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THE L^∞ ESTIMATE FOR PARABOLIC COMPLEX MONGE–AMPÈRE EQUATIONS

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Following the recent developments in Chen and Cheng (2023) and Guo et al. (2023), we derive the L^∞ estimate for Kähler–Ricci flows under certain integral assumptions. The technique also extends to some other parabolic Monge–Ampère equations derived from Kähler geometry and G_2 geometry.

1. Introduction

We will derive the L^∞ estimate for the Kähler–Ricci flow

$$\begin{cases} \partial_t \varphi = \log\Big(\dfrac{{\omega_\varphi}^n}{e^{nF}{\omega_0}^n}\Big), \\ \varphi(\,\cdot\,,0) = \varphi_0(\,\cdot\,) \end{cases} \tag{1-1}$$

under the assumption that the p-entropy $\operatorname{Ent}_p(F) = \int_M |F|^p e^{nF}{\omega_0}^n$ is bounded and $\int_M F{\omega_0}^n$ has a lower bound. Here is our main theorem.

Theorem 1.1. *Let us consider the flow equation (1-1) on $M \times [0, T)$, where M is an n-dimensional compact Kähler manifold. Let F be a space function, i.e., $F : M \to \mathbb{R}$. Assume, for some $p > n+1$, the p-entropy $\operatorname{Ent}_p(F) = \int_M |F|^p e^{nF}{\omega_0}^n$ is bounded and $\int_M nF{\omega_0}^n \geq -K$. Moreover, suppose φ is a C^2 solution, and let $\tilde{\varphi} = \varphi - \fint \varphi{\omega_0}^n$ be a normalization which has the zero integral. Then we have the L^∞ estimate*

$$\|\tilde{\varphi}\|_{L^\infty(M\times[0,T))} \leq C,$$

where C depends on n, ω_0, φ_0, p, K, and $\operatorname{Ent}_p(F)$. Most importantly, such a C does not depend on T.

Yau [1978] applied the method of Moser iteration to derive the L^∞ estimate for Monge–Ampère equations when $\|e^{nF}\|_{L^p}$ is bounded for some $p > n$. Later, Kołodziej [2003] gave another proof by using the pluripotential theory under a weaker assumption that $\|e^{nF}\|_{L^p}$ is bounded for some $p > 1$. More recently, Guo, Phong, and Tong [Guo et al. 2023] recovered Kołodziej's estimate by a PDE method which was partly motivated by the breakthrough on the cscK metric of Chen and Cheng [2021].

The Kähler–Ricci flow was firstly studied by Cao [1985] when he gave an alternative proof of Calabi's conjecture for $c_1(M) = 0$ and $c_1(M) < 0$, which investigated the estimates for the Kähler–Ricci flow instead of Monge–Ampère equations. There are abundant results on Kähler–Ricci flow, see [Eyssidieux et al. 2015; 2016; Guedj et al. 2021; Jian and Shi 2024]. Our result requires a weaker regularity on the right-hand side than Cao's L^∞ estimate and can be viewed as a parabolic analogue of [Guo et al. 2023; Wang et al. 2021].

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There are some technical improvements in our paper compared with previous results in [Chen and Cheng 2023; Guo and Phong 2023; 2024; Guo et al. 2023]. The difficulty for the flow problem is that we want to derive an L^∞ estimate independent of T. But the original auxiliary equation may not serve as a good choice. Our approach is to consider a local version of auxiliary flows instead, see Section 2. Compared with the elliptic version of L^∞ estimates, our theorem requires an extra integral condition. Rewriting (1-1) by $\omega_\varphi^n = e^{\dot\varphi + nF}{\omega_0}^n$, we could see that the L^∞ estimate of φ comes from some p-entropy bounds on $e^{\dot\varphi+nF}$. Roughly speaking, we need not only the p-entropy bound controls on e^{nF} but also some upper bounds on $\dot\varphi$. In Kähler–Ricci flow, Cao proved the supremum of $\dot\varphi$ can be controlled by the infimum of e^{nF} when F is smooth. Indeed such estimates can be generalized to general parabolic Monge–Ampère flows. However, in our theorem, we can improve the pointwise condition by some integral condition on F.

There are two directions to generalize Theorem 1.1. As in [Chen and Cheng 2023], we can replace the Monge–Ampère operator on the right-hand side by a more general nonlinear operator $\mathcal{F}$. Write $\omega_0 = \sqrt{-1}g_{j\bar m}\,dz^j \wedge d\bar z^m$ in local coordinate; then the corresponding endomorphism h_φ, which is relative to ω_φ, can be expressed by $(h_\varphi)^j_k = g^{j\bar m}(\omega_\varphi)_{\bar m k}$ in local coordinate. Let $\lambda[h_\varphi]$ be the vector of eigenvalues of h_φ, and consider the nonlinear operator $\mathcal{F} : \Gamma \subset \mathbb{R}^n \to \mathbb{R}_+$ with the following four conditions:

(1) The domain Γ is a symmetric cone with $\Gamma_n \subset \Gamma \subset \Gamma_1$, where Γ_k is defined to be the cone of vectors λ with $\sigma_j(\lambda) > 0$ for $1 \le j \le k$, where σ_j is the j-th symmetric polynomial in λ.

(2) $\mathcal{F}(\lambda)$ is symmetric in $\lambda \in \Gamma$ and it is of homogeneous degree r.

(3) $\frac{\partial \mathcal{F}}{\partial \lambda_j} > 0$ for each $j = 1, \dots, n$ and $\lambda \in \Gamma$.

(4) There is a $\gamma > 0$ such that

$$\prod_{j=1}^{n} \frac{\partial \mathcal{F}(\lambda)}{\partial \lambda_j} \ge \gamma \mathcal{F}^{n(1-1/r)} \quad \text{for all } \lambda \in \Gamma. \tag{1-2}$$

The above requirements come from [Guo et al. 2023], and there is a slight modification on the last condition since the homogeneous degree of $\mathcal{F}$ is r under our assumption. The complex Hessian operators and p-Monge–Ampère operators are examples. More examples can be found in [Harvey and Lawson 2023]. Here is our first generalization.

Theorem 1.2. *Let φ be a C^2 solution of the flow*

$$\begin{cases} \partial_t \varphi = \log\Big(\dfrac{\mathcal{F}(\lambda[h_\varphi])}{e^{rF}}\Big), \\ \varphi(\,\cdot\,, 0) = \varphi_0 \end{cases} \tag{1-3}$$

on $M \times [0, T)$, where M is an n-dimensional compact Kähler manifold. Let $F : M \to \mathbb{R}$ be a space function. Assume, for some $p > n+1$, the p-entropy $\operatorname{Ent}_p(F) = \int_M |F|^p e^{nF} {\omega_0}^n$ is bounded and $\int_M F{\omega_0}^n \ge -K$. Then we have the L^∞ estimate on the normalization $\tilde\varphi$

$$\|\tilde\varphi\|_{L^\infty(M\times[0,T))} \le C,$$

where C depends on n, ω_0, φ_0, p, K, γ, r, and $\operatorname{Ent}_p(F)$.

Another direction of generalization is motivated by Chen and Cheng [2023], who considered the L^∞ estimate for the inverse Monge–Ampère flow

$$\begin{cases}(-\partial_t u)\omega_\varphi^n = e^{nF}{\omega_0}^n,\\ \varphi(\,\cdot\,,0)=\varphi_0.\end{cases} \tag{1-4}$$

Indeed, we can consider the general complex Monge–Ampère flow

$$\begin{cases}\partial_t\varphi = \Theta\Big(\dfrac{{\omega_\varphi}^n}{e^{nF}{\omega_0}^n}\Big),\\ \varphi(\,\cdot\,,0)=\varphi_0,\end{cases} \tag{1-5}$$

where $\Theta:\mathbb{R}_+\to\mathbb{R}$ is a strictly increasing smooth function. Picard and Zhang [2020] proved the long time existence and convergence of the flow (1-5) under the assumption that $F\in C^\infty(M,\mathbb{R})$. It is the Kähler–Ricci flow when $\Theta(y)=\log y$ and the inverse Monge–Ampère flow (1-4) when $\Theta(y)=-1/y$. The general parabolic Monge–Ampère flow (1-5) also arises from many other geometric problems. For example, when $\Theta(y)=y$, this is the flow reduced from the anomaly flow with conformal Kähler initial data; see [Phong et al. 2019]. When $\Theta(y)=y^{1/3}$, this is the reduction of the G_2-Laplacian flow over a seven dimensional manifold [Picard and Suan 2024]. We can apply the new technique to prove the L^∞ estimate of the solution φ to the flow (1-5) under an analogue assumption on F.

Theorem 1.3. *Assume $\Theta(y)=-1/y,\ y,$ or $y^{1/3}$, and $\mathrm{Ent}_p(F)$ is bounded. Moreover, consider the constant K equal to $\max\big(0,\int_M\Theta(e^{-nF}){\omega_0}^n\big)$. Then, there exists a constant C depending on $n,\ \omega_0,\ \varphi_0,\ p,\ K,$ and $\mathrm{Ent}_p(F)$ such that*

$$\|\tilde\varphi\|_{L^\infty(M\times[0,T))}\le C,$$

where $\tilde\varphi$ is a normalization of a C^2 solution of the flow (1-5).

Since $\Theta<0$ in the inverse Monge–Ampère equation, we have $K\equiv 0$, which means there is no extra condition for this case.

Going forward, a constant is called *universal* if it depends only on $n,\ \omega_0,\ \varphi_0,\ p,\ \gamma,\ r,\ K,$ and $\mathrm{Ent}_p(F)$.

