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BEN HAYES, DAVID JEKEL AND SRIVATSAV KUNNAWALKAM ELAYAVALLI

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CONSEQUENCES OF THE RANDOM MATRIX SOLUTION TO THE PETERSON–THOM CONJECTURE

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We show various new structural properties of free group factors using the recent resolution (due independently to Belinschi and Capitaine, and Bordenave and Collins) of the Peterson–Thom conjecture. These results include the resolution to the coarseness conjecture due independently to the first author and Popa, a generalization of Ozawa and Popa's celebrated strong solidity result using vastly more general versions of the normalizer (and in an ultraproduct setting), a dichotomy result for intertwining of maximal amenable subalgebras of interpolated free group factors, as well as applications to ultraproduct embeddings of nonamenable subalgebras of interpolated free group factors.

1. Introduction

The structure of the group of von Neumann algebras associated to the countable free groups (also known as free group factors) has been a constant source of new results and new mysteries. Murray and von Neumann [1936] showed that the free group factors are full, i.e., they have no central sequences, and they used this structural property to distinguish the free group factors from the separable hyperfinite II_1-factor, thus giving the first example of two provably nonisomorphic separable II_1-factors. Their work left behind the now notorious open question of whether the free group factors themselves are isomorphic for different numbers of generators. Almost a century has passed in the development of II_1-factors, in which the quest to understand the structure of free group factors has been a recurring theme with several remarkable achievements.

One avenue of this research is the structure of subalgebras of free group factors. A foundational discovery of Popa [1983a] showed that every subalgebra that strictly contains the generator MASA (maximally abelian subalgebra) in a free group factor must be full and in particular nonamenable (amenability is equivalent to hyperfiniteness by fundamental work of Connes [1976]). This answered in the negative a question of Kadison at the 1967 Baton Rouge conference who asked if every self-adjoint operator in a II_1-factor is contained in a hyperfinite subfactor. The technique of asymptotic orthogonality developed by Popa to achieve the above result has been used successfully to establish this maximal amenability property for various natural subalgebras of the free group factors, such as the radial MASA [Cameron et al. 2010] (see also [Brothier and Wen 2016; Parekh et al. 2018]). Recently Boutonnet and Popa [2023] also constructed a continuum size family $(M_\alpha)_\alpha$ of interesting maximally amenable subalgebras in any free product of diffuse tracial von Neumann algebras (in particular for free group factors) with the property that M_α is not unitarily conjugate to M_β if $\alpha \neq \beta$.

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Maximal amenability can be enhanced to an absorption phenomenon as follows. We say that a diffuse $P \leq M$ has the *absorbing amenability property* if, whenever $Q \leq M$ is amenable and $P \cap Q$ is diffuse, we have $Q \leq P$. By modifying Popa's asymptotic orthogonality property, it was shown in [Houdayer 2015; Wen 2016] respectively that the generator MASA and the radial MASA admit the absorbing amenability property. This work inspired many papers establishing the absorbing amenability property (and other absorption properties such as Gamma stability) in many examples; see [Brothier and Wen 2016; Hayes et al. 2021b; Parekh et al. 2018].

Given a finite von Neumann algebra M,[1] we say that M has *unique maximal amenable extensions* if, for every diffuse, amenable subalgebra $Q \leq M$, there is a *unique* maximal amenable $P \leq M$ with $Q \subseteq P$. Peterson and Thom [2011] conjectured that any diffuse, amenable subalgebra of a free group factor has unique maximal amenable extensions, which came to be known as the *Peterson–Thom conjecture*. This conjecture was motivated by both Peterson and Thom's analogous insights on groups with positive first L^2-Betti numbers and previous work of Ozawa and Popa [2010a], Peterson [2009], and Jung [2007]. One can apply Zorn's lemma to show that, for any von Neumann algebra M and any amenable $Q \leq M$, there is *some* maximal amenable $P \leq M$ with $Q \subseteq P$. The novelty of the Peterson–Thom conjecture is that such a P should be *unique*. The Peterson–Thom conjecture is then equivalent to the statement that every maximal amenable subalgebra of a free group factor has the absorbing amenability property.

In [Hayes 2022], the first author formulated a conjecture on random matrices, which he showed implies the Peterson–Thom conjecture. Several works in random matrices made progress towards this random matrix conjecture [Bandeira et al. 2023; Collins et al. 2022; Parraud 2023]. Recent breakthroughs of Belinschi and Capitaine [2022] and of Bordenave and Collins [2023] independently prove this random matrix conjecture, thus resolving the Peterson–Thom conjecture in the positive.

The reduction of the Peterson–Thom conjecture to a random matrix problem uses Voiculescu's microstates free entropy dimension theory (see [Voiculescu 1991; 1995; 1994; 1996]), namely the 1-bounded entropy defined implicitly in [Jung 2007, Theorem 3.2] and explicitly by the first author in [Hayes 2018]. We denote the 1-bounded entropy of an algebra M by $h(M)$ (see the Appendix for the precise definition, which we will not need in the main body of the paper). The 1-bounded entropy has several permanence properties which show that the collection of algebras Q with $h(Q : M) \leq 0$ is invariant under various operations such as taking subalgebras, taking the von Neumann algebra generated by its normalizer (or other weakenings of the normalizer) of an algebra, or taking the join of two algebras with diffuse intersection; see Section 2.2 for a list of such properties. Because of the permanence properties that the 1-bounded entropy enjoys, solving the Peterson–Thom conjecture via 1-bounded entropy proves several results beyond showing that free group factors have the absorbing amenability property. This paper will explain in detail several corollaries of the recent resolution of the Peterson–Thom conjecture. As shown in [Popa 2021], solving the Peterson–Thom property via 1-bounded entropy also resolves it for the interpolated free group factors $L(\mathbb{F}_t)$, independently defined by Dykema [1994] and Rădulescu [1994],

[1] It is not necessary for M to be finite. However, it could be argued that in the general setting one should require all subalgebras to be images of normal conditional expectations. We leave it to those more versed in Tomita–Takesaki theory to work out the appropriate definition here.

for $t > 1$. We give a separate proof of this in Section 3. In fact, because of our work in Section 3, all the main results in this paper apply to interpolated free group factors and not just free group factors.

Our first main result is the positive resolution of the *coarseness conjecture* independently due to the first author [Hayes 2018, Conjecture 1.12] and Popa [2021, Conjecture 5.2]. If M is a von Neumann algebra, an M-M bimodule $\mathcal{H}$ is a Hilbert space with normal left and right actions of M which commute. We use ${}_M\mathcal{H}_M$ to mean that $\mathcal{H}$ is an M-M bimodule. If $\mathcal{H}$ and $\mathcal{K}$ are M-M bimodules, we use ${}_M\mathcal{H}_M \leq_M \mathcal{K}_M$ to mean that there is an M-bimodular isometry $\mathcal{H} \to \mathcal{K}$. If $\mathcal{H}_1 \subseteq \mathcal{H}_2$ are Hilbert spaces, we use $\mathcal{H}_2 \ominus \mathcal{H}_1$ for $\mathcal{H}_1^{\perp} \cap \mathcal{H}_2$.

Theorem 1.1. *Let $t > 1$. For any maximal amenable subalgebra $P \leq L(\mathbb{F}_t)$, we have*

$$ {}_P[L^2(L(\mathbb{F}_t)) \ominus L^2(P)]_P \leq_P (L^2(P) \otimes L^2(P))^{\oplus\infty}{}_P. $$

In [Popa 2021], this property is referred to as *coarseness* of the inclusion $P \leq L(\mathbb{F}_t)$. As explained in the introduction of that paper, we may think of coarseness as the "most random" position a subalgebra can be in.

It is of interest to specialize Theorem 1.1 to the case where P is abelian. Suppose (M, τ) is a tracial von Neumann algebra and $A \leq M$ is a maximal abelian $*$-subalgebra. Write $A = L^\infty(X, \mu)$ for some compact Hausdorff space X and some Borel probability measure μ on X. The representation

$$ \pi : C(X) \otimes C(X) \to B(L^2(M) \ominus L^2(A)), $$

given by

$$ \pi(f \otimes g)\xi = f\xi g, $$

gives rise to a spectral measure E on $X \times X$ whose marginals are Radon–Nikodym equivalent to μ. We say that $\nu \in \mathrm{Prob}(X \times X)$ is a *left/right measure* of $A \leq M$ if it is Radon–Nikodym equivalent to E. One often abuses terminology and refers to *the* left/right measure to refer to any element of this equivalence class of measures. The measure class of this measure ν together with the multiplicity function $m : X \times X \to \mathbb{N} \cup \{0\} \cup \{\infty\}$ is usually called the measure-multiplicity invariant (see [Feldman and Moore 1977; Neshveyev and Størmer 2002]), which has also been investigated in [Dykema and Mukherjee 2013; Mukherjee 2013; Popa 2021]. The essential range of the multiplicity function is called the Pukánszky invariant; this was defined in [Pukánszky 1960] and further studied in [Dykema et al. 2006; Popa 2019; Rădulescu 1991; Robertson and Steger 2010; Sinclair and Smith 2005; White 2008]. By [Ge and Popa 1998, Theorem 4.1] and [Popa 2019, Corollary 3.8 (1)], we know that every MASA in an interpolated free group factor has unbounded multiplicity function.

Theorem 1.2. *Let $M = L(\mathbb{F}_t)$ for $t > 1$. Suppose that $A \leq M$ is abelian and a maximal amenable subalgebra of M. Write $A = L^\infty(X, \mu)$ for some compact metrizable space X and some Borel probability measure on X. Then the left/right measure of $A \leq M$ is absolutely continuous with respect to $\mu \otimes \mu$.*

Our work shows that, for MASAs in interpolated free group factors which are also maximal amenable, the measure given in the measure-multiplicity invariant has to be absolutely continuous with respect to the product measure $\mu \otimes \mu$. More concretely, if we use the fact that all standard probability spaces are isomorphic to reduce to the case where (X, μ) is $[0, 1]$ with Lebesgue measure, then the measure given in the measure-multiplicity invariant has to be absolutely continuous with respect to Lebesgue measure on the unit square.

One of the landmark structural results about free group factors is the solidity property that the commutant of any diffuse subalgebra is amenable. Ozawa [2009] achieved this result first by introducing C*-algebraic boundary techniques. Popa [2007] then gave a different proof using his influential s-malleable deformations and spectral gap rigidity ideas. Peterson [2009] verified solidity for more examples using a conceptually new approach based on the theory of closable derivations. All of these results apply to algebras much more general than free group factors, for instance, von Neumann algebras of hyperbolic groups; see [Chifan and Sinclair 2013; Ding et al. 2023; Ozawa and Popa 2010b; Popa and Vaes 2014a; 2014b; Sinclair 2011].

Despite the early success of free entropy theory in establishing global structural properties of free group factors (for instance, absence of Cartan subalgebras [Voiculescu 1996] and primeness [Ge 1998]), solidity was still out of reach by free entropy methods. In this paper our first result is a proof of Ozawa's solidity theorem based on free entropy dimension techniques, which is completely different from previous arguments. In fact, for interpolated free group factors, we strengthen the celebrated strong solidity theorem of Ozawa and Popa [2010a, Corollary 1] using a vastly more general version of the normalizer. A first example is the 1-sided quasinormalizer, defined in [Izumi et al. 1998; Pimsner and Popa 1986; Popa 1999] (building off of ideas in [Popa 1983b]),

$$q^1\mathcal{N}_M(N) = \left\{ x \in M : \text{there exists } x_1, \dots, x_n \in M \text{ such that } xN \subseteq \sum_{j=1}^n N x_j \right\}.$$

We also consider the wq-normalizer, defined in [Galatan and Popa 2017; Ioana et al. 2008; Popa 2006c; 2006b],

$$\mathcal{N}_M^{wq}(N) = \{u \in \mathcal{U}(M) : uNu^* \cap N \text{ is diffuse}\}$$

and its cousin the very weak quasinormalizer

$$\mathcal{N}_M^{vwq}(N) = \{u \in \mathcal{U}(M) : \text{there exists } v \in \mathcal{U}(M) \text{ such that } uNv \cap N \text{ is diffuse}\}.$$

Since $uNv \cap N$ is not an algebra, the phrase "$uNv \cap N$ is diffuse" should be interpreted as saying that there is a sequence of unitaries $v_n \in uNv \cap N$ which tend to zero weakly. We also consider the *weak intertwining space* $wI_M(Q, Q)$ due to Popa [2005; 2021] (we restate the definition in Definition 2.1 of this paper). As shown in [Hayes 2018, Proposition 3.2] and Section 2.1, all of these are contained in the anti-coarse space

$$\mathcal{H}_{\text{anti-c}}(N \le M) = \bigcap_{T \in \operatorname{Hom}_{N-N}(L^2(M), L^2(N) \otimes L^2(N))} \ker(T).$$

Here $\operatorname{Hom}_{N-N}(L^2(M), L^2(N) \otimes L^2(N)$ is the space of bounded, linear, N-N bimodular maps

$$T : L^2(M) \to L^2(N) \otimes L^2(N).$$

Our next main result is a statement about the most general setting of the anti-coarse space; however, for the reader's sake we state it in the context of these four examples.

