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MICROLOCAL PARTITION OF ENERGY FOR FRACTIONAL-TYPE DISPERSIVE EQUATIONS

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This paper is devoted to the proof of the microlocal partition of energy for fractional-type dispersive equations including the Schrödinger equation, the linearized gravity or capillary water-wave equation and the half-Klein–Gordon equation. Roughly speaking, a quarter of the L^2 energy lies inside or outside the "light cone" $|x| = |tP'(\xi)|$ for large time. In addition, based on the study of the half-Klein–Gordon equation, the microlocal partition of energy will also be proved for the Klein–Gordon equation.

1. Introduction

1.1. *Background.* The classical partition of energy states that the energy of the solution w to the linear wave equation

$$\begin{cases} (\partial_t^2 - \Delta)w = 0, \\ w|_{t=0} = w_0 \in \dot{H}^1(\mathbb{R}^d), \\ \partial_t w|_{t=0} = w_1 \in L^2(\mathbb{R}^d), \end{cases} \tag{W}$$

inside and outside the light cone $|x| = |t|$ satisfies, in *odd* dimension d,

$$\begin{aligned} \lim_{t\to+\infty} \big(E^{\text{in}}(w_0, w_1, t) + E^{\text{in}}(w_0, w_1, -t)\big) &= \|\partial_t w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2, \\ \lim_{t\to+\infty} \big(E^{\text{out}}(w_0, w_1, t) + E^{\text{out}}(w_0, w_1, -t)\big) &= \|\partial_t w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2, \end{aligned} \tag{1-1}$$

where

$$\begin{aligned} E^{\text{in}}(w_0, w_1, t) &:= \int_{|x|<|t|} (|\nabla w|^2 + |\partial_t w|^2)\, dx, \\ E^{\text{out}}(w_0, w_1, t) &:= \int_{|x|>|t|} (|\nabla w|^2 + |\partial_t w|^2)\, dx. \end{aligned}$$

A proof of this can be found in [Duyckaerts et al. 2011; 2012], where the authors applied this result to study the soliton of the focusing energy-critical nonlinear wave equation in dimension $d = 3, 5$ via some nonlinear analysis on small data solutions. One may also refer to [Côte et al. 2015a; 2015b] for the application in equivariant wave maps. In *even* dimension d, the limits (1-1) do not hold in general settings. It is essential to add some extra corrections, which have been calculated in detail for radial solutions in [Côte et al. 2014] and for general data in [Côte and Laurent 2024]. In these references, the authors also discovered some special data such that the corrections vanish, an application of which to the 4-dimensional focusing energy-critical wave equation can be found in [Côte et al. 2018].

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The above results have been recently revisited in [Delort 2022] using the tools of microlocal analysis. Consider first a solution u of the half-wave equation

$$\begin{cases}(\partial_t/i - |D_x|)u = 0,\\ u|_{t=0} = u_0 \in L^2.\end{cases} \tag{HW}$$

Since $u = e^{it|D_x|}u_0$, the stationary phase formula shows that one expects the microlocalized energy of the solution outside a convenient neighborhood of $\{(x,\xi) : x = t(\xi/|\xi|)\}$ at time t to vanish when t goes to infinity. Because of that, it is natural to ask whether the microlocalized energy close to the preceding point, truncated outside the wave cone, gives rise to lower bound of the form (1-1). More precisely, if one defines this microlocalized truncated energy as

$$E^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(u_0,t) := \left\| \mathrm{Op}(a^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(t))u(t)\right\|^2_{L^2}, \tag{1-2}$$

$$a^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(t,x,\xi) := \chi\left(\frac{x+t(\xi/|\xi|)}{|t|^{\frac{1}{2}+\delta}}\right)\tilde{\chi}\left(\frac{|x|-|t|}{|t|^{\delta}}\right)\mathbb{1}_{|x|>|t|}, \tag{1-3}$$

where $\chi, \tilde{\chi} \in C^\infty_c(\mathbb{R}^d)$ are chosen to be real, radial and equal to 1 near zero with $\delta \in \left]0, \frac{1}{2}\right]$, then it has been proved in [Delort 2022] that in any dimension d

$$\lim_{t\to+\infty}\left(E^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(u_0,t) + E^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(u_0,-t)\right) = \|u_0\|^2_{L^2}. \tag{1-4}$$

This result may be used to recover (1-1) in odd dimension by taking $u = (-i\partial_t + |D_x|)w$. Actually, the truncated energy in (1-2) may be expressed from the microlocalized truncated energy for the solution u of the half-wave equation and from extra terms. These extra terms give a zero contribution at the limit t tending to infinity in *odd* dimensions, but not in even ones.

The heuristics underlying estimate (1-4) for the half-wave equation are as follows. Define the quantization $\mathrm{Op}(a^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta})$ of the symbol (1-3) by

$$\left(\mathrm{Op}(a^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta})f\right)(t,x) = \frac{1}{(2\pi)^d}\int e^{ix\cdot\xi} a^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(t,x,\xi)\hat{f}(t,\xi)\,d\xi.$$

Then (1-2) may be written as

$$\begin{aligned} E^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(u_0,t) &= \left\langle \mathrm{Op}(a^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta})e^{it|D_x|}u_0, \mathrm{Op}(a^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta})e^{it|D_x|}u_0\right\rangle_{L^2} \\ &= \left\langle u_0, e^{-it|D_x|}\,\mathrm{Op}(a^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta})^*\,\mathrm{Op}(a^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta})e^{it|D_x|}u_0\right\rangle_{L^2}. \end{aligned} \tag{1-5}$$

If the symbols were smooth ones, so that symbolic calculus (whose details can be found in [Zworski 2012]) could be used, one would expect the composition $\mathrm{Op}(a^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta})^*\,\mathrm{Op}(a^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta})$ to be equal, modulo negligible remainders, to $\mathrm{Op}(b)$, with $b = |a^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}|^2$, and the conjugation $e^{-it|D_x|}\mathrm{Op}(b)e^{it|D_x|}$ to be equal, up to remainders, to $\mathrm{Op}(c)$, where $c(x,\xi) = b(x - t(\xi/|\xi|))$. Applying this to (1-5), one could write this quantity as $\langle u_0, \mathrm{Op}(e)u_0\rangle$ modulo a term tending to zero when t goes to infinity, where e is given by

$$e(t,x,\xi) = \chi^2\left(\frac{x}{|t|^{\frac{1}{2}+\delta}}\right)\tilde{\chi}^2\left(\frac{|x-t(\xi/|\xi|)|-|t|}{|t|^{\delta}}\right)\mathbb{1}_{|x-t(\xi/|\xi|)|>|t|}.$$

This symbol e roughly cuts-off the phase space on the domain

$$\left\{(x,\xi) : |x| \lesssim |t|^{\frac{1}{2}+\delta},\ \left|x - t\frac{\xi}{|\xi|}\right| > |t|\right\}.$$

When $t \to +\infty$, the truncated domain tends to the half-space

$$\left\{(x,\xi) : \operatorname{sgn}(t)x \cdot \frac{\xi}{|\xi|} < 0\right\}. \tag{1-6}$$

As a consequence, the sum of truncated energy at time t and $-t$ covers the whole phase space.

We emphasize that all the arguments above are merely formal. They hold only if all the involved functions are regular enough. In fact, it has also been proved in [Delort 2022] that the cut-off operator $\mathrm{Op}(a^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta})$ may not even be bounded if the singular cut-off $|x| > |t|$ is replaced by $x \cdot (\xi/|\xi|) > t$, which seems to work formally.

The above formal point of view, though purely heuristic, suggests that the classical result (1-1) might be extended to a large class of dispersive equations. In the general system

$$\left(\frac{\partial_t}{i} - P(D_x)\right)u = 0,$$

as in the half-wave equation, one may expect that the energy concentrates in the phase space around $x + tP'(\xi) = 0$ and the partition of energy holds with generalized "light cone"

$$|x| = |tP'(\xi)|.$$

The first result for the Schrödinger equation has been given in [Delort 2022] with truncation

$$E^{\mathrm{Schr}}_{\chi,\delta}(u_0,t) := \left\|\mathrm{Op}(a^{\mathrm{Schr}}_{\chi,\delta}(t))u(t)\right\|^2_{L^2}, \tag{1-7}$$

$$a^{\mathrm{Schr}}_{\chi,\delta}(t,x,\xi) := \chi\left(\frac{x+t\xi}{|t\xi|\langle\sqrt{|t|}\,|\xi|\rangle^{-\frac{1}{2}+\delta}}\right)\mathbb{1}_{|x|>|t\xi|}. \tag{1-8}$$

The result is similar:

$$\lim_{t\to+\infty}\left(E^{\mathrm{Schr}}_{\chi,\delta}(u_0,t) + E^{\mathrm{Schr}}_{\chi,\delta}(u_0,-t)\right) = \tfrac{1}{2}\|u_0\|^2_{L^2}. \tag{1-9}$$

Here the extra factor in the cut-off χ is only for technical use, and the loss of half of the total energy $\|u_0\|^2_{L^2}$ is due to the convexity of $P(\xi) = \frac{1}{2}|\xi|^2$.

The goal of this paper is to examine if the microlocal partition of energy results (1-4), (1-9) may be extended to a large class of dispersive equations. In particular, this generalized result covers the system of linearized gravity or capillary water-wave with infinite depth

$$\frac{\partial_t u}{i} - |D_x|^{\frac{1}{2}}u = 0, \tag{LGWW}$$

$$\frac{\partial_t u}{i} - |D_x|^{\frac{3}{2}}u = 0, \tag{LCWW}$$

or with finite and constant depth h,

$$\frac{\partial_t u}{i} - |D_x|^{\frac{1}{2}}\tanh(h|D_x|)u = 0, \tag{LGWW$_h$}$$

$$\frac{\partial_t u}{i} - |D_x|^{\frac{3}{2}}\tanh(h|D_x|)u = 0. \tag{LCWW$_h$}$$

Moreover, the system of half-Klein–Gordon

$$\frac{\partial_t u}{i} - \langle D_x \rangle u = 0 \tag{HKG}$$

can also be covered by the generalized result, and the associated conclusion will further imply the microlocal partition of energy for the standard Klein–Gordon equation

$$\begin{cases} (\partial_t^2 - \Delta + 1) w = 0, \\ w|_{t=0} = w_0 \in H^1(\mathbb{R}^d), \\ \partial_t w|_{t=0} = w_1 \in L^2(\mathbb{R}^d). \end{cases} \tag{KG}$$

One may have noticed that (1-4) and (1-9) are proved only for $\delta \in \left]0, \frac{1}{2}\right]$ or $\left]0, \frac{1}{2}\right[$. The case $\delta \geqslant \frac{1}{2}$ seems useless since with such δ the cut-off χ gives no information in the concentration of energy. The critical case $\delta = 0$, however, leads to some interesting results in the limit of truncated energy. The related results will be presented in detail in the next part.

1.2. ***Main results.*** We consider the fractional-type dispersive equation

$$\begin{cases} (\partial_t / i - P(D_x)) u = 0, \\ u|_{t=0} = u_0, \end{cases} \tag{E}$$

where P is radial and smooth except at $\xi = 0$. For simplicity, P will be identified as a function of $\rho = |\xi|$ in what follows. We further assume that P is a fractional-type symbol. Namely, the following hypotheses hold for some $p_0, p_1 \neq 0$:

$$\begin{aligned} &(1) && P^{(1)} \text{ is strictly positive and monotone on }]0, +\infty[. \\ &(2.0) && \exists P_0 \geqslant 0,\ \rho \to 0+,\ \ |P^{(1)}(\rho) - P_0| \sim \rho^{p_0},\ \ |P^{(2)}(\rho)| \sim \rho^{p_0 - 1}. \\ &(2.1) && \exists P_1 \geqslant 0,\ \rho \to +\infty,\ \ |P^{(1)}(\rho) - P_1| \sim \rho^{p_1},\ \ |P^{(2)}(\rho)| \sim \rho^{p_1 - 1}. \\ &(3.0) && \forall j \in \mathbb{N}^*,\ j \geqslant 3,\ \ \forall \rho \in]0, 1[,\ \ |P^{(j)}(\rho)| \lesssim \rho^{p_0 + 1 - j}. \\ &(3.1) && \forall j \in \mathbb{N}^*,\ j \geqslant 3,\ \ \forall \rho \in]1, \infty[,\ \ |P^{(j)}(\rho)| \lesssim \rho^{p_1 + 1 - j}. \end{aligned} \tag{H_{p_0,p_1}}$$

We introduce the symbol

$$a(t, x, \xi) = a_{\chi,\delta}(t, x, \xi) = \chi\left(\frac{x + tP'(\xi)}{|t|^{\frac{1}{2}+\delta}}\right) \mathbb{1}_{|x| > |t||P'(\xi)|}, \tag{1-10}$$

where $\delta \in \mathbb{R}$, and $\chi \in C_c^\infty(\mathbb{R}^d)$ is real with $\chi(0) = 1$. The corresponding truncated energy is defined as

$$E(u_0, t) = E_{\chi,\delta}(u_0, t) = \|\operatorname{Op}(a_{\chi,\delta}(t))(u(t))\|_{L^2}^2 = \|\operatorname{Op}(a_{\chi,\delta}(t))(e^{itP(D_x)} u_0)\|_{L^2}^2, \tag{1-11}$$

where

$$\operatorname{Op}(a_{\chi,\delta}(t))(e^{itP(D_x)} u_0)(x) := \frac{1}{(2\pi)^d} \int e^{ix\xi + itP(\xi)} a_{\chi,\delta}(t, x, \xi) \hat{u}_0(\xi)\, d\xi. \tag{1-12}$$

In Propositions 2.1, 2.2, and 3.1, we shall prove that, under some extra conditions on P, the operator $\operatorname{Op}(a(t))$, together with its variations to be introduced later, is bounded on L^2, uniformly in $|t| \gg 1$. $E(u_0, t)$ is therefore a well-defined truncated energy, at least when the time $|t|$ is sufficiently large.

We first state the fundamental result which is available for fractional equations, namely (E) with $P'(\xi) = |\xi|^p(\xi/|\xi|)$, such as Schrödinger equation ($p = 1$), linearized gravity water-wave equation $\big(p = -\frac{1}{2}\big)$, and linearized capillary water-wave equation $\big(p = \frac{1}{2}\big)$.

Theorem 1.1. *Let $\chi \in C_c^\infty(\mathbb{R}^d)$ be a real function such that $\chi(0) = 1$. We further assume that P satisfies hypotheses (H_{p_0,p_1}) with $P_0 = P_1 = 0$.*

(i) *If $\delta < 0$,*

$$\lim_{t\to\pm\infty} E_{\chi,\delta}(u_0, t) = 0. \tag{1-13}$$

(ii) *If $\delta = 0$,*

$$\lim_{t\to\pm\infty} E_{\chi,\delta}(u_0, t) = \frac{1}{(2\pi)^d}\int G_\chi(\rho, \omega)|\hat{u}_0(\rho\omega)|^2\rho^{d-1}\, d\rho\, d\omega, \tag{1-14}$$

where (ρ, ω) is the polar coordinate. The function $G_\chi(\rho, \omega)$ is defined, when P is convex, by

$$G_\chi(\rho, \omega) := \frac{1}{(2\pi)^d}\left|\int_0^\infty\int_{y\cdot\omega=0} e^{i\frac{1}{2}(r^2+|y|^2)}\chi\big(\sqrt{P''(\rho)}r\omega + \sqrt{\rho^{-1}P'(\rho)}y\big)\, dy\, dr\right|^2, \tag{1-15}$$

and, when P is concave, by

$$G_\chi(\rho, \omega) := \frac{1}{(2\pi)^d}\left|\int_0^\infty\int_{y\cdot\omega=0} e^{i\frac{1}{2}(-r^2+|y|^2)}\chi\big(\sqrt{-P''(\rho)}r\omega + \sqrt{\rho^{-1}P'(\rho)}y\big)\, dy\, dr\right|^2. \tag{1-16}$$

(iii) *If $0 < \delta < \frac{1}{2}$, we further assume that χ is radial. Then*

$$\lim_{t\to\pm\infty} E_{\chi,\delta}(u_0, t) = \tfrac{1}{4}\|u_0\|_{L^2}^2. \tag{1-17}$$

Remark 1.2. In (1-17), we manage to calculate the limit of $E_{\chi,\delta}(u_0, t)$ and $E_{\chi,\delta}(u_0, -t)$ as $t \to +\infty$, instead of their sum as in (1-1) and (1-9). Notice that the heuristics discussed after (1-6) in the case of half-wave equations do not predict this fact. This shows the limitation of this formal reasoning when sharp cut-offs are involved in the symbols.

Remark 1.3. For the Schrödinger equation, the special structure of $P'(\xi) = \xi$ allows us to reduce the regularity required for χ. In Appendix C, we will show that limits (1-13) and (1-14) hold for all $\chi \in L^1$.

We will see later that our proof of Theorem 1.1 does not hold for P with nonzero P_0, P_1, such as the half-Klein–Gordon equation (HKG), where $P_1 = 1$. In order to deal with this difficulty, one way is to add some cut-off in the frequency ξ. To be precise, we introduce the modified truncated symbol

$$a^{\mathrm{mod}}(t, x, \xi) = a^{\mathrm{mod}}_{\chi,\delta}(t, x, \xi) = \chi\left(\frac{x + tP'(\xi)}{|t|^{\frac{1}{2}+\delta}}\right)\mathbb{1}_{|x|>|t||P'(\xi)|}(1 - \chi_l)\left(\frac{\xi}{|t|^{-\epsilon_0}}\right)\chi_h\left(\frac{\xi}{|t|^{\epsilon_1}}\right), \tag{1-18}$$

where ϵ_0, ϵ_1 satisfies

$$0 < \epsilon_0 \leqslant \frac{1}{p_0 + 1}, \quad 0 < \epsilon_1 \leqslant \begin{cases} +\infty & \text{if } -1 \leqslant p_1 < 0, \\ -1/(p_1 + 1) & \text{if } p_1 < -1, \end{cases} \tag{1-19}$$

and χ_l, $\chi_h \in C_c^\infty$ are radial and equal to 1 near zero. This symbol will be concerned only when $p_1 < 0 < p_0$; the reason for this will be explained later. The corresponding truncated energy is denoted by

$$E^{\mathrm{mod}}(u_0, t) = E^{\mathrm{mod}}_{\chi,\delta}(u_0, t) := \|\mathrm{Op}(a^{\mathrm{mod}}_{\chi,\delta}(t))u(t)\|_{L^2}^2.$$

Theorem 1.4. *Let $\chi \in C_c^\infty(\mathbb{R}^d)$ be a real function such that $\chi(0)=1$. We further assume that P satisfies the hypotheses (H_{p_0,p_1}) with $p_1 < 0 < p_0$, $P_0, P_1 > 0$, and that ϵ_0, ϵ_1 satisfy condition (1-19).*

(i) *If $\delta < 0$,*

$$\lim_{t\to\pm\infty} E^{\mathrm{mod}}_{\chi,\delta}(u_0,t) = 0. \tag{1-20}$$

(ii) *If $\delta = 0$,*

$$\lim_{t\to\pm\infty} E^{\mathrm{mod}}_{\chi,\delta}(u_0,t) = \frac{1}{(2\pi)^d}\int G_\chi(\rho,\omega)|\hat{u}_0(\rho\omega)|^2\rho^{d-1}\,d\rho\,d\omega, \tag{1-21}$$

where (ρ,ω) is the polar coordinate. The function $G_\chi(\rho,\omega)$ is the same one defined by (1-15) and (1-16).

(iii) *If $0 < \delta < \frac{1}{2}$, we further assume that χ is radial. Then*

$$\lim_{t\to\pm\infty} E^{\mathrm{mod}}_{\chi,\delta}(u_0,t) = \tfrac{1}{4}\|u_0\|^2_{L^2}. \tag{1-22}$$

Remark 1.5. In this theorem, we only consider the case $p_1 < 0 < p_0$, which is enough to cover all $P_0, P_1 \neq 0$. Actually, when $p_0 < 0$ (resp. $p_1 > 0$), the hypotheses (H_{p_0,p_1}) with $P_0 > 0$ (resp. $P_1 > 0$) are equivalent to (H_{p_0,p_1}) with $P_0 = 0$ (resp. $P_1 = 0$), which has been studied in Theorem 1.1.

Another way to deal with nonzero P_0, P_1 is to add an extra factor in the cut-off χ, namely, consider an alternative truncated symbol

$$a^{\mathrm{alt}}(t,x,\xi) = a^{\mathrm{alt}}_{\chi,\delta,\Lambda}(t,x,\xi) = \chi\left(\frac{x+tP'(\xi)}{|t|^{\frac{1}{2}+\delta}\Lambda(|t|^{\frac{1}{2}}\xi)}\right)\mathbb{1}_{|x|>|t||P'(\xi)|}, \tag{1-23}$$

where $\Lambda \in C^\infty(\mathbb{R}^d\setminus\{0\})$ is strictly positive, radial and satisfies the following conditions for some $\sigma_0, \sigma_1 \in \mathbb{R}$:

$$\begin{aligned}
&(1.0)\quad \rho \to 0+,\quad \Lambda(\rho)\sim\rho^{\sigma_0}.\\
&(1.1)\quad \rho \to +\infty,\quad \Lambda(\rho)\sim\rho^{\sigma_1}.\\
&(2.0)\quad \forall\rho\in\,]0,1[,\quad \Lambda^{(j)}(\rho)\lesssim\rho^{\sigma_0-j}.\\
&(2.1)\quad \forall\rho\in\,]1,\infty[,\quad \Lambda^{(j)}(\rho)\lesssim\rho^{\sigma_1-j}.\\
&(3)\quad \lim_{\rho\to+\infty}\frac{\Lambda(\rho)}{\rho^{\sigma_1}} = \lambda_1 > 0.
\end{aligned} \tag{$\mathrm{C}_{\sigma_0,\sigma_1}$}$$

The associated truncated energy is denoted by

$$E^{\mathrm{alt}}(u_0,t) = E^{\mathrm{alt}}_{\chi,\delta,\Lambda}(u_0,t) := \|\mathrm{Op}(a^{\mathrm{alt}}_{\chi,\delta,\Lambda}(t))u(t)\|^2_{L^2},$$

and the result becomes:

Theorem 1.6. *Let $\chi \in C_c^\infty(\mathbb{R}^d)$ be a real function such that $\chi(0)=1$. We assume that P satisfies the hypotheses (H_{p_0,p_1}) and Λ satisfies condition $(\mathrm{C}_{\sigma_0,\sigma_1})$ with $\sigma_0 \geqslant p_0$, $\sigma_1 \leqslant p_1$.*

(i) *If $\delta + \frac{1}{2}\sigma_1 < 0$,*

$$\lim_{t\to\pm\infty} E^{\mathrm{alt}}_{\chi,\delta,\Lambda}(u_0,t) = 0. \tag{1-24}$$

(ii) *If $\delta + \frac{1}{2}\sigma_1 = 0$,*

$$\lim_{t\to\pm\infty} E^{\mathrm{alt}}_{\chi,\delta,\Lambda}(u_0,t) = \frac{1}{(2\pi)^d}\int G^{\mathrm{alt}}_\chi(\rho,\omega)|\hat{u}_0(\rho\omega)|^2\rho^{d-1}\,d\rho\,d\omega, \tag{1-25}$$

where (ρ,ω) *is the polar coordinate. The function* $G^{\mathrm{alt}}_\chi(\rho,\omega)$ *is defined, when* P *is convex, by*

$$G^{\mathrm{alt}}_\chi(\rho,\omega) := \frac{1}{(2\pi)^d}\left|\int_0^\infty\int_{y\cdot\omega=0} e^{i\frac{1}{2}(r^2+|y|^2)}\chi\left(\frac{\sqrt{P''(\rho)}r\omega+\sqrt{\rho^{-1}P'(\rho)}y}{\lambda_1\rho^{\sigma_1}}\right)dy\,dr\right|^2, \tag{1-26}$$

and, when P *is concave, by*

$$G^{\mathrm{alt}}_\chi(\rho,\omega) := \frac{1}{(2\pi)^d}\left|\int_0^\infty\int_{y\cdot\omega=0} e^{i\frac{1}{2}(-r^2+|y|^2)}\chi\left(\frac{\sqrt{-P''(\rho)}r\omega+\sqrt{\rho^{-1}P'(\rho)}y}{\lambda_1\rho^{\sigma_1}}\right)dy\,dr\right|^2. \tag{1-27}$$

(iii) *If* $0<\delta+\frac{1}{2}\sigma_1<\frac{1}{2}$, *we further assume that* χ *is radial. Then*

$$\lim_{t\to\pm\infty} E^{\mathrm{alt}}_{\chi,\delta,\Lambda}(u_0,t) = \tfrac{1}{4}\|u_0\|^2_{L^2}. \tag{1-28}$$

Remark 1.7. The proof of Theorem 1.1 fails when $|x/t|$ is close to P_0 or P_1. The extra factor Λ together with the condition $\sigma_0 \geqslant p_0$, $\sigma_1 \leqslant p_1$ allows us to eliminate this case, and a demonstration similar to Theorem 1.1 will work when $|x/t|$ is away from P_0, P_1.

As a byproduct of Propositions 2.1 and 2.2, where the uniform-in-t boundedness on L^2 of $\mathrm{Op}(a(t))$ and $\mathrm{Op}(a^{\mathrm{mod}}(t))$ will be proved, these operators are also uniformly bounded with χ identically equal to 1, namely:

Theorem 1.8. *Let* $p_0, p_1 \neq 0$ *and* P *satisfy the hypotheses* (H_{p_0,p_1}). *There exists a constant* $C>0$ *independent of t such that,*

(i) *when* $P_0=P_1=0$,

$$\|\mathrm{Op}(\mathbb{1}_{|x|>|tP'(\xi)|})\|_{\mathcal{L}(L^2)} \leqslant C$$

holds for all $|t|>0$;

(ii) *when* $P_0, P_1>0$,

$$\left\|\mathrm{Op}\left(\mathbb{1}_{|x|>|tP'(\xi)|}(1-\chi_l)\left(\frac{\xi}{|t|^{-\epsilon_0}}\right)\chi_h\left(\frac{\xi}{|t|^{\epsilon_1}}\right)\right)\right\|_{\mathcal{L}(L^2)} \leqslant C$$

holds for all $|t|>t_0\gg 1$, *where* ϵ_0, ϵ_1 *are arbitrary parameters satisfying* (1-19) *and* $\chi_l, \chi_h \in C_c^\infty$ *are equal to* 1 *near zero.*

The boundedness of $\mathrm{Op}(\mathbb{1}_E)$ for measurable sets $E\subset\mathbb{R}^{2d}$ is of great concern in microlocal analysis, and the results above give a positive answer to some E defined via convex functions. If one changes to Weyl quantization, this problem is known as localization of the Wigner distribution. In fact, in this case, we have

$$\langle \mathrm{Op}^w(\mathbb{1}_E)u, v\rangle_{L^2} = \frac{1}{(2\pi)^{d/2}}\int_E \mathcal{W}(u,v)(x,\xi)\,dx\,d\xi,$$

where

$$\mathcal{W}(u,v)(x,\xi) = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} e^{iy\cdot\xi}u\big(x+\tfrac{1}{2}y\big)\overline{v\big(x-\tfrac{1}{2}y\big)}\,dy$$

is the Wigner distribution of (u,v). It has been found that the operator properties of $\mathrm{Op}^w(\mathbb{1}_E)$ (boundedness, positivity, spectrum, etc.) are related to the geometry of E. For example, when E is an ellipsoid, in

[Flandrin 1988; Lieb and Ostrover 2010], the authors gave some sharp estimates of the L^2-norm of $\mathrm{Op}^w(\mathbb{1}_E)$, which is related to the size of the ellipsoid. As another example, when E is a polygon on $\mathbb{R}^2$ with N sides, it was proved in [Lerner 2024] that the norm of $\mathrm{Op}^w(\mathbb{1}_E)$ can be controlled by $\sqrt{N/2}$ for $N \geqslant 3$. In the same paper, the author also proved that there exists open set E such that $\mathrm{Op}^w(\mathbb{1}_E)$ is not even bounded on L^2. The readers may refer to [Lerner 2024] for more results on this topic.

By applying the results of Theorems 1.4 and 1.8 to half-Klein–Gordon equation (HKG), we are able to obtain the following microlocal partition of energy for Klein–Gordon equation, which is an analogue of partition of energy for wave equation.

Theorem 1.9. *Let w be the unique solution to Klein–Gordon equation* (KG), *namely*

$$\begin{cases} (\partial_t^2 - \Delta + 1)w = 0, \\ w|_{t=0} = w_0 \in H^1(\mathbb{R}^d), \\ \partial_t w|_{t=0} = w_1 \in L^2(\mathbb{R}^d), \end{cases}$$

where w_0, w_1 are real, and so is w. Then, the truncated energy

$$E_\epsilon^{\mathrm{KG}}(w_0, w_1, t) := \|\mathrm{Op}(a_\epsilon^{\mathrm{KG}}(t))\partial_t w(t)\|_{L^2}^2 + \|\mathrm{Op}(a_\epsilon^{\mathrm{KG}}(t))\nabla w(t)\|_{L^2}^2 + \|\mathrm{Op}(a_\epsilon^{\mathrm{KG}}(t)) w(t)\|_{L^2}^2 \quad (1\text{-}29)$$

satisfies

$$\lim_{t\to\pm\infty} E_\epsilon^{\mathrm{KG}}(w_0, w_1, t) = \tfrac{1}{4}(\|w_0\|_{H^1}^2 + \|w_1\|_{L^2}^2), \quad (1\text{-}30)$$

where

$$a^{\mathrm{KG}}(t, x, \xi) = a_\epsilon^{\mathrm{KG}}(t, x, \xi) := \mathbb{1}_{|x|>|t\xi/\langle\xi\rangle|}\chi\left(\frac{\xi}{|t|^\epsilon}\right), \quad (1\text{-}31)$$

$0 < \epsilon < 1$, *and $\chi \in C_c^\infty(\mathbb{R}^d)$ is a real and radial function equal to* 1 *near zero.*

Remark 1.10. In view of the similarity between wave equation and Klein–Gordon equation, one may ask what (1-30) will become if we apply the same truncation $\mathbb{1}_{|x|>|t|}$ as in classical result (1-1). The answer is, for all $0 \leqslant r_0 \leqslant r_1$,

$$\lim_{t\to\pm\infty} \int_{r_0<|x/t|<r_1} (|\partial_t w|^2 + |\nabla w|^2 + |w|^2)\, dx = \|\mathbb{1}_{]\rho_0,\rho_1[}(|D_x|) w_0\|_{H^1}^2 + \|\mathbb{1}_{]\rho_0,\rho_1[}(|D_x|) w_1\|_{L^2}^2, \quad (1\text{-}32)$$

where $]\rho_0, \rho_1[= P'^{-1}(]r_0, r_1[)$. Since P' takes values in $[0, 1[$, we have in particular,

$$\lim_{t\to\pm\infty} \int_{|x|>|t|} (|\partial_t w|^2 + |\nabla w|^2 + |w|^2)\, dx = 0.$$

A detailed discussion of (1-32) will be given in Appendix D.

1.3. ***Nonnullity of the limit in the critical case.*** In Theorems 1.1, 1.4, and 1.6, we calculate the limit of energy in three cases. In the subcritical case $\delta < 0$ (or $\delta + \frac{1}{2}\sigma_1 < 0$), the truncated energy tends to 0, no matter which χ we choose. This phenomenon also exists in the supercritical case $0 < \delta < \frac{1}{2}$ (or $0 < \delta + \frac{1}{2}\sigma_1 < \frac{1}{2}$), where the limit is always half of the total energy $\|u_0\|_{L^2}$. In the critical case $\delta = 0$ (or $\delta + \frac{1}{2}\sigma_1 = 0$), however, the limit does depend on our choice of χ. If we further assume χ to be radial, it is not difficult to check that the limits (1-14), (1-21), and (1-25) are bounded and nonnegative.

In fact, when χ is radial, the function G_χ, G_χ^{alt} can be written in the form

$$\frac{1}{4}\frac{1}{(2\pi)^d}\left|\int_{\mathbb{R}^d} e^{i\frac{1}{2}(\pm x_1^2+|x'|^2)}\chi\left(\frac{\sqrt{\pm P''(\rho)}x_1+\sqrt{P'(\rho)/\rho}x'}{\lambda\rho^\sigma}\right)dx\right|^2,$$

where $+$ and $-$ stand for the convex and concave cases, respectively, $x=(x_1,x')\in\mathbb{R}\times\mathbb{R}^{d-1}$, and $\lambda>0$, $\sigma\in\mathbb{R}$. By the Plancherel theorem, it is equal to

$$\frac{1}{4}\frac{1}{(2\pi)^{2d}}\left|\int_{\mathbb{R}^d} e^{i((\pm\xi_1^2+|\xi'|^2)/(2(\lambda\rho^\sigma)^2))}\hat{\chi}\left(\frac{\xi_1}{\sqrt{\pm P''(\rho)}}+\sqrt{\frac{\rho}{P'(\rho)}}\xi'\right)\frac{1}{\sqrt{\pm P''(\rho)}}\left(\frac{\rho}{P'(\rho)}\right)^{\frac{1}{2}(d-1)}d\xi\right|^2,$$

which, after a change of variable, reads

$$\frac{1}{4}\frac{1}{(2\pi)^{2d}}\left|\int_{\mathbb{R}^d} e^{i\frac{P''(\rho)\rho\xi_1^2+P'(\rho)|\xi'|^2}{2\rho(\lambda\rho^\sigma)^2}}\hat{\chi}(\xi)\,d\xi\right|^2. \tag{1-33}$$

Therefore, G_χ can be estimated by

$$0\leqslant G_\chi(\rho,\omega)\leqslant\frac{1}{4}\frac{1}{(2\pi)^{2d}}\|\hat{\chi}\|_{L^1}^2.$$

A natural question is then whether limits (1-14), (1-21), and (1-25) are nonzero for nontrivial initial data u_0. The answer is positive for the fractional equation, i.e., with $P'(\xi)=|\xi|^{p-1}\xi,\ p\neq 0$. More precisely:

Proposition 1.11. *Under the assumption $P'(\xi)=|\xi|^{p-1}\xi$, (1-33) can be written, up to some multiple with constants, as*

$$\widetilde{G}(\rho)=\left|\int_{\mathbb{R}^d} e^{i\frac{1}{2\lambda^2}\rho^{p-1-2\sigma}(p\xi_1^2+|\xi'|^2)}\hat{\chi}(\xi)\,d\xi\right|^2.$$

If $p\neq 2\sigma+1$ and $\chi\in\mathcal{S}(\mathbb{R}^d)$ with $\chi(0)\neq 0$, $\widetilde{G}(\rho)$ is nonzero except on a set of null Lebesgue measure.

Proof. Since χ is a Schwartz function, the complex function

$$F(z):=\frac{1}{2(2\pi)^d}\int_{\mathbb{R}^d} e^{i\frac{z}{2\lambda^2}(\frac{1}{2}p\xi_1^2+\frac{1}{2}|\xi'|^2)}\hat{\chi}(\xi)\,d\xi$$

is analytic on upper half-plane $\{z\in\mathbb{C}:\operatorname{Im} z>0\}$ and continuous on its closure. In [Lusin and Priwaloff 1925], the authors proved that either the real zeros of such function form a set of zero Lebesgue measure, or it is identically zero. The same result holds thus for $\widetilde{G}(\rho)=|F(\rho^{p-1-2\sigma})|^2$. Due to the fact that $\chi(0)\neq 0$, $\widetilde{G}$ is nonzero as $\rho^{p-1-2\sigma_1}$ is small enough, and $\widetilde{G}$ is therefore nonzero almost everywhere. ☐

As a consequence, the limits (1-14), (1-21), and (1-25) are strictly positive for all nontrivial $u_0\in L^2$ under the assumption $p\neq 1$ (or $p\neq 2\sigma_1+1$). If $p=1$ (or $p=2\sigma_1+1$), the function $G_\chi(\rho)$ (or G_χ^{alt}) will no more depend on ρ and the limits (1-14), (1-21), (1-25) will take the form $c_0(\chi)\|u_0\|_{L^2}^2$, where

$$c_0(\chi)=\left|\frac{1}{2(2\pi)^d}\int_{\mathbb{R}^d} e^{i(\frac{1}{2}p\xi_1^2+\frac{1}{2}|\xi'|^2)}\hat{\chi}(\xi)\,d\xi\right|^2.$$

To obtain a nonzero limit, it suffices to choose χ such that the quantity above is nonzero. For example, one may take

- χ to be positive and supported in a sufficiently small ball centered at zero;
- χ to be Gaussian;
- χ of the form $\chi = \tilde{\chi}(\cdot/R)$, with $\tilde{\chi} \in C_c^\infty$, $\tilde{\chi}(0) \neq 0$, and $R \gg 1$.

1.4. ***Plan of this paper.*** The proofs of Theorems 1.1, 1.4, and 1.6 will be divided into two parts: uniform boundedness of truncated operator and calculation of limit. In Section 2, we will prove that $\mathrm{Op}(a(t))$ and $\mathrm{Op}(a^{\mathrm{mod}}(t))$ are uniformly bounded on L^2 in three steps. The first two steps are exactly the same, while the difference arises in the last step where one may see the difficulties caused by nonzero P_0, P_1. As a byproduct of this proof, Theorem 1.8 can be shown easily. Section 3 is devoted to the uniform boundedness of $\mathrm{Op}(a^{\mathrm{alt}}(t))$, which is much simpler than that of $\mathrm{Op}(a(t))$ and $\mathrm{Op}(a^{\mathrm{mod}}(t))$ thanks to the extra factor Λ. The uniform boundedness of truncated operators allows us to calculate the limits stated in Theorems 1.1, 1.4, and 1.6 only for some regular data u_0, which will be made precise in Section 4. In Section 5, we will prove the microlocal partition of energy for the Klein–Gordon equation by studying the half-Klein–Gordon equation.

In Appendix A, we collect technical inequalities which are frequently used in this paper, as well as some criteria of L^2-boundedness for pseudodifferential operators. Several stationary phase lemmas are presented in Appendix B; these are key techniques in calculating the limit of truncated energy. As mentioned before, our main result Theorem 1.1 can be refined for the Schrödinger equation, whose rigorous statement and proof will be given in Appendix C. Appendix D is devoted to the discussion on the classical partition of energy for the Klein–Gordon equation due to the study of the asymptotic behavior of the solution to the half-Klein–Gordon equation. The last part, Appendix E, contains some details omitted in Section 4, especially for concave P.

1.5. ***Notations and conventions.*** To end this section, we clarify some notations and conventions used in this paper.

- We say a is a symbol on $\mathbb{R}^d$ if a is a function on $\{(x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d\}$. The corresponding (pseudodifferential) operator is defined by

$$\mathrm{Op}(a) f(x) := \frac{1}{(2\pi)^d} \int e^{ix\xi} a(x,\xi) \hat{f}(\xi)\, d\xi.$$

To make this definition meaningful, we will assume in this paper that a is a measurable function with at most polynomial growth in ξ and that f belongs to the class of Schwartz functions.

- For any function $P : \mathbb{R}^d \to \mathbb{C}$, which can be regarded as a symbol independent of x, the corresponding operator will be denoted by $P(D_x)$.

- The kernel (or kernel function) of a linear operator $A : \mathcal{S}(\mathbb{R}^d) \mapsto \mathcal{S}'(\mathbb{R}^d)$ is defined as (if it exists) a tempered distribution K on $\mathbb{R}^d \times \mathbb{R}^d$ such that

$$Au(x) = \int K(x,y) u(y)\, dy.$$

For the simplicity of notation, in this paper, we will use the symbol of the kernel function to represent the operator, i.e.,

$$Ku(x) = \int K(x, y)u(y)\,dy.$$

• A function $F : \mathbb{R}^d \to \mathbb{C}$ is said to be radial if there exists a function $f : [0, \infty[\mapsto \mathbb{C}$ such that $F(x) = f(|x|)$ for all $x \in \mathbb{R}^d$. In this case, we will not distinguish the functions F and f. That is, we will write instead $F(x) = F(|x|)$ or $f(x) = f(|x|)$ for $x \in \mathbb{R}^d$.

• We will use c, C, sometimes equipped with superscripts and subscripts, to represent all the small and large constants respectively.

• For nonzero quantities ρ, r, the notation $\rho \sim r$ means that there exist constants $c, C > 0$ such that $c < \rho/r < C$.

2. L^2-boundedness of microlocal truncation operators

The goal of this section is the demonstration of following proposition, which eventually implies Theorem 1.8.

Proposition 2.1. *Let $p_0, p_1 \neq 0$, $\delta \in \mathbb{R}$, and $\chi \in C_c^\infty(\mathbb{R}^d)$. There exists a constant $C > 0$ independent of t such that, for all $|t| > 0$,*

$$\|\mathrm{Op}(a_{\chi,\delta}(t))\|_{\mathcal{L}(L^2)} \leqslant C,$$

where the symbol $a_{\chi,\delta}(t)$ is defined in (1-10) and P satisfies the hypotheses (H_{p_0,p_1}) with $P_0 = P_1 = 0$.

In parallel, we shall also prove the following result:

Proposition 2.2. *Let $p_0 > 0 > p_1$, $\delta \in \mathbb{R}$, and $\chi \in C_c^\infty(\mathbb{R}^d)$. We assume that P satisfies the hypotheses (H_{p_0,p_1}) with $P_0, P_1 > 0$. Then the modified truncated symbol*

$$a^{\mathrm{mod}}_{\chi,\delta}(t, x, \xi) = a_{\chi,\delta}(t, x, \xi)(1 - \chi_l)\left(\frac{\xi}{|t|^{-\epsilon_0}}\right)\chi_h\left(\frac{\xi}{|t|^{\epsilon_1}}\right),$$

which has already been defined in (1-18), corresponds to a bounded operator on L^2, uniformly in $|t| > t_0 \gg 1$. Here $\chi_l, \chi_h \in C_c^\infty$ are radial and equal to 1 near zero and ϵ_0, ϵ_1 satisfy the condition (1-19).

One can see in the following proof that our demonstration cannot eliminate the truncation in ξ in the definition (1-18) of a^{mod}. In fact, after some change of scaling, we will decompose the symbol a (or a^{mod}) into three components, two of which are bounded for all $P_0, P_1 \geqslant 0$, while our treatment for the last component does not hold for nonzero P_0, P_1. The complementary cut-off in ξ is used to solve this problem. Note that it is still unknown whether such restriction is essential.

To begin with, one observes that it is equivalent to study the cut-off inside the cone, namely

$$a^{\mathrm{in}}(t, x, \xi) = \chi\left(\frac{x + tP'(\xi)}{|t|^{\frac{1}{2}+\delta}}\right)\mathbb{1}_{|x|<|t||P'(\xi)|},$$

since the operator with symbol

$$\chi\left(\frac{x + tP'(\xi)}{|t|^{\frac{1}{2}+\delta}}\right)$$

is bounded uniformly in t and $\delta \in \mathbb{R}$, due to Lemma A.10 together with Lemma A.5. In this section, we will not distinguish a and a^{in} and denote both of them as a.

With a reflection in ξ, t can be assumed to be positive. The application of Lemma A.4 allows us to replace a by

$$\tilde{a}(t,x,\xi) = a\left(t, \sqrt{t}x, \frac{\xi}{\sqrt{t}}\right) = \chi\left(\frac{x/\sqrt{t} + P'(\xi/\sqrt{t})}{t^{\delta-\frac{1}{2}}}\right)\mathbb{1}_{|x/\sqrt{t}|<|P'(\xi/\sqrt{t})|}.$$

Now, we split $\tilde{a}$ into high and low frequencies, namely $\tilde{a} = \tilde{a}^\flat + \tilde{a}^\sharp$, where

$$\tilde{a}^\flat(t,x,\xi) = \tilde{a}(t,x,\xi)\tilde{\chi}(\xi),$$

and $\tilde{\chi} \in C_c^\infty(\mathbb{R}^d)$ is a radial function which equals 1 near zero. In the following, we shall treat the high- and low-frequency parts at the same time. Before entering the next step, we introduce some notations which will be frequently used in this section. In all cases, we set

$$\mu = t^{\delta-\frac{1}{2}} \in]0,+\infty[.$$

With $j=0$ for the low-frequency part and $j=1$ for the high-frequency part, we set,

- when $P_j = 0$,
$$X(t,x) := \frac{|x|}{\sqrt{t}}, \quad \Xi(t,\xi) := P'\left(\frac{|\xi|}{\sqrt{t}}\right), \quad \nu_j = +;$$
- when $P_j > 0$, $P' > P_j$,
$$X(t,x) := \frac{|x|}{\sqrt{t}} - P_j, \quad \Xi(t,\xi) := P'\left(\frac{|\xi|}{\sqrt{t}}\right) - P_j, \quad \nu_j = +;$$
- when $P_j > 0$, $P' < P_j$,
$$X(t,x) := P_j - \frac{|x|}{\sqrt{t}}, \quad \Xi(t,\xi) := P_j - P'\left(\frac{|\xi|}{\sqrt{t}}\right), \quad \nu_j = -.$$

Note that for all nonzero ξ, Ξ is strictly positive. With these notations our problem can be reduced to the uniform-in-μ, t boundedness of

$$b^\flat(t,\mu,x,\xi) = \chi\left(\frac{(P_0+\nu_0 X)\frac{x}{|x|} + (P_0+\nu_0\Xi)\frac{\xi}{|\xi|}}{\mu}\right)\mathbb{1}_{0<\frac{X}{\Xi}<1}\tilde{\chi}\left(\frac{\xi}{\sqrt{t}}\right),$$
$$b^\sharp(t,\mu,x,\xi) = \chi\left(\frac{(P_1+\nu_1 X)\frac{x}{|x|} + (P_1+\nu_1\Xi)\frac{\xi}{|\xi|}}{\mu}\right)\mathbb{1}_{0<\frac{X}{\Xi}<1}(1-\tilde{\chi})\left(\frac{\xi}{\sqrt{t}}\right).$$

We emphasize that our definition of X does not ensure its strict positivity, but one may always eliminate the part $X<0$, due to the uniform boundedness of the operator with symbol

$$\chi\left(\frac{(P_j+\nu_j X)\frac{x}{|x|} + (P_j+\nu_j\Xi)\frac{\xi}{|\xi|}}{\mu}\right),$$

which is also a consequence of Lemmas A.10 and A.5.

Now, we decompose b^ι ($\iota = \flat, \sharp$) as the sum of $b_0^\iota, \tilde{b}^\iota, b_1^\iota$ with cut-off $0 < \frac{X}{\Xi} \ll 1$ and $\frac{X}{\Xi} \sim 1$ and $0 < 1 - \frac{X}{\Xi} \ll 1$, respectively. To be precise,

$$\begin{aligned} b^\iota(t,\mu,x,\xi) &= b_0^\iota(t,\mu,x,\xi) + \tilde{b}^\iota(t,\mu,x,\xi) + b_1^\iota(t,\mu,x,\xi), \\ b_0^\iota(t,\mu,x,\xi) &= b^\iota(t,\mu,x,\xi)\chi_0(X/\Xi), \\ \tilde{b}^\iota(t,\mu,x,\xi) &= b^\iota(t,\mu,x,\xi)\Psi(X/\Xi), \\ b_1^\iota(t,\mu,x,\xi) &= b^\iota(t,\mu,x,\xi)\chi_1(1-X/\Xi), \end{aligned}$$

where χ_0, χ_1 and Ψ are radial, smooth and compactly supported. χ_0, χ_1 are supported in a small neighborhood of zero and equal to 1 near zero, while Ψ is compactly supported in $]0,1[$. By regarding μ as a t-independent parameter, we can reduce Propositions 2.1 and 2.2 to the following proposition:

Proposition 2.3. *There exist t, μ-independent constants $C > 0$ such that*

(i) *if $P_0, P_1 \geqslant 0$, for all $t, \mu > 0$,*

$$\|\mathrm{Op}(b_1^\iota(t,\mu))\|_{\mathcal{L}(L^2)} \leqslant C, \tag{2-1}$$

$$\|\mathrm{Op}(\tilde{b}^\iota(t,\mu))\|_{\mathcal{L}(L^2)} \leqslant C; \tag{2-2}$$

(ii) *if $P_0 = P_1 = 0$, for all $t, \mu > 0$,*

$$\|\mathrm{Op}(b_0^\iota(t,\mu))\|_{\mathcal{L}(L^2)} \leqslant C; \tag{2-3}$$

(iii) *if $P_0, P_1 > 0$ and $p_1 < 0 < p_0$, for all $\mu > 0$, $t > 1$,*

$$\left\|\mathrm{Op}\left(b_0^\iota(t,\mu)(1-\chi_l)\left(\frac{\xi}{t^{\frac{1}{2}-\epsilon_0}}\right)\chi_h\left(\frac{\xi}{t^{\frac{1}{2}+\epsilon_1}}\right)\right)\right\|_{\mathcal{L}(L^2)} \leqslant C. \tag{2-4}$$

Before giving the proof, we indicate below the consequence of this proposition, which implies Propositions 2.1 and 2.2, and will be used in the end of this section to conclude Theorem 1.8.

Corollary 2.4. *Let χ, χ_l, χ_h be defined as before.*

(i) *If P satisfies (H_{p_0,p_1}) with $p_0, p_1 \neq 0$ and $P_0 = P_1 = 0$, the operator*

$$\mathrm{Op}\left(\chi\left(\frac{x+tP'(\xi)}{|t|\mu}\right)\mathbb{1}_{|x|>|tP'(\xi)|}\right)$$

is bounded on L^2 uniformly in $t \neq 0$ and $\mu > 0$.

(ii) *If P satisfies (H_{p_0,p_1}) with $p_0, p_1 \neq 0$ and ϵ_0, ϵ_1 satisfy condition (1-19), the operator*

$$\mathrm{Op}\left(\chi\left(\frac{x+tP'(\xi)}{|t|\mu}\right)\mathbb{1}_{|x|>|tP'(\xi)|}(1-\chi_l)\left(\frac{\xi}{t^{\frac{1}{2}-\epsilon_0}}\right)\chi_h\left(\frac{\xi}{t^{\frac{1}{2}+\epsilon_1}}\right)\right)$$

is bounded on L^2 uniformly in $t > 1$ and $\mu > 0$.

2.1. ***Study of the symbols b_1 and $\tilde{b}$.*** In this part, we shall prove (2-1) and (2-2). One observes that both b_1^ι and $\tilde{b}^\iota$ are supported for $X \sim \Xi$, which allows us to reduce our problem via dyadic decomposition.

Proof of (2-2). We start with a homogeneous dyadic decomposition

$$1 = \sum_{k\in\mathbb{Z}} \varphi\left(\frac{\eta}{2^k}\right),$$

where $\varphi \in C_c^\infty(\mathbb{R}^d)$ is radial and supported away from zero. In this way, we may decompose $\tilde{b}^\sharp$ as $\sum_{k\geqslant 0} \tilde{b}_k$, and $\tilde{b}^\flat$ as $\sum_{k<0} \tilde{b}_k$, where

$$\tilde{b}_k(t,\mu,x,\xi) = \chi\left(\frac{(P_j+\nu_j X)\frac{x}{|x|} + (P_j+\nu_j\Xi)\frac{\xi}{|\xi|}}{\mu}\right)\Psi\left(\frac{X}{\Xi}\right)\psi(2^{-kp_j}X)\varphi\left(\frac{\xi}{2^k\sqrt{t}}\right), \tag{2-5}$$

where $\psi \in C_c^\infty(\mathbb{R}^d)$ is also radial and supported away from zero. The extra factor ψ comes from the truncation $X \sim \Xi \sim (t^{-1/2}|\xi|)^{p_j} \sim 2^{kp_j}$. This factors implies that the $\tilde{b}_k$'s are almost orthogonal so that it suffices to prove the uniform (in k, t, μ) boundedness of $\mathrm{Op}(\tilde{b}_k)$.

Note that, due to the compact support of χ and the fact that $0 < c < X/\Xi < 1-c$ for some small $c > 0$, we have

$$\begin{aligned} 2^{kp_j} \sim \Xi \lesssim |X-\Xi| &= \left|\left|(P_j+\nu_j X)\frac{x}{|x|}\right| - \left|(P_j+\nu_j\Xi)\frac{\xi}{|\xi|}\right|\right| \\ &\leqslant \left|(P_j+\nu_j X)\frac{x}{|x|} + (P_j+\nu_j\Xi)\frac{\xi}{|\xi|}\right| \lesssim \mu. \end{aligned} \tag{2-6}$$

When $t2^{k(p_j+1)} \geqslant 1$, we shall apply the Calderón–Vaillancourt theorem (see Lemma A.11). For each derivative in x, if it acts on χ, one gains $t^{-1/2}\mu^{-1} \lesssim t^{-1/2}2^{-kp_j}$ by (2-6). If ∂_x acts on Ψ or ψ, one obtains factors of size $t^{-1/2}2^{-kp_j}$. As for the derivatives in ξ, similarly, it leads to factors of size $P''(\xi/\sqrt{t})t^{-1/2}\mu^{-1}$, $\Xi^{-1}P''(\xi/\sqrt{t})t^{-1/2}$ or $t^{-1/2}2^{-k}$, which are all controlled by $t^{-1/2}2^{-k}$, as $|\xi| \sim 2^k\sqrt{t}$ and $2^{kp_j} \lesssim \mu$. Since

$$t^{-\frac12}2^{-kp_j} \times t^{-\frac12}2^{-k} = t^{-1}2^{-k(p_j+1)} \leqslant 1,$$

we may conclude by a change of scaling (Lemma A.4).

When $t2^{k(p+1)} \leqslant 1$, we shall use Lemma A.9. We first check the assumption (A-6) of this lemma with $\mu_k \in]0,+\infty[$ defined by

$$\mu_k = \begin{cases} \mu 2^{-kp_j} & \text{if } P_j+\nu_j X \sim 2^{kp_j}, \\ \mu & \text{if } P_j+\nu_j X \sim 1. \end{cases}$$

Note that we have either $P_j+\nu_j X \sim P_j+\nu_j\Xi \sim 1$ or $P_j+\nu_j X \sim P_j+\nu_j\Xi \sim 2^{kp_j}$. In fact, by definition, $P_j+\nu_j X$ and $P_j+\nu_j\Xi$ are both strictly positive. Thus, it is sufficient to consider $|k| \gg 1$. When P_j is nonzero and $kp_j < 0$, we have $X \sim \Xi \sim 2^{kp_j} \ll 1$ and then $P_j+\nu_j X \sim P_j+\nu_j\Xi \sim 1$. While P_j is nonzero and $kp_j > 0$, we have similarly $X \sim \Xi \sim 2^{kp_j} \gg 1$ and $P_j+\nu_j X \sim P_j+\nu_j\Xi \sim 2^{kp_j}$. Otherwise, P_j is equal to zero, which implies trivially $P_j+\nu_j X = X \sim 2^{kp_j}$ and $P_j+\nu_j\Xi = \Xi \sim 2^{kp_j}$. Due to observation $X \sim \Xi \sim 2^{kp_j}$, it is easy to obtain that, for all $\alpha, \beta \in \mathbb{N}^{d-1}$ and $N \in \mathbb{N}$,

$$|\partial_\omega^\alpha\partial_\theta^\beta \tilde{b}_k(t,\mu,r\omega,\rho\theta)| \leqslant C_{\alpha,\beta,N}\, g_k(r,\rho)\mu_k^{-|\alpha|-|\beta|}\left\langle\frac{d(\omega,-\theta)}{\mu_k}\right\rangle^{-N},$$

where $x = r\omega,\ \xi = \rho\theta$ are polar coordinates and

$$g_k(r,\rho) = \mathbb{1}_{\rho\sim 2^k\sqrt{t}}\mathbb{1}_{r\sim t^{1/2}2^{kp_j}}.$$

The operator of kernel g_k is controlled by

$$\|g_k\|_{L^2(drd\rho)} \lesssim (\sqrt{t}2^k \times \sqrt{t}2^{kp_j})^{\frac{1}{2}} = (t2^{k(p_j+1)})^{\frac{1}{2}} \leqslant 1,$$

which is no more than the assumption (A-5) of Lemma A.9. As a result, we may conclude (2-2) by (A-7). □

The idea of the proof of (2-1) is similar. The only difficulty is that b_1 has a singularity near $X = \Xi$. We may treat the part away from $X = \Xi$ as above and study the area near $X = \Xi$ by convexity (or concavity) of P.

Proof of (2-1). As before, we begin with the homogeneous dyadic decomposition in $\xi/\sqrt{t}$, namely $b_1^\sharp = \sum_{k\geqslant 0} b_{1,k}$ and $b_1^\flat = \sum_{k<0} b_{1,k}$, with

$$b_{1,k} = \chi\left(\frac{(P_j+\nu_j X)\frac{x}{|x|} + (P_j+\nu_j\Xi)\frac{\xi}{|\xi|}}{\mu}\right)\mathbb{1}_{\frac{X}{\Xi}<1}\chi_1\left(1-\frac{X}{\Xi}\right)\psi(2^{-kp_j}X)\varphi\left(\frac{\xi}{2^k\sqrt{t}}\right).$$

It suffices to prove that $\mathrm{Op}(b_{1,k})$ is bounded on L^2, uniformly in k, t, μ. In comparison with $\tilde{b}_k$ defined by (2-5), the main difficulty is that the nonsmooth term cannot be deleted. In the case $t2^{k(p_j+1)} \leqslant 1$, we may repeat exactly the same argument as in the study of $\tilde{b}_k$ since this argument does not require any regularity in $|x|, |\xi|$.

When $t2^{k(p_j+1)} > 1$, we will separate the singularity near $X/\Xi = 1$. Consider the decomposition

$$b_{1,k} = b'_{1,k} + b''_{1,k},$$
$$b'_{1,k} = b_{1,k}\tilde{\chi}_1(\sqrt{t}2^{k\frac{1}{2}(1-p_j)}(\Xi - X)),$$

where $\tilde{\chi}_1 \in C_c^\infty(\mathbb{R}^d)$ is radial and equal to 1 near zero.

