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ÁNGEL CASTRO, DANIEL FARACO AND BJÖRN GEBHARD

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ENTROPY SOLUTIONS TO THE MACROSCOPIC INCOMPRESSIBLE POROUS MEDIA EQUATION

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We investigate maximal potential energy dissipation as a selection criterion for subsolutions (coarse-grained solutions) in the setting of the unstable Muskat problem. We show that both (a) imposing this criterion on the level of convex integration subsolutions and (b) the strategy of Otto based on a relaxation via minimizing movements lead to the same nonlocal conservation law. Our main result shows that this equation admits an entropy solution for unstable initial data with an analytic interface.

1. Introduction

An outstanding open problem in hydrodynamics is the description of unstable interface configurations quickly leading to turbulent regimes. Examples are the thoroughly studied Saffman–Taylor [Saffman and Taylor 1958], Rayleigh–Taylor [Rayleigh 1882; Taylor 1950] and Kelvin–Helmholtz [Thomson 1871] instabilities. In these unstable regimes, Eulerian quantities — such as the velocity field — are very irregular, and the Lagrangian trajectories typically fail to be uniquely defined. Hence uniqueness is not to be expected at the microscopic level, a phenomenon that in the physics literature is known as spontaneous stochasticity [Thalabard et al. 2020], and instead it will be desirable to have a well-defined deterministic evolution at the macroscopic level. The current paper provides such a macroscopic evolution in the context of the incompressible porous medium equation derived from maximal potential energy dissipation.

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Keywords: incompressible porous media, Saffman–Taylor instability, relaxation, subsolutions, nonlocal hyperbolic conservation law, mixing zone, level sets.

1.1. ***IPM and interfaces.*** Throughout the article, we will consider the incompressible porous media (IPM) equation, given by

$$\begin{aligned}\partial_t \rho + \operatorname{div}(\rho v) &= 0,\\ \operatorname{div} v &= 0, \\ v &= -\nabla p - \rho e_2\end{aligned} \tag{1-1}$$

on the two-dimensional periodic strip $\mathbb{T} \times \mathbb{R}$, where $\mathbb{T}$ denotes the flat 1-torus of length 2π, and over a time interval $[0, T)$, $T > 0$. Here the (normalized) fluid density $\rho : [0, T) \times \mathbb{T} \times \mathbb{R} \to \mathbb{R}$, the velocity $v : [0, T) \times \mathbb{T} \times \mathbb{R} \to \mathbb{R}^2$ and the pressure $p : [0, T) \times \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ are the unknowns, and

$$-e_2 := (0, -1)^T \in \mathbb{R}^2$$

is the direction of gravity.

The model describes the evolution of a two-dimensional density-dependent incompressible fluid in an overdamped scenario (the porous medium) and under the influence of gravity. It consists of the law for mass conservation, the incompressibility condition for the velocity field and Darcy's law (see [Allaire 1989; Darcy 1856; Muskat 1934; Saffman and Taylor 1958; Sánchez-Palencia 1980] for more physical background). Constants such as mobilities (viscosities), permeability of the medium, and gravity have been set to 1. System (1-1) also models the motion of an incompressible and viscous fluid in a Hele-Shaw cell [Saffman and Taylor 1958], a different physical scenario with the same mathematical formulation.

Concerning initial conditions, we are interested in the unstable interface case, i.e.,

$$\rho_0(x) = \begin{cases} +1, & x_2 > \gamma_0(x_1), \\ -1, & x_2 < \gamma_0(x_1), \end{cases} \tag{1-2}$$

for a graph $\gamma_0 : \mathbb{T} \to \mathbb{R}$.

Generally speaking, if the initial data ρ_0 is sufficiently regular it is well known that the IPM equation has a unique regular local-in-time solution; see [Castro et al. 2009; Córdoba et al. 2007]. However, the problem of formation of singularities versus global existence is still open and only partial results are known. For example, the existence of solutions with Sobolev norms unbounded in time has recently been proven in [Kiselev and Yao 2023].

In the case of discontinuous initial data of the type (1-2), the situation is even more subtle as the following dichotomy shows: If the denser fluid is below the lighter one, then the problem is stable and the existence of solutions is well known (see Section 2.1). However, if the lighter fluid is below the heavier one, the problem is ill-posed (at least in the Muskat sense, see Section 2.1, and in the sense of bounded weak solutions, see Section 2.2).

1.2. ***Macroscopic IPM.*** In spite of this difficulty, there have been several attempts to understand the evolution of such an initial configuration at least in the coarse-grained picture. Namely, on the one hand, Felix Otto [1999] discovered that, in the Lagrangian formulation, IPM is a gradient flow, and he suggested in the unstable situation a relaxation based on the corresponding minimizing movements scheme in the Wasserstein setting (JKO scheme). On the other hand, [Córdoba et al. 2011b] showed that

(nonstrict) IPM subsolutions
Definition 2.4 (nonstrict: with "$\le$" instead of "$<$" in (2-4))
existence: [Arnaiz et al. 2021; Castro et al. 2021; Förster and Székelyhidi 2018; Noisette and Székelyhidi 2021] and boxes below
uniqueness: no

macroscopic IPM solutions
Definition 3.1
existence: boxes below
uniqueness: no

IPM solutions
Definition 2.1
existence: [Siegel et al. 2004] and induced by subsolutions via [Székelyhidi 2012]
uniqueness: no

macroscopic IPM entropy solutions
Definition 3.4
existence: Theorem 3.2
uniqueness: open

Figure 1. Relation of (sub)solutions in the unstable nonflat interface case: note that each IPM solution is indeed also a macroscopic IPM solution due to the fact that IPM solutions satisfy $\rho(t,x)^2 = 1$ for almost every (t,x). Concerning the strictness of the stated inclusions, the listed references [Arnaiz et al. 2021; Castro et al. 2021; Förster and Székelyhidi 2018; Noisette and Székelyhidi 2021] provide subsolutions different from macroscopic IPM solutions. We also believe that the other two inclusions are strict, e.g., by methods similar to the ones used in the present paper, it should be possible to construct a nonflat two-shock solution to macroscopic IPM that is neither an entropy solution nor an IPM solution.

IPM can be recast as a differential inclusion in the Tartar framework and therefore fits the adaptation of convex integration in hydrodynamics by De Lellis and Székelyhidi [2009; 2010]. Subsequently, the full relaxation of the differential inclusion has been computed in [Székelyhidi 2012] leading to a concept of coarse-grained solutions (subsolutions in the convex integration jargon). In Section 2 we present precise definitions and review the historical landmarks of the theory. As an overview the reader can also consult two diagrams: one concerning the various notions of (sub)solutions occurring in the paper and their relations, see Figure 1, and another concerning the steps of the relaxations, see Figure 2.

Let us remark that [Székelyhidi 2012] proved in the case of a flat interface that Otto's relaxation selects a convex integration subsolution, which turns out to be the global-in-time entropy solution to a

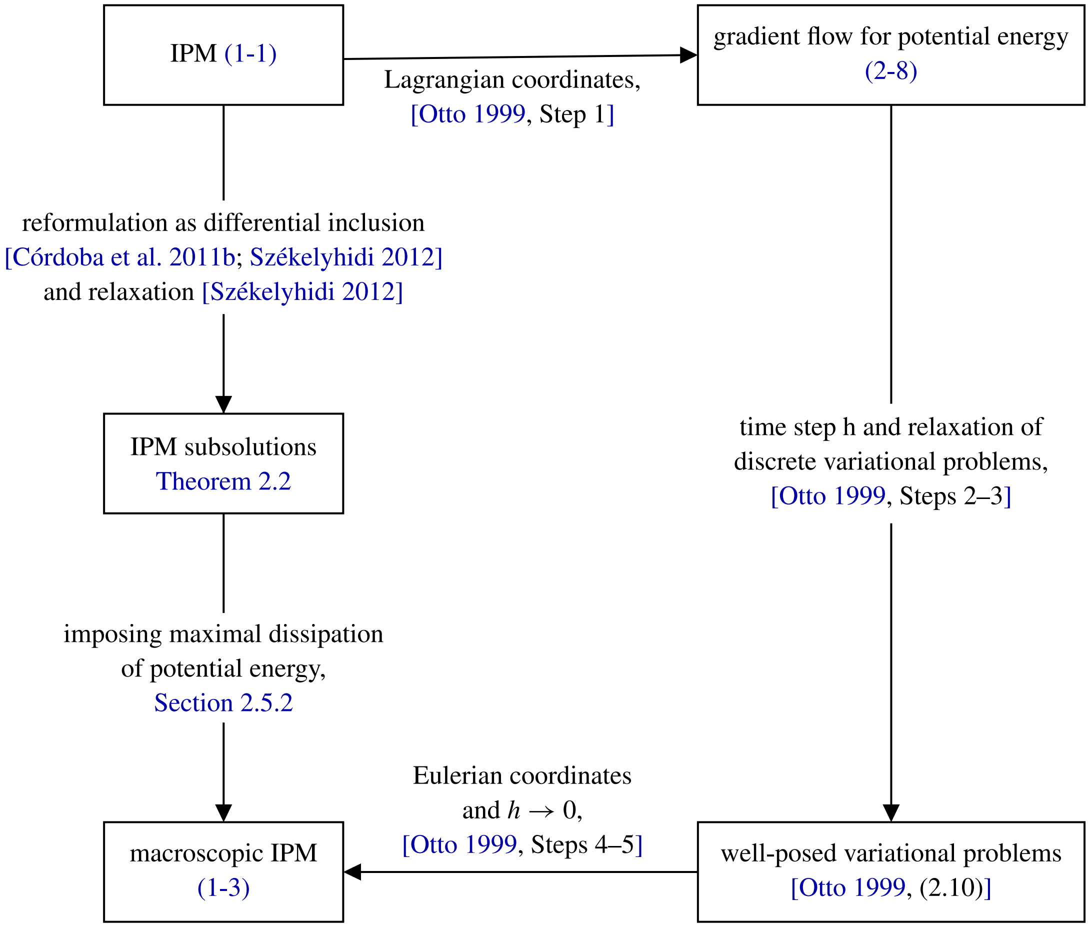

Figure 2. Relaxation of IPM in Eulerian coordinates via subsolutions on the left and in Lagrangian coordinates via minimizing movements on the right.

one-dimensional Burgers equation, reconciling both relaxation theories. In the case of a nonflat interface, the theory of convex integration starting from [Castro et al. 2021] has provided a number of subsolutions [Arnaiz et al. 2021; Castro et al. 2022; Förster and Székelyhidi 2018; Noisette and Székelyhidi 2021]. In all these situations, the starting point is an ansatz for the coarse-grained density $\bar{\rho}$ and for the growth of the mixing zone motivated in analogy to the flat case. These subsolutions show that also on a macroscopic level plenty of different evolutions are possible, such that a selection, which so far has not been available, has to be made for an attempt to claim uniqueness.

The aim of this paper is to use maximal potential energy dissipation as a selection criterion. Since, as discovered by Otto, in Lagrangian coordinates IPM is a gradient flow with respect to potential energy, this seems a natural approach. In any case, we first revisit the strategy proposed by Otto [1999] in the case of nonflat interfaces (the scheme is explained in Section 2.5.1 and Appendix B). We then reconcile it by selecting the subsolution in the convex integration terminology which at each time instant dissipates the most potential energy.

It can be shown that both (a) the relaxed minimizing movements scheme provided in [Otto 1999] (at least formally) and (b) imposing maximal potential energy dissipation among convex integration subsolutions (rigorously) lead to the equation

$$\begin{aligned} \partial_t \rho + \operatorname{div}(\rho v) + \partial_{x_2}(\rho^2) &= 0, \\ \operatorname{div} v &= 0, \\ v &= -\nabla p - \rho e_2, \end{aligned} \tag{1-3}$$

which will be referred to as *macroscopic IPM*. In Section 2 we explain in detail how the Muskat problem, the theory of subsolutions, convex integration for IPM and Otto's relaxation are connected. The derivation of (1-3) from the JKO scheme is known to experts, but as far as we are aware the arguments around maximal potential energy dissipation for subsolutions are new. In particular, it will be explained in which way (1-3) can offer a selection criterion for IPM subsolutions based on a natural extension of the gradient flow structure of IPM.

(Entropy) solutions to macroscopic IPM are subsolutions to IPM as long as they exist. By introducing a parameter $0 < \mu < 1$ in the first equation,

$$\partial_t \rho + \operatorname{div}(\rho v + \mu \rho^2 e_2) = 0,$$

macroscopic IPM produces strict subsolutions. Hence, by a suitable h-principle, see Theorem 2.2, the time of existence of microscopic solutions to IPM will be dictated by the time of existence of (1-3). This is in stark contrast to [Castro et al. 2021], where a rarefaction-like ansatz with a prescribed speed of opening of the mixing zone is made and a resulting time- and space-dependent parameter $\mu(t, x)$ is derived which is smaller than 1 just for short times.

In general we emphasize that, contrary to the procedure of [Castro et al. 2021] (and also of [Arnaiz et al. 2021; Castro et al. 2022; Förster and Székelyhidi 2018; Noisette and Székelyhidi 2021]), i.e., deriving a macroscopic equation from an ansatz, we here follow the reversed process, i.e., we consider based on a selection a fixed equation for the macroscopic evolution and derive properties of its solutions, such as the speed of opening of the mixing zone. We believe that this is a necessity when it comes to potential applications addressing for instance the prediction of a unique mixing zone evolution.

1.3. ***Existence result and idea of proof.*** The bulk of the paper is devoted to proving the existence of an entropy solution for (1-3) with (1-2) as initial data. System (1-3) can be written as a single scalar nonlocal hyperbolic conservation law,

$$\partial_t \rho + \operatorname{div}(\rho T[\rho]) + \partial_{x_2}(\rho^2) = 0, \tag{1-4}$$

where $v = T[\rho]$ is a zeroth-order singular integral operator. Contrary to other nonlocal conservation laws with a more regular nonlocal feedback — see [Amadori and Shen 2012; Amorim 2012; Betancourt et al. 2011; Blandin and Goatin 2016; Colombo et al. 2012] for examples and [Keimer and Pflug 2023] for a recent overview — a general existence and uniqueness theory for nonlocal terms as in (1-4) is not available. We bypass this by using the structure of the two-phase initial data (1-2). This approach, born out of necessity, not only provides us with the existence of a solution but in addition allows us to learn about certain properties of it. More precisely, by showing that the Burgers' term $\partial_{x_2}(\rho^2)$ is able to tear up the initial discontinuity of the density even in the presence of the incompressible velocity v, we will prove the existence of a local-in-time solution which is Lipschitz for $t > 0$. This fact is highly nontrivial and presents many technical difficulties that will be tackled in Sections 4–6, which together form the proof of our main theorem and will be described below. A careful statement of our main theorem itself, containing further properties of the solution, can be found in Section 3. We have preferred to state the existence theorem for (1-3) after the reader is hopefully convinced by Section 2 that (1-3) renders a macroscopic description for the unstable Muskat problem consistent with maximal potential energy dissipation.

One main ingredient of our proof is to look at the evolution of level sets of the density ρ in suitably scaled coordinates and to adjust properly to leading-order terms of this evolution. These steps, carried out in Section 4, reduce the initial value problem (1-2), (1-3) to a fixed-point problem of the type

$$\eta(t,y)=\frac{1}{t^{1+\alpha}}\int_0^t\int_{-2}^2\int_{\mathbb{T}}(K_s[\eta(s,\cdot)])(y,z)(h_s[\partial_{y_1}\eta(s,\cdot)])(y,z)\,dz_1\,dz_2\,ds-\frac{1}{t^{\alpha}}h_0(y) \tag{1-5}$$

for functions $\eta:[0,T)\times\mathbb{T}\times(-2,2)\to\mathbb{R}$ describing the evolution of the level sets in superlinear order with respect to $t>0$ small. The constants ± 2 for the domain of η are coming from the rarefaction speed of Burgers' equation. Moreover, here $h_0(y)$ is one of the mentioned leading-order terms — in fact the first-order term — depending on the initial graph γ_0 and $\alpha\in(0,1)$. Moreover, for each $s>0$, $y\in\mathbb{T}\times(-2,2)$ and $\xi:\mathbb{T}\times(-2,2)\to\mathbb{R}$ fixed, the function $z\mapsto(K_s[\xi])(y,z)$ is a convolution kernel of order -1 induced by the Biot–Savart law. The dependence on ξ involves both $\xi(z)$ and $\xi(y)$ in the form of the difference $\xi(y)-\xi(z)$. Similarly, the function $(y,z)\mapsto h_s[\partial_{y_1}\xi](y,z)$, again considered for a fixed s and ξ, depends on the difference $\partial_{y_1}\xi(y)-\partial_{y_1}\xi(z)$. Thus, after integration in z, the regularity of the right-hand side of (1-5) with respect to y is the regularity of $\partial_{y_1}\eta$, i.e., the right-hand side when seen as an operator looses one derivative in y_1.

In addition, as one of the main difficulties — also for potential equivalent reformulations of (1-5) where the above loss of a derivative might be avoided — we would like to point out that the kernels $K_s[\xi]$ degenerate as $s\to 0$ to a one-dimensional kernel with singularity $\sim 1/(y_1-z_1)$, i.e., to an integral kernel of order 0. Thus, estimates for $K_s[\xi](y,\cdot)$ as a kernel of order -1 cannot be obtained uniformly in s.

Regardless, in order to keep the paper enjoyable, we deal with (1-5) and its loss of derivative by considering real analytic initial interfaces. This allows us to use an adaptation of the Nirenberg–Nishida abstract Cauchy–Kovalevskaya theorem. Still, the application of it — even when we continue to ignore the so far not mentioned factor $t^{-(1+\alpha)}$ on the right-hand side — takes quite a lot of effort. It is the second main part of our proof and can be found in Section 5.

Finally, Section 6 puts everything together to give a solution to the macroscopic IPM equation. In Appendices A, B and C, we give a proof of a version of the abstract Cauchy–Kovalevskaya theorem needed for our situation, and we give some more details regarding the derivation of the macroscopic IPM equation.

1.4. ***On the entropy condition.*** We emphasize that the solution we find is an entropy solution of (1-3), or rather (1-4). The notion of an entropy solution is stated in Definition 3.4. This is consistent with the flat case $\gamma_0 = 0$ where, as said earlier, the relaxed minimizing movements scheme of Otto [1999] converges to the entropy solution of (1-4) which in that case reduces to Burgers' equation. For an extended discussion concerning the selection of the entropy solution by the minimizing movements scheme (including other gradient flows as counterexamples where a corresponding selection fails), we refer to [Gigli and Otto 2013], where the IPM relaxation is revisited in the flat setting of Otto's original work [1999]. In addition see also [Otto 2001] for a stability result in the flat case. Concerning the general, nonflat case, it was also conjectured by Otto (personal communication) that the convergence of the minimizing movements scheme to an entropy solution remains true.

Moreover, we point out that some sort of choice among solutions of (1-3) is critical in order to have a selection criterion. Indeed, already in the flat case solutions are clearly not unique, and also in the general case nonentropic solutions for (1-3) can be obtained in an easier way, for instance via (2-1) below; see Remark 3.3. We believe that the requirement of being an entropy solution leads to uniqueness for the initial value problem (1-2), (1-3), but, since the velocity v depends on ρ in a comparably singular nonlocal way, standard methods do not seem to work and uniqueness of entropy solutions to macroscopic IPM stands as an interesting open question. In any case, we emphasize that for the scheme we present there is a unique solution, and therefore our maximal dissipating subsolution is amenable to numerical calculations.

1.5. ***Further questions.*** Besides the question of uniqueness of the found entropy solution, our work opens the door to many other questions with various levels of difficulty, such as improving the regularity of the solutions, considering initial interfaces (not being analytical or not being a graph, as for example in [Castro et al. 2022]) or other densities as initial data as well. It would be interesting to see whether the JKO scheme does converge rigorously or what happens in the case of different mobilities [Mengual 2022; Otto 1999]. On a more general level, there might be other selection criteria for IPM, for example, based on surface tension [Jacobs et al. 2021] or on vanishing diffusion [Menon and Otto 2005; 2006]. Finally, we emphasize that our selection criterion ultimately is tailored to the gradient flow structure of IPM, and for other equations the reasoning necessarily must be different. In any case, we hope our work encourages the research on finding a deterministic coarse-grained evolution in the presence of instabilities.

2. Ill-posedness and relaxation

The unstable interface initial value problem considered here is highly ill-posed. In this section we explain in which sense this ill-posedness holds, as well as a strategy based on convex integration and the relaxation of [Otto 1999] to overcome it. This section, having the purpose to fully motivate equation (1-3), is mostly a review of existing results. Except for the derivation in Section 2.5.2 showing that maximal potential energy dissipating subsolutions coincide with Otto's relaxation, we do not claim any novelty. However, we are not aware that the computations in Section 2.5.1 can be found in the literature. A reader only interested in solving system (1-3) can go directly to Section 3.

2.1. ***The Muskat problem.*** If one assumes that

$$\rho(x,t)=\begin{cases}\rho_{\mathrm{up}}, & x_2>f(x_1,t),\\ \rho_{\mathrm{down}}, & x_2<f(x_1,t),\end{cases}$$

a closed equation from (1-1) can be obtained for the interface $(x_1, f(x_1,t))$. Indeed,

$$\partial_t f(x,t)=\frac{\rho_{\mathrm{down}}-\rho_{\mathrm{up}}}{4\pi}\int_{\mathbb{T}}\frac{\sin(y)(f_x(x,t)-f_x(x-y,t))}{\cosh(f(x,t)-f(x-y,t))-\cos(y)}\,dy. \tag{2-1}$$

This equation is usually known in the literature as the Muskat equation honoring M. Muskat [1934].