2. Auxiliary equations

In this section, we want to find suitable auxiliary equations as in [Guo et al. 2023] and [Chen and Cheng 2023]. To motivate what a good auxiliary equation is, we first consider the parabolic Monge–Ampère flow (1-5). To have some monotonicity properties of the auxiliary solutions, we prefer a flow with negative time derivative of ψ. Thus the inverse Monge–Ampère flow, see (1-4), is a good candidate.

Let us consider the inverse Monge–Ampère flow

$$\begin{cases}(-\dot\psi_s){\omega_{\psi_s}}^n = f_s e^{nF}{\omega_0}^n,\\ \psi_s(\,\cdot\,,0)=0,\end{cases} \tag{2-1}$$

where

$$f_s=\frac{(-\varphi-s)_+}{A_s},\qquad A_s=\int_{\Omega_s}(-\varphi-s)e^{nF}{\omega_0}^n\,\mathrm{d}t,\qquad \Omega_s=\{(z,t)\mid -\varphi(z,t)-s>0\}.$$

But such a flow has singularities, since the factor $(-\varphi - s)_+/A_s$ is not smooth in a neighborhood of $\partial\Omega_s$. Thus we need to consider a sequence of smooth functions $\tau_k(x)$ which converges uniformly to $x \cdot \chi_{\mathbb{R}^+}(x)$ and replace f_s on the right-hand side by

$$\frac{\tau_k(-\varphi - s)}{\int_\Omega \tau_k(-\varphi - s)e^{nF}}.$$

By the dominated convergence theorem, $\psi_{s,k}$ converges to ψ_s uniformly, which means we can always take a limit in the inequalities to get the desired estimates as in [Guo et al. 2023] and [Chen and Cheng 2023]. To simplify our computations, we will keep using (2-1) as our auxiliary equation.

Another crucial modification of our auxiliary flow is that we must restrict the integration over the time slices. To express our idea more clearly, we need Lemma 2.1 as well as Corollary 2.2 in [Chen and Cheng 2023], which will be stated below. For the reader's convenience, we will also include the proof from [Chen and Cheng 2023] here.

Lemma 2.1. *Consider the inverse Monge–Ampère flow*

$$\begin{cases} (-\dot{\varphi})\omega_\varphi{}^n = e^{nF}\omega_0{}^n, \\ \varphi|_{t=0} = \varphi_0. \end{cases}$$

Assuming

$$\int_{M\times[0,T]} e^{nF}\omega_0{}^n \,\mathrm{d}t = C_1 < \infty,$$

we have $|\sup_M \varphi| \le C$, *where* C *depends on* n, ω_0, C_1, *and* $\|\varphi_0\|_{L^\infty}$.

Proof. Since $\dot{\varphi} < 0$, we can get the upper bound by

$$\sup_M \varphi \le \sup_M \varphi_0 \le \|\varphi_0\|_{L^\infty}.$$

To estimate the lower bound of $\sup_M \varphi$, let us consider the I-functional

$$I(\varphi) = \frac{1}{n+1}\int_M \varphi \sum_{j=0}^{n} \omega_0{}^{n-j} \wedge \omega_\varphi{}^j$$

and its derivative

$$\frac{\mathrm{d}}{\mathrm{d}t} I(\varphi) = \int_M \partial_t\varphi\,\omega_\varphi{}^n = -\int_M e^{nF}\omega_0{}^n.$$

Therefore, for any $t' \in [0, T]$, we have

$$\begin{aligned} I(\varphi) - I(\varphi_0) = \int_0^{t'} \frac{\mathrm{d}}{\mathrm{d}t} I(\varphi)\,\mathrm{d}t &= -\int_{M\times[0,t']} e^{nF}\omega_0{}^n \,\mathrm{d}t \\ &\ge -\int_{M\times[0,T]} e^{nF}\omega_0{}^n \,\mathrm{d}t = -C_1, \end{aligned}$$

which implies $I(\varphi)$ is bounded from below on $[0, T]$.

The lower bound estimate of $\int_M \varphi {\omega_0}^n$ comes from integration by parts:

$$\begin{aligned}\int_M \varphi {\omega_0}^n - I(\varphi) &= \int_M \varphi \frac{1}{n+1}\sum_{j=0}^{n} {\omega_0}^{n-j}\wedge({\omega_0}^j - {\omega_\varphi}^j)\\ &= \frac{1}{n+1}\int_M \varphi \sum_{j=0}^{n} {\omega_0}^{n-j}\wedge \sqrt{-1}\partial\bar\partial(-\varphi)\sum_{l=0}^{j-1} {\omega_0}^{j-1-l}\wedge {\omega_\varphi}^l\\ &= \frac{1}{n+1}\int_M \sqrt{-1}\partial\varphi\wedge\bar\partial\varphi\wedge\sum_{j=0}^{n}\sum_{l=0}^{j-1} {\omega_0}^{n-1-l}\wedge{\omega_\varphi}^l \geq 0.\end{aligned}$$

Therefore, we have $\int_M \varphi {\omega_0}^n \geq -C_1$ and

$$\sup_M \varphi \geq \frac{1}{\mathrm{Vol}(M,\omega_0)}\int_M \varphi {\omega_0}^n \geq -\frac{C_1}{\mathrm{Vol}(M,\omega_0)}. \qquad \square$$

Corollary 2.2. *There exists a constant $\alpha > 0$ depending only on ω_0 such that*

$$\sup_{t\in[0,T]}\int_M e^{-\alpha\varphi}{\omega_0}^n \leq C_2,$$

where C_2 depends on M, ω_0, C_1, and $\|\varphi_0\|_{L^\infty}$.

This is a flow version of Hörmander's result; see [Hörmander 1973, Lemma 4.4] and [Tian 1987] for local and global version of such integral estimate, respectively.

Proof. Since φ is a ω_0-psh function for every $t \in [0, T]$, we have

$$\sup_{t\in[0,T]}\int_M e^{-\alpha(\varphi-\sup_M\varphi)}{\omega_0}^n \leq C.$$

From Lemma 2.1, the uniform bound of $\sup_M \varphi$ gives us the desired inequality. $\square$

The above Corollary 2.2 gives us a uniform bound on each time slice. If we apply this corollary on the space time $M\times[0, T)$, then a factor T seems unavoidable on the right-hand side. Therefore it is better to divide the whole space-time into several pieces $M\times[t_0, t_0+1]$ and try to seek an estimate independent of t_0. This idea inspires us to consider such auxiliary equations involving only local information.

To get the L^∞ estimate, we need also to consider the normalization in Theorem 1.1, which follows the same normalization in [Picard and Zhang 2020]. In conclusion, we need to use the domain

$$\widetilde\Omega_s = \{(z,t) \mid -\tilde\varphi(z,t) - s > 0\}$$

as a substitute for Ω_s.

For any $t_0 \in [0, T-1)$, let us consider a family of regions $\Omega_{s,t_0} = \widetilde\Omega_s \cap (M\times[t_0, t_0+1])$ and define a family of auxiliary equations

$$\begin{cases}(-\dot\psi_{s,t_0}){\omega_{\psi_{s,t_0}}}^n = f_{s,t_0}e^{nF}{\omega_0}^n,\\ \psi_{s,t_0}(\,\cdot\,,0) = 0,\end{cases} \tag{2-2}$$

where $A_{s,t_0} = \int_{\Omega_{s,t_0}}(-\tilde\varphi - s)e^{nF}{\omega_0}^n\,\mathrm{d}t$ and $f_{s,t_0} = (-\tilde\varphi - s)\cdot\chi_{\Omega_{s,t_0}}/A_{s,t_0}$.

The benefits appear when we apply Corollary 2.2 to such auxiliary equations. As a result of the choice of f_{s,t_0}, the integral on the right-hand side of the equation is 1. This implies

$$\int_{\Omega_{s,t_0}} e^{-\alpha\psi_{s,t_0}} {\omega_0}^n \, \mathrm{d}t \le C_2, \tag{2-3}$$

where the C_2 is universal now. The above inequality is crucial, because it integrates against t while it remains independent of T and t_0. We will use inequality (2-3) frequently in the following sections.

The family of auxiliary equations (2-2) meets the same problem as (2-1). To be precise, we also need to apply τ_k to remove the singularities. For the same reason, we will keep using (2-2) in the following sections.

The extra integral condition was chosen to make the normalization $\tilde{\varphi}$ satisfy three properties, as follows.

Lemma 2.3. *Let $\tilde{\varphi}$ be given in Theorem 1.1. Then we have*

(1) $\sup_{t\in[0,T)} \int_M \dot{\varphi}{\omega_0}^n \le C_3$,

(2) $\tilde{\varphi} \le C_3$,

(3) $\int_M |\tilde{\varphi}|{\omega_0}^n \le C_3$,

where C_3 is universal.

Proof. For (1), it comes directly from the estimates

$$\begin{aligned}\fint_M \dot{\varphi}{\omega_0}^n &= \fint_M \log\biggl(\frac{{\omega_\varphi}^n}{e^{nF}{\omega_0}^n}\biggr){\omega_0}^n = \fint \log\biggl(\frac{{\omega_\varphi}^n}{{\omega_0}^n}\biggr){\omega_0}^n - \fint \log(e^{nF}){\omega_0}^n \\ &\le \log\biggl(\fint {\omega_\varphi}^n\biggr) - \fint \log(e^{nF}){\omega_0}^n = -\fint \log(e^{nF}){\omega_0}^n \le K.\end{aligned}$$

The first estimate comes from Jensen's inequality while the second one comes from the assumption on F. The average integral is chosen with respect to $V = \int_M {\omega_0}^n$.

