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CHARACTERIZATION OF WEIGHTED HARDY SPACES ON WHICH ALL COMPOSITION OPERATORS ARE BOUNDED

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We give a complete characterization of the sequences $\beta = (\beta_n)$ of positive numbers for which all composition operators on $H^2(\beta)$ are bounded, where $H^2(\beta)$ is the space of analytic functions f on the unit disk $\mathbb{D}$ such that $\sum_{n=0}^{\infty} |a_n|^2 \beta_n < +\infty$ if $f(z) = \sum_{n=0}^{\infty} a_n z^n$. We prove that all composition operators are bounded on $H^2(\beta)$ if and only if β is essentially decreasing and slowly oscillating. We also prove that every automorphism of the unit disk induces a bounded composition operator on $H^2(\beta)$ if and only if β is slowly oscillating. We give applications of our results.

1. Introduction

Let $\beta = (\beta_n)_{n\geq 0}$ be a sequence of positive numbers such that

$$\liminf_{n\to\infty} \beta_n^{1/n} \geq 1. \tag{1-1}$$

The associated weighted Hardy space $H^2(\beta)$ is defined to be the Hilbertian space of analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ such that

$$\|f\|^2 := \sum_{n=0}^{\infty} |a_n|^2 \beta_n < \infty. \tag{1-2}$$

Condition (1-1) is equivalent to the inclusion $H^2(\beta) \subseteq \mathcal{H}ol(\mathbb{D})$. Indeed, if (1-1) holds, we have $H^2(\beta) \subseteq \mathcal{H}ol(\mathbb{D})$ since $|a_n|^2\beta_n$ is bounded and thanks to the Hadamard formula. Conversely, testing the inclusion $H^2(\beta) \subseteq \mathcal{H}ol(\mathbb{D})$ on the function $f(z) = \sum_{n=1}^{\infty} (n\sqrt{\beta(n)})^{-1} z^n \in H^2(\beta)$, we get (1-1) from the Hadamard formula.

Condition (1-1) will therefore be assumed throughout this paper, without repeating it.

When $\beta_n \equiv 1$, we recover the usual Hardy space H^2; the Bergman space corresponds to $\beta_n = 1/(n+1)$ and the Dirichlet space to $\beta_n = n+1$.

Recall that a symbol is a (nonconstant) analytic self-map $\varphi \colon \mathbb{D} \to \mathbb{D}$, and the associated composition operator $C_\varphi \colon H^2(\beta) \to \mathcal{H}ol(\mathbb{D})$ is defined as

$$C_\varphi(f) = f \circ \varphi. \tag{1-3}$$

An important question in the theory is to decide when C_φ is bounded on $H^2(\beta)$, i.e., when

$$C_\varphi \colon H^2(\beta) \to H^2(\beta).$$

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This question appears in the literature in several places. For instance, it is Problem 1 in the thesis of Nina Zorboska [1988, p. 49]. This thesis contains many interesting results, in particular Propositions 3.1 and 4.2 of the present paper (we discovered the content of Zorboska's thesis once the present paper was almost finished). See also Question 36 raised by Deddens in [Shields 1974, p. 122.c].

When $H^2(\beta)$ is the usual Hardy space H^2 (i.e., when $\beta_n \equiv 1$), it is well known, as a consequence of the Littlewood subordination principle [1925], that all symbols generate bounded composition operators [Shapiro 1993, pp. 13–17]. On the other hand, for the Dirichlet space, corresponding to $\beta_n = n+1$, not all composition operators are bounded since there exist symbols φ not belonging to the Dirichlet space (e.g., any infinite Blaschke product).

Note that, by definition of the norm of $H^2(\beta)$, all rotations R_θ, defined by $R_\theta(z) = \mathrm{e}^{i\theta} z$, with $\theta \in \mathbb{R}$, induce bounded and surjective composition operators on $H^2(\beta)$ and send isometrically $H^2(\beta)$ into itself.

Our goal in this paper is to characterize the sequences β for which all composition operators act boundedly on the space $H^2(\beta)$, i.e., send $H^2(\beta)$ into itself.

In Shapiro's presentation for the Hardy space H^2, the main point is the case $\varphi(0) = 0$ and a subordination principle for subharmonic functions (Littlewood's subordination principle). The case of automorphisms is claimed to be simple, using an integral representation for the norm and some change of variable. For general weights β, the situation is different, as we will see in this paper, and it turns out that the conditions on β for the boundedness of the composition operators C_φ on $H^2(\beta)$ are not the same depending on whether we consider the class of all symbols such that $\varphi(0) = 0$, or the class of symbols $\varphi = T_a$, where

$$T_a(z) = \frac{a+z}{1+\bar{a}z} \quad \text{for } a \in \mathbb{D}. \tag{1-4}$$

It is clear that when these two classes of composition operators are bounded, then all composition operators are bounded. Recall that every symbol φ can be written as the composition $\varphi = T_a \circ \psi$, where $\psi(0) = 0$ and $a = \varphi(0)$, and then $C_\varphi = C_\psi \circ C_{T_a}$.

In many occurrences, the weight β is defined as

$$\beta_n = \int_0^1 t^n \, d\sigma(t), \tag{1-5}$$

where σ is a positive measure on $(0, 1)$; more specifically the following definition is often used: let $G\colon (0, 1) \to \mathbb{R}_+$ be an integrable function, and let H^2_G be the space of analytic functions $f\colon \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|^2_{H^2_G} := \int_{\mathbb{D}} |f(z)|^2 G(1-|z|^2)\, dA(z) < \infty. \tag{1-6}$$

Such weighted Bergman-type spaces are used, for instance, in [Kellay and Lefèvre 2012; Kriete and MacCluer 1995; Li et al. 2014]. We have $H^2_G = H^2(\beta)$ with

$$\beta_n = 2\int_0^1 r^{2n+1} G(1-r^2)\, dr = \int_0^1 t^n G(1-t)\, dt, \tag{1-7}$$

and the sequence $\beta = (\beta_n)_n$ is *nonincreasing* (actually, the representation (1-5) is equivalent, by the Hausdorff moment theorem, to a high regularity of the sequence β, namely its *complete monotony*).

When the weight β is nonincreasing (or more generally, essentially decreasing), all the symbols vanishing at the origin induce a bounded composition operator. This was proved by C. Cowen [1990, Corollary, p. 31], using Hadamard multiplication. We can also use Kacnelson's theorem (see [Chalendar and Partington 2014] or [Lefèvre et al. 2021, Theorem 3.12]). Actually that follows from an older theorem of Goluzin [1951] (see [Duren 1983, Theorem 6.3]), which itself uses a self-refinement observed by Rogosinski of Littlewood's principle [Duren 1983, Theorem 6.2].

For weights defined as in (1-5), we have at our disposal integral representations for the norm in $H^2(\beta)$, and, as in the Hardy space case, this integral representation rather easily allows us to decide when the boundedness of C_{T_a} on $H^2(\beta)$ occurs. This is not always the case, as shown by T. Kriete and B. MacCluer in [Kriete and MacCluer 1995]. They consider spaces of Bergman-type $A^2_{\widetilde{G}} := H^2_G$, where $\widetilde{G}(r) = G(1-r^2)$, defined as the spaces of analytic functions in $\mathbb{D}$ such that

$$\int_{\mathbb{D}} |f(z)|^2 \widetilde{G}(|z|)\, dA < \infty$$

for a positive nonincreasing continuous function $\widetilde{G}$ on $[0, 1)$. They prove [Kriete and MacCluer 1995, Theorem 3] that, for

$$\widetilde{G}(r) = \exp\Big(-B\frac{1}{(1-r)^\alpha}\Big), \qquad B > 0, \quad 0 < \alpha \le 2,$$

and

$$\varphi(z) = z + t(1-z)^\gamma, \qquad 1 < \gamma \le 3, \quad 0 < t < 2^{1-\gamma},$$

φ is a symbol and C_φ is bounded on $A^2_{\widetilde{G}}$ if and only if $\gamma \ge \alpha + 1$.

Here

$$\beta_n = \int_0^1 t^n \mathrm{e}^{-B/(1-\sqrt{t})^\alpha}\, dt \lesssim \exp(-cn^{\alpha/(\alpha+1)}).$$

We point out that β is nonincreasing, so, for every symbol φ fixing the origin, the composition operator C_φ is bounded. Nevertheless, choosing $\gamma < \alpha + 1$, there exist symbols inducing an unbounded composition operator, hence not all the C_{T_a} are bounded. Actually, for every $\alpha \in (0, 2]$, no C_{T_a} is bounded because β has no polynomial lower estimate (see Proposition 4.5 below).

Contents of the paper. In Section 2, we introduce several notions of growth or regularity for a sequence β — essentially decreasing, polynomial lower and upper bounds, slow oscillation — and give some connections between them. In Section 3, we consider the composition operators whose symbol vanishes at the origin. We show that, in order for all these operators to be bounded, it is necessary that β be bounded above. We show that β is essentially decreasing if and only if all these operators are bounded and

$$\sup_{\varphi(0)=0} \|C_\varphi\| < +\infty.$$

In Theorem 3.3, we give a sufficient condition for having all the composition operators C_φ with $\varphi(0) = 0$ bounded, allowing us to give an example of a sequence β for which this happens even though $\sup_{\varphi(0)=0} \|C_\varphi\| = +\infty$ (Theorem 3.7). In Section 4, we prove that all C_{T_a} are bounded on $H^2(\beta)$ if and only if β is slowly oscillating (Theorems 4.6 and 4.9). We state our main result.

Theorem 1.1. *Let β be a sequence of positive numbers, and let*

$$T_a(z) = \frac{a+z}{1+\bar{a}z}$$

for $a \in \mathbb{D}$. The following assertions are equivalent:

(1) *For some $a \in \mathbb{D} \setminus \{0\}$, the map T_a induces a bounded composition operator C_{T_a} on $H^2(\beta)$.*

(2) *For all $a \in \mathbb{D}$, the maps T_a induce bounded composition operators C_{T_a} on $H^2(\beta)$.*

(3) *β is slowly oscillating.*

The deep implication is (2) ⇒ (3). Its proof requires some sharp estimates on the mean of Taylor coefficients of T_a for a belonging to a subinterval of (0, 1). Once we found the equivalence of (1) and (2), we realized that it already appeared in the thesis of Zorboska [1988].

In Section 5, we show (Theorem 5.1) that if β is slowly oscillating, and moreover all composition operators are bounded on $H^2(\beta)$, then β is essentially decreasing. We thus obtain the following theorem.

Theorem 1.2. *Let β be a sequence of positive numbers. Then all composition operators on $H^2(\beta)$ are bounded if and only if β is essentially decreasing and slowly oscillating.*

For the notion of essentially decreasing and slowly oscillating sequences, see Definitions 2.1 and 2.2.

We end the paper with some results about multipliers.

A first version of this paper, not including the complete characterization given here, was put on arXiv on 30 November 2020 (and a second version on 21 March 2022) under the title “Boundedness of composition operators on general weighted Hardy spaces of analytic functions”.

2. Definitions, notation, and preliminary results

The open unit disk of $\mathbb{C}$ is denoted by $\mathbb{D}$ and we write $\mathbb{T}$ for its boundary $\partial\mathbb{D}$. We set

$$e_n(z) = z^n, \quad n \geq 0.$$

The weighted Hardy space $H^2(\beta)$ defined in the introduction is a Hilbert space with the canonical orthonormal basis

$$e_n^\beta(z) = \frac{1}{\sqrt{\beta_n}} z^n, \quad n \geq 0, \tag{2-1}$$

and the reproducing kernel K_w given, for all $w \in \mathbb{D}$, by

$$K_w(z) = \sum_{n=0}^{\infty} e_n^\beta(z)\overline{e_n^\beta(w)} = \sum_{n=0}^{\infty} \frac{1}{\beta_n} \overline{w}^n z^n. \tag{2-2}$$

Note that H^2 is continuously embedded in $H^2(\beta)$ if and only if β is bounded above. In particular, this is the case when β is nonincreasing. In this paper, we need a slightly more general notion.

Definition 2.1. A sequence of positive numbers $\beta = (\beta_n)_{n\geq 0}$ is said to be *essentially decreasing* if, for some constant $C \geq 1$, we have, for all $m \geq n \geq 0$,

$$\beta_m \leq C\beta_n. \tag{2-3}$$

Note: saying that β is essentially decreasing means that the shift operator on $H^2(\beta)$ is power bounded. If β is essentially decreasing and if we set

$$\tilde{\beta}_n = \sup_{m\geq n} \beta_m,$$

the sequence $\tilde{\beta} = (\tilde{\beta}_n)$ is nonincreasing and we have $\beta_n \leq \tilde{\beta}_n \leq C\beta_n$. In particular, $H^2(\beta) = H^2(\tilde{\beta})$ (with equivalent norms) and H^2 is continuously embedded in $H^2(\beta)$.

