uncertainty principle in the case $d = 1$, $\delta > \frac{1}{2}$. The problem with the standard discretization is that a box in $V_n^0(X)$ may be too small for the fractal measure. Dyatlov and Jin [2018] addressed this issue in the case $d = 1$, $\delta \leq \frac{1}{2}$, by considering a discretization that we call the *merged discretization*. Unfortunately, if $d \geq 2$ and

$\delta \geq 1$, then the merged discretization does not satisfy desirable estimates, as intimated by the fact that such estimates have a constant of the form $O(1)^{1/(\delta(1-\delta))}$ for $\delta < 1$ in [Dyatlov and Jin 2018].

We now construct a discretization which is more appropriate to our setting. Given a compact convex set I and a real number $\alpha > 0$, we denote by $I\alpha$ the dilation of I by α from its barycenter. For sets $A, B \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$, we use the ℓ^∞ Hausdorff distance,

$$\begin{aligned} \text{dist}_\infty(x, A) &:= \inf_{a \in A} |a - x|_{\ell^\infty}, \\ \text{dist}_\infty(A, B) &:= \max\left(\sup_{a \in A} \text{dist}_\infty(a, B), \sup_{b \in B} \text{dist}_\infty(b, A)\right), \end{aligned}$$

where, for points $x = (x_i)$ and $y = (y_i)$, we have $|x - y|_{\ell^\infty} := \max_{1 \leq i \leq d} |x_i - y_i|$. Note that $\text{dist}_\infty(x, A) \neq \text{dist}_\infty(\{x\}, A)$ in general. We recall that for the Hausdorff distance we have the triangle inequalities: for every $x \in \mathbb{R}^d$ and every subset $X \subset \mathbb{R}^d$,

$$\text{dist}_\infty(x, A) \leq \text{dist}_\infty(x, B) + \text{dist}_\infty(A, B), \quad \text{dist}_\infty(X, A) \leq \text{dist}_\infty(X, B) + \text{dist}_\infty(A, B).$$

Proposition 3.3. *For every compact set $X \subset \mathbb{R}^d$, $N \in \mathbb{N}$, and $L \geq 10^3$, there is a discretization $V(X)$ of X such that, for all $I \in V_n(X)$ and $1 \leq n \leq N$,*

- there exists $I^0 \in V_n^0(X)$ such that

$$I^0(1 - L^{-2/3}) \subset I \subset I^0(1 + L^{-2/3}), \tag{3-1}$$

- and there exists a point x_0 in $X \cap I$ such that

$$\text{dist}_\infty(x_0, \partial I) \geq \frac{1}{10} L^{-2/3-n}. \tag{3-2}$$

We call this discretization the *perturbed standard discretization*, and we call elements of the perturbed standard discretization *tiles* (to emphasize that they may not be cubes).

Remark 3.4. Christ [1990] constructed dyadic cubes with similar properties for metric spaces with a doubling measure μ as in Definition 1.3. It’s possible that the construction there can also be applied to prove Theorem 1.4. Our construction is less general but does not rely on the existence of a doubling measure.

3.2. Constructing the new discretization. We prove Proposition 3.3 in this section.

3.2.1. Preliminaries. We establish some terminology and notation that we will use in the construction of the new discretization. Let I be a cube such that $\bar{I} = [a_1, b_1] \times \cdots \times [a_d, b_d]$. For $1 \leq k \leq d$, define the k -boundary

$$\partial^k I := \bigcup_{j_1, \dots, j_k} [a_1, b_1] \times \cdots \times [a_{j_i}, b_{j_i}] \times \cdots \times [a_d, b_d].$$

In other words, $\partial^k I$ is the union of all $(d-k)$ -dimensional faces of I , so that $\partial^1 I = \partial I$ and $\partial^d I$ is the set of all vertices of I . For a set $A \in \mathbb{R}^d$, $r > 0$, let the ℓ^∞ ball around A with radius r be

$$B_\infty(A, r) = \{x \in \mathbb{R}^d : \exists a \in A, |a - x|_{\ell^\infty} < r\}.$$

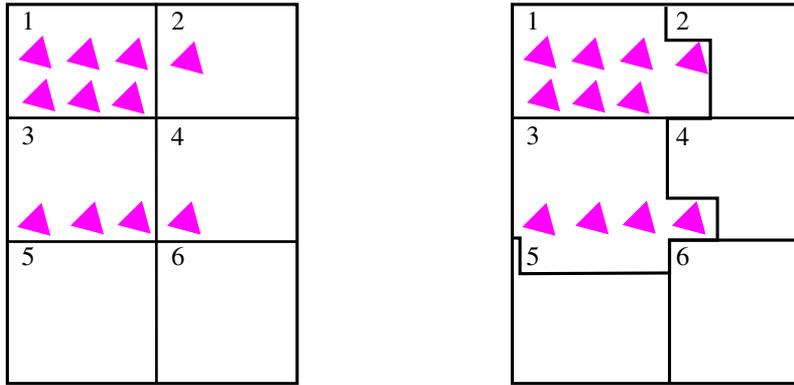


Figure 2. A standard (left) and perturbed standard (right) discretization. On the left, cube 1 is type 2, cubes 2 and 3 are type 1, cube 4 is type 0, and cubes 5 and 6 are type -1 ; on the right, tiles 1 and 3 are good and all other tiles are type -1 .

We stress that a B without a subscript refers to the ℓ^2 ball, and in particular that the balls in the definition of nonorthogonality are ℓ^2 balls!

For a subset $P \subset \partial^k I$ of the k -boundary of a cube I , suppose without loss of generality that

$$P \subseteq \{a_1, b_1\} \times \cdots \times \{a_k, b_k\} \times [a_{k+1}, b_{k+1}] \times \cdots \times [a_d, b_d].$$

In that case, we define the *tubular neighborhoods*

$$B_\infty^t(P, r) := \{x \in \mathbb{R}^d : \text{there exists } y = (y_i) \in P, \\ |x_1 - y_1| < r, \dots, |x_k - y_k| < r, x_{k+1} = y_{k+1}, \dots, x_d = y_d\} \quad (3-3)$$

and

$$B_\infty^t(P, r_1, r_2) := B_\infty^t(P, r_1) \cup B_\infty^t(P, r_2). \quad (3-4)$$

We will use these tubular neighborhoods to modify the standard cubes. Note that $B_\infty^t(P, r_1, r_2)$ has the advantage that $\text{dist}_\infty(x, \partial B_\infty^t(P, r_1, r_2)) \geq \min(r_1, r_2)$ for any $x \in P$.

Let $V^0(X)$ be the standard discretization.

Definition 3.5. For every $n \leq N$ and $I \in V_n^0(X)$, the *type* of I is defined as follows:

- I is of type d if there exists a point $x \in X \cap I$ such that $\text{dist}_\infty(x, \partial I) > \frac{1}{2}L^{-2/3-n}$.
- If $d - 1 > 0$, I is of type $d - 1$ if there exists a point $x \in X \cap I$ with $\text{dist}_\infty(x, \partial^2 I) > \frac{1}{2}L^{-2/3-n}$ but I is not of type d .
- If $d - 2 > 0$, I is of type $d - 2$ if there exists a point $x \in X \cap I$ with $\text{dist}_\infty(x, \partial^3 I) > \frac{1}{2}L^{-2/3-n}$ but I is not of type $\geq d - 1$.
-
- I is of type 0 if $X \cap I$ is nonempty and $\text{dist}_\infty(X \cap I, \partial^d I) \leq \frac{1}{2}L^{-2/3-n}$.
- I is of type -1 if $X \cap I$ is empty.

See [Figure 2](#) for an example of cube types.

We want to modify the cubes $I \in V_n^0(X)$ into tiles T , so that there exists $x_0 \in X \cap T$ satisfying

$$\text{dist}_\infty(x_0, \partial T) \geq \frac{1}{5}L^{-2/3-n}. \tag{3-5}$$

We say that a tile T is *good* if (3-5) holds and otherwise that it is *bad*. For the remainder of the proof, we assume the following.

Invariant 3.6. *If a tile T constructed from a cube I is bad, then $T \subseteq I$.*

This invariant is true at the current stage of the proof; we necessarily have $T = I$ since we have not modified any tiles yet.

We want to do induction on the type of the tiles. In order to do so, we will need a notion of “type” for a bad tile. By Invariant 3.6, in order for type to be well defined, it suffices to define the type of a tile T which was modified from a cube I such that $T \subseteq I$.

Definition 3.7. Let T be a bad tile which was modified from a cube I such that $T \subseteq I$. Assume that I has type k with respect to $X \cap T$; that is, assume that I has type k in $V_n(X \cap T)$, where $V(X \cap T)$ consists of the restriction of elements of $V(X)$ that we are already defined to T . Then the *type* of T is k .

3.2.2. Induction on type. We now induct backwards on the largest type k of a bad tile. At every stage of the induction, we iterate over all bad tiles of type k . At each stage of this iteration, either we do nothing, or we replace a tile T with $T \cup \mathcal{T}$ for a tubular neighborhood \mathcal{T} of some set. In the latter case, we replace all other tiles T' with $T' \setminus \mathcal{T}$. Here a tubular neighborhood \mathcal{T} is always of the form (3-3) or (3-4). In short, we say that we *moved* the set \mathcal{T} . It will be important that we keep track of which sets we have already moved. As such, we make the following inductive assumptions, which are vacuous at the start of the inductive process when $k = d - 1$:

Invariant 3.8. *Every bad tile has type $\leq k$.*

Invariant 3.9. *If a tile T was constructed from a cube I , then $\text{dist}_\infty(\partial T, \partial I) \leq \frac{1}{2}L^{-n-2/3}$.*

Invariant 3.10. *Let \mathcal{T} be the tubular neighborhood of a set P . If \mathcal{T} has been moved, then $\dim P \geq k$.*

Lemma 3.11. *Assume that $0 \leq k \leq d - 1$ and the above set of tiles satisfies Invariants 3.6, 3.8, 3.9, and 3.10. Then we may modify each tile to obtain a new set of tiles satisfying Invariants 3.6, 3.8, 3.9, and 3.10 but with k replaced by $k - 1$.*

Proof. Let T be a bad tile of type k modified from some cube I , and let P be a connected component of $\partial^{d-k}I \setminus B_\infty(\partial^{d-k+1}I, \frac{1}{2}L^{-2/3-n})$ such that $B^t(P, \frac{1}{5}L^{-n-2/3}) \cap X \cap T \neq \emptyset$. We modify the adjacent tiles to P :

- (1) If there is a good tile $T' \neq T$ adjacent to P , then we enlarge T' to contain the tubular neighborhood $\mathcal{T} := B_\infty^t(P, \frac{1}{2}L^{-2/3-n}) \cap (T' \cup T)$. Then:
 - (a) T' is still good.
 - (b) T no longer contains P .
 - (c) Since \mathcal{T} is contained in $T' \cup T$, no other tile is affected.

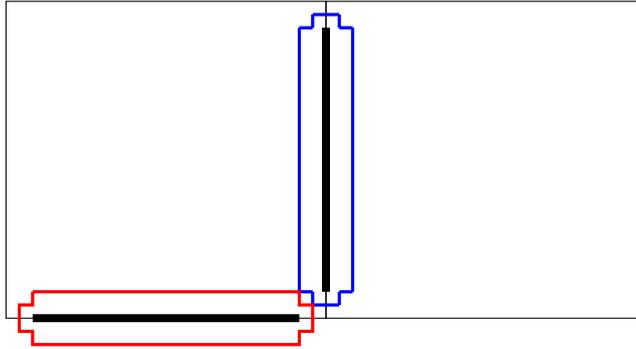


Figure 3. The proof of [Lemma 3.11](#), Case (2). The boldest black lines represent components P of the boundary. The tubular neighborhoods around them do not intersect.

- (2) Otherwise, by [Invariant 3.8](#), every tile adjacent to P has type $\leq k$. In this case, we enlarge T by a tubular neighborhood $\mathcal{T} := B_\infty^t(P, \frac{1}{2}L^{-2/3-n}, \frac{1}{4}L^{-2/3-n})$. Then:
- (a) Let \mathcal{T}' be a tubular neighborhood of a set P' which we already moved. Then:
- (i) If $\dim P' > k$, then we claim that \mathcal{T} is disjoint from \mathcal{T}' . If this is not true, then let T' be the tile containing \mathcal{T}' . Then T' is adjacent to P . By [Invariant 3.6](#), T' is good, which contradicts the fact that we are in Case (2).
 - (ii) If $\dim P' = k$, then \mathcal{T} is disjoint from \mathcal{T}' . See [Figure 3](#).
 - (iii) We cannot have $\dim P' < k$, by [Invariant 3.10](#).

