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SPECTRAL ASYMPTOTICS OF THE NEUMANN LAPLACIAN WITH VARIABLE MAGNETIC FIELD ON A SMOOTH BOUNDED DOMAIN IN THREE DIMENSIONS

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This article is devoted to the semiclassical spectral analysis of the Neumann magnetic Laplacian on a smooth bounded domain in three dimensions. Under a generic assumption on the variable magnetic field (involving a localization of the eigenfunctions near the boundary), we establish a semiclassical expansion of the lowest eigenvalues. In particular, we prove that the eigenvalues become simple in the semiclassical limit.

1. Motivation and main result

1.1. ***The operator.*** Let $\Omega \subset \mathbb{R}^3$ be a smooth connected open bounded domain. We consider $\boldsymbol{A} : \overline{\Omega} \to \mathbb{R}^3$, a smooth magnetic vector potential. The associated magnetic field is given by

$$\boldsymbol{B}(x) = \nabla \times \boldsymbol{A}(x)$$

and assumed to be nonvanishing on $\overline{\Omega}$. For $h > 0$, we consider the self-adjoint operator

$$\mathscr{L}_h = (-ih\nabla - \boldsymbol{A})^2 \tag{1-1}$$

with domain

$$\mathrm{Dom}(\mathscr{L}_h) = \{\psi \in H^2(\Omega) : \boldsymbol{n} \cdot (-ih\nabla - \boldsymbol{A})\psi = 0 \text{ on } \partial\Omega\},$$

where $\boldsymbol{n}$ is the outward pointing normal to the boundary.

The associated quadratic form is defined, for all $\psi \in H^1(\Omega)$, by

$$\mathcal{Q}_h(\psi) = \int_\Omega |(-ih\nabla - \boldsymbol{A})\psi|^2 \, \mathrm{d}x.$$

Since Ω is smooth and bounded, the operator $\mathscr{L}_h$ has compact resolvent and we can consider the nondecreasing sequence of its eigenvalues $(\lambda_n(h))_{n\geqslant 1}$ (repeated according to their multiplicities). The aim of this article is to describe the behavior of the eigenvalues $\lambda_n(h)$ in the semiclassical limit $h \to 0$.

1.2. ***The operator on a half-space with constant magnetic field.*** The boundary of Ω has an important influence on the spectral asymptotics. Let us consider $x_0 \in \partial\Omega$ and the angle $\theta(x_0) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ given by

$$\boldsymbol{B}(x_0) \cdot \boldsymbol{n}(x_0) = \|\boldsymbol{B}(x_0)\| \sin(\theta(x_0)),$$

where $\boldsymbol{n}(x_0)$ is the outward pointing normal at x_0.

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Near x_0, one will approximate Ω by the half-space $\mathbb{R}^3_+ = \{(r, s, t) \in \mathbb{R}^3 : t > 0\}$ (the variable t playing the role of the distance to the boundary). Then, this will lead us to consider the Neumann realization of

$$\mathfrak{L}_\theta = (D_r - t\cos\theta + s\sin\theta)^2 + D_s^2 + D_t^2$$

in the ambient space $L^2(\mathbb{R}^3_+)$, which already appeared in [Lu and Pan 2000] in the context of Ginzburg–Landau theory. We use the notation $D = -i\,\partial$. The corresponding magnetic field is $\boldsymbol{b}(\theta) = (0, \cos\theta, \sin\theta)$. We let

$$\boldsymbol{e}(\theta) = \inf \operatorname{sp}(\mathfrak{L}_\theta).$$

It is well known (see [Helffer and Morame 2002; Lu and Pan 2000] and also [Raymond 2017, Section 2.5.2]) that $\boldsymbol{e}$ is even, continuous and increasing on $\left[0, \frac{\pi}{2}\right]$ (from $\Theta_0 := \boldsymbol{e}(0) \in (0, 1)$ to 1) and analytic on $\left(0, \frac{\pi}{2}\right)$. Moreover, we can prove that, for all $\theta \in \left(0, \frac{\pi}{2}\right)$, $\boldsymbol{e}(\theta)$ is also the groundstate energy of the Neumann realization of the "Lu–Pan" operator, acting on $L^2(\mathbb{R}^2_+)$,

$$\mathcal{L}_\theta = (t\cos\theta - s\sin\theta)^2 + D_s^2 + D_t^2; \tag{1-2}$$

see [Raymond 2017, Section 0.1.5.4]. In this case, the groundstate energy belongs to the discrete spectrum and it is a simple eigenvalue.

These considerations lead us to introduce the function β on the boundary.

Definition 1.1. We let, for all $x \in \partial\Omega$,

$$\beta(x) = \|\boldsymbol{B}(x)\|\boldsymbol{e}(\theta(x)).$$

1.3. ***Context, known results, and main theorem.*** The function β plays a central role in the semiclassical spectral asymptotics. The one-term asymptotics of $\lambda_1(h)$ are established in [Lu and Pan 2000] (see also [Raymond 2010a] and [Fournais and Helffer 2010], where additional details are provided).

Theorem 1.2 [Lu and Pan 2000]. *We have*

$$\lambda_1(h) = h\min(b_{\min}, \beta_{\min}) + o(h),$$

where $b_{\min} = \min_{x\in\overline{\Omega}} \|\boldsymbol{B}(x)\|$ *and* $\beta_{\min} = \min_{x\in\partial\Omega} \beta(x)$.

When $\boldsymbol{B}$ is constant (or with constant norm), more accurate estimates of the groundstate energy have been obtained in [Helffer and Kachmar 2023; Helffer and Morame 2004; Raymond 2010b]. When looking at Theorem 1.2, natural questions can be asked. Can we describe more than the groundstate energy? Is the groundstate energy a simple eigenvalue? In three dimensions, most of the results in this direction have been obtained rather recently:

- When $b_{\min} < \beta_{\min}$, we can prove that the boundary is essentially not seen by the eigenfunctions with low eigenvalues and that they are localized near the minima of $\|\boldsymbol{B}\|$. Then, if the minimum is unique and nondegenerate, the analysis of [Helffer et al. 2016] applies and it can be established that

$$\lambda_n(h) = b_{\min}h + C_0h^{3/2} + (C_1(2n-1) + C_2)h^2 + o(h^2),$$

where the constants $(C_0, C_1, C_2) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}$ reflect the classical dynamics in a magnetic field.

• When $\boldsymbol{B}$ is constant (or with constant norm), we can prove that $\beta_{\min} < b_{\min}$ and that $\beta_{\min} = \Theta_0\|\boldsymbol{B}\|$. In this case, the eigenfunctions with low eigenvalues are localized near the points of the boundary where the magnetic field is tangent, that is, where $\boldsymbol{e}(\theta(x))$ is minimal. Assuming that the magnetic field becomes generically tangent to the boundary along a nice closed curve and assuming also a nondegeneracy assumption, we have, from [Hérau and Raymond 2024],

$$\lambda_n(h) = \beta_{\min} h + C_0 h^{4/3} + C_1 h^{3/2} + (C_2(2n-1) + C_3)h^{5/3} + o(h^2)$$

for some constants $(C_0, C_1, C_2, C_3) \in \mathbb{R}^2 \times \mathbb{R}_+ \times \mathbb{R}$.

The result in [Hérau and Raymond 2024] is stated in the case of a constant magnetic field, but only the fact that its norm is constant is actually used in the analysis; see Section 3.2.1 in that same work. Note that without the additional nondegeneracy assumption and stopping the analysis before Section 5.6 in that same work provides us with the two-term expansion. This observation is motivated by [Helffer and Kachmar 2023], where the two-term expansion of the groundstate energy has been obtained independently and where examples are also analyzed in detail.

When $\beta_{\min} < b_{\min}$ and when $\|\boldsymbol{B}\|$ is variable, it seems that less is known. The first estimates of the low-lying eigenvalues, and not only of the first one, are done in [Raymond 2010a] (see also [Raymond 2009]), where an upper bound is obtained under a generic assumption (see Assumption 1.3 below):

$$\lambda_n(h) \leqslant \beta_{\min} h + C_0 h^{3/2} + (C_1(2n-1) + C_2)h^2 + o(h^2) \tag{1-3}$$

for some constants $(C_0, C_1, C_2) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}$ and where C_1 is explicitly given by

$$C_1 = \frac{\sqrt{\det \operatorname{Hess}_{x_0} \beta}}{2\|\boldsymbol{B}(x_0)\| \sin\theta(x_0)}.$$

The upper bound (1-3) is obtained by means of a construction of quasimodes in local coordinates near the minimum of β and involves a number of rather subtle algebraic cancellations. At a conference in Dijon in March 2010, S. Vũ Ngọc suggested to the last author that these algebraic cancellations were the signs of a hidden normal form. At the same conference, J. Sjöstrand also suggested that a dimensional reduction in the Grushin spirit (see the remarkable survey [Sjöstrand and Zworski 2007]) could provide us with the lower bound. Retrospectively, we will see that both of them were somewhat right, but that some microlocal techniques needed to be developed further in order to tackle the problem in an efficient way.

Until now, the matching lower bound to (1-3) has only been obtained for a toy model in the case of a flat boundary with an explicit polynomial magnetic field; see [Raymond 2012]. The aim of this article is to establish a lower bound that matches (1-3) in the general case. To do so, we will, of course, work under the same assumption as in [Raymond 2010a].

Assumption 1.3. *The function β has a unique minimum, which is nondegenerate. It is attained at $x_0 \in \partial\Omega$, and we have*

$$\theta(x_0) \in \left(0, \tfrac{\pi}{2}\right). \tag{1-4}$$

Moreover, we have

$$\beta_{\min} = \beta(x_0) = \min_{x\in\partial\Omega} \beta(x) < \min_{x\in\overline{\Omega}} \|\boldsymbol{B}(x)\| = b_{\min}.$$

The main result of this article is a three-term expansion of the n-th eigenvalue of $\mathscr{L}_h$. Thereby, it completes the picture described above.

Theorem 1.4. *Under Assumption 1.3, there exist $C_0, C_1 \in \mathbb{R}$ such that, for all $n \geqslant 1$, we have*

$$\lambda_n(h) \underset{h\to 0}{=} \beta_{\min} h + C_0 h^{3/2} + \left(\frac{\sqrt{\det \operatorname{Hess}_{x_0} \beta}}{\|\boldsymbol{B}(x_0)\| \sin\theta(x_0)} \left(n - \frac{1}{2}\right) + C_1 \right) h^2 + o(h^2).$$

In particular, for all $n \geqslant 1$, $\lambda_n(h)$ becomes a simple eigenvalue as soon as h is small enough.

1.4. ***Organization and strategy of the proof.*** In Section 2, we recall the already known results of localization of the eigenfunctions near x_0. This formally reduces the spectral analysis to a neighborhood of x_0. This suggests that we should introduce local coordinates near x_0. These coordinates (r, s, t) are adapted to the geometry of the magnetic field: the coordinate s is the curvilinear coordinate along the projection of the magnetic field on the boundary (we use here that $\theta(x_0) < \frac{\pi}{2}$), the coordinate r is the geodesic coordinate transverse to s, and t is the distance to the boundary. A rather similar coordinate system has been used and described in [Hérau and Raymond 2024] (inspired from [Helffer and Morame 2004]). Then, the local action of the operator is described in Section 2.3, where we perform a Taylor expansion with respect to the normal variable t only. After a local change of gauge, this makes an approximate magnetic vector potential appear, see (2-10). In Section 2.3.2, we define a new operator on $L^2(\mathbb{R}^3_+)$ by extending the coefficients, seen as functions of (r, s) defined near $(0, 0)$, to functions on $\mathbb{R}^2$. Since this extension occurs away from the localization zone of the eigenfunctions, we get a new operator $\mathscr{L}_h^{\mathrm{app}}$ whose spectrum is close to that of $\mathscr{L}_h$, see Proposition 2.11.