To prove (2) and (3), let's consider Green's formula

$$\tilde{\varphi} = \fint_M \tilde{\varphi}{\omega_0}^n - \int_M G\Delta\tilde{\varphi}{\omega_0}^n = -\int_M G\Delta\tilde{\varphi}{\omega_0}^n = -\int_M G\Delta\varphi{\omega_0}^n,$$

where G is the Green's function with respect to ω_0.

It is well known that the Green's function G could be shifted to be nonnegative and with L^1 norm bound C'. Combining with $\operatorname{tr}_{\omega_0}\omega_\varphi = n + \Delta\varphi > 0$ and Green's formula, we have the universal estimate $\tilde{\varphi} \le nC'$.

Let I_+ and I_- be the integrals of the positive and negative parts of $\tilde{\varphi}$, respectively. Then we have

$$0 = \int_M \tilde{\varphi} = I_+ - I_- \quad \text{and} \quad I_+ \le nC'V.$$

Thus

$$\int_M |\tilde{\varphi}| = I_+ + I_- = 2I_+ \le 2nC'V.$$

The lemma follows from choosing $C_3 = \max(KV, nC', 2nC'V)$. ∎

3. Entropy bounded by energy

From Section 2, we get a good choice of a family of auxiliary equations (2-2). The following lemma is a key to the proof of Theorem 1.1,

Lemma 3.1. *Let φ be as in Theorem 1.1 and ψ_{s,t_0} be a solution of the auxiliary flow (2-2). Then there are constants β, ϵ, and Λ, with*

$$\beta=\frac{n+1}{n+2},\qquad \epsilon^{n+2}=\left(\frac{n+2}{n+1}\right)^{n+2}\Lambda,\qquad \epsilon^{n+2}=\left(\frac{C_4}{(n+1)\beta}\right)^{n+1}A_{s,t_0},$$

where C_4 is a universal constant defined below, such that

$$-\epsilon(-\psi_{s,t_0}+\Lambda)^\beta-\tilde{\varphi}-s\le 0 \tag{3-1}$$

holds on $M\times[t_0,t_0+1]$.

In the following estimates we will use ψ and f to denote ψ_{s,t_0} and f_{s,t_0}, respectively. Let us consider the test function $H=-\epsilon(-\psi+\Lambda)^\beta-\tilde{\varphi}-s$ and the linearization operator $L=-\partial/\partial t+\Delta_{\omega_{\varphi_t}}$ of the Kähler–Ricci flow (1-1). The idea of the following argument comes from [Guo et al. 2023].

Let Ω°_{s,t_0} denote the interior of Ω_{s,t_0}. Suppose the maximum of H is attained at some point $x_0=(z_0,t_0)$ outside Ω°_{s,t_0}; then we have $H\le H(x_0)\le-\tilde{\varphi}(x_0)-s\le 0$. To complete the proof, we only need to assume the maximal point x_0 of H is in Ω°_{s,t_0} and then apply the maximum principle.

The scheme of the proof is to estimate LH at x_0. The constants are chosen to make $0<\beta<1$ and $1-\beta\epsilon\Lambda^{\beta-1}=0$, which imply some cancellations. Moreover, such relations among the constants imply that LH is different from H by a positive coefficient at x_0. Roughly speaking, the lemma holds because of the facts that H is proportional to LH and $LH\le 0$. Therefore let us firstly apply the operator L to H at x_0:

$$\begin{aligned}
0\ge LH &= -\beta\epsilon(-\psi+\Lambda)^{\beta-1}\dot{\psi}+\dot{\varphi}-\textstyle\fint\dot{\varphi}\omega_0{}^n\\
&\qquad +\beta\epsilon(-\psi+\Lambda)^{\beta-1}\Delta_{\omega_\varphi}\psi+\beta(1-\beta)\epsilon(-\psi+\Lambda)^{\beta-2}|\partial\varphi|^2_{\omega_\varphi}-\Delta_{\omega_\varphi}\varphi\\
&\ge \beta\epsilon(-\psi+\Lambda)^{\beta-1}(-\dot{\psi})-(-\dot{\varphi})+\beta\epsilon(-\psi+\Lambda)^{\beta-1}\Delta_{\omega_\varphi}\psi-\Delta_{\omega_\varphi}\varphi-C_3\\
&= \beta\epsilon(-\psi+\Lambda)^{\beta-1}(-\dot{\psi})-(-\dot{\varphi})+\beta\epsilon(-\psi+\Lambda)^{\beta-1}\operatorname{tr}_{\omega_\varphi}\omega_\psi\\
&\qquad -\operatorname{tr}_{\omega_\varphi}\omega_\varphi+(1-\beta\epsilon(-\psi+\Lambda)^{\beta-1})\operatorname{tr}_{\omega_\varphi}\omega_0-C_3\\
&\ge \beta\epsilon(-\psi+\Lambda)^{\beta-1}(-\dot{\psi}+\operatorname{tr}_{\omega_\varphi}\omega_\psi)-(-\dot{\varphi}+C_3+n).
\end{aligned}$$

The last estimate comes from the choice of auxiliary equations. Since ψ solves some inverse Monge–Ampère flows with initial data being identically 0, we have $\psi\le 0$ on $M\times[0,T)$, and moreover

$$1-\beta\epsilon(-\psi+\Lambda)^{\beta-1}\ge 1-\beta\epsilon\Lambda^{\beta-1}=0.$$

Then we need to deal with the factor $-\dot{\psi}+\operatorname{tr}_{\omega_\varphi}\omega_\psi$, which is the main term of the estimate. By the geometric-arithmetic inequality, we have

$$-\dot{\psi}+\operatorname{tr}_{\omega_\varphi}\omega_\psi\ge-\dot{\psi}+n\left(\frac{\omega_\psi^n}{\omega_\varphi^n}\right)^{1/n}. \tag{3-2}$$

Combining (3-2) with the two flow equations (1-1) and (2-2) and using the geometric-arithmetic inequality again, we have

$$\begin{aligned} -\dot{\psi} + \operatorname{tr}_{\omega_\varphi} \omega_\psi &\geq -\dot{\psi} + n\Big(\frac{\omega_\psi^n}{\omega_\varphi^n}\Big)^{1/n} \\ &\geq -\dot{\psi} + n f^{1/n} \exp\Big(-\frac{1}{n}\dot{\varphi}\Big)(-\dot{\psi})^{-1/n} \\ &\geq (n+1) f^{1/(n+1)} \exp\Big(-\frac{1}{n+1}\dot{\varphi}\Big). \end{aligned} \tag{3-3}$$

Thus replacing $-\dot{\psi} + \operatorname{tr}_{\omega_\varphi} \omega_\psi$ by (3-3), we have the following estimate at x_0:

$$\begin{aligned} 0 \geq LH &\geq (n+1)\beta\epsilon(-\psi+\Lambda)^{\beta-1} f^{1/(n+1)} \exp\Big(-\frac{1}{n+1}\dot{\varphi}\Big) - (-\dot{\varphi} + C_3 + n) \\ &\geq \Big[(n+1)\beta\epsilon(-\psi+\Lambda)^{\beta-1} f^{1/(n+1)} - (-\dot{\varphi} + C_3 + n)\exp\Big(\frac{1}{n+1}\dot{\varphi}\Big)\Big]\exp\Big(-\frac{1}{n+1}\dot{\varphi}\Big). \end{aligned}$$

Since the exponential function is positive, we can simplify it by

$$0 \geq (n+1)\beta\epsilon(-\psi+\Lambda)^{\beta-1} f^{1/(n+1)} + (\dot{\varphi} - C_3 - n)\exp\Big(\frac{1}{n+1}\dot{\varphi}\Big), \tag{3-4}$$

which looks similar to the desired inequality (3-1) in Lemma 3.1. Let us consider a function

$$h(x) = (x - n - C_3)\exp\Big(\frac{1}{n+1}x\Big).$$

The function $h : \mathbb{R} \to \mathbb{R}$ has a universal lower bound $-C_4$, where

$$C_4 = (n+1)\exp\Big(\frac{C_3 - 1}{n+1}\Big).$$

Thus we have $h(\dot{\varphi}) \geq -C_4$, and moreover

$$(n+1)\beta\epsilon(-\psi+\Lambda)^{\beta-1} f^{1/(n+1)} - C_4 \leq 0. \tag{3-5}$$

Since $f = (-\tilde{\varphi} - s)/A_{s,t_0}$ at the maximal point x_0, inequality (3-5) is equivalent to

$$\Big(\frac{(n+1)\beta}{C_4}\Big)^{n+1} \epsilon^{n+1} \frac{-\tilde{\varphi} - s}{A_{s,t_0}} \leq (-\psi+\Lambda)^{(n+1)(1-\beta)}, \tag{3-6}$$

which is the test function when we chose the constants β, ϵ, and Λ as stated in Lemma 3.1. Thus we have

$$H \leq H(x_0) \leq 0,$$

which completes the proof.

From the above Lemma 3.1, we have

$$\frac{-\tilde{\varphi} - s}{A_{s,t_0}^{1/(n+2)}} \leq (c(-\psi+\Lambda))^{(n+1)/(n+2)}, \tag{3-7}$$

where

$$c = \Big(\frac{(n+1)^2\epsilon}{(n+2)C_4}\Big)^{n+2}.$$

The following estimate comes from (3-7):

$$\int_{\Omega_{s,t_0}} \exp\left[\lambda\left(\frac{-\tilde{\varphi}-s}{A_{s,t_0}^{1/(n+2)}}\right)^{(n+2)/(n+1)}\right]{\omega_0}^n\,\mathrm{d}t \le \int_{\Omega_{s,t_0}} \exp\{\lambda c(-\psi_{s,t_0}+\Lambda)\}{\omega_0}^n\,\mathrm{d}t.$$

If the universal constant λ is chosen to make $\lambda c = \alpha$, then, by Corollary 2.2 and inequality (2-3), we have

$$\int_{\Omega_{s,t_0}} e^{-\lambda c\psi_{s,t_0}}{\omega_0}^n\,\mathrm{d}t \le \int_{M\times[t_0,t_0+1]} e^{-\alpha\psi_{s,t_0}}{\omega_0}^n\,\mathrm{d}t \le C_2,$$

where C_2 is universal. In Lemma 3.1, The constant Λ is chosen to be proportional to A_{s,t_0}, i.e., $\Lambda = c' A_{s,t_0}$ for some universal constant c'. Then we can bound the right integral by $C\exp(CA_{s,t_0})$ for some universal constant C.