Definition 2.2. A sequence β is *slowly oscillating* if there are positive constants $c < 1 < C$ such that

$$c \leq \frac{\beta_m}{\beta_n} \leq C \quad \text{when } n/2 \leq m \leq 2n. \tag{2-4}$$

We may remark that this is equivalent to the existence of some function $\rho\colon (0,\infty) \to (0,\infty)$ which is bounded above on each compact subset of $(0,\infty)$ and for which $\beta_m/\beta_n \leq \rho(m/n)$, equivalently

$$\frac{1}{\rho(n/m)} \leq \frac{\beta_m}{\beta_n} \leq \rho(m/n).$$

Definition 2.3. The sequence of positive numbers $\beta = (\beta_n)$ is said to have a *polynomial lower bound* if there are positive constants c and α such that, for all integers $n \geq 1$,

$$\beta_n \geq c n^{-\alpha}. \tag{2-5}$$

This means that $H^2(\beta)$ is continuously embedded in the weighted Bergman space $\mathfrak{B}^2_{\alpha-1}$ of the analytic functions $f\colon \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|^2_{\mathfrak{B}^2_{\alpha-1}} := \alpha \int_{\mathbb{D}} |f(z)|^2 (1-|z|^2)^{\alpha-1}\, dA(z) < \infty$$

since $\mathfrak{B}^2_{\alpha-1} = H^2(\gamma)$ with $\gamma_n \approx n^{-\alpha}$.

Definition 2.4. The sequence of positive numbers $\beta = (\beta_n)$ is said to have a *polynomial upper bound* if there are positive constants C and γ such that, for all integers $n \geq 1$,

$$\beta_n \leq C n^{\gamma}. \tag{2-6}$$

The following simple proposition links these notions.

Proposition 2.5. (1) *Every slowly oscillating sequence β has polynomial lower and upper bounds.*

(2) *There are sequences that are essentially decreasing and with polynomial lower bound but are not slowly oscillating.*

(3) *There are bounded sequences that are slowly oscillating but not essentially decreasing.*

Proof. (1) This is clear because, for some $c \in (0,1)$, if $2^j \leq n < 2^{j+1}$, then

$$\beta_n \geq c\beta_{2^j} \geq c^{j+1}\beta_1 \geq c\beta_1 n^{-\alpha},$$

with $\alpha = \log(1/c)/\log 2$, and, for some $C > 1$,

$$\beta_n \leq C\beta_{2^j} \leq C^{j+1}\beta_1 \leq C\beta_1 n^{\gamma},$$

with $\gamma = \log C/\log 2$.

(2) Let $\delta > 0$. We set $\beta_0 = \beta_1 = 1$ and, for $n \geq 2$,

$$\beta_n = \frac{1}{(k!)^\delta} \quad \text{when } k! < n \leq (k+1)!.$$

The sequence β is nonincreasing.

For n and k as above, we have

$$\beta_n = \frac{1}{(k!)^\delta} \geq \frac{1}{n^\delta};$$

hence β has arbitrarily slow polynomial lower bound. However we have, for $k \geq 2$,

$$\frac{\beta_{2(k!)}}{\beta_{k!}} = \frac{(k!)^{-\delta}}{[(k-1)!]^{-\delta}} = \frac{1}{k^\delta} \xrightarrow[k\to\infty]{} 0,$$

so β is not slowly oscillating.

(3) We define β_n as follows. Let (a_k) be an increasing sequence of positive square integers such that $\lim_{k\to\infty} a_{k+1}/a_k = \infty$, for example $a_k = 4^{k^2}$, and let $b_k = \sqrt{a_k a_{k+1}}$; with our choice, this is an integer and we clearly have $a_k < b_k < a_{k+1}$. We set

$$\beta_n = \begin{cases} a_k/n & \text{for } a_k \leq n < b_k, \\ (a_k/b_k^2)n = (1/a_{k+1})n & \text{for } b_k \leq n < a_{k+1}. \end{cases}$$

The sequence (β_n) is slowly oscillating by construction. Indeed, since the other cases are obvious, it suffices to check that, for $a_k \leq n/2 < b_k \leq n < a_{k+1}$, the quotient β_m/β_n remains lower and upper bounded when $n/2 \leq m \leq n$ (it will then be automatically also satisfied when $n \leq m \leq 2n$). But, for $n/2 \leq m < b_k$, we have

$$\frac{\beta_m}{\beta_n} = \frac{a_k/m}{n/a_{k+1}} = \frac{a_k a_{k+1}}{mn} = \frac{b_k^2}{mn},$$

which is $\leq 2b_k^2/n^2 \leq 2$ and $\geq b_k^2/n^2 \geq (n/2)^2/n^2 = \frac{1}{4}$; and, for $b_k \leq m$, we have

$$\frac{\beta_m}{\beta_n} = \frac{m/a_{k+1}}{n/a_{k+1}} = \frac{m}{n} \in \left[\tfrac{1}{2}, 1\right].$$

However, even though (β_n) is bounded, since $\beta_n \leq 1$ for $a_k \leq n < b_k$ and, for $b_k \leq n < a_{k+1}$,

$$\beta_n \leq \beta_{a_{k+1}-1} = \frac{1}{a_{k+1}}(a_{k+1}-1) \leq 1,$$

it is not essentially decreasing, since

$$\frac{\beta_{a_{k+1}-1}}{\beta_{b_k}} = \frac{1}{\sqrt{a_k a_{k+1}}}(a_{k+1}-1) \sim \sqrt{\frac{a_{k+1}}{a_k}} \xrightarrow[k\to\infty]{} \infty. \qquad \square$$

Now we are going to recall some well-known facts about matrix representation of an operator T defined on a Hilbert space with an orthonormal basis $(\mathrm{e}_n)_{n\geq 0}$ and explain how they translate into our framework.

The entry $a_{m,n}$ (where $m, n \geq 0$) is defined by the m-th coordinate of $T(\mathrm{e}_n)$:

$$a_{m,n} = \mathrm{e}_m^*(T(\mathrm{e}_n)),$$

where $\mathrm{e}_k^*(x)$ stands for the k-th coordinate of the vector x.

We shall use the notation $\hat{f}(k)$ for the k-th Fourier coefficient of a function $f \in L^1(-\pi, \pi)$:

$$\hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \mathrm{e}^{-ikt} \, dt.$$

Let us point out that when the operator is the composition operator C_φ associated to the symbol φ, viewed on $H^2(\beta)$, its matrix representation in the basis $(e_n^\beta)_{n \geq 0}$ has an entry (m, n), which we write as

$$(e_m^\beta)^*(C_\varphi(e_n^\beta)) = \frac{\sqrt{\beta_m}}{\sqrt{\beta_n}} e_m^*(\varphi^n) = \frac{\sqrt{\beta_m}}{\sqrt{\beta_n}} \widehat{\varphi^n}(m)$$

since the m-th Taylor coefficient of φ^n coincides with its m-th Fourier coefficient.

We say that the reproducing kernels K_w have a *slow growth* if

$$\|K_w\| \leq \frac{C}{(1 - |w|)^s} \tag{2-7}$$

for positive constants C and s. We have the following equivalence.

Proposition 2.6. *The sequence β has polynomial lower bound if and only if the reproducing kernels K_w of $H^2(\beta)$ have a slow growth.*

Proof. Assume that the reproducing kernels have a slow growth. Since

$$\|K_w\|^2 = \sum_{k=0}^{\infty} \frac{|w|^{2k}}{\beta_k},$$

we get, for any $k \geq 2$,

$$\frac{|w|^{2k}}{\beta_k} \leq \frac{C^2}{(1 - |w|)^{2s}}.$$

Taking $w = 1 - 1/k$, we obtain $\beta_k \geq C' k^{-2s}$.

For the necessity, we only have to see that

$$\|K_w\|^2 = \frac{1}{\beta_0} + \sum_{n=1}^{\infty} \frac{|w|^{2n}}{\beta_n} \leq \frac{1}{\beta_0} + \delta^{-1} \sum_{n=1}^{\infty} n^\alpha |w|^{2n} \leq \frac{C}{(1 - |w|^2)^{\alpha+1}}. \qquad \square$$

3. Boundedness of composition operators whose symbol vanishes at the origin

3.1. ***Necessary conditions.*** We begin with this simple observation, see [Zorboska 1988, Proposition 3.1].

Proposition 3.1. *If all composition operators with symbol vanishing at* 0 *are bounded on* $H^2(\beta)$, *then the sequence* β *is bounded above.*

Proof. Let $f \in H^\infty$. Write $f = A\varphi + f(0)$, where A is a constant and φ a symbol vanishing at 0. We have $\varphi = C_\varphi(z) \in H^2(\beta)$, by hypothesis, and so $f \in H^2(\beta)$ and $H^\infty \subseteq H^2(\beta)$. It follows (by the closed graph theorem, since the convergence in norm implies pointwise convergence) that there exists a constant M such that $\|f\|_{H^2(\beta)} \leq M \|f\|_\infty$ for all $f \in H^\infty$. Testing this with $f(z) = z^n$, we get $\beta_n \leq M^2$. $\square$

Let us point out that boundedness of β_n does not suffice. For example, let (β_n) be a sequence such that $\beta_{4k+2}/\beta_{2k+1} \xrightarrow[k\to\infty]{} \infty$ (for instance $\beta_{2k} = 1$ and $\beta_{2k+1} = 1/(k+1)$); if $\varphi(z) = z^2$, then

$$\|C_\varphi(z^{2n+1})\|^2 = \|z^{2(2n+1)}\|^2 = \beta_{2(2n+1)};$$

since $\|z^{2n+1}\|^2 = \beta_{2n+1}$, the operator C_φ is not bounded on $H^2(\beta)$.

A partial characterization is given in the next proposition.

Proposition 3.2. *The following assertions are equivalent*:

(1) *All symbols φ such that $\varphi(0) = 0$ induce bounded composition operators C_φ on $H^2(\beta)$* **and**

$$\sup_{\varphi(0)=0} \|C_\varphi\| < \infty. \tag{3-1}$$

(2) *β is an essentially decreasing sequence.*

Of course, by the uniform boundedness principle, (3-1) is equivalent to

$$\sup_{\varphi(0)=0} \|f\circ\varphi\| < \infty \quad \text{for all } f \in H^2(\beta).$$

Let us point out an important fact: we shall see in Theorem 3.7 that there are weights β for which all composition operators C_φ with $\varphi(0) = 0$ are bounded but $\sup_{\varphi(0)=0}\|C_\varphi\| = +\infty$.

Proof. (2) $\Rightarrow$ (1) We may assume that β is nonincreasing. Then the Goluzin–Rogosinski theorem [Duren 1983, Theorem 6.3] gives the result; in fact, writing

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \quad \text{and} \quad (C_\varphi f)(z) = \sum_{n=0}^{\infty} d_n z^n,$$

it says that

$$\sum_{0\le k\le n} |d_k|^2 \le \sum_{0\le k\le n} |c_k|^2 \quad \text{for all } n \ge 0,$$

and hence, by Abel summation,

$$\|C_\varphi f\|^2 = \sum_{n=0}^{\infty} |d_n|^2\beta_n \le \sum_{n=0}^{\infty} |c_n|^2\beta_n = \|f\|^2,$$

leading to C_φ bounded and $\|C_\varphi\| \le 1$. This same result was also proved by Cowen [1990, Corollary of Theorem 7]. Alternatively, we can use a result of Kacnelson [1972]; see also [Chalendar and Partington 2014; 2017, Corollary 2.2; Lefèvre et al. 2021, Theorem 3.12].

(1) $\Rightarrow$ (2) Set $M = \sup_{\varphi(0)=0}\|C_\varphi\|$. Let $m > n$, and take

$$\varphi(z) = \varphi_{m,n}(z) = z\big(\tfrac{1}{2}(1 + z^{m-n})\big)^{1/n}.$$

Then $\varphi(0) = 0$ and $[\varphi(z)]^n = \frac{1}{2}(z^n + z^m)$; hence

$$\tfrac{1}{4}(\beta_n + \beta_m) = \|\varphi^n\|^2 = \|C_\varphi(e_n)\|^2 \le \|C_\varphi\|^2\|e_n\|^2 \le M^2\beta_n,$$

so β is essentially decreasing. ∎

Remark. Let us mention the following example. For $0 < r < 1$, let $\beta_n = \pi n r^{2n}$ for $n \geq 1$ and $\beta_0 = 1$. This sequence is eventually decreasing, so it is essentially decreasing. The quantity $\|f\|^2_{H^2(\beta)} - |f(0)|^2$ is the area of the part of the Riemann surface on which $r\mathbb{D}$ is mapped by f. E. Reich [1954], generalizing Goluzin's result [1951] (see [Duren 1983, Theorem 6.3]), proved that, for all symbols φ such that $\varphi(0) = 0$, the composition operator C_φ is bounded on $H^2(\beta)$ and

$$\|C_\varphi\| \leq \sup_{n\geq 1} \sqrt{n} r^{n-1} \leq \frac{1}{\sqrt{2\mathrm{e}}} \frac{1}{r\sqrt{\log(1/r)}}.$$

For $0 < r < 1/\sqrt{2}$, Goluzin's theorem asserts that $\|C_\varphi\| \leq 1$.

Note that this sequence β is not slowly oscillating, since $\beta_{2n}/\beta_n = 2r^{2n}$. Hence, from Theorem 4.9 below, we get that no composition operator C_{T_a} is bounded on $H^2(\beta)$.

However, that the weight β is essentially decreasing is not necessary for the boundedness of all composition operators C_φ, with symbol φ vanishing at 0, as we will see later (Theorem 3.7).

