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RECONSTRUCTION FOR THE CALDERÓN PROBLEM WITH LIPSCHITZ CONDUCTIVITIES

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We determine the conductivity of the interior of a body using electrical measurements on its surface. We assume only that the conductivity is bounded below by a positive constant and that the conductivity and surface are Lipschitz continuous. To determine the conductivity we first solve an associated integral equation in a ball B that properly contains the body, finding solutions in $H^1(B)$. A key ingredient is to equip this Sobolev space with an equivalent norm which depends on two auxiliary parameters that can be chosen to yield a contraction.

1. Introduction

We consider the conductivity equation in a bounded domain $\Omega \subset \mathbb{R}^n$ and place electric potentials $\phi \in H^{1/2}(\partial\Omega)$ on the Lipschitz boundary $\partial\Omega$:

$$\begin{cases} \nabla \cdot (\sigma \nabla u) = 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = \phi. \end{cases} \tag{1}$$

Throughout the article, the conductivity σ is assumed to be bounded above and below by positive constants, so that (1) has a unique weak solution u in the L^2-Sobolev space $H^1(\Omega)$. The Dirichlet-to-Neumann map Λ_σ can then be formally defined by

$$\Lambda_\sigma : \phi \mapsto \sigma \, \partial_\nu u|_{\partial\Omega}, \tag{2}$$

where ν denotes the outward unit normal vector to $\partial\Omega$. This provides us with the steady-state perpendicular currents induced by the electric potentials ϕ.

Motivated by the possibility of creating an image of the interior of a body from these noninvasive voltage-to-current measurements on its surface, Calderón [2006] asked whether the conductivity σ is uniquely determined by Λ_σ and, if so, whether σ can be calculated from Λ_σ. In two dimensions, Astala and Päivärinta answered the uniqueness part [2006b] and also provided a reconstruction algorithm [2006a]. The two-dimensional problem has distinct mathematical characteristics, so from now on we consider only $n \geqslant 3$.

With $n \geqslant 3$, it has so far been necessary to make additional regularity assumptions. Kohn and Vogelius [1984] proved uniqueness for real-analytic conductivities, and Sylvester and Uhlmann [1987] improved this to smooth conductivities. Nachman, Sylvester and Uhlmann [Nachman et al. 1988] then proved uniqueness for twice continuously differentiable conductivities, and Nachman [1988] and Novikov [1988]

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provided reconstruction algorithms. These pioneering articles provoked a great deal of interesting work, including that of Brown [1996], Päivärinta, Panchenko and Uhlmann [Päivärinta et al. 2003] and Brown and Torres [2003] for conductivities with $\frac{3}{2}$ derivatives. In the past decade, a breakthrough was made by Haberman and Tataru [2013], who proved uniqueness for continuously differentiable conductivities or Lipschitz conductivities with $\|\nabla \log \sigma\|_\infty$ sufficiently small. García and Zhang [2016] then provided a reconstruction algorithm under the same assumptions. Two of the authors removed the smallness condition from the uniqueness result in [Caro and Rogers 2016], and the purpose of this article will be to extend this work to a reconstruction algorithm that holds for all Lipschitz conductivities. We will not assume that the conductivity is constant near the boundary, nor will we extend the conductivity in order to achieve this, leading to simpler formulas than those of [García and Zhang 2016]; see Section 3.

Before we outline the proof, we remark that there are also uniqueness results for conductivities in Sobolev spaces; see [Haberman 2015; Ham et al. 2021; Ponce-Vanegas 2021]. In particular, [Haberman 2015] proved that uniqueness holds for bounded conductivities in $W^{1,n}(\overline{\Omega})$, with $n = 3$ or 4. Note that this is a strictly larger class than Lipschitz, however there are obstacles to reconstruction via their methods; see Remark 11.2 for more details. It has been conjectured that Lipschitz continuity is the sharp threshold within the scale of Hölder continuity; see for example [Brown 1996] or [Uhlmann 1998, Open Problem 1].

When σ is Lipschitz, weak solutions to (1) are in fact strong solutions; see for example [Zhang and Bao 2012, Theorem 1.3]. Defining the Dirichlet-to-Neumann map as in (2) by identifying $\sigma \, \partial_\nu u|_{\partial\Omega}$ with the normal trace of $\sigma \nabla u$, we have the divergence identity

$$\int_{\partial\Omega} \Lambda_\sigma[\phi]\psi = \int_\Omega \sigma \nabla u \cdot \nabla \psi$$

whenever $(\phi, \psi) \in H^{1/2}(\partial\Omega) \times H^1(\Omega)$; see for example [Kim and Kwon 2022, Proposition 2.4]. Given this identity, it is possible to describe the heuristic which underlies the reconstruction: For each $\xi \in \mathbb{R}^n$, one hopes to choose an oscillating pair (ϕ, ψ) so that the right-hand side becomes a nonlinear Fourier transform of σ evaluated at ξ. As the left-hand side can be calculated from the measurements, the conductivity might then be recoverable by Fourier inversion. Indeed, much of the literature, including the original work of Calderón [2006], has involved pairs $(e^{\rho \cdot x}, e^{\rho' \cdot x})$, with $\rho, \rho' \in \mathbb{C}^n$ chosen carefully, so that $\rho + \rho'$ is equal to a real constant multiple of $-i\xi$, where $i := \sqrt{-1}$. The hope is that the essentially harmonic u is not so different from $e_\rho := e^{\rho \cdot x}$, and so the complex vector ρ is chosen in such a way that $\rho \cdot \rho = 0$, so that e_ρ is harmonic.

In fact we begin by noting that u is a solution to the conductivity equation if and only if $v = \sigma^{1/2}u$ is a solution to the Schrödinger equation

$$\Delta v = qv \quad \text{in } \Omega, \tag{3}$$

where formally $q = \sigma^{-1/2}\Delta\sigma^{1/2}$. Kohn and Vogelius [1985] observed that if $\sigma|_{\partial\Omega}$ and $\nu \cdot \nabla\sigma|_{\partial\Omega}$ are known, then the Dirichlet-to-Neumann map Λ_q for the Schrödinger equation (3) can be written in terms of Λ_γ, and so the literature has mainly considered the essentially equivalent problem of recovering q from Λ_q (which is intimately connected to inverse scattering at fixed energy). We will only partially use the equivalence however: we will recover q directly from Λ_γ, circumventing the need to calculate

$\nu \cdot \nabla \sigma|_{\partial\Omega}$. This is connected to the fact that our conductivities are not regular enough to define q in a pointwise fashion. However, as noted by Brown [1996], it suffices to define $\langle qv, \psi \rangle := \langle q, v\psi \rangle$ for suitable test functions ψ, with

$$\langle q, \bullet \rangle := -\int_\Omega \nabla \sigma^{1/2} \cdot \nabla(\sigma^{-1/2} \bullet). \tag{4}$$

By the product rule and the Cauchy–Schwarz inequality, $\langle q, \bullet \rangle$ and $\langle qv, \bullet \rangle$ are bounded linear functionals on $H^1(B)$, where B is a ball that properly contains Ω, so in particular we can make sense of q and qv as distributions.

Rather than solving (3) directly, we consider solutions to the Lippmann–Schwinger-type equation

$$v = \Delta^{-1} \circ M_q[v] + e_\rho, \tag{5}$$

where $M_q : f \mapsto qf$ and the inverse of the Laplacian is defined using the Faddeev fundamental solution; see Section 2.1. Integral equations like this are usually solved globally, however we will find a $v \in H^1(B)$ which is a solution of (5) in the ball B. Writing $v = e_\rho(1 + w)$ and additionally requiring that the remainders w vanish in some sense as $|\rho| \to \infty$ gives hope that the nonlinear Fourier transform will converge to the linear Fourier transform in the limit. Solutions of this type were introduced to the problem by Sylvester and Uhlmann [1987] and have since become known as CGO solutions, where CGO stands for complex geometrical optics. Substituting into (3) and multiplying by $e_{-\rho}$, we find that

$$\Delta_\rho w = M_q[1 + w] \quad \text{in } \Omega, \tag{6}$$

where $\Delta_\rho := \Delta + 2\rho \cdot \nabla$. In much of the literature Δ_ρ is inverted using the Fourier transform and the resulting integral equation is solved globally via a contraction for $\Delta_\rho^{-1} \circ M_q$ and Neumann series. In order to reconstruct σ from Λ_σ (as opposed to just proving uniqueness), we must additionally determine which electric potentials should be placed on the boundary in order to generate the CGO solutions. A contraction for $\Delta_\rho^{-1} \circ M_q$ can also be helpful in this step; however, the need for such a contraction was circumvented in the uniqueness result of [Caro and Rogers 2016], instead solving the differential equation (6) via the method of a priori estimates.

Nachman and Street [2010] were able to recover the boundary values of CGO solutions that had been constructed via a priori estimates, however, we were unable to take advantage of their ideas; see Remark 11.1 for more details. Instead we will reprove the existence of CGO solutions, this time via Neumann series; however, we will adopt the previously mentioned intermediate approach of solving the integral equation in the ball B. That is to say, we find a $w \in H^1(B)$ such that

$$(I - \Delta_\rho^{-1} \circ M_q)w = \Delta_\rho^{-1} \circ M_q[1], \tag{7}$$

where the identity holds as elements of $H^1(B)$. This is equivalent to (5) when $\Delta^{-1} \circ M_q$ is defined appropriately; see Remark 9.3.

Most of the article will be occupied by the proof of the contraction for $\Delta_\rho^{-1} \circ M_q$ in Sections 4–9. In Section 4 we give a sketch of its proof before proving the key Carleman estimate in Section 5. In Section 6 we incorporate the associated convex weights into our localised versions of the Haberman–Tataru norms,

so that they not only depend on ρ but also on an auxiliary parameter $\lambda > 1$. The final estimate for Δ_ρ^{-1}, proved in Section 7, is somewhat weaker and easier to prove than the main estimate of [Caro and Rogers 2016], so the present article also simplifies the uniqueness result of that work. In Section 8 we bound M_q with respect to the new norms, and in Section 9 we choose the parameters in order to yield the contraction.

In Section 2 we list some of the main definitions before presenting the reconstruction algorithm in Section 3. The reconstruction formulas will not make mention of the new norms, which are only used in Section 11 to prove the validity of the formulas. In the final Section 12 we suggest some simplifications that could make the algorithm easier to implement.

2. Preliminary notation

We invert our main operator Δ_ρ initially on the space of Schwartz functions $\mathcal{S}(\mathbb{R}^n)$ using the Fourier transform defined by

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} e^{-i\xi \cdot x} f(x) \, dx$$

for all $\xi \in \mathbb{R}^n$ and $f \in \mathcal{S}(\mathbb{R}^n)$. By integration by parts, one can calculate that

$$\widehat{\Delta_\rho f}(\xi) = m_\rho(\xi) \hat{f}(\xi), \quad \text{where } m_\rho(\xi) := -|\xi|^2 + 2i\rho \cdot \xi, \tag{8}$$

for all $\xi \in \mathbb{R}^n$. The reciprocal of this Fourier multiplier is integrable on compact sets, so we can define an inverse by

$$\Delta_\rho^{-1} g(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \frac{1}{m_\rho(\xi)} \hat{g}(\xi) \, d\xi$$

for all $x \in \mathbb{R}^n$ and $g \in \mathcal{S}(\mathbb{R}^n)$.

2.1. *The Faddeev fundamental solutions.* Writing the inverse Fourier transform of the product as a convolution, we find

$$\Delta_\rho^{-1} g(x) = \int_{\mathbb{R}^n} F_\rho(x - y) g(y) \, dy \tag{9}$$

for all $x \in \mathbb{R}^n$ and $g \in \mathcal{S}(\mathbb{R}^n)$, where the fundamental solution F_ρ for Δ_ρ is defined by

$$F_\rho(x) := \lim_{r \to \infty} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \frac{1}{m_\rho(\xi)} \hat{\chi}(\xi/r) \, d\xi.$$

Here $\chi \in \mathcal{S}(\mathbb{R}^n)$ must be positive and satisfy $\hat{\chi}(0) = 1$, but the limit is insensitive to the precise choice of χ and so the integral is often written formally, taking $\hat{\chi} = 1$. This fundamental solution was first considered by Faddeev [1965] in the context of quantum inverse scattering.

We also consider the associated fundamental solution $G_\rho := e_\rho F_\rho$ for the Laplacian, and we will often write $G_\rho(x, y) := G_\rho(x - y)$. This is not so different from the usual potential-theoretic fundamental solution. Indeed, by subtracting one from the other, one obtains a harmonic function which is thus smooth by Weyl's lemma:

$$H_\rho(x) := G_\rho(x) - \frac{c_n}{(2-n)} \frac{1}{|x|^{n-2}}, \tag{10}$$

where c_n denotes the reciprocal of the measure of the unit sphere. For more details regarding the properties of Faddeev's fundamental solutions, see [Newton 1989, Section 6.1].

