# ANALYSIS & PDE

Volume 18 No. 9 2025



JONAH A. J. DUNCAN AND LUC NGUYEN


**THE $\sigma_k$-LOEWNER–NIRENBERG PROBLEM ON RIEMANNIAN MANIFOLDS FOR $k < \frac{n}{2}$**


msp

ANALYSIS AND PDE
Vol. 18 (2025), No. 9, pp. 2203–2240
DOI: 10.2140/apde.2025.18.2203

# THE $\sigma_k$-LOEWNER–NIRENBERG PROBLEM ON RIEMANNIAN MANIFOLDS FOR $k < \frac{n}{2}$

JONAH A. J. DUNCAN AND LUC NGUYEN

Let $(M^n, g_0)$ be a smooth compact Riemannian manifold of dimension $n \geq 3$ with nonempty boundary $\partial M$. Let $\Gamma \subset \mathbb{R}^n$ be a symmetric convex cone and $f$ a symmetric defining function for $\Gamma$ satisfying standard assumptions. Under an algebraic condition on $\Gamma$, which is satisfied for example by the Gårding cones $\Gamma_k^+$ when $k < \frac{1}{2}n$, we prove the existence of a locally Lipschitz viscosity solution $g_u = e^{2u} g_0$ to the fully nonlinear Loewner–Nirenberg problem associated to $(f, \Gamma)$,

$$\begin{cases} f(\lambda(-g_u^{-1}A_{g_u})) = 1, \quad \lambda(-g_u^{-1}A_{g_u}) \in \Gamma & \text{on } M \setminus \partial M, \\ u(x) \to +\infty & \text{as } \operatorname{dist}_{g_0}(x, \partial M) \to 0, \end{cases}$$

where $A_{g_u}$ is the Schouten tensor of $g_u$. Previous results on Euclidean domains show that, in general, $u$ is not differentiable. The solution $u$ is obtained as the limit of smooth solutions to a sequence of fully nonlinear Loewner–Nirenberg problems on approximating cones containing $(1, 0, \dots, 0)$, for which we also have uniqueness. In the process, we obtain an existence and uniqueness result for the corresponding Dirichlet boundary value problem with finite boundary data, which is also of independent interest. An important feature of our paper is that the existence of a conformal metric $g$ satisfying $\lambda(-g^{-1}A_g) \in \Gamma$ on $M$ is a *consequence* of our results, rather than an assumption.



## 1. Introduction

A pertinent theme in conformal geometry is to establish the existence of conformal metrics satisfying some notion of constant curvature. For example, given a compact Riemannian manifold $(M^n, g_0)$ of dimension $n \geq 3$ with nonempty boundary $\partial M$, a natural question is whether there exists a conformal metric which is complete on $M \setminus \partial M$ and has constant negative scalar curvature on $M \setminus \partial M$. In the seminal work of Loewner and Nirenberg [1974], the authors proved among other results the existence and

*MSC2020:* primary 35A01, 35A02, 35D40, 53C18, 53C21; secondary 35J60, 35J75.
*Keywords:* Loewner–Nirenberg, fully nonlinear elliptic equations, negative curvature, complete conformal metrics, Yamabe.

uniqueness of such a metric when $M \setminus \partial M$ is a bounded Euclidean domain with smooth boundary[1] and $g_0$ is the flat metric. Aviles and McOwen [1988] later extended this result to the Riemannian setting; for further related results we refer, e.g., to [Allen et al. 2018; Andersson et al. 1992; Aviles 1982; Finn 1998; Gover and Waldron 2017; Graham 2017; Han and Shen 2020; Han et al. 2024; Jiang 2021; Li 2022b; Mazzeo 1991; Véron 1981]. We note that the related problem of finding conformal metrics with constant scalar curvature on closed manifolds, known as the Yamabe problem, was solved in [Aubin 1970; Schoen 1984; Trudinger 1968; Yamabe 1960].

Since the works of Viaclovsky [2000] and Chang, Gursky and Yang [Chang et al. 2002], there has been significant interest in fully nonlinear generalisations of Yamabe-type problems, including on manifolds with boundary. Suppose that

$$\Gamma \subset \mathbb{R}^n \text{ is an open, convex, connected symmetric cone with vertex at } 0, \tag{1-1}$$

$$\Gamma_n^+ = \{\lambda \in \mathbb{R}^n : \lambda_i > 0 \ \forall\, 1 \le i \le n\} \subseteq \Gamma \subseteq \Gamma_1^+ = \{\lambda \in \mathbb{R}^n : \lambda_1 + \cdots + \lambda_n > 0\}, \tag{1-2}$$

$$f \in C^\infty(\Gamma) \cap C^0(\overline{\Gamma}) \text{ is concave, 1-homogeneous and symmetric in the } \lambda_i, \tag{1-3}$$

$$f > 0 \ \text{ in } \Gamma, \quad f = 0 \ \text{ on } \partial\Gamma, \quad f_{\lambda_i} > 0 \ \text{ in } \Gamma \text{ for } 1 \le i \le n. \tag{1-4}$$

In this paper, we study the natural generalisation of the Loewner–Nirenberg problem to the fully nonlinear setting on Riemannian manifolds. That is, for $(f, \Gamma)$ satisfying (1-1)–(1-4) and a compact Riemannian manifold $(M, g_0)$ with nonempty boundary $\partial M$, we study the existence and uniqueness of a conformal metric $g_u = e^{2u} g_0$ satisfying

$$\begin{cases} f(\lambda(-g_u^{-1} A_{g_u})) = 1, \ \ \lambda(-g_u^{-1} A_{g_u}) \in \Gamma & \text{on } M \setminus \partial M, \\ u(x) \to +\infty & \text{as } \mathrm{d}(x, \partial M) \to 0. \end{cases} \tag{1-5}$$

Here,

$$A_g = \frac{1}{n-2}\left(\mathrm{Ric}_g - \frac{R_g}{2(n-1)} g\right)$$

denotes the $(0,2)$-Schouten tensor of a Riemannian metric $g$, $\mathrm{Ric}_g$ and $R_g$ denote the Ricci curvature tensor and scalar curvature of $g$, respectively, $\lambda(T)$ denotes the vector of eigenvalues of a $(1,1)$-tensor $T$, and $\mathrm{d}(x, \partial M)$ is the distance from $x \in M$ to $\partial M$ with respect to $g_0$. Typical examples of $(f, \Gamma)$ satisfying (1-1)–(1-4) are given by $(\sigma_k^{1/k}, \Gamma_k^+)$ for $1 \le k \le n$, where $\sigma_k$ is the $k$-th elementary symmetric polynomial and $\Gamma_k^+ = \{\lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0 \ \forall\, 1 \le j \le k\}$. When $f = \sigma_1$, (1-5) reduces to the original Loewner–Nirenberg problem on Riemannian manifolds discussed above.

Much of the motivation to study (1-5) stems from the fact that, as a consequence of the Ricci decomposition, the Schouten tensor fully determines the conformal transformation properties of the full Riemann curvature tensor. We note that, for $g_u = e^{2u} g_0$, one has the transformation law

$$A_{g_u} = -\nabla_{g_0}^2 u - \tfrac{1}{2} |\nabla_{g_0} u|_{g_0}^2 g_0 + du \otimes du + A_{g_0}, \tag{1-6}$$

which demonstrates the fully nonlinear nature of (1-5) when $f \ne c\sigma_1$. Moreover, (1-5) is nonuniformly elliptic when $f \ne c\sigma_1$.

[1] Loewner and Nirenberg [1974] also considered the problem on a class of nonsmooth Euclidean domains, but we will not be concerned with such generalisations in this paper.

By the 1-homogeneity of $f$, without loss of generality we may assume

$$f\left(\tfrac{1}{2}, \dots, \tfrac{1}{2}\right) = 1. \tag{1-7}$$

As in [Li and Nguyen 2014], we define $\mu_\Gamma^+$ to be the number satisfying

$$(-\mu_\Gamma^+, 1, \dots, 1) \in \partial\Gamma.$$

We note that $\mu_\Gamma^+$ is uniquely determined by $\Gamma$ and is easily seen to satisfy $\mu_\Gamma^+ \in [0, n-1]$. When $\Gamma = \Gamma_k^+$, one has $\mu_{\Gamma_k^+}^+ = (n-k)/k$.

Our first main result concerns the solution to the Loewner–Nirenberg problem (1-5) under the assumption

$$\mu_\Gamma^+ > 1. \tag{1-8}$$

Observe that, for $\Gamma = \Gamma_k^+$, (1-8) holds if and only if $k < \frac{1}{2}n$. The role of condition (1-8) will be discussed later in the introduction.

**Theorem 1.1.** *Let $(M, g_0)$ be a smooth compact Riemannian manifold of dimension $n \geq 3$ with nonempty boundary $\partial M$, and suppose $(f, \Gamma)$ satisfies* (1-1)–(1-4), (1-7) *and* (1-8). *Then there exists a locally Lipschitz viscosity solution to* (1-5) *satisfying*

$$\lim_{\mathrm{d}(x,\partial M)\to 0} (u(x) + \ln \mathrm{d}(x, \partial M)) = 0, \tag{1-9}$$

*which is maximal in the sense that if $\tilde{u}$ is any continuous viscosity solution to* (1-5), *then $\tilde{u} \leq u$ on $M \setminus \partial M$. Moreover, when $(1, 0, \dots, 0) \in \Gamma$, $u$ is smooth and is the unique continuous viscosity solution to* (1-5).

We recall that a continuous function $u$ on $M\setminus\partial M$ is a viscosity subsolution (resp. viscosity supersolution) to the equation in (1-5) if, for any $x_0 \in M \setminus \partial M$ and $\varphi \in C^2(M \setminus \partial M)$ satisfying $u(x_0) = \varphi(x_0)$ and $u(x) \leq \varphi(x)$ near $x_0$ (resp. $u(x) \geq \varphi(x)$ near $x_0$), we have $\lambda(-g_\varphi^{-1} A_{g_\varphi})(x_0) \in \{\lambda \in \Gamma : f(\lambda) \geq 1\}$ (resp. $\lambda(-g_\varphi^{-1} A_{g_\varphi})(x_0) \in \mathbb{R}^n\setminus\{\lambda \in \Gamma : f(\lambda) > 1\}$). We say that $u$ is a viscosity solution to the equation in (1-5) if it is both a viscosity subsolution and a viscosity supersolution.

**Remark 1.2.** In previous work studying equations of the form $f(\lambda(-g_u^{-1} A_{g_u})) = 1$, it has been typical to assume that the background metric $g_0$ satisfies $\lambda(-g_0^{-1} A_{g_0}) \in \Gamma$ on $M$ (a notable exception is a result of Gursky, Streets and Warren [Gursky et al. 2011], which will be discussed later in the introduction). In contrast, one of the key points of this paper is that we do not assume the existence of such a metric in Theorem 1.1. Rather, the existence of such a metric is established as a by-product of the proof of Theorem 1.1 (see Theorem 1.6), and our proof of Theorem 1.1 would not be substantially simpler even if we were to assume the existence of such a metric from the outset. We note that after our work was submitted, Professor Rirong Yuan [2024] brought to our attention his work, where a conformal metric satisfying $\lambda(-g^{-1} A_g) \in \Gamma$ is constructed under the assumption (1-8) by an entirely different method. See also [Yuan 2022], which considers the existence problem for (1-5) assuming $\lambda(-g_0^{-1} A_{g_0}) \in \Gamma$ and $(1, 0, \dots, 0) \in \Gamma$, and addresses (1-9) and uniqueness of solutions under an even stronger assumption on $\Gamma$ (see condition (1.20) therein).

**Remark 1.3.** In the case that $M \setminus \partial M$ is a Euclidean domain, the existence of a Lipschitz viscosity solution to (1-5) was established by Gonzáles, Li and Nguyen [González et al. 2018]. It was also shown in their work that this solution is unique among continuous viscosity solutions. We note that the uniqueness of the viscosity solution obtained in Theorem 1.1 remains an open problem when $M \setminus \partial M$ is not a Euclidean domain and $(1, 0, \dots, 0) \in \partial\Gamma$.

**Remark 1.4.** In [Li and Nguyen 2021; Li et al. 2023] it was shown that if $M \setminus \partial M$ is a Euclidean domain with disconnected boundary and $\Gamma \subset \Gamma_2^+$ (in particular, this implies $(1, 0, \dots, 0) \in \partial\Gamma$), then the Lipschitz viscosity solution to (1-5) is not differentiable. Thus, in general, the Lipschitz regularity of the solution in Theorem 1.1 cannot be improved to $C^1$ regularity when $(1, 0, \dots, 0) \in \partial\Gamma$. On the other hand, the existence of a unique smooth solution to (1-5) satisfying (1-9) when $(1, 0, \dots, 0) \in \Gamma$ is new even when $M \setminus \partial M$ is a Euclidean domain. This smoothness result can be viewed as an analogue of the result in [Gursky and Viaclovsky 2003] on the existence of a smooth solution to the $\sigma_k$-Yamabe problem for the trace-modified Schouten tensor on closed manifolds.

To describe the proof of Theorem 1.1, we first introduce some notation and an equivalent formulation of the result. For $\tau \in [0, 1]$, $\lambda \in \mathbb{R}^n$ and $e = (1, \dots, 1) \in \mathbb{R}^n$, we define

$$\lambda^\tau := \tau\lambda + (1-\tau)\sigma_1(\lambda)e, \quad f^\tau(\lambda) := \frac{f(\lambda^\tau)}{\tau + n(1-\tau)} \quad \text{and} \quad \Gamma^\tau := \{\lambda : \lambda^\tau \in \Gamma\}.$$

As shown in [Duncan and Nguyen 2023, Appendix A], $\Gamma$ satisfies (1-1), (1-2) and $(1, 0, \dots, 0) \in \Gamma$ if and only if there exists $\widetilde{\Gamma}$ satisfying (1-1), (1-2) and a number $\tau < 1$ for which $\Gamma = (\widetilde{\Gamma})^\tau$. Note that (1-7) implies $f^\tau\left(\frac{1}{2}, \dots, \frac{1}{2}\right) = 1$. An equivalent formulation of Theorem 1.1 is then as follows.

**Theorem 1.1′.** *Let $(M, g_0)$ be a smooth compact Riemannian manifold of dimension $n \geq 3$ with nonempty boundary $\partial M$, and suppose $(f, \Gamma)$ satisfies* (1-1)–(1-4), (1-7) *and* (1-8). *Then, for each $\tau < 1$, there exists a smooth solution $u$ to*

$$\begin{cases} f^\tau(\lambda(-g_u^{-1}A_{g_u})) = 1, \quad \lambda(-g_u^{-1}A_{g_u}) \in \Gamma^\tau & \text{on } M \setminus \partial M, \\ u(x) \to +\infty & \text{as } \mathrm{d}(x, \partial M) \to 0, \end{cases} \tag{1-10}$$

*and moreover $u$ satisfies* (1-9) *and is the unique continuous viscosity solution to* (1-10). *When $\tau = 1$, there exists a Lipschitz viscosity solution $u$ to* (1-10) *satisfying* (1-9), *which is maximal in the sense that if $\tilde{u}$ is any continuous viscosity solution to* (1-10), *then $\tilde{u} \leq u$ on $M \setminus \partial M$.*

**Remark 1.5.** If we label the solution to (1-10) in Theorem 1.1′ as $u^\tau$ for each $\tau \leq 1$, then we will show that, for each compact set $K \subset M \setminus \partial M$, there exists a constant $C$ which is independent of $\tau$ but dependent on $M$, $g_0$, $f$, $\Gamma$ and $K$ such that

$$\|u^\tau\|_{C^{0,1}(K)} \leq C \quad \text{for all } \tau \in [0, 1].$$

In the proof of Theorem 1.1′, we will first prove the existence of a unique smooth solution to (1-10) when $\tau < 1$. The Lipschitz viscosity solution in the case $\tau = 1$ is then obtained in the limit as $\tau \to 1$. In turn, for each $\tau < 1$, the existence of a smooth solution to (1-10) is obtained as the limit of smooth solutions to Dirichlet boundary value problems with finite boundary data. Although we only need to consider constant boundary data in the proof of Theorem 1.1′, we will prove the following more general result.

**Theorem 1.6.** *Let $(M, g_0)$ be a smooth compact Riemannian manifold of dimension $n \geq 3$ with nonempty boundary $\partial M$, and suppose $(f, \Gamma)$ satisfies* (1-1)–(1-4) *and* (1-8). *Let $\psi \in C^\infty(M)$ be positive and $\xi \in C^\infty(\partial M)$. Then, for each $\tau < 1$, there exists a smooth solution $u$ to*

$$\begin{cases} f^\tau(\lambda(-g_u^{-1}A_{g_u})) = \psi, \ \ \lambda(-g_u^{-1}A_{g_u}) \in \Gamma^\tau & \text{on } M \setminus \partial M, \\ u = \xi & \text{on } \partial M, \end{cases} \tag{1-11}$$

*and moreover $u$ is the unique continuous viscosity solution to* (1-11). *When $\tau = 1$, there exists a Lipschitz viscosity solution to* (1-11).

**Remark 1.7.** If we label the solution to (1-11) in Theorem 1.6 as $u^\tau$ for each $\tau \leq 1$, then we will show that there exists a constant $C$ which is independent of $\tau$ but dependent on $M$, $g_0$, $f$, $\Gamma$, $\psi$ and $\xi$ such that

$$\|u^\tau\|_{C^{0,1}(M)} \leq C \quad \text{for all } \tau \in [0, 1].$$

The existence of a smooth solution to (1-11) when $\tau < 1$ is achieved using the continuity method, which relies on obtaining a priori estimates. To keep the introduction concise, we only discuss the $C^0$ estimates here and postpone the discussion of the other estimates to the main body of the paper. Now, if one assumes $\lambda(-g_0^{-1}A_{g_0}) \in \Gamma$ on $M$, then it is straightforward to obtain both the a priori upper and lower bounds on solutions to (1-11). Since we do not make such an assumption on $g_0$, a large portion of our work involves proving the lower bound. The a priori lower bound is obtained in two independent stages, which can be summarised as follows:

(1) First, in Section 2, we prove a local interior gradient estimate on solutions to (1-11) of the form

$$|\nabla_{g_0} u|_{g_0}(x) \leq C(r^{-1} + e^{\sup_{B_r} u}) \quad \text{for } x \in B_{r/2}, \tag{1-12}$$

where $B_r$ is a geodesic ball contained in the interior of $M$. An important feature is that the estimate (1-12) does not depend on a lower bound for $u$.

(2) Second, in Section 3.2, we construct suitable barrier functions to prove a lower bound for $u$ in a *uniform neighbourhood of* $\partial M$ — this is one of the key new ideas in this paper.

We note that the assumption $\mu_\Gamma^+ > 1$ is used in both stages above. Once the lower bound in a uniform neighbourhood of $\partial M$ is established in the second step, the local interior gradient estimate from the first step and a trivial global upper bound in Proposition 3.1 then allows one to propagate the lower bound to all of $M$ — see the proof of Proposition 3.2 for the details. As indicated in Remark 1.7 above, it is important that all estimates in the two steps above (as well as the boundary gradient estimates obtained in the main body of the paper — see Proposition 3.8) are independent of $\tau$.

In fact, the proof of Step (2) provides a purely local lower bound: if $x_0 \in \partial M$ and $u$ solves (1-11) in $M \cap B_r(x_0)$, then $u \geq C$ in $M \cap B_{r/2}(x_0)$. In our subsequent work [Duncan and Nguyen 2025], we show that this local lower bound cannot hold when (1-8) fails, that is when $\mu_\Gamma^+ \leq 1$.

We now discuss the two steps above in more detail. Our local interior gradient estimate, which is also of independent interest, is as follows.

