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ON THE KINK-KINK COLLISION PROBLEM FOR THE ϕ^6 MODEL WITH LOW SPEED

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We study the elasticity of the collision of two kinks with an incoming low speed $v \in (0, 1)$ for the nonlinear wave equation in dimension 1+1 known as the ϕ^6 model. We prove for any $k \in \mathbb{N}$ that if the incoming speed v is small enough, then, after the collision, the two solitons move away with a velocity v_f such that $|v_f - v| \leq v^k$ and the energy of the remainder will also be smaller than v^k. This manuscript is the continuation of our previous paper where we constructed a sequence ϕ_k of approximate solutions for the ϕ^6 model. The proof of our main result relies on the use of the set of approximate solutions from our previous work, modulation analysis, and a refined energy estimate method to evaluate the precision of our approximate solutions during a large time interval.

1. Introduction

1.1. ***Background.*** Considering the potential function $U(\phi) = \phi^2(1-\phi^2)^2$, the partial differential equation known as the ϕ^6 model in domain $1+1$ is defined by

$$\partial_t^2\phi(t,x) - \partial_x^2\phi(t,x) + U'(\phi(t,x)) = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}. \tag{1}$$

The solutions $\phi(t,x)$ of (1) preserve the energy given by

$$E(\phi)(t) = \int_{\mathbb{R}} \tfrac{1}{2}([\partial_t\phi(t,x)]^2 + [\partial_x\phi(t,x)]^2) + U(\phi(t,x))\,dx, \tag{energy}$$

and the momentum

$$P(\phi) = -\int_{\mathbb{R}} \partial_t\phi(t,x)\partial_x\phi(t,x)\,dx. \tag{momentum}$$

The kinetic energy and potential energy are given, respectively, by

$$E_{\text{kin}}(\phi)(t) = \int_{\mathbb{R}} \tfrac{1}{2}[\partial_t\phi(t,x)]^2\,dx, \quad E_{\text{pot}}(\phi)(t) = \int_{\mathbb{R}} \tfrac{1}{2}[\partial_x\phi(t,x)]^2 + U(\phi(t,x))\,dx.$$

The vacuum set $\mathcal{V}$ of the potential function U is the set $U^{-1}\{0\} = \{-1, 0, 1\}$. The unique constant solutions with finite energy of (1) are the functions of the form $\phi \equiv \eta$ for any $\eta \in \mathcal{V}$.

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Furthermore, it is well known that if a solution $\phi(t,x)$ of the partial differential equation (1) is in the energy space, which is the set of strong solutions with finite energy, then the solution is global-in-time (see the introduction of [Moutinho 2023] for a proof) and there exist numbers $\eta_1,\eta_2\in\mathcal{V}$ such that

$$\lim_{x\to-\infty}\phi(t,x)=\eta_1,\qquad \lim_{x\to+\infty}\phi(t,x)=\eta_2$$

for all $t\in\mathbb{R}$. The set of solutions of (1) with finite energy is invariant under space translation, time translation, space reflection, time reflection, and Lorentz transformations.

The unique nonconstant stationary solutions of (1) with finite energy are the kinks which are the space translation of either $H_{0,1}(x)$ or $H_{-1,0}(x)$ that are denoted by

$$H_{0,1}(x)=\frac{e^{\sqrt{2}x}}{\sqrt{1+e^{2\sqrt{2}x}}},\quad H_{-1,0}(x)=-H_{0,1}(-x)=\frac{-e^{-\sqrt{2}x}}{\sqrt{1+e^{-2\sqrt{2}x}}},$$

and the antikinks which are the space translation of the functions

$$H_{1,0}(x)=H_{0,1}(-x)=\frac{e^{-\sqrt{2}x}}{\sqrt{1+e^{-2\sqrt{2}x}}},\quad H_{0,-1}(x)=-H_{0,1}(x)=\frac{-e^{\sqrt{2}x}}{\sqrt{1+e^{2\sqrt{2}x}}}.$$

Using the identity

$$H_{0,1}'(x)=\sqrt{2}\frac{e^{\sqrt{2}x}}{(1+e^{2\sqrt{2}x})^{3/2}},$$

it is not difficult to verify that

$$\left\|\frac{d}{dx}H_{0,1}(x)\right\|_{L^2_x(\mathbb{R})}^2=\frac{1}{2\sqrt{2}}. \tag{2}$$

The kink $H_{0,1}$ satisfies the Bogomolny identity, which is $H_{0,1}'(x)=\sqrt{2U(H_{0,1}(x))}$, and the estimate

$$\left|\frac{d^k}{dx^k}H_{0,1}(x)\right|\lesssim_k \min(e^{\sqrt{2}x},e^{-2\sqrt{2}x}) \tag{3}$$

for any $k\geq 1$, and clearly

$$|H_{0,1}(x)|\leq e^{\sqrt{2}\min(x,0)}. \tag{4}$$

For the ϕ^6 model there are stability results for the kinks. In [Moutinho 2023], the orbital stability of two kinks with energy close to the minimal was obtained, and also the dynamics of two interacting kinks, which is a kink-kink solution with low kinetic energy and potential energy slightly bigger than the minimum possible for two kinks, was described in function of the initial data and the energy of the solution. In [Kowalczyk et al. 2021], the asymptotic stability of a kink for the ϕ^6 model was obtained, and moreover, asymptotic stability of a single kink was also obtained for a certain class of nonlinear wave equations of dimension $1+1$. There are also asymptotic stability results for a single kink in other models; for example, see [Kowalczyk et al. 2017; Delort and Masmoudi 2022] for the ϕ^4 model.

This manuscript is the sequel of [Moutinho 2024]. In this paper, we study the traveling kink-kink solutions of (1) with speed $0<v<1$ small enough. More precisely, we consider the following definition.

Definition 1. The traveling kink-kink with speed $v \in (0,1)$ are the set of solutions $\phi(t,x)$ that satisfies, for some positive constants K, c and any $t \geq K$, the decay estimate

$$\left\|(\phi(t,x), \partial_t\phi(t,x)) - \overrightarrow{H_{0,1}}\left(\frac{x-vt}{\sqrt{1-v^2}}\right) - \overrightarrow{H_{-1,0}}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right\|_{H^1_x(\mathbb{R})\times L^2_x(\mathbb{R})} \leq e^{-ct}, \tag{5}$$

where, for any $-1 < v < 1$ and any $y \in \mathbb{R}$,

$$\overrightarrow{H_{0,1}}\left(\frac{x-vt+y}{\sqrt{1-v^2}}\right) = \begin{bmatrix} H_{0,1}\left(\dfrac{x-vt+y}{\sqrt{1-v^2}}\right) \\ \dfrac{-v}{\sqrt{1-v^2}} H'_{0,1}\left(\dfrac{x-vt+y}{\sqrt{1-v^2}}\right) \end{bmatrix}, \tag{6}$$

$$\overrightarrow{H_{-1,0}}\left(\frac{x+vt-y}{\sqrt{1-v^2}}\right) = \begin{bmatrix} H_{-1,0}\left(\dfrac{x+vt-y}{\sqrt{1-v^2}}\right) \\ \dfrac{v}{\sqrt{1-v^2}} H'_{-1,0}\left(\dfrac{x+vt-y}{\sqrt{1-v^2}}\right) \end{bmatrix}. \tag{7}$$

The existence and uniqueness of solutions $\phi(t,x)$ satisfying (5) for any $0 < v < 1$ was obtained in [Chen and Jendrej 2022], but the uniqueness of the solution of (1) satisfying

$$\lim_{t\to+\infty}\left\|\vec{\phi}(t,x) - \overrightarrow{H_{0,1}}\left(\frac{x-vt}{\sqrt{1-v^2}}\right) + \overrightarrow{H_{-1,0}}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right\|_{H^1_x(\mathbb{R})\times L^2_x(\mathbb{R})} = 0$$

for $0 < v < 1$ is still an open problem. For references on the existence and uniqueness of multisoliton solutions of other nonlinear dispersive partial differential equations; see, e.g., [Martel 2005; Combet 2011].

For nonintegrable dispersive models, there exist previous results about the inelasticity of the collision of two solitons. For example, Martel and Merle [2011] verified that the collision between two solitons with nearly equal speed is not elastic. More precisely, they showed that the incoming speed of the two solitons is different to their outgoing speed after their collision.

Since the ϕ^6 model is a nonintegrable system, the collision of two kinks with low speed $0 < v < 1$ is expected to be inelastic. More precisely, we expect the existence of a value $k > 1$ such that if $0 < v \ll 1$ and $\phi(t,x)$ is a solution (1) satisfying the condition (5), then $\phi(t,x)$ should have inelasticity of order v^k, which means the existence of $t < 0$ with $|t| \gg 1$ such that

$$(\phi(t,x), \partial_t\phi(t,x)) = \overrightarrow{H_{0,1}}\left(\frac{x+v_f t+y_1(t)}{\sqrt{1-v_f^2}}\right) + \overrightarrow{H_{-1,0}}\left(\frac{x-v_f t+y_2(t)}{\sqrt{1-v_f^2}}\right) + ro(t,x), \tag{8}$$

with $v^k \ll \|ro(t)\|_{H^1_x(\mathbb{R})\times L^2_x(\mathbb{R})} \ll v$ and $v_f(t), y_1, y_2$ satisfying

$$v^k \ll |v_f(t) - v| + \max_{j\in\{1,2\}} |\dot{y}_j(t)| \ll v \tag{9}$$

for all $t < 0$ satisfying $|t| \gg 1$. Actually, in the quartic gKdV, the collision of the two solitons satisfies a similar property to our previous expectations in (8) and (9); see [Martel and Merle 2011, Theorem 1] for more details.

However, in this manuscript, we prove for the ϕ^6 model and any $k > 1$ that if $0 < v \ll 1$ and t is close to $-\infty$, both estimates (8) and (9) are not possible. Indeed, we demonstrate that if $v \ll 1$ and $\phi(t, x)$ satisfies (5), then there exists a number $e_{k,2v} \in \mathbb{R}$ satisfying, for all t close to $-\infty$,

$$\begin{gathered}(\phi(t,x), \partial_t\phi(t,x)) = \overrightarrow{H_{0,1}}\left(\frac{x + v_f t - e_{k,2v}}{\sqrt{1-v_f^2}}\right) + \overrightarrow{H_{-1,0}}\left(\frac{x - v_f t + e_{k,2v}}{\sqrt{1-v_f^2}}\right) + r_{c,v}(t,x),\\ \limsup_{t\to-\infty}\|r_{c,v}(t)\|_{H^1_x\times L^2_x} \leq v^{2k},\\ \limsup_{t\to-\infty}|v_f(v,t) - v| \leq v^{2k}.\end{gathered} \tag{10}$$

In conclusion, the inelasticity of the collision of two kinks cannot be of any order v^k for any $1 \ll k \in \mathbb{N}$, if the incoming speed v of the kinks is small enough. The problem to verify the inelasticity of the collision of kinks for the ϕ^6 model is still open. But, because of the conclusion obtained in this paper, the change $|v - v_f|$ in the speeds of each soliton is much smaller than any monomial function v^k. More precisely, for all $k > 0$,

$$\lim_{v\to0^+}\limsup_{t\to-\infty}\frac{|v_f(v,t) - v|}{v^k} = 0. \tag{11}$$

This is a new result.

The study of collision of kinks for the ϕ^6 model is important for high energy physics; see, for example, [Gani et al. 2014; Dorey et al. 2011]. Actually, in [Gani et al. 2014], it was shown numerically that there exists a critical speed v_c such that if each of the two kinks move with speed v with absolute value less than v_c and they approach each other, then they will collide and the collision will be very elastic, which is exactly the result we obtain rigorously in this paper. The study of the dynamics of multisoliton solutions of the ϕ^6 model has also applications in condensed matter physics, see [Bishop and Schneider 1978], and cosmology, see [Vilenkin and Shellard 1994].

For other nonlinear dispersive equations, there exist rigorous results of inelasticity and stability of collision of solitons. For gKdV models, the inelasticity of collision of solitons was proved for the quartic gKdV in [Martel and Merle 2011], and, for a certain class of gKdV, inelasticity of collision between solitons was also proved in [Muñoz 2010; 2012]; see also [Martel and Merle 2009]. For the nonlinear Schrödinger equation, Perelman [2011] studied the collision of two solitons of different sizes and showed that the solution does not preserve the two solitons' structure after the collision. See also [Martel and Merle 2018] for discussion on the inelasticity of the collision of two solitons for the fifth-dimensional energy critical wave equation.

1.2. *Main results.*

The main theorem obtained in this manuscript is the following result:

Theorem 2. *There exists a continuous function* $v_f : (0,1)\times\mathbb{R} \to (0,1)$ *and, for any* $0 < \theta < 1$ *and* $k \in \mathbb{N}_{\geq 2}$, *there exists* $0 < \delta(\theta, k) < 1$, *such that if* $0 < v < \delta(\theta, k)$, *and* $\phi(t, x)$ *is a traveling kink-kink solution of* (1) *with speed* v, *then there exists a number* $e_{v,k}$ *such that* $|e_{v,k}| < \ln(8/v^2)$. *Furthermore, if*

$$t \leq -\frac{\ln(1/v)^{2-\theta}}{v},$$

then $|v_f(v,t)-v|<v^k$ *and*

$$\begin{aligned}\Big\Vert\phi(t,x)-H_{0,1}\Big(\frac{x-e_{k,v}+v_f t}{\sqrt{1-v_f^2}}\Big)-H_{-1,0}\Big(\frac{x+e_{k,v}-v_f t}{\sqrt{1-v_f^2}}\Big)\Big\Vert_{H^1_x(\mathbb{R})}&\\+\Big\Vert\partial_t\phi(t,x)-\frac{v_f}{\sqrt{1-v_f^2}}H'_{0,1}\Big(\frac{x-e_{v,k}+v_f t}{\sqrt{1-v_f^2}}\Big)+\frac{v_f}{\sqrt{1-v_f^2}}H'_{-1,0}\Big(\frac{x+e_{v,k}-v_f t}{\sqrt{1-v_f^2}}\Big)\Big\Vert_{L^2_x(\mathbb{R})}&\leq v^k.\end{aligned}$$

If

$$\frac{-4\ln(1/v)^{2-\theta}}{v}\leq t\leq\frac{-\ln(1/v)^{2-\theta}}{v},$$

then

$$\begin{aligned}\Big\Vert\phi(t,x)-H_{0,1}\Big(\frac{x-e_{k,v}+vt}{\sqrt{1-v^2}}\Big)-H_{-1,0}\Big(\frac{x+e_{k,v}-vt}{\sqrt{1-v^2}}\Big)\Big\Vert_{H^1_x(\mathbb{R})}&\\+\Big\Vert\partial_t\phi(t,x)-\frac{v}{\sqrt{1-v^2}}H'_{0,1}\Big(\frac{x-e_{v,k}+vt}{\sqrt{1-v^2}}\Big)+\frac{v}{\sqrt{1-v^2}}H'_{-1,0}\Big(\frac{x+e_{v,k}-vt}{\sqrt{1-v^2}}\Big)\Big\Vert_{L^2_x(\mathbb{R})}&\leq v^k.\end{aligned}$$

Remark 3. The second inequality in Theorem 2 follows from the energy estimate method used in Section 3 to estimate the energy norm of the remainder during a large time interval.

Clearly, Theorem 2 implies (11). Actually, the first item of Theorem 2 is a consequence of the second item of this theorem and the following result about the orbital stability of two moving kinks.

Theorem 4. *There exists a constant* $c>0$ *and, for any* $\theta\in(0,1)$, *there exists* $\delta(\theta)\in(0,1)$ *such that if* $0<v<\delta(\theta)$, *and* $(\psi_0(x),\psi_1(x))\in H^1_x(\mathbb{R})\times L^2_x(\mathbb{R})$ *is an odd function satisfying*

$$\Vert(\psi_0,\psi_1)\Vert_{H^1_x\times L^2_x}<v^{2+\theta},\tag{12}$$

and $y_0\geq -4\ln v$, *then the solution* $(\phi(t,x),\partial_t\phi(t,x))$ *of the Cauchy problem*

$$\begin{cases}\partial_t^2\phi(t,x)-\partial_x^2\phi(t,x)+U'(\phi(t,x))=0,\\ \begin{bmatrix}\phi(0,x)\\ \partial_t\phi(0,x)\end{bmatrix}=\begin{bmatrix}H_{0,1}\Big(\frac{x-y_0}{\sqrt{1-v^2}}\Big)+H_{-1,0}\Big(\frac{x+y_0}{\sqrt{1-v^2}}\Big)+\psi_0(x)\\ \frac{-v}{\sqrt{1-v^2}}H'_{0,1}\Big(\frac{x-y_0}{\sqrt{1-v^2}}\Big)+\frac{v}{\sqrt{1-v^2}}H'_{-1,0}\Big(\frac{x+y_0}{\sqrt{1-v^2}}\Big)+\psi_1(x)\end{bmatrix}\end{cases}\tag{13}$$

is given for all $t\geq 0$ *by*

$$\begin{bmatrix}\phi(t,x)\\ \partial_t\phi(t,x)\end{bmatrix}=\begin{bmatrix}H_{0,1}\Big(\frac{x-y(t)}{\sqrt{1-v^2}}\Big)+H_{-1,0}\Big(\frac{x+y(t)}{\sqrt{1-v^2}}\Big)+\psi(t,x)\\ \frac{-v}{\sqrt{1-v^2}}H'_{0,1}\Big(\frac{x-y(t)}{\sqrt{1-v^2}}\Big)+\frac{v}{\sqrt{1-v^2}}H'_{-1,0}\Big(\frac{x+y(t)}{\sqrt{1-v^2}}\Big)+\partial_t\psi(t,x)\end{bmatrix},\tag{14}$$

such that

$$\begin{gathered}|y(0)-y_0|+\Vert\overrightarrow{\psi}(t,x)\Vert_{H^1_x\times L^2_x}\leq c\Vert\overrightarrow{\psi_0}(x)\Vert^{1/2}_{H^1_x\times L^2_x}+c(1+y_0)^{1/2}e^{-\sqrt{2}y_0},\\ |\dot{y}(t)-v|\leq c\Vert\overrightarrow{\psi_0}(x)\Vert_{H^1_x\times L^2_x}\end{gathered}\tag{15}$$

for all $t\in\mathbb{R}_{\geq 0}$.

Remark 5. Theorem 4 allows us to extend the description of the traveling kink-kink for all time below

$$-\frac{\ln(1/v)^{2-\theta}}{v},$$

from which we will deduce the first inequality in Theorem 2.

1.3. *Notation.*

In this subsection, we explain the notation that we are going to use in the next sections. First, for any real function $f:\mathbb{R}^2\to\mathbb{R}$ satisfying the conditions $f(t,\cdot)\in L^\infty_x(\mathbb{R})$, and $\partial_t f(t,\cdot)\in L^2_x(\mathbb{R})$, we define the function $\vec{f}:\mathbb{R}^2\to\mathbb{R}^2$ by

$$\vec{f}(t,x)=(f(t,x),\partial_t f(t,x))\quad \text{for every } (t,x)\in\mathbb{R}^2.$$

For any $k\in\mathbb{N}$ and any smooth function $f:\mathbb{R}\to\mathbb{R}$, we use the notation

$$f^{(k)}(x)=\frac{d^x}{dx^k}f(x)\quad\text{for all } x\in\mathbb{R}.$$

For any $z\in\mathbb{R}$, we use the notation $H^z_{0,1}(x)=H_{0,1}(x-z)$, $H^z_{-1,0}(x)=H_{-1,0}(x-z)$. For any subset $\mathcal{D}\subset\mathbb{R}$, any $v\in(0,1)$ and any function $y:\mathcal{D}\to\mathbb{R}$, we define the functions $\overrightarrow{H_{0,1,v,y}}:\mathcal{D}\times\mathbb{R}\to\mathbb{R}^2$, $\overrightarrow{H_{-1,0,v,y}}:\mathcal{D}\times\mathbb{R}\to\mathbb{R}^2$ by

$$\overrightarrow{H_{0,1,v,y}}(t,x)=\begin{bmatrix} H_{0,1}\left(\dfrac{x-vt+y(t)}{\sqrt{1-v^2}}\right)\\ \dfrac{-v}{\sqrt{1-v^2}}H'_{0,1}\left(\dfrac{x-vt+y(t)}{\sqrt{1-v^2}}\right)\end{bmatrix},\quad \overrightarrow{H_{-1,0,v,y}}(t,x)=\begin{bmatrix} H_{-1,0}\left(\dfrac{x+vt-y(t)}{\sqrt{1-v^2}}\right)\\ \dfrac{v}{\sqrt{1-v^2}}H'_{-1,0}\left(\dfrac{x+vt-y(t)}{\sqrt{1-v^2}}\right)\end{bmatrix}.$$

For any set $\mathcal{D}\subset\mathbb{R}$ and any nonnegative function $k:\mathcal{D}\to\mathbb{R}_{\geq 0}$, we say that $f(x)=O(k(x))$, if f has the same domain $\mathcal{D}$ as k and there is a universal constant $C>0$ such that $|f(x)|\leq Ck(x)$ for any $x\in D$. For any two nonnegative real functions $f_1(x)$ and $f_2(x)$, we have $f_1\lesssim f_2$ if there is a universal constant $C>0$ such that $f_1(x)\leq Cf_2(x)$ for any $x\in\mathbb{R}$. Furthermore, for a finite number of real variables $\alpha_1,\dots,\alpha_n$ and two nonnegative functions $f_1(\alpha_1,\dots,\alpha_n,x)$ and $f_2(\alpha_1,\dots,\alpha_n,x)$ both with domain $\mathcal{D}\times\mathbb{R}\subset\mathbb{R}^{n+1}$, we say that $f_1\lesssim_{\alpha_1,\dots,\alpha_n} f_2$ if there is a positive function $L:\mathcal{D}\to\mathbb{R}_+$ such that

$$f_1(\alpha_1,\dots,\alpha_n,x)\leq L(\alpha_1,\dots,\alpha_n)f_2(\alpha_1,\dots,\alpha_n,x)\quad\text{for all } (\alpha_1,\dots,\alpha_m,x)\in\mathcal{D}\times\mathbb{R}.$$

We write $f_1\cong f_2$ if $f_1\lesssim f_2$ and $f_2\lesssim f_1$.

We consider for any $f\in H^1_x(\mathbb{R})$ and any $g\in L^2_x(\mathbb{R})$ the norms

$$\|f\|_{H^1_x}=\|f\|_{H^1_x(\mathbb{R})}=\left(\|f\|^2_{L^2_x(\mathbb{R})}+\left\|\frac{df}{dx}\right\|^2_{L^2_x(\mathbb{R})}\right)^{1/2},\quad \|g\|_{L^2_x}=\|g\|_{L^2_x(\mathbb{R})}.$$

We also consider the norm $\|\cdot\|_{H^1_x\times L^2_x}$ given by

$$\|(f_1(x),f_2(x))\|_{H^1_x\times L^2_x}=(\|f_1\|^2_{H^1_x(\mathbb{R})}+\|f_2(x)\|^2_{L^2_x(\mathbb{R})})^{1/2}$$

for any $(f_1, f_2) \in H^1_x(\mathbb{R}) \times L^2_x(\mathbb{R})$. For any $(f_1, f_2) \in L^2_x(\mathbb{R}) \times L^2_x(\mathbb{R})$ and any $(g_1, g_2) \in L^2_x(\mathbb{R}) \times L^2_x(\mathbb{R})$, we let

$$\langle (f_1, f_2), (g_1, g_2)\rangle = \int_{\mathbb{R}} f_1(x)g_1(x) + f_2(x)g_2(x)\, dx.$$

For any functions $f_1(x), g_1(x) \in L^2_x(\mathbb{R})$, we let

$$\langle f_1, g_1\rangle = \int_{\mathbb{R}} f_1(x)g_1(x)\, dx.$$

In this manuscript, we consider the set $\mathbb{N}$ as the set of all positive integers. For any $n \in \mathbb{N}$, and any $a, b \in \mathbb{R}^n$, we denote the scalar product in the Euclidean space $\mathbb{R}^n$ by

$$\langle a : b\rangle = \sum_{j=1}^{n} a_j b_j,$$

where $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$.

1.4. ***Organization of the manuscript.*** First, from the global well-posedness of the partial differential equation (1), we recall that if ϕ is a strong solution of (1) with finite energy satisfying $\lim_{x\to\pm\infty} \phi(t_0, x) = \pm 1$ for some $t_0 \in \mathbb{R}$, then the function ϕ satisfies

$$\|\phi(t, x) - H_{0,1}(x) - H_{-1,0}(x)\|_{H^1_x(\mathbb{R})} < +\infty$$

for all $t \in \mathbb{R}$.

In Section 2.1, we will review our results from [Moutinho 2024] about the existence of a sequence of approximate solutions $(\varphi_{k,v})_{k\geq 2}$ of (1) for which there exists a set of real numbers $(y_k(v))_{k\geq 2}$ satisfying

$$\lim_{t\to+\infty} \|\overrightarrow{\varphi_k}(t, x) - \overrightarrow{H_{0,1,v,y_k}}(t, x) - \overrightarrow{H_{-1,0,v,y_k}}(t, x)\|_{H^1_x \times L^2_x} = 0,$$

and if $v \ll 1$, then $\|\partial_t^l \Lambda(\varphi_{k,v})(t, x)\|_{H^s_x} \lesssim_{s,l} v^{2k+l-1/2} e^{-2\sqrt{2}v|t|}$ for all $t \in \mathbb{R}$, $l \in \mathbb{N} \cup \{0\}$, and $s \geq 0$.

In Section 2.2, we will verify that any solution of (1) with finite energy close to a sum of two kinks can be written as

$$\phi(t, x) = \varphi_{k,v}(t, x) + \frac{y_1(t)}{\sqrt{1 - \frac{1}{4}\dot{d}(t)^2}} H'_{0,1}\left(\frac{x - \frac{1}{2}d(t) + c_k(t)}{\sqrt{1 - \frac{1}{4}\dot{d}(t)^2}}\right) + \frac{y_2(t)}{\sqrt{1 - \frac{1}{4}\dot{d}(t)^2}} H'_{0,1}\left(\frac{-x - \frac{1}{2}d(t) + c_k(t)}{\sqrt{1 - \frac{1}{4}\dot{d}(t)^2}}\right) + u(t, x), \tag{16}$$

such that, for any $t \in \mathbb{R}$, $u(t) \in H^1_x(\mathbb{R})$ satisfies the orthogonality conditions

$$\left\langle u(t, x), H'_{0,1}\left(\frac{x - \frac{1}{2}d(t) + c_k(t)}{\sqrt{1 - \frac{1}{4}\dot{d}(t)^2}}\right)\right\rangle = 0, \quad \left\langle u(t, x), H'_{0,1}\left(\frac{-x - \frac{1}{2}d(t) + c_k(t)}{\sqrt{1 - \frac{1}{4}\dot{d}(t)^2}}\right)\right\rangle = 0.$$

Moreover, using $\Lambda(\phi) \equiv 0$, we can verify that $y_1, y_2 \in C^2(\mathbb{R})$. Furthermore, using (16), we will estimate $\Lambda(\phi)(t, x)$. More precisely, we will estimate the expression $\Lambda(\phi)(t, x) - \Lambda(\varphi_{k,v})(t, x)$, in terms of $y_1(t)$, $y_2(t)$, $d(t)$, $u(t, x)$ and the estimate of the term $\Lambda(\varphi_{k,v})(t, x)$ will follow from the main results of Section 2.1 about the decay with respect to t of the approximate solutions. The function $c_k(t)$ will not appear in the evaluation of $\Lambda(\phi)(t, x)$, since we will use only its decay.

In Section 3, we will construct a function $L(t)$ to estimate $\|(u(t), \partial_t u(t))\|_{H^1_x \times L^2_x}$ during a large time interval. The main argument in this section is analogous to the ideas of Section 4 of [Moutinho 2023]. More precisely, for

$$w_{k,v}(t, x) = \frac{x - \frac{1}{2}d(t) + c_k(t)}{\sqrt{1 - \frac{1}{4}\dot{d}(t)^2}},$$

we consider first

$$L_1(t) = \int_{\mathbb{R}} \partial_t u(t, x)^2 + \partial_x u(t, x)^2 + U''\big(H_{0,1}(w_{k,v}(t, x)) - H_{0,1}(w_{k,v}(t, -x))\big)u(t, x)^2 \, dx.$$

From the orthogonality conditions satisfied by $u(t, x)$, if $v \ll 1$, we deduce the coercivity inequality

$$\|(u(t), \partial_t u(t))\|^2_{H^1_x \times L^2_x} \lesssim L_1(t).$$

The function $L(t)$ will be constructed after correction terms $L_2(t)$ and $L_3(t)$ are added to $L_1(t)$. The motivation for using the correction term $L_3(t)$ is to reduce the growth of the modulus of the expression

$$2 \int_{\mathbb{R}} \big[\partial_t^2 u(t, x) - \partial_x^2 u(t, x) + U''\big(H_{0,1}(w_{k,v}(t, x)) - H_{0,1}(w_{k,v}(t, x))\big)u(t, x)\big]\partial_t u(t, x) \, dx$$

in $\dot{L}_1(t)$. The time derivative of $L_2(t)$ will cancel with the expression

$$\int_{\mathbb{R}} \frac{\partial}{\partial t}\big[U''\big(H_{0,1}(w_{k,v}(t, x)) - H_{0,1}(w_{k,v}(t, x))\big)\big]u(t, x)^2 \, dx,$$

from $\dot{L}_1(t)$. Finally, under additional conditions in the growth of the functions $y_1(t)$, $y_2(t)$, if $0 < v \ll 1$, the function $L(t) = \sum_{j=1}^{3} L_j(t)$ will satisfy, for a constant $C(k)$ depending only on k, the estimates

$$|\dot{L}(t)| \lesssim \frac{v}{\ln(1/v)} \|(u(t), \partial_t u(t))\|^2_{H^1_x \times L^2_x},$$
$$\|(u(t), \partial_t u(t))\|^2_{H^1_x \times L^2_x} \lesssim L(t) + C(k) v^{4k} \ln(1/v)^{2n_k}$$

for all t in a large time interval, where n_k is the number described in Theorem 8. Hence, using Gronwall's lemma and the two estimates above, we will obtain an upper bound for $\|(u(t), \partial_t u(t))\|_{H^1_x \times L^2_x}$ when t belongs to a large time interval.