3.2. *Sufficient condition.*

Theorem 3.3. *Let $\beta = (\beta_n)_{n=0}^{\infty}$ be a sequence of positive numbers that is* ***weakly decreasing****, i.e.,*

$$\begin{aligned} &\text{for every } \delta > 0, \text{ there exists a positive constant } C = C(\delta) \\ &\text{such that } \beta_m \leq C\beta_n \text{ whenever } m > (1+\delta)n. \end{aligned} \tag{3-2}$$

Then, for all symbols $\varphi\colon \mathbb{D} \to \mathbb{D}$ vanishing at 0, the composition operator C_φ is bounded on $H^2(\beta)$.

Let us point out that (3-2) implies that β is bounded.

Note that Zorboska showed [1988, Example 1, pp. 14-15] that, for $\beta_n = \exp(n^a)$, with $0 < a < 1$, which is unbounded, the symbol $\varphi(z) = z^k$, $k \geq 2$, induces an unbounded composition operator on $H^2(\beta)$.

To prove Theorem 3.3, we need several lemmas.

Lemma 3.4. *Let $\varphi\colon \mathbb{D} \to \mathbb{D}$ be an analytic self-map such that $\varphi(0) = 0$ and $|\varphi'(0)| < 1$. Then there exists $\rho > 0$ such that, for all integers n and m,*

$$|\widehat{\varphi^n}(m)| \leq \exp(-[(1+\rho)n - m]).$$

Proof. Since $\varphi(0) = 0$, we can write $\varphi(z) = z\varphi_1(z)$. Since $|\varphi'(0)| < 1$, we have $\varphi_1\colon \mathbb{D} \to \mathbb{D}$. Now let $M(r) = \sup_{|z|=r} |\varphi_1(z)|$. Cauchy's inequalities say that $|\widehat{\varphi_1^n}(m)| \leq [M(r)]^n/r^m$. We have $M(r) < 1$, so there exists a positive number $\rho = \rho(r)$ such that $M(r) = r^\rho$. We get

$$|\widehat{\varphi^n}(m)| = |\widehat{\varphi_1^n}(m-n)| \leq \frac{r^{\rho n}}{r^{m-n}} = r^{(1+\rho)n-m},$$

and the result follows by taking $r = \mathrm{e}^{-1}$. □

The following result of V. È. Kacnelson [1972] was used in [Chalendar and Partington 2014; 2017, Corollary 2.2]; see also [Lefèvre et al. 2021, Theorem 3.12].

Theorem 3.5 [Kacnelson 1972]. *Let H be a separable complex Hilbert space, and let $(e_i)_{i\geq 0}$ be a fixed orthonormal basis of H. Let $M\colon H \to H$ be a bounded linear operator. We assume that the matrix of M with respect to this basis is lower-triangular: $\langle Me_j \,|\, e_i\rangle = 0$ for $i < j$.*

Let $(\gamma_j)_{j\geq 0}$ be a nondecreasing sequence of positive real numbers, and let Γ be the (possibly unbounded) diagonal operator such that $\Gamma(e_j) = \gamma_j e_j$, $j \geq 0$. Then the operator $\Gamma^{-1}M\Gamma\colon H \to H$ is bounded and, moreover,

$$\|\Gamma^{-1}M\Gamma\| \leq \|M\|.$$

We need the following generalization of Kacnelson's theorem, which is implicitly used in [Lefèvre et al. 2021, p. 13]. The matrix A only needs to be lower-triangular with respect to the order induced by the sequence $(d_n)_n$.

Lemma 3.6. *Let $A\colon \ell_2 \to \ell_2$ be a bounded operator represented by the matrix $(a_{m,n})_{m,n}$, i.e., $a_{m,n} = \langle Ae_n, e_m\rangle$, where $(e_n)_{n\geq 0}$ is the canonical basis of ℓ_2.*

Let (d_n) be a sequence of positive numbers such that, for every m and n,

$$d_m < d_n \quad \Longrightarrow \quad a_{m,n} = 0. \tag{3-3}$$

Then, D being the (possibly unbounded) diagonal operator with entries d_n, we have

$$\|D^{-1}AD\| \leq \|A\|.$$

We will propose two different proofs. The first one, using complex variables, is an adaptation of that of Kacnelson, and we reproduce it for the convenience of the reader; the second one is new and uses real variables.

Proof 1. Let $\mathbb{C}_0$ be the right-half-plane $\mathbb{C}_0 = \{z \in \mathbb{C} : \Re\mathrm{e}\, z > 0\}$. We set $H_N = \operatorname{span}\{e_n : n \leq N\}$ and

$$A_N = P_N A J_N,$$

where P_N is the orthogonal projection from ℓ_2 onto H_N and J_N is the canonical injection from H_N into ℓ_2. We consider, for $z \in \overline{C}_0$,

$$A_N(z) = D^{-z} A_N D^z\colon H_N \to H_N,$$

where $D^z(e_n) = d_n^z e_n$.

If $(a_{m,n}(z))_{m,n}$ is the matrix of $A_N(z)$ on the basis $\{e_n : n \leq N\}$ of H_N, we clearly have

$$a_{m,n}(z) = a_{m,n}(d_n/d_m)^z.$$

In particular, we have, thanks to (3-3),

$$a_{m,n}(z) = 0 \quad \text{if } d_m < d_n$$

and

$$|a_{m,n}(z)| \leq \sup_{k,l} |a_{k,l}| := M \quad \text{for all } z \in \overline{C}_0.$$

Since

$$\|A_N(z)\|^2 \leq \|A_N(z)\|_{HS}^2 = \sum_{m,n\leq N} |a_{m,n}(z)|^2 \leq (N+1)^2 M^2,$$

we get

$$\|A_N(z)\| \leq (N+1)M \quad \text{for all } z \in \overline{C}_0.$$

Let us consider the function $u_N : \overline{C}_0 \to \overline{C}_0$ defined by

$$u_N(z) = \|A_N(z)\|. \tag{3-4}$$

This function u_N is continuous on $\overline{C}_0$, bounded above by $(N+1)M$, and subharmonic in $\mathbb{C}_0$. Moreover, thanks to (3-3), the maximum principle gives

$$\sup_{\overline{C}_0} u_N(z) = \sup_{\partial\mathbb{C}_0} u_N(z).$$

Since $\|D^z\| = \|D^{-z}\| = 1$ for $z \in \partial\mathbb{C}_0$, we have

$$\|A_N(z)\| \le \|A_N\| \quad \text{for } z \in \partial\mathbb{C}_0,$$

and we get

$$\sup_{\overline{C}_0} u_N(z) \le \|A_N\| \le \|A\|.$$

In particular, $u_N(1) \le \|A\|$ and, letting N go to infinity, we obtain $\|D^{-1}AD\| \le \|A\|$.

Proof 2. Since d_n is positive, we can write $d_n = \mathrm{e}^{-\rho_n}$, where $\rho_n \in \mathbb{R}$. Let $x = (x_n)_{n\ge0}$ and $y = (y_n)_{n\ge0}$ be in ℓ^2 with finite support. We are interested in controlling the sum

$$S = \sum_{m,n} a_{m,n} \frac{d_n}{d_m} x_n \overline{y_m},$$

which can also be written

$$S = \sum_{m,n} a_{m,n} \mathrm{e}^{-|\rho_n - \rho_m|} x_n \overline{y_m}$$

since the nontrivial part of the sum runs over the pairs (m, n) such that $d_m \ge d_n$, i.e., $\rho_n \ge \rho_m$.

Now we introduce the function

$$f(t) = \frac{1}{\pi(1+t^2)} \quad \text{for } t \in \mathbb{R},$$

which is positive and belongs to the unit ball of $L^1(\mathbb{R})$. Moreover, its Fourier transform satisfies, for every $x \in \mathbb{R}$,

$$\mathcal{F}(f)(-x) = \int_{\mathbb{R}} f(t)\mathrm{e}^{ixt}\, dt = \mathrm{e}^{-|x|}.$$

We get

$$S = \int_{\mathbb{R}} f(t)\Bigg(\sum_{m,n} a_{m,n} x_n \mathrm{e}^{i\rho_n t}\, \overline{y_m \mathrm{e}^{i\rho_m t}}\Bigg)\, dt = \int_{\mathbb{R}} f(t) \langle A(x(t)), y(t)\rangle_{\ell^2}\, dt,$$

where

$$x(t) = (x_n \mathrm{e}^{i\rho_n t})_{n\ge0} \quad \text{and} \quad y(t) = (y_n \mathrm{e}^{i\rho_n t})_{n\ge0}.$$

We obtain

$$|S| \le \int_{\mathbb{R}} f(t)\|A\|\|x(t)\|\|y(t)\|\, dt = \int_{\mathbb{R}} f(t)\|A\|\|x\|\|y\|\, dt = \|A\|\|x\|\|y\|$$

since $\|f\|_{L^1(\mathbb{R})} = 1$.

Since x and y are arbitrary, this proves $\|D^{-1}AD\| \le \|A\|$. ☐

Proof of Theorem 3.3. First, if $|\varphi'(0)| = 1$, we have $\varphi(z) = \alpha z$ for some α with $|\alpha| = 1$, and the result is trivial.

So, we assume that $|\varphi'(0)| < 1$. Then, by Lemma 3.4, there exists $\rho > 0$ such that, for all m, n,

$$|\widehat{\varphi^n}(m)| \leq \exp(-[(1+\rho)n - m]).$$

It follows that, with $\delta = \frac{1}{2}\rho$, we have

$$|\widehat{\varphi^n}(m)| \leq \exp(-\delta n) \quad \text{when } m \leq (1+\delta)n.$$

Since $\varphi(0) = 0$, we also know that $\widehat{\varphi^n}(m) = 0$ if $m < n$.

Now, using property (3-2), there exists $M \geq 1$ such that

$$\beta_m \leq M\beta_n \quad \text{when } m \geq (1+\delta)n.$$

Define now a new sequence $\gamma = (\gamma_n)$ as

$$\gamma_n = \max\Big\{\beta_n, \sup_{m > (1+\delta)n} \beta_m\Big\}.$$

We have

(1) $\beta_n \leq \gamma_n \leq M\beta_n$,

(2) $\gamma_m \leq \gamma_n$ if $m \geq (1+\delta)n$.

Item (1) implies that $H^2(\gamma) = H^2(\beta)$, and we are reduced to proving that $C_\varphi \colon H^2(\gamma) \to H^2(\gamma)$ is bounded.

Let $A = (a_{m,n})_{m,n} = (\widehat{\varphi^n}(m))_{m,n}$. We have to prove that

$$B = (\gamma_m^{1/2}\gamma_n^{-1/2} a_{m,n})_{m,n}$$

represents a bounded operator on ℓ_2.

Define the matrix

$$A_1 = (a_{m,n}\mathbb{1}_{\{(m,n):m\leq(1+\delta)n\}})_{m,n},$$

and set $A_2 = A - A_1$. Define analogously B_1 and $B_2 = B - B_1$.

Then A_1 is a Hilbert–Schmidt operator because (recall that $a_{m,n} = 0$ if $m < n$) we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{(1+\delta)n} |a_{m,n}|^2 \leq \sum_{n=1}^{\infty}\sum_{m=n}^{(1+\delta)n} \exp(-2\delta n) \leq \sum_{n=1}^{\infty}(\delta n + 1)\exp(-2\delta n) < +\infty.$$

Since A is bounded, it follows that $A_2 = A - A_1$ is bounded.

We now remark that, writing $A_2 = (\alpha_{m,n})_{m,n}$, we have, with $d_n = 1/\sqrt{\gamma_n}$,

$$d_m < d_n \quad \Longrightarrow \quad \gamma_m > \gamma_n \quad \Longrightarrow \quad m < (1+\delta)n \quad \Longrightarrow \quad \alpha_{m,n} = 0.$$

Hence we can apply Lemma 3.6 to the matrix A_2, which implies that B_2 is bounded.

Now, we have $\liminf \beta_n^{1/n} \geq 1$, so $\beta_n \geq \mathrm{e}^{-\delta n}$ for n large enough; hence we have $\beta_n \geq c\mathrm{e}^{-\delta n}$ for every $n \geq 1$.

Since γ is bounded (like β is), we have, for some positive constant C,

$$\sum_{n=1}^{\infty}\sum_{m=n}^{(1+\delta)n}\frac{\gamma_m}{\gamma_n}|a_{m,n}|^2 \le \sum_{n=1}^{\infty}\sum_{m=n}^{(1+\delta)n}\frac{C}{\beta_n}\exp(-2\delta n) \le \sum_{n=1}^{\infty}\frac{C}{c}(\delta n+1)\exp(-\delta n) < +\infty,$$

meaning that B_1 is a Hilbert–Schmidt operator.

Therefore $B = B_1 + B_2$ is bounded, as desired. □

As a corollary of Theorem 3.3, we can provide the following example.

Theorem 3.7. *There exists a bounded sequence β, with polynomial lower bound, which is* ***not essentially decreasing****, and for which every composition operator with symbol vanishing at* 0 *is bounded on $H^2(\beta)$.*

We hence have $\sup_{\varphi(0)=0}\|C_\varphi\| = +\infty$.

It should be noted that, for this weight, the composition operators are not all bounded, as we will see in Proposition 4.10.