**Theorem 1.8.** *Let $(M, g_0)$ be a smooth Riemannian manifold of dimension $n \geq 3$, possibly with nonempty boundary, and suppose $(f, \Gamma)$ satisfies* (1-1)–(1-4) *and* (1-8). *Fix $\tau \in (0, 1]$, fix a positive function $\psi \in C^\infty(M)$ and suppose that $u \in C^3(B_r)$ satisfies*

$$f^\tau(\lambda(-g_u^{-1}A_{g_u})) = \psi, \quad \lambda(-g_u^{-1}A_{g_u}) \in \Gamma^\tau \tag{1-13}$$

*in a geodesic ball $B_r$ contained in the interior of $M$. Then*

$$|\nabla_{g_0} u|_{g_0}(x) \leq C(r^{-1} + e^{\sup_{B_r} u}) \quad \text{for } x \in B_{r/2}, \tag{1-14}$$

*where $C$ is a constant depending on $n$, $f$, $\Gamma$, $\|g_0\|_{C^3(B_r)}$ and $\|\psi\|_{C^1(B_r)}$ but independent of $\tau$ and $\inf_{B_r} \psi$.*

We note that Theorem 1.8 was previously obtained for $(f, \Gamma) = (\sigma_k^{1/k}, \Gamma_k^+)$ when $k < \frac{1}{2}n$ and $\tau = 1$ in the thesis of Khomrutai [2009].[2] Roughly speaking, one important observation in the thesis is as follows: if $\rho|\nabla_{g_0} u|_{g_0}^2$ attains its maximum at $x_0$ (here $\rho$ is a cutoff function satisfying standard assumptions), then in a "worst case scenario" (i.e., in a situation where the gradient estimate cannot be obtained somewhat directly), the ordered eigenvalues $\lambda_1(x_0) \geq \cdots \geq \lambda_n(x_0)$ of $(-g_0^{-1}A_{g_u})(x_0)$ are greater than or equal to a perturbation of $(1, \ldots, 1, -1)\frac{1}{2}|\nabla u|^2(x_0)$. But when $k < \frac{1}{2}n$, the vector $(1, \ldots, 1, -1)$ belongs to $\Gamma_k^+$, and so by (1-13) and homogeneity of $\sigma_k^{1/k}$, the gradient estimate follows. In our proof of Theorem 1.8, we show that this phenomenon persists for general cones satisfying $\mu_\Gamma^+ > 1$. In order to circumvent certain arguments of Khomrutai that rely on algebraic properties of the $\sigma_k$ operators, we appeal to some general cone properties recently observed by Yuan [2022].

**Remark 1.9.** For gradient estimates on solutions to equations of the form (1-13) which depend on two-sided $C^0$ bounds, see for instance [Guan 2008; Gursky and Viaclovsky 2003]. For gradient estimates for the related positive cone equation, see e.g., [Chen 2005; Guan and Wang 2003; Jin et al. 2007; Li 2009; Li and Li 2003; Viaclovsky 2002; Wang 2006].

**Remark 1.10.** We have been informed that in an upcoming work of Baozhi Chu, YanYan Li and Zongyuan Li [Chu et al. 2023], a Liouville-type theorem for a fully nonlinear, degenerate elliptic Yamabe-type equation on negative cones is proved for all $\mu_\Gamma^+ \neq 1$. As an application of this Liouville-type theorem and the method in [Li 2009] (which dealt with local gradient estimates for equations on positive cones), the authors obtain local interior gradient estimates for solutions to (1-13) depending only on one-sided $C^0$ bounds for all $\mu_\Gamma^+ \neq 1$ without assuming concavity of $f$. Counterexamples to both results are also given when $\mu_\Gamma^+ = 1$. This proof is entirely different from our proof of Theorem 1.8.

We now turn to the second step mentioned above, namely the lower bound in a neighbourhood of $\partial M$. This is achieved through constructing suitable comparison functions on small annuli; the main step here is to prove the following proposition (see Proposition 3.4 for a more precise version).

**Proposition 1.11.** *Suppose $(f, \Gamma)$ satisfies* (1-1)–(1-4) *and* (1-8), *let $g_0$ be a Riemannian metric defined on a neighbourhood $\Omega$ of the origin in $\mathbb{R}^n$, let $m \in \mathbb{R}$ and define $A_{r_-,r_+} := \{x : r_- < d_{g_0}(x, 0) < r_+\}$.*

[2] We would like to thank Baozhi Chu, YanYan Li and Zongyuan Li for bringing [Khomrutai 2009] to our attention.

*Then there exist constants $S > 1$ and $0 < R < 1$ depending on $g_0$, $f$, $\Gamma$ and $m$ such that, whenever $1 < r_+/r_- < S$ and $r_+ < R$, there exists a solution to*

$$\begin{cases} f(\lambda(-g_w^{-1}A_{g_w})) \geq 1, \ \ \lambda(-g_w^{-1}A_{g_w}) \in \Gamma & \text{on } A_{r_-,r_+}, \\ w(x) = m & \text{for } x \in \mathbb{S}_{r_-}, \\ w(x) \to -\infty & \text{as } \mathrm{d}_{g_0}(x, \mathbb{S}_{r_+}) \to 0. \end{cases}$$

Our construction of $w$ in Proposition 1.11 is modelled on the radial solutions of Chang, Han and Yang [Chang et al. 2005] to the $\sigma_k$-Yamabe equation on annular domains in $\mathbb{R}^n$ when $k < \frac{1}{2}n$. To apply Proposition 1.11 to complete the second step, we attach a collar neighbourhood $N$ to $\partial M$ and cover a neighbourhood of $\partial M$ in $M$ by sufficiently small annuli whose centres lie in $N$ and whose inner boundaries touch $\partial M$. On each of these annuli, the solutions constructed in Proposition 1.11 then serve as the desired lower bound by the comparison principle. See the proof of Proposition 3.3 for details.

**Remark 1.12.** The assumption $\mu_\Gamma^+ > 1$ plays an important role in our proof of Proposition 1.11, and in fact a similar construction is not possible when $\mu_\Gamma^+ \leq 1$. More precisely, given a smooth metric $g_0$ defined on an annulus $A_{r,R}$ and given a cone $\Gamma$ satisfying (1-1), (1-2) and $\mu_\Gamma^+ \leq 1$, there is no smooth metric $g_w = e^{2w} g_0$ satisfying $\lambda(-g_w^{-1}A_{g_w}) \in \Gamma$ on $A_{r,R}$ and for which $w \to -\infty$ at either boundary component of $A_{r,R}$. The proof of this nonexistence result uses arguments different in nature to those considered in this paper and appears in our more recent work [Duncan and Nguyen 2025].

For the remainder of the introduction, we discuss in more detail how our results and methods compare to previous work on fully nonlinear problems of Loewner–Nirenberg type. As mentioned before, when $M \setminus \partial M$ is a Euclidean domain, the existence of a Lipschitz viscosity solution to (1-5), as well as uniqueness of this solution among continuous viscosity solutions, was established in [González et al. 2018]. Moreover, counterexamples to $C^1$ regularity were given in [Li and Nguyen 2021; Li et al. 2023]. The proof in [González et al. 2018] uses Perron's method, which in turn uses canonical solutions on interior/exterior balls and a comparison principle on Euclidean domains established in [Li et al. 2018]. Since one cannot use exterior balls in the Riemannian setting and since it is not currently known whether the comparison principle in [Li et al. 2018] extends to the Riemannian setting, a different approach to that in [González et al. 2018] is required to prove Theorem 1.1′.

On the other hand, for $(f, \Gamma) = (\sigma_k^{1/k}, \Gamma_k^+)$, $2 \leq k \leq n$, Gursky, Streets and Warren [Gursky et al. 2011] proved the existence of a unique smooth solution to (1-5) with the Ricci tensor in place of the Schouten tensor (see Remark 1.13 below for the relation between this result and Theorem 1.1′, and see also [Wang 2021; Li 2022a] for some further related results). As in the present paper, the solution of Gursky, Streets and Warren is constructed as a limit of solutions with finite boundary data, and these solutions are in turn obtained using the continuity method. Their method for obtaining an a priori lower bound on solutions is different to ours and is instead based on the explicit construction of a global subsolution. Roughly speaking, the subsolution construction in [Gursky et al. 2011] uses the fact that, in the analogous formula to (1-6) for the Ricci tensor, the gradient terms are collectively nonnegative definite and so can be neglected in certain computations. In our case, the gradient terms do not have an overall sign, thus leading to our new approach for the lower bound discussed above.

**Remark 1.13.** Since $\mu^+_{\Gamma^+_k} = (n-k)/k$, it is easy to see that $\mu^+_{(\Gamma^+_k)^\tau} = (n-k)/k + (n-1)(1-\tau)$. Thus

$$\mu^+_{(\Gamma^+_k)^\tau} > 1 \quad \text{if and only if} \quad \tau < a_{n,k} := \frac{n-k+k(n-2)}{k(n-1)}.$$

On the other hand, for $\tau = (n-2)/(n-1)$, we have

$$(\sigma_k^{1/k})^\tau(\lambda(-g_u^{-1}A_{g_u})) = \frac{1}{n-1}\cdot\sigma_k^{1/k}(\lambda(-g_u^{-1}\operatorname{Ric}_{g_u})).$$

Since $(n-2)/(n-1) < a_{n,k}$ if and only if $k < n$, we therefore see that Theorem 1.1′ recovers the result of [Gursky et al. 2011] for $k < n$.

The plan of the paper is as follows: In Section 2 we prove the local interior gradient estimate stated in Theorem 1.8. In Section 3 we consider the Dirichlet boundary value problem (1-11), proving Theorem 1.6. Finally, in Section 4 we turn to the fully nonlinear Loewner–Nirenberg problem (1-10), proving Theorem 1.1′ (and hence Theorem 1.1).

*Notation.* Throughout the rest of the paper, if $X$ is a (1, 1)-tensor satisfying $\lambda(X) \in \Gamma$ then we frequently write $f(X) := f(\lambda(X))$.

## 2. Proof of Theorem 1.8: the local interior gradient estimate

In this section we prove the local interior gradient estimate stated in Theorem 1.8. Throughout the section, unless otherwise stated all derivatives and norms are taken with respect to $g_0$. Moreover, $C$ will denote a constant that may change from line to line and depends only on $n,\ f,\ \Gamma,\ \|g_0\|_{C^3(B_r)}$ and $\|\psi\|_{C^1(B_r)}$.

**2.1.** ***Set-up and main ideas of the proof.*** Our set-up for the proof of Theorem 1.8 is similar to that in the related works [Chen 2005; Guan and Wang 2003; Jin et al. 2007; Khomrutai 2009; Li 2009; Li and Li 2003; Wang 2006] on local gradient estimates. Throughout this section we write $S = A_{g_0}$ and

$$W = \nabla^2 u + \tfrac{1}{2}|\nabla u|^2 g_0 - du \otimes du - S.$$

By a standard argument, it suffices to consider the case $r = 1$ in the proof of Theorem 1.8. Suppose $\rho \in C_c^\infty(B_1)$ is a cutoff function in $B_1$ with $\rho = 1$ on $B_{1/2}$, $|\nabla\rho| \le C\rho^{1/2}$ and $|\nabla^2\rho| \le C$. Set $H = \rho|\nabla u|^2$ and suppose $H$ attains a maximum at $x_0$. We may assume that $|\nabla u| \ge 1$ at $x_0$, otherwise we are done. Choosing suitable normal coordinates centred at $x_0$, we may also assume $W = (w_{ij})$ is diagonal at $x_0$ with $w_{11} \ge \cdots \ge w_{nn}$, and hence at $x_0$ we have

$$\begin{cases} w_{ii} = u_{ii} - u_i^2 + \frac{1}{2}|\nabla u|^2 - S_{ii} & \text{for all } 1 \le i \le n, \\ u_{ij} = u_i u_j + S_{ij} & \text{for } i \ne j. \end{cases} \tag{2-1}$$

Using the fact that $H_i(x_0) = 0$ for each $i$, we obtain at $x_0$

$$\sum_{l=1}^n u_{il}u_l = -\frac{\rho_i}{2\rho}|\nabla u|^2, \tag{2-2}$$

and hence

$$\left|\sum_{l=1}^{n} u_{il}u_l\right| \le C\rho^{-1/2}|\nabla u|^2. \tag{2-3}$$

For $A_0$ a large number to be fixed later, we may assume at $x_0$ that

$$\rho^{-1/2} \le C\frac{|\nabla u|}{A_0} \quad \text{and} \quad |S| \le \frac{|\nabla u|^2}{A_0}, \tag{2-4}$$

otherwise we are done. Note that, by combining (2-3) with the first estimate in (2-4), we have

$$\left|\sum_{l=1}^{n} u_{il}u_l\right| \le C\frac{|\nabla u|^3}{A_0}. \tag{2-5}$$

Denote by $F_\tau^{ij}$ the coefficients of the linearised operator at $(g_0^{-1}W)(x_0)$, that is,

$$F_\tau^{ij} = \frac{\partial f^\tau}{\partial A_{ij}}\bigg|_{A=(g_0^{-1}W)(x_0)}.$$

Then $(F_\tau^{ij})$ is a positive definite, diagonal matrix. Also define

$$\mathcal{F}_\tau = \sum_{i=1}^{n} F_\tau^{ii} \quad \text{and} \quad \tilde{u}_{ij} := u_{ij} - S_{ij}.$$

By homogeneity and concavity of $f$, it is easy to see that $\mathcal{F}_\tau \ge 1/C > 0$: indeed, writing $\lambda = \lambda(g_0^{-1}W)(x_0)$, we have

$$\mathcal{F}_\tau = \sum_{i=1}^{n} \frac{\partial f^\tau}{\partial \lambda_i}(\lambda) = f^\tau(\lambda) + \sum_{i=1}^{n} \frac{\partial f^\tau}{\partial \lambda_i}(\lambda)(1-\lambda_i) \ge f^\tau(1,\dots,1). \tag{2-6}$$

With our set-up and notation established, we now briefly discuss the main ideas in the proof of Theorem 1.8. The first step is to obtain the following lemma.

**Lemma 2.1.** *Under the same hypotheses as Theorem 1.8 but without the restriction $\mu_\Gamma^+ > 1$, there exists a constant $C$ such that*

$$0 \ge -C\mathcal{F}_\tau(1+e^{2u})|\nabla u|^2 - C\rho\mathcal{F}_\tau\frac{|\nabla u|^4}{A_0} + \rho\sum_{i,l} F_\tau^{ii}\tilde{u}_{il}^2 \quad \text{at } x_0. \tag{2-7}$$

The proof of Lemma 2.1 is by now standard and will be given in Section 2.2.

Now, in the case that the positive term on the right-hand side of (2-7) dominates $|\nabla u|^4\mathcal{F}_\tau$, in the sense that

$$\sum_{i,l} F_\tau^{ii}\tilde{u}_{il}^2 \ge \varepsilon|\nabla u|^4\mathcal{F}_\tau \quad \text{at } x_0 \tag{2-8}$$

for a suitably chosen small constant $\varepsilon > 0$, then the desired gradient estimate is routine (the details will be given later). On the other hand, if (2-8) fails for our suitably chosen small constant $\varepsilon > 0$, we will see that the ordered eigenvalues $w_{11} \ge \cdots \ge w_{nn}$ of $W$ at $x_0$ are greater than or equal to a perturbation of $(1, \dots, 1, -1)\frac{1}{2}|\nabla u|^2$. As mentioned in the introduction, this phenomenon was previously observed in the case $(f, \Gamma) = (\sigma_k^{1/k}, \Gamma_k^+)$ when $k < \frac{1}{2}n$ in the thesis of Khomrutai [2009]. Using the fact that $(1, \dots, 1, -1) \in \Gamma$ (this is the only place in the proof of Theorem 1.8 where the assumption $\mu_\Gamma^+ > 1$ is used), the gradient estimate again follows. The details will be given in Section 2.3.

**2.2. *Proof of Lemma 2.1.*** We follow closely the proof in [Guan and Wang 2003]. In what follows, all computations are implicitly carried out at $x_0$. First observe that, by (2-2),

$$H_{ij} = \left(\rho_{ij} - \frac{2\rho_i \rho_j}{\rho}\right)|\nabla u|^2 + 2\rho \sum_{l=1}^n u_{lij} u_l + 2\rho \sum_{l=1}^n u_{il} u_{jl},$$

and hence, by positivity of $(F_\tau^{ij})$ and nonpositivity of $(H_{ij})$,

$$\begin{aligned} 0 \geq \sum_{i=1}^n F_\tau^{ii} H_{ii} &= \sum_{i=1}^n F_\tau^{ii} \left[ \left(\rho_{ii} - \frac{2\rho_i^2}{\rho}\right)|\nabla u|^2 + 2\rho \sum_{l=1}^n u_{lii} u_l + 2\rho \sum_{l=1}^n u_{il}^2 \right] \\ &= -C|\nabla u|^2 \mathcal{F}_\tau + 2\rho \sum_{i,l} F_\tau^{ii} u_{lii} u_l + 2\rho \sum_{i,l} F_\tau^{ii} u_{il}^2. \end{aligned} \tag{2-9}$$

Now, commuting derivatives yields

$$\begin{aligned} \sum_{i,l} F_\tau^{ii} u_{lii} u_l &\geq \sum_{i,l} F_\tau^{ii} u_{iil} u_l - C|\nabla u|^2 \mathcal{F}_\tau \\ &= \sum_{i,l} F_\tau^{ii} \big[ (w_{ii})_l - \big(\tfrac{1}{2}|\nabla u|^2 - u_i^2\big)_l + (S_{ii})_l \big] u_l - C|\nabla u|^2 \mathcal{F}_\tau \\ &= \sum_{l=1}^n (\psi e^{2u})_l u_l - \mathcal{F}_\tau \sum_{k,l} u_{kl} u_k u_l + 2\sum_{i,l} F_\tau^{ii} u_{il} u_i u_l + \sum_{i,l} F_\tau^{ii} (S_{ii})_l u_l - C|\nabla u|^2 \mathcal{F}_\tau, \end{aligned} \tag{2-10}$$

where to reach the last line we have used the fact that $f^\tau$ is homogeneous of degree 1 to assert that $\sum_i F_\tau^{ii} (w_{ii})_l = (f^\tau(g_0^{-1} W))_l = (\psi e^{2u})_l$. Also, since $|\nabla u| \geq 1$, we can bound the penultimate term in (2-10) from below by $-C|\nabla u|^2 \mathcal{F}_\tau$, and also observe that

$$\sum_{l=1}^n (\psi e^{2u})_l u_l = \sum_{l=1}^n e^{2u} \psi_l u_l + 2e^{2u} \psi |\nabla u|^2 \geq -Ce^{2u} |\nabla u|^2. \tag{2-11}$$

Also, by (2-5) we have

$$-\mathcal{F}_\tau \sum_{k,l} u_{kl} u_k u_l \geq -C \frac{|\nabla u|^4}{A_0} \mathcal{F}_\tau, \tag{2-12}$$

and likewise

$$2\sum_{i,l} F_\tau^{ii} u_{il} u_i u_l = 2\sum_i \left( F_\tau^{ii} u_i \sum_l u_{il} u_l \right) \geq -2\sum_i \left( |F_\tau^{ii} u_i| \left| \sum_l u_{il} u_l \right| \right) \geq -C \frac{|\nabla u|^4}{A_0} \mathcal{F}_\tau. \tag{2-13}$$

Substituting (2-11)–(2-13) back into (2-10) and recalling $\mathcal{F}_\tau \geq 1/C$, we get

$$\sum_{i,l} F_\tau^{ii} u_{lii} u_l \geq -C(1 + e^{2u})|\nabla u|^2 \mathcal{F}_\tau - C\mathcal{F}_\tau \frac{|\nabla u|^4}{A_0},$$

and, substituting this back into (2-9), we see

$$0 \geq -C\mathcal{F}_\tau(1+e^{2u})|\nabla u|^2 - C\rho\mathcal{F}_\tau \frac{|\nabla u|^4}{A_0} + 2\rho \sum_{i,l} F_\tau^{ii} u_{il}^2. \tag{2-14}$$

The desired estimate (2-7) then follows from (2-14) and the following inequality, which is a consequence of the Cauchy–Schwarz inequality and the second inequality in (2-4):

$$\sum_{i,l} F_\tau^{ii} u_{il}^2 \geq \frac{1}{2} \sum_{i,l} F_\tau^{ii} \tilde{u}_{il}^2 - \frac{1}{A_0}\mathcal{F}_\tau |\nabla u|^4. \qquad \square$$

**2.3.** ***Proof of Theorem 1.8.*** We begin this section by stating a central result in our argument, namely Proposition 2.2. The proof of Theorem 1.8 is then given assuming the validity of Proposition 2.2 — this should serve to elucidate the ideas outlined at the end of Section 2.1. The proof of Proposition 2.2 will be given later in the section and consists of a series of technical lemmas.