2.2. *The boundary integral.* For notational compactness we write the reconstruction formulas in terms of the bilinear functional $BI_{\Lambda_\sigma} : H^{1/2}(\partial\Omega) \times H^1(\Omega) \to \mathbb{C}$ defined by

$$BI_{\Lambda_\sigma}(\phi, \psi) := \int_{\partial\Omega} (\sigma^{-1/2}\Lambda_\sigma[\sigma^{-1/2}\phi] - \nu \cdot \nabla P_0[\phi])\psi, \qquad (11)$$

where $P_0[\phi]$ denotes the harmonic extension of ϕ. Brown [2001] calculated $\sigma|_{\partial\Omega}$ from Λ_σ, so the boundary integral BI_{Λ_σ} can be recovered from Λ_σ. In Lemma 10.1 we will prove that

$$BI_{\Lambda_\sigma}(\phi, G_\rho(\mathrm{x}, \bullet)) \in H^1(B \setminus \bar{\Omega}),$$

where B properly contains Ω and $f(\mathrm{x})$ denotes a function that takes the values $f(x)$ for all x in the domain. This allows us to define $\Gamma_{\Lambda_\sigma} : H^{1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$ by taking the outer trace on $\partial\Omega$:

$$\Gamma_{\Lambda_\sigma}[\phi] := BI_{\Lambda_\sigma}(\phi, G_\rho(\mathrm{x}, \bullet))|_{\partial\Omega}. \qquad (12)$$

As H_ρ is smooth, the singularity of G_ρ is the same as that of the usual potential-theoretic fundamental solution, so Γ_{Λ_σ} shares many properties with the single layer potential; see for example [Mitrea and Taylor 1999, Propositions 3.8 and 7.9]. However, we will not need these types of estimates going forward.

3. The reconstruction algorithm

Recall our a priori assumptions, that the boundary and conductivity are Lipschitz continuous and that the conductivity is bounded below by a positive constant.

The first step of the reconstruction algorithm is to determine the electric potentials that we place on the boundary in order to generate the CGO solutions. As in the previous reconstruction formulas of [García and Zhang 2016; Nachman 1988; Novikov 1988], we resort to the Fredholm alternative; however, once we have obtained the contraction, the argument will be direct, avoiding the use of generalised double layer potentials. The proof is postponed until Section 11.

Theorem 3.1. *Consider $\rho \in \mathbb{C}^n$ such that $\rho \cdot \rho = 0$ and $|\rho|^2 = \rho \cdot \bar{\rho}$ is sufficiently large. Let Γ_{Λ_σ} be defined by (12). Then*

(i) $\Gamma_{\Lambda_\sigma} : H^{1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$ *is bounded compactly,*

(ii) *if $\Gamma_{\Lambda_\sigma}[\phi] = \phi$, then $\phi = 0$,*

(iii) $I - \Gamma_{\Lambda_\sigma}$ *has a bounded inverse on $H^{1/2}(\partial\Omega)$,*

and if $v = e_\rho(1 + w)$, where $w \in H^1(B)$ is a solution to (7), then

(iv) $v|_{\partial\Omega} = (I - \Gamma_{\Lambda_\sigma})^{-1}[e_\rho|_{\partial\Omega}]$.

Next we provide a formula for the Fourier transform $\hat{q}(\xi) := \langle q, e^{-i\xi \cdot \mathrm{x}} \rangle$, where q is defined in (4). Again we postpone the proof until the penultimate section.

Theorem 3.2. *Let Π be a two-dimensional linear subspace orthogonal to $\xi \in \mathbb{R}^n$, and define*

$$S^1 := \Pi \cap \{\theta \in \mathbb{R}^n : |\theta| = 1\}.$$

For $\theta \in S^1$, let $\vartheta \in S^1$ be such that $\{\theta, \vartheta\}$ is an orthonormal basis of Π, and define

$$\rho := \tau\theta + i\left(-\frac{\xi}{2} + \left(\tau^2 - \frac{|\xi|^2}{4}\right)^{1/2}\vartheta\right), \quad \rho' := -\tau\theta + i\left(-\frac{\xi}{2} - \left(\tau^2 - \frac{|\xi|^2}{4}\right)^{1/2}\vartheta\right),$$

where $\tau > 1$. Let BI_{Λ_σ} and Γ_{Λ_σ} be defined by (11) and (12), respectively. Then

$$\hat{q}(\xi) = \lim_{T \to \infty} \frac{1}{2\pi T} \int_T^{2T} \int_{S^1} BI_{\Lambda_\sigma}((I - \Gamma_{\Lambda_\sigma})^{-1}[e_\rho|_{\partial\Omega}], e_{\rho'}) \, d\theta \, d\tau.$$

Finally, we recover σ from q using the approach of [García and Zhang 2016]. By [Brown 2001] and Plancherel's identity, we can now calculate the right-hand side of

$$\begin{cases} \Delta w + |\nabla w|^2 = q & \text{in } \Omega, \\ w|_{\partial\Omega} = \frac{1}{2}\log\sigma|_{\partial\Omega}. \end{cases} \tag{13}$$

If $w \in H^1(\Omega)$ is the unique bounded solution to (13), we then have

$$\sigma = e^{2w} \quad \text{in } \Omega.$$

This completes the reconstruction algorithm.

That $w = \log\sigma^{1/2}$ solves (13) follows directly by inspection of the definition (4) of q. For uniqueness, note that if $\tilde{w}$ also solved (13), then $u = w - \tilde{w}$ would solve

$$\begin{cases} \nabla \cdot (\gamma\nabla u) = 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where $\gamma := e^{w+\tilde{w}}$. Then $u = 0$ by uniqueness of solutions for elliptic equations; see for example [Gilbarg and Trudinger 1983, Corollary 8.2].

4. Sketch of the proof of the contraction for $\Delta_\rho^{-1} \circ M_q$

One of the main ideas of [Haberman and Tataru 2013] was to extend the domain of Δ_ρ^{-1} using Bourgain-type spaces that are adapted to the problem, instead of the usual Sobolev spaces. With $s = \frac{1}{2}$ or $-\frac{1}{2}$, their norms are defined by

$$\| \bullet \|_{\dot{X}_\rho^s} : f \in \mathcal{S}(\mathbb{R}^n) \mapsto \||m_\rho|^s \hat{f}\|_{L^2(\mathbb{R}^n)},$$

where m_ρ is the multiplier defined in (8). Then $\dot{X}_\rho^s$ is defined to be the Banach completion of $\mathcal{S}(\mathbb{R}^n)$ with respect to this norm. It is immediate from the definitions that

$$\|\Delta_\rho^{-1}g\|_{\dot{X}_\rho^{1/2}} \leqslant \|g\|_{\dot{X}_\rho^{-1/2}} \tag{14}$$

whenever $g \in \mathcal{S}(\mathbb{R}^n)$, which can be used to continuously extend the operator. For ease of reference we will call (14) *the trivial inequality*.

On the other hand, Haberman and Tataru also proved that $M_q : f \mapsto qf$ satisfies

$$\| M_q \, f \|_{\dot{X}_\rho^{-1/2}} \leqslant C \| \nabla \log \sigma \|_\infty (1 + |\rho|^{-1} \| \nabla \log \sigma \|_\infty) \| f \|_{\dot{X}_\rho^{1/2}} \tag{15}$$

whenever $f \in \dot{X}_\rho^{1/2}$; see [Haberman and Tataru 2013, Theorem 2.1]. Together these inequalities yield a contraction for $\Delta_\rho^{-1} \circ M_q$ whenever $|\rho| > 1$ and $\| \nabla \log \sigma \|_\infty$ is sufficiently small. In order to remove this smallness condition, we will alter the norms in such a way that the constant of (15) can be taken small for any Lipschitz conductivity, while maintaining a version of (14).

There is a natural gain for the higher frequencies in (15) whereas a gain for the lower frequencies can be engineered in (14) by introducing convex weights. This was the key observation of [Caro and Rogers 2016]. In order to have a gain for all frequencies, in at least one of the inequalities, we dampen the higher frequencies relative to the lower frequencies in our main norm (with the lower frequencies dampened relative to the higher frequencies in the dual norm), so that the gain for the lower frequencies in our version of (14) is passed through to our version of (15).

We prove the Carleman estimate in Section 5, we define new Banach spaces in Section 6, and then we extend the domain of Δ_ρ^{-1} via density in Section 7. We prove our version of (15) in Section 8 and then combine the estimates to obtain the contraction in Section 9.

5. Bounds for Δ_ρ^{-1} with convex weights

Let B be an open ball centred at the origin, with radius

$$R := 2 \sup_{x \in \Omega} |x|,$$

so that we comfortably have $\Omega \subset B$. The forthcoming constants will invariably depend on this R, but never on the auxiliary parameters $\rho \in \mathbb{C}^n$ or $\lambda > 1$.

5.1. The Carleman estimate. Here we will deduce our estimate for Δ_ρ^{-1} from a Carleman estimate for Δ_ρ before defining the main spaces and their duals in the following section. We improve upon the estimate

$$|\rho| \| \Delta_\rho^{-1} f \|_{L^2(B)} \leqslant C \| f \|_{L^2(\mathbb{R}^n)} \tag{16}$$

whenever $f \in C_c^\infty(B)$, which does not seem strong enough to construct CGO solutions for Lipschitz conductivities. The inequality (16) follows by combining

$$|\rho|^{1/2} \| g \|_{L^2(B)} \leqslant C \| g \|_{\dot{X}_\rho^{1/2}} \tag{17}$$

whenever $g \in \dot{X}_\rho^{1/2}$ with the trivial inequality (14), and then

$$|\rho|^{1/2} \| f \|_{\dot{X}_\rho^{-1/2}} \leqslant C \| f \|_{L^2(\mathbb{R}^n)} \tag{18}$$

whenever $f \in C_c^\infty(B)$. The constants $C > 1$ depend only on R. Away from the zero set of the Fourier multiplier m_ρ, these inequalities are obvious, and the localisation serves to blur out the effect of the zero set; see Lemma 2.2 of [Haberman and Tataru 2013] for the proof.

In the following lemma we improve the constant in (16) by introducing exponential weights that depend on the auxiliary parameter $\lambda > 1$. The extra gain in terms of λ will be key to constructing our CGO solutions for Lipschitz conductivities.

Lemma 5.1. *Consider $\rho \in \mathbb{C}^n$ such that $\rho \cdot \rho = 0$, and write $\theta := \operatorname{Re} \rho / |\operatorname{Re} \rho|$. Then*

$$\int_B |\Delta_\rho^{-1} f(x)|^2 e^{\lambda(\theta \cdot x)^2} \, dx \leqslant \frac{2}{\lambda |\rho|^2} \int_{\mathbb{R}^n} |f(x)|^2 e^{\lambda(\theta \cdot x)^2} \, dx$$

whenever $f \in C_c^\infty(\mathbb{R}^n)$ and $\lambda > 1$ satisfies $|\rho| \geqslant 4\lambda R$.

Proof. If m_ρ had been defined slightly differently at the beginning, including a superfluous $\rho \cdot \rho$ term, we could have proved a version of this lemma without the hypothesis that $\rho \cdot \rho = 0$. In fact, we begin by reducing to a purely real vector case. Indeed, letting $\operatorname{Re} \rho, \operatorname{Im} \rho \in \mathbb{R}^n$ denote the real and imaginary parts of ρ, respectively, we define $\Delta_{\operatorname{Re} \rho}^{-1}$ as in Section 2, but with m_ρ replaced by

$$m_{\operatorname{Re} \rho}(\xi) := -|\xi|^2 + 2i \operatorname{Re} \rho \cdot \xi + \operatorname{Re} \rho \cdot \operatorname{Re} \rho$$

for all $\xi \in \mathbb{R}^n$. Then, observing that

$$m_\rho(\xi) = -|\xi|^2 + 2i\rho \cdot \xi + \rho \cdot \rho = m_{\operatorname{Re} \rho}(\xi + \operatorname{Im} \rho)$$

and defining the modulation operator by $\operatorname{Mod}_{\operatorname{Im} \rho} f(x) := e^{i \operatorname{Im} \rho \cdot x} f(x)$, we find that

$$\operatorname{Mod}_{\operatorname{Im} \rho}[\Delta_\rho^{-1} f] = \Delta_{\operatorname{Re} \rho}^{-1}[\operatorname{Mod}_{\operatorname{Im} \rho} f]$$

whenever $f \in C_c^\infty(\mathbb{R}^n)$. Recalling that $|\rho|^2 = 2|\operatorname{Re} \rho|^2$ if $\rho \cdot \rho = 0$, it will therefore suffice to prove

$$\int_B |\Delta_{\operatorname{Re} \rho}^{-1} f(x)|^2 e^{\lambda(\theta \cdot x)^2} \, dx \leqslant \frac{1}{\lambda |\operatorname{Re} \rho|^2} \int_{\mathbb{R}^n} |f(x)|^2 e^{\lambda(\theta \cdot x)^2} \, dx$$

whenever $|\operatorname{Re} \rho| \geqslant 2\lambda R$. Recalling that $\theta := \operatorname{Re} \rho / |\operatorname{Re} \rho|$, by rotating to e_n, this would follow from

$$\int_B |\Delta_{\tau e_n}^{-1} f(x)|^2 e^{\lambda x_n^2} \, dx \leqslant \frac{1}{\lambda \tau^2} \int_{\mathbb{R}^n} |f(x)|^2 e^{\lambda x_n^2} \, dx \tag{19}$$

whenever $f \in C_c^\infty(\mathbb{R}^n)$ and $\tau \geqslant 2\lambda R$.