The proof of the boundedness of $b'_{1,k}$ is similar to that of the case $t2^{k(p_j+1)} \leqslant 1$. By setting $\mu_k \in]0, +\infty[$ as before, namely

$$\mu_k = \begin{cases} \mu 2^{-kp_j} & \text{if } P_j+\nu_j X \sim 2^{kp_j}, \\ \mu & \text{if } P_j+\nu_j X \sim 1, \end{cases}$$

we may have, for all $\alpha, \beta \in \mathbb{N}^{d-1}$ and $N \in \mathbb{N}$,

$$|\partial_\omega^\alpha \partial_\theta^\beta b'_{1,k}(t,\mu,r\omega,\rho\theta)| \leqslant C_{\alpha,\beta,N} h_k(r,\rho)\mu_k^{-|\alpha|-|\beta|}\left\langle\frac{d(\omega,-\theta)}{\mu_k}\right\rangle^{-N},$$

where

$$h_k(r,\rho) = \sum_{n\sim\sqrt{t}2^{k(p_j+1)/2}} \mathbb{1}_{I_n}(r)\mathbb{1}_{J_n}(\rho), \tag{2-7}$$

with

$$J_n := [2^{k\frac{1}{2}(1-p_j)}n,\ 2^{k\frac{1}{2}(1-p_j)}(n+1)],$$

$$I_n := \begin{cases} \left[\sqrt{t}P'(t^{-\frac{1}{2}}2^{k\frac{1}{2}(1-p_j)}n) - c2^{k\frac{1}{2}(p_j-1)},\ \sqrt{t}P'(t^{-\frac{1}{2}}2^{k\frac{1}{2}(1-p_j)}(n+1)) + c2^{k\frac{1}{2}(p_j-1)}\right] & \text{if } P''>0, \\ \left[\sqrt{t}P'(t^{-\frac{1}{2}}2^{k\frac{1}{2}(1-p_j)}(n+1)) - c2^{k\frac{1}{2}(p_j-1)},\ \sqrt{t}P'(t^{-\frac{1}{2}}2^{k\frac{1}{2}(1-p_j)}n) + c2^{k\frac{1}{2}(p_j-1)}\right] & \text{if } P''<0. \end{cases}$$

Note that by writing in the polar system $r = |x|,\ \rho = |\xi|$, we have that $b'_{1,k}$ is supported for $\rho \sim 2^k\sqrt{t}$ and $\sqrt{t}2^{k(1-p_j)/2}|\Xi - X| \ll 1$ due to the cut-off $\tilde{\chi}_1$. We first make a decomposition in ρ, namely

$$\rho \in [C^{-1}2^k\sqrt{t}, C2^k\sqrt{t}] \subset \bigcup_{n\sim\sqrt{t}2^{k(p_j+1)/2}} J_n,$$

and then the support $\sqrt{t}2^{k(1-p_j)/2}|\Xi - X| \ll 1$ ensures that r lies in I_n defined above, once ρ belongs to J_n. This gives the control h_k defined in (2-7).

In order to apply Lemma A.9, it suffices to check that the operator with kernel h_k is uniformly bounded on $L^2(\mathbb{R}_+)$, which can be reduced to $|I_n||J_n| \lesssim 1$ and that $\{I_n\}$ forms a uniformly finite cover. In fact, since the I_n's are pairwise disjoint (except for end points), one observes that, for all $w \in L^2(\mathbb{R}_+)$,

$$\begin{aligned}\left\|\int h_k(\,\cdot\,,\rho)w(\rho)\,d\rho\right\|^2_{L^2(\mathbb{R}_+)} &= \left\|\sum_n \mathbb{1}_{I_n}(\,\cdot\,)\int \mathbb{1}_{J_n}(\rho)w(\rho)\,d\rho\right\|^2_{L^2(\mathbb{R}_+)}\\ &= \sum_n\left\|\mathbb{1}_{I_n}(\,\cdot\,)\int \mathbb{1}_{J_n}(\rho)w(\rho)\,d\rho\right\|^2_{L^2(\mathbb{R}_+)}\\ &\leqslant \sum_n |I_n||J_n|\|\mathbb{1}_{J_n}w\|^2_{L^2(\mathbb{R}_+)} \lesssim \sum_n \|\mathbb{1}_{J_n}w\|^2_{L^2(\mathbb{R}_+)},\end{aligned}$$

where the last inequality follows from the first assertion $|I_n||J_n| \lesssim 1$. The second assertion guarantees that each point of $\mathbb{R}_+$ belongs to at most N intervals in $\{J_n\}$ for some $N \in \mathbb{N}$. This implies that $\sum_n \|\mathbb{1}_{J_n}w\|^2_{L^2(\mathbb{R}_+)} \leqslant N\|w\|^2_{L^2(\mathbb{R}_+)}$, which proves the uniform-in-t, k $L^2(\mathbb{R}_+)$-boundedness of the operator with kernel h_k.

The first assertion is obvious since $n \sim \sqrt{t}2^{k(p_j+1)/2}$ implies that

$$\begin{aligned}|J_n||I_n| &\lesssim 2^{k\frac{1}{2}(1-p_j)}\left(\left|\sqrt{t}P'(t^{-\frac{1}{2}}2^{k\frac{1}{2}(1-p_j)}(n+1)) - \sqrt{t}P'(t^{-\frac{1}{2}}2^{k\frac{1}{2}(1-p_j)}n)\right| + 2c2^{k\frac{1}{2}(p_j-1)}\right)\\ &\lesssim 2^{k\frac{1}{2}(1-p_j)}\left(\sqrt{t}2^{k(p_j-1)}t^{-\frac{1}{2}}2^{k\frac{1}{2}(1-p_j)} + 2c2^{k\frac{1}{2}(p_j-1)}\right) \lesssim 1.\end{aligned}$$

As for the second one, we observe that $I_n \cap I_{n+l} \neq \varnothing$ if and only if

$$\left|\sqrt{t}P'(t^{-\frac{1}{2}}2^{k\frac{1}{2}(1-p_j)}n) - \sqrt{t}P'(t^{-\frac{1}{2}}2^{k\frac{1}{2}(1-p_j)}(n+l))\right| \leqslant 2c2^{k\frac{1}{2}(p_j-1)}.$$

Without loss of generality, we may assume $l \geqslant 0$. Actually, the left-hand side has the following equivalence:

$$\begin{aligned}&\left|\sqrt{t}P'(t^{-\frac{1}{2}}2^{k\frac{1}{2}(1-p_j)}n) - \sqrt{t}P'(t^{-\frac{1}{2}}2^{k\frac{1}{2}(1-p_j)}(n+l))\right|\\ &\qquad = \left|2^{k\frac{1}{2}(1-p)}lP''(t^{-\frac{1}{2}}2^{k\frac{1}{2}(1-p_j)}(n+sl))\right| \qquad \text{for some } s \in [0,1]\\ &\qquad \sim 2^{k\frac{1}{2}(1-p_j)}l \times (t^{-\frac{1}{2}}2^{k\frac{1}{2}(1-p_j)}(n+sl))^{p_j-1} \qquad \text{since } |P''(\rho)| \sim \rho^{p_j-1}\\ &\qquad \sim 2^{k\frac{1}{2}(1-p_j)}l \times 2^{k(p_j-1)} = 2^{k\frac{1}{2}(p_j-1)}l.\end{aligned}$$

To prove the last equivalence, we may use the fact that $n, n+l \sim \sqrt{t}2^{k(p_j+1)/2}$, which implies that

$$c\sqrt{t}2^{k\frac{1}{2}(p_j+1)} \leqslant n \leqslant n+sl \leqslant n+l \leqslant C\sqrt{t}2^{k\frac{1}{2}(p_j+1)}.$$

In conclusion, we have that $I_n \cap I_{n+l} \neq \varnothing$ holds for finitely many l. As a result, $\mathrm{Op}(b'_{1,k})$ is bounded uniformly in t, k.

It remains to study the smooth symbol $b''_{1,k}$, which reads

$$b''_{1,k} = \chi\left(\frac{(P_j+\nu_j X)\frac{x}{|x|}+(P_j+\nu_j\Xi)\frac{\xi}{|\xi|}}{\mu}\right)\psi(2^{-kp}X)\varphi\left(\frac{\xi}{2^k\sqrt{t}}\right)\mathbb{1}_{\frac{X}{\Xi}<1}\chi_1\left(1-\frac{X}{\Xi}\right) \times(1-\tilde{\chi}_1)(\sqrt{t}2^{k\frac{1}{2}(1-p_j)}(\Xi-X)).$$

Note that this symbol is smooth, since the singularity $X/\Xi=1$ is removed by the $(1-\tilde{\chi}_1)$ factor. Under the condition $t2^{k(p_j+1)}>1$, it satisfies the condition of the Calderón–Vaillancourt theorem (see Lemma A.11). In fact, each derivative in x leads to a factor of size $t^{-1/2}\mu^{-1}$ (from χ), $t^{-1/2}2^{-kp_j}$ (from ψ and χ_1), or $2^{k(1-p_j)/2}$ (from $(1-\tilde{\chi}_1)$). The condition $t2^{k(p_j+1)}>1$ implies that $t^{-1/2}2^{-kp_j}\leqslant 2^{k(1-p_j)/2}$, while the compact support of χ and support of $(1-\tilde{\chi}_1)$ ensures that

$$t^{-\frac{1}{2}}2^{k\frac{1}{2}(p_j-1)}\lesssim|X-\Xi|\lesssim\mu,$$

i.e., $t^{-1/2}\mu^{-1}\lesssim 2^{(1-p_j)/2k}$. The same argument for ∂_ξ gives that each derivative in ξ leads to a factor of size $2^{-k(1-p_j)/2}$. The desired result thus follows from a change of scaling (Lemma A.4). □

2.2. *Study of the symbol b_0 with $P_0=P_1=0$.* In the case $P_0=P_1=0$, due to the lack of almost orthogonality as $\tilde{b}_k$'s and $b_{1,k}$'s, the remaining symbol b_0^ι will be treated via the Cotlar–Stein lemma (Lemma A.3). As before, we start with homogeneous dyadic decomposition in ξ, namely $b_0^\sharp=\mathbb{1}_{X>0}\sum_{k\geqslant 0}c_k$ and $b_0^\flat=\mathbb{1}_{X>0}\sum_{k<0}c_k$, with

$$c_k=\chi\left(\frac{(P_j+\nu_j X)\frac{x}{|x|}+(P_j+\nu_j\Xi)\frac{\xi}{|\xi|}}{\mu}\right)\chi_0\left(\frac{X}{\Xi}\right)\varphi\left(\frac{\xi}{2^k\sqrt{t}}\right). \tag{2-8}$$

It suffices to prove the (uniform-in-t) boundedness of $\sum_{k\in\mathbb{Z}}c_k$ as the multiplication with $\mathbb{1}_{X>0}$ is trivially bounded on L^2.

We first check that the $\mathrm{Op}(c_k)$'s are bounded uniformly in k, t, μ. More precisely, all the c_k's satisfy the following estimate:

Lemma 2.5. *There exists $C>0$ independent of k, t such that, for all $t>0$ and $k\in\mathbb{Z}$,*

$$\|\mathrm{Op}(c_k)\|_{\mathcal{L}(L^2)}\leqslant C\min\bigl(\max(1,(t2^{k(p_j+1)})^{-N_d}),(t2^{k(p_j+1)})^{\frac{d}{2}}\bigr)\leqslant C, \tag{2-9}$$

where $N_d\in\mathbb{N}$ depends only on dimension d.

Proof. We observe that c_k is supported for $X\ll\Xi\sim 2^{kp_j}$, which implies, on the one hand, as in (2-6),

$$2^{kp_j}\sim|X-\Xi|\leqslant\left|(P_j+\nu_j X)\frac{x}{|x|}+(P_j+\nu_j\Xi)\frac{\xi}{|\xi|}\right|\lesssim\mu,$$

and, on the another hand,

$$|x|\ll\sqrt{t}2^{kp_j},\quad |\xi|\sim\sqrt{t}2^k.$$

As a consequence of the second result, $\|\mathrm{Op}(c_k)\|_{\mathcal{L}(L^2)}$ can be trivially bounded by

$$\|\mathrm{Op}(c_k)\|_{\mathcal{L}(L^2)} \lesssim \|c_k\|_{L^2(\mathbb{R}^{2d})} \lesssim (t2^{k(p_j+1)})^{\frac{1}{2}d}.$$

It remains to check that

$$\|\mathrm{Op}(c_k)\|_{\mathcal{L}(L^2)} \leqslant C \max\bigl(1, (t2^{k(p_j+1)})^{-N_d}\bigr),$$

which can be proved via the Calderón–Vaillancourt theorem (see Lemma A.11). In fact, from each derivative in x, we may obtain extra factors of size $t^{-1/2}\mu^{-1}$ (action on χ) or $t^{-1/2}2^{-kp_j}$ (action on χ_0). As we have seen that $t^{-1/2}\mu^{-1} \lesssim t^{-1/2}2^{-kp_j}$, each derivative in x leads to a factor of size $(t^{1/2}2^{kp_j})^{-1}$. Similarly, the action of ∂_ξ on χ, χ_0, φ gives factors of size $t^{-1/2}\mu^{-1}2^{k(p_j-1)}$, $t^{-1/2}2^{-k}$, and $t^{-1/2}2^{-k}$, respectively. We may also check that $t^{-1/2}\mu^{-1}2^{k(p_j-1)} \lesssim 2^{-k}t^{-1/2}$. To sum up, c_k is smooth and satisfies

$$|\partial_x^\alpha \partial_\xi^\beta c_k(x,\xi)| \leqslant C_{\alpha,\beta}\left(\frac{1}{\sqrt{t}2^{kp_j}}\right)^{|\alpha|}\left(\frac{1}{\sqrt{t}2^{k}}\right)^{|\beta|} \quad \forall \alpha, \beta \in \mathbb{N}^d. \tag{2-10}$$

By Lemma A.4, it is equivalent to consider the rescaled symbol

$$\tilde{c}_k(x,\xi) = c_k(2^{k\frac{1}{2}(p_j-1)}x, 2^{-k\frac{1}{2}(p_j-1)}\xi),$$

which, as a result of (2-10), satisfies, for all $\gamma \in \mathbb{N}^{2d}$,

$$\|\partial_{x,\xi}^\gamma \tilde{c}_k\|_{L^\infty(\mathbb{R}^{2d})} \lesssim (t2^{k(p_j+1)})^{-|\gamma|}.$$

By applying the Calderón–Vaillancourt theorem (Lemma A.11) to $\tilde{c}_k$, we have, due to estimate (A-8), that

$$\|\mathrm{Op}(c_k)\|_{\mathcal{L}(L^2)} = \|\mathrm{Op}(\tilde{c}_k)\|_{\mathcal{L}(L^2)} \leqslant C \max\bigl(1, (t2^{k(p_j+1)})^{-N_d}\bigr). \qquad \square$$

In order to conclude (2-3) by the Cotlar–Stein lemma (Lemma A.3), it is sufficient to check conditions (A-2) and (A-3), namely:

Lemma 2.6. *There exist t, μ-independent constants C such that, for all $t, \mu > 0$,*

$$\sup_{k\in\mathbb{Z}_+}\sum_{l\in\mathbb{Z}_+}\|\mathrm{Op}(c_k)\,\mathrm{Op}(c_l)^*\|_{\mathcal{L}(L^2)}^{\frac{1}{2}} \leqslant C, \quad \sup_{k\in\mathbb{Z}_-}\sum_{l\in\mathbb{Z}_-}\|\mathrm{Op}(c_k)\,\mathrm{Op}(c_l)^*\|_{\mathcal{L}(L^2)}^{\frac{1}{2}} \leqslant C, \tag{2-11}$$

$$\sup_{k\in\mathbb{Z}_+}\sum_{l\in\mathbb{Z}_+}\|\mathrm{Op}(c_k)^*\,\mathrm{Op}(c_l)\|_{\mathcal{L}(L^2)}^{\frac{1}{2}} \leqslant C, \quad \sup_{k\in\mathbb{Z}_-}\sum_{l\in\mathbb{Z}_-}\|\mathrm{Op}(c_k)^*\,\mathrm{Op}(c_l)\|_{\mathcal{L}(L^2)}^{\frac{1}{2}} \leqslant C, \tag{2-12}$$

where $\mathbb{Z}_- = \mathbb{Z} \cap \,]-\infty, 0[$ corresponds to the low-frequency part and $\mathbb{Z}_+ = \mathbb{Z} \cap [0, +\infty[$ corresponds to the high-frequency part.

Proof of (2-11). By symbolic calculus, $\mathrm{Op}(c_k)\,\mathrm{Op}(c_l)^*$ is an operator of symbol

$$c_k \sharp c_l^*(x,\xi) = \frac{1}{(2\pi)^d}\int e^{-iy\eta} c_k(x, \xi+\eta)\overline{c_l(x+y, \xi+\eta)}\, d\eta.$$

By definition (2-8), $c_l(x,\xi)$ is supported for $|\xi| \sim 2^l\sqrt{t}$. Thus, $c_k \sharp c_l^*$ is nonzero only if $|l-k| < N_0$ for some large $N_0 \in \mathbb{N}^*$. As a consequence, (2-11) can be reduced to the uniform boundedness of $\mathrm{Op}(c_l)$, which has already been proved in Lemma 2.5. $\square$

Proof of (2-12). We apply again the symbolic calculus to obtain the following expression of the symbol of $\operatorname{Op}(c_k)^* \operatorname{Op}(c_l)$:

$$c_k^* \sharp c_l(x,\xi) = \frac{1}{(2\pi)^d} \int e^{i(x-y)\eta} \overline{c_k(y,\xi+\eta)} c_l(y,\xi)\, d\eta\, dy.$$

Recall that c_l is supported for $|x| \ll \sqrt{t} 2^{lp_j}$ and $|\xi| \sim t^{1/2} 2^l$ with the estimate (2-10). We shall check that, for all $l, k \in \mathbb{N}$,

$$\|\operatorname{Op}(c_k)^* \operatorname{Op}(c_l)\|_{\mathcal{L}(L^2)} \lesssim 2^{-\frac{1}{2}d|k-l|}, \tag{2-13}$$

which is enough to conclude (2-12). Due to (2-9), we may ignore the case $|k-l| \leqslant N_0$ for some fixed large $N_0 \in \mathbb{N}$. Note that it is sufficient to prove (2-13) only for $l \geqslant k$, since for terms with $l < k$, we have

$$\begin{aligned} \|\operatorname{Op}(c_k)^* \operatorname{Op}(c_l)\|_{\mathcal{L}(L^2)} &= \|(\operatorname{Op}(c_k)^* \operatorname{Op}(c_l))^*\|_{\mathcal{L}(L^2)} \\ &= \|\operatorname{Op}(c_l)^* \operatorname{Op}(c_k)\|_{\mathcal{L}(L^2)} \lesssim 2^{-\frac{1}{2}d|l-k|}. \end{aligned}$$

One observes that the bound of the operator with symbol $c_k^* \sharp c_l$ can be controlled by

$$\|\operatorname{Op}(c_k^* \sharp c_l)\|_{\mathcal{L}(L^2)} \lesssim \|c_k^* \sharp c_l\|_{L^2(dx\,d\xi)} \lesssim \left\| \int e^{-iy\eta} \overline{c_k(y,\xi+\eta)} c_l(y,\xi)\, dy \right\|_{L^2(d\eta\, d\xi)}.$$

The integrand of the last integral is supported for

$$|\xi+\eta| \sim 2^k \sqrt{t}, \quad |\xi| \sim 2^l \sqrt{t} \quad \text{and} \quad |y| \lesssim \min(2^{kp_j}\sqrt{t}, 2^{lp_j}\sqrt{t}). \tag{2-14}$$

Moreover, we may apply integration by parts in y to obtain some extra bounds in the estimate. To be precise, for all $N_1 \in \mathbb{N}$,

$$\begin{aligned} \int e^{-iy\eta} \overline{c_k(y,\xi+\eta)} c_l(y,\xi)\, dy &= \int \left(\frac{-\Delta_y}{|\eta|^2} \right)^{N_1} e^{-iy\eta} \overline{c_k(y,\xi+\eta)} c_l(y,\xi)\, dy \\ &= \int e^{-iy\eta} (-\Delta_y)^{N_1} \big(\overline{c_k(y,\xi+\eta)} c_l(y,\xi) \big) |\eta|^{-2N_1}\, dy \\ &= \sum_{|\alpha|+|\beta|=2N_1} C_{\alpha,\beta} \int e^{-iy\eta} \overline{\partial_y^\alpha c_k(y,\xi+\eta)} \partial_y^\beta c_l(y,\xi) |\eta|^{-2N_1}\, dy. \end{aligned}$$

Since we have reduced our problem to the case $l \geqslant k + N_0$, the integral above is supported for $|\eta| \sim 2^l \sqrt{t}$. Together with (2-10) and (2-14), we have

$$\begin{aligned} &\left| \int e^{-iy\eta} \overline{c_k(y,\xi+\eta)} c_l(y,\xi)\, dy \right| \\ &\quad \leqslant \sum_{|\alpha|+|\beta|=2N_1} C_{\alpha,\beta} \int |\partial_y^\alpha c_k(y,\xi+\eta)| \, |\partial_y^\beta c_l(y,\xi)| \, |\eta|^{-2N_1}\, dy \\ &\quad \lesssim \sum_{|\alpha|+|\beta|=2N_1} \mathbb{1}_{|\xi+\eta|\sim 2^k\sqrt{t}} \mathbb{1}_{|\xi|\sim 2^l\sqrt{t}} (2^{kp_j}\sqrt{t})^{-|\alpha|} (2^{lp_j}\sqrt{t})^{-|\beta|} (2^l\sqrt{t})^{-2N_1} \int \mathbb{1}_{|y|\lesssim \min(2^{kp_j}\sqrt{t}, 2^{lp_j}\sqrt{t})}\, dy \\ &\quad \lesssim \mathbb{1}_{|\xi+\eta|\sim 2^k\sqrt{t}} \mathbb{1}_{|\xi|\sim 2^l\sqrt{t}} (2^l\sqrt{t})^{-2N_1} \min(2^{kp_j}\sqrt{t}, 2^{lp_j}\sqrt{t})^{d-2N_1}. \end{aligned}$$

The estimate above holds for all $N_1 \in \mathbb{N}$, and thus for all $N_1 \in [0, \infty[$. In particular, we choose $N_1 = \frac{1}{2}d$, which gives

$$\begin{aligned}\|\mathrm{Op}(c_k^* \sharp c_l)\|_{\mathcal{L}(L^2)} &\lesssim \left\| \int e^{-iy\eta} \overline{c_k(y, \xi+\eta)} c_l(y,\xi)\, dy \right\|_{L^2(d\eta\, d\xi)} \\ &\lesssim (2^l \sqrt{t})^{-d} \|\mathbb{1}_{|\xi+\eta|\sim 2^k\sqrt{t}} \mathbb{1}_{|\xi|\sim 2^l\sqrt{t}}\|_{L^2(d\eta\, d\xi)} \\ &\lesssim (2^l\sqrt{t})^{-d} \times (2^k\sqrt{t} \times 2^l \sqrt{t})^{\frac{1}{2}d} = 2^{\frac{1}{2}d(k-l)} = 2^{-\frac{1}{2}d|k-l|}.\end{aligned}$$

As a conclusion, we have managed to prove that

$$\sup_{k\in\mathbb{Z}_\pm} \sum_{l\in\mathbb{Z}_\pm} \|\mathrm{Op}(c_k)^* \,\mathrm{Op}(c_l)\|_{\mathcal{L}(L^2)}^{\frac{1}{2}} \lesssim \sup_{k\in\mathbb{Z}_\pm} \sum_{l\in\mathbb{Z}_\pm} 2^{-\frac{1}{4}d|k-l|} < \infty,$$

which completes the proof. □

2.3. ***Study of the symbol b_0 with $P_0, P_1 > 0$.*** Till now, we have finished the proof of Proposition 2.1. To complete the proof of Proposition 2.2, it remains to check (2-4). Note that the argument above relies on the fact that $|X| \ll 2^{kp_j}$ implies x is supported in a region of area $(\sqrt{t}2^{kp_j})^d$, which is not true in the case where P_0, P_1 are nonzero. To overcome this problem we need the extra truncation in ξ.

Proof of (2-4). As above, we may ignore the nonsmooth factor $\mathbb{1}_{0<X/\Xi<1}$. There remain smooth symbols

$$\tilde{c}^\sharp(t,x,\xi) = \chi\left(\frac{(P_1+\nu_1 X)\frac{x}{|x|} + (P_1\nu_1\Xi)\frac{\xi}{|\xi|}}{\mu}\right)\chi_0\left(\frac{X}{\Xi}\right)(1-\tilde{\chi})\left(\frac{\xi}{\sqrt{t}}\right)\chi_h\left(\frac{\xi}{t^{\frac{1}{2}+\epsilon_1}}\right),$$

$$\tilde{c}^\flat(t,x,\xi) = \chi\left(\frac{(P_0+\nu_0 X)\frac{x}{|x|} + (P_0\nu_0\Xi)\frac{\xi}{|\xi|}}{\mu}\right)\chi_0\left(\frac{X}{\Xi}\right)\tilde{\chi}\left(\frac{\xi}{\sqrt{t}}\right)(1-\chi_l)\left(\frac{\xi}{t^{\frac{1}{2}-\epsilon_0}}\right).$$

We shall first check that $\tilde{c}^\sharp$ belongs uniformly to the Hörmander class $S^0_{1,\kappa}$ for some $\kappa \in]0,1[$, namely the collection of smooth symbols $c(x,\xi)$ such that, for all $\alpha, \beta \in \mathbb{N}^d$,

$$|\partial_x^\alpha \partial_\xi^\beta c(x,\xi)| \lesssim \langle\xi\rangle^{-|\beta|+\kappa|\alpha|}.$$

It is well known that the operators with symbol in this class are bounded on L^2, a proof of which can be found in [Hörmander 1994]. We begin with the observation that the high-frequency symbol $\tilde{c}^\sharp$ is supported for $t^{1/2} \lesssim |\xi| \lesssim t^{1/2+\epsilon_1}$. Before calculating the bounds of derivatives in x and ξ, recall that our goal is to show (2-4) under the condition $t > 1$.

For each derivative in ξ, we obtain from χ a factor of size

$$t^{-\frac{1}{2}}\mu^{-1}P''\left(\frac{|\xi|}{\sqrt{t}}\right) \sim \mu^{-1}\left(\frac{|\xi|}{\sqrt{t}}\right)^{p_1}|\xi|^{-1} \lesssim \langle\xi\rangle^{-1}.$$

The last inequality is due the support of $\tilde{c}^\sharp$. More precisely,

$$\left(\frac{|\xi|}{\sqrt{t}}\right)^{p_1} \sim \Xi \lesssim |X-\Xi| \lesssim \left|(P_1+\nu_1 X)\frac{x}{|x|} + (P_1+\nu_1\Xi)\frac{\xi}{|\xi|}\right| \lesssim \mu.$$

From the factor χ_0, one gains

$$\frac{X}{\Xi}\frac{P''(\xi/\sqrt{t})}{\Xi\sqrt{t}} \sim \frac{X}{\Xi}\frac{1}{|\xi|} \lesssim \langle\xi\rangle^{-1}.$$

Trivially, we will also obtain $\langle\xi\rangle^{-1}$ from the derivative on $(1-\tilde{\chi})$ and χ_h.

As for derivatives in x, if ∂_x acts on χ, one gains $t^{-1/2}\mu^{-1} \lesssim (\sqrt{t}\,\Xi)^{-1}$. When it acts on χ_0, the resulting factor is of size

$$\frac{1}{\sqrt{t}\,\Xi} \sim \frac{1}{t}\left(\frac{|\xi|}{\sqrt{t}}\right)^{-p_1-1}|\xi|.$$

When $0 > p_1 \geqslant -1$, we have

$$\frac{1}{t}\left(\frac{|\xi|}{\sqrt{t}}\right)^{-p_1-1}|\xi| = |\xi|^{-p_1}t^{\frac{1}{2}(p_1-1)} \lesssim \langle\xi\rangle t^{-\frac{1}{2}} \lesssim \langle\xi\rangle^{\kappa} t^{\left(\frac{1}{2}+\epsilon_1\right)(1-\kappa)-\frac{1}{2}}.$$

Thus, $\tilde{c}^{\sharp} \in S^0_{1,\kappa}$ for any $\kappa \in]0,1[$ such that

$$\left(\tfrac{1}{2}+\epsilon_1\right)(1-\kappa)-\tfrac{1}{2} \leqslant 0,$$

which is possible by choosing κ close to 1. When $p_1 < -1$, the estimate above becomes

$$\frac{1}{t}\left(\frac{|\xi|}{\sqrt{t}}\right)^{-p_1-1}|\xi| = \frac{|\xi|^{1-\kappa}}{t}\left(\frac{|\xi|}{\sqrt{t}}\right)^{-p_1-1}|\xi|^{\kappa} \lesssim t^{-1+\left(\frac{1}{2}+\epsilon_1\right)(1-\kappa)-(p_1+1)\epsilon_1}\langle\xi\rangle^{\kappa}.$$

To conclude $\tilde{c}^{\sharp} \in S^0_{1,\kappa}$, it suffices to choose $\kappa \in]0,1[$ such that

$$-1+\left(\tfrac{1}{2}+\epsilon_1\right)(1-\kappa)-(p_1+1)\epsilon_1 \leqslant 0,$$

which is equivalent to

$$\epsilon_1 \leqslant \frac{1}{-(p_1+1)}\left[1-\left(\tfrac{1}{2}+\epsilon_1\right)(1-\kappa)\right].$$

This can be realized by choosing κ close to 1, due to the definition (1-19) of ϵ_1.

To prove the uniform boundedness of $\tilde{c}^{\flat}$, which is supported for $t^{1/2-\epsilon_0} \lesssim |\xi| \lesssim t^{1/2}$, we shall apply the Calderón–Vaillancourt theorem (Lemma A.11). As above one may check easily that each ∂_ξ gives

$$|\xi|^{-1} \lesssim t^{-\frac{1}{2}+\epsilon_0},$$

while each ∂_x gives

$$\frac{1}{\sqrt{t}}\left(\frac{|\xi|}{\sqrt{t}}\right)^{-p_0} \lesssim t^{-\frac{1}{2}+p_0\epsilon_0}.$$

As a consequence, the desired result follows from Lemma A.4 and the Calderón–Vaillancourt theorem (Lemma A.11) once we have

$$t^{-\frac{1}{2}+\epsilon_0} \times t^{-\frac{1}{2}+p_0\epsilon_0} \lesssim 1 \quad \forall t > 1,$$

equivalently, $(1+p_0)\epsilon_0 \leqslant 1$, which is exactly the definition (1-19) of ϵ_0. □

2.4. ***Proof of Theorem 1.8.*** In all the proof above, we regard $\mu = t^{\delta-1/2} \in]0, +\infty[$ as a time-independent parameter. This allows us to take the limit $\mu \to +\infty$ with all the uniform estimates remaining true. Rigorously, due to Corollary 2.4, for all $f, g \in \mathcal{S}(\mathbb{R}^d)$ and $t \in \mathbb{R}$, $\mu > 0$,

$$\left|\left\langle f, \mathrm{Op}\left(\chi\left(\frac{x + tP'(\xi)}{|t|\mu}\right)\mathbb{1}_{|x|>|tP'(\xi)|}\Omega(t,\xi)\right)g\right\rangle\right| \leqslant C\|f\|_{L^2}\|g\|_{L^2}, \tag{2-15}$$

where $\Omega = 1$ when $P_0 = P_1 = 0$, and

$$\Omega(t,\xi) = (1-\chi_l)\left(\frac{\xi}{|t|^{-\epsilon_0}}\right)\chi_h\left(\frac{\xi}{|t|^{\epsilon_1}}\right)$$

when $P_0, P_1 > 0$.

The left-hand side of (2-15) is equal to

$$\left|\frac{1}{(2\pi)^d}\int f(x)e^{-ix\xi}\chi\left(\frac{x+tP'(\xi)}{|t|\mu}\right)\mathbb{1}_{|x|>|tP'(\xi)|}\Omega(t,\xi)\overline{\hat{g}(\xi)}\,d\xi\,dx\right|,$$

which, when $\mu \to +\infty$, due to the dominated convergence theorem, tends to

$$\left|\frac{1}{(2\pi)^d}\int f(x)e^{-ix\xi}\mathbb{1}_{|x|>|tP'(\xi)|}\Omega(t,\xi)\overline{\hat{g}(\xi)}\,d\xi\,dx\right|,$$

where we take $\chi(0) = 1$ without loss of generality. In conclusion, for all $f, g \in \mathcal{S}(\mathbb{R}^d)$,

$$\left|\left\langle f, \mathrm{Op}(\mathbb{1}_{|x|>|tP'(\xi)|}\Omega(t,\xi))g\right\rangle\right| \leqslant C\|f\|_{L^2}\|g\|_{L^2}.$$

Theorem 1.8 follows from the density of $\mathcal{S}(\mathbb{R}^d)$ in L^2.

3. L^2-boundedness of microlocal truncation operators: an alternative symbol

In this section, we will treat those P with nonzero P_0, P_1 in an alternative way. Instead of adding extra truncation in ξ, we shall add some extra factor in the main truncation χ. To be precise:

Proposition 3.1. *Let P, Λ satisfy conditions* (H_{p_0,p_1}) *and* ($\mathrm{C}_{\sigma_0,\sigma_1}$) *of Section 1.2 respectively, with $p_1 < 0 < p_0$, $\sigma_0 \geqslant p_0$, and $\sigma_1 \leqslant p_1$. We further assume that $\delta + \frac{1}{2}\sigma_j < \frac{1}{2}$, $j = 0, 1$. Then there exist time-independent constants $C > 0$, $t_0 \gg 1$, such that, for all $|t| > t_0$,*

$$\|\mathrm{Op}(a^{\mathrm{alt}}_{\chi,\delta,\Lambda}(t))\|_{\mathcal{L}(L^2)} \leqslant C,$$

where $a^{\mathrm{alt}}_{\chi,\delta,\Lambda}$ is defined in (1-23)

Without loss of generality, we may assume $t > 0$. Meanwhile, the change of scaling (Lemma A.4) allows us to reduce to the symbol

$$b(t,x,\xi) = \chi\left(\frac{\frac{x}{\sqrt{t}} + P'\left(\frac{\xi}{\sqrt{t}}\right)}{t^{\delta-\frac{1}{2}}\Lambda(\xi)}\right)H\left(\frac{\frac{|x|}{\sqrt{t}} - P'\left(\frac{|\xi|}{\sqrt{t}}\right)}{t^{\delta-\frac{1}{2}}\Lambda(\xi)}\right), \tag{3-1}$$

where $H \in C_b^\infty(\mathbb{R}\backslash\{0\}) \cap L^\infty(\mathbb{R})$. To recover the desired result in Proposition 3.1, it suffices to take $H = \mathbb{1}_{]0,+\infty[}$.

As in the previous section, we set,

- when $P_j = 0$,

$$X := \frac{|x|}{\sqrt{t}}, \quad \Xi := P'\left(\frac{|\xi|}{\sqrt{t}}\right), \quad \nu_j = +;$$

- when $P_j > 0,\ P' > P_j$,

$$X := \frac{|x|}{\sqrt{t}} - P_j, \quad \Xi := P'\left(\frac{|\xi|}{\sqrt{t}}\right) - P_j, \quad \nu_j = +;$$

- when $P_j > 0,\ P' < P_j$,

$$X := P_j - \frac{|x|}{\sqrt{t}}, \quad \Xi := P_j - P'\left(\frac{|\xi|}{\sqrt{t}}\right), \quad \nu_j = -.$$

With these notations, via homogeneous dyadic decomposition, we may rewrite symbol b as

$$\begin{aligned} b(t,x,\xi) &= \sum_{k\in\mathbb{Z}} b_k(t,x,\xi), \\ b_k(t,x,\xi) &= \chi\left(\frac{(P_1+\nu_1 X)\frac{x}{|x|} + (P_1+\nu_1\Xi)\frac{\xi}{|\xi|}}{t^{\delta-\frac12}\Lambda(\xi)}\right) H\left(\frac{X-\Xi}{t^{\delta-\frac12}\Lambda(\xi)}\right)\varphi\left(\frac{\xi}{\sqrt{t}2^k}\right) \quad \forall k \geqslant 0, \\ b_k(t,x,\xi) &= \chi\left(\frac{(P_0+\nu_0 X)\frac{x}{|x|} + (P_0+\nu_0\Xi)\frac{\xi}{|\xi|}}{t^{\delta-\frac12}\Lambda(\xi)}\right) H\left(\frac{X-\Xi}{t^{\delta-\frac12}\Lambda(\xi)}\right)\varphi\left(\frac{\xi}{\sqrt{t}2^k}\right) \quad \forall k < 0, \end{aligned}$$

where $\varphi \in C_c^\infty(\mathbb{R}^d)$ is radial and supported in an annulus centered at zero.

One observes that, due to factors χ and φ, b_k is supported for

$$|X-\Xi| \lesssim t^{\delta-\frac12}\Lambda(\xi) \sim t^{\delta-\frac12}(\sqrt{t}2^k)^{\sigma_j},$$

where $j=0$ when $k<0$, $j=1$ when $k \geqslant 0$. By using the fact that $\Xi \sim |\xi/\sqrt{t}|^{p_j} \sim 2^{kp_j}$, we obtain

$$\left|\frac{X}{\Xi} - 1\right| \lesssim t^{\delta+\frac12\sigma_j-\frac12} 2^{k(\sigma_j-p_j)} \leqslant t^{\delta+\frac12\sigma_j-\frac12}.$$

The last inequality is the consequence of our assumptions $\sigma_0 \geqslant p_0$ and $\sigma_1 \leqslant p_1$. Since $\delta + \frac12\sigma_j - \frac12 < 0$, if we further assume that $t \geqslant t_0 \gg 1$, the inequality above implies that $X \sim \Xi$, which allows us to add a complementary factor $\psi(2^{-kp_j}X)$ to the definition of b_k, where $\psi \in C_c^\infty(\mathbb{R})$ is radial and supported in an annulus centered at zero. In this way, we may reduce Proposition 3.1 to:

Proposition 3.2. *Under the same assumptions as in Proposition 3.1, the operator* $\mathrm{Op}(b_k)$ *is bounded, uniformly in t and k.*

In what follows, we keep using the subscript j, where $j=0$ for $k<0$ and $j=1$ for $k \geqslant 0$. By definition,

$$b_k(t,x,\xi) = \chi\left(\frac{(P_j+\nu_j X)\frac{x}{|x|} + (P_j+\nu_j\Xi)\frac{\xi}{|\xi|}}{t^{\delta-\frac12}\Lambda(\xi)}\right) H\left(\frac{X-\Xi}{t^{\delta-\frac12}\Lambda(\xi)}\right)\varphi\left(\frac{\xi}{\sqrt{t}2^k}\right)\psi\left(\frac{X}{2^{kp_j}}\right).$$

When $P_j = 0$, b_k is supported for

$$\left|\frac{x}{|x|} + \frac{\xi}{|\xi|}\right| \leqslant \frac{1}{X}\left|X\frac{x}{|x|} + \Xi\frac{\xi}{|\xi|}\right| + \frac{1}{X}\left|(X-\Xi)\frac{\xi}{|\xi|}\right|$$
$$\lesssim \frac{1}{X}t^{\delta-\frac{1}{2}}\Lambda(\xi) \sim 2^{-kp_j}t^{\delta-\frac{1}{2}}(\sqrt{t}2^k)^{\sigma_j} = t^{\delta+\frac{1}{2}\sigma_j-\frac{1}{2}}2^{k(\sigma_j-p_j)}.$$

When $P_j \neq 0$, we observe that $2^{k\sigma_j} \leqslant 2^{kp_j} \leqslant 1$, due to the choice $p_1 < 0 < p_0$. By choosing $\operatorname{Supp}\varphi$ small, which allows us to take $\operatorname{Supp}\psi$ small, we have $P_j + \nu_j X \sim 1$. Thus, b_k is supported for

$$\left|\frac{x}{|x|} + \frac{\xi}{|\xi|}\right| \leqslant \frac{1}{P_j+\nu_j X}\left|(P_j+\nu_j X)\frac{x}{|x|} + (P_j+\nu_j\Xi)\frac{\xi}{|\xi|}\right| + \frac{1}{P_j+\nu_j X}\left|(X-\Xi)\frac{\xi}{|\xi|}\right|$$
$$\lesssim \frac{1}{P_j+\nu_j X}t^{\delta-\frac{1}{2}}\Lambda(\xi) \sim t^{\delta-\frac{1}{2}}(\sqrt{t}2^k)^{\sigma_j} = t^{\delta+\frac{1}{2}\sigma_j-\frac{1}{2}}2^{k\sigma_j}.$$

If $t2^{k(p_j+1)} \leqslant 1$, by setting $\mu := t^{\delta+\sigma_j/2-1/2}2^{k(\sigma_j-p_j)} \in]0,1[$ in the case $P_j = 0$, and $\mu := t^{\delta+\sigma_j/2-1/2}2^{k\sigma_j} \in]0,1[$ in the case $P_j \neq 0$, one may check that, for all $\alpha, \beta \in \mathbb{N}^{d-1}$ and $N \in \mathbb{N}$,

$$|\partial_\omega^\alpha \partial_\theta^\beta b_k(r\omega, \rho\theta)| \lesssim \mu^{-|\alpha|-|\beta|}\mathbb{1}_{r\lesssim\sqrt{t}2^{kp}}\mathbb{1}_{\rho\sim\sqrt{t}2^k}\left\langle\frac{d(\omega,-\theta)}{\mu}\right\rangle^{-N}.$$

By applying Lemma A.9, we have

$$\|\operatorname{Op}(b_k)\|_{\mathcal{L}(L^2)} \lesssim \|\mathbb{1}_{r\lesssim\sqrt{t}2^{kp_j}}\mathbb{1}_{\rho\sim\sqrt{t}2^k}\|_{L^2(\mathbb{R}^2_+)} = C\sqrt{t2^{k(p_j+1)}} \leqslant C.$$

If $t2^{k(p_j+1)} > 1$, we decompose b_k as the sum of b_k', b_k'', which are defined by

$$b_k(t,x,\xi) = b_k'(t,x,\xi) + b_k''(t,x,\xi),$$
$$b_k'(t,x,\xi) = b_k\chi_0\big(\sqrt{t}2^{k\frac{1}{2}(1-p_j)}(X-\Xi)\big),$$

where $\chi_0 \in C_c^\infty(\mathbb{R})$ is radial, supported in a neighborhood of zero, and equal to 1 near zero.

Clearly, b_k'' is smooth on $\mathbb{R}^{2d}$. Thus, to prove the boundedness of $\operatorname{Op}(b_k'')$, we may apply the Calderón–Vaillancourt theorem (Lemma A.11). By definition, b_k'' reads

$$\chi\left(\frac{(P_j+\nu_j X)\frac{x}{|x|} + (P_j+\nu_j\Xi)\frac{\xi}{|\xi|}}{t^{\delta-\frac{1}{2}}\Lambda(\xi)}\right)H\left(\frac{X-\Xi}{t^{\delta-\frac{1}{2}}\Lambda(\xi)}\right)\varphi\left(\frac{\xi}{\sqrt{t}2^k}\right)\psi\left(\frac{X}{2^{kp_j}}\right)(1-\chi_0)\big(\sqrt{t}2^{k\frac{1}{2}(1-p_j)}(X-\Xi)\big),$$

which is supported for

$$t^{-\frac{1}{2}}2^{k\frac{1}{2}(p_j-1)} \lesssim |X-\Xi| \lesssim t^{\delta-\frac{1}{2}}\Lambda(\xi) \sim t^{\delta+\frac{1}{2}\sigma_j-\frac{1}{2}}2^{k\sigma_j}.$$

As a consequence, $t^{-(\delta+\sigma_j/2)} \lesssim 2^{k(\sigma_j+(1-p_j)/2)}$.

For each ∂_x, when it acts on χ, one obtains in its bound an extra factor of size

$$\frac{1}{\sqrt{t}}t^{\frac{1}{2}-\delta}\Lambda(\xi)^{-1} \sim t^{-(\delta+\frac{1}{2}\sigma_j)}2^{-kp_j} \lesssim 2^{k\frac{1}{2}(1-p_j)}2^{k(\sigma_j-p_j)} \leqslant 2^{k\frac{1}{2}(1-p_j)}.$$

When ∂_x acts on ψ and $(1-\chi_0)$ factors, we gain $t^{-1/2}2^{-kp_j} < 2^{k(p_j+1)/2-kp_j} = 2^{k(1-p_j)/2}$ and $2^{k(1-p_j)/2}$, respectively. For each ∂_ξ, similarly, one gains $2^{k(p_j-1)/2}$ in its bound. Namely, for all $\alpha, \beta \in \mathbb{N}^d$,

$$|\partial_x^\alpha \partial_\xi^\beta b_k''(x,\xi)| \lesssim 2^{k\frac{1}{2}(1-p_j)(|\alpha|-|\beta|)}.$$

The uniform boundedness of $\mathrm{Op}(b_k'')$ follows from Lemma A.4 and the Calderón–Vaillancourt theorem (Lemma A.11).

It remains to study the symbol b_k'. To overcome the singularity near $X = \Xi$, we will apply Lemma A.9 with the same setting of μ as in the previous paragraph, namely $\mu := t^{\delta+\sigma_j/2-1/2}2^{k(\sigma_j-p_j)} \in]0,1[$ in the case $P_j = 0$, and $\mu := t^{\delta+\sigma_j/2-1/2}2^{k\sigma_j} \in]0,1[$ in the case $P_j \neq 0$. It is easy to check that, for all $\alpha, \beta \in \mathbb{N}^{d-1}$ and $N \in \mathbb{N}$,

$$|\partial_\omega^\alpha \partial_\theta^\beta b_k(r\omega, \rho\theta)| \lesssim \mu^{-|\alpha|-|\beta|} h_k(r,\rho) \left\langle \frac{d(\omega,-\theta)}{\mu} \right\rangle^{-N},$$

where

$$h_k(r,\rho) = \sum_{n\sim\sqrt{t}2^{k(p_j+1)/2}} \mathbb{1}_{I_n}(r)\mathbb{1}_{J_n}(\rho),$$

which is exactly the same one defined in (2-7). We have seen that the operator with symbol h_k is bounded on $L^2(\mathbb{R}_+)$ uniformly in k. By Lemma A.9, we may conclude the uniform-in-k boundedness of b_k' and the proof of Proposition 3.2; hence Proposition 3.1 is completed.

4. Limit of truncated energy

In this section, we will complete the proof of Theorems 1.1, 1.4, and 1.6 by calculating the limit of truncated energy for some regular initial data u_0. These three results will follow from the proposition below:

Proposition 4.1. *Let $a_{\chi,\delta,\Lambda}(t)$ be the symbol defined by*

$$a = a_{\chi,\delta,\Lambda}(t,x,\xi) = \chi\left(\frac{x+tP'(\xi)}{|t|^{\frac{1}{2}+\delta}\Lambda(|t|^{\frac{1}{2}}\xi)}\right)\mathbb{1}_{|x|>|t||P'(\xi)|}, \tag{4-1}$$

where $P \in C^\infty(\mathbb{R}^d\backslash\{0\})$ is assumed to be a real radial function satisfying that $P''(\rho) \neq 0$ for all $\rho \in]0,\infty[$. Furthermore, we assume that $\chi \in C_c^\infty(\mathbb{R}^d)$ is real and radial with $\chi(0) = 1$ and that Λ verifies condition $(\mathrm{C}_{\sigma_0,\sigma_1})$ without any restriction in σ_0, σ_1.

With these settings, if there exists $t_0 \gg 1$ such that $\mathrm{Op}(a(t))$ is bounded on L^2 uniformly in $|t| > t_0$, for all $u_0 \in L^2$, the limits (1-24), (1-25), and (1-28) hold true.

Corollary 4.2. *We consider the same symbol a with an extra truncation in ξ, i.e.,*

$$\tilde{a} = \tilde{a}_{\chi,\delta,\Lambda}(t,x,\xi) = \chi\left(\frac{x+tP'(\xi)}{|t|^{\frac{1}{2}+\delta}\Lambda(|t|^{\frac{1}{2}}\xi)}\right)\mathbb{1}_{|x|>|t||P'(\xi)|}(1-\chi_l)\left(\frac{\xi}{|t|^{-\epsilon_0}}\right)\chi_h\left(\frac{\xi}{|t|^{\epsilon_1}}\right),$$

where $\chi_l, \chi_h \in C_c^\infty(\mathbb{R}^d)$ are equal to 1 in a neighborhood of zero, and $\epsilon_0, \epsilon_1 > 0$.

If P, Λ satisfy the same conditions as in Proposition 4.1 and $\mathrm{Op}(\tilde{a}_{\delta,\chi,\Lambda}(t))$ is bounded on L^2 uniformly in $|t| > t_0 \gg 1$, then, for all $u_0 \in L^2$, the limits (1-24), (1-25), and (1-28) hold true with $E^{\mathrm{alt}}_{\delta,\chi,\Lambda}(t)$ replaced by $\|\mathrm{Op}(\tilde{a}_{\delta,\chi,\Lambda}(t))u(t)\|^2_{L^2}$.

In order to complete the proof of Theorems 1.1 and 1.6, we may combine Propositions 2.1 and 3.1 with Proposition 4.1, where we need to take $\Lambda \equiv 1$ in the proof of Theorem 1.1. In the same way, Theorem 1.4 follows from Proposition 2.2 and Corollary 2.4 with $\Lambda \equiv 1$.

Before calculating the limit of truncated energy, we remark that, in the hypotheses of Proposition 4.1 and Corollary 4.2, $\mathrm{Op}(a(t))$ (or $\mathrm{Op}(\tilde{a}(t))$) is assumed to be bounded uniformly in $|t| > t_0 \gg 1$, which allows us to replace general $u_0 \in L^2$ by those belonging to some dense subset of L^2. In what follows, we may assume that $\hat{u}_0 \in C_c^\infty(\mathbb{R}\backslash\{0\})$. As a consequence, by taking $t_0 \gg 1$,

$$(1-\chi_l)\left(\frac{\xi}{|t|^{-\epsilon_0}}\right)\chi_h\left(\frac{\xi}{|t|^{\epsilon_1}}\right)\hat{u}_0(\xi) = \hat{u}_0(\xi),$$

which proves Corollary 4.2 from Proposition 4.1.

4.1. ***Supercritical case*** $\mathbf{0 < \delta + \frac{1}{2}\sigma_1 < \frac{1}{2}}$**.** In this part, we will study the case $\delta + \frac{1}{2}\sigma_1 \in \left]0, \frac{1}{2}\right[$ (associated to the limit (1-17), (1-22), or (1-28)) by following the same method introduced in [Delort 2022].

By definition, the truncated energy introduced in (1-11) is

$$\begin{aligned} E_{\chi,\delta,\Lambda}(\epsilon t) &= \|\mathrm{Op}(a_{\chi,\delta,\Lambda}(\epsilon t))u(\epsilon t)\|_{L^2}^2 \\ &= \frac{1}{(2\pi)^{2d}}\int e^{ix\cdot(\xi-\xi')}e^{i\epsilon t(P(\xi)-P(\xi'))}\chi\left(\frac{x+\epsilon tP'(\xi)}{t^{\frac{1}{2}+\delta}\Lambda(t^{\frac{1}{2}}\xi)}\right) \\ &\qquad \times \chi\left(\frac{x+\epsilon tP'(\xi')}{t^{\frac{1}{2}+\delta}\Lambda(t^{\frac{1}{2}}\xi)}\right)\mathbb{1}_{\frac{|x|}{t}>|P'(\xi)|,|P'(\xi')|}\hat{u}_0(\xi)\overline{\hat{u}_0(\xi')}\,dx\,d\xi\,d\xi', \end{aligned}$$

where $t \gg 1$ and $\epsilon = \pm$. In the polar system $x = r\omega$, $\xi = \rho\theta$, $\xi' = \rho'\theta'$, the integral above can be written as

$$\begin{aligned} \frac{1}{(2\pi)^{2d}}\int e^{i(r\rho\omega\cdot\theta - r\rho'\omega\cdot\theta')}e^{i\epsilon t(P(\rho)-P(\rho'))}\chi\left(\frac{r\omega+\epsilon tP'(\rho)\theta}{t^{\frac{1}{2}+\delta}\Lambda(t^{\frac{1}{2}}\rho)}\right)\chi\left(\frac{r\omega+\epsilon tP'(\rho')\theta'}{t^{\frac{1}{2}+\delta}\Lambda(t^{\frac{1}{2}}\rho')}\right) \\ \times \mathbb{1}_{\frac{r}{t}>P'(\rho),P'(\rho')}\hat{u}_0(\rho\theta)\overline{\hat{u}_0(\rho'\theta')}(r\rho\rho')^{d-1}\,d\theta\,d\theta'\,d\omega\,dr\,d\rho\,d\rho'. \end{aligned}$$

We firstly focus on the integral in θ, with integral in θ' treated in exactly the same way,

$$\int e^{ir\rho\omega\cdot\theta}\chi\left(\frac{r\omega+\epsilon tP'(\rho)\theta}{t^{\frac{1}{2}+\delta}\Lambda(t^{\frac{1}{2}}\rho)}\right)\hat{u}_0(\rho\theta)\,d\theta. \tag{4-2}$$

In this part, we always set

$$\mu = t^{\delta+\frac{1}{2}\sigma_1-\frac{1}{2}} \in]0, 1[,$$

which is strictly positive and small, since we may choose $t > t_0 \gg 1$. Due to Lemma B.5, (4-2) can be written as the sum of

$$(2\pi)^{\frac{1}{2}(d-1)}e^{i\epsilon\frac{\pi}{4}(d-1)}e^{-i\epsilon r\rho}(r\rho)^{-\frac{1}{2}(d-1)}\chi\left(\frac{r-tP'(\rho)}{t^{\frac{1}{2}+\delta}\Lambda(t^{\frac{1}{2}}\rho)}\right)\hat{u}_0(-\epsilon\rho\omega)\kappa\left(\frac{r}{t}\right)$$

and a remainder

$$e^{-i\epsilon r\rho}\mu^{d-1}S_{-\frac{1}{2}(d+1)}\left(\omega,\mu,\rho,\frac{r}{t}-P'(\rho),t;r\rho\mu^2\right)\kappa\left(\frac{r}{t}\right),$$

where $\kappa \in C_c^\infty(]0,\infty[)$ equals 1 in a neighborhood of 1, and $S_m(\omega,\mu,\rho,r',t;\zeta)$ is supported for $\zeta > c > 0$, $\rho \sim 1$ and $|r'| \lesssim \mu$ and satisfies, for all $\alpha \in \mathbb{N}^{d-1}$, $j,k,l,\gamma \in \mathbb{N}$,

$$|\partial_\omega^\alpha \partial_\mu^j \partial_\rho^k \partial_{r'}^l \partial_\zeta^\gamma S_m| \leqslant C\mu^{-(|\alpha|+j+l)}\langle\zeta\rangle^{m-\gamma}.$$

Note that it is harmless to add an extra factor κ, since the integrand of (4-2) is supported for $r \sim t$, which is a consequence of the cut-off χ together with $t \gg 1$, $\delta + \frac{1}{2}\sigma_1 < \frac{1}{2}$, and $\rho \sim 1$. We may repeat this argument for the integral in θ' and the truncated energy $E_{\chi,\delta,\Lambda}(\epsilon t)$ can be decomposed into a principal part

$$\frac{1}{(2\pi)^{d+1}}\int e^{-i\epsilon r(\rho-\rho')}e^{i\epsilon t(P(\rho)-P(\rho'))}\chi\left(\frac{r-tP'(\rho)}{t^{\frac{1}{2}+\delta}\Lambda(t^{\frac{1}{2}}\rho)}\right)\chi\left(\frac{r-tP'(\rho')}{t^{\frac{1}{2}+\delta}\Lambda(t^{\frac{1}{2}}\rho')}\right) \times \mathbb{1}_{\frac{r}{t}>P'(\rho),P'(\rho')}\hat{u}_0(-\epsilon\rho\omega)\overline{\hat{u}_0(-\epsilon\rho'\omega)}(\rho\rho')^{\frac{1}{2}(d-1)}\kappa^2\left(\frac{r}{t}\right)d\omega\,dr\,d\rho\,d\rho' \quad (4\text{-}3)$$

and remainders

$$\frac{1}{(2\pi)^{d+1}}\int e^{-i\epsilon r(\rho-\rho')}e^{i\epsilon t(P(\rho)-P(\rho'))}\mu^{2(d-1)}\kappa^2\left(\frac{r}{t}\right)S_m\left(\omega,\mu,\rho,\frac{r}{t}-P'(\rho),t;r\rho\mu^2\right) \times S_{m'}\left(\omega,\mu,\rho',\frac{r}{t}-P'(\rho'),t;r\rho'\mu^2\right)\mathbb{1}_{\frac{r}{t}>P'(\rho),P'(\rho')}(r\rho\rho')^{d-1}\,d\omega\,dr\,d\rho\,d\rho',$$

where (m,m') takes values among $\left(-\frac{d-1}{2},-\frac{d+1}{2}\right)$, $\left(-\frac{d+1}{2},-\frac{d-1}{2}\right)$, $\left(-\frac{d+1}{2},-\frac{d+1}{2}\right)$. Note that due to the condition $\delta+\frac{1}{2}\sigma_1 > \frac{1}{2}$, we have

$$r\rho\mu^2 = r\rho t^{2(\delta+\frac{1}{2}\sigma_1)-1} \sim t^{2(\delta+\frac{1}{2}\sigma_1)} > c > 0.$$

The sum of these remainders can be simplified as

$$\int e^{-i\epsilon r(\rho-\rho')}e^{i\epsilon t(P(\rho)-P(\rho'))}\frac{1}{\mu^2 r}\Sigma\left(\omega,\mu,\rho,\rho',r,t;\frac{r}{t}-P'(\rho),\frac{r}{t}-P'(\rho')\right)\mathbb{1}_{\frac{r}{t}>P'(\rho),P'(\rho')}\,d\omega\,dr\,d\rho\,d\rho', \quad (4\text{-}4)$$

where $\Sigma(\omega,\mu,\rho,\rho',r,t;s,s')$ is supported for $r\sim t$, $\rho,\rho'\sim 1$ and $|s|,|s'|\lesssim\mu$ and satisfies for all $\alpha\in\mathbb{N}^{d-1}$, $j,k,k',l,\gamma,\gamma'\in\mathbb{N}$,

$$|\partial_\omega^\alpha\partial_\mu^j\partial_\rho^k\partial_{\rho'}^{k'}\partial_r^l\partial_s^\gamma\partial_{s'}^{\gamma'}\Sigma| \lesssim \mu^{-(|\alpha|+j+\gamma+\gamma')}t^{-l}.$$

Before proceeding further, we introduce the integral

$$I(t,\epsilon_1,\epsilon_1',\epsilon_2,\epsilon_2';F) := \int e^{i[r(\epsilon_1\rho+\epsilon_1'\rho')-t(\epsilon_2P(\rho)+\epsilon_2'P(\rho'))]}\mathbb{1}_{\frac{r}{t}>P'(\rho),P'(\rho')} \times F\big(\rho,\rho',r,t;r-\epsilon_1\epsilon_2tP'(\rho),r-\epsilon_1'\epsilon_2'tP'(\rho')\big)\,dr\,d\rho\,d\rho'. \quad (4\text{-}5)$$

The limit of such integral has been studied in [Delort 2022] for strictly convex P, while the concave case can be studied with almost the same argument. To be precise:

Proposition 4.3. *Let $F(\rho,\rho',r,t;\zeta,\zeta')$ be a smooth function on $\mathbb{R}_+^4\times\mathbb{R}^2$ and $\delta'\in\left]\frac{1}{2},1\right[$. Assume that F is supported for*

$$\rho,\rho'\sim 1,\quad r\sim t,\quad |\zeta|,|\zeta'|\lesssim t^{\delta'},$$

and for all $j,j',k,\gamma,\gamma'\in\mathbb{N}$,

$$|\partial_\rho^j\partial_{\rho'}^{j'}\partial_r^k\partial_\zeta^\gamma\partial_{\zeta'}^{\gamma'}F(\rho,\rho',r,t;\zeta,\zeta')|\lesssim t^{-\delta'(k+\gamma+\gamma')}.$$

We assume further that the following pointwise limit exists:

$$\lim_{t\to+\infty}F\big(\rho,\rho',r\sqrt{t}+tP'(\rho'),t;\zeta\sqrt{t},\zeta'\sqrt{t}\big)=F_0(\rho,\rho').$$

Under all the assumptions above, we have

$$\lim_{t\to+\infty}I(t,-\epsilon,\epsilon,-\epsilon,\epsilon;F)=\frac{\pi}{2}\int_0^\infty F_0(\rho,\rho)\,d\rho$$

for all $\epsilon=\pm$ and $P\in C^\infty$ with $P''>0$ or $P''<0$.