To this end, for $1 > \delta_0 \geq A_0^{-1/10}$ a small number to be fixed later, define the set

$$\mathcal{I} = \left\{ i \in \{1, \dots, n\} : \left| w_{jj} + \tfrac{1}{2}|\nabla u|^2 \right| < 2\delta_0^2 |\nabla u|^2 \right\}.$$

We remind the reader that all computations are implicitly carried out at $x_0$, and that we have the ordering $w_{11} \geq \cdots \geq w_{nn}$. We will prove:

**Proposition 2.2.** *There exists a constant $\widetilde{C} > 1$ depending only on $n$, $f$, $\Gamma$, $\|g_0\|_{C^3(B_r)}$ and $\|\psi\|_{C^1(B_r)}$ such that if $A_0^{-1/10} \leq \delta_0 \leq \widetilde{C}^{-1}$ and*

$$\sum_{i,l} F_\tau^{ii} \tilde{u}_{il}^2 < \widetilde{C}^{-1} \delta_0^4 |\nabla u|^4 \mathcal{F}_\tau, \tag{2-15}$$

*then*:

(1) $\mathcal{I} = \{n\}$, *and*

(2) $\left| w_{n-1,n-1} - \tfrac{1}{2}|\nabla u|^2 \right| < 2\delta_0 |\nabla u|^2$.

Assuming the validity of Proposition 2.2 for now, let us complete the proof of Theorem 1.8.

*Proof of Theorem 1.8.* We start by fixing $\widetilde{C}$ sufficiently large so that Proposition 2.2 applies. Then, for $A_0 > \widetilde{C}^{10}$ to be fixed later, if $A_0^{-1/10} \leq \delta_0 \leq \widetilde{C}^{-1}$ and (2-15) is satisfied,

$$w_{n-1,n-1} = (1 + a_{n-1})\frac{|\nabla u|^2}{2} \quad \text{and} \quad w_{nn} = -(1 + a_n)\frac{|\nabla u|^2}{2}$$

for some $|a_{n-1}|, |a_n| \leq 4\delta_0$. On the other hand, since $w_{11} \geq \cdots \geq w_{nn}$ for each $\alpha = 1, \dots, n-2$, we can write $w_{\alpha\alpha} = w_{n-1,n-1} + X_\alpha$ for some $X_\alpha \geq 0$. Therefore

$$\begin{pmatrix} w_{11} \\ \vdots \\ w_{n-2,n-2} \\ w_{n-1,n-1} \\ w_{nn} \end{pmatrix} = \begin{pmatrix} X_1 \\ \vdots \\ X_{n-2} \\ 0 \\ 0 \end{pmatrix} + \frac{|\nabla u|^2}{2} \underbrace{\begin{pmatrix} 1 + a_{n-1} \\ \vdots \\ 1 + a_{n-1} \\ 1 + a_{n-1} \\ -(1 + a_n) \end{pmatrix}}_{\mathcal{B}}, \tag{2-16}$$

with the first vector on the right-hand side of (2-16) clearly belonging to $\overline{\Gamma^\tau}$ for each $\tau \le 1$ since each entry is nonnegative. We also observe that $\mathcal{B}$ is a perturbation of $\mathcal{B}_0 := (1, \dots, 1, -1)$ and that $\mathcal{B}_0 \in \Gamma^\tau$ for any $\tau \le 1$ since we assume $\mu_\Gamma^+ > 1$. Therefore, since $|a_{n-1}|, |a_n| \le 4\delta_0$, for $\widetilde{C}$ sufficiently large we will have $\mathcal{B} \in \Gamma$ with $f^\tau(\mathcal{B}) \ge \frac{1}{2} f^\tau(\mathcal{B}_0)$. Monotonicity of $f$ then implies

$$\psi e^{2u} = f^\tau(w_{11}, \dots, w_{nn}) \ge \tfrac{1}{2}|\nabla u|^2 f^\tau(\mathcal{B}) \ge \tfrac{1}{4}|\nabla u|^2 f^\tau(\mathcal{B}_0),$$

which implies the desired gradient estimate.

It remains to address the case that, for the value of $\widetilde{C}$ fixed in the foregoing argument, (2-15) is not satisfied. Then

$$\sum_{i,l} F_\tau^{ii} \tilde{u}_{il}^2 \ge \widetilde{C}^{-1} A_0^{-2/5} |\nabla u|^4 \mathcal{F}_\tau, \tag{2-17}$$

and substituting (2-17) into (2-7) we therefore have

$$0 \ge -C\mathcal{F}_\tau(1+e^{2u})|\nabla u|^2 - C\rho\mathcal{F}_\tau \frac{|\nabla u|^4}{A_0} + \widetilde{C}^{-1} A_0^{-2/5} \rho |\nabla u|^4 \mathcal{F}_\tau.$$

Multiplying through by $\widetilde{C} A_0^{2/5} \rho$ then yields the estimate

$$0 \ge -\widetilde{C} C A_0^{2/5} \rho(1+e^{2u})|\nabla u|^2 - \frac{\widetilde{C}C}{A_0^{3/5}} \rho^2 |\nabla u|^4 + \rho^2 |\nabla u|^4. \tag{2-18}$$

It follows that if we choose $A_0 \ge \max\{(2\widetilde{C}C)^{5/3}, \widetilde{C}^{10}\}$ (where $C$ and $\widetilde{C}$ are the constants in (2-18)), then we have (for a possibly different constant $C$)

$$0 \ge -C\rho(1+e^{2u})|\nabla u|^2 + \tfrac{1}{2}\rho^2 |\nabla u|^4, \tag{2-19}$$

and therefore

$$H^2 = \rho^2 |\nabla u|^4 \le C(1+e^{2u})H. \tag{2-20}$$

After dividing through by $H$ we again arrive at the desired gradient estimate. $\square$

The rest of the section is devoted to the proof of Proposition 2.2, which we obtain through a series of three lemmas. In the first of these lemmas we show that if $A_0^{-1/10} \le \delta_0 \le \widetilde{C}^{-1}$ for $\widetilde{C}$ sufficiently large, then $\mathcal{I} \ne \varnothing$.

**Lemma 2.3.** *There exists a constant $\widetilde{C} > 1$ depending only on $n$, $f$, $\Gamma$, $\|g_0\|_{C^3(B_r)}$ and $\|\psi\|_{C^1(B_r)}$ such that if $A_0^{-1/10} \le \delta_0 \le \widetilde{C}^{-1}$, then $\mathcal{I} \ne \varnothing$.*

*Proof.* It is clear that, for $\delta_0 \le \sqrt{1/n}$, there is at least one index $j \in \{1, \dots, n\}$ such that $u_j^2 \ge \delta_0^2 |\nabla u|^2$. We claim that, for such an index $j$, we have $j \in \mathcal{I}$. We follow the method in [Guan and Wang 2003]. We know that, for $l \ne j$, we have $u_{jl} = u_j u_l + S_{jl}$ and therefore

$$\sum_{l \ne j} u_{jl} u_l = \sum_{l \ne j} u_j u_l^2 + \sum_{l \ne j} S_{jl} u_l.$$

It follows that

$$\begin{aligned}\sum_{l=1}^{n} u_{jl}u_l &= \sum_{l\neq j} u_j u_l^2 + \sum_{l\neq j} S_{jl}u_l + u_{jj}u_j \\ &= u_j|\nabla u|^2 + \sum_{l\neq j} S_{jl}u_l + u_{jj}u_j - u_j^3 \\ &= \sum_{l\neq j} S_{jl}u_l - u_j((u_j^2 - |\nabla u|^2) - u_{jj}).\end{aligned}$$

Hence

$$\left| u_j((u_j^2 - |\nabla u|^2) - u_{jj}) - \sum_{l\neq j} S_{jl}u_l \right| = \left| \sum_{l=1}^{n} u_{jl}u_l \right| \overset{(2\text{-}5)}{\leq} C\frac{|\nabla u|^3}{A_0}.$$

It follows that

$$|u_j|\,|(u_j^2 - |\nabla u|^2) - u_{jj}| \leq C\frac{|\nabla u|^3}{A_0} + \left|\sum_{l\neq j} S_{jl}u_l\right| \overset{(2\text{-}4)}{\leq} C\frac{|\nabla u|^3}{A_0} \leq C\delta_0^{10}|\nabla u|^3, \tag{2-21}$$

where to reach the last inequality we have used $A_0^{-1/10} \leq \delta_0$. Substituting $|u_j| \geq \delta_0|\nabla u|$ back into (2-21) yields

$$|(u_j^2 - |\nabla u|^2) - u_{jj}| \leq C\delta_0^9|\nabla u|^2. \tag{2-22}$$

Next, substituting $u_{jj} = w_{jj} + u_j^2 - \frac{1}{2}|\nabla u|^2 + S_{jj}$ into (2-22) and again applying (2-4), we obtain

$$\left|w_{jj} + \tfrac{1}{2}|\nabla u|^2\right| \leq C\delta_0^9|\nabla u|^2 + \frac{|\nabla u|^2}{A_0} = C\delta_0^9|\nabla u|^2 + \delta_0^{10}|\nabla u|^2. \tag{2-23}$$

It is clear that one can then choose $\widetilde{C}$ sufficiently large so that the right-hand side of (2-23) is less than $2\delta_0^2|\nabla u|^2$ for $\delta_0 \leq \widetilde{C}^{-1}$. Once such a choice is made, we see that (2-23) implies $j \in \mathcal{I}$, which proves the claim and therefore the lemma. □

In our subsequent arguments we will use the following proposition, which is essentially a consequence of [Yuan 2022, Theorem 1.4] — see Appendix A for a summary of the proof.

**Proposition 2.4.** *Suppose* $\Gamma$ *satisfies* (1-1) *and* (1-2) *with* $\Gamma \neq \Gamma_n^+$ (*equivalently*, $\mu_\Gamma^+ > 0$). *Then there exists a constant* $\theta = \theta(n, \Gamma) > 0$ *such that*, *for any* $\lambda \in \Gamma$ *with* $\lambda_1 \geq \cdots \geq \lambda_n$,

$$\frac{\partial f}{\partial \lambda_i}(\lambda) \geq \theta \sum_{j=1}^{n} \frac{\partial f}{\partial \lambda_j}(\lambda) \quad \textit{if } i \in \{n-1, n\} \textit{ or } \lambda_i \leq 0. \tag{2-24}$$

We are now in a position to show that if one additionally assumes (2-15) holds for $\widetilde{C}$ sufficiently large, then $|\mathcal{I}| = \{n\}$ (recall once again the ordering $w_{11} \geq \cdots \geq w_{nn}$).

**Lemma 2.5.** *There exists a constant* $\widetilde{C} > 1$ *depending only on* $n$, $f$, $\Gamma$, $\|g_0\|_{C^3(B_r)}$ *and* $\|\psi\|_{C^1(B_r)}$ *such that if* $A_0^{-1/10} \leq \delta_0 \leq \widetilde{C}^{-1}$ *and* (2-15) *is satisfied*, *then* $|\mathcal{I}| = \{n\}$.

*Proof.* We first claim that if $\widetilde{C}$ is sufficiently large and (2-15) holds, then $u_{jj} > -2\delta_0^2|\nabla u|^2$ for $j \in \mathcal{I}$. Indeed, suppose for a contradiction that this is not the case. Then we would have

$$\sum_{i,l} F_\tau^{ii}\tilde{u}_{il}^2 \geq F_\tau^{jj}\tilde{u}_{jj}^2 \overset{(2\text{-}4)}{\geq} \frac{1}{2}F_\tau^{jj}u_{jj}^2 - F_\tau^{jj}\frac{|\nabla u|^4}{A_0^2} \geq 2F_\tau^{jj}\delta_0^4|\nabla u|^4 - F_\tau^{jj}\delta_0^{20}|\nabla u|^4$$
$$\geq F_\tau^{jj}\delta_0^4|\nabla u|^4 \geq \theta\delta_0^4|\nabla u|^4\mathcal{F}_\tau, \tag{2-25}$$

with the last inequality following from Proposition 2.4 — note that Proposition 2.4 applies in this case since $w_{jj} < 0$ by virtue of $j \in \mathcal{I}$ if $\widetilde{C}$ is sufficiently large. But this contradicts (2-15) if $\widetilde{C}$ is sufficiently large, proving the claim.

By the claim, we may therefore suppose that $\widetilde{C}$ is large enough so that $u_{jj} > -2\delta_0^2|\nabla u|^2$ whenever $j \in \mathcal{I}$. Then, for $j \in \mathcal{I}$, we therefore have

$$-2\delta_0^2|\nabla u|^2 - u_j^2 + \tfrac{1}{2}|\nabla u|^2 - S_{jj} < u_{jj} - u_j^2 + \tfrac{1}{2}|\nabla u|^2 - S_{jj} = w_{jj} < -\tfrac{1}{2}|\nabla u|^2 + 2\delta_0^2|\nabla u|^2,$$

with the last inequality following from the definition of $\mathcal{I}$. That is,

$$-u_j^2 < (-1+4\delta_0^2)|\nabla u|^2 + S_{jj} \overset{(2\text{-}4)}{<} (-1+4\delta_0^2)|\nabla u|^2 + \delta_0^{10}|\nabla u|^2 < (-1+5\delta_0^2)|\nabla u|^2. \tag{2-26}$$

Clearly (2-26) cannot hold for more than one index if $10\delta_0^2 < 1$. Hence $|\mathcal{I}| \leq 1$ for $\widetilde{C}$ sufficiently large, and after increasing $\widetilde{C}$ further if necessary so that $\mathcal{I} \neq \varnothing$ (recall that this is possible by Lemma 2.3), it must be the case that $|\mathcal{I}| = 1$, i.e., $\mathcal{I} = \{n\}$. $\square$

To finish the proof of Proposition 2.2 it remains to show (after taking $\widetilde{C}$ larger if necessary) that $\left|w_{n-1,n-1} - \frac{1}{2}|\nabla u|^2\right| < 2\delta_0|\nabla u|^2$. This is the focus of the next lemma.

**Lemma 2.6.** *There exists a constant $\widetilde{C} > 1$ depending only on $n$, $f$, $\Gamma$, $\|g_0\|_{C^3(B_r)}$ and $\|\psi\|_{C^1(B_r)}$ such that if $A_0^{-1/10} \leq \delta_0 \leq \widetilde{C}^{-1}$ and* (2-15) *is satisfied, then*

$$\left|w_{n-1,n-1} - \tfrac{1}{2}|\nabla u|^2\right| < 2\delta_0|\nabla u|^2.$$

*Proof.* **Step 1:** In this first step we show

$$w_{n-1,n-1} > \left(\tfrac{1}{2} - 2\delta_0\right)|\nabla u|^2. \tag{2-27}$$

Suppose for a contradiction that $w_{n-1,n-1} \leq \left(\frac{1}{2} - 2\delta_0\right)|\nabla u|^2$, i.e.,

$$u_{n-1,n-1} - u_{n-1}^2 - S_{n-1,n-1} \leq -2\delta_0|\nabla u|^2. \tag{2-28}$$

Either $u_{n-1}^2 < \delta_0|\nabla u|^2$ or $u_{n-1}^2 \geq \delta_0|\nabla u|^2$. In the former case, (2-28) then implies

$$u_{n-1,n-1} < -\delta_0|\nabla u|^2 + S_{n-1,n-1} \overset{(2\text{-}4)}{<} -\delta_0|\nabla u|^2 + \delta_0^{10}|\nabla u|^2 < -\tfrac{1}{2}\delta_0|\nabla u|^2 \tag{2-29}$$

if $\delta_0 < \frac{1}{2}$, and one obtains a contradiction as in (2-25) if $\widetilde{C}$ is sufficiently large — note that Proposition 2.4 is again justified since $w_{n-1,n-1}$ is the second lowest eigenvalue. If instead $u_{n-1}^2 \geq \delta_0|\nabla u|^2$, the proof of Lemma 2.3 shows that $n-1 \in \mathcal{I}$. This contradicts the conclusion $|\mathcal{I}| = \{n\}$ of Lemma 2.5 if $\widetilde{C}$ is sufficiently large. Thus (2-27) is established, which completes the proof of Step 1.

**Step 2:** In this second step we show

$$w_{n-1,n-1} < \left(\tfrac{1}{2} + 2\delta_0\right)|\nabla u|^2. \tag{2-30}$$

Indeed, we have

$$w_{n-1,n-1} = u_{n-1,n-1} - u_{n-1}^2 + \tfrac{1}{2}|\nabla u|^2 - S_{n-1,n-1} \overset{(2\text{-}4)}{\leq} |u_{n-1,n-1}| + \tfrac{1}{2}|\nabla u|^2 + \delta_0^{10}|\nabla u|^2.$$

But $|u_{n-1,n-1}| \leq \delta_0 |\nabla u|^2$, else one would obtain a contradiction as in (2-25) if $\widetilde{C}$ is sufficiently large (again we are using the fact $w_{n-1,n-1}$ is the second lowest eigenvalue, so Proposition 2.4 applies). The estimate (2-30) thus follows, which completes the proof of Step 2.

With (2-27) and (2-30) established, the proof of Lemma 2.6 is complete. □

*Proof of Proposition 2.2.* This is an immediate consequence of Lemmas 2.3, 2.5 and 2.6. □

## 3. Proof of Theorem 1.6: the Dirichlet boundary value problem

As discussed in the introduction, in the proof of Theorem 1.1′, we will first address the corresponding Dirichlet boundary value problem with finite boundary data. To this end, in this section we prove Theorem 1.6. Our proof uses the continuity method, and we proceed according to the following steps:

(1) In Section 3.1 we give a routine proof of the global upper bound on solutions for $\tau \leq 1$, independent of whether or not $\mu_\Gamma^+ > 1$.

(2) In Section 3.2 we prove the global lower bound on solutions for $\tau \leq 1$ when $\mu_\Gamma^+ > 1$. As outlined in the introduction, we use two main ingredients: our local interior gradient estimate obtained in Theorem 1.8 and a lower bound in a uniform neighbourhood of $\partial M$, which is obtained by constructing suitable comparison functions on small annuli (see Propositions 3.3 and 3.4).

(3) In Section 3.3 we prove the global gradient estimate for $\tau \leq 1$ when $\mu_\Gamma^+ > 1$. To obtain the lower bound for the normal derivative on $\partial M$, we use our comparison functions on small annuli constructed in Section 3.2, and to obtain the upper bound for the normal derivative on $\partial M$, we use comparison functions similar to that of [Guan 2008] (this latter argument does not use $\mu_\Gamma^+ > 1$). For the interior estimates we use Theorem 1.8, and for estimates near $\partial M$ we appeal to the proof of Theorem 1.8.

(4) In Section 3.4 we prove the global Hessian estimate for $\tau < 1$ following arguments of [Guan 2008]. These estimates apply whether or not $\mu_\Gamma^+ > 1$.

(5) In Section 3.5 we complete the proof of Theorem 1.6: we first prove the existence of a unique smooth solution when $\tau < 1$ using the continuity method, and we then obtain a Lipschitz viscosity solution in the case $\tau = 1$ in the limit as $\tau \to 1$.

We point out that, in order to obtain a Lipschitz viscosity solution in the limit $\tau \to 1$ in Section 3.5, it is important that our a priori $C^1$ estimates obtained in Sections 3.1–3.3 are uniform in $\tau \in [0, 1]$. On the other hand, the global Hessian estimate in Section 3.4 deteriorates as $\tau \to 1$; this is to be expected in view of the work in [Li and Nguyen 2021; Li et al. 2023], where the nonexistence of $C^2$ solutions is established for all Euclidean domains with disconnected smooth boundary when $\tau = 1$.