In Section 4, we will estimate $\|\phi(t) - \varphi_{k,v}(t)\|_{H^1_x \times L^2_x}$ during a large time interval. This estimate follows from the study of a linear ordinary differential system whose solutions $\hat{y}_1$, $\hat{y}_2$ are close to y_1, y_2 during a time interval of size much larger than $-\ln(v)/v$ and from the conclusions of the last section. Indeed, the closeness of the functions y_1, y_2 with $\hat{y}_1$, $\hat{y}_2$ during this large time interval is guaranteed because of the upper bound obtained for $\|(u(t), \partial_t u(t))\|_{H^1_x \times L^2_x}$ from the control of $L(t)$, which implies that y_1, y_2 will satisfy a ordinary differential system very close to the linear ordinary differential system satisfied by $\hat{y}_1$ and $\hat{y}_2$.

In Section 5, we will prove Theorem 4; the proof of this result is inspired by the demonstration of [Kowalczyk et al. 2021, Theorem 1; Martel et al. 2006, Theorem 1]. This result will imply in the next section the second item of Theorem 2. In addition, the main techniques used in this section are modulation techniques based on [Kowalczyk et al. 2021, §2; Martel et al. 2006], the use of conservation of energy of $\phi(t,x)$ and the monotonicity of the localized momentum given by

$$P_{+}(\phi(t),\partial_t\phi(t))=-\int_0^{+\infty}\partial_t\phi(t,x)\partial_x\phi(t,x)\,dx.$$

Finally, in Section 6, we will show that the demonstration of Theorem 2 is a direct consequence of the main results of Sections 4 and 5. For complementary information, see the Appendices.

2. Preliminaries

2.1. *Approximate solutions.*

Definition 6. We define Λ as the nonlinear operator with domain $C^2(\mathbb{R}^2,\mathbb{R})$ that satisfies

$$\Lambda(\phi_1)(t,x)=\partial_t^2\phi_1(t,x)-\partial_x^2\phi_1(t,x)+\dot{U}(\phi_1(t,x))$$

for any $\phi_1(t,x)\in C^2(\mathbb{R}^2,\mathbb{R})$.

In [Moutinho 2024], we constructed a sequence of approximate solutions $(\phi_k(v,t,x))_{k\in\mathbb{N}_{\geq 2}}$ of the partial differential equation (1) such that

$$\lim_{t\to+\infty}\left\|\phi_k(v,t,x)-H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right)-H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right\|_{H^1_x}=0,$$

$$\lim_{t\to+\infty}\left\|\partial_t\phi_k(v,t,x)+\frac{v}{\sqrt{1-v^2}}H'_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right)-\frac{v}{\sqrt{1-v^2}}H'_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right\|_{L^2_x}=0$$

More precisely, in [Moutinho 2024] we proved the following result:

Theorem 7. *There exist a sequence of functions $(\phi_k(v,t,x))_{k\geq 2}$, a sequence of real values $\delta(k)>0$ and a sequence of numbers $n_k\in\mathbb{N}$ such that, for any $0<v<\delta(k)$, $\phi_k(v,t,x)$ satisfies*

$$\lim_{t\to+\infty}\left\|\phi_k(v,t,x)-H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right)-H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right\|_{H^1_x}=0,$$

$$\lim_{t\to+\infty}\left\|\partial_t\phi_k(v,t,x)+\frac{v}{\sqrt{1-v^2}}H'_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right)-\frac{v}{\sqrt{1-v^2}}H'_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right\|_{L^2_x}=0,$$

$$\lim_{t\to-\infty}\left\|\phi_k(v,t,x)-H_{0,1}\left(\frac{x+vt-e_{v,k}}{\sqrt{1-v^2}}\right)-H_{-1,0}\left(\frac{x-vt+e_{v,k}}{\sqrt{1-v^2}}\right)\right\|_{H^1_x}=0,$$

$$\lim_{t\to-\infty}\left\|\partial_t\phi_k(v,t,x)-\frac{v}{\sqrt{1-v^2}}H'_{0,1}\left(\frac{x+vt-e_{v,k}}{\sqrt{1-v^2}}\right)+\frac{v}{\sqrt{1-v^2}}H'_{-1,0}\left(\frac{x-vt+e_{v,k}}{\sqrt{1-v^2}}\right)\right\|_{L^2_x}=0,$$

with $e_{v,k}\in\mathbb{R}$ satisfying

$$\lim_{v\to 0}\frac{\left|e_{v,k}-\frac{\ln(8/v^2)}{\sqrt{2}}\right|}{v|\ln(v)|^3}=0.$$

Moreover, if $0<v<\delta(k)$, then for any $s\geq 0$ and $l\in\mathbb{N}\cup\{0\}$, there is $C(k,s,l)>0$ such that

$$\left\|\frac{\partial^l}{\partial t^l}\Lambda(\phi_k(v,t,x))\right\|_{H^s_x(\mathbb{R})}\leq C(k,s,l)v^{2k+l}\big(|t|v+\ln(1/v^2)\big)^{n_k}e^{-2\sqrt{2}|t|v}.$$

We consider the Schwarz function $\mathcal{G}$ defined by

$$\mathcal{G}(x)=e^{-\sqrt{2}x}-\frac{e^{-\sqrt{2}x}}{(1+e^{2\sqrt{2}x})^{3/2}}+2\sqrt{2}x\frac{e^{\sqrt{2}x}}{(1+e^{2\sqrt{2}x})^{3/2}}+k_1\frac{e^{\sqrt{2}x}}{(1+e^{2\sqrt{2}x})^{3/2}} \tag{17}$$

for all $x\in\mathbb{R}$, where k_1 is the real number such that $\mathcal{G}$ satisfies $\langle\mathcal{G}(x),\,H_{0,1}'(x)\rangle_{L^2_x(\mathbb{R})}=0$. The function $\mathcal{G}$ satisfies the identity

$$-\frac{d^2}{dx^2}\mathcal{G}(x)+U''(H_{0,1}(x))\mathcal{G}(x)=[-24H_{0,1}(x)^2+30H_{0,1}(x)^4]e^{-\sqrt{2}x}+8\sqrt{2}H_{0,1}'(x); \tag{18}$$

see Lemma A.1 and Remark A.2 in the Appendix of [Moutinho 2024] for the proof.

From now on, for any $v\in(0,1)$, we consider the function $d_v:\mathbb{R}\to\mathbb{R}$ defined by

$$d_v(t)=\frac{1}{\sqrt{2}}\ln\biggl(\frac{8}{v^2}\cosh{(\sqrt{2}vt)}^2\biggr)\quad\text{for any } t\in\mathbb{R}.$$

The function d_v describes the movement between two kinks for the ϕ^6 model during a large time interval when their total energy is small and their initial speeds are both zero. For more information, see Theorem 1.11 from [Moutinho 2023].

Moreover, from the proof of Theorem 7 in [Moutinho 2024], we can construct inductively an explicit sequence of smooth functions $(\varphi_{k,v})_{k\in\mathbb{N}_{\geq 2}}$ and for each $k\in\mathbb{N}_{\geq 2}$ there exists a real number $\tau_{k,v}$ satisfying

$$|\tau_{k,v}|<\frac{\sqrt{2}}{v}\ln\biggl(\frac{8}{v^2}\biggr)$$

such that $\phi_k(v,t,x):=\varphi_{k,v}(t+\tau_{k,v},x)$ satisfies Theorem 7 for all $k\in\mathbb{N}_{\geq 2}$. More precisely, from [Moutinho 2024], we have the following theorem:

Theorem 8. *There exist a sequence of approximate solutions $\varphi_{k,v}(t,x)$, functions $r_k(v,t)$ that are smooth and even on t, and numbers $n_k\in\mathbb{N}$ such that if $0<v\ll 1$, then, for any $m\in\mathbb{N}_{\geq 1}$,*

$$|r_k(v,t)|\lesssim_k v^{2(k-1)}\ln(1/v)^{n_k},\qquad \biggl|\frac{\partial^m}{\partial t^m}r_k(v,t)\biggr|\lesssim_{k,m} v^{2(k-1)+m}\bigl[\ln(1/v)+|t|v\bigr]^{n_k}e^{-2\sqrt{2}|t|v}. \tag{19}$$

Furthermore, $\varphi_{k,v}(t,x)$ satisfies for $\rho_k(v,t)=-\frac{1}{2}d_v(t)+\sum_{j=2}^{k}r_j(v,t)$ the identity

$$\begin{aligned}\varphi_{k,v}(t,x)=H_{0,1}\biggl(\frac{x+\rho_k(v,t)}{\sqrt{1-\frac{1}{4}\dot{d}_v(t)^2}}\biggr)+H_{-1,0}\biggl(\frac{x-\rho_k(v,t)}{\sqrt{1-\frac{1}{4}\dot{d}_v(t)^2}}\biggr)&\\ +e^{-\sqrt{2}d_v(t)}\biggl[\mathcal{G}\biggl(\frac{x+\rho_k(v,t)}{\sqrt{1-\frac{1}{4}\dot{d}_v(t)^2}}\biggr)-\mathcal{G}\biggl(\frac{-x+\rho_k(v,t)}{\sqrt{1-\frac{1}{4}\dot{d}_v(t)^2}}\biggr)\biggr]&\\ +\mathcal{R}_{k,v}\biggl(vt,\frac{x+\rho_k(v,t)}{\sqrt{1-\frac{1}{4}\dot{d}_v(t)^2}}\biggr)-\mathcal{R}_{k,v}\biggl(vt,\frac{-x+\rho_k(v,t)}{\sqrt{1-\frac{1}{4}\dot{d}_v(t)^2}}\biggr)&\end{aligned} \tag{20}$$

and, for any $l \in \mathbb{N} \cup \{0\}$ *and* $s \geq 1$ *the estimates*

$$\left\| \frac{\partial^l}{\partial t^l} \Lambda(\varphi_{k,v}(t,x)) \right\|_{H^s_x(\mathbb{R})} \lesssim_{k,s,l} v^{2k+l}\big[\ln(1/v^2) + |t|v\big]^{n_k} e^{-2\sqrt{2}|t|v}, \tag{21}$$

$$\left| \frac{d^l}{dt^l} \left[\left\langle \Lambda(\varphi_{k,v})(t,x),\, H'_{0,1}\left(\frac{x+\rho_k(v,t)}{\left(1-\frac{1}{4}\dot{d}_v(t)^2\right)^{1/2}} \right) \right\rangle \right] \right| \lesssim_{k,l} v^{2k+l+2}\big[\ln(1/v^2) + |t|v\big]^{n_k+1} e^{-2\sqrt{2}|t|v}, \tag{22}$$

where $\mathcal{R}_{k,v}(t,x)$ *is a finite sum of functions* $p_{k,i,v}(t)h_{k,i}(x)$ *with* $h_{k,i} \in \mathscr{S}(\mathbb{R})$ *and each* $p_{k,i,v}(t)$ *being an even function satisfying, for all* $m \in \mathbb{N}$,

$$\left| \frac{d^m p_{k,i,v}(t)}{dt^m} \right| \lesssim_{k,m,3} v^4 \big(\ln(1/v^2) + |t|\big)^{n_{k,i}} e^{-2\sqrt{2}|t|},$$

where $n_{k,i} \in \mathbb{N}$ *depends only on* k *and* i.

Remark 9. Furthermore, Remark 5.2 of [Moutinho 2024] implies that if $v > 0$ is small enough, then the function r_2 satisfies

$$\|r_2(v,\cdot)\|_{L^\infty(\mathbb{R})} \lesssim v^2 \ln(1/v^2), \qquad \left| \frac{\partial^l}{\partial t^l} r_2(v,t) \right| \lesssim_l v^{2+l}\big[\ln(1/v^2) + |t|v\big] e^{-2\sqrt{2}|t|v}$$

for all $l \in \mathbb{N}$.

Remark 10. At first look, the statement of Theorem 8 seems to contain excessive information about the approximate solutions $\phi_k(v,t,x)$ of [Moutinho 2024]. However, we will need all of it to study the elasticity and stability of the collision of two kinks with low speed $0 < v < 1$.

2.2. ***Auxiliary estimates.*** First, we recall the Lemma 2.1 of [Moutinho 2023].

Lemma 11. *If* x_2, x_1 *are real numbers satisfying* $z = x_2 - x_1 > 0$ *and* $\alpha, \beta, m > 0$ *with* $\alpha \neq \beta$, *then*

$$\int_{\mathbb{R}} |x - x_1|^m e^{-\alpha(x-x_1)_+} e^{-\beta(x_2-x)_+} \lesssim_{m,\alpha,\beta} \max\big((1+z^m)e^{-\alpha z}, e^{-\beta z}\big),$$

Furthermore, for any $\alpha > 0$,

$$\int_{\mathbb{R}} |x - x_1|^m e^{-\alpha(x-x_1)_+} e^{-\alpha(x_2-x)_+} \lesssim_{m,\alpha} [1 + z^{m+1}] e^{-\alpha z}.$$

Actually, we will also need to use the following lemma, which we proved in [Moutinho 2024].

Lemma 12. *In the notation of Theorem 8, for* $v \in (0,1)$, *let* $w_{k,v} : \mathbb{R}^2 \to \mathbb{R}$ *be the function*

$$w_{k,v}(t,x) = \frac{x + \rho_k(v,t)}{\sqrt{1 - \frac{1}{4}\dot{d}_v(t)^2}},$$

and let $f \in L^\infty_x(\mathbb{R})$ *be a function satisfying* $f' \in \mathscr{S}(\mathbb{R})$. *Then, if* $0 < v \ll 1$, *we have for any* $l \in \mathbb{N}$ *that*

$$\frac{\partial^l}{\partial t^l} f(w_{k,v}(t,x))$$

is a finite sum of functions $q_{k,l,i,v}(t)h_i(w_{k,v}(t,x))$ *with* $h_i \in \mathscr{S}(\mathbb{R})$ *and* $q_{k,l,i,v}(t)$ *a smooth real function satisfying*

$$\|q_{k,l,i,v}\|_{L^\infty(\mathbb{R})} \lesssim v^l.$$

Furthermore, if $0 < v \ll 1$, *we have for all* $l \in \mathbb{N}$ *and any* $s \geq 0$ *that*

$$\left\| \frac{\partial^l}{\partial t^l} f(w_{k,v}(t,x)) \right\|_{H^s_x(\mathbb{R})} \lesssim_{k,s,l} v^l.$$

Moreover, we will use the following result several times in the computation of the estimates of this paper.

Lemma 13. *For any* $s \geq 1$, *we have for any functions* $f, g \in \mathscr{S}(\mathbb{R})$ *that*

$$\|fg\|_{H^s_x(\mathbb{R})} \lesssim_s \|f\|_{H^s_x(\mathbb{R})}\|g\|_{L^\infty_x(\mathbb{R})} + \|g\|_{H^s_x(\mathbb{R})}\|f\|_{L^\infty_x(\mathbb{R})} \lesssim_s \|f\|_{H^s_x(\mathbb{R})}\|g\|_{H^s_x(\mathbb{R})}.$$

As a consequence,

$$\|fg\|_{H^s_x(\mathbb{R})} \lesssim_s \|f\|_{H^{s+1}_x(\mathbb{R})}\|g\|_{H^{s+1}_x(\mathbb{R})}$$

for all $s \geq 0$.

Proof. See the proof of Lemma A.8 in [Tao 2006]. ☐

Finally, we need also Lemma 2.5 of [Moutinho 2023] which studies the coercive properties of the operator

$$-\partial_x^2 + U''(H^z_{0,1}(x) + H_{-1,0}(x))$$

when $z \gg 1$. More precisely:

Lemma 14. *There exist* $c, \delta > 0$ *such that if* $z \geq \frac{1}{\delta}$, *then for any* $g \in H^1(\mathbb{R})$ *satisfying*

$$\langle g(x), H'_{0,1}(x-z)\rangle = \langle g(x), H'_{-1,0}(x)\rangle = 0,$$

we have that

$$\left\langle -\frac{d^2}{dx^2}g(x) + U''(H_{0,1}(x-z) + H_{-1,0}(x))g(x), g(x) \right\rangle \geq c\|g\|^2_{H^1_x(\mathbb{R})}.$$

Proof. See the proof of Lemma 9 in [Moutinho 2023]. ☐

In this manuscript, to simplify our notation, we denote $d_v(t)$ by $d(t)$, which means that

$$d(t) = \frac{1}{\sqrt{2}} \ln\left(\frac{8}{v^2}\cosh\left(\sqrt{2}vt\right)^2\right). \tag{23}$$

In Lemma 3.1 of [Moutinho 2024], we have verified by induction the estimates

$$\begin{aligned} |\dot{d}(t)| &\lesssim v, \\ |d^{(l)}(t)| &\lesssim_l v^l e^{-2\sqrt{2}|t|v} \quad \text{for any } l \in \mathbb{N}_{\geq 2}. \end{aligned} \tag{24}$$

From now on, we consider for each $k \in \mathbb{N}_{\geq 2}$ the function $\phi_{k,v}(t,x)$ satisfying Theorem 8. Next, for $T_{0,k} > 0$ to be chosen later, we consider the following kind of Cauchy problem:

$$\begin{cases} \partial_t^2\phi(t,x) - \partial_x^2\phi(t,x) + U'(\phi(t,x)) = 0, \\ \|(\phi(T_{0,k},x), \partial_t\phi(T_{0,k},x)) - (\phi_{k,v}(T_{0,k},x), \partial_t\phi_{k,v}(T_{0,k},x))\|_{H^1_x(\mathbb{R})\times L^2_x(\mathbb{R})} < v^{8k}. \end{cases} \tag{25}$$

Our first objective is to prove the following theorem.

Theorem 15. *There is a constant* $C>0$ *and for any* $0<\theta<\frac{1}{4}$, $k\in\mathbb{N}_{\geq 3}$ *there exist* $C_1(k)>0$, $\delta_{k,\theta}>0$ *and* $\eta_k\in\mathbb{N}$ *such that if*

$$0<v<\delta_{k,\theta} \quad \textit{and} \quad T_{0,k}=\frac{32k}{2\sqrt{2}}\frac{\ln(1/v^2)}{v},$$

then any solution $\phi(t,x)$ *of* (25) *satisfies*

$$\|(\phi(t,x),\partial_t\phi(t,x))-(\varphi_{k,v}(t,x),\partial_t\varphi_{k,v}(t,x))\|_{H^1_x\times L^2_x}<C_1(k)v^{2k}\ln(1/v)^{\eta_k}\exp\biggl(C\frac{v|t-T_{0,k}|}{\ln(v)}\biggr) \tag{26}$$

if

$$|t-T_{0,k}|<\frac{\ln(1/v)^{2-\theta}}{v}.$$

Clearly, we can obtain from Theorems 8 and 15 the following result:

Corollary 16. *There is a constant* $C>0$ *and for any* $0<\theta<\frac{1}{4}$, $k\in\mathbb{N}_{\geq 3}$ *there exist* $C_1(k)>0$, $\delta_{k,\theta}>0$ *and* $\eta_k\in\mathbb{N}$ *such that if*

$$0<v<\delta_{k,\theta} \quad \textit{and} \quad T_{0,k}=\frac{32k}{2\sqrt{2}}\frac{\ln(1/v^2)}{v},$$

then any solution $\phi(t,x)$ *of*

$$\begin{cases}\partial_t^2\phi(t,x)-\partial_x^2\phi(t,x)+U'(\phi(t,x))=0,\\ \|(\phi(T_{0,k},x),\partial_t\phi(T_{0,k},x))-(\phi_k(v,T_{0,k},x),\partial_t\phi_k(v,T_{0,k},x))\|_{H^1_x(\mathbb{R})\times L^2_x(\mathbb{R})}<v^{8k}\end{cases}$$

satisfies

$$\|(\phi(t,x),\partial_t\phi(t,x))-(\phi_k(v,t,x),\partial_t\phi_k(v,t,x))\|_{H^1_x\times L^2_x}<C_1(k)v^{2k}\ln(1/v)^{\eta_k}\exp\biggl(C\frac{v|t-T_{0,k}|}{\ln(v)}\biggr),$$

if

$$|t-T_{0,k}|<\frac{\ln(1/v)^{2-\theta}}{v}.$$

Proof of Corollary 16. This follows from Theorems 7, 8 and 15. □

With the objective of simplifying the demonstration of Theorem 15, we will elaborate on necessary lemmas before the proof of Theorem 15. Similarly to [Moutinho 2024], using the notation of Theorem 8, we consider

$$w_{k,v}(t,x)=\frac{x-\frac{1}{2}d(t)+c_k(v,t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}. \tag{27}$$

From now on, we denote any solution $\phi(t,x)$ of the partial differential equation (25) as

$$\phi(t,x)=\varphi_{k,v}(t,x)+\frac{y_1(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}H'_{0,1}(w_{k,v}(t,x))+\frac{y_2(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}H'_{0,1}(w_{k,v}(t,-x))+u(t,x), \tag{28}$$

such that

$$\bigl\langle u(t,x),H'_{0,1}(w_{k,v}(t,x))\bigr\rangle=\bigl\langle u(t,x),H'_{0,1}(w_{k,v}(t,-x))\bigr\rangle=0. \tag{29}$$

Therefore, for $\zeta_k(t)=d(t)-2c_k(v,t)$ and from the orthogonal conditions (29) satisfied by $u(t,x)$, we deduce the identity

$$\begin{bmatrix} y_1(t) \\ y_2(t)\end{bmatrix} = M(t)^{-1}\begin{bmatrix} \langle \phi(t,x)-\varphi_{k,v}(t,x),\, H'_{0,1}(w_{k,v}(t,x))\rangle \\ \langle \phi(t,x)-\varphi_{k,v}(t,x),\, H'_{-1,0}(w_{k,v}(t,-x))\rangle\end{bmatrix}, \tag{30}$$

where, for any $t\in\mathbb{R}$, $M(t)$ is denoted by

$$M(t)=\begin{bmatrix} \|H'_{0,1}\|^2_{L^2_x} & \langle H'_{0,1}(x-\zeta_k(t)),\, H'_{-1,0}(x)\rangle \\ \langle H'_{0,1}(x-\zeta_k(t)),\, H'_{-1,0}(x)\rangle & \|H'_{0,1}\|^2_{L^2_x}\end{bmatrix}.$$

Moreover, since $\ln(1/v)\lesssim\zeta_k$, we obtain from Lemma 11 that $\langle H'_{0,1}(x-\zeta_k(t)),\, H'_{-1,0}(x)\rangle \ll 1$. Therefore, since the matrix $M(t)$ is a smooth function with domain $\mathbb{R}$, then $M(t)^{-1}$ is also smooth on $\mathbb{R}$.

Next, for $\psi(t,x)=\phi(t,x)-\varphi_{k,v}(t,x)$, we obtain from the partial differential equation (25) that $\psi(t,x)$ satisfies the partial differential equation

$$\frac{\partial^2}{\partial t^2}\psi(t,x)-\frac{\partial^2}{\partial x^2}\psi(t,x)+\Lambda(\varphi_{k,v})(t,x)+\sum_{j=2}^{6}\frac{U^{(j)}(\varphi_{k,v}(t,x))}{(j-1)!}\psi(t,x)^{j-1}=0. \tag{31}$$

Since $\varphi_{k,v}$ satisfies Theorem 8 and the partial differential equation (1) is globally well-posed in the energy space, we can verify for any initial data $(\psi_0(x),\psi_1(x))\in H^1_x(\mathbb{R})\times L^2_x(\mathbb{R})$ that there exists a unique solution $\psi(t,x)$ of (31) satisfying $(\psi(0,x),\partial_t\psi(0,x))=(\psi_0(x),\psi_1(x))$ and

$$(\psi(t,x),\partial_t\psi(t,x))\in C(\mathbb{R};H^1_x(\mathbb{R})\times L^2_x(\mathbb{R})). \tag{32}$$

Therefore, for any function $h\in\mathscr{S}(\mathbb{R})$, we deduce from (31) that

$$\begin{aligned}
&\frac{d}{dt}\langle\psi(t,x),h(x)\rangle=\langle\partial_t\psi(t,x),h(x)\rangle,\\
&\frac{d^2}{dt^2}\langle\psi(t,x),h(x)\rangle=\left\langle\frac{\partial^2}{\partial x^2}\psi(t,x)-U'(\varphi_{k,v}(t,x)+\psi(t,x))+U'(\varphi_{k,v}(t,x)),h(x)\right\rangle\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad-\langle\Lambda(\varphi_{k,v})(t,x),h(x)\rangle,
\end{aligned}$$

which implies that the real functions

$$\mathcal{P}_1(t)=\langle\psi(t,x),\,H'_{0,1}(w_{k,v}(t,x))\rangle \quad\text{and}\quad \mathcal{P}_2(t)=\langle\psi(t,x),\,H'_{-1,0}(w_{k,v}(t,-x))\rangle$$

are in $C^2(\mathbb{R})$. In conclusion, using (30) and the product rule of derivative, we deduce that $y_1,y_2\in C^2(\mathbb{R})$.

In conclusion, we obtain the following lemma:

Lemma 17. *Assuming the same hypotheses of Theorem 15, there exist functions $y_1,y_2:\mathbb{R}\to\mathbb{R}$ of class C^2 such that any solution $\phi(t,x)$ of* (25) *satisfies for any $t\in\mathbb{R}$ the identity*

$$\phi(t,x)=\varphi_{k,v}(t,x)+\frac{y_1(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}H'_{0,1}(w_{k,v}(t,x))+\frac{y_2(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}H'_{0,1}(w_{k,v}(t,-x))+u(t,x),$$

where $(u(t),\partial_t u(t))\in H^1_x(\mathbb{R})\times L^2_x(\mathbb{R})$ and the function u satisfies the orthogonality conditions

$$\langle u(t,x),\,H'_{0,1}(w_{k,v}(t,x))\rangle=0,\quad \langle u(t,x),\,H'_{0,1}(w_{k,v}(t,-x))\rangle=0.$$

Remark 18. Moreover, Theorem 8 implies that

$$\begin{aligned}
&\frac{d^2}{dt^2}\Bigg[\frac{y_j(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}H'_{0,1}(w_{k,v}(t,(-1)^{j+1}x))\Bigg]\\
&\quad=\frac{\ddot{y}_j(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}H'_{0,1}(w_{k,v}(t,(-1)^{j+1}x))+\frac{\dot{y}_j(t)\ddot{d}(t)\dot{d}(t)}{2\big(1-\frac{1}{4}\dot{d}(t)^2\big)^{3/2}}H'_{0,1}(w_{k,v}(t,(-1)^{j+1}x))\\
&\qquad+2\frac{\dot{y}_j(t)\partial_t\rho_k(v,t)}{1-\frac{1}{4}\dot{d}(t)^2}H''_{0,1}(w_{k,v}(t,(-1)^{j+1}x))\\
&\qquad+\frac{\dot{y}_j(t)\ddot{d}(t)\dot{d}(t)}{2\big(1-\frac{1}{4}\dot{d}(t)^2\big)^{2}}((-1)^{j+1}x+\rho_k(v,t))H''_{0,1}(w_{k,v}(t,(-1)^{j+1}x))\\
&\qquad+\frac{y_j(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}\frac{\partial^2}{\partial t^2}[H'_{0,1}(w_{k,v}(t,(-1)^{j+1}x))].
\end{aligned}$$

Therefore, from Theorem 8, Remark 9 and estimates (24), we deduce from the estimate above that

$$\begin{aligned}
&\frac{\partial^2}{\partial t^2}\Bigg[\frac{y_j(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}H'_{0,1}(w_{k,v}(t,(-1)^{j+1}x))\Bigg]\\
&\quad=\frac{\ddot{y}_j(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}H'_{0,1}(w_{k,v}(t,(-1)^{j+1}x))-\frac{\dot{y}_j(t)\dot{d}(t)}{1-\frac{1}{4}\dot{d}_v(t)^2}H''_{0,1}(w_{k,v}(t,(-1)^{j+1}x))\\
&\qquad+\frac{y_j(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}\frac{\partial^2}{\partial t^2}[H'_{0,1}(w_{k,v}(t,(-1)^{j+1}x))]+Q_1(t,x),
\end{aligned}$$

where $Q_1(t,\cdot)$ is a function in $H^1_x(\mathbb{R})$ satisfying

$$\|Q_1(t,x)\|_{H^1_x(\mathbb{R})}\lesssim\Big[\max_{j\in\{1,2\}}|\dot{y}_j(t)|+v\max_{j\in\{1,2\}}|y_j(t)|\Big]v^3\big(\ln(1/v^2)+|t|v\big)e^{-2\sqrt{2}|t|v}.$$

Moreover, using identities

$$\frac{d^3}{dx^3}H_{0,1}(x)=U''(H_{0,1}(x))H'_{0,1}(x),\qquad \ddot{d}(t)=16\sqrt{2}e^{-\sqrt{2}d(t)},$$

estimates (24) and the estimates of $r_j(v,t)$ in Theorem 8 and Remark 9, we obtain

$$\begin{aligned}
&\bigg(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\bigg)[H'_{0,1}(w_{k,v}(t,(-1)^{j+1}x))]\\
&\quad=-\frac{8\sqrt{2}e^{-\sqrt{2}d(t)}}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}H''_{0,1}(w_{k,v}(t,(-1)^{j+1}x))\\
&\qquad-U''(H_{0,1}(w_{k,v}(t,(-1)^{j+1}x)))H'_{0,1}(w_{k,v}(t,(-1)^{j+1}x))+Q_2(t,x),
\end{aligned}$$

where $Q_2(t,\cdot)$ is a function in $H^1_x(\mathbb{R})$ satisfying

$$\|Q_2(t,x)\|_{H^1_x(\mathbb{R})}\lesssim v^4\big(\ln(1/v^2)+|t|v\big)e^{-2\sqrt{2}|t|v}.$$

Consequently, using Lemmas 12, 13, 17 and identity $\Lambda(\phi)=0$, we conclude from Taylor's expansion theorem that

$$\begin{aligned}
&\Lambda(\varphi_{k,v})(t,x)+\partial_t^2 u(t,x)-\partial_x^2 u(t,x)+U''(\varphi_{k,v}(t,x))(\phi(t,x)-\varphi_{k,v}(t,x))\\
&\quad+\frac{\ddot{y}_1(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}H'_{0,1}(w_{k,v}(t,x))+\frac{\ddot{y}_2(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}H'_{0,1}(w_{k,v}(t,-x))\\
&\quad-\frac{y_1(t)8\sqrt{2}e^{-\sqrt{2}d(t)}}{1-\frac{1}{4}\dot{d}(t)^2}H''_{0,1}(w_{k,v}(t,x))-\frac{y_2(t)8\sqrt{2}e^{-\sqrt{2}d(t)}}{1-\frac{1}{4}\dot{d}(t)^2}H''_{0,1}(w_{k,v}(t,-x))\\
&\quad-\frac{\dot{y}_1(t)\dot{d}(t)}{1-\frac{1}{4}\dot{d}(t)^2}H''_{0,1}(w_{k,v}(x,t))-\frac{\dot{y}_2(t)\dot{d}(t)}{1-\frac{1}{4}\dot{d}(t)^2}H''_{0,1}(w_{k,v}(t,-x))\\
&\quad-y_1(t)\frac{U''(H_{0,1}(w_{k,v}(t,x)))}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}H'_{0,1}(w_{k,v}(t,x))-y_2(t)\frac{U''(H_{0,1}(w_{k,v}(t,-x)))}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}H'_{0,1}(w_{k,v}(t,-x))\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad=\mathcal{Q}(t,x),
\end{aligned}\tag{33}$$

where $\mathcal{Q}(t,\cdot)$ is a function in $H_x^1(\mathbb{R})$ satisfying, for all $t\in\mathbb{R}$,

$$\begin{aligned}
\|\mathcal{Q}(t,x)\|_{H_x^1(\mathbb{R})}&\lesssim\|u(t)\|_{H_x^1}^2+\|u(t)\|_{H_x^1}^6+\max_{j\in\{1,2\}}|y_j(t)|^2+\max_{j\in\{1,2\}}|y_j(t)|^6\\
&\quad+\Big[\max_{j\in\{1,2\}}|\dot{y}_j(t)|+v\max_{j\in\{1,2\}}|y_j(t)|\Big]v^3\big(\ln(1/v^2)+|t|v\big)e^{-2\sqrt{2}|t|v},
\end{aligned}$$

if $v>0$ is small enough.