Theorem 1.3. *Let $t > 1$ and $Q \leq L(\mathbb{F}_t)$ be a diffuse, amenable subalgebra. Then* $\mathrm{W}^*(\mathcal{H}_{\text{anti-c}}(Q \leq L(\mathbb{F}_t)))$ *remains amenable. In particular, for any*

$$X \subseteq q^1\mathcal{N}_{L(\mathbb{F}_t)}(Q) \cup \mathcal{N}^{wq}_{L(\mathbb{F}_t)}(Q) \cup wI_{L(\mathbb{F}_t)}(Q, Q) \cup \mathcal{N}^{vwq}_{L(\mathbb{F}_t)}(Q),$$

we have that $\mathrm{W}^*(X)$ *is amenable.*

We refer the reader to Section 4.2 for the precise definition of $\mathrm{W}^*(Y)$ for $Y \subseteq L^2(M)$. The case $X = \mathcal{N}_{L(\mathbb{F}_t)}(Q)$ in the above theorem recovers the strong solidity theorem of Ozawa and Popa [2010a, Corollary 1] for the special case of interpolated free group factors. Theorem 1.3 is already new for $t \in \mathbb{N}$ for and when

$$X \in \{q^1\mathcal{N}_{L(\mathbb{F}_t)}(Q), \mathcal{N}^{wq}_{L(\mathbb{F}_t)}(Q), wI_{L(\mathbb{F}_t)}(Q, Q), \mathcal{N}^{vwq}_{L(\mathbb{F}_t)}(Q)\}.$$

We may, in fact, deduce a further generalization of strong solidity for interpolated free group factors in an ultraproduct framework.

Theorem 1.4. *Let $t \in (1, +\infty)$, and let ω be a free ultrafilter on $\mathbb{N}$. Suppose that $Q \leq L(\mathbb{F}_t)^\omega$ is a diffuse, amenable subalgebra. Suppose we are given Neumann subalgebras Q_α defined for ordinals α satisfying*

- $Q_0 = Q$,
- *if α is a successor ordinal, then $Q_\alpha = \mathrm{W}^*(X_\alpha, Q_{\alpha-1})$, where*

$$X_\alpha \subseteq \mathcal{H}_{\text{anti-c}}(Q_{\alpha-1} \leq L(\mathbb{F}_t))$$

  (*for example if*

$$X_\alpha \subseteq q^1\mathcal{N}_{L(\mathbb{F}_t)^\omega}(Q_{\alpha-1}) \cup \mathcal{N}^{wq}_{L(\mathbb{F}_t)^\omega}(Q_{\alpha-1}) \cup wI_{L(\mathbb{F}_t)^\omega}(Q_{\alpha-1}, Q_{\alpha-1}) \cup \mathcal{N}^{vwq}_{M}(Q_{\alpha-1})),$$

- *if α is a limit ordinal, then $Q_\alpha = \overline{\bigcup_{\beta<\alpha} Q_\beta}^{\mathrm{SOT}}$.*

Then, for any ordinal α we have that $Q_\alpha \cap L(\mathbb{F}_t)$ is amenable. In particular, $L(\mathbb{F}_t)$ has the following Gamma stability property: if $Q \leq L(\mathbb{F}_t)^\omega$ and if $Q' \cap L(\mathbb{F}_t)^\omega$ is diffuse, then $Q \cap L(\mathbb{F}_t)$ is amenable.

This recovers the previous Gamma stability results from [Houdayer 2015] (recovering also instances of the results of [Ding and Kunnawalkam Elayavalli 2024; Ding et al. 2023]).

The case of the weak intertwining space itself leads to a dichotomy in terms of Popa's deformation/rigidity theory for maximal amenable subalgebras of free group factors. We recall the fundamental notion of intertwining introduced in [Popa 2006a; 2006c]. If M is a finite von Neumann algebra and $P, Q \leq M$, we say that *a corner of Q intertwines into P inside of M* and write $Q \prec P$ if there are nonzero projections $f \in Q$ and $e \in P$, a unital $*$-homomorphism $\Theta : fQf \to ePe$, and a nonzero partial isometry $v \in M$ such that

- $xv = v\Theta(x)$ for all $x \in fQf$,
- $vv^* \in (fQf)' \cap fMf$,
- $v^*v \in \Theta(fqf)' \cap eMe$.

This can be thought of intuitively as "Q can be unitarily embedded into P after cutting by a projection".

Theorem 1.5. *Fix $t > 1$, and let Q and P be maximal amenable subalgebras of $L(\mathbb{F}_t)$. Then exactly one of the following occurs*:

(1) *either there are nonzero projections $e \in Q$, $f \in P$ and a unitary $u \in L(\mathbb{F}_t)$ such that $u^*(ePe)u = fQf$,*

(2) *or, for any diffuse $Q_0 \leq Q$, we have that $Q_0 \not\prec P$.*

In particular, if Q, P are hyperfinite subfactors of $L(\mathbb{F}_t)$ that are maximal amenable subalgebras in $L(\mathbb{F}_t)$, then either they are unitarily conjugate or no corner of any diffuse subalgebra of one can be intertwined into the other inside of $L(\mathbb{F}_t)$.

So, given any pair P, Q of maximal amenable subalgebras of $L(\mathbb{F}_t)$, they either have unitarily conjugate corners or no diffuse subalgebra of Q can be "essentially conjugated" into P. The reader should compare this with [Popa 2006a, Theorem A.1], where a similar result is shown for MASAs in *any* tracial von Neumann algebra.

We close with an application to embeddings into matrix ultraproducts.

Corollary 1.6. *Let $t > 1$, and let $N \leq L(\mathbb{F}_t)$ be a nonamenable subfactor. Then there is a free ultrafilter ω and an embedding $\iota : N \to \prod_{k\to\omega} M_k(\mathbb{C})$, with $\iota(N)' \cap \prod_{k\to\omega} M_k(\mathbb{C}) = \mathbb{C}1$.*

The corollary is proved as follows. The results of [Hayes 2022] and [Belinschi and Capitaine 2022; Bordenave and Collins 2023] imply that every nonamenable $N \leq L(\mathbb{F}_t)$ satisfies $h(N : L(\mathbb{F}_t)) > 0$, and hence $h(N) > 0$; see Corollary 3.1 for details. Work of the second author shows that if $h(N) > 0$, then there exists some embedding of N into a matrix ultraproduct which has trivial relative commutant [Jekel 2023, Corollary 1.3, see the statement in corrigendum].

A comment on proofs. We give a few remarks on how 1-bounded entropy is used in the paper. The first is that, for any tracial von Neumann algebra, the 1-bounded entropy leads to a natural class of subalgebras called *Pinsker algebras*. To say that $P \leq M$ is Pinsker means that P is a maximal subalgebra such that $h(P : M) \leq 0$, where $h(Q : N)$ for $Q \leq N$ is the 1-bounded entropy of Q in the presence of N defined in [Hayes 2018]. Intuitively this means two things: firstly that P has "very few" embeddings into ultraproducts of matrices which extend to M, and that P is maximal with respect to inclusion among algebras which have "very few" embeddings into ultraproducts of matrices which extend to M. We refer to Section 2.3 for the precise definition of a Pinsker algebra. Given a diffuse subalgebra $Q \leq M$ with $h(Q : M) \leq 0$, general properties of 1-bounded entropy show that there is a unique Pinsker algebra $P \leq M$ with $Q \subseteq P$. In the context of the Peterson–Thom conjecture, the unique maximal amenable extension of a diffuse subalgebra can be identified exactly as the Pinsker algebra containing this subalgebra. Thus the 1-bounded entropy cannot only be used to solve the Peterson–Thom conjecture but also naturally identify the maximal amenable extensions of a given amenable subalgebra.

The second remark is that, as was previously mentioned, the 1-bounded entropy enjoys several permanence properties, which we will list in Section 2.2. For the proofs of all the theorems mentioned so far, we will only use these permanence properties as well as the fact that the results of [Belinschi and Capitaine 2022; Bordenave and Collins 2023; Hayes 2022] show that the Pinsker algebras in interpolated free group factors are precisely the maximal amenable subalgebras. In particular, we never have to work with the precise definition of 1-bounded entropy, just these permanence properties. This tells us that the axiomatic treatment of 1-bounded entropy via these general properties can be used to deduce many interesting and new results on the structure of free group factors.

Organization of the paper. In Section 2.1, we recall the anti-coarse space and expand the results in [Hayes 2018, Proposition 3.2], showing that it contains several other weakenings of the normalizer. In Section 2.2, we list the permanence properties of 1-bounded entropy we will use in this paper. For the proofs of all of our applications of [Hayes 2022] and [Belinschi and Capitaine 2022; Bordenave and Collins 2023], we will only use these permanence properties and not the precise definition of 1-bounded entropy, so these properties give an axiomatic approach to most of the proofs in this paper. In Section 2.3, we recall the notion of Pinsker algebras for 1-bounded entropy introduced in [Hayes et al. 2021b] and recast the results of [Belinschi and Capitaine 2022; Bordenave and Collins 2023] in these terms. In Section 3, we will explain how these results apply not just to maximal amenable subalgebras of free group factors, but also to those of interpolated free group factors. Section 4.1 contains the proof of the coarseness conjecture, as well as applications to maximal abelian subalgebras of free group factors. In Section 4.2, we give several generalizations of Ozawa and Popa's celebrated strong solidity theorem. In Section 4.3, we give a dichotomy for intertwining between maximal amenable subalgebras of interpolated free group factors (and more generally for Pinsker algebras in any tracial von Neumann algebra). In the Appendix, we give the definition of 1-bounded entropy and prove that it is independent of the choice of generators (a fact proved first implicitly in [Jung 2007, Theorem 3.2] and later explicitly in [Hayes 2018, Theorem A.9]). Here we give a significant conceptual and technical simplification of the proof using the noncommutative functional calculus due to the second author.

2. Preliminaries

2.1. ***The anti-coarse space.*** For an inclusion $N \leq M$ of tracial von Neumann algebras, we let

$$\mathcal{H}_{\text{anti-c}}(N \leq M) = \bigcap_{T \in \mathrm{Hom}_{N-N}(L^2(M), L^2(N) \otimes L^2(N))} \ker(T),$$

where $\mathrm{Hom}_{N-N}(L^2(M), L^2(N) \otimes L^2(N)$ is the space of bounded, linear, N-N bimodular maps

$$T : L^2(M) \to L^2(N) \otimes L^2(N).$$

It is shown in [Hayes 2018, Proposition 3.2] that this contains the following generalizations of the normalizer of $N \leq M$:

$$q^1\mathcal{N}_M(N) = \left\{ x \in M : \text{there exists } x_1, \dots, x_n \in M \text{ such that } xN \subseteq \sum_{j=1}^{n} N x_j \right\},$$

$$\mathcal{N}_M^{wq}(N) = \{u \in \mathcal{U}(M) : uNu^* \cap N \text{ is diffuse}\}.$$

We recall the following definition, due to Popa [2005; 2021] (see also [Galatan and Popa 2017; Ioana et al. 2008; Popa 2006b] for related concepts).

Definition 2.1 [Popa 2021, Definition 2.6.1]. Let (M,τ) be a tracial von Neumann algebra. For $Q, P \leq M$ diffuse, we define the *intertwining space from Q to P inside M*, denoted by $I_M(Q,P)$, to be the set of $\xi \in L^2(M)$ such that

$$\overline{\operatorname{span}\{a\xi b : a \in Q, b \in P\}}^{\|\cdot\|_2}$$

has finite dimension as a right P-module. We define the *weak intertwining space from Q to P inside M* by

$$wI_M(Q,P) = \bigcup_{Q_0 \leq Q \text{ diffuse}} I_M(Q_0, P).$$

The following is a well-known result due to [Popa 2021, Proposition 2.6.3], but we include the proof for completeness.

Proposition 2.2. *Let (M,τ) be a tracial von Neumann algebra. For $Q \leq M$ diffuse, we have*

$$wI_M(Q,Q) \subseteq \mathcal{H}_{\text{anti-c}}(Q \leq M).$$

Proof. Fix $Q_0 \leq Q$ diffuse. It suffices to show that

$$(I_M(Q_0,Q))^{\perp} \supseteq \mathcal{H}_{\text{anti-c}}(Q \leq M)^{\perp}.$$

It is a folklore result that $\mathcal{H}_{\text{anti-c}}(Q \leq M)^{\perp}$ can be embedded into an infinite direct sum of $L^2(Q) \otimes L^2(Q)$ as a Q-Q bimodule (see, e.g., [Hayes 2018, Proposition 3.3] for a complete proof). Since Q_0 is diffuse, we can find a sequence $u_n \in \mathcal{U}(Q_0)$ which tend to zero weakly. We leave it as an exercise to check that, for all $\xi, \eta \in L^2(Q) \otimes L^2(Q)$, we have

$$\lim_{n\to\infty} \sup_{y \in Q : \|y\| \leq 1} |\langle u_n \xi y, \eta\rangle| = 0 = \lim_{n\to\infty} \sup_{y \in Q : \|y\| \leq 1} |\langle y \xi u_n, \eta\rangle|$$

(to prove this one can, for instance, check it on simple tensors and then conclude using linearity and density). Since $\mathcal{H}_{\text{anti-c}}(Q \leq M)^{\perp}$ embeds into an infinite direct sum of $L^2(Q) \otimes L^2(Q)$ as a Q-Q bimodule, it follows that, for all $\xi, \eta \in \mathcal{H}_{\text{anti-c}}(Q \leq M)^{\perp}$, we have

$$\lim_{n\to\infty} \sup_{y \in Q : \|y\| \leq 1} |\langle u_n \xi y, \eta\rangle| = 0 = \lim_{n\to\infty} \sup_{y \in Q : \|y\| \leq 1} |\langle y \xi u_n, \eta\rangle|.$$

Since

$$\sup_{y \in Q : \|y\| \leq 1} |\langle u_n \xi y, \eta\rangle| = \|\mathbb{E}_Q(\eta^* u_n \xi)\|_1,$$

we have

$$0 = \lim_{n\to\infty} \|\mathbb{E}_Q(\eta^* u_n \xi)\|_1.$$

So [Popa 2019, Theorem 1.3.2] implies that $\mathcal{H}_{\text{anti-c}}(Q \leq M)^{\perp} \subseteq I_M(Q_0,Q)^{\perp}$, as desired. ☐

To further illustrate the generality of the anti-coarse space, we show that it contains the following very weak normalizer of $(N \subseteq M)$:

$$\mathcal{N}_M^{vwq}(N) = \{u \in \mathcal{U}(M) : \text{there exists } v \in \mathcal{U}(M) \text{ such that } uNv \cap N \text{ is diffuse}\}.$$

Here by "diffuse" we mean that $uNv \cap N$ contains a sequence $(u_n)_{n\in\mathbb{N}}$ of unitaries with $u_n \to 0$ in WOT as $n \to \infty$.