**3.1.** ***Upper bound.*** The global upper bound on solutions to (1-11) is routine and does not require the assumption $\mu_\Gamma^+ > 1$.

**Proposition 3.1.** *Suppose* $(f, \Gamma)$ *satisfies* (1-1)–(1-4), *and let* $\tau \le 1$. *Let* $\psi \in C^\infty(M)$ *be positive and* $\xi \in C^\infty(\partial M)$. *Then there exists a constant C which is independent of* $\tau$ *but dependent on* $g_0$, $f$, $\Gamma$, *a lower bound for* $\inf_M \psi$ *and an upper bound for* $\sup_{\partial M} \xi$ *such that any* $C^2$ *solution to* (1-11) *satisfies* $u \le C$ *on* $M$.

*Proof.* Suppose the maximum of $u$ occurs at $x_0 \in M$. If $x_0 \in \partial M$, then $u(x_0) \le \xi(x_0)$. If $x_0 \in M \setminus \partial M$, then $\nabla_{g_0}^2 u(x_0) \le 0$ and $du(x_0) = 0$, and hence

$$\psi(x_0)e^{2u(x_0)} \le f^\tau(-g_0^{-1}A_{g_0})(x_0),$$

which yields

$$u(x_0) \le \frac{1}{2}\ln\biggl(\frac{f^\tau(-g_0^{-1}A_{g_0})}{\psi}\biggr)(x_0). \qquad \square$$

**3.2.** ***Lower bound.*** In this section we obtain the global lower bound on solutions to (1-11).

**Proposition 3.2.** *Suppose* $(f, \Gamma)$ *satisfies* (1-1)–(1-4) *and* (1-8), *and let* $\tau \le 1$. *Let* $\psi \in C^\infty(M)$ *be positive and* $\xi \in C^\infty(\partial M)$. *Then there exists a constant C which is independent of* $\tau$ *but dependent on* $g_0$, $f$, $\Gamma$, *an upper bound for* $\|\psi\|_{C^1(M)}$ *and a lower bound for* $\inf_{\partial M} \xi$ *such that any* $C^3$ *solution to* (1-11) *satisfies* $u \ge C$ *on* $M$.

There are two main ingredients in our proof of Proposition 3.2: our local interior gradient estimate from Theorem 1.8 and a lower bound in a uniform neighbourhood of $\partial M$; the assumption $\mu_\Gamma^+ > 1$ plays a role at both stages. As pointed out before, a delicate point is that we do not assume that the background metric satisfies $\lambda(-g_0^{-1}A_{g_0}) \in \Gamma$ on $M$ — if such an assumption is made, then the proof of the lower bound is as straightforward as the proof of Proposition 3.1. In our case, the global lower bound requires more work and is one of the key steps in this paper.

To state our result concerning the lower bound near $\partial M$, for $\delta > 0$, we define

$$M_\delta = \{x \in M : \mathrm{d}(x, \partial M) < \delta\},$$

where $\mathrm{d}(x, \partial M)$ is the distance from $x$ to $\partial M$ with respect to $g_0$. It is well known that, for $\delta > 0$ sufficiently small, $M_\delta$ is a tubular neighbourhood of $\partial M$. We show the following.

**Proposition 3.3.** *Under the same hypotheses as Proposition 3.2, there exists a constant* $\delta > 0$ *which is independent of* $\tau$ *but dependent on* $g_0$, $f$, $\Gamma$, *an upper bound for* $\sup_M \psi$ *and a lower bound for* $\inf_{\partial M} \xi$ *such that any* $C^3$ *solution u to* (1-11) *satisfies* $u \ge \inf_{\partial M} \xi - 1$ *in* $M_\delta$.

Assuming the validity of Proposition 3.3 for now, we give the proof of Proposition 3.2.

*Proof of Proposition 3.2.* Let $\delta > 0$ be as in the statement of Proposition 3.3, so that $u$ satisfies the lower bound $u \ge \inf_{\partial M} \xi - 1$ in $M_\delta$. It follows that

$$u \ge \inf_{\partial M} \xi - 1 - \mathrm{diam}(M, g_0) \sup_{M \setminus M_\delta} |\nabla_{g_0} u|_{g_0} \quad \text{in } M. \tag{3-1}$$

On the other hand, by Theorem 1.8 and the uniform upper bound for $u$ obtained in Proposition 3.1, we have

$$|\nabla_{g_0} u|_{g_0} \leq C(\delta^{-1}+1) \quad \text{in } M \setminus M_\delta. \tag{3-2}$$

Substituting (3-2) into (3-1), the proof of Proposition 3.2 is complete. □

Roughly speaking, to prove Proposition 3.3 we cover a neighbourhood of $\partial M$ by small annuli on which we construct suitable comparison functions. The construction of such comparison functions is given in the following proposition (which is a more precise version of Proposition 1.11 stated in the introduction). For a Riemannian metric $g_0$ defined on a neighbourhood of the origin in $\mathbb{R}^n$, let $r(x) = \mathrm{d}_{g_0}(0, x)$, let $\mathbb{S}_r = \partial \mathbb{B}_r$ denote the geodesic sphere of radius $r$ centred at the origin, and denote by $A_{r_1,r_2}$ the annulus $\mathbb{B}_{r_2} \setminus \overline{\mathbb{B}}_{r_1}$. We also write

$$\beta = \frac{2}{\mu_\Gamma^+ - 1}$$

and recall the convention $g_w = e^{2w} g_0$.

**Proposition 3.4.** *Suppose* $(f, \Gamma)$ *satisfies* (1-1)–(1-4) *and* (1-8), *and let* $g_0$ *be a Riemannian metric defined on a neighbourhood* $\Omega$ *of the origin in* $\mathbb{R}^n$. *Fix a constant* $\varepsilon > 0$. *Then there exists a constant* $C > 1$ *depending only on* $g_0$, $f$ *and* $\Gamma$, *and a constant* $0 < R < 1$ *depending additionally on* $\varepsilon$, *such that*, *for each* $m \in \mathbb{R}$,

$$w(r) := (\beta+\varepsilon)\ln\left(\frac{r_+ - r}{r_+ - r_-}\right) + m \tag{3-3}$$

*satisfies*

$$\begin{cases} f(\lambda(-g_w^{-1}A_{g_w})) \geq \dfrac{f(-\mu_\Gamma^+ + C^{-1}\varepsilon, 1, \dots, 1)}{Ce^{2m}(r_+ - r_-)^2} > 0, \quad \lambda(-g_w^{-1}A_{g_w}) \in \Gamma & \text{on } A_{r_-,r_+}, \\ w(x) = m & \text{for } x \in \mathbb{S}_{r_-}, \\ w(x) \to -\infty & \text{as } \mathrm{d}(x, \mathbb{S}_{r_+}) \to 0, \end{cases} \tag{3-4}$$

*whenever* $1 < r_+/r_- < 1 + \varepsilon/(2(\beta+2))$ *and* $r_+ < R$.

**Remark 3.5.** Our choice of $w$ in (3-3) is motivated by the work of Chang, Han and Yang [Chang et al. 2005] on radial solutions to the $\sigma_k$-Yamabe equation on annular domains in $\mathbb{R}^n$. Indeed, when $\varepsilon = 0$ and $\mu_\Gamma^+ = (n-k)/k$, (3-8) corresponds to the leading order term in the solution to the $\sigma_k$-Yamabe equation in $\Gamma_k^-$ on annular domains in $\mathbb{R}^n$ for $k < \frac{1}{2}n$.

**Remark 3.6.** We reiterate that Proposition 3.4 relies crucially on the assumption $\mu_\Gamma^+ > 1$ and that a similar construction is not possible when $\mu_\Gamma^+ \leq 1$ — see Remark 1.12 in the introduction.

Assuming the validity of Proposition 3.4 for now, we first give the proof of Proposition 3.3 — the reader may wish to refer to Figure 1 in the following argument.

*Proof of Proposition 3.3.* We attach a collar neighbourhood $N$ to $\partial M$ such that $g_0$ extends smoothly to $M \cup N$; we denote this extension also by $g_0$. Let

$$D = \inf_{x \in \partial M} \mathrm{d}_{g_0}(x, \partial(M \cup N))$$

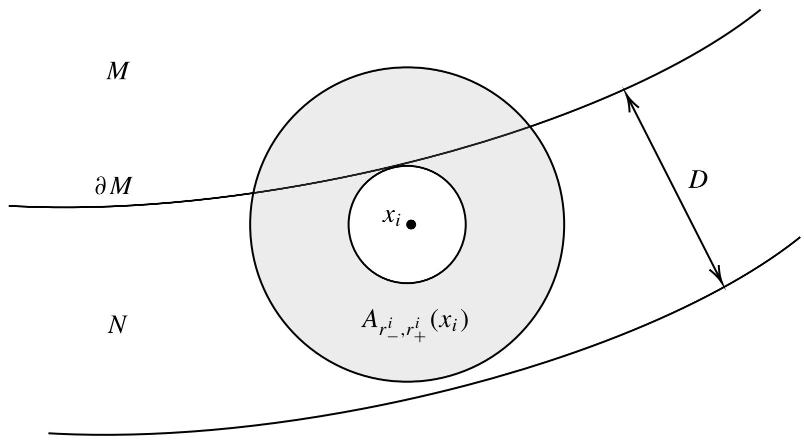


**Figure 1.** An annulus in the covering of a neighbourhood of $\partial M$ in $M$ in the proof of Proposition 3.3.

denote the thickness of $N$. Fix $\varepsilon > 0$ and let $m = \inf_{\partial M} \xi$, and cover a neighbourhood of $\partial M$ in $M$ by a finite collection of annuli $\{A_{r_-^i,r_+^i}(x_i)\}_{1\le i\le K}$ centred at $x_i$ such that the collection $\{A_{r_-^i,(r_-^i+r_+^i)/2}(x_i)\}$ still covers a neighbourhood of $\partial M$ in $M$, and such that, for each $i$,

(1) $x_i \in N$,

(2) $r_-^i + r_+^i < D$,

(3) $r_-^i = \mathrm{d}_{g_0}(x_i, \partial M)$,

(4) the closed ball $\overline{B_{r_+^i}(x_i)}$ is contained in a single normal coordinate chart $(U_i, \zeta_i)$ mapping $x_i$ to the origin,

(5) $$\frac{r_+^i}{r_-^i} \le 1 + \frac{\varepsilon}{2(\beta+2)},$$

(6) $r_i^+ < R$ is sufficiently small so that
$$\frac{f(-\mu_\Gamma^+ + C^{-1}\varepsilon, 1, \dots, 1)}{Ce^{2m}(r_+^i - r_-^i)^2} \ge \sup_M \psi$$
(here $C$ and $R$ are as in the statement of Proposition 3.4, where we are implicitly identifying the annulus $A_{r_-^i,r_+^i}(x_i)$ with its image under $\zeta_i$, which is possible by property (4)).

In what follows, we continue to implicitly make the identification between $A_{r_-^i,r_+^i}(x_i)$ and its image under $\zeta_i$.

Let $w_i$ denote the solution obtained in Proposition 3.4 on $A_{r_-^i,r_+^i}(x_i)$ with $\varepsilon > 0$ and $m = \inf_{\partial M} \xi$ as fixed above. Since $w_i$ is radially decreasing and $w_i(x) = \inf_{\partial M} \xi$ for $x \in \mathbb{S}_{r_-^i}(x_i)$, we have $w_i \le \inf_{\partial M} \xi$ on $A_{r_-^i,r_+^i}(x_i) \cap \partial M$. On the other hand, $w_i = -\infty < u$ on $\mathbb{S}_{r_+^i}(x_i)$. Therefore, the comparison principle (see Proposition 3.7 below) yields $u \ge w_i$ on $A_{r_-^i,r_+^i}(x_i) \cap M$ for each $i$. This yields a finite lower bound for $u$ on $A_{r_-^i,(r_-^i+r_+^i)/2}(x_i)$. Since we assume the collection $\{A_{r_-^i,(r_-^i+r_+^i)/2}(x_i)\}$ still covers a neighbourhood of $\partial M$ in $M$, we may piece together the estimates for $u$ on each annulus $A_{r_-^i,(r_-^i+r_+^i)/2}(x_i)$ to obtain the desired estimate for $u$ on a uniform neighbourhood of $\partial M$ in $M$. $\square$

In the above proof we made use of the following comparison principle.

**Proposition 3.7** (comparison principle). *Let $\alpha > 0$ be a positive constant and $(M, g)$ a compact Riemannian manifold with nonempty boundary $\partial M$. Suppose $u, v \in C^0(M)$ with at least one of $u$ or $v$ belonging to $C^2(M \setminus \partial M)$. If $f(-g_u^{-1}A_{g_u}) \geq f(-g_v^{-1}A_{g_v}) \geq \alpha > 0$ in the viscosity sense on $M \setminus \partial M$ and $u \leq v$ on $\partial M$, then $u \leq v$ in $M$.*

In the proof of Proposition 3.3, we only needed Proposition 3.7 in the case that both $u, v \in C^2(M \setminus \partial M)$. In this case, the proof of Proposition 3.7 is standard in light of the fact that if $f(-g_v^{-1}A_{g_v}) > 0$, $c$ is a positive constant and $w = v + c$, then $f(-g_w^{-1}A_{g_w}) < f(-g_v^{-1}A_{g_v})$. The case when $u \in C^0(M)$ in Proposition 3.7 will be needed later in the paper. When $u \in C^2(M \setminus \partial M)$, Proposition 3.7 follows from [Caffarelli et al. 2013, Theorem 2.1], since the proof on page 130 therein applies also on Riemannian manifolds with boundary. When $v \in C^2(M \setminus \partial M)$, Proposition 3.7 again follows from [Caffarelli et al. 2013, Theorem 2.1], therein considering $\widetilde{F}(x, s, p, M) := -F(x, -s, -p, -M)$ in place of $F$.

We now give the proof of Proposition 3.4.

*Proof of Proposition 3.4.* It will be more convenient to write our conformal metrics in the form $g^v = v^{-2}g_0$, so that $g_w = g^v$ for $e^{2w} = v^{-2}$. Then the (0, 2)-Schouten tensor of $g^v$ is given by

$$(A_{g^v})_{ij} = v^{-1}(\nabla^2_{g_0}v)_{ij} - \tfrac{1}{2}v^{-2}|\nabla_{g_0}v|^2_{g_0}(g_0)_{ij} + (A_{g_0})_{ij}.$$

In a fixed normal coordinate system based at the origin, it follows that if $v = v(r)$ then

$$((g^v)^{-1}A_{g^v})^p_j = v^2\left(\lambda\delta^p_j + \chi\frac{x^p x_j}{r^2}\right) + O(r^2)v|v_{rr}| + O(r)v|v_r| + O(1)v^2 \quad \text{as } r \to 0, \tag{3-5}$$

where

$$\lambda = \frac{v_r}{rv}\left(1 - \frac{rv_r}{2v}\right) \quad \text{and} \quad \chi = \frac{v_{rr}}{v} - \frac{v_r}{vr}; \tag{3-6}$$

we refer the reader to Appendix B for the derivation of (3-5). Therefore

$$(-(g^v)^{-1}A_{g^v})^p_j \geq -v^2\left(\lambda\delta^p_j + \chi\frac{x^p x_j}{r^2}\right) - |\Psi|\delta^p_j \tag{3-7}$$

in the sense of matrices, where $|\Psi| = O(r^2)v|v_{rr}| + O(r)v|v_r| + O(1)v^2$ as $r \to 0$.

**Step 1:** In this first step we compute and estimate the quantities on the right-hand side of (3-7) for our particular choice of $w$ in (3-3), i.e., for

$$v(r) = e^{-\Lambda}(r_+ - r)^{-\beta-\varepsilon}, \tag{3-8}$$

where we have written $\Lambda = m - (\beta + \varepsilon)\ln(r_+ - r_-)$. For shorthand we write $\varphi(r) = r_+ - r$. Then

$$v_r = e^{-\Lambda}(\beta + \varepsilon)\varphi^{-\beta-\varepsilon-1} \quad \text{and} \quad v_{rr} = e^{-\Lambda}(\beta + \varepsilon)(\beta + \varepsilon + 1)\varphi^{-\beta-\varepsilon-2}, \tag{3-9}$$

from which it follows that

$$\frac{v_r}{rv} = (\beta + \varepsilon)r^{-1}\varphi^{-1}, \quad \frac{rv_r}{2v} = \frac{\beta + \varepsilon}{2}r\varphi^{-1} \quad \text{and} \quad \frac{v_{rr}}{v} = (\beta + \varepsilon)(\beta + \varepsilon + 1)\varphi^{-2}.$$

Therefore

$$\lambda = \frac{v_r}{rv}\left(1 - \frac{rv_r}{2v}\right) = (\beta+\varepsilon)r^{-1}\varphi^{-1}\left(1 - \frac{\beta+\varepsilon}{2}r\varphi^{-1}\right) \tag{3-10}$$

and

$$\chi = \frac{v_{rr}}{v} - \frac{v_r}{vr} = -(\beta+\varepsilon)r^{-1}\varphi^{-1}(1-(\beta+\varepsilon+1)r\varphi^{-1}). \tag{3-11}$$

For $\Psi$ we estimate using (3-9) to get

$$\begin{aligned}|\Psi| &\le Cr^2 v|v_{rr}| + Crv|v_r| + Cv^2 \\ &\le Cre^{-2\Lambda}\varphi^{-2\beta-2\varepsilon-2}(r+\varphi+r^{-1}\varphi^2) \le C_1 re^{-2\Lambda}\varphi^{-2\beta-2\varepsilon-2} =: \eta,\end{aligned} \tag{3-12}$$

where to obtain the final estimate in (3-12) we have used the fact that $r, \varphi \le 1$ and

$$r^{-1}\varphi^2 \le \frac{r_+^2}{r_-} \le r_+\left(1 + \frac{\varepsilon}{2(\beta+2)}\right) \le C.$$

**Step 2:** We now use the computations from Step 1 to analyse the eigenvalues of the matrix on the right-hand side of (3-7), or more precisely the eigenvalues of

$$-v^2\left(\lambda\delta_j^p + \chi\frac{x^p x_j}{r^2}\right) - \eta\delta_j^p,$$

which are given by

$$-(\chi v^2 + \lambda v^2 + \eta, \lambda v^2 + \eta, \dots, \lambda v^2 + \eta).$$

We write this vector of eigenvalues more conveniently as

$$(-\lambda v^2 - \eta)\left(\frac{\chi v^2}{\lambda v^2 + \eta} + 1, 1, \dots, 1\right).$$

We make the following two claims:

Claim 1: There exist constants $c_1 > 0$ and $0 < R_1 < 1$ depending only on $g_0$, $f$ and $\Gamma$ such that

$$-\lambda v^2 - \eta > c_1 e^{-2\Lambda}\varphi^{-2\beta-2\varepsilon-2} \quad \text{in } \{r_- < r < r_+\} \tag{3-13}$$

whenever $1 < r_+/r_- < 1 + \varepsilon/(2(\beta+2))$ and $r_+ < R_1$.

Claim 2: There exists a constant $c_2 > 0$ depending only on $g_0$, $f$ and $\Gamma$, and a constant $0 < R_2 < 1$ depending additionally on $\varepsilon$ such that

$$\frac{\chi v^2}{\lambda v^2 + \eta} + 1 > -\mu_\Gamma^+ + c_2\varepsilon \quad \text{in } \{r_- < r < r_+\} \tag{3-14}$$

whenever $1 < r_+/r_- < 1 + \varepsilon/(2(\beta+2))$ and $r_+ < R_2$.