Next, from (33) of Remark 18, we consider the terms

$$Y_1(t,x)=\big[U''(\varphi_{k,v}(t,x))-U''(H_{0,1}(w_{k,v}(t,x)))\big]\frac{y_1(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}H'_{0,1}(w_{k,v}(t,x)),\tag{34}$$

$$Y_2(t,x)=\big[U''(\varphi_{k,v}(t,x))-U''(H_{0,1}(w_{k,v}(t,-x)))\big]\frac{y_2(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}H'_{0,1}(w_{k,v}(t,-x)).\tag{35}$$

Now, we will estimate the expressions

$$\langle Y_1(t),H'_{0,1}(w_{k,v}(t,x))\rangle,\quad\langle Y_2(t),H'_{0,1}(w_{k,v}(t,-x))\rangle.$$

Lemma 19. *In notation of Theorem 8 and Lemma 17, the functions $Y_1(t)$ and $Y_2(t)$ satisfy*

$$\begin{aligned}
\langle Y_1(t),H'_{0,1}(w_{k,v}(t,x))\rangle&=4\sqrt{2}e^{-\sqrt{2}d(t)}y_1(t)+y_1(t)\operatorname{Res}_1(v,t),\\
\langle Y_2(t),H'_{0,1}(w_{k,v}(t,x))\rangle&=-4\sqrt{2}e^{-\sqrt{2}d(t)}y_2(t)+y_2(t)\operatorname{Res}_2(v,t),
\end{aligned}$$

where, for any $j\in\{1,2\}$ and all $v\in(0,1)$, the function $\operatorname{Res}_j(v,t)$ is a Schwarz function on t satisfying for any $l\in\mathbb{N}\cup\{0\}$, if $0<v\ll1$, the estimate

$$\left|\frac{\partial^l}{\partial t^l}\operatorname{Res}_j(v,t)\right|\lesssim_l v^{l+4}\big[\ln(1/v^2)+|t|v\big]^{\eta_k}e^{-2\sqrt{2}|t|v}\tag{36}$$

for a number $\eta_k\geq0$ depending only on $k\in\mathbb{N}_{\geq2}$.

Proof of Lemma 19. First, we observe that

$$\left|\frac{d^l}{dt^l}e^{-\sqrt{2}d(t)}\right| = \left|\frac{d^l}{dt^l}\frac{v^2}{8}\operatorname{sech}\,(\sqrt{2}vt)^2\right| \lesssim_l v^{2+l}e^{-2\sqrt{2}|t|v}.$$

Using Taylor's expansion theorem, Theorem 8 and Lemma 13, we deduce that

$$\begin{aligned}U''(\varphi_{k,v}(t,x)) = U''(H_{0,1}(w_{k,v}(t,x)) - H_{0,1}(w_{k,v}(t,-x))) + e^{-\sqrt{2}d(t)}U^{(3)}(H_{0,1}(w_{k,v}(t,x))\\ - H_{0,1}(w_{k,v}(t,-x)))[\mathcal{G}(w_{k,v}(t,x)) - \mathcal{G}(w_{k,v}(t,-x))] + \operatorname{res}_1(v,t,x),\end{aligned}$$

where, if $0 < v \ll 1$, $\operatorname{res}_1(v,t,x)$ is a smooth function on the variables (t,x) which satisfies for some $\eta_k \in \mathbb{N}$ and any $s \geq 0$, $l \in \mathbb{N}\cup\{0\}$ the inequality

$$\left\|\frac{\partial^l}{\partial t^l}\operatorname{res}_1(v,t,x)\right\|_{H^s_x} \lesssim_{s,l} v^{4+l}\big[\ln(1/v^2) + |t|v\big]^{\eta_k}e^{-2\sqrt{2}|t|v}. \tag{37}$$

Therefore, using

$$\begin{aligned}&U''(\varphi_{k,v}(t,x)) - U''(H_{0,1}(w_{k,v}(t,x)))\\ &\quad= U''(\varphi_{k,v}(t,x)) - U''\big(H_{0,1}(w_{k,v}(t,x)) - H_{0,1}(w_{k,v}(t,-x))\big)\\ &\qquad+U''\big(H_{0,1}(w_{k,v}(t,x)) - H_{0,1}(w_{k,v}(t,-x))\big) - U''\big(H_{0,1}(w_{k,v}(t,x))\big),\end{aligned}$$

we obtain that

$$\begin{aligned}&Y_1(t,x)\sqrt{1-\tfrac{1}{4}\dot{d}(t)^2}\\ &\quad= \big[U''\big(H_{0,1}(w_{k,v}(t,x)) - H_{0,1}(w_{k,v}(t,-x))\big) - U''(H_{0,1}(w_{k,v}(t,x)))\big]y_1(t)H'_{0,1}(w_{k,v}(x,t))\\ &\qquad+ y_1(t)e^{-\sqrt{2}d(t)}U^{(3)}\big(H_{0,1}(w_{k,v}(t,x)) - H_{0,1}(w_{k,v}(t,-x))\big)\mathcal{G}(w_{k,v}(t,x))H'_{0,1}(w_{k,v}(t,x))\\ &\qquad\quad- y_1(t)e^{-\sqrt{2}d(t)}U^{(3)}\big(H_{0,1}(w_{k,v}(t,x)) - H_{0,1}(w_{k,v}(t,-x))\big)\mathcal{G}(w_{k,v}(t,-x))H'_{0,1}(w_{k,v}(t,x))\\ &\qquad\qquad+ y_1(t)\operatorname{res}_1(v,t,x).\end{aligned} \tag{38}$$

By a similar reasoning, we obtain that

$$\begin{aligned}&Y_2(t,x)\sqrt{1-\tfrac{1}{4}\dot{d}(t)^2}\\ &\quad= \big[U''\big(H_{0,1}(w_{k,v}(t,x)) - H_{0,1}(w_{k,v}(t,-x))\big) - U''(H_{0,1}(w_{k,v}(t,-x)))\big]y_2(t)H'_{0,1}(w_{k,v}(t,-x))\\ &\qquad+ y_2(t)e^{-\sqrt{2}d(t)}U^{(3)}\big(H_{0,1}(w_{k,v}(t,x)) - H_{0,1}(w_{k,v}(t,-x))\big)\mathcal{G}(w_{k,v}(t,x))H'_{0,1}(w_{k,v}(t,-x))\\ &\qquad\quad- y_2(t)e^{-\sqrt{2}d(t)}U^{(3)}\big(H_{0,1}(w_{k,v}(t,x)) - H_{0,1}(w_{k,v}(t,-x))\big)\mathcal{G}(w_{k,v}(t,-x))H'_{0,1}(w_{k,v}(t,-x))\\ &\qquad\qquad+ y_2(t)\operatorname{res}_2(v,t,x),\end{aligned} \tag{39}$$

where, if $0 < v \ll 1$, $\operatorname{res}_2(v,t,x)$ is a smooth function on t, x satisfying, for some constant $\eta_k \geq 0$, any $l \in \mathbb{N}\cup\{0\}$ and $s \geq 0$, the estimate

$$\left\|\frac{\partial^l}{\partial t^l}\operatorname{res}_2(v,t,x)\right\|_{H^s_x} \lesssim_{s,l} v^{4+l}\big[\ln(1/v^2) + |t|v\big]^{\eta_k}e^{-2\sqrt{2}|t|v}. \tag{40}$$

Next, from the fundamental theorem of calculus, we have for any $\zeta > 1$ that

$$\begin{aligned}&\big[U''(H^{\zeta}_{0,1}(x)+H_{-1,0}(x))-U''(H^{\zeta}_{0,1}(x))\big]\partial_x H^{\zeta}_{0,1}(x)\\&\quad=U^{(3)}(H^{\zeta}_{0,1}(x))H_{-1,0}(x)\partial_x H^{\zeta}_{0,1}(x)+\int_0^1 U^{(4)}(H^{\zeta}_{0,1}+\theta H_{-1,0})(1-\theta)H_{-1,0}(x)^2\partial_x H^{\zeta}_{0,1}(x)\,d\theta,\end{aligned}$$

from which with Lemma 11, estimates (3), (4) and

$$\left|\frac{d^l}{dx^l}[H_{-1,0}(x)+e^{-\sqrt{2}x}]\right|\lesssim_l \min(e^{-\sqrt{2}x},e^{-3\sqrt{2}x}),$$

we obtain that

$$\begin{aligned}&\big\langle[U''(H^{\zeta}_{0,1}(x)+H_{-1,0}(x))-U''(H^{\zeta}_{0,1}(x))]\partial_x H^{\zeta}_{0,1}(x),\,\partial_x H^{\zeta}_{0,1}(x)\big\rangle\\&\qquad=-e^{-\sqrt{2}\zeta}\int_{\mathbb{R}}U^{(3)}(H_{0,1}(x))H'_{0,1}(x)^2e^{-\sqrt{2}x}\,dx+\mathrm{res}_3(\zeta),\end{aligned}\tag{41}$$

with $\mathrm{res}_3\in C^{\infty}(\mathbb{R}_{\geq 1})$ satisfying, for all $l\in\mathbb{N}\cup\{0\}$ and $\zeta\geq 1$,

$$|\mathrm{res}_3^{(l)}(\zeta)|\lesssim_l \zeta e^{-2\sqrt{2}\zeta}.$$

Next, using $U\in C^{\infty}(\mathbb{R})$ and estimates (3), (4), we deduce for all $\zeta\geq 1$ and any $l\in\mathbb{N}\cup\{0\}$ that

$$\left|\frac{\partial^l}{\partial\zeta^l}\big[U^{(3)}(H^{\zeta}_{0,1}(x)+H_{-1,0}(x))-U^{(3)}(H^{\zeta}_{0,1}(x))\big]\right|\lesssim_l |H_{-1,0}(x)|.$$

Therefore, since $\mathcal{G}$ defined in (17) is a Schwarz function, Lemma 11 implies that

$$\mathrm{int}(\zeta)=\big\langle\big[U^{(3)}(H^{\zeta}_{0,1}(x)+H_{-1,0}(x))-U^{(3)}(H^{\zeta}_{0,1}(x))\big]\mathcal{G}(x-\zeta)\partial_x H^{\zeta}_{0,1}(x),\,\partial_x H^{\zeta}_{0,1}(x)\big\rangle$$

satisfies for all $\zeta\geq 1$ and any $l\in\mathbb{N}\cup\{0\}$ the inequality $|\mathrm{int}^{(l)}(\zeta)|\lesssim_l e^{-\sqrt{2}\zeta}$. Moreover, using the identity

$$U^{(3)}(\phi)=-48\phi+120\phi^3,\tag{42}$$

we can deduce similarly that

$$\mathrm{int}_2(\zeta)=\big\langle U^{(3)}(H^{\zeta}_{0,1}(x)+H_{-1,0}(x))\mathcal{G}(-x)H'_{-1,0}(x),\,\partial_x H^{\zeta}_{0,1}(x)\big\rangle$$

satisfies $|\mathrm{int}_2^{(l)}(\zeta)|\lesssim_l e^{-\sqrt{2}\zeta}$ for any $l\in\mathbb{N}\cup\{0\}$ and $\zeta\geq 1$. As a consequence, we deduce that there exists a real function $\mathrm{int}_3:\mathbb{R}_{\geq 1}\to\mathbb{R}$ satisfying, for any $l\in\mathbb{N}\cup\{0\}$,

$$|\mathrm{int}_3^{(l)}(\zeta)|\lesssim_l e^{-\sqrt{2}\zeta},$$

where the function int_3 satisfies the identity

$$\begin{aligned}&\big\langle U^{(3)}(H^{\zeta}_{0,1}(x)+H_{-1,0}(x))\mathcal{G}(x-\zeta)\partial_x H^{\zeta}_{0,1}(x),\,\partial_x H^{\zeta}_{0,1}(x)\big\rangle\\&\qquad-\big\langle U^{(3)}(H^{\zeta}_{0,1}(x)+H_{-1,0}(x))\mathcal{G}(-x)H'_{0,1}(-x),\,\partial_x H^{\zeta}_{0,1}(x)\big\rangle\\&\qquad\qquad=\int_{\mathbb{R}}U^{(3)}(H_{0,1}(x))H'_{0,1}(x)^2\mathcal{G}(x)\,dx+\mathrm{int}_3(\zeta).\end{aligned}\tag{43}$$

From Theorem 8, estimates (24) and the identity

$$e^{-\sqrt{2}d(t)}=\frac{v^2}{8}\operatorname{sech}{(\sqrt{2}|t|v)}^2,$$

it is not difficult to verify for any $l\in\mathbb{N}\cup\{0\}$ that if $0<v\ll 1$, then

$$\frac{d^l}{dt^l}\exp\Bigl(\frac{2\rho_{k,v}(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}\Bigr)\lesssim_l v^{2+l}e^{-2\sqrt{2}|t|v}. \tag{44}$$

In conclusion, from estimates (38), (41), (43) and Lemma 32 of Appendix A, we obtain using identity

$$w_{k,v}(t,x)=\frac{x-\frac{1}{2}d(t)+c_{k,v}}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}},$$

and Theorem 8 that $Y_1(t)$ satisfies Lemma 19.

The proof that $Y_2(t)$ satisfies Lemma 19 is similar. First, from the fundamental theorem of calculus, we have for any real number $\zeta\geq 1$ the identity

$$\begin{aligned}&\bigl[U''(H^{\zeta}_{0,1}(x)+H_{-1,0}(x))-U''(H_{-1,0}(x))\bigr]H'_{-1,0}(x)\\&\quad=[U''(H^{\zeta}_{0,1}(x))-2]H'_{-1,0}(x)+U^{(3)}(H^{\zeta}_{0,1}(x))H_{-1,0}(x)H'_{-1,0}(x)\\&\qquad+\int_0^1\bigl[U^{(4)}(H^{\zeta}_{0,1}(x)+\theta H_{-1,0}(x))-U^{(4)}(\theta H_{-1,0}(x))\bigr]H_{-1,0}(x)^2H'_{-1,0}(x)(1-\theta)\,d\theta.\end{aligned}$$

Therefore, estimates (3), (4), identity (42) and Lemma 11 imply for any $\zeta\geq 1$ the estimate

$$\Bigl|\frac{d^l}{d\zeta^l}\bigl\langle U''(H^{\zeta}_{0,1}(x)+H_{-1,0}(x))-U''(H_{-1,0}(x))-U''(H^{\zeta}_{0,1}(x))+2,\,H'_{-1,0}(x)\partial_xH^{\zeta}_{0,1}(x)\bigr\rangle\Bigr|\lesssim_l\zeta e^{-2\sqrt{2}\zeta}. \tag{45}$$

Similarly, Lemma 11 and identity (42) imply that the functions

$$\begin{aligned}\mathrm{int}_4(\zeta)&=\bigl\langle U^{(3)}(H^{\zeta}_{0,1}(x)+H_{-1,0}(x))\mathcal{G}(x-\zeta)H'_{-1,0}(x),\,\partial_xH^{\zeta}_{0,1}(x)\bigr\rangle,\\\mathrm{int}_5(\zeta)&=\bigl\langle U^{(3)}(H^{\zeta}_{0,1}(x)+H_{-1,0}(x))\mathcal{G}(-x)H'_{-1,0}(x),\,\partial_xH^{\zeta}_{0,1}(x)\bigr\rangle\end{aligned}$$

satisfy the estimates

$$|\mathrm{int}^{(l)}_4(\zeta)|+|\mathrm{int}^{(l)}_5(\zeta)|\lesssim_l e^{-\sqrt{2}\zeta} \tag{46}$$

for all $\zeta\geq 1$ and any $l\in\mathbb{N}\cup\{0\}$. Therefore, from estimates (44), (39), (45), (46), Lemma 11 and Theorem 8 imply that

$$\begin{aligned}&\bigl\langle Y_2(t,x),\,H'_{0,1}(w_{k,v}(t,x))\bigr\rangle\\&\quad=y_2(t)\int_{\mathbb{R}}[U''(H_{0,1}(x))-2]H'_{0,1}(x)H'_{-1,0}\Bigl(x+\frac{d(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}\Bigr)\,dx+y_2(t)\,\mathrm{res}_6(v,t),\end{aligned} \tag{47}$$

where $\mathrm{res}_6(v,t)$ is a real function, which satisfies for some constant $\eta_k\geq 0$, if $0<v\ll 1$,

$$\Bigl|\frac{\partial^l}{\partial t^l}\mathrm{res}_6(v,t)\Bigr|\lesssim_l v^{4+l}\bigl[\ln(1/v^2)+|t|v\bigr]^{\eta_k}e^{-2\sqrt{2}|t|v}$$

for all $l \in \mathbb{N} \cup \{0\}$. So, from identity (144) of Appendix A, estimates (24),

$$\left|\frac{d^l}{dx^l}[H_{-1,0}(x) + e^{-\sqrt{2}x}]\right| \lesssim_l \min(e^{-\sqrt{2}x}, e^{-3\sqrt{2}x}),$$

and Lemma 11, we conclude the proof of Lemma 19 for $Y_2(t)$. $\square$

Remark 20. If $v \ll 1$, using the formula $U''(\phi) = 2 - 24\phi^2 + 30\phi^4$, Lemmas 11, 12, the estimates (37), (38), (39) and (40) of the proof of Lemma 19 imply for any $s \geq 0$ that

$$\max_{j\in\{1,2\}} \|Y_j(t)\|_{H^s_x} \lesssim_s \max_{j\in\{1,2\}} |y_j(t)| v^2 e^{-2\sqrt{2}|t|v},$$
$$\max_{j\in\{1,2\}} \|\partial_t Y_j(t)\|_{H^s_x} \lesssim_s \max_{j\in\{1,2\}} |y_j(t)| v^3 e^{-2\sqrt{2}|t|v} + \max_{j\in\{1,2\}} |\dot{y}_j(t)| v^2 e^{-2\sqrt{2}|t|v},$$
$$\max_{j\in\{1,2\}} \|\partial_t^2 Y_j(t)\|_{H^s_x} \lesssim_s \max_{j\in\{1,2\}} |y_j(t)| v^4 e^{-2\sqrt{2}|t|v} + \max_{j\in\{1,2\}} |\dot{y}_j(t)| v^3 e^{-2\sqrt{2}|t|v} + \max_{j\in\{1,2\}} |y_j^{(2)}(t)| v^2 e^{-2\sqrt{2}|t|v}.$$

These estimates above don't depend on k, because from Theorem 8 we can verify for any $l \in \mathbb{N} \cup \{0\}$ the existence of $0 < \delta_{k,l} \ll 1$ such that if $0 < v < \delta_{k,l}$, then

$$\left\|\frac{\partial^l}{\partial t^l} c_k(v,t)\right\|_{L^\infty_t(\mathbb{R})} \lesssim_l v^{2+l} \ln(1/v),$$

which implies, for any $l \in \mathbb{N}$ and any $v \ll 1$,

$$\left\|\frac{\partial^l}{\partial t^l}\left[-\tfrac{1}{2}d(t) + c_k(v,t)\right]\right\|_{L^\infty_t(\mathbb{R})} \lesssim_l v^l, \quad \tfrac{1}{2}d(t) - v < \left|-\tfrac{1}{2}d(t) + c_k(v,t)\right|.$$

3. Energy estimate method

In this section, we will repeat the main argument of Section 4 of [Moutinho 2023] to construct a function $L : \mathbb{R} \to \mathbb{R}$, which is going to be used to estimate the energy norm of $(u(t), \partial_t u(t))$ during a large time interval.

First, we consider a smooth cut-off function $\chi : \mathbb{R} \to \mathbb{R}$ satisfying $0 \leq \chi \leq 1$ and

$$\chi(x) = \begin{cases} 1 & \text{if } x \leq \frac{49}{100}, \\ 0 & \text{if } x \geq \frac{1}{2}. \end{cases} \tag{48}$$

Next, using the notation of Theorem 8, we let

$$x_1(t) = -\tfrac{1}{2}d(t) + \sum_{j=2}^{k} r_j(v,t), \quad x_2(t) = \tfrac{1}{2}d(t) - \sum_{j=2}^{k} r_j(v,t). \tag{49}$$

Actually, Theorem 8 and estimates (24) imply that

$$\max_{j\in\{1,2\}} |\dot{x}_j(t)| \lesssim v, \quad \ln(1/v) \lesssim x_2(t) - x_1(t), \quad \max_{j\in\{1,2\}} |\ddot{x}_j(t)| \lesssim v^2 e^{-2\sqrt{2}|t|v}. \tag{50}$$

From now on, we define the function $\chi_1 : \mathbb{R}^2 \to \mathbb{R}$ by

$$\chi_1(t,x) = \chi\left(\frac{x - x_1(t)}{x_2(t) - x_1(t)}\right). \tag{51}$$

Clearly, using the identities

$$\frac{\partial}{\partial t}\chi_1(t,x) = \frac{-\dot{x}_1(t)}{x_2(t)-x_1(t)}\dot{\chi}\left(\frac{x-x_1(t)}{x_2(t)-x_1(t)}\right) - \frac{(\dot{x}_2(t)-\dot{x}_1(t))(x-x_1(t))}{(x_2(t)-x_1(t))^2}\dot{\chi}\left(\frac{x-x_1(t)}{x_2(t)-x_1(t)}\right),$$
$$\frac{\partial}{\partial x}\chi_1(t,x) = \frac{1}{x_2(t)-x_1(t)}\dot{\chi}\left(\frac{x-x_1(t)}{x_2(t)-x_1(t)}\right),$$

we obtain the estimates

$$\left\|\frac{\partial}{\partial t}\chi_1(t,x)\right\|_{L^\infty_x(\mathbb{R})} \lesssim \frac{v}{\ln(1/v)}, \quad \left\|\frac{\partial}{\partial x}\chi_1(t,x)\right\|_{L^\infty_x(\mathbb{R})} \lesssim \frac{1}{\ln(1/v)}. \tag{52}$$

Finally, using the notation (28) and the functions $Y_1(t)$, $Y_2(t)$ denoted respectively by (34) and (35), we define the function $A : \mathbb{R}^2 \to \mathbb{R}$ by

$$\begin{aligned} A(t,x) = {} & -\Lambda(\varphi_{k,v})(t,x)\frac{8\sqrt{2}e^{-\sqrt{2}d(t)}}{1-\frac{1}{4}\dot{d}(t)^2}\left[y_1(t)H''_{0,1}(w_{k,v}(t,x)) + y_2(t)H''_{0,1}(w_{k,v}(t,-x))\right] \\ & - Y_1(t,x) - Y_2(t,x) + \frac{\dot{y}_1(t)\dot{d}(t)}{1-\frac{1}{4}\dot{d}(t)^2}H''_{0,1}(w_{k,v}(t,x)) + \frac{\dot{y}_2(t)\dot{d}(t)}{1-\frac{1}{4}\dot{d}(t)^2}H''_{0,1}(w_{k,v}(t,-x)) \end{aligned} \tag{53}$$

for any $(t,x) \in \mathbb{R}^2$. Clearly, in the notation of Remark 18, we have the identity

$$\begin{aligned} & \partial_t^2 u(t,x) - \partial_x^2 u(t,x) + U''(H_{0,1}(w_{k,v}(t,x)) - H_{0,1}(w_{k,v}(t,-x)))u(t,x) \\ & \quad = -\frac{\ddot{y}_1(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}H'_{0,1}(w_{k,v}(t,x)) - \frac{\ddot{y}_2(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}H'_{0,1}(w_{k,v}(t,-x)) + A(t,x) + Q(t,x) \\ & \qquad + \left[U''(H_{0,1}(w_{k,v}(t,x)) - H_{0,1}(w_{k,v}(t,-x))) - U''(\varphi_{k,v}(t,x))\right]u(t,x). \end{aligned} \tag{54}$$

Next, we consider

$$\begin{aligned} L(t) = {} & \int_{\mathbb{R}} \partial_t u(t,x)^2 + \partial_x u(t,x)^2 + U''(H_{0,1}(w_{k,v}(t,x)) - H_{0,1}(w_{k,v}(t,-x)))u(t,x)^2\,dx \\ & + 2\int_{\mathbb{R}} \partial_t u(t,x)\partial_x u(t,x)[\dot{x}_1(t)\chi_1(t,x) + \dot{x}_2(t)(1-\chi_1(t,x))]\,dx - 2\int_{\mathbb{R}} u(t,x)A(t,x)\,dx. \end{aligned} \tag{55}$$

From now on, we use the notation $\vec{u}(t) = (u(t), \partial_t u(t)) \in H^1_x(\mathbb{R}) \times L^2_x(\mathbb{R})$. The main objective of Section 3 is to demonstrate the following theorem.

Theorem 21. *There exist constants* $K, c > 0$ *and, for any* $k \in \mathbb{N}_{\geq 3}$, *there exists* $0 < \delta(k) < 1$ *such that if* $0 < v \leq \delta(k)$, *then the function* $L(t)$ *given in* (55) *satisfies, while the condition*

$$\max_{j\in\{1,2\}} v^2|y_j(t)| + v|\dot{y}_j(t)| < v^{2k}\ln(1/v)^{n_k} \tag{56}$$

is true, the estimates

$$\begin{aligned} & c\|\vec{u}(t)\|^2_{H^1_x\times L^2_x} \leq L(t) + C(k)v^{4k}\ln(1/v)^{2n_k}, \\ & \quad |\dot{L}(t)| \leq K\left[\frac{v}{\ln(1/v)}\|\vec{u}(t)\|^2_{H^1_x\times L^2_x} + C(k)\|\vec{u}(t)\|_{H^1_x\times L^2_x}v^{2k+1}\ln(1/v)^{n_k}\right] \\ & \qquad\qquad + v\max_{j\in\{1,2\}}|\ddot{y}_j(t)|\|\vec{u}(t)\|_{H^1_x\times L^2_x} + K\max_{j\in\{3.7\}}\|\vec{u}(t)\|^j_{H^1_x\times L^2_x}, \end{aligned}$$

where $C(k) > 0$ is a constant depending only on k and n_k is the number defined in the statement of Theorem 8.

Proof of Theorem 21. To simplify the proof of this theorem, we describe briefly the organization of our arguments. First, we let

$$L(t) = L_1(t) + L_2(t) + L_3(t)$$

be such that

$$L_1(t) = \int_{\mathbb{R}} \partial_t u(t,x)^2 + \partial_x u(t,x)^2 + \ddot{U}(H_{0,1}(w_{k,v}(t,x) - H_{0,1}(w_{k,v}(t,-x))))u(t,x)^2\,dx, \tag{L1}$$

$$L_2(t) = 2\int_{\mathbb{R}} \partial_t u(t,x)\partial_x u(t,x)[\dot{x}_1(t)\chi_1(t,x) + \dot{x}_2(t)(1-\chi_1(t,x))]\,dx, \tag{L2}$$

$$L_3(t) = -2\int_{\mathbb{R}} u(t,x)A(t,x)\,dx. \tag{L3}$$

Next, instead of estimating the size of $|\dot{L}(t)|$, we will estimate $\dot{L}_j(t)$ for each $j \in \{1,2,3\}$. Then, using these estimates, we can evaluate with high precision

$$|\dot{L}_1(t) + \dot{L}_2(t) + \dot{L}_3(t)|,$$

and obtain the second inequality of Theorem 21. The proof of the first inequality of Theorem 21 is short and it will be done later.

