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UNIFORM BOUNDS FOR BILINEAR SYMBOLS WITH LINEAR K-QUASICONFORMALLY EMBEDDED SINGULARITY

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We prove bounds in the strict local $L^2(\mathbb{R}^d)$ range for trilinear Fourier multiplier forms with a d-dimensional singular subspace. Given a fixed parameter $K \geq 1$, we treat multipliers with nondegenerate singularity that are push-forwards by K-quasiconformal matrices of suitable symbols. As particular applications, our result recovers the uniform bounds for the one-dimensional bilinear Hilbert transforms in the strict local L^2 range, and it implies the uniform bounds for two-dimensional bilinear Beurling transforms, which are new, in the same range.

1. Introduction

Let $d \geq 1$, and let Γ_0 be the linear subspace of $\mathbb{R}^{3\times d}$ consisting of all vectors (ξ_1, ξ_2, ξ_3) with $\xi_1+\xi_2+\xi_3=0$. Trilinear Fourier multiplier forms on Γ_0 are studied in order to understand mapping properties of bilinear Fourier multiplier operators on $\mathbb{R}^d$. In the present paper, we prove bounds in the strict local L^2 range for multipliers whose singular set can be written as an image of the d-dimensional diagonal of $\mathbb{R}^{3\times d}$ under a block K-quasiconformal matrix. Our bounds depend on the matrix through the parameter K alone; in this sense we prove bounds uniform in isotropic dilations and rotations. We comment more on the motivation for such bounds after stating the main result.

We normalize the Fourier transform of a Schwartz function as

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x\cdot\xi}\, \mathrm{d}x.$$

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Let $1 < p < \infty$. We denote the L^p norm of a measurable function by

$$\|f\|_p^p := \int_{\mathbb{R}^d} |f(x)|^p \, \mathrm{d}x.$$

Let $K \geq 1$. A linear map

$$L = L_1 \oplus L_2 \oplus L_3$$

mapping $\mathbb{R}^{3\times d}$ to itself is said to be block K-quasiconformal if, for all $n \in \{1, 2, 3\}$, we have $L_n : \mathbb{R}^d \to \mathbb{R}^d$ and

$$\|L_n\|_{\mathrm{op}}^d \leq K \det L_n.$$

We say that L is nontrivial if additionally

$$L_1 + L_2 + L_3 = 0.$$

Theorem 1.1. *Let $d \geq 1$, $K \geq 1$ and*

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1, \quad 2 < p_1, p_2, p_3 < \infty.$$

There exists a constant $C = C(d, K, p_1, p_2, p_3)$ such that the following holds.

Let $m : \mathbb{R}^{3\times d} \to \mathbb{C}$ satisfy

$$|\partial_1^{\gamma_1} \partial_2^{\gamma_2} \partial_3^{\gamma_3} m(\xi)| \leq \sup\{|\xi - (\tau, \tau, \tau)|^{-|\gamma|} : \tau \in \mathbb{R}^d\} \tag{1-1}$$

for all $\gamma \in \mathbb{N}^{3\times d}$ with $|\gamma| \leq 100d$. Let L be a nontrivial block K-quasiconformal matrix. Define

$$\Lambda_m(f_1, f_2, f_3) = \int_{\mathbb{R}^{3\times d}} \delta_0(\xi_1 + \xi_2 + \xi_3) \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) \hat{f}_3(\xi_3) m(L^{-1}\xi) \, \mathrm{d}\xi,$$

where δ_0 is the Dirac mass at the origin.

Then, for all triples of Schwartz functions f_1, f_2 and f_3 on $\mathbb{R}^d$,

$$|\Lambda_m(f_1, f_2, f_3)| \leq C \prod_{n=1}^{3} \|f_n\|_{p_n}.$$

We use a symbol m defined on all of $\mathbb{R}^{3\times d}$ for convenience, but instead of that, a symbol only defined on Γ_0 with conditions stated using directional differential operators within the space Γ_0 could be used as well. Similarly, the use of the mapping L in the definition of the form is a compact way to express a set of certain anisotropic symbol estimates on m through the simple condition (1-1). We point out that the restriction of Theorem 1.1 to the strict local L^2 range is likely not to be sharp. Moreover, we do not see any obvious obstruction for an analogy of our result for higher orders of multilinearity. The only missing ingredient for the latter seems to be a suitable generalization of the uniform paraproduct estimate as in [Muscalu et al. 2002b]. However, we did not attempt any of these extensions in order to keep the technicalities in this paper more limited and have better focus on some of the key ideas of our approach. For related work in $d = 1$ extending the range of exponents of the bilinear Hilbert transform, see [Di Plinio and Thiele 2016; Li 2006; Oberlin and Thiele 2011; Thiele 2002; Uraltsev and Warchalski 2022].

The simplest interesting special case of Theorem 1.1 is $d = K = 1$, when $L = (L_1, L_2, L_3)$ is a vector of nonzero real numbers adding up to 0 and

$$m(\xi_1, \xi_2, \xi_3) = \frac{L_1\xi_1 + L_2\xi_2 + L_3\xi_3}{\sqrt{(L_1\xi_1 + L_2\xi_2 + L_3\xi_3)^2 + (\xi_1 + \xi_2 + \xi_3)^2}},$$

which restricted to the hyperplane $\xi_1 + \xi_2 + \xi_3 = 0$ reads as

$$m(\xi_1, \xi_2, \xi_3) = \operatorname{sgn}(L_1\xi_1 + L_2\xi_2 + L_3\xi_3).$$

In this case, Λ_m is a scalar multiple of the trilinear form dual to the bilinear Hilbert transform, which can be written on the spatial side as

$$\text{p.v.} \iint_{\mathbb{R}^2} f_1(x + M_1 t) f_2(x + M_2 t) f_3(x + M_3 t) \frac{\mathrm{d}x\, \mathrm{d}t}{t}, \tag{1-2}$$

where $M = (M_1, M_2, M_3)$ is a unit vector perpendicular to both $(1, 1, 1)$ and L. No two components of M are equal, because no component of L is zero. This condition is referred to as nondegeneracy of M. The case of (1-2) with two components of the unit vector M equal is called degenerate. If for example $M_3 = M_1$, we have

$$\Lambda_m(f_1, f_2, f_3) = \int_{\mathbb{R}} f_1(x) f_3(x) \left[\text{p.v.} \int_{\mathbb{R}} f_2(x + t) \frac{\mathrm{d}t}{t} \right] \mathrm{d}x.$$

One obtains L^p bounds for this form by Hölder's inequality and bounds for the linear Hilbert transform. Bounds for the nondegenerate case of the bilinear Hilbert transform require a different argument and were shown in the exponent range of Theorem 1.1 in [Lacey and Thiele 1997], albeit with constants blowing up as M tends to a degenerate value. Bounds uniform in M were later proven in [Grafakos and Li 2004] for the first time. These results are covered by Theorem 1.1.

The simplest example of our main theorem which is new is the case where $d = 2$, $K = 1$ and (L_1, L_2, L_3) is a triple of conformal matrices adding up to zero. In this case, we identify $\mathbb{R}^2$ with $\mathbb{C}$ and view the application of the matrices L_n as multiplication by complex numbers. Moreover, we set

$$m(\zeta_1, \zeta_2, \zeta_3) = \frac{(\overline{L_1\zeta_1 + L_2\zeta_2 + L_3\zeta_3})^2}{|L_1\zeta_1 + L_2\zeta_2 + L_3\zeta_3|^2 + |\zeta_1 + \zeta_2 + \zeta_3|^2}.$$

Similar computations as for the bilinear Hilbert transform identify Λ_m as a scalar multiple of what one might call the bilinear Beurling transform

$$\text{p.v.} \iint_{\mathbb{C}^2} f_1(z + M_1\zeta) f_2(z + M_2\zeta) f_3(z + M_3\zeta) \frac{\mathrm{d}A(z)\, \mathrm{d}A(\zeta)}{\zeta^2},$$

where A denotes the area measure. Thus our main theorem implies L^p bounds in the strictly locally L^2 range for the bilinear Beurling transform uniformly in M. The Beurling kernel ζ^{-2} can be replaced by any standard Calderón–Zygmund kernel arising from a Mikhlin multiplier.

In dimension $d = 1$, the cases for L allowed in Theorem 1.1 together with a small number of easily understood degenerate cases provide an exhaustive picture of all cases of L. The situation in higher dimensions is more complicated. There are completely nondegenerate cases, completely degenerate cases

in the sense that $L_n = 0$ for some n, and further there is a zoo of distinct cases that one may call partially degenerate. For fixed K, our main theorem proves uniform bounds for the nondegenerate cases as one approaches the completely degenerate cases inside a cone that stays away from the partially degenerate cases. Within the conformal context, our theorem covers all cases including the degenerate ones. In this respect, we show that the setting of one complex dimension is quite analogous to the setting of one real dimension.

Concerning the general case, a list, not exhaustive, of five partially degenerate cases for $d = 2$ was described in [Demeter and Thiele 2010], and four of the cases were shown to be bounded, albeit without any attempt to prove uniform bounds. The remaining case, called the twisted paraproduct, was later treated in [Kovač 2012] (see also [Bernicot 2012] for preliminary results and [Durcik 2015; 2017] for further work). A further partially degenerate case is the triangular Hilbert transform described in [Kovač et al. 2015], where one dimension of the kernel is integrated out because it projects to zero in the arguments of all functions. The triangular Hilbert transform is not known to satisfy any L^p bounds, and it is well understood that presently known techniques are insufficient to obtain such bounds. A version of Theorem 1.1 with uniformity in K, as opposed to our assumption on K being fixed, would imply bounds for the triangular Hilbert transform. Bounds for the triangular Hilbert transform as well as some of the known bounds for other partially degenerate cases in $d = 2$ would, in turn, imply bounds for the so-called Carleson operator in the corresponding L^p spaces, see [Carleson 1966; Fefferman 1973; Hunt 1968]. A more systematic classification of the partially degenerate cases appears in [Warchalski 2019], where also some uniform bounds are proven in a discrete model.

The main technical novelty of the current work is the application of our previous work [Fraccaroli et al. 2022], where we improved and extended the method of phase plane projections, previously studied in [Muscalu et al. 2002a] in dimension 1, to higher dimensions. In order to apply the set-up introduced in [Fraccaroli et al. 2022], we have to reformulate the standard phase space decomposition of the form Λ_m in a new way. Unlike the existing literature using either stopping times and outer measures, see [Do and Thiele 2015], or a tree-selection algorithm with various size functionals acting on families of multitiles, see [Grafakos and Li 2004; Lacey and Thiele 1997; Thiele 2002], our proof arranges the tree-selection in a different way. In particular, unlike our main inspiration [Muscalu et al. 2002a], we put emphasis on choosing the top intervals and top frequencies and let them define regions in phase space, the trees. Each tree, a region in the phase space, is then divided into a boundary and a core. The treatise of these two parts can be separated into two independent modules. The estimation of the boundaries is completely independent of paraproduct theory of any kind, just invoking Hölder's inequality. The estimation of the cores in turn relies on two real analysis lemmas, one on paraproduct estimates and one on phase space localization, which are stand-alone results that do not make any explicit reference to the notion of a tree. Clarifying the roles of the core part and the boundary part of a tree is the main insight we are communicating. Later, at the level of tree selection, we further notice that almost all nontrivial phase space interaction of the selected trees is encoded in their boundary parts. Summing up, while the paraproduct theory of boundaries is very simple and that of cores more complicated, the orders of complexity are swapped when carrying out the tree selection.

We close the introduction commenting a bit more on the background context of the study uniform bounds for multilinear operators. On one hand, one may use uniform bounds over parametrized families of singular operators to conclude bounds for superpositions of these operators as the parameter varies. While integrable rather than uniform dependence on the parameter may suffice for this purpose in some applications, even integrable dependence may need more work than the basic nonuniform bounds. We refer to [Muscalu 2014a; 2014b; 2014c] for a discussion about connections to Calderón commutators and the Cauchy integral over Lipschitz curves as the original motivation for studying the bilinear Hilbert transforms. Secondly, multilinear forms whose multipliers are characteristic functions of convex sets E are closely related to uniform bounds for multipliers which are characteristic functions of half-planes relative to tangent lines of E. This connection appears in [Demeter and Gautam 2012; Grafakos and Li 2006; Li 2008; Lie 2015; Muscalu 2000; Saari and Thiele 2023].

Finally, we describe the structure of the present paper. Section 2 contains the outline of the proof of Theorem 1.1, which is organized into four propositions. These principal propositions are proved in Sections 3, 4, 5 and 6, one proposition in each section. Theorem 1.1 is deduced from the contents of the outline Section 2 in Section 7. Sections 3–6 are independent of each other and only make reference to Section 2. Section 7 depends on arguments in Sections 3–6 only through the propositions stated in Section 2. Section 5 is slightly longer than its siblings, and it is divided further into an outline part and five further numbered subsections, which only refer to Section 2 and the overview part of Section 5.

2. Outline of the proof

We fix the dimension $d \geq 1$, dilation parameters $k_2 > k_1 > k_0 \geq 3$ with $k_i - k_j > 100d$ for $0 \leq j < i \leq 2$, and the triple of exponents (p_1, p_2, p_3) satisfying

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1, \quad 2 < p_1, p_2, p_3 < \infty.$$

Let

$$\varepsilon = \min\{p_1 - 2, p_2 - 2, p_3 - 2\}.$$

In addition, we fix a number $\alpha > 2d$, $\alpha < 8d$. We further fix linear maps L_1, L_2 and L_3 as in Theorem 1.1. For $n \in \{1, 2, 3\}$, we choose $v_n \in \mathbb{Z}$ such that

$$2^{v_n - 1} < \|L_n\|_{\mathrm{op}} \leq 2^{v_n}.$$

Fix an index $n_* \in \{1, 2, 3\}$ such that

$$v_{n_*} = \min\{v_1, v_2, v_3\}. \tag{2-1}$$

As the condition (1-1) is invariant under scaling $\xi \mapsto \lambda\xi$, we may assume that $v_{n_*} = 0$.

