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WEAKLY TURBULENT SOLUTION TO THE SCHRÖDINGER EQUATION ON THE TWO-DIMENSIONAL TORUS WITH REAL POTENTIAL DECAYING TO ZERO AT INFINITY

AMBRE CHABERT

We build a smooth time-dependent real potential on the two-dimensional torus, decaying as time tends to infinity in Sobolev norms along with all its time derivatives, and we exhibit a smooth solution to the associated Schrödinger equation on the two-dimensional torus whose H^s norms nevertheless grow logarithmically as time tends to infinity. We use Fourier decomposition in order to exhibit a discrete resonant system of interactions, which we are further able to reduce to a sequence of finite-dimensional linear systems along which the energy propagates to higher and higher frequencies. The constructions are very explicit, and we can thus obtain lower bounds on the growth rate of the solution.

1. Introduction

1.1. *Main result.* In this paper, we build an explicit $\mathcal{C}^\infty$ solution to the Schrödinger equation on the two-dimensional torus $\mathbb{T}^2 := \mathbb{R}^2/(2\pi\mathbb{Z})^2$,

$$i\partial_t u(t,x) = -\Delta u(t,x) + V(t,x)u(t,x), \quad (t,x) \in [0,+\infty) \times \mathbb{T}^2, \tag{1-1}$$

where the potential $V(t,x)$ is real, smooth on the interval $[0,+\infty)\times\mathbb{T}^2$, and decaying at infinity in Sobolev norms. With a carefully chosen V, we are able to exhibit *weakly turbulent* behaviour; that is, we are able to prove the following theorem.

Theorem 1.1. *There exist a real smooth potential $V(t,x)$ and a smooth function $u(t,x)$ such that*

$$i\partial_t u(t,x) = -\Delta u(t,x) + V(t,x)u(t,x), \quad (t,x) \in [0,+\infty) \times \mathbb{T}^2. \tag{1-2}$$

Furthermore, given any small constant $\delta > 0$ and any order $s > 0$, there exists $c_{\delta,s} > 0$ such that, as $t \to \infty$,

$$\|u(t)\|_{H^s} \geqslant c_{\delta,s}(\log t)^{s(1-\delta)}. \tag{1-3}$$

Finally, the potential V satisfies the bound,

$$\text{for all } k \in \mathbb{N}, \text{ for all } s \geqslant 0, \quad \lim_{t\to\infty} \|\partial_t^k V(t,\cdot)\|_{H^s} = 0. \tag{1-4}$$

We will in Section 5 explore possible upper bounds for the decay rate of V, which is subpolynomial; see (5-16).

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1.2. ***Earlier work.*** The first example of unbounded growth of the Sobolev norms for the Schrödinger equation (1-1) on the torus $\mathbb{T}^2$ was given in [Bourgain 1999a], although there the potential V was chosen to be *quasiperiodic*. Bourgain proves that a logarithmic growth of the Sobolev norms can be achieved in this setting and that it is optimal. Bourgain [1999b] also studied the case of a random behaviour in time with certain smoothness conditions. Furthermore, Bourgain proves in those articles that, with a bounded smooth potential V, the growth in any norm H^s is bounded by t^ε for all $\varepsilon > 0$ (with a constant that depends upon s, V, ε) and that, for a potential *analytic* in time, the bound can be refined to $(\log t)^\alpha$.

With regards to the logarithmic growth rate we are able to achieve in the present article, it is necessarily subpolynomial as V is assumed to be smooth and bounded, but we may not use the logarithmic a priori bound as $V(t)$ is not analytic in t in our construction. Still, logarithmic growth rate is nearly optimal as the optimal growth is necessarily subpolynomial.

The study of upper bounds on the possible growth rate of Sobolev norms of the solutions to the linear Schrödinger equation has a long history. The general question can be formulated as follow: consider u a regular solution to

$$i\partial_t u = Hu + P(t)u, \tag{1-5}$$

where H is either the Laplacian $-\Delta$ on a d-dimensional torus, or, more generally, when the domain is $\mathbb{R}^d$ or even a manifold, a time-independent self-adjoint nonnegative operator with some assumptions on its spectrum, and $P(t)$ is a smooth time-dependent family of pseudodifferential operators of order strictly less than 2. Then one can try and prove an upper bound on the growth rate of $\|u(t)\|_{H^s}$ as $t \to \infty$.

Maspero and Robert [2017, Theorem 1.9] proved, along with global well-posedness, t^ε upper bounds on the growth rate in the case where H has an increasing spectral gap (as is the case for the Laplacian on Zoll manifolds) and $P(t)$ is a smooth perturbation with suitable assumptions. They also proved polynomial upper bounds in broader settings. Under the increasing spectral gap assumption, the bound can be improved to $(\log t)^\gamma$ for some $\gamma > 0$ when $P(t)$ is analytic in time, which is reminiscent of Bourgain's bound. Using those results, [Bambusi et al. 2021] proved t^ε upper bounds on the growth rate of solutions to (1-5) in an abstract setting, which includes in particular the case where H is the harmonic oscillator in $\mathbb{R}^d$ and $P(t)$ is a pseudodifferential operator of order strictly lower than H depending in a quasiperiodic way on time. The first result of a t^ε upper bound with an *unbounded* $P(t)$ was obtained in [Bambusi et al. 2022] on the torus $\mathbb{T}^d$ with $H = -\Delta$. Finally, t^ε upper bounds have been proved for general hamiltonians of quantum integrable systems in [Bambusi and Langella 2022].

Regarding the dual question of exhibiting growth of Sobolev norms in solutions to (1-5), Maspero [2022; 2023] proved the existence of solutions with (unbounded) polynomial growth in the case where H has a fixed spectral gap and $P(t)$ is a potential *periodic* in time using a resonance phenomenon. Loosening the time smoothness hypothesis, Erdoğan, Killip and Schlag [Erdoğan et al. 2003] showed genericity of Sobolev norm growth when the potential is a stationary Markov process. See also [Delort 2010; Eliasson and Kuksin 2009; Nersesyan 2009; Wang 2008].

Regarding potentials whose Sobolev norms decay to 0 with time more specifically, Faou and Raphaël [2023] were able to exhibit logarithmic growth in the context where

$$H = -\Delta + |x|^2$$

is the harmonic oscillator on $\mathbb{R}^2$. Their method relies on *quasiconformal modulations* of so-called *bubble* solutions of the unperturbed Schrödinger equation. It is not surprising that we are able to exhibit logarithmic growth on the torus as the setting is similar. Indeed, both the harmonic oscillator on $\mathbb{R}^2$ and the Laplacian on the torus are operators with compact resolvent and a spectrum with geometric properties (as it is formed of points in a lattice) which allows for explicit resonance mechanism. Let us note that the author was able to prove in [Chabert 2024] that their method extends to the case where the cubic nonlinear term $u|u|^2$ is added to the equation, using an approximation scheme similar to the one found in the present article.

The method we shall use here is inspired by the seminal work [Colliander et al. 2010] refined by [Guardia and Kaloshin 2015]. Indeed, we use that, on the two-dimensional torus, eigenfunctions of the Laplacian are given by $e^{in\cdot x}$ for $n \in \mathbb{Z}^2$ with eigenvalue $|n|^2$. The lattice structure is then used to produce resonance phenomena between carefully chosen frequencies of the Fourier decomposition of the solution u. The idea is that only certain *resonant* interactions will dominate the behaviour of the solution; thus, using an arbitrarily small potential, we are able to transfer the energy of the solution to higher and higher frequencies, leading to the growth of Sobolev norms.

1.3. ***Main ideas of the proof.*** The first step of the proof is directly inspired by [Colliander et al. 2010]. In Section 2, we decompose (1-1) into Fourier frequencies, thus reducing it to an infinite-dimensional ODE on the Fourier frequencies $(a_n(t))$ of the solution. This enables us to exhibit some *resonant interactions* between Fourier frequencies, which will dominate the behaviour of the solution in terms of Sobolev norms. In that spirit, we first study a *resonant Fourier system* where we drop the nonresonant interactions. We then build a family of Fourier frequencies $(m_n)_{n\geqslant 0}$, satisfying carefully computed *orthogonality properties*, along which we are able to transfer energy to higher frequencies (as $|m_n| \to \infty$) with a well-tailored potential V for a solution $(a_n(t))$ whose Fourier frequencies are almost supported on the (m_n).