From identity (23), Remark 20 and (53) satisfied by $A(t,x)$, we deduce from the triangle inequality that

$$\|A(t,x)\|_{H^1_x(\mathbb{R})} \lesssim \|\Lambda(\varphi_{k,v})(t,x)\|_{H^1_x(\mathbb{R})} + v^2 e^{-2\sqrt{2}v|t|}\max_{j\in\{1,2\}}|y_j(t)| + v\max_{j\in\{1,2\}}|\dot{y}_j(t)|.$$

Therefore, from Theorems 7 and 8, we obtain the existence of a value $C(k) > 0$ depending only on k such that if $v \ll 1$, then

$$\|A(t,x)\|_{H^1(\mathbb{R})} \lesssim C(k)v^{2k}\big(\ln(1/v) + |t|v\big)^{n_k} e^{-2\sqrt{2}|t|v} + v^2 e^{-2\sqrt{2}|t|v}\max_{j\in\{1,2\}}|y_j(t)| + v\max_{j\in\{1,2\}}|\dot{y}_j(t)|. \tag{57}$$

In conclusion, we obtain from (L3) and the Cauchy–Schwarz inequality the existence of a value $C(k) > 0$ depending only on k satisfying

$$|L_3(t)| \lesssim \|u(t)\|_{L^2_x}\big[C(k)v^{2k}\big(\ln(1/v)+|t|v\big)^{n_k} e^{-2\sqrt{2}|t|v} + v^2 e^{-2\sqrt{2}|t|v}\max_{j\in\{1,2\}}|y_j(t)| + \max_{j\in\{1,2\}}|\dot{y}_j(t)|v\big]. \tag{58}$$

Next, Lemmas 12, 13, Remark 20 and identity (53) satisfied by $A(t,x)$ imply the inequality

$$\|\partial_t A(t,x)\|_{H^1_x(\mathbb{R})} \lesssim \left\|\frac{\partial}{\partial t}[\Lambda(\phi_k)(v,t,x)]\right\|_{H^1_x(\mathbb{R})} + \max_{j\in\{1,2\}}|y_j(t)|v^3 e^{-2\sqrt{2}|t|v} + \max_{j\in\{1,2\}}|\dot{y}_j(t)|v^2 + \max_{j\in\{1,2\}}|\ddot{y}_j(t)|v,$$

from which with Theorem 8 we conclude the existence of a new value $C(k)$ depending only on k satisfying

$$\begin{aligned}&\|\partial_t A(t,x)\|_{H^1_x}\\ &\lesssim C(k)v^{2k+1}\big(\ln(1/v)+|t|v\big)^{n_k} e^{-2\sqrt{2}|t|v} + \max_{j\in\{1,2\}}|y_j(t)|v^3 e^{-2\sqrt{2}|t|v} + \max_{j\in\{1,2\}}|\dot{y}_j(t)|v^2 + \max_{j\in\{1,2\}}|\ddot{y}_j(t)|v.\end{aligned} \tag{59}$$

In conclusion, the identity (L3), estimate (59) and Cauchy–Schwarz inequality imply the existence of a new value $C(k) > 0$ depending only on k, which satisfies

$$\begin{aligned}&\left|\dot{L}_3(t) + 2\int_{\mathbb{R}} \partial_t u(t,x) A(t,x)\,dx\right| \\ &\qquad\lesssim \|u(t,x)\|_{L^2_x}\big[C(k)v^{2k+1}\big(\ln(1/v)+|t|v\big)^{n_k} e^{-2\sqrt{2}|t|v} + \max_{j\in\{1,2\}} |y_j(t)|v^3 e^{-2\sqrt{2}|t|v}\big] \\ &\qquad\qquad + \|u(t,x)\|_{L^2_x}\big[\max_{j\in\{1,2\}} |\dot{y}_j(t)|v^2 + \max_{j\in\{1,2\}} |\ddot{y}_j(t)|v\big]. \qquad (60)\end{aligned}$$

Next, Theorem 8 implies that if $v \ll 1$, then

$$\begin{aligned}&\dot{L}_1(t) \\ &= 2\int_{\mathbb{R}} \partial_t u(t,x)\big[\partial_t^2 u(t,x) - \partial_x^2 u(t,x) + U''\big(H_{0,1}(w_{k,v}(t,x)) - H_{0,1}(w_{k,v}(t,-x))\big)u(t,x)\big]\,dx \\ &\quad - \frac{\dot{d}(t)}{2\big(1-\frac{1}{4}\dot{d}(t)^2\big)^{1/2}} \int_{\mathbb{R}} U^{(3)}\big(H_{0,1}(w_{k,v}(t,x)) - H_{0,1}(w_{k,v}(t,-x))\big) H'_{0,1}(w_{k,v}(t,x)) u(t,x)^2\,dx \\ &\quad + \frac{\dot{d}(t)}{2\big(1-\frac{1}{4}\dot{d}(t)^2\big)^{1/2}} \int_{\mathbb{R}} U^{(3)}\big(H_{0,1}(w_{k,v}(t,x)) - H_{0,1}(w_{k,v}(t,-x))\big) H'_{0,1}(w_{k,v}(t,-x)) u(t,x)^2\,dx \\ &\quad + O\left(\frac{v}{\ln(1/v)} \|(u(t), \partial_t u(t))\|^2_{H^1_x, L^2_x}\right) \qquad (61)\end{aligned}$$

Thus, from Lemma 17, identity (53), Remark 18, hypothesis (56), estimates (60), (61) and orthogonality conditions (29), we obtain the existence of a value $C(k) > 0$ depending only on k such that if $v \ll 1$, then

$$\begin{aligned}&\dot{L}_1(t) + \dot{L}_3(t) \\ &= 2\int_{\mathbb{R}} \partial_t u(t,x)\big[U''\big(H_{0,1}(w_{k,v}(t,x)) - H_{0,1}(w_{k,v}(t,-x))\big) - U''(\varphi_{k,v}(t,x))\big]u(t,x)\,dx \\ &\quad + \frac{\dot{d}(t)}{2\sqrt{1-\frac{1}{4}\dot{d}(t)^2}} \int_{\mathbb{R}} U^{(3)}\big(H_{0,1}(w_{k,v}(t,x)) - H_{0,1}(w_{k,v}(t,-x))\big) H'_{0,1}(w_{k,v}(t,-x)) u(t,x)^2\,dx \\ &\quad - \frac{\dot{d}(t)}{2\sqrt{1-\frac{1}{4}\dot{d}(t)^2}} \int_{\mathbb{R}} U^{(3)}\big(H_{0,1}(w_{k,v}(t,x)) - H_{0,1}(w_{k,v}(t,-x))\big) H'_{0,1}(w_{k,v}(t,x)) u(t,x)^2\,dx \\ &\quad + O\big(v \max_{j\in\{1,2\}} |\ddot{y}_j(t)|\|u(t)\|_{H^1_x(\mathbb{R})} + \max_{j\in\{3,7\}} \|\vec{u}(t)\|^j_{H^1_x\times L^2_x} + \|\vec{u}(t)\|_{H^1_x\times L^2_x} \max_{j\in\{1,2\}} |y_j(t)|^2\big) \\ &\quad + O\left(\|\vec{u}(t)\|_{H^1_x\times L^2_x}\big[\max_{j\in\{1,2\}} |\dot{y}_j(t)|v^2 + |y_j(t)|v^3 e^{-2\sqrt{2}|t|v}\big] + \|\vec{u}(t)\|^2_{H^1_x\times L^2_x} \frac{v}{\ln(1/v)}\right) \\ &\quad + O\big(C(k)\|\vec{u}(t)\|_{H^1_x\times L^2_x} v^{2k+1} \ln(1/v)^{n_k}\big). \qquad (62)\end{aligned}$$

Moreover, using estimates (24), Lemma 13 and identity $U(\phi) = \phi^2(1-\phi^2)^2$, we obtain from Theorem 8 that if $0 < v \ll 1$ and $s \geq 0$, then

$$\big\|\big[U''\big(H_{0,1}(w_{k,v}(t,x)) - H_{0,1}(w_{k,v}(t,-x))\big) - U''(\varphi_{k,v}(t,x))\big]\big\|_{H^s_x} \lesssim_{s,k} v^2 e^{-2\sqrt{2}|t|v}.$$

Therefore, we deduce using the Cauchy–Schwarz inequality that

$$\begin{aligned}\left|2\int_{\mathbb{R}}\partial_t u(t,x)\big[U''\big(H_{0,1}(w_{k,v}(t,x))-H_{0,1}(w_{k,v}(t,-x))\big)-U''(\varphi_{k,v}(t,x))\big]u(t,x)\,dx\right|&\\ \lesssim \big\|\big[U''(H_{0,1}(w_{k,v}(t,x))-H_{0,1}(w_{k,v}(t,x)))-U''(\varphi_{k,v}(t,x))\big]u(t,x)\big\|_{L^2_x}\|\partial_t u(t,x)\|_{L^2_x}&\\ \lesssim \big\|\big[U''(H_{0,1}(w_{k,v}(t,x))-H_{0,1}(w_{k,v}(t,-x)))-U''(\varphi_{k,v}(t,x))\big]\big\|_{H^1_x(\mathbb{R})}\|\vec{u}(t)\|^2_{H^1_x\times L^2_x}&\\ \lesssim v^2\|\vec{u}(t)\|^2_{H^1_x\times L^2_x}.&\end{aligned}$$

In conclusion,

$$\begin{aligned}&\dot{L}_1(t)+\dot{L}_3(t)\\ &\quad=\frac{\dot{d}(t)}{2\big(1-\frac{1}{4}\dot{d}(t)^2\big)^{1/2}}\int_{\mathbb{R}}U^{(3)}\big(H_{0,1}(w_{k,v}(t,x))-H_{0,1}(w_{k,v}(t,-x))\big)H'_{0,1}(w_{k,v}(t,-x))u(t,x)^2\,dx\\ &\qquad-\frac{\dot{d}(t)}{2\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}\int_{\mathbb{R}}U^{(3)}\big(H_{0,1}(w_{k,v}(t,x))-H_{0,1}(w_{k,v}(t,-x))\big)H'_{0,1}(w_{k,v}(t,x))u(t,x)^2\,dx\\ &\qquad\quad+O\big(v\max_{j\in\{1,2\}}|\ddot{y}_j(t)|\|u(t)\|_{H^1_x(\mathbb{R})}+\max_{j\in\{3,7\}}\|\vec{u}(t)\|^j_{H^1_x\times L^2_x}+\|\vec{u}(t)\|_{H^1_x\times L^2_x}\max_{j\in\{1,2\}}|y_j(t)|^2\big)\\ &\qquad\quad\;+O\bigg(\|\vec{u}(t)\|_{H^1_x\times L^2_x}\Big[\max_{j\in\{1,2\}}|\dot{y}_j(t)|v^2+|y_j(t)|v^3e^{-2\sqrt{2}|t|v}\Big]+\|\vec{u}(t)\|^2_{H^1_x\times L^2_x}\frac{v}{\ln(1/v)}\bigg)\\ &\qquad\qquad+O\big(C(k)\|\vec{u}(t)\|_{H^1_x\times L^2_x}v^{2k+1}\ln(1/v)^{n_k}\big).\end{aligned}\tag{63}$$

Based on the arguments of [Jendrej et al. 2022; Moutinho 2023], we will estimate the derivative of $L_2(t)$, for more accurate information see the third step of Lemma 4.2 in [Jendrej et al. 2022] or Theorem 4.1 of [Moutinho 2023]. Because of an argument of analogy, we only need to estimate the time derivative of

$$L_{2,1}(t)=2\dot{x}_1(t)\int_{\mathbb{R}}\chi_1(t,x)\partial_t u(t,x)\partial_x u(t,x)\,dx$$

to evaluate with high precision the derivative of $L_2(t)$. From the estimates (52), we can verify first that if $v\ll 1$, then

$$\begin{aligned}\dot{L}_{2,1}(t)=2\dot{x}_1(t)\int_{\mathbb{R}}\chi_1(t,x)\partial_t^2u(t,x)\partial_x u(t,x)\,dx+2\dot{x}_1(t)\int_{\mathbb{R}}\chi_1(t,x)\partial_t u(t,x)\partial^2_{x,t}u(t,x)\,dx&\\ +O\bigg(\frac{v}{\ln(1/v)}\|\vec{u}(t)\|^2_{H^1_x\times L^2_x}\bigg),&\end{aligned}$$

from which we deduce, using integration by parts and estimates (50), (52), that

$$\begin{aligned}\dot{L}_{2,1}(t)&=2\dot{x}_1(t)\int_{\mathbb{R}}\chi_1(t,x)\partial_t^2u(t,x)\partial_x u(t,x)\,dx+O\bigg(\frac{v}{\ln(1/v)}\|\vec{u}(t)\|^2_{H^1_x\times L^2_x}\bigg)\\ &=2\dot{x}_1(t)\int_{\mathbb{R}}\chi_1(t,x)[\partial_t^2u(t,x)-\partial_x^2u(t,x)]\partial_x u(t,x)\,dx\\ &\quad+2\dot{x}_1(t)\int_{\mathbb{R}}\chi_1(t,x)U''\big(H_{0,1}(w_{k,v}(t,x))-H_{0,1}(w_{k,v}(t,-x))\big)u(t,x)\partial_x u(t,x)\,dx\end{aligned}$$

$$
\begin{aligned}
&+2\dot{x}_1(t)\int_{\mathbb{R}}\chi_1(t,x)\partial_x^2u(t,x)\partial_xu(t,x)\,dx\\
&-2\dot{x}_1(t)\int_{\mathbb{R}}\chi_1(t,x)U''\big(H_{0,1}(w_{k,v}(t,x))-H_{0,1}(w_{k,v}(t,-x))\big)u(t,x)\partial_xu(t,x)\,dx\\
&+O\bigg(\frac{v}{\ln(1/v)}\|\vec{u}(t)\|^2_{H^1_x\times L^2_x}\bigg),
\end{aligned}
$$

and, after using integration by parts again, we deduce from (52) that

$$
\begin{aligned}
\dot{L}_{2,1}(t)={}&2\dot{x}_1(t)\int_{\mathbb{R}}\chi_1(t,x)[\partial_t^2u(t)-\partial_x^2u(t)]\partial_xu(t)\,dx\\
&+2\dot{x}_1(t)\int_{\mathbb{R}}\chi_1(t)U''(H_{0,1}(w_{k,v}(t,x))-H_{0,1}(w_{k,v}(t,-x)))u(t)\partial_xu(t)\,dx\\
&+\frac{\dot{x}_1(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}\int_{\mathbb{R}}\chi_1(t)U^{(3)}(H_{0,1}(w_{k,v}(t,x))-H_{0,1}(w_{k,v}(t,-x)))H'_{0,1}(w_{k,v}(t,x))u(t)^2\,dx\\
&+\frac{\dot{x}_1(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}\int_{\mathbb{R}}\chi_1(t)U^{(3)}(H_{0,1}(w_{k,v}(t,x))-H_{0,1}(w_{k,v}(t,-x)))H'_{0,1}(w_{k,v}(t,-x))u(t)^2\,dx\\
&+O\bigg(\frac{v}{\ln(1/v)}\|\vec{u}(t)\|^2_{H^1_x(\mathbb{R})\times L^2_x(\mathbb{R})}\bigg).
\end{aligned}
$$

Next, using estimates (3) satisfied by $H_{0,1}$, definition of $\chi_1(t,x)$, Theorem 8 and identity (27), we deduce, for $v\ll 1$, the inequality

$$
|\chi_1(t,x)H'_{0,1}(w_{k,v}(t,x))|+|(1-\chi_1(t,x))H'_{0,1}(w_{k,v}(t,-x))|\lesssim e^{-\sqrt{2}\frac{49d(t)}{100}}\lesssim v^{\frac{98}{100}}\ll\frac{1}{\ln(1/v)},
$$

from which we conclude that

$$
\begin{aligned}
\dot{L}_{2,1}(t)={}&2\dot{x}_1(t)\int_{\mathbb{R}}\chi_1(t)[\partial_t^2u(t,x)-\partial_x^2u(t,x)]\partial_xu(t,x)\,dx\\
&+2\dot{x}_1(t)\int_{\mathbb{R}}\chi_1(t)U''\big(H_{0,1}(w_{k,v}(t,x))-H_{0,1}(w_{k,v}(t,-x))\big)u(t,x)\partial_xu(t,x)\,dx\\
&+\frac{\dot{x}_1(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}\int_{\mathbb{R}}U^{(3)}\big(H_{0,1}(w_{k,v}(t,x))-H_{0,1}(w_{k,v}(t,-x))\big)H'_{0,1}(w_{k,v}(t,-x))u(t,x)^2\,dx\\
&+O\bigg(\frac{v}{\ln(1/v)}\|\vec{u}(t)\|^2_{H^1_x\times L^2_x}\bigg).
\end{aligned}
$$

Furthermore, from Remark 18, estimate (57) of $A(t,x)$ and identity (54) satisfied by $u(t,x)$, we conclude the existence of a value $C(k)>0$ depending only on k and satisfying, for any positive number $v\ll 1$,

$$
\begin{aligned}
\dot{L}_{2,1}(t)={}&\frac{\dot{x}_1(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}\int_{\mathbb{R}}U^{(3)}\big(H_{0,1}(w_{k,v}(t,x))-H_{0,1}(w_{k,v}(t,-x))\big)H'_{0,1}(w_{k,v}(t,-x))u(t,x)^2\,dx\\
&+O\big(\|\vec{u}(t)\|_{H^1_x\times L^2_x}\big[v\max_{j\in\{1,2\}}|\ddot{y}_j(t)|+C(k)v^{2k+1}\ln(1/v)^{n_k}+v\max_{j\in\{2,6\}}\|\vec{u}(t)\|^j_{H^1_x\times L^2_x}\big]\big)\\
&\quad+O\bigg(\|\vec{u}(t)\|_{H^1_x\times L^2_x}[v^3e^{-2\sqrt{2}v|t|}\max_{j\in\{1,2\}}|y_j(t)|+v^2|\dot{y}_j(t)|]+\frac{v}{\ln(1/v)}\|\vec{u}(t)\|^2_{H^1_x\times L^2_x}\bigg).
\end{aligned}
$$

Therefore, using an argument of analogy, we obtain, for any positive number $v \ll 1$, that

$$\begin{aligned}\dot{L}_2(t) ={}& \frac{\dot{x}_2(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}} \int_{\mathbb{R}} U^{(3)}\big(H_{0,1}(w_{k,v}(t,x)) - H_{0,1}(w_{k,v}(t,-x))\big) H_{0,1}'(w_{k,v}(t,x)) u(t,x)^2 \, dx \\ &+ \frac{\dot{x}_1(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}} \int_{\mathbb{R}} U^{(3)}\big(H_{0,1}(w_{k,v}(t,x)) - H_{0,1}(w_{k,v}(t,-x))\big) H_{0,1}'(w_{k,v}(t,-x)) u(t,x)^2 \, dx \\ &+ O\big(\|\vec{u}(t)\|_{H^1_x \times L^2_x}\big[v \max_{j\in\{1,2\}} |\ddot{y}_j(t)| + C(k) v^{2k+1} \ln(1/v)^{n_k}\big] + v \max_{j \in \{3,7\}} \|\vec{u}(t)\|^j_{H^1_x \times L^2_x}\big) \\ &+ O\bigg(\|\vec{u}(t)\|_{H^1_x\times L^2_x}[v^3 e^{-2\sqrt{2}v|t|} \max_{j\in\{1,2\}} |y_j(t)| + v^2 |\dot{y}_j(t)|] + \frac{v}{\ln(1/v)} \|\vec{u}(t)\|^2_{H^1_x \times L^2_x}\bigg), \qquad (64)\end{aligned}$$

where $C(k) > 0$ is a parameter depending only on k. Moreover, using (49) and Theorem 8, we deduce from estimate (64) that

$$\begin{aligned}\dot{L}_2(t) ={}& \frac{\dot{d}(t)}{\sqrt{4-\dot{d}(t)^2}} \int_{\mathbb{R}} U^{(3)}\big(H_{0,1}(w_{k,v}(t,x)) - H_{0,1}(w_{k,v}(t,-x))\big) H_{0,1}'(w_{k,v}(t,x)) u(t,x)^2 \, dx \\ &- \frac{\dot{d}(t)}{\sqrt{4-\dot{d}(t)^2}} \int_{\mathbb{R}} U^{(3)}\big(H_{0,1}(w_{k,v}(t,x)) - H_{0,1}(w_{k,v}(t,-x))\big) H_{0,1}'(w_{k,v}(t,-x)) u(t,x)^2 \, dx \\ &+ O\big(\|\vec{u}(t)\|_{H^1_x \times L^2_x}\big[v \max_{j\in\{1,2\}} |\ddot{y}_j(t)| + C(k) v^{2k+1} \ln(1/v)^{n_k}\big] + v \max_{j \in \{3,7\}} \|\vec{u}(t)\|^j_{H^1_x \times L^2_x}\big) \\ &+ O\bigg(\|\vec{u}(t)\|_{H^1_x\times L^2_x}[v^3 e^{-2\sqrt{2}v|t|} \max_{j\in\{1,2\}} |y_j(t)| + v^2 |\dot{y}_j(t)|] + \frac{v}{\ln(1/v)} \|\vec{u}(t)\|^2_{H^1_x \times L^2_x}\bigg). \qquad (65)\end{aligned}$$

Finally, the estimates (65) and (62) imply, for any $k \in \mathbb{N}_{\geq 3}$, the existence of a parameter $C(k) > 0$, depending only on k, which satisfies for any positive number $v \ll 1$ the estimate

$$\begin{aligned}|\dot{L}(t)| ={}& O\big(v \max_{j\in\{1,2\}} |\ddot{y}_j(t)| \|\vec{u}(t)\|_{H^1_x\times L^2_x} + \max_{j\in\{3,7\}} \|\vec{u}(t)\|^j_{H^1_x \times L^2_x}\big) + O\big(\|\vec{u}(t)\|_{H^1_x\times L^2_x} \max_{j\in\{1,2\}} |y_j(t)|^2\big) \\ &+ O\big(\|\vec{u}(t)\|_{H^1_x \times L^2_x}\big[\max_{j\in\{1,2\}} |\dot{y}_j(t)| v^2 + |y_j(t)| v^3 e^{-2\sqrt{2}|t|v}\big]\big) \\ &+ O\bigg(\|\vec{u}(t)\|^2_{H^1_x\times L^2_x} \frac{v}{\ln(1/v^2)} + C(k) \|\vec{u}(t)\|_{H^1_x \times L^2_x} v^{2k+1} \ln(1/v)^{n_k}\bigg), \qquad (66)\end{aligned}$$

from which we obtain the existence of a new constant $C(k) > 0$ satisfying the second inequality of Theorem 21 if the condition (56) is true and $v \ll 1$.

Now, it remains to prove the first inequality of Theorem 21. Using change of variables and Lemma 14, it is not difficult to verify that there exists $K > 0$ such that if $v \ll 1$, then

$$L_1(t) \geq K \|(u(t), \partial_t u(t))\|^2_{H^1_x \times L^2_x}.$$

Next, from the definition of $L_2(t)$ and estimates (50), we obtain that if $v \ll 1$, then

$$|L_2(t)| \ll v^{3/4} \|(u(t), \partial_t u(t))\|^2_{H^1_x \times L^2_x},$$

and while condition (56) is true, we deduce from Theorem 8 and estimate (57) the following inequality:

$$|L_3(t)| \lesssim_k \|(u(t), \partial_t u(t))\|_{H^1_x \times L^2_x} v^{2k} \ln(1/v)^{n_k}.$$

So, using Young's inequality, we can find a parameter $C_1(k) > 0$ large enough depending only on k such that

$$|L_3(t)| \leq \tfrac{1}{2} K \, \|(u(t), \partial_t u(t))\|^2_{H^1_x \times L^2_x} + C_1(k) v^{4k} \ln(1/v)^{2n_k}.$$

In conclusion, all the estimates above imply the first inequality of Theorem 21 if $0 < v \ll 1$ and condition (56) is true. □

4. Proof of Theorem 15

From the information of Theorem 21 in the last section, we are ready to start the demonstration of Theorem 15.

Proof of Theorem 15. First, for any $(t, x) \in \mathbb{R}^2$, Lemma 17 implies that $\phi(t, x)$ has the representation

$$\phi(t, x) = \varphi_{k,v}(t, x) + \frac{y_1(t)}{\sqrt{1 - \frac{1}{4}\dot{d}(t)^2}} H'_{0,1}(w_{k,v}(t, x)) + \frac{y_2(t)}{\sqrt{1 - \frac{1}{4}\dot{d}(t)^2}} H'_{0,1}(w_{k,v}(t, -x)) + u(t, x),$$

such that the function $u(t, x)$ satisfies the orthogonality conditions (29) and y_1, y_2 are functions in $C^2(\mathbb{R})$.

Step 1 (ordinary differential system of $y_1(t)$, $y_2(t)$). From Remarks 9, 18 and the definition of $A(t, x)$ in (53), we have that $u(t, x)$ is a solution of a partial differential equation of the form

$$\begin{aligned} &\partial_t^2 u(t, x) - \partial_x^2 u(t, x) + U''\big(H_{0,1}(w_{k,v}(t, x)) - H_{0,1}(w_{k,v}(t, -x))\big) u(t, x) \\ &\quad = -\frac{\ddot{y}_1(t)}{\sqrt{1 - \frac{1}{4}\dot{d}(t)^2}} H'_{0,1}(w_{k,v}(t, x)) - \frac{\ddot{y}_2(t)}{\sqrt{1 - \frac{1}{4}\dot{d}(t)^2}} H'_{0,1}(w_{k,v}(t, -x)) + A(t, x) + \mathcal{P}_1(v, t, x), \end{aligned} \tag{67}$$

where $\mathcal{P}_1(v, t, x)$ satisfies for any $0 < v \ll 1$ and any $t \in \mathbb{R}$ the inequality

$$\begin{aligned} \|\mathcal{P}_1(v, t, x)\|_{H^1_x} \lesssim{} & \|u(t)\|^2_{H^1_x} + \max_{j \in \{1,2\}} |y_j(t)|^2 + \max_{j \in \{1,2\}} |\dot{y}_j(t)| v^3 \big(\ln(1/v^2) + |t| v\big) e^{-2\sqrt{2}|t|v} \\ & + \|u(t)\|^6_{H^1_x} + \max_{j \in \{1,2\}} |y_j(t)|^6 + \max_{j \in \{1,2\}} |y_j(t)| v^4 \big(\ln(1/v^2) + |t| v\big) e^{-2\sqrt{2}|t|v}. \end{aligned}$$

With the objective of simplifying our computations, we let

$$\begin{aligned} NOL(t) = {} & \|u(t)\|^2_{H^1} + \max_{j \in \{1,2\}} |y_j(t)|^2 + v^{2(k+1)} \big(|t| v + \ln(1/v^2)\big)^{n_k + 1} e^{-2\sqrt{2}|t|v} \\ & + \|u(t)\|^6_{H^1_x} + \max_{j \in \{1,2\}} |y_j(t)|^6 + \max_{j \in \{1,2\}} |\dot{y}_j(t)| v^3 \big(\ln(1/v^2) + |t| v\big) e^{-2\sqrt{2}|t|v} \\ & + \max_{j \in \{1,2\}} |y_j(t)| v^4 \big(\ln(1/v^2) + |t| v\big)^{\max\{1, \eta_k\}} e^{-2\sqrt{2}|t|v}, \end{aligned} \tag{68}$$

where η_k is the number denoted in Lemma 19. Also, from Theorem 8, Lemma 19 and identity (53), we deduce that

$$\begin{bmatrix} \langle A(t, x), H'_{0,1}(w_{k,v}(t, x)) \rangle \\ \langle A(t, x), H'_{0,1}(w_{k,v}(t, -x)) \rangle \end{bmatrix} = e^{-\sqrt{2} d(t)} \begin{bmatrix} -4\sqrt{2} & 4\sqrt{2} \\ 4\sqrt{2} & -4\sqrt{2} \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} + \mathrm{Rest}(t), \tag{69}$$

where, if $v \ll 1$, the real function $\mathrm{Rest}(t)$ satisfies, for any $t \in \mathbb{R}$,

$$e^{2\sqrt{2}|t|v}|\mathrm{Rest}(t)| \lesssim_k v^{2(k+1)}\big(|t|v+\ln(1/v^2)\big)^{n_k+1} + \max_{j\in\{1,2\}} |y_j(t)|v^4\big(|t|v+\ln(1/v^2)\big)^{\max\{1,\eta_k\}} \\ + \max_{j\in\{1,2\}} |\dot{y}_j(t)|v^3\big(|t|v+\ln(1/v^2)\big). \tag{70}$$

From the orthogonality conditions (29), Theorem 8 and Lemma 12, we obtain the estimate

$$\big\langle \partial_t^2 u(t,x),\, H'_{0,1}(w_{k,v}(t,x))\big\rangle = \frac{\dot{d}(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}\big\langle \partial_t u(t,x),\, H''_{0,1}(w_{k,v}(t,x))\big\rangle_{L^2_x} + O(\|\vec{u}(t)\|_{H^1_x\times L^2_x}v^2). \tag{71}$$

Also, using integration by parts, identity

$$-\frac{d^3}{dx^3}H_{0,1}(x) + U''(H_{0,1}(x))H'_{0,1}(x) = 0,$$

Lemma 11 and the Cauchy–Schwarz inequality, we deduce that if $0 < v \ll 1$, then

$$\begin{aligned}
&\big\langle -\partial_x^2 u(t) + U''\big(H_{0,1}(w_{k,v}(t,x)) - H_{0,1}(w_{k,v}(t,-x))\big)u(t),\, H'_{0,1}(w_{k,v}(t,x))\big\rangle \\
&\quad = \big\langle u(t),\, \big[U''\big(H_{0,1}(w_{k,v}(t,x)) - H_{0,1}(w_{k,v}(t,-x))\big) - U''(H_{0,1}(w_{k,v}(t,x)))\big]H'_{0,1}(w_{k,v}(t,x))\big\rangle \\
&\qquad\qquad + O(v^2\|\vec{u}(t)\|_{H^1_x\times L^2_x}) \\
&\quad = O(v^2\|\vec{u}(t)\|_{H^1_x\times L^2_x}).
\end{aligned} \tag{72}$$

From now on, we denote any continuous function $f(t)$ as $O_k(NOL(t))$, if and only if f satisfies the estimate

$$|f(t)| \lesssim_k NOL(t).$$

In conclusion, applying the scalar product of the (67) with $H'_{0,1}(w_{k,v}(t,x))$ and $H'_{0,1}(w_{k,v}(t,-x))$, we obtain using Lemma 11 and estimates (71), (72) that

$$\begin{aligned}
&\begin{bmatrix} \|H'_{0,1}\|^2_{L^2_x} & O(d(t)e^{-\sqrt{2}d(t)}) \\ O(d(t)e^{-\sqrt{2}d(t)}) & \|H'_{0,1}\|^2_{L^2_x} \end{bmatrix} \begin{bmatrix} \ddot{y}_1(t) \\ \ddot{y}_2(t) \end{bmatrix} \\
&\qquad = e^{-\sqrt{2}d(t)} \begin{bmatrix} -4\sqrt{2} & 4\sqrt{2} \\ 4\sqrt{2} & -4\sqrt{2} \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} + \begin{bmatrix} O(v^2\|\vec{u}(t)\|_{H^1_x\times L^2_x}) \\ O(v^2\|\vec{u}(t)\|_{H^1_x\times L^2_x}) \end{bmatrix} \\
&\qquad\quad - \begin{bmatrix} \dfrac{\dot{d}(t)}{\big(1-\frac{1}{4}\dot{d}(t)^2\big)^{1/2}}\big\langle \partial_t u(t,x),\, H''_{0,1}(w_{k,v}(t,x))\big\rangle \\ \dfrac{\dot{d}(t)}{\big(1-\frac{1}{4}\dot{d}(t)^2\big)^{1/2}}\big\langle \partial_t u(t,x),\, H''_{0,1}(w_{k,v}(t,-x))\big\rangle \end{bmatrix} + \begin{bmatrix} O_k(NOL(t)) \\ O_k(NOL(t)) \end{bmatrix}.
\end{aligned} \tag{73}$$