In order to prove (19), we will first prove the closely related Carleman estimate

$$\|g\|_{L^2(B)}^2 \leqslant \frac{1}{\lambda \tau^2} \|e^{\lambda x_n^2/2}(\Delta + 2\tau e_n \cdot \nabla + \tau^2)(e^{-\lambda x_n^2/2} g)\|_{L^2(\mathbb{R}^n)}^2 \tag{20}$$

whenever $g \in \mathcal{S}(\mathbb{R}^n)$. Defining $\varphi(x) = \tau x_n + \frac{1}{2}\lambda x_n^2$, the integrand of the right-hand side can be rewritten as

$$e^\varphi \Delta(e^{-\varphi} g) = \Delta g - \nabla \varphi \cdot \nabla g - \nabla \cdot (\nabla \varphi g) + |\nabla \varphi|^2 g.$$

Defining the formally self-adjoint A and skew-adjoint B by

$$Ag = \Delta g + |\nabla \varphi|^2 g \quad \text{and} \quad Bg = -\nabla \varphi \cdot \nabla g - \nabla \cdot (\nabla \varphi g)$$

and integrating by parts, we have that

$$\|(A+B)g\|_{L^2(\mathbb{R}^n)}^2 = \|Ag\|_{L^2(\mathbb{R}^n)}^2 + \|Bg\|_{L^2(\mathbb{R}^n)}^2 + \int_{\mathbb{R}^n} [A,B]g\bar{g}, \tag{21}$$

where $[A,B] = AB - BA$ denotes the commutator. By the definition of φ, we have

$$Ag(x) = \Delta g(x) + (\tau + \lambda x_n)^2 g(x),$$
$$Bg(x) = -2(\tau + \lambda x_n)\,\partial_{x_n} g(x) - \lambda g(x),$$

which yields

$$[A,B]g(x) = -4\lambda \partial_{x_n}^2 g(x) + 4\lambda(\tau + \lambda x_n)^2 g(x).$$

After another integration by parts, we find

$$\int_{\mathbb{R}^n} [A,B]g\bar{g} = 4\lambda \int_{\mathbb{R}^n} |\partial_{x_n} g|^2 + 4\lambda \int_{\mathbb{R}^n} |\nabla\varphi|^2 |g|^2,$$

so that, substituting this into (21) and throwing three of the terms away, we find

$$\|e^\varphi \Delta(e^{-\varphi}g)\|_{L^2(\mathbb{R}^n)}^2 \geq 4\lambda \int_{\mathbb{R}^n} |\nabla\varphi|^2 |g|^2.$$

As $|\nabla\varphi(x)| \geq \tau - \lambda R$ whenever $|x_n| \leq R$, this yields

$$\|e^\varphi \Delta(e^{-\varphi}g)\|_{L^2(\mathbb{R}^n)}^2 \geq 4\lambda(\tau - \lambda R)^2 \|g\|_{L^2(B)}^2,$$

which implies (20) whenever $\tau \geq 2\lambda R$ and $g \in \mathcal{S}(\mathbb{R}^n)$.

Finally, by density, the inequality (20) also holds for every $g \in L^2_{\text{loc}}(\mathbb{R}^n)$ such that

$$e^{\lambda x_n^2/2}(\Delta + 2\tau e_n \cdot \nabla + \tau^2)(e^{-\lambda x_n^2/2}g) \in L^2(\mathbb{R}^n).$$

Choosing $g = e^{\lambda x_n^2/2}\Delta_{\tau e_n}^{-1} f$ with $f \in C_c^\infty(\mathbb{R}^n)$, we find that (20) implies (19). $\square$

Remark 5.2. The proof of Lemma 5.1 yields the following strengthened estimate: if $\rho \in \mathbb{C}^n$ and $\theta := \operatorname{Re}\rho/|\operatorname{Re}\rho|$, then

$$\int_{|\theta\cdot x| < |\operatorname{Re}\rho|/(2\lambda)} |\Delta_\rho^{-1} f(x)|^2 e^{\lambda(\theta\cdot x)^2}\,\mathrm{d}x \leq \frac{1}{\lambda|\operatorname{Re}\rho|^2} \int_{\mathbb{R}^n} |f(x)|^2 e^{\lambda(\theta\cdot x)^2}\,\mathrm{d}x$$

whenever $f \in \mathcal{S}(\mathbb{R}^n)$ is such that the right-hand side is finite and $\lambda > 1$.

5.2. *Estimates for derivatives.* The inequality of Lemma 5.1 has a gain in the sense of L^2, however, this is not enough to construct CGO solutions for Lipschitz conductivities since we need to control the first-order partial derivatives present in the operator M_q. For this we consider

$$\|\bullet\|_{X_{\lambda,\rho}^{1/2}} := \lambda^{1/4}|\rho|^{1/2} \|\bullet\|_{L^2(B, e^{\lambda(\theta\cdot x)^2})} + \frac{1}{\lambda^{1/4}} \|\bullet\|_{\dot{X}_\rho^{1/2}} \tag{22}$$

and combine Lemma 5.1 with the trivial inequality (14).

Lemma 5.3. *Consider $\rho \in \mathbb{C}^n$ such that $\rho \cdot \rho = 0$, and write $\theta := \operatorname{Re} \rho / |\operatorname{Re} \rho|$. Then there is a constant $C > 1$, depending only on the radius R of B, such that*

$$\|\Delta_\rho^{-1} f\|_{X_{\lambda,\rho}^{1/2}} \leqslant \frac{C}{\lambda^{1/4} |\rho|^{1/2}} \|f\|_{L^2(\mathbb{R}^n, e^{\lambda(\theta \cdot x)^2})}$$

whenever $f \in C_c^\infty(B)$ and $\lambda > 1$ satisfies $|\rho| \geqslant 4\lambda R$.

Proof. The first term in the definition (22) is bounded using Lemma 5.1, so it remains to bound the second term. Combining the trivial inequality (14) with (18), we see that

$$\|\Delta_\rho^{-1} f\|_{\dot{X}_\rho^{1/2}} \leqslant \|f\|_{\dot{X}_\rho^{-1/2}} \leqslant \frac{C}{|\rho|^{1/2}} \|f\|_{L^2(\mathbb{R}^n)} \leqslant \frac{C}{|\rho|^{1/2}} \|f\|_{L^2(\mathbb{R}^n, e^{\lambda(\theta \cdot x)^2})}$$

whenever $f \in C_c^\infty(B)$, where the constant $C > 1$ depends only on R. Dividing by $\lambda^{1/4}$ yields the desired estimate for the second term. $\qquad\square$

Lemma 5.4. *Consider $\rho \in \mathbb{C}^n$ such that $\rho \cdot \rho = 0$, and write $\theta := \operatorname{Re} \rho / |\operatorname{Re} \rho|$. Then there is a constant $C > 1$, depending only on the radius R of B, such that*

$$\|\Delta_\rho^{-1} f\|_{X_{\lambda,\rho}^{1/2}} \leqslant C\lambda^{1/4} e^{\lambda R^2/2} \|f\|_{\dot{X}_\rho^{-1/2}}$$

whenever $f \in C_c^\infty(B)$ and $\lambda > 1$.

Proof. The second term in the definition (22) can be bounded easily using the trivial inequality (14), so it remains to bound the first term. By (17), we have

$$|\rho|^{1/2} \|g\|_{L^2(B, e^{\lambda(\theta \cdot x)^2})} \leqslant e^{\lambda R^2/2} |\rho|^{1/2} \|g\|_{L^2(B)} \leqslant C e^{\lambda R^2/2} \|g\|_{\dot{X}_\rho^{1/2}}$$

whenever $g \in \dot{X}_\rho^{1/2}$, where the constant $C > 1$ depends only on R. Taking $g = \Delta_\rho^{-1} f$ and multiplying the inequality by $\lambda^{1/4}$ yields

$$\lambda^{1/4} |\rho|^{1/2} \|\Delta_\rho^{-1} f\|_{L^2(B, e^{\lambda(\theta \cdot x)^2})} \leqslant C\lambda^{1/4} e^{\lambda R^2/2} \|\Delta_\rho^{-1} f\|_{\dot{X}_\rho^{1/2}}.$$

A final application of the trivial inequality (14) yields the desired estimate. $\qquad\square$

6. The new spaces

We must extend the domain of Δ_ρ^{-1} by taking limits, so we carefully define Banach spaces using equivalence classes. We define

$$\dot{X}_\rho^{1/2}(B) := \{[f]_B : f \in \dot{X}_\rho^{1/2}\},$$

where the equivalence class $[f]_B$ is given by

$$[f]_B := \{g \in \dot{X}_\rho^{1/2} : \operatorname{ess\,supp}(f - g) \subset \mathbb{R}^n \setminus B\}.$$

The space can be endowed with the norm

$$\|[f]_B\|_{\dot{X}_\rho^{1/2}(B)} := \inf\{\|g\|_{\dot{X}_\rho^{1/2}} : g \in [f]_B\},$$

so that

$$(\dot{X}_\rho^{1/2}(B), \| \bullet \|_{\dot{X}_\rho^{1/2}(B)}) \text{ is a Banach space.}$$

We can rephrase the inequality (17) in terms of this norm. Indeed, as

$$|\rho|^{1/2}\|g\|_{L^2(B)} \leqslant C\|g\|_{\dot{X}_\rho^{1/2}}$$

whenever $g \in [f]_B$, where $C > 1$ is a constant depending only on R, we can take the infimum to find

$$|\rho|^{1/2}\|f\|_{L^2(B)} \leqslant C\|f\|_{\dot{X}_\rho^{1/2}(B)}. \tag{23}$$

Identifying the elements $[f]_B$ of $\dot{X}_\rho^{1/2}(B)$ with $f|_B$, the restriction of f to B, this yields the embedding

$$\dot{X}_\rho^{1/2}(B) \hookrightarrow L^2(B). \tag{24}$$

Moreover, we have the following equivalence of norms.

6.1. *Equivalence with the Sobolev norm.* There are constants $c, C > 0$, depending only on R, such that

$$c|\rho|^{-1/2}\|f\|_{H^1(B)} \leqslant \|f\|_{\dot{X}_\rho^{1/2}(B)} \leqslant C|\rho|^{1/2}\|f\|_{H^1(B)} \tag{25}$$

whenever $f \in H^1(B)$ and $|\rho| > 1$. To see this, note that

$$|m_\rho(\xi)| \leqslant 2(1+|\rho|)(1+|\xi|^2)$$

for all $\xi \in \mathbb{R}^n$, so that

$$\|g\|_{\dot{X}_\rho^{1/2}} \leqslant 2^{1/2}(1+|\rho|)^{1/2}\|g\|_{H^1(\mathbb{R}^n)}$$

whenever $g \in H^1(\mathbb{R}^n)$. Thus H^1-extensions are also $\dot{X}_\rho^{1/2}$-extensions, so the right-hand inequality of (25) follows by taking the infimum over H^1-extensions g of $f \in H^1(B)$.

   For the left-hand inequality, consider $g_B := \chi_B g$, where χ_B is a smooth function equal to one on B and supported on $2B$. Then, separating the low and high frequencies,

$$\|g_B\|_{H^1(\mathbb{R}^n)}^2 \leqslant \|g_B\|_{L^2(\mathbb{R}^n)}^2 + 16|\rho|^2 \int_{|\xi|\leqslant 4|\rho|} |\widehat{g_B}(\xi)|^2 \, \mathrm{d}\xi + 2 \int_{|\xi|>4|\rho|} |m_\rho(\xi)| |\widehat{g_B}(\xi)|^2 \, \mathrm{d}\xi$$

$$\leqslant C|\rho| \|g\|_{\dot{X}_\rho^{1/2}}^2$$

whenever $|\rho| > 1$, where the second inequality follows from Lemma 2.2 of [Haberman and Tataru 2013]. Restricting the left-hand side to B, we find that

$$\|g\|_{H^1(B)} \leqslant C|\rho|^{1/2}\|g\|_{\dot{X}_\rho^{1/2}}.$$

Now if g is an $\dot{X}_\rho^{1/2}$-extension of $f \in \dot{X}_\rho^{1/2}(B)$, then $f = g$ almost everywhere in B, so we can replace g on the left-hand side by f and take the infimum over g to obtain the left-hand inequality of (25).