In the case $\rho_{\mathrm{down}}>\rho_{\mathrm{up}}$, the problem is stable and local existence and regularity of solutions can be proven in different functional settings and situations [Abels and Matioc 2022; Agrawal et al. 2023; Alazard and Lazar 2020; Alazard and Nguyen 2021b; 2021a; 2023; 2022; Cameron 2019; Chen et al. 2022; Cheng et al. 2016; Choi et al. 2007; Córdoba and Gancedo 2007; Córdoba et al. 2011a; 2013; 2014; Deng et al. 2017; Escher and Matioc 2011; García-Juárez et al. 2022; 2024; Matioc 2019; Nguyen and Pausader 2020; Shi 2023], as well as global for small and medium size initial data [Alonso-Orán and Granero-Belinchón 2022; Constantin et al. 2013; 2016; 2017; Córdoba and Lazar 2021; Dong et al. 2023; Gancedo and Lazar 2022; Granero-Belinchón and Lazar 2020]. The existence of singularities for large initial data is shown in [Castro et al. 2012a; 2013] and also in [Córdoba et al. 2015; 2017].

However, if $\rho_{\mathrm{down}}<\rho_{\mathrm{up}}$, the Muskat equation is ill-posed [Córdoba and Gancedo 2007; Siegel et al. 2004]. Surprisingly, convex integration has allowed us to construct solutions to IPM starting in these kinds of unstable situations. They have been called mixing solutions and, in them, the initial interface between the two different densities disappears and a strip arises in which the two densities mix. We elaborate on these mixing solutions in the next sections. For a general picture of convex integration in the context of fluid dynamics, we refer to the surveys [Buckmaster and Vicol 2021; De Lellis and Székelyhidi 2019; 2022].

2.2. ***IPM as differential inclusion.*** The first examples of nonuniqueness of weak solutions for (1-1) using convex integration were given in [Córdoba et al. 2011b] by Córdoba, Gancedo and the second author for the initial value $\rho_0=0$. Their method bypasses the computation of the relaxation by means of so-called T_4 configurations. After this, Székelyhidi [2012] established the explicit relaxation of (1-1) for initial data of two-phase type, enabling a systematic investigation of interface problems in IPM. While the results in [Córdoba et al. 2011b; Székelyhidi 2012] established ill-posedness of IPM in the class of essentially bounded solutions, Isett and Vicol [2015] could also show the existence of compactly supported $C^{\alpha}_{t,x}$-solutions for $\alpha<\frac{1}{9}$. The starting point of our investigation is the relaxation of [Székelyhidi 2012], which we will describe in this subsection.

In the following we consider initial data with $|\rho_0|=1$ almost everywhere. The corresponding notion of weak solutions is fixed in Definition 2.1 below. Note that, for such initial data, the last condition in the definition is an additional consistency requirement coming from the continuity, or rather transport, equation in (1-1).

Definition 2.1. A pair $\rho \in L^\infty((0,T)\times\mathbb{T}\times\mathbb{R})$, $v \in L^\infty(0,T; L^2(\mathbb{T}\times\mathbb{R};\mathbb{R}^2))$ is a solution of (1-1), (1-2) provided, for any $\varphi \in \mathcal{C}_c^\infty([0,T)\times\mathbb{T}\times\mathbb{R})$, we have

$$\int_0^T\int_{\mathbb{T}\times\mathbb{R}} \rho\partial_t\varphi + \rho v\cdot\nabla\varphi\,dx\,dt + \int_{\mathbb{T}\times\mathbb{R}} \rho_0\varphi(0,\cdot\,)\,dx = 0,$$
$$\int_0^T\int_{\mathbb{T}\times\mathbb{R}} v\cdot\nabla\varphi\,dx\,dt = 0,$$
$$\int_0^T\int_{\mathbb{T}\times\mathbb{R}} (v+\rho e_2)\cdot\nabla^\perp\varphi\,dx\,dt = 0,$$

and $|\rho(t,x)| = 1$ for almost every $(t,x) \in (0,T)\times\mathbb{T}\times\mathbb{R}$.

A key step in [Córdoba et al. 2011b; Székelyhidi 2012] is to recast weak solutions as defined above as solutions to a differential inclusion, to be able to use the Murat–Tartar compensated compactness formalism [Tartar 1979].

A pair (ρ, v) is a weak solution if and only if the triple

$$(\rho, v, m) \in L^\infty((0,T)\times\mathbb{T}\times\mathbb{R}) \times (L^\infty(0,T; L^2(\mathbb{T}\times\mathbb{R})))^2$$

satisfies the linear system

$$\begin{aligned} \partial_t\rho + \operatorname{div} m &= 0,\\ \operatorname{div} v &= 0,\\ v &= -\nabla p - \rho e_2,\\ \rho(0,\cdot\,) &= \rho_0 \end{aligned} \tag{2-2}$$

distributionally, i.e., in analogy to Definition 2.1, together with

$$(\rho(t,x), v(t,x), m(t,x)) \in K := \{(\rho, v, m) \in \mathbb{R}^5 : |\rho| = 1,\ m = \rho v\} \tag{2-3}$$

for almost every $(t,x) \in (0,T)\times\mathbb{T}\times\mathbb{R}$.

Then the relaxation of the incompressible porous media equation is understood as the relaxation of the corresponding differential inclusion; i.e., in the pointwise nonlinear constraint (2-3), the set K is replaced by its convex (or more generally Λ-convex) hull. Up to technicalities, one can recover highly oscillatory solutions from this set, as the main theorem of [Székelyhidi 2012] shows.

Theorem 2.2 [Székelyhidi 2012]. *Let $\bar\rho \in L^\infty((0,T)\times\mathbb{T}\times\mathbb{R})$ and $\bar v, \bar m \in L^\infty(0,T; L^2(\mathbb{T}\times\mathbb{R}))$ satisfy (2-2) in the sense of distributions. Suppose that there exists a bounded and open set $\mathscr{U} \subset (0,T)\times\mathbb{T}\times\mathbb{R}$ such that (2-3) holds for almost every $(t,x) \notin \mathscr{U}$, while $(\bar\rho, \bar v, \bar m)$ are continuous on $\mathscr{U}$ with*

$$(\bar\rho(t,x), \bar v(t,x), \bar m(t,x)) \in \{(\rho, v, m) \in \mathbb{R}^5 : |\rho| < 1,\ |2(m-\rho v) + (1-\rho^2)e_2| < (1-\rho^2)\} \tag{2-4}$$

for every $(t,x) \in \mathscr{U}$. Then there exist infinitely many weak solutions (ρ, v) of (1-1), (1-2) that coincide with $(\bar\rho, \bar v)$ outside of $\mathscr{U}$ and are arbitrarily close to $(\bar\rho, \bar v)$ in the weak $L^2(\mathscr{U})$-topology.

In the case of the IPM system, the set on the right-hand side of (2-4) is indeed only the interior of the Λ-convex hull of K, see [Székelyhidi 2012] for a precise definition, which does not coincide with the full convex hull as opposed to the Euler equations. Still, (2-4) describes all possible weak limits of solutions to the IPM system, see [Székelyhidi 2012]. In view of that, one can therefore truly speak about the full relaxation of IPM in the context of two-phase mixtures.

This fact has been quantified in [Castro et al. 2019], where the relation between solutions and subsolutions has been made precise through an adapted h-principle. In particular, this leads to additional properties of the solutions like a degraded macroscopic behavior or the turbulent mixing at every time-slice property. The latter means that the solutions (ρ, v) induced by $(\bar{\rho}, \bar{v}, \bar{m})$ satisfy $\rho \in \mathcal{C}^0([0,T); L^2_{\text{weak}}(\mathbb{T}\times(-R,R)))$, where R is some positive number with $\mathscr{U} \subset (0,T)\times\mathbb{T}\times(-R,R)$, and

$$\left(\int_B (1-\rho(t,x))\,dx\right)\left(\int_B (1+\rho(t,x))\,dx\right) > 0 \tag{2-5}$$

for any $t \in (0,T)$ and any ball B fully contained in $\mathscr{U}_t := \{x \in \mathbb{T}\times\mathbb{R} : (t,x) \in \mathscr{U}\}$.

For later purposes, we also point out the following possible upgrade of Theorem 2.2, which is obtained by using convex integration as in [Castro et al. 2019; De Lellis and Székelyhidi 2010].

Lemma 2.3. *Let $(\bar{\rho}, \bar{v}, \bar{m})$ be as in Theorem 2.2 and $\delta : [0,T) \to \mathbb{R}$ continuous with $\delta(0) = 0$, $\delta(t) > 0$, $t > 0$. Then there exist infinitely many solutions (ρ, v) as in Theorem 2.2 with the additional property that*

$$\left|\int_{\mathbb{T}\times\mathbb{R}} (\bar{\rho}(t,x) - \rho(t,x))x_2\,dx\right| \le \delta(t)$$

for almost every $t \in [0,T)$.

Definition 2.4. Any triple $(\bar{\rho}, \bar{v}, \bar{m})$ satisfying the conditions of Theorem 2.2 is called a subsolution of (1-1), (1-2). The set $\mathscr{U}$, in other papers frequently also denoted by Ω_{mix}, is called the mixing zone of the subsolution.

Theorem 2.2 shifts the focus from a single solution to the investigation of subsolutions which are understood as possible coarse-grained or averaged solutions. As subsolutions play the central role also in the present investigation, we will frequently omit the bars in notation and instead mark solutions by $(\rho_{\text{sol}}, v_{\text{sol}})$ in case there is a chance of confusion.

2.3. *Examples of subsolutions.* The first examples of nonconstant subsolutions have been given in the same paper of Székelyhidi [2012] for the perfectly flat initial interface, $\rho_0(x) = \operatorname{sign}(x_2)$. Keeping the one-dimensional structure of the initial data, one sees that $v = 0$, $m = -\alpha(1-\rho^2)e_2$, $\alpha \in (0,1)$ reduces (2-2), (2-3) to the one-dimensional conservation law

$$\partial_t \rho + \alpha \partial_{x_2}(\rho^2) = 0,$$

which has a unique entropy solution given by

$$\rho(t,x) = \begin{cases} 1, & x_2 > 2\alpha t, \\ x_2/(2\alpha t), & -2\alpha t < x_2 < 2\alpha t, \\ -1, & x_2 < -2\alpha t. \end{cases}$$

It also has been mentioned in [Székelyhidi 2012] that the limiting case $\alpha = 1$ is in agreement with the relaxation of Otto [1999]. It coincides with (1-3) in the flat situation, see Section 4.1. In addition, this case gives an upper bound for the mixing zone. More precisely, it has been shown in [Székelyhidi 2012] that the mixing zone at time $t > 0$, $\mathscr{U}_t$, of any one-dimensional subsolution emanating from $\rho_0(x) = \text{sign}(x_2)$ is contained in the strip $[-1, 1] \times (-2t, 2t)$. A similar subsolution in the harder case of different viscosities was studied in [Mengual 2022]. Actually, the Λ-hull of IPM with different viscosities and densities is computed in that paper.

In the context of IPM and differential inclusions, we would also like to mention [Hitruhin and Lindberg 2021] which addresses the stationary, i.e., time-independent, IPM system. In that paper the lamination convex hull of that system is computed, and in addition a rigidity result for its subsolutions and an application for long-term limits of (1-1) is given.

The first examples of subsolutions giving rise to mixing solutions, i.e., solutions obtained from the subsolution via convex integration with property (2-5), for IPM starting in a nonflat interface $(x_1, f_0(x_1))$ were provided in [Castro et al. 2021]. In this paper the density ρ of the subsolution is Lipschitz and the prescribed speed of opening of the mixing zone $c(x_1)$ $(= 2\alpha$ in the flat case above) satisfies $1 \le c < 2$ and, as indicated, might depend on x_1. The result of [Castro et al. 2021] holds for initial data $f_0 \in H^5(\mathbb{R})$, i.e., in a regime where the Muskat problem cannot be solved. A numerical analysis of these subsolutions can be found in [Castro 2017], where the formation of fingers can be observed. In [Arnaiz et al. 2021], the semiclassical viewpoint developed in [Castro et al. 2021] is taken one step further (using semiclassical Sobolev spaces for example), providing an alternative proof to the main result of [Castro et al. 2021]. Indeed this later approach improves the subsolutions with respect to their regularity, as the boundary of the mixing zone is in $H^{5-1/c(x_1)}$, where $c(x_1)$ is the local speed of opening of the mixing zone, instead of merely in H^4.

Förster and Székelyhidi [2018] constructed mixing solutions with an initial interface $f_0 \in \mathcal{C}^{3+\alpha}$ relaxing the initial regularity needed in [Castro et al. 2021] but relying on subsolutions with piecewise constant density instead of Lipschitz. In this case the speed of opening of the mixing zone is $0 < c < 2$ with c uniform in x_1. Thereafter the same kind of subsolutions have been constructed in [Noisette and Székelyhidi 2021] with variable speed of opening.

As mentioned before, mixing solutions obtained via convex integration are not unique. There are two reasons for this fact: (a) different subsolutions can be found, (b) infinitely many solutions, corresponding to different distributions of the density, emanate from every fixed subsolution. In order to deal with point (b), in [Castro et al. 2019] it has been shown that all the solutions obtained from a fixed subsolution can be chosen in such a way that they share averages over large sets, i.e., they are the same as the subsolution at a macroscopic level. One of the main points of the present paper is to deal with point (a). A particular instance of this multiplicity will be illustrated in Section 2.4 below.

The constructions of the subsolutions above seem to rely on the Saffman–Taylor instability (heavy fluid on top of a lighter fluid). In [Castro et al. 2021] it was observed that there also exist mixing solutions in the stable regime (see also [Förster and Székelyhidi 2018]) which build on Kelvin–Helmholtz-type instabilities (discontinuity of the velocity field, see [Mengual 2022] for a thorough discussion of this phenomena at the level of the hulls). Actually, the analysis in [Castro et al. 2021] indicates that the mixing can be created around any point of the interface which is not both flat (with zero slope) and stable. We call points having zero slope in the stable regime fully stable points. It happens that, in an initially overhanging interface, there must be always a fully stable point. Partially unstable situations therefore require one to find compatibility between the Muskat solution and mixing solutions, see [Castro et al. 2022]. Remarkably, the construction in that paper allows one to answer the question on how to prolongate in time the singular solutions to the Muskat problem found in [Castro et al. 2012a; 2013], namely as mixing solutions.

As a last remark we would like to point out that the subsolutions constructed in [Castro et al. 2021; 2022; Förster and Székelyhidi 2018; Noisette and Székelyhidi 2021] are local in time in the sense that, although the involved functions exist over a potentially larger time interval, a small time interval has to be chosen in order to guarantee that they take values inside the convex hull, i.e., that (2-4) holds. This is in contrast to the flat cases [Mengual 2022; Székelyhidi 2012] and to the subsolution constructed in the present paper. Although here we will only prove a local-in-time existence result, the involved functions take values in (the closure of) the convex hull as long as they exist.

2.4. ***The subsolution selection problem.*** As described, the constructions from the previous subsection contain ansatzes for certain properties of the subsolution and hence for the induced mixing solutions of (1-1). To illustrate this freedom in the simplest case, let us discuss the flat interface with $\gamma_0(x_1) = 0$ in slightly more detail. As in [Székelyhidi 2012], setting $v \equiv 0$, $m = m_2(t, x_2)e_2$, $\rho = \rho(t, x_2)$, one sees that $(\rho, 0, m)$ is a subsolution if and only if

$$\partial_t \rho + \partial_{x_2} m_2 = 0, \qquad \rho(0, x) = \operatorname{sign}(x_2), \qquad |\rho| \le 1,$$
$$|2m_2 + 1 - \rho^2| < 1 - \rho^2 \ \text{ when } |\rho| < 1, \qquad m_2 = 0 \ \text{ when } |\rho| = 1,$$

and the required continuity conditions hold. Thus one could make the ansatz

$$m_2 = -\frac{1-\rho^2}{2} + \frac{1-\rho^2}{2}\xi_2 \tag{2-6}$$

with $\xi_2 : [0, T) \times \mathbb{R} \to \mathbb{R}$ satisfying $|\xi_2| < 1$ and for any such ξ_2 solve the conservation law

$$\partial_t \rho + \partial_{x_2}\left((\xi_2(t, x_2) - 1)\frac{1-\rho^2}{2} \right) = 0$$

with initial data $\rho_0(x_2) = \operatorname{sign}(x_2)$ to get plenty of subsolutions with different mixing zones and density profiles. Note that in this sense ξ_2, or rather the whole relation (2-6), plays the role of a constitutive law.

Summarizing once more, these examples show that not only does each subsolution induce infinitely many solutions of the incompressible porous media equation sharing a common coarse-grained, or averaged, behavior, but there are also infinitely many possibilities for this averaged evolution via the vast amount of possible subsolutions. This is a common problem in the construction of turbulent solutions emanating from unstable interface initial data, as for instance also for the Kelvin–Helmholtz instability

[Gebhard and Kolumbán 2022a; Mengual and Székelyhidi 2023; Székelyhidi 2011] and the Rayleigh-Taylor instability in the context of the Euler equations [Gebhard and Kolumbán 2022b; Gebhard et al. 2021; 2024]. We emphasize that, however, our criteria builds on the gradient flow structure of IPM, and therefore different ideas should be used in the case of the Euler equations, see Section 2.5.3 for a short overview of strategies used so far.

2.5. ***A selection criterion.*** We now focus in the general, not necessarily flat, case on the selection of subsolutions in terms of choosing an appropriate relation between m, ρ and v such that (2-4) holds provided $|\rho| \leq 1$.

First we will review the strategy proposed by F. Otto [1999] to relax system (1-1) based on its gradient flow structure in Lagrangian coordinates, and we will formally obtain (1-3) from this relaxation. The strategy of Otto does not rely on the notion of a subsolution in the context of differential inclusions as in Section 2. However, the solution of (1-3) will be a (nonstrict) subsolution with

$$m = \rho v - (1-\rho^2)e_2.$$

Thereafter, we will also give an argument to derive (1-3) in Eulerian coordinates directly based on subsolutions. Also, here the starting point will be the gradient flow structure of (1-1). This second argument shows that the relaxation of Otto selects among all subsolutions precisely those that maximize the dissipation of potential energy at every time instant.

The relations are summarized in Figure 2 on page 2244.

2.5.1. *Otto's relaxation.* In this section we give a very brief summary of Otto's five-step strategy leading to the macroscopic IPM equation (1-3). The discussion is not rigorous and even then we have put most of the explicit calculations in Appendix B. We adapt our notation to that of [Otto 1999], which, due to a different normalization, studies the evolution of

$$s(x,t) = \frac{1-\rho\big(x, \tfrac{1}{2}t\big)}{2}$$

instead of $\rho(x,t)$, i.e., contrary to other sections the density s is now taking values in $[0, 1]$. In these coordinates the IPM system (1-1) reads

$$\begin{aligned} \partial_t s + u\cdot\nabla s &= 0,\\ \operatorname{div} u &= 0,\\ u &= -\nabla\Pi + s e_2; \end{aligned} \tag{2-7}$$

see Appendix B.

The starting point (Step 1) of Otto's relaxation is the vital fact that, when formulated in Lagrangian coordinates, IPM can be seen as a gradient flow with respect to the potential energy

$$E[\Phi] = -\int s(x,0)\Phi(x)\cdot e_2$$

on the manifold

$$M_0 = \{\Phi \text{ one-to-one and onto, smooth, volume-preserving maps}\}.$$

More precisely, if (s, u, Π) is a solution of (2-7), then the flow $\Phi(x, t)$ induced by u satisfies

$$\int \partial_t \Phi(\,\cdot\,, t) \cdot w = -dE[\Phi(\,\cdot\,, t)]w \quad \text{for all } w \in T_{\Phi(\,\cdot\,,t)} M_0, \tag{2-8}$$

where $dE[\Phi]w$ is the Fréchet derivative of the functional E at the point $\Phi \in M_0$ in the direction

$$w \in T_\Phi M_0 = \{w \text{ smooth and such that } \nabla \cdot (w \circ \Phi^{-1}) = 0\}.$$

Fast-forwarding a bit, the next steps of Otto consist of the introduction of a time discretization with step size $h > 0$ in the form of a minimizing movements scheme (Step 2), the extension of the underlying manifold M_0 to its L^2-closure in order to turn the potentially ill-posed discrete variational problems emanating from Step 2 to well-posed ones (Step 3), and a translation of the now existing sequence of minimizers back to Eulerian coordinates (Step 4). At this point there exists a sequence of functions $\theta^{(k)}$ corresponding to $s(\,\cdot\,, t)$ at time $t = kh$, but of course potentially on a coarse-grained or "locally averaged" level, which is characterized by the following JKO scheme: $\theta^{(0)} = s(\,\cdot\,, 0)$ and, given $\theta^{(k)}$, $\theta^{(k+1)}$ is the minimizer in K of

$$\tfrac{1}{2}\,\mathrm{dist}^2(\theta^{(k)}, \theta) + \tfrac{1}{2}\,\mathrm{dist}^2(1 - \theta^{(k)}, 1 - \theta) - h \int \theta(x) x_2, \tag{2-9}$$

where the set K consists of measurable θ taking values in $[0, 1]$ and such that $\int \theta = \int s(x, 0)$, and $\mathrm{dist}^2(\theta_0, \theta_1)$ for $\theta_0, \theta_1 \in K$ is the L^2-Wasserstein distance

$$\mathrm{dist}^2(\theta_0, \theta_1) = \inf_{\Phi \in I(\theta_0, \theta_1)} \int \theta_0(x) |\Phi(x) - x|^2 \, dx$$

with

$$I(\theta_0, \theta_1) = \left\{ \Phi : \int \theta_1(y) \zeta(y) \, dy = \int \theta_0(x) \zeta(\Phi(x)) \, dx \ \ \forall \zeta \in \mathcal{C}_0^0 \right\}.$$

Notice that this indeed is a relaxation of the original problem since the densities are no longer taking values in $\{0, 1\}$ and the transport maps are not necessarily injective.