Step 2 (refined ordinary differential system). Motivated by (73), for $j \in \{1,2\}$ we define the functions

$$c_j(t) = y_j(t) - y_j(T_{0,k}) + 2\sqrt{2}\int_{T_{0,k}}^{t} \frac{\dot{d}(s)}{\big(1-\frac{1}{4}\dot{d}(s)^2\big)^{1/2}}\big\langle u(s),\, H''_{0,1}(w_{k,v}(s,(-1)^{j+1}x))\big\rangle\, ds.$$

Clearly, we can verify using (24), Lemma 12 and the Cauchy–Schwarz inequality that

$$\dot{c}_j(t)=\dot{y}_j(t)+\frac{2\sqrt{2}\dot{d}(t)}{\bigl(1-\frac{1}{4}\dot{d}(t)^2\bigr)^{1/2}}\bigl\langle u(t,x),\,H''_{0,1}(w_{k,v}(t,(-1)^{j+1}x))\bigr\rangle,$$

$$\ddot{c}_j(t)=\ddot{y}_j(t)+\frac{2\sqrt{2}\dot{d}(t)}{\bigl(1-\frac{1}{4}\dot{d}(t)^2\bigr)^{1/2}}\bigl\langle \partial_t u(t,x),\,\ddot{H}_{0,1}(w_{k,v}(t,(-1)^{j+1}x))\bigr\rangle+O(v^2\|u(t)\|_{H^1_x}).$$

In conclusion, from the ordinary differential system of equations (73) we deduce that

$$\frac{d}{dt}\begin{bmatrix}y_1(t)\\ y_2(t)\\ \dot{c}_1(t)\\ \dot{c}_2(t)\end{bmatrix}=\begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -16e^{-\sqrt{2}d(t)} & 16e^{-\sqrt{2}d(t)} & 0 & 0\\ 16e^{-\sqrt{2}d(t)} & -16e^{-\sqrt{2}d(t)} & 0 & 0\end{bmatrix}\begin{bmatrix}y_1(t)\\ y_2(t)\\ \dot{c}_1(t)\\ \dot{c}_2(t)\end{bmatrix}+\begin{bmatrix}O(v\|u(t)\|_{H^1_x})\\ O(v\|u(t)\|_{H^1_x})\\ O_k(NOL(t))+O(v^2\|\vec{u}(t)\|_{H^1_x\times L^2_x})\\ O_k(NOL(t))+O(v^2\|\vec{u}(t)\|_{H^1_x\times L^2_x})\end{bmatrix}.$$

Actually, using the change of variables

$$e_1(t)=y_1(t)-y_2(t),\quad e_2(t)=y_1(t)+y_2(t),\quad \xi_1(t)=c_1(t)-c_2(t)\quad\text{and}\quad \xi_2(t)=c_1(t)+c_2(t),$$

we obtain from the ordinary differential system of equations above that

$$\frac{d}{dt}\begin{bmatrix}e_1(t)\\ e_2(t)\\ \dot{\xi}_1(t)\\ \dot{\xi}_2(t)\end{bmatrix}=\begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -32e^{-\sqrt{2}d(t)} & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}e_1(t)\\ e_2(t)\\ \dot{\xi}_1(t)\\ \dot{\xi}_2(t)\end{bmatrix}+\begin{bmatrix}O(v\|u(t)\|_{H^1_x})\\ O(v\|u(t)\|_{H^1_x})\\ O_k(NOL(t))+O(v^2\|\vec{u}(t)\|_{H^1_x\times L^2_x})\\ O_k(NOL(t))+O(v^2\|\vec{u}(t)\|_{H^1_x\times L^2_x})\end{bmatrix}. \tag{74}$$

To simplify our notation, we let

$$M(t)=\begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -32e^{-\sqrt{2}d(t)} & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}. \tag{75}$$

It is not difficult to verify that all the solutions of linear ordinary differential equation

$$\dot{L}(t)=M(t)L(t)\qquad\text{for } L(t)\in\mathbb{R}^4$$

are the linear space generated by the functions

$$L_1(t)=\begin{bmatrix}\tanh(\sqrt{2}vt)\\ 0\\ \sqrt{2}v\,\mathrm{sech}(\sqrt{2}vt)^2\\ 0\end{bmatrix},\quad L_2(t)=\begin{bmatrix}\sqrt{2}vt\tanh(\sqrt{2}vt)-1\\ 0\\ 2v^2t\,\mathrm{sech}(\sqrt{2}vt)^2+\sqrt{2}v\tanh(\sqrt{2}vt)\\ 0\end{bmatrix},$$

$$L_3(t)=\begin{bmatrix}0\\ 1\\ 0\\ 0\end{bmatrix},\quad L_4(t)=\begin{bmatrix}0\\ t\\ 0\\ 1\end{bmatrix}.$$

Also, by elementary computation, we can verify for any $t \in \mathbb{R}$ that

$$\det[L_1(t), L_2(t), L_3(t), L_4(t)] = -\sqrt{2}v. \tag{76}$$

In conclusion, using the variation of parameters technique, we can write any C^1 solution of (74) as $L(t) = \sum_{i=1}^{4} a_i(t)L_i(t)$, such that $a_i(t) \in C^1(\mathbb{R})$ for all $1 \leq i \leq 4$ and

$$\begin{bmatrix} \tanh(\sqrt{2}vt) & \sqrt{2}vt\tanh(\sqrt{2}vt) - 1 & 0 & 0 \\ 0 & 0 & 1 & t \\ \sqrt{2}v\,\mathrm{sech}(\sqrt{2}vt)^2 & 2v^2 t\,\mathrm{sech}(\sqrt{2}vt)^2 + \sqrt{2}v\tanh(\sqrt{2}vt) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{a}_1(t) \\ \dot{a}_2(t) \\ \dot{a}_3(t) \\ \dot{a}_4(t) \end{bmatrix}$$

$$= \begin{bmatrix} O(v\|u(t)\|_{H^1_x}) \\ O(v\|u(t)\|_{H^1_x}) \\ O_k(NOL(t)) + O(v^2\|\vec{u}(t)\|_{H^1_x \times L^2_x}) \\ O_k(NOL(t)) + O(v^2\|\vec{u}(t)\|_{H^1_x \times L^2_x}) \end{bmatrix}, \tag{77}$$

with

$$\begin{bmatrix} \tanh(\sqrt{2}vT_{0,k}) & \sqrt{2}vT_{0,k}\tanh(\sqrt{2}vT_{0,k}) - 1 & 0 & 0 \\ 0 & 0 & 1 & T_{0,k} \\ \sqrt{2}v\,\mathrm{sech}(\sqrt{2}vT_{0,k})^2 & 2v^2 t\,\mathrm{sech}(\sqrt{2}vT_{0,k})^2 + \sqrt{2}v\tanh(\sqrt{2}vT_{0,k}) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1(T_{0,k}) \\ a_2(T_{0,k}) \\ a_3(T_{0,k}) \\ a_4(T_{0,k}) \end{bmatrix}$$

$$= \begin{bmatrix} y_1(T_{0,k}) - y_2(T_{0,k}) \\ y_1(T_{0,k}) + y_1(T_{0,k}) \\ \dot{c}_1(T_{0,k}) \\ \dot{c}_2(T_{0,k}) \end{bmatrix}. \tag{78}$$

Step 3 (estimate of $\|\vec{u}(t)\|_{H^1_x \times L^2_x}$). From now on, for $C_1 > 1$, $C_2 > 0$ being fixed numbers to be chosen later, we consider the set

$$B_{C_1,C_2} = \left\{ t \in \mathbb{R} \;\middle|\; \max_{j\in\{1,2\}} |y_j(t)|v^2 + |\dot{y}_j(t)|v \leq C_1 v^{2(k+1)} \ln(1/v)^{n_k+3} \exp\left(\frac{C_2 v|t - T_{0,k}|}{\ln(1/v)}\right) \right\}.$$

We also consider the set

$$D_{u,v} = \{t \in \mathbb{R} \mid \|\vec{u}(t)\|_{H^1_x \times L^2_x} < v^2\}.$$

First, if $v^2|y(T_{0,k})| + v|\dot{y}(T_{0,k})| < v^{3k}$ and $v \ll 1$, then $T_{0,k} \in B_{C_1,C_2} \cap D_{u,v}$. Indeed, this happens when

$$\|(\varphi_{k,v}(T_{0,k}), \partial_t\varphi_{k,v}(T_{0,k})) - (\phi(T_{0,k}), \partial_t\phi(T_{0,k}))\|_{H^1_x \times L^2_x} < v^{4k},$$

because, since $u(t,x)$ satisfies the orthogonality conditions (29), we can verify using Lemma 11 that

$$\|\varphi_{k,v}(T_{0,k}) - \phi(T_{0,k})\|^2_{H^1_x} \cong \max_{j\in\{1,2\}} y_j(T_{0,k})^2 + \|u(T_{0,k})\|^2_{H^1_x}. \tag{79}$$

By a similar reasoning but using now Lemma 12 and estimate (79), we can verify that if $0 < v \ll 1$, then

$$\max_{j\in\{1,2\}} \dot{y}_j(T_{0,k})^2 + \|\partial_t u(T_{0,k})\|^2_{L^2_x} \lesssim \|(\varphi_{k,v}(T_{0,k}), \partial_t\varphi_{k,}(T_{0,k})) - (\phi(T_{0,k}), \partial_t\phi(T_{0,k}))\|^2_{H^1_x \times L^2_x}, \tag{80}$$

where $T_{0,k}$ satisfies the hypothesis of Theorem 15, for more details see Appendix B in [Moutinho 2023]. Also, for any $\theta \in (0,1)$, if $v \ll 1$, then while

$$|t - T_{0,k}| < \frac{\ln(1/v)^{2-\theta}}{v},$$

and $t \in B_{C_1,C_2} \cap D_{u,v}$, we can verify the estimate

$$\max_{j\in\{1,2\}} v^2|y_j(t)| + v|\dot{y}_j(t)| < v^{2k+1}\ln(1/v)^{n_k},$$

from which with estimate (73), the definition of $NOL(t)$ at (68), the definition of $D_{u,v}$ and the assumption of $k \geq 2$, we obtain that

$$\max_{j\in\{1,2\}} |\ddot{y}_j(t)| \lesssim_k v^{2k}\ln(1/v)^{n_k} + v\|\vec{u}(t)\|_{H^1_x\times L^2_x} + \|\vec{u}(t)\|^2_{H^1_x\times L^2_x}.$$

In conclusion, if $v \ll 1$, from Theorem 21, we deduce that the functional $L(t)$ defined in last section satisfies, for a constant C_0 and a parameter $C(k)$ depending only on k, the estimates

$$\begin{aligned} |\dot{L}(t)| \lesssim v \max_{j\in\{1,2\}} |\ddot{y}_j(t)|\|\vec{u}(t)\|_{H^1_x\times L^2_x} + \|\vec{u}(t)\|^3_{H^1_x\times L^2_x} + C(k)\|\vec{u}(t)\|_{H^1_x\times L^2_x} v^{2k+1}\ln(1/v)^{n_k} \\ + \|\vec{u}(t)\|^2_{H^1_x\times L^2_x}\frac{v}{\ln(1/v^2)}, \\ C_0\|\vec{u}(t)\|^2_{H^1_x(\mathbb{R})\times L^2_x(\mathbb{R})} \leq L(t) + C(k)v^{4k}\ln(1/v)^{2n_k}. \end{aligned}$$

Therefore, from the ordinary differential system of equations defined in (73), we conclude for $v \ll 1$ that if $t \in B_{C_1,C_2} \cap D_{u,v}$ and

$$|t - T_{0,k}| < \frac{\ln(1/v)^{2-\theta}}{v}, \tag{81}$$

then there exists a constant $C(k) > 0$ depending only on k satisfying

$$|\dot{L}(t)| \lesssim C(k)\|\vec{u}(t)\|_{H^1_x\times L^2_x} v^{2k+1}\ln(1/v)^{n_k} + \|\vec{u}(t)\|^2_{H^1_x\times L^2_x}\frac{v}{\ln(1/v^2)}.$$

Therefore, by a similar argument to the proof of Theorem 4.5 in [Moutinho 2023], we can verify from Theorem 21 and the Gronwall lemma applied on $L(t)$ that there exists a constant $K > 1$, independent of k and v, such that if t satisfies condition (81) and $t \in B_{C_1,C_2} \cap D_{u,v}$, then we have the estimate

$$\|(u(t), \partial_t u(t))\|_{H^1_x\times L^2_x} \lesssim_k \max\big(\|\vec{u}(T_{0,k})\|_{H^1_x\times L^2_x}, v^{2k}\ln(1/v)^{n_k+1}\big)\exp\bigg(\frac{K|t-T_{0,k}|v}{\ln(1/v)}\bigg). \tag{82}$$

In conclusion, if $v \ll 1$, $t \in B_{C_1,C_2}$ and t satisfies (81), then $t \in D_{u,v}$ and (82) is true.

Step 4 (estimate of $y_1(t)$, $y_2(t)$). Next, we will use the estimate (82) in the ordinary differential system of equations (74) to estimate the evolution of $y_1(t)$ and $y_2(t)$ while $t \in B_{C_1,C_2}$ and t satisfies condition (81). From (68), we have that if $t \in B_{C_1,C_2}$, t satisfies condition (81) and $0 < v \ll 1$, then

$$NOL(t) \ll v^2 \max\big(\|\vec{u}(T_{0,k})\|_{H^1_x\times L^2_x}, v^{2k}\ln(1/v)^{n_k+1}\big)\exp\bigg(\frac{K|t-T_{0,k}|v}{\ln(1/v)}\bigg). \tag{83}$$

In conclusion, from the Cauchy problem (25) satisfied by ϕ, identity (76) and estimates (79), (80), and (83), we deduce from the linear system (77) the estimates

$$\begin{aligned}
|\dot{a}_1(t)| &\lesssim_k v^{2k+1}[|t|v+1]\ln(1/v)^{n_k+1}\exp\left(K\frac{v|t-T_{0,k}|}{\ln(1/v)}\right),\\
|\dot{a}_2(t)| &\lesssim_k v^{2k+1}\ln(1/v)^{n_k+1}\exp\left(K\frac{v|t-T_{0,k}|}{\ln(1/v)}\right),\\
|\dot{a}_3(t)| &\lesssim_k v^{2k+1}[|t|v+1]\ln(1/v)^{n_k+1}\exp\left(K\frac{v|t-T_{0,k}|}{\ln(1/v)}\right),\\
|\dot{a}_4(t)| &\lesssim_k v^{2k+2}\ln(1/v)^{n_k+1}\exp\left(K\frac{v|t-T_{0,k}|}{\ln(1/v)}\right).
\end{aligned}$$

In conclusion, using the initial condition (78), we deduce from the fact that $T_{0,k}$ is in B_{C_1,C_2},the fundamental theorem of calculus and the elementary estimate

$$|t|v<\ln(1/v)\exp\left(\frac{v|t|}{\ln(1/v)}\right),$$

that if $\{\theta t+(1-\theta)T_{0,k}|0<\theta<1\}\subset B_{C_1,c_2}$ and t satisfies (81), then

$$\begin{aligned}
|a_1(t)|+|a_3(t)| &\lesssim_k v^{2k}\ln(1/v)^{n_k+3}\exp\left(\frac{(K+1)|t-T_{0,k}|v}{\ln(1/v)}\right),\\
v|a_2(t)|+|a_4(t)| &\lesssim_k v^{2k+1}\ln(1/v)^{n_k+2}\exp\left(\frac{K|t-T_{0,k}|v}{\ln(1/v)}\right).
\end{aligned}$$

In conclusion from the ordinary differential system of equations (74) satisfied by $e_j(t)$ for $j\in\{1,2,3,4\}$, the fact that $e_1(t)=y_1(t)-y_2(t)$, $e_2(t)=y_1(t)+y_2(t)$ and $\xi_1(t)=c_1(t)-c_2(t)$, $\xi_2(t)=c_1(t)+c_2(t)$, we can verify by triangle inequality and the identity

$$\begin{bmatrix}e_1(t)\\e_2(t)\\e_3(t)\\e_4(t)\end{bmatrix}=\sum_{j=1}^{4}a_jL_j(t)$$

the existence of $C_1(k)>0$ depending on k such that for $C_2=K+2$ and $v\ll 1$ we have that if

$$|t-T_{0,k}|<\frac{\ln(1/v)^{2-\theta}}{v},$$

then $t\in B_{C_1(k),C_2}$. □

Remark 22. For any constants $\theta,\gamma\in(0,1)$, obviously

$$\lim_{v\to+0}v^{\gamma}\exp\left(\ln(1/v)^{\theta}\right)=0.$$

In conclusion, for fixed $k\in\mathbb{N}$ large and $0<\theta<\frac{1}{4}$, we can deduce from Theorem 15 that there is a $\Delta_{k,\theta}>0$ such that if $0<v<\Delta_{k,\theta}$, then

$$\|(\phi(t,x),\partial_t\phi(t,x))-(\phi_k(v,t,x),\partial_t\phi_k(v,t,x))\|_{H^1_x\times L^2_x}<v^{2k-1/2},$$

for all t satisfying

$$|t-T_{0,k}|<\frac{\ln(1/v)^{2-\theta}}{v}.$$

5. Proof of Theorem 4

Remark 23. The importance of this theorem is to describe the dynamics of the two solitons before the collision instant, for all $t < 0$ and $|t| \gg 1$. More precisely, if two moving kinks are coming from an infinite distance with a sufficiently low speed v satisfying $v \leq \delta(2k)$, then the inelasticity of the collision is going to be of order at most $O(v^k)$ and the kinks will move away each one with the speed of size in modulus $v + O(v^k)$ when t goes to $-\infty$.

The proof of Theorem 4 uses energy estimate techniques from [Henry et al. 1982], and the monotonicity property of the function

$$P_{+}(\phi(t), \partial_t\phi(t)) = -\int_0^{+\infty} \partial_t\phi(t,x)\partial_x\phi(t,x)\,dx, \tag{84}$$

which is nondecreasing on t when $\phi(t,\cdot)$ is odd on x. Furthermore, the demonstration of Theorem 4 is quite similar to the proof of Theorem 1 of [Kowalczyk et al. 2021] and also uses modulation techniques inspired by [Raphaël and Szeftel 2011; Kowalczyk et al. 2021].

Moreover, since the solution $\phi(t,x)$ is an odd function in the variable x for all $t \in \mathbb{R}$, we have that

$$E(\phi) = 2\left[\int_0^{+\infty} \frac{\partial_x\phi(t,x)^2 + \partial_t\phi(t,x)^2}{2} + U(\phi(t,x))\,dx\right] = 2E_{+}(\phi(t), \partial_t\phi(t)),$$

where

$$E_{+}(\phi(t), \partial_t\phi(t)) = \int_0^{+\infty} \frac{\partial_x\phi(t,x)^2 + \partial_t\phi(t,x)^2}{2} + U(\phi(t,x))\,dx \tag{85}$$

is a conserved quantity.

5.1. ***Modulation techniques.*** First, similarly to [Kowalczyk et al. 2021], we consider, for any $0 < v < 1$, $y \in \mathbb{R}$, the following function on $x \in \mathbb{R}$:

$$\overrightarrow{H_{0,1}}((v,y),x) = \begin{bmatrix} H_{0,1}\Big(\frac{x-y}{\sqrt{1-v^2}}\Big) \\ \frac{-v}{\sqrt{1-v^2}} H'_{0,1}\Big(\frac{x-y}{\sqrt{1-v^2}}\Big) \end{bmatrix}, \tag{86}$$

$$\overrightarrow{H_{-1,0}}((v,y),x) = -\overrightarrow{H_{0,1}}((v,y),-x) \quad \text{for all } x \in \mathbb{R}.$$

Next, we consider the antisymmetric map

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{87}$$

and based on [Kowalczyk et al. 2021], we consider for any $0 < v < 1$ and any $y \in \mathbb{R}$ the following functions, which were defined in Section 2.3 of [Kowalczyk et al. 2021]:

$$C_{v,y}(x) = \begin{bmatrix} \frac{1}{\sqrt{1-v^2}} H'_{0,1}\Big(\frac{x-y}{\sqrt{1-v^2}}\Big) \\ \frac{-v}{1-v^2} H''_{0,1}\Big(\frac{x-y}{\sqrt{1-v^2}}\Big) \end{bmatrix}, \tag{88}$$

$$D_{v,y}(x) = \begin{bmatrix} \frac{v}{1-v^2}\frac{x-y}{\sqrt{1-v^2}}H'_{0,1}\Big(\frac{x-y}{\sqrt{1-v^2}}\Big) \\ \frac{-1}{(1-v^2)^{3/2}}H'_{0,1}\Big(\frac{x-y}{\sqrt{1-v^2}}\Big) - \frac{v^2}{(1-v^2)^{3/2}}\frac{x-y}{\sqrt{1-v^2}}H''_{0,1}\Big(\frac{x-y}{\sqrt{1-v^2}}\Big) \end{bmatrix}. \tag{89}$$

See also [Chen and Jendrej 2019].

The following identity is going to be useful for our next results.

Lemma 24. *For any* $v \in (0, 1)$, *there holds*

$$\langle \partial_x \overrightarrow{H_{0,1}}((v, 0), x), JD_{0,v}\rangle = -(1-v^2)^{-3/2}\|H'_{0,1}\|^2_{L^2_x}.$$

Proof. See the proof of Lemma 2.4 from [Kowalczyk et al. 2021]. □

Next, for any value $y_0 \gg 1$, we will modulate any odd function (ϕ_0, ϕ_1) close to

$$\overrightarrow{H_{-1,0}}((v, y_0), x) + \overrightarrow{H_{0,1}}((v, y_0), x)$$

in the energy norm in terms of an orthogonal condition.

Lemma 25. *There exist* $K > 0$ *and* $\delta_0, \delta_1 \in (0, 1)$ *such that if* $0 < v < \delta_1$, $y_0 > 1/\delta_1$, $0 \leq \delta \leq \delta_0$ *and* $(\phi_1 - H_{0,1} - H_{-1,0}, \phi_2) \in H^1_x(\mathbb{R}) \times L^2_x(\mathbb{R})$ *is an odd function satisfying*

$$\|(\phi_1(x), \phi_2(x)) - \overrightarrow{H_{-1,0}}((v, y_0), x) - \overrightarrow{H_{0,1}}((v, y_0), x)\|_{H^1_x \times L^2_x} \leq \delta v, \tag{90}$$

then there exists a unique $\hat{y} > 1$ *such that* $|\hat{y} - y_0| \leq K\delta v$ *and the function*

$$\vec{\kappa}(x) = (\phi_1(x), \phi_2(x)) - \overrightarrow{H_{-1,0}}((v, \hat{y}), x) - \overrightarrow{H_{0,1}}((v, \hat{y}), x)$$

satisfies

$$\|\vec{\kappa}\|_{H^1_x \times L^2_x} \leq K\delta v \tag{91}$$

and $\langle \vec{\kappa}(x), J \circ D_{v,\hat{y}}(x)\rangle = 0$.

Proof of Lemma 25. The proof is completely analogous to that of Lemma 2.1 of [Kowalczyk et al. 2021]. □

Corollary 26. *In the notation of Lemma 25, there exists a constant* $C > 1$ *such that if* $v \in (0, 1)$ *is small enough, then there exists at most one number* $y \geq 2\ln\frac{1}{v}$ *satisfying*

$$\|\overrightarrow{\kappa_0}\|_{H^1_x \times L^2_x} \leq \min\Big\{\delta_0 v, \frac{K}{3C}\delta_0 v\Big\} \quad \textit{and} \quad \langle \overrightarrow{\kappa_0}(x), J \circ D_{v,y}(x)\rangle = 0,$$

where

$$\overrightarrow{\kappa_0}(x) = (\phi_1(x), \phi_2(x)) - \overrightarrow{H_{-1,0}}((v, y), x) - \overrightarrow{H_{0,1}}((v, y), x)$$

Proof of Corollary 26. Let y_1, y_2 two real numbers satisfying the results of Corollary 26. We consider the functions

$$\overrightarrow{\kappa_1}(x) = (\kappa_{1,0}(x), \kappa_{1,1}(x)) = (\phi_1(x), \phi_2(x)) - \overrightarrow{H_{-1,0}}((v, y_1), x) - \overrightarrow{H_{0,1}}((v, y_1), x),$$
$$\overrightarrow{\kappa_2}(x) = (\kappa_{2,0}(x), \kappa_{2,1}(x)) = (\phi_0(x), \phi_1(x)) - \overrightarrow{H_{-1,0}}((v, y_2), x) - \overrightarrow{H_{0,1}}((v, y_2), x).$$

Choosing $x = y_1$, we obtain the ng identity

$$H_{0,1}(0) - H_{0,1}\Big(\frac{y_1 - y_2}{\sqrt{1-v^2}}\Big) = -H_{0,1}\Big(\frac{-2y_1}{\sqrt{1-v^2}}\Big) + H_{0,1}\Big(\frac{-y_1 - y_2}{\sqrt{1-v^2}}\Big) + \kappa_{2,0}(y_1) - \kappa_{1,0}(y_1). \tag{92}$$

Since there exists a constant $c > 0$ satisfying for any $f \in H^1_x(\mathbb{R})$ the inequality

$$\|f\|_{L^\infty_x(\mathbb{R})} \leq c\|f\|_{H^1_x},$$

we deduce from (92) and the hypotheses of Corollary 26 that

$$\left|H_{0,1}(0) - H_{0,1}\left(\frac{y_1 - y_2}{\sqrt{1-v^2}}\right)\right| \leq \frac{2cK}{3C}\delta_0 v + \left|H_{0,1}\left(\frac{-2y_1}{\sqrt{1-v^2}}\right)\right| + \left|H_{0,1}\left(\frac{-y_1 - y_2}{\sqrt{1-v^2}}\right)\right|,$$

from which we deduce the estimate

$$\left|H_{0,1}(0) - H_{0,1}\left(\frac{y_1 - y_2}{\sqrt{1-v^2}}\right)\right| \leq \frac{2cK}{3C}\delta_0 v + 2v^4.$$

Consequently, since $H_{0,1}$ is an increasing function and $H'_{0,1}(0) = \frac{1}{2}$, we obtain that if $\delta_1 \ll 1$ and $0 < v < \delta_1$, then

$$|y_1 - y_2| \leq \frac{5Kc}{3C}\delta_0 v.$$

Therefore, choosing $C = 2c + 1$, from Lemma 25, we have $y_1 = y_2$ if $v > 0$ is small enough. □

Finally, using Lemma 25 and repeating the argument of the demonstration of Lemma 2.11 in [Kowalczyk et al. 2021], we can verify the following result.

Lemma 27. *There exist $K > 1$, $\delta_0 > 0$ and $\delta_1 \in (0, 1)$ such that if $0 < \delta_2 < \delta_0$, $0 < v < \delta_1$, $y_0 > \frac{7}{2}\ln\frac{1}{v}$ and the solution $(\phi(t, x), \partial_t\phi(t, x))$ of (1) satisfies, for $T > 0$,*

$$\sup_{t\in[0,T]} \inf_{y\in\mathbb{R}_{\geq y_0}} \|(\phi(t, x), \partial_t\phi(t, x)) - \overrightarrow{H_{-1,0}}((v, y), x) - \overrightarrow{H_{0,1}}((v, y), x)\|_{H^1_x\times L^2_x} \leq \delta_2 v, \tag{93}$$

then there exists a real function $y_1 : [0, T] \to \mathbb{R}_{\geq y_0/2}$ such that the solution $(\phi(t), \partial_t\phi(t))$ satisfies, for any $0 \leq t \leq T$,

$$(\phi(t), \partial_t\phi(t)) = \overrightarrow{H_{-1,0}}((v, y_1(t)), x) + \overrightarrow{H_{0,1}}((v, y_1(t)), x) + (\psi_1(t), \psi_2(t)), \tag{94}$$

$$\|(\psi_1(t), \psi_2(t))\|_{H^1_x\times L^2_x} \leq K\delta_2 v, \tag{95}$$

where $(\psi_1(t), \psi_2(t)) \in H^1_x(\mathbb{R}) \times L^2_x(\mathbb{R})$ and $y_1(t)$ satisfy the orthogonality condition of Lemma 25, and $y_1(t)$ is a function of class C^1 satisfying the inequality

$$|\dot{y}_1(t) - v| \leq K[\|(\psi_1(t), \psi_2(t))\|_{H^1_x\times L^2_x} + e^{-2\sqrt{2}y_1(t)}]. \tag{96}$$

Proof. First, from Lemma 25 and the fact that $\vec{\phi} \in C(\mathbb{R}; H^1_x(\mathbb{R}) \times L^2_x(\mathbb{R}))$, if δ_1 is small enough, we can find a constant $K > 0$ and a function $\hat{y} : [0, T] \to \left(3\ln\frac{1}{v}, +\infty\right)$ such that for

$$\vec{\kappa}(t, x) = (\phi(t, x), \partial_t\phi(t, x)) - \overrightarrow{H_{-1,0}}((v, \hat{y}(t)), x) - \overrightarrow{H_{0,1}}((v, \hat{y}(t)), x), \tag{97}$$

we have $\vec{\kappa}(t)$, $\hat{y}(t)$ satisfying the orthogonality condition of Lemma 25 and

$$\|\vec{\kappa}(t)\|_{H^1_x\times L^2_x} \leq K\delta_2 v \tag{98}$$

for all $0 \leq t \leq T$.

Next, we will construct a linear ordinary differential system of equations with solution $y_1(t)$ and we will verify that if $y_1(0)=\hat{y}(0)$, then $y_1(t)=\hat{y}(t)$ for all $t\in[0,T]$.