6.2. *The main space.* We define our main norm by

$$\| \bullet \|_{X^{1/2}_{\lambda,\rho}(B)} : f \in \dot{X}^{1/2}_{\rho}(B) \mapsto \lambda^{1/4}|\rho|^{1/2}\|f\|_{L^2(B,e^{\lambda(\theta \cdot \mathbf{x})^2})} + \frac{1}{\lambda^{1/4}}\|f\|_{\dot{X}^{1/2}_{\rho}(B)},$$

and note that by (23) it is equivalent to the homogeneous norm:

$$\lambda^{-1/4}\|f\|_{\dot{X}^{1/2}_{\rho}(B)} \leqslant \|f\|_{X^{1/2}_{\lambda,\rho}(B)} \leqslant C\lambda^{1/4}e^{\lambda R^2/2}\|f\|_{\dot{X}^{1/2}_{\rho}(B)}, \qquad (26)$$

where $C > 1$ depends only on R. Thus we can conclude that

$$(\dot{X}^{1/2}_{\rho}(B), \| \bullet \|_{X^{1/2}_{\lambda,\rho}(B)}) \text{ is a Banach space.} \qquad (27)$$

Later we will use that the constants in this norm equivalence are independent of $|\rho|$.

6.3. *A minor variant of the main space.* We also consider the norm $\| \bullet \|_{Y^{1/2}_{\lambda,-\rho}(B)}$ defined by

$$f \in \dot{X}^{1/2}_{-\rho}(B) \mapsto \max\left\{ \lambda^{1/4}|\rho|^{1/2}\|f\|_{L^2(B,e^{-\lambda(\theta \cdot \mathbf{x})^2})}, \frac{1}{\lambda^{1/4}e^{\lambda R^2/2}}\|f\|_{\dot{X}^{1/2}_{-\rho}(B)} \right\}.$$

Notice that little more than some signs have changed. As before, this norm is equivalent to the homogeneous norm:

$$\frac{1}{\lambda^{1/4}e^{\lambda R^2/2}}\|f\|_{\dot{X}^{1/2}_{-\rho}(B)} \leqslant \|f\|_{Y^{1/2}_{\lambda,-\rho}(B)} \leqslant C\lambda^{1/4}\|f\|_{\dot{X}^{1/2}_{-\rho}(B)}, \qquad (28)$$

where $C > 1$ depends only on R, and so

$$(\dot{X}^{1/2}_{-\rho}(B), \| \bullet \|_{Y^{1/2}_{\lambda,-\rho}(B)}) \text{ is a Banach space.} \qquad (29)$$

Recalling the embedding (24), this can be identified with the intersection of the spaces

$$(L^2(B), \lambda^{1/4}|\rho|^{1/2}\| \bullet \|_{L^2(B,e^{-\lambda(\theta \cdot \mathbf{x})^2})}) \quad \text{and} \quad \left(\dot{X}^{1/2}_{-\rho}(B), \frac{1}{\lambda^{1/4}e^{\lambda R^2/2}}\| \bullet \|_{\dot{X}^{1/2}_{-\rho}(B)} \right).$$

As (29) is dense in both of these spaces, we can identify the dual of their intersection with the sum of their duals; see for example [Bennett 1974, Theorem 3.1]. This provides an alternative identification of the dual of (29) which we describe now.

6.4. *The dual space.* Let $\dot{X}^{-1/2}_{\rho,c}(B)$ denote the Banach completion of $C^\infty_c(B)$ with respect to the norm

$$\| \bullet \|_{\dot{X}^{-1/2}_{\rho}} : f \in C^\infty_c(B) \mapsto \||m_\rho|^{-1/2}\hat{f}\|_{L^2(\mathbb{R}^n)}.$$

We endow $L^2(B) + \dot{X}^{-1/2}_{\rho,c}(B)$ with the norm

$$\|f\|_{Y^{-1/2}_{\lambda,\rho,c}(B)} := \inf_{f=f^\flat+f^\sharp}\left(\frac{1}{\lambda^{1/4}|\rho|^{1/2}}\|f^\flat\|_{L^2(B,e^{\lambda(\theta \cdot \mathbf{x})^2})} + \lambda^{1/4}e^{\lambda R^2/2}\|f^\sharp\|_{\dot{X}^{-1/2}_{\rho}} \right),$$

with the infimum taken over all $f^\flat \in L^2(B)$ and $f^\sharp \in \dot{X}^{-1/2}_{\rho,c}(B)$. Then

$$(L^2(B) + \dot{X}^{-1/2}_{\rho,c}(B), \| \bullet \|_{Y^{-1/2}_{\lambda,\rho,c}(B)}) \text{ is a Banach space.} \qquad (30)$$

With real-bracket pairings, Plancherel's identity takes the form

$$\langle f, g \rangle = \int_{\mathbb{R}^n} \hat{f}(\xi) \check{g}(\xi) \, d\xi = \int_{\mathbb{R}^n} \hat{f}(\xi) \hat{g}(-\xi) \, d\xi, \tag{31}$$

so that, by similar arguments to those used for Sobolev spaces, we find

$$\left(\dot{X}_{-\rho}^{1/2}(B), \frac{1}{\lambda^{1/4} e^{\lambda R^2/2}} \| \bullet \|_{\dot{X}_{-\rho}^{1/2}(B)} \right)^* \cong (\dot{X}_{\rho,c}^{-1/2}(B), \lambda^{1/4} e^{\lambda R^2/2} \| \bullet \|_{\dot{X}_{\rho}^{-1/2}});$$

see for example [Jerison and Kenig 1995, Proposition 2.9]. On the other hand, it is easy to see that

$$(L^2(B), \lambda^{1/4} |\rho|^{1/2} \| \bullet \|_{L^2(B, e^{-\lambda(\theta \cdot x)^2})})^* \cong \left(L^2(B), \frac{1}{\lambda^{1/4} |\rho|^{1/2}} \| \bullet \|_{L^2(B, e^{\lambda(\theta \cdot x)^2})} \right).$$

Thus the dual of (29) can be identified with the sum of the two dual spaces as described in (30); see for example [Bennett 1974, Theorem 3.1].

7. The locally defined extension of Δ_ρ^{-1}

We are now ready to extend the domain of Δ_ρ^{-1} by combining Lemmas 5.3 and 5.4. This extension will make no sense outside of B in contrast with the globally defined extension of $f \in C_c^\infty(B) \mapsto \Delta_\rho^{-1} f$ given by the trivial inequality (14). We denote the globally defined extension by Δ_ρ^{-1} and the locally defined extension by T_ρ^B.

Corollary 7.1. *Consider $\rho \in \mathbb{C}^n$ such that $\rho \cdot \rho = 0$ and $\lambda > 1$. Then there is a continuous linear extension T_ρ^B of*

$$f \in C_c^\infty(B) \mapsto \Delta_\rho^{-1} f|_B$$

and a constant $C > 1$, depending only on the radius R of B, such that

$$\| T_\rho^B f \|_{X_{\lambda,\rho}^{1/2}(B)} \leqslant C \| f \|_{Y_{\lambda,\rho,c}^{-1/2}(B)}$$

whenever $f \in L^2(B) + \dot{X}_{\rho,c}^{-1/2}(B)$ and $|\rho| \geqslant 4\lambda R$.

Proof. By Lemma 5.3 and the density of $C_c^\infty(B)$ in

$$\left(L^2(B), \frac{1}{\lambda^{1/4} |\rho|^{1/2}} \| \bullet \|_{L^2(B, e^{\lambda(\theta \cdot x)^2})} \right),$$

we can extend $f \in C_c^\infty(B) \mapsto \Delta_\rho^{-1} f|_B$ to a bounded linear operator T_ρ^B that satisfies

$$\| T_\rho^B f \|_{X_{\lambda,\rho}^{1/2}(B)} \leqslant \frac{C}{\lambda^{1/4} |\rho|^{1/2}} \| f \|_{L^2(B, e^{\lambda(\theta \cdot x)^2})}$$

whenever $f \in L^2(B)$. The constant $C > 1$ depends only on R. On the other hand, by Lemma 5.4 and the density of $C_c^\infty(B)$ in

$$(\dot{X}_{\rho,c}^{-1/2}(B), \lambda^{1/4} e^{\lambda R^2/2} \| \bullet \|_{\dot{X}_{\rho}^{-1/2}}),$$

we can extend $f \in C_c^\infty(B) \mapsto \Delta_\rho^{-1} f|_B$ to a bounded linear operator T_ρ^B that satisfies

$$\|T_\rho^B f\|_{X_{\lambda,\rho}^{1/2}(B)} \leqslant C\lambda^{1/4} e^{\lambda R^2/2} \|f\|_{\dot{X}_\rho^{-1/2}}$$

whenever $f \in \dot{X}_{\rho,c}^{-1/2}(B)$. Again, the constant $C > 1$ depends only on R.

Considering now $f = f^\flat + f^\sharp$ with $f^\flat \in L^2(B)$ and $f^\sharp \in \dot{X}_{\rho,c}^{-1/2}(B)$, we define

$$T_\rho^B f := T_\rho^B f^\flat + T_\rho^B f^\sharp.$$

One can show that this is well defined using the linearity of the previous extensions and the density of $C_c^\infty(B)$. Then, by the triangle inequality and the previous bounds,

$$\|T_\rho^B f\|_{X_{\lambda,\rho}^{1/2}(B)} \leqslant C\left(\frac{1}{\lambda^{1/4}|\rho|^{1/2}}\|f^\flat\|_{L^2(e^{\lambda(\theta \cdot x)^2})} + \lambda^{1/4} e^{\lambda R^2/2}\|f^\sharp\|_{\dot{X}_\rho^{-1/2}}\right),$$

where the constant C depends only on R. Since the left-hand side is independent of the representation $f = f^\flat + f^\sharp$, we can take the infimum over such representations, and the desired inequality follows. $\square$

8. The bound for M_q

With a view to further applications, we write part of this section in greater generality. Consider bounded functions $a_0, a_1, \ldots, a_n \subset L^\infty(\mathbb{R}^n)$ with compact support:

$$\operatorname{supp} a_j \subset \Omega \subset B = \{x \in \mathbb{R}^n : |x| < R\},$$

where $R := 2\sup_{x \in \Omega} |x|$. Define the bilinear form $\mathcal{B} : H^1(B) \times H^1(B) \to \mathbb{C}$ by

$$\mathcal{B}(f, g) := \int_\Omega a_0 fg + \int_\Omega A \cdot \nabla(fg),$$

where A is the vector field with components $(a_1, \ldots, a_n)$. This is well defined by an application of the product rule, followed by the Cauchy–Schwarz inequality.

Proposition 8.1. *Consider $\rho \in \mathbb{C}^n$ such that $\rho \cdot \rho = 0$ and $\lambda > 1$. Then there is a constant $C > 1$, depending only on the radius R of B, such that*

$$|\mathcal{B}(f, g)| \leqslant C\left(\frac{1}{\lambda^{1/2}|\rho|} + \frac{1}{\lambda^{1/2}} + \frac{e^{\lambda R^2/2}}{|\rho|^{1/2}}\right) \sum_{j=0}^n \|a_j\|_{L^\infty(\Omega)} \|f\|_{X_{\lambda,\rho}^{1/2}(B)} \|g\|_{Y_{\lambda,-\rho}^{1/2}(B)}$$

whenever $(f, g) \in \dot{X}_\rho^{1/2}(B) \times \dot{X}_{-\rho}^{1/2}(B)$.

Proof. For the first term, we note that, by the Cauchy–Schwarz inequality,

$$\left|\int_\Omega a_0 fg\right| \leqslant \|a_0\|_\infty \|e^{\lambda(\theta \cdot x)^2/2} f\|_{L^2(B)} \|e^{-\lambda(\theta \cdot x)^2/2} g\|_{L^2(B)} \leqslant \frac{1}{\lambda^{1/2}} \frac{1}{|\rho|} \|a_0\|_\infty \|f\|_{X_{\lambda,\rho}^{1/2}(B)} \|g\|_{Y_{\lambda,-\rho}^{1/2}(B)}$$

whenever $(f, g) \in \dot{X}_\rho^{1/2}(B) \times \dot{X}_{-\rho}^{1/2}(B)$. The second inequality follows directly from the weightings in the definition of the norms.