The proof for strictly convex P follows from that of Proposition 3.1.3 in [Delort 2022]. As for the concave case, we will give a brief proof in Appendix E. Note that we compute the limit of $I(t,-\epsilon,\epsilon,-\epsilon,\epsilon;F)$ for both signs $\epsilon=\pm1$, while in [loc. cit.] only the limit of the sum of these two terms was determined. The proof of our stronger result is not essentially different from the one in [loc. cit.] and we shall explain the modification one has to make to the argument in Appendix E.

With the notations above, the truncated energy $E_{\chi,\delta,\Lambda}(\epsilon t)$ given by the sum of (4-3) and (4-4) equals

$$E_{\chi,\delta,\Lambda}(\epsilon t)=I(t,-\epsilon,\epsilon,-\epsilon,\epsilon;F)+I(t,-\epsilon,\epsilon,-\epsilon,\epsilon;F_R),$$

where

$$F(\rho,\rho',r,t;\zeta,\zeta')=\frac{1}{(2\pi)^{d+1}}\kappa^2\Big(\frac{r}{t}\Big)\int\chi\bigg(\frac{\zeta}{t^{\frac{1}{2}+\delta}\Lambda(t^{\frac{1}{2}}\rho)}\bigg)\chi\bigg(\frac{\zeta'}{t^{\frac{1}{2}+\delta}\Lambda(t^{\frac{1}{2}}\rho')}\bigg)\times\hat{u}_0(-\epsilon\rho\omega)\overline{\hat{u}_0(-\epsilon\rho'\omega)}(\rho\rho')^{\frac{1}{2}(d-1)}\,d\omega,$$

$$F_R(\rho,\rho',r,t;\zeta,\zeta')=t^{-2(\delta+\frac{1}{2}\sigma_1)}\int\frac{t}{r}\Sigma\Big(\omega,t^{\delta+\frac{1}{2}\sigma_1-\frac{1}{2}},\rho,\rho',r,t;\frac{\zeta}{t},\frac{\zeta'}{t}\Big)\,d\omega.$$

It is easy to check that F, F_R satisfy the conditions of Proposition 4.3 with

$$\delta'=\delta+\tfrac{1}{2}\sigma_1+\tfrac{1}{2}.$$

Note that due to the condition $\delta+\frac{1}{2}\sigma_1\in\left]0,\frac{1}{2}\right[$, we have $\delta'\in\left]\frac{1}{2},1\right[$, which is required by Proposition 4.3. The corresponding limit is

$$F_0(\rho,\rho')=\frac{1}{(2\pi)^{d+1}}\int\hat{u}_0(-\epsilon\rho\omega)\overline{\hat{u}_0(-\epsilon\rho'\omega)}(\rho\rho')^{\frac{1}{2}(d-1)}\,d\omega$$

and 0, respectively. Therefore, we may conclude (1-17), (1-22), and (1-28) by Proposition 4.3. In fact, with $\epsilon = \pm$, one has

$$\begin{aligned}\lim_{t\to\pm\infty} E_{\chi,\delta}(t) = \lim_{t\to+\infty} E_{\chi,\delta}(\epsilon t) &= \lim_{t\to+\infty} I(t,-\epsilon,\epsilon,-\epsilon,\epsilon;F) + \lim_{t\to+\infty} I(t,-\epsilon,\epsilon,-\epsilon,\epsilon;F_R)\\ &= \frac{\pi}{2}\int_0^\infty F_0(\rho,\rho)\,d\rho + 0\\ &= \frac{1}{4}\frac{1}{(2\pi)^d}\int \hat{u}_0(-\epsilon\rho\omega)\overline{\hat{u}_0(-\epsilon\rho\omega)}\rho^{d-1}\,d\omega\,d\rho = \frac{1}{4}\|u_0\|_{L^2}^2.\end{aligned}$$

4.2. *Subcritical and critical case $\delta + \frac{1}{2}\sigma_1 \leqslant 0$.* In the rest of this section, we will study, under the condition $\delta + \frac{1}{2}\sigma_1 \leqslant 0$, the truncated energy $E_{\chi,\delta,\Lambda}(u_0, \epsilon t)$, with $\epsilon = \pm$, $t > t_0 \gg 1$. Here, we only write the proof of the case $P'' > 0$, while the case $P'' < 0$ can be calculated in exactly the same way.

By definition (1-11), the truncated energy $E_{\chi,\delta,\Lambda}(u_0, \epsilon t)$ equals

$$\frac{1}{(2\pi)^{2d}}\int e^{ix\cdot(\xi-\xi')}e^{i\epsilon t(P(\xi)-P(\xi'))}\chi\biggl(\frac{x+\epsilon tP'(\xi)}{t^{\frac12+\delta}\Lambda(t^{\frac12}\xi)}\biggr)\chi\biggl(\frac{x+\epsilon tP'(\xi')}{t^{\frac12+\delta}\Lambda(t^{\frac12}\xi')}\biggr) \times \mathbb{1}_{|\frac{x}{t}|>|P'(\xi)|,|P'(\xi')|}\hat{u}_0(\xi)\overline{\hat{u}_0(\xi')}\,d\xi\,d\xi'\,dx,$$

which can be rewritten in the polar system $x = r\omega,\ \xi = \rho\theta,\ \xi' = \rho'\theta'$ as

$$\frac{1}{(2\pi)^{2d}}\int e^{ir\omega\cdot(\rho\theta-\rho'\theta')}e^{i\epsilon t(P(\rho)-P(\rho'))}\chi\biggl(\frac{r\omega+\epsilon tP'(\rho)\theta}{t^{\frac12+\delta}\Lambda(t^{\frac12}\rho)}\biggr)\chi\biggl(\frac{r\omega+\epsilon tP'(\rho')\theta'}{t^{\frac12+\delta}\Lambda(t^{\frac12}\rho')}\biggr) \times \mathbb{1}_{\frac{r}{t}>P'(\rho),P'(\rho')}\hat{u}_0(\rho\theta)\overline{\hat{u}_0(\rho'\theta')}(r\rho\rho')^{d-1}\,dr\,d\theta\,d\theta'\,d\omega\,d\rho\,d\rho'.$$

We decompose $E_{\chi,\delta,\Lambda}(u_0,\epsilon t)$ as the sum of $E_\pm(\epsilon t)$, where E_+, E_- are defined as integrals over $\rho > \rho'$, $\rho' > \rho$, respectively, and the dependence on χ, δ, Λ, u_0 is omitted for the simplicity of notation. Since $E_- = \overline{E}_+$, it is enough to focus on the study of $E_+(\epsilon t)$. We first check that:

Lemma 4.4. *The integral*

$$E_+(\epsilon t) = \frac{1}{(2\pi)^{2d}}\int e^{ir\omega\cdot(\rho\theta-\rho'\theta')}e^{i\epsilon t(P(\rho)-P(\rho'))}\chi\biggl(\frac{r\omega+\epsilon tP'(\rho)\theta}{t^{\frac12+\delta}\Lambda(t^{\frac12}\rho)}\biggr)\chi\biggl(\frac{r\omega+\epsilon tP'(\rho')\theta'}{t^{\frac12+\delta}\Lambda(t^{\frac12}\rho')}\biggr) \times \mathbb{1}_{\frac{r}{t}>P'(\rho)}\mathbb{1}_{\rho>\rho'}\hat{u}_0(\rho\theta)\overline{\hat{u}_0(\rho'\theta')}(r\rho\rho')^{d-1}\,dr\,d\theta\,d\theta'\,d\omega\,d\rho\,d\rho'$$

equals, up to some $O(t^{-1/2})$ terms,

$$\begin{aligned}\frac{1}{(2\pi)^{2d}}\int_{\theta,\theta'\in\sqrt{t}(\epsilon\omega+\mathbb{S}^{d-1})} & e^{i[\sqrt{t}P'(\rho)\rho\omega\cdot(\theta-\theta')+r\rho\omega\cdot(\theta-\theta')+wP'(\rho)\theta'\cdot\omega]}e^{-\epsilon i[rw+\frac12 P''(\rho)w^2]}\\ &\times\chi\biggl(\frac{r\omega+\epsilon P'(\rho)\theta}{t^{\delta}\Lambda(t^{\frac12}\rho)}\biggr)\chi\biggl(\frac{(r+P''(\rho)w)\omega+\epsilon P'(\rho)\theta'}{t^{\delta}\Lambda(t^{\frac12}\rho)}\biggr)\\ &\times \mathbb{1}_{r>0}\mathbb{1}_{\rho>\frac{w}{\sqrt{t}}>0}|\hat{u}_0(-\epsilon\rho\omega)|^2(P'(\rho))^{d-1}\rho^{2(d-1)}\,d\theta\,d\theta'\,dr\,dw\,d\omega\,d\rho. \qquad (4\text{-}6)\end{aligned}$$

Proof. To begin with, via the change of variables,

$$r \to rt^{\frac12} + tP'(\rho), \qquad \rho' \to \rho - \frac{w}{t^{\frac12}},$$

the integral $E_+(\epsilon t)$ can be rewritten as

$$\frac{t^{d-1}}{(2\pi)^{2d}}\int e^{i[tP'(\rho)\rho\omega\cdot(\theta-\theta')+\sqrt{t}r\rho\omega\cdot(\theta-\theta')+\sqrt{t}wP'(\rho)\theta'\cdot\omega+rw\theta'\cdot\omega]}$$
$$\times e^{\epsilon it\left(P(\rho)-P(\rho-\frac{w}{\sqrt{t}})\right)}\chi\left(\frac{r\omega+\sqrt{t}P'(\rho)(\omega+\epsilon\theta)}{t^\delta\Lambda(t^{\frac12}\rho)}\right)$$
$$\times\chi\left(\frac{r\omega+\sqrt{t}P'(\rho)(\omega+\epsilon\theta')-\epsilon\sqrt{t}(P'(\rho)-P'(\rho-t^{-\frac12}w))\theta'}{t^\delta\Lambda(t^{\frac12}\rho-w)}\right)$$
$$\times\mathbb{1}_{r>0}\mathbb{1}_{\rho>\frac{w}{\sqrt{t}}>0}\hat{u}_0(\rho\theta)\overline{\hat{u}_0((\rho-t^{-\frac12}w)\theta')}$$
$$\times\left(\frac{r}{\sqrt{t}}+P'(\rho)\right)^{d-1}\rho^{d-1}(\rho-t^{-\frac12}w)^{d-1}\,d\theta\,d\theta'\,dr\,dw\,d\omega\,d\rho.$$

Due to χ, $\hat{u}_0$ factors, the integrand is supported for

$$\begin{gathered}0<r,w\lesssim t^{\delta+\frac12\sigma_1},\qquad \rho\sim 1,\\ |\omega+\epsilon\theta|,|\omega+\epsilon\theta'|\lesssim t^{\delta+\frac12\sigma_1-\frac12}.\end{gathered}\tag{4-7}$$

In fact, $\rho\sim 1$ follows directly from the fact that $\hat{u}_0$ is compactly supported away from zero. Since χ is compactly supported, the first χ factor implies that

$$\begin{aligned}r&=\big||(r+\sqrt{t}P'(\rho))\omega|-|\sqrt{t}P'(\rho)\theta|\big|\\&\leqslant|r\omega+\sqrt{t}P'(\rho)(\omega+\epsilon\theta)|\lesssim t^\delta\Lambda(t^{\frac12}\rho)\sim t^{\delta+\frac12\sigma_1},\end{aligned}$$

which further implies that

$$\sqrt{t}|\omega+\epsilon\theta|\lesssim|\sqrt{t}P'(\rho)(\omega+\epsilon\theta)|\leqslant|r\omega+\sqrt{t}P'(\rho)(\omega+\epsilon\theta)|+|r\omega|\lesssim t^\delta\Lambda(t^{\frac12}\rho)\sim t^{\delta+\frac12\sigma_1}.$$

By applying a similar argument to the second χ factor, we obtain

$$\begin{aligned}w&\sim|\sqrt{t}P'(\rho)-\sqrt{t}P'(\rho-t^{-\frac12}w)|\\&\leqslant|(r+\sqrt{t}P'(\rho))-\sqrt{t}P'(\rho-t^{-\frac12}w)|+r\\&\leqslant|(r+\sqrt{t}P'(\rho))\omega+\epsilon\sqrt{t}P'(\rho-t^{-\frac12}w)\theta'|+r\lesssim t^{\delta+\frac12\sigma_1},\end{aligned}$$

and that

$$\begin{aligned}\sqrt{t}|\omega+\epsilon\theta'|&\sim|\sqrt{t}P'(\rho)(\omega+\epsilon\theta')|\\&\leqslant|r\omega+\sqrt{t}P'(\rho)(\omega+\epsilon\theta')-\epsilon\sqrt{t}(P'(\rho)-P'(\rho-t^{-\frac12}w))\theta'|+r+|\sqrt{t}(P'(\rho)-P'(\rho-t^{-\frac12}w))|\\&\lesssim t^{\delta+\frac12\sigma_1}+r+w\lesssim t^{\delta+\frac12\sigma_1}.\end{aligned}$$

As a result, the boundedness of integrand implies that

$$E_+(\epsilon t)\lesssim t^{d-1}\times t^{2\left(\delta+\frac12\sigma_1\right)}\times\left(\frac{1}{\sqrt{t}}t^{\delta+\frac12\sigma_1}\right)^{2(d-1)}=t^{2\left(\delta+\frac12\sigma_1\right)d},$$

which tends to zero as $t\to+\infty$ when $\delta+\frac12\sigma_1<0$, and the limits (1-13), (1-20), and (1-24) follow. In the remainder of this section, we take $\delta+\frac12\sigma_1=0$.

The support of integrand also allows us to simplify $E_+(\epsilon t)$, up to some $O(t^{-1/2})$ terms, as

$$\frac{t^{d-1}}{(2\pi)^{2d}}\int e^{i[tP'(\rho)\rho\omega\cdot(\theta-\theta')+\sqrt{t}r\rho\omega\cdot(\theta-\theta')+\sqrt{t}wP'(\rho)\theta'\cdot\omega+rw\theta'\cdot\omega]} \times e^{\epsilon i[\sqrt{t}P'(\rho)w-\frac{1}{2}P''(\rho)w^2]}\chi\left(\frac{r\omega+\sqrt{t}P'(\rho)(\omega+\epsilon\theta)}{t^\delta\Lambda(t^{\frac{1}{2}}\rho)}\right) \times\chi\left(\frac{r\omega+\sqrt{t}P'(\rho)(\omega+\epsilon\theta')-\epsilon P''(\rho)w\theta'}{t^\delta\Lambda(t^{\frac{1}{2}}\rho)}\right) \times\mathbb{1}_{r>0}\mathbb{1}_{\rho>\frac{w}{\sqrt{t}}>0}\hat{u}_0(\rho\theta)\overline{\hat{u}_0(\rho\theta')}(P'(\rho))^{d-1}\rho^{2(d-1)}\,d\theta\,d\theta'\,dr\,dw\,d\omega\,d\rho.$$

Here, we use the approximations

$$t\left(P(\rho)-P\left(\rho-\frac{w}{t^{\frac{1}{2}}}\right)\right)=t\left(-P'(\rho)\left(-\frac{w}{t^{\frac{1}{2}}}\right)-\frac{P''(\rho)}{2}\frac{w^2}{t}+O\left(\frac{w^3}{t^{\frac{3}{2}}}\right)\right) =\sqrt{t}P'(\rho)w-\tfrac{1}{2}P''(\rho)w^2+O\left(\frac{1}{\sqrt{t}}\right)$$

to simplify the phase and

$$\sqrt{t}\left(P'(\rho)-P'\left(\rho-\frac{w}{\sqrt{t}}\right)\right)=\sqrt{t}\left(-P''(\rho)\left(-\frac{w}{\sqrt{t}}\right)+O\left(\frac{w^2}{t}\right)\right)=P''(\rho)w+O\left(\frac{1}{\sqrt{t}}\right)$$

to simplify the argument of the second χ.

By applying a change of variable in θ,θ',

$$\theta\mapsto t^{-\frac{1}{2}}\theta-\epsilon\omega,\qquad \theta'\mapsto t^{-\frac{1}{2}}\theta'-\epsilon\omega,$$

we can rewrite $E_+(\epsilon t)$ as

$$\frac{1}{(2\pi)^{2d}}\int_{\theta,\theta'\in\sqrt{t}(\epsilon\omega+\mathbb{S}^{d-1})} e^{i[\sqrt{t}P'(\rho)\rho\omega\cdot(\theta-\theta')+r\rho\omega\cdot(\theta-\theta')+wP'(\rho)\theta'\cdot\omega+t^{-1/2}rw\theta'\cdot\omega]} \times e^{-\epsilon i[rw+\frac{1}{2}P''(\rho)w^2]}\chi\left(\frac{r\omega+\epsilon P'(\rho)\theta}{t^\delta\Lambda(t^{\frac{1}{2}}\rho)}\right) \times\chi\left(\frac{(r+P''(\rho)w)\omega+\epsilon P'(\rho)\theta'-\epsilon t^{-\frac{1}{2}}P''(\rho)w\theta'}{t^\delta\Lambda(t^{\frac{1}{2}}\rho)}\right) \times\mathbb{1}_{r>0}\mathbb{1}_{\rho>\frac{w}{\sqrt{t}}>0}\hat{u}_0(t^{-\frac{1}{2}}\rho\theta-\epsilon\rho\omega)\overline{\hat{u}_0(t^{-\frac{1}{2}}\rho\theta'-\epsilon\rho\omega)} \times(P'(\rho))^{d-1}\rho^{2(d-1)}\,d\theta\,d\theta'\,dr\,dw\,d\omega\,d\rho+O(t^{-\frac{1}{2}}),$$

where the integrand is supported for $0<r,w\lesssim 1$, $\rho\sim 1$, and $|\theta|,|\theta'|\lesssim 1$ due to (4-7) together with $\delta+\frac{1}{2}\sigma_1=0$, which allows us to do another simplification and write $E_+(\epsilon t)$ as (4-6). ☐

Till now, we have managed to write $E_+(\epsilon t)$, up to some admissible terms, as (4-6), namely

$$\frac{1}{(2\pi)^{2d}}\int_{\theta,\theta'\in\sqrt{t}(\epsilon\omega+\mathbb{S}^{d-1})} e^{i[\sqrt{t}P'(\rho)\rho\omega\cdot(\theta-\theta')+r\rho\omega\cdot(\theta-\theta')+wP'(\rho)\theta'\cdot\omega]}e^{-\epsilon i[rw+\frac{1}{2}P''(\rho)w^2]} \times\chi\left(\frac{r\omega+\epsilon P'(\rho)\theta}{t^\delta\Lambda(t^{\frac{1}{2}}\rho)}\right)\chi\left(\frac{(r+P''(\rho)w)\omega+\epsilon P'(\rho)\theta'}{t^\delta\Lambda(t^{\frac{1}{2}}\rho)}\right) \times\mathbb{1}_{r>0}\mathbb{1}_{\rho>\frac{w}{\sqrt{t}}>0}|\hat{u}_0(-\epsilon\rho\omega)|^2(P'(\rho))^{d-1}\rho^{2(d-1)}\,d\theta\,d\theta'\,dr\,dw\,d\omega\,d\rho.$$

In the rest of this section, we will calculate the limit of this integral and conclude Proposition 4.1. Since the integral in θ, θ' is over a sphere centered at $\epsilon\omega$ with radius $\sqrt{t}$, we may write θ, θ' in local coordinates

$$\theta = h\epsilon\omega + y, \qquad h = \sqrt{t} - \sqrt{t - |y|^2},$$
$$\theta' = h'\epsilon\omega + y', \quad h' = \sqrt{t} - \sqrt{t - |y'|^2},$$

where $h, h' \in \mathbb{R},\ y, y' \in \omega^\perp := \{z \in \mathbb{R}^d : z\cdot\omega = 0\}$. The condition of support implies that $|h|, |h'|, |y|, |y'| \lesssim 1$. It is easy to check that, as $t \to +\infty$,

$$\sqrt{t}\omega\cdot(\theta-\theta') = \epsilon\sqrt{t}\big(\sqrt{t-|y'|^2} - \sqrt{t-|y|^2}\big) \to \epsilon\left(\frac{|y|^2}{2} - \frac{|y'|^2}{2}\right),$$
$$\theta\cdot\omega = \epsilon\big(\sqrt{t} - \sqrt{t-|y|^2}\big) \to 0,$$
$$\theta'\cdot\omega = \epsilon\big(\sqrt{t} - \sqrt{t-|y'|^2}\big) \to 0.$$

Therefore, by the dominated convergence theorem, as t tends to infinity, the limit of $E_+(\epsilon t)$ equals

$$\frac{1}{(2\pi)^{2d}} \int_{y,y'\in\omega^\perp, r,w>0} e^{\epsilon i[P'(\rho)\rho(\frac{1}{2}|y|^2 - \frac{1}{2}|y'|^2) - rw - \frac{1}{2}P''(\rho)w^2]} \times \chi\left(\frac{r\omega + \epsilon P'(\rho)y}{\lambda_1\rho^{\sigma_1}}\right)\chi\left(\frac{(r+P''(\rho)w)\omega + \epsilon P'(\rho)y'}{\lambda_1\rho^{\sigma_1}}\right) \times |\hat{u}_0(-\epsilon\rho\omega)|^2 (P'(\rho))^{d-1}\rho^{2(d-1)}\, dy\, dy'\, dr\, dw\, d\omega\, d\rho,$$

which, after a change of variable, is equal to

$$\frac{1}{(2\pi)^{2d}} \int_{y,y'\in\omega^\perp, r,w>0} e^{\epsilon i[(\frac{1}{2}|y|^2 - \frac{1}{2}|y'|^2) - rw - \frac{1}{2}w^2]} \chi\left(\frac{\sqrt{P''(\rho)}r\omega + \epsilon\sqrt{\rho^{-1}P'(\rho)}y}{\lambda_1\rho^{\sigma_1}}\right) \times \chi\left(\frac{\sqrt{P''(\rho)}(r+w)\omega + \epsilon\sqrt{\rho^{-1}P'(\rho)}y'}{\lambda_1\rho^{\sigma_1}}\right) \times |\hat{u}_0(-\epsilon\rho\omega)|^2 \rho^{d-1}\, dy\, dy'\, dr\, dw\, d\omega\, d\rho. \tag{4-8}$$

In order to give a compact form, we introduce the functions

$$H(r,\omega) := \frac{1}{(2\pi)^{\frac{1}{2}d}} \int_{y\cdot\omega=0} e^{\epsilon i\frac{1}{2}(r^2+|y|^2)} \chi\left(\frac{\sqrt{P''(\rho)}r\omega + \epsilon\sqrt{\rho^{-1}P'(\rho)}y}{\lambda_1\rho^{\sigma_1}}\right) dy,$$
$$F(r,\omega) := \int_r^\infty H(s,\omega)\, ds.$$

Note that since $\chi \in \mathcal{S}(\mathbb{R}^d)$, H decays rapidly at infinity, uniformly in ω. Thus, F is well-defined. With these functions, we can rewrite the integral (4-8) as

$$\frac{1}{(2\pi)^d} \iint_0^\infty H(r,\omega) \int_0^\infty \overline{H(r+w,\omega)}\, dw\, dr\, |\hat{u}_0(-\epsilon\rho\omega)|^2 \rho^{d-1}\, d\omega\, d\rho = -\frac{1}{(2\pi)^d} \iint_0^\infty \partial_r F(r,\omega)\overline{F(r,\omega)}\, dr\, |\hat{u}_0(-\epsilon\rho\omega)|^2 \rho^{d-1}\, d\omega\, d\rho.$$

As a consequence,

$$\begin{aligned}\lim_{t\to\infty} E_{\chi,\delta,\Lambda}(u_0,\epsilon t) &= \lim_{t\to\infty} 2\operatorname{Re} E_+(\epsilon t)\\ &= -\frac{2}{(2\pi)^d}\operatorname{Re}\iint_0^\infty \partial_r F(r,\omega)\overline{F(r,\omega)}\,dr\,|\hat{u}_0(-\epsilon\rho\omega)|^2\rho^{d-1}\,d\omega\,d\rho\\ &= -\frac{1}{(2\pi)^d}\iint_0^\infty \frac{\partial}{\partial r}|F|^2(r,\omega)\,dr\,|\hat{u}_0(-\epsilon\rho\omega)|^2\rho^{d-1}\,d\omega\,d\rho\\ &= \frac{1}{(2\pi)^d}\int |F(0,\omega)|^2|\hat{u}_0(-\epsilon\rho\omega)|^2\rho^{d-1}\,d\omega\,d\rho,\end{aligned}$$

where

$$|F(0,\omega)|^2 = \frac{1}{(2\pi)^d}\left|\int_0^\infty\int_{y\cdot\omega=0} e^{\epsilon i\frac{1}{2}(r^2+|y|^2)}\chi\left(\frac{\sqrt{P''(\rho)}r\omega+\epsilon\sqrt{\rho^{-1}P'(\rho)}y}{\lambda_1\rho^{\sigma_1}}\right)dy\,dr\right|^2,$$

which is exactly $G_\chi^{\mathrm{alt}}(\rho,\omega)$ defined in (1-26), or (1-15) with $\sigma_1=0$ and $\lambda_1=1$. The limits (1-14), (1-21), and (1-25) thus follow.

5. Study of the Klein–Gordon equation

In this section, we shall prove Theorem 1.9 via a study of the half-Klein–Gordon equation, i.e., (E) with $P(\xi)=\langle\xi\rangle$. Let w be the (real) solution to the Klein–Gordon equation (KG). We have then

$$0=(\partial_t^2-\Delta+1)w=-\left(\frac{\partial_t}{i}-P(D_x)\right)\left(\frac{\partial_t}{i}+P(D_x)\right)w.$$

Thus, the complex-valued function

$$u:=\left(\frac{\partial_t}{i}+P(D_x)\right)w$$

is the unique solution to the half-Klein–Gordon equation with initial data

$$u_0:=u|_{t=0}=\frac{w_1}{i}+P(D_x)w_0\in L^2.$$

Due to the fact that w is real-valued, we have the relations

$$\partial_t w=-\operatorname{Im}u,\qquad w=\langle D_x\rangle^{-1}\operatorname{Re}u.$$

In this section, we denote the truncation $\operatorname{Op}(a_\epsilon^{\mathrm{KG}}(t))$, whose symbol is defined in (1-31), as $A(t)$, and define operators $A_\pm(t)$ as $\operatorname{Op}(a_\pm^{\mathrm{KG}}(t))$, where

$$a_\pm^{\mathrm{KG}}(t)=\chi\left(\frac{x\pm tP'(\xi)}{|t|^{\frac{1}{2}+\delta}}\right)\mathbb{1}_{|x|>|tP'(\xi)|}\chi\left(\frac{\xi}{|t|^\epsilon}\right), \tag{5-1}$$

$0<\delta<\frac{1}{2}$, $0<\epsilon<1$, and $\chi\in C_c^\infty(\mathbb{R}^d)$ are the same as in (1-31). Since we are interested in the behavior as $t\to+\infty$, it is harmless to assume $t\gg 1$. Before continuing the proof, we clarify that all the involved operators are bounded uniformly in $t\gg 1$.

Lemma 5.1. *There exist a time-independent constant $C > 0$ and $t_0 \gg 1$ such that, for all $t > t_0$,*

$$\|A(t)\|_{\mathcal{L}(L^2)},\ \|A_\pm(t)\|_{\mathcal{L}(L^2)} \leqslant C.$$

Proof. It is obvious that $P(\xi) = \langle \xi \rangle$ satisfies the hypotheses (H_{p_0,p_1}) with $p_0 = 1$, $P_0 = 0$, and $p_1 = -2$, $P_1 = 1$. In what follows, we shall focus on symbols $a_\pm^{\mathrm{KG}}(t)$ (with $a_\epsilon^{\mathrm{KG}}(t)$ treated in the same way) and decompose them into high and low frequencies. Let $\tilde{\chi} \in C_c^\infty(\mathbb{R}^d)$ be a radial function which is equal to 1 in the ball $B(0, 1)$ and vanishes outside a larger ball $B(0, 2)$. We may write

$$a_{\pm,l}^{\mathrm{KG}}(t) := a_\pm^{\mathrm{KG}}(t)\tilde{\chi}(\xi), \quad a_{\pm,h}^{\mathrm{KG}}(t) := a_\pm^{\mathrm{KG}}(t)(1-\tilde{\chi})(\xi).$$

For the low-frequency part $a_{\pm,l}^{\mathrm{KG}}(t)$, it is easy to construct a symbol $P_l(\xi)$ such that P_l satisfies the hypotheses (H_{p_0,p_1}) with $p_0 = p_1 = 1$ and $P_0 = P_1 = 0$, and that $P_l(\xi) = P(\xi)$ for all $|\xi| \leqslant 2$. In this way, we have

$$a_{\pm,l}^{\mathrm{KG}}(t, x, \xi) = \chi\left(\frac{x \pm t P_l'(\xi)}{|t|^{\frac{1}{2}+\delta}}\right)\mathbb{1}_{|x|>|tP'(\xi)|}\tilde{\chi}(\xi),$$

where the factor involving t^ϵ disappears since the symbol is supported for $|\xi| \leqslant 2 \ll t^\epsilon$ by choosing $t \gg 1$. Now, we may apply Proposition 2.1 to obtain the uniform-in-t boundedness of the operator with symbol

$$\chi\left(\frac{x \pm t P_l'(\xi)}{|t|^{\frac{1}{2}+\delta}}\right)\mathbb{1}_{|x|>|tP'(\xi)|}$$

and hence boundedness of the operator $\mathrm{Op}(a_{\pm,l}^{\mathrm{KG}}(t))$.

The boundedness of $\mathrm{Op}(a_{\pm,h}^{\mathrm{KG}}(t))$ is actually an immediate consequence of Proposition 2.2. More precisely, by choosing an arbitrary ϵ' such that (ϵ', ϵ) satisfies conditions (1-19) associated to P (i.e., with $p_0 = 1$ and $p_1 = -2$), namely

$$0 < \epsilon' \leqslant \frac{1}{p_0+1} = \frac{1}{2}, \quad 0 < \epsilon \leqslant \frac{1}{-(p_1+1)} = 1,$$

we are able to apply Proposition 2.2 to obtain the uniform boundedness of the operator with symbol

$$\chi\left(\frac{x \pm t P'(\xi)}{t^{\frac{1}{2}+\delta}}\right)\mathbb{1}_{|x|>|tP'(\xi)|}(1-\chi)\left(\frac{\xi}{t^{-\epsilon'}}\right)\chi\left(\frac{\xi}{t^\epsilon}\right).$$

If we add the high-frequency truncation $(1-\tilde{\chi})(\xi)$, the symbol will be supported in $|\xi| \geqslant 1 \gg t^{-\epsilon'}$ since we have chosen $t \gg 1$. As a result, the truncation $(1-\chi)$ equals 1 and the uniform boundedness of $\mathrm{Op}(a_{\pm,h}^{\mathrm{KG}}(t))$ follows.

We have shown that $A_\pm(t) = \mathrm{Op}(a_{\pm,l}^{\mathrm{KG}}(t)) + \mathrm{Op}(a_{\pm,h}^{\mathrm{KG}}(t))$ is uniformly bounded on L^2. By repeating the same argument and replacing Propositions 2.1 and 2.2 by Theorem 1.8, we may also obtain the uniform boundedness of $A(t)$. □

Now we turn back to the proof of Theorem 1.9. By definition, the truncated energy (1-30) can be expressed as

$$E_\epsilon^{\mathrm{KG}}(\pm t) = \|A(t)\,\mathrm{Im}\,u(\pm t)\|_{L^2}^2 + \left\|A(t)\frac{D_x}{\langle D_x\rangle}\mathrm{Re}\,u(\pm t)\right\|_{L^2}^2 + \left\|A(t)\frac{1}{\langle D_x\rangle}\mathrm{Re}\,u(\pm t)\right\|_{L^2}^2.$$

The three terms on the right-hand side take the form

$$Q_{\pm}^{\mathrm{KG}}(t,\epsilon_0,R)=\tfrac{1}{4}\big\|A(t)\big(Ru(\pm t)+\epsilon_0\overline{Ru(\pm t)}\big)\big\|_{L^2}^2, \tag{5-2}$$

where $\epsilon_0\in\{+,-\}$ and R is a bounded Fourier multiplier taking values among 1, $D_x\langle D_x\rangle^{-1}$, and $\langle D_x\rangle^{-1}$. Due to the uniform boundedness of truncation operators $A(t)$, $A_{\pm}(t)$, (5-2) can be written as

$$\begin{aligned}
&\tfrac{1}{4}\big\|A_{\pm}(t)Ru(\pm t)+\epsilon_0A_{\mp}(t)\overline{Ru(\pm t)}\big\|_{L^2}^2+O(\|(A(t)-A_{\pm}(t))Ru(\pm t)\|_{L^2})\\
&\qquad\qquad\qquad\qquad+O(\|(A(t)-A_{\mp}(t))\overline{Ru(\pm t)}\|_{L^2})\\
&=\tfrac{1}{4}\|A_{\pm}(t)Ru(\pm t)\|_{L^2}^2+\tfrac{1}{4}\|A_{\mp}(t)\overline{Ru(\pm t)}\|_{L^2}^2+\epsilon_0\operatorname{Re}\langle A_{\pm}(t)Ru(\pm t),A_{\mp}(t)\overline{Ru(\pm t)}\rangle_{L^2}\\
&\qquad+O(\|(A(t)-A_{\pm}(t))Ru(\pm t)\|_{L^2})+O(\|(A(t)-A_{\mp}(t))\overline{Ru(\pm t)}\|_{L^2})\\
&=\tfrac{1}{2}\|A_{\pm}(t)Ru(\pm t)\|_{L^2}^2+\epsilon_0\operatorname{Re}\langle A_{\pm}(t)Ru(\pm t),A_{\mp}(t)\overline{Ru(\pm t)}\rangle_{L^2}+O(\|(A(t)-A_{\pm}(t))Ru(\pm t)\|_{L^2}),
\end{aligned}$$

where we use the fact that for all complex-valued functions $f\in L^2$

$$A(t)\bar f=\overline{A(t)f},\qquad A_{\pm}(t)\bar f=\overline{A_{\mp}(t)f}.$$

In order to conclude the desired limit (1-30), it suffices to prove:

Proposition 5.2. *Let v_0, $v_{1,0}$ be two functions in L^2. With $A(t)$, $A_{\pm}(t)$ as above, we have following limits*:

$$\lim_{t\to+\infty}\|A_{\pm}(t)e^{\pm itP(D_x)}v_0\|_{L^2}^2=\tfrac{1}{4}\|v_0\|_{L^2}^2, \tag{5-3}$$

$$\lim_{t\to+\infty}\langle A_{\pm}(t)e^{\pm itP(D_x)}v_0,A_{\mp}(t)e^{\mp itP(D_x)}v_{0,1}\rangle_{L^2}=0, \tag{5-4}$$

$$\lim_{t\to+\infty}\|(A(t)-A_{\pm}(t))e^{\pm itP(D_x)}v_0\|_{L^2}^2=0. \tag{5-5}$$

Once Proposition 5.2 is proved, we may apply the three limits with $v_0=Ru_0$ and $v_{0,1}=\overline{Ru_0}$ to obtain

$$\lim_{t\to+\infty}Q_{\pm}^{\mathrm{KG}}(t,\epsilon_0,R)=\tfrac{1}{8}\|Ru_0\|_{L^2}^2,$$

since, due to the definition of u, we have, for all $t\in\mathbb{R}$,

$$Ru(t)=e^{itP(D_x)}Ru_0,\qquad \overline{Ru(t)}=e^{-itP(D_x)}\overline{Ru_0}.$$

As a result, the limit (1-30) follows from

$$\begin{aligned}
\lim_{t\to\pm\infty}E_{\epsilon}^{\mathrm{KG}}(w_0,w_1,t)&=\lim_{t\to+\infty}E_{\epsilon}^{\mathrm{KG}}(w_0,w_1,\pm t)\\
&=\lim_{t\to+\infty}Q_{\pm}^{\mathrm{KG}}(t,-1,1)+Q_{\pm}^{\mathrm{KG}}(t,-1,D_x\langle D_x\rangle^{-1})+Q_{\pm}^{\mathrm{KG}}(t,1,\langle D_x\rangle^{-1})\\
&=\frac{1}{8}\left(\|u_0\|_{L^2}^2+\left\|\frac{D_x}{\langle D_x\rangle}u_0\right\|_{L^2}^2+\left\|\frac{1}{\langle D_x\rangle}u_0\right\|_{L^2}^2\right)=\frac{1}{4}\|u_0\|_{L^2}^2=\frac{1}{4}(\|w_0\|_{H^1}^2+\|w_1\|_{L^2}^2).
\end{aligned}$$

In the rest of this section, we shall prove the limits (5-3), (5-4), and (5-5). We have seen that the operators $A(t)$, $A_{\pm}(t)$ are bounded uniformly in $t>t_0\gg 1$. As a result, it suffices to calculate these limits for those v_0, $v_{0,1}$ belonging to some dense subspace of L^2. In what follows, we assume $\hat v_0$, $\hat v_{0,1}$ are smooth and supported in an annulus centered at zero.

The first limit (5-3) is no more than a consequence of (1-22) with $P(\xi)=\langle\xi\rangle$, $\epsilon_1=\epsilon$, $\chi_h=\chi$, and any $\epsilon_0\in\left]0,\frac{1}{2}\right]$. The exceptional truncation χ_l is not a problem, since it disappears when $|t|$ is large enough. For the remaining results (5-4) and (5-5), we will apply a similar argument as in the proof of Proposition 4.1 for supercritical case.

5.1. ***Limit of interaction term.*** We first calculate the limit (5-4). Clearly, via conjugation, it is enough to study the limit of

$$\left\langle A_+(t)e^{itP(D_x)}v_0,\, A_-(t)e^{-itP(D_x)}v_{0,1}\right\rangle_{L^2}, \tag{5-6}$$

since the other one can be recovered by relation

$$\begin{aligned}\overline{\left\langle A_-(t)e^{-itP(D_x)}v_0,\, A_+(t)e^{+itP(D_x)}v_{0,1}\right\rangle_{L^2}} &= \left\langle \overline{A_-(t)e^{-itP(D_x)}v_0},\, \overline{A_+(t)e^{itP(D_x)}v_{0,1}}\right\rangle_{L^2}\\ &= \left\langle A_+(t)e^{itP(D_x)}\bar v_0,\, A_-(t)e^{-itP(D_x)}\bar v_{0,1}\right\rangle_{L^2},\end{aligned}$$

where the Fourier transform of $\bar v_0$, $\bar v_{0,1}$ still belongs to the class $C_c^\infty(\mathbb{R}^d\backslash\{0\})$.

By definition (5-1) of $A_\pm(t)=\mathrm{Op}(a_\pm^{\mathrm{KG}}(t))$, (5-6) equals, in the polar system,

$$\begin{aligned}\frac{1}{(2\pi)^{2d}}\int e^{i(r\rho\omega\cdot\theta-r\rho'\omega\cdot\theta')}e^{it(P(\rho)+P(\rho'))}\chi\left(\frac{r\omega+tP'(\rho)\theta}{t^{\frac12+\delta}}\right)\chi\left(\frac{r\omega-tP'(\rho')\theta'}{t^{\frac12+\delta}}\right)&\\ \times\mathbb{1}_{\frac{r}{t}>P'(\rho),P'(\rho')}\hat v_0(\rho\theta)\overline{\hat v_{0,1}(\rho'\theta')}(r\rho\rho')^{d-1}\,d\theta\,d\theta'\,d\omega\,dr\,d\rho\,d\rho'.&\end{aligned}$$

Here we may omit the truncation in $\xi=\rho\theta$ and $\xi'=\rho'\theta'$ by taking $t\gg1$. As in previous section, we focus on the integrals in θ and θ', which are equal to, by Lemma B.5,

$$\int e^{ir\rho\omega\cdot\theta}\chi\left(\frac{r\omega+tP'(\rho)\theta}{t^{\frac12+\delta}}\right)\overline{\hat v_0(\rho\theta)}\,d\theta=e^{-ir\rho}\mu^{d-1}S^+_{-\frac12(d-1)}\left(\omega,\mu,\rho,\frac{r}{t}-P'(\rho),t;r\rho\mu^2\right)\kappa\left(\frac{r}{t}\right),$$

$$\int e^{-ir\rho'\omega\cdot\theta'}\chi\left(\frac{r\omega-tP'(\rho')\theta'}{t^{\frac12+\delta}}\right)\overline{\hat v_{0,1}(\rho'\theta')}\,d\theta'=e^{-ir\rho'}\mu^{d-1}S^-_{-\frac12(d-1)}\left(\omega,\mu,\rho',\frac{r}{t}-P'(\rho'),t;r\rho'\mu^2\right)\kappa\left(\frac{r}{t}\right),$$

respectively, where $\mu=t^{\delta-1/2}$ and $S_m^\pm(\omega,\mu,\rho,r',t;\zeta)$ is supported for $\zeta>c>0$, $\rho\sim1$ and $|r'|\lesssim\mu$ and satisfies, for all $\alpha\in\mathbb{N}^{d-1}$, $j,k,l,\gamma\in\mathbb{N}$,

$$|\partial_\omega^\alpha\partial_\mu^j\partial_\rho^k\partial_{r'}^l\partial_\zeta^\gamma S_m|\leqslant C\mu^{-(|\alpha|+j+l)}\langle\zeta\rangle^{m-\gamma}.$$

Here we add extra factor $\kappa\in C_c^\infty(]0,+\infty[)$, which equals 1 in a neighborhood of 1, due to the support of the integrand.

As a consequence, (5-6) reads

$$\begin{aligned}\frac{1}{(2\pi)^{2d}}\int e^{i[r(-\rho-\rho')-t(-P(\rho)-P(\rho'))]}S_0^+\left(\omega,\mu,\rho,\frac{r}{t}-P'(\rho),t;r\rho\mu^2\right)&\\ \times S_0^-\left(\omega,\mu,\rho',\frac{r}{t}-P'(\rho'),t;r\rho'\mu^2\right)\kappa^2\left(\frac{r}{t}\right)\mathbb{1}_{\frac{r}{t}>P'(\rho),P'(\rho')}&\\ \times\hat v_0(\rho\theta)\overline{\hat v_{0,1}(\rho'\theta')}(\rho\rho')^{\frac12(d-1)}\,d\omega\,dr\,d\rho\,d\rho',&\end{aligned}$$

which can be rewritten as $I(t,-,-,-,-;F)$ defined in (4-5), with

$$F(\rho,\rho',r,t;\zeta,\zeta')=S_0^+\left(\omega,t^{\delta-\frac{1}{2}},\rho,\frac{\zeta}{t},t;r\rho t^{2\delta-1}\right)S_0^-\left(\omega,t^{\delta-\frac{1}{2}},\rho',\frac{\zeta'}{t},t;r\rho' t^{2\delta-1}\right)\kappa^2\left(\frac{r}{t}\right). \tag{5-7}$$

In [Delort 2022], the author has proved in Proposition 3.1.1 that:

Proposition 5.3. *Let $F(\rho,\rho',r,t;\zeta,\zeta')$ be a smooth function on $\mathbb{R}_+^4\times\mathbb{R}^2$ and $\delta'\in\left]\frac{1}{2},1\right[$. Assume that F is supported for*

$$\rho,\rho'\sim 1,\quad r\sim t,\quad |\zeta|,|\zeta'|\lesssim t^{\delta'},$$

and, for all $j,j',k,\gamma,\gamma'\in\mathbb{N}$,

$$|\partial_\rho^j\partial_{\rho'}^{j'}\partial_r^k\partial_\zeta^\gamma\partial_{\zeta'}^{\gamma'}F(\rho,\rho',r,t;\zeta,\zeta')|\lesssim t^{-\delta'(k+\gamma+\gamma')}.$$

Under all the assumptions above, we have

$$\lim_{t\to+\infty} I(t,\pm,\pm,\pm,\pm;F)=0.$$

It is easy to check that the function F defined in (5-7) satisfies the conditions above with $\delta'=\delta+\frac{1}{2}$ and the limit (5-4) follows.

5.2. *Limit of the energy outside the truncation area.* It remains to prove (5-5), which requires a study of the L^2-norm of $(A(t)-A_\pm(t))e^{\pm itP(D_x)}v_0$. As in previous part, by conjugation, it suffices to focus on

$$(A(t)-A_+(t))e^{itP(D_x)}v_0(x)=\frac{1}{(2\pi)^d}\int e^{ix\xi}e^{itP(\xi)}(1-\chi)\left(\frac{x+tP'(\xi)}{t^{\frac{1}{2}+\delta}}\right)\mathbb{1}_{|x|>|tP'(\xi)|}\hat{v}_0(\xi)\,d\xi,$$

where we omit again the truncation in ξ by assuming $t\gg 1$. Actually, we have

$$\overline{(A(t)-A_-(t))e^{-itP(D_x)}v_0}=(A(t)-A_+(t))e^{itP(D_x)}\bar{v}_0,$$

with $\hat{\bar{v}}_0$ belonging to the same subspace $C_c^\infty(\mathbb{R}^d\backslash\{0\})$.

We first check that the L^2-norm of $(A(t)-A_+(t))e^{itP(D_x)}v_0$ concentrates near $|x|=t$, i.e.:

Lemma 5.4. *There exists a radial function $\kappa\in C_c^\infty(\mathbb{R}^d)$ supported in an annulus centered at zero such that, when $t\to+\infty$,*

$$(A(t)-A_+(t))e^{itP(D_x)}v_0=\kappa\left(\frac{x}{t}\right)(A(t)-A_+(t))e^{itP(D_x)}v_0+O_{L^2}(t^{-N}),$$

for any $N\in\mathbb{N}$.

Proof. Let $\kappa_0\in C_c^\infty(\mathbb{R}^d)$ be a radial function supported near zero and equal to 1 near zero. By definition,

$$(A(t)-A_+(t))e^{itP(D_x)}v_0(x)=\frac{1}{(2\pi)^d}\int e^{ix\xi}e^{itP(\xi)}(1-\chi)\left(\frac{x+tP'(\xi)}{t^{\frac{1}{2}+\delta}}\right)\mathbb{1}_{|x|>|tP'(\xi)|}\hat{v}_0(\xi)\,d\xi.$$

Due to the support of integrand, namely $0<c<|\xi|<C$ and $|x|>tP'(|\xi|)$, the function above is supported for $|x|>tc_0$, where $0<c_0<P'(c)$. That is to say, when $\operatorname{Supp}\kappa_0$ is chosen small enough,

$$\kappa_0\left(\frac{x}{t}\right)(A(t)-A_+(t))e^{itP(D_x)}v_0(x)=0\quad\forall t,x.$$

Let $\kappa_1 \in C_c^\infty(\mathbb{R}^d)$ be a radial function supported outside the unit ball. We further assume that κ_1 equals 1 away from zero. By observing that $|P'(\xi)| < 1$ for all $\xi \in \mathbb{R}^d$, we have

$$\kappa_1\left(\frac{x}{t}\right)(A(t)-A_+(t))e^{itP(D_x)}v_0(x) = \frac{\kappa_1\left(\frac{x}{t}\right)}{(2\pi)^d}\int e^{i(x\xi+tP(\xi))}(1-\chi)\left(\frac{x+tP'(\xi)}{t^{\frac{1}{2}+\delta}}\right)\hat{v}_0(\xi)\,d\xi.$$

Note that the nonsmooth term $\mathbb{1}_{|x|>t|P'(\xi)|}$ is identically 1 as we add the cut-off κ_1. By integration by parts in ξ, we may rewrite the quantity above as

$$\begin{aligned}\frac{\kappa_1\left(\frac{x}{t}\right)}{(2\pi)^d}\int e^{i(x\xi+tP(\xi))}\frac{-\partial_\xi}{i}\cdot\left[\frac{x+tP'(\xi)}{|x+tP'(\xi)|^2}(1-\chi)\left(\frac{x+tP'(\xi)}{t^{\frac{1}{2}+\delta}}\right)\hat{v}_0(\xi)\right]d\xi \\ = \frac{\kappa_1\left(\frac{x}{t}\right)}{(2\pi)^d}\int e^{i(x\xi+tP(\xi))}q_1(t,x,\xi)\hat{v}_0(\xi)\,d\xi + \frac{\kappa_1\left(\frac{x}{t}\right)}{(2\pi)^d}\int e^{i(x\xi+tP(\xi))}q_0(t,x,\xi)\cdot\hat{v}_1(\xi)\,d\xi,\end{aligned}$$

where $v_1(x) = xv_0(x)$, and

$$\begin{aligned} q_0(t,x,\xi) &= \frac{-\partial_\xi}{i}\left[\frac{x+tP'(\xi)}{|x+tP'(\xi)|^2}(1-\chi)\left(\frac{x+tP'(\xi)}{t^{\frac{1}{2}+\delta}}\right)\right], \\ q_1(t,x,\xi) &= \frac{x+tP'(\xi)}{|x+tP'(\xi)|^2}(1-\chi)\left(\frac{x+tP'(\xi)}{t^{\frac{1}{2}+\delta}}\right)\end{aligned}$$

are smooth symbols satisfying, for all $\alpha, \beta \in \mathbb{N}^d$,

$$|\partial_x^\alpha\partial_\xi^\beta q_k(t,x,\xi)| \lesssim t^{-2\delta}t^{-(\frac{1}{2}+\delta)|\alpha|}t^{(\frac{1}{2}-\delta)|\beta|}, \quad k=0,1.$$

By the Calderón–Vaillancourt theorem (Lemma A.11) and Lemma A.4, the $\mathcal{L}(L^2)$-norm of operators $\mathrm{Op}(q_k)$'s is bounded by $t^{-2\delta}$, which implies that

$$\left\|\kappa_1\left(\frac{x}{t}\right)(A(t)-A_+(t))e^{itP(D_x)}v_0\right\|_{L^2} \lesssim t^{-2\delta}(\|v_0\|_{L^2}+\|v_1\|_{L^2}) \sim t^{-2\delta}\|\langle x\rangle v_0\|_{L^2}.$$

By repeating this procedure M times, we obtain

$$\left\|\kappa_1\left(\frac{x}{t}\right)(A(t)-A_+(t))e^{itP(D_x)}v_0\right\|_{L^2} \lesssim t^{-2M\delta}\|\langle x\rangle^M v_0\|_{L^2} \lesssim t^{-2M\delta}.$$

The last estimate follows from the fact that v_0 is a Schwartz function. The proof is completed by choosing $\kappa = 1-\kappa_0-\kappa_1$. □

Thanks to Lemma 5.4, it remains to estimate the $L^2(dx)$-norm of

$$\kappa\left(\frac{x}{t}\right)(A(t)-A_+(t))e^{itP(D_x)}v_0(x),$$

or equivalently, in the polar system, the $L^2(r^{d-1}dr\,d\omega)$-norm of $I(t,r,\omega)$ defined by

$$I(t,r,\omega) = \kappa\left(\frac{r}{t}\right)\int e^{ir\rho\omega\cdot\theta}e^{itP(\rho)}(1-\chi)\left(\frac{r\omega+tP'(\rho)\theta}{t^{\frac{1}{2}+\delta}}\right)\mathbb{1}_{r>tP'(\rho)}\hat{v}_0(\rho\theta)\rho^{d-1}\,d\rho\,d\theta. \tag{5-8}$$

To begin with, we decompose (5-8) into three parts $I_+(t,r,\omega)$, $I_-(t,r,\omega)$, $I_0(t,r,\omega)$ by inserting

$$\chi_0\Big(\frac{\omega+\theta}{t^{\delta-\frac{1}{2}}}\Big),\quad \chi_0\Big(\frac{\omega-\theta}{t^{\delta-\frac{1}{2}}}\Big),\quad 1-\chi_0\Big(\frac{\omega+\theta}{t^{\delta-\frac{1}{2}}}\Big)-\chi_0\Big(\frac{\omega-\theta}{t^{\delta-\frac{1}{2}}}\Big)$$

into the integral respectively, where $\chi_0\in C_c^\infty(\mathbb{R}^d)$ is radial, supported in a small ball centered at zero, and equal to 1 near zero. The desired result (5-5) thus follows from the lemma below:

Lemma 5.5. *For all $N\in\mathbb{N}$, there exist constants C, C_N, which are independent of t, r, ω, such that*

$$\|I_+(t,r,\omega)\|^2_{L^2(r^{d-1}\,dr\,d\omega)}\leqslant C_N t^{-N}, \tag{5-9}$$

$$\|I_-(t,r,\omega)\|^2_{L^2(r^{d-1}\,dr\,d\omega)}\leqslant C t^{-1}, \tag{5-10}$$

$$\|I_0(t,r,\omega)\|^2_{L^2(r^{d-1}\,dr\,d\omega)}\leqslant C_N t^{-N}. \tag{5-11}$$

Proof of (5-9). The integral $I_+(t,r,\omega)$, by definition, reads

$$\kappa\Big(\frac{r}{t}\Big)\int e^{i[r\rho\omega\cdot\theta+tP(\rho)]}(1-\chi)\Big(\frac{r\omega+tP'(\rho)\theta}{t^{\frac{1}{2}+\delta}}\Big)\chi_0\Big(\frac{\omega+\theta}{t^{\delta-\frac{1}{2}}}\Big)\mathbb{1}_{r>tP'(\rho)}\hat{v}_0(\rho\theta)\rho^{d-1}\,d\rho\,d\theta.$$

The integrand of I_+ is supported for

$$|r-tP'(\rho)|=|-r\theta+tP'(\rho)\theta|\geqslant|r\omega+tP'(\rho)\theta|-r|\theta+\omega|\geqslant ct^{\frac{1}{2}+\delta}-Cc_0t\times t^{\delta-\frac{1}{2}}=(c-Cc_0)t^{\frac{1}{2}+\delta},$$

where $0<c_0\ll 1$ is the radius of Supp χ_0. This implies that

$$|r-tP'(\rho)|\geqslant c't^{\frac{1}{2}+\delta}.$$

As a consequence, the integrand of I_+ is smooth. This allows us to apply integration by parts in ρ, since $\rho\sim 1$ and

$$|r\omega\cdot\theta+tP'(\rho)|\geqslant|-r+tP'(\rho)|-r|\omega+\theta|\geqslant(c-2Cc_0)t^{\frac{1}{2}+\delta}\geqslant c''t^{\frac{1}{2}+\delta}.$$

More precisely, by using

$$e^{i[r\rho\omega\cdot\theta+tP(\rho)]}=\frac{-i}{r\omega\cdot\theta+tP'(\rho)}\partial_\rho e^{i[r\rho\omega\cdot\theta+tP(\rho)]},$$

one may gain $t^{-(1/2+\delta)}$ from $|r\omega\cdot\theta+tP'(\rho)|^{-1}$ and $t^{1/2-\delta}$ from each ∂_ρ. In conclusion, after integrating by parts M times in ρ we have

$$|I_+(t,r,\omega)|\lesssim t^{-2M\delta}.$$

Due to the factor κ, I_+ is supported for $r\sim t$, which implies that

$$\|I_+(t,r,\omega)\|^2_{L^2(r^{d-1}drd\omega)}\lesssim t^{-4M\delta+d}\leqslant t^{-N},$$

where M is large enough so that $4M\delta-d\geqslant N$. □

Proof of (5-10). The integral I_- will be treated as in Section 4. We first observe that the integrand of $I_-(t,r,\omega)$, which reads

$$\kappa\Big(\frac{r}{t}\Big)\int e^{i[r\rho\omega\cdot\theta+tP(\rho)]}(1-\chi)\Big(\frac{r\omega+tP'(\rho)\theta}{t^{\frac{1}{2}+\delta}}\Big)\chi_0\Big(\frac{\omega-\theta}{t^{\delta-\frac{1}{2}}}\Big)\mathbb{1}_{r>tP'(\rho)}\hat{v}_0(\rho\theta)\rho^{d-1}\,d\rho\,d\theta,$$

is supported for, as $t \gg 1$,

$$|r\omega + tP'(\rho)\theta| \geqslant |r\theta + tP'(\rho)\theta| - r|\omega-\theta| \geqslant r + tP'(\rho) - Cc_0 t^{\frac{1}{2}+\delta} \geqslant c't,$$

where $0 < c_0 \ll 1$ is the radius of Supp χ_0. That is to say, the factor $(1-\chi)$ is identically 1, and I_- becomes

$$\kappa\left(\frac{r}{t}\right)\int_{\rho_0}^{\min(\rho_1,(P')^{-1}(r/t))} e^{itP(\rho)}\int_{\mathbb{S}^d} e^{ir\rho\omega\cdot\theta}\chi_0\left(\frac{\omega-\theta}{t^{\delta-\frac{1}{2}}}\right)\hat{v}_0(\rho\theta)\,d\theta\,\rho^{d-1}\,d\rho,$$

where $\hat{v}_0(\rho\theta)$ is supported for $\rho \in [\rho_0, \rho_1]$ and we use the convention $(P')^{-1}(s) = +\infty$ when $s \geqslant 1$. Now, we may apply Lemma B.1 with $\lambda = r\rho$, $\mu = t^{\delta-\frac{1}{2}}$, and

$$F(x, y, z, \mu; \rho) = \chi_0\left(\frac{y-z}{\mu}\right)\tilde{\chi}_0\left(\frac{x-y}{\mu}\right)\hat{v}_0(\rho y).$$

Here $\tilde{\chi}_0 \in C_c^\infty(\mathbb{R}^d)$ is chosen to be equal to 1 on the support of χ_0. Note that F has to be taken at $(x, y, z) = (\omega, \theta, \omega)$ in order to recover the above integral. This corresponds in the statement of Lemma B.1 to $x = \theta$, $y = \theta'$, and $z = \omega$ (and no variable ω'). The extra term ρ is an extra parameter staying in a compact subset. Due to our choice of μ, it is clear that $r \sim t$ and $\rho \sim 1$ imply $\lambda\mu^2 \geqslant ct^{2\delta} \gg 1$. As a result, the integral in θ equals

$$\int_{\mathbb{S}^d} e^{ir\rho\omega\cdot\theta}\chi_0\left(\frac{\omega-\theta}{t^{\delta-\frac{1}{2}}}\right)\hat{v}_0(\rho\theta)\,d\theta = (2\pi)^{\frac{d-1}{2}} e^{ir\rho}\mu^{d-1} S_{-\frac{1}{2}(d-1)}(\omega, \rho, \mu; r\rho\mu^2),$$

with $S_m(\omega, \rho, \mu; \zeta)$ smooth, supported for $\rho \sim 1$, $\zeta > 1$, and satisfying, for all $\alpha, \in \mathbb{N}$, $j, k, \gamma \in \mathbb{N}$,

$$|\partial_\omega^\alpha \partial_\rho^j \partial_\mu^k \partial_\zeta^\gamma S_m^\pm(\omega, \rho, \mu; \zeta)| \leqslant C\mu^{-|\alpha|-k}\langle\zeta\rangle^{m-\gamma}.$$

This formulation allows us to rewrite $I_- r^{(d-1)/2}$, up to multiplication with constants, as

$$I_-(t, r, \omega) r^{\frac{1}{2}(d-1)} = \kappa\left(\frac{r}{t}\right)\int_{\rho_0}^{\min(\rho_1,(P')^{-1}(r/t))} e^{i[r\rho+tP(\rho)]} S_0(\omega, \rho, \mu; r\rho\mu^2)\rho^{\frac{1}{2}(d-1)}\,d\rho.$$

Now, we may apply integration by parts in ρ. Due to the fact that $r+tP'(\rho)\sim t$ and $\min(\rho_1,(P')^{-1}(r/t))\sim 1$, the boundary terms and remaining term are all bounded by t^{-1}, namely

$$|I_-(t, r, \omega) r^{\frac{1}{2}(d-1)}| \lesssim \mathbb{1}_{r\sim t} t^{-1},$$

which implies (5-10). □

Proof of (5-11). Unlike the study of $I_\pm$, I_0 will be estimated via integration by parts in the θ-variable. It is clear that the integrand of

$$I_0(t, r, \omega) = \kappa\left(\frac{r}{t}\right)\int e^{ir\rho\omega\cdot\theta} e^{itP(\rho)}(1-\chi)\left(\frac{r\omega+tP'(\rho)\theta}{t^{\frac{1}{2}+\delta}}\right)\mathbb{1}_{r>tP'(\rho)}\hat{v}_0(\rho\theta)\rho^{d-1} \times\left[1-\chi_0\left(\frac{\omega+\theta}{t^{\delta-\frac{1}{2}}}\right)-\chi_0\left(\frac{\omega-\theta}{t^{\delta-\frac{1}{2}}}\right)\right]d\rho\,d\theta$$

is supported away from $\{\theta \pm \omega = 0\}$, which allows us to rewrite the integral above in local coordinates

$$\theta = h\omega + \sqrt{1-h^2}\,y, \quad h \in [-1, 1],\ y \in \omega^\perp \cap \mathbb{S}^{d-1},$$

where $\omega^\perp$ is defined as the hyperplane $\{y \in \mathbb{R}^d : y \cdot \omega = 0\}$. In this way, we may rewrite I_0 as

$$\kappa\Big(\frac{r}{t}\Big) \int e^{ir\rho h} F_0(t, r, \rho, \omega, h\omega + \sqrt{1-h^2}\,y)(1-h^2)^{\frac{1}{2}d-1} \times \bigg[1 - \chi_0\bigg(\frac{(h+1)\omega + \sqrt{1-h^2}\,y}{t^{\delta-\frac{1}{2}}}\bigg) - \chi_0\bigg(\frac{(h-1)\omega + \sqrt{1-h^2}\,y}{t^{\delta-\frac{1}{2}}}\bigg)\bigg] dh\,dy\,d\rho, \tag{5-12}$$

where

$$F_0(t, r, \rho, \omega, \theta) = e^{itP(\rho)}(1-\chi)\bigg(\frac{r\omega + tP'(\rho)\theta}{t^{\frac{1}{2}+\delta}}\bigg)\mathbb{1}_{r > tP'(\rho)}\hat{v}_0(\rho\theta)\rho^{d-1}.$$

Note that due to the cut-off away from $\pm\omega$, the integrand of (5-12) is supported for

$$|(h \pm 1)\omega + \sqrt{1-h^2}\,y|^2 \geqslant ct^{2\delta-1}.$$

By developing the inequality above we obtain

$$\sqrt{1-h^2} \geqslant c' t^{\delta-\frac{1}{2}}.$$

Thus, F_0 is supported for $\rho \sim 1$ and satisfies, for all $k \in \mathbb{N}$,

$$\big|\partial_h^k\big(F_0(t, r, \rho, \omega, h\omega + \sqrt{1-h^2}\,y)\big)\big| \lesssim t^{(1-2\delta)k}.$$

The same estimates hold for $(1-h^2)^{d/2-1}$ and cut-off χ_0's in dimension $d \geqslant 2$, while in the trivial case $d = 1$, I_0 is identically zero.