The fifth and last step consists of passing to the limit $h \to 0$ whenever this is possible. Otto [1999] proved that this is the case for the unstable flat situation

$$s(x, 0) = \begin{cases} 0, & x_2 > 0, \\ 1, & x_2 < 0, \end{cases}$$

and that the limit of θ_h defined by

$$\theta_h(x, t) := \theta^{(k)}(x), \qquad t \in [kh, (k+1)h)$$

is the unique entropy solution of the conservation law

$$\partial_t \theta + \partial_{x_2}(\theta(1 - \theta)) = 0.$$

For a different proof of this statement we refer to the work of Gigli and Otto [2013], which in particular also contains a further examination of the relation between the minimizing movements scheme and the entropy condition.

In fact, it was conjectured by Otto (personal communication) that the described scheme, if it converges, should also lead to an entropy solution of the macroscopic IPM equation in the general, nonflat case. We refer to Section 3 for the definition of entropy solutions.

In the rest of this section we sketch how at least formally system (1-3), or rather its equivalent reformulation in terms of $s(x,t)$, arises from the JKO-characterization (2-9) of the discrete functions $\theta^{(k)}$ when *assuming* suitable convergence. Our presentation here, as well as in Appendix B which contains some more details, is devoted to conveying that the scheme indeed leads to the macroscopic IPM equation rather than to providing a rigorous proof which we defer to future work. A similar computation was derived by Otto (personal communication).

Fix t and write for simplicity $\theta^0 := \theta_h(t)$, $\theta^1 := \theta_h(t+h)$. Furthermore, let Φ^h denote the transport map corresponding to $\operatorname{dist}^2(\theta^0,\theta^1)$ and $\bar{\Phi}^h$ the transport map corresponding to $\operatorname{dist}^2(1-\theta^0, 1-\theta^1)$. Then it can be shown that there are functions a^h, $\bar{a}^h$ such that

$$\Phi^h(x) = x + (\nabla a^h \circ \Phi^h)(x),$$
$$\bar{\Phi}^h(x) = x + (\nabla \bar{a}^h \circ \bar{\Phi}^h)(x).$$

This in fact is a consequence of Brenier's theorem [1991]; still an argument is also provided in Appendix B.

Moreover, it can be deduced from first variations of the functional (2-9) that

$$a^h - \bar{a}^h = hx_2. \tag{2-10}$$

Now, we write $a^h = hp^h$, $\bar{a}^h = h\bar{p}^h$ and make the strong assumption that the introduced functions p^h, $\bar{p}^h$ have a well defined $\mathcal{C}^2$ limit denoted by p, $\bar{p}$. Moreover, we also assume that $\theta_h(t,x)$ is converging in a strong enough sense and denote the limit function by $\theta(t,x)$.

If this is the case we can pass to the limit $h \to 0$ and obtain, see Appendix B,

$$\partial_t \theta = -\operatorname{div}(\theta \nabla p), \tag{2-11}$$
$$\partial_t \theta = \Delta \bar{p} - \operatorname{div}(\theta \nabla \bar{p}). \tag{2-12}$$

Now (2-10) yields $p = \bar{p} + x_2$. Thus (2-11), (2-12) imply that

$$\Delta \bar{p} = \operatorname{div}((\nabla \bar{p} - \nabla p)\theta) = -\partial_{x_2}\theta. \tag{2-13}$$

Therefore, from (2-12) and (2-13), we deduce

$$\partial_t \theta = -\partial_{x_2}\theta - \operatorname{div}(\nabla \bar{p}\theta) = -\partial_{x_2}\theta - \operatorname{div}((\nabla \bar{p} + \theta e_2)\theta) + \operatorname{div}(\theta^2 e_2).$$

To finish we define $u = \nabla \bar{p} + \theta e_2$, which clearly satisfies $\operatorname{div} u = 0$, to get

$$\partial_t \theta + u \cdot \nabla \theta + \partial_{x_2}\theta - 2\theta \partial_{x_2}\theta = 0,$$
$$u = \nabla \bar{p} + \theta e_2,$$
$$\operatorname{div} u = 0.$$

Undoing the change of coordinates from the beginning, i.e., considering

$$\rho(t,x) = 1 - 2s(x, 2t),$$

one obtains (1-3). As said, more details can be found in Appendix B.

2.5.2. *Transfer to subsolutions.* Now we give an alternative derivation of the macroscopic system (1-3), taking a different route after Step 1 of Otto's relaxation; i.e., the starting point is again the gradient flow structure of IPM saying that solutions of (1-1) seek to maximize the dissipation of potential energy at every time instance. However, at this point we do not care in which precise sense the dissipation is maximized (in Lagrangian coordinates with respect to the L^2-metric on the manifold of area preserving diffeomorphisms). We instead simply extend the principle of maximal energy dissipation for solutions of (1-1) to its relaxation given in Theorem 2.2; i.e., we seek to investigate also subsolutions that decrease the potential energy at every time instant as much as possible.

Suppose that (ρ, v, m) is a subsolution in the sense of Definition 2.4. We define its associated relative potential energy

$$E_{\mathrm{rel}}(t) := \int_{\mathbb{T}\times\mathbb{R}} (\rho(t,x)-\rho_0(x))x_2\,dx \tag{2-14}$$

and, for now formally, compute

$$\partial_t E_{\mathrm{rel}}(t) = -\int_{\mathbb{T}\times\mathbb{R}} x_2 \operatorname{div} m(t,x)\,dx = \int_{\mathbb{T}\times\mathbb{R}} m_2(t,x)\,dx. \tag{2-15}$$

Moreover, similar to (2-6), condition (2-4) implies

$$m = \rho v - \frac{1-\rho^2}{2}e_2 + \frac{1-\rho^2}{2}\xi$$

almost everywhere for some $\xi : [0,T)\times\mathbb{T}\times\mathbb{R}\to\mathbb{R}^2$ satisfying $|\xi|<1$. Plugging this into (2-15), one deduces

$$\partial_t E_{\mathrm{rel}}(t) = \int_{\mathbb{T}\times\mathbb{R}} \rho v_2 - (1-\rho^2)\frac{1-\xi_2}{2}\,dx.$$

Hence considering $\rho(t,\cdot)$, and therefore also $v(t,\cdot)$, see Section 4.2 below, to be given, one easily sees that the energy dissipation at time t is maximized in the closure of all admissible ξ with the choice $\xi(t,x) = -e_2$.

Hence choosing constantly $\xi = -e_2$, and therefore

$$m = \rho v - (1-\rho^2)e_2, \tag{2-16}$$

we deduce that (nonstrict) subsolutions that maximize at each time instant the dissipation of potential energy are characterized as solutions of

$$\begin{aligned} \partial_t\rho + \operatorname{div}(\rho v - (1-\rho^2)e_2) &= 0,\\ \operatorname{div} v &= 0,\\ v &= -\nabla p - \rho e_2. \end{aligned} \tag{2-17}$$

The above formal computation in (2-15) can be made rigorous under mild decay assumptions, as for instance shown in Appendix C. Here, however, we would like to state some further remarks.

First of all we emphasize that, by choosing m as in (2-16), we do not obtain a subsolution in the sense of Definition 2.4, since (2-4) holds only in a nonstrict sense; thus we speak about a nonstrict subsolution.

By considering instead

$$m = \rho v - \mu(1-\rho^2)e_2, \tag{2-18}$$

i.e., $\xi = (1-2\mu)e_2$ with μ arbitrarily close to 1 but $\mu < 1$, one obtains strict subsolutions and hence actual mixing solutions via Theorem 2.2, arbitrarily close to the nonstrict ones with maximal energy dissipation. However, in the remainder of the paper we will solve (2-17) as the outstanding case and remark that a similar analysis leads to a subsolution corresponding to the system with m given by (2-18); see also Remark 3.3.

Moreover, we would like to point out that, in the flat case, where $v = 0$, system (2-17) is exactly the hyperbolic conservation law found in [Székelyhidi 2012], whose entropy solution corresponds to the maximum speed of expansion of the mixing zone; see Section 2.3.

Furthermore, we remark that, given a strict subsolution (ρ, v, m) with relative potential energy $E_{\mathrm{rel}}(t)$ defined in (2-14), one obtains infinitely many mixing solutions $(\rho_{\mathrm{sol}}, v_{\mathrm{sol}})$ as in Theorem 2.2 with the additional property that their relative potential energy at almost every time t is arbitrarily close to $E_{\mathrm{rel}}(t)$; see Lemma 2.3. In this sense there also exist actual mixing solutions with potential energy decay arbitrarily close to the maximal decay for subsolutions characterized by (2-17).

2.5.3. *Comparison to selection criteria in related problems.* As mentioned in Section 2.4, the selection of a meaningful subsolution is a general problem when studying hydrodynamic instabilities via differential inclusions. We briefly give an overview of previously applied selection criteria.

In the case of a perfectly flat interface, the selection typically is done by reducing the subsolution system to a one-dimensional hyperbolic conservation law and picking the unique entropy solution as a natural candidate. This has been done in the context of the Kelvin–Helmholtz instability for the Euler equations [Székelyhidi 2011], the Rayleigh–Taylor instability for the inhomogeneous Euler equations [Gebhard et al. 2021], and as discussed in all detail above for the flat unstable Muskat problem in IPM [Székelyhidi 2012].

Another approach, selecting the subsolution that at initial time maximizes the total energy dissipation, has been applied in the context of the nonflat Kelvin–Helmholtz instability [Mengual and Székelyhidi 2023] within the class of all subsolutions with vorticity concentrated on a finite number of sheets, and thereafter in the class of one-dimensional self-similar subsolutions emanating from the flat Rayleigh–Taylor instability modeled by the Euler equations in Boussinesq approximation [Gebhard and Kolumbán 2022b]. This strategy has been motivated by the entropy rate admissibility criterion of Dafermos [1973], which has also been investigated in [Chiodaroli and Kreml 2014; Feireisl 2014] for convex integration solutions of the compressible Euler equations. In view of Section 2.5.2, also the selection criterion considered in the present paper falls into that category. However, in contrast to [Gebhard and Kolumbán 2022b; Mengual and Székelyhidi 2023], the selection applies among all possible subsolutions (with certain natural decay at infinity) and not only within a special subclass, and it applies at all times instead of only the initial time.

Another way to select subsolutions globally in time has been studied in [Gebhard et al. 2024] in the context of the flat Rayleigh–Taylor instability for the Euler equations in Boussinesq approximation.

Similar to Section 2.5.2 above, the underlying geometric principle of the equation, in that case the least action principle, has been imposed on the level of subsolutions leading to a degenerate elliptic variational problem that turns out to be formally equivalent to the direct relaxation of the least action principle by Brenier [1989]. However, solutions obtained from this relaxation conserve the total energy, which is inconsistent with anomalous energy dissipation present in turbulent regimes. In view of that, in [Gebhard et al. 2024] an additional term, responsible for energy dissipation but subject to certain choices, has been added in the variational problem. In contrast, the relaxation of IPM considered here is not relying on any comparable choices.

3. The main result

According to the previous section, we consider on $\mathbb{T}\times\mathbb{R}$ the system

$$\begin{aligned} \partial_t\rho+\operatorname{div}(\rho v+\rho^2 e_2)&=0,\\ \operatorname{div} v&=0,\\ v&=-\nabla p-\rho e_2 \end{aligned}\tag{3-1}$$

with initial data (1-2), i.e.,

$$\rho_0(x)=\begin{cases}+1, & x_2>\gamma_0(x_1),\\ -1, & x_2<\gamma_0(x_1),\end{cases}$$

for a sufficiently regular function $\gamma_0:\mathbb{T}\to\mathbb{R}$. In fact we here consider the case of a real analytic initial interface. For completeness we also state the notion of a general weak solution to system (3-1).

Definition 3.1. A pair $\rho\in L^\infty((0,T)\times\mathbb{T}\times\mathbb{R})$, $v\in L^\infty(0,T;L^2(\mathbb{T}\times\mathbb{R};\mathbb{R}^2))$ is a solution of (3-1), (1-2) provided, for any $\varphi\in\mathcal{C}_c^\infty([0,T)\times\mathbb{T}\times\mathbb{R})$, we have

$$\int_0^T\int_{\mathbb{T}\times\mathbb{R}}\rho\partial_t\varphi+(\rho v+\rho^2e_2)\cdot\nabla\varphi\,dx\,dt+\int_{\mathbb{T}\times\mathbb{R}}\rho_0\varphi(0,\cdot\,)\,dx=0,$$
$$\int_0^T\int_{\mathbb{T}\times\mathbb{R}}v\cdot\nabla\varphi\,dx\,dt=0,$$
$$\int_0^T\int_{\mathbb{T}\times\mathbb{R}}(v+\rho e_2)\cdot\nabla^\perp\varphi\,dx\,dt=0.$$

Theorem 3.2. *Let $\gamma_0:\mathbb{T}\to\mathbb{R}$ be real analytic. Then the initial value problem (3-1), (1-2) has a local-in-time solution with the following properties*:

(i) *ρ and v are continuous on $[0,T)\times\mathbb{T}\times\mathbb{R}\setminus\{(0,x_1,\gamma_0(x_1)):x_1\in\mathbb{T}\}$.*

(ii) *$\rho(t,\cdot\,)$ is Lipschitz continuous at positive times and $v(t,\cdot\,)$ is log-Lipschitz continuous, with*

$$\begin{aligned}\|\nabla\rho(t,\cdot\,)\|_{L^\infty(\mathbb{T}\times\mathbb{R})}&\le C_0t^{-1},\\ |v(t,x)-v(t,x')|&\le C_0t^{-1}|(x-x')\log|x-x'||\end{aligned}\tag{3-2}$$

for $t\in(0,T)$, $x,x'\in\mathbb{T}\times\mathbb{R}$, $|x-x'|\le\frac12$ and a constant $C_0>0$ depending on γ_0.

(iii) *For $t \in (0, T)$, there exist two real analytic curves $\gamma_t(\,\cdot\,, \pm 1) : \mathbb{T} \to \mathbb{R}$ such that $\rho(t, x) = 1$ whenever $x_2 \geq \gamma_t(x_1, 1)$ and $\rho(t, x) = -1$ whenever $x_2 \leq \gamma_t(x_1, -1)$. Moreover, $\rho(t, \cdot\,)$ maps the remaining set into $(-1, 1)$. Also there, the level sets $\Gamma_t(h) := \{x \in \mathbb{T} \times \mathbb{R} : \rho(t, x) = h\}$, $h \in (-1, 1)$, are given by graphs of real analytic functions $\gamma_t(\,\cdot\,, h) : \mathbb{T} \to \mathbb{R}$. Furthermore, the joint map $[0, T) \times \mathbb{T} \times [-1, 1] \to \mathbb{R}$, $(t, x_1, h) \mapsto \gamma_t(x_1, h)$ belongs to the space $\mathcal{C}^1([0, T); \mathcal{C}^1(\mathbb{T} \times [-1, 1]))$, and there exists a real analytic function $s_0 : \mathbb{T} \to \mathbb{R}$ such that*

$$\gamma_t(x_1, h) = \gamma_0(x_1) + t(2h + s_0(x_1)) + o(t) \tag{3-3}$$

with respect to $\|\cdot\|_{\mathcal{C}^1(\mathbb{T}\times[-1,1])}$ as $t \to 0$.

(iv) *For any locally Lipschitz continuous $\eta : \mathbb{R} \to \mathbb{R}$, we have the balance*

$$\partial_t(\eta(\rho)) + \operatorname{div}(\eta(\rho)v + Q(\rho)e_2) = 0, \tag{3-4}$$

with initial data $\eta(\rho)(0, \cdot\,) = \eta(\rho_0)$ and flux $Q(\rho) := \int_0^\rho 2\eta'(s)s\,ds$.

Remark 3.3. (a) In fact the function $s_0 : \mathbb{T} \to \mathbb{R}$ appearing in (3-3) is precisely the normal part of the initial velocity when evaluated in $(x_1, \gamma_0(x_1))$. See Section 4.2, in particular equation (4-9), for the definition and further discussion.

(b) Note that (iii) implies that ρ is piecewise $\mathcal{C}^1$ with the exceptional set given by

$$\{(t, x_1, \gamma_t(x_1, \pm 1) : t \in [0, T), x_1 \in \mathbb{T}\}.$$

(c) Equation (3-4) is a priori understood in analogy to Definition 3.1, i.e., in a distributional sense. However, given the regularity of ρ and v, it in fact holds pointwise almost everywhere on $(0, T) \times \mathbb{T} \times \mathbb{R}$; see Section 6.

(d) Since convex functions are locally Lipschitz, the balance (3-4) in particular states that ρ is an entropy solution for the conservation law $\partial_t\rho + \operatorname{div}(\rho v +^2 e_2) = 0$, see Definition 3.4 below.

(e) We notice that, for an analytic initial interface, the Muskat equation (2-1) can be solved for short time in order to find a solution to the macroscopic IPM system (3-1), which at the same time is also a solution for IPM (see [Castro et al. 2012a] and in the case of the vortex-sheet problem [Castro et al. 2012b]). However, this solution is not an entropy solution. Moreover, piecewise constant solutions of (3-1) also could be constructed but again they would not be entropy solutions.

(f) As discussed earlier in Section 2.5.2, the solution (ρ, v) given by Theorem 3.2 induces only a nonstrict subsolution by setting $m := \rho v - (1 - \rho^2)e_2$. However, an analogous existence statement remains true when replacing the first equation of (3-1) by

$$\partial_t\rho + \operatorname{div}(\rho v + \mu\rho^2 e_2) = 0$$

corresponding to a choice of m as in (2-18) and thus to strict subsolutions when $\mu < 1$. This can be seen for instance by rescaling time and considering the nonlocal velocity field $\mu^{-1}v$ in Sections 4 and 5.

(g) Notice that (iii) describes precisely the mixing zone $\mathscr{U}$ of the subsolution, see Definition 2.4, where the corresponding solutions develop a mixing behavior. In particular, from (3-3) one can deduce the initial growth of the mixing zone, which is linear in time. When combined with [Castro et al. 2019], it also implies the observed degraded mixing property of solutions (the closer to the upper boundary, the bigger the volume fraction of the heavier fluid). In particular, by letting μ tend to 1, our method predicts a unique mixing zone selected by maximal potential energy dissipation which can be compared with experiments, as opposed to subsolutions where the mixing zone depends on an a priori ansatz.

(h) The time of existence $T > 0$ of the found solution depends on how well γ_0 can be extended holomorphically onto a complex strip, see, e.g., Lemma 5.6. In addition T is capped by 1. While the latter is an artificial bound making our proof of existence at some points slightly less technical, the former dependence is naturally appearing in proofs relying on Cauchy–Kovalevskaya theorems. The question regarding a global-in-time solution, may it be as a general entropy solution or as a solution of the level set formulation introduced in Section 4, is open.

(i) The choice of the periodic infinite strip $\mathbb{T} \times \mathbb{R}$ as our spatial domain seemed to us to be the least technical choice. Compared to the whole plane $\mathbb{R}^2$, one does not need to speak about decay/flatness at $x_1 \to \pm\infty$, still we believe that our approach can be adapted to that setting. The same is true for the bounded periodic domain $\mathbb{T} \times (0, 1)$, where the necessary estimates for the Biot–Savart kernel, see Lemma 5.5, have to be derived on a more abstract level. However, the situation in a bounded domain with vertical boundaries is more delicate and not within the scope of this paper.

For completeness we include in the following the notion of an entropy solution for equation (3-1). Note that (3-1) is a nonlocal hyperbolic conservation law. As is common for such equations, see, e.g., [Amadori and Shen 2012; Amorim 2012; Betancourt et al. 2011; Blandin and Goatin 2016; Colombo et al. 2012], the notion of an entropy solution is the one for the corresponding local conservation law where the otherwise nonlocal velocity field is considered as a fixed local one.

Definition 3.4 (entropy solution). A solution (ρ, v) in the sense of Definition 3.1 is called an entropy solution provided, for any $\varphi \in \mathcal{C}_c^\infty([0, T) \times \mathbb{T} \times \mathbb{R})$, $\varphi \geq 0$ and any convex $\eta : \mathbb{R} \to \mathbb{R}$ with induced flux $Q(\rho) := \int_0^\rho 2\eta'(s)s\,ds$, we have

$$\int_0^T \int_{\mathbb{T}\times\mathbb{R}} \eta(\rho)\partial_t\varphi + (\eta(\rho)v + Q(\rho)e_2) \cdot \nabla\varphi\,dx\,dt + \int_{\mathbb{T}\times\mathbb{R}} \eta(\rho_0)\varphi(0, \cdot\,)\,dx \geq 0.$$

We remark that typically the set of η for which the stated imbalance is required to hold is taken to be a strict subset of all convex functions, such as for instance the family $\{r \mapsto |r - c| : c \in \mathbb{R}\}$ of Kružkov [1970]; see also [Dafermos 2016]. Since our solution already satisfies the stronger property (iv), we refrain at this point from restricting the set of entropies.