For the more difficult first-order term, we consider a positive and smooth function χ, equal to 1 on the ball of radius $\frac{1}{2}$, supported in the unit ball, and bounded above by 1. Then we work with $f_B := \chi_B f$ and $g_B := \chi_B g$, where $\chi_B := \chi(\bullet / R)$ is equal to 1 on Ω and supported on B. Letting $A^\flat$ denote the vector field with components

$$a_j^\flat(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \chi\left(\frac{\xi}{16|\rho|}\right) \hat{a}_j(\xi) \, d\xi$$

for all $x \in \mathbb{R}^n$ and $j = 1, \ldots, n$, and letting $A^\sharp := A - A^\flat$, by integration by parts,

$$\int_\Omega A \cdot \nabla(fg) = -\int_{\mathbb{R}^n} \nabla \cdot A^\flat f_B g_B + \int_{\mathbb{R}^n} A^\sharp \cdot \nabla(f_B g_B).$$

Noting that $\|\nabla \cdot A^\flat\|_\infty \leqslant C|\rho|\|A\|_\infty$, the first term can be bounded as before:

$$\left| \int_{\mathbb{R}^n} \nabla \cdot A^\flat f_B g_B \right| \leqslant C \|\nabla \cdot A^\flat\|_\infty \|e^{\lambda(\theta\cdot x)^2/2} f\|_{L^2(B)} \|e^{-\lambda(\theta\cdot x)^2/2} g\|_{L^2(B)}$$

$$\leqslant C \|A\|_\infty |\rho|^{1/2} \|f\|_{L^2(B, e^{\lambda(\theta\cdot x)^2})} |\rho|^{1/2} \|g\|_{L^2(B, e^{-\lambda(\theta\cdot x)^2})}.$$

Again by the weightings in the definitions of the norms, this implies that

$$\left| \int_{\mathbb{R}^n} \nabla \cdot A^\flat f_B g_B \right| \leqslant C \frac{1}{\lambda^{1/2}} \|A\|_\infty \|f\|_{X_{\lambda,\rho}^{1/2}(B)} \|g\|_{Y_{\lambda,-\rho}^{1/2}(B)}$$

whenever $(f, g) \in \dot{X}_\rho^{1/2}(B) \times \dot{X}_{-\rho}^{1/2}(B)$.

It remains to show that

$$\left| \int_{\mathbb{R}^n} A^\sharp \cdot \nabla(f_B g_B) \right| \leqslant C \frac{e^{\lambda R^2/2}}{|\rho|^{1/2}} \|A\|_\infty \|f\|_{X_{\lambda,\rho}^{1/2}(B)} \|g\|_{Y_{\lambda,-\rho}^{1/2}(B)}. \tag{32}$$

Using the product rule, we can separate into two similar terms,

$$\int_{\mathbb{R}^n} A^\sharp \cdot \nabla(f_B g_B) = \int_{\mathbb{R}^n} A^\sharp \cdot \nabla f_B g_B + \int_{\mathbb{R}^n} A^\sharp \cdot \nabla g_B f_B, \tag{33}$$

and initially treat the first term on the right-hand side (the second term will eventually be dealt with by symmetry). We decompose the integral as

$$\int_{\mathbb{R}^n} A^\sharp \cdot \nabla f_B g_B = \int_{\mathbb{R}^n} A^\sharp \cdot \nabla L f_B L g_B + \int_{\mathbb{R}^n} A^\sharp \cdot \nabla L f_B H g_B + \int_{\mathbb{R}^n} A^\sharp \cdot \nabla H f_B g_B,$$

where L denotes the low-frequency filter defined by

$$Lf(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \chi\left(\frac{\xi}{4|\rho|}\right) \hat{f}(\xi) \, d\xi$$

and $H := I - L$. By the properties of χ, the frequency supports of $\nabla L f_B L g_B$ and $A^\sharp$ are disjoint, so that by Plancherel's identity the first term is in fact 0, yielding

$$\int_{\mathbb{R}^n} A^\sharp \cdot \nabla f_B g_B = \int_{\mathbb{R}^n} A^\sharp \cdot \nabla L f_B H g_B + \int_{\mathbb{R}^n} A^\sharp \cdot \nabla H f_B g_B.$$

Then, by the Cauchy–Schwarz inequality (writing $\| \bullet \|_2 := \| \bullet \|_{L^2(\mathbb{R}^n)}$),

$$\left| \int_{\mathbb{R}^n} A^\sharp \cdot \nabla f_B g_B \right| \leqslant \|A^\sharp\|_\infty (\|\nabla L f_B\|_2 \|H g_B\|_2 + \|\nabla H f_B\|_2 \|g_B\|_2).$$

Now as $\|A^\sharp\|_\infty \leqslant C\|A\|_\infty$ and

$$\|\nabla L f_B\|_2 \|H g_B\|_2 \leqslant C|\rho| \|L f_B\|_2 \|H g_B\|_2 \leqslant C\|f_B\|_2 \|\nabla H g_B\|_2,$$

we find that

$$\left| \int_{\mathbb{R}^n} A^\sharp \cdot \nabla f_B g_B \right| \leqslant C\|A\|_\infty (\|f_B\|_2 \|\nabla H g_B\|_2 + \|\nabla H f_B\|_2 \|g_B\|_2).$$

Since the right-hand side is symmetric in the roles of f_B and g_B, we can conclude the same bound for the second term on the right-hand side of (33), yielding

$$\left| \int_{\mathbb{R}^n} A^\sharp \cdot \nabla (f_B g_B) \right| \leqslant C\|A\|_\infty (\|f_B\|_2 \|\nabla H g_B\|_2 + \|\nabla H f_B\|_2 \|g_B\|_2). \tag{34}$$

Now clearly we have that

$$\|f_B\|_2 \leqslant \|f\|_{L^2(B, e^{\lambda (\theta \cdot x)^2})} \quad \text{and} \quad \|g_B\|_2 \leqslant e^{\lambda R^2/2} \|g\|_{L^2(B, e^{-\lambda (\theta \cdot x)^2})}.$$

On the other hand, by Lemma 2.2 of [Haberman and Tataru 2013], we have

$$\|\nabla H f_B\|_2 \leqslant C\|\tilde{f}\|_{\dot{X}_\rho^{1/2}} \quad \text{and} \quad \|\nabla H g_B\|_2 \leqslant C\|\tilde{g}\|_{\dot{X}_{-\rho}^{1/2}},$$

where $(\tilde{f}, \tilde{g}) \in \dot{X}_\rho^{1/2} \times \dot{X}_{-\rho}^{1/2}$ denotes any pair of extensions of (f, g). Substituting these inequalities into (34) and taking the infimum over extensions yields

$$\left| \int_{\mathbb{R}^n} A^\sharp \cdot \nabla (f_B g_B) \right| \leqslant C\|A\|_\infty (\|f\|_{L^2(B, e^{\lambda (\theta \cdot x)^2})} \|g\|_{\dot{X}_{-\rho}^{1/2}(B)} + e^{\lambda R^2/2} \|f\|_{\dot{X}_\rho^{1/2}(B)} \|g\|_{L^2(B, e^{-\lambda (\theta \cdot x)^2})}).$$

Recalling the weightings in the norms, this completes the proof of (32).   □

From this we can deduce our estimate for $M_q : f \mapsto qf$, where q is defined in (4).

Corollary 8.2. *Consider* $\rho \in \mathbb{C}^n$ *such that* $\rho \cdot \rho = 0$ *and* $\lambda > 1$. *Then there is a* $C > 1$, *depending on* $\|\nabla \log \sigma\|_\infty$ *and the radius* R *of* B, *such that*

$$\| M_q f \|_{Y_{\lambda, \rho, c}^{-1/2}(B)} \leqslant C \left(\frac{1}{\lambda^{1/2}|\rho|} + \frac{1}{\lambda^{1/2}} + \frac{e^{\lambda R^2/2}}{|\rho|^{1/2}} \right) \|f\|_{X_{\lambda, \rho}^{1/2}(B)}$$

whenever $f \in \dot{X}_\rho^{1/2}(B)$.

Proof. By an application of the product rule, the definition (4) can be rewritten as

$$\langle q, \psi \rangle = \frac{1}{4} \int_\Omega |\nabla \log \sigma|^2 \psi - \frac{1}{2} \int_\Omega \nabla \log \sigma \cdot \nabla \psi.$$

Using our a priori assumptions that σ is bounded below and $\nabla\sigma$ is bounded above almost everywhere (which follows from Lipschitz continuity), $\nabla\log\sigma = \sigma^{-1}\nabla\sigma$ is a vector of bounded functions. Thus, by taking

$$a_0 = \tfrac{1}{4}|\nabla\log\sigma|^2 \quad \text{and} \quad A = -\tfrac{1}{2}\nabla\log\sigma,$$

we can write

$$\langle \mathrm{M}_q\, f, g\rangle := \langle qf, g\rangle := \langle q, fg\rangle = \mathcal{B}(f, g)$$

for all $(f, g) \in \dot{X}_\rho^{1/2}(B) \times \dot{X}_{-\rho}^{1/2}(B)$. Then, by Proposition 8.1, we find that

$$|\langle \mathrm{M}_q\, f, g\rangle| \leqslant C\left(\frac{1}{\lambda^{1/2}|\rho|} + \frac{1}{\lambda^{1/2}} + \frac{e^{\lambda R^2/2}}{|\rho|^{1/2}}\right) \sum_{j=0}^{n} \|a_j\|_\infty \|f\|_{X_{\lambda,\rho}^{1/2}(B)} \|g\|_{Y_{\lambda,-\rho}^{1/2}(B)}$$

for all $(f, g) \in \dot{X}_\rho^{1/2}(B) \times \dot{X}_{-\rho}^{1/2}(B)$. Finally, using the identification

$$(\dot{X}_{-\rho}^{1/2}(B), \|\bullet\|_{Y_{\lambda,-\rho}^{1/2}(B)})^* \cong (L^2(B) + \dot{X}_{\rho,\mathrm{c}}^{-1/2}(B), \|\bullet\|_{Y_{\lambda,\rho,\mathrm{c}}^{-1/2}(B)}),$$

we obtain the desired inequality. $\qquad\square$

9. Locally defined CGO solutions via Neumann series

Let $X_{\lambda,\rho}^{1/2}(B)$ and $Y_{\lambda,\rho,\mathrm{c}}^{-1/2}(B)$ denote the Banach spaces defined in (27) and (30), respectively. Recall that $f \in C_\mathrm{c}^\infty(B) \mapsto \Delta_\rho^{-1} f|_B$ can be extended as a bounded linear operator

$$\mathrm{T}_\rho^B : Y_{\lambda,\rho,\mathrm{c}}^{-1/2}(B) \to X_{\lambda,\rho}^{1/2}(B)$$

using Corollary 7.1 and that $\mathrm{M}_q : f \mapsto qf$, with q defined in (4), is bounded as

$$\mathrm{M}_q : X_{\lambda,\rho}^{1/2}(B) \to Y_{\lambda,\rho,\mathrm{c}}^{-1/2}(B)$$

by Corollary 8.2. The contraction will follow by choosing $|\rho|$ and λ appropriately so that the product of the operator norms is small.