Let us define $E = \sup_{t_0\in[0,T-1)} \int_{M\times[t_0,t_0+1]}(-\tilde{\varphi}+C_3)e^{nF}{\omega_0}^n\,\mathrm{d}t$. Then we have

$$\begin{aligned} A_{s,t_0} &= \int_{\Omega_{s,t_0}}(-\tilde{\varphi}-s)e^{nF} \le \int_{\Omega_{s,t_0}}(-\tilde{\varphi}+C_3)e^{nF} + \int_{\Omega_{s,t_0}}(-C_3-s)e^{nF} \\ &\le \int_{\Omega_{s,t_0}}(-\tilde{\varphi}+C_3)e^{nF} = \int_{M\times[t_0,t_0+1]}(-\tilde{\varphi}+C_3)e^{nF} + \int_{M\times[t_0,t_0+1]\setminus\Omega_{s,t_0}}(\tilde{\varphi}-C_3)e^{nF} \\ &= E + \int_{M\times[t_0,t_0+1]\setminus\Omega_{s,t_0}}(\tilde{\varphi}-C_3)e^{nF} \le E, \end{aligned}$$

where the last inequality comes from $\tilde{\varphi} \le C_3$ by Lemma 2.3. In summary, we have

$$\int_{\Omega_{s,t_0}} \exp\left[\lambda\left(\frac{-\tilde{\varphi}-s}{A_{s,t_0}^{1/(n+2)}}\right)^{(n+2)/(n+1)}\right]{\omega_0}^n\,\mathrm{d}t \le C\exp(CE). \tag{3-8}$$

The E defined above is called the energy. The C_3 term inside the integral comes purely from a technical consideration that makes the inside function of the integral positive. There is no significant difference from the elliptic case where $E = \int(-\tilde{\varphi})$ since the extra integral $0 < \int e^{nF} \le \mathrm{Vol}(M,\omega_0) + \mathrm{Ent}_p(F)$ is universally bounded from both sides.

To end this section, we will use the De Giorgi iteration method to derive the C^0 estimate by assuming E is universally bounded. In the next section, we will apply the ABP estimate to get the universal bound on E which will complete the proof of Theorem 1.1. To prepare for the iteration procedure, we need such an inequality to run the iteration:

$$r\phi_{t_0}(s+r) \le A_{s,t_0} \le B_0\phi_{t_0}(s)^{1+\delta_0}, \tag{3-9}$$

where $\phi_{t_0}(s) = \int_{\Omega_{s,t_0}} e^{nF}{\omega_0}^n\,\mathrm{d}t$.

The following lemma is the De Giorgi iteration mentioned above.

Lemma 3.2. *Let $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a decreasing right continuous function with $\lim_{s\to\infty}\Phi(s) = 0$. Moreover, assume $r\Phi(s+r) \le B_0\Phi(s)^{1+\delta_0}$ for some constant $B_0 > 0$ and all $s > 0$ and $r \in [0,r]$. Then there exists a constant $S_\infty = S_\infty(\delta_0, B_0, s_0) > 0$ such that $\Phi(s) = 0$ for all $s \ge S_\infty$, where s_0 is defined during the proof.*

Proof. Fix an $s_0 > 0$ such that $\Phi(s_0)^{\delta_0} < 1/(2B_0)$. Such an s_0 exists since $\Phi(s) \to 0$ as $s \to \infty$. Define $\{s_j\}$ by

$$s_{j+1} = \sup\bigl\{s > s_j \mid \phi(s) > \tfrac{1}{2}\phi(s_j)\bigr\}.$$

Thus

$$s_{j+1} - s_j \le B_0 \Phi(s_j)^{1+\delta}/\Phi(s_{j+1}) \le 2B_0\Phi(s_j)^{\delta} \le 2B_0 2^{-j\delta_0}\Phi^{\delta_0} \le 2^{-j\delta_0}.$$

Letting

$$S_\infty = s_0 + \sum_{j\ge 0}(s_{j+1} - s_j) \le s_0 + \frac{1}{1-2^{-\delta_0}},$$

we complete the proof. □

In our application, Φ is chosen to be ϕ_{t_0}. Let us derive the two sides of (3-9) separately. The left-hand side of (3-9) can be derived by definition which is similar to the proof in [Guo et al. 2023]. The following calculations are direct:

$$\begin{aligned} A_{s,t_0} &= \int_{\Omega_{s,t_0}} (-\tilde{\varphi} - s)e^{nF}{\omega_0}^n \,\mathrm{d}t \ge \int_{\Omega_{s+r,t_0}} (-\tilde{\varphi} - s)e^{nF}{\omega_0}^n \,\mathrm{d}t \\ &\ge \int_{\Omega_{s+r,t_0}} (s + r - s)e^{nF}{\omega_0}^n \,\mathrm{d}t = r\phi(s+r). \end{aligned}$$

To get the right-hand side of (3-9), we need to apply the following inequality coming from Young's inequality:

$$\int_{\Omega_{s,t_0}} v^p e^{nF}{\omega_0}^n \,\mathrm{d}t \le \|e^{nF}\|_{L^1(\log L)^p(M\times[t_0,t_0+1])} + C_p \int_{\Omega_{s,t_0}} e^{2v}{\omega_0}^n \,\mathrm{d}t. \tag{3-10}$$

If we choose

$$v = \frac{\lambda}{2}\left(\frac{-\tilde{\varphi} - s}{A_{s,t_0}^{1/(n+2)}}\right)^{(n+2)/(n+1)},$$

then by the above inequality (3-10) we have

$$\int_{\Omega_{s,t_0}} (-\tilde{\varphi} - s)^{(n+2)p/(n+1)} e^{nF}{\omega_0}^n \,\mathrm{d}t \le C(E) A_{s,t_0}^{p/(n+1)}, \tag{3-11}$$

where the factor $C(E)$ is a constant dependent on n, ω_0, φ_0, p, γ, K, $\mathrm{Ent}_p(F)$ and additionally on E. The explicit dependence of the constant $C(E)$ on E can be expressed by combining inequalities (3-10) and (3-8).

Thus the right-hand side can be derived by the estimate

$$\begin{aligned} A_{s,t_0} &= \int_{\Omega_{s,t_0}} (-\tilde{\varphi} - s)e^{nF}{\omega_0}^n \,\mathrm{d}t \\ &\le \left(\int_{\Omega_{s,t_0}} (-\tilde{\varphi} - s)^{(n+2)p/(n+1)} e^{nF}\right)^{(n+1)/((n+2)p)} \cdot \left(\int_{\Omega_{s,t_0}} e^{nF}\right)^{1/q} \\ &\le C(E)^{(n+1)/((n+2)p)} A_{s,t_0}^{1/(n+2)} \phi_{t_0}^{1-(n+1)/(p(n+2))}, \end{aligned}$$

where the first line is by Hölder's inequality and p is the Hölder coefficient

$$\frac{n+1}{p(n+2)}+\frac{1}{q}=1.$$

In (3-9), we can choose

$$\delta_0=1+\frac{p-n-1}{p(n+1)}\quad\text{and}\quad B_0=C(E)^{1/p}.$$

To complete this section, we need to get an explicit expression on s_0. By Chebyshev's inequality

$$\phi_{t_0}(s)\le\frac{1}{s}\int_{\Omega_{s,t_0}}(-\tilde{\varphi})e^{nF}\omega_0^n\le\frac{E}{s},$$

we can choose $s_0=(2B_0)^{1/\delta_0}E$.

In conclusion, we get the following theorem from the above arguments.

Theorem 3.3. *Let $\tilde{\varphi}$ be as in Theorem 1.1. Then we have*

$$\sup_{M\times[0,T)}|\tilde{\varphi}|\le C(n,\omega_0,\varphi_0,p,\gamma,K,\mathrm{Ent}_p(F),E).$$

Moreover, if E can be controlled by a universal constant, then we have Theorem 1.1.

4. Energy bounds by the ABP estimate

In this section we use a parabolic version of the ABP estimate proved by Krylov [1976] and Tso [1985] to give us a uniform energy bound. This approach was introduced in [Chen and Cheng 2023] and is an analogue to the elliptic version in [Guo et al. 2023].

Let u be a function defined on $D=\Omega\times[0,T]$, where Ω is a bounded domain in $\mathbb{R}^n$. Then the parabolic ABP estimate says that

$$\sup_D u\le\sup_{\partial_P D}u+C_n(\operatorname{diam}\Omega)^{n/(n+1)}\left(\int_\Gamma|\partial_t u\det D_x^2u|\,\mathrm{d}x\,\mathrm{d}t\right)^{1/(n+1)},\tag{4-1}$$

where $\partial_P D$ is the parabolic boundary of D and $\Gamma=\{(x,t)\mid\partial_t u\ge0,\ D_x^2u\le0\}$.