Proof. Define $\beta_n = 1$ for $n \le 3!$, and, for $k \ge 3$,

$$\begin{cases} \beta_n = 1/k! & \text{for } k! < n \le (k+1)! - 2 \text{ and for } n = (k+1)!, \\ \beta_n = 1/(k+1)! & \text{for } n = (k+1)! - 1. \end{cases}$$

Note that, for $m > n$, we have $\beta_m > \beta_n$ only if $n = (k+1)! - 1$ and $m = (k+1)! = n+1$ for some $k \ge 3$.

However β is not essentially decreasing since, for every $k \ge 3$, we have $\beta_{n+1}/\beta_n = k+1$ if $n = (k+1)! - 1$.

The sequence β has a polynomial lower bound because $\beta_n \ge 1/(2n)$ for all $n \ge 1$. In fact, for $k \ge 3$, we have $\beta_n \ge (k+1)/n \ge 1/n$ if $k! < n \le (k+1)! - 2$ or if $n = (k+1)!$, and, for $n = (k+1)! - 1$, we have $n\beta_n = [(k+1)! - 1]/(k+1)! \ge \frac{1}{2}$. It has a polynomial upper bound since it is bounded above by 1.

Now, it remains to check (3-2) in order to apply Theorem 3.3 and finish the proof of Theorem 3.7. Note first that we have $\beta_m/\beta_n \le 1$ if $m \ge n+2$. Next, for given $\delta > 0$, there exists an integer N such that $(1+\delta)n \ge n+2$ for every $n \ge N$, so $\beta_m/\beta_n \le 1$ if $m \ge (1+\delta)n$ and $n \ge N$. It suffices to take $C = \max_{1 \le n \le N} \beta_{n+1}/\beta_n$ to obtain (3-2). The last assertion follows from Proposition 3.2. □

4. Boundedness of composition operators of the symbol T_a

Recall that, for $a \in \mathbb{D}$, we defined

$$T_a(z) = \frac{a+z}{1+\bar{a}z}, \qquad z \in \mathbb{D}. \tag{4-1}$$

It is well known that T_a is an automorphism of $\mathbb{D}$ and that $T_a(0) = a$ and $T_a(-a) = 0$.

Though we do not really need this, we remark that $(T_a)_{a\in(-1,1)}$ is a group and $(T_a)_{a\in(0,1)}$ is a semigroup. It suffices to see that $T_a \circ T_b = T_{a*b}$, with

$$a * b = \frac{a+b}{1+ab}. \tag{4-2}$$

In this section, we are going to prove a necessary and sufficient condition for the statement that all composition operators C_{T_a} for $a \in \mathbb{D}$ are bounded on $H^2(\beta)$. Namely, we have the following theorem, the proof of which will occupy Sections 4.2 and 4.3.

Theorem 4.1. *All composition operators C_{T_a}, with $a \in \mathbb{D}$, are bounded on $H^2(\beta)$ if and only if β is slowly oscillating.*

Before that, let us note the following fact; see also [Zorboska 1988, Proposition 3.6]. Recall that if φ and ψ are two symbols, then $C_\varphi \circ C_\psi = C_{\psi\circ\varphi}$.

Proposition 4.2. *If C_{T_a} is bounded on $H^2(\beta)$ for some $a \in \mathbb{D} \setminus \{0\}$, then C_{T_b} is bounded on $H^2(\beta)$ for all $b \in \mathbb{D}$.*

Moreover, the maps C_{T_b} are uniformly bounded on the compact subsets of $\mathbb{D}$.

We decompose the proof into lemmas. The first one was first proved in [Zorboska 1988] (see also [Gallardo-Gutiérrez and Partington 2013, Proposition 2.1]) and follows from the fact that if $b = \rho e^{i\theta}$ and R_θ is the rotation $R_\theta(z) = e^{i\theta} z$, which induces a unitary operator C_{R_θ} on $H^2(\beta)$, then $T_b = R_\theta \circ T_\rho \circ R_{-\theta}$ and $C_{T_b} = C_{R_{-\theta}} \circ C_{T_\rho} \circ C_{R_\theta}$.

Lemma 4.3. *The composition operator C_{T_b} is bounded if and only if $C_{T_{|b|}}$ is bounded, with equal norms.*

Lemma 4.4. *Let $r \in (0, 1)$ such that C_{T_r} is bounded. For any $b \in \mathbb{D}$ satisfying $|b| \le 2r/(1+r^2)$, C_{T_b} is bounded and we have $\|C_{T_b}\| \le \|C_{T_r}\|^2$.*

Proof. Let S be the circle $C(0, r)$ and $u \colon S \to \mathbb{R}_+$ be the continuous function defined by

$$u(s) = \left| \frac{s+r}{1+\bar{s}r} \right|. \tag{4-3}$$

By connectedness, $u(S)$ contains the segment $[0, 2r/(1+r^2)] = [u(-r), u(r)]$. Let now

$$b \in D\Big(0, \frac{2r}{1+r^2}\Big).$$

By the above, there exists $s \in S$ such that $|b| = u(s)$. This means that

$$|T_b(0)| = |b| = |u(s)| = |T_s(r)| = |(T_s \circ T_r)(0)|.$$

Therefore, $T_b(0) = e^{i\alpha}(T_s \circ T_r)(0)$ for some $\alpha \in \mathbb{R}$, and hence, by Schwarz's lemma, there is some $\theta \in \mathbb{R}$ such that $T_b = R_\alpha \circ T_s \circ T_r \circ R_\theta$. We then have $C_{T_b} = C_{R_\theta} \circ C_{T_r} \circ C_{T_s} \circ C_{R_\alpha}$. Since C_{R_θ} and C_{R_α} are unitary, we get, using Lemma 4.3 for C_{T_s},

$$\|C_{T_b}\| = \|C_{T_r} \circ C_{T_s}\| \le \|C_{T_r}\| \|C_{T_s}\| = \|C_{T_r}\|^2. \qquad \square$$

Proof of Proposition 4.2. It suffices to use Lemmas 4.3 and 4.4 and do an iteration, noting that if $r_0 = |a| > 0$ and $r_{n+1} = 2r_n/(1+r_n^2) = r_n * r_n$, then $(r_n)_{n\ge0}$ increases to 1. $\square$

4.1. ***An elementary necessary condition.*** We begin with an elementary necessary condition. It is implied by Theorem 4.9, but its statement deserves to be pointed out. Moreover, its proof is simple and highlights the role of the reproducing kernel.

Proposition 4.5. *Let $a \in (0, 1)$, and assume that T_a induces a bounded composition operator on $H^2(\beta)$. Then β has polynomial lower bound.*

Proof. Since

$$\|K_x\|^2 = \sum_{n=0}^{\infty} \frac{x^{2n}}{\beta_n},$$

we have $\|K_x\| \le \|K_y\|$ for $0 \le x \le y < 1$.

We define by induction a sequence $(u_n)_{n\ge 0}$ with

$$u_0 = 0 \quad \text{and} \quad u_{n+1} = T_a(u_n).$$

Since $T_a(1) = 1$ (recall that $a \in (0, 1)$), we have

$$1 - u_{n+1} = \int_{u_n}^{1} T_a'(t)\,dt = \int_{u_n}^{1} \frac{1-a^2}{(1+at)^2}\,dt;$$

hence

$$\frac{1-a}{1+a}(1-u_n) \le 1 - u_{n+1} \le (1-a^2)(1-u_n).$$

Let $0 < x < 1$. We can find $N \ge 0$ such that $u_N \le x < u_{N+1}$. Then

$$1 - x \le 1 - u_N \le (1-a^2)^N.$$

On the other hand, since $C_{T_a}^* K_z = K_{T_a(z)}$ for all $z \in \mathbb{D}$, we have

$$\|K_x\| \le \|K_{u_{N+1}}\| \le \|C_{T_a}\|\|K_{u_N}\| \le \|C_{T_a}\|^{N+1}\|K_{u_0}\| = \frac{1}{\sqrt{\beta_0}}\|C_{T_a}\|^{N+1}.$$

Let $s \ge 0$ such that $(1-a^2)^{-s} = \|C_{T_a}\|$. We obtain

$$\|K_x\| \le \frac{1}{\sqrt{\beta_0}(1-x)^s}\|C_{T_a}\|. \tag{4-4}$$

We get the result by using Proposition 2.6. □

Remarks. (1) For example, when $\beta_n = \exp[-c(\log(n+1))^2]$, with $c > 0$, no T_a induces a bounded composition operator on $H^2(\beta)$, even though C_φ is bounded for all symbols φ with $\varphi(0) = 0$, since β is decreasing, as we saw in Proposition 3.2.

(2) For the Dirichlet space $\mathcal{D}^2$, we have $\beta_n = n+1$, but all the maps T_a induce bounded composition operators on $\mathcal{D}^2$; see [Lefèvre et al. 2021, Remark before Theorem 3.12]. In this case β has polynomial upper bound even though it is not bounded above.

(3) However, even for decreasing sequences, a polynomial lower bound for β is not enough for some T_a to induce a bounded composition operator. Indeed, we saw in Proposition 2.5 an example of a decreasing sequence β with polynomial lower bound but not slowly oscillating, and we will see in Theorem 4.9 that this condition is needed to have some T_a induce a bounded composition operator.

(4) Gallardo-Gutiérrez and Partington [2013] give estimates for the norm of C_{T_a}, with $a \in (0, 1)$, when C_{T_a} is bounded on $H^2(\beta)$. More precisely, they proved that if β is bounded above and C_{T_a} is bounded, then

$$\|C_{T_a}\| \ge \left(\frac{1+a}{1-a}\right)^{\sigma},$$

where $\sigma = \inf\{s \geq 0 : (1-z)^{-s} \notin H^2(\beta)\}$, and

$$\|C_{T_a}\| \leq \Big(\frac{1+a}{1-a}\Big)^{\tau},$$

where $\tau = \frac{1}{2} \sup \mathfrak{Re}\, W(A)$, with A the infinitesimal generator of the continuous semigroup (S_t) defined as $S_t = C_{T_{\tanh t}}$, namely $(Af)(z) = f'(z)(1-z^2)$, and $W(A)$ its numerical range.

For $\beta_n = 1/(n+1)^{\nu}$ with $0 \leq \nu \leq 1$, the two bounds coincide, so they get

$$\|C_{T_a}\| = \Big(\frac{1+a}{1-a}\Big)^{(\nu+1)/2}.$$

4.2. *Sufficient condition.* The following sufficient condition explains in particular why all composition operators C_{T_a} are bounded on the Dirichlet space.

Theorem 4.6. *If β is slowly oscillating, then all symbols that extend analytically in a neighborhood of $\overline{\mathbb{D}}$ induce a bounded composition operator on $H^2(\beta)$.*

In particular, all C_{T_a}, for $a \in \mathbb{D}$, are bounded on $H^2(\beta)$.

To prove Theorem 4.6, we begin with a very elementary fact.

Lemma 4.7. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ have an analytic extension to an open neighborhood Ω of $\overline{\mathbb{D}}$. Then there are a constant $b > 0$ and an integer $\lambda > 1$ such that*

$$|\widehat{\varphi^n}(m)| \leq \begin{cases} \mathrm{e}^{-bn} & \text{if } n \geq \lambda m, \\ \mathrm{e}^{-bm} & \text{if } m \geq \lambda n. \end{cases}$$

Proof. Let $R > 1$ such that $\overline{D(0, R)} \subseteq \Omega$. For $0 < r \leq R$, we set

$$M(r) = \sup_{|z|=r} |\varphi(z)|.$$

Take any $r \in (0, 1)$, for instance $r = \mathrm{e}^{-1}$. We have $M(r) < 1$, so we can write $M(r) = \mathrm{e}^{-\rho}$ for some positive ρ.

Cauchy's inequalities give

$$|\widehat{\varphi^n}(m)| \leq \frac{[M(r)]^n}{r^m} = \mathrm{e}^{m-\rho n}.$$

Choose $\lambda_1 = \max(2, 2/\rho)$ and $b_1 = \rho - \lambda_1^{-1}$. Then $|\widehat{\varphi^n}(m)| \leq \mathrm{e}^{-b_1 n}$ if $n \geq \lambda_1 m$.

For the second inequality, write $R =: \mathrm{e}^{\beta}$, with $\beta > 0$. Let $\alpha > 0$ with $M(R) \leq \mathrm{e}^{\alpha}$. Cauchy's inequalities again give

$$|\widehat{\varphi^n}(m)| \leq \frac{[M(R)]^n}{R^m} \leq \mathrm{e}^{\alpha n - \beta m}.$$

Choose $\lambda_2 = \max(2, 2\alpha/\beta)$ and $b_2 = \beta - \alpha\lambda_2^{-1}$. Then $|\widehat{\varphi^n}(m)| \leq \mathrm{e}^{-b_2 m}$ if $m \geq \lambda_2 n$. We get the conclusion taking $b = \min(b_1, b_2)$ and choosing an integer $\lambda \geq \max(\lambda_1, \lambda_2)$. □

Lemma 4.8. *Let (β_n) be a slowly oscillating sequence of positive numbers. Let $A = (a_{m,n})_{m,n}$ be the matrix of a bounded operator on ℓ_2. Assume that, for some integer $\lambda > 1$ and some constants c, b, we have:*

(1) $|a_{m,n}| \leq c\mathrm{e}^{-bn}$ *when* $n \geq \lambda m$,

(2) $|a_{m,n}| \leq c\mathrm{e}^{-bm}$ *when* $m \geq \lambda n$.