In Section 3, we give a detailed construction of a potential allowing said energy transfer to higher frequencies, thanks to the crucial point that, as we only consider resonant interactions, we may light up only specific Fourier frequencies in the potential, which further reduces the resonant system to a *sequence of finite-dimensional linear systems* which we can explicitly solve.

In Sections 4 and 5, we prove that the solution to the resonant system yields a solution to the full system up to a perturbation thanks to a Cauchy sequence scheme, thus concluding that the perturbation decays to 0 as $t \to \infty$. We finally use the explicit construction of the solution to the resonant system to deduce lower bounds on the growth of the Sobolev norm of the full solution, thus concluding to the proof of Theorem 1.1.

2. Fourier decomposition and resonant system

2.1. ***Reduction to a resonant Fourier system.*** We now show how (1-1) can be heuristically approximated by an easier equation, focussing on the *resonant* interactions. Indeed, as we wish to find smooth solutions of (1-1), we may write

$$u(t,x) = \sum_{n\in\mathbb{Z}^2} a_n(t) e^{i(n\cdot x - |n|^2 t)}. \tag{2-1}$$

We now set the potential to take the form

$$V(t,x) = -\sum_{n\in\mathbb{Z}^2} 2v_n(t)\sin(|n|^2 t)e^{in\cdot x}, \tag{2-2}$$

where $v_{-n} = v_n$ is real. Thus, we need only find a solution to the l^2 system

$$\partial_t a_n = \sum_{m\in\mathbb{Z}^2} a_m(t)v_{n-m}(t)(e^{-i\omega^+_{m,n}t} - e^{-i\omega^-_{m,n}t}), \tag{$\mathcal{FS}$}$$

where we set

$$\omega^+_{m,n} := |m|^2 + |m-n|^2 - |n|^2,$$
$$\omega^-_{m,n} := |m|^2 - |m-n|^2 - |n|^2.$$

Now, in the spirit of [Colliander et al. 2010], we expect that the resonant interaction will dominate, that is, the interaction between frequencies m, n such that one of $\omega^+_{m,n}$ or $\omega^-_{m,n}$ is 0. We thus define, for $n \in \mathbb{Z}^2$,

$$\Gamma^+_{\text{res}}(n) := \{m \in \mathbb{Z}^2 : |m|^2 + |m-n|^2 - |n|^2 = 0\},$$
$$\Gamma^-_{\text{res}}(n) := \{m \in \mathbb{Z}^2 : |m|^2 - |m-n|^2 - |n|^2 = 0\}$$

and define the approximated system

$$\partial_t a_n = \sum_{m\in\Gamma^+_{\text{res}}(n)} a_m(t)v_{n-m}(t) - \sum_{m\in\Gamma^-_{\text{res}}(n)} a_m(t)v_{n-m}(t). \tag{$\mathcal{RFS}$}$$

We observe that ($\mathcal{RFS}$) conserves the l^2 norm. Indeed,

$$\frac{d}{dt}\|(a_n)\|^2_{l^2} = 2\,\text{Re}\Biggl(\sum_{n\in\mathbb{Z}^2}\sum_{m\in\Gamma^+_{\text{res}}(n)} \overline{a_n(t)}a_m(t)v_{n-m}(t) - \sum_{n\in\mathbb{Z}^2}\sum_{m\in\Gamma^-_{\text{res}}(n)} \overline{a_n(t)}a_m(t)v_{n-m}(t)\Biggr).$$

However, $m \in \Gamma^+_{\text{res}}(n)$ if and only if $n \in \Gamma^-_{\text{res}}(m)$. Using moreover that $v_{-k} = v_k$, we see that the right-hand side equals 0.

2.2. ***Geometric interpretation of the resonant frequencies.*** Now, we turn our attention to the geometric interpretation of the equation $\omega^{+/-}_{m,n} = 0$: we first see that $\omega^+_{m,n} = 0$ if and only if

$$\begin{cases} m + (n-m) = n, \\ |m|^2 + |n-m|^2 = |n|^2, \end{cases} \tag{2-3}$$

which means that m is orthogonal to $n-m$. This can be reformulated by saying that m resonates with $m+l$, where $l \in \mathbb{Z}^2$ is orthogonal to m.

Similarly we see that $\omega^-_{m,n} = 0$ if and only if $(n-m)$ is orthogonal to n, which finally means that m and n are resonant frequencies if one of m or n is the sum of the other one and an orthogonal vector. We may sum these facts up in a lemma.

Lemma 2.1. *For all $n, m \in \mathbb{Z}^2$, we have $m \in \Gamma^+_{\text{res}}(n)$ if and only if m and $n-m$ are orthogonal. Moreover, $m \in \Gamma^-_{\text{res}}(n)$ if and only if n and $n-m$ are orthogonal.*

2.3. ***Explicit family of resonant frequencies and further reduction.*** We shall now build a potential $(v_m(t))$ and a specific solution to $(\mathcal{RFS})$ by constructing two families (m_k) and (l_k), $k \geqslant 0$, of vectors of $\mathbb{Z}^2$ which satisfy good orthogonality properties. Namely, in some sense, we require that there are no exceptional resonances.

Lemma 2.2. *There exist two families $(m_k)_{k\geqslant 0}$ and $(l_k)_{k\geqslant 0}$ of vectors of $\mathbb{Z}^2$ such that*

(P1) $m_k \neq 0,\ l_k \neq 0$;

(P2) $m_k \perp l_{k'} \Leftrightarrow k = k'$;

(P3) $m_{k+1} = m_k + l_k$;

(P4) m_k *is not orthogonal to* $m_{k'}$ *and is not orthogonal to* $m_{k'} - l_{k'}$ *for all* k, k';

(P5) $m_k - l_k$ *is not orthogonal to* $l_{k'}$ *for all* k, k';

(P6) $m_{k'} - l_k$ *is not orthogonal to* l_k *for all* $k' \neq k+1$;

(P7) $m_{k'} - l_{k'} - l_k$ *is not orthogonal to* l_k *for all* k, k';

(P8) $l_k + m_{k'}$ *is not orthogonal to* l_k *for all* k, k';

(P9) $l_k + m_{k'} - l_{k'}$ *is not orthogonal to* l_k *for all* $k \neq k'$;

(P10) $|l_{k+1}| > |l_k| + 1$.

Moreover, we can find families such that there exist universal constants $C > 1 > c$ such that, for all $n \geqslant 1$, we have

$$c(n-1)! \leqslant |m_n| \leqslant C^n (n-1)!,$$
$$cn! \leqslant |l_n| \leqslant C^n n!.$$

At first glance these properties may seem overwhelming, but it follows quite directly from geometric observations that they greatly reduce the system if we choose the potential with nonzero Fourier frequencies supported in the set $\{\pm l_k\}_{k\geqslant 0}$. More precisely, before proving Lemma 2.2, we will state and prove the following lemma.

Lemma 2.3. *Set $\Lambda := \{\pm l_k,\ k \geqslant 0\}$ and $\Lambda' := \{m_k,\ k \geqslant 0\}$. Set moreover $\Sigma := \{m_k - l_k,\ k \geqslant 0\}$. Assume $(a_n(t))_{n\in\mathbb{Z}^2}$ is a solution to $(\mathcal{RFS})$ with potential $(v_n(t))_n$ such that $(a_n(0))$ is supported in $\Lambda' \cup \Sigma$ (in the sense that $a_n(0) = 0$ whenever $n \notin \Lambda' \cup \Sigma$). If $(v_n(t))_n$ is supported in Λ for all $t \geqslant 0$, then $(a_n(t))$ is supported in $\Lambda' \cup \Sigma$ for all $t \geqslant 0$.*

Moreover, define $p_k(t) := a_{m_k}(t)$, $s_k(t) := a_{m_k - l_k}(t)$ and $r_k(t) := v_{l_k}(t)$ (with the convention that $p_{-1} = r_{-1} = 0$). The system $(\mathcal{RFS})$ reduces to,

$$\text{for all } k \geqslant 0, \qquad \begin{cases} \partial_t p_k = p_{k-1} r_{k-1} - p_{k+1} r_k - s_k r_k, \\ \partial_t s_k = p_k r_k. \end{cases} \tag{2-4}$$

Proof. As $v_n(t) = 0$ whenever $n \notin \Lambda$, $(\mathcal{RFS})$ reduces to

$$\partial_t a_n = \sum_{\substack{m \in \Gamma^+_{\text{res}}(n) \\ n-m \in \Lambda}} a_m(t) v_{n-m}(t) - \sum_{\substack{m \in \Gamma^-_{\text{res}}(n) \\ n-m \in \Lambda}} a_m(t) v_{n-m}(t). \tag{2-5}$$

In order to prove the first part of the lemma, we need only show that, whenever $n \notin \Lambda' \cup \Sigma$, those m that appear on the right-hand side of (2-5) are also not in $\Lambda' \cup \Sigma$. Indeed, the system then reduces to a linear system with zero initial condition on $\mathbb{Z}^2 \backslash \Lambda' \cup \Sigma$, so by uniqueness we have $a_n(t) = 0$ for all t whenever $n \notin \Lambda' \cup \Sigma$.