In any case, due to the nature of the nonlocality of our velocity field — which is a zeroth-order singular integral operator with respect to the density ρ (see Section 4.2) — the uniqueness of the found entropy solution remains open.

4. Level set formulation

We begin our investigation with a look at the illustrative example of a perfectly flat initial interface $\gamma_0(x_1)=0$ (Section 4.1) and some known facts concerning the nonlocal velocity field v — in particular at initial time — in the nonflat case (Section 4.2). Thereafter, with the beginning of Section 4.3, we will reformulate problem (3-1), (1-2) as a suitable fixed-point problem.

4.1. ***The flat interface.*** In the perfectly flat case, $\gamma_0=0$, an x_1-independent solution of (3-1) is obtained by observing that $v=0$ and solving the Riemann problem for Burgers' equation

$$\partial_t\rho+\partial_{x_2}(\rho^2)=0,\qquad \rho(0,x_2)=\operatorname{sign}(x_2).$$

The unique entropy solution is Lipschitz continuous at positive times and explicitly given by

$$\rho(t,x)=\begin{cases}1, & x_2>2t,\\ x_2/(2t), & |x_2|\le 2t,\\ -1, & x_2<-2t.\end{cases}$$

As discussed earlier, see Section 2.3, this solution bounds the mixing zone in the class of all one-dimensional IPM subsolutions.

However, in rescaled coordinates $y\mapsto x,\ x=(y_1,ty_2)$, the solution is given by the stationary profile

$$\rho(t,y_1,ty_2)=\phi_0(y):=\begin{cases}1, & y_2>2,\\ \tfrac{1}{2}y_2, & |y_2|\le 2,\\ -1, & y_2<-2,\end{cases}\tag{4-1}$$

or in other words the level sets $\rho(t,\cdot)^{-1}(\{h\}),\ h\in(-1,1)$, are given by flat lines $\{x:x_2=2ht\}$ that as time evolves are pulled apart with speed $2h$.

Of course these are simple reformulations, but a key point in our analysis is an appropriate extension of this principle to the general, nonflat case where the velocity field does not vanish. This will be done by keeping the profile $\phi_0(y)$ on the right-hand side of (4-1) and allowing the transformation $y\mapsto x$ to be of the type $x=(y_1,ty_2+f(t,y))$, i.e., we keep the "pulling"-term ty_2 dealing with the Burgers' term $\partial_{x_2}(\rho^2)$ in the equation and allow the level sets to have a general form reacting to the nonlocal velocity field. The details in terms of induced equations for f are in Sections 4.3–4.5.

4.2. ***Biot–Savart and the initial velocity field.*** The flat case discussed in the previous subsection is a very special case in the sense that $v=0$ and the resulting equation is local. In the general case a key feature of both systems, IPM and the relaxation, is the nonlocal relation between the density ρ and the velocity field v. More precisely, the last two equations in (1-1), (3-1), respectively, i.e., the incompressibility condition and Darcy's law, can be understood by means of a zeroth-order convolution operator. Indeed, taking the curl of Darcy's law, one sees that, at each time, $v(t,\cdot)$ is an incompressible vector field with vorticity given by

$$\partial_{x_1}v_2(t,x)-\partial_{x_2}v_1(t,x)=-\partial_{x_1}\rho(t,x).\tag{4-2}$$

Thus, when requiring decay as $|x_2| \to \infty$, the velocity field v is, at least in the case of our interest, uniquely determined in terms of the Biot–Savart operator

$$v(t,x) = (K * (-\partial_{x_1}\rho(t,\cdot)))(x) = \int_{\mathbb{T}\times\mathbb{R}} K(x-z)(-\partial_{x_1}\rho(t,z))\,dz. \tag{4-3}$$

On $\mathbb{T}\times\mathbb{R}$ the kernel K is given by

$$K(z) := \frac{1}{4\pi}\frac{(-\sinh(z_2), \sin(z_1))^T}{\cosh(z_2)-\cos(z_1)}, \tag{4-4}$$

and, as usual, K is the orthogonal gradient of the corresponding Green's function

$$G(z) := \frac{1}{4\pi}\log(\cosh(z_2)-\cos(z_1)). \tag{4-5}$$

Relation (4-3) has to be interpreted accordingly at initial time $t = 0$ due to the fact that $-\partial_{x_1}\rho_0$ is only a measure supported on the interface

$$\Gamma_0 := \{(x_1, \gamma_0(x_1)) : x_1 \in \mathbb{T}\}.$$

Thus, the initial velocity field $v_0(x)$ is the one of a vortex-sheet and therefore discontinuous across the interface.

Lemma 4.1. *The unique square integrable solution of*

$$v = -\nabla p - \rho_0 e_2, \quad \operatorname{div} v = 0 \quad \text{on } \mathbb{T}\times\mathbb{R} \tag{4-6}$$

is given by

$$v_0(x) = \int_{\mathbb{T}} K\begin{pmatrix} x_1 - z_1 \\ x_2 - \gamma_0(z_1)\end{pmatrix} 2\gamma_0'(z_1)\,dz_1 \tag{4-7}$$

for $x \notin \Gamma_0$, while the one-sided limits at Γ_0 are given by

$$\lim_{\substack{\pm(y_2-\gamma_0(y_1))>0\\ y\to(x_1,\gamma_0(x_1))}} v_0(y) = \text{p.v.}\int_{\mathbb{T}} K\begin{pmatrix} x_1 - z_1 \\ \gamma_0(x_1) - \gamma_0(z_1)\end{pmatrix} 2\gamma_0'(z_1)\,dz_1 \mp \frac{\gamma_0'(x_1)}{1+\gamma_0'(x_1)^2}\begin{pmatrix} 1 \\ \gamma_0'(x_1)\end{pmatrix}. \tag{4-8}$$

Proof. First of all one can check that the right-hand side of (4-7) defines a locally integrable solution of (4-6) with exponential decay as $|x_2| \to \infty$. Thus standard elliptic estimates imply that this is the only solution with these properties.

In order to compute the one-sided limits, we write

$$K(z) = \frac{1}{2\pi}\frac{z^\perp}{|z|^2}\eta(z_1) + K_{\text{reg}}(z),$$

where $\eta : \mathbb{T}\times\mathbb{R}$ is a smooth periodic cutoff function with $\eta(z_1) = 1$ for $|z_1| \le 1$ and $\eta(z_1) = 0$ for $|z_1| \ge 2$, and the regular part $K_{\text{reg}} : \mathbb{T}\times\mathbb{R} \to \mathbb{R}^2$,

$$K_{\text{reg}}(z) := K(z) - \frac{1}{2\pi}\frac{z^\perp}{|z|^2}\eta(z_1),$$

is smooth. In fact K_{reg} is harmonic where $\eta(z_1) = 1$. Furthermore, using complex notation, we write $z^\perp/|z|^2 = (1/(iz))^*$, where z^* denotes complex conjugation.

Then, denoting by $v_{0,\mathrm{reg}}$ the contribution from the regular part K_{reg}, we have

$$v_0(y) - v_{0,\mathrm{reg}}(y) = \left(\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{2\gamma_0'(z_1)}{y-(z_1+i\gamma_0(z_1))}\,dz_1\right)^* = -\left(\frac{1}{2\pi i}\int_{\Gamma_0} \frac{1}{\xi - y}\,\frac{2\gamma_0'(\xi_1)}{1+i\gamma_0'(\xi_1)}\,d\xi\right)^*$$

for $y \notin \Gamma_0$. Now taking one sided limits $y \to x \in \Gamma_0$, expression (4-8) follows from the Sokhotski–Plemelj formula; see [Muskhelishvili 1972]. □

Formulas (4-8) show that the initial velocity field is still continuous across the interface in the normal direction. Therefore the (not normalized) normal velocity at the interface $s_0 : \mathbb{T} \to \mathbb{R}$,

$$s_0(x_1) := v_0(x_1, \gamma_0(x_1)) \cdot \begin{pmatrix} -\gamma_0'(x_1) \\ 1 \end{pmatrix} = \mathrm{p.v.}\int_{\mathbb{T}} K\begin{pmatrix} x_1 - z_1 \\ \gamma_0(x_1) - \gamma_0(z_1) \end{pmatrix} 2\gamma_0'(z_1)\,dz_1 \cdot \begin{pmatrix} -\gamma_0'(x_1) \\ 1 \end{pmatrix}, \tag{4-9}$$

is well-defined. It will play an important role in our further analysis as it dictates the motion of Lagrangian particles at the interface to first order when ignoring the Burgers' term $\partial_{x_2}(\rho^2)$.

4.3. ***Rescaling and level set function.*** We now transform problem (3-1), (1-2) in terms of level sets. The reformulation here is understood on a formal level. We will solve the derived fixed-point problem in Section 5 and a posteriori justify the transformations in Section 6.

The starting point is the following ansatz for ρ capturing the effect of the Burgers' part described in Section 4.1. Assume that there exists $f : [0, T) \times \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ sufficiently regular with

$$f(0, y) = \gamma_0(y_1) \tag{4-10}$$

and such that, for every $t \in (0, T)$, $y_1 \in \mathbb{T}$, the map $\mathbb{R} \to \mathbb{R}$, $y_2 \mapsto ty_2 + f(t, y_1, y_2)$ is a monotone diffeomorphism.

Then each of the transformations $X_t : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$, $t \in (0, T)$,

$$X_t(y) = \begin{pmatrix} y_1 \\ ty_2 + f(t, y) \end{pmatrix},$$

is a diffeomorphism as well.

We now seek to find a solution of (3-1), (1-2) on $[0, T)$ having the property that

$$\rho(t, X_t(y)) = \phi_0(y_2) = \begin{cases} +1, & y_2 \geq 2, \\ \frac{1}{2}y_2, & y_2 \in (-2, +2), \\ -1, & y_2 \leq -2. \end{cases} \tag{4-11}$$

For $t > 0$, we compute

$$DX_t(y) = \begin{pmatrix} 1 & 0 \\ \partial_{y_1} f(t, y) & t + \partial_{y_2} f(t, y) \end{pmatrix}, \tag{4-12}$$

$$DX_t(y)^{-1} = \begin{pmatrix} 1 & 0 \\ \frac{-\partial_{y_1} f(t,y)}{t+\partial_{y_2} f(t,y)} & \frac{1}{t+\partial_{y_2} f(t,y)} \end{pmatrix}, \tag{4-13}$$

$$\nabla\rho(t, X_t(y)) = \frac{1}{2(t + \partial_{y_2} f(t, y))} \begin{pmatrix} -\partial_{y_1} f(t, y) \\ 1 \end{pmatrix} \mathbb{1}_{(-2,2)}(y_2), \tag{4-14}$$

so that the first equation of (3-1) — when written in nondivergence form — under the ansatz (4-11) is equivalent to

$$0 = \mathbb{1}_{(-2,2)}(y_2)\begin{pmatrix}-\partial_{y_1} f(t,y)\\ 1\end{pmatrix}\cdot\left(\begin{pmatrix}0\\ y_2+\partial_t f(t,y)-2\phi_0(y_2)\end{pmatrix}-v(t,X_t(y))\right).$$

Since $2\phi_0(y_2)=\mathbb{1}_{(-2,2)}(y_2)=y_2$, expanding the above equation leads to

$$\partial_t f(t,y) = v(t,y_1,ty_2+f(t,y))\cdot\begin{pmatrix}-\partial_{y_1} f(t,y)\\ 1\end{pmatrix} \tag{4-15}$$

for $(t,y_1,y_2)\in(0,T)\times\mathbb{T}\times(-2,2)$.

Note that, in view of (4-14), the velocity field in (4-15) is always considered in directions normal to the level sets of ρ.

4.4. ***Transformation of the velocity field.*** For $t>0$, we have that $v(t,\cdot)$ (in all reasonable scenarios) is given by the Biot–Savart law (4-3); see Section 4.2.

Applying the transformation $X_t(y)$, we compute the velocity field

$$v(t,y_1,ty_2+f(t,y)) = v(t,X_t(y))$$

occurring in (4-15). First of all, formulas (4-12) and (4-14) imply

$$\begin{aligned} v(t,X_t(y)) &= -\int_{\mathbb{T}\times\mathbb{R}} K(X_t(y)-z)\partial_{x_1}\rho(t,z)\,dz \\ &= -\int_{\mathbb{T}\times\mathbb{R}} K(X_t(y)-X_t(z))\partial_{x_1}\rho(t,X_t(z))\det DX_t(z)\,dz \\ &= \frac{1}{2}\int_{-2}^{2}\int_{\mathbb{T}} K(X_t(y)-X_t(z))\partial_{y_1} f(t,z)\,dz_1\,dz_2. \end{aligned}$$

Next we compute the full right-hand side of (4-15) and exploit the fact that the velocity field $v(t,X_t(y))$ is only needed in normal directions. More precisely, for $z\neq y$, we have

$$\begin{aligned} &\partial_{y_1} f(t,z)K(X_t(y)-X_t(z))\cdot\begin{pmatrix}-\partial_{y_1} f(t,y)\\ 1\end{pmatrix} \\ &\quad= \partial_{y_1} f(t,z)\nabla G(X_t(y)-X_t(z))\cdot\begin{pmatrix}1\\ \partial_{y_1} f(t,y)\end{pmatrix} \\ &\quad= \partial_{y_1} f(t,y)\nabla G(X_t(y)-X_t(z))\cdot\begin{pmatrix}1\\ \partial_{y_1} f(t,z)\end{pmatrix} - \partial_1 G(X_t(y)-X_t(z))(\partial_{y_1} f(t,y)-\partial_{y_1} f(t,z)) \\ &\quad= -\partial_{y_1} f(t,y)\frac{d}{dz_1}(G(X_t(y)-X_t(z))) - K_2(X_t(y)-X_t(z))(\partial_{y_1} f(t,y)-\partial_{y_1} f(t,z)). \end{aligned}$$

Thus after integration we obtain an additional cancelation in the convolution, i.e.,

$$v(t,X_t(y))\cdot\begin{pmatrix}-\partial_{y_1} f(t,y)\\ 1\end{pmatrix} = -\frac{1}{2}\int_{-2}^{2}\int_{\mathbb{T}} K_2(X_t(y)-X_t(z))(\partial_{y_1} f(t,y)-\partial_{y_1} f(t,z))\,dz_1\,dz_2. \tag{4-16}$$

4.5. ***Equation for f.*** Combining (4-16) with (4-15), we see that (3-1) can — after our ansatz — be written in the closed form

$$\partial_t f(t,y) = -\frac{1}{2}\int_{-2}^{2}\int_{\mathbb{T}} K_2(\tilde{\Delta} X_t(y,z))(\partial_{y_1} f(t,y) - \partial_{y_1} f(t,z))\,dz_1\,dz_2, \tag{4-17}$$

where

$$\tilde{\Delta} X_t(y,z) := X_t(y) - X_t(z) = \begin{pmatrix} y_1 - z_1 \\ t(y_2 - z_2) + f(t,y) - f(t,z) \end{pmatrix}$$

also depends on f. Via translation in z_1, equation (4-17) can also be written as

$$\partial_t f(t,y) = -\frac{1}{2}\int_{-2}^{2}\int_{\mathbb{T}} K_2(\Delta X_t(y,z))\Delta\partial_{y_1} f_t(y,z)\,dz_1\,dz_2, \tag{4-18}$$

where we have used the abbreviation

$$\begin{aligned} \Delta X_t(y,z) &:= \begin{pmatrix} z_1 \\ t(y_2 - z_2) + f(t,y_1,y_2) - f_t(t,y_1 - z_1, z_2) \end{pmatrix}, \\ \Delta\partial_{y_1} f_t(y,z) &:= \partial_{y_1} f(t,y_1,y_2) - \partial_{y_1} f(t,y_1 - z_1, z_2). \end{aligned} \tag{4-19}$$

The latter form turns out to be more convenient to work with.

4.6. ***One more ansatz.*** One important assumption in the above derivation is the invertibility of the maps $(X_t)_{t>0}$. In order to guarantee this, we further make the ansatz

$$f(t,y) = \gamma_0(y_1) + t s_0(y_1) + \tfrac{1}{2} t^{1+\alpha}\eta(t,y), \tag{4-20}$$

where $\alpha \in (0,1)$ and the functions $s_0 : \mathbb{T} \to \mathbb{R}$, $\eta : (0,T) \times \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ are sufficiently regular. In order to avoid potential confusion, we emphasize that the function η has nothing to do with an entropy; compare with the η appearing in Definition 3.4. Furthermore, we remark that the particular choice of $\alpha \in (0,1)$ is not important; see Section 5.7 for further discussion.

By this ansatz f satisfies (4-10), and the desired invertibility can be assumed true for a small time interval (depending on $\|\partial_{y_2}\eta\|_{L^\infty}$). Moreover, since at $t = 0$ we have $\partial_t f = s_0$, $\partial_{y_1} f = \gamma_0'$, passing formally to the limit on the right-hand side of (4-18), one sees that s_0 necessarily is given by

$$-\frac{1}{2}\int_{-2}^{2}\int_{\mathbb{T}} K_2(\Delta X_0(y_1,z_1))\Delta\gamma_0'(y_1,z_1)\,dz_1\,dz_2, \tag{4-21}$$

where

$$\begin{aligned} \Delta X_0(y_1,z_1) &:= \begin{pmatrix} z_1 \\ \gamma_0(y_1) - \gamma_0(y_1 - z_1) \end{pmatrix}, \\ \Delta\gamma_0'(y_1,z_1) &:= \gamma_0'(y_1) - \gamma_0'(y_1 - z_1). \end{aligned}$$

A quick computation similar to the one in Section 4.4 and comparison with (4-9) shows that the above expression is precisely the normal component of the initial velocity evaluated at $(y_1, \gamma_0(y_1))$, i.e., (4-21). This shows that the function $s_0(y_1)$ is indeed forced to be the normal component of $v_0(y_1, \gamma_0(y_1))$.

Finally we integrate (4-18) in time and use (4-20), (4-21) in order to deduce that, for f to be a solution to (4-18), η must be a solution of the fixed-point problem

$$\eta(t,y) = -\frac{1}{t^{1+\alpha}} \int_0^t \int_{-2}^2 \int_{\mathbb{T}} K_2(\Delta X_s(y,z)) \Delta \partial_{y_1} f_s(y,z) - K_2(\Delta X_0(y_1,z_1)) \Delta \gamma_0'(y_1,z_1)\, dz_1\, dz_2\, ds. \tag{4-22}$$

Note that η and $\partial_{y_1}\eta$ enter the right-hand side through (4-19), (4-20).

5. Existence of a solution for analytic graphs

Our goal is to show that, for a real analytic $\gamma_0 : \mathbb{T} \to \mathbb{R}$, there exists a unique local-in-time solution η of problem (4-22). The proof relies on the following version of the abstract Cauchy–Kovalevskaya theorem based on the formulation of Nishida [1977]; see also [Nirenberg 1972].

In order to avoid confusion we emphasize that, throughout Section 5, every symbol ρ, ρ', $\bar{\rho}$, ρ_0 denotes a positive constant referring to the size of the domain of analyticity. This is done in analogy to [Nishida 1977]. At no time in Section 5 do we mention the density function $\rho(t,x)$, which we seek to construct, or the initial density $\rho_0(x)$.

Theorem 5.1. *Let $(B_\rho)_{\rho\in(0,\rho_0)}$, $\rho_0 > 0$, be a scale of Banach spaces with $\|\cdot\|_{\rho'} \le \|\cdot\|_\rho$ for $0 < \rho' < \rho < \rho_0$, and consider the integral equation*

$$u(t) = \frac{1}{a(t)} \int_0^t F(u(s), s)\, ds \tag{5-1}$$

for a given continuous function $a : [0,\infty) \to \mathbb{R}$ with $a(t) > 0$ for $t > 0$. If F is such that

(i) *there exist $R > 0$, $T > 0$ such that, for every $0 < \rho' < \rho < \rho_0$, the map*

$$\{u \in B_\rho : \|u\|_\rho < R\} \times [0,T) \to B_{\rho'}, \qquad (u,t) \mapsto F(u,t),$$

is well-defined and continuous,

(ii) *there exists $b : [0,T) \to [0,\infty)$ continuous such that, for any $0 < \rho' < \rho < \rho_0$ and all $u, v \in B_\rho$, $\|u\|_\rho < R$, $\|v\|_\rho < R$, $t \in [0,T)$, we have*

$$\|F(u,t) - F(v,t)\|_{\rho'} \le \frac{b(t)}{\rho - \rho'} \|u - v\|_\rho,$$

(iii) *$F(0,\cdot) \in L^1(0,T;B_\rho)$ for any $\rho \in (0,\rho_0)$, and there exists $c : [0,T) \to [0,\infty)$ continuously differentiable on $(0,T)$ and continuous on $[0,T)$ with $c(0) = 0$ as well as $c'(t) > 0$ for $t > 0$ such that, for all $\rho \in (0,\rho_0)$, $t \in (0,T)$, we have*

$$\frac{1}{a(t)} \int_0^t \|F(0,s)\|_\rho\, ds \le \frac{c(t)}{\rho_0 - \rho},$$

(iv) *for a constant $K > 0$, the functions $a(t)$, $b(t)$, $c(t)$ appearing in (5-1), (ii), (iii) satisfy the relation*

$$\sup_{s\in(0,t)} \left| \frac{b(s)c(s)}{c'(s)} \right| \le K a(t) c(t), \qquad t \in (0,T), \tag{5-2}$$

then there exists a constant $\bar{a} = \bar{a}(K, R) > 0$ *and a unique* $u(t)$ *which, for any* $\rho \in (0, \rho_0)$, *maps the interval* $\{t \in [0, T) : c(t) < \bar{a}(\rho_0 - \rho)\}$ *continuously into the* R*-ball of* B_ρ. *Moreover,* u *satisfies* (5-1) *and*

$$\|u(t)\|_\rho = O\left(\frac{c(t)}{\rho_0 - \rho}\right) \quad \text{as } t \to 0.$$

In particular, $u(0) = 0$.