Denote by $B(x, r)$ the open ball centered at $x \in \mathbb{R}^d$ and with radius r. For $\xi \in \mathbb{R}^{3\times d}$, $r > 0$, and $n \in \{1, 2, 3\}$, define $Q_n(\xi, r) \subset \mathbb{R}^d$ to be the minimal open rectangular box with sides parallel to the coordinate axes containing $B(\xi_n, 2^{v_n} r)$. Let

$$Q(\xi, r) = Q_1(\xi, r) \times Q_2(\xi, r) \times Q_3(\xi, r).$$

Let

$$\Gamma = \{(L_1\tau, L_2\tau, L_3\tau) : \tau \in \mathbb{R}^d\}.$$

Let $\mathcal{W}$ be a maximal set of pairwise disjoint rectangles of the form $Q(\xi, 2^{-j})$ with $\xi \in \mathbb{R}^{3\times d}$ and $j \in \mathbb{Z}$ with

$$\Gamma \cap Q(\xi, 2^{k_0-j}) = \varnothing$$

and

$$\Gamma \cap Q(\xi, 2^{k_0+1-j}) \neq \varnothing.$$

For all $N > 0$, let $\mathcal{W}_N$ be the finite subset of $\mathcal{W}$ defined by

$$\mathcal{W}_N = \{Q(\xi, 2^{-j}) \in \mathcal{W} : |\xi|, |j| \leq N\}.$$

For a cube with sides parallel to the coordinate axes $I \subset \mathbb{R}^d$, define the mollified distance ρ_I by

$$\rho_I(x) = \inf\{r > 1 : x \in (2r-1)I\},$$

where aI denotes the cube with the same center as I and a times the side-length. Moreover, for a Borel set $F \subset \mathbb{R}^d$, define

$$\rho_I(F) = \inf\{\rho_I(x) : x \in F\}.$$

Definition 2.1 (frequency cut-offs). Let $E \subset \mathbb{R}^{3\times d}$ be bounded with open interior. Define $\Phi_n^\alpha(E)$ to be the set of continuous complex-valued functions ϕ on $\mathbb{R}^d$ with

$$|\phi(x)| \leq 2^{(v_n-j)d}\rho_{[0,2^{j-v_n})^d}^{-\alpha}(x)$$

for all $x \in \mathbb{R}^d$ and

$$\operatorname{supp}\hat{\phi} \subset \{\xi_n : \xi \in E\},$$

where $j \in \mathbb{Z}$ is maximal such that there exists $\xi \in \mathbb{R}^{3\times d}$ with $E \subset Q(\xi, 2^{-j})$.

In Section 7, Theorem 1.1 is reduced to Proposition 2.2 below, where the multiplier is replaced by a sum of tensor multipliers.

Proposition 2.2 (weak estimate for tensor model). *Let*

$$\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = 1, \quad 2 < q_1, q_2, q_3 < \infty.$$

There exists a constant $C = C(d, \alpha, k_0, q_1, q_2, q_3)$ such that the following holds.

For $Q \in \mathcal{W}$ and $n \in \{1, 2, 3\}$, let $\phi_{Q,n} \in \Phi_n^{4\alpha}(Q)$. For each $n \in \{1, 2, 3\}$, let $f_n \in L^2(\mathbb{R}^d)$ be a function such that

$$\|f_n\|_\infty \leq 2.$$

Then, for all $N > 0$,

$$\Big|\sum_{Q\in\mathcal{W}_N}\int_{\mathbb{R}^d}\prod_{n=1}^{3}[\phi_{Q,n} * f_n(x)]\,\mathrm{d}x\Big| \leq C\prod_{n=1}^{3}\|f_n\|_2^{2/q_n}. \tag{2-2}$$

The proof of Proposition 2.2 can be found in Section 6. It requires several intermediate results, which we state next. The following frequency-localized version of Proposition 2.2 will play a role inside the proof. While the singularity of the bilinear multiplier in Proposition 2.2 can still be truly d-dimensional, Proposition 2.3 only deals with a point singularity in the spirit of more classical Coifman–Meyer multilinear multipliers. Proposition 2.3 will be proven in Section 3.

Proposition 2.3 (frequency-localized estimate). *Let k be a positive integer and*

$$\frac{1}{q_1}+\frac{1}{q_2}+\frac{1}{q_3}=1, \quad 2<q_1,q_2,q_3<\infty.$$

There exists a constant $C=C(d,\alpha,k_0,k,q_1,q_2,q_3)$ such that the following holds.

Let $\eta\in\Gamma$. For $Q\in\mathcal{W}$ and $n\in\{1,2,3\}$, let $\phi_{Q,n}\in\Phi_n^{4\alpha}(Q)$. For each $n\in\{1,2,3\}$, let $f_n\in L^{q_n}(\mathbb{R}^d)$. Then, for all $N>0$,

$$\Bigg|\sum_{\substack{Q\in\mathcal{W}_N\\ \eta\in 2^kQ}}\int_{\mathbb{R}^d}\prod_{n=1}^{3}[\phi_{Q,n}*f_n(x)]\,\mathrm{d}x\Bigg|\le C\prod_{n=1}^{3}\|f_n\|_{q_n}.$$

The reduction of Proposition 2.2 to Proposition 2.3 features a stopping-time argument, which introduces spatial truncations in addition to the mere frequency localization discussed so far and utilizes the notion of trees defined below.

For $k\in\mathbb{Z}$, let $\mathcal{D}_k=\{2^k([0,1)^d+l):l\in\mathbb{Z}^d\}$ and $\mathcal{D}=\bigcup_{k\in\mathbb{Z}}\mathcal{D}_k$. An element of $\mathcal{D}$ is called a dyadic cube.

Definition 2.4 (multitile, n-tile). A product $I\times Q$ is called a multitile if $I\in\mathcal{D}$ and $Q\in\mathcal{W}$ and $|Q_{n_*}|^{-1}=|I|$. For a multitile $I\times Q$ and $n\in\{1,2,3\}$, we call the product $I\times Q_n$ an n-tile. If $P=I\times Q$ is a multitile, we write I_P for I and Q_P for Q.

Definition 2.5 (tree). Let $\mathcal{V}$ be a finite subset of multitiles, let $\xi\in\Gamma$, and let $I_0\in\mathcal{D}$. Assume there exists at least one $P\in\mathcal{V}$ with $I_P=I_0$ and $\xi\in 2^{k_2+1}Q_P$. Then the triple $(\xi,I_0,\mathcal{V})$ defines a tree T. We write ξ_T for ξ, I_T for I_0, $\mathcal{V}_T$ for $\mathcal{V}$, and j_T for the top scale $\log_2|I_0|^{1/d}$. Attached to the tree T are the following objects:

- The family $\mathcal{P}_T$ of multitiles in $\mathcal{V}$ with $I_P\subset I_T$ and
$$\xi_T\in 2^{k_2+1}Q_P.$$
- The family $\mathcal{B}_T$ of multitiles $P\in\mathcal{P}_T$ with
$$\xi_T\in 2^{k_2+1}Q_P\setminus 2^{k_1+1}Q_P.$$
- The family $\mathcal{I}_T$ of dyadic cubes $I\in\mathcal{D}$ such that there exist P and P' in $\mathcal{P}_T\setminus\mathcal{B}_T$ with
$$I_P\subset I\subset I_{P'}.$$

The following definition gives a gauge to the size of a function near a tree.

Definition 2.6 (main sizes). Let $1 \le p \le \infty$, $n \in \{1, 2, 3\}$, and $f \in L^p(\mathbb{R}^d)$. Let T be a tree. We define

$$\Sigma^{\mathrm{bdr}}_{n,p,f}(T) = \sup_{P \in \mathcal{B}_T} \sup_{\phi \in \Phi^{4\alpha}_n(Q_P)} \frac{\|\rho_{I_P}^{-\alpha}[\phi * f]\|_p}{|I_P|^{1/p}},$$

$$\Sigma^{\mathrm{sum}}_{n,f}(T) = \left(\frac{1}{|I_T|} \sum_{P \in \mathcal{B}_T} \sup_{\phi \in \Phi^{4\alpha}_n(Q_P)} \|1_{I_P}[\phi * f]\|_2^2 \right)^{1/2},$$

$$\Sigma^{\mathrm{cor}}_{n,p,f}(T) = \sup_{i \in \mathbb{Z}} \sup_{I \in \mathcal{D}_i \cap \mathcal{I}_T} \sup_{\phi \in \Phi^{4\alpha}_n(Q_{T,i})} \frac{\|\rho_I^{-\alpha}[\phi * f]\|_p}{|I|^{1/p}},$$

where $Q_{T,i} = Q(\xi_T, 2^{k_1+5d-i})$ and $1/\infty$ is understood to be 0.

Heuristically, the core size is large enough to control a phase space paraproduct, but it is slightly too imprecise in terms of phase space localization. In order to maintain the information about frequency localization of a tree, the frequencies seen as peripheral with respect to the top frequency must be measured with a different kind of size, the sum size. The pair of sum size and core size are together strong enough to control the paraproduct and maintain the phase space localization, but in order to sum together the trees of different amplitudes, this couple still fails by a logarithmic blowup. To adjust this last piece, a multiplicative fraction of the sum size is replaced by the boundary size, which is a sup size again, but of nature lacunary with respect to the top frequency. After this last adjustment, the triple of sizes succeeds in the task of controlling the paraproduct, maintaining phase space localization and recovering summability over amplitudes. In the following proposition, we control the phase space paraproduct by the sizes. The proof can be found in Section 4.

Proposition 2.7 (phase space-localized estimate). *Let*

$$\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = 1, \quad 2 < q_1, q_2, q_3 < \infty.$$

There exists a constant $C = C(d, \alpha, k_0, k, q_1, q_2, q_3)$ such that the following holds.

Let T be a tree. For each $P \in \mathcal{P}_T$ and $n \in \{1, 2, 3\}$, let $\phi_{P,n} \in \Phi^{4\alpha}_n(Q_P)$. Then, for any $n' \in \{1, 2, 3\}$,

$$\left| \sum_{P \in \mathcal{B}_T} \int_{\mathbb{R}^d} 1_{I_P}(x) \prod_{n=1}^{3} [\phi_{P,n} * f_n(x)] \, \mathrm{d}x \right| \le C|I_T| \Sigma^{\mathrm{bdr}}_{n',\infty,f_{n'}}(T) \prod_{n \ne n'} \Sigma^{\mathrm{sum}}_{n,f_n}(T), \tag{2-3}$$

$$\left| \sum_{P \in \mathcal{P}_T \setminus \mathcal{B}_T} \int_{\mathbb{R}^d} 1_{I_P}(x) \prod_{n=1}^{3} [\phi_{P,n} * f_n(x)] \, \mathrm{d}x \right| \le C|I_T| \prod_{n=1}^{3} \Sigma^{\mathrm{cor}}_{n,q_n,f}(T). \tag{2-4}$$

The remaining ingredient of the proof of Proposition 2.2 is a partition of the set of all multitiles into trees, to which Proposition 2.7 can be applied. This last proposition will be proved in Section 5.

Proposition 2.8 (decomposition of the phase space). *There exists a constant $C = C(d, \alpha, k_0, k_1, k_2)$ such that the following holds.*

Let $N, N' > 0$. Let $\mathcal{V}$ be the finite subset of multitiles defined by

$$\mathcal{V} = \{P : Q_P \in \mathcal{W}_N,\ I_P \subset [-N'2^N, N'2^N]^{3\times d}\},$$

with $\mathcal{W}$ as in Proposition 2.2. For each $M \in \mathbb{Z} \cup \{-\infty\}$, there exists a family of trees $\mathcal{T}_M$ such that

$$\mathcal{V} = \bigcup_{M \in \mathbb{Z} \cup \{-\infty\}} \bigcup_{T \in \mathcal{T}_M} \mathcal{P}_T,$$

and the following hold for each $n \in \{1, 2, 3\}$:

- *For each tree $T \in \mathcal{T}_M$ for which there exists $P \in \mathcal{P}_T$ with $2^{k_1+1} Q_P \ni \xi_T$, we have*

$$\Sigma^{\mathrm{cor}}_{n,2,f_n}(T) \le 2^{M/2} \| f_n \|_2.$$

- *For every tree $T \in \mathcal{T}_M$, we have*

$$\Sigma^{\mathrm{bdr}}_{n,2,f_n}(T) + \Sigma^{\mathrm{sum}}_{n,f_n}(T) \le 2^{M/2} \| f_n \|_2.$$

- *For every tree T with $\mathcal{V}_T \subset \mathcal{V}$, we have*

$$\Sigma^{\mathrm{cor}}_{n,2,f_n}(T) + \Sigma^{\mathrm{bdr}}_{n,2,f_n}(T) + \Sigma^{\mathrm{sum}}_{n,f_n}(T) \le C \| f_n \|_\infty. \tag{2-5}$$

- *We have*

$$\sum_{T \in \mathcal{T}_M} 2^M |I_T| \le C. \tag{2-6}$$

Complementary notation. We conclude the section outlining the proof by listing some notational conventions that we intentionally omitted when describing the strategy of the proof but which will be helpful for understanding the proofs. In what follows, a constant C will depend on d, α, ε, k_0, k_1, and k_2. The exact dependence will be implicit in our arguments. We occasionally use the shorthand notation $A \lesssim B$ when $A \le CB$ for such a constant C.

Concerning the frequency cut-offs, see Definition 2.1, we use the following shorthand notations:

- Given $\xi \in \mathbb{R}^{3\times d}$ and $j \in \mathbb{Z}$, we define

$$\Phi^\alpha_{n,j}(\xi) = \Phi^\alpha_n(Q(\xi, 2^{-j})).$$

- Given $\xi \in \mathbb{R}^{3\times d}$ and $j \in \mathbb{Z}$, we define

$$\Psi^\alpha_{n,j}(\xi) = \Phi^\alpha_n(Q(\xi, 2^{-j}) \setminus Q(\xi, 2^{-j-2})).$$

- Given $\xi \in \mathbb{R}^{3\times d}$, we denote by $M_n(\xi, E)$ the set of ϕ such that

$$\sup_{\tau \in \mathbb{R}^d \setminus \{\xi_n\}} |(\tau - \xi_n)^\beta \partial^\beta \hat{\phi}(\tau)| \le 2^{-\upsilon_n |\beta|}, \quad \operatorname{supp} \hat{\phi} \subset E$$

  for all $\beta \in \mathbb{N}^d$ with $|\beta| \le 100d$. We call such a ϕ a normalized n-Mikhlin cut-off to E at ξ.