Once the claims are proved, Proposition 3.4 is obtained as follows. First fix $r_+$ and $r_-$ such that $1 < r_+/r_- < 1 + \varepsilon/(2(\beta+2))$ and $r_+ < \min\{R_1, R_2\}$. By Claim 2 and the definition of $\mu_\Gamma^+$,

$$f\left(\frac{\chi v^2}{\lambda v^2 + \eta} + 1, 1, \dots, 1\right) > f(-\mu_\Gamma^+ + c_2\varepsilon, 1, \dots, 1) > 0 \quad \text{in } \{r_- < r < r_+\}.$$

Then, by Claim 1, it follows that

$$f\left((-\lambda v^2-\eta)\left(\frac{\chi v^2}{\lambda v^2+\eta}+1, 1, \dots, 1\right)\right) > c_1 e^{-2\Lambda}\varphi^{-2\beta-2\varepsilon-2} f(-\mu_\Gamma^+ + c_2\varepsilon, 1, \dots, 1) \quad \text{in } \{r_- < r < r_+\},$$

from which (3-4) follows. To complete the proof of Proposition 3.4, it therefore remains to prove Claims 1 and 2.

**Note:** We will use at various stages the fact that

$$1 < \frac{r_+}{r_-} < 1 + \frac{\varepsilon}{2(\beta+2)} \quad \iff \quad 0 < \varphi r^{-1} < \frac{\varepsilon}{2(\beta+2)} \quad \text{in } \{r_- < r < r_+\}. \tag{3-15}$$

*Proof of Claim 1.* Suppose $1 < r_+/r_- < 1+\varepsilon/(2(\beta+2))$ and $r_+ < 1$. We start by computing

$$-\lambda v^2 = e^{-2\Lambda}\varphi^{-2\beta-2\varepsilon-2}(\beta+\varepsilon)\left(\frac{\beta+\varepsilon}{2} - \varphi r^{-1}\right). \tag{3-16}$$

By (3-15) and (3-16), it follows that

$$-\lambda v^2 \geq \frac{1}{C} e^{-2\Lambda}\varphi^{-2\beta-2\varepsilon-2} \quad \text{in } \{r_- < r < r_+\}. \tag{3-17}$$

Recalling also that

$$\eta = C_1 r e^{-2\Lambda}\varphi^{-2\beta-2\varepsilon-2}, \tag{3-18}$$

we see that (3-17) and (3-18) imply

$$-\lambda v^2 - \eta \geq (C^{-1} - C_1 r) e^{-2\Lambda}\varphi^{-2\beta-2\varepsilon-2} \quad \text{in } \{r_- < r < r_+\}. \tag{3-19}$$

The inequality (3-13) then follows from (3-19) after taking $r_+$ sufficiently small. This completes the proof of Claim 1. □

*Proof of Claim 2.* Suppose $1 < r_+/r_- < 1+\varepsilon/(2(\beta+2))$ and $r_+ < 1$. By (3-16) and the fact that $\mu_\Gamma^+ = (2+\beta)/\beta$, we have

$$-\lambda v^2 - \mu_\Gamma^+\lambda v^2 = -\frac{2+2\beta}{\beta}\lambda v^2 = \frac{2+2\beta}{\beta} e^{-2\Lambda}\varphi^{-2\beta-2\varepsilon-2}(\beta+\varepsilon)\left(\frac{\beta+\varepsilon}{2} - \varphi r^{-1}\right), \tag{3-20}$$

and, by the formula for $\chi$ in (3-11), we have

$$-\chi v^2 = e^{-2\Lambda}(\beta+\varepsilon)\varphi^{-2\beta-2\varepsilon-2}(\varphi r^{-1} - (\beta+\varepsilon+1)). \tag{3-21}$$

It follows from (3-20) and (3-21) that

$$-\chi v^2 - \lambda v^2 - \mu_\Gamma^+\lambda v^2 = e^{-2\Lambda}\varphi^{-2\beta-2\varepsilon-2}\frac{\beta+\varepsilon}{\beta}(\varepsilon - (\beta+2)r^{-1}\varphi). \tag{3-22}$$

On the other hand, by (3-15), we have

$$\frac{\beta+\varepsilon}{\beta}(\varepsilon - (\beta+2)r^{-1}\varphi) > \frac{\varepsilon}{2} \quad \text{in } \{r_- < r < r_+\},$$

which when substituted into (3-22) yields

$$-\chi v^2 - \lambda v^2 - \mu_\Gamma^+ \lambda v^2 > \frac{\varepsilon}{2} e^{-2\Lambda} \varphi^{-2\beta-2\varepsilon-2} \quad \text{in } \{r_- < r < r_+\}. \tag{3-23}$$

Recalling (3-18), the estimate (3-23) therefore implies

$$-\chi v^2 - \lambda v^2 - \mu_\Gamma^+ \lambda v^2 - \eta - \mu_\Gamma^+ \eta \geq \Big(\frac{\varepsilon}{2} - Cr\Big) e^{-2\Lambda} \varphi^{-2\beta-2\varepsilon-2} \quad \text{in } \{r_- < r < r_+\}. \tag{3-24}$$

After taking $r_+$ smaller if necessary (but in a way that only depends on $\varepsilon$ and the constant $C$ in (3-24)), we therefore have

$$-\chi v^2 - \lambda v^2 - \mu_\Gamma^+ \lambda v^2 - \eta - \mu_\Gamma^+ \eta \geq \frac{\varepsilon}{4} e^{-2\Lambda} \varphi^{-2\beta-2\varepsilon-2} \quad \text{in } \{r_- < r < r_+\},$$

or equivalently

$$\frac{\chi v^2}{\lambda v^2 + \eta} + 1 \geq -\mu_\Gamma^+ + \frac{\frac{\varepsilon}{4} e^{-2\Lambda} \varphi^{-2\beta-2\varepsilon-2}}{-\lambda v^2 - \eta}. \tag{3-25}$$

On the other hand, by (3-16), we have

$$0 < -\lambda v^2 - \eta \leq -\lambda v^2 \leq C e^{-2\Lambda} \varphi^{-2\beta-2\varepsilon-2} \quad \text{in } \{r_- < r < r_+\}.$$

Thus, if $r_+$ is chosen sufficiently small (but depending only on $g_0$, $f$, $\Gamma$ and $\varepsilon$), we see

$$\frac{\chi v^2}{\lambda v^2 + \eta} + 1 \geq -\mu_\Gamma^+ + c\varepsilon \quad \text{in } \{r_- < r < r_+\},$$

as required. This completes the proof of Claim 2. □

As explained above, with Claims 1 and 2 established, the proof of Proposition 3.4 is complete. □

**3.3.** ***Gradient estimate.*** In this section we prove the global gradient estimate.

**Proposition 3.8.** *Suppose* $(f, \Gamma)$ *satisfies* (1-1)–(1-4) *and* (1-8), *and let* $\tau \leq 1$. *Let* $\psi \in C^\infty(M)$ *be positive and* $\xi \in C^\infty(\partial M)$. *Then there exists a constant C which is independent of* $\tau$ *but dependent on* $g_0$, $f$, $\Gamma$ *and upper bounds for* $\|\psi\|_{C^1(M)}$, $\|\xi\|_{C^2(\partial M)}$ *and* $\|u\|_{C^0(M)}$ *such that any* $C^3$ *solution to* (1-11) *satisfies* $|\nabla_{g_0} u|_{g_0} \leq C$ *on* $M$.

*Proof.* By a conformal change of background metric, we may assume without loss of generality that $\xi \equiv 0$.

By our interior local gradient estimate in Theorem 1.8, we only need to prove the gradient estimate near the boundary, say in $B_{1/2}(y_0) \cap M$, where $y_0 \in \partial M$ is arbitrary. Consider $H = \rho |\nabla_{g_0} u|_{g_0}^2$, where $\rho$ is a smooth cutoff function satisfying $\rho = 1$ on $B_{1/2}(y_0)$, $\rho = 0$ outside $B_1(y_0)$, $|\nabla_{g_0} \rho|_{g_0} \leq C\rho^{1/2}$ and $|\nabla_{g_0}^2 \rho|_{g_0} \leq C$. Suppose that $H$ attains its maximum at $x_0 \in M$. If $x_0 \notin B_1(y_0) \cap M$, then $\nabla_{g_0} u = 0$ in $B_{1/2}(y_0) \cap M$ and we are done. If $x_0 \in B_1(y_0) \cap (M \setminus \partial M)$, then our proof of Theorem 1.8 applies and we again obtain the desired estimate. It remains to consider the case that $x_0 \in B_1(y_0) \cap \partial M$.

We first observe that, since $\xi \equiv 0$ on $\partial M$, the tangential derivatives of $u$ on $\partial M$ vanish. Therefore, we only need to bound the normal derivative $\nabla_\nu u(x_0)$, where $\nu$ denotes the inward pointing unit normal to $\partial M$ at $x_0$. We first consider the lower bound for $\nabla_\nu u(x_0)$. With the same setup and notation as in the proof of Proposition 3.3, except now with $m = u(x_0) = 0$, consider an annulus $A_{r_-,r_+}(y)$ satisfying $\mathbb{S}_{r_-}(y) \cap \partial M = \{x_0\}$ and conditions (1)–(6) in the proof of Proposition 3.3. Then the function $w$ on $A_{r_-,r_+}(y)$, as defined in (3-3), satisfies $w \le u$ on $A_{r_-,r_+}(y) \cap \partial M$ since $w(x_0) = u(x_0)$ and $w$ is radially decreasing. By the comparison principle stated in Proposition 3.7, it follows that $w \le u$ on $A_{r_-,r_+}(y) \cap M$. Thus, for $x \in A_{r_-,r_+}(y) \cap M$, we have

$$\frac{u(x) - u(x_0)}{\mathrm{d}(x, x_0)} = \frac{u(x) - w(x_0)}{\mathrm{d}(x, x_0)} \ge \frac{w(x) - w(x_0)}{\mathrm{d}(x, x_0)},$$

which implies

$$\nabla_\nu u(x_0) \ge \nabla_\nu w(x_0).$$

For the upper bound for $\nabla_\nu u(x_0)$, we use a barrier function constructed in [Guan 2008]. First observe that, since $\Gamma \subset \Gamma_1^+$, we have

$$0 < \sigma_1(-g_0^{-1} A_{g_u}) = \Delta_{g_0} u + \frac{n-2}{2} |\nabla_{g_0} u|_{g_0}^2 - \sigma_1(g_0^{-1} A_{g_0}).$$

Now let $d(x) = \mathrm{d}(x, \partial M)$ and recall $M_\delta = \{x \in M : d(x) < \delta\}$. It is well known that, for sufficiently small $\delta > 0$, $d$ is smooth in $M_\delta$ with $|\nabla_{g_0} d|_{g_0} = 1$. To obtain an upper bound for $\nabla_\nu u(x_0)$, it suffices to find a function $\bar{u} \in C^3(M_\delta)$ satisfying

$$\begin{cases} \sigma_1(-g_0^{-1} A_{g_{\bar{u}}}) \le 0 & \text{in } M_\delta, \\ \bar{u} = u & \text{on } \partial M, \\ \bar{u} \ge u & \text{on } \partial M_\delta \setminus \partial M. \end{cases} \tag{3-26}$$

Indeed, once such a function $\bar{u}$ is obtained, the maximum principle implies $\bar{u} \ge u$ on $M_\delta$, and it follows that, for any $x \in M_\delta$, we have

$$\frac{u(x) - u(x_0)}{\mathrm{d}(x, x_0)} = \frac{u(x) - \bar{u}(x_0)}{\mathrm{d}(x, x_0)} \le \frac{\bar{u}(x) - \bar{u}(x_0)}{\mathrm{d}(x, x_0)},$$

which implies $\nabla_\nu u(x_0) \le \nabla_\nu \bar{u}(x_0)$.

To this end, we define as in [Guan 2008]

$$\bar{u}(x) = \frac{1}{n-2} \ln \frac{d(x) + \delta^2}{\delta^2}.$$

We first observe that $\bar{u}|_{\partial M} = 0 = u|_{\partial M}$. Next we calculate $\sigma_1(-g_0^{-1} A_{g_{\bar{u}}})$. In what follows, we denote by $\nabla d$ the differential of $d$ (whereas $\nabla_{g_0} d$ will continue to denote the gradient of $d$ with respect to $g_0$). Routine computations yield

$$\nabla_{g_0} \bar{u}(x) = \frac{1}{n-2} \frac{\nabla_{g_0} d(x)}{d(x) + \delta^2}$$

and

$$\nabla_{g_0}^2 \bar{u}(x) = \frac{1}{n-2} \left( \frac{\nabla_{g_0}^2 d(x)}{d(x) + \delta^2} - \frac{\nabla d(x) \otimes \nabla d(x)}{(d(x) + \delta^2)^2} \right),$$

from which it follows that

$$\begin{aligned}\sigma_1(-g_0^{-1}A_{g_{\bar u}}) &= \Delta_{g_0}\bar u + \frac{n-2}{2}|\nabla_{g_0}\bar u|_{g_0}^2 - \sigma_1(g_0^{-1}A_{g_0})\\ &\le C - \frac{1}{2(n-2)}\frac{1}{(d(x)+\delta^2)^2} + \frac{C}{d(x)+\delta^2},\end{aligned} \tag{3-27}$$

where we have used the fact that $|\nabla_{g_0}d|_{g_0}=1$ and $|\Delta_{g_0}d|\le C$ in $M_\delta$ for $\delta$ sufficiently small. We then see that the negative term on the last line of (3-27) dominates the remaining terms for $\delta>0$ sufficiently small. Therefore, for $\delta>0$ sufficiently small, we have $\sigma_1(-g_0^{-1}A_{g_{\bar u}})\le 0$ in $M_\delta$.

Finally, we observe that on $\partial M_\delta\setminus\partial M$ we have

$$\bar u = \frac{1}{n-2}\ln\left(\frac{\delta+\delta^2}{\delta^2}\right) \ge \frac{1}{n-2}\ln(1/\delta).$$

Choosing $\delta$ smaller if necessary so that

$$\frac{1}{n-2}\ln(1/\delta)\ge \max_M u \quad \text{on } \partial M_\delta\setminus\partial M,$$

the construction of $\bar u$ is complete. This completes the proof of Proposition 3.8. □

**3.4. *Hessian estimate.*** In this section we give the global Hessian estimate assuming $\tau<1$.

**Proposition 3.9.** *Suppose $(f,\Gamma)$ satisfies* (1-1)–(1-4), *and let $\tau<1$. Let $\psi\in C^\infty(M)$ be positive and $\xi\in C^\infty(\partial M)$. Then there exists a constant $C$ depending on $g_0$, $f$, $\Gamma$, $(1-\tau)^{-1}$ and upper bounds for $\|\psi\|_{C^2(M)}$, $\|\xi\|_{C^2(M)}$ and $\|u\|_{C^1(M)}$ such that any solution to* (1-11) *satisfies $|\nabla_{g_0}^2 u|_{g_0}\le C$ on $M$.*

We point out that we do not require $\mu_\Gamma^+>1$ in Proposition 3.9.

*Proof.* If the maximum of $|\nabla_{g_0}^2u|_{g_0}$ occurs in $M\setminus\partial M$, then one can appeal to the proof of the global estimate in [Gursky and Viaclovsky 2003] if $f=\sigma_k^{1/k}$, or the proof of the global estimate in [Guan 2008] for general $(f,\Gamma)$ satisfying (1-1)–(1-4). So we suppose that the maximum occurs at a point $x_0\in\partial M$. Let $e_n$ denote the interior unit normal vector field on $\partial M$, and fix an orthonormal frame $\{e_1,\dots,e_{n-1}\}$ for the tangent bundle of $\partial M$ near $x_0$. By parallel transporting along geodesics normal to $\partial M$, we may extend this to an orthonormal frame $\{e_1,\dots,e_n\}$ for the tangent bundle of $M$ near $x_0$. Since $(\nabla_{g_0}^2u)_{ij}(x_0)=(\nabla_{g_0}^2\xi)_{ij}(x_0)$ for $i,j\ne n$, we only need to estimate $(\nabla_{g_0}^2u)_{ij}(x_0)$ when at least one of $i$ or $j$ are equal to $n$. The proof is almost identical to that in [Guan 2008], but for the convenience of the reader we summarise the argument here. In what follows, all computations are carried out in a neighbourhood of $x_0$ on which the frame $\{e_1,\dots,e_n\}$ is defined.

Still with the convention $g_u=e^{2u}g_0$, it will be convenient to write the equation in (1-11) in the equivalent form

$$f(\lambda(-g_0^{-1}A_{g_u}^\tau))=\psi e^{2u},\quad \lambda(-g_u^{-1}A_{g_u}^\tau)\in\Gamma \quad \text{on } M\setminus\partial M, \tag{3-28}$$

where

$$\begin{aligned}A_{g_u}^\tau &= \tau A_{g_u} + (1-\tau)\sigma_1(-g_u^{-1}A_{g_u})g_u\\ &= -\tau\nabla_{g_0}^2u-(1-\tau)\Delta_{g_0}u g_0 - b_{n,\tau}|\nabla_{g_0}u|_{g_0}^2 g_0 + \tau\, du\otimes du + A_{g_0}^\tau\end{aligned}$$

and $b_{n,\tau} = \frac{1}{2}(n-2-(n-3)\tau)$. Writing $F[u] = f(\lambda(-g_0^{-1}A_{g_u}^{\tau}))$ and

$$F^{ij} = \frac{\partial f}{\partial A_{ij}}\bigg|_{A=-g_0^{-1}A_{g_u}^{\tau}},$$

the linearisation of $F$ at $u$ in the direction $\eta$ (excluding zero-order terms) is given by

$$\begin{aligned}\mathcal{L}\eta &= F^{ij}(\tau(\nabla_{g_0}^2\eta)_{ij} + (1-\tau)\Delta_{g_0}\eta(g_0)_{ij} + 2b_{n,\tau}\langle\nabla_{g_0}u, \nabla_{g_0}\eta\rangle_{g_0}(g_0)_{ij} - 2\tau\,\partial_i u, \partial_j\eta)\\ &= F^{ij}(\tau(\nabla_{g_0}^2\eta)_{ij} - 2\tau\,\partial_i u\,\partial_j\eta) + ((1-\tau)\Delta_{g_0}\eta + 2b_{n,\tau}\langle\nabla_{g_0}u, \nabla_{g_0}\eta\rangle_{g_0})\sum_i F^{ii}.\end{aligned} \tag{3-29}$$

Now suppose $\delta > 0$ is sufficiently small so that $d(x) = \mathrm{d}(x, \partial M)$ is smooth in $M_\delta = \{x \in M : d(x) < \delta\}$. For a positive constant $N$ to be determined later, define

$$v = \frac{N}{2}d^2 - d. \tag{3-30}$$

A routine computation shows that, for $\delta > 0$ sufficiently small,

$$|\mathcal{L}d| \le C_0 \sum_i F^{ii} \quad \text{in } M_\delta, \tag{3-31}$$

where $C_0$ is a constant independent of $\tau$ but depending on $g_0$ and an upper bound for $\|u\|_{C^1(M)}$. It follows that

$$\begin{aligned}\mathcal{L}d^2 &= 2d\mathcal{L}d + 2(1-\tau)|\nabla_{g_0}d|_{g_0}^2\sum_i F^{ii} + 2F^{ij}\,\partial_i d\,\partial_j d\\ &\ge 2d\mathcal{L}d + 2(1-\tau)\sum_i F^{ii}\\ &\ge 2((1-\tau) - C_0 d)\sum_i F^{ii} \quad \text{in } M_\delta.\end{aligned} \tag{3-32}$$

Choosing $N \ge 4(1+C_0)/(1-\tau)$ and subsequently $\delta \le \min\{N^{-1}, C_0^{-1}\}$, one sees from (3-31) and (3-32) that the function $v$ defined in (3-30) satisfies

$$\mathcal{L}v \ge \sum_i F^{ii} \quad \text{and} \quad v \le -\frac{d}{2} \quad \text{in } M_\delta. \tag{3-33}$$

With (3-33) in hand, one can then show the following.