For the choices $a(t) = 1,\ b(t) = c_1,\ c(t) = c_2 t$ with some constants $c_1, c_2 > 0$, the above theorem is the abstract Cauchy–Kovalevskaya theorem in the formulation of Nishida [1977]. The proof of Theorem 5.1 requires indeed just some minor modifications which are presented in Appendix A. For a related generalization of the abstract Cauchy–Kovalevskaya theorem, see also [Reissig 1987; 1988].

We will apply Theorem 5.1 in the following situation.

Lemma 5.2. *Let* $c_1, c_2 > 0$ *and* $\alpha \in (0, 1)$. *There exist* $T = T(\alpha),\ K = K(c_1, c_2) > 0$ *such that* $a(t) := t^{1+\alpha}$, $b(t) := c_1 t^{1+\alpha}|\log t|,\ c(t) := c_2 t^{1-\alpha}|\log t|$ *satisfy* (5-2).

Proof. Consider $T \in (0, 1)$ such that

$$(1-\alpha)|\log t| \geq 2, \qquad |\log t| t^\alpha \leq 1$$

for all $t \in (0, T)$. Then, for $0 < s < t < T$, we have

$$\frac{b(s)c(s)}{c'(s)} = c_1 \frac{s^{2+\alpha}|\log s|^2}{(1-\alpha)|\log s| - 1} \leq c_1 s^2 |\log s| \leq c_1 t^2 |\log t| = \frac{c_1}{c_2} a(t)c(t).$$

Thus (5-2) holds with $K := c_1 c_2^{-1}$. □

5.1. *Banach spaces.*

Set

$$\Omega_0 := \mathbb{T} \times (-2, 2)$$

as well as

$$U_\rho := \{z \in \mathbb{C} : |\mathrm{Im}(z)| < \rho\}, \qquad \Omega_\rho := U_\rho \times (-2, 2)$$

for $\rho > 0$.

We define the space B_ρ to consist of all continuous functions $\eta : \Omega_0 \to \mathbb{R},\ y \mapsto \eta(y)$, which satisfy

(i) for every $y_2 \in (-2, 2)$, the function $\eta(\,\cdot\,, y_2)$ extends to a holomorphic function $U_\rho \to \mathbb{C}$ which is again denoted by $\eta(\,\cdot\,, y_2)$,

(ii) the derivative $\partial_{y_2}\eta : \Omega_\rho \to \mathbb{C}$ exists and is uniformly continuous, and $\partial_{y_2}\eta(\,\cdot\,, y_2)$ is holomorphic on U_ρ for every $y_2 \in (-2, 2)$,

(iii) the norm

$$\|\eta\|_\rho := \|\eta\|_{L^\infty(\Omega_\rho)} + \|\partial_{y_1}\eta\|_{L^\infty(\Omega_\rho)} + \|\partial_{y_2}\eta\|_{L^\infty(\Omega_\rho)}$$

is finite.

For clarification, the extension in (i) strictly speaking is the extension of the 2π-periodic function $\eta(\,\cdot\,, y_2) : \mathbb{R} \to \mathbb{R}$. The extension $U_\rho \to \mathbb{C},\ y_1 \mapsto \eta(y_1, y_2)$, therefore is periodic in the real part of y_1. Moreover, $\partial_{y_1}\eta$ denotes the complex derivative in the first component, while $\partial_{y_2}\eta$ is the real partial derivative with respect to the second component. Although the two derivatives are of slightly different nature, we still use a gradient notation $\nabla_y \eta := (\partial_{y_1}\eta, \partial_{y_2}\eta)^T$.

Clearly each B_ρ is a Banach space and $B_\rho \subset B_{\rho'}$, $\|\cdot\|_{\rho'} \le \|\cdot\|_\rho$ whenever $\rho' < \rho$. Moreover, for the introduced scale of spaces, we have the following lemma, which is a direct consequence of Cauchy's integral formula for analytic functions.

Lemma 5.3 (Cauchy). *Let $0 < \rho' < \rho$ and $\eta \in B_\rho$. Then, for $j = 1, 2$, we have*

$$\|\partial_{y_1}\partial_{y_j}\eta\|_{L^\infty(\Omega_{\rho'})} \le \frac{C}{\rho-\rho'}\|\eta\|_\rho$$

for $C = (2\pi)^{-1}$.

In particular, $\partial_{y_1}\eta$ is — as is η itself — Lipschitz continuous on Ω_0. This together with the assumed uniform continuity of $\partial_{y_2}\eta$ implies the following.

Lemma 5.4. *Let $\rho > 0$ and $\eta \in B_\rho$. Then $\eta : \Omega_0 \to \mathbb{R}$ extends to $\mathcal{C}^1(\bar{\Omega}_0)$, and $\eta(\,\cdot\,, y_2)$ and $\partial_{y_2}\eta(\,\cdot\,, y_2)$ are real analytic for each $y_2 \in [-2, 2]$.*

Also note that $\partial_{y_2}\partial_{y_1}\eta(y) = \partial_{y_1}\partial_{y_2}\eta(y)$ for $\eta \in B_\rho,\ y \in \Omega_\rho$, for instance, by means of Cauchy's integral formula.

5.2. ***Notation.*** From now on we fix $\alpha \in (0, 1)$ and a real analytic initial datum $\gamma_0 : \mathbb{T} \to \mathbb{R}$. Clearly γ_0 can be extended to a holomorphic function defined on $U_{2\rho_0}$ for some $\rho_0 > 0$ small.

Hence all (complex) derivatives are uniformly bounded on U_{ρ_0}, e.g., there exist a constant $C_0 > 0$ such that

$$\|\gamma_0'\|_{L^\infty(U_{\rho_0})} \le C_0. \tag{5-3}$$

More generally, henceforth, $C_0 > 0$ always denotes a constant depending solely on the $L^\infty(U_{\rho_0})$-norm of a fixed finite amount of derivatives of γ_0. (A detailed look at the proof reveals that the first five derivatives of γ_0 are sufficient. However, the precise number is not important.) In contrast $C > 0$ usually denotes a constant not depending on γ_0. Both constants typically change from line to line. Also we point out that distinguishing C_0 from C is not essential for the proof of Theorem 3.2.

For a pair $a = (a_1, a_2) \in \mathbb{R} \times \mathbb{C}$, we define

$$|a|_* := (|a_1|^2 + |a_2|^2)^{1/2} = (a_1^2 + a_2 a_2^*)^{1/2}. \tag{5-4}$$

Moreover, whenever we write $|z_1|$ for $z_1 \in \mathbb{T}$, we mean the absolute value of the unique representative of z_1 in $[-\pi, \pi)$. In particular, we will also use $|a|_*$ for pairs $a \in \mathbb{T} \times \mathbb{C}$.

For any function $g : \Omega_{\rho_0} \to \mathbb{C}^n$ or $h : U_{\rho_0} \to \mathbb{C}^n$ we use the abbreviation

$$\Delta g(y, z) := g(y) - g(y_1 - z_1, z_2) \quad \text{or} \quad \Delta h(y_1, z_1) := h(y_1) - h(y_1 - z_1) \tag{5-5}$$

for $y = (y_1, y_2) \in \Omega_{\rho_0},\ z = (z_1, z_2) \in \Omega_0$ and $y_1 \in U_{\rho_0},\ z_1 \in \mathbb{T}$, respectively. In the proofs we will usually omit the points (y, z) and simply write Δg and Δh.

Furthermore, for $t \ge 0$ and $\eta \in B_{\rho_0}$, we define $f_t^\eta : \Omega_{\rho_0} \to \mathbb{C},\ X_t^\eta : \Omega_{\rho_0} \to \mathbb{C}^2$,

$$f_t^\eta(y) := \gamma_0(y_1) + t s_0(y_1) + \tfrac{1}{2} t^{1+\alpha}\eta(y), \quad X_t^\eta(y) := \begin{pmatrix} y_1 \\ t y_2 + f_t^\eta(y) \end{pmatrix}. \tag{5-6}$$

The function $s_0 : U_{\rho_0} \to \mathbb{C}$ will be introduced in Lemma 5.6 below. At time $t = 0$, we simply write $X_0(y_1)$ instead of $X_0^\eta(y)$. The second component of $X_t^\eta(y)$ is denoted by $X_{t,2}^\eta(y)$. There is no need to distinguish the first component, since it is just given by y_1.

5.3. ***Preliminary lemmas.*** In order to define the function F as a complex extension of the functional appearing in (4-22), we need some preparation.

Recall that the second component K_2 of the Biot–Savart kernel on $\mathbb{T} \times \mathbb{R}$ is given by

$$K_2(a) = K_2(a_1, a_2) = \frac{1}{4\pi} \frac{\sin(a_1)}{\cosh(a_2) - \cos(a_1)}.$$

Thus, for fixed $a_1 \in \mathbb{T}$, the canonical extension of $K_2(a_1, \cdot)$ to $a_2 \in \mathbb{C}$ is holomorphic on the open set $\{a_2 \in \mathbb{C} : \cosh(a_2) - \cos(a_1) \neq 0\}$. We define

$$\mathcal{U} := \{a \in \mathbb{T} \times \mathbb{C} : \cosh(a_2) - \cos(a_1) \neq 0\}.$$

Lemma 5.5. *Let $\kappa \in \left(0, \frac{1}{2}\right)$. The sets*

$$\mathcal{U}^\kappa := \left\{(a_1, a_2) \in \mathbb{T} \times \mathbb{C} : |\mathrm{Im}(a_2)| < \kappa(|a_1| + |\mathrm{Re}(a_2)|),\ |\mathrm{Im}(a_2)| < \tfrac{\pi}{2}\right\}$$

are subsets of $\mathcal{U}$ with $\partial\mathcal{U}^\kappa \cap \partial\mathcal{U} = \{0\}$. Moreover, there exists a constant $C > 0$ depending on κ such that, for all $a \in \mathcal{U}^\kappa$, $j = 0, 1, 2$, we have

$$|\partial_{a_2}^j K_2(a)| \le C|a|_*^{-(1+j)}. \tag{5-7}$$

Proof. Let $a \in \overline{\mathcal{U}}^\kappa \cap \partial\mathcal{U}$, $a_2 = u + iv$. Then

$$0 = \cosh(a_2) - \cos(a_1) = \cosh(u)\cos(v) - \cos(a_1) + i\sinh(u)\sin(v)$$

implies $v = 0$, and thus $\cosh(u) = \cos(a_1)$, which is only possible for $a_1 = u = 0$; or $u = 0$ and $\cos(v) = \cos(a_1)$, which in the closure of $\mathcal{U}^\kappa$ is again only possible for $a_1 = u = 0$. Thus $\mathcal{U}^\kappa \subset \mathcal{U}$ and $\partial\mathcal{U}^\kappa \cap \partial\mathcal{U} = \{0\}$.

For the second part we split the analysis into three regions: $a \in \mathcal{U}^\kappa$, $|a|_*$ close to 0; $a \in \mathcal{U}^\kappa$, $|a|_*$ large; and the remaining subset of $\mathcal{U}^\kappa$.

Let us start with $a \in \mathcal{U}^\kappa$, $|a|_*$ close to 0. Writing again $a = (a_1, u + iv)$ and using that $v^2 \le 2\kappa^2(a_1^2 + u^2)$, we have

$$\begin{aligned} |a_1^2 + a_2^2| &= (a_1^4 + 2a_1^2(u^2 - v^2) + (u^2 - v^2)^2 + 4u^2v^2)^{1/2} \\ &\ge ((1 - 4\kappa^2)a_1^4 + 2a_1^2(1 - 2\kappa^2)u^2 + (u^2 + v^2)^2)^{1/2} \\ &\ge (1 - 4\kappa^2)(a_1^4 + |a_2|^4)^{1/2} \ge \tfrac{1}{2}(1 - 4\kappa^2)|a|_*^2. \end{aligned}$$

Then, for $a \in \mathcal{U}^\kappa$, a small, it follows that

$$|K_2(a)| \le \frac{1}{2\pi} \frac{|a_1| + O(|a_1|^3)}{|a_1^2 + a_2^2| - O(|a_1|^4) - O(|a_2|^4)} \le \frac{1}{\pi} \frac{|a|_* + O(|a|_*^3)}{(1 - 4\kappa^2)|a|_*^2 - O(|a|_*^4)} \le \frac{C}{|a|_*}.$$

Doing the same for higher-order derivatives, it follows that there exists $\varepsilon > 0$ such that (5-7) holds for all $a \in \mathcal{U}^\kappa$ with $|a|_* < \varepsilon$. We fix such an ε.

Next let us consider the opposite regime $a = u + iv \in \mathcal{U}^\kappa$, $|a|_*$ large. Note that this necessarily means that $|u|$ has to be large since $|a_1| \le \pi$, $|v| \le \frac{\pi}{2}$ by definition of $\mathcal{U}^\kappa$. We then estimate

$$\begin{aligned}|\cosh(a_2) - \cos(a_1)| &\ge |\cosh(u)\cos(v) + i\sinh(u)\sin(v)| - 1\\ &= (\cosh^2(u) - \sin^2(v))^{1/2} - 1\\ &\ge \left(\frac{u^2}{2} - 1\right)^{1/2} - 1.\end{aligned}$$

Consequently one can find constants $C > 0$ and $R > 0$ such that (5-7) with $j = 0$ holds for all $a \in \mathcal{U}^\kappa$ with $|a|_* > R$. Again this procedure can be extended to higher-order derivatives giving an R as above but with (5-7) valid for $j = 0, 1, 2$ for $|a|_* > R$. Let us also fix such an R.

Let now a be in the remaining set, i.e., $a \in \mathcal{U}^\kappa$ with $\varepsilon \le |a|_* \le R$. The closure of this set is compact and bounded away from $\partial\mathcal{U}$, where the denominator of K_2 vanishes. Therefore the existence of $C > 0$ such that (5-7) holds also on this set follows just by continuity of $\partial^j_{a_2} K_2$, $j = 0, 1, 2$. This finishes the proof of the lemma. □

Lemma 5.6. *Let $\rho_0 > 0$ be chosen such that γ_0 extends holomorphically to $U_{2\rho_0}$ with*

$$4\|\mathrm{Im}(\gamma_0')\|_{L^\infty(U_{\rho_0})} < 1. \tag{5-8}$$

Then the complex extension of the initial normal velocity $s_0 : \Omega_0 \to \mathbb{R}$,

$$s_0(y_1) := -2\int_{\mathbb{T}} K_2(\Delta X_0(y_1, z_1))\Delta\gamma_0'(y_1, z_1)\,dz_1,$$

is holomorphic on U_{ρ_0}. Moreover, the $L^\infty(U_{\rho_0})$-norm of any finite number of derivatives of s_0 can be bounded by C_0. In particular, all derivatives of s_0 are given by differentiation under the integral.

Proof. By (5-8) one estimates

$$|\mathrm{Im}(\gamma_0(y_1) - \gamma_0(y_1 - z_1))| < \tfrac{1}{4}|z_1|$$

for $z_1 \in [-\pi, \pi)$, $z_1 \ne 0$, $y_1 \in U_{\rho_0}$. Thus, by Lemma 5.5 the composition of K_2 with $y_1 \mapsto \Delta X_0(y_1, z_1)$ is holomorphic for every $z_1 \ne 0$. Moreover, again by Lemma 5.5, for such z_1, we have

$$|\partial_{y_1}(K_2(\Delta X_0(y_1, z_1)))| \le C\left(\frac{|\Delta\gamma_0'(y_1, z_1)|^2}{|\Delta X_0(y_1, z_1)|_*^2} + \frac{|\Delta\gamma_0''(y_1, z_1)|}{|\Delta X_0(y_1, z_1)|_*}\right) \le C_0.$$

It follows that s_0 is holomorphic and that $\|s_0'\|_{L^\infty(U_{\rho_0})} \le C_0$. The same can be shown for higher-order derivatives. □

The following two lemmas provide careful estimates needed for the compensation of various terms appearing in the definition of our nonlinear map F below. We are also careful with the uniform integrability as we need to be able to neglect what happens in some small sets.

Lemma 5.7. *Let $\rho_0 > 0$ be as in Lemma 5.6, and let $R > 0$. There exists $T = T(R, C_0, \alpha) \in (0, 1)$ such that, for all $\eta \in B_\rho$, $\|\eta\|_\rho < R$, $\rho \in (0, \rho_0)$ and $t \in [0, T)$, $y \in \Omega_\rho$, $z \in \Omega_0$, we have $\Delta X^\eta_t(y, z) \in \overline{\mathcal{U}^{3/8}}$ and*

$$t|y_2 - z_2| \le C_0|\Delta X^\eta_t(y, z)|_*. \tag{5-9}$$

Proof. First of all chose $T \in (0,1)$ with $T^\alpha R \le 1$. Then, omitting the (y,z) dependence in the notation, see Section 5.2, we have

$$\begin{aligned}\tfrac{1}{2}t|y_2-z_2| &\le t|y_2-z_2| - \tfrac{1}{2}t^{1+\alpha}R|y_2-z_2| \\ &\le \big|t(y_2-z_2)+\tfrac{1}{2}t^{1+\alpha}\,\mathrm{Re}(\eta(y)-\eta(y_1,z_2))\big| \\ &= \big|\mathrm{Re}\big(\Delta X^\eta_{t,2}-\Delta\gamma_0 - t\Delta s_0 - \tfrac{1}{2}t^{1+\alpha}(\eta(y_1,z_2)-\eta(y_1-z_1,z_2))\big)\big| \\ &\le |\mathrm{Re}(\Delta X^\eta_{t,2})| + (C_0(1+T)+T^{1+\alpha}R)|z_1| \le |\mathrm{Re}(\Delta X^\eta_{t,2})| + C_0|z_1|.\end{aligned}$$

This implies (5-9).

In order to see that $\Delta X^\eta_t \in \overline{\mathcal{U}^{3/8}}$, we use (5-8) as well as the just shown inequality to deduce

$$\begin{aligned}|\mathrm{Im}(\Delta X^\eta_{t,2})| &= \big|\mathrm{Im}\big(\Delta\gamma_0 + t\Delta s_0 + \tfrac{1}{2}t^{1+\alpha}\Delta\eta\big)\big| \\ &\le \big(\tfrac{1}{4}+TC_0+T^{1+\alpha}R\big)|z_1| + T^\alpha Rt|y_2-z_2| \\ &\le \big(\tfrac{1}{4}+T(C_0+1)\big)|z_1| + T^\alpha R|\mathrm{Re}(\Delta X^\eta_{t,2})| + T^\alpha RC_0|z_1|.\end{aligned}$$

Thus by choosing $T>0$ even smaller, we have the desired inequality

$$|\mathrm{Im}(\Delta X^\eta_{t,2})| \le \tfrac{3}{8}(|z_1| + |\Delta\,\mathrm{Re}(X^\eta_{t,2})|). \qquad \square$$

Lemma 5.8. *Let $\rho_0,\ R,\ T>0$ be as in Lemma 5.7. For $\eta\in B_\rho$, $\|\eta\|_\rho<R$, $\rho\in(0,\rho_0)$ and $y\in\Omega_\rho$, $t\in(0,T)$, we have*

$$\int_{\Omega_0} \frac{1}{|\Delta X^\eta_t(y,z)|_*}\,dz \le C_0|\log t|. \tag{5-10}$$

The integrability of $|\Delta X^\eta_t(y,\cdot)|_*^{-1}$ is uniform with respect to $y\in\Omega_0$ and with respect to t considered on any interval of the form $[t_0,T)$ with $t_0>0$.

Proof. In view of (5-9), we have

$$\begin{aligned}\int_{\Omega_0}\frac{1}{|\Delta X^\eta_t|_*}\,dz &\le C_0\int_{\Omega_0}\frac{1}{|z_1|+t|y_2-z_2|}\,dz = C_0\int_{\mathbb{T}}\int_{y_2-2}^{y_2+2}\frac{1}{|z_1|+t|z_2|}\,dz_2\,dz_1 \\ &\le C_0\int_0^\pi\int_0^4\frac{1}{z_1+tz_2}\,dz_2\,dz_1 = C_0\Big(\frac{\pi}{t}\log\Big(1+\frac{4t}{\pi}\Big)+4\log\Big(\frac{\pi}{4t}+1\Big)\Big),\end{aligned}$$

which is of order $|\log t|$. Note here that $t<1$ since T is assumed to be less than 1.