In Section 3, we perform the analysis of $\mathscr{L}_h^{\mathrm{app}}$ with the help of the change of coordinates $(r, s) \mapsto \mathscr{J}(r, s) = (u_1, u_2)$, whose geometric role is to make the normal component of the magnetic field constant (here, we use $\theta(x_0) > 0$). This idea is reminiscent of [Morin et al. 2023] in two dimensions; see Proposition 2.2 in that work. We are reduced to the spectral analysis of the operator $\mathscr{N}_h$, see (3-1). Then, we conjugate $\mathscr{N}_h$ by a tangential Fourier transform (in the direction u_1) and a translation/dilation T (after these transforms, the variable u_1 becomes z). After these explicit transforms, we get a new operator $\mathscr{N}_\hbar^{\sharp}$, which can be seen as a differential operator of order 2 in the variables (z, t) with coefficients that are h-pseudodifferential operators (with an expansion in powers of $\hbar = h^{1/2}$) in the variable u_2 only, see (3-10). Its eigenfunctions are localized in (z, t), see Proposition 3.3 and Remark 3.4.

In Section 4, this localization with respect to z suggests that we should insert cutoff functions in the coefficients of our operator. By doing this, we get the operator $\mathscr{N}_\hbar^{\flat}$, see (4-1). The advantage of $\mathscr{N}_\hbar^{\flat}$ is that it can be considered as a pseudodifferential operator with operator-valued symbol in a reasonable class $S(\mathbb{R}^2, N)$, see Proposition 4.2. The principal operator symbol $n_0(u, \upsilon)$ is unitarily equivalent to the Lu–Pan operator $\|\boldsymbol{B}(\upsilon, -u)\|\mathcal{L}_{\theta(\upsilon,-u)}$ (where we make a slight abuse of notation by forgetting the reference to the local coordinates on the boundary), see Proposition 4.4. Then, we may construct an inverse for $n_0 - \Lambda$ by means of the so-called Grushin formalism as soon as Λ is close to $\beta_{\min}$, see Lemma 4.5. This is the first step in the approximate parametrix construction for $\mathscr{N}_\hbar^{\flat} - \Lambda$ given in Proposition 4.7, which is the key of the proof of Theorem 1.4. Let us emphasize that this parametrix construction is inspired by [Keraval 2018] and based on ideas developed by A. Martinez and J. Sjöstrand. This formalism has recently been used in [Hérau and Raymond 2024] in three dimensions (see also [Bonnaillie-Noël et al. 2022; Fahs et al. 2024; Fournais et al. 2023] in the case of two dimensions). At a formal level, this parametrix construction relates the kernel of $\mathscr{N}_\hbar^\flat - \Lambda$ to that of an effective pseudodifferential operator $Q_\hbar^\pm(\Lambda)$, see (4-7).

In Section 5 we relate the spectrum of $\mathscr{N}_\hbar^\sharp$ to that of the effective operator $(p_\hbar^{\mathrm{eff}})^W$, see (5-1). Note: the effective operator is an operator in one dimension. This contrasts with [Hérau and Raymond 2024], where a double Grushin reduction is used: here this reduction is done in one step with the help of the Lu–Pan operator. The quasi-parametrix in Proposition 4.7 is the bridge between the spectra of $\mathscr{N}_\hbar^\sharp$ and $(p_\hbar^{\mathrm{eff}})^W$.

We emphasize that we have to be very careful when studying this connection since the symbol of the effective operator is not necessarily real-valued (only its principal symbol p_0 is a priori real). This again contrasts with [Hérau and Raymond 2024] and all the previous works on the subject. This non-self-adjointness comes from the fact that $\mathscr{N}_h$ is not self-adjoint on the canonical L^2-space but on a weighted L^2-space. That is why a short detour into the world of non-self-adjoint operators is used in Section 5. In fact, one will not need the operator $(p_\hbar^{\mathrm{eff}})^W$ more than its approximation $(p_\hbar^{\mathrm{mod}})^W$ near the minimum of p_0, see Section 5.1. This approximation is a complex perturbation of the harmonic oscillator. Its spectrum is well known as well as the behavior of its resolvent.

In Section 5.2.1, we use rescaled Hermite functions to construct quasimodes for $\mathscr{N}_\hbar^\sharp$. This shows that the spectrum of the model operator is in fact real, and we get an accurate upper bound of $\lambda_n(\mathscr{N}_\hbar^\sharp)$ in (5-5). This reproves in a much shorter way (1-3) (see [Raymond 2010a, Theorem 1.5], where the convention $\|\boldsymbol{B}(x_0)\| = 1$ is used). Section 5.2.2 is devoted to establishing the corresponding lower bound (by using in particular that the eigenvalues of the non-self-adjoint operator $(p_\hbar^{\mathrm{mod}})^W$ have algebraic multiplicity 1).

Remark 1.5. The above analysis explains the presence of $\beta_{\min}$, attached to the lowest eigenvalue of the Lu–Pan operator, as the leading term in the semiclassical asymptotics. Similarly, the constant

$$\frac{\sqrt{\det \mathrm{Hess}_{x_0} \beta}}{\|\boldsymbol{B}(x_0)\| \sin\theta(x_0)}$$

appears as the uncertainty constant attached to the effective harmonic oscillator $(p_\hbar^{\mathrm{mod}})^W$ after the Grushin reduction to a one-dimensional problem. This spectral gap combines the normal component of the magnetic field with the spectrum of the Lu–Pan operator (and thus it has no obvious dynamical interpretation). The latter is deeply related to the nondegeneracy assumption on β. However, the geometric interpretation of the constants C_0 and C_1 is not clear since they come from the non-self-adjoint linear part of $(p_\hbar^{\mathrm{mod}})^W$.

2. Localization near x_0 and consequences

2.1. ***Localization estimates.*** In this section, we gather some already-known localization properties of the eigenfunctions; see [Raymond 2009].

Proposition 2.1 (localization near the boundary). *Again under Assumption 1.3, for all $\epsilon > 0$ such that $\beta_{\min} + \epsilon < b_{\min}$, there exist α, C, $h_0 > 0$ such that, for all $h \in (0, h_0)$ and all eigenfunctions ψ of $\mathscr{L}_h$ associated with an eigenvalue $\lambda \leqslant (\beta_{\min} + \epsilon)h$, we have*

$$\int_\Omega e^{2\alpha\, \mathrm{dist}(x,\partial\Omega)/\sqrt{h}} |\psi|^2 \,\mathrm{d}x \leqslant C\|\psi\|^2. \tag{2-1}$$

For $\delta > 0$, we consider the δ-neighborhood of the boundary given by

$$\Omega_\delta := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \delta\}.$$

Due to Proposition 2.1, in the following, we take

$$\delta = h^{1/2-\eta}$$

for $\eta \in \left(0, \frac{1}{2}\right)$. We consider $\mathscr{L}_{h,\delta} = (-ih\nabla - \boldsymbol{A})^2$, the operator with magnetic Neumann condition on $\partial\Omega$ and Dirichlet condition on $\partial\Omega_\delta \setminus \partial\Omega$.

Corollary 2.2. *Let $n \geqslant 1$. There exist $C, h_0 > 0$ such that, for all $h \in (0, h_0)$,*

$$\lambda_n(\mathscr{L}_{h,\delta}) - Ce^{-Ch^{-\eta}} \leqslant \lambda_n(\mathscr{L}_h) \leqslant \lambda_n(\mathscr{L}_{h,\delta}).$$

Note that the upper bound in Corollary 2.2 easily follows from the min-max theorem, whereas the lower bound is obtained by using Proposition 2.1.

Thanks to Corollary 2.2, we may focus on the spectral analysis of $\mathscr{L}_{h,\delta}$. The following proposition can be found in [Fournais and Helffer 2010, Chapter 9] and [Helffer and Morame 2002, Theorem 4.3] (see also the proof of [Hérau and Raymond 2024, Proposition 2.9]).

Proposition 2.3 (localization near x_0). *Let $M > 0$. There exist $C, h_0 > 0$ and $\alpha > 0$ such that, for all $h \in (0, h_0)$ and all eigenfunctions ψ of $\mathscr{L}_{h,\delta}$ associated with an eigenvalue λ such that $\lambda \leqslant \beta_{\min} h + M h^{3/2}$, we have*

$$\int_{\Omega_\delta} e^{2\alpha \operatorname{dist}(x,\partial\Omega)/\sqrt{h}} |\psi(x)|^2 \, dx + \int_{\Omega_\delta} e^{2\alpha \|x - x_0\|^2/h^{1/4}} |\psi(x)|^2 \, dx \leqslant C \|\psi\|^2. \tag{2-2}$$

Proposition 2.3 invites us to consider a local chart near x_0 and to write the operator in the corresponding coordinates. In order to simplify our analysis, we construct below a system of coordinates compatible with the geometry of the magnetic field.

2.2. ***Adapted coordinates near x_0.*** This section is devoted to introducing coordinates adapted to the magnetic field. Most of the properties of our coordinates system have been established in [Hérau and Raymond 2024].

2.2.1. *Coordinate in the direction of the magnetic field on the boundary.* We set

$$\boldsymbol{b}(x) = \frac{\boldsymbol{B}(x)}{\|\boldsymbol{B}(x)\|},$$

and we consider its projection on the tangent plane at $x \in \partial\Omega$:

$$\boldsymbol{b}^{\parallel}(x) = \boldsymbol{b}(x) - \langle \boldsymbol{b}(x), \boldsymbol{n}(x) \rangle \boldsymbol{n}(x),$$

where $\boldsymbol{n}$ is the outward pointing normal.

Due to Assumption 1.3, near x_0, the vector field $\boldsymbol{b}^{\parallel}$ does not vanish. This allows us to consider the unit vector field

$$\boldsymbol{f}(x) = \frac{\boldsymbol{b}^{\parallel}(x)}{\|\boldsymbol{b}^{\parallel}(x)\|}$$

and the associated integral curve γ given by

$$\gamma'(s) = \boldsymbol{f}(\gamma(s)), \quad \gamma(0) = x_0,$$

which is well-defined on $(-s_0, s_0)$ for some $s_0 > 0$. Clearly, γ is smooth and with values in $\partial\Omega$.

2.2.2. *Coordinates on the boundary.* Denoting by K the second fundamental form of $\partial\Omega$ associated to the Weingarten map defined by,

$$\text{for all } U, V \in T_x\partial\Omega, \quad K_x(U, V) = \langle \mathrm{d}\boldsymbol{n}_x(U), V\rangle,$$

we can consider the ODE with parameter s of unknown $r \mapsto \gamma(r, s)$,

$$\partial_r^2\gamma(r, s) = -K(\partial_r\gamma(r, s), \partial_r\gamma(r, s))\boldsymbol{n}(\gamma(r, s)),$$

with initial conditions

$$\gamma(0, s) = \gamma(s), \quad \partial_r\gamma(0, s) = -\gamma'(s)^{\perp},$$

where $\perp$ is taken in the tangent space and such that $(\gamma', \gamma'^{\perp}, \boldsymbol{n})$ is a direct orthonormal basis. The minus is here so that $(\partial_r\gamma, \partial_s\gamma, \boldsymbol{n})$ is also a direct orthonormal basis along $\gamma(\,\cdot\,)$. The curve $\gamma(r, \cdot\,)$ is the image of $\gamma(\,\cdot\,)$ under the geodesic flow on $\partial\Omega$ (with initial velocity orthogonal to $\gamma(\,\cdot\,)$) at time r.

This ODE has a unique smooth solution $(-r_0, r_0) \times (-s_0, s_0) \ni (r, s) \mapsto \gamma(r, s)$, where $r_0 > 0$ is chosen small enough. Let us gather the important properties of $(r, s) \mapsto \gamma(r, s)$. Their proofs may be found in [Hérau and Raymond 2024].