Now, we may apply M times integration by parts in h for (5-12). As $r \sim t$, $\rho \sim 1$, each ∂_h in the amplitude gives $t^{1-2\delta}$, we have

$$|I_0(t, r, \omega)| \lesssim \mathbb{1}_{r \sim t} t^{-2M\delta},$$

and (5-11) follows from

$$\|I_0(t, r, \omega)\|^2_{L^2(r^{d-1}dr\,d\omega)} \lesssim t^{-4M\delta+d} \lesssim t^{-N},$$

by choosing $4M\delta - d \geqslant N$. □

Appendix A: Technical lemmas

This appendix is a collection of technical lemmas which are used in previous sections.

A1. *Some technical inequalities.*

Lemma A.1. *For any real number $m > d-1$, there exists a constant $C = C(m, d) > 0$ such that*

$$\sup_{\xi \in \mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \langle R\omega - \xi\rangle^{-m}\, d\omega \leqslant CR^{-(d-1)}.$$

Proof. By a change of variable, it is equivalent to study the boundedness of

$$\int_{R\mathbb{S}^{d-1}} K(\omega - \xi)\, d\omega,$$

where $K(x) = (1+|x|^m)^{-1}$. This integral can be regarded as a convolution of K and the Borel measure μ_R, which is defined by,

$$\forall \phi \in C_c(\mathbb{R}^d), \quad \langle \mu_R, \phi \rangle := \int_{R\mathbb{S}^{d-1}} \phi(\omega)\, d\omega.$$

We introduce the α-dimensional density of a Borel measure μ:

$$M_\mu^{(\alpha)}(x) := \sup_{r>0} \frac{\mu(B(x,r))}{r^\alpha}.$$

Note that $M_{\mu_R}^{(d-1)}(\xi) = M_{\mu_1}^{(d-1)}(\xi/R)$ and that $M_{\mu_1}^{(d-1)}$ is a bounded function. Thus, $M_{\mu_R}^{(d-1)}(\xi)$ is bounded uniformly in $R > 0$ and $\xi \in \mathbb{R}^d$.

Now, it suffices to show that there exists some constant $c = c(m, d)$ such that, for any Borel measure μ on $\mathbb{R}^d$,

$$K * \mu(\xi) \leqslant c M_\mu^{(d-1)}(\xi).$$

By applying a translation, the problem can be reduced to the case $\xi = 0$:

$$K * \mu(0) = \int_{\mathbb{R}^d} K(-y)\, d\mu(y) = \int_{\mathbb{R}^d} K(y)\, d\mu(y) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \mu(B(0, r_j^{(n)})),$$

where the last equality follows from dominated convergence theorem, with $r_j^{(n)}$ defined by $K^{-1}(1-j/n) = r_j^{(n)}\mathbb{S}^{d-1}$. We may calculate $r_j^{(n)}$ explicitly:

$$r_j^{(n)} = \left(\frac{j/n}{1-j/n}\right)^{\frac{1}{m}}.$$

Therefore, by the definition of $(d-1)$-dimensional density, we have

$$\begin{aligned} K * \mu(0) &\leqslant M_\mu^{(d-1)}(0) \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} (r_j^{(n)})^{d-1} \\ &= M_\mu^{(d-1)}(0) \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \left(\frac{j/n}{1-j/n}\right)^{\frac{d-1}{m}} = M_\mu^{(d-1)}(0) \int_0^1 \left(\frac{x}{1-x}\right)^{\frac{d-1}{m}} dx. \end{aligned}$$

The last quantity is finite since $m > d-1$. □

Lemma A.2. *Let $S_{-1} \in C^\infty(\mathbb{R})$ with $|S_{-1}^{(\alpha)}(\xi)| \leqslant C_\alpha \langle \xi \rangle^{-1-\alpha}$ for any $\alpha \in \mathbb{N}$. Then, for any $N \in \mathbb{N}$, there exists some constant C such that*

$$\left| \int_{\mathbb{R}} e^{i\lambda\xi} S_{-1}(\xi)\, d\xi \right| \leqslant C \langle \lambda \rangle^{-N} (1 + \log_-(|\lambda|)) \quad \forall \lambda \neq 0, \tag{A-1}$$

where the integral on the left-hand side should be understood as an oscillatory integral and the function $\log_-$ *is defined by*

$$\log_-(t) := \begin{cases} |\log(t)| & \text{if } t \in \,]0,1[, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For simplicity, we may assume $\lambda > 0$ and that S_{-1} is supported in $[c_0, \infty[$ for some $c_0 > 0$.

When $\lambda \geqslant c$ for some small constant $c > 0$, we may apply integration by parts on ξ; then the left-hand side of (A-1) can be controlled by

$$\frac{C}{\lambda}\int_{\mathbb{R}} \langle\xi\rangle^{-2}\,d\xi \lesssim \langle\lambda\rangle^{-1}.$$

To obtain arbitrary polynomial decrease for large λ, we only need to apply integration by parts several times.

When $\lambda < c$, we introduce a cut-off $\chi \in C_c^\infty(\mathbb{R})$ such that $\chi = 1$ in a large neighborhood of 0. For the part where $|\lambda\xi|$ is small,

$$\left|\int_{\mathbb{R}} e^{i\lambda\xi} S_{-1}(\xi)\chi(\lambda\xi)\,d\xi\right| \leqslant C\int_{c_1}^{c_2/\lambda} \frac{d\xi}{\xi} \leqslant C'(1+\log_-(\lambda)).$$

For the remaining part where $|\lambda\xi|$ is large, we need to estimate

$$\int_{\mathbb{R}} e^{i\lambda\xi} S_{-1}(\xi)(1-\chi)(\lambda\xi)\,d\xi.$$

By applying integration by parts in ξ, we will obtain two integrals. The first one is of the form

$$\int_{\mathbb{R}} e^{i\lambda\xi} S_{-1}(\xi)\chi'(\lambda\xi)\,d\xi,$$

which can be treated as above. The second one takes the form

$$\frac{1}{\lambda}\int_{\mathbb{R}} e^{i\lambda\xi} S_{-2}(\xi)(1-\chi)(\lambda\xi)\,d\xi,$$

which is bounded by

$$\frac{1}{\lambda}\int_{|\xi|>c'/\lambda} \langle\xi\rangle^{-2}\,d\xi \leqslant C. \qquad \square$$

Lemma A.3 (Cotlar–Stein lemma). *Let H be a Hilbert space and $\{T_j\}_{j\in\mathbb{N}}$ be a series of bounded operators on H. If*

$$A = \sup_{j\in\mathbb{N}} \sum_{k\in\mathbb{N}} \|T_j T_k^*\|_{\mathcal{L}(H)}^{\frac{1}{2}} < +\infty, \tag{A-2}$$

$$B = \sup_{j\in\mathbb{N}} \sum_{k\in\mathbb{N}} \|T_j^* T_k\|_{\mathcal{L}(H)}^{\frac{1}{2}} < +\infty, \tag{A-3}$$

the operator $T = \sum_{j\in\mathbb{N}} T_j$ is well-defined via the pointwise limit,

$$\forall u \in H, \quad Tu := \lim_{J\to+\infty} \sum_{j=0}^{J} T_j u \quad \text{in } H.$$

Moreover, T is bounded on H with estimate

$$\|T\|_{\mathcal{L}(H)} \leqslant \sqrt{AB}. \tag{A-4}$$

The inequality (A-4) was first given in [Cotlar 1955] for finite sum with $\|T_j T_k^*\|_{\mathcal{L}(H)}$ and $\|T_j^* T_k\|_{\mathcal{L}(H)}$ decreasing exponentially in $|j-k|$. The generalized version stated above and its proof can be found, for example, in Theorem 1, Chapter VII of [Stein 1993].

A2. *Criteria on L^2-boundedness of pseudodifferential operators.*

Lemma A.4. *Let a be a symbol on $\mathbb{R}^d$ and $\lambda > 0$. The rescaled symbol*

$$a_\lambda(x, \xi) := a\left(\lambda x, \frac{\xi}{\lambda}\right)$$

satisfies

$$\|\mathrm{Op}(a_\lambda)\|_{\mathcal{L}(L^2)} = \|\mathrm{Op}(a)\|_{\mathcal{L}(L^2)}.$$

Lemma A.5. *Let a be a symbol on $\mathbb{R}^d$ and the symbol $\tilde{a}$ is defined as*

$$\tilde{a}(x, \xi) := a(\xi, x).$$

Then we have

$$\|\mathrm{Op}(a)\|_{\mathcal{L}(L^2)} = \|\mathrm{Op}(\tilde{a})\|_{\mathcal{L}(L^2)}.$$

Lemma A.6. *Let K be a kernel function of operator $\mathcal{K}$ defined as*

$$\mathcal{K}u(x) := \int K(x, x-y)u(y)\,dy.$$

If there exists $K_0 \in L^1(\mathbb{R}^d)$ such that

$$|K(x, z)| \leqslant K_0(z) \ \forall x, z \in \mathbb{R}^d,$$

we have

$$\|\mathcal{K}\|_{\mathcal{L}(L^2)} \leqslant \|K_0\|_{L^1}.$$

Lemma A.7. *Let a and b be symbols satisfying*

$$|a(x, \xi)| \leqslant b(x, \xi),$$

and $B \in \mathcal{L}(L^2)$ be an operator defined by

$$Bu(x) = \int b(x, \xi)v(\xi)\,d\xi.$$

Then

$$\|\mathrm{Op}(a)\|_{\mathcal{L}(L^2)} \leqslant \|B\|_{\mathcal{L}(L^2)}.$$

Lemma A.8. *Let $m : \mathbb{S}^{d-1} \times \mathbb{S}^{d-1} \times]0, \infty[\to \mathbb{C}$ be a smooth function satisfying for all $\alpha, \alpha' \in \mathbb{N}^{d-1}$ and $N \in \mathbb{N}$*

$$|\partial_\omega^\alpha \partial_{\omega'}^{\alpha'} m(\omega, \omega', \mu)| \leqslant C_{\alpha,\alpha',N} A \mu^{-|\alpha|-|\alpha'|} \left\langle \frac{d(\omega, -\omega')}{\mu} \right\rangle^{-N},$$

where A is a quantity independent of ω and ω', d is the distance on the sphere $\mathbb{S}^{d-1}$. For all $\lambda > 0$, the operator T_λ is defined by

$$T_\lambda u(\omega') = \int_{\mathbb{S}^{d-1}} e^{i\lambda\omega\omega'} m(\omega, \omega', \mu) u(\omega)\,d\omega.$$

Then there exists a constant $C > 0$ such that

$$\|T_\lambda\|_{\mathcal{L}(L^2(\mathbb{S}^{d-1}))} \leqslant C A \lambda^{-\frac{1}{2}(d-1)}.$$

When $\mu \in [1, \infty[$, the condition on m becomes $m(\cdot, \cdot, \mu) \in C_b^\infty(\mathbb{S}^{d-1} \times \mathbb{S}^{d-1})$ uniformly in $\mu \geqslant 1$. By using the local coordinate of the unit sphere $\mathbb{S}^{d-1}$ (which is compact and thus can be covered by finite local patches), one can reduce the problem to the $\mathbb{R}^{d-1}$ case without considering the parameter μ. The proof of this reduced case can be found in, for example, [Sogge 2017, Theorem 2.1.1]. As for the case $\mu \in]0, 1]$, one may refer to [Delort 2022, Proposition A.1.7]. A direct consequence is stated as following:

Lemma A.9. *Let $a(x, \xi, \mu)$ be a smooth symbol on $\mathbb{R}^d \times \mathbb{R}^d$ depending on a parameter $\mu \in]0, \infty[$. Let b be a function on $\mathbb{R}_+^2$ such that the associated operator B defined below is bounded on $L^2(\mathbb{R}_+)$, namely*

$$\|B\|_{\mathcal{L}(L^2(\mathbb{R}_+))} < \infty, \quad Bf(r) = \int_0^\infty b(r, \rho) f(\rho)\, d\rho. \tag{A-5}$$

If, in the polar system $x = r\omega$, $\xi = \rho\theta$, $a(x, \xi, \mu)$ satisfies, for all $\alpha, \beta \in \mathbb{N}^{d-1}$ and $N \in \mathbb{N}$,

$$|\partial_\omega^\alpha \partial_\theta^\beta a(r\omega, \rho\theta, \mu)| \leqslant C_{\alpha,\beta,N} b(r, \rho) \mu^{-|\alpha|-|\beta|} \left\langle \frac{d(\omega, -\theta)}{\mu} \right\rangle^{-N}, \tag{A-6}$$

then we have

$$\|\mathrm{Op}(a)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leqslant C \|B\|_{\mathcal{L}(L^2(\mathbb{R}_+))}, \tag{A-7}$$

where $C > 0$ is a universal constant.

Proof. By the definition of $\mathrm{Op}(a)$, we have

$$\mathrm{Op}(a)u(r\omega) = \frac{1}{(2\pi)^d} \int_0^\infty \int_{\mathbb{S}^{d-1}} e^{ir\rho\omega\theta} a(r\omega, \rho\theta) \hat{u}(\rho\theta) \rho^{d-1}\, d\theta\, d\rho.$$

It is easy to check that $\|f(\rho, \theta)\|_{L^2(d\rho\, d\theta)} = (2\pi)^{d/2} \|u\|_{L^2(\mathbb{R}^d)}$, with $f(\rho, \theta) = \hat{u}(\rho\theta)\rho^{(d-1)/2}$. Thus,

$$\begin{aligned}
\|\mathrm{Op}(a)u\|_{L^2(\mathbb{R}^d)} &= \|\mathrm{Op}(a)u(r\omega) r^{\frac{1}{2}(d-1)}\|_{L^2(dr\, d\omega)} \\
&= \left\| \frac{1}{(2\pi)^d} \int_0^\infty \int_{\mathbb{S}^{d-1}} e^{ir\rho\omega\theta} a(r\omega, \rho\theta) f(\rho, \theta) (r\rho)^{\frac{1}{2}(d-1)}\, d\theta\, d\rho \right\|_{L^2(dr\, d\omega)} \\
&\leqslant C_0 \left\| \int_0^\infty (r\rho)^{\frac{1}{2}(d-1)} \left\| \int_{\mathbb{S}^{d-1}} e^{ir\rho\omega\theta} a(r\omega, \rho\theta) f(\rho, \theta)\, d\theta \right\|_{L^2(d\omega)} d\rho \right\|_{L^2(dr)}.
\end{aligned}$$

By applying the previous lemma with $m(\omega, \theta, \mu) = a(r\omega, \rho\theta, \mu)$, where r, ρ should be regarded as parameters, $A = b(r, \rho)$, and $\lambda = r\rho$, we obtain

$$\left\| \int_{\mathbb{S}^{d-1}} e^{ir\rho\omega\theta} a(r\omega, \rho\theta) f(\rho, \theta)\, d\theta \right\|_{L^2(d\omega)} \leqslant C_1 (r\rho)^{-\frac{1}{2}(d-1)} b(r, \rho) \|f(\theta, \rho)\|_{L^2(d\theta)}.$$

Thus,

$$\begin{aligned}
\|\mathrm{Op}(a)u\|_{L^2(\mathbb{R}^d)} &\leqslant C_0 C_1 \left\| \int_0^\infty b(r, \rho) \|f(\theta, \rho)\|_{L^2(d\theta)}\, d\rho \right\|_{L^2(dr)} \\
&\leqslant C_0 C_1 \|B\|_{\mathcal{L}(L^2(\mathbb{R}_+))} \|f(\theta, \rho)\|_{L^2(d\theta\, d\rho)} = C \|B\|_{\mathcal{L}(L^2(\mathbb{R}_+))} \|u\|_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

□

Lemma A.10. *Let a be a symbol on* $\mathbb{R}^d$, *depending on some parameter* $\lambda \in]0, \infty[$. *If, for all* $\alpha \in \mathbb{N}^d$,

$$\sup_{x\in\mathbb{R}^d} \|\partial_\xi^\alpha a(x,\cdot)\|_{L^1_\xi} \leqslant C_\alpha \lambda^{d-|\alpha|},$$

the operator $\operatorname{Op}(a)$ *is bounded on* L^2, *uniformly in* λ.

Proof. Due to Lemma A.4, the problem can be reduced to the case $\lambda = 1$. The kernel of the operator $\operatorname{Op}(a)$ is $K(x, y) = J(x, x - y)$, where

$$J(x,z) = \frac{1}{(2\pi)^d} \int e^{iz\cdot\xi} a(x,\xi)\, d\xi.$$

We first observe that, for all $x \in \mathbb{R}^d$, $|J(x,z)| \leqslant (2\pi)^{-d} \|a(x,\cdot)\|_{L^1_\xi}$, which is uniformly bounded. Then, by integration by parts, we have, for all $N \in \mathbb{N}$,

$$||z|^{2N} J(x,z)| \lesssim \sum_{|\alpha|=2N} \left| \int e^{iz\cdot\xi} \partial_\xi^\alpha a(x,\xi)\, d\xi \right| \lesssim \sum_{|\alpha|=2N} \sup_{x\in\mathbb{R}^d} \|\partial_\xi^\alpha a(x,\cdot)\|_{L^1_\xi} \leqslant C_\alpha.$$

That is to say, J is bounded and has any polynomial decay in z, uniformly in x. In particular,

$$|K(x,y)| = |J(x,x-y)| \lesssim \langle x-y\rangle^{-(d+1)}.$$

The conclusion follows from Schur's lemma. □

Lemma A.11 (Calderón–Vaillancourt theorem). *For smooth symbol* $a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, *the following estimate holds*:

$$\|\operatorname{Op}(a)\|_{\mathcal{L}(L^2)} \lesssim \sup_{|\alpha|,|\beta|\leqslant N_d} \|\partial_x^\alpha \partial_\xi^\beta a\|_{L^\infty(\mathbb{R}^d_x\times\mathbb{R}^d_\xi)}, \tag{A-8}$$

where N_d *is a universal constant depending only on dimension* d.

The earliest version of (A-8) was given in [Calderón and Vaillancourt 1971], where α_j, β_j are required to be no more than 3 for all $j = 1, 2, \dots, d$. Then, in [Coifman and Meyer 1978], the authors optimized it to $N_d = \left[\frac{1}{2}d\right] + 1$, while some other assumptions in α, β are given in the same paper. Readers may also find an alternative proof via the Gabor transform in [Hwang 1987].

Appendix B: Stationary phase lemmas

Lemma B.1. *Let*

$$F : (\mathbb{S}^{d-1})^4 \times]0,1] \to \mathbb{C},$$
$$(\theta, \theta', \omega, \omega', \mu) \mapsto F(\theta, \theta', \omega, \omega', \mu),$$

be a smooth function supported for $d(\theta', \theta) + d(\theta', \omega') < \delta'$, *where d is the metric on the sphere* $\mathbb{S}^{d-1}$ *and* $\delta' > 0$ *is a small constant. We assume that F satisfies, for all* $\alpha, \alpha', \beta, \beta' \in \mathbb{N}$, $j, N \in \mathbb{N}$,

$$|\partial_\theta^\alpha \partial_{\theta'}^{\alpha'} \partial_\omega^\beta \partial_{\omega'}^{\beta'} \partial_\mu^j F| \leqslant C \mu^{-|\alpha|-|\alpha'|-|\beta|-j} \left\langle \frac{d(\theta',\omega)}{\mu} \right\rangle^{-N}.$$

For any parameter $\lambda > 0$, *we define the integral*

$$I_{\pm}(\theta, \omega, \omega'; \lambda, \mu) = \int_{\mathbb{S}^{d-1}} e^{\pm i\lambda\theta\theta'} F(\theta, \theta', \omega, \omega', \mu)\, d\theta'.$$

If δ' *is small enough (depending on the sphere* $\mathbb{S}^{d-1}$*), and* $\lambda\mu^2 \geqslant c > 0$, *then we can write*

$$I_{\pm}(\theta, \omega, \omega'; \lambda, \mu) = e^{\pm i\lambda} \mu^{d-1} S^{\pm}_{-\frac{1}{2}(d-1)}(\theta, \omega, \omega', \mu; \lambda\mu^2),$$

where $S^{\pm}_{-(d-1)/2}(\theta, \omega, \omega', \mu; \zeta)$ *is a smooth function supported for* $d(\theta, \omega') \leqslant 2\delta'$, *satisfying, for all* $\alpha, \beta, \beta' \in \mathbb{N},\ j, \gamma, N \in \mathbb{N}$,

$$|\partial_\theta^\alpha \partial_\omega^\beta \partial_{\omega'}^{\beta'} \partial_\mu^j \partial_\zeta^\gamma S^{\pm}_{-\frac{1}{2}(d-1)}| \leqslant C\mu^{-|\alpha|-|\beta|-j} \langle\zeta\rangle^{-\frac{1}{2}(d-1)-\gamma} \left\langle \frac{d(\theta, \omega)}{\mu} \right\rangle^{-N}. \tag{B-1}$$

Moreover, for $|\zeta| \geqslant c > 0$, $S^{\pm}_{-(d-1)/2}(\theta, \omega, \omega', \mu; \zeta)$ *can be decomposed as*

$$(2\pi)^{\frac{1}{2}(d-1)} e^{\mp i\frac{\pi}{4}(d-1)} F(\theta, \theta, \omega, \omega', \mu) \zeta^{-\frac{1}{2}(d-1)} + S^{\pm}_{-\frac{1}{2}(d+1)}(\theta, \omega, \omega', \mu; \zeta), \tag{B-2}$$

where $S^{\pm}_{-(d+1)/2}(\theta, \omega, \omega', \mu; \zeta)$ *is smooth and supported for* $d(\theta, \omega') \leqslant 2\delta'$, *satisfying the estimate* (B-1) *with* $d-1$ *replaced by* $d+1$.

Lemma B.2. *Let*

$$F_0 : (\mathbb{S}^{d-1})^4 \times]0, 1] \to \mathbb{C},$$
$$(\theta, \theta', \omega, \omega', \mu) \mapsto F(\theta, \theta', \omega, \omega', \mu),$$

be a smooth function supported for $\min(d(\theta', \theta), d(\theta', -\theta)) > \delta' > 0$, *where* d *is the metric on the sphere* $\mathbb{S}^{d-1}$ *and* δ' *is constant. We assume that* F *satisfies, for all* $\alpha, \alpha', \beta, \beta' \in \mathbb{N},\ j, N \in \mathbb{N}$,

$$|\partial_\theta^\alpha \partial_{\theta'}^{\alpha'} \partial_\omega^\beta \partial_{\omega'}^{\beta'} \partial_\mu^j F_0| \leqslant C\mu^{-|\alpha|-|\alpha'|-|\beta|-j} \left\langle \frac{d(\theta', \omega)}{\mu} \right\rangle^{-N}.$$

For any parameter $\lambda > 0$, *the integral*

$$I_{\pm}(\theta, \omega, \omega'; \lambda, \mu) = \int_{\mathbb{S}^{d-1}} e^{\pm i\lambda\theta\theta'} F_0(\theta, \theta', \omega, \omega', \mu)\, d\theta'$$

can be written as

$$I_{\pm}(\theta, \omega, \omega'; \lambda, \mu) = e^{\pm i\lambda\theta\omega} \mu^{d-1} R^{\pm}(\theta, \omega, \omega', \mu; \lambda\mu),$$

where $R^{\pm}(\theta, \omega, \omega', \mu; \zeta)$ *is a smooth function satisfying, for all* $\alpha, \beta, \beta' \in \mathbb{N},\ j, \gamma, N \in \mathbb{N}$,

$$|\partial_\theta^\alpha \partial_\omega^\beta \partial_{\omega'}^{\beta'} \partial_\mu^j \partial_\zeta^\gamma R^{\pm}| \leqslant C\mu^{-|\alpha|-|\beta|-j} \langle\zeta\rangle^{-N}. \tag{B-3}$$

For the proof of Lemmas B.1 and B.2, one may find a general result in [Delort 2022, Proposition A.1.1].

Remark B.3. In Lemmas B.1 and B.2, the dependence on ω' is not crucial in the proof; we may eliminate the conditions and results involving ω'. Meanwhile, if F (resp. F_0) depends on some other parameters, the same condition can be inherited by $S_{-(d-1)/2}$ (resp. R) from F (resp. F_0).

Lemmas B.1 and B.2, together with Remark B.3, result in the following lemmas, which are important techniques used in the main text.

Lemma B.4. *Let*

$$G : (\mathbb{S}^{d-1})^2 \times]0,1] \to \mathbb{C},$$
$$(\theta', \omega, \mu) \mapsto G(\theta', \omega, \mu),$$

be a smooth function supported for $d(\theta', \omega) < \delta'$, where d is the metric on $\mathbb{S}^{d-1}$ and $\delta' > 0$ is a small constant. We further assume that, for all $\alpha', \beta \in \mathbb{N}^{d-1}$, $j, N \in \mathbb{N}$,

$$|\partial_{\theta'}^{\alpha'} \partial_{\omega}^{\beta} \partial_{\mu}^{j} G| \lesssim \mu^{-|\alpha'|-|\beta|-j} \left\langle \frac{d(\theta', \omega)}{\mu} \right\rangle^{-N}.$$

For $\lambda > 0$, $\epsilon \in \{\pm\}$, we define

$$I(\theta, \omega, \mu; \lambda) := \int_{\mathbb{S}^{d-1}} e^{\epsilon i \lambda \theta \cdot \theta'} G(\theta', \omega, \mu)\, d\theta'.$$

Then, under the conditions that δ' is small enough and that $\lambda\mu^2 > c > 0$ for some constant c, the integral I may be written as the sum of principal terms

$$e^{\pm \epsilon i \lambda} \mu^{d-1} S^{\pm}_{-\frac{1}{2}(d-1)}(\theta, \omega, \mu; \lambda\mu^2),$$

and a remainder

$$e^{\epsilon i \lambda \theta \cdot \omega} \mu^{d-1} R(\theta, \omega, \mu; \lambda\mu),$$

where $S^{\pm}_m(\theta, \omega, \mu; \zeta)$ is a smooth function supported on $d(\theta, \pm\omega) < 2\delta'$, $\zeta > c > 0$, satisfying, for all $\alpha, \beta \in \mathbb{N}^{d-1}$, $j, n, N \in \mathbb{N}$,

$$|\partial_{\theta}^{\alpha} \partial_{\omega}^{\beta} \partial_{\mu}^{j} \partial_{\zeta}^{n} S^{\pm}_m| \lesssim \mu^{-|\alpha|-|\beta|-j} \langle \zeta \rangle^{m-n} \left\langle \frac{d(\theta, \pm\omega)}{\mu} \right\rangle^{-N},$$

and $R(\theta, \omega, \mu; \zeta)$ is a smooth function satisfying, for all $N \in \mathbb{N}$,

$$|R| \lesssim \langle \zeta \rangle^{-N}.$$

Moreover, if G depends on some extra parameters, the same bound for G (and its derivatives in parameters) can be inherited by $S^{\pm}_{-(d-1)/2}$ and R.

Proof. We only give the proof for $\epsilon = +$, while the other one can be treated in the same way. To begin with, we introduce the functions

$$F_{\pm}(\theta, \theta', \omega, \mu) := G(\pm\theta', \pm\omega, \mu)\chi(\theta - \theta'),$$
$$F_0(\theta, \theta', \omega, \mu) := G(\theta', \omega, \mu)\big(1 - \chi(\theta - \theta') - \chi(\theta + \theta')\big),$$

where $\chi \in C_c^\infty(\mathbb{R}^d)$ is radial and supported in a small neighborhood of zero, with value 1 near zero. By using these functions, we may rewrite the integral I as $I_+ + I_- + I_0$ with

$$I_{\pm} = \int e^{\pm i \lambda \theta \cdot \theta'} F_{\pm}(\theta, \theta', \pm\omega, \mu)\, d\theta', \quad I_0 = \int e^{i \lambda \theta \cdot \theta'} F_0(\theta, \theta', \omega, \mu)\, d\theta'.$$

It is clear that $F_\pm$, F_0 satisfy the conditions in Lemmas B.1 and B.2, respectively. Then, by applying these lemmas, we can obtain the desired expressions and corresponding estimates. There remain two points to check: the support of $S^\pm_{-(d-1)/2}$ and the dependence on extra parameters. The latter is merely an application of Remark B.3, while the former can be shown by observing that, when $\operatorname{Supp}\chi$ is taken to be small enough, the integrand of integrals $I_\pm$ is supported on $d(\pm\theta',\omega)<\delta'$ and $d(\theta,\theta')<\delta'$. □

Lemma B.5. *Let $\chi\in C_c^\infty(\mathbb{R}^d)$ and $f\in C_c^\infty(\mathbb{R}^d\backslash\{0\})$. Assume that P is smooth on $]0,\infty[$, and Λ is a positive function such that*

$$\begin{aligned} \Lambda(\rho)&\sim\rho^\sigma && \text{as } \rho\to+\infty,\\ |\Lambda^{(j)}(\rho)|&\lesssim\rho^{\sigma-j} && \forall\rho>\rho_0>0,\ j\in\mathbb{N},\\ \frac{\Lambda(\rho)}{\rho^\sigma}&\to\lambda_0 && \text{as } \rho\to+\infty \end{aligned}$$

hold for some $\sigma\in\mathbb{R}$ and $\lambda_0>0$.

Then there exists $t_0\gg1$ depending on f, P', and Λ such that, for $\epsilon,\epsilon'\in\{+1,-1\}$, $\delta+\frac12\sigma\in\left[0,\frac12\right[$, $r>c>0$, and $t>t_0$, the integral

$$\int_{\mathbb{S}^{d-1}}e^{i\epsilon' r\rho\omega\theta}\chi\left(\frac{r\omega+\epsilon tP'(\rho)\theta}{t^{\frac12+\delta}\Lambda(t^{\frac12}\rho)}\right)f(\rho\theta)\,d\theta$$

can be decomposed as a principal term

$$(2\pi)^{\frac12(d-1)}e^{i\epsilon\epsilon'\frac{\pi}{4}(d-1)}e^{-i\epsilon\epsilon' r\rho}(r\rho)^{-\frac12(d-1)}\chi\left(\frac{r-tP'(\rho)}{t^{\frac12+\delta}\Lambda(t^{\frac12}\rho)}\right)f(-\epsilon\rho\omega)$$

and a remainder

$$e^{-i\epsilon\epsilon' r\rho}\mu^{d-1}S_{-(d+1)/2}\left(\omega,\mu,\rho,\frac{r}{t}-P'(\rho),t;r\rho\mu^2\right),$$

where $\mu=t^{\delta+\sigma/2-1/2}$, $S_m(\omega,\mu,\rho,r',t;\zeta)$ is supported for $\zeta>c>0$, $\rho\sim1$ and $|r'|\lesssim\mu$ and satisfies, for all $\alpha\in\mathbb{N}^{d-1}$, $j,k,l,\gamma\in\mathbb{N}$,

$$|\partial_\omega^\alpha\partial_\mu^j\partial_\rho^k\partial_{r'}^l\partial_\zeta^\gamma S_{-\frac12(d+1)}|\leqslant C\mu^{-(|\alpha|+j+l)}\langle\zeta\rangle^{m-\gamma}.$$

Proof. Using the notation $r=t(r'+P'(\rho))$, where $|r'|\lesssim\mu=t^{\delta+\sigma/2-1/2}$, we can rewrite the integral as

$$\int_{\mathbb{S}^{d-1}}e^{i\epsilon' t(r'+P'(\rho))\rho\omega\theta}\chi\left(\frac{r'\omega+P'(\rho)(\omega+\epsilon\theta)}{t^{-\frac12+\delta}\Lambda(t^{\frac12}\rho)}\right)f(\rho\theta)\,d\theta. \tag{B-4}$$

Consider the function

$$F(x,y,z,\mu;\rho,r',t)=\chi\left(\frac{r'x+P'(\rho)(x-y)}{\mu t^{-\frac12\sigma}\Lambda(t^{\frac12}\rho)}\right)\tilde\chi\left(\frac{x-y}{\mu}\right)\tilde\chi\left(\frac{y-z}{\mu}\right)f(\rho y),$$

with $\tilde\chi\in C_c^\infty(\mathbb{R}^d)$ taking value 1 in a large ball centered at the origin. By setting $\lambda=t(r'+P'(\rho))\rho>c>0$, we may rewrite the integral (B-4) as

$$\int_{\mathbb{S}^{d-1}}e^{-i\epsilon\epsilon'\lambda(-\epsilon\omega)\theta}F(-\epsilon\omega,\theta,-\epsilon\omega,\mu;\rho,r',t)\,d\theta.$$

Note that the integrand in (B-4) ensures that $|\omega+\epsilon\theta|\lesssim\mu\ll 1$, which allows us to add $\tilde{\chi}$ factors in the definition of F.

By using the fact that $\rho\sim 1$, it is easy to verify that when $x, y, z\in\mathbb{S}^{d-1}$, for all $\alpha,\alpha',\beta\in\mathbb{N}^{d-1}$, $j, k, l, N\in\mathbb{N}$,

$$|\partial_x^\alpha\partial_y^{\alpha'}\partial_z^\beta\partial_\mu^j\partial_\rho^k\partial_{r'}^l F(x,y,z,\mu;\rho,r')|\leqslant C\mu^{-|\alpha|-|\alpha'|-|\beta|-j-l}\left\langle\frac{d(y,z)}{\mu}\right\rangle^{-N},$$

and that F is supported for $\rho\sim 1$, $|r'|\lesssim\mu$, $d(x,y)\leqslant C_0\mu\ll 1$, which ensures that

$$\lambda\mu^2=t^{2(\delta+\frac{1}{2}\sigma)}(r'+P'(\rho))\rho>c>0.$$

Therefore, the conclusion follows from Lemma B.1 with extra parameters ρ, r', t but without the ω'-variable. □

Appendix C: A refined result for the Schrödinger equation

In the case of the Schrödinger equation $P(\xi)=\frac{1}{2}|\xi|^2$, the structure of $P'(\xi)=\xi$ allows us to prove parts (i) and (ii) of Theorem 1.1 for nonsmooth χ, namely:

Theorem C.1. *Let $a_{\chi,\delta}$ and $E_{\chi,\delta}$ be as defined in (1-10) and (1-11), respectively. We assume, as in Theorem 1.1, that Λ is identically equal to 1. Then, for any $u_0\in L^2$, we have:*

(i) *If $\chi\in L^1$ and $\delta<0$,*

$$\lim_{t\to+\infty}E_{\chi,\delta}(u_0,t)=\lim_{t\to+\infty}E_{\chi,\delta}(u_0,-t)=0. \tag{C-1}$$

(ii) *If $\chi\in L^1$ and $\delta=0$,*

$$\lim_{t\to+\infty}E_{\chi,\delta}(u_0,t)=\lim_{t\to+\infty}E_{\chi,\delta}(u_0,-t)=\frac{1}{(2\pi)^d}\int G_\chi(\omega)|\hat{u}_0(\rho\omega)|^2\rho^{d-1}\,d\rho\,d\omega, \tag{C-2}$$

where (ρ,ω) is the polar coordinate, and the function $G_\chi(\omega)$ is defined as

$$G_\chi(\omega):=\frac{1}{(2\pi)^d}\left|\int_{x\cdot\omega>0}e^{i\frac{1}{2}|x|^2}\chi(x)\,dx\right|^2. \tag{C-3}$$

In order to prove the limits (C-1) and (C-2), we indicate that, compared with Theorem 1.1, the only difficulty is the loss of regularity in χ, which can be overcome by finding a bound of the operator $\mathrm{Op}(a(t))$ depending merely on $\|\chi\|_{L^1}$.

C1. ***An alternative bound of the truncated operator.*** The goal of this subsection is to find a uniform bound of $\mathrm{Op}(a(t))$, which involves less regularity of χ than Proposition 2.1. Via Lemma A.4, it suffices to study the symbol

$$b(x,\xi)=\chi\left(\frac{x+\xi}{\lambda}\right)\mathbb{1}_{|x|>|\xi|},$$

where $\lambda=|t|^\delta\in\,]0,\infty[$. To be precise, we shall prove the following proposition.

Proposition C.2. *There exists a constant $C > 0$ independent of λ such that, for all $\lambda > 0$,*

$$\|\mathrm{Op}(b)\|_{\mathcal{L}(L^2)} \leqslant C\lambda^d \|\chi\|_{L^1}.$$

Proof of Proposition C.2. It is easy to calculate that

$$\int |b(x,\xi)|\,dx \leqslant \int \left|\chi\left(\frac{x+\xi}{\lambda}\right)\right| dx = \lambda^d \|\chi\|_{L^1},$$
$$\int |b(x,\xi)|\,d\xi \leqslant \int \left|\chi\left(\frac{x+\xi}{\lambda}\right)\right| d\xi = \lambda^d \|\chi\|_{L^1}.$$

Thus, the desired estimate follows from Schur's lemma and Lemma A.7. □

C2. ***Calculation of the limit.*** By Proposition C.2, when $\delta < 0$, we have

$$\|\mathrm{Op}(a)\|_{\mathcal{L}(L^2)} \leqslant Ct^{\delta d}\|\chi\|_{L^1} \to 0 \quad \text{as } t \to \infty,$$

which implies (C-1).

In the critical case $\delta = 0$, when χ is smooth, compactly supported, and constant near zero, the limit (C-2) follows from the limit (1-14). If χ is no more than an L^1 function, we may approximate χ by some regular function in L^1. To be precise, for all $n \in \mathbb{N}$, there exist $\chi_n \in C_c^\infty(\mathbb{R}^d)$, which are constant near zero, such that

$$\|\chi - \chi_n\|_{L^1} < \frac{1}{n}.$$

To highlight the dependence on χ, in the rest of this section, we will add the subscript χ for concerning terms, for example,

$$a_\chi(t,x,\xi) = \chi\left(\frac{x+t\xi}{|t|^{\frac{1}{2}}}\right)\mathbb{1}_{|x|>|t||\xi|}.$$

Proposition C.2 implies that, for all $u_0 \in L^2$ and $t \neq 0$,

$$\begin{aligned}\|\mathrm{Op}(a_\chi)e^{itP(D_x)}u_0 - \mathrm{Op}(a_{\chi_n})e^{itP(D_x)}u_0\|_{L^2} &= \|\mathrm{Op}(a_{\chi-\chi_n})e^{itP(D_x)}u_0\|_{L^2}\\ &\leqslant C\|\chi-\chi_n\|_{L^1}\|e^{itP(D_x)}u_0\|_{L^2} < C\frac{1}{n}\|u_0\|_{L^2}.\end{aligned}$$

Therefore, for fixed $u_0 \in L^2$, the limit

$$\mathrm{Op}(a_\chi)e^{itP(D_x)}u_0 \to \mathrm{Op}(a_{\chi_n})e^{itP(D_x)}u_0 \text{ in } L^2 \quad \text{as } n \to \infty$$

is uniform in t. We may conclude (C-2) by passing to the limit $t \to \pm\infty$:

$$\begin{aligned}\lim_{t\to\pm\infty} E_{\chi,\delta}(u_0,t) &= \lim_{t\to\pm\infty}\|\mathrm{Op}(a_\chi)e^{itP(D_x)}u_0\|_{L^2}^2\\ &= \lim_{t\to\pm\infty}\lim_{n\to\infty}\|\mathrm{Op}(a_{\chi_n})e^{itP(D_x)}u_0\|_{L^2}^2 = \lim_{n\to\infty}\lim_{t\to\pm\infty}\|\mathrm{Op}(a_{\chi_n})e^{itP(D_x)}u_0\|_{L^2}^2\\ &= \lim_{n\to\infty}\frac{1}{(2\pi)^d}\int G_{\chi_n}(\omega)|\hat{u}_0(\rho\omega)|^2\rho^{d-1}\,d\rho\,d\omega = \frac{1}{(2\pi)^d}\int G_\chi(\omega)|\hat{u}_0(\rho\omega)|^2\rho^{d-1}\,d\rho\,d\omega.\end{aligned}$$

The last equality follows from the dominated convergence theorem and the continuity of G on $\chi \in L^1$, which is obvious due to the definition (C-3) of G_χ.

Appendix D: Partition of energy for the Klein–Gordon equation — a classical setting

In this part, we shall give a proof of (1-32) via an alternative study of asymptotic behavior of the solution to the half-Klein–Gordon equation (HKG), namely

$$\begin{cases} (\partial_t/i - P(D_x))u = 0, \\ u|_{t=0} = u_0, \end{cases}$$

where $P(\xi) = \langle \xi \rangle$ is a smooth symbol. As in Section 5, instead of studying the solution w to the Klein–Gordon equation (KG), we turn to

$$u = \left(\frac{\partial_t}{i} + P(D_x)\right) w,$$

which is a solution to the half-Klein–Gordon equation with initial data

$$u_0 = u|_{t=0} = \frac{w_1}{i} + P(D_x) w_0 \in L^2.$$

Using this notation, we may rewrite the integral on the left-hand side of (1-32) as

$$\begin{aligned}
\int_{r_0<|\frac{x}{t}|<r_1} & (|\partial_t w|^2 + |\nabla w|^2 + |w|^2)\, dx \\
&= \int \left(\left|\mathbb{1}_{r_0<|\frac{x}{t}|<r_1} \partial_t w\right|^2 + \left|\mathbb{1}_{r_0<|\frac{x}{t}|<r_1} \nabla w\right|^2 + \left|\mathbb{1}_{r_0<|\frac{x}{t}|<r_1} w\right|^2\right) dx \\
&= \int \left(\left|\mathbb{1}_{r_0<|\frac{x}{t}|<r_1} \operatorname{Im} u\right|^2 + \left|\mathbb{1}_{r_0<|\frac{x}{t}|<r_1} i \frac{D_x}{\langle D_x \rangle} \operatorname{Re} u\right|^2 + \left|\mathbb{1}_{r_0<|\frac{x}{t}|<r_1} \langle D_x \rangle^{-1} \operatorname{Re} u\right|^2\right) dx \\
&= \int \left(\left|\operatorname{Im} \mathbb{1}_{r_0<|\frac{x}{t}|<r_1} u\right|^2 + \left|\operatorname{Im} \mathbb{1}_{r_0<|\frac{x}{t}|<r_1} \frac{D_x}{\langle D_x \rangle} u\right|^2 + \left|\operatorname{Re} \mathbb{1}_{r_0<|\frac{x}{t}|<r_1} \langle D_x \rangle^{-1} u\right|^2\right) dx \\
&= \int \Bigg(\left|\operatorname{Im} \mathbb{1}_{r_0<|\frac{x}{t}|<r_1} e^{itP(D_x)} u_0\right|^2 + \left|\operatorname{Im} \mathbb{1}_{r_0<|\frac{x}{t}|<r_1} e^{itP(D_x)} \frac{D_x}{\langle D_x \rangle} u_0\right|^2 \\
&\qquad\qquad + \left|\operatorname{Re} \mathbb{1}_{r_0<|\frac{x}{t}|<r_1} e^{itP(D_x)} \langle D_x \rangle^{-1} u_0\right|^2\Bigg) dx.
\end{aligned}$$

Notice that the term on the right-hand side takes the form of

$$\int \left| \mathcal{A} g\left(\frac{x}{t}\right) v(t,x) \right|^2 dx, \tag{D-1}$$

where $\mathcal{A} \in \{\operatorname{Re}, \operatorname{Im}\}$, $g(y) = \mathbb{1}_{r_0<|y|<r_1}$, and v is a solution to the half-Klein–Gordon equation, which can be written as $v(t) = e^{itP(D_x)} v_0$, with

$$v_0 = u_0, \quad \frac{D_x}{\langle D_x \rangle} u_0, \quad \langle D_x \rangle^{-1} u_0,$$

which all belong to L^2, since $u_0 \in L^2$. In order to calculate the limit of such quantity, we need to study the asymptotic behavior of v, which will be given in the next part.

D1. *Asymptotic behavior.* In this part, we shall state our problem in a general setting. Let u be the unique solution to (E) with initial data $u_0 \in L^2$. We assume the symbol P is smooth except at zero, which covers all the fractional-type equations. Since P' is well-defined except at zero, we may introduce ν the push-forward of Lebesgue measure under P', i.e.,

$$\nu(E) := \mathrm{Leb}(P'^{-1}(E)) \quad \forall E \subset \mathbb{R}^d \text{ measurable.} \tag{D-2}$$

For all functions $g \in L^\infty$, we are interested in the asymptotic formula

$$g\left(\frac{x}{t}\right)u(t) = g(-P'(D_x))u(t) + o_{L^2}(1) \quad \text{as } t \to \pm\infty. \tag{D-3}$$

We denote by $\mathcal{G}$ the collection of those functions g satisfying this formula for all $u_0 \in L^2$, namely

$$\mathcal{G} := \{g \in L^\infty : \text{(D-3) holds for all } u_0 \in L^2\} := \{g \in L^\infty : \text{(D-3) holds for all } u_0 \text{ with } \hat{u}_0 \in C_c^\infty(\mathbb{R}^d)\}.$$

The two definitions given above are equivalent since the multiplication with $g(x/t)$ and Fourier multiplier $g(-P'(D_x))$ are both bounded on L^2 uniformly in time. The equivalence then follows from the fact that the subspace $\mathcal{F}^{-1}C_c^\infty(\mathbb{R}^d \backslash \{0\})$ is dense in L^2, where $\mathcal{F}$ is the Fourier transform.

In order to prove (1-32), we may use the following lemma:

Lemma D.1. *Let $E \subset \mathbb{R}^d$ be any measurable set whose boundary has null Lebesgue measure, namely*

$$\mathrm{Leb}(\partial E) = 0.$$

If the measure ν defined in (D-2) is absolutely continuous with respect to Lebesgue measure, we have

$$\mathbb{1}_E \in \mathcal{G}.$$

We assume this lemma is true and prove it later. It is easy to check that the assumptions in Lemma D.1 hold true for $P(\xi) = \langle \xi \rangle$ and $E = \{r_0 < |y| < r_1\}$. As a result, (D-1) can be written as

$$\begin{aligned}\int \left|\mathcal{A}\mathbb{1}_E\left(\frac{x}{t}\right)v(t,x)\right|^2 dx &= \int |\mathcal{A}\mathbb{1}_E(-P'(D_x))v(t,x)|^2\,dx + o(1) \\ &= \int |\mathcal{A}\mathbb{1}_{]\rho_0,\rho_1[}(|D_x|)v(t,x)|^2\,dx + o(1),\end{aligned}$$

when $t \to \pm\infty$. Recall that $]\rho_0, \rho_1[= P'^{-1}(]r_0, r_1[)$. Actually, this formula implies (1-32), since as $t \to \pm\infty$, we have

$$\begin{aligned}&\int_{r_0<|\frac{x}{t}|<r_1} (|\partial_t w|^2 + |\nabla w|^2 + |w|^2)\,dx \\ &\quad= \int \left(\left|\mathrm{Im}\,\mathbb{1}_{r_0<|\frac{x}{t}|<r_1} u(t,x)\right|^2 + \left|\mathrm{Im}\,\mathbb{1}_{r_0<|\frac{x}{t}|<r_1} \frac{D_x}{\langle D_x\rangle} u(t,x)\right|^2 + \left|\mathrm{Re}\,\mathbb{1}_{r_0<|\frac{x}{t}|<r_1}\langle D_x\rangle^{-1}u(t,x)\right|^2\right) dx \\ &\quad= \int \Bigg(\left|\mathrm{Im}\,\mathbb{1}_{]\rho_0,\rho_1[}(|D_x|)u(t,x)\right|^2 + \left|\mathrm{Im}\,\mathbb{1}_{]\rho_0,\rho_1[}(|D_x|)\frac{D_x}{\langle D_x\rangle}u(t,x)\right|^2 \\ &\qquad\qquad + \left|\mathrm{Re}\,\mathbb{1}_{]\rho_0,\rho_1[}(|D_x|)\langle D_x\rangle^{-1}u(t,x)\right|^2\Bigg) dx + o(1) \\ &\quad= \left\|\mathbb{1}_{]\rho_0,\rho_1[}(|D_x|)\partial_t w\right\|_{L^2}^2 + \left\|\mathbb{1}_{]\rho_0,\rho_1[}(|D_x|)\nabla w\right\|_{L^2}^2 + \left\|\mathbb{1}_{]\rho_0,\rho_1[}(|D_x|)w\right\|_{L^2}^2 + o(1) \\ &\quad= \left\|\mathbb{1}_{]\rho_0,\rho_1[}(|D_x|)\partial_t w\right\|_{L^2}^2 + \left\|\mathbb{1}_{]\rho_0,\rho_1[}(|D_x|)w\right\|_{H^1}^2 + o(1) \\ &\quad= \|\mathbb{1}_{]\rho_0,\rho_1[}(|D_x|)w_1\|_{H^1}^2 + \|\mathbb{1}_{]\rho_0,\rho_1[}(|D_x|)w_0\|_{L^2}^2 + o(1).\end{aligned}$$

In order to complete the proof of (1-32), we give now the proof of Lemma D.1.

Proof of Lemma D.1. Without loss of generality, we may assume in what follows that $\hat{u}_0 \in C_c^\infty(\mathbb{R}^d)$. Now, we fix $\chi \in C_c^\infty(\mathbb{R}^d)$ which equals 1 in a ball centered at zero and consider the symbol

$$a_0(t,x,\xi) = \chi\left(\frac{x + tP'(\xi)}{|t|^{\frac{1}{2}+\delta}}\right), \tag{D-4}$$

where $\delta \in \left]0, \frac{1}{2}\right[$. We have seen that Lemmas A.5 and A.10 imply the uniform-in-t boundedness of $\mathrm{Op}(a_0(t))$. Moreover, by integration by parts, we can prove that for all $u_0 \in L^2$

$$\|u(t) - \mathrm{Op}(a_0(t))u(t)\|_{L^2} \to 0 \quad \text{as } t \to \pm\infty.$$

By writing

$$E_t := E + B(0, ct^{-\frac{1}{2}+\delta}), \qquad \widetilde{E}_t := E^c + B(0, ct^{-\frac{1}{2}+\delta}),$$

where $c > 0$ is a constant determined by $\operatorname{Supp}\chi$, we may apply the uniform-in-time L^2-boundedness of multiplication with $\mathbb{1}_E(x/t)$ and Fourier multiplier $\mathbb{1}_E(-P'(D_x))$ to obtain that, as $t \to \pm\infty$,

$$\begin{aligned}
&\mathbb{1}_E\left(\frac{x}{t}\right)u(t,x) - \mathbb{1}_E(-P'(D_x))u(t,x)\\
&\quad= \mathbb{1}_E\left(\frac{x}{t}\right)\mathrm{Op}(a_0(t))u(t,x) - \mathrm{Op}(a_0(t))\mathbb{1}_E(-P'(D_x))u(t,x) + o_{L^2}(1)\\
&\quad= \mathbb{1}_E\left(\frac{x}{t}\right)\mathrm{Op}(a_0(t))\mathbb{1}_{E_t}(-P'(D_x))u(t,x) - \mathrm{Op}(a_0(t))\mathbb{1}_E(-P'(D_x))u(t,x) + o_{L^2}(1)\\
&\quad= \mathbb{1}_E\left(\frac{x}{t}\right)\mathrm{Op}(a_0(t))\mathbb{1}_{E_t\setminus E}(-P'(D_x))u(t,x) - \mathbb{1}_{E^c}\left(\frac{x}{t}\right)\mathrm{Op}(a_0(t))\mathbb{1}_E(-P'(D_x))u(t,x) + o_{L^2}(1)\\
&\quad= \mathbb{1}_E\left(\frac{x}{t}\right)\mathrm{Op}(a_0(t))\mathbb{1}_{E_t\setminus E}(-P'(D_x))u(t,x) - \mathbb{1}_{E^c}\left(\frac{x}{t}\right)\mathrm{Op}(a_0(t))\mathbb{1}_{E\cap\widetilde{E}_t}(-P'(D_x))u(t,x) + o_{L^2}(1).
\end{aligned}$$

Note that in the calculation above, it is possible to add extra cut-off in $-P'(D_x)$ since the symbol $a_0(t)$ is supported for

$$\left|\frac{x}{t} - (-P'(\xi))\right| \leqslant ct^{-\frac{1}{2}+\delta}.$$

As a consequence, the uniform boundedness of multiplication with $\mathbb{1}_E(x/t)$, $\mathbb{1}_{E^c}$ and $\mathrm{Op}(a_0(t))$ implies

$$\begin{aligned}
&\limsup_{t\to+\infty}\left\|\mathbb{1}_E\left(\frac{x}{t}\right)u(t) - \mathbb{1}_E(-P'(D_x))u(t)\right\|_{L^2}\\
&\qquad\lesssim \limsup_{t\to+\infty}\left(\left\|\mathbb{1}_{E_t\setminus E}(-P'(D_x))u(t)\right\|_{L^2} + \left\|\mathbb{1}_{E\cap\widetilde{E}_t}(-P'(D_x))u(t)\right\|_{L^2}\right)\\
&\qquad= (2\pi)^{-\frac{1}{2}d}\limsup_{t\to+\infty}\left(\left\|\mathbb{1}_{E_t\setminus E}(-P')\hat{u}_0\right\|_{L^2} + \left\|\mathbb{1}_{E\cap\widetilde{E}_t}(-P')\hat{u}_0\right\|_{L^2}\right) && \text{(Plancherel theorem)}\\
&\qquad= (2\pi)^{-\frac{1}{2}d}\left(\left\|\mathbb{1}_{\bigcap_{t>0}E_t\setminus E}(-P')\hat{u}_0\right\|_{L^2} + \left\|\mathbb{1}_{\bigcap_{t>0}E\cap\widetilde{E}_t}(-P')\hat{u}_0\right\|_{L^2}\right) && \text{(DCT)}\\
&\qquad\lesssim \|\mathbb{1}_{\partial E}(-P')\hat{u}_0\|_{L^2}.
\end{aligned}$$

The last quantity is actually zero since ∂E has zero Lebesgue measure and ν is absolutely continuous with respect to Lebesgue measure, which ensures that $\mathbb{1}_{\partial E}(-P'(\xi))$ vanishes almost everywhere. $\square$

D2. ***Further remarks on $\mathcal{G}$.*** In the proof of (1-32), we only study the function

$$g(y) = \mathbb{1}_{r_0<|y|<r_1}.$$

A natural question is whether (1-32) holds true for other functions g, or equivalently, which type of function is contained in the class $\mathcal{G}$. In this part, we shall indicate an error in a classical result and prove that $\mathcal{G}$ contains at least the continuous functions.