Proposition 2.3. *Let (M,τ) be a tracial von Neumann algebra. For $N\leq M$ diffuse, we have*

$$\mathcal{N}_M^{vwq}(N)\subseteq\mathcal{H}_{\text{anti-c}}(N\leq M).$$

Proof. The argument proceeds exactly as in [Hayes 2018, proof of Proposition 3.2], but we include it for the reader's convenience. Let $u\in\mathcal{N}_M^{wqv}(N)$. By definition, it suffices to show that, for every

$$T\in\operatorname{Hom}_{N-N}(L^2(M),L^2(N)\otimes L^2(N)),$$

we have $T(u)=0$. Let $v\in\mathcal{U}(M)$ and $u_n\in\mathcal{U}(N)\cap uNv$ be such that $u_n\to_{n\to\infty}^{\text{WOT}}0$. Write

$$w_n=u^*u_nv^*\in\mathcal{U}(N)$$

and observe that $w_n\to 0$ in WOT. Then, using that T is N-N bimodular and that $u_n,w_n\in\mathcal{U}(N)$,

$$\begin{aligned}\|T(u)\|_2^2=\|T(u)w_n\|_2^2&=\langle T(u)w_n,T(u)w_n\rangle=\langle T(u)w_n,T(uw_n)\rangle\\&=\langle T(u)w_n,T(u_nv^*)\rangle=\langle u_n^*T(u)w_n,T(v^*)\rangle.\end{aligned}$$

Since $u_n^*\to_{n\to\infty}^{\text{WOT}}0$ and $w_n\to_{n\to\infty}^{\text{WOT}}0$, it follows as in Proposition 2.2 that $\langle u_n^*\xi w_n,\eta\rangle\to_{n\to\infty}0$ for all $\xi,\eta\in L^2(N)\otimes L^2(N)$. Taking limits as $n\to\infty$ above thus shows that $T(u)=0$. □

2.2. 1-*bounded entropy.* One of the main ideas going into the proof of the Peterson–Thom conjecture is the 1-bounded entropy h of a tracial von Neumann algebra, a numerical invariant which appeared implicitly in [Jung 2007] and was made explicit by the first author in [Hayes 2018]. We will need the more general notion of 1-bounded entropy in the presence, which is defined for inclusions $N\leq M$ of tracial von Neumann algebras and is denoted by $h(N:M)$. For detailed expositions and recent work on this topic, see [Charlesworth et al. 2023; Chifan et al. 2023; Hayes et al. 2021a; 2021b; 2024; Jekel 2023; Kunnawalkam Elayavalli 2023].

For the applications in this paper, we will not need to use the definition of h directly, only the properties listed below. We include the precise definition of h in the Appendix, along with a streamlined proof that the definition is independent of the choice of the generating sets for the von Neumann algebras. The name "1-bounded entropy" derives from the following result, connecting the 1-bounded entropy to the strong 1-boundedness of Jung [2007] .

Theorem 2.4 [Hayes 2018, Proposition A.16]. *A tracial von Neumann algebra M is strongly* 1*-bounded* (in the sense of [Jung 2007]) *if and only if M is finitely generated and diffuse and satisfies $h(M)<\infty$.*

For this reason, we say that any tracial von Neumann algebra (M,τ) (even if M is not finitely generated or diffuse) is *strongly* 1*-bounded* if $h(M)<+\infty$.

Let (M,τ) be a tracial von Neumann algebra. The 1-bounded entropy in the presence enjoys the following properties:

(P1) $h(M)=h(M:M)$ for every tracial von Neumann algebra (M,τ).

(P2) Suppose $N\leq M$. Then $h(N:M)\geq 0$ if M embeds into an ultrapower of $\mathcal{R}$, and $h(N:M)=-\infty$ if M does not embed into an ultrapower of $\mathcal{R}$. (We leave this as an exercise.)

(P3) $h(N_1:M_1)\leq h(N_2:M_2)$ if $N_1\leq N_2\leq M_2\leq M_1$. (We leave this as an exercise.)

(P4) $h(N : M) \leq 0$ if $N \leq M$ and N is hyperfinite. (We leave this as an exercise.)

(P5) $h(M) = \infty$ if M is diffuse, and $M = \mathrm{W}^*(x_1, \dots, x_n)$, where $x_j \in M_{\mathrm{sa}}$ for all $1 \leq j \leq n$ and $\delta_0(x_1, \dots, x_n) > 1$. For example, this applies to $M = L(\mathbb{F}_n)$ for $n > 1$ because of [Voiculescu 1994; 1996] together with Theorem 2.4 and [Jung 2007, Corollary 3.5].

(P6) $h(N_1 \vee N_2 : M) \leq h(N_1 : M) + h(N_2 : M)$ if $N_1, N_2 \leq M$ and $N_1 \cap N_2$ is diffuse. (See [Hayes 2018, Lemma A.12].)

(P7) Suppose that $(N_\alpha)_\alpha$ is an increasing chain of diffuse von Neumann subalgebras of a von Neumann algebra M. Then

$$h\Big(\bigvee_\alpha N_\alpha : M\Big) = \sup_\alpha h(N_\alpha : M).$$

(See [Hayes 2018, Lemma A.10].)

(P8) $h(N : M) = h(N : M^\omega)$ if ω is a free ultrafilter on an infinite set. (See [Hayes 2018, Proposition 4.5].)

(P9) $h(\mathrm{W}^*(\mathcal{H}_{\text{anti-c}}(N \leq M)) : M) = h(N : M)$ if $N \leq M$ is diffuse. (See [Hayes 2018, Theorem 3.8]. See also Section 4.2 for the definition of $\mathrm{W}^*(Y)$ for $Y \subseteq L^2(M)$.)

(P10) Let I be a countable set and $M = \bigoplus_{i \in I} M_i$ with M_i diffuse for all i. Suppose that τ is a faithful trace on M and that λ_i is the trace of the identity on M_i. Endow M_i with the trace $\tau_i = \tau|_{M_i}/\lambda_i$. Fix $N_i \leq M_i$ for all $i \in I$. Then

$$h(N : M, \tau) \leq \sum_i \lambda_i^2 h(N_i : M_i, \tau_i).$$

(See the proof of [Hayes 2018, Proposition A.13 (i)].)

(P11) If $z \in \mathrm{Proj}(Z(M))$, $N \leq M$, and $h(N : M) \leq 0$, then $h(Nz : Mz) \leq 0$. (See [Hayes et al. 2021a, Lemma 4.2].)

(P12) $h(pNp : pMp) = (1/\tau(p)^2) h(N : M)$ if $N \leq M$ is diffuse, p is a nonzero projection in N, and M is a factor. (This follows from modifying the proofs of [Hayes 2018, Proposition A.13 (ii)] and [Hayes et al. 2021a, Proposition 4.6].)

2.3. *Pinsker algebras.*

Definition 2.5. Let (M, τ) be a tracial von Neumann algebra. We say that $P \leq M$ is *Pinsker* if $h(P : M) \leq 0$ and, for any $P \leq Q \leq M$ with $P \neq Q$, we have $h(Q : M) > 0$.

By properties (P6) and (P7), if $Q \leq M$ is diffuse and $h(Q : M) \leq 0$, then there is a unique Pinsker algebra $P \leq M$ with $Q \subseteq P$. E.g.,

$$P = \bigvee_{N \leq M, N \supseteq Q, h(N:M) \leq 0} N.$$

We call P the *Pinsker algebra of* $Q \subseteq M$. By [Hayes 2022] and the recent breakthrough result in [Belinschi and Capitaine 2022] and [Bordenave and Collins 2023], we have the following classification of Pinsker subalgebras of free group factors.

Theorem 2.6. *Fix $r \in \mathbb{N} \cup \{\infty\}$. Then*

(i) *$Q \leq L(\mathbb{F}_r)$ is amenable if and only if $h(Q : L(\mathbb{F}_r)) = 0$,*

(ii) *$P \leq L(\mathbb{F}_r)$ is Pinsker if and only if it is maximal amenable.*

As remarked in [Hayes et al. 2021b], we may think of 1-bounded entropy as analogous to the Kolmogorov–Sinaĭ entropy in the context of probability measure-preserving actions of groups. Entropy for probability measure-preserving actions of groups was first developed in the case of $\mathbb{Z}$ by [Kolmogorov 1958; Sinaĭ 1959], for amenable groups in [Kieffer 1975; Ornstein and Weiss 1987], and then for sofic groups in [Bowen 2010; Kerr and Li 2011]. See also [Seward 2019] for a potential approach to entropy for arbitrary acting groups, called Rokhlin entropy. A probability measure-preserving action $G \curvearrowright (X, \mu)$ with G sofic is said to have *complete positive entropy* if every nontrivial quotient probability measure-preserving quotient action has positive entropy. Dually, this is equivalent to saying that if B is a G-invariant von Neumann subalgebra B of $L^\infty(X, \mu)$ with $B \neq \mathbb{C}1$, then the action of G on B has positive entropy.

Motivated by the sofic entropy case, one could naively define a tracial von Neumann algebra (M, τ) to have completely positive 1-bounded entropy if any nontrivial subalgebra has positive 1-bounded entropy. However, this will never be satisfied, as any hyperfinite subalgebra will have vanishing 1-bounded entropy. Thus any tracial von Neumann algebra will have many subalgebras with vanishing 1-bounded entropy (these subalgebras can be chosen to be diffuse if M is). Instead, we should think of a result saying that the only subalgebras with vanishing 1-bounded entropy are ones that can be quickly deduced to vanishing have 1-bounded entropy from the properties (P1)–(P12) listed above (e.g., hyperfinite algebras, property Gamma algebras, nonprime algebras, algebras with a Cartan, etc.) as a complete positive entropy result. We may thus think of Theorem 2.6 as a complete positive entropy result for 1-bounded entropy. Since free group factors may be thought of as the free probability analogue of Bernoulli shifts (e.g., because $L(\mathbb{F}_\infty)$ is the crossed product algebra associated to a free Bernoulli shift), Theorem 2.6 should be compared with previous results establishing complete positive entropy of Bernoulli shifts (see [Kerr 2014; Rudolph and Weiss 2000]).

As discussed in [de Santiago et al. 2021, Section 5], Pinsker algebras of measure-preserving dynamical systems are analogous to the maximal rigid subalgebras of s-malleable deformations in that work. This allows for an exchange of ideas and methods between deformation/rigidity theory, free probability theory, and ergodic theory. This will be exploited in Section 4.3, where we adapt arguments in the aforementioned work to show that Pinsker algebras do not have any weak intertwiners between them unless they have corners which are unitarily conjugate.

3. Pinsker algebras of interpolated free group factors

As mentioned before, the combined results of [Hayes 2022] and [Belinschi and Capitaine 2022; Bordenave and Collins 2023] prove that, for $r \in \mathbb{N}$, we have that $Q \leq L(\mathbb{F}_r)$ is amenable if and only if $h(Q : L(\mathbb{F}_r)) = 0$. The main goal of this section is to explain how this automatically generalizes to interpolated free group factors.

Corollary 3.1. *Fix $t > 1$. Then $Q \le L(\mathbb{F}_t)$ is amenable if and only if $h(Q : L(\mathbb{F}_t)) = 0$.*

By rephrasing the above corollary we obtain the following.

Corollary 3.2. *Fix $t > 1$. Let $P \le L(\mathbb{F}_t)$. Then $P \le L(\mathbb{F}_t)$ is Pinsker if and only if P is maximal amenable.*

In order to obtain this result, we will study the relationship between Pinsker algebras and compression. First, we generalize property (P12) to the case where M is not a factor.

Lemma 3.3. *Let (M, τ) be a tracial von Neumann algebra and $P \le M$. If $h(P : M) \le 0$, then, for every projection $p \in P$, we have that $h(pPp : pMp) \le 0$.*

Proof. By decomposing the center of M into atomic and diffuse pieces, we can find a central projection $z_0 \in M$ (potentially zero), a countable set I (potentially empty), and central projections $(z_i)_{i \in I}$ such that

- $1 = z_0 + \sum_i z_i$,
- in the case $z_0 \neq 0$, we have that Mz_0 has diffuse center,
- Mz_i is a factor for all $i \in I$.