Then the matrix $\tilde{A} = (a_{m,n}\sqrt{\beta_m/\beta_n})_{m,n}$ *also defines a bounded operator on* ℓ_2.

Proof. In the sequel $\|\cdot\|$ stands for the ℓ^2-norm.

Since β is slowly oscillating, it has polynomial lower and upper bounds: for some $\alpha, \gamma > 0$ and $\delta \in (0,1)$, we have $\delta(n+1)^{-\alpha} \leq \beta_n \leq \delta^{-1}(n+1)^{\gamma}$.

The matrix $\tilde{A}$ is Hilbert–Schmidt far from the diagonal since

$$\sum_{n=1}^{\infty}\sum_{\lambda m<n}|a_{m,n}|^2\frac{\beta_m}{\beta_n} \lesssim \sum_{n=1}^{\infty}\sum_{\lambda m<n}(n+1)^{\alpha+\gamma}|a_{m,n}|^2 \lesssim \sum_{n=1}^{\infty}(n+1)^{\alpha+\gamma+1}\mathrm{e}^{-2bn} < +\infty$$

and

$$\sum_{n=0}^{\infty}\sum_{m>\lambda n}|a_{m,n}|^2\frac{\beta_m}{\beta_n} \lesssim \sum_{n=0}^{\infty}\sum_{m>\lambda n}(n+1)^{\alpha+\gamma}|a_{m,n}|^2 \lesssim \sum_{n=0}^{\infty}(n+1)^{\alpha+\gamma}\Bigg(\sum_{m>\lambda n}\mathrm{e}^{-2bm}\Bigg) < +\infty.$$

Since β_m/β_n remains bounded from above and below around the diagonal, the matrix $\tilde{A}$ behaves like A near the diagonal. More precisely, if I, J are blocks of integers such that $(m,n) \in I \times J$ implies that $n/\lambda^2 \leq m \leq \lambda^2 n$, then, with obvious notation (e.g., P_I is the orthogonal projection on $\mathrm{span}(e_n, n \in I)$), the slow oscillation of β gives, for some $C > 0$,

$$\Bigg|\sum_{(m,n)\in I\times J} a_{m,n}x_n\overline{y_m}\sqrt{\frac{\beta_m}{\beta_n}}\Bigg| \leq \|A\|\Bigg(\sum_{(m,n)\in I\times J}|x_n|^2|y_m|^2\frac{\beta_m}{\beta_n}\Bigg)^{1/2} \leq C^{1/2}\|A\|\|P_J x\|\|P_I y\|.$$

For $k = 0, 1, 2, \ldots$, let $J_k = [\lambda^k, \lambda^{k+1}[$ and, for $k = 1, 2, \ldots$, we define $I_k = [\lambda^{k-1}, \lambda^{k+2}[$. We also define $I_0 = [0, \lambda^2[$.

We define the matrix R to have entries

$$r_{m,n} = \begin{cases} \sqrt{\beta_m/\beta_n}\,a_{m,n} & \text{if } (m,n) \in \bigcup_{k=0}^{\infty}(I_k \times J_k), \\ 0 & \text{elsewhere.} \end{cases}$$

Let H_k be the subspace of the sequences $(x_n)_{n\geq 0}$ in ℓ_2 such that $x_n = 0$ for $n \notin I_k$, i.e.,

$$H_k = \mathrm{span}\{e_n : n \in I_k\} \quad \text{and} \quad \widetilde{H}_k = \mathrm{span}\{e_n : n \in J_k\}.$$

Let P_k be (the matrix of) the orthogonal projection of ℓ_2 with range H_k and Q_k that with range $\widetilde{H}_k$. Then $R_k = P_k A Q_k$ is the matrix with entries $a_{m,n}$ when $(m,n) \in I_k \times J_k$ and 0 elsewhere. By the above discussion, we have

$$|(R_k x \,|\, y)| \leq C^{1/2}\|A\|\|Q_k x\|\|P_k y\|.$$

We point out that, for every $y \in \ell^2$, we have $\sum \|P_k y\|^2 \leq 3\|y\|^2$ since each integer belongs to at most three intervals I_k.

In the same way, for every $x \in \ell^2$, we have $\sum \|Q_k x\|^2 \leq \|x\|^2$ since the subspaces $\widetilde{H}_k$ are orthogonal. Summing up over k, we get the boundedness of $R = \sum_{k=0}^{\infty} R_k$.

Now let us check when the entries of R do not coincide with the entries of $\tilde{A}$. Actually, it happens when (m, n) does not belong to the union of the $I_k \times J_k$. When $n \geq 1$, it means that n belongs to some J_p but $m \notin I_p$: either $m < \lambda^{p-1}$ or $m \geq \lambda^{p+2}$, and hence either $m/n < \lambda^{-1}$ or $m/n > \lambda$. Therefore the nonzero entries (m, n) of $\tilde{A} - R$ satisfy either $n > \lambda m$ or $m > \lambda n$.

That ends the proof since we have seen at the beginning that $\tilde{A} - R$ is Hilbert–Schmidt. □

Remark. The proof shows that, instead of (1) and (2), it is enough to have

$$\sum_{m<C_1 n} n^{\alpha+1}|a_{m,n}|^2 < \infty \quad \text{and} \quad \sum_{m>C_2 n} m^{\alpha}|a_{m,n}|^2 < \infty.$$

Moreover, the proof also shows that, when β is slowly oscillating, if we set

$$E = \{(m, n) : C_1 n \leq m \leq C_2 n\} \quad \text{for some } C_1, C_2 > 0,$$

then the matrix $(\sqrt{\beta_m/\beta_n}\,\mathbb{1}_E(m, n))$ is a Schur multiplier over *all* the bounded matrices, while Kacnelson's theorem (Theorem 3.5) says that, if $\gamma = (\gamma_n)$ is nonincreasing, the matrix (γ_m/γ_n) is a Schur multiplier of all bounded *lower-triangular* matrices.

Proof of Theorem 4.6. Thanks to Lemma 4.7, the hypotheses of Lemma 4.8 are fulfilled by the matrix whose entries are $a_{m,n} = \widehat{\varphi^n}(m)$. It follows (with the notation of Lemma 4.8) that $\tilde{A}$ is bounded on ℓ^2, which means exactly that T_a is bounded on $H^2(\beta)$. □

4.3. ***Necessary condition.*** The main theorem of this section is the following.

Theorem 4.9. *If the composition operator C_{T_a} is bounded on $H^2(\beta)$ for some $a \in \mathbb{D} \setminus \{0\}$, then β is slowly oscillating.*

Let us give a corollary of this result.

Proposition 4.10. *For the weight β constructed in the proof of Theorem 3.7, no automorphism T_a with $0 < a < 1$ can be bounded.*

Proof. Indeed, it is clear that β is not slowly oscillating, since

$$\frac{\beta_{(k+1)!-1}}{\beta_{(k+1)!}} = \frac{1}{k+1} \xrightarrow[k\to\infty]{} 0. \qquad \square$$

To prove Theorem 4.9, we need estimates on the Taylor coefficients of T_a^n. Actually, the Taylor coefficients of T_a^n are the Fourier coefficients of $x \in \mathbb{R} \mapsto T_a^n(\mathrm{e}^{ix})$, and we shall denote them with the same notation $\widehat{T_a^n}$. Sharp such estimates are given in [Szehr and Zarouf 2020; 2021], and we thank R. Zarouf for interesting information on this subject (see also [Borichev et al. 2024]). Our method, using stationary phase and the van der Corput lemma, is a variant of that used in [Szehr and Zarouf 2020; 2021] and goes back at least to [Girard 1973]. However, we need minorizations of $|\widehat{T_a^n}(m)|$ when m is close to n, and Szehr and Zarouf's estimates show that this quantity oscillates and, for individual a, can be too small for our purpose, so we cannot use them and have to prove an estimate *in mean* for a in some subinterval of $(0, 1)$.

We begin with a standard fact, which we give with its proof for the convenience of the reader.

Lemma 4.11. *Let $a \in (0, 1)$, and let*

$$P_{-a}(x) = \frac{1-a^2}{1+2a\cos x + a^2}$$

be the Poisson kernel at the point $-a$. Then, for all $x \in [-\pi, \pi]$,

$$T_a(\mathrm{e}^{ix}) = \exp[i V_a(x)], \tag{4-5}$$

where

$$V_a(x) = \int_0^x P_{-a}(t)\, dt. \tag{4-6}$$

Proof. For $t \in [-\pi, \pi]$, write

$$\psi(t) := \frac{\mathrm{e}^{it}+a}{1+a\mathrm{e}^{it}} = \exp(iv(t)),$$

with v a real-valued, $\mathcal{C}^1$-function on $[-\pi, \pi]$ such that $v(0) = 0$. This is possible since $|\psi(\mathrm{e}^{it})| = 1$ and $\psi(0) = 1$. Differentiating both sides with respect to t, we get

$$i\mathrm{e}^{it}\frac{1-a^2}{(1+a\mathrm{e}^{it})^2} = iv'(t)\frac{\mathrm{e}^{it}+a}{1+a\mathrm{e}^{it}}.$$

This implies

$$v'(t) = \frac{1-a^2}{|1+a\mathrm{e}^{it}|^2} = P_{-a}(t),$$

and the result follows since $v(0) = 0 = V_a(0)$. □

Let us note that, with V_a the function of Lemma 4.11, the Fourier formulas give, since $\widehat{T_a^n}(m)$ is real or since $nV_a(x) - mx$ is odd,

$$2\pi\widehat{T_a^n}(m) = \int_{-\pi}^{\pi} \exp(i[nV_a(x) - mx])\, dx = 2\,\Re\mathrm{e}\, I_{m,n}, \tag{4-7}$$

where

$$I_{m,n} = \int_0^{\pi} \exp i[nV_a(x) - mx]\, dx. \tag{4-8}$$

Now the main ingredient for proving Theorem 4.9 is the following.

Proposition 4.12. *Let $I := \left[\frac{1}{2}, \frac{2}{3}\right]$. There exist constants $\alpha > 1$, e.g., $\alpha = \frac{5}{4}$, and $\delta \in \left(0, \frac{1}{2}\right)$ such that, for n large enough ($n \geq n_0$), we have*

$$\int_I |\widehat{T_a^n}(m)|^2\, da \geq \frac{\delta}{n} \quad \text{for all } m \in [\alpha^{-1}n, \alpha n]. \tag{4-9}$$

Proof. We will set once and for all

$$q = \frac{m}{n}, \tag{4-10}$$

so that $\alpha^{-1} \leq q \leq \alpha$ where $\alpha = \frac{5}{4}$ (say). We will only consider pairs (a, q) satisfying

$$a \in I = \left[\tfrac{1}{2}, \tfrac{2}{3}\right], \quad q \in J := \left[\tfrac{4}{5}, \tfrac{5}{4}\right]. \tag{4-11}$$

Such pairs will be called *admissible*.

With this notation, we set, for $0 \le x \le \pi$,

$$F_q(x) = V_a(x) - \frac{m}{n}x = \int_0^x P_{-a}(t)\,dt - qx, \tag{4-12}$$

where P_{-a} is the Poisson kernel at $-a$. We have

$$F_q'(x) = \frac{(1-a^2)}{1+2a\cos x + a^2} - q,$$

and the unique (if it exists) critical point $x_q = x_q(a)$ of F_q in $[0, \pi]$ is given by $P_{-a}(x_q) = q$, that is,

$$\cos x_q = \frac{1}{q}\frac{1-a^2}{2a} - \frac{1+a^2}{2a} =: h_q(a). \tag{4-13}$$

We now proceed through a series of simple lemmas and begin by estimates on h_q and x_q.

Lemma 4.13. *There are positive constants $C > 1$ and $\delta \in \left(0, \frac{1}{2}\right)$ such that, for every admissible pair (a, q), we have*

$$|h_q(a)| \le 1 - \delta \quad \textit{and} \quad |h_q'(a)| \le C, \tag{4-14}$$

so there is one critical point $x_q(a)$ satisfying

$$\delta \le x_q(a) \le \pi - \delta \quad \textit{and} \quad \sin x_q(a) \ge \delta; \tag{4-15}$$

moreover,

$$|x_q'(a)| \le C \quad \textit{and} \quad \delta \le |P_{-a}'(x_q)| \le C. \tag{4-16}$$

Proof. We have

$$h_q(a) = \left(\frac{1}{q}\frac{1-a^2}{2a}\right) + \left(-\frac{1+a^2}{2a}\right) =: u(a) + v(a),$$

with u and v respectively decreasing and increasing on $[0, 1]$ and with $v \le 0$, so that we have, for $q \in J$,

$$h_q(a) \le u\left(\frac{1}{2}\right) = \frac{3}{4q} \le \frac{15}{16}.$$

Similarly:

$$h_q(a) \ge u\left(\frac{2}{3}\right) + v\left(\frac{1}{2}\right) = \frac{5}{12q} - \frac{5}{4} \ge \frac{1}{3} - \frac{5}{4} = -\frac{11}{12}.$$

Next, $2h_q'(a) = (1-1/q)1/a^2 - (1+1/q)$; hence $|h_q'(a)| \le C$. So, writing $x_q = x_q(a) = \arccos h_q(a)$, we get, with another constant $C > 0$,

$$|x_q'(a)| = \frac{|h_q'(a)|}{\sqrt{1-h_q(a)^2}} \le C$$

since $h_q(a)^2 \le 1-\delta$. Finally, $\frac{1}{9} \le (1-a)^2 \le 1 + 2a\cos x_q + a^2 \le 4$, and since

$$P_{-a}'(x_q) = \frac{2a(1-a^2)\sin x_q}{(1+2a\cos x_q + a^2)^2},$$

we get the final estimates, ending the proof. □

Back to Proposition 4.12.