Therefore \mathcal{T} is disjoint from all tubular neighborhoods which were already moved.

- (b) T becomes good.
(c) Every tile $T' \neq T$ adjacent to P no longer contains P .

We iterate the above procedure over all possible components P , stopping once there are no more components to consider. This happens after finitely many stages because of the following facts:

- (1) If a tubular neighborhood of a component P is absorbed by a tile T of type k and its other neighboring tile is T' , then T becomes good, and P can no longer witness that T' has type $\geq k$. Therefore we will not iterate over P again.
- (2) At each stage, no new bad tiles are created, and no bad tiles are given more points and remain bad. Therefore [Invariants 3.6](#) and [3.8](#) are preserved.
- (3) [Invariant 3.9](#) is preserved because if T was constructed from I then we only modify T in a neighborhood of distance $\frac{1}{2}L^{-n-2/3}$ of ∂I .
- (4) [Invariant 3.10](#) is preserved because we only moved tubular neighborhoods of sets of dimension k .

After iterating over all possible components P , [Invariant 3.8](#) is improved, so that every bad tile has type $\leq k - 1$. Indeed, if T is still bad and was type k , then every tubular neighborhood of a component P which could witness that T had type k was absorbed into a neighboring tile, so T must have type $\leq k - 1$. \square

After stage $k = 0$, every bad tile has type -1 by [Invariant 3.8](#). However, if T is a tile of type -1 , then by definition $X \cap T \cap I$ is empty. Then, by [Invariant 3.6](#), $X \cap T$ is empty, and we may discard the tile T entirely.

Let $\tilde{V}_n(X)$ be the set of good tiles that were constructed from $V_n(X)$ by the above procedure. Then every tile in $\tilde{V}_n(X)$ satisfies [\(3-5\)](#) and

$$X = \bigsqcup_{T \in \tilde{V}_n(X)} T \cap X.$$

However, $\tilde{V}(X)$ may not have a tree structure, so it is not a discretization.

3.2.3. Obtaining a tree structure. We now modify $\tilde{V}(X)$ to be a discretization $V(X)$. We again proceed by induction. For $n > N$, let $V_n(X) = V_n^0(X)$. Now suppose that $n \leq N$ and we have constructed $(V_m(X))_{m \geq n+1}$ to be a discretization of X . For each element $T \in \tilde{V}_n(X)$, we define subsets $\mathcal{C}(T)$ of $V_{n+1}(X)$ as follows:

- The subsets $\mathcal{C}(T)$ are all disjoint and their disjoint union is $V_{n+1}(X)$.
- If $S \in V_{n+1}(X)$ and $S \subseteq T$, then $S \in \mathcal{C}(T)$.
- If $S \in V_{n+1}(X)$ intersects multiple T , then we pick one T for which S lies in $\mathcal{C}(T)$.

We now define $V_n(X) = \{\bigcup_{S \in \mathcal{C}(T)} S : T \in \tilde{V}_n(X)\}$. Thus, for each $I \in V_n(X)$, there exists an element $T \in \tilde{V}_n(X)$ such that

$$\text{dist}_\infty(\partial T, \partial I) \leq 2L^{-n-1} \leq \frac{1}{10}L^{-n-2/3}$$

(where the second inequality is because $L \geq 10^3$), and, for $x \in T$ satisfying [\(3-5\)](#), $x \in I$. Then, for every $x \in X$, there exists a unique $I' \in V_{n+1}(X)$ containing x by our inductive assumption, and a unique $I \in V_n(X)$ which is a superset of I' by the fact that $\{\mathcal{C}(T) : T \in \tilde{V}_n(X)\}$ is a partition of $V_{n+1}(X)$. It follows that $(V_m(X))_{m \geq n}$ is a discretization of X .

By construction, there exists $x_0 \in X \cap T$ satisfying [\(3-5\)](#); hence

$$\text{dist}_\infty(x_0, \partial I) \geq \text{dist}_\infty(x_0, \partial T) - \text{dist}_\infty(\partial I, \partial T) \geq \left(\frac{1}{5} - \frac{1}{10}\right)L^{-n-2/3} = \frac{1}{10}L^{-n-2/3},$$

and hence x_0 satisfies [\(3-2\)](#). If we denote by I^0 the cube that we modified to create T , then, by [Invariant 3.9](#),

$$\text{dist}_\infty(\partial I^0, \partial I) \leq \text{dist}_\infty(\partial I^0, \partial T) + \text{dist}_\infty(\partial T, \partial I) \leq \left(\frac{1}{2} + \frac{1}{10}\right)L^{-n-2/3},$$

which one can use to show [\(3-1\)](#). This completes the proof of [Proposition 3.3](#).

3.3. Regularity of the discretization. We now show that if the compact set X is the support of a doubling measure then its perturbed standard discretization $V(X)$ satisfies regularity conditions similar to those established in [\[Dyatlov and Jin 2018, Lemma 2.1\]](#) for the merged discretization in the case $d = 1$.

We begin by showing that every pair of tiles $(I, J) \in V_n(X) \times V_m(Y)$ has children which contain points for which the estimate

$$|\Phi(x_0, y_0) - \Phi(x_0, y_1) - \Phi(x_1, y_0) + \Phi(x_1, y_1)| \gtrsim |x_0 - x_1| \cdot |y_0 - y_1|$$

holds. This is the key new estimate needed in the higher-dimensional case.

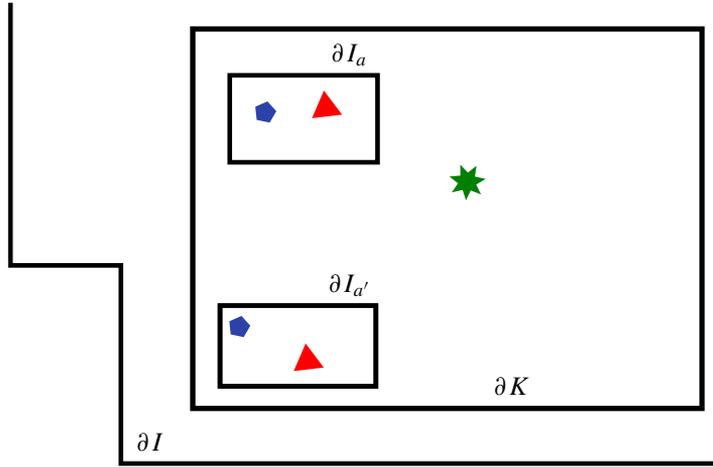


Figure 4. A typical situation in the proof of Lemma 3.12. The child tiles $I_a, I_{a'}$ are contained in the cube $K \subset I$ and are much smaller than I . The green 7-pointed star denotes \underline{x} , the red triangles denote $\tilde{x}_a, \tilde{x}_{a'}$, and the blue pentagons denote $x_a, x_{a'}$.

Lemma 3.12. *Let $\Phi \in C^3(\mathbb{R}^d \times \mathbb{R}^d)$, and let $X, Y \subseteq \mathbb{R}^d$ be Φ -nonorthogonal with constant c_N from scales (L^{-K_X}, L^{-K_Y}) to 1. Let $V(X), V(Y)$ be the perturbed standard discretizations of X, Y . Then, for*

$$L \geq \max(180^3, 10^{10} c_N^{-3} \|\partial_{xy}^2 \Phi\|_{C^1}^3) d^{3/2} \tag{3-6}$$

and every $n < K_X, m < K_Y, I \in V_n(X), J \in V_m(Y)$, there exist children $I_a, I_{a'}$ of I and $J_b, J_{b'}$ of J such that, for every $x_\alpha \in I_\alpha, y_\beta \in J_\beta$, and $\omega_{\alpha\beta} := \Phi(x_\alpha, y_\beta)$, we have

$$\frac{c_N}{1000} \leq L^{m+n+4/3} |\omega_{ab} - \omega_{a'b} - \omega_{ab'} + \omega_{a'b'}| \leq \frac{\|\partial_{xy}^2 \Phi\|_{C^0}}{20} \tag{3-7}$$

and

$$L^{n+2/3} |x_a - x_{a'}|, L^{m+2/3} |y_b - y_{b'}| \leq \frac{1}{2}. \tag{3-8}$$

Moreover, we may assume:

$$\text{for any } x_a \in I_a \text{ and } x_{a'} \in I_{a'}, \text{ the line segment } \overline{x_a x_{a'}} \text{ always lies in } I. \tag{3-9}$$

Proof. By Proposition 3.3, we may choose $\underline{x} \in X \cap I$ and $\underline{y} \in Y \cap J$ such that

$$\min(L^{n+2/3} \text{dist}_\infty(\underline{x}, \partial I), L^{m+2/3} \text{dist}_\infty(\underline{y}, \partial J)) \geq \frac{1}{10}.$$

Let $r_X = \frac{1}{20} L^{-n-2/3}$ and $r_Y = \frac{1}{20} L^{-m-2/3}$. One can show that, if (3-6) holds, then $L \geq 20^3$ and

$$(1 + 2L^{-2/3})^4 \leq \frac{9}{8}. \tag{3-10}$$

Since $n \leq K_X - 1$ and $L \geq 20^3$,

$$r_X = \frac{1}{20} L^{-n-2/3} \geq \frac{1}{20} L^{-K_X} L^{1/3} \geq L^{-K_X},$$

and similarly $r_Y \geq L^{-K_Y}$. By nonorthogonality, there exist $\tilde{x}_a, \tilde{x}_{a'} \in X \cap B(\underline{x}, r_X)$ and $\tilde{y}_b, \tilde{y}_{b'} \in Y \cap B(\underline{y}, r_Y)$ such that for $\tilde{\omega}_{\alpha\beta} := \Phi(\tilde{x}_\alpha, \tilde{y}_\beta)$,

$$|\tilde{\omega}_{ab} - \tilde{\omega}_{a'b} - \tilde{\omega}_{ab'} + \tilde{\omega}_{a'b'}| \geq c_N r_X r_Y. \tag{3-11}$$

In the other direction, (2-3) and the triangle inequality gives

$$|\tilde{\omega}_{ab} - \tilde{\omega}_{a'b} - \tilde{\omega}_{ab'} + \tilde{\omega}_{a'b'}| \leq \|\partial_{xy}^2 \Phi\|_{C^0} \cdot |\tilde{x}_a - \tilde{x}_{a'}| \cdot |\tilde{y}_b - \tilde{y}_{b'}|. \tag{3-12}$$

Let I_α be the child of I containing \tilde{x}_α and J_β be the child of J containing \tilde{y}_β . Pick arbitrary points $x_\alpha \in I_\alpha$ and $y_\beta \in J_\beta$. We first use (3-1), (3-10), and (3-6) to bound

$$\begin{aligned} |x_a - x_{a'}| &\leq 2r_X + \text{diam } I_a + \text{diam } I_{a'} \\ &\leq \frac{1}{10}L^{-n-2/3} + 2d^{1/2}L^{-n-1}(1 + 2L^{-2/3})^2 \\ &\leq \frac{1}{10}L^{-n-2/3} + 5d^{1/2}L^{-n-1} \\ &\leq \frac{1}{2}L^{-n-2/3}. \end{aligned}$$

A similar estimate holds on $|y_b - y_{b'}|$, which proves the upper bound in (3-8).