Theorem 9.1. *Consider $\rho \in \mathbb{C}^n$ such that $\rho \cdot \rho = 0$ and $\lambda > 1$. Then there is a $C_0 > 1$, depending on $\|\nabla\log\sigma\|_\infty$ and the radius R of B, such that*

$$\|\mathrm{T}_\rho^B \circ \mathrm{M}_q\|_{\mathcal{L}(X_{\lambda,\rho}^{1/2}(B))} \leqslant \tfrac{1}{2} \tag{35}$$

whenever $|\rho| > \lambda e^{\lambda R^2}$ and $\lambda = 36C_0^2$. For all $f \in Y_{\lambda,\rho,\mathrm{c}}^{-1/2}(B)$, there is a $w \in X_{\lambda,\rho}^{1/2}(B)$ such that

$$(\mathrm{I} - \mathrm{T}_\rho^B \circ \mathrm{M}_q)w = \mathrm{T}_\rho^B[f]. \tag{36}$$

Moreover, there is a $C > 1$, depending only on R, such that

$$\|w\|_{X_{\lambda,\rho}^{1/2}(B)} \leqslant C\|f\|_{Y_{\lambda,\rho,\mathrm{c}}^{-1/2}(B)}. \tag{37}$$

Proof. By combining Corollaries 7.1 and 8.2, we have that $T_\rho^B \circ M_q$ is a bounded operator whenever $|\rho| \geqslant 4\lambda R$. Furthermore, there exists a constant $C_0 > 1$ such that

$$\|T_\rho^B \circ M_q\|_{\mathcal{L}(X_{\lambda,\rho}^{1/2}(B))} \leqslant C_0\left(\frac{1}{\lambda^{1/2}|\rho|} + \frac{1}{\lambda^{1/2}} + \frac{e^{\lambda R^2/2}}{|\rho|^{1/2}}\right) \leqslant \frac{1}{2}$$

whenever $|\rho|^{1/2} > 6C_0 e^{\lambda R^2/2}$ and $\lambda^{1/2} = 6C_0$. Then, by Neumann series, $I - T_\rho^B \circ M_q$ has a bounded inverse,

$$(I - T_\rho^B \circ M_q)^{-1} = \sum_{k\geqslant 0}(T_\rho^B \circ M_q)^k$$

on $X_{\lambda,\rho}^{1/2}(B)$, and so $w = (I - T_\rho^B \circ M_q)^{-1}T_\rho^B[f]$ satisfies (36). Moreover,

$$\|w\|_{X_{\lambda,\rho}^{1/2}(B)} \leqslant \sum_{k\geqslant 0}\|(T_\rho^B \circ M_q)^k T_\rho^B[f]\|_{X_{\lambda,\rho}^{1/2}(B)} \leqslant 2\|T_\rho^B[f]\|_{X_{\lambda,\rho}^{1/2}(B)}$$

by the triangle inequality, the contraction (35), and summing the geometric series. Then (37) follows by a final application of Corollary 7.1. $\qquad\square$

Recall that we can also use the trivial inequality (14) to extend $f \in C_c^\infty(B) \mapsto \Delta_\rho^{-1} f$ as a bounded linear operator

$$\Delta_\rho^{-1} : \dot{X}_\rho^{-1/2} \to \dot{X}_\rho^{1/2}.$$

In the following corollary we clarify that the restriction of this extension to the ball B and the previous locally defined extension T_ρ^B are the same. We also record the properties of our CGO solutions that we will need in the remaining sections.

Corollary 9.2. *Consider $\rho \in \mathbb{C}^n$ and $\lambda > 1$ as in Theorem 9.1. Then*

$$\|\Delta_\rho^{-1} \circ M_q\|_{\mathcal{L}(X_{\lambda,\rho}^{1/2}(B))} \leqslant \frac{1}{2}, \tag{38}$$

there is a $w \in H^1(B)$ that solves (7), and there is a $C > 1$, depending on $\|\nabla \log \sigma\|_\infty$ and the radius R of B, such that

$$\|w\|_{\dot{X}_\rho^{1/2}(B)} \leqslant C\|q\|_{\dot{X}_\rho^{-1/2}}. \tag{39}$$

Moreover, $v = e_\rho(1 + w) \in H^1(B)$ solves the Lippmann–Schwinger-type equation

$$(I - S_q)v = e_\rho, \quad \text{where } S_q := e_\rho \Delta_\rho^{-1} \circ M_q[e_{-\rho} \bullet] \tag{40}$$

as elements of $H^1(B)$, and is also a weak solution to the Schrödinger equation (3).

Proof. By Corollary 7.1, the equivalence of norms (26), and the trivial inequality (14),

$$\|T_\rho^B g - \Delta_\rho^{-1} g\|_{X_{\lambda,\rho}^{1/2}(B)} \leqslant \|T_\rho^B[g - g_j]\|_{X_{\lambda,\rho}^{1/2}(B)} + \|\Delta_\rho^{-1}[g_j - g]\|_{X_{\lambda,\rho}^{1/2}(B)}$$

$$\leqslant C(\|g - g_j\|_{Y_{\lambda,\rho,c}^{-1/2}(B)} + \|g_j - g\|_{\dot{X}_\rho^{-1/2}}),$$

and given that the dual norms are also equivalent, by (28), we can choose $g_j \in C_c^\infty(B)$ such that the right-hand side converges to 0. Then, combining with Corollary 8.2, the contraction (38) follows directly from the previous contraction (35).

Taking $f = M_q[1]$ in Theorem 9.1, we find $w \in X_{\lambda,\rho}^{1/2}(B)$ solving

$$w = T_\rho^B \circ M_q[1 + w].$$

Again by Corollary 8.2, we have $M_q[1 + w] \in Y_{\lambda,\rho,c}^{-1/2}(B)$, so that, taking this as the function g above, we can also write

$$w = \Delta_\rho^{-1} \circ M_q[1 + w] \tag{41}$$

as elements of $X_{\lambda,\rho}^{1/2}(B)$. Thus, combining with the norm equivalences (25) and (26), we find that $w \in H^1(B)$ solves (7). Moreover, the inequality (39) follows from the previous inequality (37) combined with (26) and the dual version of (28).

Finally, writing $v = e_\rho(1 + w)$, we can multiply (41) by e_ρ to find

$$v - e_\rho = e_\rho \Delta_\rho^{-1} \circ M_q[e_{-\rho} v] =: S_q[v]$$

as elements of $H^1(B)$. Then, by integration by parts and Plancherel's identity (31), cancelling the Fourier multipliers,

$$-\int_{\mathbb{R}^n} \nabla S_q[v] \cdot \nabla \psi = \int_{\mathbb{R}^n} \Delta_\rho^{-1} \circ M_q[e_{-\rho} v] e_\rho \Delta[e_{-\rho} e_\rho \psi]$$

$$= \int_{\mathbb{R}^n} m_\rho^{-1} \widehat{M_q[e_{-\rho} v]} m_\rho (e_\rho \psi)^\vee$$

$$= \langle qv, \psi \rangle \tag{42}$$

whenever $\psi \in C_c^\infty(B)$. Given that e_ρ is harmonic, we see that $v \in H^1(B)$ is also a weak solution to the Schrödinger equation (3). □

Remark 9.3. The CGO solutions $v = e_\rho(1 + w)$ given by Corollary 9.2 also satisfy

$$v = \sum_{k \geqslant 0} e_\rho (\Delta_\rho^{-1} \circ M_q)^k[1], \tag{43}$$

with convergence in $H^1(B)$. On the other hand, we have that

$$e_\rho (\Delta_\rho^{-1} \circ M_q)^k[1] = (e_\rho \Delta_\rho^{-1} \circ M_q[e_{-\rho} \bullet])^k[e_\rho] = S_q^k[e_\rho]$$

as elements of $H^1(B)$. Substituting this into (43), we find that

$$v = \sum_{k \geqslant 0} S_q^k[e_\rho]$$

again in the $H^1(B)$-sense. If we had proven that S_q is contractive on $H^1(B)$, we could have solved (40) more directly by Neumann series, and the solution would have taken this form.

10. The boundary integral identities

Here we use the divergence theorem to equate the boundary integral to an integral over the domain. Identities similar to the first identity of Lemma 10.1, often known as Alessandrini identities, are foundational for the Calderón problem. Recall that our boundary integral $BI_{\Lambda_\sigma} : H^{1/2}(\partial\Omega) \times H^1(\Omega) \to \mathbb{C}$ is defined by

$$BI_{\Lambda_\sigma}(\phi, \psi) := \int_{\partial\Omega} (\sigma^{-1/2}\Lambda_\sigma[\sigma^{-1/2}\phi] - \nu \cdot \nabla P_0[\phi])\psi, \tag{44}$$

where $P_0[\phi]$ denotes the harmonic extension of ϕ. A key idea of [Nachman 1988] and [Novikov 1988] was to take the Faddeev fundamental solution within boundary integrals similar to this, yielding similar formulas to the second identity in Lemma 10.1.

Lemma 10.1. *Let q be defined by* (4), *and let BI_{Λ_σ} be defined by* (44). *Then*

$$BI_{\Lambda_\sigma}(v|_{\partial\Omega}, \psi) = \langle qv, \psi \rangle$$

whenever ψ is harmonic on Ω and $v \in H^1(\Omega)$ solves the Schrödinger equation (3). *Moreover,*

$$BI_{\Lambda_\sigma}(v|_{\partial\Omega}, G_\rho(\mathbf{x}, \bullet)) = S_q[v](\mathbf{x})$$

as elements of $H^1(B \setminus \overline{\Omega})$, where S_q is defined in (40).

Proof. For the first identity, consider the weak solution to the conductivity equation given by $u = \sigma^{-1/2}v$. Recalling that $\nabla\sigma$ is bounded almost everywhere, by an application of the product rule, we find that $\Delta u = -\sigma^{-1}\nabla\sigma \cdot \nabla u \in L^2(\Omega)$. Thus the normal traces can be defined so that the divergence theorem can be applied to $\sigma\nabla u \sigma^{-1/2}\psi - \nabla P_0[\phi]\psi$, yielding

$$BI_{\Lambda_\sigma}(v|_{\partial\Omega}, \psi) = \int_\Omega (\sigma\nabla u \cdot \nabla(\sigma^{-1/2}\psi) - \nabla P_0[v|_{\partial\Omega}] \cdot \nabla\psi); \tag{45}$$

see for example [Kim and Kwon 2022, Proposition 2.4]. Now, as ψ is harmonic on Ω, we have

$$\int_\Omega \nabla(P_0[v|_{\partial\Omega}] - \sigma^{1/2}u) \cdot \nabla\psi = 0,$$

which can be substituted in (45) to find that

$$BI_{\Lambda_\sigma}(v|_{\partial\Omega}, \psi) = \int_\Omega (\sigma\nabla u \cdot \nabla(\sigma^{-1/2}\psi) - \nabla(\sigma^{1/2}u) \cdot \nabla\psi).$$

Then, after applying the product rule again, terms cancel and one finds that the right-hand side of this identity is equal to $\langle q\sigma^{1/2}u, \psi \rangle = \langle qv, \psi \rangle$, as desired.

For the second identity, recall that $G_{-\rho} := e_{-\rho}F_{-\rho}$ is a fundamental solution for the Laplacian. In particular $\Delta G_{-\rho}(\bullet, x) = 0$ on Ω for all $x \in B \setminus \overline{\Omega}$. On the other hand, G_ρ inherits a skew symmetry from F_ρ,

$$G_{-\rho}(y, x) := e_{-\rho}(y - x)F_{-\rho}(y - x) = e_\rho(x - y)F_\rho(x - y) =: G_\rho(x, y), \tag{46}$$

so we can reinterpret this as $\Delta G_\rho(x, \bullet) = 0$ on Ω for all $x \in B \setminus \overline{\Omega}$. Thus, we can substitute this into the first identity to find that

$$BI_{\Lambda_\sigma}(v|_{\partial\Omega}, G_\rho(x, \bullet)) = \langle qv, G_\rho(x, \bullet) \rangle \tag{47}$$

for all $x \in B \setminus \overline{\Omega}$.

Now, for any $f \in H^1(\Omega)$ and any smooth ψ_x supported in a small ball centred at x and properly contained in $B \setminus \overline{\Omega}$, we have that

$$\int_B \langle qf, G_\rho(y, \bullet) \rangle \psi_x(y) \, \mathrm{d}y = \left\langle qf, \int_B G_\rho(y, \bullet) \psi_x(y) \, \mathrm{d}y \right\rangle. \tag{48}$$

This follows by interchanging the integral and the gradient, using Lebesgue's dominated convergence theorem, and applying Fubini's theorem. Then using the skew symmetry (46) again and the kernel representation (9) of Δ_ρ^{-1}, the right-hand side of (48) can be rewritten as

$$\langle qf, e_{-\rho} \Delta_{-\rho}^{-1}[e_\rho \psi_x] \rangle = \int_B e_\rho \Delta_\rho^{-1}[qfe_{-\rho}](y) \psi_x(y) \, \mathrm{d}y. \tag{49}$$

Here we have considered Δ_ρ^{-1} to be the globally defined extension given by (14), and the identity follows by moving the Fourier multiplier m_ρ^{-1} from one term to the other after an application of Plancherel's identity (31). Combining (48) with (49) and recalling the definition (40) of S_q, we find that

$$\int_B \langle qf, G_\rho(y, \bullet) \rangle \psi_x(y) \, \mathrm{d}y = \int_B S_q[f](y) \psi_x(y) \, \mathrm{d}y.$$

Now by the bounds of the previous section, we have that $S_q[f] \in H^1(B)$, and so, letting ψ_x approximate the Dirac delta δ_x, we find that

$$\langle qf, G_\rho(x, \bullet) \rangle = S_q[f](x) \tag{50}$$

for almost every $x \in B \setminus \overline{\Omega}$ by a suitable version of the Lebesgue differentiation theorem; see for example [Muscalu and Schlag 2013, Theorem 2.12]. Taking $f = v$ and combining (47) with (50) yields the second identity. $\square$

11. The proofs of Theorems 3.1 and 3.2

The second identity of Lemma 10.1 allows us to define $\Gamma_{\Lambda_\sigma} : H^{1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$ by taking the outer trace $T_{\partial\Omega} : H^1(B \setminus \overline{\Omega}) \to H^{1/2}(\partial\Omega)$ of the boundary integral:

$$\Gamma_{\Lambda_\sigma}[\phi] := T_{\partial\Omega}[BI_{\Lambda_\sigma}(\phi, G_\rho(\mathrm{x}, \bullet))] \tag{51}$$

for all $\phi \in H^{1/2}(\partial\Omega)$. Moreover, it gives us the alternative representation

$$\Gamma_{\Lambda_\sigma}[\phi] = T_{\partial\Omega} \circ S_q \circ P_q[\phi], \tag{52}$$

where S_q is defined in (40) and $P_q[\phi]$ denotes the solution to (3) with Dirichlet data ϕ.