Take $n \notin \Lambda' \cup \Sigma$. We claim that if $m \in \Lambda' \cup \Sigma$ satisfies $n - m \in \Lambda$, then $m \notin \Gamma^+_{\text{res}}(n) \cup \Gamma^-_{\text{res}}(n)$. Indeed, assume first that $m = m_k$ for some k and $n - m \in \Lambda$. Then there exists $k' \geqslant 0$ such that $n - m = \pm l_{k'}$.

(i) If $n = m_k + l_{k'}$, then $k \neq k'$ as otherwise $n = m_{k+1} \in \Lambda'$; but then m_k is not orthogonal to $n - m_k = l_{k'}$ thanks to (P2), thus $m_k \notin \Gamma^+_{\text{res}}(n)$. Similarly, $n - m_k = l_k$ is not orthogonal to $n = m_k + l_{k'}$ thanks to (P8).

(ii) If $n = m_k - l_{k'}$, then $k' \neq k$ as otherwise $n \in \Sigma$ and $k' \neq k - 1$ as otherwise $n = m_{k-1} \in \Lambda'$. Thus, $m_k \notin \Gamma^+_{\text{res}}(n)$ as m_k is not orthogonal to $-l_{k'}$ by (P2), and $m_k \notin \Gamma^-_{\text{res}}(n)$ as $l_{k'} = n - m_k$ is not orthogonal to $m_k - l_{k'} = n$ thanks to (P6).

Now, assume that $m = m_k - l_k$ for some $k \geqslant 0$ and $n - m = \pm l_{k'}$ for some $k' \geqslant 0$.

(i) If $n = m_k - l_k + l_{k'}$, then $k \neq k'$ as $n \notin \Lambda'$, so $m_k - l_k$ is not orthogonal to $l_{k'}$ thanks to (P5) — thus $m_k - l_k \notin \Gamma^+_{\text{res}}(n)$ — and $l_{k'}$ is not orthogonal to $m_k - l_k + l_{k'}$ thanks to (P9) — thus $m_k - l_k \notin \Gamma^-_{\text{res}}(n)$.

(ii) Finally, if $n = m_k - l_k - l_{k'}$, then — as $m_k - l_k$ is not orthogonal to $-l_{k'}$ thanks to (P5) — we find that $m_k - l_k \notin \Gamma^+_{\text{res}}(n)$ and — as $m_k - l_k - l_{k'}$ is not orthogonal to $-l_{k'}$ thanks to (P7) — we also find that $m_k - l_k \notin \Gamma^-_{\text{res}}(n)$.

In order to prove the second part of the lemma, we follow the same steps. Take $k \geqslant 0$. First, let $m \in \Gamma^+_{\text{res}}(m_k) \cap (\Lambda' \cup \Sigma)$ such that $m_k - m = \pm l_{k'}$. As m is orthogonal to $l_{k'}$, properties (P2) and (P5) yield that $m = m_{k'}$, and thus $m_k = m_{k'} \pm l_{k'}$. As m_k is orthogonal to l_k, (P5) yields that necessarily $m_k = m_{k'} + l_{k'} = m_{k'+1}$, and thus $k' = k - 1$ (as from (P3) and (P10) we have $|m_{i+1}| > |m_i|$), which yields the contribution $p_{k-1} r_{k-1}$ to the right-hand side of the first equation.

Now, let $m \in \Gamma^-_{\text{res}}(m_k) \cap (\Lambda' \cup \Sigma)$ such that $m_k - m = \pm l_{k'}$. As $l_{k'}$ is orthogonal to m_k, (P2) yields that $k = k'$ — thus $m = m_k \pm l_k$ — and both are in $\Gamma^-_{\text{res}}(m_k) \cap (\Lambda' \cup \Sigma)$. This yields the contribution $-p_{k+1} r_k - s_k r_k$ to the right-hand side of the first equation.

Finally, take $k \geqslant 0$ and $m \in \Gamma^+_{\text{res}}(m_k - l_k) \cap (\Lambda' \cup \Sigma)$ such that $m_k - l_k - m = \pm l_{k'}$: as m is orthogonal to $l_{k'}$, we find again that $m = m_{k'}$, and thus that $m_k = l_k + m_{k'} \pm l_{k'}$. If the sign is a minus, property (P9) yields that $k' = k$, and thus $m = m_k$, which gives the contribution $p_k r_k$ to the right-hand side of the second equation (as we recall that $v_{-n} = v_n$ for all n). If the sign is a plus, we find that $m_k - l_k = m_{k'+1}$ is orthogonal to $l_{k'+1}$, which contradicts property (P5).

We see moreover that there is not a $m \in \Gamma^-_{\text{res}}(m_k - l_k) \cap (\Lambda' \cup \Sigma)$ such that $m_k - l_k - m \in \Lambda$. Indeed by definition of Γ^-_{res}, this would mean that there is a k' such that $m_k - l_k$ is orthogonal to $l_{k'}$ thus contradicting property (P5). □

We now turn to the proof of Lemma 2.2. Choose $m_0 \in \mathbb{Z}^2 \backslash \{0\}$ arbitrarily; for example, $m_0 = (1, 0)$. As $m_{k+1} = m_k + l_k$, we need only construct the l_k for $k \geqslant 0$. We will do so by induction. Assume the sequence (m_k) is constructed up to $k = n$ and satisfies properties (P1)–(P10) (which means that $l_0, \ldots, l_{n-1}$ have been constructed). We need to construct $l_n \in \mathbb{Z}^2$ (and thus $m_{n+1} = m_n + l_n$) such that the properties still hold up to $k = n + 1$. Define m the vector obtained from m_n by applying a rotation by angle $\pi/2$ (which is orthogonal to m_n and which has the same Euclidean norm). We will show that there is $a \in \mathbb{N}$ such that $n + 1 \leqslant a \leqslant C(n + 1)$, with C a universal constant such that setting $l_n := am$ will suffice.

- (P1) always holds, and (P3) holds by definition.

- In order for (P2) to hold, observe first that, by construction, $l_n = am$ is orthogonal to m_n. Moreover, we need, on the one hand, m_k to be nonorthogonal to l_n for $k \leqslant n - 1$. However, since $m_k \perp l_k$ up to $k = n$ by induction and since we are in dimension 2, this amounts to asking l_k to be nonorthogonal to m_n for $k \leqslant n - 1$, which is true by induction from (P2). On the other hand, we need m_{n+1} to be nonorthogonal to l_k up to $k = n$; that is, since $l_k \perp m_k$ and since we are in dimension 2, we need only prove that $m_{n+1} = m_n + am$ is not parallel to m_k for $k \leqslant n$. It is always true for $k = n$ as $a > 0$, and, for each $k \leqslant n - 1$, there is at most one value of a for which m_{n+1} could be parallel to m_k (as m is not parallel to m_k, otherwise m_k would be orthogonal to m_n, thus contradicting (P4)). This excludes at most n possible values for a.

- In order for (P4) to hold, we need $m_{n+1} \cdot m_k \neq 0$ for $k \leqslant n$. It is always true for $k = n$, and for $k < n$ it means that $m_n \cdot m_k + am \cdot m_k \neq 0$. Now, $m \cdot m_k \neq 0$, otherwise this would contradict (P2). Thus, at most n possible values of a are to be excluded. We also need $m_{n+1} \cdot (m_k - l_k) \neq 0$ for $k \leqslant n$, which is always true for $k = n$ if we set $a \geqslant 2$, and it follows from the construction of (P5) that $m \cdot (m_k - l_k) \neq 0$, as m_n is not parallel to $m_k - l_k$; hence this excludes at most n values of a. We finally need $m_k \cdot (m_n - am) \neq 0$ for $k \leqslant n - 1$, which excludes at most n values of a, as $m \cdot m_k \neq 0$.