As mentioned in Section 2, we want to construct a family of local auxiliary equations. The auxiliary equations in this section are chosen to be

$$\begin{cases}(-\partial_t\psi_{t_0})\omega_{\psi_{t_0}}{}^n=\dfrac{(|F|^p+1)\cdot\chi_{M\times[t_0,t_0+1]}}{\int_{M\times[t_0,t_0+1]}(|F|^p+1)e^{nF}\omega_0{}^n\,\mathrm{d}t}e^{nF}\omega_0{}^n,\\ \psi_{t_0}(\,\cdot\,,0)=0.\end{cases}\tag{4-2}$$

We will use ψ to denote ψ_{t_0} for convenience and will skip the computation involved with τ_k for the same reason we did in Section 2. Moreover, define a universal constant

$$\Psi=\int_{M\times[t_0,t_0+1]}(|F|^p+1)e^{nF}\omega_0{}^n\,\mathrm{d}t.$$

Parallel with Lemma 3.1, the following lemma plays a key role in this section.

Lemma 4.1. *Let φ be as in Theorem 1.1 and ψ be a solution of (4-2). For any $0<\beta<1$, there exists a constant C which depends on n, ω_0, φ_0, p, γ, K, $\mathrm{Ent}_p(F)$, and additionally on β, such that the following holds on $M\times[t_0, t_0+1]$:*

$$-\epsilon(-\psi+\Lambda)^\beta-\tilde{\varphi}\le C, \tag{4-3}$$

where the constants $\epsilon>0$ and $\Lambda>0$ are defined as

$$\beta\epsilon\Lambda^{\beta-1}=\tfrac{1}{4},\quad \Lambda=\left(\frac{n^p(2n+1)^p2^p4^{n+1}9^{n+1}(n+1)^{n+1}C_4^{n+1}\Psi}{10^{n+1}\alpha^p}\right)^{1/((n+1)(1-\beta))}. \tag{4-4}$$

The constants ϵ and Λ depend additionally on β.

Let ρ be the test function defined by

$$\rho=-\epsilon(-\psi+\Lambda)^\beta-\tilde{\varphi}$$

and L be the linearization as above. To prove Lemma 4.1, we only need to restrict ρ to its positive part. More precisely, consider $h_s(x)=x+\sqrt{x^2+s}$ and use $h_s(\rho)$ to approximate $2\rho_+$. Therefore we have

$$2\sup\rho\le 2\sup\rho_+\le\sup h_s(\rho),$$

and in addition the upper bounds of h_s imply an upper bound of ρ.

Let us consider $h_s(\rho)^b$, where

$$b=1+\frac{1}{(2n+2)(2n+1)},$$

and assume $h_s(\rho)^b$ attains its maximal value Q at some point $x_0\in M\times[0,T)$. Moreover, we can assume $Q>1$, otherwise there is nothing to prove. Let us apply the parabolic ABP estimate for $H=h_s(\rho)^b\cdot\eta$, where η is a cut-off function defined below.

Assume $r_0=\min\{1,\mathrm{inj}(M,\omega_0)\}$, where $\mathrm{inj}(M,\omega_0)$ is the injectivity radius of (M,ω_0). The cut-off function $\eta: M\to\mathbb{R}$ is defined in the following way:

$$\eta\equiv 1 \quad \text{on } B_{\omega_0}\big(x_0,\tfrac{1}{2}r_0\big), \tag{4-5}$$

$$\eta\equiv 1-\theta \quad \text{on } \big\{M\backslash B_{\omega_0}\big(x_0,\tfrac{3}{4}r_0\big)\big\}, \tag{4-6}$$

$$1-\theta\le\eta\le 1 \quad \text{on } \big\{B_{\omega_0}\big(x_0,\tfrac{3}{4}r_0\big)\backslash B_{\omega_0}\big(x_0,\tfrac{1}{2}r_0\big)\big\}, \tag{4-7}$$

$$|\nabla\eta|^2_{\omega_0}\le 10\theta^2/r_0^2, \tag{4-8}$$

$$|\nabla^2\eta|_{\omega_0}\le 10\theta/r_0^2, \tag{4-9}$$

where $0<\theta<1$ is a small constant defined by

$$\theta=\min\left\{\frac{r_0^2}{100Q^{1/b}},\frac{1}{2(2n+1)(2n+2)}\right\}\le\frac{1}{10}.$$

Since η is a space function, it has vanishing time derivative, which reduce our later computations.

Proof of Lemma 4.1. The following inequality can be derived directly by applying the operator L on H:

$$LH \geq bh'h^{b-1}(-\partial_t\rho)\eta + (\Delta_{\omega_\varphi}h^b)\eta + 2\,\mathrm{Re}\langle\nabla h^b, \overline{\nabla}\eta\rangle_{\omega_\varphi} + h^b\Delta_{\omega_\varphi}\eta. \tag{4-10}$$

We will consider each of the terms separately to get good controls. Let us bound the last two terms:

$$h^b\Delta_{\omega_\varphi}\eta \geq -\frac{10\theta}{r_0^2}h^b\,\mathrm{tr}_{\omega_\varphi}\omega_0, \tag{4-11}$$

$$2\,\mathrm{Re}\langle\nabla h^b, \overline{\nabla}\eta\rangle \geq -\frac{b(b-1)}{2}|\nabla h|^2_{\omega_\varphi}h^{b-2} - \frac{2b}{b-1}h^b|\nabla\eta|^2_{\omega_\varphi}. \tag{4-12}$$

Then, we expand the second term and get

$$(\Delta_{\omega_\varphi}h^b)\eta = b(b-1)|\nabla h|^2_{\omega_\varphi}h^{b-2}\eta + bh'h^{b-1}(\Delta_{\omega_\varphi}\rho)\eta + b|\nabla\rho|^2_{\omega_\varphi}h''h^{b-1}\eta. \tag{4-13}$$

Combining (4-10)–(4-13), and noticing that the first term in (4-13) can be absorbed into the first term in (4-12) and the third term of (4-13) is positive, we have

$$\begin{aligned}LH &\geq bh'h^{b-1}(-\partial_t\rho)\eta + bh'h^{b-1}(\Delta_{\omega_\varphi}\rho)\eta - \frac{2b}{b-1}h^b|\nabla\eta|^2_{\omega_\varphi} - \frac{10\theta}{r_0^2}h^b\,\mathrm{tr}_{\omega_\varphi}\omega_0\\ &\geq bh'h^{b-1}(L\rho)\eta - \frac{2b}{b-1}\frac{10\theta^2}{r_0^2}h^b\,\mathrm{tr}_{\omega_\varphi}\omega_0 - \frac{10\theta}{r_0^2}h^b\,\mathrm{tr}_{\omega_\varphi}\omega_0.\end{aligned} \tag{4-14}$$

As we mentioned, the derivatives of the cut-off function η will produce $\mathrm{tr}_{\omega_\varphi}\omega_0$ terms in (4-14) which will be absorbed in the later estimates. The $L\rho$ term is the main term of (4-14), and it has the same structure as the main term of the test function appearing in Lemma 3.1. This fact motivates the following argument.

Let us compute $L\rho$ and drop the positive term $\beta(1-\beta)\epsilon(-\psi+\Lambda)^{\beta-2}|\nabla\psi|^2$. Then we have

$$\begin{aligned}L\rho &\geq -\beta\epsilon\dot{\psi}(-\psi+\Lambda)^{\beta-1} + \dot{\tilde{\varphi}} + \beta\epsilon(-\psi+\Lambda)^{\beta-1}\Delta_{\omega_\varphi}\psi + \beta(1-\beta)\epsilon(-\psi+\Lambda)^{\beta-2}|\nabla\psi|^2 - \Delta_{\omega_\varphi}\tilde{\varphi}\\ &\geq -\beta\epsilon\dot{\psi}(-\psi+\Lambda)^{\beta-1} + \dot{\varphi} + \beta\epsilon(-\psi+\Lambda)^{\beta-1}\Delta_{\omega_\varphi}\psi - \Delta_{\omega_\varphi}\varphi - C_3.\end{aligned}$$

Since $\Delta_{\omega_\varphi}\varphi + \mathrm{tr}_{\omega_\varphi}\omega_0 = n$ and $\Delta_{\omega_\varphi}\psi + \mathrm{tr}_{\omega_\varphi}\omega_0 = \mathrm{tr}_{\omega_\varphi}\omega_\psi$, we have

$$L\rho \geq \beta\epsilon(-\psi+\Lambda)^{\beta-1}(-\dot{\psi} + \mathrm{tr}_{\omega_\varphi}\omega_\psi) + \dot{\varphi} + (1-\beta\epsilon\Lambda^{\beta-1})\,\mathrm{tr}_{\omega_\varphi}\omega_0 - C_3 - n. \tag{4-15}$$

The $\mathrm{tr}_{\omega_\varphi}\omega_0$ term in (4-15) will serve as a good term to absorb the last two terms in (4-14). The estimate for the rest of the terms in (4-15) follows the same idea in (3-2)–(3-4).

$$\begin{aligned}&\beta\epsilon(-\psi+\Lambda)^{\beta-1}(-\dot{\psi} + \mathrm{tr}_{\omega_\varphi}\omega_\psi) + \dot{\varphi} - C_3 - n\\ &\quad\geq (n+1)\beta\epsilon(-\psi+\Lambda)^{\beta-1}\tilde{f}^{1/(n+1)}\exp\Big(-\frac{1}{n+1}\dot{\varphi}\Big) + \dot{\varphi} - C_3 - n\\ &\quad= \Big[(n+1)\beta\epsilon(-\psi+\Lambda)^{\beta-1}\tilde{f}^{1/(n+1)} + (\dot{\varphi} - C_3 - n)\exp\Big(\frac{1}{n+1}\dot{\varphi}\Big)\Big]\exp\Big(-\frac{1}{n+1}\dot{\varphi}\Big)\\ &\quad\geq [(n+1)\beta\epsilon(-\psi+\Lambda)^{\beta-1}\tilde{f}^{1/(n+1)} - C_4]\exp\Big(-\frac{1}{n+1}\dot{\varphi}\Big),\end{aligned} \tag{4-16}$$

where

$$\tilde{f}=\frac{(|F|^p+1)\chi_{M\times[t_0,t_0+1]}}{\int_{M\times[t_0,t_0+1]}(|F|^p+1)e^{nF}{\omega_0}^n\,\mathrm{d}t}.$$