To prove (3-7), let $c_x := 2r_X$, $c_y := 2r_Y$, $\varepsilon_x := \max(\text{diam } I_a, \text{diam } I_{a'})$, $\varepsilon_y := \max(\text{diam } J_b, \text{diam } J_{b'})$. Then, by (2-3), Lemma 2.2, (3-1), (3-10), and (3-6),

$$\begin{aligned} |\omega_{ab} - \omega_{ab'} - \omega_{a'b} + \omega_{a'b'} - \tilde{\omega}_{ab} + \tilde{\omega}_{ab'} + \tilde{\omega}_{a'b} - \tilde{\omega}_{a'b'}| &\leq 7\|\partial_{xy}^2 \Phi\|_{C^1}(c_x \varepsilon_y + c_y \varepsilon_x) \\ &\leq \frac{7}{5}\|\partial_{xy}^2 \Phi\|_{C^1}(d^{1/2}L^{-n-m-5/3}(1 + 2L^{-2/3})^2) \\ &\leq 2\|\partial_{xy}^2 \Phi\|_{C^1}d^{1/2}L^{-n-m-5/3}. \end{aligned}$$

Combining this estimate with (3-11) and (3-6), we obtain

$$\begin{aligned} |\omega_{ab} - \omega_{ab'} - \omega_{a'b} + \omega_{a'b'}| &\geq |\tilde{\omega}_{ab} - \tilde{\omega}_{a'b} - \tilde{\omega}_{ab'} + \tilde{\omega}_{a'b'}| - |\omega_{ab} - \omega_{ab'} - \omega_{a'b} + \omega_{a'b'} - \tilde{\omega}_{ab} + \tilde{\omega}_{ab'} + \tilde{\omega}_{a'b} - \tilde{\omega}_{a'b'}| \\ &\geq \frac{1}{400}c_N L^{-n-m-4/3} - 2\|\partial_{xy}^2 \Phi\|_{C^1}d^{1/2}L^{-n-m-5/3} \\ &\geq \frac{1}{1000}c_N L^{-n-m-4/3}, \end{aligned}$$

which is the desired lower bound in (3-7). For the upper bound, since $\tilde{x}_a, \tilde{x}_{a'} \in B(\underline{x}, r_X)$ and $\tilde{y}_b, \tilde{y}_{b'} \in B(\underline{y}, r_Y)$ we use (2-3):

$$\begin{aligned} |\omega_{ab} - \omega_{ab'} - \omega_{a'b} + \omega_{a'b'}| &\leq 4\|\partial_{xy}^2 \Phi\|_{C^0}(r_X + \sqrt{d}L^{-n-1})(r_Y + \sqrt{d}L^{-m-1}) \\ &\leq \frac{1}{20}\|\partial_{xy}^2 \Phi\|_{C^0}L^{-n-m-4/3}. \end{aligned}$$

Finally we prove (3-9). We use (3-1), (3-10), (3-6), and the fact that $\text{dist}_\infty(a, b) \leq |a - b|$ to estimate

$$\text{dist}_\infty(x_a, \underline{x}) \leq \text{dist}_\infty(x_a, \tilde{x}_a) + \text{dist}_\infty(\tilde{x}_a, \underline{x}) \leq 2L^{-n-1} + r_X \leq \frac{1}{15}L^{-n-2/3}.$$

The same bound holds for $x_{a'}$, and it follows that $x_a, x_{a'}$ are contained in the convex set

$$K := B_\infty(\underline{x}, \frac{1}{15}L^{-n-2/3}).$$

In particular, $\ell := \overline{x_a x_{a'}}$ satisfies $\ell \subset K$. This implies $\ell \subset I$ since

$$\text{dist}_\infty(\partial K, \partial I) \geq \text{dist}_\infty(\underline{x}, \partial I) - \frac{1}{15}L^{-n-2/3} \geq \frac{1}{30}L^{-n-2/3},$$

so that $K \subseteq I$. □

We now give a probabilistic interpretation of the above lemmas. To establish notation, suppose that $I \in V_n(X)$ for some compact set X and some n . We write $\{I_a : a \in A\}$ for the set of children of I . This induces the structure of a probability space on A ; namely,

$$\Pr(a) := \frac{\mu_X(I_a)}{\mu_X(I)}.$$

Proposition 3.13. *Let $\Phi \in C^3(\mathbb{R}^d \times \mathbb{R}^d)$, and suppose that L satisfies (3-6). Let (X, μ_X) be doubling with constant $C_D(X)$ on scales $[L^{-K_X}, 1]$, let (Y, μ_Y) be doubling with constant $C_D(Y)$ on scales $[L^{-K_Y}, 1]$, let $V(X), V(Y)$ be their perturbed standard discretizations, and assume that (X, Y) is Φ -nonorthogonal with constant c_N from scales (L^{-K_X}, L^{-K_Y}) to 1, $n < K_X, m < K_Y, I \in V_n(X)$, and $J \in V_m(Y)$, and $\{I_a : a \in A\}$ and $\{J_b : b \in B\}$ are the sets of children of I, J . Furthermore, choose, for each $a \in A$ and $b \in B, x_a \in I_a$ and $y_b \in J_b$, and set $\omega_{ab} := \Phi(x_a, y_b)$.*

Draw independent random outcomes $a, a' \in A$ and $b, b' \in B$. Then, with probability

$$\rho \geq C_D(X)^{-2\lceil \log_2(20L^{5/3}) \rceil} C_D(Y)^{-2\lceil \log_2(20L^{5/3}) \rceil}, \tag{3-13}$$

we have

$$\frac{1}{1000}c_N L^{-1/3} \leq L^{n+m+1} |\omega_{ab} - \omega_{a'b} - \omega_{ab'} + \omega_{a'b'}| \leq \pi \tag{3-14}$$

and

$$L^{n+2/3} |x_a - x_{a'}|, L^{m+2/3} |y_b - y_{b'}| \leq \frac{1}{2}. \tag{3-15}$$

Moreover, we may assume,

$$\text{for any } x_a \in I_a \text{ and } x_{a'} \in I_{a'}, \text{ the line segment } \overline{x_a x_{a'}} \text{ always lies in } I. \tag{3-16}$$

Proof. By Lemma 3.12, there exist a, b, a', b' satisfying (3-7) and (3-8). By definition of the perturbed standard discretization, there exists $x_* \in I_a \cap X$ with $I_* := \frac{1}{10}B_\infty(x_*, L^{-n-5/3}) \subset I_a$. Moreover,

$$I \subset B_\infty(x_0, 2L^{-n}) = I_*(20L^{5/3}).$$

Therefore,

$$\Pr(a) = \frac{\mu_X(I_a)}{\mu_X(I)} \geq \frac{\mu_X(I_*)}{\mu_X(I_*(20L^{5/3}))} \geq C_D(X)^{-\lceil \log_2(20L^{5/3}) \rceil}.$$

We have analogous lower bounds on $\Pr(b), \Pr(a')$, and $\Pr(b')$. Then, by independence,

$$\rho \geq \Pr(a) \Pr(a') \Pr(b) \Pr(b'),$$

which gives (3-13), and (3-7) and (3-8) clearly imply (3-15) and the lower bound on (3-14). The condition (3-16) comes from (3-9). For the upper bound we apply (3-7) and (3-6). □

4. The induction on scales

We now begin the proof of [Theorem 1.4](#). Let $\Phi \in C^3(\mathbb{R}^d \times \mathbb{R}^d)$ and $p \in C^1(\mathbb{R}^d \times \mathbb{R}^d)$ be the phase and symbol of \mathcal{B}_h , and let $K := \lfloor -\log_L h \rfloor$.

Let $(X, \mu_X), (Y, \mu_Y)$ be doubling with constants $C_D(X), C_D(Y)$ on scales $\geq h$, let $V(X), V(Y)$ be their perturbed standard discretizations, and assume that (X, Y) is Φ -nonorthogonal with constant c_N from scales (h, h) to 1.

For $I \in V_n(X)$ and $J \in V_m(Y)$, where $n + m + 1 = K$, we set

$$F_J(x) = \frac{1}{\mu_Y(J)} \int_J \exp\left(i \frac{\Phi(x, y) - \Phi(x, y_J)}{h}\right) p(x, y) f(y) d\mu_Y(y).$$

Here y_J is the center of J^0 , the box in the standard discretization associated to J . Let $\{I_a : a \in A\}$ and $\{J_b : b \in B\}$ be sets of children with their usual probability measures. Let $x_a := \arg \max_{I_a} |F_J|$ and $y_b := y_{J_b}$.

4.1. Mean value space. We need to generalize the space $C_\theta(I)$ where $d = 1$ (see [\[Dyatlov and Jin 2018, §2.2\]](#) and also [\[Naud 2005, Lemma 5.4\]](#)), which is supposed to locally measure oscillation on I whilst also being “scale-invariant”.³ This will allow us to get some gain out of the cancellation obtained from nonorthogonality while performing induction on scales.

Definition 4.1. Given $I \in V_n(X)$ and $\theta \in (0, 1)$, we define the $C_\theta(I)$ norm for functions $f \in C^1(I)$ by

$$\|f\|_{C_\theta(I)} := \max(\|f\|_{C^0(I)}, \theta \operatorname{diam}(I) \|\nabla f\|_{C^0(I)}).$$

Given $J \in V_m(Y)$, we define $\Psi_b : I \rightarrow \mathbb{R}$ as

$$\Psi_b(x) := \frac{\Phi(x, y_{J_b}) - \Phi(x, y_J)}{h}.$$

Lemma 4.2. *Let*

$$\theta \leq \frac{1}{8 \max(1, \|\partial_{xy}^2 \Phi\|_{C^0(I_{\text{conv}})})}$$

(where I_{conv} is the convex hull of I) and $L \geq 10$. Then, for $f \in C_\theta(I)$,

$$\|e^{i\Psi_b} f\|_{C_\theta(I_a)} \leq \|f\|_{C_\theta(I)}. \tag{4-1}$$

Proof. Observe that if ψ is a smooth function on $I_{a,\text{conv}}$ then any $f \in C_\theta(I_a)$ satisfies

$$|\nabla(e^{i\psi} f)| = |ie^{i\psi} f \nabla \psi + e^{i\psi} \nabla f| \leq |f \nabla \psi| + |\nabla f|.$$

Hence

$$\theta \operatorname{diam}(I_a) |\nabla(e^{i\Psi_b} f)(x)| \leq \theta \operatorname{diam}(I_a) \|\nabla \Psi_b\|_{C^0(I_a)} \|f\|_{C^0(I_a)} + \theta \operatorname{diam}(I_a) \|\nabla f\|_{C^0(I_a)}.$$

³We cannot use the space $C^1(I)$ with its norm $\|f\|_{C^1(I)} := \|f\|_{C^0(I)} + \|\nabla f\|_{C^0(I)}$, because the first and second terms in the norm will scale differently if we rescale I .

We estimate that

$$|\nabla \Psi_b(x)| = \frac{1}{h} |\partial_x(\Phi(x, y_{J_b}) - \Phi(x, y_J))| \leq \frac{1}{h} |y_{J_b} - y_J| \|\partial_{xy}^2 \Phi\|_{C^0(I_{\text{conv}})} \leq \frac{\text{diam}(J)}{h} \|\partial_{xy}^2 \Phi\|_{C^0(I_{\text{conv}})}.$$

So, by hypothesis on θ and L ,

$$\begin{aligned} \theta \text{diam}(I_a) \|\nabla \Psi_b\|_{C^0(I_a)} \|f\|_{C^0(I_a)} &\leq \theta \frac{\text{diam}(I_a) \text{diam}(J)}{h} \|\partial_{xy}^2 \Phi\|_{C^0(I_{\text{conv}})} \|f\|_{C^0(I)} \\ &\leq \theta (1 + L^{-2/3})^2 \|\partial_{xy}^2 \Phi\|_{C^0(I_{\text{conv}})} \|f\|_{C_\theta(I)} \\ &\leq \frac{1}{4} \|f\|_{C_\theta(I)}. \end{aligned}$$

In addition, by hypothesis on L ,

$$\theta \text{diam}(I_a) \|\nabla f\|_{C^0(I_a)} \leq \frac{2}{L} \theta \text{diam}(I) \|\nabla f\|_{C^0(I)} \leq \frac{1}{4} \|f\|_{C_\theta(I)}.$$

Summing up,

$$\theta \text{diam}(I_a) \|\nabla(e^{i\Psi_b} f)\|_{C^0(I_a)} \leq \|f\|_{C_\theta(I)}.$$

We also trivially have

$$\|f\|_{C^0(I_a)} \leq \|f\|_{C^0(I)} \leq \|f\|_{C_\theta(I)},$$

which proves (4-1). □

4.2. Inductive step. Our next task is to prove the following analogue of [Dyatlov and Jin 2018, Lemma 3.2].

Proposition 4.3. *Let $I \in V_n(X)$, $J \in V_m(Y)$, where $n + m + 1 = K$. Draw a random $b \in B$, and assume that (3-14) and (3-15) hold with probability ρ . Assume that*

$$L \geq \max\left(\frac{10^{12}d^3}{c_N^3\theta^{3/2}}, \frac{10^{10}\|\partial_{xy}^2 \Phi\|_{C^1}^3 d^{3/2}}{c_N^3}\right), \tag{4-2}$$

$$\varepsilon_1 \leq \frac{\rho^2 c_N^2}{10^9 d^2 L^{2/3}}. \tag{4-3}$$

Then we have the improvement

$$\mathbb{E}_{a \in A} \|F_J\|_{C_\theta(I_a)}^2 \leq (1 - \varepsilon_1) \mathbb{E}_{b \in B} \|F_{J_b}\|_{C_\theta(I)}^2. \tag{4-4}$$

By Proposition 3.13, we can always choose L and $\varepsilon_1 > 0$ such that the hypotheses of this proposition are met.