Proposition 2.4. *The function $(r, s) \mapsto \gamma(r, s)$ is valued in $\partial\Omega$. Moreover, we have*

$$|\partial_r\gamma(r, s)| = 1, \quad \langle \partial_r\gamma, \partial_s\gamma\rangle = 0.$$

In this chart γ, the first fundamental form on $\partial\Omega$ is given by the matrix

$$g(r, s) = \begin{pmatrix} 1 & 0 \\ 0 & \alpha(r, s) \end{pmatrix}, \quad \alpha(r, s) = |\partial_s\gamma(r, s)|^2.$$

For all $s \in (-s_0, s_0)$, we have $\alpha(0, s) = 1$ and $\partial_s\alpha(0, s) = 0$.

2.2.3. *Coordinates near the boundary.* We consider the tubular coordinates associated with the chart γ:

$$y = (r, s, t) \mapsto \Gamma(r, s, t) = \gamma(r, s) - t\boldsymbol{n}(\gamma(r, s)) = x. \tag{2-3}$$

The map Γ is a smooth diffeomorphism from $Q_0 := (-r_0, r_0) \times (-s_0, s_0) \times (0, t_0)$ to $\Gamma(Q_0)$, as soon as $t_0 > 0$ is chosen small enough. The differential of Γ can be written as

$$\mathrm{d}\Gamma_y = [(\mathrm{Id} - t\,\mathrm{d}\boldsymbol{n})(\partial_r\gamma), (\mathrm{Id} - t\,\mathrm{d}\boldsymbol{n})(\partial_s\gamma), -\boldsymbol{n}], \tag{2-4}$$

and the Euclidean metric becomes

$$\boldsymbol{G} = (\mathrm{d}\Gamma)^{\mathrm{T}}\,\mathrm{d}\Gamma = \begin{pmatrix} \boldsymbol{g} & 0 \\ 0 & 1 \end{pmatrix}, \tag{2-5}$$

with

$$\boldsymbol{g}(r,s,t)=\begin{pmatrix} \|(\mathrm{Id}-t\,\mathrm{d}\boldsymbol{n})(\partial_r\gamma)\|^2 & \langle(\mathrm{Id}-t\,\mathrm{d}\boldsymbol{n})(\partial_r\gamma),(\mathrm{Id}-t\,\mathrm{d}\boldsymbol{n})(\partial_s\gamma)\rangle \\ \langle(\mathrm{Id}-t\,\mathrm{d}\boldsymbol{n})(\partial_r\gamma),(\mathrm{Id}-t\,\mathrm{d}\boldsymbol{n})(\partial_s\gamma)\rangle & \|(\mathrm{Id}-t\,\mathrm{d}\boldsymbol{n})(\partial_s\gamma)\|^2 \end{pmatrix}.$$

We have $g(r,s)=\boldsymbol{g}(r,s,0)$, where g is defined in Proposition 2.4.

2.2.4. *The magnetic form in tubular coordinates.* In this section, we discuss the expression of the magnetic field in the coordinates induced by Γ. This discussion can be found in [Raymond 2017, Section 0.1.2.2] and [Hérau and Raymond 2024, Section 3.2]. We consider the 1-form

$$\sigma=\boldsymbol{A}\cdot\mathrm{d}x=\sum_{\ell=1}^{3}A_\ell\,\mathrm{d}x_\ell.$$

Its exterior derivative is the magnetic 2-form

$$\omega=\mathrm{d}\sigma=\sum_{1\leqslant k<\ell\leqslant 3}(\partial_k A_\ell-\partial_\ell A_k)\,\mathrm{d}x_k\wedge\mathrm{d}x_\ell,$$

which can also be written as

$$\omega=B_3\,\mathrm{d}x_1\wedge\mathrm{d}x_2-B_2\,\mathrm{d}x_1\wedge\mathrm{d}x_3+B_1\,\mathrm{d}x_2\wedge\mathrm{d}x_3.$$

Note also that,

$$\text{for all } U,V\in\mathbb{R}^3,\quad \omega(U,V)=\det(U,V,\boldsymbol{B})=\langle U\times V,\boldsymbol{B}\rangle.$$

Let us now consider the effect of the change of variables $\Gamma(y)=x$. We have

$$\Gamma^*\sigma=\sum_{j=1}^{3}\tilde{A}_j\,\mathrm{d}y_j,\qquad \tilde{\boldsymbol{A}}=(\mathrm{d}\Gamma)^{\mathrm{T}}\circ\boldsymbol{A}\circ\Gamma, \tag{2-6}$$

and

$$\Gamma^*\omega=\Gamma^*\,\mathrm{d}\sigma=\mathrm{d}(\Gamma^*\sigma)=[\,\cdot\,,\,\cdot\,,\nabla\times\tilde{\boldsymbol{A}}].$$

Here we use the notation Γ^* for the pullback by Γ. This also gives that, for all $U,V\in\mathbb{R}^3$,

$$\det(\mathrm{d}\Gamma(U),\mathrm{d}\Gamma(V),\boldsymbol{B})=\det(U,V,\nabla\times\tilde{\boldsymbol{A}})\quad\text{or}\quad \det\mathrm{d}\Gamma(\,\cdot\,,\,\cdot\,,\mathrm{d}\Gamma^{-1}(\boldsymbol{B}))=\det(\,\cdot\,,\,\cdot\,,\nabla\times\tilde{\boldsymbol{A}}),$$

so that,

$$\nabla\times\tilde{\boldsymbol{A}}=(\det\mathrm{d}\Gamma)\,\mathrm{d}\Gamma^{-1}(\boldsymbol{B}).$$

Note then that, using (2-5), we get

$$|\boldsymbol{g}|^{-1/2}\nabla\times\tilde{\boldsymbol{A}}=\mathcal{B}, \tag{2-7}$$

where $\mathcal{B}(y):=\mathrm{d}\Gamma_y^{-1}(\boldsymbol{B}(x))$ corresponds to the coordinates of $\boldsymbol{B}(y)$ in the image of the canonical basis by $\mathrm{d}\Gamma_y$. With our specific change of coordinates (2-3), we have

$$\boldsymbol{B}=\mathrm{d}\Gamma(\mathcal{B})=\mathcal{B}_1(\mathrm{Id}-t\,\mathrm{d}\boldsymbol{n})(\partial_r\gamma)+\mathcal{B}_2(\mathrm{Id}-t\,\mathrm{d}\boldsymbol{n})(\partial_s\gamma)-\mathcal{B}_3\boldsymbol{n}.$$

For all $x \in \partial\Omega$, i.e., $t = 0$, we have

$$\begin{aligned} \boldsymbol{B}(x) &= \mathcal{B}_1(r, s, 0)\, \partial_r \gamma + \mathcal{B}_2(r, s, 0)\, \partial_s \gamma - \mathcal{B}_3(r, s, 0) \boldsymbol{n}(\gamma(r, s)), \\ \|\boldsymbol{B}(x)\|^2 &= \mathcal{B}_1^2(r, s, 0) + \alpha(r, s) \mathcal{B}_2^2(r, s, 0) + \mathcal{B}_3^2(r, s, 0). \end{aligned} \tag{2-8}$$

Moreover, we have

$$\mathcal{B}_1(r, s, 0) = \langle \boldsymbol{B}, \partial_r \gamma \rangle, \quad \alpha(r, s) \mathcal{B}_2(r, s, 0) = \langle \boldsymbol{B}, \partial_s \gamma \rangle, \quad \mathcal{B}_3(r, s, 0) = -\langle \boldsymbol{B}, \boldsymbol{n} \rangle.$$

Note that our choice of coordinate s (along the projection of the magnetic field on the tangent plane) and of transverse coordinate r implies that

$$\mathcal{B}_1(0, s, 0) = 0, \quad \mathcal{B}_2(0, s, 0) > 0,$$

thanks to Assumption 1.3.

Definition 2.5. In a neighborhood of $(0, 0)$, we can consider the unique smooth function θ such that

$$\boldsymbol{B}(\gamma(r, s)) \cdot \boldsymbol{n}(\gamma(r, s)) = \|\boldsymbol{B}(\gamma(r, s))\| \sin\theta(r, s)$$

and satisfying $\theta(r, s) \in \left(0, \frac{\pi}{2}\right)$. With a slight abuse of notation, we let

$$\beta(r, s) = \|\boldsymbol{B}(\gamma(r, s))\| \boldsymbol{e}(\theta(r, s)).$$

Remark 2.6. We have

$$\mathcal{B}_3(r, s) = -\|\boldsymbol{B}(\gamma(r, s))\| \sin(\theta(r, s)).$$

Moreover, since $\mathcal{B}_2 > 0$ and $\alpha(0, s) = 1$,

$$\mathcal{B}_2(0, s, 0) = \|\boldsymbol{B}(\gamma(0, s))\| \cos\theta(0, s), \quad \mathcal{B}_3(0, s, 0) = -\|\boldsymbol{B}(\gamma(0, s))\| \sin\theta(0, s).$$

In fact, we can choose a suitable explicit $\tilde{\boldsymbol{A}}$ such that (2-7) holds in a neighborhood of $(0, 0, 0)$.

Lemma 2.7. *Considering*

$$\begin{aligned} \tilde{A}_1(r, s, t) &= \int_0^t [|\boldsymbol{g}|^{1/2} \mathcal{B}_2](r, s, \tau)\, \mathrm{d}\tau, \\ \tilde{A}_2(r, s, t) &= -\int_0^t [|\boldsymbol{g}|^{1/2} \mathcal{B}_1](r, s, \tau)\, \mathrm{d}\tau + \int_0^r [|\boldsymbol{g}|^{1/2} \mathcal{B}_3](u, s, 0)\, \mathrm{d}u, \\ \tilde{A}_3(r, s, t) &= 0, \end{aligned}$$

we have $\nabla \times \tilde{\boldsymbol{A}}(r, s, t) = |\boldsymbol{g}|^{1/2} \mathcal{B}(r, s, t)$.

Proof. This follows from a straightforward computation and the fact that $|\boldsymbol{g}|^{1/2} \mathcal{B}$ is divergence-free. □

Remark 2.8. Note that the proof of Lemma 2.7 does not involve global geometric quantities on the boundary as in [Hérau and Raymond 2024, Proposition 3.3], since our analysis is local near x_0.

2.3. ***First approximation of the magnetic Laplacian in local coordinates.*** If the support of ψ is close enough to x_0, we may express $\mathcal{Q}_h(\psi)$ in the local chart given by $\Gamma(y)=x$. Letting $\tilde{\psi}(y)=\psi\circ\Gamma(y)$, we have then

$$\mathcal{Q}_h(\psi)=\int\langle \boldsymbol{G}^{-1}(-ih\nabla_y-\tilde{\boldsymbol{A}}(y))\tilde{\psi},(-ih\nabla_y-\tilde{\boldsymbol{A}}(y))\tilde{\psi}\rangle|\boldsymbol{g}|^{1/2}\,\mathrm{d}y.$$

In the Hilbert space $L^2(|\boldsymbol{g}|^{1/2}\,\mathrm{d}y)$, the operator locally takes the form

$$|\boldsymbol{g}|^{-1/2}(-ih\nabla_y-\tilde{\boldsymbol{A}}(y))\cdot|\boldsymbol{g}|^{1/2}\boldsymbol{G}^{-1}(-ih\nabla_y-\tilde{\boldsymbol{A}}(y)),\tag{2-9}$$

where $\boldsymbol{G}$ is defined in (2-5). From now on, the analysis deviates from [Hérau and Raymond 2024].