What we have now is the following inequality, noting that the cut-off function satisfies $\frac{9}{10}\le\eta\le 1$ for any points in the space-time $M\times[0,T)$:

$$LH\ge(n+1)bh'h^{b-1}\left[\frac{9}{10}\beta\epsilon(-\psi+\Lambda)^{\beta-1}\tilde{f}^{1/(n+1)}-\frac{C_4}{n+1}\right]\exp\Big(-\frac{1}{n+1}\dot{\varphi}\Big)$$
$$+bh^{b-1}\left[\frac{9}{10}h'(1-\beta\epsilon\Lambda^{\beta-1})-\frac{20\theta^2}{(b-1)r_0^2}h-\frac{10\theta}{br_0^2}h\right]\operatorname{tr}_{\omega_\varphi}\omega_0. \quad (4\text{-}17)$$

Although the core of the lemma is to control ρ_+, the choice of h_s makes the negative part of ρ involved in the above inequality. So we will estimate on sets $\Omega_+=\{\rho>0\}$ and $\Omega_-=\{\rho\le 0\}$ separately.

On Ω_-, we have

$$0\le h_s(\rho)=\rho+\sqrt{\rho^2+s}=\frac{s}{\sqrt{\rho^2+s}-\rho}\le\sqrt{s}$$

and

$$0\le h_s'(\rho)=1+\frac{\rho}{\sqrt{\rho^2+s}}\le 1.$$

Combining the two bounds on h and h' and inequality (4-17), we have

$$LH\ge bs^{(b-1)/2}\left[-C\exp\Big(-\frac{1}{n+1}\dot{\varphi}\Big)-\frac{20\theta^2}{(b-1)r_0^2}h\operatorname{tr}_{\omega_\varphi}\omega_0-\frac{10\theta}{br_0^2}h\operatorname{tr}_{\omega_\varphi}\omega_0\right]$$
$$=bs^{(b-1)/2}\left[-C\exp\Big(-\frac{1}{n+1}\dot{\varphi}\Big)-c(b,\theta,r_0)h\operatorname{tr}_{\omega_\varphi}\omega_0\right]$$

on Ω_-, where C is universal and

$$c(b,\theta,r_0)=\frac{20\theta^2}{(b-1)r_0^2}+\frac{10\theta}{br_0^2}.$$

On the other hand, $1\le h'\le 2$ on Ω_+. By the choice of the constants Λ and ϵ in (4-4), the coefficient of the $\operatorname{tr}_{\omega_\varphi}\omega_0$ term in (4-17) is positive.

Therefore, on the set Ω_+, we have

$$LH\ge Cbh^{b-1}\left[\beta\epsilon(-\psi+\Lambda)^{\beta-1}\tilde{f}^{1/(n+1)}-\frac{10C_4}{9(n+1)}\right]\exp\Big(-\frac{1}{n+1}\dot{\varphi}\Big).$$

Combining the above two estimates, we obtain

$$LH\ge bs^{(b-1)/2}\left[-C\exp\Big(-\frac{1}{n+1}\dot{\varphi}\Big)-c(b,\theta,r_0)h\operatorname{tr}_{\omega_\varphi}\omega_0\right]\chi_{\Omega_-}$$
$$+Cbh^{b-1}\left(\beta\epsilon(-\psi+\Lambda)^{\beta-1}\tilde{f}^{1/(n+1)}-\frac{10C_4}{9(n+1)}\right)\exp\Big(-\frac{1}{n+1}\dot{\varphi}\Big)\chi_{\Omega_+}=:R. \quad (4\text{-}18)$$

To apply the parabolic ABP estimate for the test function H, we define the domain

$$\widetilde{\Gamma}=\{(z,t)\mid\partial_t H\ge 0,\ D_z^2H\le 0\}.$$

We need also discuss how to control the operator L on $\widetilde{\Gamma}$. The estimate can be derived by

$$LH = -\frac{\partial}{\partial t} H + \Delta_{\omega_\varphi} H \leq -(2n+1)\biggl(\biggl|\frac{\partial}{\partial t} H \cdot \det D^2 H\biggr| \cdot \biggl(\frac{{\omega_0}^n}{{\omega_\varphi}^n}\biggr)^2\biggr)^{1/(2n+1)},$$

which connects our operator with the parabolic ABP estimate. In conclusion,

$$\biggl(|\partial_t H \cdot \det D^2 H| \cdot \biggl(\frac{{\omega_0}^n}{{\omega_\varphi}^n}\biggr)^2\biggr)^{1/(2n+1)} \leq -\frac{R}{2n+1} \leq \frac{R_-}{2n+1}.$$

Note that the Hessian matrix $D^2 H$ is the real Hessian of H instead of the complex Hessian of H. Therefore we have

$$|\partial_t H \cdot \det D^2 H| \leq c_n {R_-}^{2n+1} \biggl(\frac{{\omega_\varphi}^n}{{\omega_0}^n}\biggr)^2.$$

The factor ${\omega_\varphi}^n / {\omega_0}^n$ is hard to control since it appears as a quadratic term. We can use different strategies to bound such term on Ω_+ and Ω_-. On the domain Ω_+, we have

$$\begin{aligned} h^{(2n+1)(b-1)} \biggl(\beta\epsilon(-\psi+\Lambda)^{\beta-1} \tilde{f}^{1/(n+1)} - \frac{10C_4}{9(n+1)}\biggr)_-^{2n+1} \exp\biggl(-\frac{2n+1}{n+1}\dot{\varphi}\biggr) \biggl(\frac{{\omega_\varphi}^n}{{\omega_0}^n}\biggr)^2 \\ \leq h^{(2n+1)(b-1)} \biggl(\frac{10C_4}{9(n+1)}\biggr)^{2n+1} e^{(n(2n+1)/(n+1))F} \biggl(\frac{{\omega_\varphi}^n}{{\omega_0}^n}\biggr)^{2-(2n+1)/(n+1)} \\ = \biggl(\frac{10C_4}{9(n+1)}\biggr)^{2n+1} h^{(2n+1)(b-1)} e^{(n(2n+1)/(n+1))F} \biggl(\frac{{\omega_\varphi}^n}{{\omega_0}^n}\biggr)^{1/(n+1)}, \end{aligned} \tag{4-19}$$

while on Ω_-, we have

$$\begin{aligned} s^{(2n+1)(b-1)/2}\biggl(\exp\biggl(-\frac{1}{n+1}\dot{\varphi}\biggr) + c(b,\theta,r_0) h \operatorname{tr}_{\omega_\varphi} \omega_0\biggr)^{2n+1} \biggl(\frac{{\omega_\varphi}^n}{{\omega_0}^n}\biggr)^2 \\ \leq C_5 s^{(2n+1)(b-1)/2} = C_5 s^{1/(4n+4)}, \end{aligned} \tag{4-20}$$

where C_5 is not universal since it depends additionally on φ, t, and t_0.

Defining $D = B_r(x_0) \times [t_0, t_0+1]$ and combining (4-19) and (4-20), the parabolic estimate (4-1) tells us that

$$\begin{aligned} &\sup_D(H) - \sup_{\partial_P D}(H) \\ &\leq C_6 \biggl(\int_{D\cap\Omega_+} h^{(2n+1)(b-1)} e^{(n(2n+1)/(n+1))F} \biggl(\frac{{\omega_\varphi}^n}{{\omega_0}^n}\biggr)^{1/(n+1)} {\omega_0}^n \, dt + \int_{D\cap\Omega_-} C_5 s^{1/(4n+4)} {\omega_0}^n \, dt\biggr)^{1/(2n+1)} \\ &\leq C_6 \biggl(\int_{D\cap\Omega_+} \frac{1}{n+1}\biggl(n h^{(2n+1)(n+1)(b-1)} e^{n(2n+1)F} + \frac{{\omega_\varphi}^n}{{\omega_0}^n}\biggr) {\omega_0}^n \, dt + C_5 s^{1/(4n+4)}\biggr)^{1/(2n+1)} \\ &\leq C_6 \biggl(\int_{D\cap\Omega_+} h^{1/2} e^{n(2n+1)F} + \int_{M\times[t_0,t_0+1]} {\omega_\varphi}^n \, dt + C_5 s^{1/(4n+4)}\biggr)^{1/(2n+1)} \\ &\leq C_6 \biggl(\int_{D\cap\Omega_+} h^{1/2} e^{n(2n+1)F} + 1 + C_5 s^{1/(4n+4)}\biggr)^{1/(2n+1)}, \end{aligned} \tag{4-21}$$

where C_6 is universal. Moreover, C_6 changes line by line as it absorbs all universal coefficients derived from the estimates. We use the volume of M to absorb the bad factor on Ω_+ and C_5 to absorb the same bad factor on Ω_-

The integral over the set $D \cap \Omega_+$ is in fact integrated over the set

$$\{\rho > 0\} \cap \left\{ \beta\epsilon(-\psi + \Lambda)^{\beta-1} \tilde{f}^{1/(n+1)} - \frac{10C_4}{9(n+1)} < 0 \right\}.$$