4.2.1. The contradiction assumption. We set up the proof of Proposition 4.3 by first recording

$$F_J = \mathbb{E}_{b \in B} e^{i\Psi_b} F_{J_b}. \tag{4-5}$$

We have the following lemma which is nearly identical to [Dyatlov and Jin 2018, Lemma 3.3].

Lemma 4.4. *For each $a \in A$,*

$$\|F_J\|_{C_\theta(I_a)}^2 \leq \left(\mathbb{E}_{b \in B} \|F_{J_b}\|_{C_\theta(I)}\right)^2 \leq \mathbb{E}_{b \in B} \|F_{J_b}\|_{C_\theta(I)}^2. \tag{4-6}$$

Proof. By (4-1),

$$\|e^{i\Psi_b} F_{J_b}\|_{C_\theta(I_a)} \leq \|F_{J_b}\|_{C_\theta(I)}.$$

The assertions of (4-6) now follow from (4-5) and the Cauchy–Schwarz inequality. □

We set $R := \mathbb{E}_{b \in B} \|F_{J_b}\|_{C_\theta(I)}^2$. Draw $a \in A$ independently of b . Taking expectations in (4-6), we obtain

$$\sigma^2 := \mathbb{E}_{b \in B} \|F_{J_b}\|_{C_\theta(I)}^2 - \mathbb{E}_{a \in A} \|F_J\|_{C_\theta(I_a)}^2 \geq \text{Var}_{b \in B} \|F_{J_b}\|_{C_\theta(I)}. \tag{4-7}$$

In particular, (4-7) can be written as

$$\sigma^2 = R - \mathbb{E}_{a \in A} \|F_J\|_{C_\theta(I_a)}^2.$$

If we knew that $\sigma^2 \geq \varepsilon_1 R$, then the improvement (4-4) would follow. So, we assume towards a contradiction that

$$\sigma^2 < \varepsilon_1 R. \tag{4-8}$$

Let

$$F_{ab} := F_{J_b}(x_a), \quad \omega_{ab} := \Psi_b(x_a), \quad f_{ab} := e^{i\omega_{ab}} F_{ab}. \tag{4-9}$$

Note carefully that ω_{ab} disagrees with the phase in Proposition 3.13 by a factor of h . We compute

$$F_J(x_a) = \mathbb{E}_{b \in B} f_{ab} \tag{4-10}$$

and, for each $a \in A$,

$$\mathbb{E}_{b \in B} |F_{ab}|^2 \leq \mathbb{E}_{b \in B} \|F_{J_b}\|_{C_\theta(I)}^2 = R. \tag{4-11}$$

4.2.2. Outline of the proof. By our contradiction assumption (4-8) and variance bound (4-7), the $C_\theta(I)$ norms of the functions F_{J_b} are all almost independent of b . One can show that f_{ab} is almost independent of b (see (4-12)). By the mean value theorem, F_{ab} does not vary too much in a (see (4-23)). However, the events (3-14) and (3-15) have positive probability, so we may condition on them without losing too much, and, after conditioning, the phases of f_{ab} and $f_{a'b'}$ cannot be too correlated by (3-14) and (3-15). So we expect cancellation between f_{ab} and $f_{a'b'}$ whenever a, a', b, b' are drawn at random by the square-root cancellation heuristic. This cancellation implies that the conditional expectation of $|F_{ab}|^2$ is both very small and comparable to R , a contradiction.

4.2.3. Two unconditional moment estimates. We now make two unconditional moment estimates; we shall later use Cantelli’s inequality to show that weaker versions of the same moment estimates hold even when we condition on the events (3-14) and (3-15).

Lemma 4.5. *One has*

$$\mathbb{E}_{a \in A} \text{Var}_{b \in B} f_{ab} \leq \mathbb{E}_{\substack{a \in A \\ b \in B}} |F_{ab}|^2 - R + 2\sigma^2 \leq 2\sigma^2, \tag{4-12}$$

$$\mathbb{E}_{\substack{a \in A \\ b \in B}} |F_{ab}| \geq (1 - 2\varepsilon_1)\sqrt{R}. \tag{4-13}$$

Proof. We follow [Dyatlov and Jin 2018, Lemma 3.5]. By Lemma 4.2, for each a, b ,

$$\theta \|\nabla(e^{i\Psi_b} F_{J_b})\|_{C^0(I_a)} \text{diam } I_a \leq \frac{1}{2} \|F_{J_b}\|_{C_\theta(I)}.$$

From the definition of $C_\theta(I_a)$, (4-5), and the triangle inequality, for each $a \in A$,

$$\begin{aligned} \|F_J\|_{C_\theta(I_a)} &= \max(\|F_J\|_{C^0(I_a)}, \theta \|\nabla F_J\|_{C^0(I_a)} \text{diam } I_a) \\ &\leq \max\left(\|F_J\|_{C^0(I_a)}, \theta \text{diam } I_a \mathbb{E}_{b \in B} \|\nabla(e^{i\Psi_b} F_{J_b})\|_{C^0(I_a)}\right) \\ &\leq \max\left(\|F_J\|_{C^0(I_a)}, \frac{1}{2} \mathbb{E}_{b \in B} \|F_{J_b}\|_{C_\theta(I)}\right). \end{aligned}$$

We estimate the squares of the two terms in the maximum using (4-6):

$$\|F_J\|_{C^0(I_a)}^2 \leq \frac{1}{2} (\|F_J\|_{C^0(I_a)}^2 + \|F_J\|_{C_\theta(I_a)}^2) \leq \frac{1}{2} (\|F_J\|_{C^0(I_a)}^2 + R)$$

and

$$\left(\frac{1}{2} \mathbb{E}_{b \in B} \|F_{J_b}\|_{C_\theta(I)}\right)^2 \leq \frac{1}{4} \mathbb{E}_{b \in B} \|F_{J_b}\|_{C_\theta(I)}^2 \leq \frac{1}{4} R \leq \frac{1}{2} (\|F_J\|_{C^0(I_a)}^2 + R).$$

In summary, we have

$$\|F_J\|_{C_\theta(I_a)}^2 \leq \frac{1}{2} (\|F_J\|_{C^0(I_a)}^2 + R). \tag{4-14}$$

After taking expectations and applying (4-7), we get

$$\mathbb{E}_{a \in A} \|F_J\|_{C^0(I_a)}^2 \geq 2 \mathbb{E}_{a \in A} \|F_J\|_{C_\theta(I_a)}^2 - R = R - 2\sigma^2. \tag{4-15}$$

We also record that, by (4-5), (4-10), and the fact that x_a maximizes $|F_J|$,

$$\mathbb{E}_{a \in A} \left| \mathbb{E}_{b \in B} f_{ab} \right| = \mathbb{E}_{a \in A} \left| \mathbb{E}_{b \in B} e^{i\Psi_b(x_a)} F_{J_b}(x_a) \right| = \mathbb{E}_{a \in A} |F_J(x_a)| = \mathbb{E}_{a \in A} \|F_J\|_{C^0(I_a)}. \tag{4-16}$$

Combining this fact with (4-15),

$$\mathbb{E}_{a \in A} \left| \mathbb{E}_{b \in B} f_{ab} \right|^2 \geq R - 2\sigma^2.$$

Therefore,

$$\mathbb{E}_{\substack{a \in A \\ b \in B}} |F_{ab}|^2 = \mathbb{E}_{\substack{a \in A \\ b \in B}} |f_{ab}|^2 = \mathbb{E}_{a \in A} \left(\left| \mathbb{E}_{b \in B} f_{ab} \right|^2 + \text{Var}_{b \in B} f_{ab} \right) \geq R - 2\sigma^2 + \mathbb{E}_{a \in A} \text{Var}_{b \in B} f_{ab}. \tag{4-17}$$

Rearranging, we obtain

$$\mathbb{E}_{a \in A} \text{Var}_{b \in B} f_{ab} \leq \mathbb{E}_{\substack{a \in A \\ b \in B}} |F_{ab}|^2 - R + 2\sigma^2. \tag{4-18}$$

Then (4-12) follows from (4-11).

To obtain (4-13), we first estimate

$$\mathbb{E}_{\substack{a \in A \\ b \in B}} |F_{ab}| = \mathbb{E}_{\substack{a \in A \\ b \in B}} |f_{ab}| \geq \mathbb{E}_{a \in A} \left| \mathbb{E}_{b \in B} f_{ab} \right| = \mathbb{E}_{a \in A} \sqrt{\mathbb{E}_{b \in B} |f_{ab}|^2 - \text{Var}_{b \in B} f_{ab}}. \tag{4-19}$$

From (4-17) and (4-11), and the contradiction assumption (4-8),

$$\begin{aligned} \mathbb{E}_{a \in A} \sqrt{\mathbb{E}_{b \in B} |f_{ab}|^2 - \text{Var}_{b \in B} f_{ab}} &\geq \frac{1}{\sqrt{\max_{a \in A} \mathbb{E}_{b \in B} |f_{ab}|^2}} \mathbb{E}_{a \in A} \left(\mathbb{E}_{b \in B} |f_{ab}|^2 - \text{Var}_{b \in B} f_{ab} \right) \\ &\geq \mathbb{E}_{a \in A} \frac{\mathbb{E}_{b \in B} |f_{ab}|^2 - \text{Var}_{b \in B} f_{ab}}{\sqrt{R}} \geq \frac{R - 2\sigma^2}{\sqrt{R}} \\ &\geq (1 - 2\varepsilon_1)\sqrt{R}. \end{aligned} \quad \square$$

4.2.4. Drawing random nonorthogonal tiles. By (4-6) and the Cauchy–Schwarz inequality,

$$\mathbb{E}_{b \in B} \|F_{J_b}\|_{C_\theta(I)} \leq \sqrt{R}. \tag{4-20}$$

Let T be the event that $\|F_{J_b}\|_{C_\theta(I)} \leq 2\sqrt{R}$. By the moment bounds (4-20) and (4-7), the contradiction assumption (4-8), and Cantelli’s inequality (2-2),

$$\Pr(T) > 1 - \varepsilon_1. \tag{4-21}$$

We let T' be the respective event for b' , where a', b' are drawn independently from a, b . From (4-12), (4-21), and (2-1), we obtain

$$\begin{aligned} \mathbb{E}_{\substack{a \in A \\ b, b' \in B}} (|f_{ab} - f_{ab'}|^2 \mid T \cap T') &\leq \frac{1}{\Pr(T \cap T')} \mathbb{E}_{\substack{a \in A \\ b, b' \in B}} |f_{ab} - f_{ab'}|^2 \\ &\leq \frac{2}{\Pr(T)^2} \mathbb{E}_{a \in A} \text{Var}_{b \in B} f_{ab} \leq 2.5 \cdot 2\sigma^2 = 5\sigma^2. \end{aligned} \tag{4-22}$$

If T and (3-16) hold, then, by Lemma 4.2,

$$|F_{ab} - F_{a'b}| \leq \frac{2\sqrt{R}}{\theta} L^{H(I)} |x_a - x_{a'}|. \tag{4-23}$$

Let S be the intersection of T, T' , and the events (3-14), (3-15) and (3-16). By (4-3), $\varepsilon_1 \leq \frac{1}{10}\rho$, so by (4-21),

$$\frac{\Pr(S)}{\Pr(T)^2} \geq \frac{\rho - 2(1 - \Pr(T))}{\Pr(T)^2} \geq \frac{\rho - 2\varepsilon_1}{(1 - \varepsilon_1)^2} \geq \frac{1}{2}\rho. \tag{4-24}$$

If S holds, then by (4-23) and (3-15),

$$|F_{ab} - F_{a'b}| \leq \frac{\sqrt{R}}{L^{2/3}\theta}. \tag{4-25}$$

4.2.5. Conditional second moment bounds. We now use (4-24) and (4-25) to obtain lower and upper bounds on $\mathbb{E}(|F_{ab}|^2 \mid S)$ which are not both tenable.