3. Proof of Proposition 2.3: paraproduct

Let η and $\phi_{Q,n}$ be given as in Proposition 2.3. By a translation on the Fourier transform side we may assume $\eta = 0$. By definition of $\mathcal{W}$, for each $Q \in \mathcal{W}$ we have $0 \notin 2Q$. Hence there exists $n \in \{1, 2, 3\}$ such that $0 \notin 2Q_n$. By splitting into three cases and estimating (2-2) in each case separately, we may assume without loss of generality that $0 \notin 2Q_1$ for all $Q \in \mathcal{W}$. Further, for each $j \in \mathbb{Z}$, there exists at

most $C(d,k)$ distinct elements $Q \in \mathcal{W}$ with $2^k Q \ni 0$ and $|Q_{n_*}| = 2^{-jd}$. By splitting into $C(d,k)$ further subcases, we may assume there exists at most one such Q. Even further, for each Q with $|Q_{n_*}| = 2^{-jd}$, there exist $C(d,k)$, $\widetilde{C}(d,k)$, and $\{c_{j',n} : |c_{j',n}| \le \widetilde{C}(d,k), |j'-j| \le C(d,k)\}$ such that

$$\phi_{Q,n} = \sum_{j':|j'-j|\le C(d,k)} c_{j',n}\phi_{j',n}, \quad \phi_{j',n} \in \Phi^{4\alpha}_{n,j'}(0).$$

Hence we may further reduce the study to the case where $\phi_{Q,n}$ is replaced by $\phi_{j,n}$ as above and v_n is replaced by v_n' with $|v_n - v_n'| \le C(d,k)$. Hence we aim at bounding

$$\Big|\sum_{j\in\mathcal{N}} \int_{\mathbb{R}^d} c_j \prod_{n=1}^{3} [\phi_{j,n} * f_n(x)]\,\mathrm{d}x\Big|,$$

where $\mathcal{N} \subset \mathbb{Z}$ is finite, $\phi_{j,1} \in \Psi^{4\alpha}_{1,j}(0)$, and $\phi_{j,n} \in \Phi^{4\alpha}_{n,j}(0)$ for $n \in \{2,3\}$.

Let χ be a Schwartz function on $\mathbb{R}^d$ such that $\hat{\chi}(\tau) = 0$ for $|\tau| \ge 2$ and $\hat{\chi}(\tau) = 1$ for $|\tau| \le 1$. Define, for $l \in \mathbb{Z}$,

$$\chi_l(x) = 2^{-ld}\chi(2^{-l}x),$$

and, for each $j \in \mathcal{N}$ and $n \in \{2,3\}$, define

$$\hat{\rho}_{j,n} = \hat{\phi}_{j,n} - \hat{\phi}_{j,n}(0)\hat{\chi}_{j-v_n}.$$

By the triangle inequality, it suffices to prove, for any collection

$$\{c_j : |c_j| \le 1,\ j \in \mathcal{N}\},$$

bounds for the tree expressions

$$\mathrm{I} = \Big|\sum_{j\in\mathcal{N}} \int_{\mathbb{R}^d} [c_j\phi_{j,1} * f_1(x)][\rho_{j,2} * f_2(x)][\phi_{j,3} * f_3(x)]\,\mathrm{d}x\Big|, \tag{3-1}$$

$$\mathrm{II} = \Big|\sum_{j\in\mathcal{N}} \int_{\mathbb{R}^d} [c_j\phi_{j,1} * f_1(x)][\chi_{j-v_2} * f_2(x)][\rho_{j,3} * f_3(x)]\,\mathrm{d}x\Big|, \tag{3-2}$$

$$\mathrm{III} = \Big|\sum_{j\in\mathcal{N}} \int_{\mathbb{R}^d} [c_j\phi_{j,1} * f_1(x)][\chi_{j-v_2} * f_2(x)][\chi_{j-v_3} * f_3(x)]\,\mathrm{d}x\Big| \tag{3-3}$$

separately.

We begin with (3-1). We estimate it with Cauchy–Schwartz in $\mathcal{N}$ and Hölder in $\mathbb{R}^d$ by

$$\Big\|\Big(\sum_{j\in\mathcal{N}} |\phi_{j,1} * f_1|^2\Big)^{1/2}\Big\|_{q_1} \Big\|\Big(\sum_{j\in\mathcal{N}} |\rho_{j,2} * f_2|^2\Big)^{1/2}\Big\|_{q_2} \Big\|\sup_{j\in\mathcal{N}} |\phi_{j,3} * f_3|\Big\|_{q_3}.$$

The term (3-2) is estimated similarly by

$$\Big\|\Big(\sum_{j\in\mathcal{N}} |\phi_{j,1} * f_1|^2\Big)^{1/2}\Big\|_{q_1} \Big\|\sup_{j\in\mathcal{N}} |\chi_{j-v_2} * f_2|\Big\|_{q_2} \Big\|\Big(\sum_{j\in\mathcal{N}} |\rho_{j,3} * f_3|^2\Big)^{1/2}\Big\|_{q_3}.$$

In both cases, we can apply the standard square function estimate (see Theorem 5.1.2 in [Grafakos 2008]) and maximal function estimates to obtain the desired bound. This completes the proof for I and II.

It remains to estimate (3-3). We telescope χ_{j-v_2} and χ_{j-v_3} into functions $\psi_l := \chi_{l-1} - \chi_l$ and thus write

$$\mathrm{III} \lesssim \sum_{m_1=-v_1-10}^{-v_1+10} \Bigg| \sum_{\substack{m_2\geq -v_2\\ m_3\geq -v_3}} \sum_{j\in\mathcal{N}} \int_{\mathbb{R}^d} [\phi_{m_1+j} * f_1(x)] \prod_{n=2}^{3} [\psi_{m_n+j} * f_n(x)]\,\mathrm{d}x \Bigg|, \tag{3-4}$$

where $\phi_{m_1+j} = \phi_{j,1} * \psi_{m_1+j}$.

We fix a triple $(\kappa_1, \kappa_2, \kappa_3) \in \mathbb{Z}^3$ and restrict the sums to $m_n \in \kappa_n + 1000d\mathbb{Z}$ for $n \in \{2, 3\}$ and $j \in \widetilde{\mathcal{N}} = \kappa_1 + 1000d\mathbb{Z}$. By the triangle inequality and summation over the $(1000d)^3$ values of $(\kappa_1, \kappa_2, \kappa_3)$, it suffices to bound the restricted sum. Consider then a fixed term in the sum (3-4). Such a term is nonzero only if

$$0 \in (\operatorname{supp} \hat{\phi}_{m_1+j} + \operatorname{supp} \hat{\psi}_{m_2+j} + \operatorname{supp} \hat{\psi}_{m_3+j}).$$

Recalling that we work with indices modulo $1000d$, this happens only if two of the numbers in $\{m_1, m_2, m_3\}$ are equal and the remaining one is larger.

We first assume $m_1 = m_n \leq m_{n'}$ for fixed $n, n' \in \{2, 3\}$. Then, for $\mathfrak{m} = \max(m_1, -v_{n'})$, we bound (3-4) by

$$\begin{aligned}
&\Bigg| \sum_{m_{n'}\geq \mathfrak{m}} \sum_{j\in\widetilde{\mathcal{N}}} \int_{\mathbb{R}^d} [\phi_{m_1+j} * f_1(x)][\psi_{m_1+j} * f_n(x)][\psi_{m_{n'}+j} * f_{n'}(x)]\,\mathrm{d}x \Bigg| \\
&\qquad \leq \int_{\mathbb{R}^d} \sum_{j\in\widetilde{\mathcal{N}}} |\phi_{m_1+j} * f_1(x)|\,|\psi_{m_1+j} * f_n(x)| \Bigg| \sum_{m_{n'}\geq \mathfrak{m}} \psi_{m_{n'}+j} * f_{n'}(x) \Bigg| \,\mathrm{d}x \\
&\qquad \leq \Bigg\| \Bigg(\sum_{j\in\widetilde{\mathcal{N}}} |\phi_{m_1+j} * f_1|^2 \Bigg)^{1/2} \Bigg\|_{q_1} \Bigg\| \Bigg(\sum_{j\in\widetilde{\mathcal{N}}} |\psi_{m_1+j} * f_n|^2 \Bigg)^{1/2} \Bigg\|_{q_n} \Bigg\| \sup_{j\in\widetilde{\mathcal{N}}} \Bigg| \sum_{m_{n'}\geq \mathfrak{m}+j} \psi_{m_{n'}} * f_{n'} \Bigg| \Bigg\|_{q_{n'}}.
\end{aligned}$$

These factors are bounded by the square function estimate and maximally truncated singular integral estimate, which completes the proof in this case.

Assume then that $m_2 = m_3 \leq m_1$. Now, for $\mathfrak{m} = \max(-v_2, -v_3)$, we bound (3-4) by

$$\begin{aligned}
&\Bigg| \sum_{j\in\widetilde{\mathcal{N}}} \int_{\mathbb{R}^d} [\phi_{m_1+j} * f_1(x)] \sum_{k=j+\mathfrak{m}}^{j+m_1} [\psi_k * f_2(x)][\psi_k * f_3(x)]\,\mathrm{d}x \Bigg| \\
&\qquad = \Bigg| \int_{\mathbb{R}^d} \sum_{k\in\mathbb{Z}} [\psi_k * f_2(x)][\psi_k * f_3(x)] \sum_{j\in\widetilde{\mathcal{N}}\cap\{k-m_1,\dots,k-\mathfrak{m}\}} [\phi_{m_1+j} * f_1(x)] \Bigg| \,\mathrm{d}x \\
&\qquad \leq \Bigg\| \Bigg(\sum_{k\in\mathbb{Z}} |\psi_k * f_2|^2 \Bigg)^{1/2} \Bigg\|_{q_2} \Bigg\| \Bigg(\sum_{k\in\mathbb{Z}} |\psi_k * f_3|^2 \Bigg)^{1/2} \Bigg\|_{q_3} \Bigg\| \sup_{k\in\mathbb{Z}} \Bigg| \sum_{j\in\widetilde{\mathcal{N}}\cap\{k-m_1,\dots,k-\mathfrak{m}\}} \phi_{m_1+j} * f_1 \Bigg| \Bigg\|_{q_1}.
\end{aligned}$$

Again, the bound follows by the square function estimate and maximally truncated singular integral estimate, and the proof is complete. □

4. Proof of Proposition 2.7: tree estimate

Boundary part. Given any family of multitiles $\mathcal{F} \subset \mathcal{P}_T$, we define

$$\Lambda_{\mathcal{F}}(f_1, f_2, f_3) = \sum_{P \in \mathcal{F}} \int_{\mathbb{R}^d} 1_{I_P}(x) \prod_{n=1}^{3} [\phi_{P,n} * f_n(x)] \, \mathrm{d}x.$$

We start with the easier bound (2-3).

Proposition 4.1. *There exists a constant C such that, for any $n' \in \{1, 2, 3\}$,*

$$|\Lambda_{\mathcal{B}_T}(f_1, f_2, f_3)| \le C |I_T| \Sigma^{\mathrm{bdr}}_{n,\infty,f_{n'}}(T) \prod_{n \ne n'} \Sigma^{\mathrm{sum}}_{n,f_n}(T).$$

Proof. By Hölder's inequality in $\mathbb{R}^d$ and the Cauchy–Schwartz inequality in $\mathcal{B}_T$,

$$|\Lambda_{\mathcal{B}_T}(f_1, f_2, f_3)| \le \sup_{P \in \mathcal{B}_T} \|1_P[\phi_{P,n'} * f_{n'}]\|_\infty \prod_{n \ne n'} \Big(\sum_{P \in \mathcal{B}_T} \|1_P[\phi_{P,n} * f_n]\|_2^2 \Big)^{1/2}. \qquad \square$$

We turn to estimating the form $\Lambda_{\mathcal{B}_T \setminus \mathcal{P}_T}$, which is the main source of difficulty in the proof. Here we will need several auxiliary tools, including Proposition 2.3 and some results from [Fraccaroli et al. 2022].

Phase space projections. Define, for $j \in \mathbb{Z}$,

$$\mathcal{I}_{T,j} := \{I_P : P \in \mathcal{P}_T \setminus \mathcal{B}_T\} \cap \mathcal{D}_j, \quad E_j^0 := \bigcup \mathcal{I}_{T,j}.$$

Define further, for each integer $k \ge 1$,

$$\mathcal{I}^k_{T,j} := \{I \in \mathcal{D}_j : \rho_I(E_j^0) \le k\}, \quad E_j^k := \bigcup \mathcal{I}^k_{T,j}.$$

Finally, for $\xi \in \mathbb{R}^{3\times d}$ and $n \in \{1, 2, 3\}$, we let $\mathrm{Mod}_{n,\xi}$ be the mapping such that

$$\mathrm{FT}(\mathrm{Mod}_{n,\xi} f)(\tau) = \hat{f}(\tau + \xi_n),$$

where FT is the Fourier transform. We define the phase space localization by using the construction from [Fraccaroli et al. 2022].

Definition 4.2 (phase plane projection). Let $v \ge 0$ be an integer, $n \in \{1, 2, 3\}$, and T be a tree. Let h be a Schwartz function. We define $\Pi_{T,n} h = \mathrm{Mod}_{n,-\xi}\, g$, where g is the output of Theorem 1.1 in [Fraccaroli et al. 2022] based on the input parameter $m = v_n$, input function $f = \mathrm{Mod}_{n,\xi}\, h$, input cube $U = I_T$, and the input M being the family of minimal cubes in $\bigcup_{j\in\mathbb{Z}} \mathcal{I}_{T,j}$.

By scaling, we can now quote the following result from [Fraccaroli et al. 2022].

Theorem 4.3 [Fraccaroli et al. 2022, Theorem 1.1]. *Let $1 \le p \le \infty$ and $1/p + 1/p' = 1$. Let $\alpha > d$ and $0 \le k \le k_1 + 4d$. There exists a constant $C = C(d, \alpha, p, k_0, k_1)$ such that the following holds.*

Let T be a tree and fix $n \in \{1, 2, 3\}$. Then, for every $j \le j_T$ and $J \in \mathcal{D}_j$,

$$\|\Pi_{T,n} f\|_p \le C \Sigma^{\mathrm{cor}}_{n,p,f}(T) |I_T|^{1/p} \tag{4-1}$$

and

$$\sum_{i\le j_T}\sum_{\substack{I\in\mathcal{I}_{T,i}\\ I\subset J}}\sup_{\phi\in\Phi^{4\alpha}_{n,i-k}(\xi)}|I|^{1/p'}\|\rho_I^{-3\alpha}[\phi*(f-\Pi_{T,n}f)]\|_p\le C\Sigma^{\mathrm{cor}}_{n,p,f}(T)|J|. \tag{4-2}$$

For every $j\le j_T$ and $J\in\mathcal{D}_j$ such that $I\not\subset 3J$ for any $I\in\mathcal{I}_{T,j}$,

$$\sum_{i\le j_T}\sup_{\substack{I\in\mathcal{D}_i\setminus\mathcal{I}_T\\ I\subset J}}\sup_{\psi\in\Psi^{4\alpha}_{n,i-k}(\xi)}|I|^{-1/p}\|\rho_I^{-3\alpha}[\psi*\Pi_{T,n}f]\|_p\le C\Sigma^{\mathrm{cor}}_{n,p,f}(T)\|1_{7I_T}\rho_J^{-\alpha}\|_\infty. \tag{4-3}$$

Proof of Proposition 2.7. It remains to prove

$$|\Lambda_{\mathcal{P}_T\setminus\mathcal{B}_T}(f_1,f_2,f_3)|\le C|I_T|\prod_{n=1}^{3}\Sigma^{\mathrm{cor}}_{n,q_n,f_n}(T),$$

as by Proposition 4.1 we already know (2-3) to hold.