2.3.1. *Expansion with respect to t.* Due to the localization near the boundary at the scale $h^{1/2}$, we are led to replace $\tilde{\boldsymbol{A}}$ by its Taylor expansion $\tilde{\boldsymbol{A}}^{[3]}$ at order 3 and $\boldsymbol{g}$ and $\boldsymbol{G}$ by their Taylor expansions at order 2. We let

$$\begin{aligned}\tilde{A}_1^{[3]}(r,s,t)&=t[|\boldsymbol{g}|^{1/2}\mathcal{B}_2](r,s,0)+C_2\hat{t}^2+C_3\hat{t}^3,\\ \tilde{A}_2^{[3]}(r,s,t)&=-t[|\boldsymbol{g}|^{1/2}\mathcal{B}_1](r,s,0)+F(r,s)+E_2\hat{t}^2+E_3\hat{t}^3,\\ \tilde{A}_3^{[3]}(r,s,t)&=0,\end{aligned}\tag{2-10}$$

where $\hat{t}=t\chi(h^{-1/2+\eta}t)$ for some smooth cutoff function χ equal to 1 near 0 and where

$$F(r,s)=\int_0^r[|\boldsymbol{g}|^{1/2}\mathcal{B}_3](\ell,s,0)\,\mathrm{d}\ell,\tag{2-11}$$

and the functions $C_j(r,s)$ and $E_j(r,s)$ are smooth. We emphasize that we only truncate the terms of order at least 2 in t in the above expression.

Due to Assumption 1.3, $(r,s)\mapsto(F(r,s),s)$ is a smooth diffeomorphism on a neighborhood of $(0,0)$.

We also consider the expansions

$$\begin{aligned}|\boldsymbol{g}|^{1/2}(r,s,t)&=a_0(r,s)+ta_1(r,s)+t^2a_2(r,s)+\mathscr{O}(t^3),\\ \boldsymbol{G}^{-1}&=(M_0(r,s)+tM_1(r,s)+t^2M_2(r,s))^{-1}+\mathscr{O}(t^3),\end{aligned}$$

and we let

$$m(r,s,t)=a_0(r,s)+\hat{t}a_1(r,s)+\hat{t}^2a_2(r,s),\quad M(r,s,t)=M_0(r,s)+\hat{t}M_1(r,s)+\hat{t}^2M_2(r,s).\tag{2-12}$$

Recall that $|\boldsymbol{g}|(r,s,0)=\alpha(r,s)$.

2.3.2. *Extension of the functions of the tangential variables.* It will be convenient to work on the half-space $\mathbb{R}^3_+$ instead of a neighborhood of $(0,0,0)$.

Given $\epsilon_0>0$, consider a smooth, odd, and nondecreasing function $\zeta:\mathbb{R}\to\mathbb{R}$ such that $\zeta(x)=x$ on $[0,\epsilon_0]$ and $\zeta(x)=2\epsilon_0$ for all $x\geqslant 2\epsilon_0$. In particular, $\|\zeta\|_\infty=2\epsilon_0$. We let

$$Z(r,s)=(\zeta(r),\zeta(s)).$$

The following lemma is a straightforward consequence of Assumption 1.3.

Lemma 2.9. *For ϵ_0 small enough, the function $\hat{\beta} = \beta \circ Z : \mathbb{R}^2 \to \mathbb{R}_+$ is smooth and has a unique minimum (at $(0,0)$), which is nondegenerate and not attained at infinity.*

Let us now replace the function $\mathscr{B} : (r,s) \mapsto \alpha(r,s)^{1/2}\mathcal{B}(r,s,0)$ by $\mathscr{B} \circ Z$ in (2-10) and (2-11). We replace the other coefficients C_j and E_j by $C_j \circ Z$ and $E_j \circ Z$. Note that we have the following.

Lemma 2.10. *For ϵ_0 small enough, the function*

$$\mathscr{J} : \mathbb{R}^2 \ni (r,s) \mapsto \left(\int_0^r [|\boldsymbol{g}|^{1/2}\mathcal{B}_3](Z(\ell,s),0)\,\mathrm{d}\ell, s \right) = u = (u_1,u_2) \in \mathbb{R}^2$$

is smooth, and it is a global diffeomorphism.

This leads us to consider the new vector potential

$$\begin{aligned}
\hat{A}_1(r,s,t) &= t\mathring{C}_1 + \mathring{C}_2\hat{t}^2 + \mathring{C}_3\hat{t}^3,\\
\hat{A}_2(r,s,t) &= -t\mathring{E}_1 + \mathscr{J}_1(r,s) + \mathring{E}_2\hat{t}^2 + \mathring{E}_3\hat{t}^3,\\
\hat{A}_3(r,s,t) &= 0,
\end{aligned} \tag{2-13}$$

where $C_1 = \alpha^{1/2}\mathcal{B}_2$, $E_1 = \alpha^{1/2}\mathcal{B}_1$ and with the notation $\mathring{f} = f \circ Z$.

The rest of the article will be devoted to the spectral analysis of the operator associated with the new quadratic form

$$\mathcal{Q}_h^{\mathrm{app}}(\varphi) = \int_{\mathbb{R}^3_+} \langle (\mathring{M})^{-1}(-ih\nabla_y - \hat{\boldsymbol{A}}(y))\varphi, (-ih\nabla_y - \hat{\boldsymbol{A}}(y))\varphi\rangle \mathring{m}\,\mathrm{d}y.$$

This self-adjoint operator $\mathscr{L}_h^{\mathrm{app}}$ is acting as

$$\mathring{m}^{-1}(-ih\nabla_y - \hat{\boldsymbol{A}}) \cdot \mathring{m}(\mathring{M})^{-1}(-ih\nabla_y - \hat{\boldsymbol{A}})$$

in the ambient Hilbert space $L^2(\mathbb{R}^3_+, \mathring{m}\,\mathrm{d}y)$. We recall that m and M are given in (2-12). This spectral analysis is motivated by the fact that the low-lying spectra of $\mathscr{L}_h$ and $\mathscr{L}_h^{\mathrm{app}}$ coincide modulo $o(h^2)$ in the sense of the following proposition.

Proposition 2.11. *We have, for all $n \geqslant 1$,*

$$\lambda_n(h) = \lambda_n(\mathscr{L}_h^{\mathrm{app}}) + o(h^2).$$

We omit the proof. It follows from Corollary 2.2, the localization estimates given in Proposition 2.3 (which are also true in the coordinates (r,s,t) for the eigenfunctions of $\mathscr{L}_h^{\mathrm{app}}$ by using the same arguments), and the min-max theorem. These localization estimates allow us to remove the cutoff functions up to remainders of order $\mathscr{O}(h^\infty)$ and to control the remainders of the expansion in t.

3. Change of coordinates and metaplectic transform

In order to perform the spectral analysis of $\mathscr{L}_h^{\mathrm{app}}$, it is convenient to use the change of variable $\mathscr{J}$ given in Lemma 2.10. More precisely, we will use the unitary transform induced by $\mathscr{J}$ defined by

$$U : L^2(\mathbb{R}^3_+, \mathring{m}\,\mathrm{d}y) \to L^2(\mathbb{R}^3_+, \check{m}|\mathrm{Jac}\,\mathscr{J}^{-1}|\,\mathrm{d}u\,\mathrm{d}t), \quad \varphi \mapsto \check{\varphi},$$

where we use the notation $\breve{f}(u,t) = f(\mathscr{J}^{-1}(u),t)$ and the slight abuse of notation $\breve{\mathring{f}} = \breve{f}$. Then, we focus on the operator $\mathscr{N}_h = U\mathscr{L}_h^{\mathrm{app}}U^{-1}$ acting in $L^2(\mathbb{R}^3_+, \breve{m}|\mathrm{Jac}\,\mathscr{J}^{-1}|\,\mathrm{d}u\,\mathrm{d}t)$. The operator $\mathscr{N}_h$ is acting as

$$\mathscr{N}_h = U\mathscr{L}_h^{\mathrm{app}}U^{-1} = \breve{m}^{-1}\mathscr{D}_h \cdot \breve{m}(\breve{M})^{-1}\mathscr{D}_h, \tag{3-1}$$

where

$$\mathscr{D}_h = \begin{pmatrix} -ih\breve{C}_0\,\partial_{u_1} - t\breve{C}_1 - \hat{t}^2\breve{C}_2 - \hat{t}^3\breve{C}_3 \\ -ih\,\partial_{u_2} - u_1 - ih\breve{E}_0\,\partial_{u_1} + t\breve{E}_1 - \hat{t}^2\breve{E}_2 - \hat{t}^3\breve{E}_3 \\ -ih\,\partial_t \end{pmatrix}$$

and

$$C_0 = \partial_r \mathscr{J}_1 = \alpha^{1/2}\mathcal{B}_3, \qquad E_0 = \partial_s \mathscr{J}_1. \tag{3-2}$$

Notation 3.1. We will use the following classical notation for the semiclassical Weyl quantization of a symbol $a = a(u,\upsilon)$. We let

$$a^W\psi(u) = \frac{1}{(2\pi h)^2}\int_{\mathbb{R}^4} e^{i(u-x)\cdot\upsilon/h}\, a\Big(\frac{u+x}{2},\upsilon\Big)\psi(x)\,\mathrm{d}x\,\mathrm{d}\upsilon.$$

Proposition 3.2. *Let $K > 0$ and $\eta \in \big(0,\frac{1}{2}\big)$. Let Ξ be a smooth function of the real variable equal to* 0 *near* 0 *and* 1 *away from a compact neighborhood of* 0*. There exists $h_0 > 0$ such that, for all $h \in (0,h_0)$ and for all normalized eigenfunctions ψ of $\mathscr{N}_h$ associated with an eigenvalue λ such that $\lambda \leqslant Kh$, we have, in $L^2(\mathbb{R}^3_+)$,*

$$\bigg[\Xi\Big(\frac{u_1-\upsilon_2}{h^{1/2-\eta}}\Big)\bigg]^W \psi = \mathscr{O}(h^\infty).$$

Proof. To simplify the notation, we write

$$\Xi_h = \Xi\Big(\frac{u_1-\upsilon_2}{h^{1/2-\eta}}\Big).$$

Note that Ξ_h^W is a bounded operator by virtue of the Calderón–Vaillancourt theorem (see [Zworski 2012, Theorem 4.23]).

Let ψ be a normalized eigenfunction of $\mathscr{N}_h$ associated with an eigenvalue λ such that $\lambda \leqslant Kh$. The eigenvalue equation gives us

$$\langle \mathscr{N}_h \Xi_h^W\psi, \Xi_h^W\psi\rangle = \lambda\|\Xi_h^W\psi\|^2 + \langle[\mathscr{N}_h, \Xi_h^W]\psi, \Xi_h^W\psi\rangle, \tag{3-3}$$

where $\langle\cdot,\cdot\rangle$ is the scalar product in $L^2(\mathbb{R}^3_+, \breve{m}|\mathrm{Jac}\,\mathscr{J}^{-1}|\,\mathrm{d}u\,\mathrm{d}t)$.

According to the localization at the scale $h^{1/2}$ with respect to t, we can insert a cutoff function supported in $\{t \leqslant h^{(1-\eta)/2}\}$, and we obtain, for $j = 2, 3$,

$$\|t^j\,\Xi_h^W\psi\| \leqslant Ch^{1-\eta}\|\Xi_h^W\psi\| + \mathscr{O}(h^\infty)\|\psi\|. \tag{3-4}$$

Then, we write

$$\begin{aligned}\langle \mathscr{N}_h \Xi_h^W\psi, \Xi_h^W\psi\rangle_{L^2(\mathbb{R}^3_+,\breve{m}|\mathrm{Jac}\,\mathscr{J}^{-1}|\,\mathrm{d}u\,\mathrm{d}t)} &= \langle \mathscr{D}_h\cdot\breve{m}(\breve{M})^{-1}\mathscr{D}_h\Xi_h^W\psi, |\mathrm{Jac}\,\mathscr{J}^{-1}|\,\Xi_h^W\psi\rangle_{L^2(\mathbb{R}^3_+,\,\mathrm{d}u\,\mathrm{d}t)} \\ &= \langle \breve{m}(\breve{M})^{-1}\mathscr{D}_h\Xi_h^W\psi, \mathscr{D}_h(|\mathrm{Jac}\,\mathscr{J}^{-1}|\,\Xi_h^W\psi)\rangle_{L^2(\mathbb{R}^3_+,\,\mathrm{d}u\,\mathrm{d}t)}.\end{aligned}$$

We notice that $[\mathscr{D}_h, |\mathrm{Jac}\,\mathscr{J}^{-1}|] = \mathscr{O}(h)$ and that $\check{m}(\check{M})^{-1} \geqslant c > 0$. This implies that

$$\begin{aligned}\langle \mathscr{N}_h \Xi_h^W \psi, \Xi_h^W \psi \rangle_{L^2(\mathbb{R}^3_+, \check{m}|\mathrm{Jac}\,\mathscr{J}^{-1}|\,\mathrm{d}u\,\mathrm{d}t)} &\geqslant c\|\mathscr{D}_h \Xi_h^W \psi\|^2 - Ch\|\mathscr{D}_h \Xi_h^W \psi\|\|\Xi_h^W \psi\| \\ &\geqslant \tfrac{1}{2}c\|\mathscr{D}_h \Xi_h^W \psi\|^2 - Ch^2\|\Xi_h^W \psi\|^2,\end{aligned}$$

where we use the Young inequality to get the last estimate.