We restate the main theorems from Section 3 before proving them. The proof of the second part of the following theorem bears some resemblance to the argument of [Astala et al. 2016, Theorem 3.1], allowing us to avoid the use of double layer potentials.

Theorem 3.1. *Consider $\rho \in \mathbb{C}^n$ such that $\rho \cdot \rho = 0$ and $|\rho|^2 = \rho \cdot \bar{\rho}$ is sufficiently large. Let Γ_{Λ_σ} be defined by* (12). *Then*

(i) $\Gamma_{\Lambda_\sigma} : H^{1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$ *is bounded compactly,*

(ii) *if $\Gamma_{\Lambda_\sigma}[\phi] = \phi$, then $\phi = 0$,*

(iii) $I - \Gamma_{\Lambda_\sigma}$ *has a bounded inverse on $H^{1/2}(\partial\Omega)$,*

and if $v = e_\rho(1 + w)$, where $w \in H^1(B)$ is a solution to (7), *then*

(iv) $v|_{\partial\Omega} = (I - \Gamma_{\Lambda_\sigma})^{-1}[e_\rho|_{\partial\Omega}]$.

Proof. By hypothesis $(I - S_q)v = e_\rho$, so part (iv) follows from the alternative representation (52) and part (iii), which in turn will follow from parts (i) and (ii) by the Fredholm alternative.

To see (i), note first that the trace operator $T_{\partial\Omega} : H^1(B \setminus \bar{\Omega}) \to H^{1/2}(\partial\Omega)$ and solution operator $P_q : H^{1/2}(\partial\Omega) \to H^1(\Omega)$ are bounded. Combining this with the alternative representation (52), it will suffice to show that $S_q : H^1(\Omega) \to H^1(B \setminus \bar{\Omega})$ is bounded compactly. For this we recall that, on $B \setminus \bar{\Omega}$, we have the representation (50), and so by applications of the product rule we can divide the operator into three parts $S_q = S_1 + S_2 + S_3$, where

$$S_1[f] := \frac{1}{4} \int_\Omega |\nabla \log \sigma(y)|^2 f(y) G_\rho(\bullet - y) \, dy,$$

$$S_2[f] := -\frac{1}{2} \int_\Omega \nabla \log \sigma(y) \cdot \nabla f(y) G_\rho(\bullet - y) \, dy,$$

$$S_3[f] := \frac{1}{2} \int_\Omega \nabla \log \sigma(y) \cdot \nabla G_\rho(\bullet - y) f(y) \, dy.$$

By our a priori assumptions, $\nabla \log \sigma = \sigma^{-1} \nabla \sigma \in L^\infty(\Omega)^n$, and on the other hand G_ρ and ∇G_ρ are locally integrable by (10). Thus, by Young's convolution inequality,

$$S_1 : L^2(\Omega) \to L^2(B \setminus \bar{\Omega}), \quad S_2 : H^1(\Omega) \to L^2(B \setminus \bar{\Omega}), \quad \text{and} \quad S_3 : L^2(\Omega) \to L^2(B \setminus \bar{\Omega})$$

are bounded. Moreover, by Lebesgue's dominated convergence theorem, we can take derivatives under the integral, and by (10) we have that

$$\partial_{x_j} \partial_{x_i} G_\rho(x - y) = c_n n \frac{(x_j - y_j)(x_i - y_i)}{|x - y|^{n+2}} + \partial_{x_j} \partial_{x_i} H_\rho(x - y).$$

On the one hand, the second-order Riesz transforms are easily bounded in L^2 noting that the Fourier multipliers $-\xi_j \xi_i / |\xi|^2$ are uniformly bounded; see for example [Muscalu and Schlag 2013, Section 7.2]. On the other hand, the operator corresponding to the second term can be bounded in $L^2(B \setminus \bar{\Omega})$ by Young's inequality again. Together we find that

$$S_1 : L^2(\Omega) \to H^2(B \setminus \bar{\Omega}), \quad S_2 : H^1(\Omega) \to H^2(B \setminus \bar{\Omega}), \quad \text{and} \quad S_3 : L^2(\Omega) \to H^1(B \setminus \bar{\Omega})$$

are bounded. Thus, by Rellich's theorem, all three operators are bounded from $H^1(\Omega)$ to $H^1(B \setminus \bar{\Omega})$ compactly. Altogether we find that S_q maps $H^1(\Omega)$ to $H^1(B \setminus \bar{\Omega})$ compactly, which completes the proof of (i).

In order to see (ii), we combine its hypothesis with the alternative representation (52), obtaining

$$\phi = \Gamma_{\Lambda_\sigma}[\phi] = T_{\partial\Omega} \circ S_q \circ P_q[\phi]. \tag{53}$$

By the bounds of Section 9, we know that $S_q \circ P_q[\phi] = e_\rho \Delta_\rho^{-1} \circ M_q[e_{-\rho} P_q[\phi]] \in H^1(B)$, so we can replace the outer trace on $\partial\Omega$ with the inner trace as they both extend the restriction to $\partial\Omega$ of smooth functions, which are dense in $H^1(B)$. On the other hand, combining the calculation (42) with the defining property of $P_q[\phi]$, we have that

$$-\int_{\mathbb{R}^n} \nabla(S_q \circ P_q[\phi]) \cdot \nabla\psi = \langle q P_q[\phi], \psi \rangle = -\int_{\mathbb{R}^n} \nabla P_q[\phi] \cdot \nabla\psi$$

whenever $\psi \in C_c^\infty(\Omega)$, and so $\Delta[S_q \circ P_q[\phi] - P_q[\phi]] = 0$ in Ω in the weak sense. Combining this with our hypothesis (53) and the uniqueness of solutions for the Dirichlet problem with zero boundary data, we find that

$$S_q \circ P_q[\phi] = P_q[\phi] \quad \text{in } \Omega. \tag{54}$$

If we had a contraction for S_q, it would be easier to conclude that $\phi = 0$. In any case, we can use the contraction we have by considering

$$\eta := e_{-\rho} S_q \circ P_q[\phi] = \Delta_\rho^{-1} \circ M_q[e_{-\rho} P_q[\phi]] = \Delta_\rho^{-1} \circ M_q[\eta],$$

where the final identity follows from the definition of η and (54). Then our contraction (38) implies that η must be the zero element of $X_{\lambda,\rho}^{1/2}(B)$, so, by the equivalence of the norms, $e_\rho \eta$ must be the zero element of $H^1(B)$. Then, by the definition of η and (54) again, $P_q[\phi]$ is the zero element of $H^1(\Omega)$. Finally, by uniqueness of the Dirichlet problem, ϕ is the zero element of $H^{1/2}(\partial\Omega)$, which completes the proof of the injectivity. □

Remark 11.1. Much of the previous argument is insensitive to the choice of fundamental solutions used to invert Δ and Δ_ρ. Rather than troubling ourselves to invert Δ_ρ using the Faddeev fundamental solution, we could have more easily inverted the operator using the a priori estimates proved in the uniqueness result of [Caro and Rogers 2016]. Indeed, we were able to use those estimates to find a different fundamental solution K_ρ and w such that

$$w(x) - \langle qw, K_\rho(x, \bullet) \rangle = \langle q, K_\rho(x, \bullet) \rangle \quad \text{in } B \setminus \overline{\Omega}.$$

The associated CGO solutions $v = e_\rho(1 + w)$ satisfy

$$v(x) - \langle qv, L_\rho(x, \bullet) \rangle = e_\rho(x) \quad \text{in } B \setminus \overline{\Omega},$$

where $L_\rho(x, y) := e_\rho(x - y) K_\rho(x, y)$ as before. However, not only are these fundamental solutions less explicitly defined, they also fail to satisfy the skew symmetry law (46): that is $K_{-\rho}(x, y) = K_\rho(y, x)$. Thus, even though we know that $L_\rho(\bullet, y)$ is harmonic on $\mathbb{R}^n \setminus \{y\}$, one is unable to conclude that $L_\rho(x, \bullet)$ is harmonic on $\mathbb{R}^n \setminus \{x\}$, which is what allowed us to take it in the boundary integral identity. We attempted to modify the fundamental solution so that the skew symmetry law is satisfied as in [Nachman and Street 2010]; however, we were unable to do this while maintaining the contraction.

We are now ready to complete the formula for the Fourier transform $\hat{q}(\xi) := \langle q, e^{-i\xi \cdot x} \rangle$, with q defined in (4). The proof makes use of the boundary integral identity again combined with the averaging argument due to [Haberman and Tataru 2013].

Theorem 3.2. *Let Π be a two-dimensional linear subspace orthogonal to $\xi \in \mathbb{R}^n$, and define*

$$S^1 := \Pi \cap \{\theta \in \mathbb{R}^n : |\theta| = 1\}.$$

For $\theta \in S^1$, let $\vartheta \in S^1$ be such that $\{\theta, \vartheta\}$ is an orthonormal basis of Π, and define

$$\rho := \tau\theta + i\left(-\frac{\xi}{2} + \left(\tau^2 - \frac{|\xi|^2}{4}\right)^{1/2}\vartheta\right), \quad \rho' := -\tau\theta + i\left(-\frac{\xi}{2} - \left(\tau^2 - \frac{|\xi|^2}{4}\right)^{1/2}\vartheta\right),$$

where $\tau > 1$. Let BI_{Λ_σ} and Γ_{Λ_σ} be defined by (11) and (12), respectively. Then

$$\hat{q}(\xi) = \lim_{T \to \infty} \frac{1}{2\pi T} \int_T^{2T}\!\!\int_{S^1} BI_{\Lambda_\sigma}((I - \Gamma_{\Lambda_\sigma})^{-1}[e_\rho|_{\partial\Omega}], e_{\rho'}) \, d\theta \, d\tau.$$

Proof. Noting that $\rho \cdot \rho = \rho' \cdot \rho' = 0$, we can take the CGO solution $v = e_\rho(1 + w) \in H^1(B)$ given by Corollary 9.2 and $\psi = e_{\rho'}$ in the first boundary integral identity of Lemma 10.1. Noting also that $\rho + \rho' = -i\xi$, the right-hand side of the identity can be written as $\hat{q}(\xi)$ plus a remainder term. Indeed, we find that

$$BI_{\Lambda_\sigma}(v|_{\partial\Omega}, e_{\rho'}) = \hat{q}(\xi) + \langle qw, e^{-i\xi \cdot x} \rangle. \tag{55}$$

Now, for any extension $\tilde{w} \in \dot{X}_\rho^{1/2}$ of w and smooth χ_B equal to 1 on Ω and supported on B, by duality we have that

$$|\langle qw, e^{-i\xi \cdot x} \rangle| \leqslant \|q\|_{\dot{X}_\rho^{-1/2}} \|\chi_B e^{-i\xi \cdot x}\tilde{w}\|_{\dot{X}_\rho^{1/2}} \leqslant C\|q\|_{\dot{X}_\rho^{-1/2}} \|\tilde{w}\|_{\dot{X}_\rho^{1/2}},$$

where the constant $C > 1$ depends on $|\xi|$ and R; see [Haberman and Tataru 2013, Lemma 2.2] or [Caro et al. 2013, (3.17)]. Taking the infimum over extensions, we find

$$|\langle qw, e^{-i\xi \cdot x} \rangle| \leqslant C\|q\|_{\dot{X}_\rho^{-1/2}} \|w\|_{\dot{X}_\rho^{1/2}(B)}.$$

Then, using the estimate (39) for the remainder in Corollary 9.2 and taking an average over ρ, we find that

$$\frac{1}{2\pi T} \int_T^{2T}\!\!\int_{S^1} |\langle qw, e^{-i\xi \cdot x} \rangle| \, d\theta \, d\tau \leqslant \frac{C}{2\pi T} \int_T^{2T}\!\!\int_{S^1} \|q\|^2_{\dot{X}_\rho^{-1/2}} \, d\theta \, d\tau,$$

where $C > 1$ depends on $|\xi|$, the radius R, and $\|\nabla \log \sigma\|_\infty$. Now, [Haberman and Tataru 2013, Lemma 3.1] proved that the right-hand side converges to 0 as $T \to \infty$. Combining with (55), noting that $\hat{q}(\xi)$ is unchanged by the average, yields

$$\hat{q}(\xi) = \lim_{T \to \infty} \frac{1}{2\pi T} \int_T^{2T}\!\!\int_{S^1} BI_{\Lambda_\sigma}(v|_{\partial\Omega}, e_{\rho'}) \, d\theta \, d\tau.$$

Finally, we can use our formula for the values of v on the boundary given by Theorem 3.1, which completes the proof. $\qquad\square$

Remark 11.2. In [Haberman 2015; Ham et al. 2021; Ponce-Vanegas 2021], the contraction was found after taking similar averages over ρ, which yields the existence of a sequence of CGO solutions

$$\{v_j = e_{\rho_j}(1 + w_j)\}_{j \geqslant 1} \quad \text{with } |\rho_j| \to \infty \text{ as } j \to \infty.$$

The authors of the aforementioned works were able to take advantage of the existence of these solutions to prove uniqueness; however, in order to reconstruct in terms of these solutions, one would need to know which values of $\rho_j \in \mathbb{C}^n$ to take.