Core part. By decomposing $\Lambda_{\mathcal{P}_T\setminus\mathcal{B}_T}$ into $C(d,k_0,k_1)$ many distinct sums, we can assume that, for each $j\in\mathbb{Z}$, there is at most one $Q\in\mathcal{W}$ such that $Q_P=Q$ and $|I_P|=2^{jd}$ for some $P\in\mathcal{P}_T\setminus\mathcal{B}_T$. We pick a sequence of functions

$$\phi_{j,n}\in\Phi^{4\alpha}_n(Q)$$

such that

$$\begin{aligned}|\Lambda_{\mathcal{P}_T\setminus\mathcal{B}_T}(f_1,f_2,f_3)|&\le\sum_{j\in\mathbb{Z}}\max_{P:Q_P=Q}\left|\int_{\mathbb{R}^d}1_{E^1_j}(x)\prod_{n=1}^{3}[\phi_{P,n}*f_n(x)]\,\mathrm{d}x\right|\\&\le C\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^d}1_{E^1_j}(x)\prod_{n=1}^{3}[\phi_{j,n}*f_n(x)]\,\mathrm{d}x.\end{aligned}$$

We define

$$\Lambda_{\mathcal{C}_T}(f_1,f_2,f_3):=\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^d}1_{E^1_j}(x)\prod_{n=1}^{3}[\phi_{j,n}*f_n(x)]\,\mathrm{d}x,$$

$$\Lambda_{\mathcal{C}_T,c}(f_1,f_2,f_3):=\sum_{\substack{j\in\mathbb{Z}\\ E^0_j\neq\varnothing}}\int_{\mathbb{R}^d}1_{(E^1_j)^c}(x)\prod_{n=1}^{3}[\phi_{j,n}*f_n(x)]\,\mathrm{d}x.$$

We compute

$$\begin{aligned}|\Lambda_{\mathcal{C}_T}(f_1,f_2,f_3)|&\le|\Lambda_{\mathcal{C}_T,c}(\Pi_{T,1}f_1,\Pi_{T,2}f_2,\Pi_{T,3}f_3)|\\&\quad+|\Lambda_{\mathcal{C}_T}(\Pi_{T,1}f_1,\Pi_{T,2}f_2,\Pi_{T,3}f_3)+\Lambda_{\mathcal{C}_T,c}(\Pi_{T,1}f_1,\Pi_{T,2}f_2,\Pi_{T,3}f_3)|\\&\quad+|\Lambda_{\mathcal{C}_T}(f_1-\Pi_{T,1}f_1,\Pi_{T,2}f_2,\Pi_{T,3}f_3)|\\&\quad+|\Lambda_{\mathcal{C}_T}(f_1,f_2-\Pi_{T,2}f_2,\Pi_{T,3}f_3)|\\&\quad+|\Lambda_{\mathcal{C}_T}(f_1,f_2,f_3-\Pi_{T,3}f_3)|\\&=:\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V}.\end{aligned}\tag{4-4}$$

For clarity, we state three auxiliary facts before estimating the five terms above.

Lemma 4.4. *For each $j \in \mathbb{Z}$, there is $n_j \in \{1, 2, 3\}$ and coefficients $c_{i,j}$ and functions $\phi_{i,j} \in \Psi^{4\alpha}_{n,j+i}(\xi_T)$ such that*

$$\phi_{P,n_j} = \sum_{i=-k_1-3d}^{-k_0+1} c_{i,j}\phi_{i,j}, \qquad \sum_{i=-k_1-3d}^{-k_0+1} |c_{i,j}| \le C(d).$$

Proof. For each $j \in \mathbb{Z}$ and $P \in \mathcal{P}_T \setminus \mathcal{B}_T$ with $|I_P| = 2^{jd}$, we know that $\xi_T \notin 2^{k_0} Q_P$. Hence there exists at least one $n_j \in \{1, 2, 3\}$ such that $(\xi_T)_{n_j} \notin 2^{k_0} Q_{n_j}$. The claim follows from this. □

Lemma 4.5. *Let $\mathcal{A}$ be the set of dyadic cubes I maximal with $|I| \le |I_T|$ and $J \subset 3I$ for no $J \in \mathcal{I}_T$ with $|J| \le |I|$. Then*

$$\mathcal{A}_j = \{J \in \mathcal{A} : |J| \ge 2^{jd}\}$$

is a partition of $\mathbb{R}^d \setminus E^1_j$.

Proof. Disjointness follows from maximality. If $x \in \mathbb{R}^d \setminus \bigcup \mathcal{A}_j$, then $J \in \mathcal{D}_j$ with $x \in J$ satisfies $3J \supset I$ for some $I \in \mathcal{I}_T$ with $|I| \le |J|$. Then $\hat{I} \in \mathcal{D}_j$ with $\hat{I} \supset I$ satisfies $\hat{I} \in \mathcal{I}_{T,j}$ and $J \subset 3\hat{I}$. Hence $J \subset E^1_j$. The inclusion $\bigcup \mathcal{A}_j \subset \mathbb{R}^d \setminus E^1_j$ follows by definition. □

Lemma 4.6. *Let $j \in \mathbb{Z}$ and $J \in \mathcal{D}_j$ be such that $5J \supset I$ for some $I \in \mathcal{I}_{T,j}$. Then*

$$\|1_J[\phi_{j,n} * \Pi_{T,n} f_n]\|_{q_n} \le C|J|^{1/q_n} \Sigma^{\mathrm{cor}}_{n,q_n,f_n}(T).$$

Proof. This follows by applying (4-2) to J and restricting the sum on the left-hand side to a single term as

$$\begin{aligned}\|1_J[\phi_{j,n} * \Pi_{T,n} f_n]\|_{q_n} &\le \|1_J[\phi_{j,n} * (\Pi_{T,n} f_n - f_n)]\|_{q_n} + \|1_J[\phi_{j,n} * f_n]\|_{q_n} \\ &\le C|J|^{1/q_n} \Sigma^{\mathrm{cor}}_{n,q_n,f_n}(T).\end{aligned}$$ □

Now we can estimate the five terms in (4-4). To estimate I, we recall that, for each $j \in \mathbb{Z}$, there exists $n_j \in \{1, 2, 3\}$ as in Lemma 4.4. We fix n_j to be one of them so that the three sets

$$\mathcal{N}_n = \{j \in \mathbb{Z} : E^1_j \ne \varnothing,\ n_j = n\}$$

partition the subset of $\mathbb{Z}$ appearing in the definition of I. Then

$$\begin{aligned}\mathrm{I} = |\Lambda_{\mathcal{C}_T,c}(\Pi_{T,1} f_1, \Pi_{T,2} f_2, \Pi_{T,3} f_3)| &\le \sum_{\nu=1}^{3} \int_{\mathbb{R}^d} \sum_{j \in \mathcal{N}_\nu} 1_{(E^1_j)^c} \prod_{n=1}^{3} |\phi_{j,n} * \Pi_{T,n} f_n(x)|\, \mathrm{d}x \\ &\le \sum_{\nu=1}^{3} \Big(\prod_{n \ne \nu} \|M_{\mathrm{HL}} \Pi_{T,n} f_n\|_{q_n} \Big) \Big\| \sum_{j \in \mathcal{N}_\nu} 1_{(E^1_j)^c} |\phi_{j,\nu} * \Pi_{T,\nu} f_\nu| \Big\|_{q_\nu},\end{aligned}$$

where M_{HL} is the Hardy–Littlewood maximal function. By the maximal function theorem and (4-1) from Theorem 4.3,

$$\|M_{\mathrm{HL}} \Pi_{T,n} f_n\|_{q_n} \le C|I_T|^{1/q_n} \Sigma^{\mathrm{cor}}_{n,q_n,f_n}(T).$$

By Lemma 4.5 and Minkowski's inequality,

$$\Big\| \sum_{j\in\mathcal{N}_\nu} 1_{(E_j^1)^c} |\phi_{j,\nu} * \Pi_{T,\nu} f_\nu| \Big\|_{q_\nu}^{q_\nu} = \sum_{J\in A} \Big\| 1_J \sum_{j\in\mathcal{N}_\nu} \sum_{I\in\mathcal{D}_j\setminus\mathcal{I}_{T,j}^1} 1_I |\phi_{j,\nu} * \Pi_{T,\nu} f_\nu| \Big\|_{q_\nu}^{q_\nu}$$
$$\le \sum_{J\in A} \Big(\sum_{j\in\mathcal{N}_\nu} \Big(\sum_{\substack{I\in\mathcal{D}_j\setminus\mathcal{I}_{T,j}^1 \\ I\subset J}} \|1_I[\phi_{j,\nu} * \Pi_{T,\nu} f_\nu]\|_{q_\nu}^{q_\nu} \Big)^{1/q_\nu} \Big)^{q_\nu}.$$

By Lemma 4.4 and (4-3) from Theorem 4.3,

$$\sum_{j\in\mathcal{N}_\nu} \Big(\sum_{\substack{I\in\mathcal{D}_j\setminus\mathcal{I}_{T,j}^1 \\ I\subset J}} \|1_I[\phi_{j,\nu} * \Pi_{T,\nu} f_\nu]\|_{q_\nu}^{q_\nu} \Big)^{1/q_\nu} \le |J|^{1/q_n} \sum_{i=-k_1-3d}^{-k_0+1} \sum_{j\in\mathcal{N}_\nu} \sup_{\substack{I\in\mathcal{D}_j\setminus\mathcal{I}_{T,j}^1 \\ I\subset J}} \sup_{\psi\in\Psi_{\nu,j+i}^{4\alpha}(\xi_T)} \frac{\|1_I[\psi * \Pi_{T,\nu} f_\nu]\|_{q_\nu}}{|I|^{1/q_\nu}}$$
$$\le C|J|^{1/q_\nu} \Sigma_{\nu,q_\nu,f_\nu}^{\mathrm{cor}}(T) \|1_{7I_T} \rho_J^{-\alpha}\|_\infty.$$

Summing the q_ν-th power over J concludes the proof.

To estimate

$$\mathrm{II} = |\Lambda_{\mathcal{C}_T}(\Pi_{T,1} f_1, \Pi_{T,2} f_2, \Pi_{T,3} f_3) + \Lambda_{\mathcal{C}_T,c}(\Pi_{T,1} f_1, \Pi_{T,2} f_2, \Pi_{T,3} f_3)|,$$

it suffices to apply Proposition 2.3 (the global paraproduct estimate) and (4-1) in Theorem 4.3 (the L^p estimate for the phase space projection). The desired bound follows.

We move to estimate $\mathrm{III} + \mathrm{IV} + \mathrm{V}$. Note that, for $n \in \{1, 2, 3\}$ and $J \in \mathcal{I}_{T,j}^1$, by definition of $\Sigma_{n,p,f}^{\mathrm{cor}}(T)$,

$$\|1_J[\phi_{j,n} * f_n]\|_{q_n} \le |J|^{1/q_n} \Sigma_{n,q_n,f_n}^{\mathrm{cor}}(T),$$

and further, by Lemma 4.6,

$$\|1_J[\phi_{j,n} * \Pi_{T,n} f_n]\|_{q_n} \le C|J|^{1/q_n} \Sigma_{n,q_n,f_n}^{\mathrm{cor}}(T).$$

By these estimates and Hölder's inequality,

$$\mathrm{III} + \mathrm{IV} + \mathrm{V} \le C \max_{n\in\{1,2,3\}} \Big\{ \Big(\prod_{n'\neq n} \Sigma_{n',q_{n'},f_{n'}}^{\mathrm{cor}}(T) \Big) \sum_{j\in\mathbb{Z}} \sum_{J\in\mathcal{I}_{T,j}^1} |J|^{1-1/q_n} \|1_J[\phi_{j,n} * (f_n - \Pi_{T,n} f_n)]\|_{q_n} \Big\},$$

from which the claim follows by (4-2) of Theorem 4.3. □

5. Proof of Proposition 2.8: tree selection

We start by defining two auxiliary sizes that are needed to complement those in Definition 2.6.

Definition 5.1. Under the set-up of Definition 2.6, define

$$\Sigma_{n,p,f}^{\mathrm{bdr,top}}(T) = \sup_{\substack{P\in\mathcal{B}_T \\ I_P = I_T}} \sup_{\phi\in\Phi_n^{4\alpha}(Q_P)} \frac{\|\rho_{I_T}^{-\alpha}[\phi * f]\|_p}{|I_T|^{1/p}},$$

$$\Sigma_{n,p,f}^{\mathrm{cor,top}}(T) = \sup_{\phi\in\Phi_{n,j_T-k_1-5d}^{4\alpha}(\xi_T)} \frac{\|\rho_I^{-\alpha}[\phi * f]\|_p}{|I_T|^{1/p}}.$$

We formalize the idea of greedy selection by stating the following definition.

Definition 5.2 (selection). Let $\mathcal{V}$ be a finite set of multitiles. Let $\mathcal{T}$ be the family of all trees in any of the subsets of $\mathcal{V}$. Let S be a positive integer. A selection is a mapping $\sigma : \{1, \dots, S\} \to \mathcal{T}$ such that

- $\sigma(1)$ is a tree in $\mathcal{V}_{\sigma(1)} = \mathcal{V}$,
- $\sigma(i+1)$ is a tree in $\mathcal{V}_{\sigma(i+1)} = \mathcal{V}_{\sigma(i)} \setminus \mathcal{P}_{\sigma(i)}$ for all $i \in \{1, \dots, S-1\}$.