By using again the Young inequality and (3-4) to deal with the powers $\hat{t}^2$ and $\hat{t}^3$ in $\mathscr{D}_h$, this yields, for some $c, C > 0$,

$$\langle \mathscr{N}_h \Xi_h^W \psi, \Xi_h^W \psi \rangle_{L^2(\mathbb{R}^3_+, \check{m}|\mathrm{Jac}\,\mathscr{J}^{-1}|\,\mathrm{d}u\,\mathrm{d}t)} \geqslant c Q_h^0(\Xi_h^W \psi) - Ch^{1-\eta}\|\Xi_h^W \psi\|^2 + \mathscr{O}(h^\infty)\|\psi\|^2, \tag{3-5}$$

where

$$Q_h^0(\varphi) = \|h\,\partial_t \varphi\|^2 + \|(h\check{C}_0 D_{u_1} - t\check{C}_1)\varphi\|^2 + \|(hD_{u_2} - u_1 + h\check{E}_0 D_{u_1} + t\check{E}_1)\varphi\|^2.$$

Then, using again the Young inequality, we find that

$$Q_h^0(\varphi) \geqslant \|h\,\partial_t \varphi\|^2 + \tfrac{1}{2}\|h\check{C}_0 D_{u_1}\varphi\|^2 + \tfrac{1}{2}\|(hD_{u_2} - u_1)\varphi\|^2 - 2\|h\check{E}_0 D_{u_1}\varphi\|^2 - C\|t\varphi\|^2.$$

Notice that there exists $c > 0$ such that

$$|\check{C}_0| \geqslant c, \quad |\check{E}_0| \leqslant \tfrac{1}{4}c, \tag{3-6}$$

where we recall (3-2) and Lemma 2.10. Indeed, we have $E_0(0,0) = 0$ and, for some $c_0 > 0$, we have $C_0 \geqslant c_0 > 0$. In particular, ϵ_0 can be chosen small enough in the extension procedure in Section 2.3.2 that (3-6) holds. This shows that, for some $c_0 > 0$,

$$Q_h^0(\varphi) \geqslant \|h\,\partial_t \varphi\|^2 + c_0\|hD_{u_1}\varphi\|^2 + \tfrac{1}{2}\|(hD_{u_2} - u_1)\varphi\|^2 - C\|t\varphi\|^2. \tag{3-7}$$

On the support of Ξ_h, we have $(\upsilon_2 - u_1)^2 \geqslant ch^{1-2\eta}$ for some $c > 0$. Thus (3-4), (3-5), (3-7), and again the localization in t yield

$$\langle \mathscr{N}_h \Xi_h^W \psi, \Xi_h^W \psi \rangle_{L^2(\mathbb{R}^3_+, \check{m}|\mathrm{Jac}\,\mathscr{J}^{-1}|\,\mathrm{d}u\,\mathrm{d}t)} \geqslant \tfrac{1}{2}\tilde{c}h^{1-2\eta}\|\Xi_h^W \psi\|^2 + \mathscr{O}(h^\infty)\|\psi\|^2. \tag{3-8}$$

Using classical results of composition of pseudo-differential operators, we have

$$\langle [\mathscr{N}_h, \Xi_h^W]\psi, \Xi_h^W \psi \rangle \leqslant Ch^{1+\eta}\|\underline{\Xi}_h^W \psi\|^2 + \mathscr{O}(h^\infty)\|\psi\|^2, \tag{3-9}$$

where $\underline{\Xi}$ has a support slightly larger than that of Ξ_h. Here we use the energy estimate

$$\|\mathscr{D}_h \Xi_h^W \psi\| = \mathscr{O}(h^{1/2})\|\underline{\Xi}_h^W \psi\| + \mathscr{O}(h^\infty)\|\psi\|,$$

which follows from rough estimates of (3-3).

Thus, by combining (3-3), (3-8), and (3-9) with the fact that $\lambda \leqslant Kh$, we obtain

$$\|\Xi_h^W \psi\|^2 \leqslant Mh^\eta\|\underline{\Xi}_h^W \psi\|^2 + \mathscr{O}(h^\infty)\|\psi\|^2.$$

Finally, by an induction argument on the size of the support of Ξ, we get

$$\|\Xi_h^W \psi\| = \mathscr{O}(h^\infty)\|\psi\|. \qquad \square$$

Let us consider the partial semiclassical Fourier transform $\mathscr{F}_2$ with respect to u_2 and the translation/dilation $T : u_1 \mapsto (u_1 - \upsilon_2)h^{-1/2} = z$. Slightly abusing notation, we identify T with $\varphi \mapsto \varphi \circ T$. We mention that $\mathscr{F}_2$ is the metaplectic transform associated with the linear symplectic application $(u_2, \upsilon_2) \mapsto (\upsilon_2, -u_2)$; see, for instance, [Martinez 2002, Section 3.4]. Letting $V = \mathscr{F}_2^{-1} T$, we have

$$V^*(-ih\,\partial_{u_2} - u_1)V = -h^{1/2} z.$$

For the following it is pertinent to introduce the new semiclassical parameter

$$\hbar = h^{1/2},$$

keeping in mind that we continue to deal not only with h-pseudodifferential operators but also with asymptotic expansions in $\hbar$. The preceding equality then becomes

$$V^*(-ih\,\partial_{u_2} - u_1)V = -\hbar z.$$

Similarly, with the dilation $W : t \mapsto h^{-1/2} t = \hbar^{-1} t$, we get

$$W^* V^* \mathscr{D}_h V W = \hbar \mathscr{D}_\hbar^\sharp,$$

with

$$\mathscr{D}_\hbar^\sharp = \begin{pmatrix} -iC_0^\sharp\,\partial_z - tC_1^\sharp - \hbar t^2 \chi(\hbar^{2\eta} t)^2 C_2^\sharp - \hbar^2 t^3 \chi(\hbar^{2\eta} t)^3 C_3^\sharp \\ -z - iE_0^\sharp\,\partial_z + tE_1^\sharp - \hbar t^2 \chi(\hbar^{2\eta} t)^2 E_2^\sharp - \hbar^2 t^3 \chi(\hbar^{2\eta} t)^3 E_3^\sharp \\ -i\,\partial_t \end{pmatrix}^W,$$

where the coefficients of the conjugated operator $\mathscr{D}_\hbar^\sharp$ are now given by $P^\sharp = \breve{P}(\upsilon_2 + \hbar z, -u_2)$. Here the Weyl quantization can be considered only in the variables (u_2, υ_2) since z is now a "space variable". We let

$$\mathscr{N}_\hbar^\sharp = [m_\hbar{}^{-1}]^\sharp \mathscr{D}_\hbar^\sharp \cdot [m_\hbar (M_\hbar)^{-1}]^\sharp \mathscr{D}_\hbar^\sharp,$$

where $m_\hbar(\cdot\,, t) = m(\cdot\,, \hbar t)$ and $M_\hbar(\cdot\,, t) = M(\cdot\,, \hbar t)$. The operator $\mathscr{N}_\hbar^\sharp$ is equipped with the domain $(VW)^{-1}\,\mathrm{Dom}\,\mathscr{N}_h$ (which is still made of functions satisfying the Neumann boundary condition). Note that $\mathscr{N}_h$ and $\hbar^2 \mathscr{N}_\hbar^\sharp$ are unitarily equivalent since

$$W^* V^* \mathscr{N}_h V W = \hbar^2 \mathscr{N}_\hbar^\sharp. \tag{3-10}$$

After all these elementary transforms, Proposition 3.2 can be reformulated as follows.

Proposition 3.3. *Let $K > 0$ and $\eta \in \left(0, \tfrac{1}{2}\right)$. Let Ξ be a smooth function of the real variable equal to* 0 *near* 0 *and* 1 *away from a compact neighborhood of* 0*. There exists $\hbar_0 > 0$ such that, for all $\hbar \in (0, \hbar_0)$ and for all normalized eigenfunctions ψ of $\mathscr{N}_\hbar^\sharp$ associated with an eigenvalue λ such that $\lambda \leqslant K$, we have*

$$\Xi(\hbar^{2\eta} z)\psi = \mathscr{O}(\hbar^\infty).$$

Remark 3.4. As a consequence of the Agmon estimates and working in the coordinates (u_1, u_2, t), we notice that the eigenfunctions are also roughly localized in "frequency" in the sense that, for all $(\alpha, \beta, \gamma) \in \mathbb{N}^3$ and all $\eta \in \left(0, \tfrac{1}{2}\right)$, there exist $C, \hbar_0 > 0$ such that, for all $\hbar \in (0, \hbar_0)$,

$$\|t^\alpha z^\beta D_z^\gamma \psi\| + \|t^\alpha z^\beta D_t^\gamma \psi\| \leqslant C \hbar^{-2\eta(\alpha+\beta+\gamma)} \|\psi\|.$$

4. A pseudodifferential operator with operator symbol

Proposition 3.3 invites us to insert cutoff functions in the coefficients of the operator $\mathscr{N}_\hbar^\sharp$. Working from now on with the semiclassical parameter $\hbar$, we therefore consider

$$\mathscr{N}_\hbar^\flat = ([m_\hbar{}^{-1}]^\flat)^W \mathscr{D}_\hbar^\flat \cdot ([m_\hbar (M_\hbar)^{-1}]^\flat)^W \mathscr{D}_\hbar^\flat, \tag{4-1}$$

where

$$\mathscr{D}_\hbar^\flat = \begin{pmatrix} -iC_0^\flat\,\partial_z - tC_1^\flat - \hbar t^2\chi(\hbar^{2\eta}t)^2 C_2^\flat - \hbar^2 t^3\chi(\hbar^{2\eta}t)^3 C_3^\flat \\ -z - iE_0^\flat\,\partial_z + tE_1^\flat - \hbar t^2\chi(\hbar^{2\eta}t)^2 E_2^\flat - \hbar^2 t^3\chi(\hbar^{2\eta}t)^3 E_3^\flat \\ -i\,\partial_t \end{pmatrix}^W, \tag{4-2}$$

with $P^\flat = \breve{P}(\upsilon_2 + \hbar\chi_\eta(z)z, -u_2)$, where $\chi_\eta(z) = \chi_0(\hbar^{2\eta}z)$, the function χ_0 being smooth, with a compact support, and equal to 1 on a neighborhood of the support of $1 - \Xi$.