Step 1 (construction of the ordinary differential equation satisfied by y_1). The argument of the demonstration of the remaining part of Lemma 27 is completely analogous to the proof of Lemma 2.11 of [Kowalczyk et al. 2021]. More precisely, similarly to Lemma 2.11 of [Kowalczyk et al. 2021], we will construct an ordinary differential equation with solution $y_1(t)$, which, during their time of existence, preserves the orthogonality conditions

$$\langle(\psi_1(t,x),\psi_2(t,x)),JD_{v,y_1(t)}(x)\rangle=0, \tag{99}$$

where J is defined in (87), and we will verify that if $y_1(0)=\hat{y}(0)$, then $y_1(t)=\hat{y}(t)$ for all $0\leq t\leq T$. From the global well-posedness of the partial differential (1) in the energy space, we have for any $T_0>0$ that $\phi(t,x)-H_{0,1}(x)-H_{-1,0}(x)\in C([-T_0,T_0],H^1_x(\mathbb{R}))$ and $\partial_t\phi(t,x)\in C([-T_0,T_0],L^2_x(\mathbb{R}))$. Therefore, if there exists a interval $[0,T_1]\subset[0,T]$ such that $y_1\in C^1([0,T_1])$ when restricted to this interval and

$$(\phi(t),\partial_t\phi(t))=\overrightarrow{H_{-1,0}}((v,y_1(t)),x)+\overrightarrow{H_{0,1}}((v,y_1(t)),x)+(\psi_1(t),\psi_2(t))\quad\text{for any } t\in[0,T_1], \tag{100}$$

then $(\psi_1(t),\psi_2(t))=(\psi_1(t,x),\psi_2(t,x))$ satisfies, for any functions $h_1,h_2\in\mathscr{S}(\mathbb{R})$, the identity

$$\frac{d}{dt}\big\langle(\psi_1(t,x),\psi_2(t,x)),(h_1(x),h_2(x))\big\rangle=\big\langle\partial_t(\psi_1(t,x),\psi_2(t,x)),(h_1(x),h_2(x))\big\rangle$$

if $t\in[0,T_1]$.

Consequently, if we derive the (99) in time, we obtain the following linear ordinary differential equation satisfied by $y_1(t)$:

$$\dot{y}_1(t)\big\langle(\psi_1(t,x),\psi_2(t,x)),J\partial_{y_1}D_{v,y_1(t)}(x)\big\rangle+\big\langle\partial_t(\psi_1(t,x),\psi_2(t,x)),JD_{v,y_1(t)}(x)\big\rangle=0. \tag{101}$$

Since $x^mH'_{0,1}(x)\in\mathscr{S}(\mathbb{R})$ for all $m\in\mathbb{N}\cup\{0\}$, we have that the functions $\omega_1,\omega_2:[0,T]\times(1,+\infty)\to\mathbb{R}$ defined by

$$\omega_1(t,y)=\big\langle(\psi_1(t,x),\psi_2(t,x)),J\partial_yD_{v,y}(x)\big\rangle,\ \omega_2(t,y)=\big\langle\partial_t(\psi_1(t,x),\psi_2(t,x)),JD_{v,y}(x)\big\rangle$$

are continuous and, for any $t\in[0,T]$, $\omega_1(t,\cdot),\omega_2(t,\cdot):(1,+\infty)\to\mathbb{R}$ are smooth.

Step 2 (partial differential equation satisfied by $\vec{\psi}$). First, we consider the self-adjoint operator

$$\operatorname{Hess}(y_1(t),x):H^2_x(\mathbb{R})\subset L^2_x(\mathbb{R})\to\mathbb{R},$$

which satisfies, for all $t\in[0,T]$,

$$\operatorname{Hess}(y_1(t),x)=\begin{bmatrix}-\partial_x^2+U''\Big(H_{0,1}\Big(\frac{x-y_1(t)}{\sqrt{1-v^2}}\Big)-H_{0,1}\Big(\frac{-x-y_1(t)}{\sqrt{1-v^2}}\Big)\Big) & 0\\ 0 & 1\end{bmatrix}, \tag{102}$$

and the self-adjoint operator $\operatorname{Hess}_1(y_1(t),x):H^2_x(\mathbb{R})\subset L^2_x(\mathbb{R})\to\mathbb{R}$ denoted by

$$\operatorname{Hess}_1(y_1(t),x)=\begin{bmatrix}-\partial_x^2+U''\Big(H_{0,1}\Big(\frac{x-y_1(t)}{\sqrt{1-v^2}}\Big)\Big) & 0\\ 0 & 1\end{bmatrix}. \tag{103}$$

Next, we consider the maps $\operatorname{Int}:\mathbb{R}^2\to\mathbb{R}^2$ and $\mathcal{T}:\mathbb{R}^2\times H^1_x(\mathbb{R})\to\mathbb{R}^2$, which we denote by

$$\operatorname{Int}(y,x)=\begin{bmatrix}0\\ U'\Big(-H_{0,1}\Big(\frac{-x-y_1}{\sqrt{1-v^2}}\Big)\Big)+U'\Big(H_{0,1}\Big(\frac{x-y}{\sqrt{1-v^2}}\Big)\Big)\end{bmatrix} \\ -\begin{bmatrix}0\\ U'\Big(H_{0,1}\Big(\frac{x-y}{\sqrt{1-v^2}}\Big)-H_{0,1}\Big(\frac{-x-y}{\sqrt{1-v^2}}\Big)\Big)\end{bmatrix}, \tag{104}$$

$$\mathcal{T}(y,x,\psi)=\begin{bmatrix}0\\ -\sum_{j=3}^{6}U^{(j)}\Big(H_{0,1}\Big(\frac{x-y}{\sqrt{1-v^2}}\Big)-H_{0,1}\Big(\frac{-x-y}{\sqrt{1-v^2}}\Big)\Big)\frac{\psi(x)^{j-1}}{(j-1)!}\end{bmatrix} \tag{105}$$

for any $(y,x)\in\mathbb{R}^2$ and $\psi\in H^1_x(\mathbb{R})$. Therefore, if $[0,T_1]\subset[0,T]$, $y_1\in C^1([0,T_1])$ and $y_1\geq 1$, $0<v_1<1$ then, from the partial differential equation (1) and identity (100), we deduce that $(\psi_1(t,x),\psi_2(t,x))$ is a solution in the space $C([0,T_1],H^1_x(\mathbb{R})\times L^2_x(\mathbb{R}))$ of the partial differential equation

$$\partial_t(\psi_1(t,x),\psi_2(t,x))=(\dot{y}_1(t)-v)[C_{v,y_1(t)}(x)-C_{v,y_1(t)}(-x)]+J\operatorname{Hess}(y_1(t),x)(\psi_1(t,x),\psi_2(t,x)) \\ +\operatorname{Int}(y_1(t),x)+\mathcal{T}(y_1(t),x,\psi_1(t)), \tag{106}$$

where J is the antisymmetric operator defined in (87).

In the next step, we will assume the existence of $0\leq T_1\leq T$ such that y_1 is of class C^1 in the interval $[0,T_1]$, and $y_1\geq 1$ for any $t\in[0,T_1]$. Moreover, we will prove that when this condition is true, then $|\dot{y}_1(t)-v|$ is sufficiently small for all $t\in[0,T_1]$.

Step 3 (estimate of $|\dot{y}_1(t)-v|$). Uniquely in this step, for any continuous nonnegative function $f:[0,T_1]\times(0,1)\times(1,+\infty)\to\mathbb{R}$, we say that a function $g:[0,T_1]\times(0,1)\times(1,+\infty)\to\mathbb{R}$ is $O(f)$ if and only if g is a continuous function satisfying the following properties:

- There is a constant $c>0$ such that $|g(t,v,y)|<cf(t,v,y)$ for all (t,v,y) in $[0,T_1]\times(0,1)\times(1,+\infty)$.
- $g(t,\cdot):(0,1)\times(1,+\infty)\to\mathbb{R}$ is smooth for all $t\in[0,T_1]$.

We recall that J, $C_{v,y_1(t)}$ and $D_{v,y_1(t)}$ are defined, respectively, in (87), (88) and (89). Using Lemma 11, we obtain that if $y_1(t)\geq 1$ and $v\in(0,1)$ is small enough, then

$$|\langle C_{v,y_1(t)}(x),J\circ D_{v,y_1(t)}(-x)\rangle|+|\langle C_{v,y_1(t)}(x),JC_{v,y_1(t)}(-x)\rangle|+|\langle D_{v,y_1(t)}(x),JD_{v,y_1(t)}(-x)\rangle| \\ \lesssim y_1(t)^4e^{-2\sqrt{2}y_1(t)}. \tag{107}$$

Furthermore, using the partial differential equation (106) satisfied by $(\psi_1(t,x),\psi_2(t,x))$, we deduce for any $t\in[0,T_1]\subset[0,T]$ the identity

$$\langle\partial_t(\psi_1(t,x),\psi_2(t,x)),JD_{v,y_1(t)}(x)\rangle \\ =(\dot{y}_1(t)-v)\langle C_{v,y_1(t)}(x),JD_{v,y_1(t)}(x)\rangle-(\dot{y}_1(t)-v)\langle C_{v,y_1(t)}(-x),JD_{v,y_1(t)}(x)\rangle \\ +\big\langle J\operatorname{Hess}(y_1(t),x)(\psi_1(t,x),\psi_2(t,x)),JD_{v,y_1(t)}(x)\big\rangle \\ +\big\langle\mathcal{T}(y_1(t),x,\psi_1(t))+\operatorname{Int}(y_1(t),x),JD_{v,y_1(t)}(x)\big\rangle. \tag{108}$$

Moreover, from Lemma 24 and identity $J^* = -J$, we have

$$\langle JD_{v,y_1(t)}(x), C_{v,y_1(t)}(x)\rangle = -\langle D_{v,y_1(t)}(x), JC_{v,y_1(t)}(x)\rangle = (1-v^2)^{-3/2}\|H'_{0,1}\|^2_{L^2_x}. \tag{109}$$

Therefore, using (108), estimates (107) and Lemma 11, we deduce the following estimate

$$\begin{aligned}&\big\langle \partial_t(\psi_1(t,x), \psi_2(t,x)), JD_{v,y_1(t)}(x)\big\rangle\\ &\quad= (\dot{y}_1(t)-v)[(1-v^2)^{-3/2}\|H'_{0,1}\|^2_{L^2_x} + O(y_1(t)^4 e^{-2\sqrt{2}y_1(t)})]\\ &\qquad\quad+\big\langle J\operatorname{Hess}(y_1(t),x)(\psi_1(t,x),\psi_2(t,x)), JD_{v,y_1(t)}\big\rangle\\ &\qquad\qquad+\big\langle \mathcal{T}(y_1(t),x,\psi_1(t)), JD_{v,y_1(t)}(x)\big\rangle + \big\langle \operatorname{Int}(y_1(t),x), JD_{v,y_1(t)}(x)\big\rangle.\end{aligned}$$

Furthermore, since for any $\zeta\in\mathbb{R}$ we have the identity

$$\begin{aligned}&U'(H^{\zeta}_{0,1}(x)+H_{-1,0}(x)) - U'(H^{\zeta}_{0,1}(x)) - U'(H_{-1,0}(x))\\ &\quad= -24H_{-1,0}(x)H^{\zeta}_{0,1}(x)(H_{-1,0}(x)+H^{\zeta}_{0,1}(x)) + \sum_{j=1}^{4}\binom{5}{j}H_{-1,0}(x)^j H^{\zeta}_{0,1}(x)^{5-j},\end{aligned}$$

we deduce from Lemma 11 and the definition of function Int that $\|\operatorname{Int}(y_1(t),x,\psi(t))\|_{L^2_x} \lesssim e^{-2\sqrt{2}y_1(t)}$. Next, since $\|U^{(l)}\|_{L^\infty[-1,1]} < +\infty$ for any $l\in\mathbb{N}\cup\{0\}$, we deduce using Lemma 13 and the definition of function $\mathcal{T}$ that

$$\|\mathcal{T}(y_1(t),x,\psi_1(t))\|_{L^2_x} \le \|\mathcal{T}(y_1(t),x,\psi_1(t))\|_{H^1_x} \lesssim \|\psi_1(t,x)\|^2_{H^1_x}.$$

As a consequence,

$$\begin{aligned}&\big\langle \partial_t(\psi_1(t,x),\psi_2(t,x)), JD_{v,y_1(t)}(x)\big\rangle\\ &\quad= (\dot{y}_1(t)-v)\big[(1-v^2)^{-3/2}\|H'_{0,1}\|^2_{L^2_x} + O(y_1(t)^4e^{-2\sqrt{2}y_1(t)})\big]\\ &\qquad+\big\langle J\operatorname{Hess}(y_1(t),x)(\psi_1(t,x),\psi_2(t,x)), JD_{v_1(t),y_1(t)}(x)\big\rangle + O(e^{-2\sqrt{2}y_1(t)} + \|\vec{\psi}(t)\|^2_{H^1_x\times L^2_x})\end{aligned} \tag{110}$$

for any $t\in[0,T_1]$.

Furthermore, using identities (102), (103), the formula of $D_{v,y}$ in (89) and Lemma 11, we can deduce the estimate

$$\|[\operatorname{Hess}(y_1(t),x) - \operatorname{Hess}_1(y_1(t),x)]D_{v,y_1(t)}(x)\|_{L^2_x(\mathbb{R};\mathbb{R}^2)} \lesssim e^{-2\sqrt{2}y_1(t)}$$

for all $t\in[0,T_1]$. Thus, after using integration by parts and the Cauchy–Schwarz inequality, we deduce for all $t\in[0,T_1]$ that

$$\big|\big\langle J[\operatorname{Hess}(y_1(t),x) - \operatorname{Hess}_1(y_1(t),x)]\vec{\psi}(t), JD_{v_1(t),y_1(t)}(x)\big\rangle\big| \lesssim \|\vec{\psi}(t)\|_{H^1_x\times L^2_x}e^{-2\sqrt{2}y_1(t)}.$$

Consequently, since $\langle j(a), a\rangle = 0$ for all $a\in\mathbb{R}^2$, we obtain that if y_1 is a function of class C^1 in the interval $[0,T_1]$ and $v\in(0,1)$ is small enough, then

$$\begin{aligned}&\big\langle \partial_t(\psi_1(t,x),\psi_2(t,x)), JD_{v,y_1(t)}(x)\big\rangle\\ &= (\dot{y}_1(t)-v)\bigg[-\frac{\|H'_{0,1}\|^2_{L^2_x}}{(1-v^2)^{3/2}} + O(y_1(t)^4e^{-2\sqrt{2}y_1(t)})\bigg] + \big\langle J\operatorname{Hess}_1(y_1(t),x)(\psi_1(t,x),\psi_2(t,x)), JD_{v,y_1(t)}(x)\big\rangle\\ &\qquad + O(e^{-2\sqrt{2}y_1(t)} + \|\vec{\psi}(t)\|^2_{H^1_x\times L^2_x})\end{aligned} \tag{111}$$

for any $t\in[0,T_1]$.

Next, using (103), it is not difficult to verify the identity

$$\mathrm{Hess}_1(y_1(t),x)D_{v,y_1(t)}(x) - vJ[\partial_x D_{v,y_1(t)}(x)] = JC_{v,y_1(t)}(x);$$

see Lemma 2.4 of [Kowalczyk et al. 2021] for the proof. Consequently, we have for any $t \in [0, T_1]$ that

$$\begin{aligned}&\big\langle J\,\mathrm{Hess}_1(y_1(t),x)(\psi_1(t,x),\psi_2(t,x)),\, JD_{v,y_1(t)}(x)\big\rangle\\ &\qquad\qquad = -v\big\langle(\psi_1(t,x),\psi_2(t,x)),\, J\partial_{y_1}D_{v,y_1(t)}(x)\big\rangle + \big\langle(\psi_1(t,x),\psi_2(t,x)),\, JC_{v,y_1(t)}(x)\big\rangle.\end{aligned}$$

In conclusion, estimate (111) and identity (101) imply that

$$\begin{aligned}(\dot{y}_1(t)-v)\left[\frac{-\|H'_{0,1}\|^2_{L^2_x}}{(1-v^2)^{3/2}} + O\big(\|(\psi_1(t),\psi_2(t))\|_{H^1_x\times L^2_x} + y_1(t)^4e^{-2\sqrt{2}y_1(t)}\big)\right]&\\ = O\big(e^{-2\sqrt{2}y_1(t)} + \|(\psi_1(t),\psi_2(t))\|_{H^1_x\times L^2_x}\big)& \qquad (112)\end{aligned}$$

for all $t \in [0, T_1]$.

Step 4 (proof that $y_1 \in C^1$). Equations (101) and (108) imply that y_1 satisfies the ordinary differential equation

$$\begin{aligned}&(\dot{y}_1(t)-v)\big[\big\langle C_{v,y_1(t)}(x), JD_{v,y_1(t)}(x)\big\rangle - \big\langle C_{v,y_1(t)}(-x), JD_{v,y_1(t)}(x)\big\rangle + \big\langle(\psi_1(t),\psi_2(t)), J\partial_{y_1}D_{v,y_1(t)}(x)\big\rangle\big]\\ &\quad = -v\big\langle(\psi_1(t,x),\psi_2(t,x)), J\partial_{y_1}D_{v,y_1(t)}(x)\big\rangle\\ &\qquad - \big\langle J\,\mathrm{Hess}(y_1(t),x)(\psi_1(t,x),\psi_2(t,x)) + \mathcal{T}(y_1(t),x,\psi_1(t)) + \mathrm{Int}(y_1(t),x), JD_{v,y_1(t)}(x)\big\rangle, \qquad (113)\end{aligned}$$

which is a first-order nonautonomous differential system of the form

$$(\dot{y}_1(t)-v)\alpha_v(t,y_1(t)) = \beta_v(t,y_1(t)),$$

where the functions $\alpha_v, \beta_v : [0,T]\times\mathbb{R} \to \mathbb{R}$ are continuous when $v \in (0,1)$.

Moreover, from the hypotheses of Lemma 27, Lemma 11 and identities (102), (104), (105), we can deduce for any $t \in [0,T]$ that the restrictions of $\alpha_v(t,\cdot)$ and $\beta_v(t,\cdot)$ in the set $\big(3\ln\frac{1}{v}, +\infty\big)$ are locally Lipschitz when v is small enough.

Furthermore, from the first step, we have $y_1(0) = \hat{y}(0) > 3\ln\frac{1}{v}$ which implies $y_1(0)^4e^{-2\sqrt{2}y_1(0)} < v^3$ if v is small enough. Moreover, we deduce from (97) and (98) that $\|(\psi_1(0),\psi_2(0))\|_{H^1_x\times L^2_x} \leq K\delta_2 v$ and we also have

$$\alpha_v(0,y_1(0)) = \frac{-\|H'_{0,1}\|^2_{L^2_x}}{(1-v^2)^{3/2}} + O(v) > 0,$$

because of the estimate (112) when v is small enough.

Consequently, the Picard–Lindelöf theorem implies the existence of an interval $[0,T_1] \subset [0,T]$ such that $y_1 : [0,T_1] \to \mathbb{R}_{>2\ln(1/v)}$ is a C^1 function and since y_1 satisfies (101), we have for any $t \in [0,T_1]$ that

$$\big\langle(\psi_1(t,x),\psi_2(t,x)), JD_{v,y_1(t)}(x)\big\rangle = \langle\overrightarrow{\psi}(0,x), JD_{v,y_1(0)}(x)\rangle = 0. \qquad (114)$$

Furthermore, since $\hat{y}(t) \geq 3\ln\frac{1}{v}$, we can deduce from the continuity of function y_1, Lemma 25 and Corollary 26 the identity $y_1(t) = \hat{y}(t)$ for all $t \in [0,T_1]$. Consequently, $y_1(t) \geq 3\ln\frac{1}{v}$ for all $t \in [0,T_1]$ and

$$\|(\psi_1(t),\psi_2(t))\|_{H^1_x\times L^2_x} = \big\|\overrightarrow{\phi}(t,x) - \overrightarrow{H_{-1,0}}((v,y_1(t)),x) - \overrightarrow{H_{0,1}}((v,y_1(t)),x)\big\|_{H^1_x\times L^2_x} \leq K\delta_2 v \qquad (115)$$

for all $t \in [0,T_1]$, because of estimate (97) and identity (98).

Therefore, using a bootstrap argument and estimate (112), we can conclude that the function y_1 is in $C^1[0, T]$ and satisfies (114) for all $t \in [0, T]$. Finally, estimate (96) is a direct consequence of (112), (115) and the fact that $y_1 \geq 3 \ln \frac{1}{v}$. ☐

5.2. *Orbital stability of the parameter y.* In this subsection, we consider $\phi(t, x)$ as a solution of (1) having finite energy and with an initial data $(u_1(x), u_2(x))$ satisfying the hypotheses of Theorem 4. Moreover, if v is small enough, from the local well-posedness of the partial differential equation (1) in the space of solutions with finite energy, we can deduce from Lemma 25 the existence of a constant $C > 0$ and a positive number ϵ such that, for all $t \in [0, \epsilon]$,

$$(\phi(t, x), \partial_t \phi(t, x)) = \overrightarrow{H_{-1,0}}((v, y(t)), x) + \overrightarrow{H_{0,1}}((v, y(t)), x) + (\psi_1(t, x), \psi_2(t, x)),$$

where $(\psi_1(t, x), \psi_2(t, x))$ is an odd function in x, and $y(t)$, $(\psi_1(t, x), \psi_2(t, x))$ satisfy the orthogonality conditions in Lemma 25 and the inequality

$$|y(t) - y_0| + \|(\psi_1(t, x), \psi_2(t, x))\|_{H^1_x \times L^2_x} \leq 2C \|(u_1, u_2)\|_{H^1_x \times L^2_x}. \tag{116}$$

Finally, we are ready to start the proof of Theorem 4

Remark 28 (main argument). The main techniques of the demonstration of Theorem 4 are inspired by the proof of Theorem 1 of [Kowalczyk et al. 2021].

More precisely, recalling the functions E_+ and P_+ from (85) and (84), we will analyze the function

$$M(\phi(t)) = E_+(\phi(t)) - v P_+(\phi(t)). \tag{117}$$

First, from the local well-posedness of the partial differential equation (1) in the energy space, it is enough to verify Theorem 4 in the case where $(u_1(x), u_2(x))$ is a smooth odd function because the estimate (15) and the density of smooth functions in Sobolev spaces would imply that (15) would be true for any $(u_1(x), u_2(x)) \in H^1_x \times L^2_x$ satisfying the hypothesis of Theorem 4.

Since $P_+(t)$ is not necessarily a conserved quantity, $M(t)$ is not necessarily a constant function given any smooth initial data of $(\phi(0, x), \partial_t \phi(0, x))$ satisfying the hypotheses of Theorem 4.

However, $P_+(t)$ is a nonincreasing function in time, more precisely, for smooth solutions $\phi(t, x)$ of (13), we can verify using integration by parts, from the fact that $\phi(t, x)$ is an odd function in x for any $t \in \mathbb{R}$, the estimate

$$\frac{d}{dt}\left[-\int_0^{+\infty} \partial_t \phi(t, x) \partial_x \phi(t, x)\, dx\right] = \tfrac{1}{2}\phi(t, 0)^2 \geq 0. \tag{118}$$

In conclusion, since it was verified before that $E_+(t)$ is a conserved quantity, we have that

$$M(\phi(t)) \leq M(\phi(0)) \quad \text{for any } t \geq 0,$$

and using Lemma 25, we will verify that $M(\phi(0)) - M(\phi(t))$ satisfies a coercive inequality, from which we will deduce (15).

Proof of Theorem 4. From the observations in Remark 28, it is enough to prove Theorem 4 for the case where $\overrightarrow{\psi_0}(x)$ is a smooth odd function. To simplify our proof, we separate the argument into different steps.

Step 1 (local description of solution $\phi(t,x)$). From the observation of inequality (116) and from the Lemma 25, we can verify the existence of an interval $[0,\epsilon]$ such that if $t\in[0,\epsilon]$, then

$$(\phi(t,x),\partial_t\phi(t,x))=\overrightarrow{H_{-1,0}}((v,y(t)),x)+\overrightarrow{H_{0,1}}((v,y(t)),x)+(\psi_1(t,x),\psi_2(t,x)),\tag{119}$$

with $v(t)$, $y(t)$, $(\psi_1(t,x),\psi_2(t,x))$ satisfying all the conditions of Lemma 25.

Step 2 (estimate of $E_+(\phi(t),\partial_t\phi(t))$ around the kinks). We recall the definition of $E_+(\phi(t),\partial_t\phi(t))$ in (85) given by

$$E_+(\phi(t),\partial_t\phi(t))=\int_0^{+\infty}\frac{\partial_x\phi(t,x)^2+\partial_t\phi(t,x)^2}{2}+U(\phi(t,x))\,dx.$$

Next, we substitute $\phi(t,x)$ and $\partial_t\phi(t,x)$ in the equation above by the formula of $(\phi(t,x),\partial_t\phi(t,x))$ in Step 1. Using (4), (3) and the fact that $y(t)>1$ for $0\leq t\leq\epsilon$, we obtain for all $x\geq 0$ that

$$\left|\frac{\partial^l}{\partial x^l}H_{-1,0}\left(\frac{x+y(t)}{\sqrt{1-v^2}}\right)\right|\lesssim_l(1-v^2)^{-l/2}e^{-\sqrt{2}(y(t)+x)}\quad\text{for any } l\in\mathbb{N}\cup\{0\},\tag{120}$$

from which we also deduce, using Lemma 11, the estimate

$$\int_{\mathbb{R}}H'_{0,1}\left(\frac{x-y(t)}{\sqrt{1-v^2}}\right)H'_{-1,0}\left(\frac{x+y(t)}{\sqrt{1-v^2}}\right)\lesssim(1-v^2)^{1/2}y(t)e^{-2\sqrt{2}y(t)}.\tag{121}$$

In addition, since $\|U^{(l)}\|_{L^\infty[-1,1]}<+\infty$ for any $l\in\mathbb{N}$, we can deduce using Lemma 13 the inequality

$$\left\|U^{(l)}\left(H_{0,1}\left(\frac{x-y(t)}{\sqrt{1-v^2}}\right)+H_{-1,0}\left(\frac{x+y(t)}{\sqrt{1-v^2}}\right)\right)\psi_1(t,x)^l\right\|_{H^1_x}\lesssim_l\|\psi_1(t,x)\|^l_{H^1_x}.$$

In conclusion, since

$$\phi(t,x)=H_{0,1}\left(\frac{x-y(t)}{\sqrt{1-v^2}}\right)+H_{-1,0}\left(\frac{x+y(t)}{\sqrt{1-v^2}}\right)+\psi_1(t,x),\tag{122}$$

$$\partial_t\phi(t,x)=-\frac{v}{\sqrt{1-v^2}}H'_{0,1}\left(\frac{x-y(t)}{\sqrt{1-v^2}}\right)+\frac{v}{\sqrt{1-v^2}}H'_{-1,0}\left(\frac{x+y(t)}{\sqrt{1-v^2}}\right)+\psi_2(t,x),\tag{123}$$

we deduce from the formula (85), estimates (120), (121) and Taylor's expansion theorem that

$$\begin{aligned}
&E_+(\phi(t),\partial_t\phi(t))\\
&\quad=\int_0^{+\infty}\frac{1+v^2}{2(1-v^2)}H'_{0,1}\left(\frac{x-y(t)}{\sqrt{1-v^2}}\right)^2+U\left(H_{0,1}\left(\frac{x-y(t)}{\sqrt{1-v^2}}\right)\right)dx\\
&\qquad-\frac{1}{\sqrt{1-v^2}}\int_0^{+\infty}vH'_{0,1}\left(\frac{x-y(t)}{\sqrt{1-v^2}}\right)\psi_2(t,x)\,dx-H'_{0,1}\left(\frac{x-y(t)}{\sqrt{1-v^2}}\right)\partial_x\psi_1(t,x)\\
&\qquad\quad+\int_0^{+\infty}U'\left(H_{0,1}\left(\frac{x-y(t)}{\sqrt{1-v^2}}\right)\right)\psi_1(t,x)\,dx\\
&\qquad\qquad+\frac{1}{2}\left[\int_0^{+\infty}\partial_x\psi_1(t,x)^2+U''\left(H_{0,1}\left(\frac{x-y(t)}{\sqrt{1-v^2}}\right)\right)\psi_1(t,x)^2+\psi_2(t,x)^2\right]dx\\
&\qquad\qquad\quad+O\left((1-v^2)^{-1/2}y(t)e^{-2\sqrt{2}y(t)}\right)+O\left(\|\overrightarrow{\psi}(t)\|_{H^1_x\times L^2_x}e^{-\sqrt{2}y(t)}+\|\psi_1(t,x)\|^3_{H^1_x(\mathbb{R})}\right),
\end{aligned}\tag{124}$$

while $(\phi(t,x),\partial_t\phi(t,x))$ satisfies identities (122) and (123). Moreover, from (122), we can obtain from (124), while $(\phi(t),\partial_t\phi(t))$ satisfies (122) and (123), that

$$\begin{aligned}
&E_+(\phi(t),\partial_t\phi(t))\\
&=\int_{-\infty}^{+\infty}\frac{1+v^2}{2(1-v^2)}H'_{0,1}\Big(\frac{x-y(t)}{\sqrt{1-v^2}}\Big)^2+U\Big(H_{0,1}\Big(\frac{x-y(t)}{\sqrt{1-v^2}}\Big)\Big)\,dx\\
&\quad-\frac{1}{\sqrt{1-v^2}}\int_{-\infty}^{+\infty}vH'_{0,1}\Big(\frac{x-y(t)}{\sqrt{1-v^2}}\Big)\psi_2(t,x)-H'_{0,1}\Big(\frac{x-y(t)}{\sqrt{1-v^2}}\Big)\partial_x\psi_1(t,x)\\
&\qquad+\int_{-\infty}^{+\infty}U'\Big(H_{0,1}\Big(\frac{x-y(t)}{\sqrt{1-v^2}}\Big)\Big)\psi_1(t,x)\,dx\\
&\qquad\quad+\frac{1}{2}\Big[\int_0^{+\infty}\partial_x\psi_1(t,x)^2+U''\Big(H_{0,1}\Big(\frac{x-y(t)}{\sqrt{1-v^2}}\Big)\Big)\psi_1(t,x)^2+\psi_2(t,x)^2\,dx\Big]\\
&\qquad\qquad+O\big((1-v^2)^{-1/2}y(t)e^{-2\sqrt{2}y(t)}\big)+O\big(\|\overrightarrow{\psi}(t)\|_{H^1_x\times L^2_x}e^{-\sqrt{2}y(t)}+\|\psi_1(t,x)\|^3_{H^1_x(\mathbb{R})}\big),
\end{aligned}\tag{125}$$

We also recall the Bogomolny identity $H'_{0,1}(x)=\sqrt{2U(H_{0,1}(x))}$, from which we deduce with change of variables that

$$\frac{1}{2}\int_{\mathbb{R}}H'_{0,1}\Big(\frac{x}{\sqrt{1-v^2}}\Big)^2\,dx=\int_{\mathbb{R}}U\Big(H_{0,1}\Big(\frac{x}{\sqrt{1-v^2}}\Big)\Big)\,dx=\sqrt{1-v^2}\frac{\|H'_{0,1}\|^2_{L^2_x}}{2}.\tag{126}$$

Step 3 (conclusion of the estimate of $E_+(t)$). Since $\overrightarrow{H_{0,1}}((v,y(t)),x)$ is defined by

$$\overrightarrow{H_{0,1}}((v,y(t)),x)=\begin{bmatrix}H_{0,1}\big(\frac{x-y(t)}{\sqrt{1-v(t)^2}}\big)\\ -\frac{v}{\sqrt{1-v^2}}H'_{0,1}\big(\frac{x-y(t)}{\sqrt{1-v^2}}\big)\end{bmatrix},$$

and we can verify by similar reasoning to (124) the identity

$$E(\overrightarrow{H_{0,1}}((v,y(t)),x))=\int_{-\infty}^{+\infty}\frac{1+v^2}{2(1-v^2)}H'_{0,1}\Big(\frac{x-y(t)}{\sqrt{1-v^2}}\Big)^2+U\Big(H_{0,1}\Big(\frac{x-y(t)}{\sqrt{1-v^2}}\Big)\Big)\,dx.$$

We conclude that $E(\overrightarrow{H_{0,1}}((v,y(t)),x))=(1/\sqrt{1-v^2})\|H'_{0,1}\|^2_{L^2_x}$. In conclusion, using (125), we obtain that

$$\begin{aligned}
&E_+(\phi(t),\partial_t\phi(t))\\
&\quad=\frac{1}{\sqrt{1-v^2}}\|H'_{0,1}\|^2_{L^2_x}-\int_{-\infty}^{+\infty}\frac{v}{\sqrt{1-v^2}}H'_{0,1}\Big(\frac{x-y(t)}{\sqrt{1-v^2}}\Big)\psi_2(t,x)\,dx\\
&\qquad+\int_{-\infty}^{+\infty}\frac{1}{\sqrt{1-v^2}}H'_{0,1}\Big(\frac{x-y(t)}{\sqrt{1-v^2}}\Big)\partial_x\psi_1(t,x)+\int_{-\infty}^{+\infty}U'\Big(H_{0,1}\Big(\frac{x-y(t)}{\sqrt{1-v^2}}\Big)\Big)\psi_1(t,x)\,dx\\
&\qquad\quad+\frac{1}{2}\Big[\int_0^{+\infty}\partial_x\psi_1(t,x)^2+U''\Big(H_{0,1}\Big(\frac{x-y(t)}{\sqrt{1-v^2}}\Big)\Big)\psi_1(t,x)^2+\psi_2(t,x)^2\Big]\\
&\qquad\qquad+O\big((1-v^2)^{-1/2}y(t)e^{-2\sqrt{2}y(t)}+\|(\psi_1(t),\psi_2(t))\|_{H^1_x\times L^2_x}e^{-\sqrt{2}y(t)}\big)+O(\|\psi_1(t)\|^3_{H^1_x(\mathbb{R})}).
\end{aligned}$$

From this using integration by parts we conclude that

$$\begin{aligned}
&E_+(\phi(t), \partial_t\phi(t))\\
&\quad= \frac{1}{\sqrt{1-v^2}}\|H_{0,1}'\|_{L_x^2}^2 + v\langle J \circ C_{v,y(t)}, \overrightarrow{\psi(t)}\rangle\\
&\qquad+\frac{1}{2}\left[\int_0^{+\infty} \psi_2(t,x)^2 + \partial_x\psi_1(t,x)^2 + U''\left(H_{0,1}\left(\frac{x-y(t)}{\sqrt{1-v^2}}\right)\right)\psi_1(t,x)^2\right]\\
&\qquad\quad+O\big((1-v^2)^{-1/2}y(t)e^{-2\sqrt{2}y(t)}\big) + O\big(\|(\psi_1(t),\psi_2(t))\|_{H_x^1\times L_x^2}e^{-\sqrt{2}y(t)} + \|\psi_1(t)\|_{H_x^1}^3\big),
\end{aligned}\tag{127}$$

where the function $C_{v,y}(x)$ is defined in (88).