The asymptotic formula (D-3) was, to our knowledge, first studied in [Strichartz 1981], where the author claimed in Corollary 2.2 that, when ν is absolutely continuous with respect to Lebesgue measure,

$$L^\infty = \mathcal{G}. \tag{D-5}$$

However, the original proof given in [loc. cit.] is false. Actually, the author managed to prove in Theorem 2.1 that for general ν

$$\{\text{Fourier transform of finite measure}\} \subset \mathcal{G}$$

and reduce the conjecture to

$$\mathbb{1}_E \subset \mathcal{G} \quad \text{for bounded and measurable } E \subset \mathbb{R}^d. \tag{D-6}$$

The method used in [loc. cit.] is that, by regularity of Lebesgue measure, we may choose a series of compact sets $\{K_n\}_{n\in\mathbb{N}}$ and bounded open sets $\{U_n\}_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$

$$K_n \subset K_{n+1}, \quad U_{n+1} \subset U_n, \quad K_n \subset E \subset U_n, \quad \lim_{n\to+\infty} \mathrm{Leb}(U_n \backslash K_n) = 0.$$

Then it is easy to find smooth functions $g_n \in C_c^\infty(U_n)$ which are nonnegative, range in [0, 1], and equal to 1 on K_n. Clearly, $\mathbb{1}_E - g_n$ is supported in $U_n \backslash K_n$, whose measure tends to zero. By the dominated convergence theorem and the fact that $0 \leqslant \mathbb{1}_E - g_n \leqslant 1$, we have

$$\limsup_{t\to\pm\infty} \|(\mathbb{1}_E - g_n)(-P'(D_x))u\|_{L^2}^2 = (2\pi)^{-d} \|(\mathbb{1}_E - g_n)(-P'(\xi))\hat{u}_0(\xi)\|_{L^2_\xi}^2 \xrightarrow{n\to+\infty} 0.$$

Note that, to apply the dominated convergence theorem, we need $(\mathbb{1}_E - g_n)(-P'(\xi))$ converges to zero almost everywhere, which is a consequence of absolute continuity of ν with respect to Lebesgue measure. Therefore, in order to prove (D-6), it suffices to check that

$$\limsup_{t\to\pm\infty} \|(\mathbb{1}_E - g_n)(x/t)u\|_{L^2_x}^2 \xrightarrow{n\to+\infty} 0. \tag{D-7}$$

Once it holds true, one has, for all $n \in \mathbb{N}$,

$$\begin{aligned}
&\limsup_{t\to\pm\infty} \|\mathbb{1}_E(x/t)u - \mathbb{1}_E(-P'(D_x))u\|_{L^2_x} \\
&\quad \leqslant \limsup_{t\to\pm\infty} \|g_n(x/t)u - g_n(-P'(D_x))u\|_{L^2_x} + \limsup_{t\to\pm\infty} \|(\mathbb{1}_E - g_n)(x/t)u\|_{L^2_x} \\
&\qquad\qquad + \limsup_{t\to\pm\infty} \|(\mathbb{1}_E - g_n)(-P'(D_x))u\|_{L^2} \\
&\quad = \limsup_{t\to\pm\infty} \|(\mathbb{1}_E - g_n)(x/t)u\|_{L^2_x} + \limsup_{t\to\pm\infty} \|(\mathbb{1}_E - g_n)(-P'(D_x))u\|_{L^2},
\end{aligned}$$

since g_n is obviously the Fourier transform of some finite measure. The two quantities on the right-hand side vanish as n tends to infinity, and the desired conclusion $\mathbb{1}_E \in \mathcal{G}$ follows.

The proof of (D-7) given in [Strichartz 1981] is to find $h_n \in \mathcal{G}$ such that

$$C \geqslant h_n \geqslant \mathbb{1}_{O_n} \quad \text{and} \quad h_n \to 0 \text{ a.e.},$$

where $O_n = U_n \backslash K_n$ is an open set. If such h_n exists, one obtains immediately that

$$\begin{aligned}\lim_{n\to+\infty} \limsup_{t\to\pm\infty} \|(\mathbb{1}_E - g_n)(x/t)u(t)\|_{L^2}^2 &\leqslant \lim_{n\to+\infty} \limsup_{t\to\pm\infty} \|h_n(x/t)u(t)\|_{L^2}^2 \\ &\leqslant \lim_{n\to+\infty} \limsup_{t\to\pm\infty} \|h_n(-P'(D_x))u(t)\|_{L^2}^2 \\ &= \lim_{n\to+\infty} (2\pi)^{-d} \int h_n^2(-\nabla P(\xi))|\hat{u}_0(\xi)|^2 \, d\xi = 0.\end{aligned}$$

Note that the second inequality is a consequence of $h_n \in \mathcal{G}$ and the last equality follows from the dominated convergence theorem.

Since we only know that $\mathcal{G}$ contains a subset of continuous functions (Fourier transform of finite measures), it is essential to assume the h_n's to be continuous. However, for general decreasing bounded open sets O_n, even if their measures decrease to zero, such continuous h_n's do *not* exist. Otherwise, it is harmless to assume h_n's are supported in the same large ball. Then, for one thing by the dominated convergence theorem,

$$\lim_{n\to+\infty} \int h_n(y)\, dy = 0,$$

and for another thing, we have

$$\int h_n(y)\, dy \geqslant \int_{\overline{O}_n} h_n(y)\, dy = \operatorname{Leb}(\overline{O}_n) \geqslant 0,$$

where $\overline{O}_n$ is the closure of open set O_n. Here the first inequality is due to the continuity of the h_n's. As a result,

$$\lim_{n\to+\infty} \operatorname{Leb}(\overline{O}_n) = 0.$$

The contradiction arises from the fact that $\operatorname{Leb}(O_n)$ tends to zero does not imply that $\operatorname{Leb}(\overline{O}_n)$ tends to zero. For example, let $\{r_j\}_{j\in\mathbb{N}}$ be a sequence of rational numbers in the unit ball $B = B(0, 1)$ of $\mathbb{R}^d$ centered at zero. Consider the series of open sets

$$O_n := \bigcup_{j\in\mathbb{N}} B(r_j, 2^{-j-n}).$$

Clearly the O_n's are open as the union of open sets and

$$\operatorname{Leb}(O_n) \leqslant \sum_{j\in\mathbb{N}} \operatorname{Leb}(B(r_j, 2^{-j-n})) \lesssim \sum_{j\in\mathbb{N}} 2^{-(j+n)d} \sim 2^{-nd} \to 0 \quad \text{as } n \to +\infty.$$

Since $\{r_j\}_{j\in\mathbb{N}}$ is dense in $B = B(0, 1)$, the closure of each O_n contains at least the unit ball B. As a consequence,

$$\lim_{n\to+\infty} \operatorname{Leb}(\overline{O}_n) \geqslant \lim_{n\to+\infty} \operatorname{Leb}(B) = C_d > 0.$$

We emphasize that the argument above does not falsify (D-5) and it is still unknown whether this conjecture is true. Here we shall prove rigorously that all bounded continuous functions belong to $\mathcal{G}$.

Proposition D.2. *If the measure ν defined in (D-2) is absolutely continuous with respect to Lebesgue measure, we have*

$$C_b^0(\mathbb{R}^d) \subset \mathcal{G}.$$

Proof. As in the proof of Lemma D.1, we assume that $\hat{u}_0 \in C_c^\infty(\mathbb{R}^d)$. To begin with, we check that the Schwartz class $\mathcal{S} \subset \mathcal{G}$. For any $g \in \mathcal{S}$,

$$\begin{aligned}
&\left(g\left(\frac{x}{t}\right) - g(-P'(D_x))\right)u(t,x) \\
&\quad = \frac{1}{(2\pi)^d}\int e^{i(x\cdot\xi + tP(\xi))}\left(g\left(\frac{x}{t}\right) - g(-P'(\xi))\right)\hat{u}_0(\xi)\,d\xi \\
&\quad = \frac{1}{(2\pi)^d}\int e^{i(x\cdot\xi + tP(\xi))}\left(\frac{x}{t} + P'(\xi)\right)\int_0^1 g'\left(\frac{\tau}{t}x - (1-\tau)P'(\xi)\right)d\tau\,\hat{u}_0(\xi)\,d\xi \\
&\quad = \frac{i}{(2\pi)^d t}\int e^{i(x\cdot\xi + tP(\xi))}\partial_\xi\left[\int_0^1 g'\left(\frac{\tau}{t}x - (1-\tau)P'(\xi)\right)d\tau\,\hat{u}_0(\xi)\right]d\xi \\
&\quad = \frac{i}{(2\pi)^d t}\int e^{i(x\cdot\xi + tP(\xi))}\int_0^1 g''\left(\frac{\tau}{t}x - (1-\tau)P'(\xi)\right)(\tau - 1)\,d\tau\,P''(\xi)\hat{u}_0(\xi)\,d\xi \\
&\qquad\qquad + \frac{i}{(2\pi)^d t}\int e^{i(x\cdot\xi + tP(\xi))}\int_0^1 g'\left(\frac{\tau}{t}x - (1-\tau)P'(\xi)\right)d\tau\,\partial_\xi\hat{u}_0(\xi)\,d\xi.
\end{aligned}$$

Notice that, by Lemma A.5 and A.10, the operator of symbol

$$g''\left(\frac{\tau}{t}x - (1-\tau)P'(\xi)\right), \quad g'\left(\frac{\tau}{t}x - (1-\tau)P'(\xi)\right)$$

is bounded uniformly in t and τ and that functions

$$P''(\xi)\hat{u}_0(\xi), \quad \partial_\xi\hat{u}_0(\xi)$$

belong to L^2, since we have assumed $\hat{u}_0 \in C_c^\infty(\mathbb{R}^d)$. As a consequence,

$$\left\|\left(g\left(\frac{x}{t}\right) - g(-P'(D_x))\right)u(t,x)\right\|_{L_x^2} \lesssim |t|^{-1} \xrightarrow{t\to\pm\infty} 0.$$

By noticing that $\mathcal{G}$ is closed under L^∞-norm, we have

$$C_0^0 := \left\{g \in C^0 : \lim_{|y|\to+\infty} g(y) = 0\right\} = \bar{\mathcal{S}} \subset \mathcal{G},$$

where $\bar{\mathcal{S}}$ is the closure of $\mathcal{S}$ with respect to L^∞-norm.

It remains to pass to general continuous function g. In fact, we only need to consider those $g \geqslant 0$, since once may always write g as the difference of two nonnegative continuous functions, which are both bounded. Let us fix $\chi \in C_c^\infty$ which equals 1 near zero and define, for all $R > 0$,

$$\chi_R(y) := \chi\left(\frac{y}{R}\right).$$

For arbitrary $R > 0$, we have

$$\begin{aligned}\limsup_{t\to\pm\infty} &\|(g(x/t) - g(-P'(D_x)))u(t)\|_{L^2} \\ &\leqslant \limsup_{t\to\pm\infty} \|((g\chi_R)(x/t) - (g\chi_R)(-P'(D_x)))u(t)\|_{L^2} \\ &\qquad + \limsup_{t\to\pm\infty} \|(g(1-\chi_R))(x/t)u(t)\|_{L^2} + \limsup_{t\to\pm\infty} \|(g(1-\chi_R))(-P'(D_x))u(t)\|_{L^2} \\ &= \limsup_{t\to\pm\infty} \|(g(1-\chi_R))(x/t)u(t)\|_{L^2} + \limsup_{t\to\pm\infty} \|(g(1-\chi_R))(-P'(D_x))u(t)\|_{L^2},\end{aligned}$$

since $g\chi_R \in C_0^0$. The second term on the right-hand side can be calculated as

$$\limsup_{t\to\pm\infty} \|(g(1-\chi_R))(-P'(D_x))u(t)\|_{L^2} = (2\pi)^{-\frac{1}{2}d} \|(g(1-\chi_R))(-P')\hat{u}_0\|_{L^2},$$

which, due to the dominated convergence theorem and the absolute continuity of ν with respect to Lebesgue measure, converges to zero as $R \to +\infty$. As for the cut-off in x, we observe that

$$\limsup_{t\to\pm\infty} \|(g(1-\chi_R))(x/t)u(t)\|_{L^2} \leqslant \|g\|_{L^\infty} \limsup_{t\to\pm\infty} \|(1-\chi_R)(x/t)u(t)\|_{L^2}.$$

Note that since $\chi_R \in \mathcal{S} \subset \mathcal{G}$, $(1-\chi_R)(x/t)u(t)$ can be written, when $t \to \pm\infty$, as

$$u(t) - \chi_R(-P'(D_x))u(t) + o_{L^2}(1).$$

As a result,

$$\begin{aligned}\limsup_{t\to\pm\infty} \|(1-\chi_R)(x/t)u(t)\|_{L^2} &= \limsup_{t\to\pm\infty} \|(1-\chi_R)(-P'(D_x))u(t)\|_{L^2} \\ &= (2\pi)^{-\frac{1}{2}d} \|(1-\chi_R)\hat{u}_0\|_{L^2}.\end{aligned}$$

We have seen that the last quantity tends to zero as $R \to +\infty$. In conclusion, we have proved that, for all $u_0 \in \mathcal{F}^{-1}C_c^\infty(\mathbb{R}^d\backslash\{0\})$,

$$\lim_{R\to+\infty} \limsup_{t\to\pm\infty} \left\| \big(g(x/t) - g(-P'(D_x))\big)u(t) \right\|_{L^2} = 0,$$

and thus $g \in \mathcal{G}$. □

D3. ***Proof of (D-5) for the dispersive system.*** Before ending this section, we give a proof of (D-5) for the dispersive system. To be precise, we assume that there exists some dense subspace $\mathcal{D}_0 \subset L^2$ such that

$$\|e^{itP(D_x)}u_0\|_{L^\infty} \leqslant C(u_0)|t|^{-\frac{1}{2}d} \quad \forall u_0 \in \mathcal{D}_0 \quad \forall |t| > 1, \tag{D-8}$$

where $C(u_0)$ is a constant depending on u_0.

Theorem D.3. *Assume that P is smooth except at zero and ν defined in (D-2) is absolutely continuous with respect to Lebesgue measure. Then the dispersion estimate (D-8), associated with some dense subspace $\mathcal{D}_0 \subset L^2$, implies (D-5).*

Proof. Recall that, as mentioned in previous section, it has been proved in [Strichartz 1981] that, when ν is absolutely continuous with respect to Lebesgue measure, (D-5) is equivalent to (D-6). Therefore, it suffices to check that, for all $u_0 \in \mathcal{D}_0$,

$$\left\| \mathbb{1}_E\left(\frac{x}{t}\right)u(t) - \mathbb{1}_E(D_x)u(t)\right\|_{L^2} \to 0 \quad \text{as } t \to \pm\infty,$$

where E is any bounded measurable set.

In Lemma D.1, we have proved this result for those E whose boundary has zero Lebesgue measure. The idea of treatment of general E is to approximate it by a finite union of cubes and control the remaining part via a dispersion estimate. To begin with, we fix an arbitrarily small $\epsilon > 0$. By outer regularity of Lebesgue measure, there exists an open set $\widetilde{E}_\epsilon \supset E$ such that

$$\operatorname{Leb}(\widetilde{E}_\epsilon \backslash E) < \tfrac{1}{2}\epsilon.$$

Since any open set can be expressed as the union of almost disjoint closed cubes, we may find finitely many closed cubes $\{K_j\}_{j=1}^N$ such that $K_j \subset \widetilde{E}_\epsilon$ and

$$\operatorname{Leb}(\widetilde{E}_\epsilon \backslash E_\epsilon) < \tfrac{1}{2}\epsilon, \quad \text{where } E_\epsilon = \textstyle\bigcup_{j=1}^N K_j.$$

One may observe that

$$\operatorname{Leb}(E\backslash E_\epsilon \sqcup E_\epsilon \backslash E) < \epsilon.$$

Now, for any $u_0 \in \mathcal{D}_0$, we have

$$\begin{aligned}
&\left\| \mathbb{1}_E\left(\frac{x}{t}\right)u(t) - \mathbb{1}_E(D_x)u(t)\right\|_{L^2} \\
&\leqslant \left\|(\mathbb{1}_{E_\epsilon} - \mathbb{1}_E)\left(\frac{x}{t}\right)u(t)\right\|_{L^2} + \left\|\mathbb{1}_{E_\epsilon}\left(\frac{x}{t}\right)u(t) - \mathbb{1}_{E_\epsilon}(D_x)u(t)\right\|_{L^2} + \left\|(\mathbb{1}_{E_\epsilon} - \mathbb{1}_E)(D_x)u(t)\right\|_{L^2} \\
&\leqslant C(u_0)\left\|(\mathbb{1}_{E_\epsilon} - \mathbb{1}_E)\left(\frac{x}{t}\right)\right\|_{L^2}|t|^{-\frac{1}{2}d} + \left\|\mathbb{1}_{E_\epsilon}\left(\frac{x}{t}\right)u(t) - \mathbb{1}_{E_\epsilon}(D_x)u(t)\right\|_{L^2} + (2\pi)^{-\frac{1}{2}d}\left\|(\mathbb{1}_{E_\epsilon} - \mathbb{1}_E)\hat{u}_0\right\|_{L^2} \\
&\leqslant C(u_0)\operatorname{Leb}(E\backslash E_\epsilon \sqcup E_\epsilon\backslash E)^{\frac{1}{2}} + \left\|\mathbb{1}_{E_\epsilon}\left(\frac{x}{t}\right)u(t) - \mathbb{1}_{E_\epsilon}(D_x)u(t)\right\|_{L^2} + (2\pi)^{-\frac{1}{2}d}\left\|\mathbb{1}_{E\backslash E_\epsilon \sqcup E_\epsilon\backslash E}(\xi)\hat{u}_0(\xi)\right\|_{L^2_\xi} \\
&< C(u_0)\epsilon^{\frac{1}{2}} + \left\|\mathbb{1}_{E_\epsilon}\left(\frac{x}{t}\right)u(t) - \mathbb{1}_{E_\epsilon}(D_x)u(t)\right\|_{L^2} + (2\pi)^{-\frac{1}{2}d}\left\|\mathbb{1}_{E\backslash E_\epsilon \sqcup E_\epsilon\backslash E}(\xi)\hat{u}_0(\xi)\right\|_{L^2_\xi}.
\end{aligned}$$

Due to the fact that E_ϵ is the union of finitely many closed cubes, the boundary of E_ϵ has zero Lebesgue measure. As a result, the second term on the right-hand side tends to zero as $t \to \pm\infty$, i.e.,

$$\limsup_{t\to\pm\infty}\left\| \mathbb{1}_E\left(\frac{x}{t}\right)u(t) - \mathbb{1}_E(D_x)u(t)\right\|_{L^2} \lesssim \epsilon^{\frac{1}{2}} + \left\|\mathbb{1}_{E\backslash E_\epsilon \sqcup E_\epsilon\backslash E}(\xi)\hat{u}_0(\xi)\right\|_{L^2_\xi}.$$

Since $\hat{u}_0 \in L^2$ and $\operatorname{Leb}(E\backslash E_\epsilon \sqcup E_\epsilon\backslash E) < \epsilon$, the right-hand side becomes arbitrarily small, if $\epsilon > 0$ is taken small enough. The desired result thus follows. □

The dispersion estimate (D-8) holds for all symbols P we deal with in the present paper, as a consequence of stationary phase lemma.

Proposition D.4. *Let P be radial and smooth except at zero. If $P'' > 0$ or $P'' < 0$, the dispersion estimate (D-8) holds with $\mathcal{D}_0 = \mathcal{F}^{-1}C_c^\infty(\mathbb{R}^d \backslash \{0\})$.*

Proof. Without loss of generality, we assume that $t > 1$. Let u_0 be any function in $\mathcal{D}_0 = \mathcal{F}^{-1}C_c^\infty(\mathbb{R}^d \backslash \{0\})$. To prove the inequality (D-8), it is equivalent by definition to prove that

$$\left\| \int e^{i(x\cdot\xi + tP(\xi))} \hat{u}_0(\xi)\, d\xi \right\|_{L^\infty(dx)} \lesssim t^{-\frac{1}{2}d}.$$

Since P is radial, we may write the inequality above in the polar system $\xi = \rho\theta$ as

$$\left\| \int e^{it\left(\frac{x\cdot\theta}{t}\rho + P(\rho)\right)} \hat{u}_0(\rho\theta)\rho^{d-1}\, d\rho\, d\theta \right\|_{L^\infty(dx)} \lesssim t^{-\frac{1}{2}d}.$$

By letting $s = x\cdot\theta/t \in \mathbb{R}$, it suffices to prove that

$$\sup_{s\in\mathbb{R},\theta\in\mathbb{S}^{d-1}} \left| \int e^{it(s\rho + P(\rho))} \hat{u}_0(\rho\theta)\rho^{d-1}\, d\rho \right| \lesssim t^{-\frac{1}{2}d}.$$

Since ρ stays between two positive constants and $P'' \neq 0$ on $]0, +\infty[$, this inequality follows from stationary phase lemma. □

Corollary D.5. *Equation (D-5) holds for all P satisfying condition (H_{p_0,p_1}) with $p_0, p_1 \neq 0$.*

Appendix E: Proof of Proposition 4.3

In [Delort 2022], the author proved Proposition 4.3 for strictly convex P. In this part, we will explain how the same argument works for strictly concave P and how to calculate the limit for $\epsilon = \pm$ respectively. Since most of the calculations were done in Section 3 of [loc. cit.], we will omit these details.

By definition $I(t, -\epsilon, \epsilon, -\epsilon, \epsilon; F)$ equals

$$\int e^{i\epsilon[r(-\rho+\rho')-t(-P(\rho)+P(\rho'))]} \mathbb{1}_{\frac{r}{t} > P'(\rho), P'(\rho')} F\big(\rho, \rho', r, t; r - tP'(\rho), r - tP'(\rho')\big)\, dr\, d\rho\, d\rho',$$

which can be split into I_+ and I_-, with domain of integral $\rho - \rho' > 0$ and $\rho - \rho' < 0$, respectively. Namely,

$$I_+ = \int e^{i\epsilon[r(-\rho+\rho')-t(-P(\rho)+P(\rho'))]} \mathbb{1}_{r > tP'(\rho')} \mathbb{1}_{\rho-\rho'>0} F\big(\rho, \rho', r, t; r - tP'(\rho), r - tP'(\rho')\big)\, dr\, d\rho\, d\rho',$$

$$I_- = \int e^{i\epsilon[r(-\rho+\rho')-t(-P(\rho)+P(\rho'))]} \mathbb{1}_{r > tP'(\rho')} \mathbb{1}_{\rho-\rho'<0} F\big(\rho, \rho', r, t; r - tP'(\rho), r - tP'(\rho')\big)\, dr\, d\rho\, d\rho'.$$

In what follows, we study mainly the integral I_+, with I_- manipulated in the same way. By the change of variables $r \to tr + tP'(\rho')$, $\rho' \to \rho - w$, I_+ reads

$$t \int e^{i\epsilon t[-(r+P'(\rho-w))w-(-P(\rho)+P(\rho-w))]} \mathbb{1}_{r>0} \mathbb{1}_{w>0} \\ \times F(\rho, \rho - w, t(r + P'(\rho - w)), t; tr - tP'(\rho) + tP'(\rho - w), tr)\, dr\, d\rho\, dw.$$

We introduce the notation

$$P(\rho') - P(\rho) = P'(\rho)(\rho' - \rho) + g(\rho, \rho')(\rho' - \rho)^2,$$

where g is strictly negative since P is concave;

$$\widetilde{F}(\rho,\rho',r,t;\zeta,\zeta')=F(\rho,\rho',tr,t;t\zeta,t\zeta'),$$

which is smooth, supported for

$$\rho,\rho',r\sim 1,\quad |\zeta|,|\zeta'|\lesssim t^{\delta'-1},$$

and satisfies, for all $j,j',k,\gamma,\gamma'\in\mathbb{N}$,

$$\left|\partial_\rho^j\partial_{\rho'}^{j'}\partial_r^k\partial_\zeta^\gamma\partial_{\zeta'}^{\gamma'}\widetilde{F}(\rho,\rho',r,t;\zeta,\zeta')\right|\lesssim t^{(1-\delta')(k+\gamma+\gamma')}.$$

Moreover, we have the pointwise limit

$$\lim_{t\to\infty}\widetilde{F}\left(\rho,\rho',\frac{r}{\sqrt{t}}+P'(\rho'),t;\frac{\zeta}{\sqrt{t}},\frac{\zeta'}{\sqrt{t}}\right)=F_0(\rho,\rho').$$

With these notations, I_+ can be expressed as

$$t\int e^{-i\epsilon t[rw-g(\rho-w,\rho)w^2]}\mathbb{1}_{r>0}\mathbb{1}_{w>0}\widetilde{F}\big(\rho,\rho-w,r+P'(\rho-w),t;r-P'(\rho)+P'(\rho-w),r\big)\,dr\,d\rho\,dw.$$

The same calculus as in Lemma 3.1.4 of [Delort 2022] shows that, up to some terms tending to zero, I_+ equals

$$\begin{aligned}t\int e^{-i\epsilon trw}\mathbb{1}_{r>0}e^{i\epsilon tg(\rho-w,\rho)w^2}\mathbb{1}_{w>0}\widetilde{F}\big(\rho,\rho-w,r+P'(\rho),t;r,r\big)\,dr\,dw\,d\rho\\ =\int e^{-i\epsilon rw}\mathbb{1}_{r>0}e^{i\epsilon g(\rho-w/\sqrt{t},\rho)w^2}\mathbb{1}_{w>0}\widetilde{F}\left(\rho,\rho-\frac{w}{\sqrt{t}},\frac{r}{\sqrt{t}}+P'(\rho),t;\frac{r}{\sqrt{t}},\frac{r}{\sqrt{t}}\right)dr\,dw\,d\rho,\end{aligned}$$

whose formal limit by taking the pointwise limit of the integrand is

$$\int e^{-i\epsilon rw}\mathbb{1}_{r>0}e^{i\epsilon g(\rho,\rho)w^2}\mathbb{1}_{w>0}F_0(\rho,\rho)\,dr\,dw\,d\rho.$$

The error between this integral and I_+ is actually $o(1)$, due the calculation of the proof of Proposition 3.1.3 of [loc. cit.]. In conclusion, we have

$$\begin{aligned}I_+&=\int e^{-i\epsilon rw}\mathbb{1}_{r>0}e^{i\epsilon g(\rho,\rho)w^2}\mathbb{1}_{w>0}F_0(\rho,\rho)\,dr\,dw\,d\rho+o(1),\\ I_-&=\int e^{-i\epsilon rw}\mathbb{1}_{r>0}e^{-i\epsilon g(\rho,\rho)w^2}\mathbb{1}_{w<0}F_0(\rho,\rho)\,dr\,dw\,d\rho+o(1).\end{aligned}$$

As a consequence, the limit of $I(t,-\epsilon,\epsilon,-\epsilon,\epsilon;F)$ is

$$\begin{aligned}\int e^{-i\epsilon rw}\mathbb{1}_{r>0}\,dr\big(e^{i\epsilon g(\rho,\rho)w^2}\mathbb{1}_{w>0}+e^{-i\epsilon g(\rho,\rho)w^2}\mathbb{1}_{w<0}\big)\,dw\,F_0(\rho,\rho)\,d\rho\\ =\int -i\epsilon(w-i\epsilon 0)^{-1}\big(\cos(\epsilon g(\rho,\rho)w^2)+\operatorname{sgn}(w)i\sin(\epsilon g(\rho,\rho)w^2)\big)\,dw\;F_0(\rho,\rho)\,d\rho\\ =\int -i\epsilon(w-i\epsilon 0)^{-1}\cos(g(\rho,\rho)w^2)\,dw\;F_0(\rho,\rho)\,d\rho+\int\frac{\sin(g(\rho,\rho)w^2)}{|w|}\,dw\;F_0(\rho,\rho)\,d\rho,\end{aligned}$$

where the integrals in r and w should be understood in the sense of oscillatory integral and distribution, respectively.

Due to the assumption that P is strictly concave, or equivalently $P'' < 0$, g is negative. Since $(w - i\epsilon 0)^{-1}$ is homogeneous of degree -1, the right-hand side of the equality above can be rewritten as

$$\int -i\epsilon(w - i\epsilon 0)^{-1}\cos(w^2)\,dw\,F_0(\rho,\rho)\,d\rho - \int \frac{\sin(w^2)}{|w|}\,dw\,F_0(\rho,\rho)\,d\rho.$$

For one thing, the distribution $(w \pm i0)^{-1}$ can be expressed as

$$(w \pm i0)^{-1} = \mp\pi i\delta_0 + \text{P.V.}\,\frac{1}{w}.$$

And for another thing, due to the Dirichlet integral $\int_0^\infty (\sin y/y)\,dy = \frac{\pi}{2}$, we have

$$\int \frac{\sin(w^2)}{|w|}\,dw = \frac{\pi}{2}.$$

These identities imply

$$\begin{aligned}\lim_{t\to+\infty} I(t,-\epsilon,\epsilon,-\epsilon,\epsilon;F) &= \int\Big(\pi\delta_0 - i\epsilon\,\text{P.V.}\,\frac{1}{w}\Big)\cos(w^2)\,dw\,F_0(\rho,\rho)\,d\rho + \text{sgn}(g)\frac{\pi}{2}\int F_0(\rho,\rho)\,d\rho\\ &= \pi\int F_0(\rho,\rho)\,d\rho - \frac{\pi}{2}\int F_0(\rho,\rho)\,d\rho\\ &= \frac{\pi}{2}\int F_0(\rho,\rho)\,d\rho.\end{aligned}$$

In the case of strictly convex P, it has been proved in [Delort 2022] that

$$\begin{aligned}&\lim_{t\to+\infty} I(t,-\epsilon,\epsilon,-\epsilon,\epsilon;F)\\ &\qquad= \int i\epsilon(w + i\epsilon 0)^{-1}\cos(g(\rho,\rho)w^2)\,dw\,F_0(\rho,\rho)\,d\rho - \int \frac{\sin(g(\rho,\rho)w^2)}{|w|}\,dw\,F_0(\rho,\rho)\,d\rho,\end{aligned}$$

where g is positive. We may repeat the argument above and conclude that

$$\begin{aligned}\lim_{t\to+\infty} I(t,-\epsilon,\epsilon,-\epsilon,\epsilon;F) &= \int\Big(\pi\delta_0 + i\epsilon\,\text{P.V.}\,\frac{1}{w}\Big)\cos(w^2)\,dw\,F_0(\rho,\rho)\,d\rho - \frac{\pi}{2}\int F_0(\rho,\rho)\,d\rho\\ &= \pi\int F_0(\rho,\rho)d\rho - \frac{\pi}{2}\int F_0(\rho,\rho)\,d\rho\\ &= \frac{\pi}{2}\int F_0(\rho,\rho)\,d\rho.\end{aligned}$$

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ON THE KINK-KINK COLLISION PROBLEM FOR THE ϕ^6 MODEL WITH LOW SPEED

ABDON MOUTINHO

We study the elasticity of the collision of two kinks with an incoming low speed $v \in (0, 1)$ for the nonlinear wave equation in dimension 1+1 known as the ϕ^6 model. We prove for any $k \in \mathbb{N}$ that if the incoming speed v is small enough, then, after the collision, the two solitons move away with a velocity v_f such that $|v_f - v| \leq v^k$ and the energy of the remainder will also be smaller than v^k. This manuscript is the continuation of our previous paper where we constructed a sequence ϕ_k of approximate solutions for the ϕ^6 model. The proof of our main result relies on the use of the set of approximate solutions from our previous work, modulation analysis, and a refined energy estimate method to evaluate the precision of our approximate solutions during a large time interval.

1. Introduction

1.1. *Background.* Considering the potential function $U(\phi) = \phi^2(1-\phi^2)^2$, the partial differential equation known as the ϕ^6 model in domain $1+1$ is defined by

$$\partial_t^2\phi(t,x) - \partial_x^2\phi(t,x) + U'(\phi(t,x)) = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}. \tag{1}$$

The solutions $\phi(t,x)$ of (1) preserve the energy given by

$$E(\phi)(t) = \int_{\mathbb{R}} \tfrac{1}{2}([\partial_t\phi(t,x)]^2 + [\partial_x\phi(t,x)]^2) + U(\phi(t,x))\,dx, \tag{energy}$$

and the momentum

$$P(\phi) = -\int_{\mathbb{R}} \partial_t\phi(t,x)\partial_x\phi(t,x)\,dx. \tag{momentum}$$

The kinetic energy and potential energy are given, respectively, by

$$E_{\text{kin}}(\phi)(t) = \int_{\mathbb{R}} \tfrac{1}{2}[\partial_t\phi(t,x)]^2\,dx, \quad E_{\text{pot}}(\phi)(t) = \int_{\mathbb{R}} \tfrac{1}{2}[\partial_x\phi(t,x)]^2 + U(\phi(t,x))\,dx.$$

The vacuum set $\mathcal{V}$ of the potential function U is the set $U^{-1}\{0\} = \{-1, 0, 1\}$. The unique constant solutions with finite energy of (1) are the functions of the form $\phi \equiv \eta$ for any $\eta \in \mathcal{V}$.

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Furthermore, it is well known that if a solution $\phi(t,x)$ of the partial differential equation (1) is in the energy space, which is the set of strong solutions with finite energy, then the solution is global-in-time (see the introduction of [Moutinho 2023] for a proof) and there exist numbers $\eta_1, \eta_2 \in \mathcal{V}$ such that

$$\lim_{x\to-\infty}\phi(t,x)=\eta_1, \qquad \lim_{x\to+\infty}\phi(t,x)=\eta_2$$

for all $t\in\mathbb{R}$. The set of solutions of (1) with finite energy is invariant under space translation, time translation, space reflection, time reflection, and Lorentz transformations.

The unique nonconstant stationary solutions of (1) with finite energy are the kinks which are the space translation of either $H_{0,1}(x)$ or $H_{-1,0}(x)$ that are denoted by

$$H_{0,1}(x)=\frac{e^{\sqrt{2}x}}{\sqrt{1+e^{2\sqrt{2}x}}}, \quad H_{-1,0}(x)=-H_{0,1}(-x)=\frac{-e^{-\sqrt{2}x}}{\sqrt{1+e^{-2\sqrt{2}x}}},$$

and the antikinks which are the space translation of the functions

$$H_{1,0}(x)=H_{0,1}(-x)=\frac{e^{-\sqrt{2}x}}{\sqrt{1+e^{-2\sqrt{2}x}}}, \quad H_{0,-1}(x)=-H_{0,1}(x)=\frac{-e^{\sqrt{2}x}}{\sqrt{1+e^{2\sqrt{2}x}}}.$$

Using the identity

$$H_{0,1}'(x)=\sqrt{2}\frac{e^{\sqrt{2}x}}{(1+e^{2\sqrt{2}x})^{3/2}},$$

it is not difficult to verify that

$$\left\|\frac{d}{dx}H_{0,1}(x)\right\|^2_{L^2_x(\mathbb{R})}=\frac{1}{2\sqrt{2}}. \tag{2}$$

The kink $H_{0,1}$ satisfies the Bogomolny identity, which is $H_{0,1}'(x)=\sqrt{2U(H_{0,1}(x))}$, and the estimate

$$\left|\frac{d^k}{dx^k}H_{0,1}(x)\right| \lesssim_k \min(e^{\sqrt{2}x}, e^{-2\sqrt{2}x}) \tag{3}$$

for any $k\geq 1$, and clearly

$$|H_{0,1}(x)|\leq e^{\sqrt{2}\min(x,0)}. \tag{4}$$

For the ϕ^6 model there are stability results for the kinks. In [Moutinho 2023], the orbital stability of two kinks with energy close to the minimal was obtained, and also the dynamics of two interacting kinks, which is a kink-kink solution with low kinetic energy and potential energy slightly bigger than the minimum possible for two kinks, was described in function of the initial data and the energy of the solution. In [Kowalczyk et al. 2021], the asymptotic stability of a kink for the ϕ^6 model was obtained, and moreover, asymptotic stability of a single kink was also obtained for a certain class of nonlinear wave equations of dimension $1+1$. There are also asymptotic stability results for a single kink in other models; for example, see [Kowalczyk et al. 2017; Delort and Masmoudi 2022] for the ϕ^4 model.

This manuscript is the sequel of [Moutinho 2024]. In this paper, we study the traveling kink-kink solutions of (1) with speed $0<v<1$ small enough. More precisely, we consider the following definition.

Definition 1. The traveling kink-kink with speed $v \in (0,1)$ are the set of solutions $\phi(t,x)$ that satisfies, for some positive constants K, c and any $t \geq K$, the decay estimate

$$\left\| (\phi(t,x), \partial_t \phi(t,x)) - \overrightarrow{H_{0,1}}\left(\frac{x - vt}{\sqrt{1-v^2}}\right) - \overrightarrow{H_{-1,0}}\left(\frac{x + vt}{\sqrt{1-v^2}}\right) \right\|_{H^1_x(\mathbb{R}) \times L^2_x(\mathbb{R})} \leq e^{-ct}, \tag{5}$$

where, for any $-1 < v < 1$ and any $y \in \mathbb{R}$,

$$\overrightarrow{H_{0,1}}\left(\frac{x - vt + y}{\sqrt{1-v^2}}\right) = \begin{bmatrix} H_{0,1}\left(\frac{x - vt + y}{\sqrt{1-v^2}}\right) \\ \frac{-v}{\sqrt{1-v^2}} H'_{0,1}\left(\frac{x - vt + y}{\sqrt{1-v^2}}\right) \end{bmatrix}, \tag{6}$$

$$\overrightarrow{H_{-1,0}}\left(\frac{x + vt - y}{\sqrt{1-v^2}}\right) = \begin{bmatrix} H_{-1,0}\left(\frac{x + vt - y}{\sqrt{1-v^2}}\right) \\ \frac{v}{\sqrt{1-v^2}} H'_{-1,0}\left(\frac{x + vt - y}{\sqrt{1-v^2}}\right) \end{bmatrix}. \tag{7}$$

The existence and uniqueness of solutions $\phi(t,x)$ satisfying (5) for any $0 < v < 1$ was obtained in [Chen and Jendrej 2022], but the uniqueness of the solution of (1) satisfying

$$\lim_{t \to +\infty} \left\| \vec{\phi}(t,x) - \overrightarrow{H_{0,1}}\left(\frac{x - vt}{\sqrt{1-v^2}}\right) + \overrightarrow{H_{-1,0}}\left(\frac{x + vt}{\sqrt{1-v^2}}\right) \right\|_{H^1_x(\mathbb{R}) \times L^2_x(\mathbb{R})} = 0$$

for $0 < v < 1$ is still an open problem. For references on the existence and uniqueness of multisoliton solutions of other nonlinear dispersive partial differential equations; see, e.g., [Martel 2005; Combet 2011].

For nonintegrable dispersive models, there exist previous results about the inelasticity of the collision of two solitons. For example, Martel and Merle [2011] verified that the collision between two solitons with nearly equal speed is not elastic. More precisely, they showed that the incoming speed of the two solitons is different to their outgoing speed after their collision.

Since the ϕ^6 model is a nonintegrable system, the collision of two kinks with low speed $0 < v < 1$ is expected to be inelastic. More precisely, we expect the existence of a value $k > 1$ such that if $0 < v \ll 1$ and $\phi(t,x)$ is a solution (1) satisfying the condition (5), then $\phi(t,x)$ should have inelasticity of order v^k, which means the existence of $t < 0$ with $|t| \gg 1$ such that

$$(\phi(t,x), \partial_t \phi(t,x)) = \overrightarrow{H_{0,1}}\left(\frac{x + v_f t + y_1(t)}{\sqrt{1-v_f^2}}\right) + \overrightarrow{H_{-1,0}}\left(\frac{x - v_f t + y_2(t)}{\sqrt{1-v_f^2}}\right) + ro(t,x), \tag{8}$$

with $v^k \ll \|ro(t)\|_{H^1_x(\mathbb{R}) \times L^2_x(\mathbb{R})} \ll v$ and $v_f(t)$, y_1, y_2 satisfying

$$v^k \ll |v_f(t) - v| + \max_{j \in \{1,2\}} |\dot{y}_j(t)| \ll v \tag{9}$$

for all $t < 0$ satisfying $|t| \gg 1$. Actually, in the quartic gKdV, the collision of the two solitons satisfies a similar property to our previous expectations in (8) and (9); see [Martel and Merle 2011, Theorem 1] for more details.

However, in this manuscript, we prove for the ϕ^6 model and any $k > 1$ that if $0 < v \ll 1$ and t is close to $-\infty$, both estimates (8) and (9) are not possible. Indeed, we demonstrate that if $v \ll 1$ and $\phi(t, x)$ satisfies (5), then there exists a number $e_{k,2v} \in \mathbb{R}$ satisfying, for all t close to $-\infty$,

$$(\phi(t, x), \partial_t \phi(t, x)) = \overrightarrow{H_{0,1}}\left(\frac{x + v_f t - e_{k,2v}}{\sqrt{1 - v_f^2}}\right) + \overrightarrow{H_{-1,0}}\left(\frac{x - v_f t + e_{k,2v}}{\sqrt{1 - v_f^2}}\right) + r_{c,v}(t, x),$$
$$\limsup_{t \to -\infty} \|r_{c,v}(t)\|_{H^1_x \times L^2_x} \le v^{2k},$$
$$\limsup_{t \to -\infty} |v_f(v, t) - v| \le v^{2k}. \tag{10}$$

In conclusion, the inelasticity of the collision of two kinks cannot be of any order v^k for any $1 \ll k \in \mathbb{N}$, if the incoming speed v of the kinks is small enough. The problem to verify the inelasticity of the collision of kinks for the ϕ^6 model is still open. But, because of the conclusion obtained in this paper, the change $|v - v_f|$ in the speeds of each soliton is much smaller than any monomial function v^k. More precisely, for all $k > 0$,

$$\lim_{v \to 0^+} \limsup_{t \to -\infty} \frac{|v_f(v, t) - v|}{v^k} = 0. \tag{11}$$

This is a new result.

The study of collision of kinks for the ϕ^6 model is important for high energy physics; see, for example, [Gani et al. 2014; Dorey et al. 2011]. Actually, in [Gani et al. 2014], it was shown numerically that there exists a critical speed v_c such that if each of the two kinks move with speed v with absolute value less than v_c and they approach each other, then they will collide and the collision will be very elastic, which is exactly the result we obtain rigorously in this paper. The study of the dynamics of multisoliton solutions of the ϕ^6 model has also applications in condensed matter physics, see [Bishop and Schneider 1978], and cosmology, see [Vilenkin and Shellard 1994].

For other nonlinear dispersive equations, there exist rigorous results of inelasticity and stability of collision of solitons. For gKdV models, the inelasticity of collision of solitons was proved for the quartic gKdV in [Martel and Merle 2011], and, for a certain class of gKdV, inelasticity of collision between solitons was also proved in [Muñoz 2010; 2012]; see also [Martel and Merle 2009]. For the nonlinear Schrödinger equation, Perelman [2011] studied the collision of two solitons of different sizes and showed that the solution does not preserve the two solitons' structure after the collision. See also [Martel and Merle 2018] for discussion on the inelasticity of the collision of two solitons for the fifth-dimensional energy critical wave equation.

1.2. ***Main results.*** The main theorem obtained in this manuscript is the following result:

Theorem 2. *There exists a continuous function $v_f : (0, 1) \times \mathbb{R} \to (0, 1)$ and, for any $0 < \theta < 1$ and $k \in \mathbb{N}_{\ge 2}$, there exists $0 < \delta(\theta, k) < 1$, such that if $0 < v < \delta(\theta, k)$, and $\phi(t, x)$ is a traveling kink-kink solution of (1) with speed v, then there exists a number $e_{v,k}$ such that $|e_{v,k}| < \ln(8/v^2)$. Furthermore, if*

$$t \le -\frac{\ln(1/v)^{2-\theta}}{v},$$

then $|v_f(v,t)-v|<v^k$ *and*

$$\left\|\phi(t,x)-H_{0,1}\left(\frac{x-e_{k,v}+v_f t}{\sqrt{1-v_f^2}}\right)-H_{-1,0}\left(\frac{x+e_{k,v}-v_f t}{\sqrt{1-v_f^2}}\right)\right\|_{H^1_x(\mathbb{R})} + \left\|\partial_t\phi(t,x)-\frac{v_f}{\sqrt{1-v_f^2}}H'_{0,1}\left(\frac{x-e_{v,k}+v_f t}{\sqrt{1-v_f^2}}\right)+\frac{v_f}{\sqrt{1-v_f^2}}H'_{-1,0}\left(\frac{x+e_{v,k}-v_f t}{\sqrt{1-v_f^2}}\right)\right\|_{L^2_x(\mathbb{R})}\leq v^k.$$

If

$$\frac{-4\ln(1/v)^{2-\theta}}{v}\leq t\leq\frac{-\ln(1/v)^{2-\theta}}{v},$$

then

$$\left\|\phi(t,x)-H_{0,1}\left(\frac{x-e_{k,v}+vt}{\sqrt{1-v^2}}\right)-H_{-1,0}\left(\frac{x+e_{k,v}-vt}{\sqrt{1-v^2}}\right)\right\|_{H^1_x(\mathbb{R})} + \left\|\partial_t\phi(t,x)-\frac{v}{\sqrt{1-v^2}}H'_{0,1}\left(\frac{x-e_{v,k}+vt}{\sqrt{1-v^2}}\right)+\frac{v}{\sqrt{1-v^2}}H'_{-1,0}\left(\frac{x+e_{v,k}-vt}{\sqrt{1-v^2}}\right)\right\|_{L^2_x(\mathbb{R})}\leq v^k.$$

Remark 3. The second inequality in Theorem 2 follows from the energy estimate method used in Section 3 to estimate the energy norm of the remainder during a large time interval.

Clearly, Theorem 2 implies (11). Actually, the first item of Theorem 2 is a consequence of the second item of this theorem and the following result about the orbital stability of two moving kinks.

Theorem 4. *There exists a constant* $c>0$ *and, for any* $\theta\in(0,1)$, *there exists* $\delta(\theta)\in(0,1)$ *such that if* $0<v<\delta(\theta)$, *and* $(\psi_0(x),\psi_1(x))\in H^1_x(\mathbb{R})\times L^2_x(\mathbb{R})$ *is an odd function satisfying*

$$\|(\psi_0,\psi_1)\|_{H^1_x\times L^2_x}<v^{2+\theta}, \tag{12}$$

and $y_0\geq -4\ln v$, *then the solution* $(\phi(t,x),\partial_t\phi(t,x))$ *of the Cauchy problem*

$$\begin{cases}\partial_t^2\phi(t,x)-\partial_x^2\phi(t,x)+U'(\phi(t,x))=0,\\ \begin{bmatrix}\phi(0,x)\\ \partial_t\phi(0,x)\end{bmatrix}=\begin{bmatrix}H_{0,1}\left(\frac{x-y_0}{\sqrt{1-v^2}}\right)+H_{-1,0}\left(\frac{x+y_0}{\sqrt{1-v^2}}\right)+\psi_0(x)\\ \frac{-v}{\sqrt{1-v^2}}H'_{0,1}\left(\frac{x-y_0}{\sqrt{1-v^2}}\right)+\frac{v}{\sqrt{1-v^2}}H'_{-1,0}\left(\frac{x+y_0}{\sqrt{1-v^2}}\right)+\psi_1(x)\end{bmatrix}\end{cases} \tag{13}$$

is given for all $t\geq 0$ *by*

$$\begin{bmatrix}\phi(t,x)\\ \partial_t\phi(t,x)\end{bmatrix}=\begin{bmatrix}H_{0,1}\left(\frac{x-y(t)}{\sqrt{1-v^2}}\right)+H_{-1,0}\left(\frac{x+y(t)}{\sqrt{1-v^2}}\right)+\psi(t,x)\\ \frac{-v}{\sqrt{1-v^2}}H'_{0,1}\left(\frac{x-y(t)}{\sqrt{1-v^2}}\right)+\frac{v}{\sqrt{1-v^2}}H'_{-1,0}\left(\frac{x+y(t)}{\sqrt{1-v^2}}\right)+\partial_t\psi(t,x)\end{bmatrix}, \tag{14}$$

such that

$$|y(0)-y_0|+\|\overrightarrow{\psi}(t,x)\|_{H^1_x\times L^2_x}\leq c\|\overrightarrow{\psi_0}(x)\|^{1/2}_{H^1_x\times L^2_x}+c(1+y_0)^{1/2}e^{-\sqrt{2}y_0}, \tag{15}$$
$$|\dot{y}(t)-v|\leq c\|\overrightarrow{\psi_0}(x)\|_{H^1_x\times L^2_x}$$

for all $t\in\mathbb{R}_{\geq 0}$.

Remark 5. Theorem 4 allows us to extend the description of the traveling kink-kink for all time below

$$-\frac{\ln{(1/v)}^{2-\theta}}{v},$$

from which we will deduce the first inequality in Theorem 2.

1.3. ***Notation.*** In this subsection, we explain the notation that we are going to use in the next sections. First, for any real function $f:\mathbb{R}^2\to\mathbb{R}$ satisfying the conditions $f(t,\cdot)\in L^{\infty}_x(\mathbb{R})$, and $\partial_t f(t,\cdot)\in L^2_x(\mathbb{R})$, we define the function $\overrightarrow{f}:\mathbb{R}^2\to\mathbb{R}^2$ by

$$\overrightarrow{f}(t,x)=(f(t,x),\partial_t f(t,x))\quad\text{for every } (t,x)\in\mathbb{R}^2.$$

For any $k\in\mathbb{N}$ and any smooth function $f:\mathbb{R}\to\mathbb{R}$, we use the notation

$$f^{(k)}(x)=\frac{d^x}{dx^k}f(x)\quad\text{for all } x\in\mathbb{R}.$$

For any $z\in\mathbb{R}$, we use the notation $H^z_{0,1}(x)=H_{0,1}(x-z)$, $H^z_{-1,0}(x)=H_{-1,0}(x-z)$. For any subset $\mathcal{D}\subset\mathbb{R}$, any $v\in(0,1)$ and any function $y:\mathcal{D}\to\mathbb{R}$, we define the functions $\overrightarrow{H_{0,1,v,y}}:\mathcal{D}\times\mathbb{R}\to\mathbb{R}^2$, $\overrightarrow{H_{-1,0,v,y}}:\mathcal{D}\times\mathbb{R}\to\mathbb{R}^2$ by

$$\overrightarrow{H_{0,1,v,y}}(t,x)=\begin{bmatrix} H_{0,1}\Big(\frac{x-vt+y(t)}{\sqrt{1-v^2}}\Big)\\ \frac{-v}{\sqrt{1-v^2}}H'_{0,1}\Big(\frac{x-vt+y(t)}{\sqrt{1-v^2}}\Big)\end{bmatrix},\quad \overrightarrow{H_{-1,0,v,y}}(t,x)=\begin{bmatrix} H_{-1,0}\Big(\frac{x+vt-y(t)}{\sqrt{1-v^2}}\Big)\\ \frac{v}{\sqrt{1-v^2}}H'_{-1,0}\Big(\frac{x+vt-y(t)}{\sqrt{1-v^2}}\Big)\end{bmatrix}.$$

For any set $\mathcal{D}\subset\mathbb{R}$ and any nonnegative function $k:\mathcal{D}\to\mathbb{R}_{\geq 0}$, we say that $f(x)=O(k(x))$, if f has the same domain $\mathcal{D}$ as k and there is a universal constant $C>0$ such that $|f(x)|\leq Ck(x)$ for any $x\in D$. For any two nonnegative real functions $f_1(x)$ and $f_2(x)$, we have $f_1\lesssim f_2$ if there is a universal constant $C>0$ such that $f_1(x)\leq Cf_2(x)$ for any $x\in\mathbb{R}$. Furthermore, for a finite number of real variables $\alpha_1,\ldots,\alpha_n$ and two nonnegative functions $f_1(\alpha_1,\ldots,\alpha_n,x)$ and $f_2(\alpha_1,\ldots,\alpha_n,x)$ both with domain $\mathcal{D}\times\mathbb{R}\subset\mathbb{R}^{n+1}$, we say that $f_1\lesssim_{\alpha_1,\ldots,\alpha_n} f_2$ if there is a positive function $L:\mathcal{D}\to\mathbb{R}_+$ such that

$$f_1(\alpha_1,\ldots,\alpha_n,x)\leq L(\alpha_1,\ldots,\alpha_n)f_2(\alpha_1,\ldots,\alpha_n,x)\quad\text{for all } (\alpha_1,\ldots,\alpha_m,x)\in\mathcal{D}\times\mathbb{R}.$$

We write $f_1\cong f_2$ if $f_1\lesssim f_2$ and $f_2\lesssim f_1$.

We consider for any $f\in H^1_x(\mathbb{R})$ and any $g\in L^2_x(\mathbb{R})$ the norms

$$\|f\|_{H^1_x}=\|f\|_{H^1_x(\mathbb{R})}=\bigg(\|f\|^2_{L^2_x(\mathbb{R})}+\bigg\|\frac{df}{dx}\bigg\|^2_{L^2_x(\mathbb{R})}\bigg)^{1/2},\quad \|g\|_{L^2_x}=\|g\|_{L^2_x(\mathbb{R})}.$$

We also consider the norm $\|\cdot\|_{H^1_x\times L^2_x}$ given by

$$\|(f_1(x),f_2(x))\|_{H^1_x\times L^2_x}=(\|f_1\|^2_{H^1_x(\mathbb{R})}+\|f_2(x)\|^2_{L^2_x(\mathbb{R})})^{1/2}$$

for any $(f_1, f_2) \in H^1_x(\mathbb{R}) \times L^2_x(\mathbb{R})$. For any $(f_1, f_2) \in L^2_x(\mathbb{R}) \times L^2_x(\mathbb{R})$ and any $(g_1, g_2) \in L^2_x(\mathbb{R}) \times L^2_x(\mathbb{R})$, we let

$$\langle (f_1, f_2), (g_1, g_2) \rangle = \int_{\mathbb{R}} f_1(x) g_1(x) + f_2(x) g_2(x)\, dx.$$

For any functions $f_1(x), g_1(x) \in L^2_x(\mathbb{R})$, we let

$$\langle f_1, g_1 \rangle = \int_{\mathbb{R}} f_1(x) g_1(x)\, dx.$$

In this manuscript, we consider the set $\mathbb{N}$ as the set of all positive integers. For any $n \in \mathbb{N}$, and any $a, b \in \mathbb{R}^n$, we denote the scalar product in the Euclidean space $\mathbb{R}^n$ by

$$\langle a : b \rangle = \sum_{j=1}^{n} a_j b_j,$$

where $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$.

1.4. ***Organization of the manuscript.*** First, from the global well-posedness of the partial differential equation (1), we recall that if ϕ is a strong solution of (1) with finite energy satisfying $\lim_{x \to \pm\infty} \phi(t_0, x) = \pm 1$ for some $t_0 \in \mathbb{R}$, then the function ϕ satisfies

$$\|\phi(t, x) - H_{0,1}(x) - H_{-1,0}(x)\|_{H^1_x(\mathbb{R})} < +\infty$$

for all $t \in \mathbb{R}$.

In Section 2.1, we will review our results from [Moutinho 2024] about the existence of a sequence of approximate solutions $(\varphi_{k,v})_{k \geq 2}$ of (1) for which there exists a set of real numbers $(y_k(v))_{k \geq 2}$ satisfying

$$\lim_{t \to +\infty} \|\overrightarrow{\varphi_k}(t, x) - \overrightarrow{H_{0,1,v,y_k}}(t, x) - \overrightarrow{H_{-1,0,v,y_k}}(t, x)\|_{H^1_x \times L^2_x} = 0,$$

and if $v \ll 1$, then $\|\partial_t^l \Lambda(\varphi_{k,v})(t, x)\|_{H^s_x} \lesssim_{s,l} v^{2k+l-1/2} e^{-2\sqrt{2} v |t|}$ for all $t \in \mathbb{R}$, $l \in \mathbb{N} \cup \{0\}$, and $s \geq 0$.