- In order for (P5) to hold, we need, on the one hand, $m_k - l_k$ to be nonparallel to m_{n+1} for $k \leqslant n$, which excludes at most n values of a, as this is always true for $k = n$ and as $m_k - l_k$ is not parallel to m for $k < n$ thanks to (P4). On the other hand, we need $m_n - l_n = m_n - am$ to be nonparallel to m_k for $k \leqslant n$, which again excludes at most n values for a as m is not parallel to m_k for $k < n$ thanks to (P2).

- In order for (P6) to hold, we need, on the one hand, $(m_{n+1} - l_k) \cdot l_k \neq 0$ for $k \leqslant n - 1$, which is equivalent to $am \cdot l_k \neq$ constant. As we know that $m \cdot l_k \neq 0$ (otherwise m_k is orthogonal to m_n), this excludes at most n values for a. One the other hand, we need $m_k - am$ to be nonorthogonal to am for $k \leqslant n - 1$, which is ensured by the fact that $|m| > |m_k|$.

- In order for (P7) to hold, we need, on the one hand, $m_n - am - l_k$ to be nonorthogonal to l_k for $k \leqslant n - 1$, which excludes at most n values for a, as $m \cdot l_k \neq 0$. On the other hand, we need $m_k - l_k - am$ to be nonorthogonal to am for $k \leqslant n - 1$, and once again this excludes at most n values for a.

- In order for (P8) to hold, we need, on the one hand, $am + m_k$ to be nonorthogonal to am for $k \leqslant n$, thus excluding at most n values for a, and, on the other hand, $l_k + m_n + am$ to be nonorthogonal to l_k for $k \leqslant n - 1$, which excludes at most n values for a, as $m \cdot l_k \neq 0$.

- In order for (P9) to hold, we need, on the one hand, $am + m_k - l_k$ to be nonorthogonal to am for $k \leqslant n - 1$, thus excluding at most n values for a, and, on the other hand, $l_k + m_n - am$ to be nonorthogonal to l_k, excluding once again at most n values for a.

We thus finally see that any $a \geqslant 1$ except maybe at most $C(n+1)$ values can be chosen, where $C \geqslant 2$. Up to taking C a little larger we may thus find $n+1 \leqslant a \leqslant C(n+1)$ such that setting $l_n = am$ enables the induction hypothesis to be satisfied.

By this procedure we are able to construct sequences for which the desired properties hold. Moreover, we have $n|m_n| \leqslant |l_n| \leqslant Cn|m_n|$, and thus

$$n|m_n| \leqslant |m_{n+1}| \leqslant C'n|m_n|$$

for $C' = \sqrt{C+1}$; thus proving the last part of the lemma.

3. Solution to the resonant system

Thanks to the previous section, we are now able to exhibit explicit potential $(r_k(t))$ and an explicit solution $(p_k(t))$, $(s_k(t))$ to (2-4) for which we control precisely the energy transfer between Fourier frequencies. We turn to the explicit study of the mechanism that will allow energy transfer between frequencies. We start at $t = 0$ with well-chosen values for p_0, p_1, s_0 and set the other p_k and s_k to be 0. The idea is then to locally fully transfer the energy from (p_k, s_k, p_{k+1}) to $(p_{k+1}, s_{k+1}, p_{k+2})$ in finite time, thus ensuring that, for all given n, after a time T_n, we have $p_k = s_k = 0$ for all $k \leqslant n$. Now, as $(\mathcal{RFS})$ conserves the l^2 norm, this ensures that the Sobolev H^s norm is greater than $|m_n|^s$ for $t \geqslant T_n$.

3.1. ***General form of the solution to the linear system.*** Explicitly, fix an interval $I = [t_0, t_1]$ and a smooth function ϕ on I. Fix $k \geqslant 1$. We look at the system

$$\begin{cases} \partial_t p_{k+1} = \phi(t) p_k, \\ \partial_t p_k = -\phi(t) p_{k+1} - \phi(t) s_k, \\ \partial_t s_k = \phi(t) p_k, \end{cases} \tag{3-1}$$

which corresponds to (2-4) when we only light up $r_k(t) = \phi(t)$; that is, we set $r_{k'}(t) = 0$ for $k' \neq k$ on I. In comparison, (2-4) is a system on *all* different values of k, whereas in system (3-1) we have *fixed* a particular value for k. Hence, the equation for p_{k+1} corresponds to the first line of (2-4), where k is replaced by $k+1$ and $r_l = 0$ for $l \neq k$. The system can then be written in the form of a simple linear system:

$$\partial_t \begin{pmatrix} p_{k+1} \\ p_k \\ s_k \end{pmatrix} = \phi(t) A \begin{pmatrix} p_{k+1} \\ p_k \\ s_k \end{pmatrix}, \tag{3-2}$$

where we set

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{3-3}$$

Now, the solution with initial condition

$$\begin{pmatrix} p_{k+1}(t_0) \\ p_k(t_0) \\ s_k(t_0) \end{pmatrix}$$

is given by

$$\begin{pmatrix} p_{k+1}(t) \\ p_k(t) \\ s_k(t) \end{pmatrix} = \exp\left(\left(\int_{t_0}^{t} \phi(s)\,ds\right) A\right) \begin{pmatrix} p_{k+1}(t_0) \\ p_k(t_0) \\ s_k(t_0) \end{pmatrix}. \tag{3-4}$$

Now, one can compute

$$\exp(TA) = \begin{pmatrix} \frac{1}{2}(\cos(T\sqrt{2})+1) & \frac{1}{\sqrt{2}}\sin(T\sqrt{2}) & \frac{1}{2}(\cos(T\sqrt{2})-1) \\ -\frac{1}{\sqrt{2}}\sin(T\sqrt{2}) & \cos(T\sqrt{2}) & -\frac{1}{\sqrt{2}}\sin(T\sqrt{2}) \\ \frac{1}{2}(\cos(T\sqrt{2})-1) & \frac{1}{\sqrt{2}}\sin(T\sqrt{2}) & \frac{1}{2}(\cos(T\sqrt{2})+1) \end{pmatrix}. \tag{3-5}$$

This explicit matrix allows us to build three moves in order to transfer a specific configuration from (p_k, s_k, p_{k+1}) to $(p_{k+1}, s_{k+1}, p_{k+2})$ in finite time.

3.1.1. *First move.* Start with

$$\begin{pmatrix} p_{k+1}(t_0) \\ p_k(t_0) \\ s_k(t_0) \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{pmatrix}. \tag{3-6}$$

We set ϕ, a nonnegative $\mathcal{C}^\infty$ function with support in $[t_0, t_1]$ such that $\int \phi = 7\pi/(4\sqrt{2})$. We have

$$\begin{pmatrix} p_{k+1}(t_1) \\ p_k(t_1) \\ s_k(t_1) \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}. \tag{3-7}$$

3.1.2. *Second move.* We now set

$$\begin{pmatrix} p_{k+1}(t_0) \\ p_k(t_0) \\ s_k(t_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{3-8}$$

With the integral of ϕ being $\pi/(2\sqrt{2})$, we have

$$\begin{pmatrix} p_{k+1}(t_1) \\ p_k(t_1) \\ s_k(t_1) \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}. \tag{3-9}$$

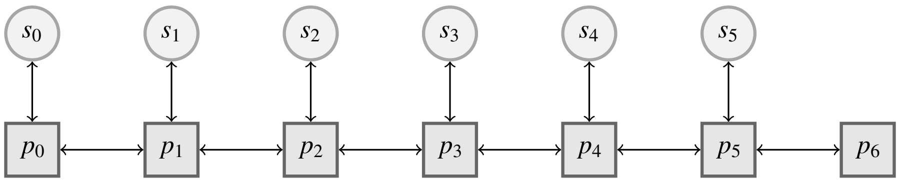
3.1.3. *Third move.* If finally we set

$$\begin{pmatrix} p_{k+1}(t_0) \\ p_k(t_0) \\ s_k(t_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{3-10}$$

and set the integral of ϕ to be $\pi/\sqrt{2}$, we have

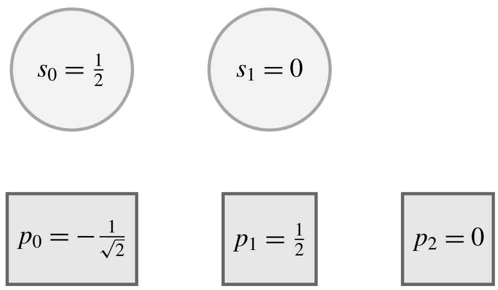
$$\begin{pmatrix} p_{k+1}(t_1) \\ p_k(t_1) \\ s_k(t_1) \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}. \tag{3-11}$$