Lemma 4.6. For $M := 8000000$,

$$\mathbb{E}_{\substack{a \in A \\ b \in B}} (|F_{ab}|^2 \mid S) \leq Md^2 \left(\frac{R}{c_N^2 L^{2/3}\theta} + 2 \frac{L^{2/3}\sigma^2}{c_N^2 \rho} \right). \tag{4-26}$$

Proof. We take all expectations and probabilities over a, a', b, b' . Write

$$\tau := \omega_{ab} - \omega_{ab'} - \omega_{a'b} + \omega_{a'b'};$$

so if S holds then

$$|e^{i\tau} - 1|^2 \geq |\tau|^2 \geq 10^{-6} c_N^2 L^{-2/3}$$

by (3-14) and [Dyatlov and Jin 2018, Lemma 2.6]. Following [Dyatlov and Jin 2018, 19], we rewrite (recalling the notation set in (4-9))

$$\begin{aligned} |(e^{i\tau} - 1)F_{ab}| &= |e^{i(\omega_{ab} - \omega_{ab'})} F_{ab} - e^{i(\omega_{a'b} - \omega_{a'b'})} F_{ab}| \\ &= |e^{-i\omega_{ab'}}(f_{ab} - f_{ab'}) + F_{ab'} - F_{a'b'} - e^{-i\omega_{a'b'}}(f_{a'b} - f_{a'b'}) + e^{i(\omega_{a'b} - \omega_{a'b'})}(F_{a'b} - F_{ab})|. \end{aligned}$$

So by the triangle inequality in L^2 ,

$$\mathbb{E}(|(e^{i\tau} - 1)F_{ab}|^2 | S) \leq 4 \mathbb{E}(|F_{ab} - F_{a'b}|^2 + |F_{a'b'} - F_{ab'}|^2 + |f_{ab} - f_{ab'}|^2 + |f_{a'b'} - f_{a'b}|^2 | S).$$

So

$$\begin{aligned} \mathbb{E}(|F_{ab}|^2 | S) &\leq 10^6 \cdot \frac{d^2 L^{2/3}}{c_N^2} \mathbb{E}(|(e^{i\tau} - 1)F_{ab}|^2 | S) \\ &\leq \frac{Md^2 L^{2/3}}{2c_N^2} \mathbb{E}(|F_{ab} - F_{a'b}|^2 + |F_{a'b'} - F_{ab'}|^2 | S) + \frac{Md^2 L^{2/3}}{2c_N^2} \mathbb{E}(|f_{ab} - f_{ab'}|^2 + |f_{a'b'} - f_{a'b}|^2 | S). \end{aligned}$$

Applying (4-25),

$$|F_{ab} - F_{a'b}|^2 + |F_{a'b'} - F_{ab'}|^2 \leq \frac{2R}{L^{4/3}\theta}.$$

Since S implies $T \cap T'$ and a, a' are independent,

$$\mathbb{E}(|f_{ab} - f_{ab'}|^2 + |f_{a'b'} - f_{a'b}|^2 | S) \leq 2 \frac{\Pr(T)^2}{\Pr(S)} \mathbb{E}(|f_{ab} - f_{ab'}|^2 | T \cap T').$$

By (4-24), $\Pr(T)^2 / \Pr(S) \leq 2/\rho$. Summing all this up and applying (4-22), we conclude (4-26). □

Lemma 4.7. *One has*

$$\mathbb{E}_{\substack{a \in A \\ b \in B}} (|F_{ab}|^2 | S) \geq \frac{1}{6} R. \tag{4-27}$$

Proof. By (4-13), we conclude that

$$\Pr_{a \in A} \left(\mathbb{E}_{b \in B} |F_{ab}| < (1 - 2\sqrt{\epsilon_1})\sqrt{R} \right) \leq \sqrt{\epsilon_1}.$$

By Cantelli's inequality (2-2),

$$\Pr_{b \in B} \left(|F_{ab}| \leq \mathbb{E}_{b \in B} |F_{ab}| - \frac{1}{2}\sqrt{R} \right) \leq \frac{\text{Var}_{b \in B} |F_{ab}|}{\text{Var}_{b \in B} |F_{ab}| + \frac{1}{4}R}.$$

Since $|F_{ab}| = |f_{ab}|$, it follows from (4-12) and (4-8) that

$$\begin{aligned} \Pr(|F_{ab}|^2 \leq \frac{1}{5}R) &\leq \Pr\left(\mathbb{E}_{a \in A} \mathbb{E}_{b \in B} |F_{ab}| < (1 - 2\sqrt{\epsilon_1})\sqrt{R}\right) + \Pr\left(\mathbb{E}_{b \in B} |F_{ab}| \geq (1 - 2\sqrt{\epsilon_1})\sqrt{R}, |F_{ab}|^2 \leq \frac{1}{5}R\right) \\ &\leq \sqrt{\epsilon_1} + \mathbb{E}_{a \in A} \Pr\left(|F_{ab}| \leq \mathbb{E}_{b \in B} |F_{ab}| - \frac{1}{2}\sqrt{R}\right) \\ &\leq \sqrt{\epsilon_1} + \frac{4 \mathbb{E}_{a \in A} \text{Var}_{b \in B} f_{ab}}{R} \\ &\leq \sqrt{\epsilon_1} + \frac{8\sigma^2}{R} < 2\sqrt{\epsilon_1}. \end{aligned}$$

But by (4-24),

$$\Pr(|F_{ab}|^2 \leq \frac{1}{5}R \mid S) = \frac{\Pr((|F_{ab}|^2 \leq \frac{1}{5}R) \cap S)}{\Pr(S)} \leq \frac{2 \Pr(|F_{ab}|^2 \leq \frac{1}{5}R)}{\rho}.$$

The definition (4-3) of ϵ_1 then implies

$$\Pr(|F_{ab}|^2 \leq \frac{1}{5}R \mid S) \leq \frac{4\sqrt{\epsilon_1}}{\rho} < L^{-1/3}.$$

Therefore

$$\Pr(|F_{ab}|^2 \geq \frac{1}{5}R \mid S) \geq 1 - L^{-1/3},$$

so by Markov's inequality and the assumption (4-2),

$$\mathbb{E}_{\substack{a \in A \\ b \in B}} (|F_{ab}|^2 \mid S) \geq \frac{1}{5}R \Pr(|F_{ab}|^2 \geq \frac{1}{5}R \mid S) \geq \frac{1}{6}R. \quad \square$$

4.2.6. Deriving a contradiction. The two above conditional second moment bounds contradict (4-2) and (4-3) and the contradiction assumption (4-8). To be more precise, combining (4-26) with (4-27) and (4-8), we obtain

$$\frac{1}{6}R \leq \mathbb{E}_{\substack{a \in A \\ b \in B}} (|F_{ab}|^2 \mid S) \leq Md^2 \left(\frac{R}{c_N^2 L^{2/3} \theta} + \frac{2L^{2/3} \sigma^2}{c_N^2 \rho} \right) < Md^2 \left(\frac{R}{c_N^2 L^{2/3} \theta} + \frac{2L^{2/3} \epsilon_1 R}{c_N^2 \rho} \right).$$

Dividing both sides by RM and applying (4-2) and (4-3), we obtain

$$2 \cdot 10^{-8} < \frac{1}{48 \cdot 10^6} = \frac{1}{6M} \leq \frac{d^2}{c_N^2 L^{2/3} \theta} + \frac{2d^2 L^{2/3} \epsilon_1}{c_N^2 \rho} \leq \frac{1}{10^8} + \frac{2}{10^9} = 1.2 \cdot 10^{-8}.$$

This is a contradiction that proves that $\sigma^2 \geq \epsilon_1 R$ and so completes the proof of Proposition 4.3.

4.3. Proof of main theorem. To prove Theorem 1.4, we iterate Proposition 4.3. For each J , we define

$$E_J : V_{K-H(J)}(X) \rightarrow \mathbb{R}, \quad I \mapsto \|F_J\|_{C_\theta(I)}.$$

We endow $V_n(X)$ with the discrete measure induced by μ_X , namely $\mu_X(\{I\}) = \mu_X(I)$, and J with the restricted fractal measure μ_J .

First suppose that $J \in V_K(Y)$. Then, by the Cauchy–Schwarz inequality, it follows that

$$\begin{aligned} |\nabla F_J(x)| &= \frac{1}{\mu_Y(J)} \int_J i \partial_x \Psi_J(x, y) \exp(i(\Psi_J(x, y))) p(x, y) f(x, y) \\ &\quad + \exp(i(\Psi_J(x, y))) \partial_x p(x, y) f(y) \, d\mu_Y(y) \\ &\leq \frac{1}{\sqrt{\mu_Y(J)}} \left(\frac{\text{diam } J}{h} \|\partial_{xy}^2 \Phi\|_{C^0} \|f\|_{L^2(J)} \|p\|_{C^0} + \|\partial_x p\|_{C^0} \|f\|_{L^2(J)} \right) \end{aligned}$$

and

$$\|F_J\|_{C^0} \leq \frac{\|p\|_{C^0} \|f\|_{L^2(J)}}{\sqrt{\mu_Y(J)}}.$$

Thus,

$$E_J(I) = \|F_J\|_{C_\theta(I)} \leq \frac{\|p\|_{C^1} \|f\|_{L^2(J)}}{\sqrt{\mu_Y(J)}}. \tag{4-28}$$

Taking L^2 norms of both sides of (4-28), we get

$$\|E_J\|_{L^2}^2 \leq \frac{\|p\|_{C^1}^2 \mu_X(X)}{\mu_Y(J)} \|f\|_{L^2(J)}^2. \tag{4-29}$$

If we take L^2 norms of both sides of (4-4), we get

$$\|E_J\|_{L^2}^2 \leq (1 - \varepsilon_1) \mathbb{E}_{b \in B} \|E_{J_b}\|_{L^2}^2. \tag{4-30}$$

Inducting backwards on $H(J)$ with (4-29) as base case and (4-30) as inductive case, we conclude that, if J is a tile in Y such that $H(J) = 0$,

$$\|E_J\|_{L^2}^2 \leq \frac{\|p\|_{C^1}^2 \mu_X(X)}{\mu_Y(J)} (1 - \varepsilon_1)^K \|f\|_{L^2(J)}^2.$$

Summing both sides in J , we obtain (note $\|\mathcal{B}_h(1_J f)\|_{L^2} \leq \mu_Y(J) \|E_J\|_{L^2}$ and $\mu_X(X) = \mu_Y(Y) = 1$)

$$\|\mathcal{B}_h f\|_{L^2}^2 \lesssim \|p\|_{C^1}^2 \mu_X(X) \mu_Y(Y) (1 - \varepsilon_1)^K \|f\|_{L^2}^2 \lesssim (1 - \varepsilon_1)^K \|f\|_{L^2}^2.$$

We now can set

$$\varepsilon_0 := \frac{\varepsilon_1}{6 \log L} \leq \frac{\log(1 - \varepsilon_1)^{-1}}{2 \log L}$$

and plug in θ in (4-2) to obtain (1-5) and (1-6). Then $(1 - \varepsilon_1)^{K/2} \leq h^{\varepsilon_0}$, so

$$\|\mathcal{B}_h\|_{L^2(\mu_Y) \rightarrow L^2(\mu_X)} \lesssim h^{\varepsilon_0},$$

which completes the proof of [Theorem 1.4](#).

5. Applications

5.1. Classical fractal uncertainty principle. We now prove [Theorem 1.5](#) following [[Dyatlov and Jin 2018](#), [Theorem 1](#), [Remarks 1](#)]. The classical version of the fractal uncertainty principle, [Theorem 1.5](#), uses Lebesgue measure. In order to use our main [Theorem 1.4](#), we need to define a rescaling of the Lebesgue measure. This is the content of the following lemma.

Lemma 5.1. *Let (X, μ) be δ -regular on scales $[h, 1]$, $h > 0$, where $\delta \in [0, d]$ and μ is the δ -dimensional Hausdorff measure. Let $X_h := X + B_h$ and*

$$\mu_h(A) := h^{\delta-d} |X_h \cap A|.$$

Then (X_h, μ_h) is δ -regular on scales $[2h, 1]$ with constant

$$C_R(X_h) := 6^d \max(|\mathbb{B}^d|, |\mathbb{B}^d|^{-1}) C_R(X)^2.$$

Here $|X_h \cap A|$ is the Lebesgue measure of $X_h \cap A$ and \mathbb{B}^d is the unit ball in \mathbb{R}^d .