4.1. *The symbol and its properties.* Expanding the operator $\mathscr{N}_\hbar^\flat$ with respect to $\hbar$ (say first at a formal level) suggests that we should consider the following self-adjoint operator, depending on the parameters (u_2, υ_2) and acting in the variables (z, t) as

$$\begin{aligned} &n_0(u_2, \upsilon_2) \\ &= (-i\breve{C}_0(\upsilon_2, -u_2)\,\partial_z - t\breve{C}_1(\upsilon_2, -u_2))^2 + \alpha^{-1}(\upsilon_2, -u_2)(-z - i\breve{E}_0(\upsilon_2, -u_2)\,\partial_z + t\breve{E}_1(\upsilon_2, -u_2))^2 - \partial_t^2, \end{aligned}$$

with the domain

$$\mathrm{Dom}(n_0) = \{\psi \in L^2(\mathbb{R}^2_+) : n_0(u_2, \upsilon_2)\psi \in L^2(\mathbb{R}^2_+),\ \partial_t\psi(z, 0) = 0\},$$

and where we recall that C_1 and E_1 are given in (2-13). The domain of $n_0(u_2, \upsilon_2)$ depends on (u_2, υ_2). However, we can check that it is unitarily equivalent to a self-adjoint operator with domain independent of (u_2, υ_2), see the proof of Proposition 4.4 below. In the following, we will use a class of operator symbols of the form

$$S(\mathbb{R}^2, \mathcal{L}(\mathscr{A}_1, \mathscr{A}_2)) = \{a \in \mathscr{C}^\infty(\mathbb{R}^2, \mathcal{L}(\mathscr{A}_1, \mathscr{A}_2)) : \forall\gamma \in \mathbb{N}^2, \exists C_\gamma > 0 : \|\partial^\gamma a\|_{\mathcal{L}(\mathscr{A}_1, \mathscr{A}_2)} \leqslant C_\gamma\},$$

where $\mathscr{A}_1$ and $\mathscr{A}_2$ are (fixed) Hilbert spaces. We also introduce

$$\mathscr{B}_k = \{\psi \in L^2(\mathbb{R}^2_+) : \forall\alpha \in \mathbb{N}^2, |\alpha| \leqslant k \Rightarrow (\langle t\rangle^k + \langle z\rangle^k)\,\partial^\alpha\psi \in L^2(\mathbb{R}^2_+)\} \tag{4-3}$$

and the class of symbols

$$S(\mathbb{R}^2, N) = \bigcap_{k \geqslant N} S(\mathbb{R}^2, \mathcal{L}(\mathscr{B}_k, \mathscr{B}_{k-N})),$$

and we notice that $n_0 \in S(\mathbb{R}^2, 2)$.

Remark 4.1. Note that these classes of symbols are not algebras. However, the classical Moyal product of symbols in $S(\mathbb{R}^2, N)$ and $S(\mathbb{R}^2, M)$ is well-defined and belongs to $S(\mathbb{R}^2, N+M)$; see [Keraval 2018, Theorem 2.1.12].

In fact, for $N \geqslant 2$, by using a classical trace theorem, we may also define

$$\mathscr{B}_N^{\mathrm{Neu}} = \{\psi \in \mathscr{B}_N : \partial_t \psi(z, 0) = 0\} \quad (\subset \mathrm{Dom}\, n_0)$$

and the associated class $S^{\mathrm{Neu}}(\mathbb{R}^2, N)$. We can also write $n_0 \in S^{\mathrm{Neu}}(\mathbb{R}^2, 2)$ to remember that the domain of n_0 is equipped with the Neumann condition.

By expanding $\mathscr{N}_\hbar^\flat$ in powers of $\hbar$ and by using the composition theorem for pseudodifferential operators [Keraval 2018, Theorem 2.1.12], we get the following.

Proposition 4.2. *The operator $\mathscr{N}_\hbar^\flat$ is an h-pseudodifferential operator with symbol in the class $S^{\mathrm{Neu}}(\mathbb{R}^2, 2)$. Moreover, we can write the expansion*

$$\mathscr{N}_\hbar^\flat = n_0^W + \hbar n_1^W + \hbar^2 n_2^W + \hbar^3 r_\hbar^W, \tag{4-4}$$

with n_1, n_2, and $r_\hbar$ in the class $S^{\mathrm{Neu}}(\mathbb{R}^2, 8)$.

Proof. Let us recall that $\mathscr{N}_\hbar^\flat$ is given in (4-1). Let us notice that the operator $\mathscr{D}_\hbar^\flat$, defined in (4-2), is indeed a pseudodifferential operator with operator-valued symbol. With respect to the variables z and t, it is a differential operator of order 1 whose symbol is

$$\begin{pmatrix} -iC_0^\flat \partial_z - tC_1^\flat - \hbar t^2 \chi(\hbar^{2\eta} t)^2 C_2^\flat - \hbar^2 t^3 \chi(\hbar^{2\eta} t)^3 C_3^\flat \\ -z - iE_0^\flat \partial_z + tE_1^\flat - \hbar t^2 \chi(\hbar^{2\eta} t)^2 E_2^\flat - \hbar^2 t^3 \chi(\hbar^{2\eta} t)^3 E_3^\flat \\ -i\,\partial_t \end{pmatrix} \tag{4-5}$$

and belongs to $S(\mathbb{R}^2, 1)$. The functions/symbols $[m_\hbar{}^{-1}]^\flat$ and $[m_\hbar(M_\hbar)^{-1}]^\flat$ belong to $S(\mathbb{R}^2, 0)$. Combining these considerations with (4-1), it remains to apply the composition theorem for pseudodifferential operators with operator symbols, see Remark 4.1.

To get (4-4), it is sufficient to use the Taylor expansions in $\hbar$ of the symbol (4-5), $[m_\hbar{}^{-1}]^\flat$, and $[m_\hbar(M_\hbar)^{-1}]^\flat$, and to apply again the composition theorem (the worst remainders being roughly of order 8 in (z, t)). □

Remark 4.3. We will see that the accurate descriptions of n_1 and n_2 in (4-4) are not necessary to prove our main theorem. The use of the more restrictive class $S^{\mathrm{Neu}}(\mathbb{R}^2, 8)$ allows us to deal with the uniformity in the semiclassical expansions in $\hbar$.

Let us describe the groundstate energy of the principal symbol n_0. From now on, we lighten the notation by setting $(u_2, \upsilon_2) = (u, \upsilon)$.

Proposition 4.4. *For all $(u, \upsilon) \in \mathbb{R}^2$, the bottom of the spectrum of n_0 belongs to the discrete spectrum and it is a simple eigenvalue that equals $\breve{\beta}(\upsilon, -u)$. The corresponding normalized eigenfunction $\mathfrak{f}_{u,\upsilon}$ belongs to the Schwartz class and depends on (u, υ) smoothly.*

Moreover, there exists $c > 0$ such that, by possibly choosing ϵ_0 smaller in Lemma 2.9, we have, for all $(u, \upsilon) \in \mathbb{R}^2$,

$$\inf \mathrm{sp}(n_0(u, \upsilon)|_{\mathfrak{f}_{u,\upsilon}^\perp}) \geqslant \beta_{\min} + c \geqslant \breve{\beta}(\upsilon, -u).$$

Proof. By using the Fourier transform in z and then a change of gauge, we are reduced to the case when $E_0 = 0$. With a rescaling in z, n_0 is unitarily equivalent to

$$(-i\,\partial_z - t\check{C}_1)^2 + \alpha^{-1}(-\check{C}_0 z + t\check{E}_1)^2 - \partial_t^2 = (-i\,\partial_z - tb_2)^2 + (b_3 z + tb_1)^2 - \partial_t^2,$$

with

$$b_1 = \check{\mathcal{B}}_1, \quad b_2 = (\alpha^{1/2}\mathcal{B}_2)\check{}, \quad b_3 = -\check{\mathcal{B}}_3,$$

where the functions are evaluated at $(\upsilon_2, -u_2)$. Recalling (2-8), we see that the Euclidean norm of $b = (b_1, b_2, b_3)$ is

$$\|b\|_2 = \|\check{\boldsymbol{B}}\|,$$

with a slight abuse of notation. By homogeneity, we can easily scale out $\|\check{\boldsymbol{B}}\|$ and consider the operator

$$(-i\,\partial_z - tb_2)^2 - \partial_t^2 + (tb_1 + b_3 z)^2,$$

with

$$b_1 = \cos\theta\cos\varphi, \quad b_2 = \cos\theta\sin\varphi, \quad b_3 = \sin\theta.$$

Completing a square leads to the identity

$$(-i\,\partial_z - tb_2)^2 - \partial_t^2 + (tb_1 + b_3 z)^2 = -\partial_t^2 + (t\cos\theta - \sin\varphi D_z - z\sin\theta\cos\varphi)^2 + (\cos\varphi D_z - z\sin\theta\sin\varphi)^2.$$

This shows, thanks to the rescaling $z = \tilde{z}\sin\varphi$ (since $\sin\varphi$ is nonzero) and the change of gauge

$$\psi \mapsto e^{-i\frac{\tilde{z}^2}{2}\sin\theta\cos\varphi}\psi,$$

that the operator is unitarily equivalent to

$$D_t^2 + (t\cos\theta - D_{\tilde{z}})^2 + (\cot\varphi D_{\tilde{z}} - \tilde{z}\sin\theta)^2$$

and then, by the Fourier transform, to

$$D_t^2 + (t\cos\theta - \zeta)^2 + (\zeta\cot\varphi + \sin\theta D_\zeta)^2.$$

Thanks to the change of gauge

$$\psi \mapsto e^{-i\frac{\zeta^2\cot\varphi}{2\sin\theta}}\psi$$

(which is well-defined since $\sin\theta \neq 0$), this last operator is unitarily equivalent to

$$D_t^2 + D_z^2 + (t\cos\theta - z\sin\theta)^2,$$

which is nothing but the Lu–Pan operator defined in (1-2), which is unitarily equivalent to

$$\cos^2\theta D_t^2 + \sin^2\theta D_z^2 + (t - z)^2$$

(whose domain is independent of θ).

The eigenfunction $\mathfrak{f}_{u,\upsilon}$ belongs to the Schwartz class by virtue of [Raymond 2009, Corollaire 5.1.2] and the stability of the Schwartz class under Fourier and gauge transforms. □

4.2. ***An approximate parametrix.***

4.2.1. *Inverting the principal symbol.*

Lemma 4.5. *Consider $\epsilon > 0$ and $\Lambda \leqslant \beta_{\min} + \epsilon$. We let*

$$\mathscr{P}_0(\Lambda) = \begin{pmatrix} n_0(u,\upsilon) - \Lambda & \cdot\, \mathfrak{f}_{u,\upsilon} \\ \langle \cdot\,, \mathfrak{f}_{u,\upsilon}\rangle & 0 \end{pmatrix}.$$

For ϵ small enough, $\mathscr{P}_0(\Lambda) : \operatorname{Dom} n_0 \times \mathbb{C} \to L^2(\mathbb{R}^2_+) \times \mathbb{C}$ is bijective. Its inverse is denoted by $\mathscr{Q}_0$ and is given by

$$\mathscr{Q}_0 = \mathscr{Q}_0(\Lambda) = \begin{pmatrix} (n_0(u,\upsilon) - \Lambda)_\perp^{-1} & \cdot\, \mathfrak{f}_{u,\upsilon} \\ \langle \cdot\,, \mathfrak{f}_{u,\upsilon}\rangle & \Lambda - \breve{\beta}(\upsilon, -u) \end{pmatrix},$$

where $(n_0(u,\upsilon) - \Lambda)_\perp^{-1}$ is the regularized resolvent on $(\operatorname{span} \mathfrak{f}_{u,\upsilon})^\perp$.

Moreover, we have $\mathscr{Q}_0 \in S(\mathbb{R}^2, 0)$.