Over this set we have, from the choice of constants,

$$\begin{aligned} n(2n+1)|F| &\le \left(\frac{10C_4}{9(n+1)}\right)^{(n+1)/p} \Psi^{1/p} (\beta\epsilon)^{-(n+1)/p} (-\psi + \Lambda)^{(1-\beta)(n+1)/p} \\ &= \frac{\alpha}{2}(-\psi + \Lambda)^{(1-\beta)(n+1)/p}. \end{aligned} \tag{4-22}$$

Moreover, $h(\rho) \le 2\rho + \sqrt{s}$. Then by combining with the inequality (4-21), we have

$$\begin{aligned} &\sup_D(H) - \sup_{\partial_P D}(H) \\ &\quad \le C_6 \left(\int_{D\cap\Omega_+} (-\tilde{\varphi} + \sqrt{s})^{1/2} \exp\left(\frac{\alpha}{2}(-\psi + \Lambda)^{(1-\beta)(n+1)/p}\right) \omega_0{}^n \, \mathrm{d}t + 1 + C_5 s^{1/(4n+4)} \right)^{1/(2n+1)} \\ &\quad \le C_6 \left(\int_{D\cap\Omega_+} (-\tilde{\varphi} + \exp(\alpha(-\psi + \Lambda)^{(1-\beta)(n+1)/p})) \omega_0{}^n \, \mathrm{d}t + 1 + s^{1/2} + C_5 s^{1/(4n+4)} \right)^{1/(2n+1)} \\ &\quad \le C + C_5 s^{1/(4n+2)}, \end{aligned} \tag{4-23}$$

where C has the same dependencies as Λ. Moreover C_5 changes line by line.

The last inequality is derived based on the following two inequalities: the integral $\int_M (-\varphi)\omega_0{}^n \, \mathrm{d}t$ is uniformly bounded by Lemma 2.3, and the fact that $0 < (1-\beta)(n+1)/p < (n+1)/p < 1$ since $0 < \beta < 1$ and $p > n+1$. Therefore the second integral is bounded by Corollary 2.2 and inequality (2-3).

By (4-23) and the definition of θ, we have

$$cQ^{1-1/b} \le \theta Q \le \sup_D(H) - \sup_{\partial_P D}(H) \le C + C_5 s^{1/(4n+2)},$$

where c is universal. In addition, $\sup \rho$ can be controlled by

$$2 \sup \rho_+ \le \sup h_s(\rho) \le Q^{1/b}.$$

The proof of Lemma 4.1 follows from taking the limit $s \to 0^+$. □

Once we have Lemma 4.1, the following theorem is a direct application of Jensen's inequality.

Theorem 4.2. *Let φ be the C^2 solution defined in Theorem 1.1. For any $\beta \in (0, 1)$ and $t_0 \in [0, T-1)$, we have the energy estimate*

$$\int_{M\times[t_0, t_0+1]} (-\tilde{\varphi} + C_3)^{1/\beta} e^{nF} \omega_0{}^n \, \mathrm{d}t \le C,$$

where the constant C has the same dependencies as Λ defined in Lemma 4.1.

Proof. Similar to the application of Lemma 3.1 and (3-7), we have

$$\int_{M\times[t_0,t_0+1]} \exp(c_\beta(-\tilde{\varphi}+C_3)^{1/\beta})\omega_0{}^n\,\mathrm{d}t \le C_\beta,$$

where c_β and C_β both have the same dependencies as Λ.

Let $\widetilde{V}=\int_M e^{nF}\omega_0{}^n\,\mathrm{d}t$ be the volume on the weighted volume form $e^{nF}\omega_0{}^n\,\mathrm{d}t$. Taking logarithms of both sides and applying Jensen's inequality, we have

$$\begin{aligned}\widetilde{V}\log\left(\frac{C_\beta}{\widetilde{V}}\right) &\ge \widetilde{V}\log\left(\frac{1}{\widetilde{V}}\int_{M\times[t_0,t_0+1]}\exp(c_\beta(-\tilde{\varphi}+C_3)^{1/\beta}-nF)e^{nF}\omega_0{}^n\,\mathrm{d}t\right)\\ &\ge \int_{M\times[t_0,t_0+1]}(c_\beta(-\tilde{\varphi}+C_3)^{1/\beta}-nF)e^{nF}\omega_0{}^n\,\mathrm{d}t\\ &\ge c_\beta\int_{M\times[t_0,t_0+1]}(-\tilde{\varphi}+C_3)^{1/\beta}e^{nF}\omega_0{}^n\,\mathrm{d}t-\mathrm{Ent}_p(F).\end{aligned}$$

The theorem follows from $\widetilde{V}\le \mathrm{Ent}_p(F)+V(M,\omega_0)$ and the fact $y\log(y)>-1/e$ for $y>0$. □

Proof of Theorem 1.1. Theorem 3.3 tells us the result follows directly from a uniform control on E. By Hölder's inequality, we have

$$\int_{M\times[t_0,t_0+1]}(-\tilde{\varphi}+C_3)e^{nF}\omega_0{}^n\,\mathrm{d}t \le \widetilde{V}^{1/n}\left(\int_{M\times[t_0,t_0+1]}(-\tilde{\varphi}+C_3)^{n/(n-1)}e^{nF}\omega_0{}^n\,\mathrm{d}t\right)^{(n-1)/n}.$$

If we fix $\beta=1-1/n$ in Theorem 4.2, then the integral estimate is universal and independent of t_0. Thus we complete the proof of Theorem 1.1. □

5. Some generalizations

In this section, we derive some generalizations, Theorems 1.2 and 1.3, of Theorem 1.1.

The idea of Theorem 1.2 comes from the result of Chen and Cheng [2023] for general parabolic Hessian equations. Recall that r denotes the homogeneous degree of the operator $\mathcal{F}$ and the linearization of the flow (1-3) is

$$Lu=-\partial_t u+G^{i\bar{j}}u_{i\bar{j}},$$

where

$$G^{i\bar{j}}=\frac{1}{\mathcal{F}}\frac{\partial\mathcal{F}(\lambda[h_\varphi])}{\partial h_{i\bar{j}}}.$$

To prove Theorem 1.2, we will use the family of auxiliary equations (2-2) and follow the same argument as in Sections 3 and 4.

The proof of main estimate (3-1) is tedious, and we will only show the essential differences compared to previous sections. When we apply the operator L to the test function

$$-\epsilon(-\psi+\Lambda)^\beta-\tilde{\varphi}-s,$$

the Laplacian operator will be replaced by the trace operator $\operatorname{tr}_G v = G^{i\bar{j}} v_{i\bar{j}}$. More precisely, we have the estimates

$$\begin{aligned}
0 &\geq L(-\epsilon(-\psi+\Lambda)^\beta - \tilde{\varphi} - s) \\
&\geq -\beta\epsilon(-\psi+\Lambda)^{\beta-1}\dot{\psi} + \dot{\tilde{\varphi}} + \beta\epsilon(-\psi+\Lambda)^{\beta-1} \operatorname{tr}_G(\sqrt{-1}\partial\bar{\partial}\psi) \\
&\qquad\qquad + \beta(1-\beta)\epsilon(-\varphi+\Lambda)^{\beta-2}|\partial\varphi|_G^2 - \operatorname{tr}_G(\sqrt{-1}\partial\bar{\partial}\varphi) \\
&\geq \beta\epsilon(-\psi+\Lambda)^{\beta-1}(-\dot{\psi}) - (-\dot{\varphi}) + \beta\epsilon(-\psi+\Lambda)^{\beta-1} \operatorname{tr}_G(\sqrt{-1}\partial\bar{\partial}\psi) - \operatorname{tr}_G(\sqrt{-1}\partial\bar{\partial}\varphi) - C_3 \\
&\geq \beta\epsilon(-\psi+\Lambda)^{\beta-1}(-\dot{\psi}) - (-\dot{\varphi}) + \beta\epsilon(-\psi+\Lambda)^{\beta-1} \operatorname{tr}_G \omega_\psi - \operatorname{tr}_G \omega_\varphi - C_3 \\
&\geq \beta\epsilon(-\psi+\Lambda)^{\beta-1}(-\dot{\psi} + \operatorname{tr}_G \omega_\psi) - (-\dot{\varphi} + r + C_3).
\end{aligned}$$

We also must deal with the factor $-\dot{\psi} + \operatorname{tr}_G \omega_\psi$ as in inequalities (3-2)–(3-4). The lower bound of the determinant on the condition of $\mathcal{F}$ will give us the lower bound $\det G^{i\bar{j}} \geq \gamma \mathcal{F}^{-n/r}$. By the flows (2-2) and (1-3) and the homogeneous degree r condition, we have

$$\begin{aligned}
-\dot{\psi} + \operatorname{tr}_G \omega_\psi &\geq (n+1) \sqrt[n+1]{\frac{f e^{nF} {\omega_0}^n}{{\omega_\psi}^n} \cdot {\omega_\psi}^n \det G^{i\bar{j}}} \\
&\geq C_7 f^{1/(n+1)} \sqrt[n+1]{\left(\frac{e^{rF}}{\mathcal{F}}\right)^{n/r}} \geq C_7 f^{1/(n+1)} \exp\left(-\frac{n}{r(n+1)}\dot{\varphi}\right),
\end{aligned} \tag{5-1}$$

where C_7 is a universal constant.