For $i \in \{0\} \cup I$, let $P_i = (pPp)z_i$ (even though z_i may not be in P, we still have that P_i is a von Neumann subalgebra of M as z_i is central). Set $\hat{P} = \overline{\sum_i P_i}^{\text{WOT}}$. Then, by (P3) and (P10),

$$h(pPp : pMp) \le h(\hat{P} : pMp) \le \tau(pz_0)^2 h(P_0 : (pMp)z_0) + \sum_i \tau(pz_i)^2 h(P_i : (pMp)z_i)$$
$$\le \tau(pz_0)^2 h((pMp)z_0) + \sum_i \tau(pz_i)^2 h(P_i : (pMp)z_i). \tag{1}$$

So it suffices to show each term on the right-hand side of this inequality is nonpositive.

We first show that $\tau(pz_0)^2 h((pMp)z_0) \le 0$. If $pz_0 = 0$, the claim is true. Otherwise, since M has diffuse center and $Z(pMp) = pZ(M)p$, we have that $(pMp)z_0$ has diffuse center. Thus $(pMp)z_0 = \mathrm{W}^*(\mathcal{N}_{(pMp)z_0}(pZ(M)pz_0))$, and so the combination of properties (P4) and (P9) implies $h((pMp)z_0) \le 0$.

Now consider $h(P_i : (pMp)z_i)$ for $i \in I$. By property (P11), we know that

$$h(Pz_i : Mz_i) \le 0$$

for all $i \in I$. Thus, by property (P12), we have that

$$h(P_i : (pMp)z_i) = \frac{1}{\tau(pz_i)^2} h(Pz_i : Mz_i) \le 0.$$

Thus we have shown that all terms on the right-hand side of (1) are nonpositive, and this completes the proof. □

We now show that being a Pinsker algebra is preserved under taking corners and amplifications.

Proposition 3.4. *Let (M, τ) be a tracial von Neumann algebra, and suppose that $P \le M$ is Pinsker. Then*

(i) *we have $Z(M) \subseteq P$,*

(ii) *for any nonzero projection $p \in P$, we have that pPp is a Pinsker subalgebra of pMp,*

(iii) *for any $n \in \mathbb{N}$, we have that $M_n(P)$ is a Pinsker subalgebra of $M_n(M)$.*

Proof. (i) Note that $Z(M) \vee P \subseteq \mathrm{W}^*(\mathcal{N}_M(P))$, and thus, by (P9),

$$h(Z(M) \vee P : M) \leq h(P : M) = 0,$$

and so P being Pinsker forces $Z(M) \vee P \subseteq P$. That is, $Z(M) \subseteq P$.

(ii) Let z be the central support of $p \in M$. Then, by (i), we know that z is under the central support of p in P. So there exists a collection $\{v_i\}_{i \in I}$ of nonzero partial isometries in P such that $v_i^* v_i \leq p$ and $z = \sum_i v_i v_i^*$. Set $p_i = v_i^* v_i$. We may, and will, assume that there is some i_0 such that $v_{i_0} = p$. By Lemma 3.3, $h(pPp : pMp) \leq 0$, and so there exists a Pinsker subalgebra Q of pMp containing pPp. Set

$$\hat{Q} = \overline{Q + \sum_{i \in I \setminus \{i_0\}} v_i v_i^* P v_i v_i^*}^{\mathrm{WOT}}.$$

Thus, by (P10),

$$\begin{aligned} h(Q : M) &\leq h\Bigg(\hat{Q} : \overline{pMp + \sum_{i \in I \setminus \{i_0\}} v_i v_i^* M v_i v_i^*}^{\mathrm{WOT}}\Bigg) \\ &\leq \tau(p)^2 h(Q : pMp) + \sum_{i \in I \setminus \{i_0\}} \tau(v_i v_i^*)^2 h(v_i v_i^* P v_i v_i^* : v_i v_i^* M v_i v_i^*) \\ &\leq \sum_{i \in I \setminus \{i_0\}} \tau(v_i v_i^*)^2 h(v_i v_i^* P v_i v_i^* : v_i v_i^* M v_i v_i^*). \end{aligned}$$

For all $i \in I$, we have that $x \mapsto v_i x v_i^*$ gives a trace-preserving isomorphism from $p_i M p_i$ to $v_i v_i^* M v_i v_i^*$, which takes $p_i P p_i$ to $v_i v_i^* P v_i v_i^*$. Hence, for all $i \in I$,

$$h(v_i v_i^* P v_i v_i^* : v_i v_i^* M v_i v_i^*) = h(p_i P p_i : p_i M p_i) \leq 0$$

by Lemma 3.3. Altogether we have shown that $h(\hat{Q} : M) \leq 0$. Note that

$$\hat{Q} \cap P \supseteq \overline{pPp + \sum_{i \in I \setminus \{i_0\}} v_i v_i^* P v_i v_i^*}^{\mathrm{WOT}},$$

which is diffuse. Hence, by property (P6), we have that $h(\hat{Q} \vee P : M) \leq 0$, and, by maximality, we have that $\hat{Q} \vee P \subseteq P$. It follows that $\hat{Q} \subseteq P$. So

$$Q = p\hat{Q}p \subseteq pPp.$$

(iii) Consider $M \leq M_n(M)$ by identifying it with $M \otimes 1 \leq M \otimes M_n(\mathbb{C}) \cong M_n(M)$. Under this identification, $\mathcal{N}_{M_n(M)}(P) \supseteq \mathcal{U}(M_n(\mathbb{C})) \cup \mathcal{U}(P)$, so $\mathrm{W}^*(\mathcal{N}_{M_n(M)}(P)) \supseteq M_n(P)$. Thus, by properties (P9) and (P3),

$$h(M_n(P) : M_n(M)) \leq h(P : M_n(M)) \leq h(P : M) \leq 0.$$

So we can let Q be the Pinsker algebra of $M_n(M)$ containing $M_n(P)$. Let e_{ij} be the standard matrix units of $M_n(\mathbb{C})$ viewed as elements of $M_n(M)$. Then, by (ii), we have that $e_{11} Q e_{11} = P$. But then, for all $x \in Q$, we have that

$$x = \sum_{i,j} e_{ii} x e_{jj} = \sum_{i,j} e_{i1} e_{11} x e_{11} e_{1j} \in M_n(P).$$

So $Q \leq M_n(P)$. $\square$

Proof of Corollary 3.1. The forward implication is (P4) of the main properties of 1-bounded entropy we listed above. For the reverse implication, suppose for the sake of contradiction that Q is not amenable. Then by Connes–Haagerup characterization of amenability (see Lemma 2.2 in [Haagerup 1985]), there is a projection $p \in Z(Q)$ and $u_1, \dots, u_r \in \mathcal{U}(Qp)$ such that

$$\left\| \frac{1}{r} \sum_{j=1}^{r} u_j \otimes \bar{u}_j \right\| < 1,$$

where $\bar{u}_j = (u_j^*)^{\mathrm{op}}$ and the norm is computed in $Qp \otimes_{\min} Qp$. Let $P \leq pL(\mathbb{F}_t)p$ be the Pinsker algebra of $L(\mathbb{F}_t)$ which contains Q. By fundamental results of Dykema and Rădulescu (see [Dykema 1994; Rădulescu 1994]), we may choose $s > 0$ such that $(pL(\mathbb{F}_t)p)^s \cong L(\mathbb{F}_2)$. Fix an integer $n > s$, and let q be a projection in $M_n(P)$ such that $\operatorname{Tr} \otimes \tau(q) = s$. Observe that

$$\left\| \frac{1}{r} \sum_{j=1}^{r} (1_n \otimes u_j) \otimes \overline{1_n \otimes u_j} \right\| < 1,$$

where 1_n is the identity of $M_n(\mathbb{C})$. Hence, it follows that $M_n(P)$ also has no nonzero amenable direct summands. We leave it as an exercise to show that this implies that $M_n(P)$ has no nonzero amenable corners. By Proposition 3.4, we know that

$$qM_n(P)q \leq qM_n(pL(\mathbb{F}_t)p)q \cong L(\mathbb{F}_2)$$

is Pinsker. It follows from Theorem 2.6 that $qM_n(P)q$ is amenable. This contradicts our previous observation that $M_n(P)$ has no nonzero amenable corners. □

4. Main results

4.1. *Orthocomplement bimodule structure for maximal amenable subalgebras.* We start with the following consequence of Theorem 2.6 on the structure of the orthocomplement bimodule for any maximal amenable $P \leq L(\mathbb{F}_t)$. Note that this verifies the coarseness conjecture, independently due to the first author [Hayes 2018, Conjecture 1.12] and Popa [2021, Conjecture 5.2].

Corollary 4.1. *Let $M = L(\mathbb{F}_t)$ for some $t > 1$. For any maximal amenable $P \leq L(\mathbb{F}_t)$, we have that*

$${}_P(L^2(M) \ominus L^2(P))_P \leq [L^2(P) \otimes L^2(P)]^{\oplus\infty}.$$

Proof. As noted in Proposition 2.2, we have that $\mathcal{H}_{\text{anti-c}}(P \leq M)^{\perp}$ embeds into $[L^2(P) \otimes L^2(P)]^{\oplus\infty}$ as a P-P bimodule. Since P is Pinsker by Theorem 2.6, property (P9) implies that

$$\mathcal{H}_{\text{anti-c}}(P \leq M) = L^2(P).$$

Thus,

$$L^2(M) \ominus L^2(P) = \mathcal{H}_{\text{anti-c}}(P \leq M)^{\perp}$$

embeds into $[L^2(P) \otimes L^2(P)]^{\oplus\infty}$. □

Suppose (M, τ) is a tracial von Neumann algebra and $A \leq M$ is a maximal abelian $*$-subalgebra. Write $A = L^\infty(X, \mu)$ for some compact Hausdorff space X and some Borel probability measure μ on X. Let $\pi : C(X) \otimes C(X) \to B(L^2(M) \ominus L^2(A))$ be as in the definition of the left/right measure given in the introduction. Note that if ν is a left/right measure and if $\phi : C(X) \otimes C(X) \to L^\infty(X \times X, \nu)$ is the map sending an element of $C(X) \otimes C(X) \cong C(X \times X)$ to its $L^\infty(\nu)$-equivalence class, then there is a unique normal $*$-isomorphism $\rho : L^\infty(X \times X, \nu) \to \overline{\pi(C(X) \otimes C(X))}^{\mathrm{SOT}}$ such that $\pi = \rho \circ \phi$.

Corollary 4.2. *Let $M = L(\mathbb{F}_t)$ for some $t > 1$. Suppose that $A \leq M$ is abelian and a maximal amenable subalgebra of M. Write $A = L^\infty(X, \mu)$ for some compact metrizable space X and some Borel probability measure μ on X. Then the left/right measure of $A \leq M$ is absolutely continuous with respect to $\mu \otimes \mu$.*

Proof. Let $E : X \times X \to \mathrm{Proj}(L^2(M) \ominus L^2(A))$ be the spectral measure corresponding to the representation π defined as in the paragraph before Corollary 4.2. For a bounded, Borel map $\phi : X \times X \to \mathbb{C}$, we let

$$\tilde{\pi}(\phi) = \int \phi \, dE.$$

By Corollary 4.1, we know that $L^2(M) \ominus L^2(A)$ embeds into an infinite direct sum of $L^2(A) \otimes L^2(A)$ as an A-A bimodule. Thus, for any vector $\xi \in L^2(M) \ominus L^2(A)$, we may find a sequence $(k_n)_n \in L^2(X \times X)$ such that $\sum_n \int \|k_n\|_2^2 < +\infty$ and

$$\langle \pi(\phi)\xi, \xi \rangle = \sum_n \langle \phi k_n, k_n \rangle = \sum_n \iint \phi(x, y) |k_n(x, y)|^2 \, d\mu(x) \, d\mu(y) \quad \text{for all } \phi \in C(X \times X).$$

Set $K = \sum_n |k_n|^2$. Then, for every bounded, Borel $\phi : X \times X \to \mathbb{C}$, we have

$$\langle \tilde{\pi}(\phi)\xi, \xi \rangle = \iint \phi(x, y) K(x, y) \, d\mu(x) \, d\mu(y).$$

In particular, if $B \subseteq \mathbb{C}$ is Borel and $(\mu \otimes \mu)(B) = 0$, then

$$\|\tilde{\pi}(1_B)\xi\|_2^2 = \langle \tilde{\pi}(1_B)\xi, \xi \rangle = 0,$$

the first equality holding as $\tilde{\pi}(1_B)$ is a projection. Since this holds for every $\xi \in L^2(M) \ominus L^2(A)$, we see that $E(B) = \tilde{\pi}(1_B) = 0$. So we have shown that E is absolutely continuous with respect to $\mu \otimes \mu$. □

4.2. ***Generalizations of strong solidity.*** Throughout this section, we will be interested in properties of $\mathrm{W}^*(\mathcal{H}_{\text{anti-c}}(Q \leq M))$ for an inclusion $Q \leq M$ of tracial von Neumann algebras. Since $\mathcal{H}_{\text{anti-c}}(Q \leq M)$ is a subset of $L^2(M)$ and not of M, we need to explain what we mean by $\mathrm{W}^*(\mathcal{H}_{\text{anti-c}}(Q \leq M))$. Every $\xi \in L^2(M)$ may be identified with the densely defined, closed, unbounded operator L_ξ on $L^2(M)$; this L_ξ is the closure of the operator L_ξ^o which has $\mathrm{dom}(L_\xi^o) = M$ and is defined by $L_\xi^o(x) = \xi x$ for all $x \in M$. For $\xi \in L^2(M)$, we let $L_\xi = V_\xi |L_\xi|$ be its polar decomposition. For $X \subseteq L^2(M)$, we then define

$$\mathrm{W}^*(X) = \mathrm{W}^*\big(\{V_\xi : \xi \in X\} \cup \{\phi(|L_\xi|) : \xi \in X, \phi : [0, \infty) \to \mathbb{C} \text{ is bounded and Borel}\}\big).$$

Throughout this section, given a tracial von Neumann algebra (M, τ), we view $M \leq M^\omega$ by identifying it with the image of the constant sequences.