3.2. ***Idea of the construction of the potential and the resonant solution.*** These easy observations yield the construction both of the potential $(r_k(t))$ and of the solution $(p_k(t)),\ (s_k(t))$. We may represent the solution $(p_k(t)),\ (s_k(t))$ as points in the semi-infinite chain

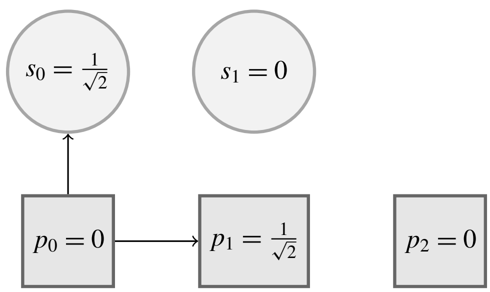


where the arrows represent the possible interactions between the Fourier frequencies induced by the potential $(r_k(t))$.

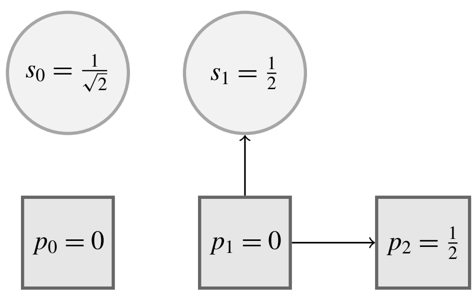
Assume that, at $t = 0$, we start with the configuration



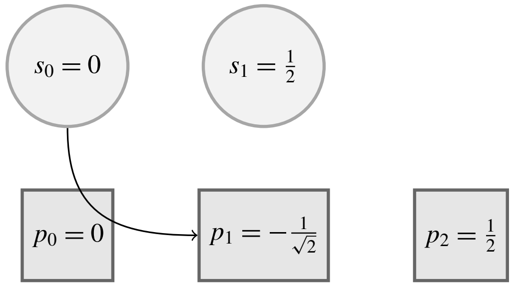
Then, using the first move, if we light up only r_0 during an appropriate time, we may fully transfer the mass from p_0 to s_0 and p_1 equally:



Now, we clear p_1 using the second move; that is, lighting up only $r_1(t)$, we can fully transfer the mass from p_1 to s_1 and p_2 equally:



Finally, we use the third move to transfer fully the remaining mass from s_0 to p_1 lighting only r_0 again:



Thus, we find exactly the same situation as at the start, with indexes incremented by 1. This enables us to start a recursive scheme so that, as time goes by, we repeat these three moves to transfer the mass to higher frequencies. The strategy to ensure that the potential V decreases in Sobolev norms as $t \to \infty$ is that, up to lighting r_k for a longer time, we may at each step choose it to be arbitrarily small.

3.3. ***Explicit computation of the potential and of the resonant solution.*** We now make the previous argument rigorous. We first find a smooth function ϕ on $\mathbb{R}$, nonnegative and nondecreasing, such that $\phi = 0$ on $(-\infty, 0]$, $\phi = 1$ on $[1, +\infty)$, and we set $\alpha = \int_0^1 \phi$. Take $(\beta_k)_{k \geqslant 0}$ to be a sequence of positive real numbers such that $\beta_k \ll 1$. The (β_k) will control the amplitude to which we light up r_k, and we will fix them later in order to control the decay of the potential V in Sobolev norms.

3.3.1. *Initialising the induction.* We choose, at $t = 0$,

$$\begin{pmatrix} p_1(0) \\ p_0(0) \\ s_0(0) \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{pmatrix} \tag{3-12}$$

(and the other p_k, s_k are set to 0). We now set

$$r_0(t) = \begin{cases} \frac{7\pi}{4\sqrt{2}\alpha}\beta_0\phi(t), & 0 \leqslant t \leqslant 1, \\ \frac{7\pi}{4\sqrt{2}\alpha}\beta_0, & 1 \leqslant t \leqslant 1 + t_0, \\ \frac{7\pi}{4\sqrt{2}\alpha}\beta_0\phi(t_0 + 2 - t), & 1 + t_0 \leqslant t \leqslant t_0 + 2, \end{cases} \tag{3-13}$$

where we set t_0 such that $\int_0^{t_0+2} r_0 = 7\pi/(4\sqrt{2})$, which means $t_0 = \alpha(\beta_0^{-1} - 2)$. We set $r_k(t) = 0$ on $[0, t_0 + 2]$ for all $k \geqslant 1$.

Now, at $t = t_0 + 2$, we find

$$\begin{pmatrix} p_1(t_0+2) \\ p_0(t_0+2) \\ s_0(t_0+2) \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}. \tag{3-14}$$

Set now

$$r_1(t) = \begin{cases} \frac{\pi}{2\sqrt{2}\alpha}\beta_1\phi(t - (t_0 + 2)), & t_0 + 2 \leqslant t \leqslant t_0 + 3, \\ \frac{\pi}{2\sqrt{2}\alpha}\beta_1, & t_0 + 3 \leqslant t \leqslant t_0 + 3 + t_1, \\ \frac{\pi}{2\sqrt{2}\alpha}\beta_1\phi(4 + t_0 + t_1 - t), & 3 + t_0 + t_1 \leqslant t \leqslant 4 + t_0 + t_1, \end{cases} \tag{3-15}$$

with t_1 such that the integral of r_1 on $[2+t_0, 4+t_0+t_1]$ is equal to $\pi/(2\sqrt{2})$, which means $t_1 = \alpha(\beta_1^{-1}-2)$. Set $r_k(t) = 0$ on $[2+t_0, 4+t_0+t_1]$ for all $k \neq 1$. Now, at $t = 4+t_0+t_1$, we have

$$\begin{pmatrix} p_2(4+t_0+t_1) \\ p_1(4+t_0+t_1) \\ s_1(4+t_0+t_1) \\ p_0(4+t_0+t_1) \\ s_0(4+t_0+t_1) \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \\ 1/\sqrt{2} \end{pmatrix}, \tag{3-16}$$

and the other p_k, s_k are equal to 0. To finish the cycle, we need to transfer all the mass from s_0 to p_1, and we will end up with $(p_2, p_1, s_1) = (1/2, -1/\sqrt{2}, 1/2)$ which was exactly the initial state on (p_1, p_0, s_0). This enables us to start a recursive process. More precisely, set

$$r_0(t) = \begin{cases} \frac{\pi}{\sqrt{2}\alpha}\beta_0\phi(t-(4+t_0+t_1)), & 4+t_0+t_1 \leqslant t \leqslant 5+t_0+t_1, \\ \frac{\pi}{\sqrt{2}\alpha}\beta_0, & 5+t_0+t_1 \leqslant t \leqslant 5+t_0+t_1+t_0, \\ \frac{\pi}{\sqrt{2}\alpha}\beta_0\phi(6+t_0+t_1+t_0-t), & 5+t_0+t_1+t_0 \leqslant t \leqslant 6+t_0+t_1+t_0, \end{cases} \tag{3-17}$$

with once again $t_0 = \alpha(\beta_0^{-1}-2)$, and $r_k(t) = 0$ on $[4+t_0+t_1, 6+t_0+t_1+t_0]$ for $k \neq 1$. We have, at $t = 6+2t_0+t_1$,

$$\begin{pmatrix} p_2(t) \\ p_1(t) \\ s_1(t) \\ p_0(t) \\ s_0(t) \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \\ 0 \\ 0 \end{pmatrix}, \tag{3-18}$$

as was expected.