We saw in (4-7) that the value of $a_{m,n} := \widehat{T_a^n}(m)$ is given by the formula

$$a_{m,n} = \frac{1}{\pi} \,\mathfrak{Re}\, I_{m,n}. \tag{4-17}$$

We have the following estimate, whose proof is postponed (recall that $q = m/n$ and $x_q = x_q(a)$).

Proposition 4.14. *We have*

$$I_{m,n} = \sqrt{2\pi} n^{-1/2} \frac{e^{i[nF_q(x_q)+\pi/4]}}{\sqrt{|P'_{-a}(x_q)|}} + O(n^{-3/5}), \tag{4-18}$$

where the O only depends on a and so is absolute as long as (a, q) is admissible.

Note that $\frac{3}{5} > \frac{1}{2}$. We hence have

$$a_{m,n} = \sqrt{\frac{2}{\pi}} n^{-1/2} \frac{\cos[\pi/4 + nF_q(x_q)]}{\sqrt{|P'_{-a}(x_q)|}} + O(n^{-3/5}). \tag{4-19}$$

It will be convenient to introduce $\varphi_q(a)$, which we do by setting

$$F_q(x_q(a)) = \varphi_q(a). \tag{4-20}$$

Then, since $\frac{1}{2} + \frac{3}{5} = \frac{11}{10}$, $\cos^2(\pi/4 + x) = \frac{1}{2}(1 - \sin 2x)$ and $|P'_{-a}(x_q)| \geq \delta$ by Lemma 4.13, we have

$$a^2_{m,n} = \frac{1}{\pi} n^{-1} \frac{1 - \sin[2n\varphi_q(a)]}{|P'_{-a}(x_q)|} + O(n^{-11/10})$$

implying, since $|P'_{-a}(x_q)| \leq C$ by Lemma 4.13 (again for (a, q) admissible) and changing δ,

$$a^2_{m,n} \geq \delta n^{-1}(1 - \sin[2n\varphi_q(a)]) + O(n^{-11/10}). \tag{4-21}$$

We will also need estimates on the derivatives of $\varphi_q(a)$.

Lemma 4.15. *If (a, q) is admissible, then φ_q decreases on I and, moreover,*

(1) $|\varphi'_q(a)| \geq \delta$,

(2) $|\varphi''_q(a)| \leq C$.

Proof. Note, in passing, that, with $x = x_q(a) \in [0, \pi]$ (thanks to (4-12)),

$$\varphi_q(a) = \int_0^x [P_{-a}(t) - P_{-a}(x)]\, dt \leq 0$$

since the integrand is negative. Next, if f and g are real C^1-functions and

$$\Phi(a) = \int_0^{f(a)} g(a, t)\, dt,$$

the chain rule gives

$$\Phi'(a) = f'(a) g(a, f(a)) + \int_0^{f(a)} \frac{\partial g}{\partial a}(a, t)\, dt.$$

With $g(a, t) = P_{-a}(t)$ and $f(a) = x_q(a)$, we get, remembering that $x_q(a)$ is critical for F_q,

$$\varphi'_q(a) = [P_{-a}(x_q(a)) - q] x'_q(a) + \int_0^{x_q(a)} \frac{\partial P_{-a}}{\partial a}(a, t)\, dt = \int_0^{x_q(a)} \frac{\partial P_{-a}}{\partial a}(a, t)\, dt.$$

But $P_{-a}(t) = 1 + 2\sum_{k=1}^{\infty}(-a)^k \cos kt$, so we have

$$\varphi_q'(a) = \int_0^{x_q(a)} \Big(-2\sum_{k=1}^{\infty} k(-a)^{k-1}\cos kt\Big)\,dt = \frac{2}{a}\sum_{k=0}^{\infty}(-a)^k \sin[kx_q(a)],$$

that is,

$$\varphi_q'(a) = \frac{2}{a}\,\mathfrak{Im}\,\frac{1}{1+a\mathrm{e}^{ix_q(a)}} = \frac{-2\sin x_q(a)}{1+2a\cos x_q(a)+a^2} < 0. \tag{4-22}$$

Now, (4-15) gives (1).

Since $|x_q'(a)| \le C$ by Lemma 4.13, the chain rule and (4-22) clearly give the uniform boundedness of $|\varphi_q''(a)|$ when (a, q) is admissible, and this ends the proof. □

Lemmas 4.13 and 4.15 will now be exploited through a simple variant of the van der Corput inequalities.

Lemma 4.16. *Let $f\colon [A, B] \to \mathbb{R}$, with $A < B$, be a $\mathcal{C}^2$-function satisfying $|f'| \ge \delta$ and $|f''| \le C$, and let us put $M = \int_A^B \mathrm{e}^{inf(x)}\,dx$. Then*

$$|M| \le \frac{2}{n\delta} + \frac{C(B-A)}{n\delta^2}.$$

Proof. Write

$$\mathrm{e}^{inf} = \frac{(\mathrm{e}^{inf})'}{inf'}$$

and integrate by parts to get

$$M = \left[\frac{\mathrm{e}^{inf}}{inf'}\right]_A^B - \frac{i}{n}\int_A^B \mathrm{e}^{inf(x)}\frac{f''(x)}{[f'(x)]^2}\,dx =: M_1 + M_2,$$

with $|M_1| \le 2/(n\delta)$ and $|M_2| \le ((B-A)/n)\cdot C/\delta^2$. □

End of proof of Proposition 4.12. The preceding lemma can be applied with $A = \frac{1}{2}$, $B = \frac{2}{3}$, $f = \varphi_q$ and n changed into $2n$, since Lemma 4.15 shows that this f meets the assumptions of Lemma 4.16. This gives us, uniformly with respect to (a, q) admissible,

$$\left|\int_I \sin[2n\varphi_q(a)]\,da\right| \le \left|\int_I \mathrm{e}^{2in\varphi_q(a)}\,da\right| \le \frac{C}{n}. \tag{4-23}$$

Now, integrating (4-21) on I and using (4-23) gives, for some numerical $\delta \in \big(0, \frac{1}{2}\big)$,

$$\int_I |\widehat{T_a^n}(m)|^2\,da \ge \delta n^{-1} + O(n^{-2}) + O(n^{-11/10}) \ge \tfrac{1}{2}\delta n^{-1}$$

for $n \ge n_0$ and $\alpha^{-1} \le m/n \le \alpha$ (recall that $a_{m,n} = \widehat{T_a^n}(m)$). This ends the proof of Proposition 4.12. □

Proof of Theorem 4.9. By Proposition 4.2, C_{T_a} is bounded for all $a \in \mathbb{D}$, and, thanks to Lemma 4.4,

$$K := \sup_{1/2\le a\le 2/3} \|C_{T_a}\| < +\infty.$$

Matricially, this can be written, for all $a \in \big(\frac{1}{2}, \frac{2}{3}\big)$,

$$\left\|\left(\widehat{T_a^n}(m)\sqrt{\frac{\beta_m}{\beta_n}}\right)_{m,n}\right\| \le K.$$

In particular, for every $n \geq 1$, we have, considering the columns and rows of the previous matrix,

$$\sum_{m=1}^{\infty} |\widehat{T_a^n}(m)|^2 \frac{\beta_m}{\beta_n} \leq K^2, \quad \text{i.e.,} \quad \sum_{m=1}^{\infty} |\widehat{T_a^n}(m)|^2 \beta_m \leq K^2 \beta_n,$$

and, for every $m \geq 1$,

$$\sum_{n=1}^{\infty} |\widehat{T_a^n}(m)|^2 \frac{\beta_m}{\beta_n} \leq K^2, \quad \text{i.e.,} \quad \sum_{n=1}^{\infty} |\widehat{T_a^n}(m)|^2 \frac{1}{\beta_n} \leq \frac{K^2}{\beta_m}.$$

In particular, for every $n \geq 1$,

$$\sum_{(4/5)n \leq j \leq (5/4)n} |\widehat{T_a^n}(j)|^2 \beta_j \leq K^2 \beta_n \tag{4-24}$$

and, for every $m \geq 1$,

$$\sum_{(4/5)m \leq k \leq (5/4)m} |\widehat{T_a^k}(m)|^2 \frac{1}{\beta_k} \leq \frac{K^2}{\beta_m}. \tag{4-25}$$

Integrating on $a \in \left(\frac{1}{2}, \frac{2}{3}\right)$ and using Proposition 4.12, we get, from (4-24), for n large enough,

$$\frac{\delta}{n} \sum_{(4/5)n \leq j \leq (5/4)n} \beta_j \leq \frac{K^2}{6} \beta_n \tag{4-26}$$

and, from (4-25), for m large enough, we have both

$$\frac{\delta}{m} \sum_{(4/5)m \leq k \leq m} \frac{1}{\beta_k} \leq \frac{5K^2}{24} \frac{1}{\beta_m} \tag{4-27}$$

and

$$\frac{\delta}{m} \sum_{m \leq k \leq (5/4)m} \frac{1}{\beta_k} \leq \frac{5K^2}{24} \frac{1}{\beta_m}. \tag{4-28}$$

Since the harmonic mean (over the sets of integers $\left[\frac{4}{5}m, m\right]$ and $\left[m, \frac{5}{4}m\right]$, which have cardinality $\approx n \approx m$) is less than the arithmetical mean, we obtain, from (4-27) and (4-28), both

$$\beta_m \leq \frac{125}{24\delta} \frac{K^2}{m} \sum_{(4/5)m \leq k \leq m} \beta_k \tag{4-29}$$

and

$$\beta_m \leq \frac{10}{3\delta} \frac{K^2}{m} \sum_{m \leq k \leq (5/4)m} \beta_k. \tag{4-30}$$

Now assume that $n \leq m \leq \frac{5}{4}n$. From (4-29), we have

$$\beta_m \lesssim \frac{1}{m} \sum_{(4/5)m \leq k \leq m} \beta_k \lesssim \frac{1}{n} \sum_{(4/5)n \leq k \leq (5/4)n} \beta_k \lesssim \beta_n$$

thanks to (4-26). From (4-30) and (4-26), we treat the case $\frac{4}{5}n \leq m \leq n$ in the same way. We conclude that, for some constant $c > 0$, we have, for n and m large enough satisfying $\frac{4}{5}n \leq m \leq \frac{5}{4}n$,

$$\beta_m \leq c\beta_n, \tag{4-31}$$

which means that β is slowly oscillating. □

Proof of Proposition 4.14. We will use a variant of [Titchmarsh 1986, Lemma 4.6, p. 72] on the van der Corput's version of the stationary phase method. A careful reading of the proof in [Titchmarsh 1986, p. 72] gives the version below, which only needs local estimates on the second derivative F'', as occurs in our situation. For the sake of completeness, we will give a proof, postponed to the Appendix.

Proposition 4.17 (stationary phase). *Let F be a real function with continuous derivatives up to the third order on the interval $[A, B]$ and $F'' > 0$ throughout $]A, B[$. Assume that there is a (unique) point c in $]A, B[$ such that $F'(c) = 0$ and that, for some positive numbers λ_2, λ_3, and η, the following assertions hold*:

(1) $[c-\eta, c+\eta] \subseteq [A, B]$,

(2) $F''(x) \geq \lambda_2$ *for all* $x \in [c-\eta, c+\eta]$,

(3) $|F'''(x)| \leq \lambda_3$ *for all* $x \in [A, B]$.

Then

$$\int_A^B e^{iF(x)}\,dx = \sqrt{2\pi}\,\frac{e^{i[F(c)+\pi/4]}}{F''(c)^{1/2}} + O\Big(\frac{1}{\eta\lambda_2} + \eta^4\lambda_3\Big), \tag{4-32}$$

where the O involves an absolute constant.

We will show that Proposition 4.17 is applicable with $F = nF_q$ and

$$[A, B] = [0, \pi], \quad c = x_q, \quad \lambda_2 = \kappa_0 n, \quad \lambda_3 = C_0 n, \quad \eta = (\lambda_2\lambda_3)^{-1/5}.$$

The parameter η is chosen to make both error terms in Proposition 4.17 equal: $(\eta\lambda_2)^{-1} = \eta^4\lambda_3$; so

$$\eta = \kappa n^{-2/5}$$

and

$$\frac{1}{\eta\lambda_2} + \eta^4\lambda_3 = \tilde{\kappa} n^{-3/5} = O(n^{-3/5}) \tag{4-33}$$

(with $\kappa = (\kappa_0 C_0)^{-1/5}$ and $\tilde{\kappa} = 2/\kappa_0\kappa$).