Since the function

$$h(x) = (x - r - C_3)\exp\left(\frac{n}{r(n+1)}x\right) > c,$$

where

$$c = -\frac{r(n+1)}{n}\exp\left(\frac{nC_3 - r}{r(n+1)}\right),$$

we have the same estimate

$$0 \geq C_7\beta\epsilon(-\psi+\Lambda)^{\beta-1} f^{1/(n+1)} - \frac{r(n+1)}{n}\exp\left(\frac{nC_3 - r}{r(n+1)}\right).$$

To derive the ABP estimate and (4-16) for this case, we need to calculate the estimate of the operator L on $\widetilde{\Gamma}$. We have

$$LH \leq -(2n+1)\left(|H_t \cdot \det D^2 H| \cdot \left(\frac{1}{(\det G)^2}\right)\right)^{1/(2n+1)}.$$

Moreover the bad factor in the integration over $D \cap \Omega_+$ is

$$\begin{aligned}
\exp\left(-\frac{n(2n+1)}{r(n+1)}\dot{\varphi}\right)\frac{1}{(\det G)^2} &= e^{(n^2(2n+1)/(r(n+1)))F} \mathcal{F}^{-n(2n+1)/(r(n+1))} \frac{1}{(\det G)^2} \\
&\leq \frac{1}{\gamma^2} e^{(n^2(2n+1)/(r(n+1)))F} \mathcal{F}^{n/(r(n+1))}.
\end{aligned}$$

The exponent of $\mathcal{F}$ is $n/(r(n+1))$, which is less than or equal to 1 if we assume the degree satisfies $1 \leq r \leq n$, and so we have a similar control as in (4-21). Theorem 1.2 follows from an analogue of Theorem 4.2.

We can also consider the much more general flow equation (1-5). Let us list the linearization operators for different choices of Θ firstly:

$$Lu = \begin{cases} -\frac{\partial}{\partial t}u + \dot{\varphi}\Delta_\varphi u, & \Theta(x) = x, \\ -\frac{\partial}{\partial t}u + \frac{1}{3}\dot{\varphi}\Delta_\varphi u, & \Theta(x) = x^{1/3}, \\ -\frac{\partial}{\partial t}u - \dot{\varphi}\Delta_\varphi u, & \Theta(x) = -1/x. \end{cases} \tag{5-2}$$

To get Lemmas 3.1 and 4.1 under the new setting, we need to reprove Lemma 2.3 to get the upper bound of the integral. The following arguments are divided into two cases.

When $\Theta(y) = -1/y$, we have

$$\int_M \dot{\varphi} = \int_M -e^{nF}\frac{{\omega_0}^n}{{\omega_\varphi}^n} < 0.$$

When $\Theta(y) = y^a$ for $a > 0$, we have

$$\ddot{\varphi} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{{\omega_\varphi}^n}{e^{nF}{\omega_0}^n}\right)^a = a\Delta_\varphi\dot{\varphi}\frac{{\omega_\varphi}^n}{e^{nF}{\omega_0}^n}\left(\frac{{\omega_\varphi}^n}{e^{nF}{\omega_0}^n}\right)^{a-1} = a\dot{\varphi}\Delta_\varphi\dot{\varphi}.$$

Consider the first variation of the functional $\int_M \dot{\varphi}{\omega_\varphi}^n$, given by

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_M \dot{\varphi}{\omega_\varphi}^n &= \int_M \ddot{\varphi}{\omega_\varphi}^n + \int_M \dot{\varphi}\frac{\mathrm{d}}{\mathrm{d}t}({\omega_\varphi}^n) \\ &= \int_M \ddot{\varphi}{\omega_\varphi}^n + \int_M \dot{\varphi}\Delta_\varphi\dot{\varphi}{\omega_\varphi}^n \\ &= (a+1)\int_M \dot{\varphi}\Delta_\varphi\dot{\varphi}{\omega_\varphi}^n \\ &= -(a+1)\int_M |\nabla\dot{\varphi}|^2_{\omega_\varphi}{\omega_\varphi}^n \le 0. \end{aligned}$$

Then the estimate of $\int_M \dot{\varphi}{\omega_0}^n$ follows from

$$\begin{aligned} \int_M \dot{\varphi}{\omega_0}^n &\le \int_M \dot{\varphi}{\omega_0}^n - \int_M \dot{\varphi}{\omega_\varphi}^n + \int_M \dot{\varphi}(\,\cdot\,,0){\omega_{\varphi_0}}^n \\ &\le \int_M \dot{\varphi}({\omega_0}^n - {\omega_\varphi}^n) + \int_M e^{-anF}\left(\frac{{\omega_{\varphi_0}}^n}{{\omega_0}^n}\right)^a {\omega_{\varphi_0}}^n \\ &\le \int_M \dot{\varphi}(1 - e^{nF}\dot{\varphi}^{1/a}){\omega_0}^n + C\int_M e^{-anF}{\omega_0}^n, \end{aligned}$$

where C is universal.

Consider a function $A(y) = y - ly^{1+1/a}$ defined on $y \in [0,\infty)$, where l is positive. Using calculus, we have

$$A(y) \le A\left(\frac{a^a}{l^a(a+1)^a}\right) = \frac{a^a}{(a+1)^{a+1}}l^{-a}.$$

This yields the estimate

$$\int_M \dot{\varphi}{\omega_0}^n \le \frac{a^a}{(a+1)^{a+1}}\int_M e^{-anF}{\omega_0}^n + C\int_M e^{-anF}{\omega_0}^n \le CK.$$

Once we have the above results, we need to prove Lemmas 3.1 and 4.1 for the flow equation (1-5). However the operator L has an extra factor $-\dot{\varphi}$ in the Laplacian term which requires slightly different calculations. We will only discuss the case when $\Theta = -1/y$; the other two cases can be treated similarly. To start with, we have

$$L(-\epsilon(-\psi+\Lambda)^{\beta} - \tilde{\varphi} - s) \geq \beta\epsilon(-\psi+\Lambda)^{\beta-1}(-\dot{\psi} - \dot{\varphi}\,\mathrm{tr}_{\omega_\psi}\,\omega_\varphi) + \dot{\varphi} + n\dot{\varphi}. \tag{5-3}$$

There is no constant C_3 since $\int \dot{\varphi} < 0$, and there is a term $n\dot{\varphi}$ since we have the extra factor when we compute the second derivative. By applying the geometric-arithmetic inequality, we have

$$-\dot{\psi} - \dot{\varphi}\,\mathrm{tr}_{\omega_\psi}\,\omega_\varphi \geq (n+1)\biggl(\frac{f e^{nF}\omega_0{}^n}{\omega_\psi{}^n}(-\dot{\varphi})^n\frac{\omega_\psi{}^n}{\omega_\varphi{}^n}\biggr)^{1/(n+1)} = (n+1) f^{1/(n+1)}(-\dot{\varphi})$$

and

$$L(-\epsilon(-\psi+\Lambda)^{\beta} - \tilde{\varphi} - s) \geq (n+1)(\beta\epsilon(-\psi+\Lambda)^{\beta-1} f^{1/(n+1)} - 1)(-\dot{\varphi}).$$

Therefore we can drop the positive factor $-\dot{\varphi}$ and evaluate the inequality at the maximal point.

To derive an analogue of Lemma 4.1, we need to consider

$$\rho = -\epsilon(-\psi+\Lambda)^{\beta} - \tilde{\varphi} - (n+1)(t-t_0),$$

where the last two terms $-(t-t_0)$ do not affect the result since we only estimate locally on $M \times [t_0, t_0+1]$. If the new ρ has an upper bound which has the same dependencies as the constants in Lemma 4.1, then $-\epsilon(-\psi+\Lambda)^{\beta} - \tilde{\varphi}$ does as well. Following a similar calculation, we have

$$L\rho \geq (n+1)\biggl(\beta\epsilon(-\psi+\Lambda)^{\beta-1}\tilde{f}^{1/(n+1)} - 1 + \frac{1}{-\dot{\varphi}}\biggr)(-\dot{\varphi}),$$

where $1/-\dot{\varphi}$ comes from $L(t-t_0) = -1$. Applying the ABP estimate, we have

$$\biggl(|\partial_t H \cdot \det D^2 H| \cdot (-\dot{\varphi})^{2n}\biggl(\frac{\omega_0{}^n}{\omega_\varphi{}^n}\biggr)^2\biggr)^{1/(2n+1)} \leq \frac{R_-}{2n+1}. \tag{5-4}$$

In conclusion, the main term in the ABP estimate is

$$\begin{aligned}\biggl(\beta\epsilon(-\psi+\Lambda)^{\beta-1}\tilde{f}^{1/(n+1)} - 1 + \frac{1}{-\dot{\varphi}}\biggr)_-^{2n+1}(-\dot{\varphi})^{2n+1-2n}\biggl(\frac{\omega_\varphi{}^n}{\omega_0{}^n}\biggr)^2 \\ = \biggl(\beta\epsilon(-\psi+\Lambda)^{\beta-1}\tilde{f}^{1/(n+1)} - 1 + \frac{1}{-\dot{\varphi}}\biggr)_-^{2n+1} e^{2nF}\frac{1}{-\dot{\varphi}}.\end{aligned} \tag{5-5}$$

The term $1/-\dot{\varphi}$ can be controlled pointwisely on the domain $D \cap \Omega_+$. More specifically,

$$\beta\epsilon(-\psi+\Lambda)^{\beta-1}\tilde{f}^{1/(n+1)} - 1 + \frac{1}{-\dot{\varphi}} < 0$$

implies both inequalities

$$\beta\epsilon(-\psi+\Lambda)^{\beta-1}\tilde{f}^{1/(n+1)} - 1 < 0 \quad \text{and} \quad \frac{1}{-\dot{\varphi}} - 1 < 0.$$

The rest of the proof for the $\Theta = -1/y$ case follows the same procedure.

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