3.3.2. *Recursive scheme.* Set $t_n := \alpha(\beta_n^{-1}-2)$. Suppose, for $T_n = 6n+2t_0+3t_1+3t_2+\cdots+3t_{n-1}+t_n$, we have

$$\begin{pmatrix} p_{n+1} \\ p_n \\ s_n \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{pmatrix}, \tag{3-19}$$

with the other p_k, s_k being equal to 0. We set now

$$r_n(t) = \begin{cases} \frac{7\pi}{8\sqrt{2}\alpha}\beta_n\phi(t), & T_n \leqslant t \leqslant 1+T_n, \\ \frac{7\pi}{8\sqrt{2}\alpha}\beta_n, & 1+T_n \leqslant t \leqslant 1+T_n+t_n, \\ \frac{7\pi}{8\sqrt{2}\alpha}\beta_n\phi(2+T_n+t_n-t), & 1+T_n+t_n \leqslant t \leqslant T_n+2+t_n, \end{cases} \tag{3-20}$$

all the other r_k being set to 0 on $[T_n, T_n+2+t_n]$. Now we have, at $t = T_n+2+t_n$,

$$\begin{pmatrix} p_{n+1} \\ p_n \\ s_n \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}. \tag{3-21}$$

Now set

$$r_{n+1}(t)=\begin{cases}\frac{\pi}{2\sqrt{2}\alpha}\beta_{n+1}\phi(t-(T_n+2+t_n)), & T_n+2+t_n\leqslant t\leqslant T_n+3+t_n,\\ \frac{\pi}{2\sqrt{2}\alpha}\beta_{n+1}, & T_n+3+t_n\leqslant t\leqslant T_n+3+t_n+t_{n+1},\\ \frac{\pi}{2\sqrt{2}\alpha}\beta_{n+1}\phi(T_n+4+t_n+t_{n+1}-t), & T_n+3+t_n+t_{n+1}\leqslant t\leqslant T_n+4+t_n+t_{n+1},\end{cases}\tag{3-22}$$

the other r_k being set to 0 on $[T_n+2+t_n, T_n+4+t_n+t_{n+1}]$. We have, at $t=T_n+4+t_n+t_{n+1}$,

$$\begin{pmatrix}p_{n+2}(T_n+4+t_n+t_{n+1})\\ p_{n+1}(T_n+4+t_n+t_{n+1})\\ s_{n+1}(T_n+4+t_n+t_{n+1})\\ p_n(T_n+4+t_n+t_{n+1})\\ s_n(T_n+4+t_n+t_{n+1})\end{pmatrix}=\begin{pmatrix}1/2\\ 0\\ 1/2\\ 0\\ 1/\sqrt{2}\end{pmatrix}.\tag{3-23}$$

Set finally

$$r_n(t)=\begin{cases}\frac{\pi}{\sqrt{2}\alpha}\beta_n\phi(t-(T_n+4+t_n+t_{n+1})), & T_n+4+t_n+t_{n+1}\leqslant t\leqslant T_n+5+t_n+t_{n+1},\\ \frac{\pi}{\sqrt{2}\alpha}\beta_n, & T_n+5+t_n+t_{n+1}\leqslant t\leqslant T_n+5+t_n+t_{n+1}+t_n,\\ \frac{\pi}{\sqrt{2}\alpha}\beta_n\phi(T_n+6+t_n+t_{n+1}+t_n-t), & T_n+5+t_n+t_{n+1}+t_n\leqslant t\leqslant T_n+6+t_n+t_{n+1}+t_n.\end{cases}$$

We now have, at $T_{n+1}=T_n+6+t_n+t_{n+1}+t_n$,

$$\begin{pmatrix}p_{n+2}(t)\\ p_{n+1}(t)\\ s_{n+1}(t)\\ p_n(t)\\ s_n(t)\end{pmatrix}=\begin{pmatrix}1/2\\ -1/\sqrt{2}\\ 1/2\\ 0\\ 0\end{pmatrix}.\tag{3-24}$$

We may now induce this construction for all $n\geqslant 1$, which yields a solution $(p_k(t), s_k(t))$ to (2-4), thus leading to a solution $(a_n(t))$ of $(\mathcal{RFS})$ which we control very explicitly.

Remark 3.1. Provided the β_k are small enough, the explicit construction yields firstly that $|a_n(t)|\leqslant 1$ for all $n,\ t$, and secondly the following behaviour for $(a_n(t))$: for each n, observe that $a_n(t)=0$ outside of a finite interval. Moreover, this interval can be divided into a bounded number of subintervals, so that either those subintervals are of length 2 (corresponding to the time we take to light up an r_k or turn it off), or $a_n(t)$ is a finite linear combination of oscillating factors e^{ift}, where the frequency f is of the order of β_k for some k and hence is arbitrarily small.

3.4. ***Explicit choice for β_k in order for V to decay.*** In order to prove Theorem 1.1, we need to ensure that V and all its derivatives decay with respect to all Sobolev norms as $t\to\infty$. Now, from the construction, we see that, for all $t\geqslant 0$, there is a unique $k(t)$ such that $v_n=0$ for all $n\neq\pm l_{k(t)}$. Now, for any $m\in\mathbb{N}$ and any $s\geqslant 0$, we have

$$\|\partial_t^m V(t,\cdot)\|_{H^s}\simeq\beta_{k(t)}|l_{k(t)}|^{s+2m}.\tag{3-25}$$

As $k(t) \to +\infty$ when $t \to +\infty$, and thus as $|l_{k(t)}| \to +\infty$, we need to ensure that β_k decays faster with respect to k than any power of l_k. A natural choice is

$$\beta_k := |l_k|^{-|l_k|}, \tag{3-26}$$

and we will see that this choice indeed enables us to close the estimates.

4. Approximation

4.1. ***Resonant solution and perturbation decomposition.*** In order to construct a solution to the full system $(\mathcal{FS})$, we try and approximate it by the solution $(a_n(t))$ built in the previous section. In this spirit, we set the solution $(b_n(t))$ to have the a priori form $b_n(t) = a_n(t) + c_n(t)$, where $a_n(t)$ is the solution to $(\mathcal{RFS})$ built above and c_n is a perturbation. We may thus write

$$\partial_t(a_n + c_n) = \sum_{m\in\mathbb{Z}^2} (a_m + c_m)(t) v_{n-m}(t)(e^{-i\omega^+_{m,n}t} - e^{-i\omega^-_{m,n}t}), \tag{4-1}$$

and we already know that

$$\partial_t a_n = \sum_{m\in\Gamma^+_{\text{res}}(n)} a_m(t) v_{n-m}(t) - \sum_{m\in\Gamma^-_{\text{res}}(n)} a_m(t) v_{n-m}(t). \tag{4-2}$$

Thus we need (c_n) to solve

$$\partial_t c_n = \sum_{m\in\mathbb{Z}^2} c_m(t) v_{n-m}(t)(e^{-i\omega^+_{m,n}t} - e^{-i\omega^-_{m,n}t}) + \sum_{m\notin\Gamma^+_{\text{res}}(n)} a_m(t) v_{n-m}(t) e^{-i\omega^+_{m,n}t} - \sum_{m\notin\Gamma^-_{\text{res}}(n)} a_m(t) v_{n-m}(t) e^{-i\omega^-_{m,n}t}. \tag{4-3}$$

Our goal is now to build a solution (c_n) to (4-3) which decays as $t \to \infty$. We will use a Cauchy sequence method: equation (4-3) is globally well-posed in $l^1(\mathbb{Z})$, so we may set, for a given integer $N > 0$, the solution (c_n^N) on $\mathbb{R}_+$ with initial condition $c^N(T_N) = 0$. We have

$$c_n^N(t) = -\sum_{m\in\mathbb{Z}^2} \int_t^{T_N} c_m^N(s) v_{n-m}(s)(e^{-i\omega^+_{m,n}s} - e^{-i\omega^-_{m,n}s})\, ds - \sum_{m\notin\Gamma^+_{\text{res}}(n)} \int_t^{T_N} a_m(s) v_{n-m}(s) e^{-i\omega^+_{m,n}s}\, ds + \sum_{m\notin\Gamma^-_{\text{res}}(n)} \int_t^{T_N} a_m(s) v_{n-m}(s) e^{-i\omega^-_{m,n}s}\, ds,$$

from which we infer, for $t \leqslant T_N$,

$$\|(c_n^N(t))\|_{l^1} \leqslant 2\int_t^{T_N} \|(c_n^N(s))\|_{l^1} \|(v_n(s))\|_{l^1}\, ds + \sum_n \sum_{m\notin\Gamma^+_{\text{res}}(n)} \left| \int_t^{T_N} a_m(s) v_{n-m}(s) e^{i\omega^+_{m,n}s}\, ds \right| + \sum_n \sum_{m\notin\Gamma^-_{\text{res}}(n)} \left| \int_t^{T_N} a_m(s) v_{n-m}(s) e^{i\omega^-_{m,n}s}\, ds \right|,$$

which we rewrite as the inequality, for $t \leqslant T_N$,

$$\|(c_n^N(t))\|_{l^1} \leqslant \alpha(t) + \int_t^{T_N} \|(c_n^N(s))\|_{l^1} \beta(s)\, ds. \tag{4-4}$$