12. Reconstruction in practice

There is an extensive literature dedicated to the real-world practicalities of the Calderón problem, such as stability, partial data and numerical implementation; see for example [Caro et al. 2016; Delbary et al. 2012; Kenig et al. 2007]. Here we suggest some simplifications that would make things easier to measure and calculate without dwelling on how much the simplifications would corrupt the image.

12.1. *What to measure.* An approximation of the conductivity on the surface $\sigma|_{\partial\Omega}$ could be measured directly by placing real potential differences over pairs of adjacent electrodes, measuring the induced current, and applying Ohm's law. Earlier reconstruction algorithms also required the perpendicular gradient of the conductivity on the surface, which seems harder to measure directly. We would also need to measure an approximation of

$$\text{Meas}_T(\xi) := \frac{1}{2\pi T} \int_T^{2T} \int_{S^1} \int_{\partial\Omega} \Lambda_\sigma[\sigma^{-1/2} e_\rho] \sigma^{-1/2} e_{\rho'} \, d\theta \, d\tau$$

for all $\xi \in R^{-1}\mathbb{Z}^n \cap [-cT, cT]^n$, where $cT > 1$ and R is approximately twice the diameter of Ω. For the complex integrand one can place two separate real electric potentials. Given sufficient access to a large enough part of the surface, one would hope to approximate the inner integral with some accuracy; however, applying the oscillating electric potentials could prove to be the more difficult technical challenge. The outer averaged integrals seem less important and a more rudimentary finite sum approximation could be sufficient.

12.2. *What to calculate.* Given Meas_T and $\sigma|_{\partial\Omega}$, one could then employ a triangular finite element method to calculate an approximate solution to

$$\begin{cases} \Delta v = (\text{Re } q_T)v & \text{in } \Omega, \\ v = \sigma|_{\partial\Omega}^{1/2} & \text{on } \partial\Omega, \end{cases}$$

where, letting $\mathbf{1}_\Omega$ denote the characteristic function of the domain, q_T is defined by

$$q_T(x) := \frac{1}{(2\pi R)^n} \sum_{\xi \in R^{-1}\mathbb{Z}^n \cap [-cT, cT]^n} e^{ix \cdot \xi} \left(\text{Meas}_T(\xi) + \frac{|\xi|^2}{2} \widehat{\mathbf{1}_\Omega}(\xi) \right).$$

Then the grayscale image is given by v^2, taking T as large as is practicable.

12.3. *Justification of the simplifications.* A loose interpretation of Theorem 3.1 is that $v|_{\partial\Omega}$ is not so different from $e_\rho|_{\partial\Omega}$ (this is known as the Born approximation; see [Delbary et al. 2012; Knudsen and Mueller 2011; Siltanen et al. 2000] for numerical implementations). Indeed, if the conductivity were constant, then Γ_{Λ_σ} would be identically 0 and so part of the reconstruction integral from Theorem 3.2 could be rewritten using the divergence theorem:

$$\int_{\partial\Omega} \partial_\nu P_0[e_\rho] e_{\rho'} = \int_\Omega \nabla e_\rho \cdot \nabla e_{\rho'} = \rho \cdot \rho' \int_\Omega e^{-i\xi\cdot x} = -\frac{|\xi|^2}{2}\widehat{\mathbf{1}_\Omega}(\xi).$$

Note also that, by the uncertainty principle, $\hat{q}$ and $\widehat{\mathbf{1}_\Omega}$ are essentially constant at scale R^{-1}. Thus the reconstruction formula approximately simplifies to $\hat{q} \approx \lim_{T\to\infty} \hat{q}_T$ pointwise. Note that the cutoff of the frequencies serves to mollify, so that q_T is a function even though it approximately converges to q in the distributional sense. Finally, one observes that $\sigma^{1/2}$ is the unique solution to the Schrödinger equation with $v|_{\partial\Omega} = \sigma|_{\partial\Omega}^{1/2}$.

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References

[Astala and Päivärinta 2006a] K. Astala and L. Päivärinta, "A boundary integral equation for Calderón's inverse conductivity problem", *Collect. Math.* (2006), 127–139. MR

[Astala and Päivärinta 2006b] K. Astala and L. Päivärinta, "Calderón's inverse conductivity problem in the plane", *Ann. of Math.* (2) **163**:1 (2006), 265–299. MR

[Astala et al. 2016] K. Astala, D. Faraco, and K. M. Rogers, "Unbounded potential recovery in the plane", *Ann. Sci. Éc. Norm. Supér.* (4) **49**:5 (2016), 1027–1051. MR

[Bennett 1974] C. Bennett, "Banach function spaces and interpolation methods, I: The abstract theory", *J. Functional Analysis* **17** (1974), 409–440. MR

[Brown 1996] R. M. Brown, "Global uniqueness in the impedance-imaging problem for less regular conductivities", *SIAM J. Math. Anal.* **27**:4 (1996), 1049–1056. MR

[Brown 2001] R. M. Brown, "Recovering the conductivity at the boundary from the Dirichlet to Neumann map: a pointwise result", *J. Inverse Ill-Posed Probl.* **9**:6 (2001), 567–574. MR

[Brown and Torres 2003] R. M. Brown and R. H. Torres, "Uniqueness in the inverse conductivity problem for conductivities with 3/2 derivatives in L^p, $p > 2n$", *J. Fourier Anal. Appl.* **9**:6 (2003), 563–574. MR

[Calderón 2006] A. P. Calderón, "On an inverse boundary value problem", *Comput. Appl. Math.* **25**:2-3 (2006), 133–138. MR

[Caro and Rogers 2016] P. Caro and K. M. Rogers, "Global uniqueness for the Calderón problem with Lipschitz conductivities", *Forum Math. Pi* **4** (2016), art. id. e2. MR

[Caro et al. 2013] P. Caro, A. García, and J. M. Reyes, "Stability of the Calderón problem for less regular conductivities", *J. Differential Equations* **254**:2 (2013), 469–492. MR

[Caro et al. 2016] P. Caro, D. Dos Santos Ferreira, and A. Ruiz, "Stability estimates for the Calderón problem with partial data", *J. Differential Equations* **260**:3 (2016), 2457–2489. MR

[Delbary et al. 2012] F. Delbary, P. C. Hansen, and K. Knudsen, "Electrical impedance tomography: 3D reconstructions using scattering transforms", *Appl. Anal.* **91**:4 (2012), 737–755. MR

[Faddeev 1965] L. Faddeev, "Increasing solutions of the Schrödinger equation", *Dokl. Akad. Nauk SSSR* **165** (1965), 514–517. In Russian; translated in *Sov. Phys. Dokl.* **10** (1966), 1033–1035.

[García and Zhang 2016] A. García and G. Zhang, "Reconstruction from boundary measurements for less regular conductivities", *Inverse Problems* **32**:11 (2016), art. id. 115015. MR

[Gilbarg and Trudinger 1983] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Grundl. Math. Wissen. **224**, Springer, 1983. Corrected reprint, 1998. MR

[Haberman 2015] B. Haberman, "Uniqueness in Calderón's problem for conductivities with unbounded gradient", *Comm. Math. Phys.* **340**:2 (2015), 639–659. MR

[Haberman and Tataru 2013] B. Haberman and D. Tataru, "Uniqueness in Calderón's problem with Lipschitz conductivities", *Duke Math. J.* **162**:3 (2013), 497–516. MR

[Ham et al. 2021] S. Ham, Y. Kwon, and S. Lee, "Uniqueness in the Calderón problem and bilinear restriction estimates", *J. Funct. Anal.* **281**:8 (2021), art. id. 109119. MR

[Jerison and Kenig 1995] D. Jerison and C. E. Kenig, "The inhomogeneous Dirichlet problem in Lipschitz domains", *J. Funct. Anal.* **130**:1 (1995), 161–219. MR

[Kenig et al. 2007] C. E. Kenig, J. Sjöstrand, and G. Uhlmann, "The Calderón problem with partial data", *Ann. of Math.* (2) **165**:2 (2007), 567–591. MR

[Kim and Kwon 2022] H. Kim and H. Kwon, "Dirichlet and Neumann problems for elliptic equations with singular drifts on Lipschitz domains", *Trans. Amer. Math. Soc.* **375**:9 (2022), 6537–6574. MR

[Knudsen and Mueller 2011] K. Knudsen and J. L. Mueller, "The Born approximation and Calderón's method for reconstruction of conductivities in 3-D", *Discrete Contin. Dyn. Syst.* (2011), 844–853. MR

[Kohn and Vogelius 1984] R. Kohn and M. Vogelius, "Determining conductivity by boundary measurements", *Comm. Pure Appl. Math.* **37**:3 (1984), 289–298. MR

[Kohn and Vogelius 1985] R. V. Kohn and M. Vogelius, "Determining conductivity by boundary measurements, II: Interior results", *Comm. Pure Appl. Math.* **38**:5 (1985), 643–667. MR

[Mitrea and Taylor 1999] M. Mitrea and M. Taylor, "Boundary layer methods for Lipschitz domains in Riemannian manifolds", *J. Funct. Anal.* **163**:2 (1999), 181–251. MR

[Muscalu and Schlag 2013] C. Muscalu and W. Schlag, *Classical and multilinear harmonic analysis*, *I*, Cambridge Studies in Advanced Mathematics **137**, Cambridge University Press, 2013. MR

[Nachman 1988] A. I. Nachman, "Reconstructions from boundary measurements", *Ann. of Math.* (2) **128**:3 (1988), 531–576. MR

[Nachman and Street 2010] A. Nachman and B. Street, "Reconstruction in the Calderón problem with partial data", *Comm. Partial Differential Equations* **35**:2 (2010), 375–390. MR

[Nachman et al. 1988] A. Nachman, J. Sylvester, and G. Uhlmann, "An n-dimensional Borg–Levinson theorem", *Comm. Math. Phys.* **115**:4 (1988), 595–605. MR

[Newton 1989] R. G. Newton, *Inverse Schrödinger scattering in three dimensions*, Springer, 1989. MR

[Novikov 1988] R. G. Novikov, "A multidimensional inverse spectral problem for the equation $-\Delta\psi + (v(x) - Eu(x))\psi = 0$", *Funktsional. Anal. i Prilozhen.* **22**:4 (1988), 11–22. In Russian; translated in *Funct. Anal. Appl.* **22**:4 (1988), 263–272. MR

[Päivärinta et al. 2003] L. Päivärinta, A. Panchenko, and G. Uhlmann, "Complex geometrical optics solutions for Lipschitz conductivities", *Rev. Mat. Iberoamericana* **19**:1 (2003), 57–72. MR

[Ponce-Vanegas 2021] F. Ponce-Vanegas, "A bilinear strategy for Calderón's problem", *Rev. Mat. Iberoam.* **37**:6 (2021), 2119–2160. MR

[Siltanen et al. 2000] S. Siltanen, J. Mueller, and D. Isaacson, "An implementation of the reconstruction algorithm of A Nachman for the 2D inverse conductivity problem", *Inverse Problems* **16**:3 (2000), 681–699. MR

[Sylvester and Uhlmann 1987] J. Sylvester and G. Uhlmann, "A global uniqueness theorem for an inverse boundary value problem", *Ann. of Math.* (2) **125**:1 (1987), 153–169. MR

[Uhlmann 1998] G. Uhlmann, "Inverse boundary value problems for partial differential equations", pp. 77–86 in *Proceedings of the International Congress of Mathematicians*, *III* (Berlin, 1998), edited by G. Fischer and U. Rehmann, Deutsche Mathematiker Vereinigung, Berlin, 1998. MR

[Zhang and Bao 2012] W. Zhang and J. Bao, "Regularity of very weak solutions for elliptic equation of divergence form", *J. Funct. Anal.* **262**:4 (2012), 1867–1878. MR

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