Step 4 (estimate of $-vP_+(\phi(t), \partial_t\phi(t))$). First, we recall from (84) that $P_+(\phi(t), \partial_t\phi(t))$ is given by

$$P_+(\phi(t), \partial_t\phi(t)) = -\int_0^{+\infty} \partial_t\phi(t,x)\partial_x\phi(t,x)\,dx.$$

Then, while $(\phi(t,x), \partial_t\phi(t,x))$ satisfies the formula

$$(\phi(t,x), \partial_t\phi(t,x)) = \overrightarrow{H_{-1,0}}((v, y(t)), x) + \overrightarrow{H_{0,1}}((v, y(t)), x) + (\psi_1(t,x), \psi_2(t,x)),$$

using the estimates (120) and (121), we obtain by similar reasoning to the estimate of (2.12) of Lemma 2.3 in [Kowalczyk et al. 2021] that

$$\begin{aligned}
-vP_+(\phi(t), \partial_t\phi(t)) = -\frac{v^2}{\sqrt{1-v^2}}\|H_{0,1}'\|_{L_x^2}^2 - v\langle J\circ C_{v,y(t)}, \overrightarrow{\psi}(t)\rangle&\\
+v\int_0^{+\infty}\partial_x\psi_1(t,x)\psi_2(t,x)\,dx + O\left(\frac{v^2}{(1-v^2)}y(t)e^{-2\sqrt{2}y(t)}\right)&\\
+O\left(\frac{v}{\sqrt{1-v^2}}e^{-\sqrt{2}y(t)}\|(\psi_1(t),\psi_2(t))\|_{H_x^1\times L_x^2}\right)&.
\end{aligned}\tag{128}$$

More precisely, the errors in the estimate (128) come from estimate (120) and the Cauchy–Schwarz inequality applied to

$$\int_0^{+\infty}\left|H_{-1,0}'\left(\frac{x+y(t)}{\sqrt{1-v^2}}\right)\right|[|\partial_x\psi_1(t,x)| + |\psi_2(t,x)|]\,dx,$$

from Lemma 11 applied to the integral

$$\int_0^{+\infty} H_{0,1}'\left(\frac{x-y(t)}{\sqrt{1-v^2}}\right)H_{-1,0}'\left(\frac{x+y(t)}{\sqrt{1-v^2}}\right)dx,$$

and from the elementary estimate

$$\int_{-\infty}^0 H_{0,1}'\left(\frac{x-y(t)}{\sqrt{1-v^2}}\right)^2 dx + \int_0^{+\infty} H_{-1,0}'\left(\frac{x+y(t)}{\sqrt{1-v^2}}\right)^2 dx \lesssim e^{-2\sqrt{2}y(t)},$$

which can be obtained from (120).

Step 5 (estimate and monotonicity of $M(\phi(t),\partial_t\phi(t))$). From estimates (127) and (128), we deduce

$$\begin{aligned}
&M(\phi(t),\partial_t\phi(t))\\
&\quad= E_+(\phi(t),\partial_t\phi(t)) - vP_+(\phi(t),\partial_t\phi(t))\\
&\quad= \sqrt{1-v^2}\|H_{0,1}'\|_{L^2_x}^2 + \frac{1}{2}\left[\int_0^{+\infty}\psi_2(t,x)^2 + \partial_x\psi_1(t,x)^2 + U''\left(H_{0,1}\left(\frac{x-y(t)}{\sqrt{1-v(t)^2}}\right)\right)\psi_1(t,x)^2\,dx\right]\\
&\qquad\qquad + O\left(v\|(\psi_1(t),\psi_2(t))\|_{H^1_x\times L^2_x}^2 + \|(\psi_1(t),\psi_2(t))\|_{H^1_x\times L^2_x}e^{-\sqrt{2}y(t)}\right)\\
&\qquad\qquad + O\left(\|\psi_1(t)\|_{H^1_x}^3 + +y(t)e^{-2\sqrt{2}y(t)}\right). \qquad (129)
\end{aligned}$$

Furthermore, using estimate (127) and Lemma 11, we can also verify the estimates

$$\begin{aligned}
E_+\left(\overrightarrow{H_{0,1}}(v,y(t)) + \overrightarrow{H_{-1,0}}(v,y(t))\right) &= \frac{1}{\sqrt{1-v^2}}\|H_{0,1}'\|_{L^2_x}^2 + O(y(t)e^{-2\sqrt{2}y(t)}),\\
P_+\left(\overrightarrow{H_{0,1}}(v,y(t)) + \overrightarrow{H_{-1,0}}(v,y(t))\right) &= \frac{v}{\sqrt{1-v^2}}\|H_{0,1}'\|_{L^2_x}^2 + O(y(t)e^{-2\sqrt{2}y(t)}).
\end{aligned}$$

Therefore, we obtain that

$$M\left(\overrightarrow{H_{0,1}}(v,y(t)) + \overrightarrow{H_{-1,0}}(v,y(t))\right) = \sqrt{1-v^2}\|H_{0,1}'\|_{L^2_x}^2 + O(y(t)e^{-2\sqrt{2}y(t)}), \qquad (130)$$

from which we deduce

$$\begin{aligned}
&M(\phi(t),\partial_t\phi(t))\\
&\quad= M(\overrightarrow{H_{0,1}}(v,y(0)) + \overrightarrow{H_{-1,0}}(v,y(0)))\\
&\qquad+\frac{1}{2}\left[\int_0^{+\infty}\psi_2(t,x)^2 + \partial_x\psi_1(t,x)^2 + U''\left(H_{0,1}\left(\frac{x-y(t)}{\sqrt{1-v^2}}\right)\right)\psi_1(t,x)^2\,dx\right]\\
&\qquad+O\left(\max\{y(t)e^{-2\sqrt{2}y(t)}, y(0)e^{-2\sqrt{2}y(0)}\}\right)+O\left(v\|(\psi_1(t),\psi_2(t))\|_{H^1_x\times L^2_x}^2+\|(\psi_1(t),\psi_2(t))\|_{H^1_x\times L^2_x}^3\right).
\end{aligned}$$

Consequently, since $M(\phi(0),\partial_t\phi(0)) \geq M(\phi(t),\partial_t\phi(t))$ for all $t\geq 0$ and

$$(\phi(0),\partial_t\phi(0)) = \overrightarrow{H_{0,1}}(v,y(0)) + \overrightarrow{H_{-1,0}}(v,y(0)) + (\psi_1(0),\psi_2(0)),$$

we have for every $t\geq 0$ the estimate

$$\begin{aligned}
&\int_0^{+\infty}\psi_2(t,x)^2 + \partial_x\psi_1(t,x)^2 + U''\left(H_{0,1}\left(\frac{x-y(t)}{\sqrt{1-v^2}}\right)\right)\psi_1(t,x)^2\,dx\\
&\qquad\lesssim y(t)e^{-2\sqrt{2}y(t)} + y(0)e^{-2\sqrt{2}y(0)} + v\|(\psi_1(t),\psi_2(t))\|_{H^1_x\times L^2_x}^2 + \|(\psi_1(t),\psi_2(t))\|_{H^1_x\times L^2_x}^3\\
&\qquad\qquad + \|(\psi_1(0),\psi_2(0))\|_{H^1_x\times L^2_x},
\end{aligned}$$

from which with Lemma 34 we deduce for all $t\geq 0$ that

$$\|(\psi_1(t),\psi_2(t))\|_{H^1_x\times L^2_x}^2 \lesssim y(t)e^{-2\sqrt{2}y(t)} + y(0)e^{-2\sqrt{2}y(0)} + \|(\psi_1(0),\psi_2(0))\|_{H^1_x\times L^2_x}, \qquad (131)$$

if $v\ll 1$.

Step 6 (final argument). The last argument is to prove that the set denoted by

$$BO = \left\{t \in \mathbb{R}_{\geq 0} \mid \|(\psi_1(t), \psi_2(t))\|_{H^1_x \times L^2_x} \leq v^{1+\theta/4},\, y(t) \geq y(0) \text{ and (119) is true}\right\} \tag{132}$$

is the proper $\mathbb{R}_{\geq 0}$. From the hypotheses of Theorem 4 and Step 1, we can verify that $0 \in BO$.

Furthermore, from Step 1, we have obtained that there exists $\epsilon > 0$ such that if $0 \leq t \leq \epsilon$, then

$$(\phi(t,x), \partial_t \phi(t,x)) = \overrightarrow{H}_{-1,0}((v, y(t)), x) + \overrightarrow{H}_{0,1}((v, y(t)), x) + (\psi_1(t,x), \psi_2(t,x))$$

and

$$|y(t) - y_0| + \|(\psi_1(t), \psi_2(t))\|_{H^1_x \times L^2_x} \leq 2C \|(u_1, u_2)\|_{H^1_x \times L^2_x}. \tag{133}$$

Since $\|(u_1, u_2)\|_{H^1_x \times L^2_x} \leq v^{2+\theta}$ and Lemma 25 implies the estimate

$$\|(\psi_1(0), \psi_2(0))\|_{H^1_x \times L^2_x} \lesssim \|(u_1, u_2)\|_{H^1_x \times L^2_x},$$

from (133) and Lemma 27, we deduce the existence of a constant $0 < K$ independent of ϵ and v such that $y(t)$ is a function of class C^1 in $[0, \epsilon]$ and for any $t \in [0, \epsilon]$, the inequality

$$|\dot{y}(t) - v| \leq K[\|(\psi_1(t), \psi_2(t))\|_{H^1_x \times L^2_x} + e^{-2\sqrt{2}y(t)}] \tag{134}$$

is true. Therefore,

$$\dot{y}(t) \geq v - K[\|(\psi_1(t), \psi_2(t))\|_{H^1_x \times L^2_x} + e^{-2\sqrt{2}y(t)}] \tag{135}$$

while $t \in [0, \epsilon]$. Moreover, from inequality (133) and the observations done before, to prove that $[0, \epsilon] \subset BO$ it is only needed to verify that $y(t) \geq y(0)$ for all $t \in [0, \epsilon]$.

First, since $y(t)$ is continuous for $t \in [0, \epsilon]$, there exists $\epsilon_2 \in (0, \epsilon)$ such that if $0 \leq t \leq \epsilon_2$, then

$$y(t) \geq \tfrac{3}{4} y(0),$$

so (133), (135) and the estimate $\|(\psi_1(0), \psi_2(0))\|_{H^1_x \times L^2_x} \lesssim \|(u_1, u_2)\|_{H^1_x \times L^2_x} \leq v^{2+\theta}$ imply that if $0 \leq t \leq \epsilon_2$ and $0 < v \ll 1$, then

$$\dot{y}(t) \geq v - v^2 - K e^{-3\sqrt{2}y(0)/2} \geq \tfrac{4}{5} v. \tag{136}$$

In conclusion, estimate (133), the hypothesis of $y_0 \geq 4 \ln \frac{1}{v}$ and inequality (136) imply for $v \ll 1$ that if $0 \leq t \leq \epsilon_2$, then $y(t) \geq y(0) + \frac{4}{5} v t$ and $[0, \epsilon_2] \subset BO$.

If $t \in [\epsilon_2, \epsilon]$, it is not difficult to verify that $y(t) \geq y(0)$ in this region. Indeed, the continuity of the function y would imply otherwise the existence of t_i satisfying $\epsilon_2 < t_i \leq \epsilon$, $y(t_i) = y(0)$ and $y(s) > y(0)$ for any $\epsilon_2 \leq s < t_i$, which implies that estimate (136) is true for $t \in [\epsilon_2, t_1]$. But, repeating the argument above, we would conclude that $y(t_i) \geq y(0) + \frac{4}{5} v t_i$, which is a contradiction. In conclusion, the interval $[0, \epsilon]$ is contained in the set BO.

Similarly, from Lemma 27, we can use inequality (135) to verify that $y(t) \geq y(0) + \frac{4}{5} v t$ always when $[0, t] \subset BO$. Therefore, estimate (131) implies

$$\|(\psi_1(t), \psi_2(t))\|_{H^1_x \times L^2_x(x)} \lesssim \|(u_1, u_2)\|^{1/2}_{H^1_x \times L^2_x} + y(0)^{1/2} e^{-\sqrt{2}y(0)} \ll v^{1+\theta/4} \tag{137}$$

if $[0, t] \in BO$.

In conclusion, $BO = \mathbb{R}_{\geq 0}$ and estimates (134), (137) imply the result of Theorem 4 for all $t \geq 0$. $\square$

6. Proof of Theorem 2

First, from Theorem 1.3 in [Chen and Jendrej 2022], we know for any $0 < v < 1$ that there exist $\delta(v) > 0$, $T(v) > 0$ and a solution $\phi(t, x)$ of (1) with finite energy satisfying the identity

$$\phi(t,x) = H_{0,1}\left(\frac{x - vt}{(1-v^2)^{1/2}}\right) + H_{-1,0}\left(\frac{-x - vt}{(1-v^2)^{1/2}}\right) + \psi(t,x), \tag{138}$$

and the decay estimate

$$\sup_{t\geq T} \|(\psi(t,x), \partial_t\psi(t,x))\|_{H^1_x\times L^2_x} e^{\delta t} < +\infty \tag{139}$$

for any $T \geq T(v)$ and $\delta \leq \delta(v)$. Moreover, we can find $\delta(v), T(v) > 0$ such that

$$\sup_{t\geq T(v)} \|(\psi(t,x), \partial_t\psi(t,x))\|_{H^1_x\times L^2_x} e^{\delta(v)t} < 1. \tag{140}$$

Indeed, in [Chen and Jendrej 2022] it was proved using fixed point theorem that for any $0 < v < 1$ there is a unique solution of (1) that satisfies (139) for some $T, \delta > 0$.

Next, if we restrict the argument of the proof of Proposition 3.6 of [Chen and Jendrej 2022] to the traveling kink-kink of the ϕ^6 model, we can find explicitly the values of $\delta(v)$ and $T(v)$. More precisely, we have:

Theorem 29. *There is* $\delta_0 > 0$ *such that if* $0 < v < \delta_0$ *there exists a unique solution* $\phi(t,x)$ *of* (1) *with*

$$h(t,x) = \phi(t,x) - H_{0,1}\left(\frac{x - vt}{(1-v^2)^{1/2}}\right) - H_{-1,0}\left(\frac{x + vt}{(1-v^2)^{1/2}}\right),$$

satisfying (139) *for some* $0 < \delta < 1$ *and* $T > 0$. *Furthermore, we have if*

$$t \geq \frac{4\ln(1/v)}{v}$$

that

$$\|(h(t,x), \partial_t h(t,x))\|_{H^1_x\times L^2_x} \leq e^{-vt}. \tag{141}$$

This solution is also an odd function on x.

Proof. See Appendix B. ☐

Finally, we have obtained all the framework necessary to start the demonstration of Theorem 2.

Proof of Theorem 2. First, from Theorem 29, for any $k \in \mathbb{N}$ bigger than 2 and $0 < v \leq \delta_0$, we have that the traveling kink-kink with speed v satisfies for

$$T_{0,k} = \frac{32k\ln(1/v^2)}{2\sqrt{2}v}$$

the estimate

$$\|(h(T_{0,k}), \partial_t h(T_{0,k}))\|_{H^1_x\times L^2_x} \leq v^{16\sqrt{2}k} \tag{142}$$

for $h(t,x)$ the function denoted in Theorem 29. Now, we start the proof of the second item of Theorem 2.

Step 1 (proof of second item of Theorem 2). First, in the notation of Theorem 8, we consider

$$\phi_k(v,t,x) = \varphi_{k,v}(t, x + \tau_{k,v}).$$

For the $T_{0,k}$ given before, we can verify using Theorems 7 and 8 that

$$\begin{aligned}\Big\|\phi_k(v,T_{0,k},x)-H_{0,1}\Big(\frac{x-vT_{0,k}}{\sqrt{1-v^2}}\Big)-H_{-1,0}\Big(\frac{x+vT_{0,k}}{\sqrt{1-v^2}}\Big)\Big\|_{H^1_x}\\+\Big\|\partial_t\phi_k(v,T_{0,k},x)+\frac{v}{\sqrt{1-v^2}}H'_{0,1}\Big(\frac{x-vT_{0,k}}{\sqrt{1-v^2}}\Big)-\frac{v}{\sqrt{1-v^2}}H'_{-1,0}\Big(\frac{x+vT_{0,k}}{\sqrt{1-v^2}}\Big)\Big\|_{H^1_x}\leq v^{15k}.\end{aligned}$$

In conclusion, Theorem 15 and Remark 22 imply that there is $\Delta_{k,\theta}>0$ such that if also $v<\Delta_{k,\theta}$, then

$$\|(\phi(t,x),\partial_t\phi(t,x))-(\phi_k(v,t,x),\partial_t\phi_k(v,t,x))\|_{H^1_x\times L^2_x}<v^{2k-1/2},$$

while

$$|t-T_{0,k}|<\frac{\ln(1/v)^{2-\theta/2}}{v}.$$

Also, Theorems 7 and 8 implies that if $v\ll 1$ and

$$-4\frac{\ln(1/v)^{2-\theta}}{v}\leq t\leq-\frac{\ln(1/v)^{2-\theta}}{v},$$

then there exist $e_{k,v}$ satisfying

$$\Big|e_{v,k}-\frac{1}{\sqrt{2}}\ln\Big(\frac{8}{v^2}\Big)\Big|\ll 1$$

such that

$$\begin{aligned}\Big\|\phi_k(v,t,x)-H_{0,1}\Big(\frac{x-e_{k,v}+vt}{\sqrt{1-v^2}}\Big)-H_{-1,0}\Big(\frac{x+e_{k,v}-vt}{\sqrt{1-v^2}}\Big)\Big\|_{H^1_x}\\+\Big\|\partial_t\phi_k(v,t,x)-\frac{v}{\sqrt{1-v^2}}H'_{0,1}\Big(\frac{x-e_{k,v}+vt}{\sqrt{1-v^2}}\Big)+\frac{v}{\sqrt{1-v^2}}H'_{-1,0}\Big(\frac{x+e_{k,v}-vt}{\sqrt{1-v^2}}\Big)\Big\|_{H^1_x}\ll v^{2k-1/2}.\end{aligned}\tag{143}$$

In conclusion, the second item of Theorem 2 follows from the observation above and Remark 22.

Step 2 (proof of first item of Theorem 2). From Step 1, for

$$t_0=-\frac{\ln(1/v)^{2-\theta}}{v},$$

we obtained that $\phi(t_0,x)$ satisfies (143). Next, we will study the behavior of $\phi(t,x)$ for $t\leq t_0$, which is equivalent to studying the function $\phi_1(t,x)=\phi(-(t+t_0),x)$ for $t\geq 0$.

However, from the estimate (143), we can verify that $(\phi_1(0,x),\partial_t\phi_1(0,x))$ satisfies the hypotheses of Theorem 4, if we consider $y_0=e_{k,v}-vt_0$ and $0<v<\ll 1$. Therefore, using the result of Theorem 4 and the identity $\phi_1(t,x)=\phi(-(t+t_0),x)$, we obtain the first item of Theorem 2. □

Appendix A: Auxiliary estimates

In this appendix, we complement our article by demonstrating complementary estimates.

Lemma 30. *For*

$$\mathcal{G}(x)=e^{-\sqrt{2}x}-\frac{e^{-\sqrt{2}x}}{(1+e^{2\sqrt{2}x})^{3/2}}+x\frac{e^{\sqrt{2}x}}{(1+e^{2\sqrt{2}x})^{3/2}}+k_1\frac{e^{\sqrt{2}x}}{(1+e^{2\sqrt{2}x})^{3/2}},$$

we have that

$$\int_{\mathbb{R}} U^{(3)}(H_{0,1}(x))H_{0,1}'(x)^2\mathcal{G}(x)\,dx = \int_{\mathbb{R}} U^{(3)}(H_{0,1}(x))H_{0,1}'(x)^2 e^{-\sqrt{2}x}\,dx - \sqrt{2}\int_{\mathbb{R}}[U''(H_{0,1}(x))-2]H_{0,1}'(x)e^{-\sqrt{2}x}\,dx.$$

Remark 31. Indeed, the value k_1 in Lemma 30 can be replaced by zero, since

$$\int_{\mathbb{R}} U^{(3)}(H_{0,1}(x))H_{0,1}'(x)^3\,dx = 0.$$

Proof of Lemma 30. First, from identity $H_{0,1}''(x) = U'(H_{0,1}(x))$ and integration by parts, we can verify the identity

$$\int_{\mathbb{R}} U^{(3)}(H_{0,1}(x))H_{0,1}'(x)^2\mathcal{G}(x)\,dx = \int_{\mathbb{R}} U'(H_{0,1}(x))[\mathcal{G}''(x) - U''(H_{0,1})\mathcal{G}(x)]\,dx.$$

Also, since $-\mathcal{G}''(x)+U''(H_{0,1}(x))\mathcal{G}(x) = [U''(H_{0,1}(x))-2]e^{-\sqrt{2}x} + 8\sqrt{2}H_{0,1}'(x)$ and $\langle H_{0,1}', U'(H_{0,1})\rangle = 0$, we conclude using integration by parts that

$$\begin{aligned}\int_{\mathbb{R}} U^{(3)}(H_{0,1}(x))H_{0,1}'(x)^2\mathcal{G}(x)\,dx &= -\int_{\mathbb{R}} U'(H_{0,1}(x))[U''(H_{0,1}(x))-2]e^{-\sqrt{2}x}\,dx\\ &= -\int_{\mathbb{R}} H_{0,1}''(x)[U''(H_{0,1}(x))-2]e^{-\sqrt{2}x}\,dx,\\ &= \int_{\mathbb{R}} U^{(3)}(H_{0,1}(x))H_{0,1}'(x)^2 e^{-\sqrt{2}x}\,dx - \sqrt{2}\int_{\mathbb{R}}[U''(H_{0,1}(x))-2]H_{0,1}'(x)e^{-\sqrt{2}x}\,dx. \quad \square\end{aligned}$$

Now, using integration by parts and identity (27) of [Moutinho 2023], we have that

$$-\int_{\mathbb{R}}[U''(H_{0,1}(x))-2]e^{-\sqrt{2}x}H_{0,1}'(x)\,dx = -\sqrt{2}\int_{\mathbb{R}}[6H_{0,1}(x)^5 - 8H_{0,1}(x)^3]e^{-\sqrt{2}x}\,dx = 4, \tag{144}$$

from which we deduce the following lemma.

Lemma 32. $\displaystyle\int_{\mathbb{R}} U^{(3)}(H_{0,1}(x))H_{0,1}'(x)^2\mathcal{G}(x)\,dx - \int_{\mathbb{R}} U^{(3)}(H_{0,1}(x))H_{0,1}'(x)^2 e^{-\sqrt{2}x}\,dx = 4\sqrt{2}.$

Lemma 33. *There is* $\delta > 0$, $c > 0$ *such that if*

$$0 < v < \delta, \quad d(t) = \frac{1}{\sqrt{2}}\ln\biggl(\frac{8}{v^2}\cosh(\sqrt{2}vt)^2\biggr),$$

then for

$$H_{0,1}^{+}(x,t) = H_{0,1}\biggl(\frac{x-\frac{1}{2}d(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}\biggr), \quad H_{0,1}^{-}(x,t) = H_{-1,0}\biggl(\frac{x+\frac{1}{2}d(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}\biggr),$$

and any $g \in H_x^1(\mathbb{R})$ *such that*

$$\langle g(x), \partial_x H_{0,1}^{+}(x,t)\rangle = 0, \quad \langle g(x), \partial_x H_{0,1}^{-}(x,t)\rangle = 0,$$

we have

$$c\|g\|_{H_x^1}^2 \le \bigl\langle -\partial_x^2 g(x) + U''(H_{0,1}^{+}(x,t) + H_{0,1}^{-}(x,t))g(x), g(x)\bigr\rangle. \tag{145}$$

Proof of Lemma 33. First, to simplify our computations we let

$$\gamma_{d(t)} = \frac{1}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}.$$

Next, we can verify using a change of variables that

$$\langle U''(H^{+}_{0,1}(x,t))g(x), g(x)\rangle = \sqrt{1-\tfrac{1}{4}\dot{d}(t)^2}\int_{\mathbb{R}} U''(H_{0,1}(y))\big[g\big(\big(y+\tfrac{1}{2}d(t)\gamma_{d(t)}\big)\gamma_{d(t)}^{-1}\big)\big]^2\,dy,$$

and

$$\int_{\mathbb{R}} \frac{dg(x)}{dx}^2\,dx = \frac{1}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}\int_{\mathbb{R}}\left[\frac{d}{dy}[g(y\gamma_{d(t)}^{-1})]\right]^2 dy. \tag{146}$$

We now let

$$g_1(t,y) = g\big(y\sqrt{1-\tfrac{1}{4}\dot{d}(t)^2}\big) = g(y\gamma_{d(t)}^{-1}).$$

Moreover, $L = -\partial_x^2 + U''(H_{0,1}(x))$ is a positive operator in $L^2(\mathbb{R})$ when it is restricted to the orthogonal complement of $H'_{0,1}(x)$ in $L^2_x(\mathbb{R})$; see [Jendrej et al. 2022] *or* [Moutinho 2023] *for the proof. In conclusion, we deduce that there is a constant $C > 0$ independent of $v > 0$ such that*

$$\left\langle -\frac{d^2}{dx^2}g(x) + U''(H^{+}_{0,1}(x,t))g(x), g(x)\right\rangle \geq C\sqrt{1-\tfrac{1}{4}\dot{d}(t)^2}\,\|g_1(t,y)\|^2_{H^1_y(\mathbb{R})}, \tag{147}$$

so, from $\dot{d}(t) = v\tanh(\sqrt{2}vt)$ and identity (146), *we deduce that there is a constant $C_1 > 0$ such that if $v \ll 1$, then*

$$\left\langle -\frac{d^2}{dx^2}g(x) + U''(H^{+}_{0,1}(x,t))g(x), g(x)\right\rangle \geq C_1\|g(x)\|^2_{H^1(\mathbb{R})}. \tag{148}$$

Similarly, we can verify for the same constant $C_1 > 0$ that if $\langle g(x), \partial_x H^{-}_{-1,0}(x,t)\rangle_{L^2_x} = 0$ and $v \ll 1$, then

$$\left\langle -\frac{d^2}{dx^2}g(x) + U''(H^{-}_{0,1}(x,t))g(x), g(x)\right\rangle \geq C_1\|g(x)\|^2_{H^1(\mathbb{R})}. \tag{149}$$

The remaining part of the proof proceeds exactly as the proof of Lemma 2.6 *of* [Moutinho 2023]. ☐

Lemma 34. *There exist $C > 1$, $c > 0\delta > 0$ such that if $0 < v < \delta$, then for any $(\varphi_1, \varphi_2) \in H^1_x(\mathbb{R}) \times L^2_x(\mathbb{R})$ we have that*

$$\int_{\mathbb{R}} \varphi_2^2 + \partial_x\varphi_1^2 + U''\left(H_{0,1}\left(\frac{x}{\sqrt{1-v^2}}\right)\right)\varphi_1(x)^2\,dx \geq c\|(\varphi_1,\varphi_2)\|^2_{H^1_x\times L^2_x} - C\langle(\varphi_1,\varphi_2), JD_{v,0}(x)\rangle^2.$$

Proof. The proof is completely analogous to that of property (2) of [Kowalczyk et al. 2021, Lemma 2.8]. ☐

Appendix B: Proof of Theorem 29

We start by letting

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

and we consider for $x \in \mathbb{R}$ and $-1 < v < 1$ the functions

$$\psi^{0}_{-1,0}(x, v) = J \begin{bmatrix} H'_{-1,0}\Big(\frac{x}{\sqrt{1-v^2}}\Big) \\ \frac{v}{1-v^2} H^{(2)}_{-1,0}\Big(\frac{x}{\sqrt{1-v^2}}\Big) \end{bmatrix}, \tag{150}$$

$$\psi^{1}_{-1,0}(x, v) = J \begin{bmatrix} vxH'_{-1,0}\Big(\frac{x}{\sqrt{1-v^2}}\Big) \\ \frac{1}{\sqrt{1-v^2}} H'_{-1,0}\Big(\frac{x}{\sqrt{1-v^2}}\Big) + \frac{v^2 x}{1-v^2} H^{(2)}_{-1,0}\Big(\frac{x}{\sqrt{1-v^2}}\Big) \end{bmatrix}, \tag{151}$$

and we write, for $j \in \{0, 1\}$, $\psi^{j}_{0,1}(x, v) = \psi^{j}_{-1,0}(-x, -v)$.