By Gronwall's lemma,

$$\text{for all } t \leqslant T_N, \quad \|(c_n^N(t))\|_{l^1} \leqslant \alpha(t) + \int_t^{T_N} \alpha(s)\beta(s)\exp\left(\int_t^s \beta(\sigma)\,d\sigma\right) ds. \tag{4-5}$$

4.2. *Estimates on $\alpha(t)$.* First, let us study $\alpha(t)$. The set of pairs $(m, n-m)$, $n \in Z^2$ and $m \notin \Gamma_{\text{res}}^+(n)$ (resp. $m \notin \Gamma_{\text{res}}^-(n)$), is equal to the set of pairs $(n_1, n_2) \in \mathbb{Z}^2$ such that n_1 and n_2 aren't orthogonal (resp. n_2 and $n_1 + n_2$ aren't orthogonal). Moreover, we have $v_n(s) = 0$ for all $n \neq \pm l_k$ for a given $k \geqslant 0$, and we recall that $v_{-n} = v_n$. Finally, we know that $a_n(s)v_{l_k}(s) = 0$ as soon as

$$n \notin \{m_k,\, m_k - l_k,\, m_{k+1},\, m_{k+1} - l_{k+1},\, m_{k+2}\} =: E_k.$$

We may then write

$$\alpha(t) = \sum_{k\geqslant 0}\sum_{n\in E_k} I(k, n, t), \tag{4-6}$$

where $I(k, n)$ is a sum of at most four quantities of the form

$$J(k, n, \omega, t) := \left|\int_t^{T_n} a_n(s)r_k(s)e^{i\omega s}\,ds\right| \tag{4-7}$$

and ω is a frequency belonging to $\mathbb{Z}\backslash\{0\}$, thus ensuring $|\omega| \geqslant 1$. (It is here that we use the nonresonance of the interactions).

We may now write

$$\int_t^{T_n} a_n(s)r_k(s)e^{i\omega s}\,ds = \left[\left(\int_s^t a_n(\sigma)e^{i\omega\sigma}\,d\sigma\right) r_k(s)\right]_t^{T_N} - \int_t^{T_N}\left(\int_s^t a_n(\sigma)e^{i\omega\sigma}\,d\sigma\right) r_k'(s)\,ds.$$

The bracket term is equal to 0 as r_k is 0 at T_N for all k. Moreover, we may infer from the construction of r_k that

$$\int_{\mathbb{R}_+} |r_k'(s)|\,ds \leqslant C\beta_k, \tag{4-8}$$

with C a universal constant independent of k (indeed, we use that r_k is a constant except maybe on a finite number of intervals of length 2 where its derivative is bounded by $c\beta_k\|\phi'\|_\infty$).

Finally, we have

$$\left|\int_s^t a_n(\sigma)e^{i\omega\sigma}\,d\sigma\right| \leqslant C, \tag{4-9}$$

with C a universal constant independent of s, n, t, ω. Indeed, for any n, using Remark 3.1, we know that, on the one hand, $|a_n| \leqslant 1$ on $\mathbb{R}_+$, and, on the other hand, that, outside of a fixed finite number of intervals of length 2 (yielding a bounded contribution to the integral), a_n is either equal to 0 or equal to a finite linear combination with a bounded number of terms of oscillating exponentials e^{ift}, with frequency $f = C'\beta_l$, where C' is a universal constant and $l \geqslant 0$. Thus, up to choosing $|m_0|$ larger, we can require that we always have $|f| < 1/2$. Hence, we are left with integrating oscillating exponentials $e^{i(f+\omega)\sigma}$ where $|f + \omega| \geqslant 1/2$ (since $|\omega| > 1$). A simple integration is enough to conclude the proof of the claim.

This yields the bound

$$J(k,n,\omega,t)\leqslant C\beta_k, \tag{4-10}$$

where C is a universal constant. Moreover, we see that $r_k(s)=0$ for all $s\geqslant T_{k+1}$; thus we have

$$J(k,n,\omega,t)=0 \quad \text{for all } t\geqslant T_{k+1}. \tag{4-11}$$

From this we may infer the bound

$$\alpha(t)\leqslant C\sum_{k\geqslant k(t)}\beta_k, \tag{4-12}$$

where we set $k(t)$ to be the smallest nonnegative integer such that $t\leqslant T_{k+1}$. Using moreover the fast decay of β_k, we may further bound, up to taking a larger C,

$$\alpha(t)\leqslant C\beta_{k(t)}. \tag{4-13}$$

4.3. ***Estimates on $\beta(t)$.*** As for $\beta(t)$, we see that, for all t, there is a unique $l(t)$ such that $r_k(t)=0$ as soon as $k\neq l(t)$; thus we find that

$$\beta(t)=4r_{l(t)}(t). \tag{4-14}$$

This yields the bound

$$\int_t^s \beta(\sigma)\,d\sigma\leqslant 4\int_0^s r_{l(\sigma)}(\sigma)\,d\sigma\leqslant C(k(s)+1);$$

indeed, we see that the integral of r_k over $\mathbb{R}_+$ is a constant independent of k.

4.4. ***Conclusion of the estimates on c^N.*** We may thus bound, for $t\leqslant T_N$,

$$\|(c_n^N(t))\|_{l^1}\leqslant C\biggl(\beta_{k(t)}+\int_t^{T_N}\beta_{k(s)}\beta_{l(s)}\exp(C(k(s)+1))\,ds\biggr). \tag{4-15}$$

Now, from the construction of $(r_k(s))$, we have $l(s)\geqslant k(s)$, and thus $\beta_{l(s)}\leqslant\beta_{k(s)}$. Therefore, for $t\leqslant T_N$, we have

$$\|(c_n^N(t))\|_{l^1}\leqslant C\biggl(\beta_{k(t)}+\int_t^{T_N}\beta_{k(s)}^2\exp(C(k(s)+1))\,ds\biggr). \tag{4-16}$$

Now, $k(s)$ is equal to k on an interval with measure l_k such that $l_k\beta_k$ is equal to a constant, yielding the bound

$$\|(c_n^N(t))\|_{l^1}\leqslant C\biggl(\beta_{k(t)}+\sum_{k\geqslant k(t)}\beta_k e^{Ck}\biggr). \tag{4-17}$$

As β_k is decaying faster than a double exponential, we finally have,

$$\text{for all } t\leqslant T_N,\quad \|(c_n^N(t))\|_{l^1}\leqslant C\beta_{k(t)}e^{Ck(t)}. \tag{4-18}$$

5. Cauchy sequence and conclusion

5.1. ***Cauchy sequence.*** We now prove that (c^N) is a Cauchy sequence in $l^1(\mathbb{Z})$. Set $M > N$. We look at the equation satisfied by $c^M - c^N$:

$$(c_n^M - c_n^N)(t) = -\sum_{m\in\mathbb{Z}^2}\int_t^{T_N}(c_m^M - c_m^N)(s)v_{n-m}(s)(e^{i\omega_{m,n}^+ s} - e^{i\omega_{m,n}^- s})\,ds + c_n^M(T_N)$$
$$-\sum_{m\notin\Gamma_{\text{res}}^+(n)}\int_{T_N}^{T_M}a_m(s)v_{n-m}(s)e^{-i\omega_{m,n}^+ s}\,ds + \sum_{m\notin\Gamma_{\text{res}}^-(n)}\int_{T_N}^{T_M}a_m(s)v_{n-m}(s)e^{-i\omega_{m,n}^- s}\,ds.$$