The slight technical difficulty encountered here is that $F_q''(x)$ vanishes at 0 and π. But Proposition 4.17 covers this case. We have

$$F''(x) = nF_q''(x) = nP'_{-a}(x) = 2a(1-a^2)\frac{\sin x}{(1+2a\cos x + a^2)^2}n,$$

and there are some positive (and absolute) constants κ_0 and σ such that

$$F''(x) \geq \kappa_0 n = \lambda_2 \quad \text{for } x \in [\sigma, \pi-\sigma]. \tag{4-34}$$

Now (for n large enough), we have $[x_q - \eta, x_q + \eta] \subseteq [\sigma, \pi - \sigma]$. Hence assumptions (1) and (2) of Proposition 4.17 are satisfied.

Finally, since $F(x) = nF_q(x) = n[V_a(x) - qx]$ and $F''' = nF_q''' = nV_a''' = nP''_{-a}$, we have, for all $x \in [0, \pi]$ and (a, q) admissible,

$$|F'''(x)| \leq C_0 n = \lambda_3,$$

where C_0 is absolute and assertion (3) of Proposition 4.17 holds.

With (4-33) this ends the proof of (4-18), once we note that $nV_a''(x_q) = F''(x_q)$. □

5. Boundedness of all composition operators

In this section, we characterize all the sequences β for which all composition operators are bounded on $H^2(\beta)$. The main remaining step is the following theorem.

Theorem 5.1. *Assume that all composition operators C_φ are bounded on $H^2(\beta)$. Then β is essentially decreasing.*

As an immediate consequence, we obtain Theorem 1.2.

Proof of Theorem 1.2. Assume that β is essentially decreasing and slowly oscillating. All composition operators C_ψ with $\psi(0) = 0$ are bounded on $H^2(\beta)$ (see the introduction or Proposition 3.2). Since β is slowly oscillating, all the composition operators C_{T_a}, with $a \in \mathbb{D}$, are bounded thanks to Theorem 4.6. Now it is very classical that we can get the boundedness of every composition operators. Indeed given a symbol φ, the symbol $\psi = T_a \circ \varphi$ fixes the origin for $a = -\varphi(0)$. Since $C_\varphi = C_\psi \circ C_{T_{-a}}$, the conclusion follows.

Assume that all composition operators are bounded on $H^2(\beta)$; in particular, the C_{T_a} ones are bounded on $H^2(\beta)$, and β is slowly oscillating, thanks to Theorem 4.9. It also follows from Theorem 5.1 that β is essentially decreasing. □

We will use the following elementary, but crucial, lemma.

Lemma 5.2. *Let u be a function analytic in an open neighborhood Ω of $\overline{\mathbb{D}}$. Then, for every $\varepsilon > 0$, there exists an integer $N \geq 1$ such that*

$$\sum_{j=Np}^{\infty} |\widehat{u^p}(j)|^2 \leq \varepsilon \quad \textit{for all } p \geq 1. \tag{5-1}$$

Proof. From Lemma 4.7, we know that there exist some integer $\lambda > 1$ and a constant $b > 0$ such that $|\widehat{u^p}(j)| \leq \mathrm{e}^{-bj}$ when $j \geq \lambda p$. Therefore, for any $N \geq \lambda$, we have

$$\sum_{j=Np}^{\infty} |\widehat{u^p}(j)|^2 \leq (1-\mathrm{e}^{-2b})^{-1}\mathrm{e}^{-2bNp} \leq (1-\mathrm{e}^{-2b})^{-1}\mathrm{e}^{-2bN} \leq \varepsilon$$

as soon as N is chosen large enough. □

Proof of Theorem 5.1. Thanks to Theorem 4.9, we know that β is slowly oscillating.

Now, assume that the sequence β is not essentially decreasing.

We are going to construct an analytic function $\varphi\colon \mathbb{D} \to \mathbb{D}$ such that the composition operator C_φ is not bounded on $H^2(\beta)$. This function φ will be a Blaschke product of the form

$$\varphi(z) = \prod_{k=1}^{\infty} T_{a_k}(z^{n_k}) = \prod_{k=1}^{\infty} \frac{z^{n_k} + a_k}{1 + a_k z^{n_k}}$$

for a sequence of numbers $a_k \in (0, 1)$ such that $\sum_{k\geq 1}(1-a_k) < +\infty$ and a sequence of positive integers n_k increasing to infinity.

Observe that φ will be indeed a convergent Blaschke product, with n_k zeroes of modulus a_k^{1/n_k}, $k = 1, 2, \dots$, because, for $T_a(z) = (z+a)/(1+az)$, with $0 < a < 1$, we have

$$|T_{a_k}(z^{n_k}) - 1| \leq \frac{2(1-a_k)}{1-|z|}$$

and, setting $a_k = \mathrm{e}^{-\varepsilon_k}$, we get

$$\sum_k n_k(1 - a_k^{1/n_k}) \leq \sum_k n_k(\varepsilon_k/n_k) = \sum_k \varepsilon_k < +\infty.$$

These sequences will be constructed by induction, together with another sequence of integers $(m_k)_{k\geq 1}$.

Since β is not essentially decreasing, there exist integers $n_1 > m_1 \geq 4$ such that $\beta_{n_1} \geq 2\beta_{m_1}$. We start with

$$a_1 = 1 - \frac{1}{m_1} \geq \frac{3}{4}.$$

Using Lemma 5.2 with $u = T_{a_1}$, we get $N_0 \geq 1$ such that

$$\sum_{j=N_0 m}^{\infty} |\widehat{T_{a_1}^m}(j)|^2 \leq 2^{-15} \quad \text{for all } m \geq 1.$$

Assume now that we have constructed increasing sequences of integers

$$m_1, m_2, \dots, m_k, \quad n_1, n_2, \dots, n_k, \quad N_0, N_1, \dots, N_{k-1}$$

such that, for $1 \leq l \leq k-1$, we have

$$m_{l+1} \geq 4m_l \quad \text{and} \quad n_{l+1} \geq 4n_l$$

and, for $1 \leq l \leq k$,

$$n_l \geq N_{l-1} m_l \quad \text{and} \quad \beta_{n_l} \geq 2^l \beta_{m_l}$$

and

$$\sum_{j=N_{l-1} m_l}^{\infty} |\widehat{\varphi_l^m}(j)|^2 \leq 2^{-15},$$

where

$$a_l = 1 - \frac{1}{m_l} \quad \text{and} \quad \varphi_l(z) = T_{a_l}(z^{n_l}).$$

We then apply Lemma 5.2 again to the function $u = u_k = \varphi_1 \cdots \varphi_k$. We get $N_k > N_{k-1}$ such that

$$\sum_{j=N_k m}^{\infty} |\widehat{u_k^m}(j)|^2 \leq 2^{-15} \quad \text{for all } m \geq 1. \tag{5-2}$$

Since β is not essentially decreasing but is slowly oscillating, there exist $m_{k+1} \geq 4m_k$ and $n_{k+1} \geq 4n_k$ such that

$$n_{k+1} \geq N_k m_{k+1} \quad \text{and} \quad \beta_{n_{k+1}} \geq 2^{k+1} \beta_{m_{k+1}}.$$

We set

$$a_{k+1} = 1 - \frac{1}{m_{k+1}} \quad \text{and} \quad \varphi_{k+1}(z) = T_{a_{k+1}}(z^{n_{k+1}}).$$

This ends the induction.

It remains to check that

$$\sum_{k=1}^{\infty}(1-a_k)=\sum_{k=1}^{\infty}\frac{1}{m_k}\le\sum_{k=1}^{\infty}4^{-k}=\frac{1}{3}<+\infty$$

to get that the infinite product $\varphi=\prod_{k\ge 1}\varphi_k$ converges uniformly on compact subsets of $\mathbb{D}$.

To show that the composition operator C_φ is not bounded on $H^2(\beta)$, it suffices to show that, for some constant $c_1>0$, we have, for all $k\ge 2$,

$$\sum_{j=n_k}^{2n_k}|\widehat{\varphi^{m_k}}(j)|^2\ge c_1. \tag{5-3}$$

Indeed, since β is slowly oscillating, there is a positive constant $\delta<1$ such that

$$\beta_j\ge\delta\beta_{n_k}\quad\text{for } j=n_k,n_k+1,\dots,2n_k.$$

Then, if we set $e_k(z)=z^{m_k}$, we have, since $C_\varphi(e_k)=\varphi^{m_k}$,

$$\frac{\|C_\varphi(e_k)\|^2_{H^2(\beta)}}{\|e_k\|^2_{H^2(\beta)}}\ge\frac{\sum_{j=n_k}^{2n_k}|\widehat{\varphi^{m_k}}(j)|^2\beta_j}{\beta_{m_k}}\ge\frac{c_1\delta\beta_{n_k}}{\beta_{m_k}}\ge 2^k c_1\delta\xrightarrow[k\to\infty]{}+\infty,$$

and so C_φ is not bounded on $H^2(\beta)$.

We now have to show (5-3). Let us agree to write formally, for an analytic function $f(z)=\sum_{k=0}^{\infty}f_kz^k$ and an arbitrary positive integer p,

$$f(z)=\sum_{k=0}^{p}f_kz^k+O(z^{p+1}).$$

For that, we set

$$G_k(z)=\prod_{l=k+1}^{\infty}\varphi_l(z)=\prod_{l=k+1}^{\infty}a_l+O(z^{n_{k+1}}).$$

We have, for $k\ge 2$,

$$\varphi(z)=v_k(z)\varphi_k(z)G_k(z),$$

where $v_k=\varphi_1\cdots\varphi_{k-1}$.

Remark now that, for $0<a<1$, we have

$$T_a(z)=a+(1-a^2)z+O(z^2),$$

so

$$\varphi_k(z)=T_{a_k}(z^{n_k})=a_k+(1-a_k^2)z^{n_k}+O(z^{2n_k}).$$

Then

$$[G_k(z)]^{m_k}=\Bigl(\prod_{l=k+1}^{\infty}a_l\Bigr)^{m_k}+O(z^{n_{k+1}}) \tag{5-4}$$

and

$$[\varphi_k(z)]^{m_k}=a_k^{m_k}+(1-a_k^2)m_ka_k^{m_k-1}z^{n_k}+O(z^{2n_k}). \tag{5-5}$$

But

$$a_k^{m_k-1} = \Big(1 - \frac{1}{m_k}\Big)^{m_k-1} \geq \mathrm{e}^{-1} := c_2 \tag{5-6}$$

and

$$(1-a_k^2) m_k a_k^{m_k-1} \geq (1-a_k) m_k a_k^{m_k-1} \geq c_2. \tag{5-7}$$

Moreover, since $1 - x \geq \mathrm{e}^{-2x}$ for $0 \leq x \leq \frac{1}{2}$, we have

$$\Big(\prod_{l=k+1}^{\infty} a_l\Big)^{m_k} \geq \exp\Big(-2\Big(\sum_{l=k+1}^{\infty} \frac{1}{m_l}\Big) m_k\Big) \geq \exp\Big(-2\sum_{l=1}^{\infty} 4^{-l}\Big) = \exp\big(-\tfrac{2}{3}\big) := c_3. \tag{5-8}$$

Afterwards, by (5-2), we have

$$\sum_{j=N_{k-1}m_k}^{\infty} |\widehat{v_k^{m_k}}(j)|^2 \leq 2^{-15}. \tag{5-9}$$

Set $v_k^{m_k} = g_1 + g_2$, with

$$\begin{cases} g_1(z) = \sum_{j=0}^{N_{k-1}m_k} \widehat{v_k^{m_k}}(j) z^j, \\ g_2(z) = \sum_{j>N_{k-1}m_k} \widehat{v_k^{m_k}}(j) z^j. \end{cases}$$

By (5-9), we have, with $\|\cdot\|_2 = \|\cdot\|_{L^2(\mathbb{T})}$,

$$\|g_2\|_2^2 = \sum_{j>N_{k-1}m_k} |\widehat{v_k^{m_k}}(j)|^2 \leq 2^{-15}.$$

Besides, since φ_k is inner as a product of inner functions, we have $|v_k(z)| = 1$ for all $z \in \mathbb{T}$, so

$$\|g_1\|_2^2 = \|v_k\|_2^2 - \|g_2\|_2^2 \geq 1 - 2^{-15}.$$

Now, $\varphi^{m_k} = v_k^{m_k} \varphi_k^{m_k} G_k^{m_k} = F_1 + F_2$, with

$$F_1 = g_1 \varphi_k^{m_k} G_k^{m_k} \quad \text{and} \quad F_2 = g_2 \varphi_k^{m_k} G_k^{m_k}.$$

Using (5-4), (5-5), (5-7) and (5-8), we get

$$\sum_{j=n_k}^{n_k+N_{k-1}m_k} |\widehat{F_1}(j)|^2 = \Big(\prod_{l=k+1}^{\infty} a_l\Big)^{2m_k} [(1-a_k^2) m_k a_k^{m_k-1}]^2 \sum_{j=0}^{N_{k-1}m_k} |\widehat{g_1}(j)|^2 \geq (1-2^{-15}) c_2^2 c_3^2.$$

As

$$\|F_2\|_2^2 \leq \|g_2\|_2^2 \|\varphi_k^{m_k}\|_\infty^2 \|G_k^{m_k}\|_\infty^2 \leq 2^{-15},$$

we get, using the inequality $|a+b|^2 \geq \frac{1}{2}|a|^2 - |b|^2$,

$$\begin{aligned} \sum_{j=n_k}^{2n_k} |\widehat{\varphi^{m_k}}(j)|^2 &\geq \sum_{j=n_k}^{n_k+N_{k-1}m_k} |\widehat{F_1}(j) + \widehat{F_2}(j)|^2 \geq \frac{1}{2} \sum_{j=n_k}^{n_k+N_{k-1}m_k} |\widehat{F_1}(j)|^2 - \sum_{j=n_k}^{n_k+N_{k-1}m_k} |\widehat{F_2}(j)|^2 \\ &\geq \frac{1}{2}(1-2^{-15}) c_2^2 c_3^2 - 2^{-15} = \frac{1}{2}(1-2^{-15}) \mathrm{e}^{-10/3} - 2^{-15} \geq 2^{-9} - 2^{-15} > 0. \end{aligned}$$

$\square$

6. Some results on multipliers

In this section, we give some results on the multipliers on $H^2(\beta)$, which show how the different notions of regularity for β come into play.