**Lemma 3.10.** *Fix $\delta > 0$ sufficiently small as in the foregoing argument. If $h \in C^2(\overline{M}_\delta)$ satisfies $h \le 0$ on $\partial M$, $h(z_0) = 0$ for some $z_0 \in \partial M$ and*

$$-\mathcal{L}h \le C_1 \sum_i F^{ii} \quad \textit{in } M_\delta \tag{3-34}$$

*for some constant $C_1$, then*

$$(\nabla_{g_0}h)_n(z_0) \le C, \tag{3-35}$$

*where $C$ is a constant depending on $g_0$, $C_1$, $(1-\tau)^{-1}$ and upper bounds for $\|h\|_{C^0(\overline{M}_\delta)}$ and $\|u\|_{C^1(M)}$.*

*Proof.* It is clear from the definition of $v$ that we can choose $A > 0$ large (depending on $\|h\|_{C^0(\overline{M}_\delta)}$) such that $-Av - h \geq 0$ on $\partial M_\delta$. On the other hand, using (3-33) and (3-34), we have

$$\mathcal{L}(-Av - h) \leq (-A + C_1) \sum_i F^{ii} \quad \text{in } M_\delta,$$

and hence $\mathcal{L}(-Av - h) \leq 0$ in $M_\delta$ for $A$ sufficiently large. Thus, for $A$ sufficiently large the maximum principle yields $-Av - h \geq 0$ in $M_\delta$, and since $(-Av - h)(z_0) = 0$, it follows that $(\nabla_{g_0}(-Av - h))_n(z_0) \geq 0$, i.e., $(\nabla_{g_0} h)_n(z_0) \leq -A(\nabla_{g_0} v)_n(z_0)$. The estimate (3-35) then follows. □

We now continue the proof of Proposition 3.9. Suppose $i \in \{1, \dots, n-1\}$ and define $h = \pm(\nabla_{g_0}(u - \bar{\xi}))_i$, where (as in the proof of Proposition 3.8) $\bar{\xi}$ denotes the extension of $\xi$ to $M_\delta$ such that $\bar{\xi}$ is constant along geodesics normal to $\partial M$. By differentiating (3-28), one can show directly that $|\mathcal{L}(\nabla_{g_0} u)_i| \leq C \sum_i F^{ii}$, and by (2-6) we also have $|\mathcal{L}\bar{\xi}| \leq C \leq C \sum_i F^{ii}$. Therefore $h$ satisfies the assumptions of Lemma 3.10, and it follows from Lemma 3.10 that

$$|(\nabla^2_{g_0} u)_{in}(x_0)| \leq C.$$

It remains to estimate the double normal derivative $(\nabla^2_{g_0} u)_{nn}(x_0)$. Note that, since $\{e_1, \dots, e_n\}$ is an orthonormal frame and $(\nabla^2_{g_0} u)_{ii}(x_0) = (\nabla^2_{g_0} \xi)_{ii}(x_0)$ for $i \in \{1, \dots, n-1\}$, obtaining an upper (resp. lower) bound for $(\nabla^2_{g_0} u)_{nn}(x_0)$ is equivalent to obtaining an upper (resp. lower) bound for $\Delta_{g_0} u(x_0)$. Now, since $\Gamma \subseteq \Gamma_1^+$, the lower bound $\Delta_{g_0} u \geq -C$ in $M$ is immediate. To obtain the upper bound for $(\nabla^2_{g_0} u)_{nn}(x_0)$, we may assume $(\nabla^2_{g_0} u)_{nn}(x_0) \geq 1$, otherwise we are done. We may also assume that, with respect to the frame $\{e_1, \dots, e_n\}$, the Hessian of $u$ at $x_0$ is given by $\nabla^2_{g_0} u(x_0) = \operatorname{diag}((\nabla^2_{g_0} u)_{11}(x_0), \dots, (\nabla^2_{g_0} u)_{nn}(x_0))$. Then, by (3-28), monotonicity of $f$ and our estimates for $(\nabla^2_{g_0} u)_{ij}(x_0)$ when $i$ and $j$ are not both equal to $n$, we have

$$\psi(x_0) e^{2u(x_0)} = f(-g_0^{-1} A^\tau_{g_u}(x_0)) \geq f((1-\tau)(\nabla^2_{g_0} u)_{nn}(x_0) g_0 + B), \tag{3-36}$$

where $B$ is a symmetric matrix bounded in terms of $\|u\|_{C^1(M)}$. Observing that, by homogeneity of $f$,

$$\frac{1}{t} f(t g_0 + B) = f(g_0 + t^{-1} B) \to f(g_0) \quad \text{as } t \to \infty,$$

we see that (3-36) implies an upper bound for $(\nabla^2_{g_0} u)_{nn}(x_0)$. □

**3.5.** ***Proof of Theorem 1.6.*** We first prove the existence of a smooth solution to (1-11) when $\tau < 1$. Fix $\varepsilon > 0$, and let $S_\varepsilon = \{\tau \in [0, 1-\varepsilon] \colon \text{(1-11) admits a solution in } C^{2,\alpha}(M)\}$. Since (1-11) admits a unique smooth solution when $\tau = 0$, $S_\varepsilon$ is nonempty. A computation as in (3-29) (but now including zero-order terms) shows that the linearised operator is invertible as a mapping from $C^{2,\alpha}(M)$ to $C^\alpha(M)$, from which openness of $S_\varepsilon$ follows. By Propositions 3.1 and 3.2, solutions to (1-11) admit a global $C^0$ estimate. By Proposition 3.8, solutions to (1-11) therefore admit a global $C^1$ estimate. Note that, at this point, the estimates are independent of $\varepsilon$. By Proposition 3.9, one then obtains the global $C^2$ estimate on solutions to (1-11), which do now depend on $\varepsilon$. With the $C^2$ estimate established, (1-11) becomes uniformly elliptic, and the regularity theory of Evans and Kyrlov [Evans 1982; Krylov 1982; 1983] then implies a $C^{2,\alpha}$ estimate. Thus $S_\varepsilon$ is also closed, and so $S_\varepsilon = [0, 1-\varepsilon]$. Since $\varepsilon > 0$ was arbitrary, existence of a $C^{2,\alpha}$

solution to (1-11) for any $\tau < 1$ then follows. Higher regularity then follows from classical Schauder theory, and uniqueness is a consequence of the comparison principle in Proposition 3.7.

Now, since the solutions obtained to (1-11) are uniformly bounded in $C^1(M)$ as $\tau \to 1$, along a sequence $\tau_i \to 1$ these solutions converge uniformly to some $u \in C^{0,1}(M)$. The proof that $u$ is a viscosity solution to (1-11) when $\tau = 1$ is exactly the same as in the proof of Theorem 1.3 in [Li and Nguyen 2021] and is omitted here. ☐

## 4. Proof of Theorem 1.1′: the fully nonlinear Loewner–Nirenberg problem

In this section we prove Theorem 1.1′. Our proof proceeds according to the following steps:

(1) In Section 4.1 we construct a smooth solution to (1-10) when $\tau < 1$. The solution is obtained as the limit of solutions with constant finite boundary data $m \in \mathbb{R}$ (which we know to exist by Theorem 1.6) as $m \to \infty$.

(2) In Section 4.2 we prove that there exists a smooth solution $u$ to (1-10) when $\tau < 1$ satisfying the asymptotics stated in (1-9).

(3) In Section 4.3 we prove that *any* smooth solution to (1-10) must satisfy (1-9) when $\tau < 1$. When combined with the maximum principle, this will imply that the solution $u$ obtained to (1-10) is unique when $\tau < 1$.

(4) In Section 4.4 we complete the proof of Theorem 1.1′.

**4.1.** ***Existence of a smooth solution to* (1-10) *when* $\tau < 1$.** Fix $\tau < 1$, and suppose that $(f, \Gamma)$ satisfies (1-1)–(1-4), (1-7) and (1-8). By Theorem 1.6, we know that, for each $m \in \mathbb{R}$, there exists a unique smooth solution $u_m$ to

$$\begin{cases} f^\tau(\lambda(-g_{u_m}^{-1}A_{g_{u_m}})) = 1, \quad \lambda(-g_{u_m}^{-1}A_{g_{u_m}}) \in \Gamma^\tau & \text{on } M \setminus \partial M, \\ u_m = m & \text{on } \partial M. \end{cases} \tag{4-1}$$

In this section we show that, in the limit $m \to \infty$, one obtains a smooth solution $u$ to (1-10).

**Proposition 4.1.** *Fix $\tau < 1$, and suppose that $(f, \Gamma)$ satisfies* (1-1)–(1-4), (1-7) *and* (1-8). *Let $u_m$ denote the unique smooth solution to* (4-1). *Then a subsequence of $\{u_m\}_m$ converges locally uniformly as $m \to \infty$ to a solution $u \in C^\infty(M \setminus \partial M)$ of* (1-10). *Moreover, given any constant $\alpha > 0$, there exists a constant $\delta > 0$ independent of $\tau$ but dependent on $g_0$, $\alpha$, $f$ and $\Gamma$ such that $u \geq \alpha$ in $M_\delta \setminus \partial M$.*

*Proof.* Since the comparison principle in Proposition 3.7 implies $u_{m+1} \geq u_m$, to prove the existence of a limit $u \in C^\infty(M \setminus \partial M)$ solving (1-10), it suffices to show that, for each compact set $K \subset M \setminus \partial M$, there exists a constant $C$ independent of $m$ such that $\|u_m\|_{C^2(K)} \leq C$; higher order estimates then follow from the work of Evans and Krylov [Evans 1982; Krylov 1982] and classical Schauder theory.

The lower bound is trivial (and in fact global) since $u_m \geq u_1$ for all $m$. Next we address the local upper bound — note that whilst we obtained a global upper bound in Proposition 3.1, the bound therein depends on $m$, which is insufficient for our current purposes. Recalling the normalisation $f\left(\frac{1}{2}, \dots, \frac{1}{2}\right) = 1$, we have by concavity and homogeneity of $f$

$$f(\lambda) \le f\left(\frac{\sigma_1(\lambda)}{n}e\right) + \nabla f\left(\frac{\sigma_1(\lambda)}{n}e\right)\cdot\left(\lambda - \frac{\sigma_1(\lambda)}{n}e\right) = \frac{f(e)}{n}\sigma_1(\lambda) = \frac{2}{n}\sigma_1(\lambda) \tag{4-2}$$

for $\lambda \in \Gamma$, and thus any solution to the equation in (4-1) satisfies $R_{g_{u_m}} \le -n(n-1)$. On the other hand, by [Aviles and McOwen 1988], there exists a smooth metric $g_w = e^{2w}g_0$ satisfying

$$\begin{cases} R_{g_w} = -n(n-1) & \text{on } M \setminus \partial M, \\ w(y) \to +\infty & \text{as } \mathrm{d}(y, \partial M) \to 0. \end{cases} \tag{4-3}$$

By the comparison principle for the semilinear equation (4-3), $u_m \le w$ in $M \setminus \partial M$ for each $m$, which yields a finite upper bound for $u_m$ on any compact subset of $M \setminus \partial M$ which is independent of $m$. The local gradient estimate then follows from Theorem 1.8, or alternatively one can appeal to [Guan 2008, Theorem 2.1] since we have the two-sided $C^0$ bound at this point. For the local Hessian estimate, we appeal to [Guan 2008, Theorem 3.1]. We therefore obtain the full $C^2$ estimate $\|u_m\|_{C^2(K)} \le C(K)$ on any compact set $K \subset M \setminus \partial M$, as required.

It remains to prove the second assertion in the statement of Proposition 4.1. Fix $\alpha > 0$ and consider the solution $u_{\alpha+1}$ to (4-1) with $m = \alpha + 1$. Since $u_{\alpha+1}$ admits a global $C^0$ estimate depending only $g_0$, $\alpha$, $f$ and $\Gamma$, there exists a constant $\delta > 0$ depending only on $g_0$, $\alpha$, $f$ and $\Gamma$ such that $u_{\alpha+1} \ge \alpha$ in $M_\delta$. By the comparison principle in Proposition 3.7, $u \ge u_{\alpha+1}$ in $M \setminus \partial M$, and in particular $u \ge \alpha$ in $M_\delta \setminus \partial M$, as required. □

**4.2.** ***Asymptotics.*** Fix $\tau < 1$ and suppose that $(f, \Gamma)$ satisfies (1-1)–(1-4), (1-7) and (1-8). In this section we show that there exists a smooth solution $u$ to (1-10) satisfying (1-9), that is

$$\lim_{\mathrm{d}(x,\partial M)\to 0} (u(x) + \ln \mathrm{d}(x, \partial M)) = 0. \tag{4-4}$$

**Remark 4.2.** At this point of the argument, we do not know that this constructed solution coincides with the one obtained in Section 4.1, although we will later see in Section 4.3 that this is the case.

We start by proving an upper bound on the growth of any smooth solution to the equation in (1-10), irrespective of the boundary data or whether $\tau < 1$ or $\mu_\Gamma^+ > 1$.

**Proposition 4.3.** *Let $(M, g_0)$ be a smooth Riemannian manifold with nonempty boundary and suppose that $(f, \Gamma)$ satisfies* (1-1)–(1-4) *and* (1-7). *Then there exist constants $\delta > 0$ and $C > 0$ depending only on $g_0$ such that any continuous metric $g_u = e^{2u}g_0$ satisfying*

$$f(\lambda(-g_u^{-1}A_{g_u})) \ge 1, \quad \lambda(-g_u^{-1}A_{g_u}) \in \Gamma \quad \textit{in the viscosity sense on } M \setminus \partial M \tag{4-5}$$

*satisfies*

$$u(x) + \ln \mathrm{d}(x, \partial M) \le C\mathrm{d}(x, \partial M)^{1/2} \quad \textit{in } M_\delta \setminus \partial M. \tag{4-6}$$

*In particular, any continuous metric $g_u = e^{2u}g_0$ satisfying* (4-5) *satisfies*

$$\limsup_{\mathrm{d}(x,\partial M)\to 0} (u(x) + \ln \mathrm{d}(x, \partial M)) \le 0. \tag{4-7}$$

*Proof.* By (4-2), the comparison principle for viscosity sub- and supersolutions to uniformly elliptic equations implies that if $g_w = e^{2w} g_0$ satisfies

$$\begin{cases} \sigma_1(-g_w^{-1} A_{g_w}) \le \frac{1}{2} n & \text{in } \Omega \Subset M \setminus \partial M, \\ w(x) \to +\infty & \text{as } \mathrm{d}(x, \partial\Omega) \to 0, \end{cases} \tag{4-8}$$

then $u \le w$ in $\Omega$. Since $\sigma_1(-g_w^{-1} A_{g_w}) = -(2(n-1))^{-1} R_{g_w}$, the transformation law for scalar curvature implies that the equation in (4-8) is equivalent to

$$-\frac{S_{g_0}}{n-1} + 2\Delta_{g_0} w + (n-2)|\nabla_{g_0} w|_{g_0}^2 \le n e^{2w}. \tag{4-9}$$

We follow an argument of Gursky, Streets and Warren [Gursky et al. 2011], in turn based on the original argument of Loewner and Nirenberg [1974], to construct such local supersolutions near $\partial M$. For a point $x_0$ a distance $d$ from $\partial M$, consider a point $z_0$ a distance $R > d$ from $\partial M$, which lies along the shortest path geodesic from $x_0$ to $\partial M$. We may assume $R$ is small enough so that $\Delta_{g_0} d^2(z_0, \cdot) \ge 1$ on $B_R(z_0)$, and so that there exists a function $h$ defined on $[0, R^2]$ satisfying

$$(n-2)(h')^2 + 2h'' \le 0, \quad h' > \max_M |S_{g_0}| + \widetilde{C}(g_0), \quad h(0) = 0, \tag{4-10}$$

where $\widetilde{C}(g_0)$ is a sufficiently large constant to be fixed in the proof. Indeed, once $\widetilde{C}(g_0)$ is fixed, the function $h(t) = \sqrt{t + \varepsilon^2} - \varepsilon$ satisfies (4-10) for $\varepsilon$ sufficiently small and $t$ in a sufficiently small interval $[0, R^2]$.

Let $r$ denote the distance from $z_0$, and define on $B_R(z_0)$ the radial function

$$w(r) = -\ln(R^2 - r^2) + h(R^2 - r^2) + \ln \alpha,$$

where $\alpha > 0$ is to be determined. Exactly as in the proof of Lemma 5.2 in [Gursky et al. 2011], a direct computation shows that, for $R$ sufficiently small and $\widetilde{C}(g_0)$ sufficiently large, the left-hand side of (4-9) satisfies

$$-\frac{S_{g_0}}{n-1} + 2\Delta_{g_0} w + (n-2)|\nabla_{g_0} w|_{g_0}^2 \le \frac{4nR^2}{(R^2 - r^2)^2} e^{2h} = \frac{4nR^2}{\alpha^2} e^{2w}. \tag{4-11}$$

Therefore, if we take $\alpha = 2R$, we see $w$ indeed satisfies (4-9). We then obtain

$$\begin{aligned} u(x_0) \le w(x_0) &= -\ln(R^2 - (R-d)^2) + h(R^2 - (R-d)^2) + \ln(2R) \\ &= -\ln(d(2R-d)) + h(d(2R-d)) + \ln(2R) \\ &= -\ln d - \ln\Big(1 - \frac{d}{2R}\Big) + h(d(2R-d)). \end{aligned}$$

But $h(d(2R-d)) = \sqrt{d(2R-d) + \varepsilon^2} - \varepsilon \le \sqrt{d(2R-d)} \le C\sqrt{d}$ and

$$\ln\Big(1 - \frac{d}{2R}\Big) \ge -\frac{d}{2R} \ge -C\sqrt{d}$$

for sufficiently small $d$, and thus (4-6) follows. The inequality (4-7) is a clear consequence of (4-6). $\square$

We are now in a position to prove the existence of a smooth solution to (1-10) when $\tau < 1$ with the desired asymptotic behaviour in (4-4).

**Proposition 4.4.** *Fix $\tau < 1$, and suppose that $(f, \Gamma)$ satisfies* (1-1)–(1-4), (1-7) *and* (1-8). *Then there exists a smooth solution $g_v = e^{2v} g_0$ to* (1-10) *and a constant $C$ independent of $\tau$ but dependent on $g_0$, $f$ and $\Gamma$ such that the following holds: for each $\varepsilon > 0$ sufficiently small, there exists a constant $a \gg 0$ independent of $\tau$ but dependent on $g_0$, $\varepsilon$, $C$, $f$ and $\Gamma$ such that*

$$v(x) + \ln \mathrm{d}(x, \partial M) \geq \ln \sqrt{1 - 2\varepsilon} - \ln(1 + a\mathrm{d}(x, \partial M)) \quad in\ A_\varepsilon^a \subset M, \tag{4-12}$$

*where*

$$A_\varepsilon^a = \Big\{ x \in M \setminus \partial M : d(x) + ad(x)^2 \leq \frac{\varepsilon}{C} \Big\}.$$

*In particular,*

$$\lim_{\mathrm{d}(x,\partial M) \to 0} (v(x) + \ln \mathrm{d}(x, \partial M)) = 0. \tag{4-13}$$

*Proof.* Consider an exhaustion of $M$ by smooth compact manifolds with boundary defined by

$$M_{(j)} = \{x \in M : \mathrm{d}(x, \partial M) \geq j^{-1}\}.$$

By Proposition 4.1, for each $j$, there exists a smooth solution $g_{v_{(j)}} = e^{2v_{(j)}} g_0$ to

$$\begin{cases} f^\tau(-g_{v_{(j)}}^{-1} A_{g_{v_{(j)}}}) = 1, \ \ \lambda(-g_{v_{(j)}}^{-1} A_{g_{v_{(j)}}}) \in \Gamma^\tau & \text{on } M_{(j)} \setminus \partial M_{(j)}, \\ v_{(j)}(x) \to +\infty & \text{as } \mathrm{d}(x, \partial M_{(j)}) \to 0. \end{cases}$$

(Note that we put parentheses around the index $j$ to avoid confusion with the solutions $u_m$ to (4-1)). Since $v_{(j)}(x) \to +\infty$ as $\mathrm{d}(x, \partial M_{(j)}) \to 0$, the comparison principle in Proposition 3.7 implies that if $j < m$, then

$$v_{(m)}|_{M_{(j)}} < v_{(j)}. \tag{4-14}$$

Now, as justified in the proof of Proposition 4.1, a subsequence of $\{v_{(j)}\}_j$ converges locally uniformly to some $v \in C^\infty(M \setminus \partial M)$. We claim that $v$ is our desired function. It is clear that $v$ solves the equation in (1-10). We now establish (4-12), which we split into two steps: in the first step we show $v(x) \to +\infty$ as $\mathrm{d}(x, \partial M) \to 0$, and in the second step we prove (4-12).