The uniform integrability follows from

$$\frac{1}{|z_1|+t_0|z_2|} \in L^1(\mathbb{T}\times(-4,4)). \qquad \square$$

5.4. ***Definition of F.*** Let us fix $\rho_0>0$ as in Lemma 5.6. Take $R=1$ and a corresponding $T\in(0,1)$ from Lemma 5.7.

We define the application $(\eta,t)\mapsto F(\eta,t)=F_t(\eta)$ by setting

$$F_t(\eta)(y) := -\int_{\Omega_0} K_2(\Delta X^\eta_t(y,z))\Delta\partial_{y_1}f^\eta_t(y,z) - K_2(\Delta X_0(y_1,z_1))\Delta\gamma_0'(y_1,z_1)\,dz$$

for $t>0$ and $F_0(\eta)(y)=0$.

Lemma 5.9. *F when seen as a map*

$$\{\eta \in B_\rho : \|\eta\|_\rho < 1\} \times [0, T) \to B_{\rho'}$$

is well-defined for all $0 < \rho' < \rho < \rho_0$. Moreover, for $\eta \in B_\rho$, $\|\eta\|_\rho < 1$, the map $[0, T) \to B_{\rho'}$, $t \mapsto F_t(\eta)$, is continuous.

Proof. Let $\eta \in B_\rho$, $\|\eta\|_\rho < 1$ and $t \in (0, T)$. In view of Lemma 5.6, it remains to look at

$$\widetilde{F}_t(\eta)(y) := F_t(\eta)(y) + 2s_0(y_1) = -\int_{\Omega_0} K_2(\Delta X_t^\eta(y, z)) \Delta \partial_{y_1} f_t^\eta(y, z)\, dz.$$

For $y \in \Omega_\rho$ and $z \in \Omega_0$ with $z_1 \neq 0$, one computes

$$\partial_{y_1}(K_2(\Delta X_t^\eta)\Delta\partial_{y_1} f_t^\eta) = \partial_{a_2} K_2(\Delta X_t^\eta)(\Delta\partial_{y_1} f_t^\eta)^2 + K_2(\Delta X_t^\eta)\Delta\partial_{y_1}^2 f_t^\eta, \tag{5-11}$$

$$\partial_{y_2}(K_2(\Delta X_t^\eta)\Delta\partial_{y_1} f_t^\eta) = \partial_{a_2} K_2(\Delta X_t^\eta)\big(t + \tfrac{1}{2}t^{1+\alpha}\partial_{y_2}\eta(y)\big)\Delta\partial_{y_1} f_t^\eta + K_2(\Delta X_t^\eta)\tfrac{1}{2}t^{1+\alpha}\partial_{y_2}\partial_{y_1}\eta(y), \tag{5-12}$$

where, as usual, we have omitted the (y, z) dependence in the Δ-notation.

In order to get uniform integrability, we use (5-9) to estimate

$$\begin{aligned} |\Delta\partial_{y_1}^j f_t^\eta| &\le C_0|z_1| + \|\partial_{y_1}^{j+1}\eta\|_{L^\infty(\Omega_{\rho'})}|z_1| + \|\partial_{y_2}\partial_{y_1}^j\eta\|_{L^\infty(\Omega_{\rho'})} t|y_2 - z_2| \\ &\le C_0(1 + \|\partial_{y_1}^j \nabla_y \eta\|_{L^\infty(\Omega_{\rho'})})|\Delta X_t^\eta|_* \end{aligned} \tag{5-13}$$

for $y \in \Omega_{\rho'}$, $j = 0, 1, 2$. Now (5-13) and Lemmas 5.5 and 5.7 imply

$$|K_2(\Delta X_t^\eta)\Delta\partial_{y_1} f_t^\eta| \le C_0(1 + \|\partial_{y_1}\nabla_y\eta\|_{L^\infty(\Omega_{\rho'})}). \tag{5-14}$$

As a consequence $\widetilde{F}_t$, and thus F_t, maps at least into $L^\infty(\Omega_{\rho'})$. Moreover, combining similarly (5-7), (5-13) for $j = 1, 2$, and (5-9) to estimate (5-11), (5-12), one sees that

$$|\partial_{y_1}(K_2(\Delta X_t^\eta)\Delta\partial_{y_1} f_t^\eta)| \le C_0(1 + \|\partial_{y_1}\nabla_y\eta\|^2_{L^\infty(\Omega_{\rho'})} + \|\partial_{y_1}^2\nabla_y\eta\|_{L^\infty(\Omega_{\rho'})}) \tag{5-15}$$

and, recalling $t < 1$, $|\partial_{y_2}\eta(y)| < 1$, that

$$|\partial_{y_2}(K_2(\Delta X_t^\eta)\Delta\partial_{y_1} f_t^\eta)| \le C_0\frac{t}{|\Delta X_t^\eta|_*}(1 + \|\partial_{y_1}\nabla_y\eta\|_{L^\infty(\Omega_{\rho'})}). \tag{5-16}$$

It follows that the complex derivative $\partial_{y_1} F_t(\eta)$ exists and is bounded on $\Omega_{\rho'}$. Moreover, in view of Lemma 5.8, the same is true for the (real) derivative $\partial_{y_2} F_t(\eta)$.

Next we turn to the required uniform continuity of $\partial_{y_2} F_t(\eta)$ on $\Omega_{\rho'}$. First of all observe that the corresponding integrant (5-12) as a function of $(z, y) \in \Omega_0 \times \Omega_{\rho'}$ is uniformly continuous on subsets which have their z_1-component bounded away from 0. Here one uses the Cauchy integral formula and the assumed uniform continuity of $\partial_{y_2}\eta$ on the larger set Ω_ρ in order to conclude the uniform continuity of $\partial_{y_2}\partial_{y_1}\eta(y) = \partial_{y_1}\partial_{y_2}\eta(y)$. This together with the uniform integrability of the majorant given in (5-16) via Lemma 5.8 implies that $\partial_{y_2} F_t(\eta)$ is uniformly continuous on $\Omega_{\rho'}$; see also the argument below for continuity in time.

Moreover, in a similar way as above for (5-14)–(5-16), one can check that, for any $y \in \Omega_{\rho'}$, $z_1 \neq 0$,

$$|\partial_{y_1}\partial_{y_2}(K_2(\Delta X_t^\eta)\Delta\partial_{y_1} f_t^\eta)| \leq \frac{C_0 t}{|\Delta X_t^\eta|_*}(1 + \|\partial_{y_1}\nabla_y \eta\|_{L^\infty(\Omega_{\rho'})}^2 + \|\partial_{y_1}^2\nabla_y\eta\|_{L^\infty(\Omega_{\rho'})}),$$

which implies that also $\partial_{y_2} F_t(\eta)$ is complex-differentiable in y_1.

In order to conclude $F_t(\eta) \in B_{\rho'}$, it therefore only remains to observe that $F_t(y) \in \mathbb{R}$ for $y \in \Omega_0$.

It remains to prove the continuity of $[0, T) \ni t \mapsto F_t(\eta) \in B_{\rho'}$. Let $t, t_0 \in (0, T)$ and take $\delta > 0$ sufficiently small. For $z \in \Omega_0$ with $|z_1| > \delta$ as well as $y \in \Omega_{\rho'}$, we have

$$|K_2(\Delta X_t^\eta)\Delta\partial_{y_1} f_t^\eta - K_2(\Delta X_{t_0}^\eta)\Delta\partial_{y_1} f_{t_0}^\eta| \leq \frac{C_0}{\delta^2}|t - t_0|$$

due to Lemmas 5.5 and 5.7. On the set $\{z \in \Omega_0 : |z_1| < \delta\}$, one uses the uniform majorant given in (5-14) to conclude the continuity of $(0, T) \ni t \mapsto F_t(\eta)$ with respect to $\|\cdot\|_{L^\infty(\Omega_{\rho'})}$.

For the corresponding continuity of $\partial_{y_1} F_t(\eta)$, $\partial_{y_2} F_t(\eta)$ with respect to $\|\cdot\|_{L^\infty(\Omega_{\rho'})}$, one uses a similar combination of Lipschitz continuity on $|z_1| > \delta$ and uniform integrability on the strip $|z_1| < \delta$ induced by (5-15), (5-16) and Lemma 5.8.

Finally, continuity at $t_0 = 0$ can be shown in the exact same way by noting that, compared to Lemma 5.8, the additional factor t in (5-16) for $\partial_{y_2} F_t(\eta)$ causes $t|\Delta X_t^\eta|_*^{-1}$ to be uniformly integrable with respect to t taken from the open interval $t \in (0, T)$. □

Remark 5.10. The continuity of F as stated in (i) of Theorem 5.1 will follow from Lemma 5.9 when combined with the Lipschitz property of Lemma 5.11 below.

5.5. ***Contraction property.*** Next we will verify (ii) of Theorem 5.1, with $b(t) = C_0 t^{1+\alpha}|\log t|$. Let ρ_0, R, $T > 0$ be as in Section 5.4. Recall that $R = 1$ and $T = T(R, C_0, \alpha) < 1$. Without loss of generality we also assume $\rho_0 < 1$.

Lemma 5.11. *For all* $0 < \rho' < \rho < \rho_0$, $\eta, \zeta \in B_\rho$, $\|\eta\|_\rho < 1$, $\|\zeta\|_\rho < 1$ *and* $t \in [0, T)$, *we have*

$$\|F_t(\eta) - F_t(\zeta)\|_{\rho'} \leq \frac{C_0 t^{1+\alpha}|\log t|}{\rho - \rho'}\|\eta - \zeta\|_\rho.$$

For the proof of Lemma 5.11 we first of all state some estimates implied by the lemmas in Section 5.3.

Lemma 5.12. *Let* $0 < \rho' < \rho < \rho_0$ *and* $\eta, \xi, \zeta \in B_\rho$ *with* $\|\eta\|_\rho$, $\|\xi\|_\rho$, $\|\zeta\|_\rho < 1$. *For* $y \in \Omega_{\rho'}$, $z \in \Omega_0$, $t \in [0, T)$, *we have*

$$t|\Delta\zeta(y, z)| \leq C_0\|\zeta\|_{\rho'}|\Delta X_t^\xi(y, z)|_*, \tag{5-17}$$

$$t|\Delta\partial_{y_1}\zeta(y, z)| \leq \frac{C_0}{\rho - \rho'}\|\zeta\|_\rho|\Delta X_t^\xi(y, z)|_*, \tag{5-18}$$

$$|\Delta\partial_{y_1} f_t^\eta(y, z)| \leq \frac{C_0}{\rho - \rho'}|\Delta X_t^\xi(y, z)|_*, \tag{5-19}$$

$$|\Delta\zeta(y, z)\Delta\partial_{y_1} f_t^\eta(y, z)| \leq C_0\|\zeta\|_{\rho'}|\Delta X_t^\xi(y, z)|_*, \tag{5-20}$$

$$|\Delta\zeta(y, z)\Delta\partial_{y_1}^2 f_t^\eta(y, z)| \leq \frac{C_0}{\rho - \rho'}\|\zeta\|_{\rho'}|\Delta X_t^\xi(y, z)|_*. \tag{5-21}$$

Proof. By (5-9) in Lemma 5.7, one deduces

$$t|\Delta\zeta| \le \|\nabla_y \zeta\|_{L^\infty(\Omega_{\rho'})}(|z_1| + t|y_2 - z_2|) \le C_0 \|\zeta\|_{\rho'} |\Delta X_t^{\xi}(y,z)|_*.$$

This shows (5-17). Inequality (5-18) is obtained in the same way by additionally applying Cauchy's Lemma 5.3.

Next, (5-19) follows from

$$|\Delta\partial_{y_1} f_t^{\eta}| \le C_0|z_1| + t^{1+\alpha}|\Delta\partial_{y_1}\eta|$$

and (5-18), while (5-20) is a consequence of

$$|\Delta\zeta\,\Delta\partial_{y_1} f_t^{\eta}| \le C_0 \|\zeta\|_{L^\infty(\Omega_{\rho'})}|z_1| + t^{1+\alpha}|\Delta\zeta|\|\partial_{y_1}\eta\|_{L^\infty(\Omega_{\rho'})}$$

and (5-17).

Finally, (5-21) is achieved in the same way as (5-20) but with an additional use of Lemma 5.3. □

Proof of Lemma 5.11. Let $0 < \rho' < \rho < \rho_0$ and $\eta, \zeta \in B_\rho$ be as stated. For $\lambda \in [0,1]$, define

$$\xi_\lambda := \lambda\eta + (1-\lambda)\zeta.$$

Then $\xi_\lambda \in B_\rho$ and $\|\xi_\lambda\|_\rho < 1$.

Now for $y \in \Omega_{\rho'}$ we write

$$\begin{aligned}|F_t(\eta)(y) - F_t(\zeta)(y)| \\ \le \int_{\Omega_0} |(K_2(\Delta X_t^{\eta}) - K_2(\Delta X_t^{\zeta}))\Delta\partial_{y_1} f_t^{\eta}|\,dz + \int_{\Omega_0} |K_2(\Delta X_t^{\zeta})(\Delta\partial_{y_1} f_t^{\eta} - \Delta\partial_{y_1} f_t^{\zeta})|\,dz.\end{aligned} \tag{5-22}$$

In order to estimate the first term, we first use the fundamental theorem of calculus to write

$$\int_{\Omega_0} |K_2(\Delta X_t^{\eta}) - K_2(\Delta X_t^{\zeta})||\Delta\partial_{y_1} f_t^{\eta}|\,dz = \int_{\Omega_0} \left| \int_0^1 \partial_{a_2} K_2(\Delta X_t^{\xi_\lambda}) \tfrac{1}{2} t^{1+\alpha}(\Delta\eta - \Delta\zeta)\,d\lambda \right| |\Delta\partial_{y_1} f_t^{\eta}|\,dz.$$

Now, $|\Delta\partial_{y_1} f_t^{\eta}|$ is dealt with by (5-19) with $\xi = \xi_\lambda$, Lemma 5.5 and its equation (5-7) are used to deal with $\partial_{a_2} K_2(\Delta X_t^{\xi_\lambda})$ and by definition of $\|\cdot\|_{\rho'}$ we arrive at the estimate

$$\begin{aligned}\int_{\Omega_0} |K_2(\Delta X_t^{\eta}) - K_2(\Delta X_t^{\zeta})||\Delta\partial_{y_1} f_t^{\eta}|\,dz \\ = \int_{\Omega_0} \left| \int_0^1 \partial_{a_2} K_2(\Delta X_t^{\xi_\lambda}) \tfrac{1}{2} t^{1+\alpha}(\Delta\eta - \Delta\zeta)\,d\lambda \right| |\Delta\partial_{y_1} f_t^{\eta}|\,dz \\ \le C_0 t^{1+\alpha}\|\eta - \zeta\|_{\rho'} \int_{\Omega_0}\int_0^1 \frac{1}{|\Delta X_t^{\xi_\lambda}|_*^2} \frac{|\Delta X_t^{\xi_\lambda}|_*}{\rho - \rho'}\,d\lambda\,dz \le \frac{C_0 t^{1+\alpha}|\log t|}{\rho - \rho'}\|\eta - \zeta\|_\rho,\end{aligned}$$

where the last inequality is a direct application of Lemma 5.8. Again by Lemmas 5.5 and 5.8, the second term in (5-22) is bounded by

$$\int_{\Omega_0} |K_2(\Delta X_t^{\zeta})(\Delta\partial_{y_1} f_t^{\eta} - \Delta\partial_{y_1} f_t^{\zeta})|\,dz \le C \int_{\Omega_0} \frac{1}{|\Delta X_t^{\zeta}|_*} t^{1+\alpha}|\Delta\partial_{y_1}\eta - \Delta\partial_{y_1}\zeta|\,dz \le C_0 t^{1+\alpha}|\log t|\|\eta - \zeta\|_\rho.$$

Thus,

$$\|F_t(\eta)-F_t(\zeta)\|_{L^\infty(\Omega_{\rho'})} \le \frac{C_0 t^{1+\alpha}|\log t|}{\rho-\rho'}\|\eta-\zeta\|_\rho.$$

Let us now turn to the corresponding inequality with ∂_{y_1}. In a similar way as before we write the decomposition

$$\partial_{y_1}F_t(\eta)(y)-\partial_{y_1}F_t(\zeta)(y) = -\int_{\Omega_0} A_1+A_2+A_3+A_4\,dz,$$

where

$$\begin{aligned}
A_1 &:= (\partial_{a_2}K_2(\Delta X_t^\eta)-\partial_{a_2}K_2(\Delta X_t^\zeta))(\Delta\partial_{y_1}f_t^\eta)^2,\\
A_2 &:= \partial_{a_2}K_2(\Delta X_t^\zeta)((\Delta\partial_{y_1}f_t^\eta)^2-(\Delta\partial_{y_1}f_t^\zeta)^2),\\
A_3 &:= (K_2(\Delta X_t^\eta)-K_2(\Delta X_t^\zeta))\Delta\partial_{y_1}^2 f_t^\eta,\\
A_4 &:= K_2(\Delta X_t^\zeta)(\Delta\partial_{y_1}^2 f_t^\eta-\Delta\partial_{y_1}^2 f_t^\zeta),
\end{aligned}$$

see (5-11). Regarding A_1, we use (5-19) and (5-20) to deduce

$$\begin{aligned}
\int_{\Omega_0}|A_1|\,dz &\le C\int_{\Omega_0}\int_0^1 \frac{1}{|\Delta X_t^{\xi_\lambda}|_*^3}t^{1+\alpha}|\Delta(\eta-\zeta)\Delta\partial_{y_1}f_t^\eta||\Delta\partial_{y_1}f_t^\eta|\,dz\,d\lambda\\
&\le \frac{C_0t^{1+\alpha}}{\rho-\rho'}\|\eta-\zeta\|_{\rho'}\int_0^1\int_{\Omega_0}\frac{1}{|\Delta X_t^{\xi_\lambda}|_*}\,dz\,d\lambda \le \frac{C_0t^{1+\alpha}|\log t|}{\rho-\rho'}\|\eta-\zeta\|_\rho.
\end{aligned}$$

By making use of (5-21) instead of (5-20), one can bound $\int_{\Omega_0}|A_3|\,dz$ in a similar way. We omit the details.

Next for A_2, inequality (5-19) implies

$$\int_{\Omega_0}|A_2|\,dz \le C\int_{\Omega_0}\frac{1}{|\Delta X_t^\zeta|_*^2}|\Delta\partial_{y_1}f_t^\eta+\Delta\partial_{y_1}f_t^\zeta|t^{1+\alpha}|\Delta\partial_{y_1}\eta-\Delta\partial_{y_1}\zeta|\,dz \le \frac{C_0t^{1+\alpha}|\log t|}{\rho-\rho'}\|\eta-\zeta\|_\rho.$$

Finally, the estimate for $\int_{\Omega_0}|A_4|\,dz$ is a straightforward consequence of Lemmas 5.5, 5.7, 5.8 and Cauchy's Lemma 5.3.

Summarizing, we have shown

$$\|\partial_{y_1}F_t(\eta)-\partial_{y_1}F_t(\zeta)\|_{L^\infty(\Omega_{\rho'})} \le \frac{C_0t^{1+\alpha}|\log t|}{\rho-\rho'}\|\eta-\zeta\|_\rho.$$

It therefore remains to check ∂_{y_2}. Again we write the decomposition

$$\partial_{y_2}F_t(\eta)(y)-\partial_{y_2}F_t(\zeta)(y) = -\int_{\Omega_0} B_1+B_2+B_3+B_4\,dz,$$

where, see (5-12),

$$\begin{aligned}
B_1 &:= (\partial_{a_2}K_2(\Delta X_t^\eta)-\partial_{a_2}K_2(\Delta X_t^\zeta))\big(t+\tfrac12 t^{1+\alpha}\partial_{y_2}\eta(y)\big)\Delta\partial_{y_1}f_t^\eta,\\
B_2 &:= \partial_{a_2}K_2(\Delta X_t^\zeta)\big[\big(t+\tfrac12 t^{1+\alpha}\partial_{y_2}\eta(y)\big)\Delta\partial_{y_1}f_t^\eta-\big(t+\tfrac12 t^{1+\alpha}\partial_{y_2}\zeta(y)\big)\Delta\partial_{y_1}f_t^\zeta\big],\\
B_3 &:= (K_2(\Delta X_t^\eta)-K_2(\Delta X_t^\zeta))\tfrac12 t^{1+\alpha}\partial_{y_2}\partial_{y_1}\eta(y),\\
B_4 &:= K_2(\Delta X_t^\zeta)\tfrac12 t^{1+\alpha}(\partial_{y_2}\partial_{y_1}\eta(y)-\partial_{y_2}\partial_{y_1}\zeta(y)).
\end{aligned}$$

Since $t < 1$ and $|\partial_{y_2}\eta(y)| < 1$, we get

$$\int_{\Omega_0} |B_1|\,dz \le C t^{1+\alpha} \int_0^1 \int_{\Omega_0} \frac{1}{|\Delta X_t^{\xi_\lambda}|_*^3} t|\Delta\eta - \Delta\zeta||\Delta\partial_{y_1} f_t^\eta|\,dz\,d\lambda \le \frac{C_0 t^{1+\alpha}|\log t|}{\rho-\rho'}\|\eta-\zeta\|_\rho$$

by (5-17) and (5-19).