The set $\mathcal{M}(H^2(\beta))$ of multipliers of $H^2(\beta)$ is by definition the vector space of functions h analytic on $\mathbb{D}$ such that $hf \in H^2(\beta)$ for all $f \in H^2(\beta)$. When $h \in \mathcal{M}(H^2(\beta))$, the operator M_h of multiplication by h is bounded on $H^2(\beta)$ by the closed graph theorem. The space $\mathcal{M}(H^2(\beta))$ equipped with the operator norm is a Banach space. We note the obvious property

$$\mathcal{M}(H^2(\beta)) \hookrightarrow H^\infty \text{ contractively.} \tag{6-1}$$

Indeed, if $h \in \mathcal{M}(H^2(\beta))$, we easily get, for all $w \in \mathbb{D}$,

$$M_h^*(K_w) = \overline{h(w)} K_w,$$

and so by taking norms and simplifying, we are left with $|h(w)| \leq \|M_h\|$, showing that $h \in H^\infty$ with $\|h\|_\infty \leq \|M_h\|$.

Proposition 6.1. *We have $\mathcal{M}(H^2(\beta)) = H^\infty$ isomorphically if and only if β is essentially decreasing.*

Proof. The sufficient condition is proved in [Lefèvre et al. 2021, beginning of the proof of Proposition 3.16]. For the necessity, we then have $\|M_h\| \approx \|h\|_\infty$ for every $h \in H^\infty$ by the Banach isomorphism theorem. Now, for $m > n$ (recall that $e_n(z) = z^n$),

$$e_m(z) = z^{m-n} z^n = (M_{e_{m-n}} e_n)(z);$$

so, since $\|M_{e_{m-n}}\| \leq C \|e_{m-n}\|_\infty = C$ for some positive constant C,

$$\beta_m = \|e_m\|^2 \leq C^2 \|e_n\|^2 = C^2 \beta_n. \qquad \square$$

In [Lefèvre et al. 2021, Section 3.6], we gave the following notion of an *admissible* Hilbert space of analytic functions.

Definition 6.2. A Hilbert space H of analytic functions on $\mathbb{D}$, containing the constants, and with reproducing kernels K_a, $a \in \mathbb{D}$, is said to be *admissible* if

(i) H^2 is continuously embedded in H,

(ii) $\mathcal{M}(H) = H^\infty$,

(iii) the automorphisms of $\mathbb{D}$ induce bounded composition operators on H,

(iv) $\dfrac{\|K_a\|_H}{\|K_b\|_H} \leq h\Big(\dfrac{1-|b|}{1-|a|}\Big)$ for $a, b \in \mathbb{D}$, where $h \colon \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing function.

We proved in that paper that every weighted Hilbert space $H^2(\beta)$ with β nonincreasing is admissible under the additional hypothesis that the automorphisms of $\mathbb{D}$ induce bounded composition operators. In view of Theorem 4.6, we get the following result.

Proposition 6.3. *Let β be a weight.*

(1) *If β is essentially decreasing, then we have* (i), (ii), (iii) *in Definition 6.2.*

(2) *If β is slowly oscillating, then we have* (iv) *in Definition 6.2.*

Let us give a different proof from the one in [Lefèvre et al. 2021].

Proof. (1) Let us assume that β is essentially decreasing. Then item (i) holds, as well as item (ii), by Proposition 6.1. Item (iii) is Theorem 4.6.

(2) Now we assume that β is slowly oscillating.

Let $0 < s < r < 1$.

Without loss of generality, we may assume that $r, s \geq \frac{1}{2}$. It is enough to prove

$$\|K_r\|^2 \leq C \|K_{r^2}\|^2 \tag{6-2}$$

for some constant $C > 1$. Indeed, iteration of (6-2) gives

$$\|K_r\|^2 \leq C^k \|K_{r^{2^k}}\|^2,$$

and if k is the smallest integer such that $r^{2^k} \leq s$, we have

$$2^{k-1} \log r > \log s$$

and

$$2^k \leq D \frac{1-s}{1-r},$$

where D is a numerical constant. Writing $C = 2^\alpha$ with $\alpha > 1$, we obtain

$$\left(\frac{\|K_r\|}{\|K_s\|}\right)^2 \leq C^k = (2^k)^\alpha \leq D^\alpha \left(\frac{1-s}{1-r}\right)^\alpha.$$

To prove (6-2), we pick some $M > 1$ such that

$$\beta_{2n} \geq M^{-1} \beta_n$$

and

$$\beta_{2n-1} \geq M^{-1} \beta_n$$

for all $n \geq 1$, since β is slowly oscillating. Write $t = r^2$. We have

$$\|K_r\|^2 = \frac{1}{\beta_0} + \sum_{n=1}^{\infty} \frac{t^{2n}}{\beta_{2n}} + \sum_{n=1}^{\infty} \frac{t^{2n-1}}{\beta_{2n-1}},$$

implying, since $t^{2n-1} \leq 4t^{2n}$,

$$\|K_r\|^2 \leq \frac{1}{\beta_0} + M \sum_{n=1}^{\infty} \frac{t^{2n}}{\beta_n} + 4M \sum_{n=1}^{\infty} \frac{t^{2n}}{\beta_n} \leq 5M \|K_t\|^2.$$ □

The notion of an admissible Hilbert space H is useful for the set of conditional multipliers:

$$\mathcal{M}(H, \varphi) = \{w \in H : w(f \circ \varphi) \in H \text{ for all } f \in H\}.$$

As a corollary of [Lefèvre et al. 2021, Theorem 3.18], we get the following.

Corollary 6.4. *Let β be essentially decreasing and slowly oscillating. Then*

(1) $\mathcal{M}(H^2, \varphi) \subseteq \mathcal{M}(H^2(\beta), \varphi)$,

(2) $\mathcal{M}(H^2(\beta), \varphi) = H^2(\beta)$ *if and only if* $\|\varphi\|_\infty < 1$,

(3) $\mathcal{M}(H^2(\beta), \varphi) = H^\infty$ *if and only if* φ *is a finite Blaschke product.*

We add here as another application of our results an answer to a question appearing in Problem 5 in the thesis of Zorboska [1988].

Theorem 6.5. *Let β be a weight such that $H^2(\beta)$ is disc-automorphism-invariant, and let φ be a symbol inducing a compact composition operator on $H^2(\beta)$. Then the Denjoy–Wolff point of φ must be in $\mathbb{D}$.*

In other words, φ has a fixed point in $\mathbb{D}$.

In the statement, "$H^2(\beta)$ is disc-automorphism-invariant" means that, for all the automorphisms T_a, where $a \in \mathbb{D}$, we have that C_{T_a} is bounded on $H^2(\beta)$ (equivalently it is bounded for at least one $a \in \mathbb{D} \setminus \{0\}$).

For the definition of the Denjoy–Wolff point, we refer to [Shapiro 1993].

Proof. From Theorem 4.9, we know that β is slowly oscillating, and from Proposition 6.3, we know that

$$\frac{\|K_a\|_{H^2(\beta)}}{\|K_b\|_{H^2(\beta)}} \leq h\left(\frac{1-|b|}{1-|a|}\right) \quad \text{for every } a, b \in \mathbb{D}, \tag{6-3}$$

where $h \colon \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing function.

Now we split the proof into two cases:

- If $\sum 1/\beta_n < \infty$, then $H^2(\beta) \subset A(\mathbb{D})$ (continuously) thanks to the Cauchy–Schwarz inequality. It follows from [Shapiro 1987, Theorem 2.1] that $\|\varphi\|_\infty < 1$, and the conclusion follows obviously.

- If $\sum 1/\beta_n = \infty$, then the normalized reproducing kernel $K_z/\|K_z\|$ is weakly converging to 0 when $|z| \to 1^-$ since $\|K_z\| \to +\infty$.

Since C_φ is compact, C_φ^* is compact as well, and we get

$$\frac{K_{\varphi(z)}}{\|K_z\|} \to 0 \quad \text{when } |z| \to 1^-$$

and equivalently

$$\frac{\|K_z\|}{\|K_{\varphi(z)}\|} \to +\infty \quad \text{when } |z| \to 1^-.$$

But, from (6-3), we get

$$h\left(\frac{1-|\varphi(z)|}{1-|z|}\right) \to +\infty \quad \text{when } |z| \to 1^-;$$

hence, since h is nondecreasing,

$$\frac{1-|\varphi(z)|}{1-|z|} \to +\infty \quad \text{when } |z| \to 1^-.$$

By the Denjoy–Wolff theorem [Shapiro 1993], the conclusion follows in this case too. $\square$

Appendix

In this appendix, we give the proof of Proposition 4.17.

The following lemma can be found in [Montgomery 1994, Lemma 1, p. 47].

Lemma A.1. *Let $F\colon [u, v] \to \mathbb{R}$, with $u < v$, be a $\mathcal{C}^2$-function with $F'' > 0$ and F' not vanishing on $[u, v]$. Let*

$$J = \int_u^v \mathrm{e}^{iF(x)}\, dx.$$

Then:

(a) *if $F' > 0$ on $[u, v]$, we have $|J| \le 2/F'(u)$,*

(b) *if $F' < 0$ on $[u, v]$, we have $|J| \le 2/|F'(v)|$.*

Proof of Proposition 4.17. Write now the integral I of Proposition 4.17 on $[A, B]$ as $I = I_1 + I_2 + I_3$, with

$$I_1 = \int_A^{c-\eta} \mathrm{e}^{iF(x)}\, dx, \quad I_2 = \int_{c-\eta}^{c+\eta} \mathrm{e}^{iF(x)}\, dx, \quad I_3 = \int_{c+\eta}^{B} \mathrm{e}^{iF(x)}\, dx.$$

Lemma A.1 with $u = A$ and $v = c - \eta$ implies, since $F' < 0$ on $[A, c - \eta]$,

$$|I_1| \le \frac{2}{|F'(c-\eta)|} \le \frac{2}{\eta\lambda_2}, \tag{A-1}$$

where, for the last inequality, we just have to write

$$|F'(c-\eta)| = F'(c) - F'(c-\eta) = \eta F''(\xi)$$

for some $\xi \in [c - \eta, c]$ and to note that $F''(\xi) \ge \lambda_2$, by hypothesis.

Similarly, Lemma A.1 with $u = c + \eta$ and $v = B$ implies

$$|I_3| \le \frac{2}{F'(c+\eta)} \le \frac{2}{\eta\lambda_2}. \tag{A-2}$$

We can now estimate I_2. The Taylor formula shows that

$$F(x) = F(c) + \tfrac{1}{2}(x-c)^2 F''(c) + R,$$

with

$$|R| \le \tfrac{1}{6}|x-c|^3\lambda_3.$$

Hence

$$I_2 = \mathrm{e}^{iF(c)} \int_0^\eta 2\exp\bigl(\tfrac{1}{2}ix^2F''(c)\bigr)\, dx + S,$$

with

$$|S| \le \lambda_3 \int_0^\eta \tfrac{1}{3}x^3\, dx = \tfrac{1}{12}\eta^4\lambda_3.$$

Finally, set

$$K = \int_0^\eta 2\exp\bigl(\tfrac{1}{2}ix^2F''(c)\bigr)\, dx.$$

We make the change of variable $x = \sqrt{2/F''(c)}\sqrt{t}$. Recall that $\int_0^\infty e^{it}/\sqrt{t}\,dt = \sqrt{\pi}e^{i\pi/4}$ is the classical Fresnel integral and that an integration by parts gives, for $m > 0$,

$$\left|\int_m^\infty \frac{e^{it}}{\sqrt{t}}\,dt\right| \le \frac{2}{\sqrt{m}}.$$

Therefore, with $m = \frac{1}{2}\eta^2 F''(c)$,

$$K = \sqrt{\frac{2}{F''(c)}}\int_0^m \frac{e^{it}}{\sqrt{t}}\,dt = \sqrt{\frac{2\pi}{F''(c)}}e^{i\pi/4} + R_m,$$

with

$$|R_m| \le C\sqrt{\frac{1}{F''(c)}}\frac{1}{\sqrt{m}} \le \frac{C}{\eta\lambda_2}.$$

All in all, we proved that

$$I_2 = \sqrt{\frac{2\pi}{F''(c)}}\exp[i(F(c)+\pi/4)] + O\Big(\frac{1}{\eta\lambda_2} + \eta^4\lambda_3\Big), \tag{A-3}$$

and the same estimate holds for I, thanks to (A-1) and (A-2).

We have hence proved Proposition 4.17. □

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