Proof. Using the same algebraic computations as in [Keraval 2018] and the spectral gap in Proposition 4.4, we get the announced inverse. Moreover, it is also clear that $\mathscr{Q}_0$ is bounded from $L^2(\mathbb{R}^2_+)$ to $L^2(\mathbb{R}^2_+)$ uniformly in (u,υ). The fact that it belongs to the class $S(\mathbb{R}^2, 0)$ follows from weighted resolvent estimates similar to [Raymond 2009, pp. 100-101]; see also [Fahs et al. 2024, Appendix]. □

We let

$$\mathscr{P}_\hbar(\Lambda) = \begin{pmatrix} n_0 + \hbar n_1 + \hbar^2 n_2 + \hbar^3 r_\hbar - \Lambda & \cdot\, \mathfrak{f}_{u,\upsilon} \\ \langle \cdot\,, \mathfrak{f}_{u,\upsilon}\rangle & 0 \end{pmatrix} = \mathscr{P}_0(\Lambda) + \hbar \mathscr{P}_1 + \hbar^2 \mathscr{P}_2 + \hbar^3 \mathscr{R}_\hbar,$$

where n_0, n_1, n_2, and $r_\hbar$ are given in Proposition 4.2.

4.2.2. *The approximate parametrix.* Let us now construct an approximate (at order 2) inverse of $\mathscr{P}_\hbar^W$ when it acts on the Schwartz class (with Neumann condition). We consider

$$\mathscr{Q}_\hbar = \mathscr{Q}_0 + \hbar \mathscr{Q}_1 + \hbar^2 \mathscr{Q}_2 = \begin{pmatrix} Q_\hbar & Q_\hbar^+ \\ Q_\hbar^- & Q_\hbar^\pm \end{pmatrix},$$

where

$$\mathscr{Q}_1 = -\mathscr{Q}_0 \mathscr{P}_1 \mathscr{Q}_0, \quad \mathscr{Q}_2 = -\mathscr{Q}_0 \mathscr{P}_2 \mathscr{Q}_0 + \mathscr{Q}_0 \mathscr{P}_1 \mathscr{Q}_0 \mathscr{P}_1 \mathscr{Q}_0 - \frac{1}{i}\{\mathscr{Q}_0, \mathscr{P}_0\}\mathscr{Q}_0. \tag{4-6}$$

By Remark 4.1, the symbols $\mathscr{Q}_1$ and $\mathscr{Q}_2$ belong to $S(\mathbb{R}^2, M)$ for some $M \geqslant 8$. By computing products of matrices and using the exponential decay of $\mathfrak{f}_{u,\upsilon}$, we get

$$Q_\hbar^\pm(\Lambda) = \Lambda - (p_0 + \hbar p_1 + \hbar^2 p_{2,\Lambda}), \tag{4-7}$$

with $p_0 = \breve{\beta}(\upsilon, -u)$ and $p_1, p_{2,\Lambda} \in S_{\mathbb{R}^2}(1)$, where

$$S_{\mathbb{R}^2}(1) = \{a \in \mathscr{C}^\infty(\mathbb{R}^2, \mathbb{C}) : \forall \alpha \in \mathbb{N}^2, \exists C_\alpha > 0 : |\partial^\alpha a| \leqslant C_\alpha\}.$$

In addition, $\Lambda \mapsto p_{2,\Lambda} \in S_{\mathbb{R}^2}(1)$ is analytic in a neighborhood of $\beta_{\min}$.

Remark 4.6. Let us emphasize here that nothing a priori ensures that the subprincipal symbols p_1 and $p_{2,E}$ are real-valued since our formal operator is not self-adjoint on the canonical L^2-space.

The reason to consider the expressions (4-6) simply comes from the semiclassical expansion of the product $\mathscr{Q}_\hbar^W \mathscr{P}_\hbar^W$ by means of the composition theorem [Keraval 2018, Theorem 2.1.12]. These explicit choices, with the Calderón–Vaillancourt theorem [Keraval 2018, Theorem 2.1.16] to estimate the remainders, imply the following proposition.

Proposition 4.7. *There exists $N \geqslant 2$ such that the following holds. We have*

$$\mathscr{Q}_\hbar^W \mathscr{P}_\hbar^W = \mathrm{Id}_{\mathscr{S}^{\mathrm{Neu}}(\overline{\mathbb{R}}^3_+)\times\mathscr{S}(\mathbb{R})} + \hbar^3 \mathscr{R}^W_{\hbar,\ell}, \quad \mathscr{P}_\hbar^W \mathscr{Q}_\hbar^W = \mathrm{Id}_{\mathscr{S}(\overline{\mathbb{R}}^3_+)\times\mathscr{S}(\mathbb{R})} + \hbar^3 \mathscr{R}^W_{\hbar,r},$$

where $\mathscr{R}_{\hbar,\ell}$ and $\mathscr{R}_{\hbar,r}$ belong to $S(\mathbb{R}^2, N)$ and where $\mathscr{S}^{\mathrm{Neu}}(\overline{\mathbb{R}}^3_+)$ denotes the Schwartz class on $\mathbb{R}^2_+$ with Neumann condition at $t = 0$.

In particular, we have, for all $\psi \in \mathscr{S}^{\mathrm{Neu}}(\overline{\mathbb{R}}^3_+)$,

$$\begin{aligned} Q_\hbar^W(\mathscr{N}_\hbar^\flat - \Lambda)\psi + (Q_\hbar^+)^W \mathfrak{P}\psi &= \psi + \mathscr{O}(\hbar^3)\|\psi\|_{L^2(\mathbb{R},\mathscr{B}_N)}, \\ (Q_\hbar^-)^W(\mathscr{N}_\hbar^\flat - \Lambda)\psi + (Q_\hbar^\pm)^W \mathfrak{P}\psi &= \mathscr{O}(\hbar^3)\|\psi\|_{L^2(\mathbb{R},\mathscr{B}_N)} \end{aligned} \tag{4-8}$$

and, for all $\varphi \in \mathscr{S}(\mathbb{R})$,

$$\begin{aligned} (\mathscr{N}_\hbar^\flat - \Lambda)(Q_\hbar^+)^W\varphi + \mathfrak{P}^*(Q_\hbar^\pm)^W\varphi &= \mathscr{O}(\hbar^3)\|\varphi\|, \\ \mathfrak{P}(Q_\hbar^+)^W\varphi &= \varphi + \mathscr{O}(\hbar^3)\|\varphi\|. \end{aligned} \tag{4-9}$$

Here, $\mathfrak{P} = (\langle\,\cdot\,, \mathfrak{f}_{u,\upsilon}\rangle)^W$, $\mathscr{B}_N$ is given in (4-3), *and $\|\cdot\|_{L^2(\mathbb{R},\mathscr{B}_N)}$ is the L^2-norm defined thanks to the Bochner integral valued in the Banach space $\mathscr{B}_N$.*

5. Spectral consequences

This last section is devoted to the proof of Theorem 1.4, with the help of Proposition 4.7. The spectrum of $\mathscr{N}_\hbar^\sharp$ will be compared to the spectrum of a model operator, derived from an effective h-pseudodifferential operator whose symbol has the following expansion in powers of $\hbar = h^{1/2}$:

$$p_\hbar^{\mathrm{eff}} = p_0 + \hbar p_1 + \hbar^2 p_{2,\beta_{\min}}, \tag{5-1}$$

see (4-7).

5.1. ***A model operator.*** Let us consider

$$p_\hbar^{\mathrm{mod}}(U) = p_\hbar^{\mathrm{eff}}(0) + \tfrac{1}{2}\,\mathrm{Hess}_{(0,0)}\, p_0(U,U) + \hbar p_1^{\mathrm{lin}}(U), \quad U = (u,\upsilon),$$

where p_1^{lin} is the linear approximation of p_1 at $(0,0)$. The corresponding h-pseudodifferential operator $(p_\hbar^{\mathrm{mod}})^W$ is not self-adjoint due to the linear part. However, this operator still has compact resolvent, and we can compute its spectrum and estimate its resolvent. Let us explain this. Thanks to Assumption 1.3, the quadratic form $\mathrm{Hess}_{(0,0)}\, p_0(U,U)$ can be diagonalized with a rotation (which is a symplectic transformation in two dimensions). Thus, by using a metaplectic transformation (or by means of an explicit linear transformation in u), we may assume that the symbol is

$$p_\hbar^{\mathrm{mod}} = p_\hbar^{\mathrm{eff}}(0) + \tfrac{1}{2}d_0(u^2+\upsilon^2) + \hbar(\alpha u + \beta\upsilon)$$

for some $d_0 > 0$ and $(\alpha,\beta) \in \mathbb{C}^2$.

Remark 5.1. In fact, we have

$$d_0 = \sqrt{\det \mathrm{Hess}_{(0,0)}\, p_0} = \sqrt{\det \mathrm{Hess}_{(0,0)}\, \breve{\beta}(\upsilon, -u)} = \sqrt{\frac{\det \mathrm{Hess}_{x_0}\, \beta}{\|\boldsymbol{B}(x_0)\|^2 \sin^2 \theta(x_0)}},$$

where we use the notation introduced at the beginning of Section 3, the change of variable $\mathscr{J}$ in Lemma 2.10, and Remark 2.6.

By completing the square and recalling that we deal with h-quantizations and that we have let $\hbar = h^{1/2}$, we get

$$(p_\hbar^{\mathrm{mod}})^W = \tilde{p}_\hbar^{\mathrm{eff}}(0) + \frac{d_0}{2}\left(\left(u + \frac{\hbar\alpha}{d_0}\right)^2 + \left(\hbar^2 D_u + \frac{\hbar\beta}{d_0}\right)^2\right), \quad \tilde{p}_\hbar^{\mathrm{eff}}(0) = p_\hbar^{\mathrm{eff}}(0) - \frac{\alpha^2+\beta^2}{2d_0}\hbar^2.$$

For all $n \geqslant 1$, we let

$$\begin{aligned} f_n(u) &= [e^{-i\beta\cdot/d_0} H_n(\cdot)]\left(u + \frac{\alpha}{d_0}\right), \\ f_{n,\hbar}(u) &= \hbar^{-1/2} f_n(\hbar^{-1}u), \end{aligned}$$

where H_n is the n-th normalized Hermite function.

The family $(f_{n,\hbar})_{n\geqslant 1}$ is a total family in $L^2(\mathbb{R})$ (but not necessarily orthogonal). It satisfies

$$\begin{aligned} (p_\hbar^{\mathrm{mod}})^W f_{n,\hbar} &= \lambda_n^{\mathrm{mod}}(\hbar) f_{n,\hbar}, \\ \lambda_n^{\mathrm{mod}}(\hbar) &= \tfrac{1}{2} d_0(2n-1)\hbar^2 + \tilde{p}_\hbar^{\mathrm{eff}}(0). \end{aligned} \tag{5-2}$$

By the analytic perturbation theory (see [Kato 1995, Chapter VII]), the spectrum of $(p_\hbar^{\mathrm{mod}})^W$ is made of eigenvalues of algebraic multiplicity 1, and it is given by

$$\mathrm{sp}((p_\hbar^{\mathrm{mod}})^W) = \left\{\tfrac{1}{2} d_0(2n-1)\hbar^2 + \tilde{p}_\hbar^{\mathrm{eff}}(0),\ n \geqslant 1\right\}.$$

Moreover, for all compact $K \subset \mathbb{C}$, there exists $C_K > 0$ such that, for all $\mu \in K$,

$$\|((p_\hbar^{\mathrm{mod}})^W - \tilde{p}_\hbar^{\mathrm{eff}}(0) - \hbar^2\mu)^{-1}\| \leqslant \frac{C_K}{\mathrm{dist}(\tilde{p}_\hbar^{\mathrm{eff}}(0) + \hbar^2\mu, \mathrm{sp}((p_\hbar^{\mathrm{mod}})^W))}. \tag{5-3}$$

To see this, consider the operator

$$\mathscr{A}_\hbar = (p_\hbar^{\mathrm{mod}})^W - \tilde{p}_\hbar^{\mathrm{eff}}(0) = \left(u + \frac{\hbar\alpha}{d_0}\right)^2 + \left(\hbar^2 D_u + \frac{\hbar\beta}{d_0}\right)^2.$$

When $\hbar = 1$, we have the estimate

$$\|(\mathscr{A}_1 - \mu)^{-1}\| \leqslant \frac{C_K}{\mathrm{dist}(\mu, \mathrm{sp}(\mathscr{A}_1))}, \tag{5-4}$$

which follows from the fact that the eigenvalues have algebraic multiplicity 1 (the Riesz projectors associated with the finite number of eigenvalues in K have rank 1). To get (5-3), we use the rescaling $u = \hbar\tilde{u}$ and (5-4).