Definition 4.3. Let (M, τ) be a tracial von Neumann algebra. For a free ultrafilter $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$, we say that M is *ω-strongly solid* if $\mathrm{W}^*(N_{M^\omega}(Q)) \cap M$ is amenable for all diffuse, amenable $Q \leq M^\omega$. We say that M is *ultrasolid* if it is ω-strongly solid for every free ultrafilter ω. We say that M is *spectrally solid* if, for any diffuse, amenable $Q \leq M$, we have that $\mathrm{W}^*(\mathcal{H}_{\text{anti-c}}(Q \leq M))$ is amenable. Given a free ultrafilter $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$, we say that M is *spectrally ω-solid* if, for any diffuse, amenable $Q \leq M^\omega$, we have that $\mathrm{W}^*(\mathcal{H}_{\text{anti-c}}(Q \leq M^\omega)) \cap M$ is amenable. We say that M is *spectrally ultrasolid* if it is *spectrally ω-solid* for every free ultrafilter ω.

Corollary 4.4. *Fix $t > 1$.*

(i) *$L(\mathbb{F}_t)$ is spectrally ultrasolid.*

(ii) *If $Q \leq L(\mathbb{F}_t)$, $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ is a free ultrafilter and $Q' \cap L(\mathbb{F}_t)^\omega$ is diffuse, then Q is amenable.*

Proof. For notational simplicity, set $M = L(\mathbb{F}_t)$.

(i) Fix $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$. Let $Q \leq M^\omega$ be diffuse and amenable. Then, by properties (P8), (P3), (P9), and (P4),

$$h(\mathrm{W}^*(\mathcal{H}_{\text{anti-c}}(Q \leq M^\omega)) \cap M : M) = h(\mathrm{W}^*(\mathcal{H}_{\text{anti-c}}(Q \leq M^\omega)) \cap M : M^\omega) \leq h(\mathrm{W}^*(\mathcal{H}_{\text{anti-c}}(Q \leq M^\omega)) : M^\omega) = h(Q : M^\omega) \leq 0.$$

So $h(\mathrm{W}^*(\mathcal{H}_{\text{anti-c}}(Q \leq M^\omega)) \cap M : M) \leq 0$, which implies (Corollary 3.2) that $\mathrm{W}^*(\mathcal{H}_{\text{anti-c}}(Q \leq M^\omega)) \cap M$ is amenable.

(ii) Fix $A \leq Q' \cap M^\omega$ diffuse and abelian. Then $Q \leq W^*(\mathcal{N}_{M^\omega}(A)) \leq \mathrm{W}^*(\mathcal{H}_{\text{anti-c}}(A \leq M^\omega))$, and the result this follows from (i). □

In particular, Corollary 4.4 and Proposition 2.2 imply that if $P \leq L(\mathbb{F}_t)$ is a maximal amenable subalgebra, then $wI_{L(\mathbb{F}_t)}(P, P) \subseteq L^2(P)$, and so P is *strongly malnormal* in the sense of [Popa 2021].

We can take several iterations of this procedure in the ultraproduct setting and it will still have amenable intersection with the diagonal copy of $L(\mathbb{F}_t)$.

Corollary 4.5. *Fix $t > 1$ and a free ultrafilter $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$. Suppose that $Q \leq L(\mathbb{F}_t)^\omega$ is diffuse and amenable. Suppose we are given von Neumann subalgebras Q_α defined for ordinals α which satisfy the following properties*:

- $Q_0 = Q$.
- *If α is a successor ordinal, then $Q_{\alpha-1} \leq Q_\alpha \leq \mathrm{W}^*(\mathcal{H}_{\text{anti-c}}(Q_{\alpha-1} \leq L(\mathbb{F}_t)^\omega))$.*
- *If α is a limit ordinal, then $Q_\alpha = \overline{\bigcup_{\beta<\alpha} Q_\beta}^{\mathrm{SOT}}$.*

Then, for any ordinal α, we have that $Q_\alpha \cap L(\mathbb{F}_t)$ is amenable.

Proof. One applies properties (P8), (P3), (P9), (P7), and transfinite induction to see that

$$h(Q_\alpha : L(\mathbb{F}_t)^\omega) = 0$$

for any α. It then follows by property (P8) that

$$h(Q_\alpha \cap L(\mathbb{F}_t) : L(\mathbb{F}_t) = h(Q_\alpha \cap L(\mathbb{F}_t) : L(\mathbb{F}_t)^\omega) = h(Q_\alpha \cap L(\mathbb{F}_t) : L(\mathbb{F}_t)^\omega) \leq h(Q_\alpha : L(\mathbb{F}_t)^\omega) = 0.$$

The corollary now follows from Theorem 2.6. □

Corollary 4.6. *Fix $t > 1$. Then $L(\mathbb{F}_t)$ has the following Gamma stability property. Fix a free ultrafilter $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$. If $Q \leq L(\mathbb{F}_t)^\omega$ and $Q' \cap L(\mathbb{F}_t)^\omega$ is diffuse, then $Q \cap L(\mathbb{F}_t)$ is amenable.*

Proof. Fix $A \leq Q' \cap L(\mathbb{F}_t)^\omega$ diffuse and abelian. Note that

$$Q \leq \mathrm{W}^*(\mathcal{N}_{L(\mathbb{F}_t)^\omega}(A)) \leq W^*(\mathcal{H}_{\text{anti-c}}(A \leq L(\mathbb{F}_t)^\omega)).$$

Applying Corollary 4.4 with $Q_0 = A$ and $Q_\alpha = Q \vee A$ for all $\alpha \geq 1$, we see that $Q \cap L(\mathbb{F}_t) \leq (A \vee Q) \cap L(\mathbb{F}_t)$ is amenable. □

4.3. *Intertwining properties for Pinsker algebras.* In this section, we explore how Pinsker algebras behave with respect to intertwining properties in the sense of [Popa 2006a; 2006c].

Theorem 4.7. *Let (M, τ) be a tracial von Neumann algebra, and let $P, Q \leq M$ be Pinsker. Then exactly one of the following occurs*:

- *either there are nonzero projections $e \in P$, $f \in Q$ and a unitary $u \in \mathcal{U}(M)$ such that $u(ePe)u^* = fQf$*,
- *or $wI_M(Q, P) = \{0\}$.*

Proof. Suppose that $wI_M(Q, P) \neq \{0\}$, so there is a diffuse $Q_0 \leq Q$ with $Q_0 \prec P$. This means there are nonzero projections $f_0 \in Q_0$, $e_0 \in P$, a unital $*$-homomorphism $\Theta : f_0 Q_0 f_0 \to e_0 P e_0$, and a nonzero partial isometry $v \in M$ such that

- $xv = v\Theta(x)$ for all $x \in f_0 Q_0 f_0$,
- $vv^* \in (f_0 Q_0 f_0)' \cap f_0 M f_0$,
- $v^*v \in \Theta(f_0 Q_0 f_0)' \cap e_0 M e_0$.

By property (P9), we have

$$h\big(\mathrm{W}^*(\mathcal{N}_{f_0 M f_0}(f_0 Q_0 f_0)) : f_0 M f_0\big) \leq h(f_0 Q f_0 : f_0 M f_0) \leq 0,$$

and since $f_0 Q f_0 \leq f_0 M f_0$ is Pinsker by Proposition 3.4, we know that $\mathcal{N}_{f_0 M f_0}(f_0 Q_0 f_0) \subseteq \mathcal{U}(f_0 Q f_0)$. Similarly, $\mathcal{N}_{e_0 M e_0}(e_0 P e_0) = \mathcal{U}(e_0 P e_0)$. It then follows as in [de Santiago et al. 2021, Theorem 6.8] that $f = vv^* \in Q$ and $e = v^*v \in P$.

Since $f \in Q$, we have that $v^*(fQf)v$ is a subalgebra of eMe. Moreover, conjugation by v implements an isomorphism between the inclusion $fQf \leq fMf$ and the inclusion $v^*(fQf)v \leq eMe$, which implies that $h(v^*(fQf)v : eMe) \leq 0$. Moreover,

$$v^*(fQf)v \cap ePe \supseteq e\Theta(f_0 Q f_0).$$

Since $\Theta(f_0 Q f_0)$ is the image of a diffuse subalgebra under a normal $*$-homomorphism, it follows that it is diffuse. Since ePe is Pinsker by Proposition 3.4 (ii) and $h(v^*(fQf)v : eMe) \leq 0$, this forces $v^*(fQf)v = ePe$ by property (P6). Since M is finite, there is a unitary $u \in \mathcal{U}(M)$ such that $fu = v$. Then $u^*(fQf)u = ePe$, as desired. □

In the case where P and Q are factors, the first option in the dichotomy can be strengthened to saying that P and Q are unitarily conjugate.

Corollary 4.8. *Let (M, τ) be a tracial von Neumann algebra. Let P and Q be Pinsker subalgebras of M such that P and Q are factors. Then, either P and Q are unitarily conjugate, or $wI_M(P, Q) = 0$.*

This follows from the general fact that if M is a tracial von Neumann algebra and P and $Q \leq M$ are factors with unitarily conjugate corners, then they are unitarily conjugate. This is a folklore result and we include the proof here for completeness.

Proposition 4.9. *If Q, P are subalgebras of a tracial von Neumann algebra (M, τ) with unitarily conjugate corners and if P, Q are factors, then P, Q are unitarily conjugate.*

Proof. Choose nonzero projections $p \in P$, $q \in Q$ and a unitary partial isometry $v \in M$, with $v^*v = p$, $vv^* = q$, and

$$vPv^* = qQq.$$

Since P is a factor, we may shrink p, q if necessary to assume that $\tau(p) = 1/n$ for some integer n. Choose projections $p_1, \ldots, p_n$ in P which are pairwise orthogonal with $\tau(p_j) = 1/n$ for all j and with $p_1 = p$. Choose analogous projections $q_1, \ldots, q_n$ in Q with $q_1 = q$. Since P, Q are factors for $2 \leq j \leq n$, we may choose partial isometries a_j, b_j in P, Q such that $a_j^* a_j = p$, $a_j a_j^* = p_j$, $b_j^* b_j = q$, $b_j b_j^* = q_j$, and set $a_1 = p$, $b_1 = q$. Finally, let

$$u = \sum_i b_i v a_i^*.$$

Then u is a unitary, and if $x \in P$, then $u(a_i a_i^* x a_j a_j^*)u^* \in Q$ for all $1 \leq i, j \leq n$. Using that any $x \in P$ is equal to $\sum_{i,j} a_i a_i^* x a_j a_j^*$, we see that $uPu^* \subseteq Q$. By a symmetric argument, $u^*Qu \subseteq P$. □

The combination of the above results allows us to deduce Theorem 1.5 as follows.

Proof of Theorem 1.5. The fact that either (1) or (2) of Theorem 1.5 holds follows from Theorems 4.7 and 2.6. The "in particular" part follows from Corollary 4.8 and Theorem 2.6. □

Example 4.10. Dykema [1993] implies that $L(\mathbb{F}_2) \cong L^\infty[0, 1] * \mathcal{R}$. By Popa [1983a], $L^\infty[0, 1]$ and $\mathcal{R}$ are maximal amenable subalgebras in $L(\mathbb{F}_2)$. Hence, they are Pinsker subalgebras by Theorem 2.6. (Alternatively, [Hayes et al. 2021b] shows directly that they are Pinsker subalgebras.) Therefore, given any automorphism ϕ of $L(\mathbb{F}_2)$, by Theorem 4.7, $L^\infty[0, 1]$ and $\phi(\mathcal{R})$ in $L(\mathbb{F}_2)$ either have zero intertwiners or have unitarily conjugate corners. They do not have isomorphic corners; therefore $wI(L^\infty[0, 1], \phi(\mathcal{R})) = 0$.

Since the free product of any two amenable separable tracial von Neumann algebras is $L(\mathbb{F}_2)$, we can generalize this example quite a bit. To handle these more general examples, we want a strengthening of Theorem 4.7 along the lines of Corollary 4.8 that does not assume that P and Q are factors. First, we use a maximality argument to make the projection e in Theorem 4.7 as large as possible.