Moreover,

$$\int_{\Omega_0} |B_2|\,dz \le C\int_{\Omega_0} \frac{1}{|\Delta X_t^\zeta|_*^2}\big[t^{1+\alpha}|\partial_{y_2}\eta(y)-\partial_{y_2}\zeta(y)||\Delta\partial_{y_1} f_t^\eta| + (t+t^{1+\alpha}|\partial_{y_2}\zeta(y)|)t^{1+\alpha}|\Delta\partial_{y_1}\eta - \Delta\partial_{y_1}\zeta|\big]\,dz \le \frac{C_0 t^{1+\alpha}|\log t|}{\rho-\rho'}\|\eta-\zeta\|_\rho$$

by use of (5-19) in the first term as well as (5-18) in the second.

The estimate for $\int_{\Omega_0}|B_3|\,dz$ follows in analogy to $\int_{\Omega_0}|B_1|\,dz$ utilizing (5-17) and Cauchy's Lemma 5.3, whereas the estimate for $\int_{\Omega_0}|B_4|\,dz$ relies solely on Lemma 5.3.

This finishes the proof of Lemma 5.11. □

5.6. *The affine term.* In order to complete the list of ingredients of Theorem 5.1, we investigate $F_t(0)$. As usual, we consider $\rho_0 \in (0,1)$ to be fixed according to Lemma 5.6 and $R = 1$, $T = T(R, C_0, \alpha) \in (0,1)$ given by Lemma 5.7.

Lemma 5.13. *For any $\rho \in (0,\rho_0)$, $t \in (0,T)$, we have*

$$\|F_t(0)\|_\rho \le C_0 t|\log t|.$$

Proof. Let $y \in \Omega_\rho$. Recall that

$$\Delta X_t^0 = \begin{pmatrix} z_1 \\ \Delta\gamma_0 + t\Delta s_0 + t(y_2 - z_2)\end{pmatrix} = \Delta X_0 + \begin{pmatrix} 0 \\ t\Delta s_0 + t(y_2-z_2)\end{pmatrix}, \quad z \in \Omega_0.$$

In view of Lemmas 5.5, 5.7, 5.8 and the boundedness of Ω_0, we have

$$\begin{aligned}
|F_t(0)(y)| &\le \int_{\Omega_0} |K_2(\Delta X_t^0) - K_2(\Delta X_0)||\Delta\gamma_0'| + t|K_2(\Delta X_t^0)|A|\Delta s_0'|\,dz \\
&\le \int_0^1\int_{\Omega_0} |\partial_{a_2}K_2(\Delta X_{\lambda t}^0)t(y_2 - z_2 + \Delta s_0)||\Delta\gamma_0'|\,dz\,d\lambda + C_0 t \\
&\le C_0 t\Big(1 + \int_0^1\int_{\Omega_0}\frac{1}{|\Delta X_{\lambda t}^0|_*}\,dz\,d\lambda\Big) \\
&\le C_0 t\Big(1 + \int_0^1 |\log(\lambda t)|\,d\lambda\Big) \\
&\le C_0 t|\log t|.
\end{aligned}$$

Next we estimate $\partial_{y_1} F_t(0)$. One has

$$|\partial_{y_1} F_t(0)(y)| \le \int_{\Omega_0} |\partial_{a_2} K_2(\Delta X_t^0) - \partial_{a_2} K_2(\Delta X_0)||\Delta\gamma_0'|^2 + |K_2(\Delta X_t^0)|t|\Delta s_0''| \\ + |K_2(\Delta X_t^0) - K_2(\Delta X_0)||\Delta\gamma_0''| + |\partial_{a_2} K_2(\Delta X_t^0)|(2t\Delta\gamma_0'\Delta s_0' + t^2|\Delta s_0'|^2)\, dz.$$

The terms appearing on the right-hand side can be dealt with in a similar way as above.

Finally, we also state

$$|\partial_{y_2} F_t(0)(y)| \le \int_{\Omega_0} |\partial_{a_2} K_2(\Delta X_t^0)|t|\Delta\gamma_0' + t\Delta s_0'|\, dz \le C_0 t|\log t|.$$

This finishes the proof of Lemma 5.13. □

Remark 5.14. Note that Lemma 5.13 implies

$$\frac{1}{t^{1+\alpha}} \int_0^t \|F_s(0)\|_\rho\, ds \le C_0 t^{1-\alpha}|\log t| \le \frac{C_0}{\rho_0 - \rho} t^{1-\alpha}|\log t|;$$

i.e., Theorem 5.1 (iii) holds with $a(t) = t^{1+\alpha},\ c(t) = C_0 t^{1-\alpha}|\log t|$.

5.7. *Conclusion and additional remarks.* In Sections 5.1–5.6 we have verified all the conditions of Theorem 5.1. As a consequence we deduce the following statement.

Proposition 5.15. *Let $\rho_0 > 0$ be as in Lemma 5.6. There exists $\bar a = \bar a(C_0) > 0,\ T = T(C_0, \alpha) > 0$ and a unique function $t \mapsto \eta_t$ with the properties that, for every $\rho \in (0, \rho_0)$, the map*

$$I_\rho := \{t \in [0, T) : C_0 t^{1-\alpha}|\log t| < \bar a(\rho_0 - \rho)\} \ni t \mapsto \eta_t \in B_\rho$$

is continuous with $\|\eta_t\|_\rho < 1,\ t \in I_\rho$, and such that, for all $y \in \Omega_\rho,\ t \in I_\rho$, we have

$$\eta_0(y) = 0, \qquad \eta_t(y) = \frac{1}{t^{1+\alpha}} \int_0^t F_s(\eta_s)(y)\, ds. \tag{5-23}$$

We finish the investigation of the fixed-point problem (4-22) with some accompanying remarks concerning properties of the solution η_t given by Proposition 5.15.

The first addresses regularity. In contrast to the analyticity of η_t in y_1, we only know that η_t is continuously differentiable in y_2. Using (5-23) it seems possible to upgrade the regularity with respect to y_2. However, since $F_s(\eta_s)(y)$ involves the integration over the finite interval $(-2, 2)$ with respect to z_2, in contrast to $\mathbb{T}$ for the integration in z_1, the maximal regularity for $\eta_t(y_1, \cdot) : [-2, 2] \to \mathbb{C}$ is expected to be finite. In any case, since a higher regularity of η_t with respect to y_2 would only improve the regularity of our subsolution inside the mixing zone and not across its boundary, we have not pursued this topic any further.

Next we turn to the role of the parameter α. Suppose that we set up problem (4-22) with respect to two different choices $0 < \alpha < \beta < 1$ leading to two different right-hand sides involving $F_t^\alpha(\eta),\ F_t^\beta(\eta)$. Our previous analysis gives two solutions $\eta_t^\alpha,\ \eta_t^\beta$ with corresponding intervals $I_\rho^\alpha \subset [0, T^\alpha),\ I_\rho^\beta \subset [0, T^\beta)$, $\rho \in (0, \rho_0)$. Note that the intervals $I_\rho^\alpha,\ I_\rho^\beta$ are defined with the same $\bar a$ and recall that $T^\alpha, T^\beta \in (0, 1)$.

Lemma 5.16. *We have* $t^{\beta-\alpha}\eta_t^\beta = \eta_t^\alpha$ *on* $[0, \min\{T^\alpha, T^\beta\})$.

Proof. Define $J_\rho^\alpha := I_\rho^\alpha \cap [0, T^\beta)$ and $J_\rho^\beta := I_\rho^\beta \cap [0, T^\alpha)$. Then $J_\rho^\beta \subset J_\rho^\alpha$ due to the fact that $\beta > \alpha$ and $t < 1$. Both functions $t^{\beta-\alpha}\eta_t^\beta$, η_t^α are continuous maps from J_ρ^β into the unit ball of B_ρ, $\rho \in (0, \rho_0)$, and they both vanish at $t = 0$. Moreover, it is easy to check that

$$t^{\beta-\alpha}\eta_t^\beta(y) = t^{\beta-\alpha}\frac{1}{t^{1+\beta}}\int_0^t F_s^\beta(\eta_s^\beta)(y)\,ds = \frac{1}{t^{1+\alpha}}\int_0^t F_s^\alpha(s^{\beta-\alpha}\eta_s^\beta)(y)\,ds.$$

Thus Proposition 5.15 implies $t^{\beta-\alpha}\eta_t^\beta = \eta_t^\alpha$ as long as both are defined. □

Both solutions $t^{\beta-\alpha}\eta_t^\beta$, η_t^α of (5-23) then extend uniquely to a common maximal solution of (5-23) enjoying the properties of Proposition 5.15. Moreover, Lemma 5.16 shows that $t^{1+\alpha}\eta_t^\alpha$ is independent of the considered $\alpha \in (0, 1)$. Hence the induced function $f(t, y)$, defined in Section 6 below, is independent of $\alpha \in (0, 1)$.

Finally we remark that, for the choice $\alpha = 1$ in ansatz (4-20), a more careful analysis would have been required. In that case the initial value $\eta_0(y)$ is not expected to be given by 0 and the estimate given in Lemma 5.13 does not even lead to boundedness of $t^{-2}\int_0^t \|F_s(0)\|_\rho\,ds$. However, since this analysis has not been needed in order to prove existence of a Lipschitz solution of (3-1), we leave the case $\alpha = 1$ as a possible future improvement.

6. Justification of ansatzes

We will now verify that η provided by Proposition 5.15 indeed induces — when undoing the transformations stated in Section 4 — a solution of the macroscopic IPM system (3-1).

Given η from Proposition 5.15, we first of all define $f : [0, T) \times \bar{\Omega}_0 \to \mathbb{R}$,

$$f(t, y) := f_t^{\eta_t}(y) = \gamma_0(y_1) + ts_0(y_1) + \tfrac{1}{2}t^{1+\alpha}\eta_t(y),$$

where $T = T(C_0, \alpha) > 0$ can be taken as the endpoint of the interval $I_{\rho_0/2}$ for instance. Also recall Lemma 5.4 if needed for the extension to the closure of Ω_0.

Lemma 6.1. *We have* $f \in \mathcal{C}^1([0, T); \mathcal{C}^1(\bar{\Omega}_0))$, *with*

$$\|\partial_{y_2} f(t, \cdot)\|_{L^\infty(\Omega_0)} \le \tfrac{1}{2}t^{1+\alpha}, \quad t \in (0, T). \tag{6-1}$$

Moreover, the functions $f(t, \cdot, y_2)$, $\partial_t f(t, \cdot, y_2)$, $\partial_{y_2} f(t, \cdot, y_2)$, $t \in [0, T)$, $y_2 \in [-2, 2]$, *are real analytic, and* f *satisfies the initial value problem* $f(0, y) = \gamma_0(y_1)$,

$$\partial_t f(t, y) = -\frac{1}{2}\int_{\Omega_0} K_2(\Delta X_t^{\eta_t}(y, z))(\partial_{y_1} f(t, y) - \partial_{y_1} f(t, y_1 - z_1, z_2))\,dz \tag{6-2}$$

for $t \in [0, T)$, $y \in \bar{\Omega}_0$.

Proof. As a direct consequence of Lemma 5.9 and Proposition 5.15, one sees that f belongs to $\mathcal{C}^1([0, T); B_{\rho_0/2})$ and satisfies (6-1), (6-2) for $t \in [0, T)$, $y \in \Omega_{\rho_0/2}$. The statement follows from the definition of the spaces B_ρ and Lemma 5.4. □

We are now able to prove our main result.

Proof of Theorem 3.2. Let f be as in Lemma 6.1. Define the open space-time set

$$\mathscr{U} := \{(t,x) \in (0,T) \times \mathbb{T} \times \mathbb{R} : -2t + f(t,x_1,-2) < x_2 < 2t + f(t,x_1,2)\}$$

as well as the slices

$$\mathscr{U}_t := \{x \in \mathbb{T} \times \mathbb{R} : (t,x) \in \mathscr{U}\}, \quad t \in (0,T).$$

As a consequence of (6-1), the maps $X_t : \Omega_0 \to \mathscr{U}_t$,

$$X_t(y) := \begin{pmatrix} y_1 \\ ty_2 + f(t,y) \end{pmatrix}, \quad t \in (0,T),$$

are $\mathcal{C}^1$ diffeomorphisms with the property that the joint maps $(0,T) \times \Omega_0 \to \mathbb{T} \times \mathbb{R}$, $(t,y) \mapsto X_t(y)$, and $\mathscr{U} \to \mathbb{T} \times \mathbb{R}$, $(t,x) \mapsto X_t^{-1}(x)$, are also of class $\mathcal{C}^1$.

In view of (4-11), we thus can indeed define the density $\rho : [0,T) \times \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ by setting

$$\rho(t,x) := \begin{cases} 1, & x_2 \geq 2t + f(t,x_1,2), \\ \frac{1}{2}(X_t^{-1}(x))_2, & x \in \mathscr{U}_t, \\ -1, & x_2 \leq -2t + f(t,x_1,-2), \end{cases}$$

for $t \in (0,T)$ and $\rho(0,x) := \rho_0(x)$. Here $(X_t^{-1}(x))_2$ denotes the second component of $X_t^{-1}(x)$. Observe that ρ is continuous except at points $(0,x_1,\gamma_0(x_1))$, $x_1 \in \mathbb{T}$, and piecewise $\mathcal{C}^1$ with the exceptional set being $\partial\mathscr{U} \subset [0,T) \times \mathbb{T} \times \mathbb{R}$. Moreover, as long as t is positive, $\rho(t,\cdot)$ is Lipschitz continuous and there exists a constant $C_0 > 0$ depending on the initial data such that

$$|\nabla\rho(t,x)| \leq \frac{C_0}{t} \mathbb{1}_{\mathscr{U}_t}(x) \tag{6-3}$$

for all $(t,x) \notin \partial\mathscr{U}$.

Moreover, standard elliptic estimates show that v defined through (4-3), (4-7) and (4-8) is the unique L^2 solution of the last two equations of (3-1).

The stated log-Lipschitz continuity of $v(t,\cdot)$, $t > 0$, is a consequence of the Biot–Savart operator acting on a compactly supported L^∞-vorticity; see [Marchioro and Pulvirenti 1994]. In addition, it is also easy to see that $v : [0,T) \times \mathbb{T} \times \mathbb{R} \to \mathbb{R}^2$ is continuous except at the one-dimensional set $\{(0,x_1,\gamma_0(x_1)) : x_1 \in \mathbb{T}\}$.

Hence we have shown properties (i), (ii) of Theorem 3.2. Moreover, observe that property (iii) holds by construction, with γ_t given by

$$\gamma_t(x_1,h) := t2h + f(t,x_1,2h), \quad x_1 \in \mathbb{T}, \quad h \in [-1,1].$$

It thus remains to show that the first equation of (3-1) and the entropy balances (3-4) are satisfied.

The regularity of ρ implies

$$\int_{\mathbb{T}\times\mathbb{R}} \rho(t,\cdot)v(t,\cdot) \cdot \nabla\varphi \, dx = -\int_{\mathbb{T}\times\mathbb{R}} v(t,\cdot) \cdot \nabla\rho(t,\cdot)\varphi \, dx$$

for all $t \in (0, T)$, $\varphi \in \mathcal{C}^\infty(\mathbb{T} \times \mathbb{R})$. It follows that (ρ, v) is a solution in the sense of Definition 3.1 if and only if the transport form

$$\partial_t \rho + v \cdot \nabla \rho + 2\rho \partial_{x_2} \rho = 0 \tag{6-4}$$

of the equation is satisfied pointwise in $(0, T) \times \mathbb{T} \times \mathbb{R} \setminus \partial \mathscr{U}$.

At points $(t, x) \notin \overline{\mathscr{U}}$, equation (6-4) trivially holds. Inside $\mathscr{U}$ one can check that the computations in Section 4.3 are possible showing that (6-4) is equivalent to (4-15). Note that in Section 4.3 we have formally assumed that the X_t are global diffeomorphisms mapping $\mathbb{T} \times \mathbb{R}$ to itself, but as the reader can easily see, it is enough to have transformations from Ω_0 to the corresponding $\mathscr{U}_t$.

Observing also that the computations in Section 4.4 are legal in our scenario, one sees that (6-4) on $\mathscr{U}$ is indeed equivalent to (6-2).

Finally, let $\eta : \mathbb{R} \to \mathbb{R}$ be an arbitrary Lipschitz continuous function, and define the function $Q : \mathbb{R} \to \mathbb{R}$,

$$Q(u) := \int_0^u 2\eta'(s) s \, ds,$$

which is also Lipschitz continuous when restricted to any compact interval of $\mathbb{R}$. Consequently we have enough regularity to deduce (3-4) by multiplying (6-4) with $\eta'(\rho(t, x))$ and applying the chain rule. This finishes the proof of Theorem 3.2. □

Appendix A: The abstract Cauchy–Kovalevskaya theorem

Proof of Theorem 5.1. As indicated in Section 5, the proof of Theorem 5.1 is a slight modification of the original proof in [Nishida 1977].

Let $a_0 > 0$ and set $a_{k+1} := a_k(1 - (k+2)^{-2})$, $k = 0, 1, \dots$. Then

$$a := \lim_{k \to \infty} a_k > 0.$$

For $\rho \in (0, \rho_0)$ and $k = 0, 1, \dots$, we define the intervals

$$I_{k,\rho} := \{t \in [0, T) : c(t) < a_k(\rho_0 - \rho)\}.$$

We also define for a function u with $u : I_{k,\rho} \to B_\rho$ continuous for any $\rho \in (0, \rho_0)$ the norm

$$M_k[u] := \sup\left\{ \|u(t)\|_\rho \left(\frac{a_k(\rho_0 - \rho)}{c(t)} - 1 \right) : \rho \in (0, \rho_0),\, t \in I_{k,\rho} \right\}.$$

Note that, for $c(t) = t$, one recovers Nishida's setup. Now one recursively constructs the sequence

$$u_0(t) := 0, \qquad u_{k+1}(t) := \begin{cases} \frac{1}{a(t)} \int_0^t F(u_k(s), s)\, ds, & t \in (0, T), \\ 0, & t = 0. \end{cases}$$

We claim that, for a_0 chosen sufficiently small, the recursion is well-defined, that each $u_k : I_{k,\rho} \to B_\rho$ is continuous with $\|u_k(t)\|_\rho < \frac{1}{2} R$ for $t \in I_{k,\rho}$, $\rho \in (0, \rho_0)$, and that

$$\lambda_{k-1} := M_k[u_k - u_{k-1}] \le (4Ka_0)^{k-1} a_0, \tag{A-1}$$

where $K > 0$ is the constant appearing in (5-2).

We first of all note that $u_1(t)$ exists and satisfies the stated continuity condition due to assumptions (i) and (iii). Moreover, for $t \in I_{0,\rho}$, we have $\|u_1(t)\|_\rho < a_0$. Thus, we pick a_0 at least as small as $\frac{1}{2}R$. One also easily checks that $\lambda_0 \le a_0$.

From now on we proceed by induction. Assume that the recursion with the above stated properties is possible up to some $k \ge 1$. Then it is also clear that $u_{k+1} : I_{k+1,\rho} \to B_\rho$ is well-defined as well as continuous on the open interval $I_{k+1,\rho} \setminus \{0\}$ for any $\rho \in (0, \rho_0)$.

If we assume for now that (A-1) also holds for λ_k, then, for $t \in I_{k+1,\rho}$, we obtain in analogy to [Nishida 1977] the estimate

$$\|u_{k+1}(t)\|_\rho \le \sum_{j=0}^{k} \lambda_j \left(\frac{a_j(\rho_0 - \rho)}{c(t)} - 1 \right)^{-1} \le \sum_{j=0}^{k} \lambda_j \left(\frac{a_j}{a_{j+1}} - 1 \right)^{-1}$$
$$\le a_0 \sum_{j=0}^{k} (4K a_0)^j (j+2)^2 < \tfrac{1}{2} R$$

by choice of a_0 independent of k. Moreover, the first inequality in the above line of estimates applied at times $t > 0$ with $c(t) < \frac{1}{2} a(\rho_0 - \rho)$ also gives

$$\|u_{k+1}(t)\|_\rho \le c(t) \sum_{j=0}^{k} \frac{\lambda_j}{a_j(\rho_0 - \rho) - c(t)} \le \frac{2a_0 c(t)}{a(\rho_0 - \rho)} \sum_{j=0}^{k} (4K a_0)^j,$$

which shows that u_{k+1} is also continuous with respect to $\|\cdot\|_\rho$ at $t = 0$.

To finish the induction it thus remains to show (A-1) for λ_k. The clever move is to use the contraction property with a different Banach space at each time τ inside the integral. Namely, exactly as in [Nishida 1977, p. 631], the contraction property of F (Theorem 5.1 (ii)) with

$$\rho(\tau) := \frac{1}{2} \left(\rho_0 - \frac{c(\tau)}{a_k} + \rho \right)$$

and the definition of λ_{k-1} lead to

$$\|u_{k+1}(t) - u_k(t)\|_\rho \le \frac{4\lambda_{k-1} a_k}{a(t)} \int_0^t \frac{b(\tau) c(\tau)}{(a_k(\rho_0 - \rho) - c(\tau))^2} \, d\tau$$

for $t \in I_{k,\rho}$. At this point we use (5-2) and a change of variables to obtain

$$\|u_{k+1}(t) - u_k(t)\|_\rho \le 4\lambda_{k-1} a_k K c(t) \int_0^{c(t)} \frac{1}{(a_k(\rho_0 - \rho) - \xi)^2} \, d\xi$$

from where one can conclude $\lambda_k \le 4K\lambda_{k-1} a_0$ by following [Nishida 1977] again.