Proof. Let $N = N_X(x, r, h)$ be the cardinality of a maximal h -separated subset of $X \cap B(x, r)$ for $x \in X$ and $r \geq 2h$. By [Dyatlov and Zahl 2016, Lemma 7.4], we have

$$C_R(X)^{-2} \frac{r^\delta}{h^\delta} \leq N_X(x, r, h) \leq C_R(X)^2 \left(1 + \frac{2r}{h}\right)^\delta.$$

If $\{x_1, \dots, x_N\}$ is such a maximal set and $I_n := B(x_n, 2h)$, then $X_h \cap B(x, r) \subseteq \bigcup_{n=1}^N I_n$, so we have

$$\mu_h(B(x, r)) \leq h^{\delta-d} \sum_{n=1}^N |I_n| \leq 2^d h^\delta |\mathbb{B}^d| N \leq 2^d |\mathbb{B}^d| C_R(X)^2 (h + 2r)^\delta \leq C_R(X_h) r^\delta.$$

Conversely, if $J_n := B(x_n, \frac{1}{2}h)$, then J_n and J_m are disjoint and $\bigcup_{n=1}^N J_n \subseteq X_h \cap B(x, r)$, so we have

$$\mu_h(B(x, r)) \geq \sum_{n=1}^N h^{\delta-d} |J_n| \geq N \frac{h^\delta}{2^d} |\mathbb{B}^d| \geq C_R(X)^{-2} 2^{-d} |\mathbb{B}^d| r^\delta \geq C_R(X_h)^{-1} r^\delta. \quad \square$$

We now show that the nonorthogonality condition also holds for (X_h, Y_h) .

Lemma 5.2. *Let (X, Y) be Φ -nonorthogonal on scales $[h, 1]$, $h > 0$. Then (X_h, Y_h) is Φ -nonorthogonal on scales $[2h, 1]$ with constant $c_N(X_h, Y_h) := \frac{1}{4} c_N(X, Y)$.*

Proof. Let $x_0 \in X_h$, $y_0 \in Y_h$, and $r_X, r_Y \geq 2h$; then there exist $\tilde{x}_0 \in X$ and $\tilde{y}_0 \in Y$ with

$$\max(|x_0 - \tilde{x}_0|, |y_0 - \tilde{y}_0|) \leq h.$$

Putting $\tilde{r}_X := r_X - h$ and $\tilde{r}_Y := r_Y - h$, we can find by Φ -nonorthogonality of (X, Y) points

$$x_1, x_2 \in X \cap B(\tilde{x}_0, \tilde{r}_X) \subseteq X \cap B(x_0, r_X)$$

and

$$y_1, y_2 \in Y \cap B(\tilde{y}_0, \tilde{r}_Y) \subseteq Y \cap B(y_0, r_Y)$$

such that

$$|\Phi(x_1, y_1) - \Phi(x_1, y_2) - \Phi(x_2, y_1) + \Phi(x_2, y_2)| \geq c_N(X) \tilde{r}_X \tilde{r}_Y \geq c_N(X_h) r_X r_Y. \quad \square$$

Proof of Theorem 1.5. We define $(X_h, \mu_{X,h})$, $(Y_h, \mu_{Y,h})$ as in Lemma 5.1 and introduce the Fourier integral operator

$$\mathcal{B}_h f(\xi) := \int_{Y_h} e^{ix \cdot \xi / h} f(x) d\mu_{Y,h}(x).$$

By the above lemmas, $(X_h, \mu_{X,h})$ is δ -regular, $(Y_h, \mu_{Y,h})$ is δ' -regular, and (X_h, Y_h) is Φ -nonorthogonal. Thus, by [Theorem 1.4](#),⁴ there exists $\varepsilon_0 > 0$ such that

$$\|1_{X_h} \mathcal{F}_h 1_{Y_h}\|_{L^2 \rightarrow L^2} = \frac{h^{(d-\delta-\delta')/2}}{(2\pi)^{d/2}} \|\mathcal{B}_h\|_{L^2(\mu_{Y,h}) \rightarrow L^2(\mu_{X,h})} \lesssim h^{(d-\delta-\delta')/2+\varepsilon_0}. \quad \square$$

5.2. Convex cocompact hyperbolic manifolds. In this section we prove [Theorem 1.6](#). First we recall some preliminaries for convex cocompact hyperbolic manifolds.

Let \mathbb{H}^{d+1} be the $(d+1)$ -dimensional hyperbolic space (with constant curvature -1). The orientation preserving isometry group is given by $G = \text{SO}(d+1, 1)_0$. Let $K = \text{SO}(d+1)$ be a maximal compact subgroup, so that $\mathbb{H}^{d+1} = G/K$. We are interested in infinite volume hyperbolic manifolds given by $M = \Gamma \backslash G/K$, where $\Gamma \subset G$ is a convex cocompact Zariski-dense torsion-free discrete subgroup.

The *limit set* is defined as $\Lambda(\Gamma) := \overline{\Gamma x} \cap \partial_\infty(\mathbb{H}^{d+1}) \subset \overline{\mathbb{H}^{d+1}}$ for any $x \in \mathbb{H}^{d+1}$ (one can show that the definition is independent of the choice of x). Let $\text{Hull}(\Lambda(\Gamma))$ be the convex hull of $\Lambda(\Gamma)$ in \mathbb{H}^{d+1} . Then Γ is called *convex cocompact* if the convex core $\text{Core}(M) := \Gamma \backslash \text{Hull}(\Lambda(\Gamma)) \subset M$ is compact, and Γ is *Zariski dense* if Γ is not contained in the zero set of some nontrivial polynomial on $\text{SO}(d+1, 1)_0$. We identify the sphere \mathbb{S}^d with $\mathbb{R}^d \cup \{\infty\}$. In the Poincaré upper half-space model, the limit set $\Lambda(\Gamma) \subset \mathbb{S}^d$ is a compact set of dimension $\delta_\Gamma \in (0, d)$ (see [[Sarkar and Winter 2021](#), §2]), and we may assume that $\Lambda(\Gamma)$ is a compact subset of \mathbb{R}^d .

We recall the following nonconcentration property from [[Sarkar and Winter 2021](#), Proposition 6.6].

Proposition 5.3. *Let $\Gamma \subset G$ be a convex cocompact subgroup such that Γ is Zariski dense in G . Then there exists $c_0 > 0$ such that, for any $x \in \Lambda(\Gamma) \cap \mathbb{R}^d$, $\varepsilon \in (0, 1)$, and $w \in \mathbb{R}^d$ with $|w| = 1$, there exists $y \in \Lambda(\Gamma) \cap B(x, \varepsilon)$ such that*

$$|\langle y - x, w \rangle| > c_0 \varepsilon. \tag{5-1}$$

As a corollary we have the following.

Corollary 5.4. *Let M be a convex cocompact hyperbolic $(d+1)$ -dimensional manifold such that Γ is Zariski dense in G . Then, for any $\Phi \in C^3(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R})$ such that $\partial_{xy}^2 \Phi(x, y)$ is nonvanishing, the pair $(\Lambda(\Gamma), \Lambda(\Gamma))$ is Φ -nonorthogonal with some constant $c_N > 0$ from scales 0 to 1.*

Proof. By the mean value theorem, for $x_1, x_2 \in B(x_0, r_X)$ and $y_1, y_2 \in B(y_0, r_Y)$,

$$|\Phi(x_0, y_0) - \Phi(x_1, y_0) - \Phi(x_0, y_1) + \Phi(x_1, y_1) - \langle \partial_{xy}^2 \Phi(x_0, y_0)(x_1 - x_0), y_1 - y_0 \rangle| \leq \|\Phi\|_{C^3} r_X r_Y (r_X + r_Y).$$

Let $H = \ker(\partial_{xy}^2 \Phi(x_0, y_0))$ and v be a unit normal vector to H (if $H = \{0\}$, then we choose v arbitrarily). By [Proposition 5.3](#), there exists $x_1 \in \Lambda(\Gamma) \cap B(x_0, r_X)$ such that $|\langle x_1 - x_0, v \rangle| > c_0 r_X$. This would imply, for some $c_1 \in (0, 1)$, that

$$|\partial_{xy}^2 \Phi(x_0, y_0)(x_1 - x_0)| > c_1 c_0 r_X.$$

By [Proposition 5.3](#) again, there exists $y_1 \in \Lambda(\Gamma) \cap B(y_0, r_Y)$ such that

$$|\langle \partial_{xy}^2 \Phi(x_0, y_0)(x_1 - x_0), y_1 - y_0 \rangle| > c_1 c_0^2 r_X r_Y.$$

⁴The fact that regularity and nonorthogonality only hold up to scale $2h$ causes us to incur a loss of a power of 2, but this is irrelevant.

Thus we may choose $r_X, r_Y \leq \frac{1}{10} c_1 c_0^2 \|\Phi\|_{C^3}^{-1}$ such that

$$|\Phi(x_0, y_0) - \Phi(x_1, y_0) - \Phi(x_0, y_1) + \Phi(x_1, y_1)| > \frac{1}{2} c_1 c_0^2 r_X r_Y,$$

i.e., nonorthogonality holds with

$$c_N = \frac{c_1^3 c_0^6}{200(1 + \|\Phi\|_{C^3})^2} > 0. \quad \square$$

Proof of Theorem 1.6. As before, we may assume that $\Lambda(\Gamma)$ is contained in a compact region of $\mathbb{R}^d \subset \mathbb{S}^d$. On such a region, the stereographic projection $\varphi : \mathbb{S}^d \setminus \{\infty\} \rightarrow \mathbb{R}^d$ is bounded in C^3 with C^3 -bounded inverse. Consider the measure

$$\mu_h(A) := h^{\delta_\Gamma - d} |\Lambda(\Gamma)_h \cap A|,$$

which is defined using the Euclidean metric but is comparable to the measure defined using the spherical metric due to the bounds on the stereographic projection. By Lemma 5.1, μ_h is δ_Γ -regular on scales $[2h, 1]$.

Let $\chi \in C_0^\infty(\mathbb{S}^d \times \mathbb{S}^d \setminus \{(x, x) : x \in \mathbb{S}^d\})$, and define $B_\chi(h) : L^2(\mathbb{S}^d) \rightarrow L^2(\mathbb{S}^d)$ by

$$B_\chi(h)u(x) = (2\pi h)^{-d/2} \int_{\mathbb{S}^d} |x - y|^{2i/h} \chi(x, y)u(y) dy.$$

Then $1_{\Lambda(\Gamma)_h} B_\chi(h) 1_{\Lambda(\Gamma)_h}$ can be rewritten as $(2\pi h)^{-d/2} h^{d - \delta_\Gamma} \mathcal{B}_h$, where \mathcal{B}_h is the operator studied in Theorem 1.4 with $p(y') dy' = \chi(\varphi^{-1}(y))\varphi_*(dy)$, $\mu_X = \mu_Y = \mu_h$, and $\Phi(x, y) = 2 \log |\varphi^{-1}(x) - \varphi^{-1}(y)|$. Combining Theorem 1.4 with the bounds on the stereographic projection and Corollary 5.4, we conclude the fractal uncertainty bound

$$\|1_{\Lambda(\Gamma)_h} B_\chi(h) 1_{\Lambda(\Gamma)_h}\|_{L^2(\mathbb{S}^d) \rightarrow L^2(\mathbb{S}^d)} \leq Ch^{d/2 - \delta_\Gamma + \varepsilon_0}.$$

By a covering argument as in [Bourgain and Dyatlov 2018, Proposition 4.2], we have, for $\rho \in (0, 1)$,

$$\|1_{\Lambda(\Gamma)_{h\rho}} B_\chi(h) 1_{\Lambda(\Gamma)_{h\rho}}\|_{L^2(\mathbb{S}^d) \rightarrow L^2(\mathbb{S}^d)} \leq Ch^{d/2 - \delta_\Gamma + \varepsilon_0 - 2(1-\rho)}.$$

Thus, $\Lambda(\Gamma)$ satisfies the fractal uncertainty principle with exponent $\beta = \frac{1}{2}d - \delta_\Gamma + \varepsilon_0$ in the sense of [Dyatlov and Zahl 2016, Definition 1.1]. Applying [Dyatlov and Zahl 2016, Theorem 3], we conclude that the Laplacian on M has only finitely many resonances in $\{\text{Im } \lambda > \delta_\Gamma - \frac{1}{2}d - \varepsilon_0 + \varepsilon\}$ for any $\varepsilon > 0$, proving Theorem 1.6. \square

Appendix: The nonorthogonality constant of a classical Schottky group

In this appendix we demonstrate a simple way to estimate the nonorthogonality constant for classical Schottky groups Γ in $\text{SO}(3, 1)_0 = \text{PSL}(2, \mathbb{C})$, pointed out to us by Qiuyu Ren. The key idea is to use the fact that Möbius transformations are conformal maps and preserve circles in order to derive (5-1).

We illustrate this by considering Schottky groups of genus 2. Let D_1, D_2, D_3, D_4 be four disjoint closed disks in $\mathbb{C}\mathbb{P}^1 = \partial_\infty \mathbb{H}^3$, and let $\gamma_1, \gamma_2 \in \text{PSL}(2, \mathbb{C})$ such that

$$\gamma_1(\overline{D_3^c}) = D_1, \quad \gamma_2(\overline{D_4^c}) = D_2, \quad \gamma_3 = \gamma_1^{-1}, \quad \gamma_4 = \gamma_2^{-1}.$$

Let $\Gamma = \langle \gamma_1, \gamma_2 \rangle$ be the free group generated by γ_1 and γ_2 . Thus, Γ is a Schottky group of genus 2.