Theorem 4.11. *Let (M, τ) be a tracial von Neumann algebra, and let $P, Q \leq M$ be Pinsker subalgebras. There exist projections $e \in P$ and $f \in Q$ and a partial isometry $v \in M$ from e to f such that*

(1) $v(ePe)v^* = fQf$,

(2) *if $e \neq 1$ and u is a unitary such that $v = ue = fu$, then*

$$wI_{(1-f)M(1-f)}((1-f)Q(1-f), u(1-e)P(1-e)u^*) = 0.$$

Proof. Step 1: We show that there exists a choice of e and f satisfying (1) that is maximal, in the sense that no strictly larger projections satisfy (1). To this end, we will apply Zorn's lemma to the set of triples (e, f, v), where $e \in P$ and $f \in Q$ are projections and v is a partial isometry from e to f such that $v(ePe)v^* = fQf$. The partial order on this set will be given by $(e, f, v) \leq (e', f', v')$ if $e \leq e'$, $f \leq f'$, and $fv'e = v$. Note that $fv'e = v$ implies that $e = v^*v = e(v')^* fv'e$, which in turn implies

$$|v'e - v|^2 = e + e(v')^*v'e - 2\,\mathrm{Re}(e(v')^*v) = e + e(v')^*v'e - 2\,\mathrm{Re}(e(v')^* fv'e) = e(v')^*v'e - e \leq 0.$$

So $v'e = v$ and similarly $fv' = v$. One checks readily that the above order is a partial order. So it remains to show that every increasing chain $\{(e_\alpha, f_\alpha, v_\alpha)\}_{\alpha\in I}$ has an upper bound. Let $e = \sup_\alpha e_\alpha$ and $f = \sup_\alpha f_\alpha$. Note that, for $\alpha \leq \beta$, we have

$$v_\beta - v_\alpha = v_\beta e_\beta - v_\alpha = v_\beta(e_\beta - e_\alpha),$$

using our previous observation that $\alpha \leq \beta$ implies $v_\alpha = v_\beta e_\alpha$. Since e_α converges to e, it follows that $(e_\alpha)_{\alpha\in I}$ is Cauchy in $L^2(M)$; this in turn implies that $(v_\alpha)_{\alpha\in I}$ is Cauchy in $L^2(M)$ and hence converges to some $v \in L^2(M)$. This v is necessarily also a partial isometry, and $v_\alpha = f_\alpha v e_\alpha$. Moreover, we have $v(ePe)v^* \subseteq fQf$ because, for each $x \in P$, we have $v(e_\alpha x e_\alpha)v^* = v_\alpha(e_\alpha x e_\alpha)v_\alpha^* \in f_\alpha Q f_\alpha \subseteq fQf$. So taking the limit over α, we get $v(exe)v^* \in fQf$. The same reasoning shows that $v^* fQfv \subseteq ePe$, and hence $v(ePe)v^* = fQf$. Hence, by Zorn's lemma, there exists a maximal choice of e, f, and v.

Step 2: Now we will apply Theorem 4.7 to show that the maximal e, f, and v satisfy (2). Let u be a unitary such that $v = ue = fu$. Suppose for contradiction that

$$wI_{(1-f)M(1-f)}((1-f)Q(1-f), u(1-e)P(1-e)u^*) \neq \{0\}.$$

By Proposition 3.4 (ii), $(1-f)Q(1-f)$ is Pinsker in $(1-f)M(1-f)$. Note uPu^* is Pinsker in M, and hence $(1-f)uPu^*(1-f) = u(1-e)P(1-e)u^*$ is Pinsker in $(1-f)M(1-f)$. By Theorem 4.7,

$$wI_{(1-f)M(1-f)}((1-f)Q(1-f), u(1-e)P(1-e)u^*) \neq \{0\}$$

implies that there exist projections $e_0 \in u(1-e)P(1-e)u^*$ and $f_0 \in (1-f)Q(1-f)$ and a partial isometry v_0 from e_0 to f_0 that conjugates $e_0u(1-e)P(1-e)u^*e_0$ to $f_0(1-f)Q(1-f)f_0$. Write $e' = u^*e_0u$, so that e' is a projection in P with $e' \leq 1-e$. Let $f' = f_0$, which is a projection in Q with $f' \leq 1-f$. Let $v' = v_0u$, which is a partial isometry in M sending e' to f'. Then

$$v'e'Pe'(v')^* = v_0e_0u(1-e)P(1-e)u^*e_0v_0^* = f_0(1-f)Q(1-f)f_0 = f'Qf'.$$

By Proposition 3.4 (ii), $(f+f')Q(f+f')$ is Pinsker in $(f+f')M(f+f')$, and $(e+e')P(e+e')$ is Pinsker in $(e+e')M(e+e')$, so that

$$(v+v')(e+e')P(e+e')(v+v')^* = (f+f')(v+v')P(v+v')^*(f+f')$$

is Pinsker in $(f+f')M(f+f')$. Now $(f+f')Q(f+f')$ and $(v+v')(e+e')P(e+e')(v+v')^*$ contain the common diffuse subalgebra

$$fQf \oplus f'Qf' = (v+v')[ePe \oplus e'Pe'](v+v')^*.$$

Since $(f+f')Q(f+f')$ and $(v+v')(e+e')P(e+e')(v+v')^*$ are both Pinsker in $(f+f')Q(f+f')$ and intersect diffusely, they must be equal. This contradicts the maximality of (e, f, v) and thus establishes that $wI_{(1-f)M(1-f)}((1-f)Q(1-f), u(1-e)P(1-e)u^*) = \{0\}$. □

Theorem 4.11 implies the following corollary about projections in the Pinsker algebras P and Q. Note that, in the case where P and Q are both factors, (2) below reduces to saying e is either 0 or 1, which yields Corollary 4.8. Thus, the following corollary can be understood as a generalization of Corollary 4.8.

Corollary 4.12. *Let P and Q be Pinsker subalgebras of a tracial von Neumann algebra (M, τ). Let e, f, and v be as in the previous theorem. Then the following hold*:

(1) *Let $e_1, e_2 \in P$ with $e_1 \le 1-e$, $e_2 \le e$, and $e_1 \sim_P e_2$. Let f_1 and f_2 satisfy the analogous conditions for Q and f. Then $ve_2v^* \wedge f_2 = 0$.*

(2) *In particular, if Q is a factor, then e is central in P.*

Proof. (1) Suppose $e_1, e_2 \in P$ with $e_1 \le 1-e$, $e_2 \le e$, and $e_1 \sim_P e_2$. Let f_1 and f_2 satisfy the analogous conditions for Q and f. Suppose for contradiction that $ve_2v^* \wedge f_2 \ne 0$. Let $f_2' = ve_2v^* \wedge f_2$. Let f_1' be the corresponding subprojection of f_1. Let $e_2' = v^* f_2' v$, and let e_1' be the corresponding subprojection of e_1. Then $e_1' P e_1'$ is unitarily conjugate to $e_2' P e_2'$ using the partial isometry that transforms e_1 to e_2, and similarly $f_1' Q f_1'$ is unitarily conjugate to $f_2' Q f_2'$, and $e_2' P e_2'$ is unitarily conjugate to $f_2' Q f_2'$ using v. This implies that

$$wI_{(1-f)M(1-f)}((1-f)Q(1-f), v(1-e)P(1-e)v^*) \ne \{0\}$$

(or alternatively, it directly contradicts the maximality in Step 2 of the previous proof).

(2) Suppose Q is a factor, and assume for contradiction that e is not central in P. Then there must exist some projections $e_1, e_2 \in P$ with $e_1 \le 1-e$, $e_2 \le e$, and $e_1 \sim_P e_2$. Because Q is a factor, there exist projections f_1 and f_2 satisfying the analogous conditions for Q and f and with $f_2 = ve_2v^*$. Hence, we get a contradiction from (1). □

Example 4.13. Dykema [1993, Theorem 4.6] showed that if A and B are SOT-separable diffuse amenable tracial von Neumann algebras, then $A * B \cong L(\mathbb{F}_2)$. Taking two such pairs, there is an isomorphism $\alpha : A_1 * B_1 \to A_2 * B_2 = M$; let α be any such isomorphism. Note that A_1 and A_2 are Pinsker subalgebras by [Hayes et al. 2021b], and hence Theorem 4.7 applies to A_1 and A_2.

- In particular, suppose that $A_1 = \mathcal{R}$ and $A_2 = \mathcal{R}$. Then either $\alpha(A_1)$ and A_2 are unitarily conjugate, or $wI_M(A_2, \alpha(A_1)) = \{0\}$.
- Suppose that $A_1 = \mathcal{R} \oplus \mathcal{R}$ and $A_2 = \mathcal{R}$. Then the projection e from Theorem 4.11 must be central in A_1, and hence there are only four possible choices of e, resulting in a tetrachotomy.
- Generalizing Example 4.10, suppose $A_1 = L^\infty[0, 1]$ and $A_2 = \mathcal{R}$. Then, for any nonzero projections $e \in \alpha(A_1)$ and $f \in A_2$, the von Neumann algebras $e\alpha(A_1)e$ and fA_2f are not isomorphic. Hence the projection e in Theorem 4.11 must be zero, and hence $wI_M(A_2, \alpha(A_1)) = \{0\}$.

Appendix: Invariance of 1-bounded entropy via noncommutative functional calculus

A.1. ***Microstate spaces and definition of h.*** Here we recall definitions of the space of noncommutative laws. Let $\mathbb{C}\langle t_i : i \in I\rangle$ be the algebra of noncommutative complex polynomials in $(t_i)_{i\in I}$ (i.e., the free $\mathbb{C}$-algebra on the set I). We give $\mathbb{C}\langle t_i : i \in I\rangle$ the unique $*$-algebra structure which makes the t_i self-adjoint. If $\mathcal{M}$ is a W*-algebra and $\boldsymbol{x} = (x_i)_{i\in I} \in \mathcal{M}_{\text{sa}}^I$, then there is a unique $*$-homomorphism

$$\text{ev}_{\boldsymbol{x}} : \mathbb{C}\langle t_i : i \in I\rangle \to \mathcal{M}$$

such that $\text{ev}_{\boldsymbol{x}}(t_i) = x_i$. For a noncommutative polynomial $p \in \mathbb{C}\langle t_i : i \in I\rangle$, we define $p(\boldsymbol{x}) = p((x_i)_{i\in I})$ to be $\text{ev}_{\boldsymbol{x}}(p)$.

A tracial *noncommutative law* of a self-adjoint I-tuple is a linear functional $\lambda : \mathbb{C}\langle t_i : i \in I\rangle \to \mathbb{C}$ that is

(1) unital, that is, $\lambda(1) = 1$;

(2) positive, that is, $\lambda(p^* p) \geq 0$;

(3) tracial, that is, $\lambda(pq) = \lambda(qp)$;

(4) exponentially bounded, that is, for some $(R_i)_{i\in I} \in (0, +\infty)^I$, we have

$$|\lambda(t_{i(1)} \cdots t_{i(\ell)})| \leq R_{i(1)} \cdots R_{i(\ell)}$$

for all ℓ and all $i(1), \ldots, i(\ell) \in I$.

Given $\boldsymbol{R} = (R_i)_{i\in I} \in (0, +\infty)^I$, we define

$$\Sigma_{\boldsymbol{R}} = \Sigma_{(R_i)_{i\in I}}$$

to be the set of noncommutative laws satisfying (4) for our given choice of $(R_i)_{i\in I}$. We equip $\Sigma_{\boldsymbol{R}}$ with the topology of pointwise convergence on $\mathbb{C}\langle t_i : i \in I\rangle$, which makes it a compact Hausdorff space.

For a tracial W*-algebra $(\mathcal{M}, \tau)$, a tuple $\boldsymbol{x} = (x_i)_{i\in I} \in \mathcal{M}_{\text{sa}}^I$, and $\boldsymbol{R} = (R_i)_{i\in I} \in (0, +\infty)^I$ satisfying $\|x_i\| \leq R_i$, we define the *noncommutative law of* $\boldsymbol{x}$ as the map

$$\lambda_{\boldsymbol{x}} : \mathbb{C}\langle t_i : i \in I\rangle \to \mathbb{C}, \quad p \mapsto \tau(p(x)).$$

It is straightforward to verify that $\lambda_{\boldsymbol{x}}$ is in $\Sigma_{\boldsymbol{R}}$. Conversely, given any $\lambda \in \Sigma_{\boldsymbol{R}}$, there exists some $(\mathcal{M}, \tau)$ and $\boldsymbol{x} \in \mathcal{M}_{\text{sa}}^I$ such that $\lambda_{\boldsymbol{x}} = \lambda$ and $\|x_i\| \leq R_i$ for all $i \in I$; see either [Belinschi and Nica 2008, Proposition 4.2] or the proof of [Anderson et al. 2010, Proposition 5.2.14 (d)]. We also remark that $(\mathcal{M}, \tau)$ could be $M_n(\mathbb{C})$ with the normalized trace $\tau_n = (1/n)\,\text{Tr}$. Thus, if $\boldsymbol{x} \in M_n(\mathbb{C})_{\text{sa}}^I$, then $\lambda_{\boldsymbol{x}}$ is a well-defined noncommutative law.

The 1-bounded entropy h is defined in terms of Voiculescu's microstate spaces.

Definition A.1 (microstate space). Let $\mathcal{M}$ be a tracial von Neumann algebra and I an index set. Let $\boldsymbol{R} \in (0, +\infty)^I$, let $\boldsymbol{y} \in \mathcal{M}_{\text{sa}}^I$ be a self-adjoint tuple with $\|y_i\| \leq R_i$, and let $\mathcal{O} \subseteq \Sigma_{\boldsymbol{R}}$ be a neighborhood of $\lambda_{\boldsymbol{y}}$. Then we define the microstate space

$$\Gamma_{\boldsymbol{R}}^{(n)}(\mathcal{O}) := \{\boldsymbol{Y} \in M_n(\mathbb{C})_{\text{sa}}^I : \|Y_i\| \leq R_i \text{ for all } i \in I \text{ and } \lambda_{\boldsymbol{Y}} \in \mathcal{O}\}.$$

Definition A.2 (orbital covering numbers). Let I be an index set, and let $\Omega \subset M_n(\mathbb{C})^I_{\mathrm{sa}}$. For $F \subseteq I$ finite and $\varepsilon > 0$, we define the *orbital* (F, ε)*-neighborhood* of Ω as the set

$$N^{\mathrm{orb}}_{F,\varepsilon}(\Omega) = \{\boldsymbol{Y} \in M_n(\mathbb{C})^I_{\mathrm{sa}} : \text{there exists } \boldsymbol{Y}' \in \Omega,\ U \in \mathcal{U}(M_n(\mathbb{C})),\ \|Y_i - UY_i'U^*\|_2 < \varepsilon \text{ for } i \in F\}.$$

Moreover, we define the orbital covering number $K^{\mathrm{orb}}_{F,\varepsilon}(\Omega)$ as the minimal cardinality of a set Ω' such that $\Omega \subseteq N^{\mathrm{orb}}_{F,\varepsilon}(\Omega')$.