To prove Proposition 2.8, we will construct several selections over the initial set of multitiles. We first show that selections based on top size defined above have good orthogonality properties and as a second step we show that convexity properties allow us to infer estimates for main sizes of Definition 2.6 from those for the auxiliary top sizes of Definition 5.1. There will be three different selection processes. The first selection serves to identify the trees with large core size. The following proposition shows that they have controlled overlap.

Proposition 5.3. *There exists a constant C such that the following holds.*

Let $D > 1$. Let $f \in L^2(\mathbb{R}^d)$. Let $\mathcal{V}$ be a finite set of multitiles and let σ be a selection in $\mathcal{V}$. Let $M > 0$. Assume the following properties of the selection:

- *If I_i is the top cube of $\sigma(i)$ and if I_{i+1} is the top cube of $\sigma(i+1)$, then $|I_{i+1}| \le |I_i|$ for all $i \in \{1, \dots, S-1\}$.*
- *For each $i \in \{1, \dots, S\}$, there exists $A_i \in \mathcal{P}_{\sigma(i)}$ with $2^{k_1+1} Q_{A_i} \ni \xi_{\sigma(i)}$.*
- *For each $i \in \{1, \dots, S\}$, we have $M \le (\Sigma^{\mathrm{cor,top}}_{n,2,f} \circ \sigma)(i) \le DM$.*

Then

$$\left(\sum_{i=1}^{S} M^2 |I_{\sigma(i)}| \right)^{1/2} \le CD \|f\|_2.$$

The next selection serves to remove the trees that contain a lacunary multitile, not treated by the core size, that however happens to give a large contribution.

Proposition 5.4. *There exists a constant C such that the following holds.*

Let $D > 1$. Let $f \in L^2(\mathbb{R}^d)$. Let $\mathcal{V}$ be a finite set of multitiles and let σ be a selection in $\mathcal{V}$. Let $M > 0$. Assume the following properties of the selection:

- *If I_i is the top cube of $\sigma(i)$ and if I_{i+1} is the top cube of $\sigma(i+1)$, then $|I_{i+1}| \le |I_i|$ for all $i \in \{1, \dots, S-1\}$.*
- *For each $i \in \{1, \dots, S\}$, we have $M \le (\Sigma^{\mathrm{bdr,top}}_{n,2,f} \circ \sigma)(i) \le DM$.*

Then

$$\left(\sum_{i=1}^{S} M^2 |I_{\sigma(i)}| \right)^{1/2} \le CD \|f\|_2.$$

The third selection removes the trees whose boundaries are contributing a lot to the right-hand side of Proposition 2.7. While the choice order of the previous selections was based on metric geometry, only using the size of the top cube, the treatise of the lacunary parts of the trees requires us to carry out a cone decomposition and consider an order of selection based on that.

Proposition 5.5. *There exists a constant C such that the following holds.*

Let $D > 1$, and let e be a unit vector orthogonal to $d-1$ coordinate axes. Let $f \in L^2(\mathbb{R}^d)$. Let $\mathcal{V}$ be a finite set of multitiles, and let σ be a selection in $\mathcal{V}$. Let $M > 0$. For each $i \in \{1, \dots, S\}$, define

$$C_i = \{\xi \in \mathbb{R}^d : |\xi - \xi_{\sigma(i)}| \le 2(\xi - \xi_{\sigma(i)}) \cdot e\},$$

and let $\mu_i \in M_n(\xi_{\sigma(i)}, C_i)$. Assume the following properties of the selection:

- *For all $i \in \{1, \dots, S-1\}$, assume that $\xi_{\sigma(i)} \cdot e \ge \xi_{\sigma(i+1)} \cdot e$.*
- *For each $i \in \{1, \dots, S\}$, we have $M \le (\Sigma^{\mathrm{sum}}_{n,\mu_i * f} \circ \sigma)(i)$ and $(\Sigma^{\mathrm{bdr}}_{n,2,\mu_i * f} \circ \sigma)(i) \le DM$.*

Then

$$\left(\sum_{i=1}^{S} M^2 |I_{\sigma(i)}|\right)^{1/2} \le CD\|f\|_2.$$

To apply the propositions stated above, we still have to solve the discrepancy between the definitions of sizes in Definitions 2.6 and 5.1. This is the content of the last proposition of this section. We need one more definition.

Definition 5.6 (convex collection). A finite family of multitiles $\mathcal{V}$ is a convex collection if, for any tree T on $\mathcal{V}$ and

$$j_{min} = \min_{P \in \mathcal{P}_T} \log_2 |I_P|^{1/d},$$

the condition $j \in \mathbb{Z} \cap \{i : j_{min} \le i \le j_T\}$ implies that there exist $P \in \mathcal{P}_T$ with $|I_P| = 2^{jd}$ and the condition that $2^{k_1+1} Q_P \ni \xi_T$ for some $P \in \mathcal{P}_T$ implies $2^{k_1+1} Q_{P'}$ for a $P' \in \mathcal{P}_T$ with $I_{P'} = I_T$.

For the purpose of the proof of our main theorem, the convex collections are the only ones that matter. The importance of the convex collections lies in the fact that every tree on a convex collection has a subtree whose size is attained by one of its top multitiles.

Moreover, for a tree T, we set

$$\Theta(T) = \begin{cases} 1 & \text{if there exists } P \in \mathcal{P}_T \text{ with } 2^{k_1+1} Q_P \ni \xi_T. \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 5.7. *Let $\mathcal{V}$ be a convex family. Let $\{e_\delta : 1 \le \delta \le 2d\}$ be the unit vectors orthogonal to the $(d-1)$-dimensional coordinate hyperplanes. Let*

$$C_e = \{\xi \in \mathbb{R}^d : |\xi - \xi_{\sigma(i)}| \le 2(\xi - \xi_{\sigma(i)}) \cdot e\}.$$

Let $\mu^{\delta,n} \in M_n(C_{e_\delta})$ with

$$\sum_{\delta=1}^{2d} \hat{\mu}^{\delta,n}(\xi) = 1, \quad \xi \ne 0.$$

For a tree T on $\mathcal{V}$, we set

$$\hat{\mu}^{\delta}_{T,n}(\xi) = \hat{\mu}^{\delta,n}(\xi - (\xi_T)_n).$$

Let $M \in \mathbb{Z}$ be such that, for all trees T on $\mathcal{V}$,

$$\max_{\substack{1\le\delta\le 2d\\ n\in\{1,2,3\}}} \max\{\Sigma^{\mathrm{bdr}}_{n,2,f_n}(T),\ \Sigma^{\mathrm{sum}}_{n,\mu_T^{\delta,n}*f_n}(T),\ \Theta(T)\Sigma^{\mathrm{cor}}_{n,2,f_n}(T)\} \le 2^{M/2}\|f_n\|_2.$$

Then there exists a selection σ on $\mathcal{V}$ such that

$$\widetilde{\mathcal{V}}_{M-1} = \mathcal{V}\setminus\bigcup_{i=1}^{S}\mathcal{P}_{\sigma(i)}$$

is a convex family such that, for all trees on $\widetilde{\mathcal{V}}_{M-1}$,

$$\max_{\substack{1\le\delta\le 2d\\ n\in\{1,2,3\}}} \max\{\Sigma^{\mathrm{bdr}}_{n,2,f_n}(T),\ \Sigma^{\mathrm{sum}}_{n,\mu_T^{\delta,n}*f_n}(T),\ \Theta(T)\Sigma^{\mathrm{cor}}_{n,2,f_n}(T)\} \le 2^{(M-10d)/2}\|f_n\|_2 \tag{5-1}$$

and

$$\sum_{i=1}^{S} 2^M |I_{\sigma(i)}| \lesssim 1. \tag{5-2}$$

Auxiliary propositions for almost orthogonality. In this subsection, we prove two additional estimates that are needed in the proofs of Propositions 5.3, 5.4 and 5.5.

Proposition 5.8. *Let $\alpha > 2d$. There exists a constant C such that the following holds.*

Let $j \in \mathbb{Z}$, $k \ge 0$, $\xi \in \mathbb{R}^d$, and $f \in L^\infty(\mathbb{R}^d)$. Let $\varphi \in \Phi^{4\alpha}_{n,j-k}(\xi)$ and I be a cube with $|I| = 2^{jd}$. Denote by $\mathcal{M}_{\mathrm{HL}}$ the Hardy–Littlewood maximal function. Then, for all $x \in \mathbb{R}^d$,

$$|\varphi * (\rho_I^{-\alpha} f)(x)| \le C\rho_I(x)^{-\alpha}\mathcal{M}_{\mathrm{HL}} f(x).$$

Proof. As for $j' \le j$ we have $\rho_{[0,2^{j'})^d} \ge \rho_{[0,2^j)^d}$, then, for any $\varphi \in \Phi^{4\alpha}_{n,j-k}(\xi)$ and $x \in \mathbb{R}^d$, we have

$$|\rho^{\alpha}_{[0,2^{j-k})^d}(x)\varphi(x)| \le 2^{(j-k+v_n)d}\rho^{-3\alpha}_{[0,2^{j-k-v_n})^d}(x).$$

We also have

$$\rho^{-\alpha}_{[0,2^j)^d}(x-y)\rho_I^{-\alpha}(y) \le C\rho_I^{-\alpha}(x)$$

for all $x, y \in \mathbb{R}^d$. Indeed, if $2\rho_I(y) \ge \rho_I(x)$, this is clear, and if $2\rho_I(y) \le \rho_I(x)$, then

$$\rho_{[0,2^j)^d}(x-y) \ge \rho_I(x) - \rho_I(y) \ge \frac{\rho_I(x)}{2}.$$

In conclusion,

$$\begin{aligned}|\varphi * (\rho_I^{-\alpha} f)(x)| &= \left|\int_{\mathbb{R}^d} \rho^{\alpha}_{[0,2^j)^d}(x-y)\varphi(x-y)\rho^{-\alpha}_{[0,2^j)^d}(x-y)\rho_I^{-\alpha}(y) f(y)\,\mathrm{d}y\right| \\ &\le C\rho_I(x)^{-\alpha}[2^{(j-k+v_n)d}\rho^{-3\alpha}_{[0,2^{j-k-v_n})^d} * (\rho_I^{-\alpha}|f|)](x).\end{aligned}$$

□

The second auxiliary proposition is essentially a restatement of Lemmata 5.1, 5.2 and 5.3 in [Muscalu et al. 2002a]. Also this estimate is needed in the proofs of Propositions 5.3, 5.4 and 5.5.

Proposition 5.9. *Let A_1 be a positive constant, and let $\alpha > d$. Then there exists a constant A_2 such that the following holds.*

Let $J \in \mathcal{D}$. Let $\mathcal{I} \subset \mathcal{D}$ be a family of cubes satisfying

$$\sum_{\substack{I\in\mathcal{I}\\ I\subset I'}} |I| \le A_1 |I'|$$

for all cubes I' and $|I| \le |J|$ for all $I \in \mathcal{I}$. For each $I \in \mathcal{I}$, let $g_I \in L^2(\mathbb{R})$ be given. Then

$$\Big\| \rho_J^{-\alpha} \sum_{I\in\mathcal{I}} |I|^{1/2} g_I \rho_I^{-\alpha} \Big\|_2 \le A_2 |J|^{1/2} \sup_{I\in\mathcal{I}} \|g_I\|_2. \tag{5-3}$$

Proof. We first prove the reminiscent inequality

$$\Big\| \sum_{I\in\mathcal{I}} |I|^{1/2} g_I 1_{DI} \Big\|_2 \le 2D^d \sqrt{5A_1} \sup_{I\in\mathcal{I}} \|g_I\|_2 \Big(\sum_{I\in\mathcal{I}} |I| \Big)^{1/2} \tag{5-4}$$

for all odd numbers $D \ge 3$. Here the nonlocal cut-off functions are replaced by sharp cut-off functions.

Fix a family $\mathcal{I}$ and the corresponding functions g_I. Let $\mathcal{I}' \subset \mathcal{I}$ be finite. Let A be the sharp constant for the inequality (5-4) when considered over all finite subfamilies of $\mathcal{I}'$. Then

$$\begin{aligned}
\Big\| \sum_{I\in\mathcal{I}'} |I|^{1/2} g_I 1_{DI} \Big\|_2^2 &\le 2 \sum_{I\in\mathcal{I}'} \sum_{\substack{J\in\mathcal{I}'\\ DJ\subset 5DI}} \langle |I|^{1/2} g_I 1_{DI}, |J|^{1/2} g_J 1_{DJ} \rangle \le 2 \sum_{I\in\mathcal{I}'} |I|^{1/2} \|g_I\|_2 \Big\| \sum_{\substack{J\in\mathcal{I}'\\ DJ\subset 5DI}} |J|^{1/2} g_J 1_{DJ} \Big\|_2 \\
&\le 2AD^{d/2} \sup_{I\in\mathcal{I}'} \|g_I\|_2^2 \sum_{I\in\mathcal{I}'} |I|^{1/2} \Big(\sum_{\substack{J\in\mathcal{I}'\\ J\subset 5DI}} |J| \Big)^{1/2} \le 2\sqrt{5A_1} AD^d \sup_{I\in\mathcal{I}'} \|g_I\|_2^2 \Big(\sum_{I\in\mathcal{I}'} |I| \Big).
\end{aligned}$$

Consequently, $A \le 2D^d \sqrt{5A_1}$. As this constant is independent of $\mathcal{I}'$ and the functions g_I, the proof of (5-4) is complete.

To prove (5-3), we write

$$\rho_I^{-\alpha} \le \sum_{k=1}^{\infty} k^{-\alpha} 1_{(2k-1)I} \quad \text{and} \quad \rho_J^{-\alpha} \le 1_J + \sum_{l=1}^{\infty} l^{-\alpha} 1_{(2l+1)J\setminus(2l-1)J}.$$

Set $\mathcal{I}_{k,l} = \{I \in \mathcal{I} : (2k-1)I \cap (2l+1)J \ne \varnothing\}$. Then

$$\begin{aligned}
\Big\| \rho_J^{-\alpha} \sum_{I\in\mathcal{I}} |I|^{1/2} g_I \rho_I^{-\alpha} \Big\|_2 &\le \sum_{k,l=1}^{\infty} k^{-\alpha} l^{-\alpha} \Big\| 1_{(2l+1)J\setminus(2l-1)J} \sum_{I\in\mathcal{I}_{k,l}} |I|^{1/2} g_I 1_{(2k-1)I} \Big\|_2 \\
&\le \sum_{k,l=1}^{\infty} k^{-\alpha} l^{-\alpha} \Big\| 1_{(2\max\{l,k\}+2)J} \sum_{I\in\mathcal{I}_{k,l}} |I|^{1/2} g_I 1_{(2k-1)I} \Big\|_2 \\
&\lesssim \sum_{k,l=1}^{\infty} 2^{-d/2} k^{-\alpha-d/2} l^{-\alpha} |(2\max\{l,k\}+2)J|^{1/2} \lesssim |J|^{1/2}.
\end{aligned}$$

□

Finally, we state as a separate proposition the obvious fact that elements of $\mathcal{W}$ with overlapping projections are close to each other in the product space too, something that is a direct consequence of the defining inequalities of $\mathcal{W}$.