Now Theorem 5.1 follows as in [Nirenberg 1972; Nishida 1977]. □

Appendix B: More on Otto's relaxation

We here add some more details regarding the fifth step of Otto's relaxation [1999] in the general nonflat case, which has only been sketched in Section 2.5.1.

Before doing that we will quickly convince ourselves that the setting in [Otto 1999] is indeed equivalent to the formulation of IPM considered in our paper. Otto considers the equations

$$\begin{aligned} \partial_t s + u \cdot \nabla s &= 0, \\ \nabla \cdot u &= 0, \\ u &= -\nabla p + s e_2, \end{aligned} \tag{B-1}$$

which correspond with [Otto 1999, (1.1)–(1.2)] and the first equation on page 875 of [Otto 1999] with $\lambda = 1$. The parameter λ in that paper is the quotient of the mobilities. In our case, we have taken both mobilities equal to one and then $\lambda = 1$. More importantly, in that paper,

$$s = \{0, 1\}, \tag{B-2}$$

however

$$\rho = \{-1, 1\} \tag{B-3}$$

in our case.

Let us see how we can go from (1-1), (B-3) to (B-1)–(B-2). Firstly we define

$$\bar{s} = \tfrac{1}{2}(1 - \rho), \qquad \rho = 1 - 2\bar{s},$$

and thus

$$\begin{aligned} \partial_t \bar{s} + v \cdot \nabla \bar{s} &= 0, \\ \nabla \cdot v &= 0, \\ v &= -\nabla(p + x_2) + 2\bar{s} e_2, \end{aligned}$$

with

$$\bar{s} = \{0, 1\}.$$

We define $\bar{u} = \frac{1}{2} v$ and $\overline{\Pi} = \frac{1}{2}(p + x_2)$, which yields

$$\begin{aligned} \partial_t \bar{s} + 2\bar{u} \cdot \nabla \bar{s} &= 0, \\ \nabla \cdot \bar{u} &= 0, \\ \bar{u} &= -\nabla \overline{\Pi} + \bar{s} e_2. \end{aligned}$$

Finally we take $s(x, t) = \bar{s}\big(x, \frac{1}{2}t\big)$, $u\big(x, \frac{1}{2}t\big) = \bar{u}\big(x, \frac{1}{2}t\big)$ and $\Pi(x, t) = \overline{\Pi}\big(x, \frac{1}{2}t\big)$; thus

$$\begin{aligned} \partial_t s + u \cdot \nabla s &= 0, \\ \nabla \cdot u &= 0, \\ u &= -\nabla \Pi + s e_2, \end{aligned}$$

with $s = \{0, 1\}$, which agrees with (B-1)–(B-2) (up to a relabeling of the pressure). Therefore, if we show that (B-1)–(B-2) relaxes to

$$\begin{aligned} \partial_t s + u \cdot \nabla s + \partial_{x_2} s - 2s \partial_{x_2} s &= 0, \\ \nabla \cdot u &= 0, \\ u &= -\nabla \Pi + s e_2, \end{aligned} \tag{B-4}$$

with $s \in [0, 1]$, by undoing the previous transformations, we see that (1-1), (B-3) relaxes to

$$\begin{aligned} \partial_t \rho + v \cdot \nabla \rho + 2\rho \partial_{x_2} \rho &= 0, \\ \nabla \cdot v &= 0, \\ v &= -\nabla p - \rho e_2, \end{aligned} \tag{B-5}$$

with $\rho \in [-1, 1]$.

Next, we begin our formal discussion with the outcome of the fourth step of Otto, after which there exists for each $h > 0$ a sequence of "coarse-grained" functions $\{\theta^k\}_{k=0}^{N(h)}$ that are characterized by the following JKO scheme (which we understand as a minimizing movements scheme with respect to the Wasserstein distance):

$\theta^{(k+1)}$ is the minimizer in K of

$$\tfrac{1}{2}\operatorname{dist}^2(\theta^{(k)}, \theta) + \tfrac{1}{2}\operatorname{dist}^2(1-\theta^{(k)}, 1-\theta) - h\int \theta(x) x_2, \tag{B-6}$$

where the set K consists of measurable θ taking values in $[0, 1]$ and such that $\int \theta = \int s(x, 0)$, and $\operatorname{dist}^2(\theta_0, \theta_1)$ is the L^2-Wasserstein distance

$$\operatorname{dist}^2(\theta_0, \theta_1) = \inf_{\Phi \in I(\theta_0, \theta_1)} \int \theta_0(x) |\Phi(x) - x|^2 \, dx,$$

with

$$I(\theta_0, \theta_1) = \left\{ \Phi : \int \theta_1(y) \zeta(y) \, dy = \int \theta_0(x) \zeta(\Phi(x)) \, dx \ \ \forall \zeta \in \mathcal{C}_0^0 \right\}.$$

In the definition of $I(\theta_0, \theta_1)$, we have been deliberately imprecise and defer the reader to [Otto 1999] for the proper definition. Even more, in order to make the exposition clearer, in the following we will assume that the minimizer exists, that it is smooth and that it satisfies pointwise the corresponding Monge–Ampere equation; i.e.,

$$I(\theta_0, \theta_1) = \{\Phi \text{ diffeomorphism} : (\theta_1 \circ \Phi)(x) J_\Phi(x) = \theta_0(x)\}.$$

Here J_Φ denotes the Jacobian determinant $\det D\Phi$.

As explained in Section 2.5.1, our goal is to show, on a formal level, that the limit as $h \to 0$ — we will assume that it exists in the first place — of the functions

$$\theta_h(x, t) := \theta^{(k)}(x), \quad t \in [kh, (k+1)h)),$$

is characterized by system (B-4).

We begin with the Euler–Lagrange equation of (B-6). For a given $\theta_0 \in K$, let θ_1 be the minimizer in K of

$$F[\theta] \equiv \tfrac{1}{2}\operatorname{dist}^2(\theta_0, \theta) + \tfrac{1}{2}\operatorname{dist}^2(1-\theta_0, 1-\theta) - h\int \theta(x) x_2.$$

Then we have that

$$D_\theta F[\theta_1]\psi = \frac{d}{d\tau} F[\theta_1 + \tau\psi]\Big|_{\tau=0} = 0,$$

where we simply assume that $\theta_1 + \tau\psi \in K$; i.e., we in particular consider ψ with $\int \psi = 0$. In order to compute $D_\theta F[\theta_1]\psi$, we first look at $D_\theta \operatorname{dist}^2(\theta_0, \theta_1)\psi$. Let $\Phi_0^\tau \in I(\theta_0, \theta_1 + \tau\psi)$ be such that

$$\operatorname{dist}^2(\theta_0, \theta_1 + \tau\psi) = \inf_{\Phi \in I(\theta_0, \theta_1+\tau\psi)} \int \theta_0(x)|\Phi(x) - x|^2\, dx = \int \theta_0(x)|\Phi_0^\tau(x) - x|^2\, dx.$$

We define

$$w \circ \Phi_0^0 = \frac{d\Phi_0^\tau}{d\tau}\bigg|_{\tau=0},$$

and thus

$$\tfrac{1}{2} D_\theta \operatorname{dist}^2(\theta_0, \theta_1)\psi = \int \theta_0(x)(\Phi_0^0(x) - x)\cdot(w \circ \Phi_0^0)(x)\, dx. \tag{B-7}$$

We next compute for which w we have $\Phi_0^\tau \in I(\theta_0, \theta_1 + \tau\psi)$. We have

$$J_{\Phi_0^\tau}(x)((\theta_1 + \tau\psi)\circ\Phi_0^\tau)(x) = \theta_0(x). \tag{B-8}$$

Taking a τ-derivative in (B-8) and evaluating at $\tau = 0$ yields

$$J_{\Phi_0^0} \operatorname{div} w \circ \Phi_0^0 \theta_1 \circ \Phi_0^0 + J_{\Phi_0^0} w \circ \Phi_0^0 \cdot \nabla\theta_1 \circ \Phi_0^0 + J_{\Phi_0^0} \psi \circ \Phi_0^0 = 0,$$

which reduces to

$$\operatorname{div}(w\theta_1) + \psi = 0. \tag{B-9}$$

In addition, Φ_0^0 minimizes

$$\int \theta_0(x)|\Phi(x) - x|^2\, dx$$

in $I(\theta_0, \theta_1)$. So, for every family of flows $(\Phi_\delta^0) \in I(\theta_0, \theta_1)$, we have that

$$\frac{d}{d\delta}\int \theta_0(x)|\Phi_\delta^0(x) - x|^2\, dx\bigg|_{\delta=0} = 0.$$

That is,

$$\int \theta_0(x)(\Phi_0^0(x) - x)\cdot(\overline{w} \circ \Phi_0^0)(x)\, dx = 0,$$

where if Φ_δ is the flow of a vector field $\overline{w}$,

$$\overline{w} \circ \Phi_0^0 = \frac{d\Phi_\delta^0}{d\delta}\bigg|_{\delta=0}, \tag{B-10}$$

$$\operatorname{div}(\theta_1 \overline{w}) = 0. \tag{B-11}$$

The condition (B-11), equivalent to $\Phi_\delta \in I(\theta_0, \theta_1)$, is deduced by differentiating

$$J_{\Phi_\delta^0}(x)(\theta_1 \circ \Phi_\delta^0)(x) = \theta_0(x) \tag{B-12}$$

with respect to δ.

Therefore, we have

$$0 = \int \theta_0(x)(\Phi_0^0(x) - x)\cdot(\overline{w} \circ \Phi_0^0)(x)\, dx = \int \theta_1(x)\overline{w}(x)\cdot(x - (\Phi_0^0)^{-1}(x))\, dx,$$

where in the last equality we have used the definition of $I(\theta_0,\theta_1)$. Since $\overline{w}$ is an arbitrary vector field, Hodge decomposition implies that

$$x-(\Phi_0^0)^{-1}(x)=\nabla a(x) \tag{B-13}$$

for some function a. In order to avoid technicalities, we here have implicitly assumed that θ_1 does not vanish.

From (B-7), (B-9) and (B-13), we see that

$$\tfrac{1}{2}D_\theta \operatorname{dist}^2(\theta_0,\theta_1)\psi=\int \theta_1(x)w(x)\cdot\nabla a(x)\,dx=-\int \nabla\cdot(\theta_1 w)(x)a(x)\,dx=\int \psi(x)a(x).$$

We have obtained that

$$\tfrac{1}{2}D_\theta \operatorname{dist}^2(\theta_0,\theta_1)\psi=\int \psi(x)a(x)\Phi_0^0(x)=x+(\nabla a\circ\Phi_0^0)(x).$$

Similar computations yield

$$\tfrac{1}{2}D_\theta \operatorname{dist}^2(1-\theta_0,1-\theta_1)\psi=-\int \psi(x)\bar{a}(x)\overline{\Phi}_0^0(x)=x+(\nabla\bar{a}\circ\overline{\Phi}_0^0)(x),$$

and putting everything together we arrive at

$$D_\theta F[\theta_1]\psi=\int (a(x)-\bar{a}(x)-hx_2)\psi(x)\,dx=0 \tag{B-14}$$

for all ψ with $\int\psi=0$. Moreover, since $\Phi_0^0\in I(\theta_0,\theta_1)$ and $\overline{\Phi}_0^0\in I(1-\theta_0,1-\theta_1)$,

$$\theta_1(x)=J_{(\Phi_0^0)^{-1}}(x)\theta_0(x-\nabla a(x)), \tag{B-15}$$

$$(1-\theta_1)(x)=J_{(\overline{\Phi}_0^0)^{-1}}(x)(1-\theta_0)(x-\nabla\bar{a}(x)). \tag{B-16}$$

Note that so far we have omitted the h-dependence of the functions a, $\bar{a}$, Φ_0^0, $\overline{\Phi}_0^0$ in our notation. We continue doing so when introducing $p=a/h$, $\bar{p}=\bar{a}/h$ which, up to a constant, satisfy

$$p-\bar{p}=x_2$$

by (B-14). Note that the constant is irrelevant since only derivatives of p and $\bar{p}$ will play a role. To obtain a formal limit as $h\to 0$, we will assume in the following that the h-dependent functions p and $\bar{p}$ have a well-defined $\mathcal{C}^2$ limit, which will again be denoted by p and $\bar{p}$.

Now we take said limit. On one hand we have from (B-15) that

$$\begin{aligned}\frac{\theta_1(x)-\theta_0(x)}{h}&=\frac{J_{(\Phi_0^0)^{-1}}(x)\theta_0(x-h\nabla p(x))-\theta_0(x)}{h}\\&=\frac{(J_{(\Phi_0^0)^{-1}}(x)-1)\theta_0(x-h\nabla p(x))+\theta_0(x-h\nabla p(x))-\theta_0(x)}{h}.\end{aligned}$$

Recall that Φ_0^0 is linked to p via (B-13), and thus, since $\Phi_0^0(x)\to x$, we have

$$J_{(\Phi_0^0)^{-1}}(x)-1=-h\Delta p(x)$$

at first order in h. Thus, when letting $h \to 0$ in the difference quotient, we arrive at

$$\partial_t \theta = -\Delta p \theta - \nabla p \cdot \nabla \theta. \tag{B-17}$$

On the other hand, we have from (B-16) that

$$\frac{\theta_1(x) - \theta_0(x)}{h} = \frac{1 - J_{(\bar{\Phi}_0^0)^{-1}}(x) + J_{(\bar{\Phi}_0^0)^{-1}}(x)\theta_0(x - h\nabla \bar{p}(x)) - \theta_0(x)}{h}.$$

Passing to the limit yields

$$\partial_t \theta = \Delta \bar{p} - \Delta \bar{p} \theta - \nabla \theta \cdot \nabla \bar{p}. \tag{B-18}$$

In order for (B-17) and (B-18) to agree, we have

$$\Delta \bar{p} - \nabla \cdot (\nabla \bar{p} \theta) = -\nabla \cdot (\nabla p \theta),$$

and since $p = \bar{p} + x_2$ we have

$$\Delta \bar{p} = -\nabla \cdot (\nabla x_2 \theta) = -\partial_{x_2} \theta. \tag{B-19}$$

Therefore, from (B-18) and (B-19),

$$\partial_t \theta = -\partial_{x_2} \theta - \nabla \cdot (\nabla \bar{p} \theta) = -\partial_{x_2} \theta - \nabla \cdot ((\nabla \bar{p} + \theta e_2)\theta) + \nabla \cdot (\theta^2 e_2).$$

To finish we define $u = \nabla \bar{p} + \theta e_2$, which clearly satisfies $\nabla \cdot u = 0$, to get

$$\begin{aligned} \partial_t \theta + u \cdot \nabla \theta + \partial_{x_2} \theta - 2\theta \partial_{x_2} \theta &= 0, \\ u &= \nabla \bar{p} + \theta e_2, \\ \nabla \cdot u &= 0, \end{aligned}$$

which agrees with (B-4).

Appendix C: Rigorous energy dissipation

In Section 2.5.2 equation (2-15), we have formally computed the decay rate of the total potential energy. For completeness we give sufficient conditions when this computation is justified. Also for completeness, we show that the subsolution given by Theorem 3.2 indeed satisfies the sufficient conditions.

Lemma C.1. *Let $\rho_0 \in L^1_{\mathrm{loc}}(\mathbb{T} \times \mathbb{R})$ be some initial data, and further suppose that the pair of functions $(\rho, m) \in L^1_{\mathrm{loc}}((0, T) \times \mathbb{T} \times \mathbb{R}; \mathbb{R} \times \mathbb{R}^2)$ satisfies*

$$\partial_t \rho + \operatorname{div} m = 0, \qquad \rho(0, \cdot) = \rho_0$$

on $(0, T) \times \mathbb{T} \times \mathbb{R}$ in the sense of distributions. If there exists $\alpha > 0$ such that

$$m_2,\ (\rho - \rho_0) x_2,\ (\rho - \rho_0)|x_2|^{1+\alpha} \in \mathcal{C}^0([0, T); L^1(\mathbb{T} \times \mathbb{R})), \qquad m_2 |x_2|^\alpha \in L^1((0, T) \times \mathbb{T} \times \mathbb{R}),$$

then the relative potential energy defined in (2-14) belongs to $\mathcal{C}^1([0, T))$, and we have

$$\frac{d}{dt} E_{\mathrm{rel}}(t) = \int_{\mathbb{T} \times \mathbb{R}} m_2(t, x)\, dx.$$

Proof. Let $R > 0$ and $\varphi_R : \mathbb{R} \to [0, 1]$ be a cutoff function with $\varphi_R(x_2) = 1$ for $|x_2| \le R$, $\varphi_R(x_2) = 0$ for $|x_2| \ge 2R$ and $|\varphi_R'(x_2)| \le 2R^{-1}$, $x_2 \in \mathbb{R}$.

We use the abbreviation $E(t) = E_{\mathrm{rel}}(t)$ and define

$$E_R(t) := \int_{\mathbb{T}\times\mathbb{R}} (\rho(t,x) - \rho_0(x)) x_2 \varphi_R(x_2)\, dx.$$

Note that $E(t)$, $E_R(t)$ are well-defined at every time $t \in [0, T)$, and we have

$$|E(t) - E_R(t)| \le \|(\rho(t,\cdot) - \rho_0)|x_2|^{1+\alpha}\|_{L^1(\mathbb{T}\times\mathbb{R})} \frac{1}{R^\alpha} = O(R^{-\alpha})$$

uniformly in time as $R \to \infty$. Thus

$$h^{-1}(E(t+h) - E(t)) = h^{-1}(E_R(t+h) - E_R(t)) + h^{-1} O(R^{-\alpha}).$$

Moreover, the assumed continuity conditions and approximation of the indicator function of $[t, t+h]$ imply

$$\begin{aligned} E_R(t+h) - E_R(t) &= \int_t^{t+h}\int_{\mathbb{T}\times\mathbb{R}} m \cdot \nabla(x_2\varphi_R(x_2))\, dx\, ds \\ &= \int_t^{t+h}\int_{\mathbb{T}\times\mathbb{R}} m_2\, dx\, ds + \int_t^{t+h}\int_{\mathbb{T}\times\mathbb{R}} m_2(\varphi_R(x_2) - 1 + x_2\varphi_R'(x_2))\, dx\, ds. \end{aligned}$$

Now the latter term can be bounded by $5R^{-\alpha}\|m_2|x_2|^\alpha\|_{L^1((0,T)\times\mathbb{T}\times\mathbb{R})}$, implying that

$$h^{-1}(E(t+h) - E(t)) = h^{-1}\int_t^{t+h}\int_{\mathbb{T}\times\mathbb{R}} m_2\, dx\, ds + h^{-1} O(R^{-\alpha}).$$

The statement follows. □

Let (ρ, v) be the solution constructed in Theorem 3.2 and set

$$m = \rho v - (1 - \rho^2) e_2.$$

Lemma C.2. *In addition to the properties stated in Theorem 3.2, the velocity field v satisfies*

$$|v(t,x)| \le C e^{-|x_2|}$$

whenever $|x_2| \ge R$ for constants $C, R > 0$ independent of t. The pair (ρ, m) in particular satisfies the conditions of Lemma C.1.

Proof. Regarding the second component, one easily sees that

$$|v_2(t,x)| \le |\mathscr{U}_t| \|\partial_{x_1}\rho(t,\cdot)\|_{L^\infty(\mathbb{T}\times\mathbb{R})} \|K_2(x - \cdot)\|_{L^\infty(\mathscr{U}_t)},$$

which can be bounded by $Ce^{-|x_2|}$ for $|x_2| \ge R$ with constants $C, R > 0$ independent of time.

Regarding the first component, we cannot exploit the decay of the kernel, since $K_1(z) \to \mp 1$ as $z_2 \to \pm\infty$. Still by subtracting vanishing horizontal averages, we deduce

$$\begin{aligned} |v_1(t,x)| &= \left| \int_{\mathbb{T}\times\mathbb{R}} \partial_{x_1}\rho(t,y)(K_1(x-y) - K_1(x-(0,y_2))\,dy \right| \\ &\le |\mathscr{U}_t| \|\partial_{x_1}\rho(t,\cdot)\|_{L^\infty(\mathbb{T}\times\mathbb{R})} \|\partial_{z_1} K_1(x-\cdot)\|_{L^\infty(\mathscr{U}_t^*)} \pi, \end{aligned}$$

where $\mathscr{U}_t^*$ is the set of points obtained by taking all segments between $y \in \mathscr{U}_t$ and $(0, y_2)$. It is only important that those sets are bounded uniformly in time, which allows us to argue as above for v_2, since $\partial_{z_1} K_1$ now has the required decay. □

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ÁNGEL CASTRO: angel_castro@icmat.es
Consejo Superior de Investigaciones Científicas, Instituto de Ciencias Matemáticas-CSIC, ICMAT-CSIC-UAM-UCM-UC3M, Madrid, Spain

DANIEL FARACO: daniel.faraco@uam.es
Departamento de Mathemátics, Universidad Autónoma de Madrid, Instituto de Ciencias Matemáticas, CSIC-UAM-UC3M-UCM, Madrid, Spain

BJÖRN GEBHARD: bjoern.gebhard@uni-muenster.de
Angewandte Mathematik Münster, Universität Münster, Münster, Germany

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