Definition A.3. Let (M, τ) be a tracial von Neumann algebra, let I and J be index sets, let $\boldsymbol{R} \in (0,\infty)^I$ and $\boldsymbol{S} \in (0,\infty)^J$, and let $\boldsymbol{x} \in M^I_{\mathrm{sa}}$ and $\boldsymbol{y} \in M^J_{\mathrm{sa}}$, with $\|x_i\| \le R_i$ for $i \in I$ and $\|y_j\| \le S_j$ for $j \in J$. For a neighborhood $\mathcal{O}$ of $\lambda_{(\boldsymbol{x},\boldsymbol{y})}$ in $\Sigma_{(\boldsymbol{R},\boldsymbol{S})}$, consider the microstate space $\Gamma^{(n)}_{(\boldsymbol{R},\boldsymbol{S})}(\mathcal{O}) \subseteq M_n(\mathbb{C})^{I\sqcup J}_{\mathrm{sa}}$, and let $\pi_I(\Gamma^{(n)}_{(\boldsymbol{R},\boldsymbol{S})}(\mathcal{O}))$ be its projection onto the I-indexed coordinates. Then define

$$h_{\boldsymbol{R},\boldsymbol{S}}(\boldsymbol{x} : \boldsymbol{y}) = \sup_{\substack{F \subseteq I \text{ finite} \\ \varepsilon > 0}} \inf_{\mathcal{O} \ni \lambda_{(\boldsymbol{x},\boldsymbol{y})}} \limsup_{n\to\infty} \frac{1}{n^2} \log K^{\mathrm{orb}}_{F,\varepsilon}(\pi_I(\Gamma^{(n)}_{(\boldsymbol{R},\boldsymbol{S})}(\mathcal{O}))).$$

In this appendix we will give an argument showing, at the same time, that this computation yields the same result for every $\boldsymbol{R}$ and $\boldsymbol{S}$ with $R_i \ge \|x_i\|$ and $S_j \ge \|y_j\|$, and that $h(\boldsymbol{x} : \boldsymbol{y})$ only depends on $\mathrm{W}^*(\boldsymbol{x})$ and $\mathbf{W}^*(\boldsymbol{x}, \boldsymbol{y})$ (and the restriction of the trace to these algebras).

A.2. ***L^2-continuous functional calculus.*** Here we recall the construction of a certain space of noncommutative functions given as L^2-limits of trace polynomials, uniformly over all noncommutative laws. Trace polynomials have been studied in many previous works such as [Cébron 2013; Dabrowski et al. 2021; Driver et al. 2013; Jing 2015; Kemp 2016; 2017; Procesi 1976; Razmyslov 1974; 1985; Rains 1997; Sengupta 2008]. The uniform L^2-completion of trace polynomials was first introduced in [Jekel 2020a; 2020b; 2022], and its relationship with continuous model theory was addressed in [Jekel 2023, §3.5]. Moreover, [Hayes et al. 2021b, Remark 3.5] described how this space is an example of the tracial completions of C*-algebras studied in [Ozawa 2013, p. 351-352] and [Bosa et al. 2019] and implicitly in [Carrión et al. 2023, §6]. Here we follow the version of the construction in [Hayes et al. 2021b, §3].

Definition A.4. Fix an index set I and $\boldsymbol{R} \in (0, +\infty)^I$. Consider the space

$$\mathcal{A}_{\boldsymbol{R}} = C(\Sigma_{\boldsymbol{R}}) \otimes \mathbb{C}\langle t_i : i \in I\rangle.$$

Given $(\mathcal{M}, \tau)$ and $\boldsymbol{x} \in \mathcal{M}^I_{\mathrm{sa}}$ with $\|x_i\| \le R_i$, we define the evaluation map

$$\mathrm{ev}_{\boldsymbol{x}} : \mathcal{A}_{\boldsymbol{R}} \to \mathcal{M}, \quad \phi \otimes p \mapsto \phi(\lambda_{\boldsymbol{x}}) p(\boldsymbol{x}).$$

Then we define a semi-norm on $C(\Sigma_{\boldsymbol{R}}) \otimes \mathbb{C}\langle t_i : i \in I\rangle$ by

$$\|f\|_{\boldsymbol{R},2} = \sup_{(\mathcal{M},\tau),\boldsymbol{x}} \|\mathrm{ev}_{\boldsymbol{x}}(f)\|_{L^2(\mathcal{M})},$$

where the supremum is over all tracial W*-algebras $(\mathcal{M}, \tau)$ and all $\boldsymbol{x} \in \mathcal{M}^I_{\mathrm{sa}}$ with $\|x_i\| \le R_i$. Denote by $\mathcal{F}_{\boldsymbol{R},2}$ the completion of $\mathcal{A}_{\boldsymbol{R}}/\{f \in \mathcal{A}_{\boldsymbol{R}} : \|f\|_{\boldsymbol{R},2} = 0\}$.

It is immediate that, for every $(\mathcal{M},\tau)$, for every self-adjoint tuple $\boldsymbol{x} \in \mathcal{M}_{\mathrm{sa}}^I$ with $\|x_i\| \le R_i$, the evaluation map $\mathrm{ev}_{\boldsymbol{x}} : \mathcal{A}_{\boldsymbol{R}} \to \mathcal{M}$ passes to a well-defined map $\mathcal{F}_{\boldsymbol{R},2} \to L^2(\mathcal{M})$, which we continue to denote by $\mathrm{ev}_{\boldsymbol{x}}$, and we will also define $f(\boldsymbol{x}) = \mathrm{ev}_{\boldsymbol{x}}(f)$. Moreover, it is clear that $f(\boldsymbol{x}) = \mathrm{ev}_{\boldsymbol{x}}(f)$ always lies in $L^2(\mathrm{W}^*(\boldsymbol{x}))$ because this holds when f is a simple tensor.

It will be convenient often to restrict our attention to elements of $\mathcal{F}_{\boldsymbol{R},2}$ that are bounded in operator norm. For $f \in \mathcal{F}_{\boldsymbol{R},2}$, let us define

$$\|f\|_{\boldsymbol{R},\infty} = \sup_{(\mathcal{M},\tau),\boldsymbol{x}} \|\mathrm{ev}_{\boldsymbol{x}}(f)\|$$

and then set

$$\mathcal{F}_{\boldsymbol{R},\infty} = \{f \in \mathcal{F}_{\boldsymbol{R},2} : \|f\|_{\boldsymbol{R},\infty} < +\infty\}.$$

It was shown in [Hayes et al. 2021b, Lemma 3.3] that $\mathcal{F}_{\boldsymbol{R},\infty}$ is a C*-algebra with respect to the norm $\|\cdot\|_{\boldsymbol{R},\infty}$ and the multiplication and $*$-operation arising from the natural ones on simple tensors.

One of the most useful properties of $\mathcal{F}_{\boldsymbol{R},\infty}$ is that it allows all the elements of a von Neumann algebra to be expressed as a function of the generators. More precisely, [Hayes et al. 2021b, Proposition 3.4] showed the following.

Proposition A.5. *Given $(\mathcal{M},\tau)$ and $\boldsymbol{x} \in \mathcal{M}_{\mathrm{sa}}^I$ with $\|x_i\| \le R_i$, the evaluation map $\mathrm{ev}_{\boldsymbol{x}} : \mathcal{F}_{\boldsymbol{R},2} \to L^2(\mathrm{W}^*(\boldsymbol{x}))$ is surjective. It also restricts to a surjective $*$-homomorphism $\mathcal{F}_{\boldsymbol{R},\infty} \to \mathrm{W}^*(\boldsymbol{x})$.*

This surjectivity property on its own is not too significant because, for instance, the double dual of the C*-universal free product of $C[-R_i, R_i]$ over $i \in I$ can be used to define a functional calculus that is surjective. The benefit of the construction used here is that it achieves surjectivity at the *same time* as relatively strong continuity properties.

First, we show that the noncommutative law of the output will depend continuously on the noncommutative law of the input. As we will see later, this property allows these functions to transform between microstate spaces for different generators of a von Neumann algebra. To fix notation, let I and I' be index sets. Let $\boldsymbol{R} \in (0,+\infty)^I$ and $\boldsymbol{R}' \in (0,+\infty)^{I'}$. We define

$$\mathcal{F}_{\boldsymbol{R},\boldsymbol{R}'} = \{\boldsymbol{f} = (f_i)_{i \in I'} \in (\mathcal{F}_{\boldsymbol{R},\infty})_{\mathrm{sa}}^{I'} : \|f_i\|_{\boldsymbol{R},\infty} \le R_i' \text{ for all } i \in I'\}.$$

Proposition A.6 [Hayes et al. 2021b, Proposition 3.7]. *Let $\boldsymbol{R} \in (0,+\infty)^I$ and $\boldsymbol{R}' \in (0,+\infty)^{I'}$. Let $\boldsymbol{f} = (f_i)_{i \in I'} \in \mathcal{F}_{\boldsymbol{R},\boldsymbol{R}'}$.*

(1) *Given $(\mathcal{M},\tau)$ and $\boldsymbol{x} \in \mathcal{M}_{\mathrm{sa}}^I$ with $\|x_i\| \le R_i$, we set $\boldsymbol{f}(\boldsymbol{x}) = (f_i(\boldsymbol{x}))_{i \in I'}$. Then $\lambda_{\boldsymbol{f}(\boldsymbol{x})}$ is uniquely determined by $\lambda_{\boldsymbol{x}}$.*

(2) *Let $\boldsymbol{f}_*$ be the "push-forward" mapping $\Sigma_{\boldsymbol{R}} \to \Sigma_{\boldsymbol{R}'}$ defined by $\boldsymbol{f}_*\lambda_{\boldsymbol{x}} = \lambda_{\boldsymbol{f}(\boldsymbol{x})}$ for all such tuples $\boldsymbol{x}$. Then $\boldsymbol{f}_*$ is continuous.*

Another consequence of this is that these spaces of functions are closed under composition.

Proposition A.7. *Fix index sets I, I', and I'' and corresponding tuples $\boldsymbol{R}$, $\boldsymbol{R}'$, and $\boldsymbol{R}''$ from $(0,\infty)$. Let $\boldsymbol{f} \in \mathcal{F}_{\boldsymbol{R},\boldsymbol{R}'}$ and $\boldsymbol{g} \in \mathcal{F}_{\boldsymbol{R}',\boldsymbol{R}''}$. Then there exists a unique $\boldsymbol{h} \in \mathcal{F}_{\boldsymbol{R},\boldsymbol{R}'}$ such that $\boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{f}(\boldsymbol{x}))$ for all M and $\boldsymbol{x} \in M^I$ with $\|x_i\| \le R_i$.*

Proof. First, we consider $g \in \mathcal{F}_{\boldsymbol{R}',2}$ and show that $g \circ \boldsymbol{f}$ is a well-defined element of $\mathcal{F}_{\boldsymbol{R}',2}$. If g is a simple tensor $\phi \otimes p$, then $\phi(\lambda_{\boldsymbol{f}(\boldsymbol{x})}) = \phi \circ \boldsymbol{f}_* \lambda_{\boldsymbol{x}}$ defines a continuous function on the space of laws by the previous proposition. Also, since $\mathcal{F}_{\mathcal{R},\infty}$ is a C*-algebra, $p \circ \boldsymbol{f} \in \mathcal{F}_{\boldsymbol{R},\infty}$ and hence so is $g \circ \boldsymbol{f} = (\phi \circ \boldsymbol{f}_* \otimes 1)(p \circ \boldsymbol{f})$.

Next, if g is a linear combination of simple tensors, one can check directly that $\|g \circ \boldsymbol{f}\|_{\boldsymbol{R},2} \leq \|g\|_{\boldsymbol{R}',2}$ by considering evaluations on all possible tuples. This estimate allows us to pass to the completion, showing that if $g \in \mathcal{F}_{\boldsymbol{R}',2}$, then $g \circ \boldsymbol{f} \in \mathcal{F}_{\boldsymbol{R},2}$.

Again, by evaluating on points, we see that $\|g \circ \boldsymbol{f}\|_{\boldsymbol{R},\infty} \leq \|g\|_{\boldsymbol{R}',\infty}$. Replacing the single function g by an I''-tuple yields the asserted result. □

The second continuity property that we need for $\mathcal{F}_{\boldsymbol{R},\boldsymbol{R}'}$ is uniform continuity with respect to the L^2 norm. This will allow us to use the functions in $\mathcal{F}_{\boldsymbol{R},\boldsymbol{R}'}$ to "push forward" ε-coverings from one microstate space to another.