**Step 1:** In this first step we show that $v(x) \to +\infty$ as $\mathrm{d}(x, \partial M) \to 0$. To this end, let $d(x) = \mathrm{d}(x, \partial M)$, and define $\varphi = -\ln(B(d + ad^2))$, $g_\varphi = e^{2\varphi} g_0$, where $a$ and $B$ are positive constants to be determined. Writing $e^{2\varphi} = \psi^{-2}$, so that $\psi = B(d + ad^2)$, we compute near $\partial M$

$$|\nabla_{g_0} \psi|_{g_0}^2 = B^2(1 + 2ad)^2 \quad \text{and} \quad \nabla_{g_0}^2 \psi = B(1 + 2ad) \nabla_{g_0}^2 d + 2aB \nabla d \otimes \nabla d,$$

where $\nabla d$ denotes the differential of $d$. It follows that, near $\partial M$,

$$\begin{aligned} -g_\varphi^{-1} A_{g_\varphi} &= g_0^{-1}\big(-\psi \nabla_{g_0}^2 \psi + \tfrac{1}{2} |\nabla_{g_0} \psi|_{g_0}^2 g_0 - \psi^2 A_{g_0}\big) \\ &= B^2 g_0^{-1}\big(\tfrac{1}{2} g_0 + 2a^2 d^2 [g_0 - \nabla d \otimes \nabla d - d \nabla_{g_0}^2 d] - d(1 + 3ad) \nabla_{g_0}^2 d \\ &\qquad + 2ad[g_0 - \nabla d \otimes \nabla d] - d^2(1 + ad)^2 A_{g_0}\big). \end{aligned} \tag{4-15}$$

Taking for instance $a = 1$, we then see that, for $\delta$ fixed sufficiently small and $B$ fixed sufficiently large,

$$f^\tau(-g_\varphi^{-1} A_{g_\varphi}) \geq 1 \quad \text{in } M_\delta \setminus \partial M. \tag{4-16}$$

To use (4-16) to show $v(x) \to +\infty$ as $\mathrm{d}(x, \partial M) \to 0$, we follow the proof of [Loewner and Nirenberg 1974, Theorem 5]. For $m \gg 1$, denote by $S_m$ the set where $\varphi(x) = -\ln(B(d+d^2)) \geq m$. We may assume (by taking $m$ sufficiently large) that $S_m$ is a tubular neighbourhood of $\partial M$ contained in $M_\delta$. Let $\Sigma_m = \partial S_m \setminus \partial M$ and $D_m = \min_{\Sigma_m} v$, and suppose $J$ is sufficiently large so that $\Sigma_m \subset M_{(j)}$ for all $j \geq J$. Then $\varphi = m$ and $v \geq D_m$ on $\Sigma_m$, and, by the monotonicity in (4-14), we also have $v_{(j)} \geq D_m$ on $\Sigma_m$ for each $j \geq J$. Therefore

$$v_{(j)} + \max\{0, m - D_m\} \geq m = \varphi \quad \text{on } \Sigma_m \tag{4-17}$$

and

$$v_{(j)} + \max\{0, m - D_m\} = \infty > \varphi \quad \text{on } \partial M_{(j)}. \tag{4-18}$$

In light of (4-16)–(4-18), the comparison principle in Proposition 3.7 implies $v_{(j)} + \max\{0, m - D_m\} \geq \varphi$ on $M_{(j)} \cap S_m$. Sending $j \to \infty$, it follows that $v + \max\{0, m - D_m\} \geq \varphi$ in $S_m$, and in particular $v(x) \to +\infty$ as $\mathrm{d}(x, \partial M) \to 0$.

**Step 2:** In this second step we show that $v$ satisfies (4-12). The method is essentially a quantitative version of Step 1, requiring a more careful choice of parameters $a$ and $B$ in the definition of $\varphi$.

We first claim that the two quantities in the square parentheses in (4-15) are nonnegative definite for sufficiently small $d$. Indeed, observe that $g_0(x) - \nabla d(x) \otimes \nabla d(x)$ is the induced metric on $\partial M_{d(x)} \setminus \partial M$ and is therefore nonnegative definite. Moreover, $\nabla^2_{g_0} d$ is a bounded tensor near $\partial M$ whose kernel contains $\nabla d$. Hence $\nabla^2_{g_0} d$ is bounded from above by $C(g_0 - \nabla d \otimes \nabla d)$ for some constant $C$ depending only on $(M, g_0)$. Therefore, $g_0 - \nabla d \otimes \nabla d - d\nabla^2_{g_0} d$ is nonnegative definite for $d$ sufficiently small, as claimed.

In light of (4-15) and the above claim, we see that, for $\delta$ chosen sufficiently small independently of $a$ (but depending on $(M, g_0)$) and $\widehat{C} \geq 1$ a constant such that $|A_{g_0}|_{g_0}, |\nabla^2_{g_0} d|_{g_0} \leq \widehat{C}$ on $M_\delta$, we have

$$\begin{aligned} -g_\varphi^{-1} A_{g_\varphi} &\geq B^2 g_0^{-1}\bigl(\tfrac{1}{2} g_0 - d(1+3ad)\nabla^2_{g_0} d - d^2(1+ad)^2 A_{g_0}\bigr) \\ &\geq B^2 g_0^{-1}\bigl(\tfrac{1}{2} - \widehat{C} d - \widehat{C}(1+3a)d^2 - 2\widehat{C} a d^3 - \widehat{C} a^2 d^4\bigr) g_0 \end{aligned} \tag{4-19}$$

in $M_\delta \setminus \partial M$. Since we will eventually take $a$ large, we may assume $a \geq 1$, in which case (4-19) implies

$$-g_\varphi^{-1} A_{g_\varphi} \geq B^2\bigl(\tfrac{1}{2} - \widehat{C}[d + 4ad^2 + 2ad^3 + a^2 d^4]\bigr)\,\mathrm{Id} \quad \text{in } M_\delta \setminus \partial M. \tag{4-20}$$

Now fix $\varepsilon > 0$ small, define $B = 1/\sqrt{1-2\varepsilon}$ and denote by $\hat{A}^a_\varepsilon$ the set

$$\hat{A}^a_\varepsilon = \left\{x \in M \setminus \partial M : \varphi(x) = -\ln(B(d + ad^2)) \geq -\ln \frac{\varepsilon}{100\widehat{C}}\right\} = \left\{x \in M \setminus \partial M : d + ad^2 \leq \frac{\varepsilon\sqrt{1-2\varepsilon}}{100\widehat{C}}\right\},$$

where $\widehat{C}$ is the constant in (4-20). It is easily verified that, in $\hat{A}^a_\varepsilon$, we have $\widehat{C}(d + 4ad^2 + 2ad^3 + a^2 d^4) \leq \varepsilon$. Moreover, if we define

$$\Sigma^a_\varepsilon = \partial \hat{A}^a_\varepsilon \setminus \partial M,$$

then $\Sigma^a_\varepsilon$ converges to $\partial M$ as $a$ increases. It follows from these two facts and (4-20) that, for $a$ sufficiently large (depending only on $(M, g_0)$),

$$-g_\varphi^{-1} A_{g_\varphi} \geq B^2 \operatorname{diag}\bigl(\tfrac{1}{2} - \varepsilon, \dots, \tfrac{1}{2} - \varepsilon\bigr) = \operatorname{diag}\bigl(\tfrac{1}{2}, \dots, \tfrac{1}{2}\bigr) \quad \text{in } \hat{A}^a_\varepsilon.$$

It then follows from our normalisation $f\left(\frac{1}{2}, \dots, \frac{1}{2}\right) = 1$ that

$$f^\tau(-g_\varphi^{-1} A_{g_\varphi}) \geq 1 \quad \text{in } \hat{A}_\varepsilon^a. \tag{4-21}$$

We now let

$$C_\varepsilon^a = \min_{\Sigma_\varepsilon^a} v.$$

Since $v(x) \to +\infty$ as $\mathrm{d}(x, \partial M) \to 0$ (by Step 1) and since $\Sigma_\varepsilon^a$ converges to $\partial M$ as $a$ increases, we can choose $a$ large enough so that $C_\varepsilon^a \geq -\ln(\varepsilon/(100\widehat{C}))$. Moreover, this choice of $a$ depends only on $g_0$, $\varepsilon$, $\widehat{C}$, $f$ and $\Gamma$: since each $v_{(j)}$ was constructed according to the procedure in the proof of Proposition 4.1, we know from the second statement in Proposition 4.1 that there exists $\delta = \delta(g_0, \varepsilon, \widehat{C}, f, \Gamma) > 0$ such that $v_{(j)} \geq -\ln(\varepsilon/(100\widehat{C}))$ in $(M_{(j)})_\delta \setminus \partial M_{(j)}$ for each $j$. Taking $j \to \infty$, we see $v \geq -\ln(\varepsilon/(100\widehat{C}))$ in $M_\delta \setminus \partial M$. Therefore, to ensure $C_\varepsilon^a \geq -\ln(\varepsilon/(100\widehat{C}))$, one only needs to pick $a$ large depending on $\delta = \delta(g_0, \varepsilon, \widehat{C}, f, \Gamma)$.

We now fix such a value of $a$ and suppose $J$ is sufficiently large so that $\Sigma_\varepsilon^a \subset M_{(j)}$ for all $j \geq J$. Then $\varphi = -\ln(\varepsilon/(100\widehat{C}))$ and $v \geq C_\varepsilon^a$ on $\Sigma_\varepsilon^a$, and, by the monotonicity in (4-14), we also have $v_{(j)} \geq C_\varepsilon^a$ on $\Sigma_\varepsilon^a$ for each $j \geq J$. Therefore,

$$v_{(j)} \geq -\ln \frac{\varepsilon}{100\widehat{C}} = \varphi \quad \text{on } \Sigma_\varepsilon^a \tag{4-22}$$

and

$$v_{(j)} = \infty > \varphi \quad \text{on } \partial M_{(j)}. \tag{4-23}$$

In light of (4-21)–(4-23), the comparison principle in Proposition 3.7 then yields

$$v_{(j)} \geq \varphi \quad \text{in } \hat{A}_\varepsilon^a \cap M_{(j)}.$$

Sending $j \to \infty$, it follows that $v \geq \varphi$ in $\hat{A}_\varepsilon^a$, i.e.,

$$v \geq \varphi = -\ln(B(d + ad^2)) = \ln\sqrt{1 - 2\varepsilon} - \ln d - \ln(1 + ad) \quad \text{in } \hat{A}_\varepsilon^a.$$

This is precisely (4-12) after relabelling constants, and thus the second step is complete.

To complete the proof of the proposition, we observe that (4-12) implies

$$\liminf_{\mathrm{d}(x,\partial M)\to 0} (v(x) + \ln \mathrm{d}(x, \partial M)) \geq \ln\sqrt{1 - 2\varepsilon},$$

and, since $\varepsilon > 0$ is arbitrary, it follows that

$$\liminf_{\mathrm{d}(x,\partial M)\to 0} (v(x) + \ln \mathrm{d}(x, \partial M)) \geq 0. \tag{4-24}$$

By (4-24) and Proposition 4.3, we therefore see that $v$ satisfies (4-13). □

**4.3. *Uniqueness.*** Having just established the existence of a smooth solution to (1-10) satisfying (4-4) when $\tau < 1$ and $\mu_\Gamma^+ > 1$, we now turn to uniqueness of solutions. We start with the following.

**Proposition 4.5.** *Fix $\tau < 1$, and suppose that $(f, \Gamma)$ satisfies* (1-1)–(1-4), (1-7) *and* (1-8). *Then any continuous viscosity solution $g_u = e^{2u} g_0$ to* (1-10) *satisfies* (4-4).

*Proof.* Let $u$ be a continuous viscosity solution to (1-10). By Proposition 4.3, we know that $u$ satisfies $\limsup_{\mathrm{d}(x,\partial M)\to 0}(u(x)+\ln \mathrm{d}(x,\partial M)) \le 0$, so it remains to show

$$\liminf_{\mathrm{d}(x,\partial M)\to 0}(u(x)+\ln \mathrm{d}(x,\partial M)) \ge 0. \tag{4-25}$$

To prove (4-25), we attach a collar neighbourhood $N$ to $\partial M$, extend $g_0$ smoothly to $M\cup N$ and consider the sequence $\{M^{(j)}\}_j$ of smooth compact manifolds with boundary given by

$$M^{(j)} = \{x \in M\cup N : \mathrm{d}(x, M) \le j^{-1}\}.$$

Note that for $x\in M$ and $j$ sufficiently large, $\mathrm{d}(x,\partial M^{(j)}) = \mathrm{d}(x,\partial M)+j^{-1}$. Fix $\varepsilon > 0$. By Proposition 4.4, there exist constants $\delta > 0$ and $a > 0$ depending on $g_0$, $\varepsilon$, $f$, $\Gamma$ but independent of $j$, and a smooth metric $g_{u^{(j)}} = e^{2u^{(j)}} g_0$ for each $j$ such that

$$f^\tau(-g_{u^{(j)}}^{-1}A_{g_{u^{(j)}}}) = 1,\quad \lambda(-g_{u^{(j)}}A_{g_{u^{(j)}}}) \in \Gamma^\tau \quad \text{on } M^{(j)}\setminus\partial M^{(j)}$$

and

$$u^{(j)}(x)+\ln \mathrm{d}(x,\partial M^{(j)}) \ge \ln\sqrt{1-2\varepsilon} - \ln(1+a\mathrm{d}(x,\partial M^{(j)})) \quad \text{in } (M^{(j)})_\delta\setminus\partial M^{(j)}.$$

In particular, for $j$ sufficiently large so that $(M^{(j)})_\delta \cap M \ne \varnothing$, we have

$$u^{(j)}(x)+\ln\left(\mathrm{d}(x,\partial M)+\frac{1}{j}\right) \ge \ln\sqrt{1-2\varepsilon} - \ln\left(1+a\mathrm{d}(x,\partial M)+\frac{a}{j}\right) \quad \text{in } M_{\delta-1/j}. \tag{4-26}$$

Now, by the comparison principle in Proposition 3.7, $u^{(j)}|_M \le u$ for each $j$, and thus (4-26) implies

$$u(x)+\ln\left(\mathrm{d}(x,\partial M)+\frac{1}{j}\right) \ge \ln\sqrt{1-2\varepsilon} - \ln\left(1+a\mathrm{d}(x,\partial M)+\frac{a}{j}\right) \quad \text{in } M_{\delta-1/j}\setminus\partial M. \tag{4-27}$$

After taking $j\to\infty$ in (4-27), it follows that

$$\liminf_{\mathrm{d}(x,\partial M)\to 0}(u(x)+\ln \mathrm{d}(x,\partial M)) \ge \ln\sqrt{1-2\varepsilon},$$

and, since $\varepsilon > 0$ is arbitrary, we obtain (4-25). □

Finally we prove uniqueness of solutions to (1-10) when $\tau < 1$.

**Proposition 4.6.** *Fix $\tau < 1$, suppose that $(f,\Gamma)$ satisfies* (1-1)–(1-4), (1-7) *and* (1-8), *and let $v$ denote the smooth solution to* (1-10) *obtained in Proposition 4.4. Then $v$ is the unique continuous viscosity solution to* (1-10).

*Proof.* Suppose that $w$ is a continuous viscosity solution to (1-10). By Proposition 4.5, both $v$ and $w$ satisfy (4-4). For $\delta \ge 0$, define $\Sigma_\delta = \{d = \delta\}$. Then, for each $\varepsilon > 0$, there exists a minimal $\delta_\varepsilon > 0$ such that $w \le v+\varepsilon$ on $\Sigma_{\delta_\varepsilon}$. Writing $v_\varepsilon = v+\varepsilon$, we have

$$f^\tau(-g_0^{-1}A_{g_{v_\varepsilon}}) = f^\tau(-g_0^{-1}A_{g_v}) = e^{2v} < e^{2v_\varepsilon},$$

and thus $v_\varepsilon$ is a supersolution of the equation in (1-10). By the comparison principle in Proposition 3.7, it follows that $w \le v+\varepsilon$ on $M\setminus M_{\delta_\varepsilon}$. By minimality of $\delta_\varepsilon$, we have $\delta_\varepsilon \to 0$ as $\varepsilon\to 0$, and thus $w\le v$ on $M\setminus\partial M$. Reversing the roles of $w$ and $v$, we see that $w \ge v$ on $M\setminus\partial M$, and therefore $w = v$. □

**4.4.** ***Proof of Theorem 1.1′.*** The existence of a smooth solution to (1-10) for each $\tau < 1$, the asymptotic behaviour stated in (1-9) and uniqueness in the class of continuous viscosity solutions follow from Propositions 4.4 and 4.6. Let us denote these solutions by $u^\tau$. As observed previously, these solutions $u^\tau$ satisfy a locally uniform $C^1$ estimate which is independent of $\tau$; i.e., for each compact set $K \subset M \setminus \partial M$, there exists a constant $C$ independent of $\tau$ but dependent on $g_0$, $f$, $\Gamma$ and $K$ such that

$$\|u^\tau\|_{C^1(K)} \le C.$$

It follows that a subsequence of $\{u^\tau\}$ converges locally uniformly in $C^{0,\alpha}$ to some $u \in C^{0,1}_{\mathrm{loc}}(M, g_0)$ for each $\alpha \in (0, 1)$. As noted in the proof of Theorem 1.6 in Section 3.5, the fact that $u$ is a viscosity solution to (1-10) when $\tau = 1$ follows from exactly the same argument as in the proof of [Li and Nguyen 2021, Theorem 1.4]. It remains to show that $u$ satisfies the asymptotics in (1-9) and is maximal.

To this end, first note that, since we only require $u$ to be a viscosity subsolution in Proposition 4.3,

$$\limsup_{\mathrm{d}(x,\partial M)\to 0} (u(x) + \ln \mathrm{d}(x, \partial M)) \le 0. \tag{4-28}$$

To show that

$$\liminf_{\mathrm{d}(x,\partial M)\to 0} (u(x) + \ln \mathrm{d}(x, \partial M)) \ge 0, \tag{4-29}$$

we first recall that $u$ is the $C^{0,\alpha}$ limit of the solutions $u^\tau$ as $\tau \to 1$. By Proposition 4.4, for each $\varepsilon > 0$ sufficiently small, there exist constants $\delta > 0$ and $a > 0$ independent of $\tau$ (but dependent on $g_0$, $\varepsilon$, $f$ and $\Gamma$) such that

$$u^\tau(x) + \ln \mathrm{d}(x, \partial M) \ge \ln \sqrt{1 - 2\varepsilon} - \ln(1 + a\mathrm{d}(x, \partial M)) \quad \text{in } M_\delta \setminus \partial M. \tag{4-30}$$

Taking $\tau \to 1$ in (4-30), we obtain

$$u(x) + \ln \mathrm{d}(x, \partial M) \ge \ln \sqrt{1 - 2\varepsilon} - \ln(1 + a\mathrm{d}(x, \partial M)) \quad \text{in } M_\delta \setminus \partial M,$$

and (4-29) then follows exactly as in the proof of Proposition 4.4.