In Section 2.2, we will verify that any solution of (1) with finite energy close to a sum of two kinks can be written as

$$\phi(t, x) = \varphi_{k,v}(t, x) + \frac{y_1(t)}{\sqrt{1 - \frac{1}{4}\dot{d}(t)^2}} H'_{0,1}\left(\frac{x - \frac{1}{2}d(t) + c_k(t)}{\sqrt{1 - \frac{1}{4}\dot{d}(t)^2}}\right) + \frac{y_2(t)}{\sqrt{1 - \frac{1}{4}\dot{d}(t)^2}} H'_{0,1}\left(\frac{-x - \frac{1}{2}d(t) + c_k(t)}{\sqrt{1 - \frac{1}{4}\dot{d}(t)^2}}\right) + u(t, x), \quad (16)$$

such that, for any $t \in \mathbb{R}$, $u(t) \in H^1_x(\mathbb{R})$ satisfies the orthogonality conditions

$$\left\langle u(t, x), H'_{0,1}\left(\frac{x - \frac{1}{2}d(t) + c_k(t)}{\sqrt{1 - \frac{1}{4}\dot{d}(t)^2}}\right)\right\rangle = 0, \quad \left\langle u(t, x), H'_{0,1}\left(\frac{-x - \frac{1}{2}d(t) + c_k(t)}{\sqrt{1 - \frac{1}{4}\dot{d}(t)^2}}\right)\right\rangle = 0.$$

Moreover, using $\Lambda(\phi) \equiv 0$, we can verify that $y_1, y_2 \in C^2(\mathbb{R})$. Furthermore, using (16), we will estimate $\Lambda(\phi)(t, x)$. More precisely, we will estimate the expression $\Lambda(\phi)(t, x) - \Lambda(\varphi_{k,v})(t, x)$, in terms of $y_1(t), y_2(t), d(t), u(t, x)$ and the estimate of the term $\Lambda(\varphi_{k,v})(t, x)$ will follow from the main results of Section 2.1 about the decay with respect to t of the approximate solutions. The function $c_k(t)$ will not appear in the evaluation of $\Lambda(\phi)(t, x)$, since we will use only its decay.

In Section 3, we will construct a function $L(t)$ to estimate $\|(u(t), \partial_t u(t))\|_{H_x^1 \times L_x^2}$ during a large time interval. The main argument in this section is analogous to the ideas of Section 4 of [Moutinho 2023]. More precisely, for

$$w_{k,v}(t, x) = \frac{x - \frac{1}{2}d(t) + c_k(t)}{\sqrt{1 - \frac{1}{4}\dot{d}(t)^2}},$$

we consider first

$$L_1(t) = \int_{\mathbb{R}} \partial_t u(t, x)^2 + \partial_x u(t, x)^2 + U''\big(H_{0,1}(w_{k,v}(t, x)) - H_{0,1}(w_{k,v}(t, -x))\big)u(t, x)^2 \, dx.$$

From the orthogonality conditions satisfied by $u(t, x)$, if $v \ll 1$, we deduce the coercivity inequality

$$\|(u(t), \partial_t u(t))\|_{H_x^1 \times L_x^2}^2 \lesssim L_1(t).$$

The function $L(t)$ will be constructed after correction terms $L_2(t)$ and $L_3(t)$ are added to $L_1(t)$. The motivation for using the correction term $L_3(t)$ is to reduce the growth of the modulus of the expression

$$2 \int_{\mathbb{R}} \big[\partial_t^2 u(t, x) - \partial_x^2 u(t, x) + U''\big(H_{0,1}(w_{k,v}(t, x)) - H_{0,1}(w_{k,v}(t, x))\big)u(t, x)\big]\partial_t u(t, x) \, dx$$

in $\dot{L}_1(t)$. The time derivative of $L_2(t)$ will cancel with the expression

$$\int_{\mathbb{R}} \frac{\partial}{\partial t}\big[U''\big(H_{0,1}(w_{k,v}(t, x)) - H_{0,1}(w_{k,v}(t, x))\big)\big]u(t, x)^2 \, dx,$$

from $\dot{L}_1(t)$. Finally, under additional conditions in the growth of the functions $y_1(t)$, $y_2(t)$, if $0 < v \ll 1$, the function $L(t) = \sum_{j=1}^{3} L_j(t)$ will satisfy, for a constant $C(k)$ depending only on k, the estimates

$$|\dot{L}(t)| \lesssim \frac{v}{\ln(1/v)} \|(u(t), \partial_t u(t))\|_{H_x^1 \times L_x^2}^2,$$
$$\|(u(t), \partial_t u(t))\|_{H_x^1 \times L_x^2}^2 \lesssim L(t) + C(k)v^{4k} \ln(1/v)^{2n_k}$$

for all t in a large time interval, where n_k is the number described in Theorem 8. Hence, using Gronwall's lemma and the two estimates above, we will obtain an upper bound for $\|(u(t), \partial_t u(t))\|_{H_x^1 \times L_x^2}$ when t belongs to a large time interval.

In Section 4, we will estimate $\|\phi(t) - \varphi_{k,v}(t)\|_{H_x^1 \times L_x^2}$ during a large time interval. This estimate follows from the study of a linear ordinary differential system whose solutions $\hat{y}_1$, $\hat{y}_2$ are close to y_1, y_2 during a time interval of size much larger than $-\ln(v)/v$ and from the conclusions of the last section. Indeed, the closeness of the functions y_1, y_2 with $\hat{y}_1$, $\hat{y}_2$ during this large time interval is guaranteed because of the upper bound obtained for $\|(u(t), \partial_t u(t))\|_{H_x^1 \times L_x^2}$ from the control of $L(t)$, which implies that y_1, y_2 will satisfy a ordinary differential system very close to the linear ordinary differential system satisfied by $\hat{y}_1$ and $\hat{y}_2$.

In Section 5, we will prove Theorem 4; the proof of this result is inspired by the demonstration of [Kowalczyk et al. 2021, Theorem 1; Martel et al. 2006, Theorem 1]. This result will imply in the next section the second item of Theorem 2. In addition, the main techniques used in this section are modulation techniques based on [Kowalczyk et al. 2021, §2; Martel et al. 2006], the use of conservation of energy of $\phi(t, x)$ and the monotonicity of the localized momentum given by

$$P_+(\phi(t), \partial_t\phi(t)) = -\int_0^{+\infty} \partial_t\phi(t, x)\partial_x\phi(t, x)\, dx.$$

Finally, in Section 6, we will show that the demonstration of Theorem 2 is a direct consequence of the main results of Sections 4 and 5. For complementary information, see the Appendices.

2. Preliminaries

2.1. *Approximate solutions.*

Definition 6. We define Λ as the nonlinear operator with domain $C^2(\mathbb{R}^2, \mathbb{R})$ that satisfies

$$\Lambda(\phi_1)(t, x) = \partial_t^2\phi_1(t, x) - \partial_x^2\phi_1(t, x) + \dot{U}(\phi_1(t, x))$$

for any $\phi_1(t, x) \in C^2(\mathbb{R}^2, \mathbb{R})$.

In [Moutinho 2024], we constructed a sequence of approximate solutions $(\phi_k(v, t, x))_{k\in\mathbb{N}_{\geq 2}}$ of the partial differential equation (1) such that

$$\lim_{t\to+\infty}\left\|\phi_k(v, t, x) - H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right) - H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right\|_{H_x^1} = 0,$$

$$\lim_{t\to+\infty}\left\|\partial_t\phi_k(v, t, x) + \frac{v}{\sqrt{1-v^2}}H_{0,1}'\left(\frac{x-vt}{\sqrt{1-v^2}}\right) - \frac{v}{\sqrt{1-v^2}}H_{-1,0}'\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right\|_{L_x^2} = 0$$

More precisely, in [Moutinho 2024] we proved the following result:

Theorem 7. *There exist a sequence of functions $(\phi_k(v, t, x))_{k\geq 2}$, a sequence of real values $\delta(k) > 0$ and a sequence of numbers $n_k \in \mathbb{N}$ such that, for any $0 < v < \delta(k)$, $\phi_k(v, t, x)$ satisfies*

$$\lim_{t\to+\infty}\left\|\phi_k(v, t, x) - H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right) - H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right\|_{H_x^1} = 0,$$

$$\lim_{t\to+\infty}\left\|\partial_t\phi_k(v, t, x) + \frac{v}{\sqrt{1-v^2}}H_{0,1}'\left(\frac{x-vt}{\sqrt{1-v^2}}\right) - \frac{v}{\sqrt{1-v^2}}H_{-1,0}'\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right\|_{L_x^2} = 0,$$

$$\lim_{t\to-\infty}\left\|\phi_k(v, t, x) - H_{0,1}\left(\frac{x+vt-e_{v,k}}{\sqrt{1-v^2}}\right) - H_{-1,0}\left(\frac{x-vt+e_{v,k}}{\sqrt{1-v^2}}\right)\right\|_{H_x^1} = 0,$$

$$\lim_{t\to-\infty}\left\|\partial_t\phi_k(v, t, x) - \frac{v}{\sqrt{1-v^2}}H_{0,1}'\left(\frac{x+vt-e_{v,k}}{\sqrt{1-v^2}}\right) + \frac{v}{\sqrt{1-v^2}}H_{-1,0}'\left(\frac{x-vt+e_{v,k}}{\sqrt{1-v^2}}\right)\right\|_{L_x^2} = 0,$$

with $e_{v,k} \in \mathbb{R}$ satisfying

$$\lim_{v\to 0}\frac{\left|e_{v,k} - \frac{\ln(8/v^2)}{\sqrt{2}}\right|}{v|\ln(v)|^3} = 0.$$

Moreover, if $0 < v < \delta(k)$, then for any $s \geq 0$ and $l \in \mathbb{N}\cup\{0\}$, there is $C(k, s, l) > 0$ such that

$$\left\|\frac{\partial^l}{\partial t^l}\Lambda(\phi_k(v, t, x))\right\|_{H_x^s(\mathbb{R})} \leq C(k, s, l)v^{2k+l}\left(|t|v + \ln(1/v^2)\right)^{n_k}e^{-2\sqrt{2}|t|v}.$$

We consider the Schwarz function $\mathcal{G}$ defined by

$$\mathcal{G}(x) = e^{-\sqrt{2}x} - \frac{e^{-\sqrt{2}x}}{(1+e^{2\sqrt{2}x})^{3/2}} + 2\sqrt{2}x\frac{e^{\sqrt{2}x}}{(1+e^{2\sqrt{2}x})^{3/2}} + k_1\frac{e^{\sqrt{2}x}}{(1+e^{2\sqrt{2}x})^{3/2}} \tag{17}$$

for all $x \in \mathbb{R}$, where k_1 is the real number such that $\mathcal{G}$ satisfies $\langle \mathcal{G}(x), H'_{0,1}(x)\rangle_{L^2_x(\mathbb{R})} = 0$. The function $\mathcal{G}$ satisfies the identity

$$-\frac{d^2}{dx^2}\mathcal{G}(x) + U''(H_{0,1}(x))\mathcal{G}(x) = [-24H_{0,1}(x)^2 + 30H_{0,1}(x)^4]e^{-\sqrt{2}x} + 8\sqrt{2}H'_{0,1}(x); \tag{18}$$

see Lemma A.1 and Remark A.2 in the Appendix of [Moutinho 2024] for the proof.

From now on, for any $v \in (0, 1)$, we consider the function $d_v : \mathbb{R} \to \mathbb{R}$ defined by

$$d_v(t) = \frac{1}{\sqrt{2}}\ln\left(\frac{8}{v^2}\cosh{(\sqrt{2}vt)}^2\right) \quad \text{for any } t \in \mathbb{R}.$$

The function d_v describes the movement between two kinks for the ϕ^6 model during a large time interval when their total energy is small and their initial speeds are both zero. For more information, see Theorem 1.11 from [Moutinho 2023].

Moreover, from the proof of Theorem 7 in [Moutinho 2024], we can construct inductively an explicit sequence of smooth functions $(\varphi_{k,v})_{k\in\mathbb{N}_{\geq 2}}$ and for each $k \in \mathbb{N}_{\geq 2}$ there exists a real number $\tau_{k,v}$ satisfying

$$|\tau_{k,v}| < \frac{\sqrt{2}}{v}\ln\left(\frac{8}{v^2}\right)$$

such that $\phi_k(v, t, x) := \varphi_{k,v}(t + \tau_{k,v}, x)$ satisfies Theorem 7 for all $k \in \mathbb{N}_{\geq 2}$. More precisely, from [Moutinho 2024], we have the following theorem:

Theorem 8. *There exist a sequence of approximate solutions $\varphi_{k,v}(t, x)$, functions $r_k(v, t)$ that are smooth and even on t, and numbers $n_k \in \mathbb{N}$ such that if $0 < v \ll 1$, then, for any $m \in \mathbb{N}_{\geq 1}$,*

$$|r_k(v,t)| \lesssim_k v^{2(k-1)}\ln(1/v)^{n_k}, \qquad \left|\frac{\partial^m}{\partial t^m}r_k(v,t)\right| \lesssim_{k,m} v^{2(k-1)+m}\big[\ln(1/v) + |t|v\big]^{n_k}e^{-2\sqrt{2}|t|v}. \tag{19}$$

Furthermore, $\varphi_{k,v}(t, x)$ satisfies for $\rho_k(v,t) = -\frac{1}{2}d_v(t) + \sum_{j=2}^{k} r_j(v,t)$ the identity

$$\begin{aligned}\varphi_{k,v}(t,x) = H_{0,1}\left(\frac{x+\rho_k(v,t)}{\sqrt{1-\frac{1}{4}\dot{d}_v(t)^2}}\right) &+ H_{-1,0}\left(\frac{x-\rho_k(v,t)}{\sqrt{1-\frac{1}{4}\dot{d}_v(t)^2}}\right)\\ &+ e^{-\sqrt{2}d_v(t)}\left[\mathcal{G}\left(\frac{x+\rho_k(v,t)}{\sqrt{1-\frac{1}{4}\dot{d}_v(t)^2}}\right) - \mathcal{G}\left(\frac{-x+\rho_k(v,t)}{\sqrt{1-\frac{1}{4}\dot{d}_v(t)^2}}\right)\right]\\ &+ \mathcal{R}_{k,v}\left(vt, \frac{x+\rho_k(v,t)}{\sqrt{1-\frac{1}{4}\dot{d}_v(t)^2}}\right) - \mathcal{R}_{k,v}\left(vt, \frac{-x+\rho_k(v,t)}{\sqrt{1-\frac{1}{4}\dot{d}_v(t)^2}}\right)\end{aligned} \tag{20}$$

and, for any $l \in \mathbb{N} \cup \{0\}$ *and* $s \geq 1$ *the estimates*

$$\left\| \frac{\partial^l}{\partial t^l} \Lambda(\varphi_{k,v}(t,x)) \right\|_{H^s_x(\mathbb{R})} \lesssim_{k,s,l} v^{2k+l} \big[\ln(1/v^2) + |t|v\big]^{n_k} e^{-2\sqrt{2}|t|v}, \tag{21}$$

$$\left| \frac{d^l}{dt^l} \left[\left\langle \Lambda(\varphi_{k,v})(t,x),\, H'_{0,1}\left(\frac{x+\rho_k(v,t)}{\big(1-\frac{1}{4}\dot{d}_v(t)^2\big)^{1/2}} \right) \right\rangle \right] \right| \lesssim_{k,l} v^{2k+l+2} \big[\ln(1/v^2) + |t|v\big]^{n_k+1} e^{-2\sqrt{2}|t|v}, \tag{22}$$

where $\mathcal{R}_{k,v}(t,x)$ *is a finite sum of functions* $p_{k,i,v}(t) h_{k,i}(x)$ *with* $h_{k,i} \in \mathscr{S}(\mathbb{R})$ *and each* $p_{k,i,v}(t)$ *being an even function satisfying, for all* $m \in \mathbb{N}$,

$$\left| \frac{d^m p_{k,i,v}(t)}{dt^m} \right| \lesssim_{k,m,3} v^4 \big(\ln(1/v^2) + |t|\big)^{n_{k,i}} e^{-2\sqrt{2}|t|},$$

where $n_{k,i} \in \mathbb{N}$ *depends only on* k *and* i.

Remark 9. Furthermore, Remark 5.2 of [Moutinho 2024] implies that if $v > 0$ is small enough, then the function r_2 satisfies

$$\|r_2(v,\cdot)\|_{L^\infty(\mathbb{R})} \lesssim v^2 \ln(1/v^2), \qquad \left| \frac{\partial^l}{\partial t^l} r_2(v,t) \right| \lesssim_l v^{2+l} \big[\ln(1/v^2) + |t|v\big] e^{-2\sqrt{2}|t|v}$$

for all $l \in \mathbb{N}$.

Remark 10. At first look, the statement of Theorem 8 seems to contain excessive information about the approximate solutions $\phi_k(v,t,x)$ of [Moutinho 2024]. However, we will need all of it to study the elasticity and stability of the collision of two kinks with low speed $0 < v < 1$.

2.2. *Auxiliary estimates.*

First, we recall the Lemma 2.1 of [Moutinho 2023].

Lemma 11. *If* x_2, x_1 *are real numbers satisfying* $z = x_2 - x_1 > 0$ *and* $\alpha, \beta, m > 0$ *with* $\alpha \neq \beta$, *then*

$$\int_{\mathbb{R}} |x - x_1|^m e^{-\alpha(x-x_1)_+} e^{-\beta(x_2-x)_+} \lesssim_{m,\alpha,\beta} \max((1+z^m)e^{-\alpha z}, e^{-\beta z}),$$

Furthermore, for any $\alpha > 0$,

$$\int_{\mathbb{R}} |x - x_1|^m e^{-\alpha(x-x_1)_+} e^{-\alpha(x_2-x)_+} \lesssim_{m,\alpha} [1 + z^{m+1}] e^{-\alpha z}.$$

Actually, we will also need to use the following lemma, which we proved in [Moutinho 2024].

Lemma 12. *In the notation of Theorem 8, for* $v \in (0,1)$, *let* $w_{k,v} : \mathbb{R}^2 \to \mathbb{R}$ *be the function*

$$w_{k,v}(t,x) = \frac{x + \rho_k(v,t)}{\sqrt{1 - \frac{1}{4}\dot{d}_v(t)^2}},$$

and let $f \in L^\infty_x(\mathbb{R})$ *be a function satisfying* $f' \in \mathscr{S}(\mathbb{R})$. *Then, if* $0 < v \ll 1$, *we have for any* $l \in \mathbb{N}$ *that*

$$\frac{\partial^l}{\partial t^l} f(w_{k,v}(t,x))$$

is a finite sum of functions $q_{k,l,i,v}(t)h_i(w_{k,v}(t,x))$ *with* $h_i \in \mathscr{S}(\mathbb{R})$ *and* $q_{k,l,i,v}(t)$ *a smooth real function satisfying*

$$\|q_{k,l,i,v}\|_{L^\infty(\mathbb{R})} \lesssim v^l.$$

Furthermore, if $0 < v \ll 1$, *we have for all* $l \in \mathbb{N}$ *and any* $s \geq 0$ *that*

$$\left\| \frac{\partial^l}{\partial t^l} f(w_{k,v}(t,x)) \right\|_{H_x^s(\mathbb{R})} \lesssim_{k,s,l} v^l.$$

Moreover, we will use the following result several times in the computation of the estimates of this paper.

Lemma 13. *For any* $s \geq 1$, *we have for any functions* $f, g \in \mathscr{S}(\mathbb{R})$ *that*

$$\|fg\|_{H_x^s(\mathbb{R})} \lesssim_s \|f\|_{H_x^s(\mathbb{R})}\|g\|_{L_x^\infty(\mathbb{R})} + \|g\|_{H_x^s(\mathbb{R})}\|f\|_{L_x^\infty(\mathbb{R})} \lesssim_s \|f\|_{H_x^s(\mathbb{R})}\|g\|_{H_x^s(\mathbb{R})}.$$

As a consequence,

$$\|fg\|_{H_x^s(\mathbb{R})} \lesssim_s \|f\|_{H_x^{s+1}(\mathbb{R})}\|g\|_{H_x^{s+1}(\mathbb{R})}$$

for all $s \geq 0$.

Proof. See the proof of Lemma A.8 in [Tao 2006]. □

Finally, we need also Lemma 2.5 of [Moutinho 2023] which studies the coercive properties of the operator

$$-\partial_x^2 + U''(H_{0,1}^z(x) + H_{-1,0}(x))$$

when $z \gg 1$. More precisely:

Lemma 14. *There exist* $c, \delta > 0$ *such that if* $z \geq \frac{1}{\delta}$, *then for any* $g \in H^1(\mathbb{R})$ *satisfying*

$$\langle g(x), H_{0,1}'(x-z)\rangle = \langle g(x), H_{-1,0}'(x)\rangle = 0,$$

we have that

$$\left\langle -\frac{d^2}{dx^2} g(x) + U''(H_{0,1}(x-z) + H_{-1,0}(x))g(x), g(x) \right\rangle \geq c\|g\|_{H_x^1(\mathbb{R})}^2.$$

Proof. See the proof of Lemma 9 in [Moutinho 2023]. □

In this manuscript, to simplify our notation, we denote $d_v(t)$ by $d(t)$, which means that

$$d(t) = \frac{1}{\sqrt{2}} \ln\left(\frac{8}{v^2}\cosh(\sqrt{2}vt)^2\right). \tag{23}$$

In Lemma 3.1 of [Moutinho 2024], we have verified by induction the estimates

$$\begin{aligned} |\dot{d}(t)| &\lesssim v, \\ |d^{(l)}(t)| &\lesssim_l v^l e^{-2\sqrt{2}|t|v} \quad \text{for any } l \in \mathbb{N}_{\geq 2}. \end{aligned} \tag{24}$$

From now on, we consider for each $k \in \mathbb{N}_{\geq 2}$ the function $\phi_{k,v}(t,x)$ satisfying Theorem 8. Next, for $T_{0,k} > 0$ to be chosen later, we consider the following kind of Cauchy problem:

$$\begin{cases} \partial_t^2\phi(t,x) - \partial_x^2\phi(t,x) + U'(\phi(t,x)) = 0, \\ \|(\phi(T_{0,k},x), \partial_t\phi(T_{0,k},x)) - (\phi_{k,v}(T_{0,k},x), \partial_t\phi_{k,v}(T_{0,k},x))\|_{H_x^1(\mathbb{R})\times L_x^2(\mathbb{R})} < v^{8k}. \end{cases} \tag{25}$$

Our first objective is to prove the following theorem.

Theorem 15. *There is a constant $C > 0$ and for any $0 < \theta < \frac{1}{4}$, $k \in \mathbb{N}_{\geq 3}$ there exist $C_1(k) > 0$, $\delta_{k,\theta} > 0$ and $\eta_k \in \mathbb{N}$ such that if*

$$0 < v < \delta_{k,\theta} \quad \textit{and} \quad T_{0,k} = \frac{32k}{2\sqrt{2}} \frac{\ln(1/v^2)}{v},$$

then any solution $\phi(t,x)$ of (25) satisfies

$$\|(\phi(t,x), \partial_t \phi(t,x)) - (\varphi_{k,v}(t,x), \partial_t \varphi_{k,v}(t,x))\|_{H^1_x \times L^2_x} < C_1(k) v^{2k} \ln(1/v)^{\eta_k} \exp\left(C \frac{v|t-T_{0,k}|}{\ln(v)}\right) \tag{26}$$

if

$$|t - T_{0,k}| < \frac{\ln(1/v)^{2-\theta}}{v}.$$

Clearly, we can obtain from Theorems 8 and 15 the following result:

Corollary 16. *There is a constant $C > 0$ and for any $0 < \theta < \frac{1}{4}$, $k \in \mathbb{N}_{\geq 3}$ there exist $C_1(k) > 0$, $\delta_{k,\theta} > 0$ and $\eta_k \in \mathbb{N}$ such that if*

$$0 < v < \delta_{k,\theta} \quad \textit{and} \quad T_{0,k} = \frac{32k}{2\sqrt{2}} \frac{\ln(1/v^2)}{v},$$

then any solution $\phi(t,x)$ of

$$\begin{cases} \partial_t^2 \phi(t,x) - \partial_x^2 \phi(t,x) + U'(\phi(t,x)) = 0, \\ \|(\phi(T_{0,k},x), \partial_t \phi(T_{0,k},x)) - (\phi_k(v,T_{0,k},x), \partial_t \phi_k(v,T_{0,k},x))\|_{H^1_x(\mathbb{R}) \times L^2_x(\mathbb{R})} < v^{8k} \end{cases}$$

satisfies

$$\|(\phi(t,x), \partial_t \phi(t,x)) - (\phi_k(v,t,x), \partial_t \phi_k(v,t,x))\|_{H^1_x \times L^2_x} < C_1(k) v^{2k} \ln(1/v)^{\eta_k} \exp\left(C \frac{v|t-T_{0,k}|}{\ln(v)}\right),$$

if

$$|t - T_{0,k}| < \frac{\ln(1/v)^{2-\theta}}{v}.$$

Proof of Corollary 16. This follows from Theorems 7, 8 and 15. □

With the objective of simplifying the demonstration of Theorem 15, we will elaborate on necessary lemmas before the proof of Theorem 15. Similarly to [Moutinho 2024], using the notation of Theorem 8, we consider

$$w_{k,v}(t,x) = \frac{x - \frac{1}{2}d(t) + c_k(v,t)}{\sqrt{1 - \frac{1}{4}\dot{d}(t)^2}}. \tag{27}$$

From now on, we denote any solution $\phi(t,x)$ of the partial differential equation (25) as

$$\phi(t,x) = \varphi_{k,v}(t,x) + \frac{y_1(t)}{\sqrt{1 - \frac{1}{4}\dot{d}(t)^2}} H'_{0,1}(w_{k,v}(t,x)) + \frac{y_2(t)}{\sqrt{1 - \frac{1}{4}\dot{d}(t)^2}} H'_{0,1}(w_{k,v}(t,-x)) + u(t,x), \tag{28}$$

such that

$$\langle u(t,x), H'_{0,1}(w_{k,v}(t,x)) \rangle = \langle u(t,x), H'_{0,1}(w_{k,v}(t,-x)) \rangle = 0. \tag{29}$$

Therefore, for $\zeta_k(t) = d(t) - 2c_k(v, t)$ and from the orthogonal conditions (29) satisfied by $u(t, x)$, we deduce the identity

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = M(t)^{-1} \begin{bmatrix} \langle \phi(t, x) - \varphi_{k,v}(t, x), H'_{0,1}(w_{k,v}(t, x)) \rangle \\ \langle \phi(t, x) - \varphi_{k,v}(t, x), H'_{-1,0}(w_{k,v}(t, -x)) \rangle \end{bmatrix}, \tag{30}$$

where, for any $t \in \mathbb{R}$, $M(t)$ is denoted by

$$M(t) = \begin{bmatrix} \|H'_{0,1}\|^2_{L^2_x} & \langle H'_{0,1}(x - \zeta_k(t)), H'_{-1,0}(x) \rangle \\ \langle H'_{0,1}(x - \zeta_k(t)), H'_{-1,0}(x) \rangle & \|H'_{0,1}\|^2_{L^2_x} \end{bmatrix}.$$

Moreover, since $\ln(1/v) \lesssim \zeta_k$, we obtain from Lemma 11 that $\langle H'_{0,1}(x - \zeta_k(t)), H'_{-1,0}(x) \rangle \ll 1$. Therefore, since the matrix $M(t)$ is a smooth function with domain $\mathbb{R}$, then $M(t)^{-1}$ is also smooth on $\mathbb{R}$.

Next, for $\psi(t, x) = \phi(t, x) - \varphi_{k,v}(t, x)$, we obtain from the partial differential equation (25) that $\psi(t, x)$ satisfies the partial differential equation

$$\frac{\partial^2}{\partial t^2}\psi(t, x) - \frac{\partial^2}{\partial x^2}\psi(t, x) + \Lambda(\varphi_{k,v})(t, x) + \sum_{j=2}^{6} \frac{U^{(j)}(\varphi_{k,v}(t, x))}{(j-1)!}\psi(t, x)^{j-1} = 0. \tag{31}$$

Since $\varphi_{k,v}$ satisfies Theorem 8 and the partial differential equation (1) is globally well-posed in the energy space, we can verify for any initial data $(\psi_0(x), \psi_1(x)) \in H^1_x(\mathbb{R}) \times L^2_x(\mathbb{R})$ that there exists a unique solution $\psi(t, x)$ of (31) satisfying $(\psi(0, x), \partial_t\psi(0, x)) = (\psi_0(x), \psi_1(x))$ and

$$(\psi(t, x), \partial_t\psi(t, x)) \in C(\mathbb{R}; H^1_x(\mathbb{R}) \times L^2_x(\mathbb{R})). \tag{32}$$

Therefore, for any function $h \in \mathscr{S}(\mathbb{R})$, we deduce from (31) that

$$\frac{d}{dt}\langle \psi(t, x), h(x) \rangle = \langle \partial_t\psi(t, x), h(x) \rangle,$$
$$\frac{d^2}{dt^2}\langle \psi(t, x), h(x) \rangle = \left\langle \frac{\partial^2}{\partial x^2}\psi(t, x) - U'(\varphi_{k,v}(t, x) + \psi(t, x)) + U'(\varphi_{k,v}(t, x)), h(x) \right\rangle - \langle \Lambda(\varphi_{k,v})(t, x), h(x) \rangle,$$

which implies that the real functions

$$\mathcal{P}_1(t) = \langle \psi(t, x), H'_{0,1}(w_{k,v}(t, x)) \rangle \quad \text{and} \quad \mathcal{P}_2(t) = \langle \psi(t, x), H'_{-1,0}(w_{k,v}(t, -x)) \rangle$$

are in $C^2(\mathbb{R})$. In conclusion, using (30) and the product rule of derivative, we deduce that $y_1, y_2 \in C^2(\mathbb{R})$.

In conclusion, we obtain the following lemma:

Lemma 17. *Assuming the same hypotheses of Theorem 15, there exist functions $y_1, y_2 : \mathbb{R} \to \mathbb{R}$ of class C^2 such that any solution $\phi(t, x)$ of (25) satisfies for any $t \in \mathbb{R}$ the identity*

$$\phi(t, x) = \varphi_{k,v}(t, x) + \frac{y_1(t)}{\sqrt{1 - \frac{1}{4}\dot{d}(t)^2}} H'_{0,1}(w_{k,v}(t, x)) + \frac{y_2(t)}{\sqrt{1 - \frac{1}{4}\dot{d}(t)^2}} H'_{0,1}(w_{k,v}(t, -x)) + u(t, x),$$

where $(u(t), \partial_t u(t)) \in H^1_x(\mathbb{R}) \times L^2_x(\mathbb{R})$ and the function u satisfies the orthogonality conditions

$$\langle u(t, x), H'_{0,1}(w_{k,v}(t, x)) \rangle = 0, \quad \langle u(t, x), H'_{0,1}(w_{k,v}(t, -x)) \rangle = 0.$$

Remark 18. Moreover, Theorem 8 implies that

$$\begin{aligned}\frac{d^2}{dt^2}\Big[\frac{y_j(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}H'_{0,1}(w_{k,v}(t,(-1)^{j+1}x))\Big]\\ &= \frac{\ddot{y}_j(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}H'_{0,1}(w_{k,v}(t,(-1)^{j+1}x)) + \frac{\dot{y}_j(t)\ddot{d}(t)\dot{d}(t)}{2\big(1-\frac{1}{4}\dot{d}(t)^2\big)^{3/2}}H'_{0,1}(w_{k,v}(t,(-1)^{j+1}x))\\ &\quad + 2\frac{\dot{y}_j(t)\partial_t\rho_k(v,t)}{1-\frac{1}{4}\dot{d}(t)^2}H''_{0,1}(w_{k,v}(t,(-1)^{j+1}x))\\ &\quad + \frac{\dot{y}_j(t)\ddot{d}(t)\dot{d}(t)}{2\big(1-\frac{1}{4}\dot{d}(t)^2\big)^2}((-1)^{j+1}x+\rho_k(v,t))H''_{0,1}(w_{k,v}(t,(-1)^{j+1}x))\\ &\quad + \frac{y_j(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}\frac{\partial^2}{\partial t^2}[H'_{0,1}(w_{k,v}(t,(-1)^{j+1}x))].\end{aligned}$$

Therefore, from Theorem 8, Remark 9 and estimates (24), we deduce from the estimate above that

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\Big[\frac{y_j(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}H'_{0,1}(w_{k,v}(t,(-1)^{j+1}x))\Big]\\ &= \frac{\ddot{y}_j(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}H'_{0,1}(w_{k,v}(t,(-1)^{j+1}x)) - \frac{\dot{y}_j(t)\dot{d}(t)}{1-\frac{1}{4}\dot{d}_v(t)^2}H''_{0,1}(w_{k,v}(t,(-1)^{j+1}x))\\ &\quad + \frac{y_j(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}\frac{\partial^2}{\partial t^2}[H'_{0,1}(w_{k,v}(t,(-1)^{j+1}x))] + \mathcal{Q}_1(t,x),\end{aligned}$$

where $\mathcal{Q}_1(t,\cdot)$ is a function in $H^1_x(\mathbb{R})$ satisfying

$$\|\mathcal{Q}_1(t,x)\|_{H^1_x(\mathbb{R})} \lesssim [\max_{j\in\{1,2\}}|\dot{y}_j(t)| + v\max_{j\in\{1,2\}}|y_j(t)|]v^3\big(\ln(1/v^2)+|t|v\big)e^{-2\sqrt{2}|t|v}.$$

Moreover, using identities

$$\frac{d^3}{dx^3}H_{0,1}(x) = U''(H_{0,1}(x))H'_{0,1}(x), \quad \ddot{d}(t) = 16\sqrt{2}e^{-\sqrt{2}d(t)},$$

estimates (24) and the estimates of $r_j(v,t)$ in Theorem 8 and Remark 9, we obtain

$$\begin{aligned}\Big(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\Big)[H'_{0,1}(w_{k,v}(t,(-1)^{j+1}x))]\\ &= -\frac{8\sqrt{2}e^{-\sqrt{2}d(t)}}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}H''_{0,1}(w_{k,v}(t,(-1)^{j+1}x))\\ &\quad - U''(H_{0,1}(w_{k,v}(t,(-1)^{j+1}x)))H'_{0,1}(w_{k,v}(t,(-1)^{j+1}x)) + \mathcal{Q}_2(t,x),\end{aligned}$$

where $\mathcal{Q}_2(t,\cdot)$ is a function in $H^1_x(\mathbb{R})$ satisfying

$$\|\mathcal{Q}_2(t,x)\|_{H^1_x(\mathbb{R})} \lesssim v^4\big(\ln(1/v^2)+|t|v\big)e^{-2\sqrt{2}|t|v}.$$

Consequently, using Lemmas 12, 13, 17 and identity $\Lambda(\phi)=0$, we conclude from Taylor's expansion theorem that

$$\begin{aligned}
&\Lambda(\varphi_{k,v})(t,x)+\partial_t^2u(t,x)-\partial_x^2u(t,x)+U''(\varphi_{k,v}(t,x))(\phi(t,x)-\varphi_{k,v}(t,x))\\
&+\frac{\ddot{y}_1(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}H'_{0,1}(w_{k,v}(t,x))+\frac{\ddot{y}_2(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}H'_{0,1}(w_{k,v}(t,-x))\\
&-\frac{y_1(t)8\sqrt{2}e^{-\sqrt{2}d(t)}}{1-\frac{1}{4}\dot{d}(t)^2}H''_{0,1}(w_{k,v}(t,x))-\frac{y_2(t)8\sqrt{2}e^{-\sqrt{2}d(t)}}{1-\frac{1}{4}\dot{d}(t)^2}H''_{0,1}(w_{k,v}(t,-x))\\
&-\frac{\dot{y}_1(t)\dot{d}(t)}{1-\frac{1}{4}\dot{d}(t)^2}H''_{0,1}(w_{k,v}(x,t))-\frac{\dot{y}_2(t)\dot{d}(t)}{1-\frac{1}{4}\dot{d}(t)^2}H''_{0,1}(w_{k,v}(t,-x))\\
&-y_1(t)\frac{U''(H_{0,1}(w_{k,v}(t,x)))}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}H'_{0,1}(w_{k,v}(t,x))-y_2(t)\frac{U''(H_{0,1}(w_{k,v}(t,-x)))}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}H'_{0,1}(w_{k,v}(t,-x))\\
&=\mathcal{Q}(t,x), \qquad (33)
\end{aligned}$$

where $\mathcal{Q}(t,\cdot)$ is a function in $H^1_x(\mathbb{R})$ satisfying, for all $t\in\mathbb{R}$,

$$\begin{aligned}
\|\mathcal{Q}(t,x)\|_{H^1_x(\mathbb{R})}\lesssim{}&\|u(t)\|^2_{H^1_x}+\|u(t)\|^6_{H^1_x}+\max_{j\in\{1,2\}}|y_j(t)|^2+\max_{j\in\{1,2\}}|y_j(t)|^6\\
&+\big[\max_{j\in\{1,2\}}|\dot{y}_j(t)|+v\max_{j\in\{1,2\}}|y_j(t)|\big]v^3\big(\ln(1/v^2)+|t|v\big)e^{-2\sqrt{2}|t|v},
\end{aligned}$$

if $v>0$ is small enough.

Next, from (33) of Remark 18, we consider the terms

$$Y_1(t,x)=\big[U''(\varphi_{k,v}(t,x))-U''(H_{0,1}(w_{k,v}(t,x)))\big]\frac{y_1(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}H'_{0,1}(w_{k,v}(t,x)), \qquad (34)$$

$$Y_2(t,x)=\big[U''(\varphi_{k,v}(t,x))-U''(H_{0,1}(w_{k,v}(t,-x)))\big]\frac{y_2(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}H'_{0,1}(w_{k,v}(t,-x)). \qquad (35)$$

Now, we will estimate the expressions

$$\langle Y_1(t),H'_{0,1}(w_{k,v}(t,x))\rangle,\quad \langle Y_2(t),H'_{0,1}(w_{k,v}(t,-x))\rangle.$$

Lemma 19. *In notation of Theorem 8 and Lemma 17, the functions $Y_1(t)$ and $Y_2(t)$ satisfy*

$$\langle Y_1(t),H'_{0,1}(w_{k,v}(t,x))\rangle=4\sqrt{2}e^{-\sqrt{2}d(t)}y_1(t)+y_1(t)\operatorname{Res}_1(v,t),$$
$$\langle Y_2(t),H'_{0,1}(w_{k,v}(t,x))\rangle=-4\sqrt{2}e^{-\sqrt{2}d(t)}y_2(t)+y_2(t)\operatorname{Res}_2(v,t),$$

where, for any $j\in\{1,2\}$ and all $v\in(0,1)$, the function $\operatorname{Res}_j(v,t)$ is a Schwarz function on t satisfying for any $l\in\mathbb{N}\cup\{0\}$, if $0<v\ll1$, the estimate

$$\left|\frac{\partial^l}{\partial t^l}\operatorname{Res}_j(v,t)\right|\lesssim_l v^{l+4}\big[\ln(1/v^2)+|t|v\big]^{\eta_k}e^{-2\sqrt{2}|t|v} \qquad (36)$$

for a number $\eta_k\geq0$ depending only on $k\in\mathbb{N}_{\geq2}$.

Proof of Lemma 19. First, we observe that

$$\left|\frac{d^l}{dt^l}e^{-\sqrt{2}d(t)}\right| = \left|\frac{d^l}{dt^l}\frac{v^2}{8}\operatorname{sech}(\sqrt{2}vt)^2\right| \lesssim_l v^{2+l}e^{-2\sqrt{2}|t|v}.$$

Using Taylor's expansion theorem, Theorem 8 and Lemma 13, we deduce that

$$U''(\varphi_{k,v}(t,x)) = U''(H_{0,1}(w_{k,v}(t,x)) - H_{0,1}(w_{k,v}(t,-x))) + e^{-\sqrt{2}d(t)}U^{(3)}(H_{0,1}(w_{k,v}(t,x)) - H_{0,1}(w_{k,v}(t,-x)))[\mathcal{G}(w_{k,v}(t,x)) - \mathcal{G}(w_{k,v}(t,-x))] + \operatorname{res}_1(v,t,x),$$

where, if $0 < v \ll 1$, $\operatorname{res}_1(v,t,x)$ is a smooth function on the variables (t,x) which satisfies for some $\eta_k \in \mathbb{N}$ and any $s \geq 0$, $l \in \mathbb{N}\cup\{0\}$ the inequality

$$\left\|\frac{\partial^l}{\partial t^l}\operatorname{res}_1(v,t,x)\right\|_{H^s_x} \lesssim_{s,l} v^{4+l}\left[\ln(1/v^2) + |t|v\right]^{\eta_k}e^{-2\sqrt{2}|t|v}. \tag{37}$$

Therefore, using

$$\begin{aligned}U''(\varphi_{k,v}(t,x)) &- U''(H_{0,1}(w_{k,v}(t,x)))\\ &= U''(\varphi_{k,v}(t,x)) - U''\big(H_{0,1}(w_{k,v}(t,x)) - H_{0,1}(w_{k,v}(t,-x))\big)\\ &\quad + U''\big(H_{0,1}(w_{k,v}(t,x)) - H_{0,1}(w_{k,v}(t,-x))\big) - U''\big(H_{0,1}(w_{k,v}(t,x))\big),\end{aligned}$$

we obtain that

$$\begin{aligned}&Y_1(t,x)\sqrt{1-\tfrac{1}{4}\dot{d}(t)^2}\\ &= \big[U''\big(H_{0,1}(w_{k,v}(t,x)) - H_{0,1}(w_{k,v}(t,-x))\big) - U''(H_{0,1}(w_{k,v}(t,x)))\big]y_1(t)H'_{0,1}(w_{k,v}(x,t))\\ &\quad + y_1(t)e^{-\sqrt{2}d(t)}U^{(3)}\big(H_{0,1}(w_{k,v}(t,x)) - H_{0,1}(w_{k,v}(t,-x))\big)\mathcal{G}(w_{k,v}(t,x))H'_{0,1}(w_{k,v}(t,x))\\ &\quad - y_1(t)e^{-\sqrt{2}d(t)}U^{(3)}\big(H_{0,1}(w_{k,v}(t,x)) - H_{0,1}(w_{k,v}(t,-x))\big)\mathcal{G}(w_{k,v}(t,-x))H'_{0,1}(w_{k,v}(t,x))\\ &\quad + y_1(t)\operatorname{res}_1(v,t,x).\end{aligned} \tag{38}$$

By a similar reasoning, we obtain that

$$\begin{aligned}&Y_2(t,x)\sqrt{1-\tfrac{1}{4}\dot{d}(t)^2}\\ &= \big[U''\big(H_{0,1}(w_{k,v}(t,x)) - H_{0,1}(w_{k,v}(t,-x))\big) - U''(H_{0,1}(w_{k,v}(t,-x)))\big]y_2(t)H'_{0,1}(w_{k,v}(t,-x))\\ &\quad + y_2(t)e^{-\sqrt{2}d(t)}U^{(3)}\big(H_{0,1}(w_{k,v}(t,x)) - H_{0,1}(w_{k,v}(t,-x))\big)\mathcal{G}(w_{k,v}(t,x))H'_{0,1}(w_{k,v}(t,-x))\\ &\quad - y_2(t)e^{-\sqrt{2}d(t)}U^{(3)}\big(H_{0,1}(w_{k,v}(t,x)) - H_{0,1}(w_{k,v}(t,-x))\big)\mathcal{G}(w_{k,v}(t,-x))H'_{0,1}(w_{k,v}(t,-x))\\ &\quad + y_2(t)\operatorname{res}_2(v,t,x),\end{aligned} \tag{39}$$

where, if $0 < v \ll 1$, $\operatorname{res}_2(v,t,x)$ is a smooth function on t, x satisfying, for some constant $\eta_k \geq 0$, any $l \in \mathbb{N}\cup\{0\}$ and $s \geq 0$, the estimate

$$\left\|\frac{\partial^l}{\partial t^l}\operatorname{res}_2(v,t,x)\right\|_{H^s_x} \lesssim_{s,l} v^{4+l}\left[\ln(1/v^2) + |t|v\right]^{\eta_k}e^{-2\sqrt{2}|t|v}. \tag{40}$$

Next, from the fundamental theorem of calculus, we have for any $\zeta > 1$ that

$$\big[U''(H^{\zeta}_{0,1}(x)+H_{-1,0}(x)) - U''(H^{\zeta}_{0,1}(x))\big]\partial_x H^{\zeta}_{0,1}(x) \\ = U^{(3)}(H^{\zeta}_{0,1}(x))H_{-1,0}(x)\partial_x H^{\zeta}_{0,1}(x) + \int_0^1 U^{(4)}(H^{\zeta}_{0,1}+\theta H_{-1,0})(1-\theta)H_{-1,0}(x)^2\partial_x H^{\zeta}_{0,1}(x)\,d\theta,$$

from which with Lemma 11, estimates (3), (4) and

$$\left|\frac{d^l}{dx^l}[H_{-1,0}(x)+e^{-\sqrt{2}x}]\right| \lesssim_l \min(e^{-\sqrt{2}x}, e^{-3\sqrt{2}x}),$$

we obtain that

$$\big\langle[U''(H^{\zeta}_{0,1}(x)+H_{-1,0}(x)) - U''(H^{\zeta}_{0,1}(x))]\partial_x H^{\zeta}_{0,1}(x), \partial_x H^{\zeta}_{0,1}(x)\big\rangle \\ = -e^{-\sqrt{2}\zeta}\int_{\mathbb{R}} U^{(3)}(H_{0,1}(x))H'_{0,1}(x)^2 e^{-\sqrt{2}x}\,dx + \mathrm{res}_3(\zeta), \tag{41}$$

with $\mathrm{res}_3 \in C^{\infty}(\mathbb{R}_{\geq 1})$ satisfying, for all $l \in \mathbb{N}\cup\{0\}$ and $\zeta \geq 1$,

$$|\mathrm{res}_3^{(l)}(\zeta)| \lesssim_l \zeta e^{-2\sqrt{2}\zeta}.$$

Next, using $U \in C^{\infty}(\mathbb{R})$ and estimates (3), (4), we deduce for all $\zeta \geq 1$ and any $l \in \mathbb{N}\cup\{0\}$ that

$$\left|\frac{\partial^l}{\partial\zeta^l}\big[U^{(3)}(H^{\zeta}_{0,1}(x)+H_{-1,0}(x)) - U^{(3)}(H^{\zeta}_{0,1}(x))\big]\right| \lesssim_l |H_{-1,0}(x)|.$$

Therefore, since $\mathcal{G}$ defined in (17) is a Schwarz function, Lemma 11 implies that

$$\mathrm{int}(\zeta) = \big\langle\big[U^{(3)}(H^{\zeta}_{0,1}(x)+H_{-1,0}(x)) - U^{(3)}(H^{\zeta}_{0,1}(x))\big]\mathcal{G}(x-\zeta)\partial_x H^{\zeta}_{0,1}(x), \partial_x H^{\zeta}_{0,1}(x)\big\rangle$$

satisfies for all $\zeta \geq 1$ and any $l \in \mathbb{N}\cup\{0\}$ the inequality $|\mathrm{int}^{(l)}(\zeta)| \lesssim_l e^{-\sqrt{2}\zeta}$. Moreover, using the identity

$$U^{(3)}(\phi) = -48\phi + 120\phi^3, \tag{42}$$

we can deduce similarly that

$$\mathrm{int}_2(\zeta) = \big\langle U^{(3)}(H^{\zeta}_{0,1}(x)+H_{-1,0}(x))\mathcal{G}(-x)H'_{-1,0}(x), \partial_x H^{\zeta}_{0,1}(x)\big\rangle$$

satisfies $|\mathrm{int}_2^{(l)}(\zeta)| \lesssim_l e^{-\sqrt{2}\zeta}$ for any $l \in \mathbb{N}\cup\{0\}$ and $\zeta \geq 1$. As a consequence, we deduce that there exists a real function $\mathrm{int}_3 : \mathbb{R}_{\geq 1} \to \mathbb{R}$ satisfying, for any $l \in \mathbb{N}\cup\{0\}$,

$$|\mathrm{int}_3^{(l)}(\zeta)| \lesssim_l e^{-\sqrt{2}\zeta},$$

where the function int_3 satisfies the identity

$$\big\langle U^{(3)}(H^{\zeta}_{0,1}(x)+H_{-1,0}(x))\mathcal{G}(x-\zeta)\partial_x H^{\zeta}_{0,1}(x), \partial_x H^{\zeta}_{0,1}(x)\big\rangle \\ - \big\langle U^{(3)}(H^{\zeta}_{0,1}(x)+H_{-1,0}(x))\mathcal{G}(-x)H'_{0,1}(-x), \partial_x H^{\zeta}_{0,1}(x)\big\rangle \\ = \int_{\mathbb{R}} U^{(3)}(H_{0,1}(x))H'_{0,1}(x)^2\mathcal{G}(x)\,dx + \mathrm{int}_3(\zeta). \tag{43}$$

From Theorem 8, estimates (24) and the identity

$$e^{-\sqrt{2}d(t)} = \frac{v^2}{8}\operatorname{sech}(\sqrt{2}|t|v)^2,$$

it is not difficult to verify for any $l \in \mathbb{N}\cup\{0\}$ that if $0 < v \ll 1$, then

$$\frac{d^l}{dt^l}\exp\left(\frac{2\rho_{k,v}(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}\right) \lesssim_l v^{2+l}e^{-2\sqrt{2}|t|v}. \tag{44}$$

In conclusion, from estimates (38), (41), (43) and Lemma 32 of Appendix A, we obtain using identity

$$w_{k,v}(t,x) = \frac{x-\frac{1}{2}d(t)+c_{k,v}}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}},$$

and Theorem 8 that $Y_1(t)$ satisfies Lemma 19.

The proof that $Y_2(t)$ satisfies Lemma 19 is similar. First, from the fundamental theorem of calculus, we have for any real number $\zeta \geq 1$ the identity

$$\begin{aligned}
&\big[U''(H^{\zeta}_{0,1}(x)+H_{-1,0}(x))-U''(H_{-1,0}(x))\big]H'_{-1,0}(x)\\
&\qquad = [U''(H^{\zeta}_{0,1}(x))-2]H'_{-1,0}(x)+U^{(3)}(H^{\zeta}_{0,1}(x))H_{-1,0}(x)H'_{-1,0}(x)\\
&\qquad\qquad + \int_0^1 \big[U^{(4)}(H^{\zeta}_{0,1}(x)+\theta H_{-1,0}(x))-U^{(4)}(\theta H_{-1,0}(x))\big]H_{-1,0}(x)^2H'_{-1,0}(x)(1-\theta)\,d\theta.
\end{aligned}$$

Therefore, estimates (3), (4), identity (42) and Lemma 11 imply for any $\zeta \geq 1$ the estimate

$$\left|\frac{d^l}{d\zeta^l}\big\langle U''(H^{\zeta}_{0,1}(x)+H_{-1,0}(x))-U''(H_{-1,0}(x))-U''(H^{\zeta}_{0,1}(x))+2,\, H'_{-1,0}(x)\partial_x H^{\zeta}_{0,1}(x)\big\rangle\right| \lesssim_l \zeta e^{-2\sqrt{2}\zeta}. \tag{45}$$

Similarly, Lemma 11 and identity (42) imply that the functions

$$\begin{aligned}
\operatorname{int}_4(\zeta) &= \big\langle U^{(3)}(H^{\zeta}_{0,1}(x)+H_{-1,0}(x))\mathcal{G}(x-\zeta)H'_{-1,0}(x),\, \partial_x H^{\zeta}_{0,1}(x)\big\rangle,\\
\operatorname{int}_5(\zeta) &= \big\langle U^{(3)}(H^{\zeta}_{0,1}(x)+H_{-1,0}(x))\mathcal{G}(-x)H'_{-1,0}(x),\, \partial_x H^{\zeta}_{0,1}(x)\big\rangle
\end{aligned}$$

satisfy the estimates

$$|\operatorname{int}^{(l)}_4(\zeta)| + |\operatorname{int}^{(l)}_5(\zeta)| \lesssim_l e^{-\sqrt{2}\zeta} \tag{46}$$

for all $\zeta \geq 1$ and any $l \in \mathbb{N}\cup\{0\}$. Therefore, from estimates (44), (39), (45), (46), Lemma 11 and Theorem 8 imply that

$$\begin{aligned}
&\big\langle Y_2(t,x),\, H'_{0,1}(w_{k,v}(t,x))\big\rangle\\
&\qquad = y_2(t)\int_{\mathbb{R}}[U''(H_{0,1}(x))-2]H'_{0,1}(x)H'_{-1,0}\left(x+\frac{d(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}\right)dx + y_2(t)\operatorname{res}_6(v,t),
\end{aligned} \tag{47}$$

where $\operatorname{res}_6(v,t)$ is a real function, which satisfies for some constant $\eta_k \geq 0$, if $0 < v \ll 1$,

$$\left|\frac{\partial^l}{\partial t^l}\operatorname{res}_6(v,t)\right| \lesssim_l v^{4+l}\big[\ln(1/v^2)+|t|v\big]^{\eta_k}e^{-2\sqrt{2}|t|v}$$

for all $l \in \mathbb{N} \cup \{0\}$. So, from identity (144) of Appendix A, estimates (24),

$$\left| \frac{d^l}{dx^l}[H_{-1,0}(x) + e^{-\sqrt{2}x}] \right| \lesssim_l \min(e^{-\sqrt{2}x}, e^{-3\sqrt{2}x}),$$

and Lemma 11, we conclude the proof of Lemma 19 for $Y_2(t)$. □

Remark 20. If $v \ll 1$, using the formula $U''(\phi) = 2 - 24\phi^2 + 30\phi^4$, Lemmas 11, 12, the estimates (37), (38), (39) and (40) of the proof of Lemma 19 imply for any $s \geq 0$ that

$$\max_{j\in\{1,2\}} \|Y_j(t)\|_{H^s_x} \lesssim_s \max_{j\in\{1,2\}} |y_j(t)| v^2 e^{-2\sqrt{2}|t|v},$$

$$\max_{j\in\{1,2\}} \|\partial_t Y_j(t)\|_{H^s_x} \lesssim_s \max_{j\in\{1,2\}} |y_j(t)| v^3 e^{-2\sqrt{2}|t|v} + \max_{j\in\{1,2\}} |\dot{y}_j(t)| v^2 e^{-2\sqrt{2}|t|v},$$

$$\max_{j\in\{1,2\}} \|\partial_t^2 Y_j(t)\|_{H^s_x} \lesssim_s \max_{j\in\{1,2\}} |y_j(t)| v^4 e^{-2\sqrt{2}|t|v} + \max_{j\in\{1,2\}} |\dot{y}_j(t)| v^3 e^{-2\sqrt{2}|t|v} + \max_{j\in\{1,2\}} |y_j^{(2)}(t)| v^2 e^{-2\sqrt{2}|t|v}.$$

These estimates above don't depend on k, because from Theorem 8 we can verify for any $l \in \mathbb{N} \cup \{0\}$ the existence of $0 < \delta_{k,l} \ll 1$ such that if $0 < v < \delta_{k,l}$, then

$$\left\| \frac{\partial^l}{\partial t^l} c_k(v,t) \right\|_{L^\infty_t(\mathbb{R})} \lesssim_l v^{2+l} \ln(1/v),$$

which implies, for any $l \in \mathbb{N}$ and any $v \ll 1$,

$$\left\| \frac{\partial^l}{\partial t^l}\left[-\tfrac{1}{2}d(t) + c_k(v,t)\right] \right\|_{L^\infty_t(\mathbb{R})} \lesssim_l v^l, \quad \tfrac{1}{2}d(t) - v < \left|-\tfrac{1}{2}d(t) + c_k(v,t)\right|.$$

3. Energy estimate method

In this section, we will repeat the main argument of Section 4 of [Moutinho 2023] to construct a function $L : \mathbb{R} \to \mathbb{R}$, which is going to be used to estimate the energy norm of $(u(t), \partial_t u(t))$ during a large time interval.

First, we consider a smooth cut-off function $\chi : \mathbb{R} \to \mathbb{R}$ satisfying $0 \leq \chi \leq 1$ and

$$\chi(x) = \begin{cases} 1 & \text{if } x \leq \frac{49}{100}, \\ 0 & \text{if } x \geq \frac{1}{2}. \end{cases} \tag{48}$$

Next, using the notation of Theorem 8, we let

$$x_1(t) = -\tfrac{1}{2}d(t) + \sum_{j=2}^{k} r_j(v,t), \quad x_2(t) = \tfrac{1}{2}d(t) - \sum_{j=2}^{k} r_j(v,t). \tag{49}$$

Actually, Theorem 8 and estimates (24) imply that

$$\max_{j\in\{1,2\}} |\dot{x}_j(t)| \lesssim v, \quad \ln(1/v) \lesssim x_2(t) - x_1(t), \quad \max_{j\in\{1,2\}} |\ddot{x}_j(t)| \lesssim v^2 e^{-2\sqrt{2}|t|v}. \tag{50}$$

From now on, we define the function $\chi_1 : \mathbb{R}^2 \to \mathbb{R}$ by

$$\chi_1(t,x) = \chi\left(\frac{x - x_1(t)}{x_2(t) - x_1(t)}\right). \tag{51}$$

Clearly, using the identities

$$\frac{\partial}{\partial t}\chi_1(t,x) = \frac{-\dot{x}_1(t)}{x_2(t)-x_1(t)}\dot{\chi}\left(\frac{x-x_1(t)}{x_2(t)-x_1(t)}\right) - \frac{(\dot{x}_2(t)-\dot{x}_1(t))(x-x_1(t))}{(x_2(t)-x_1(t))^2}\dot{\chi}\left(\frac{x-x_1(t)}{x_2(t)-x_1(t)}\right),$$
$$\frac{\partial}{\partial x}\chi_1(t,x) = \frac{1}{x_2(t)-x_1(t)}\dot{\chi}\left(\frac{x-x_1(t)}{x_2(t)-x_1(t)}\right),$$

we obtain the estimates

$$\left\|\frac{\partial}{\partial t}\chi_1(t,x)\right\|_{L^\infty_x(\mathbb{R})} \lesssim \frac{v}{\ln(1/v)}, \quad \left\|\frac{\partial}{\partial x}\chi_1(t,x)\right\|_{L^\infty_x(\mathbb{R})} \lesssim \frac{1}{\ln(1/v)}. \tag{52}$$

Finally, using the notation (28) and the functions $Y_1(t)$, $Y_2(t)$ denoted respectively by (34) and (35), we define the function $A : \mathbb{R}^2 \to \mathbb{R}$ by

$$\begin{aligned} A(t,x) = {}& -\Lambda(\varphi_{k,v})(t,x)\frac{8\sqrt{2}e^{-\sqrt{2}d(t)}}{1-\frac{1}{4}\dot{d}(t)^2}\left[y_1(t)H''_{0,1}(w_{k,v}(t,x)) + y_2(t)H''_{0,1}(w_{k,v}(t,-x))\right] \\ & - Y_1(t,x) - Y_2(t,x) + \frac{\dot{y}_1(t)\dot{d}(t)}{1-\frac{1}{4}\dot{d}(t)^2}H''_{0,1}(w_{k,v}(t,x)) + \frac{\dot{y}_2(t)\dot{d}(t)}{1-\frac{1}{4}\dot{d}(t)^2}H''_{0,1}(w_{k,v}(t,-x)) \end{aligned} \tag{53}$$

for any $(t,x) \in \mathbb{R}^2$. Clearly, in the notation of Remark 18, we have the identity

$$\begin{aligned} &\partial_t^2 u(t,x) - \partial_x^2 u(t,x) + U''(H_{0,1}(w_{k,v}(t,x)) - H_{0,1}(w_{k,v}(t,-x)))u(t,x) \\ &\quad = -\frac{\ddot{y}_1(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}H'_{0,1}(w_{k,v}(t,x)) - \frac{\ddot{y}_2(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}H'_{0,1}(w_{k,v}(t,-x)) + A(t,x) + \mathcal{Q}(t,x) \\ &\qquad + \left[U''(H_{0,1}(w_{k,v}(t,x)) - H_{0,1}(w_{k,v}(t,-x))) - U''(\varphi_{k,v}(t,x))\right]u(t,x). \end{aligned} \tag{54}$$

Next, we consider

$$\begin{aligned} L(t) = {}& \int_{\mathbb{R}} \partial_t u(t,x)^2 + \partial_x u(t,x)^2 + U''(H_{0,1}(w_{k,v}(t,x)) - H_{0,1}(w_{k,v}(t,-x)))u(t,x)^2\,dx \\ & + 2\int_{\mathbb{R}} \partial_t u(t,x)\partial_x u(t,x)[\dot{x}_1(t)\chi_1(t,x) + \dot{x}_2(t)(1-\chi_1(t,x))]\,dx - 2\int_{\mathbb{R}} u(t,x)A(t,x)\,dx. \end{aligned} \tag{55}$$

From now on, we use the notation $\vec{u}(t) = (u(t), \partial_t u(t)) \in H^1_x(\mathbb{R}) \times L^2_x(\mathbb{R})$. The main objective of Section 3 is to demonstrate the following theorem.