Thus

$$\begin{aligned}\|((c_n^M - c_n^N)(t))\|_{l^1} &\leqslant 2\int_t^{T_N}\|((c_n^M - c_n^N)(s))\|_{l^1}\|(v_n(s))\|_{l^1}\,ds + \|(c_n^M(T_N))\|_{l^1}\\ &\quad + \sum_n\sum_{m\notin\Gamma_{\text{res}}^+(n)}\left|\int_{T_N}^{T_M}a_m(s)v_{n-m}(s)e^{-i\omega_{m,n}^+ s}\,ds\right|\\ &\quad + \sum_n\sum_{m\notin\Gamma_{\text{res}}^-(n)}\left|\int_{T_N}^{T_M}a_m(s)v_{n-m}(s)e^{-i\omega_{m,n}^- s}\,ds\right|\\ &\leqslant 2\int_t^{T_N}\|((c_n^M - c_n^N)(s))\|_{l^1}\|(v_n(s))\|_{l^1}\,ds + C\beta_{k(T_N)}e^{Ck(T_N)} + C\beta_{k(T_N)}\\ &\leqslant 2\int_t^{T_N}\|((c_n^M - c_n^N)(s))\|_{l^1}\|(v_n(s))\|_{l^1}\,ds + C\beta_{N-1}e^{C(N-1)}.\end{aligned}$$

Using the backward Gronwall lemma,

$$\|((c_n^M - c_n^N)(t))\|_{l^1} \leqslant C\beta_{N-1}e^{C(N-1)}\left(1 + \int_t^{T_N}\beta(s)\exp\left(\int_t^s\beta(\sigma)\,d\sigma\right)ds\right), \tag{5-1}$$

where $\beta(s) = 2\|(v_n(s))\|_{l^1}$. We know that $\beta(s) = 4r_{l(s)}(s)$, and thus $\int_s^t\beta(\sigma)\,d\sigma \leqslant C(k(s)+1)$. We have

$$\|((c_n^M - c_n^N)(t))\|_{l^1} \leqslant C\beta_{N-1}e^{C(N-1)}\left(1 + \int_t^{T_N}\beta_{k(s)}\exp(Ck(s))\,ds\right). \tag{5-2}$$

This upper bound decays to 0 as $N, M \to \infty$ if we fix t. This shows that $(c^N(t))$ is a Cauchy sequence in $l^1(\mathbb{Z}^2)$, and it thus converges to a $c(t)$ such that, using integral form of the differential equation, $b = a + c$ is a solution to $(\mathcal{RFS})$. We have, moreover,

$$\|(c_n(t))\|_{l^1} \leqslant C\beta_{k(t)}e^{Ck(t)}, \tag{5-3}$$

and this upper bound decays to 0 as $t \to +\infty$, as expected.

5.2. ***Growth of the Sobolev norm: qualitative result.*** In order to conclude, we recall that $\|(a_n(t))\|_{l^2}$ is preserved and that, for all $t \geqslant 0$, there are at most five of the a_n that are nonzero. Therefore, we have, on the one hand, that $a_k = 0$ for $|k| < |m_n|$ and for all $t \geqslant T_n$, and, on the other hand, that there exists $|k| \geqslant |m_n|$ such that $|a_k(t)| \geqslant \varepsilon$, where $\varepsilon > 0$ is a universal constant. Now, if we set N large enough, we can ensure that $\|(c_n(t))\|_{l^1} \leqslant \varepsilon/2$ for $t \geqslant T_N$. Therefore, for all $t \geqslant T_N$ with N large enough, there exists $|k| \geqslant |m_N|$ such that $b_k = a_k + c_k$ satisfies $|b_k| \geqslant \varepsilon/2$. Now, this ensures that,

$$\text{for all } t \geqslant T_N, \quad \|(b_n(t))\|_{H^s} \geqslant |k|^s |b_k| \geqslant (\varepsilon/2)|m_N|^s. \tag{5-4}$$

This already yields a qualitative result for Theorem 1.1, as we already proved in Section 3.4 that the potential V along with all its time derivatives are decaying in all Sobolev norms when $t \to +\infty$.

5.3. ***Quantitative estimates on the growth rate.*** We now investigate the quantitative bounds that we can hope to get on the rate of growth.

We first see that $T_n \leqslant C\beta_n^{-1}$ using the fast decay of β_n. Moreover, as $|l_n| \leqslant C^n n!$, we find that

$$T_n \leqslant \exp(C^n n! \log(C^n n!)). \tag{5-5}$$

This yields the lower bound

$$\|(b_n(t))\|_{H^s} \geqslant \delta |m_{n(t)}|^s, \tag{5-6}$$

where $\delta > 0$ is a constant and $n(t)$ is the largest integer n such that $\exp(C^n n! \log(C^n n!)) \leqslant t$. Now, we know moreover that $|m_n| \geqslant c(n-1)!$, thus leading to the lower bound

$$\|u(t)\|_{H^s} \geqslant \varepsilon c^s ((n(t)-1)!)^s. \tag{5-7}$$

In order to obtain better bounds, take $\eta > 0$. We first use Stirling's formula

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}, \tag{5-8}$$

which ensures that, provided n is large enough,

$$C^n n! \log(C^n n!) \leqslant ((1+\eta)n)^{(1+\eta)n}. \tag{5-9}$$

Now set $f(x) := x^x$. We find that, provided

$$f((1+\eta)n) \leqslant \log t \tag{5-10}$$

and provided n is large enough, we have $n \leqslant n(t)$. Now, provided n is large enough, we also have

$$(n-1)! \geqslant ((1-\eta)n)^{(1-\eta)n} = f((1-\eta)n). \tag{5-11}$$

Thus, setting $E(x)$ to be the largest integer k such that $k \leqslant x$, we can find a lower bound of the form

$$\begin{aligned}
\|u(t)\|_{H^s} &\geqslant \left(cf\left(\frac{1-\eta}{1+\eta} E(f^{-1}(\log t))\right)\right)^s \\
&\geqslant c_{s,\eta} \exp\left(s\frac{(1-\eta)^2}{1+\eta} f^{-1}(\log t) \log\left(\frac{(1-\eta)^2}{1+\eta} f^{-1}(\log t)\right)\right) && \text{(provided } t \text{ is large enough)} \\
&\geqslant c_{s,\eta} \exp\left(s\frac{(1-\eta)^3}{1+\eta} f^{-1}(\log t) \log(f^{-1}(\log t))\right) && \text{(provided } t \text{ is large enough)} \\
&\geqslant c_{s,\eta} (\log t)^{s(1-\eta)^3/(1+\eta)}.
\end{aligned}$$

As we may choose η arbitrarily, we find that, given any δ, $s > 0$, there exists $c_{\delta,s} > 0$ such that, for $t > 1$,

$$\|u(t)\|_{H^s} \geqslant c_{\delta,s} (\log t)^{s(1-\delta)}, \tag{5-12}$$

thus concluding the proof of Theorem 1.1.

5.4. ***Estimates on the decay rate of V.*** We now prove similar upper bounds on the decay rate of the potential $V(t)$. Fix $s \geqslant 0$ and $m \in \mathbb{N} \cup \{0\}$. Thanks to (1-4), we may bound

$$\|\partial_t^m V(t,\cdot)\|_{H^s} \leqslant c |l_{k(t)}|^{M-|l_{k(t)}|}, \tag{5-13}$$

where $M = M_{m,s} > 0$ and $k(t)$ is the unique $k \geqslant 0$ such that $r_{k(t)} \neq 0$. We may furthermore infer from the previous subsection that, given $\delta > 0$, there exists $c_\delta > 0$ such that

$$|l_{k(t)}| \geqslant c_\delta (\log t)^{1-\delta}. \tag{5-14}$$

Thus

$$\|\partial_t^m V(t,\cdot)\|_{H^s} \leqslant C_\delta \exp((M_{m,s} - (\log t)^{1-\delta})(1-\delta) \log\log t). \tag{5-15}$$

As this holds for all $\delta > 0$, we may conclude that, for all $\delta > 0$, there exists $C_{\delta,m,s}$ such that

$$\|\partial_t^m V(t,\cdot)\|_{H^s} \leqslant C_{\delta,m,s} \exp(-(\log t)^{1-\delta}) \log\log t). \tag{5-16}$$

As this yields a quantitative bounds for the decay of V, it should be noted that it is subpolynomial in the sense that the upper bound decays slower than $t^{-\varepsilon}$ for all $\varepsilon > 0$. It doesn't seem that we can improve the bound, as, on $[T_N, T_{N+1}]$, $\|V(t)\|_{H^1}$ is of order β_N and T_{N+1} is of order β_{N+1}^{-1}. As for all $\varepsilon > 0$ asymptotically we have $\beta_{N+1}^{\varepsilon} \ll \beta_N$, we thus cannot hope for a better bound.

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