Finally, to see that $u$ is maximal, suppose that $\tilde{u}$ is another continuous viscosity solution to (1-10). By Proposition 4.3, (4-28) holds with $\tilde{u}$ in place of $u$, and we also know that (1-9) is satisfied with $u^\tau$ in place of $u$ for each $\tau \le 1$. Combining these facts, it follows that, for each $\tau \le 1$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\tilde{u} \le u^\tau_\varepsilon := u^\tau + \varepsilon \quad \text{in } M_\delta \setminus \partial M.$$

On the other hand, $f^\tau(-g^{-1}_{u^\tau_\varepsilon} A_{g_{u^\tau_\varepsilon}}) = e^{-2\varepsilon} f^\tau(-g^{-1}_{u^\tau} A_{g_{u^\tau}}) < 1$ on $M \setminus \partial M$ and $f^\tau(-g^{-1}_{\tilde{u}} A_{g_{\tilde{u}}}) \ge 1$ in the viscosity sense on $M \setminus \partial M$; to see this latter inequality, observe

$$\begin{aligned} f^\tau(\lambda) = \frac{1}{\tau + n(1-\tau)} f(\tau\lambda + (1-\tau)\sigma_1(\lambda)e) &\ge \frac{1}{\tau + n(1-\tau)} (\tau f(\lambda) + (1-\tau)\sigma_1(\lambda) f(e)) \\ &\overset{(4\text{-}2)}{\ge} \frac{1}{\tau + n(1-\tau)} \left( \tau f(\lambda) + (1-\tau) \frac{n f(\lambda)}{f(e)} f(e) \right) = f(\lambda). \end{aligned}$$

By the comparison principle in Proposition 3.7, it follows that $\tilde{u} \le u^\tau_\varepsilon$ in $M \setminus M_\delta$, and therefore $\tilde{u} \le u^\tau_\varepsilon$ in $M \setminus \partial M$. Taking $\varepsilon \to 0$ and then $\tau \to 1$, it follows that $\tilde{u} \le u$ in $M \setminus \partial M$, as claimed. □

## Appendix A: Proof of Proposition 2.4: a cone property

Proposition 2.4 is essentially a consequence of [Yuan 2022, Theorem 1.4]. We summarise the details here for the convenience of the reader. Let $\Gamma$ be any cone satisfying (1-1) and (1-2), and define

$$\kappa_\Gamma = \max\{k : (\underbrace{0, \dots, 0}_{k}, \underbrace{1, \dots, 1}_{n-k}) \in \Gamma\}.$$

Assume for now that there exists a constant $\theta = \theta(n, \Gamma) > 0$ such that, whenever $\lambda \in \Gamma$ with $\lambda_1 \geq \cdots \geq \lambda_n$,

$$\frac{\partial f}{\partial \lambda_i}(\lambda) \geq \theta \sum_{j=1}^{n} \frac{\partial f}{\partial \lambda_j}(\lambda) \quad \text{for } i \geq n - \kappa_\Gamma. \tag{A-1}$$

Since $\kappa_\Gamma = 0$ if and only if $\Gamma = \Gamma_n^+$, we see that $\kappa_\Gamma \geq 1$ whenever $\Gamma \neq \Gamma_n^+$, and thus (2-24) holds for $i \in \{n-1, n\}$. Also, it is easy to see that $\kappa_\Gamma$ is equal to the maximum number of negative entries a vector in $\Gamma$ can have; i.e.,

$$\kappa_\Gamma = \max\{k : (-\alpha_1, \dots, -\alpha_k, \alpha_{k+1}, \dots, \alpha_n) \in \Gamma,\ \alpha_j > 0 \text{ for all } 1 \leq j \leq n\}.$$

Thus (2-24) also holds if $\lambda_i \leq 0$.

It remains to justify (A-1), for which we follow [Yuan 2022]. By concavity, $f_i(\lambda) \geq f_j(\lambda)$ whenever $\lambda_i \leq \lambda_j$. In particular, our ordering implies

$$\frac{\partial f}{\partial \lambda_n}(\lambda) \geq \frac{1}{n} \sum_{j=1}^{n} \frac{\partial f}{\partial \lambda_j}(\lambda),$$

which establishes (A-1) for $\Gamma = \Gamma_n^+$.

On the other hand, for a general cone $\Gamma$ satisfying (1-1) and (1-2), we have

$$\sum_{i=1}^{n} f_i(\lambda)\mu_i > 0 \quad \text{whenever } \lambda, \mu \in \Gamma. \tag{A-2}$$

Suppose $\Gamma \neq \Gamma_n^+$, in which case it is clear that $\kappa_\Gamma > 0$, and fix any $\alpha_1, \dots, \alpha_n > 0$ such that

$$(-\alpha_1, \dots, -\alpha_{\kappa_\Gamma}, \alpha_{\kappa_\Gamma+1}, \dots, \alpha_n) \in \Gamma.$$

Then (A-2) implies

$$\sum_{i=\kappa_\Gamma+1}^{n} \alpha_i f_{n-i+1}(\lambda) - \sum_{i=1}^{\kappa_\Gamma} \alpha_i f_{n-i+1}(\lambda) > 0. \tag{A-3}$$

We may assume $\alpha_1 \geq \cdots \geq \alpha_{\kappa_\Gamma}$, in which case (A-3) implies

$$f_{n-\kappa_\Gamma}(\lambda) > \frac{\alpha_1}{\sum_{i=\kappa_\Gamma+1}^{n} \alpha_i} f_n(\lambda).$$

The desired estimate then follows for all $i \geq n - \kappa_\Gamma$, again by our ordering. $\square$

## Appendix B: The Schouten tensor for a radial conformal factor

In this appendix we prove the formula (3-5). In normal coordinates, $r = \sqrt{x_1^2 + \cdots + x_n^2}$, and therefore $\partial_i v(r) = (x_i/r)v_r$. It follows that

$$|\nabla_{g_0} v|_{g_0}^2 = g_0^{ij} \partial_i v \partial_j v = \frac{g_0^{ij} x_i x_j}{r^2} v_r^2 = v_r^2,$$

where we have used the fact that
$$\frac{\partial}{\partial r}=\frac{x_i}{\sqrt{x_1^2+\cdots+x_n^2}}\frac{\partial}{\partial x_i}$$
has unit magnitude. Moreover,
$$(\nabla^2_{g_0}v)_{ij}=\partial_i\partial_j v-\Gamma^k_{ij}\partial_k v=\frac{\delta_{ij}}{r}v_r+\frac{x_ix_j}{r}\left(\frac{v_{rr}}{r}-\frac{v_r}{r^2}\right)-\Gamma^k_{ij}\partial_k v.$$
Combining the above, we therefore see that
$$\begin{aligned}(g_v^{-1}A_{g_v})^p_j&=v^2(g_0^{-1}A_{g_v})^p_j=v^2g_0^{pi}(A_{g_v})_{ij}\\&=v^2\left[\frac{g_0^{pi}\delta_{ij}}{vr}v_r+g_0^{pi}\frac{x_ix_j}{vr}\left(\frac{v_{rr}}{r}-\frac{v_r}{r^2}\right)-g_0^{pi}\frac{\Gamma^k_{ij}x_kv_r}{vr}-\frac{v_r^2}{2v^2}\delta^p_j+(g_0^{-1}A_{g_0})^p_j\right].\end{aligned}$$
Now write $g_0^{pi}=\delta^{pi}+\chi^{pi}$, where $\chi=O(r^2)$ as $r\to 0$. Then
$$\begin{aligned}(g_v^{-1}A_{g_v})^p_j&=v^2\left[\frac{\delta^p_j}{vr}v_r+\frac{x^px_j}{vr}\left(\frac{v_{rr}}{r}-\frac{v_r}{r^2}\right)-\frac{v_r^2}{2v^2}\delta^p_j\right]\\&\qquad+\underbrace{v^2\left[\frac{\chi^{pi}\delta_{ij}}{vr}v_r+\chi^{pi}\frac{x_ix_j}{vr}\left(\frac{v_{rr}}{r}-\frac{v_r}{r^2}\right)-g_0^{pi}\frac{\Gamma^k_{ij}x_kv_r}{vr}+(g_0^{-1}A_{g_0})^p_j\right]}_{=\Psi^p_j}\\&=v^2\left(\lambda\delta^p_j+\chi\frac{x^px_j}{r^2}\right)+\Psi^p_j,\end{aligned}$$
where $\lambda$ and $\chi$ are as in (3-6). Now, since $\chi=O(r^2)$, we have
$$v^2\frac{\chi^{pi}\delta_{ij}}{vr}v_r=O(r)v|v_r|,\quad v^2\chi^{pi}\frac{x_ix_j}{vr}\left(\frac{v_{rr}}{r}-\frac{v_r}{r^2}\right)=O(r^2)v|v_{rr}|+O(r)v|v_r|,$$
and, since $\Gamma^k_{ij}=O(r)$ and $(g_0^{-1}A_{g_0})^p_j=O(1)$, we also have
$$v^2g_0^{pi}\frac{\Gamma^k_{ij}x_kv_r}{vr}=O(r)v|v_r|\quad\text{and}\quad v^2(g_0^{-1}A_{g_0})^p_j=O(1)v^2.$$
The claim (3-5) then follows.

## Acknowledgements

The authors would like to thank Professor YanYan Li for stimulating discussions and his constant support, and the anonymous reviewers for their careful reading of the manuscript and helpful suggestions.

## References

[Allen et al. 2018] P. T. Allen, J. Isenberg, J. M. Lee, and I. S. Allen, "Weakly asymptotically hyperbolic manifolds", *Comm. Anal. Geom.* **26**:1 (2018), 1–61. MR

[Andersson et al. 1992] L. Andersson, P. T. Chruściel, and H. Friedrich, "On the regularity of solutions to the Yamabe equation and the existence of smooth hyperboloidal initial data for Einstein's field equations", *Comm. Math. Phys.* **149**:3 (1992), 587–612. MR

[Aubin 1970] T. Aubin, "Métriques riemanniennes et courbure", *J. Differential Geometry* **4** (1970), 383–424. MR

[Aviles 1982] P. Aviles, “A study of the singularities of solutions of a class of nonlinear elliptic partial differential equations”, *Comm. Partial Differential Equations* **7**:6 (1982), 609–643. MR

[Aviles and McOwen 1988] P. Aviles and R. C. McOwen, “Complete conformal metrics with negative scalar curvature in compact Riemannian manifolds”, *Duke Math. J.* **56**:2 (1988), 395–398. MR

[Caffarelli et al. 2013] L. Caffarelli, Y. Li, and L. Nirenberg, “Some remarks on singular solutions of nonlinear elliptic equations, III: Viscosity solutions including parabolic operators”, *Comm. Pure Appl. Math.* **66**:1 (2013), 109–143. MR

[Chang et al. 2002] S.-Y. A. Chang, M. J. Gursky, and P. C. Yang, “An equation of Monge–Ampère type in conformal geometry, and four-manifolds of positive Ricci curvature”, *Ann. of Math.* (2) **155**:3 (2002), 709–787. MR

[Chang et al. 2005] S.-Y. A. Chang, Z.-C. Han, and P. Yang, “Classification of singular radial solutions to the $\sigma_k$ Yamabe equation on annular domains”, *J. Differential Equations* **216**:2 (2005), 482–501. MR

[Chen 2005] S.-y. S. Chen, “Local estimates for some fully nonlinear elliptic equations”, *Int. Math. Res. Not.* **2005**:55 (2005), 3403–3425. MR

[Chu et al. 2023] B. Chu, Y. Li, and Z. Li, “Liouville theorems for conformally invariant fully nonlinear equations, I”, 2023. arXiv 2311.07542

[Duncan and Nguyen 2023] J. A. J. Duncan and L. Nguyen, “Differential inclusions for the Schouten tensor and nonlinear eigenvalue problems in conformal geometry”, *Adv. Math.* **432** (2023), art. id. 109263. MR

[Duncan and Nguyen 2025] J. A. J. Duncan and L. Nguyen, “The fully nonlinear Loewner–Nirenberg problem: Liouville theorems and counterexamples to local boundary estimates”, preprint, 2025. arXiv 2507.16383

[Evans 1982] L. C. Evans, “Classical solutions of fully nonlinear, convex, second-order elliptic equations”, *Comm. Pure Appl. Math.* **35**:3 (1982), 333–363. MR

[Finn 1998] D. L. Finn, “Existence of positive solutions to $\Delta_g u = u^q + Su$ with prescribed singularities and their geometric implications”, *Comm. Partial Differential Equations* **23**:9-10 (1998), 1795–1814. MR

[González et al. 2018] M. d. M. González, Y. Li, and L. Nguyen, “Existence and uniqueness to a fully nonlinear version of the Loewner–Nirenberg problem”, *Commun. Math. Stat.* **6**:3 (2018), 269–288. MR

[Gover and Waldron 2017] A. R. Gover and A. Waldron, “Renormalized volume”, *Comm. Math. Phys.* **354**:3 (2017), 1205–1244. MR

[Graham 2017] C. R. Graham, “Volume renormalization for singular Yamabe metrics”, *Proc. Amer. Math. Soc.* **145**:4 (2017), 1781–1792. MR

[Guan 2008] B. Guan, “Complete conformal metrics of negative Ricci curvature on compact manifolds with boundary”, *Int. Math. Res. Not.* **2008** (2008), art. id. rnn105. MR

[Guan and Wang 2003] P. Guan and G. Wang, “Local estimates for a class of fully nonlinear equations arising from conformal geometry”, *Int. Math. Res. Not.* **2003**:26 (2003), 1413–1432. MR

[Gursky and Viaclovsky 2003] M. J. Gursky and J. A. Viaclovsky, “Fully nonlinear equations on Riemannian manifolds with negative curvature”, *Indiana Univ. Math. J.* **52**:2 (2003), 399–419. MR

[Gursky et al. 2011] M. Gursky, J. Streets, and M. Warren, “Existence of complete conformal metrics of negative Ricci curvature on manifolds with boundary”, *Calc. Var. Partial Differential Equations* **41**:1-2 (2011), 21–43. MR

[Han and Shen 2020] Q. Han and W. Shen, “The Loewner–Nirenberg problem in singular domains”, *J. Funct. Anal.* **279**:6 (2020), art. id. 108604. MR

[Han et al. 2024] Q. Han, X. Jiang, and W. Shen, “The Loewner–Nirenberg problem in cones”, *J. Funct. Anal.* **287**:8 (2024), art. id. 110566. MR

[Jiang 2021] X. Jiang, “Boundary expansion for the Loewner–Nirenberg problem in domains with conic singularities”, *J. Funct. Anal.* **281**:7 (2021), art. id. 109122. MR

[Jin et al. 2007] Q. Jin, A. Li, and Y. Y. Li, “Estimates and existence results for a fully nonlinear Yamabe problem on manifolds with boundary”, *Calc. Var. Partial Differential Equations* **28**:4 (2007), 509–543. MR

[Khomrutai 2009] S. Khomrutai, *Regularity of singular solutions to $\sigma_k$-Yamabe problems*, Ph.D. thesis, University of Notre Dame, 2009, available at https://www.proquest.com/docview/851148151. MR

[Krylov 1982] N. V. Krylov, “Boundedly inhomogeneous elliptic and parabolic equations”, *Izv. Akad. Nauk SSSR Ser. Mat.* **46**:3 (1982), 487–523. In Russian. MR

[Krylov 1983] N. V. Krylov, "Boundedly inhomogeneous elliptic and parabolic equations in a domain", *Izv. Akad. Nauk SSSR Ser. Mat.* **47**:1 (1983), 75–108. In Russian. MR

[Li 2009] Y. Li, "Local gradient estimates of solutions to some conformally invariant fully nonlinear equations", *Comm. Pure Appl. Math.* **62**:10 (2009), 1293–1326. MR

[Li 2022a] G. Li, "A flow approach to the generalized Loewner–Nirenberg problem of the $\sigma_k$-Ricci equation", *Calc. Var. Partial Differential Equations* **61**:5 (2022), art. id. 169. MR

[Li 2022b] G. Li, "Two flow approaches to the Loewner–Nirenberg problem on manifolds", *J. Geom. Anal.* **32**:1 (2022), art. id. 7. MR

[Li and Li 2003] A. Li and Y. Li, "On some conformally invariant fully nonlinear equations", *Comm. Pure Appl. Math.* **56**:10 (2003), 1416–1464. MR

[Li and Nguyen 2014] Y. Li and L. Nguyen, "Harnack inequalities and Bôcher-type theorems for conformally invariant, fully nonlinear degenerate elliptic equations", *Comm. Pure Appl. Math.* **67**:11 (2014), 1843–1876. MR

[Li and Nguyen 2021] Y. Li and L. Nguyen, "Solutions to the $\sigma_k$-Loewner–Nirenberg problem on annuli are locally Lipschitz and not differentiable", *J. Math. Study* **54**:2 (2021), 123–141. MR

[Li et al. 2018] Y. Li, L. Nguyen, and B. Wang, "Comparison principles and Lipschitz regularity for some nonlinear degenerate elliptic equations", *Calc. Var. Partial Differential Equations* **57**:4 (2018), art. id. 96. MR

[Li et al. 2023] Y. Li, L. Nguyen, and J. Xiong, "Regularity of viscosity solutions of the $\sigma_k$-Loewner–Nirenberg problem", *Proc. Lond. Math. Soc.* (3) **127**:1 (2023), 1–34. MR

[Loewner and Nirenberg 1974] C. Loewner and L. Nirenberg, "Partial differential equations invariant under conformal or projective transformations", pp. 245–272 in *Contributions to analysis* (*a collection of papers dedicated to Lipman Bers*), edited by L. V. Ahlfors et al., Academic Press, New York-London, 1974. MR

[Mazzeo 1991] R. Mazzeo, "Regularity for the singular Yamabe problem", *Indiana Univ. Math. J.* **40**:4 (1991), 1277–1299. MR

[Schoen 1984] R. Schoen, "Conformal deformation of a Riemannian metric to constant scalar curvature", *J. Differential Geom.* **20**:2 (1984), 479–495. MR

[Trudinger 1968] N. S. Trudinger, "Remarks concerning the conformal deformation of Riemannian structures on compact manifolds", *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (3) **22** (1968), 265–274. MR

[Véron 1981] L. Véron, "Singularités éliminables d'équations elliptiques non linéaires", *J. Differential Equations* **41**:1 (1981), 87–95. MR

[Viaclovsky 2000] J. A. Viaclovsky, "Conformal geometry, contact geometry, and the calculus of variations", *Duke Math. J.* **101**:2 (2000), 283–316. MR

[Viaclovsky 2002] J. A. Viaclovsky, "Estimates and existence results for some fully nonlinear elliptic equations on Riemannian manifolds", *Comm. Anal. Geom.* **10**:4 (2002), 815–846. MR

[Wang 2006] X.-J. Wang, "A priori estimates and existence for a class of fully nonlinear elliptic equations in conformal geometry", *Chinese Ann. Math. Ser. B* **27**:2 (2006), 169–178. MR

[Wang 2021] Y. Wang, "Boundary expansions of complete conformal metrics with negative Ricci curvatures", *Calc. Var. Partial Differential Equations* **60**:4 (2021), art. id. 131. MR

[Yamabe 1960] H. Yamabe, "On a deformation of Riemannian structures on compact manifolds", *Osaka Math. J.* **12** (1960), 21–37. MR

[Yuan 2022] R. Yuan, "The partial uniform ellipticity and prescribed problems on the conformal classes of complete metrics", preprint, 2022. arXiv 2203.13212

[Yuan 2024] R. Yuan, "Notes on conformal metrics of negative curvature on manifolds with boundary", *J. Math. Study* **57**:3 (2024), 373–378. MR


Received 31 Oct 2023. Revised 6 Sep 2024. Accepted 29 Oct 2024.

JONAH A. J. DUNCAN: jonah.duncan@ucl.ac.uk
*Department of Mathematics, University College London, London, United Kingdom*

LUC NGUYEN: luc.nguyen@maths.ox.ac.uk
*Mathematical Institute, University of Oxford, Oxford, United Kingdom*

# Analysis & PDE
msp.org/apde




Cover image: Eric J. Heller: “Linear Ramp”

---

See inside back cover or msp.org/apde for submission instructions.

---


The subscription price for 2025 is US $475/year for the electronic version, and $735/year (+$70, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscriber address should be sent to MSP.

---

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online.

---

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

**mathematical sciences publishers**

**nonprofit scientific publishing**

http://msp.org/

# ANALYSIS & PDE

Volume 18 No. 9 2025