Proposition A.8 [Hayes et al. 2021b, Proposition 3.9]. *Let I be an index set and $\boldsymbol{R} \in (0, +\infty)^I$, and let $f \in \mathcal{F}_{\boldsymbol{R},2}$. Then, for every $\varepsilon > 0$, there exists a finite $F \subseteq I$ and a $\delta > 0$ such that, for every $(\mathcal{M}, \tau)$ and $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{M}_{\mathrm{sa}}^I$ with $\|x_i\|, \|y_i\| \leq R_i$, if $\|x_i - y_i\|_2 < \delta$ for all $i \in F$, then $\|f(\boldsymbol{x}) - f(\boldsymbol{y})\|_2 < \varepsilon$.*

A.3. *Proof of monotonicity and invariance.*

Theorem A.9. *Let (M, τ) be a tracial von Neumann algebra. Let I and J be index sets, let $\boldsymbol{R} \in (0, \infty)^I$ and $\boldsymbol{S} \in (0, \infty)^J$, and let $\boldsymbol{x} \in M_{\mathrm{sa}}^I$ and $\boldsymbol{y} \in M_{\mathrm{sa}}^J$, with $\|x_i\| \leq R_i$ for $i \in I$ and $\|y_j\| \leq S_j$ for $j \in J$. Let I', J', $\boldsymbol{R}'$, $\boldsymbol{S}'$, $\boldsymbol{x}' \in M_{\mathrm{sa}}^{I'}$, $\boldsymbol{y}' \in M_{\mathrm{sa}}^{J'}$ satisfy similar conditions. Suppose that*

$$\mathrm{W}^*(\boldsymbol{x}, \boldsymbol{y}) \supseteq \mathrm{W}^*(\boldsymbol{x}', \boldsymbol{y}') \quad \textit{and} \quad \mathrm{W}^*(\boldsymbol{x}) \subseteq \mathrm{W}^*(\boldsymbol{x}').$$

Then

$$h_{\boldsymbol{R},\boldsymbol{S}}(\boldsymbol{x} : \boldsymbol{y}) \leq h_{\boldsymbol{R}',\boldsymbol{S}'}(\boldsymbol{x}' : \boldsymbol{y}').$$

Proof. Unwinding the suprema and infima in the definition of h, it suffices to show that, for every $F \subseteq I$ finite and $\varepsilon > 0$, there exists $F' \subseteq I'$ finite and $\varepsilon' > 0$ such that, for every neighborhood $\mathcal{O}'$ of $\lambda_{(\boldsymbol{x}',\boldsymbol{y}')}$ in $\Sigma_{(\boldsymbol{R}',\boldsymbol{S}')}$, there exists a neighborhood $\mathcal{O}$ of $\lambda_{(\boldsymbol{x},\boldsymbol{y})}$ in $\Sigma_{(\boldsymbol{R},\boldsymbol{S})}$ such that

$$K_{F,\varepsilon}^{\mathrm{orb}}(\pi_I(\Gamma_{(\boldsymbol{R},\boldsymbol{S})}^{(n)}(\mathcal{O}))) \leq K_{F',\varepsilon'}^{\mathrm{orb}}(\pi_{I'}(\Gamma_{(\boldsymbol{R}',\boldsymbol{S}')}^{(n)}(\mathcal{O}'))).$$

Fix F and ε. Because $\mathrm{W}^*(\boldsymbol{x}) \subseteq \mathrm{W}^*(\boldsymbol{x}')$, there exists $\boldsymbol{f} \in \mathcal{F}_{\boldsymbol{R}',\boldsymbol{R}}$ such that $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{x}')$. By Proposition A.8, there exists $\varepsilon' > 0$ and $F' > 0$ such that, for all tracial von Neumann algebras N and all $\boldsymbol{z}, \boldsymbol{w} \in N^{I'}$ with $\|z_i\|, \|w_i\| \leq R_i'$, we have that $\max_{i \in F'} \|z_i - w_i\|_2 < 2\varepsilon'$ implies $\max_{i \in F} \|f_i(\boldsymbol{z}) - f_i(\boldsymbol{w})\|_2 < \frac{1}{2}\varepsilon$.

Now fix a neighborhood $\mathcal{O}'$ of $\lambda_{\boldsymbol{x}',\boldsymbol{y}'}$ in $\Sigma_{(\boldsymbol{R}',\boldsymbol{S}')}$. We claim that

$$K_{F,\varepsilon/2}^{\mathrm{orb}}(\boldsymbol{f}(\pi_{I'}(\Gamma_{\boldsymbol{R}',\boldsymbol{S}'}^{(n)}(\mathcal{O}')))) \leq K_{F',\varepsilon'}^{\mathrm{orb}}(\pi_{I'}(\Gamma_{\boldsymbol{R}',\boldsymbol{S}'}^{(n)}(\mathcal{O}'))).$$

Indeed, let Ω be a set of cardinality $K_{F',\varepsilon'}^{\mathrm{orb}}(\pi_I(\Gamma_{\boldsymbol{R},\boldsymbol{S}}^{(n)}(\mathcal{O}')))$ such that $\pi_I(\Gamma_{\boldsymbol{R},\boldsymbol{S}}^{(n)}(\mathcal{O}')) \subseteq N_{F',\varepsilon'}^{\mathrm{orb}}(\Omega)$. Let $\Omega' \subseteq \pi_I(\Gamma_{\boldsymbol{R},\boldsymbol{S}}^{(n)}(\mathcal{O}'))$ be chosen to have one element within ε' of each element of Ω, so that

$$\pi_I(\Gamma_{\boldsymbol{R},\boldsymbol{S}}^{(n)}(\mathcal{O}')) \subseteq N_{F',2\varepsilon'}^{\mathrm{orb}}(\Omega').$$

Then each $\boldsymbol{X}' \in \Omega'$, and more generally in $\pi_I(\Gamma^{(n)}_{\boldsymbol{R},\boldsymbol{S}}(\mathcal{O}'))$ satisfies $\|X_i'\| \leq R_i$, so that it is valid to apply the uniform continuity estimate for $\boldsymbol{f}$ to such points $\boldsymbol{X}'$. The choice of (F, ε) thus implies that

$$\boldsymbol{f}(\pi_{I'}(\Gamma^{(n)}_{\boldsymbol{R}',\boldsymbol{S}'}(\mathcal{O}'))) \subseteq N^{\mathrm{orb}}_{F',\varepsilon'/2}(\boldsymbol{f}(\Omega')),$$

which proves our claim about the covering numbers.

Next, we describe how to choose $\mathcal{O}$. Since $\mathrm{W}^*(\boldsymbol{x}', \boldsymbol{y}') \subseteq \mathrm{W}^*(\boldsymbol{x}, \boldsymbol{y})$, there exists $\boldsymbol{g} \in \mathcal{F}_{(\boldsymbol{R},\boldsymbol{S}),(\boldsymbol{R}',\boldsymbol{S}'))}$ such that $(\boldsymbol{x}', \boldsymbol{y}') = \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y})$. By continuity of $\boldsymbol{g}_* : \Sigma_{(\boldsymbol{R},\boldsymbol{S})} \to \Sigma_{(\boldsymbol{R}',\boldsymbol{S}')}$, the set

$$\mathcal{O}_1 = (\boldsymbol{g}_*)^{-1}(\mathcal{O}')$$

is open. Let

$$\mathcal{O}_2 = \left\{\lambda_{(\boldsymbol{z},\boldsymbol{w})} \in \Sigma_{(\boldsymbol{R},\boldsymbol{S})} : \max_{i \in F} \| f_i \circ \pi_{I'} \circ \boldsymbol{g}(\boldsymbol{z}, \boldsymbol{w}) - z_i \|_2 < \tfrac{1}{2}\varepsilon\right\}.$$

The set $\mathcal{O}_2$ is open using Propositions A.7 and A.6. It also contains $\lambda_{(\boldsymbol{x},\boldsymbol{y})}$ because

$$\boldsymbol{f}(\pi_{I'}(\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}))) = \boldsymbol{f}(\pi_{I'}(\boldsymbol{x}', \boldsymbol{y}')) = \boldsymbol{f}(\boldsymbol{x}') = \boldsymbol{x}.$$

Let $\mathcal{O} = \mathcal{O}_1 \cap \mathcal{O}_2$. We claim that

$$\pi_I \Gamma^{(n)}_{\boldsymbol{R},\boldsymbol{S}}(\mathcal{O}) \subseteq N^{\mathrm{orb}}_{F,\varepsilon/2}(\boldsymbol{f}(\pi_{I'}(\Gamma^{(n)}_{\boldsymbol{R}',\boldsymbol{S}'}(\mathcal{O}')))).$$

Indeed, if $\boldsymbol{X}$ is in the set on the left-hand side, then there exists $\boldsymbol{Y}$ such that $\lambda_{(\boldsymbol{X},\boldsymbol{Y})} \in \mathcal{O}$. In particular, this means that $\lambda_{\boldsymbol{g}(\boldsymbol{X},\boldsymbol{Y})} \in \mathcal{O}'$, or in other words $\boldsymbol{g}(\boldsymbol{X}, \boldsymbol{Y}) \in \Gamma^{(n)}_{\boldsymbol{R}',\boldsymbol{S}'}(\mathcal{O}')$. Moreover,

$$\max_{i \in F} \| X_i - f_i \circ \pi_{I'} \circ \boldsymbol{g}(\boldsymbol{X}, \boldsymbol{Y}) \|_2 < \tfrac{1}{2}\varepsilon.$$

Therefore, $\boldsymbol{X}$ is in the $(F, \varepsilon/2)$-neighborhood of $\boldsymbol{f} \circ \pi_{I'}$ of some point in $\Gamma^{(n)}_{\boldsymbol{R}',\boldsymbol{S}'}(\mathcal{O}')$, which proves the claimed inclusion.

This inclusion $\pi_I \Gamma^{(n)}_{\boldsymbol{R},\boldsymbol{S}}(\mathcal{O}) \subseteq N^{\mathrm{orb}}_{F,\varepsilon/2}(\boldsymbol{f}(\pi_{I'}(\Gamma^{(n)}_{\boldsymbol{R}',\boldsymbol{S}'}(\mathcal{O}'))))$ in turn implies that

$$K^{\mathrm{orb}}_{F,\varepsilon}(\pi_I \Gamma^{(n)}_{\boldsymbol{R},\boldsymbol{S}}(\mathcal{O})) \leq K^{\mathrm{orb}}_{F,\varepsilon/2}(\boldsymbol{f}(\pi_{I'}(\Gamma^{(n)}_{\boldsymbol{R}',\boldsymbol{S}'}(\mathcal{O}')))) \leq K^{\mathrm{orb}}_{F',\varepsilon'}(\pi_{I'}(\Gamma^{(n)}_{(\boldsymbol{R}',\boldsymbol{S}')}(\mathcal{O}'))),$$

where the second inequality is the earlier claim that we proved. □

This theorem implies the following:

- In the case where $\boldsymbol{x} = \boldsymbol{x}'$ and $\boldsymbol{y} = \boldsymbol{y}'$, the theorem shows that $h_{\boldsymbol{S},\boldsymbol{R}}(\boldsymbol{x} : \boldsymbol{y})$ is independent of $\boldsymbol{R}$ and $\boldsymbol{S}$ so long as $\|x_i\| \leq R_i$ for $i \in I$ and $\|y_j\| \leq S_j$ for $j \in J$. Thus, we may unambiguously write $h(\boldsymbol{x} : \boldsymbol{y})$.
- In the case where $\mathrm{W}^*(\boldsymbol{x}, \boldsymbol{y}) = \mathrm{W}^*(\boldsymbol{x}', \boldsymbol{y}')$ and $\mathrm{W}^*(\boldsymbol{x}) = \mathrm{W}^*(\boldsymbol{x}')$, the theorem shows that $h(\boldsymbol{x} : \boldsymbol{y}) = h(\boldsymbol{x}' : \boldsymbol{y}')$. Hence, for $N \leq M$, we may unambiguously define $h(N : M)$ as $h(\boldsymbol{x} : \boldsymbol{y})$ for some tuples $\boldsymbol{x}$ and $\boldsymbol{y}$ such that $N = \mathrm{W}^*(\boldsymbol{x})$ and $M = \mathrm{W}^*(\boldsymbol{x}, \boldsymbol{y})$.
- Now suppose that $P \leq N \leq M$. Applying the theorem in the case where $\mathrm{W}^*(\boldsymbol{x}, \boldsymbol{y}) = \mathrm{W}^*(\boldsymbol{x}', \boldsymbol{y}') = M$ and $P = \mathrm{W}^*(\boldsymbol{x}) \subseteq \mathrm{W}^*(\boldsymbol{x}') = N$, we obtain $h(P : M) \leq h(N : M)$.
- Again, suppose $P \leq N \leq M$. Applying the theorem in the case where $\mathrm{W}^*(\boldsymbol{x}, \boldsymbol{y}) = M \supseteq N = \mathrm{W}^*(\boldsymbol{x}', \boldsymbol{y}')$ and $P = \mathrm{W}^*(\boldsymbol{x}) = \mathrm{W}^*(\boldsymbol{x}')$, we obtain $h(P : M) \leq h(P : N)$.

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BEN HAYES: brh5c@virginia.edu
Department of Mathematics, University of Virginia, Charlottesville, VA, United States

DAVID JEKEL: daj@math.ku.dk
Department of Mathematics, University of Copenhagen, Copenhagen, Denmark

SRIVATSAV KUNNAWALKAM ELAYAVALLI: skunnawalkamelayaval@ucsd.edu
Department of Mathematics, University of California, San Diego, La Jolla, CA, United States

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