Theorem 21. *There exist constants $K, c > 0$ and, for any $k \in \mathbb{N}_{\geq 3}$, there exists $0 < \delta(k) < 1$ such that if $0 < v \leq \delta(k)$, then the function $L(t)$ given in (55) satisfies, while the condition*

$$\max_{j\in\{1,2\}} v^2|y_j(t)| + v|\dot{y}_j(t)| < v^{2k}\ln(1/v)^{n_k} \tag{56}$$

is true, the estimates

$$\begin{aligned} & c\|\vec{u}(t)\|^2_{H^1_x\times L^2_x} \leq L(t) + C(k)v^{4k}\ln(1/v)^{2n_k}, \\ & |\dot{L}(t)| \leq K\left[\frac{v}{\ln(1/v)}\|\vec{u}(t)\|^2_{H^1_x\times L^2_x} + C(k)\|\vec{u}(t)\|_{H^1_x\times L^2_x}v^{2k+1}\ln(1/v)^{n_k}\right] \\ & \qquad + v\max_{j\in\{1,2\}}|\ddot{y}_j(t)|\|\vec{u}(t)\|_{H^1_x\times L^2_x} + K\max_{j\in\{3.7\}}\|\vec{u}(t)\|^j_{H^1_x\times L^2_x}, \end{aligned}$$

where $C(k)>0$ is a constant depending only on k and n_k is the number defined in the statement of Theorem 8.

Proof of Theorem 21. To simplify the proof of this theorem, we describe briefly the organization of our arguments. First, we let

$$L(t)=L_1(t)+L_2(t)+L_3(t)$$

be such that

$$L_1(t)=\int_{\mathbb{R}}\partial_t u(t,x)^2+\partial_x u(t,x)^2+\ddot{U}(H_{0,1}(w_{k,v}(t,x)-H_{0,1}(w_{k,v}(t,-x))))u(t,x)^2\,dx, \tag{L1}$$

$$L_2(t)=2\int_{\mathbb{R}}\partial_t u(t,x)\partial_x u(t,x)[\dot{x}_1(t)\chi_1(t,x)+\dot{x}_2(t)(1-\chi_1(t,x))]\,dx, \tag{L2}$$

$$L_3(t)=-2\int_{\mathbb{R}}u(t,x)A(t,x)\,dx. \tag{L3}$$

Next, instead of estimating the size of $|\dot{L}(t)|$, we will estimate $\dot{L}_j(t)$ for each $j\in\{1,2,3\}$. Then, using these estimates, we can evaluate with high precision

$$|\dot{L}_1(t)+\dot{L}_2(t)+\dot{L}_3(t)|,$$

and obtain the second inequality of Theorem 21. The proof of the first inequality of Theorem 21 is short and it will be done later.

From identity (23), Remark 20 and (53) satisfied by $A(t,x)$, we deduce from the triangle inequality that

$$\|A(t,x)\|_{H^1_x(\mathbb{R})}\lesssim\|\Lambda(\varphi_{k,v})(t,x)\|_{H^1_x(\mathbb{R})}+v^2e^{-2\sqrt{2}v|t|}\max_{j\in\{1,2\}}|y_j(t)|+v\max_{j\in\{1,2\}}|\dot{y}_j(t)|.$$

Therefore, from Theorems 7 and 8, we obtain the existence of a value $C(k)>0$ depending only on k such that if $v\ll 1$, then

$$\|A(t,x)\|_{H^1(\mathbb{R})}\lesssim C(k)v^{2k}\big(\ln(1/v)+|t|v\big)^{n_k}e^{-2\sqrt{2}|t|v}+v^2e^{-2\sqrt{2}|t|v}\max_{j\in\{1,2\}}|y_j(t)|+v\max_{j\in\{1,2\}}|\dot{y}_j(t)|. \tag{57}$$

In conclusion, we obtain from (L3) and the Cauchy–Schwarz inequality the existence of a value $C(k)>0$ depending only on k satisfying

$$|L_3(t)|\lesssim\|u(t)\|_{L^2_x}\Big[C(k)v^{2k}\big(\ln(1/v)+|t|v\big)^{n_k}e^{-2\sqrt{2}|t|v}+v^2e^{-2\sqrt{2}|t|v}\max_{j\in\{1,2\}}|y_j(t)|+\max_{j\in\{1,2\}}|\dot{y}_j(t)|v\Big]. \tag{58}$$

Next, Lemmas 12, 13, Remark 20 and identity (53) satisfied by $A(t,x)$ imply the inequality

$$\|\partial_t A(t,x)\|_{H^1_x(\mathbb{R})}\lesssim\left\|\frac{\partial}{\partial t}[\Lambda(\phi_k)(v,t,x)]\right\|_{H^1_x(\mathbb{R})}+\max_{j\in\{1,2\}}|y_j(t)|v^3e^{-2\sqrt{2}|t|v}+\max_{j\in\{1,2\}}|\dot{y}_j(t)|v^2+\max_{j\in\{1,2\}}|\ddot{y}_j(t)|v,$$

from which with Theorem 8 we conclude the existence of a new value $C(k)$ depending only on k satisfying

$$\|\partial_t A(t,x)\|_{H^1_x}$$
$$\lesssim C(k)v^{2k+1}\big(\ln(1/v)+|t|v\big)^{n_k}e^{-2\sqrt{2}|t|v}+\max_{j\in\{1,2\}}|y_j(t)|v^3e^{-2\sqrt{2}|t|v}+\max_{j\in\{1,2\}}|\dot{y}_j(t)|v^2+\max_{j\in\{1,2\}}|\ddot{y}_j(t)|v. \tag{59}$$

In conclusion, the identity (L3), estimate (59) and Cauchy–Schwarz inequality imply the existence of a new value $C(k) > 0$ depending only on k, which satisfies

$$\begin{aligned}\Big|\dot{L}_3(t)+2\int_{\mathbb{R}}\partial_t u(t,x)A(t,x)\,dx\Big| &\lesssim \|u(t,x)\|_{L^2_x}\big[C(k)v^{2k+1}\big(\ln(1/v)+|t|v\big)^{n_k}e^{-2\sqrt{2}|t|v}+\max_{j\in\{1,2\}}|y_j(t)|v^3e^{-2\sqrt{2}|t|v}\big]\\ &\quad+\|u(t,x)\|_{L^2_x}\big[\max_{j\in\{1,2\}}|\dot{y}_j(t)|v^2+\max_{j\in\{1,2\}}|\ddot{y}_j(t)|v\big]. \qquad (60)\end{aligned}$$

Next, Theorem 8 implies that if $v \ll 1$, then

$$\begin{aligned}\dot{L}_1(t) &= 2\int_{\mathbb{R}}\partial_t u(t,x)\big[\partial_t^2 u(t,x)-\partial_x^2 u(t,x)+U''\big(H_{0,1}(w_{k,v}(t,x))-H_{0,1}(w_{k,v}(t,-x))\big)u(t,x)\big]\,dx\\ &\quad-\frac{\dot{d}(t)}{2\big(1-\frac{1}{4}\dot{d}(t)^2\big)^{1/2}}\int_{\mathbb{R}}U^{(3)}\big(H_{0,1}(w_{k,v}(t,x))-H_{0,1}(w_{k,v}(t,-x))\big)H'_{0,1}(w_{k,v}(t,x))u(t,x)^2\,dx\\ &\quad+\frac{\dot{d}(t)}{2\big(1-\frac{1}{4}\dot{d}(t)^2\big)^{1/2}}\int_{\mathbb{R}}U^{(3)}\big(H_{0,1}(w_{k,v}(t,x))-H_{0,1}(w_{k,v}(t,-x))\big)H'_{0,1}(w_{k,v}(t,-x))u(t,x)^2\,dx\\ &\quad+O\left(\frac{v}{\ln(1/v)}\|(u(t),\partial_t u(t))\|^2_{H^1_x,L^2_x}\right) \qquad (61)\end{aligned}$$

Thus, from Lemma 17, identity (53), Remark 18, hypothesis (56), estimates (60), (61) and orthogonality conditions (29), we obtain the existence of a value $C(k) > 0$ depending only on k such that if $v \ll 1$, then

$$\begin{aligned}\dot{L}_1(t)+\dot{L}_3(t) &= 2\int_{\mathbb{R}}\partial_t u(t,x)\big[U''\big(H_{0,1}(w_{k,v}(t,x))-H_{0,1}(w_{k,v}(t,-x))\big)-U''(\varphi_{k,v}(t,x))\big]u(t,x)\,dx\\ &\quad+\frac{\dot{d}(t)}{2\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}\int_{\mathbb{R}}U^{(3)}\big(H_{0,1}(w_{k,v}(t,x))-H_{0,1}(w_{k,v}(t,-x))\big)H'_{0,1}(w_{k,v}(t,-x))u(t,x)^2\,dx\\ &\quad-\frac{\dot{d}(t)}{2\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}\int_{\mathbb{R}}U^{(3)}\big(H_{0,1}(w_{k,v}(t,x))-H_{0,1}(w_{k,v}(t,-x))\big)H'_{0,1}(w_{k,v}(t,x))u(t,x)^2\,dx\\ &\quad+O\big(v\max_{j\in\{1,2\}}|\ddot{y}_j(t)|\|u(t)\|_{H^1_x(\mathbb{R})}+\max_{j\in\{3,7\}}\|\vec{u}(t)\|^j_{H^1_x\times L^2_x}+\|\vec{u}(t)\|_{H^1_x\times L^2_x}\max_{j\in\{1,2\}}|y_j(t)|^2\big)\\ &\quad+O\left(\|\vec{u}(t)\|_{H^1_x\times L^2_x}\big[\max_{j\in\{1,2\}}|\dot{y}_j(t)|v^2+|y_j(t)|v^3e^{-2\sqrt{2}|t|v}\big]+\|\vec{u}(t)\|^2_{H^1_x\times L^2_x}\frac{v}{\ln(1/v)}\right)\\ &\quad+O\big(C(k)\|\vec{u}(t)\|_{H^1_x\times L^2_x}v^{2k+1}\ln(1/v)^{n_k}\big). \qquad (62)\end{aligned}$$

Moreover, using estimates (24), Lemma 13 and identity $U(\phi)=\phi^2(1-\phi^2)^2$, we obtain from Theorem 8 that if $0 < v \ll 1$ and $s \geq 0$, then

$$\big\|\big[U''\big(H_{0,1}(w_{k,v}(t,x))-H_{0,1}(w_{k,v}(t,-x))\big)-U''(\varphi_{k,v}(t,x))\big]\big\|_{H^s_x}\lesssim_{s,k} v^2e^{-2\sqrt{2}|t|v}.$$

Therefore, we deduce using the Cauchy–Schwarz inequality that

$$\begin{aligned}\left|2\int_{\mathbb{R}}\partial_t u(t,x)\big[U''\big(H_{0,1}(w_{k,v}(t,x))-H_{0,1}(w_{k,v}(t,-x))\big)-U''(\varphi_{k,v}(t,x))\big]u(t,x)\,dx\right|\\ \lesssim \big\|\big[U''(H_{0,1}(w_{k,v}(t,x))-H_{0,1}(w_{k,v}(t,x)))-U''(\varphi_{k,v}(t,x))\big]u(t,x)\big\|_{L^2_x}\|\partial_t u(t,x)\|_{L^2_x}\\ \lesssim \big\|\big[U''(H_{0,1}(w_{k,v}(t,x))-H_{0,1}(w_{k,v}(t,-x)))-U''(\varphi_{k,v}(t,x))\big]\big\|_{H^1_x(\mathbb{R})}\|\vec{u}(t)\|^2_{H^1_x\times L^2_x}\\ \lesssim v^2\|\vec{u}(t)\|^2_{H^1_x\times L^2_x}.\end{aligned}$$

In conclusion,

$$\begin{aligned}&\dot{L}_1(t)+\dot{L}_3(t)\\ &=\frac{\dot{d}(t)}{2\big(1-\frac{1}{4}\dot{d}(t)^2\big)^{1/2}}\int_{\mathbb{R}}U^{(3)}\big(H_{0,1}(w_{k,v}(t,x))-H_{0,1}(w_{k,v}(t,-x))\big)H'_{0,1}(w_{k,v}(t,-x))u(t,x)^2\,dx\\ &\quad-\frac{\dot{d}(t)}{2\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}\int_{\mathbb{R}}U^{(3)}\big(H_{0,1}(w_{k,v}(t,x))-H_{0,1}(w_{k,v}(t,-x))\big)H'_{0,1}(w_{k,v}(t,x))u(t,x)^2\,dx\\ &\quad+O\big(v\max_{j\in\{1,2\}}|\ddot{y}_j(t)|\|u(t)\|_{H^1_x(\mathbb{R})}+\max_{j\in\{3,7\}}\|\vec{u}(t)\|^j_{H^1_x\times L^2_x}+\|\vec{u}(t)\|_{H^1_x\times L^2_x}\max_{j\in\{1,2\}}|y_j(t)|^2\big)\\ &\quad+O\bigg(\|\vec{u}(t)\|_{H^1_x\times L^2_x}\big[\max_{j\in\{1,2\}}|\dot{y}_j(t)|v^2+|y_j(t)|v^3e^{-2\sqrt{2}|t|v}\big]+\|\vec{u}(t)\|^2_{H^1_x\times L^2_x}\frac{v}{\ln(1/v)}\bigg)\\ &\quad+O\big(C(k)\|\vec{u}(t)\|_{H^1_x\times L^2_x}v^{2k+1}\ln(1/v)^{n_k}\big).\end{aligned}\tag{63}$$

Based on the arguments of [Jendrej et al. 2022; Moutinho 2023], we will estimate the derivative of $L_2(t)$, for more accurate information see the third step of Lemma 4.2 in [Jendrej et al. 2022] or Theorem 4.1 of [Moutinho 2023]. Because of an argument of analogy, we only need to estimate the time derivative of

$$L_{2,1}(t)=2\dot{x}_1(t)\int_{\mathbb{R}}\chi_1(t,x)\partial_t u(t,x)\partial_x u(t,x)\,dx$$

to evaluate with high precision the derivative of $L_2(t)$. From the estimates (52), we can verify first that if $v\ll 1$, then

$$\dot{L}_{2,1}(t)=2\dot{x}_1(t)\int_{\mathbb{R}}\chi_1(t,x)\partial_t^2u(t,x)\partial_x u(t,x)\,dx+2\dot{x}_1(t)\int_{\mathbb{R}}\chi_1(t,x)\partial_t u(t,x)\partial^2_{x,t}u(t,x)\,dx\\ +O\bigg(\frac{v}{\ln(1/v)}\|\vec{u}(t)\|^2_{H^1_x\times L^2_x}\bigg),$$

from which we deduce, using integration by parts and estimates (50), (52), that

$$\begin{aligned}\dot{L}_{2,1}(t)&=2\dot{x}_1(t)\int_{\mathbb{R}}\chi_1(t,x)\partial_t^2u(t,x)\partial_x u(t,x)\,dx+O\bigg(\frac{v}{\ln(1/v)}\|\vec{u}(t)\|^2_{H^1_x\times L^2_x}\bigg)\\ &=2\dot{x}_1(t)\int_{\mathbb{R}}\chi_1(t,x)[\partial_t^2u(t,x)-\partial_x^2u(t,x)]\partial_x u(t,x)\,dx\\ &\quad+2\dot{x}_1(t)\int_{\mathbb{R}}\chi_1(t,x)U''\big(H_{0,1}(w_{k,v}(t,x))-H_{0,1}(w_{k,v}(t,-x))\big)u(t,x)\partial_x u(t,x)\,dx\end{aligned}$$

$$
\begin{aligned}
&+2\dot{x}_1(t)\int_{\mathbb{R}}\chi_1(t,x)\partial_x^2 u(t,x)\partial_x u(t,x)\,dx\\
&\quad-2\dot{x}_1(t)\int_{\mathbb{R}}\chi_1(t,x)U''\big(H_{0,1}(w_{k,v}(t,x))-H_{0,1}(w_{k,v}(t,-x))\big)u(t,x)\partial_x u(t,x)\,dx\\
&\qquad+O\bigg(\frac{v}{\ln(1/v)}\|\vec{u}(t)\|^2_{H^1_x\times L^2_x}\bigg),
\end{aligned}
$$

and, after using integration by parts again, we deduce from (52) that

$$
\begin{aligned}
\dot{L}_{2,1}(t)&=2\dot{x}_1(t)\int_{\mathbb{R}}\chi_1(t,x)[\partial_t^2 u(t)-\partial_x^2 u(t)]\partial_x u(t)\,dx\\
&\quad+2\dot{x}_1(t)\int_{\mathbb{R}}\chi_1(t)U''(H_{0,1}(w_{k,v}(t,x))-H_{0,1}(w_{k,v}(t,-x)))u(t)\partial_x u(t)\,dx\\
&\quad+\frac{\dot{x}_1(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}\int_{\mathbb{R}}\chi_1(t)U^{(3)}(H_{0,1}(w_{k,v}(t,x))-H_{0,1}(w_{k,v}(t,-x)))H'_{0,1}(w_{k,v}(t,x))u(t)^2\,dx\\
&\quad+\frac{\dot{x}_1(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}\int_{\mathbb{R}}\chi_1(t)U^{(3)}(H_{0,1}(w_{k,v}(t,x))-H_{0,1}(w_{k,v}(t,-x)))H'_{0,1}(w_{k,v}(t,-x))u(t)^2\,dx\\
&\quad+O\bigg(\frac{v}{\ln(1/v)}\|\vec{u}(t)\|^2_{H^1_x(\mathbb{R})\times L^2_x(\mathbb{R})}\bigg).
\end{aligned}
$$

Next, using estimates (3) satisfied by $H_{0,1}$, definition of $\chi_1(t,x)$, Theorem 8 and identity (27), we deduce, for $v\ll 1$, the inequality

$$
|\chi_1(t,x)H'_{0,1}(w_{k,v}(t,x))|+|(1-\chi_1(t,x))H'_{0,1}(w_{k,v}(t,-x))|\lesssim e^{-\sqrt{2}\frac{49d(t)}{100}}\lesssim v^{\frac{98}{100}}\ll\frac{1}{\ln(1/v)},
$$

from which we conclude that

$$
\begin{aligned}
\dot{L}_{2,1}(t)&=2\dot{x}_1(t)\int_{\mathbb{R}}\chi_1(t)[\partial_t^2 u(t,x)-\partial_x^2 u(t,x)]\partial_x u(t,x)\,dx\\
&\quad+2\dot{x}_1(t)\int_{\mathbb{R}}\chi_1(t)U''\big(H_{0,1}(w_{k,v}(t,x))-H_{0,1}(w_{k,v}(t,-x))\big)u(t,x)\partial_x u(t,x)\,dx\\
&\quad+\frac{\dot{x}_1(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}\int_{\mathbb{R}}U^{(3)}\big(H_{0,1}(w_{k,v}(t,x))-H_{0,1}(w_{k,v}(t,-x))\big)H'_{0,1}(w_{k,v}(t,-x))u(t,x)^2\,dx\\
&\quad+O\bigg(\frac{v}{\ln(1/v)}\|\vec{u}(t)\|^2_{H^1_x\times L^2_x}\bigg).
\end{aligned}
$$

Furthermore, from Remark 18, estimate (57) of $A(t,x)$ and identity (54) satisfied by $u(t,x)$, we conclude the existence of a value $C(k)>0$ depending only on k and satisfying, for any positive number $v\ll 1$,

$$
\begin{aligned}
\dot{L}_{2,1}(t)&=\frac{\dot{x}_1(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}\int_{\mathbb{R}}U^{(3)}\big(H_{0,1}(w_{k,v}(t,x))-H_{0,1}(w_{k,v}(t,-x))\big)H'_{0,1}(w_{k,v}(t,-x))u(t,x)^2\,dx\\
&\quad+O\Big(\|\vec{u}(t)\|_{H^1_x\times L^2_x}\Big[v\max_{j\in\{1,2\}}|\ddot{y}_j(t)|+C(k)v^{2k+1}\ln(1/v)^{n_k}+v\max_{j\in\{2,6\}}\|\vec{u}(t)\|^j_{H^1_x\times L^2_x}\Big]\Big)\\
&\qquad+O\bigg(\|\vec{u}(t)\|_{H^1_x\times L^2_x}[v^3e^{-2\sqrt{2}v|t|}\max_{j\in\{1,2\}}|y_j(t)|+v^2|\dot{y}_j(t)|]+\frac{v}{\ln(1/v)}\|\vec{u}(t)\|^2_{H^1_x\times L^2_x}\bigg).
\end{aligned}
$$

Therefore, using an argument of analogy, we obtain, for any positive number $v \ll 1$, that

$$\begin{aligned}\dot{L}_2(t) &= \frac{\dot{x}_2(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}\int_{\mathbb{R}} U^{(3)}\big(H_{0,1}(w_{k,v}(t,x)) - H_{0,1}(w_{k,v}(t,-x))\big)H_{0,1}'(w_{k,v}(t,x))u(t,x)^2\,dx \\ &\quad + \frac{\dot{x}_1(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}\int_{\mathbb{R}} U^{(3)}\big(H_{0,1}(w_{k,v}(t,x)) - H_{0,1}(w_{k,v}(t,-x))\big)H_{0,1}'(w_{k,v}(t,-x))u(t,x)^2\,dx \\ &\quad + O\big(\|\vec{u}(t)\|_{H^1_x\times L^2_x}\big[v\max_{j\in\{1,2\}}|\ddot{y}_j(t)| + C(k)v^{2k+1}\ln(1/v)^{n_k}\big] + v\max_{j\in\{3,7\}}\|\vec{u}(t)\|^j_{H^1_x\times L^2_x}\big) \\ &\quad + O\bigg(\|\vec{u}(t)\|_{H^1_x\times L^2_x}[v^3e^{-2\sqrt{2}v|t|}\max_{j\in\{1,2\}}|y_j(t)| + v^2|\dot{y}_j(t)|] + \frac{v}{\ln(1/v)}\|\vec{u}(t)\|^2_{H^1_x\times L^2_x}\bigg), \end{aligned} \tag{64}$$

where $C(k)>0$ is a parameter depending only on k. Moreover, using (49) and Theorem 8, we deduce from estimate (64) that

$$\begin{aligned}\dot{L}_2(t) &= \frac{\dot{d}(t)}{\sqrt{4-\dot{d}(t)^2}}\int_{\mathbb{R}} U^{(3)}\big(H_{0,1}(w_{k,v}(t,x)) - H_{0,1}(w_{k,v}(t,-x))\big)H_{0,1}'(w_{k,v}(t,x))u(t,x)^2\,dx \\ &\quad - \frac{\dot{d}(t)}{\sqrt{4-\dot{d}(t)^2}}\int_{\mathbb{R}} U^{(3)}\big(H_{0,1}(w_{k,v}(t,x)) - H_{0,1}(w_{k,v}(t,-x))\big)H_{0,1}'(w_{k,v}(t,-x))u(t,x)^2\,dx \\ &\quad + O\big(\|\vec{u}(t)\|_{H^1_x\times L^2_x}\big[v\max_{j\in\{1,2\}}|\ddot{y}_j(t)| + C(k)v^{2k+1}\ln(1/v)^{n_k}\big] + v\max_{j\in\{3,7\}}\|\vec{u}(t)\|^j_{H^1_x\times L^2_x}\big) \\ &\quad + O\bigg(\|\vec{u}(t)\|_{H^1_x\times L^2_x}[v^3e^{-2\sqrt{2}v|t|}\max_{j\in\{1,2\}}|y_j(t)| + v^2|\dot{y}_j(t)|] + \frac{v}{\ln(1/v)}\|\vec{u}(t)\|^2_{H^1_x\times L^2_x}\bigg). \end{aligned} \tag{65}$$

Finally, the estimates (65) and (62) imply, for any $k\in\mathbb{N}_{\geq 3}$, the existence of a parameter $C(k)>0$, depending only on k, which satisfies for any positive number $v\ll 1$ the estimate

$$\begin{aligned}|\dot{L}(t)| &= O\big(v\max_{j\in\{1,2\}}|\ddot{y}_j(t)|\|\vec{u}(t)\|_{H^1_x\times L^2_x} + \max_{j\in\{3,7\}}\|\vec{u}(t)\|^j_{H^1_x\times L^2_x}\big) + O\big(\|\vec{u}(t)\|_{H^1_x\times L^2_x}\max_{j\in\{1,2\}}|y_j(t)|^2\big) \\ &\qquad + O\big(\|\vec{u}(t)\|_{H^1_x\times L^2_x}\big[\max_{j\in\{1,2\}}|\dot{y}_j(t)|v^2 + |y_j(t)|v^3e^{-2\sqrt{2}|t|v}\big]\big) \\ &\qquad + O\bigg(\|\vec{u}(t)\|^2_{H^1_x\times L^2_x}\frac{v}{\ln(1/v^2)} + C(k)\|\vec{u}(t)\|_{H^1_x\times L^2_x}v^{2k+1}\ln(1/v)^{n_k}\bigg), \end{aligned} \tag{66}$$

from which we obtain the existence of a new constant $C(k)>0$ satisfying the second inequality of Theorem 21 if the condition (56) is true and $v\ll 1$.

Now, it remains to prove the first inequality of Theorem 21. Using change of variables and Lemma 14, it is not difficult to verify that there exists $K>0$ such that if $v\ll 1$, then

$$L_1(t) \geq K\|(u(t),\partial_t u(t))\|^2_{H^1_x\times L^2_x}.$$

Next, from the definition of $L_2(t)$ and estimates (50), we obtain that if $v\ll 1$, then

$$|L_2(t)| \ll v^{3/4}\|(u(t),\partial_t u(t))\|^2_{H^1_x\times L^2_x},$$

and while condition (56) is true, we deduce from Theorem 8 and estimate (57) the following inequality:

$$|L_3(t)| \lesssim_k \|(u(t),\partial_t u(t))\|_{H^1_x\times L^2_x}v^{2k}\ln(1/v)^{n_k}.$$

So, using Young's inequality, we can find a parameter $C_1(k)>0$ large enough depending only on k such that

$$|L_3(t)| \leq \tfrac{1}{2}K\|(u(t),\partial_t u(t))\|^2_{H^1_x\times L^2_x} + C_1(k)v^{4k}\ln(1/v)^{2n_k}.$$

In conclusion, all the estimates above imply the first inequality of Theorem 21 if $0<v\ll 1$ and condition (56) is true. □

4. Proof of Theorem 15

From the information of Theorem 21 in the last section, we are ready to start the demonstration of Theorem 15.

Proof of Theorem 15. First, for any $(t,x)\in\mathbb{R}^2$, Lemma 17 implies that $\phi(t,x)$ has the representation

$$\phi(t,x) = \varphi_{k,v}(t,x) + \frac{y_1(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}H'_{0,1}(w_{k,v}(t,x)) + \frac{y_2(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}H'_{0,1}(w_{k,v}(t,-x)) + u(t,x),$$

such that the function $u(t,x)$ satisfies the orthogonality conditions (29) and y_1, y_2 are functions in $C^2(\mathbb{R})$.

Step 1 (ordinary differential system of $y_1(t)$, $y_2(t)$). From Remarks 9, 18 and the definition of $A(t,x)$ in (53), we have that $u(t,x)$ is a solution of a partial differential equation of the form

$$\begin{aligned}\partial_t^2 u(t,x) - \partial_x^2 u(t,x) + U''\big(H_{0,1}(w_{k,v}(t,x)) - H_{0,1}(w_{k,v}(t,-x))\big)u(t,x)\\ = -\frac{\ddot{y}_1(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}H'_{0,1}(w_{k,v}(t,x)) - \frac{\ddot{y}_2(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}H'_{0,1}(w_{k,v}(t,-x)) + A(t,x) + \mathcal{P}_1(v,t,x),\end{aligned} \tag{67}$$

where $\mathcal{P}_1(v,t,x)$ satisfies for any $0<v\ll 1$ and any $t\in\mathbb{R}$ the inequality

$$\begin{aligned}\|\mathcal{P}_1(v,t,x)\|_{H^1_x} \lesssim \|u(t)\|^2_{H^1_x} + \max_{j\in\{1,2\}}|y_j(t)|^2 + \max_{j\in\{1,2\}}|\dot{y}_j(t)|v^3\big(\ln(1/v^2)+|t|v\big)e^{-2\sqrt{2}|t|v}\\ + \|u(t)\|^6_{H^1_x} + \max_{j\in\{1,2\}}|y_j(t)|^6 + \max_{j\in\{1,2\}}|y_j(t)|v^4\big(\ln(1/v^2)+|t|v\big)e^{-2\sqrt{2}|t|v}.\end{aligned}$$

With the objective of simplifying our computations, we let

$$\begin{aligned}NOL(t) = \|u(t)\|^2_{H^1} + \max_{j\in\{1,2\}}|y_j(t)|^2 + v^{2(k+1)}\big(|t|v+\ln(1/v^2)\big)^{n_k+1}e^{-2\sqrt{2}|t|v}\\ + \|u(t)\|^6_{H^1_x} + \max_{j\in\{1,2\}}|y_j(t)|^6 + \max_{j\in\{1,2\}}|\dot{y}_j(t)|v^3\big(\ln(1/v^2)+|t|v\big)e^{-2\sqrt{2}|t|v}\\ + \max_{j\in\{1,2\}}|y_j(t)|v^4\big(\ln(1/v^2)+|t|v\big)^{\max\{1,\eta_k\}}e^{-2\sqrt{2}|t|v},\end{aligned} \tag{68}$$

where η_k is the number denoted in Lemma 19. Also, from Theorem 8, Lemma 19 and identity (53), we deduce that

$$\begin{bmatrix}\langle A(t,x), H'_{0,1}(w_{k,v}(t,x))\rangle\\ \langle A(t,x), H'_{0,1}(w_{k,v}(t,-x))\rangle\end{bmatrix} = e^{-\sqrt{2}d(t)}\begin{bmatrix}-4\sqrt{2} & 4\sqrt{2}\\ 4\sqrt{2} & -4\sqrt{2}\end{bmatrix}\begin{bmatrix}y_1(t)\\ y_2(t)\end{bmatrix} + \text{Rest}(t), \tag{69}$$

where, if $v \ll 1$, the real function $\mathrm{Rest}(t)$ satisfies, for any $t \in \mathbb{R}$,

$$e^{2\sqrt{2}|t|v}|\mathrm{Rest}(t)| \lesssim_k v^{2(k+1)}\big(|t|v+\ln(1/v^2)\big)^{n_k+1} + \max_{j\in\{1,2\}} |y_j(t)|v^4\big(|t|v+\ln(1/v^2)\big)^{\max\{1,\eta_k\}} + \max_{j\in\{1,2\}} |\dot{y}_j(t)|v^3\big(|t|v+\ln(1/v^2)\big). \quad (70)$$

From the orthogonality conditions (29), Theorem 8 and Lemma 12, we obtain the estimate

$$\big\langle \partial_t^2 u(t,x),\, H_{0,1}'(w_{k,v}(t,x))\big\rangle = \frac{\dot{d}(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}\big\langle \partial_t u(t,x),\, H_{0,1}''(w_{k,v}(t,x))\big\rangle_{L_x^2} + O(\|\vec{u}(t)\|_{H_x^1\times L_x^2} v^2). \quad (71)$$

Also, using integration by parts, identity

$$-\frac{d^3}{dx^3}H_{0,1}(x) + U''(H_{0,1}(x))H_{0,1}'(x) = 0,$$

Lemma 11 and the Cauchy–Schwarz inequality, we deduce that if $0 < v \ll 1$, then

$$\begin{aligned}&\big\langle -\partial_x^2 u(t) + U''\big(H_{0,1}(w_{k,v}(t,x)) - H_{0,1}(w_{k,v}(t,-x))\big)u(t),\, H_{0,1}'(w_{k,v}(t,x))\big\rangle \\ &\quad = \big\langle u(t), \big[U''\big(H_{0,1}(w_{k,v}(t,x)) - H_{0,1}(w_{k,v}(t,-x))\big) - U''(H_{0,1}(w_{k,v}(t,x)))\big]H_{0,1}'(w_{k,v}(t,x))\big\rangle \\ &\qquad + O(v^2\|\vec{u}(t)\|_{H_x^1\times L_x^2}) \\ &\quad = O(v^2\|\vec{u}(t)\|_{H_x^1\times L_x^2}). \end{aligned} \quad (72)$$

From now on, we denote any continuous function $f(t)$ as $O_k(NOL(t))$, if and only if f satisfies the estimate

$$|f(t)| \lesssim_k NOL(t).$$

In conclusion, applying the scalar product of the (67) with $H_{0,1}'(w_{k,v}(t,x))$ and $H_{0,1}'(w_{k,v}(t,-x))$, we obtain using Lemma 11 and estimates (71), (72) that

$$\begin{aligned}&\begin{bmatrix} \|H_{0,1}'\|_{L_x^2}^2 & O(d(t)e^{-\sqrt{2}d(t)}) \\ O(d(t)e^{-\sqrt{2}d(t)}) & \|H_{0,1}'\|_{L_x^2}^2 \end{bmatrix}\begin{bmatrix}\ddot{y}_1(t)\\ \ddot{y}_2(t)\end{bmatrix} \\ &\quad = e^{-\sqrt{2}d(t)}\begin{bmatrix} -4\sqrt{2} & 4\sqrt{2} \\ 4\sqrt{2} & -4\sqrt{2}\end{bmatrix}\begin{bmatrix} y_1(t) \\ y_2(t)\end{bmatrix} + \begin{bmatrix} O(v^2\|\vec{u}(t)\|_{H_x^1\times L_x^2}) \\ O(v^2\|\vec{u}(t)\|_{H_x^1\times L_x^2})\end{bmatrix} \\ &\qquad - \begin{bmatrix} \frac{\dot{d}(t)}{(1-\frac{1}{4}\dot{d}(t)^2)^{1/2}}\big\langle \partial_t u(t,x),\, H_{0,1}''(w_{k,v}(t,x))\big\rangle \\ \frac{\dot{d}(t)}{(1-\frac{1}{4}\dot{d}(t)^2)^{1/2}}\big\langle \partial_t u(t,x),\, H_{0,1}''(w_{k,v}(t,-x))\big\rangle \end{bmatrix} + \begin{bmatrix} O_k(NOL(t)) \\ O_k(NOL(t))\end{bmatrix}. \end{aligned} \quad (73)$$

Step 2 (refined ordinary differential system). Motivated by (73), for $j \in \{1,2\}$ we define the functions

$$c_j(t) = y_j(t) - y_j(T_{0,k}) + 2\sqrt{2}\int_{T_{0,k}}^t \frac{\dot{d}(s)}{(1-\frac{1}{4}\dot{d}(s)^2)^{1/2}}\big\langle u(s),\, H_{0,1}''(w_{k,v}(s,(-1)^{j+1}x))\big\rangle\, ds.$$

Clearly, we can verify using (24), Lemma 12 and the Cauchy–Schwarz inequality that

$$\dot{c}_j(t) = \dot{y}_j(t) + \frac{2\sqrt{2}\dot{d}(t)}{\left(1-\frac{1}{4}\dot{d}(t)^2\right)^{1/2}}\left\langle u(t,x), H''_{0,1}(w_{k,v}(t,(-1)^{j+1}x))\right\rangle,$$

$$\ddot{c}_j(t) = \ddot{y}_j(t) + \frac{2\sqrt{2}\dot{d}(t)}{\left(1-\frac{1}{4}\dot{d}(t)^2\right)^{1/2}}\left\langle \partial_t u(t,x), \ddot{H}_{0,1}(w_{k,v}(t,(-1)^{j+1}x))\right\rangle + O(v^2\|u(t)\|_{H^1_x}).$$

In conclusion, from the ordinary differential system of equations (73) we deduce that

$$\frac{d}{dt}\begin{bmatrix} y_1(t)\\ y_2(t)\\ \dot{c}_1(t)\\ \dot{c}_2(t)\end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -16e^{-\sqrt{2}d(t)} & 16e^{-\sqrt{2}d(t)} & 0 & 0\\ 16e^{-\sqrt{2}d(t)} & -16e^{-\sqrt{2}d(t)} & 0 & 0\end{bmatrix}\begin{bmatrix} y_1(t)\\ y_2(t)\\ \dot{c}_1(t)\\ \dot{c}_2(t)\end{bmatrix} + \begin{bmatrix} O(v\|u(t)\|_{H^1_x})\\ O(v\|u(t)\|_{H^1_x})\\ O_k(NOL(t)) + O(v^2\|\vec{u}(t)\|_{H^1_x\times L^2_x})\\ O_k(NOL(t)) + O(v^2\|\vec{u}(t)\|_{H^1_x\times L^2_x})\end{bmatrix}.$$

Actually, using the change of variables

$$e_1(t) = y_1(t) - y_2(t),\quad e_2(t) = y_1(t) + y_2(t),\quad \xi_1(t) = c_1(t) - c_2(t)\quad \text{and}\quad \xi_2(t) = c_1(t) + c_2(t),$$

we obtain from the ordinary differential system of equations above that

$$\frac{d}{dt}\begin{bmatrix} e_1(t)\\ e_2(t)\\ \dot{\xi}_1(t)\\ \dot{\xi}_2(t)\end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -32e^{-\sqrt{2}d(t)} & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix} e_1(t)\\ e_2(t)\\ \dot{\xi}_1(t)\\ \dot{\xi}_2(t)\end{bmatrix} + \begin{bmatrix} O(v\|u(t)\|_{H^1_x})\\ O(v\|u(t)\|_{H^1_x})\\ O_k(NOL(t)) + O(v^2\|\vec{u}(t)\|_{H^1_x\times L^2_x})\\ O_k(NOL(t)) + O(v^2\|\vec{u}(t)\|_{H^1_x\times L^2_x})\end{bmatrix}. \tag{74}$$

To simplify our notation, we let

$$M(t) = \begin{bmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -32e^{-\sqrt{2}d(t)} & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}. \tag{75}$$

It is not difficult to verify that all the solutions of linear ordinary differential equation

$$\dot{L}(t) = M(t)L(t)\quad \text{for } L(t)\in\mathbb{R}^4$$

are the linear space generated by the functions

$$L_1(t) = \begin{bmatrix} \tanh(\sqrt{2}vt)\\ 0\\ \sqrt{2}v\,\mathrm{sech}(\sqrt{2}vt)^2\\ 0\end{bmatrix},\quad L_2(t) = \begin{bmatrix} \sqrt{2}vt\tanh(\sqrt{2}vt) - 1\\ 0\\ 2v^2t\,\mathrm{sech}(\sqrt{2}vt)^2 + \sqrt{2}v\tanh(\sqrt{2}vt)\\ 0\end{bmatrix},$$

$$L_3(t) = \begin{bmatrix} 0\\ 1\\ 0\\ 0\end{bmatrix},\quad L_4(t) = \begin{bmatrix} 0\\ t\\ 0\\ 1\end{bmatrix}.$$

Also, by elementary computation, we can verify for any $t \in \mathbb{R}$ that

$$\det\left[L_1(t), L_2(t), L_3(t), L_4(t)\right] = -\sqrt{2}v. \tag{76}$$

In conclusion, using the variation of parameters technique, we can write any C^1 solution of (74) as $L(t) = \sum_{i=1}^{4} a_i(t)L_i(t)$, such that $a_i(t) \in C^1(\mathbb{R})$ for all $1 \leq i \leq 4$ and

$$\begin{bmatrix} \tanh(\sqrt{2}vt) & \sqrt{2}vt\tanh(\sqrt{2}vt) - 1 & 0 & 0 \\ 0 & 0 & 1 & t \\ \sqrt{2}v\,\mathrm{sech}(\sqrt{2}vt)^2 & 2v^2t\,\mathrm{sech}(\sqrt{2}vt)^2 + \sqrt{2}v\tanh(\sqrt{2}vt) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{a}_1(t) \\ \dot{a}_2(t) \\ \dot{a}_3(t) \\ \dot{a}_4(t) \end{bmatrix}$$
$$= \begin{bmatrix} O(v\|u(t)\|_{H^1_x}) \\ O(v\|u(t)\|_{H^1_x}) \\ O_k(NOL(t)) + O(v^2\|\vec{u}(t)\|_{H^1_x \times L^2_x}) \\ O_k(NOL(t)) + O(v^2\|\vec{u}(t)\|_{H^1_x \times L^2_x}) \end{bmatrix}, \tag{77}$$

with

$$\begin{bmatrix} \tanh(\sqrt{2}vT_{0,k}) & \sqrt{2}vT_{0,k}\tanh(\sqrt{2}vT_{0,k}) - 1 & 0 & 0 \\ 0 & 0 & 1 & T_{0,k} \\ \sqrt{2}v\,\mathrm{sech}(\sqrt{2}vT_{0,k})^2 & 2v^2t\,\mathrm{sech}(\sqrt{2}vT_{0,k})^2 + \sqrt{2}v\tanh(\sqrt{2}vT_{0,k}) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1(T_{0,k}) \\ a_2(T_{0,k}) \\ a_3(T_{0,k}) \\ a_4(T_{0,k}) \end{bmatrix}$$
$$= \begin{bmatrix} y_1(T_{0,k}) - y_2(T_{0,k}) \\ y_1(T_{0,k}) + y_1(T_{0,k}) \\ \dot{c}_1(T_{0,k}) \\ \dot{c}_2(T_{0,k}) \end{bmatrix}. \tag{78}$$

Step 3 (estimate of $\|\vec{u}(t)\|_{H^1_x \times L^2_x}$). From now on, for $C_1 > 1$, $C_2 > 0$ being fixed numbers to be chosen later, we consider the set

$$B_{C_1,C_2} = \left\{ t \in \mathbb{R} \;\middle|\; \max_{j\in\{1,2\}} |y_j(t)|v^2 + |\dot{y}_j(t)|v \leq C_1 v^{2(k+1)} \ln(1/v)^{n_k+3} \exp\left(\frac{C_2 v|t - T_{0,k}|}{\ln(1/v)}\right) \right\}.$$

We also consider the set

$$D_{u,v} = \{t \in \mathbb{R} \mid \|\vec{u}(t)\|_{H^1_x \times L^2_x} < v^2\}.$$

First, if $v^2|y(T_{0,k})| + v|\dot{y}(T_{0,k})| < v^{3k}$ and $v \ll 1$, then $T_{0,k} \in B_{C_1,C_2} \cap D_{u,v}$. Indeed, this happens when

$$\|(\varphi_{k,v}(T_{0,k}), \partial_t\varphi_{k,v}(T_{0,k})) - (\phi(T_{0,k}), \partial_t\phi(T_{0,k}))\|_{H^1_x \times L^2_x} < v^{4k},$$

because, since $u(t, x)$ satisfies the orthogonality conditions (29), we can verify using Lemma 11 that

$$\|\varphi_{k,v}(T_{0,k}) - \phi(T_{0,k})\|^2_{H^1_x} \cong \max_{j\in\{1,2\}} y_j(T_{0,k})^2 + \|u(T_{0,k})\|^2_{H^1_x}. \tag{79}$$

By a similar reasoning but using now Lemma 12 and estimate (79), we can verify that if $0 < v \ll 1$, then

$$\max_{j\in\{1,2\}} \dot{y}_j(T_{0,k})^2 + \|\partial_t u(T_{0,k})\|^2_{L^2_x} \lesssim \|(\varphi_{k,v}(T_{0,k}), \partial_t\varphi_{k,}(T_{0,k})) - (\phi(T_{0,k}), \partial_t\phi(T_{0,k}))\|^2_{H^1_x \times L^2_x}, \tag{80}$$

where $T_{0,k}$ satisfies the hypothesis of Theorem 15, for more details see Appendix B in [Moutinho 2023]. Also, for any $\theta \in (0,1)$, if $v \ll 1$, then while

$$|t - T_{0,k}| < \frac{\ln(1/v)^{2-\theta}}{v},$$

and $t \in B_{C_1,C_2} \cap D_{u,v}$, we can verify the estimate

$$\max_{j\in\{1,2\}} v^2|y_j(t)| + v|\dot{y}_j(t)| < v^{2k+1}\ln(1/v)^{n_k},$$

from which with estimate (73), the definition of $NOL(t)$ at (68), the definition of $D_{u,v}$ and the assumption of $k \geq 2$, we obtain that

$$\max_{j\in\{1,2\}} |\ddot{y}_j(t)| \lesssim_k v^{2k}\ln(1/v)^{n_k} + v\|\vec{u}(t)\|_{H^1_x\times L^2_x} + \|\vec{u}(t)\|^2_{H^1_x\times L^2_x}.$$

In conclusion, if $v \ll 1$, from Theorem 21, we deduce that the functional $L(t)$ defined in last section satisfies, for a constant C_0 and a parameter $C(k)$ depending only on k, the estimates

$$|\dot{L}(t)| \lesssim v \max_{j\in\{1,2\}} |\ddot{y}_j(t)|\|\vec{u}(t)\|_{H^1_x\times L^2_x} + \|\vec{u}(t)\|^3_{H^1_x\times L^2_x} + C(k)\|\vec{u}(t)\|_{H^1_x\times L^2_x} v^{2k+1}\ln(1/v)^{n_k} + \|\vec{u}(t)\|^2_{H^1_x\times L^2_x}\frac{v}{\ln(1/v^2)},$$

$$C_0\|\vec{u}(t)\|^2_{H^1_x(\mathbb{R})\times L^2_x(\mathbb{R})} \leq L(t) + C(k)v^{4k}\ln(1/v)^{2n_k}.$$

Therefore, from the ordinary differential system of equations defined in (73), we conclude for $v \ll 1$ that if $t \in B_{C_1,C_2} \cap D_{u,v}$ and

$$|t - T_{0,k}| < \frac{\ln(1/v)^{2-\theta}}{v}, \tag{81}$$

then there exists a constant $C(k) > 0$ depending only on k satisfying

$$|\dot{L}(t)| \lesssim C(k)\|\vec{u}(t)\|_{H^1_x\times L^2_x} v^{2k+1}\ln(1/v)^{n_k} + \|\vec{u}(t)\|^2_{H^1_x\times L^2_x}\frac{v}{\ln(1/v^2)}.$$

Therefore, by a similar argument to the proof of Theorem 4.5 in [Moutinho 2023], we can verify from Theorem 21 and the Gronwall lemma applied on $L(t)$ that there exists a constant $K > 1$, independent of k and v, such that if t satisfies condition (81) and $t \in B_{C_1,C_2} \cap D_{u,v}$, then we have the estimate

$$\|(u(t), \partial_t u(t))\|_{H^1_x\times L^2_x} \lesssim_k \max\big(\|\vec{u}(T_{0,k})\|_{H^1_x\times L^2_x}, v^{2k}\ln(1/v)^{n_k+1}\big)\exp\bigg(\frac{K|t-T_{0,k}|v}{\ln(1/v)}\bigg). \tag{82}$$

In conclusion, if $v \ll 1$, $t \in B_{C_1,C_2}$ and t satisfies (81), then $t \in D_{u,v}$ and (82) is true.

Step 4 (estimate of $y_1(t)$, $y_2(t)$). Next, we will use the estimate (82) in the ordinary differential system of equations (74) to estimate the evolution of $y_1(t)$ and $y_2(t)$ while $t \in B_{C_1,C_2}$ and t satisfies condition (81). From (68), we have that if $t \in B_{C_1,C_2}$, t satisfies condition (81) and $0 < v \ll 1$, then

$$NOL(t) \ll v^2 \max\big(\|\vec{u}(T_{0,k})\|_{H^1_x\times L^2_x}, v^{2k}\ln(1/v)^{n_k+1}\big)\exp\bigg(\frac{K|t-T_{0,k}|v}{\ln(1/v)}\bigg). \tag{83}$$

In conclusion, from the Cauchy problem (25) satisfied by ϕ, identity (76) and estimates (79), (80), and (83), we deduce from the linear system (77) the estimates

$$|\dot{a}_1(t)| \lesssim_k v^{2k+1}[|t|v+1]\ln(1/v)^{n_k+1}\exp\left(K\frac{v|t-T_{0,k}|}{\ln(1/v)}\right),$$
$$|\dot{a}_2(t)| \lesssim_k v^{2k+1}\ln(1/v)^{n_k+1}\exp\left(K\frac{v|t-T_{0,k}|}{\ln(1/v)}\right),$$
$$|\dot{a}_3(t)| \lesssim_k v^{2k+1}[|t|v+1]\ln(1/v)^{n_k+1}\exp\left(K\frac{v|t-T_{0,k}|}{\ln(1/v)}\right),$$
$$|\dot{a}_4(t)| \lesssim_k v^{2k+2}\ln(1/v)^{n_k+1}\exp\left(K\frac{v|t-T_{0,k}|}{\ln(1/v)}\right).$$

In conclusion, using the initial condition (78), we deduce from the fact that $T_{0,k}$ is in B_{C_1,C_2},the fundamental theorem of calculus and the elementary estimate

$$|t|v<\ln(1/v)\exp\left(\frac{v|t|}{\ln(1/v)}\right),$$

that if $\{\theta t+(1-\theta)T_{0,k}|0<\theta<1\}\subset B_{C_1,c_2}$ and t satisfies (81), then

$$|a_1(t)|+|a_3(t)| \lesssim_k v^{2k}\ln(1/v)^{n_k+3}\exp\left(\frac{(K+1)|t-T_{0,k}|v}{\ln(1/v)}\right),$$
$$v|a_2(t)|+|a_4(t)| \lesssim_k v^{2k+1}\ln(1/v)^{n_k+2}\exp\left(\frac{K|t-T_{0,k}|v}{\ln(1/v)}\right).$$

In conclusion from the ordinary differential system of equations (74) satisfied by $e_j(t)$ for $j\in\{1,2,3,4\}$, the fact that $e_1(t)=y_1(t)-y_2(t)$, $e_2(t)=y_1(t)+y_2(t)$ and $\xi_1(t)=c_1(t)-c_2(t)$, $\xi_2(t)=c_1(t)+c_2(t)$, we can verify by triangle inequality and the identity

$$\begin{bmatrix} e_1(t)\\ e_2(t)\\ e_3(t)\\ e_4(t)\end{bmatrix}=\sum_{j=1}^{4}a_jL_j(t)$$

the existence of $C_1(k)>0$ depending on k such that for $C_2=K+2$ and $v\ll 1$ we have that if

$$|t-T_{0,k}|<\frac{\ln(1/v)^{2-\theta}}{v},$$

then $t\in B_{C_1(k),C_2}$. □

Remark 22. For any constants $\theta,\gamma\in(0,1)$, obviously

$$\lim_{v\to +0} v^{\gamma}\exp\big(\ln(1/v)^{\theta}\big)=0.$$

In conclusion, for fixed $k\in\mathbb{N}$ large and $0<\theta<\frac{1}{4}$, we can deduce from Theorem 15 that there is a $\Delta_{k,\theta}>0$ such that if $0<v<\Delta_{k,\theta}$, then

$$\|(\phi(t,x),\partial_t\phi(t,x))-(\phi_k(v,t,x),\partial_t\phi_k(v,t,x))\|_{H^1_x\times L^2_x}<v^{2k-1/2},$$

for all t satisfying

$$|t-T_{0,k}|<\frac{\ln(1/v)^{2-\theta}}{v}.$$

5. Proof of Theorem 4

Remark 23. The importance of this theorem is to describe the dynamics of the two solitons before the collision instant, for all $t<0$ and $|t|\gg 1$. More precisely, if two moving kinks are coming from an infinite distance with a sufficiently low speed v satisfying $v\leq \delta(2k)$, then the inelasticity of the collision is going to be of order at most $O(v^k)$ and the kinks will move away each one with the speed of size in modulus $v+O(v^k)$ when t goes to $-\infty$.

The proof of Theorem 4 uses energy estimate techniques from [Henry et al. 1982], and the monotonicity property of the function

$$P_{+}(\phi(t),\partial_t\phi(t)) = -\int_0^{+\infty} \partial_t\phi(t,x)\partial_x\phi(t,x)\,dx, \tag{84}$$

which is nondecreasing on t when $\phi(t,\cdot)$ is odd on x. Furthermore, the demonstration of Theorem 4 is quite similar to the proof of Theorem 1 of [Kowalczyk et al. 2021] and also uses modulation techniques inspired by [Raphaël and Szeftel 2011; Kowalczyk et al. 2021].

Moreover, since the solution $\phi(t,x)$ is an odd function in the variable x for all $t\in\mathbb{R}$, we have that

$$E(\phi)=2\left[\int_0^{+\infty}\frac{\partial_x\phi(t,x)^2+\partial_t\phi(t,x)^2}{2}+U(\phi(t,x))\,dx\right]=2E_{+}(\phi(t),\partial_t\phi(t)),$$

where

$$E_{+}(\phi(t),\partial_t\phi(t))=\int_0^{+\infty}\frac{\partial_x\phi(t,x)^2+\partial_t\phi(t,x)^2}{2}+U(\phi(t,x))\,dx \tag{85}$$

is a conserved quantity.

5.1. ***Modulation techniques.*** First, similarly to [Kowalczyk et al. 2021], we consider, for any $0<v<1$, $y\in\mathbb{R}$, the following function on $x\in\mathbb{R}$:

$$\overrightarrow{H_{0,1}}((v,y),x)=\begin{bmatrix} H_{0,1}\left(\frac{x-y}{\sqrt{1-v^2}}\right)\\ \frac{-v}{\sqrt{1-v^2}}H_{0,1}'\left(\frac{x-y}{\sqrt{1-v^2}}\right)\end{bmatrix}, \tag{86}$$

$$\overrightarrow{H_{-1,0}}((v,y),x)=-\overrightarrow{H_{0,1}}((v,y),-x)\quad\text{for all } x\in\mathbb{R}.$$

Next, we consider the antisymmetric map

$$J=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}, \tag{87}$$

and based on [Kowalczyk et al. 2021], we consider for any $0<v<1$ and any $y\in\mathbb{R}$ the following functions, which were defined in Section 2.3 of [Kowalczyk et al. 2021]:

$$C_{v,y}(x)=\begin{bmatrix}\frac{1}{\sqrt{1-v^2}}H_{0,1}'\left(\frac{x-y}{\sqrt{1-v^2}}\right)\\ \frac{-v}{1-v^2}H_{0,1}''\left(\frac{x-y}{\sqrt{1-v^2}}\right)\end{bmatrix}, \tag{88}$$

$$D_{v,y}(x)=\begin{bmatrix}\dfrac{v}{1-v^2}\dfrac{x-y}{\sqrt{1-v^2}}H'_{0,1}\Big(\dfrac{x-y}{\sqrt{1-v^2}}\Big)\\ \dfrac{-1}{(1-v^2)^{3/2}}H'_{0,1}\Big(\dfrac{x-y}{\sqrt{1-v^2}}\Big)-\dfrac{v^2}{(1-v^2)^{3/2}}\dfrac{x-y}{\sqrt{1-v^2}}H''_{0,1}\Big(\dfrac{x-y}{\sqrt{1-v^2}}\Big)\end{bmatrix}. \tag{89}$$

See also [Chen and Jendrej 2019].

The following identity is going to be useful for our next results.

Lemma 24. *For any $v \in (0, 1)$, there holds*

$$\langle \partial_x \overrightarrow{H_{0,1}}((v, 0), x), JD_{0,v}\rangle = -(1-v^2)^{-3/2}\|H'_{0,1}\|^2_{L^2_x}.$$

Proof. See the proof of Lemma 2.4 from [Kowalczyk et al. 2021]. □

Next, for any value $y_0 \gg 1$, we will modulate any odd function (ϕ_0, ϕ_1) close to

$$\overrightarrow{H_{-1,0}}((v, y_0), x) + \overrightarrow{H_{0,1}}((v, y_0), x)$$

in the energy norm in terms of an orthogonal condition.

Lemma 25. *There exist $K > 0$ and $\delta_0, \delta_1 \in (0, 1)$ such that if $0 < v < \delta_1$, $y_0 > 1/\delta_1$, $0 \le \delta \le \delta_0$ and $(\phi_1 - H_{0,1} - H_{-1,0}, \phi_2) \in H^1_x(\mathbb{R}) \times L^2_x(\mathbb{R})$ is an odd function satisfying*

$$\|(\phi_1(x), \phi_2(x)) - \overrightarrow{H_{-1,0}}((v, y_0), x) - \overrightarrow{H_{0,1}}((v, y_0), x)\|_{H^1_x \times L^2_x} \le \delta v, \tag{90}$$

then there exists a unique $\hat{y} > 1$ such that $|\hat{y} - y_0| \le K\delta v$ and the function

$$\vec{\kappa}(x) = (\phi_1(x), \phi_2(x)) - \overrightarrow{H_{-1,0}}((v, \hat{y}), x) - \overrightarrow{H_{0,1}}((v, \hat{y}), x)$$

satisfies

$$\|\vec{\kappa}\|_{H^1_x \times L^2_x} \le K\delta v \tag{91}$$

and $\langle \vec{\kappa}(x), J \circ D_{v,\hat{y}}(x)\rangle = 0$.