Proposition 5.10. *Let $a \geq 0$. Let $Q, Q' \in \mathcal{W}$ satisfy $|Q| \geq |Q'|$. Assume that, for some $n \in \{1, 2, 3\}$, there exist*

$$\xi \in 2^a Q_n \cap 2^a Q'_n, \quad \eta \in 2^a Q \cap \Gamma, \quad \zeta \in 2^a Q' \cap \Gamma.$$

Then $2^{a+4} Q \supset Q'$.

Proof. First we note that the projection $\mathfrak{P} : \Gamma \to \mathbb{R}^d$ defined through $\mathfrak{P}\xi = \xi_n$ is a bijection. This follows from the regularity of L and the fact $\Gamma = \{L(\tau, \tau, \tau) : \tau \in \mathbb{R}^d\}$. Consider the metrics

$$\operatorname{dist}_{\text{full}}(\xi, \eta) = \inf\{r : \eta \in Q(\xi, r)\},$$
$$\operatorname{dist}_n(\xi_n, \eta_n) = \inf\{r : \eta_n \in Q_n(\xi_n, r)\}.$$

The left inverse $\mathfrak{P}^{-1}$ is a 2-Lipschitz mapping $(\mathbb{R}^d, \operatorname{dist}_n) \to (\Gamma, \operatorname{dist}_{\text{full}})$ following directly from the definition of the metrics. We infer that

$$\operatorname{dist}_{\text{full}}(\mathfrak{P}^{-1}\xi, \mathfrak{P}^{-1}\mathfrak{P}\eta) \leq 2 \operatorname{dist}_n(\xi, \mathfrak{P}\eta),$$
$$\operatorname{dist}_{\text{full}}(\mathfrak{P}^{-1}\xi, \mathfrak{P}^{-1}\mathfrak{P}\zeta) \leq 2 \operatorname{dist}_n(\xi, \mathfrak{P}\zeta),$$

so that

$$\operatorname{dist}_{\text{full}}(\mathfrak{P}^{-1}\mathfrak{P}\zeta, \mathfrak{P}^{-1}\mathfrak{P}\eta) \leq 2^{a+2} \operatorname{diam} Q_n,$$

where the diameter diam is computed with respect to d_n. As $|Q| \geq |Q'|$, we conclude $2^{a+3} Q \cap Q' \neq \varnothing$, and the claim follows. □

As an immediate corollary of Proposition 5.10, we conclude that the multitiles $P \in \mathcal{B}_T$ have all their frequency projections supported far from the projections of the top frequency. This will imply important L^2 orthogonality properties for the sum size.

Proposition 5.11. *Given $P \in \mathcal{B}_T$ with $|I_P| = 2^{jd}$ for some $j \in \mathbb{Z}$, we have, for all $n \in \{1, 2, 3\}$,*

$$(Q_P)_n \subset \mathbb{R}^d \setminus Q_n(\xi_T, 2^{-j-k_2}).$$

Proof. Define $Q = Q_P$. By construction, $2^{k_0+1} Q \cap \Gamma \neq \varnothing$. On the other hand, as $P \in \mathcal{B}_T$, we know that $\xi_T \notin 2^{k_1+1} Q$. Set $Q' = Q(\xi_T, 2^{-j+k_1/50})$. Then $Q' \cap Q = \varnothing$. It follows by Proposition 5.10 that $Q'_n \cap Q_n = \varnothing$. □

5.1. *Proof of Proposition 5.3*: *core size.* For each $i \in \{1, \dots, S\}$, we find

$$\phi_i \in \Phi^{4\alpha}_{n, j_{\sigma(i)} - k_1 - 5d}(\xi_{\sigma(i)})$$

such that

$$c_i = \|\rho^{-\alpha}_{I_{\sigma(i)}}[\phi_i * f]\|_2, \quad M\sqrt{|I_{\sigma(i)}|} \leq c_i \leq DM\sqrt{|I_{\sigma(i)}|}.$$

Let

$$g_i = \frac{\rho^{-\alpha}_{I_{\sigma(i)}}[\phi_i * f]}{\|\rho^{-\alpha}_{I_{\sigma(i)}}[\phi_i * f]\|_2}.$$

Now

$$\sum_{i=1}^{S} M^2 |I_{\sigma(i)}| \lesssim \sum_{i=1}^{S} c_i |I_{\sigma(i)}|^{1/2} \langle f, \bar{\phi}_i * (\rho_{I_{\sigma(i)}}^{-\alpha} g_i)\rangle \le \|f\|_2 \Big\| \sum_{i=1}^{S} c_i |I_{\sigma(i)}|^{1/2} [\phi_i * (\rho_{I_{\sigma(i)}}^{-\alpha} g_i)] \Big\|_2 .$$

Expanding the square and using the symmetry, we obtain

$$\Big\| \sum_{i=1}^{S} |I_{\sigma(i)}|^{1/2} c_i [\phi_i * (\rho_{I_T}^{-\alpha} g_i)] \Big\|_2^2 \lesssim D^2 M^2 \sum_{i=1}^{S} |I_{\sigma(i)}|^{1/2} \sum_{l=i}^{S} |I_{\sigma(l)}|^{1/2} \langle \rho_{I_{\sigma(l)}}^{-\alpha} g_l, \phi_l * [\phi_i * (\rho_{I_{\sigma(i)}}^{-\alpha} g_i)]\rangle. \quad (5\text{-}5)$$

Let

$$\mathcal{A}_i = \{l \in \{i, \dots, S\} : \operatorname{supp} \hat{\phi}_l \cap \operatorname{supp} \hat{\phi}_i \neq \varnothing\}.$$

By Proposition 5.8, for all $l \in \mathcal{A}_i$,

$$\phi_l * [\phi_i * (\rho_{I_{\sigma(i)}}^{-\alpha} g_i)] \le C \rho_{I_{\sigma(i)}}^{-\alpha} \mathcal{M}_{\mathrm{HL}} g_i,$$

where $\mathcal{M}_{\mathrm{HL}}$ is the Hardy–Littlewood maximal function. Using this estimate, the Cauchy–Schwarz inequality and the Hardy–Littlewood maximal function theorem, we bound the right-hand side of (5-5) by

$$C D^2 M^2 \sum_{i=1}^{S} |I_{\sigma(i)}| \Big\| \frac{1}{|I_{\sigma(i)}|^{1/2}} \sum_{l \in \mathcal{A}_i} |I_{\sigma(l)}|^{1/2} g_l \rho_{I_{\sigma(l)}}^{-\alpha} \rho_{I_{\sigma(i)}}^{-\alpha} \Big\|_2 .$$

By hypothesis, for each $l \in \{1, \dots, S\}$, there exists a top multitile $A_l \in \mathcal{P}_{\sigma(l)}$ with $\xi_{\sigma(l)} \in 2^{k_1+1} Q_{A_l}$. Hence, given $l, j \in \mathcal{A}_i$ with $l > j$, we have

$$(2 \operatorname{supp} \hat{\phi}_l) \cap (2 \operatorname{supp} \hat{\phi}_j) \neq \varnothing.$$

Therefore, we have

$$(2 \operatorname{supp} \hat{\phi}_l) \cap (2 \operatorname{supp} \hat{\phi}_j) \neq \varnothing \quad \text{and} \quad I_{\sigma(l)} \cap I_{\sigma(j)} \neq \varnothing$$

only if $|I_{\sigma(l)}| \gtrsim |I_{\sigma(j)}|$, as otherwise Proposition 5.10 would imply

$$2^{k_2+1} Q_{A_l} \supset 2^{k_1+5} Q_{A_l} \supset 2^{k_1+1} Q_{A_j} \ni \xi_{\sigma(j)},$$

which in turn would contradict $A_l \in \mathcal{V}_{\sigma(l)}$. By the definition of the selection, $|I_{\sigma(l)}| \le |I_{\sigma(j)}|$. Moreover, for every fixed $I_{\sigma(j)}$, there are only up to $C(d, k_0)$ elements $l \in \mathcal{A}_i$ such that $I_{\sigma(j)} = I_{\sigma(l)}$, so we can conclude that, for any $i \in \{1, \dots, S\}$,

$$\Big\| \sum_{l \in \mathcal{A}_i} 1_{I_{\sigma(l)}} \Big\|_\infty \lesssim 1;$$

hence $\{I_{\sigma(l)} : l \in \mathcal{A}_i\}$ is a Carleson family. By Proposition 5.9,

$$\Big\| \frac{1}{|I_{\sigma(i)}|^{1/2}} \sum_{l \in \mathcal{A}_i} |I_{\sigma(l)}|^{1/2} g_l \rho_{I_{\sigma(l)}}^{-\alpha} \rho_{I_{\sigma(i)}}^{-\alpha} \Big\|_2 \lesssim 1,$$

and we have shown the claim for the sum over all $i \in \{1, \dots, S\}$. □

5.2. *Proof of Proposition 5.4: boundary size.* For each $i \in \{1, \dots, S\}$, we find

$$\phi_i \in \Phi_n^{4\alpha}(Q_i),$$

where $Q_i = Q_{P_i}$ and P_i is a top multitile of $\sigma(i)$ such that

$$c_i = \|\rho_{I_{\sigma(i)}}^{-\alpha}[\phi_i * f]\|_2, \quad M\sqrt{|I_{\sigma(i)}|} \le c_i \le DM\sqrt{|I_{\sigma(i)}|}.$$

Let

$$g_i = \frac{\rho_{I_T}^{-\alpha}[\phi_i * f]}{\|\rho_{I_T}^{-\alpha}[\phi_i * f]\|_2}.$$

Now

$$\sum_{i=1}^{S} M^2|I_{\sigma(i)}| \lesssim \sum_{i=1}^{S} c_i|I_{\sigma(i)}|^{1/2}\langle f, \bar{\phi}_i * (\rho_{I_{\sigma(i)}}^{-\alpha} g_i)\rangle \le \|f\|_2 \Big\| \sum_{i=1}^{S} c_i|I_{\sigma(i)}|^{1/2}[\phi_i * (\rho_{I_{\sigma(i)}}^{-\alpha} g_i)]\Big\|_2.$$

Fix $\kappa \in \{0, \dots, 99\}$. Write $\mathcal{L} = \{i \in \{1, \dots, S\} : \log_2|I_i|^{1/d} \in \kappa + 100\mathbb{Z}\}$. Expanding the square and using the symmetry, we obtain

$$\Big\| \sum_{i\in\mathcal{L}} |I_{\sigma(i)}|^{1/2} c_i[\phi_i * (\rho_{I_T}^{-\alpha} g_i)]\Big\|_2^2 \lesssim D^2M^2 \sum_{i\in\mathcal{L}} |I_{\sigma(i)}|^{1/2} \sum_{\substack{l\in\mathcal{L}\\ |I_l|\le|I_i|}} |I_{\sigma(l)}|^{1/2}\langle g_l\rho_{I_{\sigma(l)}}^{-\alpha}, \phi_l * [\phi_i * (\rho_{I_{\sigma(i)}}^{-\alpha} g_i)]\rangle.$$

Let

$$\mathcal{A}_i = \{l \in \mathcal{L} : l \in \{i, \dots, S\}, \operatorname{supp}\hat{\phi}_l \cap \operatorname{supp}\hat{\phi}_i \neq \varnothing\}.$$

By Proposition 5.8, the Cauchy–Schwarz inequality and the estimates for the Hardy–Littlewood maximal function as above, it suffices to prove a bound by constant of

$$\Big\| \frac{1}{|I_{\sigma(i)}|^{1/2}} \sum_{l\in\mathcal{A}_i} |I_{\sigma(l)}|^{1/2} g_l \rho_{I_{\sigma(l)}}^{-\alpha} \rho_{I_{\sigma(i)}}^{-\alpha}\Big\|_2.$$

Indeed, by the triangle inequality we can then sum over $\kappa \in \{0, \dots, 99\}$ to conclude the proof. By Proposition 5.9, it hence remains to show that $\{I_{\sigma(l)} : l \in \mathcal{A}_i\}$ is a Carleson family.

Given $l, j \in \mathcal{A}_i$ with $l > j \ge i$ and hence $|I_{\sigma(l)}| \le |I_{\sigma(j)}|$, we have

$$(Q_j)_n \cap (Q_i)_n \neq \varnothing, \quad (Q_l)_n \cap (Q_i)_n \neq \varnothing.$$

Therefore, we have

$$I_{\sigma(l)} \cap I_{\sigma(j)} \neq \varnothing$$

only if $|I_{\sigma(l)}| = |I_{\sigma(j)}|$, as otherwise Proposition 5.10 would imply

$$2^{k_2+1}Q_l \supset 2^{k_2+1}Q_j \ni \xi_{\sigma(j)},$$

which in turn would contradict $P_l \in \mathcal{V}_{\sigma(l)}$. Therefore $I_{\sigma(j)}$ and $I_{\sigma(l)}$ are pairwise disjoint unless $I_{\sigma(j)} = I_{\sigma(l)}$. However, as above, for every fixed $I_{\sigma(j)}$, there are only up to $C(d, k_0)$ elements $l \in \mathcal{A}_i$ such that $I_{\sigma(j)} = I_{\sigma(l)}$. Hence $\{I_{\sigma(l)} : l \in \mathcal{A}_i\}$ is a Carleson family, and the proof is complete. □

5.3. ***Proof of Proposition 5.5*: *sum size.*** Consider an index $i \in \{1, \dots, S\}$. For $\theta \geq 1$, we define

$$C_i(\theta) = \{\xi \in \mathbb{R}^d : |\xi - \xi_{\sigma(i)}| \leq \theta(\xi - \xi_{\sigma(i)}) \cdot e\}.$$

Write

$$a_i(j, \theta) = C_i(\theta) \cap (Q_n(\xi_{\sigma(i)}, 2^{-j+1}) \setminus Q_n(\xi_{\sigma(i)}, 2^{-j})).$$

We let

$$\mathcal{B}^i_j = \{P \in \mathcal{B}_{\sigma(i)} : (Q_P)_n \cap a_i(j, 2) \neq \varnothing\}.$$

We note that if $P \in \mathcal{B}^i_j$, then by Proposition 5.11,

$$(Q_P)_n \subset \bigcup_{k=j-50k_2}^{j+50k_2} a_i(k, 10).$$

For each $P \in \mathcal{B}^i_j$, we find $\phi_P \in M_n(\xi_{\sigma(i)}, (Q_P)_n)$ with $\hat{\phi}_P \subset (Q_P)_n$ such that, for

$$c_P = \|\rho_{I_P}^{-\alpha}[\phi_P * f]\|_2,$$

we have

$$\sum_{j\in\mathbb{Z}} \sum_{P\in\mathcal{B}^i_j} c_P^2 \gtrsim M^2 |I_{\sigma(i)}|, \quad c_P \leq DM\sqrt{|I_P|}.$$

Let

$$g_P = \frac{\rho_{I_P}^{-\alpha}[\phi_P * f]}{\|\rho_{I_P}^{-\alpha}[\phi_P * f]\|_2}$$

if $P \in \mathcal{B}^i_j$ for some i and j, and $g_P = 0$ otherwise.