We identify $\mathbb{C}\mathbb{P}^1$, \mathbb{S}^2 , and $\mathbb{R}^2 \cup \{\infty\}$. We may assume that the D_i do not contain ∞ and hence are contained in \mathbb{R}^2 . Given vectors $v, w \in \mathbb{R}^2$, let $\angle(v, w)$ denote the angle between v, w . The notion of a circle is not completely invariant under conformal transformations of $\mathbb{C}\mathbb{P}^1$. We recall that a *generalized circle* is either a circle or a line; conformal transformations map generalized circles to generalized circles. We will choose the disks D_1, D_2, D_3, D_4 such that

$$\text{no generalized circle passes through all four disks.} \tag{A-1}$$

The circle taken here is not necessarily a great circle.

Let $\bar{a} \equiv a + 2 \pmod{4}$ for $a \in \mathcal{A} = \{1, 2, 3, 4\}$, so that $\bar{1} = 3, \bar{2} = 4$. The limit set $\Lambda(\Gamma)$ is given by the Cantor-like procedure

$$\Lambda(\Gamma) = \bigcap_{n=1}^{\infty} \bigsqcup_{a \in \mathcal{W}^n} D_a, \quad \mathcal{W}^n = \{a_1 a_2 \cdots a_n \in \mathcal{A}^n : \bar{a}_i \neq a_{i+1}\},$$

where $D_a = \gamma_{a_1}(\gamma_{a_2}(\cdots(\gamma_{a_{n-1}}(D_{a_n}))))$.

The nonorthogonality condition (1-3) follows from the nonconcentration property (5-1). Thus it suffices to find absolute constants $0 < c_1 < 1$ and $\kappa = \kappa(\Gamma) > 0$ such that, for each $x \in \Lambda(\Gamma)$, $\epsilon > 0$, and unit vector $w \in \mathbb{R}^2$, there exists an element $y \in \Lambda(\Gamma) \cap B(x, \epsilon) \setminus B(x, c_1\epsilon)$ such that

$$|\cos \angle(x - y, w)| \geq \kappa.$$

Suppose $x \in D_a = D_{a_0 b}$ and $B(x, \epsilon)$ is roughly of the size of D_{a_0} . Then there are two other disks in D_{a_0} , which we call $D_{a_0 c}$ and $D_{a_0 d}$. By condition (A-1) and conformal invariance of the action of Γ , we know that, for any $y_c \in D_{a_0 c} \cap \Lambda(\Gamma)$ and $y_d \in D_{a_0 d} \cap \Lambda(\Gamma)$,

$$\text{the circle passing through } x, y_c, y_d \text{ lies inside } D_{a_0}. \tag{A-2}$$

A Möbius transformation preserving the unit disk is a composition of a rotation and the map

$$z \mapsto \frac{a - z}{1 - \bar{a}z}.$$

A simple computation shows the angles of the triangle $\Delta(x, y_c, y_d)$ are uniformly lower bounded under conformal maps preserving D_{a_0} if we assume (A-2). This implies that

$$\theta < \angle(y_c - x, y_d - x) < \pi - \theta$$

for some constant θ depending on the initial angles between $\gamma_a(D_b)$, $a \neq \bar{b}$. Thus, by the pigeonhole principle,

$$\max(|\cos \angle(y_c - x, w)|, |\cos \angle(y_d - x, w)|) \geq \cos\left(\frac{\pi - \theta}{2}\right).$$

If we assume, moreover,

for any $b \neq \bar{a} \neq c$, there exist $a' \neq a, b' \neq \bar{a}'$ such that

$$\text{no circle passes through } \gamma_a(D_b), \gamma_a(D_c), \gamma_{a'}(D_{b'}), \text{ and } D_{\bar{a}} \tag{A-3}$$

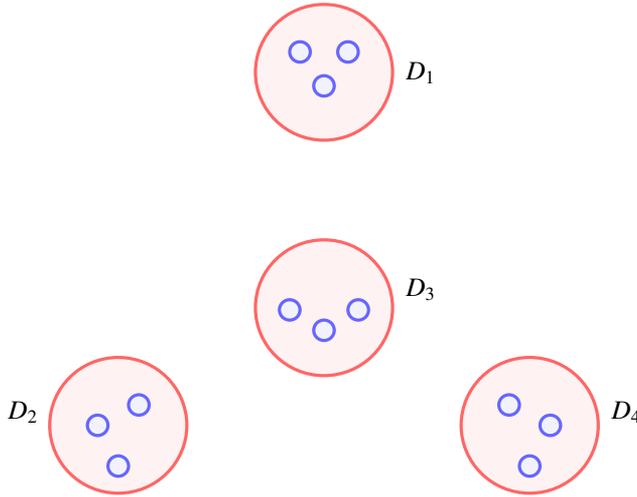


Figure 5. Iteration of disks under a Schottky group.

(which can be achieved if we choose the disks D_a to be small and with generic centers), then we can derive a lower bound on c_1 in a similar way. To be more precise, let $x \in D_a = D_{a_0b} = S_{a_1ab}$ as before; then by assumption (A-3), there exists $a' \neq a$ and $b' \neq \bar{a}'$ such that

$$\text{the circle passing through } D_{a_0b}, D_{a_0c}, \text{ and } D_{a_1a'b'} \text{ lies inside } D_{a_1}. \tag{A-4}$$

In particular, for any $y_{a'b'} \in D_{a_1a'b'}$, the angles of the triangle $\Delta(x, y_c, y_{a'b'})$ are lower bounded. This in particular implies that the length of $\overline{xy_c}$ is comparable to the length of $\overline{y_cy_{a'b'}}$, which by the previous step is comparable with the size of D_{a_0} . This allows us to compute a lower bound of c_1 .

If one runs this procedure carefully, then it would be possible to compute an explicit nonorthogonality constant in terms of the angles between the disks $\gamma_a(D_b)$ in the initial step and the uniform constants in doing conformal transformations.

We do not bother to do the computation here, but we include Figure 5 to indicate how the procedure works. Conformal invariance ensures that the small blue disks always have an angle that lies in $[\theta, \pi - \theta]$.

While one needs to compute the above parameters κ and θ for any given Zariski-dense classical Schottky group Γ , we claim that this is always possible in principle, at least after passing to a finer scale. We say that a pair of words $\mathbf{a}, \mathbf{b} \in \mathcal{W}^n$, $n \in \mathbb{N} \cup \{+\infty\}$, is ε -separated if their weighted Hamming distance satisfies

$$\sum_{i=1}^n \frac{1_{a_i \neq b_i}}{2^i} \geq \varepsilon.$$

Lemma A.1. *Let Γ be a classical Schottky group which is Zariski dense in $\text{PSL}(2, \mathbb{C})$. For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that, for every $n \geq N$ and every triple of words $\mathbf{a}^n, \mathbf{b}^n, \mathbf{c}^n \in \mathcal{W}^n$ which are pairwise ε -separated, there exists $\mathbf{d}^n \in \mathcal{W}^n$ such that, for every circle X which meets all three disks $D_{\mathbf{a}^n}, D_{\mathbf{b}^n}, D_{\mathbf{c}^n}$, we have that X does not meet $D_{\mathbf{d}^n}$.*

Proof. We first prove an analogous result for the set of infinite words \mathcal{W}^∞ and then reduce the finite case to the infinite case. To formulate it, let x_a be the unique point in $\lim_n D_{a_1 \dots a_n}$ (so $a \mapsto x_a$ is a homeomorphism $\mathcal{W}^\infty \rightarrow \Lambda(\Gamma)$, where \mathcal{W}^∞ is given the product topology).

Let $a, b, c \in \mathcal{W}^\infty$ be distinct. Then there is a unique circle $X_{abc} \subset \mathbb{C}\mathbb{P}^1$ passing through x_a, x_b, x_c . We claim that there exists $d \in \mathcal{W}^\infty$ such that $x_d \notin X_{abc}$. Otherwise $\Lambda(\Gamma)$ is contained in a circle, which contradicts [Proposition 5.3](#).

We now address the finite case. Suppose that the lemma fails on some $a^n, b^n, c^n \in \mathcal{W}^n$ for each $n \in \mathbb{N}$ which are ε -separated, so, for every $d^n \in \mathcal{W}^n$, there exists a circle $X(d^n)$ which meets all disks $D_{a^n}, D_{b^n}, D_{c^n}, D_{d^n}$. Let $a, b, c \in \mathcal{W}^\infty$ be the limits of a^n , etc., and let $d \in \mathcal{W}^\infty$ be given. Then $d = \lim_n d^n$ for some sequence $d^n \in \mathcal{W}^n$, and we can define

$$X := \lim_n X(d^n)$$

in Hausdorff distance. Then, $x_a, x_b, x_c, x_d \in X$, and a, b, c are ε -separated and hence distinct. Moreover, X is the limit of circles in $\mathbb{C}\mathbb{P}^1$ whose radii are bounded from below (by ε -separation), so X is a circle, and hence $X = X_{abc}$. This contradicts the infinite case. □

Assuming [Lemma A.1](#), for $D_a = D_{a_1 \dots a_{2n}}$, we can find $b, c \in \mathcal{W}^{2n}$ such that any circle passing through D_a, D_b , and D_c lies in the disk D_{a_1} . This is because, given $D_{a_1 \dots a_{2n}}$ and $\overline{D_{a_1}^c}$, we have

$$\gamma_{\bar{a}_n} \cdots \gamma_{\bar{a}_2} \gamma_{\bar{a}_1}(D_{a_1 \dots a_{2n}}) = D_{a_{n+1} \dots a_{2n}}, \quad \gamma_{\bar{a}_n} \cdots \gamma_{\bar{a}_2} \gamma_{\bar{a}_1}(\overline{D_{a_1}^c}) = D_{\bar{a}_n \dots \bar{a}_2 \bar{a}_1}.$$

By [Lemma A.1](#), there exists $b_0, c_0 \in \mathcal{W}^n$ such that no circle passes through $D_{a_{n+1} \dots a_{2n}}, D_{\bar{a}_n \dots \bar{a}_2 \bar{a}_1}, D_{b_0}$, and D_{c_0} . Applying $\gamma_{a_1} \cdots \gamma_{a_n}$, we conclude any circle passing through

$$D_{a_1 \dots a_{2n}}, \quad D_{a_1 \dots a_n b_0}, \quad D_{a_1 \dots a_n c_0}$$

lies inside D_{a_1} (there might be cancellations for the words $a_1 \cdots a_n b_0$ and $a_1 \cdots a_n c_0$ but one can always pass to a smaller disk). This allows us to compute the angle θ as before for general Zariski-dense classical Schottky groups.

Acknowledgments

The authors would like to thank Semyon Dyatlov for suggesting this problem and for helpful comments on earlier drafts. We would also like to thank Frédéric Naud for suggesting the references [[Sarkar and Winter 2021](#); [Stoyanov 2008](#); [2011](#)], Pratyush Sarkar for suggesting the references [[Guillopé and Zworski 1995](#); [Mazzeo and Melrose 1987](#); [Sarkar 2022](#)], Terence Tao for helpful discussions and for suggesting the reference [[Christ 1990](#)], Qiuyu Ren for proposing the method we use in the [Appendix](#), and Long Jin and Ruixiang Zhang for helpful discussions. We thank the anonymous referees for helpful suggestions to improve the paper.

Backus was supported by the National Science Foundation’s Graduate Research Fellowship Program under Grant No. DGE-2040433, Leng was supported by the NSF’s GRFP under Grant No. DGE-2034835, and Tao was partially supported by the NSF grant DMS-1952939 and Simons Targeted Grant award no. 896630.