5.2. *Refined estimates.*

5.2.1. *From the model operator to $\mathscr{N}_\hbar^\sharp$.* The functions $(f_{n,\hbar})$ can serve as quasimodes for $\mathscr{N}_\hbar^\sharp$ with the help of (4-9). Indeed, by taking $\Lambda = \lambda_n^{\mathrm{mod}}(\hbar)$ and $\varphi = f_{n,\hbar}$, we see that

$$(\mathscr{N}_\hbar^\flat - \lambda_n^{\mathrm{mod}}(\hbar))(Q_\hbar^+)^W f_{n,\hbar} = \mathscr{O}(\hbar^3).$$

Since $(Q_\hbar^+)^W f_{n,\hbar}$ is localized near $(z,t) = (0,0)$ (due to the exponential decay of $\mathfrak{f}_{u,\upsilon}$, which is uniform in (u,υ)), we get

$$(\mathscr{N}_\hbar^\sharp - \lambda_n^{\mathrm{mod}}(\hbar))(Q_\hbar^+)^W f_{n,\hbar} = \mathscr{O}(\hbar^3).$$

By using the inverse Fourier transform and translation/dilation, $(Q_\hbar^+)^W f_{n,\hbar}$ becomes a quasimode for $\mathscr{N}_h$, see (3-1) and the end of Section 3. But the operator $\mathscr{N}_h$ is unitarily equivalent to a self-adjoint operator for a suitable scalar product on the usual L^2-space. Therefore, we can apply the spectral theorem, and we deduce that

$$\operatorname{dist}(\lambda_n^{\mathrm{mod}}(\hbar), \operatorname{sp}(\mathscr{N}_\hbar^\sharp)) \leqslant C\hbar^3.$$

In particular, this implies that, for $\hbar$ small enough, $\lambda_n^{\mathrm{mod}}(\hbar)$ is real. This shows that we necessarily have

$$p_1(0) \in \mathbb{R}, \quad p_2(0) - \frac{\alpha^2 + \beta^2}{2d_0} \in \mathbb{R}.$$

This also implies that

$$\lambda_n(\mathscr{N}_\hbar^\sharp) \leqslant \lambda_n^{\mathrm{mod}}(\hbar) + C\hbar^3. \tag{5-5}$$

5.2.2. *From $\mathscr{N}_\hbar^\sharp$ to the model operator.* Let $n \geqslant 1$. Let us consider an eigenfunction ψ of $\mathscr{N}_\hbar^\sharp$ associated with the eigenvalue $\lambda_n(\mathscr{N}_\hbar^\sharp)$.

We know that $\lambda_n(\mathscr{N}_\hbar^\sharp) = \beta_{\min} + o(1)$ and that the corresponding eigenfunctions are localized in (z,t) (due to the Agmon estimates and Proposition 3.3). Thus, in (4-8), we can replace $\mathscr{N}_\hbar^\flat$ by $\mathscr{N}_\hbar^\sharp$, and we deduce that

$$((p_\hbar^{\mathrm{eff}})^W - \lambda_n(\mathscr{N}_\hbar^\sharp))\mathfrak{P}\psi = \mathscr{O}(\hbar^{3-\eta})\|\psi\|, \quad \|\psi\| \leqslant C\|\mathfrak{P}\psi\|, \tag{5-6}$$

for $\eta > 0$ as small as we want. We use Remark 3.4 to control the remainders $\|\psi\|_{L^2(\mathbb{R},\mathscr{B}_N)}$ by $\mathscr{O}(\hbar^{-\eta})\|\psi\|$. By taking the scalar product with $\mathfrak{P}\psi$, taking the real part and using the min-max principle, we get that

$$\lambda_n(\mathscr{N}_\hbar^\sharp) \geqslant \beta_{\min} + p_1(0)\hbar - C\hbar^2.$$

This establishes the two-term asymptotic estimate

$$\lambda_n(\mathscr{N}_\hbar^\sharp) = \beta_{\min} + p_1(0)\hbar + \mathscr{O}(\hbar^2).$$

Therefore, we can focus on the description of the eigenvalues of the form

$$\lambda_n(\mathscr{N}_\hbar^\sharp) = \beta_{\min} + p_1(0)\hbar + \mu_n(\hbar)\hbar^2$$

for $\mu_n(\hbar) \in D(0,R)$ with a given $R > 0$. We have

$$((p_\hbar^{\mathrm{eff}})^W - (\beta_{\min} + p_1(0)\hbar + \mu_n(\hbar)\hbar^2))\mathfrak{P}\psi_n = \mathscr{O}(\hbar^{3-\eta})\|\mathfrak{P}\psi_n\|, \tag{5-7}$$

where ψ_n denotes a normalized eigenfunction associated to the n-th eigenvalue of $\mathscr{N}_\hbar^\sharp$. In fact, by considering (5-7) and again Proposition 4.7, the function $\mathfrak{P}\psi_n$ is microlocalized near (0, 0), the minimum of the principal symbol p_0. Since this minimum is nondegenerate, the quadratic approximation of the symbol shows that $\mathfrak{P}\psi_n$ is microlocalized near $(u, \upsilon) = (0, 0)$ at the scale $\hbar^{1-\eta}$ for any $\eta \in \left(0, \frac{1}{2}\right)$. In particular, we deduce that

$$((p_\hbar^{\mathrm{mod}})^W - (\beta_{\min} + p_1(0)\hbar + \mu_n(\hbar)\hbar^2))\mathfrak{P}\psi_n = \mathscr{O}(\hbar^{3-3\eta})\|\mathfrak{P}\psi_n\|.$$

From the resolvent estimate (5-3), this implies that

$$\mu_n(\hbar) \in \bigcup_{j\geqslant 1} D\left(\frac{d_0}{2}(2j-1) + d_1, C\hbar^{1-3\eta}\right), \quad d_1 = p_2(0) - \frac{\alpha^2+\beta^2}{2d_0},$$

where $D(z, r)$ denotes the disc of center $z \in \mathbb{C}$ and radius $r > 0$. In particular, we have

$$\mu_1(\hbar) \geqslant \tfrac{1}{2}d_0 + d_1 - C\hbar^{1-3\eta}.$$

This shows that

$$\lambda_1(\mathscr{N}_\hbar^\sharp) \geqslant \beta_{\min} + p_1(0)\hbar + \left(\tfrac{1}{2}d_0 + d_1\right)\hbar^2 - C\hbar^{3-3\eta},$$

and thus, with (5-5), we get

$$\mu_1(\hbar) = \tfrac{1}{2}d_0 + d_1 + \mathscr{O}(\hbar^{1-3\eta})$$

and

$$\lambda_1(\mathscr{N}_\hbar^\sharp) = \lambda_1^{\mathrm{mod}}(\hbar) + \mathscr{O}(\hbar^{3-3\eta}).$$

Let us now deal with $\lambda_2(\mathscr{N}_\hbar^\sharp)$ and recall (5-5). Assume by contradiction that $\mu_2(\hbar) \in D\left(\frac{1}{2}d_0 + d_1, C\hbar^{1-3\eta}\right)$. Then, we have

$$|\mu_2(\hbar) - \mu_1(\hbar)| \leqslant C\hbar^{1-3\eta}.$$

We infer that

$$((p_\hbar^{\mathrm{mod}})^W - \lambda_1^{\mathrm{mod}}(\hbar))\mathfrak{P}\psi = \mathscr{O}(\hbar^{3-3\eta})\|\mathfrak{P}\psi\|$$

for all $\psi \in \operatorname{span}(\psi_1, \psi_2)$. Moreover, coming back to (4-8) (see also (5-7)), we also get that $\|\psi\| \leqslant C\|\mathfrak{P}\psi\|$ for all $\psi \in \operatorname{span}(\psi_1, \psi_2)$. In particular, $\mathfrak{P}(\operatorname{span}(\psi_1, \psi_2))$ is of dimension 2. Let us consider the Riesz projector (in the characteristic subspace of $(p_\hbar^{\mathrm{mod}})^W$ associated with the smallest eigenvalue)

$$\Pi = \frac{1}{2i\pi}\int_{\mathscr{C}(\lambda_1^{\mathrm{mod}}(\hbar), \hbar^{3-4\eta})} (\zeta - (p_\hbar^{\mathrm{mod}})^W)^{-1}\,\mathrm{d}\zeta,$$

which is of rank 1. Then, for all $\varphi \in \mathfrak{P}(\operatorname{span}(\psi_1, \psi_2))$, we write, with the Cauchy formula,

$$\Pi\varphi = \varphi + \frac{1}{2i\pi}\int_{\mathscr{C}(\lambda_1^{\mathrm{mod}}(\hbar), \hbar^{3-4\eta})} ((\zeta - (p_\hbar^{\mathrm{mod}})^W)^{-1} - (\zeta - \lambda_1^{\mathrm{mod}}(\hbar))^{-1})\varphi\,\mathrm{d}\zeta.$$

But, we have

$$(\zeta - (p_\hbar^{\mathrm{mod}})^W)^{-1} - (\zeta - \lambda_1^{\mathrm{mod}}(\hbar))^{-1} = (\zeta - \lambda_1^{\mathrm{mod}}(\hbar))^{-1}(\zeta - (p_\hbar^{\mathrm{mod}})^W)^{-1}((p_\hbar^{\mathrm{mod}})^W - \lambda_1^{\mathrm{mod}}(\hbar)),$$

so that, by using the resolvent estimate (5-3), we get

$$\|\Pi\varphi-\varphi\|\leqslant C\hbar^{3-4\eta}\hbar^{-3+4\eta}\hbar^{-3+4\eta}\hbar^{3-3\eta}\|\varphi\|=C\hbar^{\eta}\|\varphi\|.$$

This shows that the range of Π is of dimension at least 2 as soon as $\hbar$ is small enough. This is a contradiction. Therefore, we must have $\mu_2(\hbar)\in D\big(3\big(\tfrac{1}{2}d_0\big)+d_1, C\hbar^{1-3\eta}\big)$. In particular, we have

$$\mu_2(\hbar)=3\big(\tfrac{1}{2}d_0\big)+d_1+\mathscr{O}(\hbar^{1-3\eta}),\quad \lambda_2(\mathscr{N}_\hbar^\sharp)=\lambda_2^{\mathrm{mod}}(\hbar)+\mathscr{O}(\hbar^{3-3\eta}).$$

We proceed by induction to get that, for all $n\geqslant 1$,

$$\mu_n(\hbar)=(2n-1)\big(\tfrac{1}{2}d_0\big)+d_1+\mathscr{O}(\hbar^{1-3\eta}),\quad \lambda_n(\mathscr{N}_\hbar^\sharp)=\lambda_n^{\mathrm{mod}}(\hbar)+\mathscr{O}(\hbar^{3-3\eta}). \tag{5-8}$$

5.2.3. *End of the proof of Theorem 1.4.* Proposition 2.11 shows that the first eigenvalues of $\mathscr{L}_h$ coincide with those of $\mathscr{L}_h^{\mathrm{app}}$ modulo $o(h^2)$. Then, by (3-1), $\mathscr{L}_h^{\mathrm{app}}$ is unitarily equivalent to $\mathscr{N}_h$. The operator $\mathscr{N}_h$ is unitarily equivalent to $\hbar^2\mathscr{N}_\hbar^\sharp$, see (3-10). Theorem 1.4 follows from (5-8) and (5-2) (see also Remark 5.1 for the explicit formula for d_0).

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