Now

$$\begin{aligned}\sum_{i=1}^{S} M^2 |I_{\sigma(i)}| &\lesssim \sum_{i=1}^{S} \sum_{j\in\mathbb{Z}} \sum_{P\in\mathcal{B}^i_j} c_P \langle f, \bar{\phi}_P * (\rho_{I_P}^{-\alpha} g_P)\rangle \\ &\leq \|f\|_2 \Big\| \sum_{i=1}^{S} \sum_{j\in\mathbb{Z}} \sum_{P\in\mathcal{B}^i_j} c_P [\phi_P * (\rho_{I_P}^{-\alpha} g_P)] \Big\|_2 .\end{aligned}$$

By the triangle inequality, we may restrict the sum over $j \in \mathbb{Z}$ to a sum over $j \in \kappa + 1000k_2\mathbb{Z}$ and integer κ. For fixed κ and every $i \in \{1, \dots, S\}$ we define

$$\mathcal{E}^i_\kappa = \bigcup_{j\in\kappa+1000k_2\mathbb{Z}} \mathcal{B}^i_j.$$

Squaring the second factor and using symmetry, we compute

$$\begin{aligned}&\Big\| \sum_{i=1}^{S} \sum_{P\in\mathcal{E}^i_\kappa} c_P [\phi_P * (\rho_{I_P}^{-\alpha} g_P)] \Big\|_2^2 \\ &\quad \lesssim \Big(\sup_{P\in\bigcup_{i=1}^{S} \mathcal{E}^i_\kappa} \frac{c_P^2}{|I_P|} \Big) \sum_{s=1}^{S} \sum_{P\in\mathcal{E}^s_\kappa} |I_P|^{1/2} \sum_{l=1}^{S} \sum_{\substack{P'\in\mathcal{E}^l_\kappa \\ |I_{P'}|\leq|I_P|}} |I_{P'}|^{1/2} \langle \rho_{I_{P'}}^{-\alpha} g_{P'}, \phi_{P'} * [\phi_P * (\rho_{I_P}^{-\alpha} g_P)]\rangle. \qquad (5\text{-}6)\end{aligned}$$

Fix s and $P \in \mathcal{P}_{\sigma(s)}$, and let

$$\mathcal{A}_P = \Big\{ P' \in \bigcup_{j \le 0} \mathcal{B}^s_{j_{\sigma(s)}+1000j} : \operatorname{supp} \hat{\phi}_P \cap \operatorname{supp} \hat{\phi}_{P'} \neq \varnothing \Big\}.$$

By Proposition 5.11, we may apply Proposition 5.8 to bound

$$\phi_{P'} * [\phi_P * (\rho_{I_P}^{-\alpha} g_P)] \lesssim \rho_{I_P}^{-\alpha} \mathcal{M}_{\mathrm{HL}} g_P.$$

By the Cauchy–Schwarz inequality and the Hardy–Littlewood maximal function theorem as above, we hence obtain

$$\begin{aligned} &\sum_{P' \in \mathcal{A}_P} |I_{P'}|^{1/2} \langle \rho_{I_{P'}}^{-\alpha} g_{P'}, \phi_{P'} * [\phi_P * (\rho_{I_P}^{-\alpha} g_P)] \rangle \\ &\lesssim \sum_{P' \in \mathcal{A}_P} |I_{P'}|^{1/2} \langle \rho_{I_P}^{-\alpha} \rho_{I_{P'}}^{-\alpha} g_{P'}, \mathcal{M}_{\mathrm{HL}} g_P \rangle \lesssim \Big\| \sum_{P' \in \mathcal{A}_P} |I_{P'}|^{1/2} \rho_{I_P}^{-\alpha} \rho_{I_{P'}}^{-\alpha} g_{P'} \Big\|_2 \lesssim |I_P|^{1/2} \rho_{I_P}(\mathbb{R}^d \setminus I_{\sigma(s)}), \end{aligned} \tag{5-7}$$

where the last inequality follows by Proposition 5.12 below, the fact that, for every fixed I_P, there are only up to $C(d, k_0)$ elements $P' \in \mathcal{A}_P$ such that $I_P = I_{P'}$, and Proposition 5.9.

Proposition 5.12. *Assume that $L \in \sigma(l)$, $H \in \sigma(h)$, and that $L, H \in \mathcal{A}_P$. Assume additionally that $|I_H| < |I_L| < |I_P|$. Then*

$$I_L \cap I_H = \varnothing, \quad (I_L \cup I_H) \cap I_P = \varnothing.$$

Proof. Because $L, H \in \mathcal{A}_P$,

$$(Q_L)_n \cap (Q_P)_n \neq \varnothing, \quad (Q_H)_n \cap (Q_P)_n \neq \varnothing.$$

As $2^{2000k_2}|I_H| \le 2^{1000k_2}|I_L| \le |I_P|$, this implies, by Proposition 5.10,

$$\xi_{\sigma(s)} \in 2^{k_2+1} Q_P \subset 2^{k_2+1} Q_L \subset 2^{k_2+1} Q_H. \tag{5-8}$$

Further,

$$a_l(j, 10) \cap a_h(j', 10) \neq \varnothing,$$

with $j' \le j - 10k_2$ only if $\xi_{\sigma(l)} \cdot e > \xi_{\sigma(h)} \cdot e$. Hence $s < l < h$, and the claim follows by (5-8), as otherwise it would contradict $L \in \mathcal{V}_{\sigma(l)}$ and $H \in \mathcal{V}_{\sigma(h)}$. □

Applying the estimate (5-7) to the second factor on the right-hand side of (5-6), we obtain

$$\begin{aligned} \sum_{s=1}^{S} \sum_{P \in \mathcal{E}_k^s} |I_P|^{1/2} \Big\| \sum_{P' \in \mathcal{A}_P} |I_{P'}|^{1/2} \rho_{I_P}^{-\alpha} \rho_{I_{P'}}^{-\alpha} g_{P'} \Big\|_2 &\lesssim \sum_{s=1}^{S} \sum_{P \in \mathcal{P}_{\sigma(s)}} |I_P| \rho_{I_P}^{-\alpha}(\mathbb{R}^d \setminus I_{\sigma(s)}) \\ &\lesssim \sum_{s=1}^{S} |I_{\sigma(s)}| \sum_{k=0}^{\infty} k 2^{-k} \lesssim \sum_{s=1}^{S} |I_{\sigma(s)}|. \end{aligned}$$

This concludes the proof of Proposition 5.5. □

5.4. ***Proof of Proposition 5.7*: *recursion.*** To streamline the language, we introduce the following definition.

Definition 5.13 (admissible tree). Let $\mathcal{V}$ be a finite subset of multitiles. A tree $T = T(\xi, I_0, \mathcal{V})$ is said to be n-admissible with respect to boundary size if

$$\Sigma^{\mathrm{bdr}}_{n,2,f_n}(T) \le \Sigma^{\mathrm{bdr,top}}_{n,2,f_n}(T).$$

It is said to be n-admissible with respect to core size if

$$\Sigma^{\mathrm{cor}}_{n,2,f_n}(T) \le \Sigma^{\mathrm{cor,top}}_{n,2,f_n}(T).$$

Proposition 5.14. *Let $N, N' > 0$. The family of multitiles*

$$\mathcal{V} = \{P : Q_P \in \mathcal{W}_N,\ I_P \subset [-N'2^N, N'2^N]^{3\times d}\},$$

with $\mathcal{W}$ in Proposition 2.2, is a convex collection. If σ is a selection on a convex collection, then $\mathcal{V}_{\sigma(i)}$ is a convex collection for all $i \in \{1, \dots, S\}$.

Proof. The proof is clear. □

Now we can proceed to the actual proof. We define the selection on $\mathcal{V}$ as follows. For notational purposes, we set $I_{\sigma(0)} = \mathbb{R}^d$, $\mathcal{P}_{\sigma(0)} = \varnothing$, and $\mathcal{V}_{\sigma(0)} = \mathcal{V}$. Finally, without loss of generality and only for notational convenience, assume $\|f_n\|_2 = 1$ for all n.

Suppose $\sigma(i-1)$ has been defined. A tree T is called an X-tree if $X(T) \ge 2^{(M-10d)/2}$ for some size X. We first define the selection by choosing repeatedly $\Sigma^{\mathrm{cor,top}}_{n,2,f_n}$-trees with $n = 1$ such that trees with larger top cubes are chosen first, only admissible trees are chosen, and only trees T with $\Theta(T) \ne 0$ are chosen. We denote by s_n the number of steps at which we reach the last tree chosen. This number is finite as there are only finitely many multitiles in the original collection $\mathcal{V}$.

We replace $\mathcal{V}$ with $\mathcal{V}_{\sigma(s_n)} \setminus \mathcal{P}_{\sigma(s_n)}$. We repeat the same process with X replaced by $\Sigma^{\mathrm{cor,top}}_{n,2,f_n}$, first with $n = 2$ and then with $n = 3$. This way, we create three selections. The first is σ_1 on $\mathcal{V}$. The second is σ_2 on $\mathcal{V}_{\sigma(s_1)} \setminus \mathcal{P}_{\sigma(s_1)}$, and the third is σ_3 on $\mathcal{V}_{\sigma(s_2)} \setminus \mathcal{P}_{\sigma(s_2)}$. By Proposition 5.3, each of these selections satisfies (5-2). Set

$$\mathcal{V}_1 = \mathcal{V} \setminus \bigcup_{n=1}^{3} \bigcup_{i=s_{n-1}+1}^{s_n} \mathcal{P}_{\sigma(i)}.$$

A tree T on $\mathcal{V}_1$ is either inadmissible or satisfies

$$\Theta(T)\Sigma^{\mathrm{cor,top}}_{n,2,f_n}(T) \le 2^{(M-10d)/2}.$$

For admissible trees, the latter condition is the desired size bound. On the other hand, if an inadmissible tree violates the size bound, then, by convexity of the reference family, it contains an admissible subtree violating the size bound. But this possibility was just ruled out. This concludes the treatise with respect to the core size.

We repeat the same selection process with $\Sigma^{\mathrm{bdr,top}}_{n,2,f_n}$ instead of $\Theta\Sigma^{\mathrm{cor,top}}_{n,2,f_n}$, and this gives us three more selections σ_4, σ_5, and σ_6 and a family $\mathcal{V}_2$ such that the trees in the selections satisfy (5-2) by Proposition 5.4 and, by the argument as in the case of the core size, the size bound is also valid.

It remains to treat the sum size. Let $\{e_\delta : 1 \le \delta \le 2d\}$ be the collection of unit vectors orthogonal to the $(d-1)$-dimensional hyperplanes. Let $\mu^{\delta,n} \in M_n(C_{e_\delta})$, with

$$\sum_{\delta=1}^{2d} \hat{\mu}^{\delta,n}(\xi) = 1, \quad \xi \neq 0.$$

For a tree T on $\mathcal{V}_2$ or any of its subfamilies, we set $\hat{\mu}^{\delta}_{T,n}(\xi) = \hat{\mu}^{\delta,n}(\xi - (\xi_T)_n)$. We run the selection choosing trees T such that

$$\Sigma^{\mathrm{sum}}_{n,2,\mu^{\delta,n}*f_n}(T) \ge 2^{(M-10d)/2},$$

so that those with maximal $e_\delta \cdot (\xi_T)_n$ are chosen first. Again, we repeat this process for each n and each δ. Each of the $6d$ selections satisfies the hypotheses of Proposition 5.5, and the trees not chosen satisfy (5-1). Collecting the trees in all of the selections constructed so far, we obtain the family $\mathcal{T}_M$, and the proof is complete. □

5.5. *Conclusion of the proof of Proposition 2.8.* Because the family of multitiles $\mathcal{V}$ in the hypothesis of the proposition is finite, there exists M such that the hypothesis of Proposition 5.7 holds. The claim except for (2-5) follows by induction.

To prove (2-5), we first note that, for any tree T and any $p \in [1, \infty]$,

$$\Sigma^{\mathrm{cor}}_{n,p,f_n}(T) + \Sigma^{\mathrm{bdr}}_{n,p,f_n}(T) \le C(d)\|f_n\|_\infty$$

is obvious. It remains to bound the sum size.