References

- [Bourgain and Dyatlov 2017] J. Bourgain and S. Dyatlov, “Fourier dimension and spectral gaps for hyperbolic surfaces”, *Geom. Funct. Anal.* **27**:4 (2017), 744–771. [MR](#) [Zbl](#)
- [Bourgain and Dyatlov 2018] J. Bourgain and S. Dyatlov, “Spectral gaps without the pressure condition”, *Ann. of Math. (2)* **187**:3 (2018), 825–867. [MR](#) [Zbl](#)
- [Bunke and Olbrich 1999] U. Bunke and M. Olbrich, “Group cohomology and the singularities of the Selberg zeta function associated to a Kleinian group”, *Ann. of Math. (2)* **149**:2 (1999), 627–689. [MR](#) [Zbl](#)
- [Chernov 1998] N. I. Chernov, “Markov approximations and decay of correlations for Anosov flows”, *Ann. of Math. (2)* **147**:2 (1998), 269–324. [MR](#) [Zbl](#)
- [Chow and Sarkar 2022] M. Chow and P. Sarkar, “Exponential mixing of frame flows for convex cocompact locally symmetric spaces”, preprint, 2022. [arXiv 2211.14737](#)
- [Christ 1990] M. Christ, “A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral”, *Colloq. Math.* **60/61**:2 (1990), 601–628. [MR](#) [Zbl](#)
- [Cladek and Tao 2021] L. Cladek and T. Tao, “Additive energy of regular measures in one and higher dimensions, and the fractal uncertainty principle”, *Ars Inven. Anal.* (2021), art. id. 1. [MR](#) [Zbl](#)
- [Cohen 2023] A. Cohen, “Fractal uncertainty in higher dimensions”, preprint, 2023. [arXiv 2305.05022](#)
- [Cohen 2025] A. Cohen, “Fractal uncertainty for discrete two-dimensional Cantor sets”, *Anal. PDE* **18**:3 (2025), 743–772. [MR](#) [Zbl](#)
- [Dolgopyat 1998] D. Dolgopyat, “On decay of correlations in Anosov flows”, *Ann. of Math. (2)* **147**:2 (1998), 357–390. [MR](#) [Zbl](#)
- [Dyatlov 2019] S. Dyatlov, “An introduction to fractal uncertainty principle”, *J. Math. Phys.* **60**:8 (2019), art. id. 081505. [MR](#) [Zbl](#)
- [Dyatlov and Jin 2017] S. Dyatlov and L. Jin, “Resonances for open quantum maps and a fractal uncertainty principle”, *Comm. Math. Phys.* **354**:1 (2017), 269–316. [MR](#) [Zbl](#)
- [Dyatlov and Jin 2018] S. Dyatlov and L. Jin, “Dolgopyat’s method and the fractal uncertainty principle”, *Anal. PDE* **11**:6 (2018), 1457–1485. [MR](#) [Zbl](#)
- [Dyatlov and Zahl 2016] S. Dyatlov and J. Zahl, “Spectral gaps, additive energy, and a fractal uncertainty principle”, *Geom. Funct. Anal.* **26**:4 (2016), 1011–1094. [MR](#) [Zbl](#)
- [Guillarmou 2005] C. Guillarmou, “Meromorphic properties of the resolvent on asymptotically hyperbolic manifolds”, *Duke Math. J.* **129**:1 (2005), 1–37. [MR](#) [Zbl](#)
- [Guillopé and Zworski 1995] L. Guillopé and M. Zworski, “Polynomial bounds on the number of resonances for some complete spaces of constant negative curvature near infinity”, *Asymptotic Anal.* **11**:1 (1995), 1–22. [MR](#) [Zbl](#)
- [Han and Schlag 2020] R. Han and W. Schlag, “A higher-dimensional Bourgain–Dyatlov fractal uncertainty principle”, *Anal. PDE* **13**:3 (2020), 813–863. [MR](#) [Zbl](#)
- [Jin and Zhang 2020] L. Jin and R. Zhang, “Fractal uncertainty principle with explicit exponent”, *Math. Ann.* **376**:3–4 (2020), 1031–1057. [MR](#) [Zbl](#)
- [Khalil 2023] O. Khalil, “Exponential mixing via additive combinatorics”, preprint, 2023. [arXiv 2305.00527](#)
- [Khalil 2024] O. Khalil, “Polynomial fourier decay for Patterson–Sullivan measures”, preprint, 2024. [arXiv 2404.09424](#)
- [Li and Pan 2023] J. Li and W. Pan, “Exponential mixing of geodesic flows for geometrically finite hyperbolic manifolds with cusps”, *Invent. Math.* **231**:3 (2023), 931–1021. [MR](#) [Zbl](#)
- [Li et al. 2021] J. Li, F. Naud, and W. Pan, “Kleinian Schottky groups, Patterson–Sullivan measures, and Fourier decay”, *Duke Math. J.* **170**:4 (2021), 775–825. [MR](#) [Zbl](#)
- [Li et al. 2023] J. Li, W. Pan, and P. Sarkar, “Exponential mixing of frame flows for geometrically finite hyperbolic manifolds”, preprint, 2023. [arXiv 2302.03798](#)
- [Liverani 2004] C. Liverani, “On contact Anosov flows”, *Ann. of Math. (2)* **159**:3 (2004), 1275–1312. [MR](#) [Zbl](#)

- [Lugosi 2009] G. Lugosi, “Desigualtats de concentració”, *Butl. Soc. Catalana Mat.* **24**:2 (2009), 97–136, 209. [MR](#)
- [Mazzeo and Melrose 1987] R. R. Mazzeo and R. B. Melrose, “Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature”, *J. Funct. Anal.* **75**:2 (1987), 260–310. [MR](#) [Zbl](#)
- [Naud 2005] F. Naud, “Expanding maps on Cantor sets and analytic continuation of zeta functions”, *Ann. Sci. École Norm. Sup.* (4) **38**:1 (2005), 116–153. [MR](#) [Zbl](#)
- [Oh and Winter 2016] H. Oh and D. Winter, “Uniform exponential mixing and resonance free regions for convex cocompact congruence subgroups of $SL_2(\mathbb{Z})$ ”, *J. Amer. Math. Soc.* **29**:4 (2016), 1069–1115. [MR](#) [Zbl](#)
- [Patterson 1976] S. J. Patterson, “The limit set of a Fuchsian group”, *Acta Math.* **136**:3–4 (1976), 241–273. [MR](#) [Zbl](#)
- [Patterson 1988] S. J. Patterson, “On a lattice-point problem in hyperbolic space and related questions in spectral theory”, *Ark. Mat.* **26**:1 (1988), 167–172. [MR](#) [Zbl](#)
- [Patterson and Perry 2001] S. J. Patterson and P. A. Perry, “The divisor of Selberg’s zeta function for Kleinian groups”, *Duke Math. J.* **106**:2 (2001), 321–390. [MR](#) [Zbl](#)
- [Rössel 2008] J. Rössel, *Image of Sierpinski’s carpet*, 2008, available at [https://w.wiki/EGk\\$](https://w.wiki/EGk$).
- [Sarkar 2022] P. Sarkar, “Generalization of Selberg’s 3/16 theorem for convex cocompact thin subgroups of $SO(n, 1)$ ”, *Adv. Math.* **409** (2022), art. id. 108610. [MR](#) [Zbl](#)
- [Sarkar and Winter 2021] P. Sarkar and D. Winter, “Exponential mixing of frame flows for convex cocompact hyperbolic manifolds”, *Compos. Math.* **157**:12 (2021), 2585–2634. [MR](#) [Zbl](#)
- [Stoyanov 2008] L. Stoyanov, “Spectra of Ruelle transfer operators for contact flows”, *Serdica Math. J.* **34**:1 (2008), 219–252. [MR](#) [Zbl](#)
- [Stoyanov 2011] L. Stoyanov, “Spectra of Ruelle transfer operators for axiom A flows”, *Nonlinearity* **24**:4 (2011), 1089–1120. [MR](#) [Zbl](#)
- [Sullivan 1979] D. Sullivan, “The density at infinity of a discrete group of hyperbolic motions”, *Inst. Hautes Études Sci. Publ. Math.* **50** (1979), 171–202. [MR](#) [Zbl](#)
- [Thurston 1979] W. P. Thurston, “The geometry and topology of three-manifolds”, lecture notes, Princeton University, 1979, available at <https://url.msp.org/gt3m>.
- [Tsujii and Zhang 2023] M. Tsujii and Z. Zhang, “Smooth mixing Anosov flows in dimension three are exponentially mixing”, *Ann. of Math. (2)* **197**:1 (2023), 65–158. [MR](#) [Zbl](#)
- [Vacossin 2023] L. Vacossin, “Resolvent estimates in strips for obstacle scattering in 2D and local energy decay for the wave equation”, *Pure Appl. Anal.* **5**:4 (2023), 1009–1039. [MR](#) [Zbl](#)
- [Vasy 2013a] A. Vasy, “Microlocal analysis of asymptotically hyperbolic and Kerr-de Sitter spaces”, *Invent. Math.* **194**:2 (2013), 381–513. [MR](#) [Zbl](#)
- [Vasy 2013b] A. Vasy, “Microlocal analysis of asymptotically hyperbolic spaces and high-energy resolvent estimates”, pp. 487–528 in *Inverse problems and applications: inside out, II*, edited by G. Uhlmann, Math. Sci. Res. Inst. Publ. **60**, Cambridge Univ. Press, 2013. [MR](#) [Zbl](#)

Received 14 Apr 2023. Revised 27 May 2024. Accepted 20 Jul 2024.

AIDAN BACKUS: aidan_backus@brown.edu

Department of Mathematics, Brown University, Providence, RI, United States

JAMES LENG: jamesleng@math.ucla.edu

Department of Mathematics, University of California, Los Angeles, Los Angeles, CA, United States

ZHONGKAI TAO: ztao@math.berkeley.edu

Department of Mathematics, University of California, Berkeley, Berkeley, CA, United States

Analysis & PDE

msp.org/apde

EDITOR-IN-CHIEF

Clément Mouhot Cambridge University, UK
c.mouhot@dpmms.cam.ac.uk

BOARD OF EDITORS

Massimiliano Berti	Scuola Intern. Sup. di Studi Avanzati, Italy berti@sissa.it	William Minicozzi II	Johns Hopkins University, USA minicozz@math.jhu.edu
Zbigniew Blocki	Uniwersytet Jagielloński, Poland zbigniew.blocki@uj.edu.pl	Werner Müller	Universität Bonn, Germany mueller@math.uni-bonn.de
Charles Fefferman	Princeton University, USA cf@math.princeton.edu	Igor Rodnianski	Princeton University, USA irod@math.princeton.edu
David Gérard-Varet	Université de Paris, France david.gerard-varet@imj-prg.fr	Yum-Tong Siu	Harvard University, USA siu@math.harvard.edu
Colin Guillarmou	Université Paris-Saclay, France colin.guillarmou@universite-paris-saclay.fr	Terence Tao	University of California, Los Angeles, USA tao@math.ucla.edu
Ursula Hamenstaedt	Universität Bonn, Germany ursula@math.uni-bonn.de	Michael E. Taylor	Univ. of North Carolina, Chapel Hill, USA met@math.unc.edu
Peter Hintz	ETH Zurich, Switzerland peter.hintz@math.ethz.ch	Gunther Uhlmann	University of Washington, USA gunther@math.washington.edu
Vadim Kaloshin	Institute of Science and Technology, Austria vadim.kaloshin@gmail.com	András Vasy	Stanford University, USA andras@math.stanford.edu
Izabella Laba	University of British Columbia, Canada ilaba@math.ubc.ca	Dan Virgil Voiculescu	University of California, Berkeley, USA dvv@math.berkeley.edu
Anna L. Mazzucato	Penn State University, USA alm24@psu.edu	Jim Wright	University of Edinburgh, UK j.r.wright@ed.ac.uk
Richard B. Melrose	Massachusetts Inst. of Tech., USA rbm@math.mit.edu	Maciej Zworski	University of California, Berkeley, USA zworski@math.berkeley.edu
Frank Merle	Université de Cergy-Pontoise, France merle@ihes.fr		

PRODUCTION

production@msp.org

Silvio Levy, Scientific Editor

Cover image: Eric J. Heller: “Linear Ramp”

See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2025 is US \$475/year for the electronic version, and \$735/year (+\$70, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscriber address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2025 Mathematical Sciences Publishers

ANALYSIS & PDE

Volume 18 No. 7 2025

Regularized Brascamp–Lieb inequalities NEAL BEZ and SHOHEI NAKAMURA	1567
Cosmic censorship near FLRW spacetimes with negative spatial curvature DAVID FAJMAN and LIAM URBAN	1615
Spectral estimates for free boundary minimal surfaces via Montiel–Ros partitioning methods ALESSANDRO CARLOTTO, MARIO B. SCHULZ and DAVID WIYGUL	1715
The fractal uncertainty principle via Dolgopyat’s method in higher dimensions AIDAN BACKUS, JAMES LENG and ZHONGKAI TAO	1769
Consequences of the random matrix solution to the Peterson–Thom conjecture BEN HAYES, DAVID JEKEL and SRIVATSAV KUNNAWALKAM ELAYAVALLI	1805