Next, we will use Lemma 2.6 of [Chen and Jendrej 2022].

Lemma 35. *The functions*

$$Y^{0}_{-1,0}(v; x, t) = -J\psi^{0}_{-1,0}(x + vt, v), \tag{152}$$

$$Y^{1}_{-1,0}(v; x, t) = -J\psi^{1}_{-1,0}(x + vt, v) + t\sqrt{1 - v^2}\,Y^{0}_{-1,0}(v; x + vt, t) \tag{153}$$

are solutions of the linear differential system

$$\frac{d}{dt}\begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} = J \begin{bmatrix} -\frac{\partial^2}{\partial x^2} + U''\Big(H_{-1,0}\Big(\frac{x+vt}{\sqrt{1-v^2}}\Big)\Big) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}, \tag{154}$$

and the functions

$$Y^{0}_{0,1}(v; x, t) = -J\psi^{0}_{0,1}(x - vt, v), \tag{155}$$

$$Y^{1}_{0,1}(v; x, t) = -J\psi^{1}_{0,1}(x - vt, v) + t\sqrt{1 - v^2}\,Y^{0}_{0,1}(v; x - vt, t) \tag{156}$$

are solutions of the linear differential system

$$\frac{d}{dt}\begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} = J \begin{bmatrix} -\frac{\partial^2}{\partial x^2} + U''\Big(H_{0,1}\Big(\frac{x-vt}{\sqrt{1-v^2}}\Big)\Big) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}. \tag{157}$$

Now, similarly to [Chen and Jendrej 2022], we consider the linear operator $L_{+,-}(v, t)$ defined by

$$L_{+,-}(v, t) = \begin{bmatrix} -\frac{\partial^2}{\partial x^2} + U''\Big(H_{0,1}\Big(\frac{x-vt}{\sqrt{1-v^2}}\Big) + H_{-1,0}\Big(\frac{x+vt}{\sqrt{1-v^2}}\Big)\Big) & 0 \\ 0 & 1 \end{bmatrix}. \tag{158}$$

We recall that

$$H_{0,1}(x) = \frac{e^{\sqrt{2}x}}{\sqrt{1 + e^{2\sqrt{2}x}}}, \quad \text{and} \quad \left|\frac{d^l}{dx^l} H_{0,1}(x)\right| \lesssim \min(e^{\sqrt{2}x}, e^{-2\sqrt{2}x}) \quad \text{for any } l \in \mathbb{N}.$$

From now on, we let $\psi^{j}_{-1,0}(v; t, x) = \psi^{j}_{-1,0}(x + vt, v)$ and $\psi^{j}_{0,1}(v; t, x) = \psi^{j}_{-1,0}(x - vt, v)$ for any $j \in \{0, 1\}$. Furthermore, using Lemma 11, we can verify similarly to the proof of Proposition 2.8 of [Chen and Jendrej 2022] the following result.

Lemma 36. *There exists $C > 0$, such that for any $0 < v < 1$, we have for all $t \in \mathbb{R}_{\geq 1}$ that*

$$\left\| \frac{\partial}{\partial t}\psi^0_{0,1}(v;t,x) - L_{+,-}J\psi^0_{0,1}(v;t,x)\right\|_{L^2_x} \leq C\exp\left(\frac{-2\sqrt{2}v|t|}{\sqrt{1-v^2}}\right),$$

$$\left\| \frac{\partial}{\partial t}\psi^0_{-1,0}(v;t,x) - L_{+,-}J\psi^0_{-1,0}(v;t,x)\right\|_{L^2_x} \leq C\exp\left(\frac{-2\sqrt{2}v|t|}{\sqrt{1-v^2}}\right),$$

$$\left\| \frac{\partial}{\partial t}\psi^1_{0,1}(v;t,x) - L_{+,-}J\psi^1_{0,1}(v;t,x) + \sqrt{1-v^2}\psi^0_{0,1}(v;t,x)\right\|_{L^2_x} \leq C(|t|v+1)v\exp\left(\frac{-2\sqrt{2}v|t|}{\sqrt{1-v^2}}\right),$$

$$\left\| \frac{\partial}{\partial t}\psi^1_{-1,0}(v;t,x) - L_{+,-}J\psi^1_{-1,0}(v;t,x) + \sqrt{1-v^2}\psi^0_{-1,0}(v;t,x)\right\|_{L^2_x} \leq C(|t|v+1)v\exp\left(\frac{-2\sqrt{2}v|t|}{\sqrt{1-v^2}}\right).$$

Next, we consider a smooth cut function $0 \leq \chi(x) \leq 1$ that satisfies

$$\chi(x) = \begin{cases} 1 & \text{if } x \leq 2(1-10^{-3}), \\ 0 & \text{if } x \geq 2. \end{cases}$$

From now on, for each $0 < v < 1$, we consider $p(v) = \frac{1}{2}v(1-10^{-3})$ and we also let

$$\chi_1(v;t,x) = \chi\left(\frac{x+vt}{p(v)t}\right), \quad \chi_2(v;t,x) = 1-\chi\left(\frac{x+vt}{p(v)t}\right).$$

Lemma 37. *There is $c, \delta_0 > 0$ such that if $0 < v < \delta_0$, then*

$$Q(t,r) = \frac{1}{2}\left[\int_{\mathbb{R}} \partial_t r(t,x)^2 + \partial_x r(t,x)^2 + U''\left(H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right) + H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right)r(t,x)^2\,dx\right] + \sum_{j=1}^{2} v\int_{\mathbb{R}} \chi_j(v;t,x)(-1)^j\partial_t r(t,x)\partial_x r(t,x)\,dx$$

satisfies, for any $t \geq \frac{\ln(1/v)}{v}$,

$$Q(t,r) \geq c\|\vec{r}(t)\|^2_{H^1_x\times L^2_x} - \frac{1}{c}\left[\sum_{j=0}^{1}\langle\vec{r}(t),\psi^j_{-1,0}(v;t)\rangle^2 + \langle\vec{r}(t),\psi^j_{0,1}(v;t)\rangle^2\right].$$

Proof. From the definitions of $\psi^1_{-1,0}$ and $\psi^1_{0,1}$, we can verify that there is a constant $C > 0$ such that if $v \ll 1$, then

$$\left|\left\langle r(t), H'_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right)\right\rangle^2\right| \leq C\big[\langle(r(t),\partial_t r(t)),\psi^1_{0,1}(v;t)\rangle^2 + v^2\|(r(t),\partial_t r(t))\|^2_{H^1_x\times L^2_x}\big], \tag{159}$$

$$\left|\left\langle r(t), \dot{H}_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right\rangle^2\right| \leq C\big[\langle(r(t),\partial_t r(t)),\psi^1_{-1,0}(v;t)\rangle^2 + v^2\|(r(t),\partial_t r(t))\|^2_{H^1_x\times L^2_x}\big]. \tag{160}$$

Then, using the estimates (159) and (160), the proof of Lemma 37 is analogous to the demonstration of Lemma 2.3 of [Jendrej et al. 2022] or the proof of Lemma 2.5 in [Moutinho 2023] or the demonstration of Lemma 33 in Appendix A ☐

Remark 38. Proposition 2.10 of [Chen and Jendrej 2022] implies that for any $0 < v < 1$, there is T_v and c_v such that Lemma 37 holds with c_v in the place of c for all $t \geq T_v$.

Lemma 39. *There exists $C > 0$ such that, for any $0 < v < 1$, if $f(t,x) \in L^{\infty}_t(\mathbb{R}; H^1_x(\mathbb{R}))$ and $h(t,x) \in L^{\infty}_t(\mathbb{R}_{\geq 1}; H^1_x(\mathbb{R})) \cap C^1_t(\mathbb{R}_{\geq 1}; L^2_x(\mathbb{R}))$ is a solution of the integral equation associated to the partial differential equation*

$$\partial_t^2 h(t,x) - \partial_x^2 h(t,x) + U''\biggl(H_{0,1}\Bigl(\frac{x-vt}{\sqrt{1-v^2}}\Bigr) + H_{-1,0}\Bigl(\frac{x+vt}{\sqrt{1-v^2}}\Bigr)\biggr)h(t,x) = f(t,x),$$

for some boundary condition $(h(t_0), \partial_t h(t_0)) \in H^1_x(\mathbb{R}) \times L^2_x(\mathbb{R})$, then

$$Q(t,h) = \frac{1}{2}\biggl[\int_{\mathbb{R}} \partial_t h(t,x)^2 + \partial_x h(t,x)^2 + U''\biggl(H_{0,1}\Bigl(\frac{x-vt}{\sqrt{1-v^2}}\Bigr) + H_{-1,0}\Bigl(\frac{x+vt}{\sqrt{1-v^2}}\Bigr)\biggr)h(t,x)^2\,dx\biggr] \\ + \sum_{j=1}^{2} v \int_{\mathbb{R}} \chi_j(v;t,x)(-1)^j \partial_t h(t,x)\partial_x h(t,x)\,dx$$

satisfies

$$\left|\frac{\partial}{\partial t} Q(t,h)\right| \\ \leq C\biggl[\|f(t)\|_{L^2_x}\|(h(t),\partial_t h(t))\|_{H^1_x \times L^2_x} + \|(h(t),\partial_t h(t))\|^2_{H^1_x \times L^2_x}\biggl(v\exp\Bigl(\frac{-\sqrt{2}vt(1-10^{-3})^2}{\sqrt{1-v^2}}\Bigr) + \frac{1}{t}\biggr)\biggr]$$

for all $t \geq 1$.

Proof. First, from the equation satisfied by $h(t,x)$, we obtain that

$$\int_{\mathbb{R}}\biggl[\partial_t^2 h(t,x) - \partial_x^2 h(t,x) + U''\biggl(H_{0,1}\Bigl(\frac{x-vt}{\sqrt{1-v^2}}\Bigr) + H_{-1,0}\Bigl(\frac{x+vt}{\sqrt{1-v^2}}\Bigr)\biggr)h(t,x)^2\biggr]\partial_t h(t,x)\,dx \\ = \int_{\mathbb{R}} f(t,x)\partial_t h(t,x)\,dx. \quad (161)$$

As a consequence, we deduce by integration by parts that

$$\frac{d}{dt}\biggl[\int_{\mathbb{R}} \partial_t h(t)^2 + \partial_x h(t)^2 + U''\biggl(H_{0,1}\Bigl(\frac{x-vt}{\sqrt{1-v^2}}\Bigr) + H_{-1,0}\Bigl(\frac{x+vt}{\sqrt{1-v^2}}\Bigr)\biggr)h(t)^2\,dx\biggr] \\ = -\frac{v}{\sqrt{1-v^2}}\int_{\mathbb{R}} U^{(3)}\biggl(H_{0,1}\Bigl(\frac{x-vt}{\sqrt{1-v^2}}\Bigr) + H_{-1,0}\Bigl(\frac{x+vt}{\sqrt{1-v^2}}\Bigr)\biggr)H'_{0,1}\Bigl(\frac{x-vt}{\sqrt{1-v^2}}\Bigr)h(t)^2\,dx \\ + \frac{v}{\sqrt{1-v^2}}\int_{\mathbb{R}} U^{(3)}\biggl(H_{0,1}\Bigl(\frac{x-vt}{\sqrt{1-v^2}}\Bigr) + H_{-1,0}\Bigl(\frac{x+vt}{\sqrt{1-v^2}}\Bigr)\biggr)H'_{-1,0}\Bigl(\frac{x+vt}{\sqrt{1-v^2}}\Bigr)h(t)^2\,dx \\ + 2\int_{\mathbb{R}} f(t,x)h(t,x)\,dx. \quad (162)$$

Next, from the definition of $\chi_1(v;t,x)$ and $\chi_2(v;t,x)$, we can verify for each $j \in \{1,2\}$ that

$$\begin{aligned}\frac{d}{dt}\Big[v\int_{\mathbb{R}}\chi_j(v;t,x)(-1)^j\partial_t h(t,x)\partial_x h(t,x)\,dx\Big]\\ &= v\int_{\mathbb{R}}\chi_j(v;t,x)(-1)^j\partial_t^2 h(t,x)\partial_x h(t,x)\,dx + v\int_{\mathbb{R}}\chi_j(v;t,x)(-1)^j\partial_t h(t,x)\partial^2_{t,x}h(t,x)\,dx\\ &\quad + O\Big(\|\dot\chi\|_{L^\infty_x(\mathbb{R})}\frac{v}{t}\|(h(t),\partial_t h(t))\|^2_{H^1_x\times L^2_x}\Big),\end{aligned}$$

from which we deduce using integration by parts that

$$\begin{aligned}\frac{d}{dt}\Big[v\int_{\mathbb{R}}\chi_j(v;t,x)(-1)^j\partial_t h(t,x)\partial_x h(t,x)\,dx\Big]\\ &= v\int_{\mathbb{R}}\chi_j(v;t,x)(-1)^j\partial_t^2 h(t,x)\partial_x r(t,x)\,dx + O\Big(\|\dot\chi\|_{L^\infty_x(\mathbb{R})}\frac{1}{t}\|(h(t),\partial_t h(t))\|^2_{H^1_x\times L^2_x}\Big).\end{aligned} \tag{163}$$

From the equation satisfied by $h(t,x)$, we have that

$$\begin{aligned}v\int_{\mathbb{R}}\chi_j(v;t,x)(-1)^j\partial_t^2 h(t,x)\partial_x h(t,x)\,dx\\ &= v\int_{\mathbb{R}}\chi_j(v;t,x)(-1)^j f(t,x)\partial_x h(t,x)\,dx + v\int_{\mathbb{R}}\chi_j(v;t,x)(-1)^j\partial_x^2 h(t,x)\partial_x h(t,x)\,dx\\ &\quad - v\int_{\mathbb{R}}\chi_j(v;t,x)(-1)^j U''\Big(H_{0,1}\Big(\frac{x-vt}{\sqrt{1-v^2}}\Big)+H_{-1,0}\Big(\frac{x+vt}{\sqrt{1-v^2}}\Big)\Big)h(t,x)\partial_x h(t,x)\,dx.\end{aligned}$$

So, using integration by parts, we obtain for any $j \in \{1,2\}$ that

$$\begin{aligned}2\sqrt{1-v^2}\int_{\mathbb{R}}\chi_j(v;t,x)\partial_t^2 h(t,x)\partial_x h(t,x)\,dx\\ &= \int_{\mathbb{R}}\chi_j(v;t,x)U^{(3)}\Big(H_{0,1}\Big(\frac{x-vt}{\sqrt{1-v^2}}\Big)+H_{-1,0}\Big(\frac{x+vt}{\sqrt{1-v^2}}\Big)\Big)H'_{0,1}\Big(\frac{x-vt}{\sqrt{1-v^2}}\Big)h(t,x)^2\,dx\\ &\quad + \int_{\mathbb{R}}\chi_j(v;t,x)U^{(3)}\Big(H_{0,1}\Big(\frac{x-vt}{\sqrt{1-v^2}}\Big)+H_{-1,0}\Big(\frac{x+vt}{\sqrt{1-v^2}}\Big)\Big)H'_{-1,0}\Big(\frac{x+vt}{\sqrt{1-v^2}}\Big)h(t,x)^2\,dx\\ &\quad + O\Big(\|\chi'\|_{L^\infty_x(\mathbb{R})}\frac{1}{vt}\|(h(t),\partial_t h(t))\|^2_{H^1_x\times L^2_x} + \|f(t)\|_{L^2_x}\|(h(t),\partial_t h(t))\|_{H^1_x\times L^2_x}\Big).\end{aligned}$$

From the definitions of $\chi_1(v;t,x)$ and $\chi_2(v;t,x)$, we can verify for all $t > 1$ that

$$\begin{aligned}H'_{0,1}\Big(\frac{x-vt}{\sqrt{1-v^2}}\Big)\chi_1(v;t,x) &< \sqrt{2}\exp\Big(-\frac{\sqrt{2}vt(1+2\times10^{-3})}{\sqrt{1-v^2}}\Big),\\ \dot H_{-1,0}\Big(\frac{x+vt}{\sqrt{1-v^2}}\Big)\chi_2(v;t,x) &< \sqrt{2}\exp\Big(-\frac{\sqrt{2}vt(1-10^{-3})^2}{\sqrt{1-v^2}}\Big).\end{aligned}$$

In conclusion, we obtain that

$$\begin{aligned}\sum_{j=1}^{2} v\int_{\mathbb{R}} &\chi_j(v;t,x)(-1)^j \partial_t^2 h(t,x)\partial_x h(t,x)\,dx\\ =&\frac{v}{2\sqrt{1-v^2}}\int_{\mathbb{R}} U^{(3)}\Big(H_{0,1}\Big(\frac{x-vt}{\sqrt{1-v^2}}\Big)+H_{-1,0}\Big(\frac{x+vt}{\sqrt{1-v^2}}\Big)\Big)H_{0,1}'\Big(\frac{x-vt}{\sqrt{1-v^2}}\Big)h(t,x)^2\,dx\\ &-\frac{v}{2\sqrt{1-v^2}}\int_{\mathbb{R}} U^{(3)}\Big(H_{0,1}\Big(\frac{x-vt}{\sqrt{1-v^2}}\Big)+H_{-1,0}\Big(\frac{x+vt}{\sqrt{1-v^2}}\Big)\Big)H_{-1,0}'\Big(\frac{x+vt}{\sqrt{1-v^2}}\Big)h(t,x)^2\,dx\\ &\qquad+O\Big(\|\dot{\chi}\|_{L^\infty_x(\mathbb{R})}\frac{1}{t}\|(h(t),\partial_t h(t))\|^2_{H^1_x\times L^2_x}+v\|f(t)\|_{L^2_x}\|(h(t),\partial_t h(t))\|_{H^1_x\times L^2_x}\Big)\\ &\qquad+O\Big(v\exp\Big(-\frac{\sqrt{2}vt(1-10^{-3})^2}{(1-v^2)^{1/2}}\Big)\|h(t,x)\|^2_{H^1_x(\mathbb{R})}\Big).\end{aligned}\tag{164}$$

So, using estimate (164), Lemma 39 will follow from the sum of (162) and (163). ☐

Lemma 40. *There is* $C>0$, *such that, for any* $0<v<1$, *if* $f(t,x)\in L^\infty_t(\mathbb{R};H^1_x(\mathbb{R}))$ *and* $h(t,x)\in L^\infty_t(\mathbb{R}_{\geq 1};H^1_x(\mathbb{R}))\cap C^1_t(\mathbb{R}_{\geq 1};L^2_x(\mathbb{R}))$ *is a solution of the integral equation associated to the partial differential equation*

$$\partial_t^2 h(t,x)-\partial_x^2 h(t,x)+U''\Big(H_{0,1}\Big(\frac{x-vt}{\sqrt{1-v^2}}\Big)+H_{-1,0}\Big(\frac{x+vt}{\sqrt{1-v^2}}\Big)\Big)h(t,x)=f(t,x)$$

for some boundary condition $(h(t_0),\partial_t h(t_0))\in H^1_x(\mathbb{R})\times L^2_x(\mathbb{R})$, *then for* $\vec{h}(t)=(h(t,x),\partial_t h(t,x))$ *we have*

$$\Big|\frac{d}{dt}\langle\vec{h}(t),\psi^0_{-1,0}(v;t)\rangle\Big|\leq C\Big[\|f(t)\|_{L^2_x(\mathbb{R})}+\|\overrightarrow{h(t)}\|_{H^1_x(\mathbb{R})\times L^2_x(\mathbb{R})}\exp\Big(\frac{-2\sqrt{2}vt}{(1-v^2)^{1/2}}\Big)\Big],$$

$$\Big|\frac{d}{dt}\langle\vec{h}(t),\psi^0_{0,1}(v;t)\rangle\Big|\leq C\Big[\|f(t)\|_{L^2_x(\mathbb{R})}+\|\overrightarrow{h(t)}\|_{H^1_x(\mathbb{R})\times L^2_x(\mathbb{R})}\exp\Big(\frac{-2\sqrt{2}vt}{(1-v^2)^{1/2}}\Big)\Big],$$

$$\begin{aligned}&\Big|\frac{d}{dt}\langle\vec{h}(t),\psi^1_{-1,0}(v;t)\rangle+(1-v^2)^{1/2}\langle\vec{h}(t),\psi^0_{-1,0}(v;t)\rangle\Big|\\ &\qquad\leq C\Big[\|f(t)\|_{L^2_x}+\|\overrightarrow{h(t)}\|_{H^1_x\times L^2_x}(|t|v+1)\exp\Big(\frac{-2\sqrt{2}vt}{(1-v^2)^{1/2}}\Big)\Big],\end{aligned}$$

$$\begin{aligned}&\Big|\frac{d}{dt}\langle\vec{h}(t),\psi^1_{0,1}(v;t)\rangle+(1-v^2)^{1/2}\langle\vec{h}(t),\psi^0_{0,1}(v;t)\rangle\Big|\\ &\qquad\leq C\Big[\|f(t)\|_{L^2_x}+\|\overrightarrow{h(t)}\|_{H^1_x\times L^2_x}(|t|v+1)\exp\Big(\frac{-2\sqrt{2}vt}{(1-v^2)^{1/2}}\Big)\Big].\end{aligned}$$

Proof of Lemma 40. This follows directly from the identity

$$\frac{d}{dt}\vec{h}(t)=JL_{+,-}\vec{h}(t)+\begin{bmatrix}0\\ f(t,x)\end{bmatrix},\tag{165}$$

and from Lemma 36. ☐

Proof of Theorem 29. For

$$T_0 \geq \frac{4\ln(1/v)}{v},$$

we consider similarly to [Chen and Jendrej 2022] the norms

$$\|u\|_{L^2_{v,T_0}} = \sup_{t\geq T_0} e^{vt}\|u(t,x)\|_{L^2_x(\mathbb{R})}, \quad \|u\|_{H^1_{v,T_0}} = \sup_{t\geq T_0} e^{vt}[\|u(t,x)\|^2_{H^1_x(\mathbb{R})} + \|\partial_t u(t,x)\|^2_{L^2_x(\mathbb{R})}]^{1/2}.$$

Next, from Lemma 40, we can verify using the fundamental theorem of calculus that there is a constant $C > 1$ such that if $v \ll 1$, then for any $t \geq T_0$ we have that

$$|\langle \vec{h}(t), \psi^0_{-1,0}(v;t)\rangle| \leq C\left[\|f\|_{L^2_{v,T_0}}\frac{e^{-vt}}{v} + \|h\|_{H^1_{v,T_0}}\frac{e^{-(2\sqrt{2}+1)vt}}{v}\right], \tag{166}$$

$$|\langle \vec{h}(t), \psi^1_{-1,0}(v;t)\rangle| \leq C\left[\|f\|_{L^2_{v,T_0}}\frac{e^{-vt}}{v^2} + \|h\|_{H^1_{v,T_0}} te^{-(2\sqrt{2}+1)vt} + \|h\|_{H^1_{v,T_0}}\frac{e^{-(2\sqrt{2}+1)vt}}{v^2}\right], \tag{167}$$

$$|\langle \vec{h}(t), \psi^0_{0,1}(v;t)\rangle| \leq C\left[\|f\|_{L^2_{v,T_0}}\frac{e^{-vt}}{v} + \|h\|_{H^1_{v,T_0}}\frac{e^{-(2\sqrt{2}+1)vt}}{v}\right], \tag{168}$$

$$|\langle \vec{h}(t), \psi^1_{0,1}(v;t)\rangle| \leq C\left[\|f\|_{L^2_{v,T_0}}\frac{e^{-vt}}{v^2} + \|h\|_{H^1_{v,T_0}} te^{-(2\sqrt{2}+1)vt} + \|h\|_{H^1_{v,T_0}}\frac{e^{-(2\sqrt{2}+1)vt}}{v^2}\right]. \tag{169}$$

Also, from Lemma 39, we can verify using the fundamental theorem of calculus for any $t \geq T_0$ that there is a constant $K \geq 1$ such that if $v \ll 1$, then

$$\int_t^{+\infty}\left|\frac{d}{ds}Q(s,h)\right| ds \leq K\left[\frac{e^{-2vt}}{v}\|f\|_{L^2_{v,T_0}}\|h\|_{H^1_{v,T_0}} + \|h\|^2_{H^1_{v,T_0}}\left(\frac{e^{-2vt}}{vt} + e^{-t(2v+\sqrt{2}v(1-10^{-3})^2)}\right)\right]. \tag{170}$$

In conclusion, similarly to Step 1 in the proof of Lemma 3.1 of [Chen and Jendrej 2022], we deduce using the estimates (166)–(170) with Lemma 37 that there exists a new constant $C > 1$ such that for any $t \geq T_0$ and $v \ll 1$ we have

$$\|h\|^2_{H^1_{v,T_0}} \leq \frac{C}{v^4}\|f\|^2_{L^2_{v,T_0}}. \tag{171}$$

The fact that the constant C in (171) is independent of v follows from

$$T_0 \geq \frac{4\ln(1/v)}{v},$$

which implies that

$$\frac{e^{-2vt}}{v^4} + \frac{e^{-2vt}}{vt} \ll v^4.$$

We also observe that if $(g_1(t,x), \partial_t g_1(t,x))$ and $(g_2(t,x), \partial_t g_2(t,x))$ are in the space $(g(t), \partial_t g(t)) \in H^1_x(\mathbb{R}) \times L^2_x(\mathbb{R})$ such that

$$\|(g(t), \partial_t g(t))\|_{L^\infty([T_0,+\infty], H^1_x \times L^2_x)} \leq 1, \tag{172}$$

then, since $U \in C^\infty$, we can verify that the function

$$\begin{aligned} N(v,\vec{g})(t,x) = {} & U'\left(H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right) + H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right) + g(t,x)\right) - U'\left(H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right) \\ & - U'\left(H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right)\right) - U''\left(H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right) + H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right)\right)g(t,x) \end{aligned} \tag{173}$$

satisfies, for some new constant $C \geq 1$ and any $v \ll 1$,

$$\|N(v, \overrightarrow{g_1(t)}) - N(v, \overrightarrow{g_2(t)})\|_{H^1_x} \leq C[\|g_1(t)\|_{H^1_x} + \|g_2(t)\|_{H^1_x}]\|g_1(t) - g_2(t)\|_{H^1_x},$$

which implies

$$\|N(v, \overrightarrow{g_1(t)}) - N(v, \overrightarrow{g_2(t)})\|_{H^1_{v,T_0}} \leq Ce^{-vt}[\|g_1\|_{H^1_{v,T_0}} + \|g_2\|_{H^1_{v,T_0}}]\|g_1 - g_2\|_{H^1_{v,T_0}}. \tag{174}$$

In conclusion, by repeating the argument of the proof of Proposition 3.6 of [Chen and Jendrej 2022], we can verify using the Lipschitz estimate of (174) and estimate (171) that if

$$T_0 \geq \frac{4\ln(1/v)}{v} \quad \text{and} \quad v \ll 1,$$

then there exists a map

$$S : \{u \in H^1_{v,T_0} \mid \|u\|_{H^1_{v,T_0}} \leq 1\} \to \{u \in H^1_{v,T_0} \mid \|u\|_{H^1_{v,T_0}} \leq 1\} \tag{175}$$

such that $\mu(t, x) = S(u)(t, x)$ is the unique solution of the equation

$$\partial_t^2 \mu(t, x) - \partial_x^2 \mu(t, x) + U''\left(H_{-1,0}\left(\frac{x + vt}{\sqrt{1 - v^2}}\right) + H_{0,1}\left(\frac{x - vt}{\sqrt{1 - v^2}}\right)\right)\mu(t, x) = N(v, \vec{\mu})(t, x), \tag{176}$$

such that $\mu \in H^1_{v,T_0}$. Indeed, the uniqueness is guaranteed by estimate (171) and from estimates (171) and (174) we have that the map S is a contraction in the set

$$B = \{u \in H^1_{v,T_0} \mid \|u\|_{H^1_{v,T_0}} \leq 1\},$$

and so Theorem 29 follows similarly to the proof of Proposition 3.6 of [Chen and Jendrej 2022] by using Banach's fixed point theorem. ☐

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