Let η be a smooth function with $\eta \gtrsim 1_{I_T}$ and $\operatorname{supp}\hat{\eta} \subset Q_{n_*}(0, 2^{-j_T})$. Let $\{\varphi_P \in \Phi^{4\alpha}_n(Q_P) : P \in \mathcal{B}_T\}$ be functions that almost achieve the supremum in the definition of the sum size. First, we note

$$\sum_{P\in\mathcal{B}_T} \|1_{I_P}[\varphi_P * (1_{\mathbb{R}^d\setminus 3I_T} f_n)]\|_2^2 \lesssim \|f_n\|_\infty \sum_{P\in\mathcal{B}_T} |I_{I_P}|\rho^{-\alpha}_{I_P}(\mathbb{R}^d \setminus 3I_T) \lesssim |I_T|\|f_n\|_\infty.$$

Second, we note

$$\sum_{P\in\mathcal{B}_T} \|1_{I_P}[\varphi_P * (1_{3I_T} f_n)]\|_2^2 \lesssim \sum_{k=k_1}^{k_2} \sum_{j\le j_T} \|1_T[\varphi_{j,k} * (1_{3I_T} f_n)]\|_2^2 \tag{5-9}$$

for a family of sequences $\{\{\varphi_{j,k} \in \Psi^{4\alpha}_{n,j-k}(\xi_T) : j \le j_T\} : k \in \{k_1, \dots, k_2\}\}$. Without loss of generality, we fix k and we drop it from notation. For terms with $j_T - j \le 100$, we estimate

$$\|1_T[\varphi_j * (1_{3I_T} f_n)]\|_2^2 \lesssim |I_T|\|f_n\|_\infty$$

as in the cases of core and boundary sizes. For terms with $j_T - j > 100$, we note that

$$[\mathbb{R}^d \setminus Q_n(\xi_T, 2^{-j+k-100})] \supset (\operatorname{supp}\hat{\eta} + \operatorname{supp}\hat{\varphi}_j) \supset \operatorname{supp}(\hat{\eta} * \hat{\varphi}_j).$$

Consequently, we have $\operatorname{supp}(\hat{\eta}*\hat{\varphi}_j)\cap\operatorname{supp}(\hat{\eta}*\hat{\varphi}_{j'})=\varnothing$ whenever $|j-j'|\geq 100$, and $\sum_{j<j_T-100}\hat{\eta}*\hat{\varphi}_j$ is a Mikhlin multiplier with bounds only depending on the dimension.

We bound each inner sum in (5-9) by

$$\sum_{j<j_T-100}\|\eta[\varphi_j*(1_{3I_T}f_n)]\|_2^2 \lesssim \sum_{l=1}^{100}\sum_{\substack{j\in l+100\mathbb{Z}\\ j<j_T-100}}\|\eta[\varphi_j*(1_{3I_T}f_n)]\|_2^2 \lesssim \sum_{l=1}^{100}\Big\|\eta\sum_{\substack{j\in l+100\mathbb{Z}\\ j<j_T-100}}[\varphi_j*(1_{3I_T}f_n)]\Big\|_2^2 \lesssim \|1_{3I_T}f_n\|_2^2,$$

where the last step followed by the Mikhlin multiplier theorem. The claim then follows. □

6. Proof of Proposition 2.2: tensorized model form

By dominated convergence, there exists $N'=N'(d,\alpha,\varepsilon,k_0,f_n,N)>0$ such that, for the finite subset of multitiles $\mathcal{V}=\{P:Q_P\in\mathcal{W}_N,\ I_P\subset[-N'2^N,N'2^N]^{3\times d}\}$, we have

$$\Big|\sum_{Q\in\mathcal{W}_N}\int_{\mathbb{R}^d}\prod_{n=1}^{3}[\phi_{Q,n}*f_n(x)]\,\mathrm{d}x\Big|\leq 2\Big|\sum_{P\in\mathcal{V}}\int_{\mathbb{R}^d}1_{I_P}(x)\prod_{n=1}^{3}[\phi_{P,n}*f_n(x)]\,\mathrm{d}x\Big|.$$

Consider the families of trees $\mathcal{T}_M$ as in Proposition 2.8. By the triangle inequality, we get the upper bound

$$\sum_{M\in\mathbb{N}\cup\{-\infty\}}\sum_{T\in\mathcal{T}_M}\Big|\sum_{P\in\mathcal{P}_T}\int_{\mathbb{R}^d}1_{I_P}(x)\prod_{n=1}^{3}[\phi_{P,n}*f_n(x)]\,\mathrm{d}x\Big|.$$

By Proposition 2.7, we get the upper bound

$$\sum_{M\in\mathbb{N}\cup\{-\infty\}}\sum_{T\in\mathcal{T}_M}|I_T|\Big(\Sigma^{\mathrm{bdr}}_{n_*,\infty,f_{n_*}}(T)\prod_{n\neq n_*}\Sigma^{\mathrm{sum}}_{n,f_n}(T)+\prod_{n=1}^{3}\Sigma^{\mathrm{cor}}_{n,q_n,f_n}(T)\Big),\tag{6-1}$$

where the second summand in brackets appears if and only if $\mathcal{P}_T\setminus\mathcal{B}_T\neq\varnothing$, i.e., if there exists $P\in\mathcal{P}_T$ with $2^{k_1+1}Q_P\ni\xi_T$.

By log-convexity and (2-5) from Proposition 2.8, we have

$$\Sigma^{\mathrm{cor}}_{n,q_n,f_n}(T)\lesssim\Sigma^{\mathrm{cor}}_{n,2,f_n}(T)^{2/q_n},\qquad \Sigma^{\mathrm{sum}}_{n,f_n}(T)\lesssim\Sigma^{\mathrm{sum}}_{n,f_n}(T)^{2/q_n}.$$

By the local Bernstein's inequality (see, e.g., Proposition 1.2 in [Fraccaroli et al. 2022]),

$$\Sigma^{\mathrm{bdr}}_{n,\infty,f_n}(T)\lesssim 2^{dv_n/2}\Sigma^{\mathrm{bdr}}_{n,2,f_n}(T),\qquad \Sigma^{\mathrm{cor}}_{n,q_n,f_n}(T)\lesssim 2^{dv_n(1/2-1/q_n)}\Sigma^{\mathrm{cor}}_{n,2,f_n}(T).$$

In addition, we know by Proposition 2.8 that, for all $n\in\{1,2,3\}$ and $T\in\mathcal{T}_M$,

$$\Sigma^{\mathrm{bdr}}_{n,2,f_n}(T)\lesssim\min\{1,2^{M/2}\|f_n\|_2\},\qquad \Sigma^{\mathrm{sum}}_{n,f_n}(T)\leq 2^{M/2}\|f_n\|_2,\qquad \Sigma^{\mathrm{cor}}_{n,2,f_n}(T)\lesssim 1.$$

Moreover, if there exists $P\in\mathcal{P}_T$ with $2^{k_1+1}Q_P\ni\xi_T$, we also have

$$\Sigma^{\mathrm{cor}}_{n,2,f_n}(T)\leq 2^{M/2}\|f_n\|_2.$$

Recalling that there is n_* such that $v_{n_*} = 0$, we hence bound (6-1) by

$$\sum_{M\in\mathbb{N}\cup\{-\infty\}} \sum_{T\in\mathcal{T}_M} |I_T| \min\{1, 2^{M/2}\|f_{n_*}\|_2\} \prod_{n\neq n_*} 2^{M/q_n}\|f_n\|_2^{2/q_n}$$
$$\lesssim \sum_{M\in\mathbb{N}\cup\{-\infty\}} \min\{2^{-M/q_{n_*}}, 2^{M(1/2-1/q_{n_*})}\|f_{n_*}\|_2\} \prod_{n\neq n_*} \|f_n\|_2^{2/q_n} \lesssim \prod_{n=1}^{3} \|f_n\|_2^{2/q_n},$$

where the first inequality used equation (2-6) from Proposition 2.8. This concludes the proof of Proposition 2.2. □

7. Proof of Theorem 1.1

The integral

$$\int_{\mathbb{R}^{3\times d}} \delta_0(\xi_1+\xi_2+\xi_3)\hat{f}_1(\xi_1)\hat{f}_2(\xi_2)\hat{f}_3(\xi_3)m(L^{-1}\xi)\,\mathrm{d}\xi \tag{7-1}$$

is absolutely convergent for Schwartz functions f_1, f_2, and f_3. Approximating m with a symbol supported in a compact set not meeting $\{(\tau,\tau,\tau) : \tau\in\mathbb{R}^d\}$, we conclude by the Lebesgue dominated convergence theorem, boundedness of m and absolute convergence of the integral that it suffices to assume m is compactly supported.

By multilinear interpolation [Janson 1988], it suffices to prove a bound for the integral (7-1) by

$$C\prod_{n=1}^{3}\|f_n\|_{q_n}$$

when $f_n = 1_{E_n}$ for measurable sets E_n of finite measure and where C is a constant independent of all E_n and m. Because m is assumed to be compactly supported, the integral (7-1) is absolutely convergent even with $f_n = 1_{E_n}$. By standard convolution approximation and the dominated convergence theorem, we see that it suffices to bound the integral (7-1) by

$$C\prod_{n=1}^{3}|E_n|^{1/q_n}$$

whenever f_n is a smooth function with

$$\|f_n\|_\infty \le 2, \quad \|f_n\|_2 \le 2|E_n|^{1/2}.$$

Indeed, the convolution mollification converges in all L^p norms with p finite, in particular with $p\in\{2, q_n\}$. We see that, for each $n\in\{1,2,3\}$, the function f_n satisfies the assumptions on Proposition 2.2.

Next we form a Whitney-type decomposition of $\mathbb{R}^{3\times d}\setminus\Gamma$. For each $\xi\in\mathbb{R}^{3\times d}\setminus\Gamma$, set

$$r_\xi = \tfrac{3}{4}\inf\{r>0 : Q(\xi,r)\cap\Gamma\neq\varnothing\},$$

where $Q(\xi,r)\subset\mathbb{R}^{3\times d}$ is the open rectangular box defined in Section 2. Let

$$\mathcal{A} = \{Q(\xi,r) : r = 2^{-k_0}r_\xi\}.$$

We let $\mathcal{W}$ be a maximal pairwise disjoint family of $Q \in \mathcal{A}$ such that $Q \cap [-2^N, 2^N]^{3\times d} \neq \varnothing$, where $N \geq 100$ is an integer such that $\operatorname{supp} m \subset L^{-1}([-2^N, 2^N]^{3\times d})$. It is clear that, for each fixed $k \leq k_0$,

$$\{2^k Q : Q \in \mathcal{W}\}$$

has bounded overlap. Note that

$$\operatorname{supp} m \subset L^{-1}\Big(\bigcup_{Q\in\mathcal{W}} 5Q\Big).$$

For $Q \in \mathcal{W}$, we define $Q_n = \{\xi_n \in \mathbb{R}^d : \xi \in Q\}$. Let $\{\eta_Q : Q \in \mathcal{W}\}$ form a partition of unity adapted to $\mathcal{W}$, meaning that, for each $Q \in \mathcal{W}$, the smooth function $\eta_Q \geq 0$ is supported in $6Q$ and satisfies the bounds

$$|\partial_1^{\gamma_1}\partial_2^{\gamma_2}\partial_3^{\gamma_3}\eta_Q(\xi)| \leq C_\gamma |Q_1|^{-|\gamma_1|/d}|Q_2|^{-|\gamma_2|/d}|Q_3|^{-|\gamma_3|/d}$$

for constants C_γ only depending on $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{N}^{3\times d}$.

Let $\chi_{Q,n}$ be a smooth function with

$$1_{7Q_n} \leq \chi_{Q,n} \leq 1_{8Q_n}, \quad |\partial^\gamma \chi_{Q,n}(\tau)| \leq C_\gamma |Q_n|^{-|\gamma|/d}$$

for all $\gamma \in \mathbb{N}^d$ and $|\gamma| \leq 100d$. Let $\chi_Q(\xi) = \chi_{Q,1}(\xi_1)\chi_{Q,2}(\xi_2)\chi_{Q,3}(\xi_3)$ for $\xi \in \mathbb{R}^{3\times d}$.

Let A_Q be a linear mapping sending $7Q - \mathfrak{P}(\xi_Q)$ into $[0, 2\pi)^{3\times d}$ and such that

$$A_Q([8Q - \mathfrak{P}(\xi_Q)]) \setminus (-2\pi, 2\pi)^{3\times d} \neq \varnothing,$$

where ξ_Q is such that $Q = Q(\xi_Q, r)$ and $\mathfrak{P}$ is the orthogonal projection of $\mathbb{R}^{3\times d}$ onto Γ. Such a matrix is of block form: $A_Q = A_{Q,1} \oplus A_{Q,2} \oplus A_{Q,3}$. We expand as a Fourier series

$$m_Q(L^{-1}\xi) := \eta_Q(\xi)m(L^{-1}\xi) = \chi_Q(\xi) \sum_{k\in\mathbb{Z}^{3\times d}} a_{Q,k} \prod_{n=1}^{3} e^{2\pi i k_n \cdot A_n \xi_n},$$

so that $m = \sum_{Q\in\mathcal{W}} m_Q$.

Write

$$m_{Q,k,n}(L_n^{-1}\xi_n) = \chi_{Q,n}(\xi_n)e^{2\pi i k_n \cdot A_n \xi_n} \quad \text{and} \quad a_k = \sup_{Q\in\mathcal{W}} |a_{Q,k}|.$$

For the function $\phi_{Q,k,n}$ defined by

$$\hat{\phi}_{Q,k,n}(\tau) = (1 + |k_n|)^{-4\alpha} m_{Q,k,n}(L_n^{-1}\tau),$$

we have $c\phi_{Q,k,n} \in \Phi_n^{4\alpha}(Q)$ up to a bounded multiplicative constant c independent of Q, k, and n.

Now we can write

$$\begin{aligned}
&\left|\int_{\mathbb{R}^{3\times d}} \delta_0(\xi_1 + \xi_2 + \xi_3)\hat{f}_1(\xi_1)\hat{f}_2(\xi_2)\hat{f}_3(\xi_3)m(L^{-1}\xi)\,\mathrm{d}\xi\right| \\
&\qquad \leq \sum_{k\in\mathbb{Z}^{3\times d}} |a_k| \left|\sum_{Q\in\mathcal{W}} \int_{\mathbb{R}^{3\times d}} \delta_0(\xi_1 + \xi_2 + \xi_3) \prod_{n=1}^{3} m_{Q,k,n}(L_n^{-1}\xi_n)\hat{f}_n(\xi_n)\,\mathrm{d}\xi\right| \\
&\qquad \lesssim \sum_{k\in\mathbb{Z}^{3\times d}} |a_k|(1 + |k|)^{12\alpha} \left|\sum_{Q\in\mathcal{W}} \int_{\mathbb{R}^d} \prod_{n=1}^{3} [\phi_{Q,k,n} * f_n(x)]\,\mathrm{d}x\right|.
\end{aligned}$$

By Proposition 2.2, this is bounded by

$$C \prod_{n=1}^{3} |E_n|^{1/q_n} \sum_{k \in \mathbb{Z}^{3\times d}} |a_k|(1+|k|)^{12\alpha}.$$

By smoothness of the symbol m and the upper bound on α, we know $|a_k| \le C|k|^{-12\alpha-3d-1}$, and hence the proof is complete. □

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