Proof of Lemma 25. The proof is completely analogous to that of Lemma 2.1 of [Kowalczyk et al. 2021]. □

Corollary 26. *In the notation of Lemma 25, there exists a constant $C > 1$ such that if $v \in (0, 1)$ is small enough, then there exists at most one number $y \ge 2\ln\frac{1}{v}$ satisfying*

$$\|\vec{\kappa_0}\|_{H^1_x \times L^2_x} \le \min\left\{\delta_0 v, \frac{K}{3C}\delta_0 v\right\} \quad \text{and} \quad \langle \vec{\kappa_0}(x), J \circ D_{v,y}(x)\rangle = 0,$$

where

$$\vec{\kappa_0}(x) = (\phi_1(x), \phi_2(x)) - \overrightarrow{H_{-1,0}}((v, y), x) - \overrightarrow{H_{0,1}}((v, y), x)$$

Proof of Corollary 26. Let y_1, y_2 two real numbers satisfying the results of Corollary 26. We consider the functions

$$\vec{\kappa_1}(x) = (\kappa_{1,0}(x), \kappa_{1,1}(x)) = (\phi_1(x), \phi_2(x)) - \overrightarrow{H_{-1,0}}((v, y_1), x) - \overrightarrow{H_{0,1}}((v, y_1), x),$$
$$\vec{\kappa_2}(x) = (\kappa_{2,0}(x), \kappa_{2,1}(x)) = (\phi_0(x), \phi_1(x)) - \overrightarrow{H_{-1,0}}((v, y_2), x) - \overrightarrow{H_{0,1}}((v, y_2), x).$$

Choosing $x = y_1$, we obtain the ng identity

$$H_{0,1}(0) - H_{0,1}\left(\frac{y_1 - y_2}{\sqrt{1-v^2}}\right) = -H_{0,1}\left(\frac{-2y_1}{\sqrt{1-v^2}}\right) + H_{0,1}\left(\frac{-y_1 - y_2}{\sqrt{1-v^2}}\right) + \kappa_{2,0}(y_1) - \kappa_{1,0}(y_1). \tag{92}$$

Since there exists a constant $c > 0$ satisfying for any $f \in H^1_x(\mathbb{R})$ the inequality

$$\|f\|_{L^\infty_x(\mathbb{R})} \leq c\|f\|_{H^1_x},$$

we deduce from (92) and the hypotheses of Corollary 26 that

$$\left|H_{0,1}(0) - H_{0,1}\left(\frac{y_1 - y_2}{\sqrt{1-v^2}}\right)\right| \leq \frac{2cK}{3C}\delta_0 v + \left|H_{0,1}\left(\frac{-2y_1}{\sqrt{1-v^2}}\right)\right| + \left|H_{0,1}\left(\frac{-y_1-y_2}{\sqrt{1-v^2}}\right)\right|,$$

from which we deduce the estimate

$$\left|H_{0,1}(0) - H_{0,1}\left(\frac{y_1 - y_2}{\sqrt{1-v^2}}\right)\right| \leq \frac{2cK}{3C}\delta_0 v + 2v^4.$$

Consequently, since $H_{0,1}$ is an increasing function and $H'_{0,1}(0) = \frac{1}{2}$, we obtain that if $\delta_1 \ll 1$ and $0 < v < \delta_1$, then

$$|y_1 - y_2| \leq \frac{5Kc}{3C}\delta_0 v.$$

Therefore, choosing $C = 2c + 1$, from Lemma 25, we have $y_1 = y_2$ if $v > 0$ is small enough. □

Finally, using Lemma 25 and repeating the argument of the demonstration of Lemma 2.11 in [Kowalczyk et al. 2021], we can verify the following result.

Lemma 27. *There exist $K > 1$, $\delta_0 > 0$ and $\delta_1 \in (0, 1)$ such that if $0 < \delta_2 < \delta_0$, $0 < v < \delta_1$, $y_0 > \frac{7}{2}\ln\frac{1}{v}$ and the solution $(\phi(t, x), \partial_t\phi(t, x))$ of (1) satisfies, for $T > 0$,*

$$\sup_{t\in[0,T]}\inf_{y\in\mathbb{R}_{\geq y_0}} \|(\phi(t, x), \partial_t\phi(t, x)) - \overrightarrow{H_{-1,0}}((v, y), x) - \overrightarrow{H_{0,1}}((v, y), x)\|_{H^1_x\times L^2_x} \leq \delta_2 v, \tag{93}$$

then there exists a real function $y_1 : [0, T] \to \mathbb{R}_{\geq y_0/2}$ such that the solution $(\phi(t), \partial_t\phi(t))$ satisfies, for any $0 \leq t \leq T$,

$$(\phi(t), \partial_t\phi(t)) = \overrightarrow{H_{-1,0}}((v, y_1(t)), x) + \overrightarrow{H_{0,1}}((v, y_1(t)), x) + (\psi_1(t), \psi_2(t)), \tag{94}$$

$$\|(\psi_1(t), \psi_2(t))\|_{H^1_x\times L^2_x} \leq K\delta_2 v, \tag{95}$$

where $(\psi_1(t), \psi_2(t)) \in H^1_x(\mathbb{R}) \times L^2_x(\mathbb{R})$ and $y_1(t)$ satisfy the orthogonality condition of Lemma 25, and $y_1(t)$ is a function of class C^1 satisfying the inequality

$$|\dot{y}_1(t) - v| \leq K[\|(\psi_1(t), \psi_2(t))\|_{H^1_x\times L^2_x} + e^{-2\sqrt{2}y_1(t)}]. \tag{96}$$

Proof. First, from Lemma 25 and the fact that $\vec{\phi} \in C(\mathbb{R}; H^1_x(\mathbb{R}) \times L^2_x(\mathbb{R}))$, if δ_1 is small enough, we can find a constant $K > 0$ and a function $\hat{y} : [0, T] \to \left(3\ln\frac{1}{v}, +\infty\right)$ such that for

$$\vec{\kappa}(t, x) = (\phi(t, x), \partial_t\phi(t, x)) - \overrightarrow{H_{-1,0}}((v, \hat{y}(t)), x) - \overrightarrow{H_{0,1}}((v, \hat{y}(t)), x), \tag{97}$$

we have $\vec{\kappa}(t)$, $\hat{y}(t)$ satisfying the orthogonality condition of Lemma 25 and

$$\|\vec{\kappa}(t)\|_{H^1_x\times L^2_x} \leq K\delta_2 v \tag{98}$$

for all $0 \leq t \leq T$.

Next, we will construct a linear ordinary differential system of equations with solution $y_1(t)$ and we will verify that if $y_1(0)=\hat{y}(0)$, then $y_1(t)=\hat{y}(t)$ for all $t\in[0,T]$.

Step 1 (construction of the ordinary differential equation satisfied by y_1). The argument of the demonstration of the remaining part of Lemma 27 is completely analogous to the proof of Lemma 2.11 of [Kowalczyk et al. 2021]. More precisely, similarly to Lemma 2.11 of [Kowalczyk et al. 2021], we will construct an ordinary differential equation with solution $y_1(t)$, which, during their time of existence, preserves the orthogonality conditions

$$\langle(\psi_1(t,x),\psi_2(t,x)),\,JD_{v,y_1(t)}(x)\rangle=0, \tag{99}$$

where J is defined in (87), and we will verify that if $y_1(0)=\hat{y}(0)$, then $y_1(t)=\hat{y}(t)$ for all $0\leq t\leq T$. From the global well-posedness of the partial differential (1) in the energy space, we have for any $T_0>0$ that $\phi(t,x)-H_{0,1}(x)-H_{-1,0}(x)\in C([-T_0,T_0],H^1_x(\mathbb{R}))$ and $\partial_t\phi(t,x)\in C([-T_0,T_0],L^2_x(\mathbb{R}))$. Therefore, if there exists a interval $[0,T_1]\subset[0,T]$ such that $y_1\in C^1([0,T_1])$ when restricted to this interval and

$$(\phi(t),\partial_t\phi(t))=\overrightarrow{H_{-1,0}}((v,y_1(t)),x)+\overrightarrow{H_{0,1}}((v,y_1(t)),x)+(\psi_1(t),\psi_2(t))\quad\text{for any } t\in[0,T_1], \tag{100}$$

then $(\psi_1(t),\psi_2(t))=(\psi_1(t,x),\psi_2(t,x))$ satisfies, for any functions $h_1,h_2\in\mathscr{S}(\mathbb{R})$, the identity

$$\frac{d}{dt}\big\langle(\psi_1(t,x),\psi_2(t,x)),(h_1(x),h_2(x))\big\rangle=\big\langle\partial_t(\psi_1(t,x),\psi_2(t,x)),(h_1(x),h_2(x))\big\rangle$$

if $t\in[0,T_1]$.

Consequently, if we derive the (99) in time, we obtain the following linear ordinary differential equation satisfied by $y_1(t)$:

$$\dot{y}_1(t)\big\langle(\psi_1(t,x),\psi_2(t,x)),\,J\partial_{y_1}D_{v,y_1(t)}(x)\big\rangle+\big\langle\partial_t(\psi_1(t,x),\psi_2(t,x)),\,JD_{v,y_1(t)}(x)\big\rangle=0. \tag{101}$$

Since $x^mH'_{0,1}(x)\in\mathscr{S}(\mathbb{R})$ for all $m\in\mathbb{N}\cup\{0\}$, we have that the functions $\omega_1,\omega_2:[0,T]\times(1,+\infty)\to\mathbb{R}$ defined by

$$\omega_1(t,y)=\big\langle(\psi_1(t,x),\psi_2(t,x)),\,J\partial_yD_{v,y}(x)\big\rangle,\ \omega_2(t,y)=\big\langle\partial_t(\psi_1(t,x),\psi_2(t,x)),\,JD_{v,y}(x)\big\rangle$$

are continuous and, for any $t\in[0,T]$, $\omega_1(t,\cdot),\omega_2(t,\cdot):(1,+\infty)\to\mathbb{R}$ are smooth.

Step 2 (partial differential equation satisfied by $\overrightarrow{\psi}$). First, we consider the self-adjoint operator

$$\operatorname{Hess}(y_1(t),x):H^2_x(\mathbb{R})\subset L^2_x(\mathbb{R})\to\mathbb{R},$$

which satisfies, for all $t\in[0,T]$,

$$\operatorname{Hess}(y_1(t),x)=\begin{bmatrix}-\partial_x^2+U''\Big(H_{0,1}\Big(\frac{x-y_1(t)}{\sqrt{1-v^2}}\Big)-H_{0,1}\Big(\frac{-x-y_1(t)}{\sqrt{1-v^2}}\Big)\Big) & 0\\ 0 & 1\end{bmatrix}, \tag{102}$$

and the self-adjoint operator $\operatorname{Hess}_1(y_1(t),x):H^2_x(\mathbb{R})\subset L^2_x(\mathbb{R})\to\mathbb{R}$ denoted by

$$\operatorname{Hess}_1(y_1(t),x)=\begin{bmatrix}-\partial_x^2+U''\Big(H_{0,1}\Big(\frac{x-y_1(t)}{\sqrt{1-v^2}}\Big)\Big) & 0\\ 0 & 1\end{bmatrix}. \tag{103}$$

Next, we consider the maps $\operatorname{Int}:\mathbb{R}^2\to\mathbb{R}^2$ and $\mathcal{T}:\mathbb{R}^2\times H^1_x(\mathbb{R})\to\mathbb{R}^2$, which we denote by

$$\operatorname{Int}(y,x)=\begin{bmatrix}0\\ U'\Big(-H_{0,1}\Big(\frac{-x-y_1}{\sqrt{1-v^2}}\Big)\Big)+U'\Big(H_{0,1}\Big(\frac{x-y}{\sqrt{1-v^2}}\Big)\Big)\end{bmatrix}-\begin{bmatrix}0\\ U'\Big(H_{0,1}\Big(\frac{x-y}{\sqrt{1-v^2}}\Big)-H_{0,1}\Big(\frac{-x-y}{\sqrt{1-v^2}}\Big)\Big)\end{bmatrix}, \tag{104}$$

$$\mathcal{T}(y,x,\psi)=\begin{bmatrix}0\\ -\sum_{j=3}^{6}U^{(j)}\Big(H_{0,1}\Big(\frac{x-y}{\sqrt{1-v^2}}\Big)-H_{0,1}\Big(\frac{-x-y}{\sqrt{1-v^2}}\Big)\Big)\frac{\psi(x)^{j-1}}{(j-1)!}\end{bmatrix} \tag{105}$$

for any $(y,x)\in\mathbb{R}^2$ and $\psi\in H^1_x(\mathbb{R})$. Therefore, if $[0,T_1]\subset[0,T]$, $y_1\in C^1([0,T_1])$ and $y_1\geq 1$, $0<v_1<1$ then, from the partial differential equation (1) and identity (100), we deduce that $(\psi_1(t,x),\psi_2(t,x))$ is a solution in the space $C([0,T_1],H^1_x(\mathbb{R})\times L^2_x(\mathbb{R}))$ of the partial differential equation

$$\partial_t(\psi_1(t,x),\psi_2(t,x))=(\dot y_1(t)-v)[C_{v,y_1(t)}(x)-C_{v,y_1(t)}(-x)]+J\operatorname{Hess}(y_1(t),x)(\psi_1(t,x),\psi_2(t,x))\\ +\operatorname{Int}(y_1(t),x)+\mathcal{T}(y_1(t),x,\psi_1(t)), \tag{106}$$

where J is the antisymmetric operator defined in (87).

In the next step, we will assume the existence of $0\leq T_1\leq T$ such that y_1 is of class C^1 in the interval $[0,T_1]$, and $y_1\geq 1$ for any $t\in[0,T_1]$. Moreover, we will prove that when this condition is true, then $|\dot y_1(t)-v|$ is sufficiently small for all $t\in[0,T_1]$.

Step 3 (estimate of $|\dot y_1(t)-v|$). Uniquely in this step, for any continuous nonnegative function $f:[0,T_1]\times(0,1)\times(1,+\infty)\to\mathbb{R}$, we say that a function $g:[0,T_1]\times(0,1)\times(1,+\infty)\to\mathbb{R}$ is $O(f)$ if and only if g is a continuous function satisfying the following properties:

- There is a constant $c>0$ such that $|g(t,v,y)|<cf(t,v,y)$ for all (t,v,y) in $[0,T_1]\times(0,1)\times(1,+\infty)$.
- $g(t,\cdot):(0,1)\times(1,+\infty)\to\mathbb{R}$ is smooth for all $t\in[0,T_1]$.

We recall that J, $C_{v,y_1(t)}$ and $D_{v,y_1(t)}$ are defined, respectively, in (87), (88) and (89). Using Lemma 11, we obtain that if $y_1(t)\geq 1$ and $v\in(0,1)$ is small enough, then

$$|\langle C_{v,y_1(t)}(x),J\circ D_{v,y_1(t)}(-x)\rangle|+|\langle C_{v,y_1(t)}(x),JC_{v,y_1(t)}(-x)\rangle|+|\langle D_{v,y_1(t)}(x),JD_{v,y_1(t)}(-x)\rangle|\\ \lesssim y_1(t)^4e^{-2\sqrt{2}y_1(t)}. \tag{107}$$

Furthermore, using the partial differential equation (106) satisfied by $(\psi_1(t,x),\psi_2(t,x))$, we deduce for any $t\in[0,T_1]\subset[0,T]$ the identity

$$\langle\partial_t(\psi_1(t,x),\psi_2(t,x)),JD_{v,y_1(t)}(x)\rangle\\ =(\dot y_1(t)-v)\langle C_{v,y_1(t)}(x),JD_{v,y_1(t)}(x)\rangle-(\dot y_1(t)-v)\langle C_{v,y_1(t)}(-x),JD_{v,y_1(t)}(x)\rangle\\ +\big\langle J\operatorname{Hess}(y_1(t),x)(\psi_1(t,x),\psi_2(t,x)),JD_{v,y_1(t)}(x)\big\rangle\\ +\big\langle\mathcal{T}(y_1(t),x,\psi_1(t))+\operatorname{Int}(y_1(t),x),JD_{v,y_1(t)}(x)\big\rangle. \tag{108}$$

Moreover, from Lemma 24 and identity $J^*=-J$, we have

$$\langle JD_{v,y_1(t)}(x),\,C_{v,y_1(t)}(x)\rangle=-\langle D_{v,y_1(t)}(x),\,JC_{v,y_1(t)}(x)\rangle=(1-v^2)^{-3/2}\|H'_{0,1}\|^2_{L^2_x}. \tag{109}$$

Therefore, using (108), estimates (107) and Lemma 11, we deduce the following estimate

$$\begin{aligned}&\big\langle\partial_t(\psi_1(t,x),\,\psi_2(t,x)),\,JD_{v,y_1(t)}(x)\big\rangle\\&\quad=(\dot y_1(t)-v)[(1-v^2)^{-3/2}\|H'_{0,1}\|^2_{L^2_x}+O(y_1(t)^4e^{-2\sqrt{2}y_1(t)})]\\&\qquad+\big\langle J\,\mathrm{Hess}(y_1(t),x)(\psi_1(t,x),\,\psi_2(t,x)),\,JD_{v,y_1(t)}\big\rangle\\&\qquad\quad+\big\langle\mathcal{T}(y_1(t),x,\psi_1(t)),\,JD_{v,y_1(t)}(x)\big\rangle+\big\langle\mathrm{Int}(y_1(t),x),\,JD_{v,y_1(t)}(x)\big\rangle.\end{aligned}$$

Furthermore, since for any $\zeta\in\mathbb{R}$ we have the identity

$$\begin{aligned}&U'(H^{\zeta}_{0,1}(x)+H_{-1,0}(x))-U'(H^{\zeta}_{0,1}(x))-U'(H_{-1,0}(x))\\&\quad=-24H_{-1,0}(x)H^{\zeta}_{0,1}(x)(H_{-1,0}(x)+H^{\zeta}_{0,1}(x))+\sum_{j=1}^{4}\binom{5}{j}H_{-1,0}(x)^jH^{\zeta}_{0,1}(x)^{5-j},\end{aligned}$$

we deduce from Lemma 11 and the definition of function Int that $\|\mathrm{Int}(y_1(t),x,\psi(t))\|_{L^2_x}\lesssim e^{-2\sqrt{2}y_1(t)}$. Next, since $\|U^{(l)}\|_{L^\infty[-1,1]}<+\infty$ for any $l\in\mathbb{N}\cup\{0\}$, we deduce using Lemma 13 and the definition of function $\mathcal{T}$ that

$$\|\mathcal{T}(y_1(t),x,\psi_1(t))\|_{L^2_x}\le\|\mathcal{T}(y_1(t),x,\psi_1(t))\|_{H^1_x}\lesssim\|\psi_1(t,x)\|^2_{H^1_x}.$$

As a consequence,

$$\begin{aligned}&\big\langle\partial_t(\psi_1(t,x),\,\psi_2(t,x)),\,JD_{v,y_1(t)}(x)\big\rangle\\&\quad=(\dot y_1(t)-v)\big[(1-v^2)^{-3/2}\|H'_{0,1}\|^2_{L^2_x}+O(y_1(t)^4e^{-2\sqrt{2}y_1(t)})\big]\\&\qquad+\big\langle J\,\mathrm{Hess}(y_1(t),x)(\psi_1(t,x),\,\psi_2(t,x)),\,JD_{v_1(t),y_1(t)}(x)\big\rangle+O(e^{-2\sqrt{2}y_1(t)}+\|\overrightarrow{\psi}(t)\|^2_{H^1_x\times L^2_x})\end{aligned} \tag{110}$$

for any $t\in[0,T_1]$.

Furthermore, using identities (102), (103), the formula of $D_{v,y}$ in (89) and Lemma 11, we can deduce the estimate

$$\|[\mathrm{Hess}(y_1(t),x)-\mathrm{Hess}_1(y_1(t),x)]D_{v,y_1(t)}(x)\|_{L^2_x(\mathbb{R};\mathbb{R}^2)}\lesssim e^{-2\sqrt{2}y_1(t)}$$

for all $t\in[0,T_1]$. Thus, after using integration by parts and the Cauchy–Schwarz inequality, we deduce for all $t\in[0,T_1]$ that

$$\Big|\big\langle J[\mathrm{Hess}(y_1(t),x)-\mathrm{Hess}_1(y_1(t),x)]\overrightarrow{\psi}(t),\,JD_{v_1(t),y_1(t)}(x)\big\rangle\Big|\lesssim\|\overrightarrow{\psi}(t)\|_{H^1_x\times L^2_x}e^{-2\sqrt{2}y_1(t)}.$$

Consequently, since $\langle j(a),a\rangle=0$ for all $a\in\mathbb{R}^2$, we obtain that if y_1 is a function of class C^1 in the interval $[0,T_1]$ and $v\in(0,1)$ is small enough, then

$$\begin{aligned}&\big\langle\partial_t(\psi_1(t,x),\,\psi_2(t,x)),\,JD_{v,y_1(t)}(x)\big\rangle\\&=(\dot y_1(t)-v)\Big[-\frac{\|H'_{0,1}\|^2_{L^2_x}}{(1-v^2)^{3/2}}+O(y_1(t)^4e^{-2\sqrt{2}y_1(t)})\Big]+\big\langle J\,\mathrm{Hess}_1(y_1(t),x)(\psi_1(t,x),\psi_2(t,x)),\,JD_{v,y_1(t)}(x)\big\rangle\\&\qquad+O(e^{-2\sqrt{2}y_1(t)}+\|\overrightarrow{\psi}(t)\|^2_{H^1_x\times L^2_x})\end{aligned} \tag{111}$$

for any $t\in[0,T_1]$.

Next, using (103), it is not difficult to verify the identity

$$\mathrm{Hess}_1(y_1(t),x)D_{v,y_1(t)}(x) - vJ[\partial_x D_{v,y_1(t)}(x)] = JC_{v,y_1(t)}(x);$$

see Lemma 2.4 of [Kowalczyk et al. 2021] for the proof. Consequently, we have for any $t\in[0,T_1]$ that

$$\begin{aligned}\big\langle J\,\mathrm{Hess}_1(y_1(t),x)(\psi_1(t,x),\psi_2(t,x)),\, JD_{v,y_1(t)}(x)\big\rangle \\ = -v\big\langle(\psi_1(t,x),\psi_2(t,x)),\, J\partial_{y_1}D_{v,y_1(t)}(x)\big\rangle + \big\langle(\psi_1(t,x),\psi_2(t,x)),\, JC_{v,y_1(t)}(x)\big\rangle.\end{aligned}$$

In conclusion, estimate (111) and identity (101) imply that

$$\begin{aligned}(\dot y_1(t)-v)\left[\frac{-\|H_{0,1}'\|_{L^2_x}^2}{(1-v^2)^{3/2}} + O(\|(\psi_1(t),\psi_2(t))\|_{H^1_x\times L^2_x} + y_1(t)^4 e^{-2\sqrt{2}y_1(t)})\right] \\ = O(e^{-2\sqrt{2}y_1(t)} + \|(\psi_1(t),\psi_2(t))\|_{H^1_x\times L^2_x}) \qquad (112)\end{aligned}$$

for all $t\in[0,T_1]$.

Step 4 (proof that $y_1\in C^1$). Equations (101) and (108) imply that y_1 satisfies the ordinary differential equation

$$\begin{aligned}(\dot y_1(t)-v)\big[\big\langle C_{v,y_1(t)}(x),\, JD_{v,y_1(t)}(x)\big\rangle - \big\langle C_{v,y_1(t)}(-x),\, JD_{v,y_1(t)}(x)\big\rangle + \big\langle(\psi_1(t),\psi_2(t)),\, J\partial_{y_1}D_{v,y_1(t)}(x)\big\rangle\big] \\ = -v\big\langle(\psi_1(t,x),\psi_2(t,x)),\, J\partial_{y_1}D_{v,y_1(t)}(x)\big\rangle \\ -\big\langle J\,\mathrm{Hess}(y_1(t),x)(\psi_1(t,x),\psi_2(t,x)) + \mathcal{T}(y_1(t),x,\psi_1(t)) + \mathrm{Int}(y_1(t),x),\, JD_{v,y_1(t)}(x)\big\rangle, \qquad (113)\end{aligned}$$

which is a first-order nonautonomous differential system of the form

$$(\dot y_1(t)-v)\alpha_v(t,y_1(t)) = \beta_v(t,y_1(t)),$$

where the functions $\alpha_v,\beta_v:[0,T]\times\mathbb{R}\to\mathbb{R}$ are continuous when $v\in(0,1)$.

Moreover, from the hypotheses of Lemma 27, Lemma 11 and identities (102), (104), (105), we can deduce for any $t\in[0,T]$ that the restrictions of $\alpha_v(t,\cdot)$ and $\beta_v(t,\cdot)$ in the set $\big(3\ln\frac{1}{v},+\infty\big)$ are locally Lipschitz when v is small enough.

Furthermore, from the first step, we have $y_1(0)=\hat y(0)>3\ln\frac{1}{v}$ which implies $y_1(0)^4e^{-2\sqrt{2}y_1(0)}<v^3$ if v is small enough. Moreover, we deduce from (97) and (98) that $\|(\psi_1(0),\psi_2(0))\|_{H^1_x\times L^2_x}\le K\delta_2 v$ and we also have

$$\alpha_v(0,y_1(0)) = \frac{-\|H_{0,1}'\|_{L^2_x}^2}{(1-v^2)^{3/2}} + O(v) > 0,$$

because of the estimate (112) when v is small enough.

Consequently, the Picard–Lindelöf theorem implies the existence of an interval $[0,T_1]\subset[0,T]$ such that $y_1:[0,T_1]\to\mathbb{R}_{>2\ln(1/v)}$ is a C^1 function and since y_1 satisfies (101), we have for any $t\in[0,T_1]$ that

$$\big\langle(\psi_1(t,x),\psi_2(t,x)),\, JD_{v,y_1(t)}(x)\big\rangle = \langle\overrightarrow{\psi}(0,x),\, JD_{v,y_1(0)}(x)\rangle = 0. \qquad (114)$$

Furthermore, since $\hat y(t)\ge 3\ln\frac{1}{v}$, we can deduce from the continuity of function y_1, Lemma 25 and Corollary 26 the identity $y_1(t)=\hat y(t)$ for all $t\in[0,T_1]$. Consequently, $y_1(t)\ge 3\ln\frac{1}{v}$ for all $t\in[0,T_1]$ and

$$\|(\psi_1(t),\psi_2(t))\|_{H^1_x\times L^2_x} = \big\|\overrightarrow{\phi}(t,x) - \overrightarrow{H_{-1,0}}((v,y_1(t)),x) - \overrightarrow{H_{0,1}}((v,y_1(t)),x)\big\|_{H^1_x\times L^2_x} \le K\delta_2 v \qquad (115)$$

for all $t\in[0,T_1]$, because of estimate (97) and identity (98).

Therefore, using a bootstrap argument and estimate (112), we can conclude that the function y_1 is in $C^1[0,T]$ and satisfies (114) for all $t \in [0,T]$. Finally, estimate (96) is a direct consequence of (112), (115) and the fact that $y_1 \geq 3\ln\frac{1}{v}$. ☐

5.2. *Orbital stability of the parameter y.* In this subsection, we consider $\phi(t,x)$ as a solution of (1) having finite energy and with an initial data $(u_1(x), u_2(x))$ satisfying the hypotheses of Theorem 4. Moreover, if v is small enough, from the local well-posedness of the partial differential equation (1) in the space of solutions with finite energy, we can deduce from Lemma 25 the existence of a constant $C>0$ and a positive number ϵ such that, for all $t\in[0,\epsilon]$,

$$(\phi(t,x), \partial_t\phi(t,x)) = \overrightarrow{H_{-1,0}}((v, y(t)), x) + \overrightarrow{H_{0,1}}((v, y(t)), x) + (\psi_1(t,x), \psi_2(t,x)),$$

where $(\psi_1(t,x), \psi_2(t,x))$ is an odd function in x, and $y(t)$, $(\psi_1(t,x), \psi_2(t,x))$ satisfy the orthogonality conditions in Lemma 25 and the inequality

$$|y(t)-y_0| + \|(\psi_1(t,x), \psi_2(t,x))\|_{H^1_x\times L^2_x} \leq 2C\|(u_1,u_2)\|_{H^1_x\times L^2_x}. \tag{116}$$

Finally, we are ready to start the proof of Theorem 4

Remark 28 (main argument). The main techniques of the demonstration of Theorem 4 are inspired by the proof of Theorem 1 of [Kowalczyk et al. 2021].

More precisely, recalling the functions E_+ and P_+ from (85) and (84), we will analyze the function

$$M(\phi(t)) = E_+(\phi(t)) - vP_+(\phi(t)). \tag{117}$$

First, from the local well-posedness of the partial differential equation (1) in the energy space, it is enough to verify Theorem 4 in the case where $(u_1(x), u_2(x))$ is a smooth odd function because the estimate (15) and the density of smooth functions in Sobolev spaces would imply that (15) would be true for any $(u_1(x), u_2(x)) \in H^1_x \times L^2_x$ satisfying the hypothesis of Theorem 4.

Since $P_+(t)$ is not necessarily a conserved quantity, $M(t)$ is not necessarily a constant function given any smooth initial data of $(\phi(0,x), \partial_t\phi(0,x))$ satisfying the hypotheses of Theorem 4.

However, $P_+(t)$ is a nonincreasing function in time, more precisely, for smooth solutions $\phi(t,x)$ of (13), we can verify using integration by parts, from the fact that $\phi(t,x)$ is an odd function in x for any $t\in\mathbb{R}$, the estimate

$$\frac{d}{dt}\left[-\int_0^{+\infty}\partial_t\phi(t,x)\partial_x\phi(t,x)\,dx\right] = \tfrac{1}{2}\phi(t,0)^2 \geq 0. \tag{118}$$

In conclusion, since it was verified before that $E_+(t)$ is a conserved quantity, we have that

$$M(\phi(t)) \leq M(\phi(0)) \quad \text{for any } t\geq 0,$$

and using Lemma 25, we will verify that $M(\phi(0)) - M(\phi(t))$ satisfies a coercive inequality, from which we will deduce (15).

Proof of Theorem 4. From the observations in Remark 28, it is enough to prove Theorem 4 for the case where $\overrightarrow{\psi}_0(x)$ is a smooth odd function. To simplify our proof, we separate the argument into different steps.

Step 1 (local description of solution $\phi(t,x)$). From the observation of inequality (116) and from the Lemma 25, we can verify the existence of an interval $[0,\epsilon]$ such that if $t\in[0,\epsilon]$, then

$$(\phi(t,x),\partial_t\phi(t,x)) = \overrightarrow{H_{-1,0}}((v,y(t)),x)+\overrightarrow{H_{0,1}}((v,y(t)),x)+(\psi_1(t,x),\psi_2(t,x)), \tag{119}$$

with $v(t)$, $y(t)$, $(\psi_1(t,x),\psi_2(t,x))$ satisfying all the conditions of Lemma 25.

Step 2 (estimate of $E_+(\phi(t),\partial_t\phi(t))$ around the kinks). We recall the definition of $E_+(\phi(t),\partial_t\phi(t))$ in (85) given by

$$E_+(\phi(t),\partial_t\phi(t)) = \int_0^{+\infty}\frac{\partial_x\phi(t,x)^2+\partial_t\phi(t,x)^2}{2}+U(\phi(t,x))\,dx.$$

Next, we substitute $\phi(t,x)$ and $\partial_t\phi(t,x)$ in the equation above by the formula of $(\phi(t,x),\partial_t\phi(t,x))$ in Step 1. Using (4), (3) and the fact that $y(t)>1$ for $0\leq t\leq\epsilon$, we obtain for all $x\geq 0$ that

$$\left|\frac{\partial^l}{\partial x^l}H_{-1,0}\left(\frac{x+y(t)}{\sqrt{1-v^2}}\right)\right| \lesssim_l (1-v^2)^{-l/2}e^{-\sqrt{2}(y(t)+x)} \quad \text{for any } l\in\mathbb{N}\cup\{0\}, \tag{120}$$

from which we also deduce, using Lemma 11, the estimate

$$\int_{\mathbb{R}} H_{0,1}'\left(\frac{x-y(t)}{\sqrt{1-v^2}}\right)H_{-1,0}'\left(\frac{x+y(t)}{\sqrt{1-v^2}}\right) \lesssim (1-v^2)^{1/2}y(t)e^{-2\sqrt{2}y(t)}. \tag{121}$$

In addition, since $\|U^{(l)}\|_{L^\infty[-1,1]}<+\infty$ for any $l\in\mathbb{N}$, we can deduce using Lemma 13 the inequality

$$\left\|U^{(l)}\left(H_{0,1}\left(\frac{x-y(t)}{\sqrt{1-v^2}}\right)+H_{-1,0}\left(\frac{x+y(t)}{\sqrt{1-v^2}}\right)\right)\psi_1(t,x)^l\right\|_{H^1_x} \lesssim_l \|\psi_1(t,x)\|^l_{H^1_x}.$$

In conclusion, since

$$\phi(t,x) = H_{0,1}\left(\frac{x-y(t)}{\sqrt{1-v^2}}\right)+H_{-1,0}\left(\frac{x+y(t)}{\sqrt{1-v^2}}\right)+\psi_1(t,x), \tag{122}$$

$$\partial_t\phi(t,x) = -\frac{v}{\sqrt{1-v^2}}H_{0,1}'\left(\frac{x-y(t)}{\sqrt{1-v^2}}\right)+\frac{v}{\sqrt{1-v^2}}H_{-1,0}'\left(\frac{x+y(t)}{\sqrt{1-v^2}}\right)+\psi_2(t,x), \tag{123}$$

we deduce from the formula (85), estimates (120), (121) and Taylor's expansion theorem that

$$\begin{aligned}
&E_+(\phi(t),\partial_t\phi(t))\\
&= \int_0^{+\infty}\frac{1+v^2}{2(1-v^2)}H_{0,1}'\left(\frac{x-y(t)}{\sqrt{1-v^2}}\right)^2+U\left(H_{0,1}\left(\frac{x-y(t)}{\sqrt{1-v^2}}\right)\right)dx\\
&\quad -\frac{1}{\sqrt{1-v^2}}\int_0^{+\infty} vH_{0,1}'\left(\frac{x-y(t)}{\sqrt{1-v^2}}\right)\psi_2(t,x)\,dx - H_{0,1}'\left(\frac{x-y(t)}{\sqrt{1-v^2}}\right)\partial_x\psi_1(t,x)\\
&\quad +\int_0^{+\infty}U'\left(H_{0,1}\left(\frac{x-y(t)}{\sqrt{1-v^2}}\right)\right)\psi_1(t,x)\,dx\\
&\quad +\frac{1}{2}\left[\int_0^{+\infty}\partial_x\psi_1(t,x)^2+U''\left(H_{0,1}\left(\frac{x-y(t)}{\sqrt{1-v^2}}\right)\right)\psi_1(t,x)^2+\psi_2(t,x)^2\right]dx\\
&\quad +O\left((1-v^2)^{-1/2}y(t)e^{-2\sqrt{2}y(t)}\right)+O\left(\|\overrightarrow{\psi}(t)\|_{H^1_x\times L^2_x}e^{-\sqrt{2}y(t)}+\|\psi_1(t,x)\|^3_{H^1_x(\mathbb{R})}\right),
\end{aligned} \tag{124}$$

while $(\phi(t,x), \partial_t\phi(t,x))$ satisfies identities (122) and (123). Moreover, from (122), we can obtain from (124), while $(\phi(t), \partial_t\phi(t))$ satisfies (122) and (123), that

$$
\begin{aligned}
&E_+(\phi(t), \partial_t\phi(t))\\
&= \int_{-\infty}^{+\infty} \frac{1+v^2}{2(1-v^2)} H_{0,1}'\Big(\frac{x-y(t)}{\sqrt{1-v^2}}\Big)^2 + U\Big(H_{0,1}\Big(\frac{x-y(t)}{\sqrt{1-v^2}}\Big)\Big)\,dx\\
&\quad - \frac{1}{\sqrt{1-v^2}}\int_{-\infty}^{+\infty} vH_{0,1}'\Big(\frac{x-y(t)}{\sqrt{1-v^2}}\Big)\psi_2(t,x) - H_{0,1}'\Big(\frac{x-y(t)}{\sqrt{1-v^2}}\Big)\partial_x\psi_1(t,x)\\
&\qquad + \int_{-\infty}^{+\infty} U'\Big(H_{0,1}\Big(\frac{x-y(t)}{\sqrt{1-v^2}}\Big)\Big)\psi_1(t,x)\,dx\\
&\qquad + \frac{1}{2}\Big[\int_0^{+\infty} \partial_x\psi_1(t,x)^2 + U''\Big(H_{0,1}\Big(\frac{x-y(t)}{\sqrt{1-v^2}}\Big)\Big)\psi_1(t,x)^2 + \psi_2(t,x)^2\,dx\Big]\\
&\qquad\quad + O\big((1-v^2)^{-1/2}y(t)e^{-2\sqrt{2}y(t)}\big) + O\big(\|\overrightarrow{\psi}(t)\|_{H_x^1\times L_x^2}e^{-\sqrt{2}y(t)} + \|\psi_1(t,x)\|_{H_x^1(\mathbb{R})}^3\big),
\end{aligned}
\tag{125}
$$

We also recall the Bogomolny identity $H_{0,1}'(x) = \sqrt{2U(H_{0,1}(x))}$, from which we deduce with change of variables that

$$
\frac{1}{2}\int_{\mathbb{R}} H_{0,1}'\Big(\frac{x}{\sqrt{1-v^2}}\Big)^2\,dx = \int_{\mathbb{R}} U\Big(H_{0,1}\Big(\frac{x}{\sqrt{1-v^2}}\Big)\Big)\,dx = \sqrt{1-v^2}\,\frac{\|H_{0,1}'\|_{L_x^2}^2}{2}. \tag{126}
$$

Step 3 (conclusion of the estimate of $E_+(t)$). Since $\overrightarrow{H_{0,1}}((v,y(t)),x)$ is defined by

$$
\overrightarrow{H_{0,1}}((v,y(t)),x) = \begin{bmatrix} H_{0,1}\Big(\frac{x-y(t)}{\sqrt{1-v(t)^2}}\Big) \\ -\frac{v}{\sqrt{1-v^2}}H_{0,1}'\Big(\frac{x-y(t)}{\sqrt{1-v^2}}\Big)\end{bmatrix},
$$

and we can verify by similar reasoning to (124) the identity

$$
E(\overrightarrow{H_{0,1}}((v,y(t)),x)) = \int_{-\infty}^{+\infty} \frac{1+v^2}{2(1-v^2)} H_{0,1}'\Big(\frac{x-y(t)}{\sqrt{1-v^2}}\Big)^2 + U\Big(H_{0,1}\Big(\frac{x-y(t)}{\sqrt{1-v^2}}\Big)\Big)\,dx.
$$

We conclude that $E(\overrightarrow{H_{0,1}}((v,y(t)),x)) = (1/\sqrt{1-v^2})\|H_{0,1}'\|_{L_x^2}^2$. In conclusion, using (125), we obtain that

$$
\begin{aligned}
&E_+(\phi(t), \partial_t\phi(t))\\
&\quad= \frac{1}{\sqrt{1-v^2}}\|H_{0,1}'\|_{L_x^2}^2 - \int_{-\infty}^{+\infty} \frac{v}{\sqrt{1-v^2}} H_{0,1}'\Big(\frac{x-y(t)}{\sqrt{1-v^2}}\Big)\psi_2(t,x)\,dx\\
&\qquad + \int_{-\infty}^{+\infty} \frac{1}{\sqrt{1-v^2}} H_{0,1}'\Big(\frac{x-y(t)}{\sqrt{1-v^2}}\Big)\partial_x\psi_1(t,x) + \int_{-\infty}^{+\infty} U'\Big(H_{0,1}\Big(\frac{x-y(t)}{\sqrt{1-v^2}}\Big)\Big)\psi_1(t,x)\,dx\\
&\qquad\quad + \frac{1}{2}\Big[\int_0^{+\infty} \partial_x\psi_1(t,x)^2 + U''\Big(H_{0,1}\Big(\frac{x-y(t)}{\sqrt{1-v^2}}\Big)\Big)\psi_1(t,x)^2 + \psi_2(t,x)^2\Big]\\
&\qquad\qquad + O\big((1-v^2)^{-1/2}y(t)e^{-2\sqrt{2}y(t)} + \|(\psi_1(t),\psi_2(t))\|_{H_x^1\times L_x^2}e^{-\sqrt{2}y(t)}\big) + O(\|\psi_1(t)\|_{H_x^1(\mathbb{R})}^3).
\end{aligned}
$$

From this using integration by parts we conclude that

$$\begin{aligned}
&E_+(\phi(t), \partial_t\phi(t)) \\
&\quad= \frac{1}{\sqrt{1-v^2}}\|H_{0,1}'\|_{L_x^2}^2 + v\langle J \circ C_{v,y(t)}, \overrightarrow{\psi(t)}\rangle \\
&\qquad+ \frac{1}{2}\Big[\int_0^{+\infty} \psi_2(t,x)^2 + \partial_x\psi_1(t,x)^2 + U''\Big(H_{0,1}\Big(\frac{x-y(t)}{\sqrt{1-v^2}}\Big)\Big)\psi_1(t,x)^2\Big] \\
&\qquad\quad+ O\big((1-v^2)^{-1/2}y(t)e^{-2\sqrt{2}y(t)}\big) + O\big(\|(\psi_1(t), \psi_2(t))\|_{H_x^1\times L_x^2}e^{-\sqrt{2}y(t)} + \|\psi_1(t)\|_{H_x^1}^3\big), \qquad (127)
\end{aligned}$$

where the function $C_{v,y}(x)$ is defined in (88).

Step 4 (estimate of $-vP_+(\phi(t), \partial_t\phi(t))$). First, we recall from (84) that $P_+(\phi(t), \partial_t\phi(t))$ is given by

$$P_+(\phi(t), \partial_t\phi(t)) = -\int_0^{+\infty} \partial_t\phi(t,x)\partial_x\phi(t,x)\,dx.$$

Then, while $(\phi(t,x), \partial_t\phi(t,x))$ satisfies the formula

$$(\phi(t,x), \partial_t\phi(t,x)) = \overrightarrow{H_{-1,0}}((v, y(t)), x) + \overrightarrow{H_{0,1}}((v, y(t)), x) + (\psi_1(t,x), \psi_2(t,x)),$$

using the estimates (120) and (121), we obtain by similar reasoning to the estimate of (2.12) of Lemma 2.3 in [Kowalczyk et al. 2021] that

$$\begin{aligned}
-vP_+(\phi(t), \partial_t\phi(t)) = -\frac{v^2}{\sqrt{1-v^2}}\|H_{0,1}'\|_{L_x^2}^2 - v\langle J \circ C_{v,y(t)}, \vec{\psi}(t)\rangle& \\
+ v\int_0^{+\infty} \partial_x\psi_1(t,x)\psi_2(t,x)\,dx + O\Big(\frac{v^2}{(1-v^2)}y(t)e^{-2\sqrt{2}y(t)}\Big)& \\
+ O\Big(\frac{v}{\sqrt{1-v^2}}e^{-\sqrt{2}y(t)}\|(\psi_1(t), \psi_2(t))\|_{H_x^1\times L_x^2}\Big).& \qquad (128)
\end{aligned}$$

More precisely, the errors in the estimate (128) come from estimate (120) and the Cauchy–Schwarz inequality applied to

$$\int_0^{+\infty} \Big|H_{-1,0}'\Big(\frac{x+y(t)}{\sqrt{1-v^2}}\Big)\Big|[|\partial_x\psi_1(t,x)| + |\psi_2(t,x)|]\,dx,$$

from Lemma 11 applied to the integral

$$\int_0^{+\infty} H_{0,1}'\Big(\frac{x-y(t)}{\sqrt{1-v^2}}\Big)H_{-1,0}'\Big(\frac{x+y(t)}{\sqrt{1-v^2}}\Big)\,dx,$$

and from the elementary estimate

$$\int_{-\infty}^0 H_{0,1}'\Big(\frac{x-y(t)}{\sqrt{1-v^2}}\Big)^2\,dx + \int_0^{+\infty} H_{-1,0}'\Big(\frac{x+y(t)}{\sqrt{1-v^2}}\Big)^2\,dx \lesssim e^{-2\sqrt{2}y(t)},$$

which can be obtained from (120).

Step 5 (estimate and monotonicity of $M(\phi(t), \partial_t\phi(t))$). From estimates (127) and (128), we deduce

$$\begin{aligned} &M(\phi(t), \partial_t\phi(t)) \\ &\quad = E_+(\phi(t), \partial_t\phi(t)) - vP_+(\phi(t), \partial_t\phi(t)) \\ &\quad = \sqrt{1-v^2}\|H_{0,1}'\|_{L_x^2}^2 + \frac{1}{2}\Big[\int_0^{+\infty} \psi_2(t,x)^2 + \partial_x\psi_1(t,x)^2 + U''\Big(H_{0,1}\Big(\frac{x-y(t)}{\sqrt{1-v(t)^2}}\Big)\Big)\psi_1(t,x)^2\,dx\Big] \\ &\qquad + O\big(v\|(\psi_1(t),\psi_2(t))\|_{H_x^1\times L_x^2}^2 + \|(\psi_1(t),\psi_2(t))\|_{H_x^1\times L_x^2} e^{-\sqrt{2}y(t)}\big) \\ &\qquad + O\big(\|\psi_1(t)\|_{H_x^1}^3 + +y(t)e^{-2\sqrt{2}y(t)}\big). \qquad (129) \end{aligned}$$

Furthermore, using estimate (127) and Lemma 11, we can also verify the estimates

$$E_+\big(\overrightarrow{H_{0,1}}(v,y(t)) + \overrightarrow{H_{-1,0}}(v,y(t))\big) = \frac{1}{\sqrt{1-v^2}}\|H_{0,1}'\|_{L_x^2}^2 + O(y(t)e^{-2\sqrt{2}y(t)}),$$

$$P_+\big(\overrightarrow{H_{0,1}}(v,y(t)) + \overrightarrow{H_{-1,0}}(v,y(t))\big) = \frac{v}{\sqrt{1-v^2}}\|H_{0,1}'\|_{L_x^2}^2 + O(y(t)e^{-2\sqrt{2}y(t)}).$$

Therefore, we obtain that

$$M\big(\overrightarrow{H_{0,1}}(v,y(t)) + \overrightarrow{H_{-1,0}}(v,y(t))\big) = \sqrt{1-v^2}\|H_{0,1}'\|_{L_x^2}^2 + O(y(t)e^{-2\sqrt{2}y(t)}), \qquad (130)$$

from which we deduce

$$\begin{aligned} &M(\phi(t), \partial_t\phi(t)) \\ &\quad = M(\overrightarrow{H_{0,1}}(v,y(0)) + \overrightarrow{H_{-1,0}}(v,y(0))) \\ &\qquad + \frac{1}{2}\Big[\int_0^{+\infty} \psi_2(t,x)^2 + \partial_x\psi_1(t,x)^2 + U''\Big(H_{0,1}\Big(\frac{x-y(t)}{\sqrt{1-v^2}}\Big)\Big)\psi_1(t,x)^2\,dx\Big] \\ &\qquad + O\big(\max\{y(t)e^{-2\sqrt{2}y(t)}, y(0)e^{-2\sqrt{2}y(0)}\}\big) + O\big(v\|(\psi_1(t),\psi_2(t))\|_{H_x^1\times L_x^2}^2 + \|(\psi_1(t),\psi_2(t))\|_{H_x^1\times L_x^2}^3\big). \end{aligned}$$

Consequently, since $M(\phi(0), \partial_t\phi(0)) \geq M(\phi(t), \partial_t\phi(t))$ for all $t \geq 0$ and

$$(\phi(0), \partial_t\phi(0)) = \overrightarrow{H_{0,1}}(v,y(0)) + \overrightarrow{H_{-1,0}}(v,y(0)) + (\psi_1(0), \psi_2(0)),$$

we have for every $t \geq 0$ the estimate

$$\begin{aligned} &\int_0^{+\infty} \psi_2(t,x)^2 + \partial_x\psi_1(t,x)^2 + U''\Big(H_{0,1}\Big(\frac{x-y(t)}{\sqrt{1-v^2}}\Big)\Big)\psi_1(t,x)^2\,dx \\ &\qquad \lesssim y(t)e^{-2\sqrt{2}y(t)} + y(0)e^{-2\sqrt{2}y(0)} + v\|(\psi_1(t),\psi_2(t))\|_{H_x^1\times L_x^2}^2 + \|(\psi_1(t),\psi_2(t))\|_{H_x^1\times L_x^2}^3 \\ &\qquad\quad + \|(\psi_1(0),\psi_2(0))\|_{H_x^1\times L_x^2}, \end{aligned}$$

from which with Lemma 34 we deduce for all $t \geq 0$ that

$$\|(\psi_1(t),\psi_2(t))\|_{H_x^1\times L_x^2}^2 \lesssim y(t)e^{-2\sqrt{2}y(t)} + y(0)e^{-2\sqrt{2}y(0)} + \|(\psi_1(0),\psi_2(0))\|_{H_x^1\times L_x^2}, \qquad (131)$$

if $v \ll 1$.

Step 6 (final argument). The last argument is to prove that the set denoted by

$$BO = \left\{t \in \mathbb{R}_{\geq 0} \mid \|(\psi_1(t), \psi_2(t))\|_{H^1_x \times L^2_x} \leq v^{1+\theta/4},\ y(t) \geq y(0) \text{ and (119) is true}\right\} \tag{132}$$

is the proper $\mathbb{R}_{\geq 0}$. From the hypotheses of Theorem 4 and Step 1, we can verify that $0 \in BO$.

Furthermore, from Step 1, we have obtained that there exists $\epsilon > 0$ such that if $0 \leq t \leq \epsilon$, then

$$(\phi(t,x), \partial_t \phi(t,x)) = \vec{H}_{-1,0}((v, y(t)), x) + \vec{H}_{0,1}((v, y(t)), x) + (\psi_1(t,x), \psi_2(t,x))$$

and

$$|y(t) - y_0| + \|(\psi_1(t), \psi_2(t))\|_{H^1_x \times L^2_x} \leq 2C\|(u_1, u_2)\|_{H^1_x \times L^2_x}. \tag{133}$$

Since $\|(u_1, u_2)\|_{H^1_x \times L^2_x} \leq v^{2+\theta}$ and Lemma 25 implies the estimate

$$\|(\psi_1(0), \psi_2(0))\|_{H^1_x \times L^2_x} \lesssim \|(u_1, u_2)\|_{H^1_x \times L^2_x},$$

from (133) and Lemma 27, we deduce the existence of a constant $0 < K$ independent of ϵ and v such that $y(t)$ is a function of class C^1 in $[0, \epsilon]$ and for any $t \in [0, \epsilon]$, the inequality

$$|\dot{y}(t) - v| \leq K[\|(\psi_1(t), \psi_2(t))\|_{H^1_x \times L^2_x} + e^{-2\sqrt{2}y(t)}] \tag{134}$$

is true. Therefore,

$$\dot{y}(t) \geq v - K[\|(\psi_1(t), \psi_2(t))\|_{H^1_x \times L^2_x} + e^{-2\sqrt{2}y(t)}] \tag{135}$$

while $t \in [0, \epsilon]$. Moreover, from inequality (133) and the observations done before, to prove that $[0, \epsilon] \subset BO$ it is only needed to verify that $y(t) \geq y(0)$ for all $t \in [0, \epsilon]$.

First, since $y(t)$ is continuous for $t \in [0, \epsilon]$, there exists $\epsilon_2 \in (0, \epsilon)$ such that if $0 \leq t \leq \epsilon_2$, then

$$y(t) \geq \tfrac{3}{4} y(0),$$

so (133), (135) and the estimate $\|(\psi_1(0), \psi_2(0))\|_{H^1_x \times L^2_x} \lesssim \|(u_1, u_2)\|_{H^1_x \times L^2_x} \leq v^{2+\theta}$ imply that if $0 \leq t \leq \epsilon_2$ and $0 < v \ll 1$, then

$$\dot{y}(t) \geq v - v^2 - Ke^{-3\sqrt{2}y(0)/2} \geq \tfrac{4}{5} v. \tag{136}$$

In conclusion, estimate (133), the hypothesis of $y_0 \geq 4 \ln \frac{1}{v}$ and inequality (136) imply for $v \ll 1$ that if $0 \leq t \leq \epsilon_2$, then $y(t) \geq y(0) + \frac{4}{5} vt$ and $[0, \epsilon_2] \subset BO$.

If $t \in [\epsilon_2, \epsilon]$, it is not difficult to verify that $y(t) \geq y(0)$ in this region. Indeed, the continuity of the function y would imply otherwise the existence of t_i satisfying $\epsilon_2 < t_i \leq \epsilon$, $y(t_i) = y(0)$ and $y(s) > y(0)$ for any $\epsilon_2 \leq s < t_i$, which implies that estimate (136) is true for $t \in [\epsilon_2, t_1]$. But, repeating the argument above, we would conclude that $y(t_i) \geq y(0) + \frac{4}{5} vt_i$, which is a contradiction. In conclusion, the interval $[0, \epsilon]$ is contained in the set BO.

Similarly, from Lemma 27, we can use inequality (135) to verify that $y(t) \geq y(0) + \frac{4}{5} vt$ always when $[0, t] \subset BO$. Therefore, estimate (131) implies

$$\|(\psi_1(t), \psi_2(t))\|_{H^1_x \times L^2_x(x)} \lesssim \|(u_1, u_2)\|^{1/2}_{H^1_x \times L^2_x} + y(0)^{1/2} e^{-\sqrt{2}y(0)} \ll v^{1+\theta/4} \tag{137}$$

if $[0, t] \in BO$.

In conclusion, $BO = \mathbb{R}_{\geq 0}$ and estimates (134), (137) imply the result of Theorem 4 for all $t \geq 0$. □

6. Proof of Theorem 2

First, from Theorem 1.3 in [Chen and Jendrej 2022], we know for any $0 < v < 1$ that there exist $\delta(v) > 0$, $T(v) > 0$ and a solution $\phi(t, x)$ of (1) with finite energy satisfying the identity

$$\phi(t,x) = H_{0,1}\left(\frac{x - vt}{(1-v^2)^{1/2}}\right) + H_{-1,0}\left(\frac{-x - vt}{(1-v^2)^{1/2}}\right) + \psi(t,x), \tag{138}$$

and the decay estimate

$$\sup_{t \geq T} \|(\psi(t,x), \partial_t \psi(t,x))\|_{H_x^1 \times L_x^2} e^{\delta t} < +\infty \tag{139}$$

for any $T \geq T(v)$ and $\delta \leq \delta(v)$. Moreover, we can find $\delta(v), T(v) > 0$ such that

$$\sup_{t \geq T(v)} \|(\psi(t,x), \partial_t \psi(t,x))\|_{H_x^1 \times L_x^2} e^{\delta(v) t} < 1. \tag{140}$$

Indeed, in [Chen and Jendrej 2022] it was proved using fixed point theorem that for any $0 < v < 1$ there is a unique solution of (1) that satisfies (139) for some $T, \delta > 0$.

Next, if we restrict the argument of the proof of Proposition 3.6 of [Chen and Jendrej 2022] to the traveling kink-kink of the ϕ^6 model, we can find explicitly the values of $\delta(v)$ and $T(v)$. More precisely, we have:

Theorem 29. *There is $\delta_0 > 0$ such that if $0 < v < \delta_0$ there exists a unique solution $\phi(t, x)$ of (1) with*

$$h(t,x) = \phi(t,x) - H_{0,1}\left(\frac{x - vt}{(1-v^2)^{1/2}}\right) - H_{-1,0}\left(\frac{x + vt}{(1-v^2)^{1/2}}\right),$$

satisfying (139) for some $0 < \delta < 1$ and $T > 0$. Furthermore, we have if

$$t \geq \frac{4\ln(1/v)}{v}$$

that

$$\|(h(t,x), \partial_t h(t,x))\|_{H_x^1 \times L_x^2} \leq e^{-vt}. \tag{141}$$

This solution is also an odd function on x.

Proof. See Appendix B. □

Finally, we have obtained all the framework necessary to start the demonstration of Theorem 2.

Proof of Theorem 2. First, from Theorem 29, for any $k \in \mathbb{N}$ bigger than 2 and $0 < v \leq \delta_0$, we have that the traveling kink-kink with speed v satisfies for

$$T_{0,k} = \frac{32k\ln(1/v^2)}{2\sqrt{2}v}$$

the estimate

$$\|(h(T_{0,k}), \partial_t h(T_{0,k}))\|_{H_x^1 \times L_x^2} \leq v^{16\sqrt{2}k} \tag{142}$$

for $h(t, x)$ the function denoted in Theorem 29. Now, we start the proof of the second item of Theorem 2.

Step 1 (proof of second item of Theorem 2). First, in the notation of Theorem 8, we consider

$$\phi_k(v,t,x) = \varphi_{k,v}(t, x + \tau_{k,v}).$$

For the $T_{0,k}$ given before, we can verify using Theorems 7 and 8 that

$$\left\|\phi_k(v, T_{0,k}, x) - H_{0,1}\left(\frac{x - vT_{0,k}}{\sqrt{1-v^2}}\right) - H_{-1,0}\left(\frac{x+vT_{0,k}}{\sqrt{1-v^2}}\right)\right\|_{H^1_x} + \left\|\partial_t\phi_k(v, T_{0,k}, x) + \frac{v}{\sqrt{1-v^2}}H'_{0,1}\left(\frac{x - vT_{0,k}}{\sqrt{1-v^2}}\right) - \frac{v}{\sqrt{1-v^2}}H'_{-1,0}\left(\frac{x+vT_{0,k}}{\sqrt{1-v^2}}\right)\right\|_{H^1_x} \leq v^{15k}.$$

In conclusion, Theorem 15 and Remark 22 imply that there is $\Delta_{k,\theta} > 0$ such that if also $v < \Delta_{k,\theta}$, then

$$\|(\phi(t,x), \partial_t\phi(t,x)) - (\phi_k(v,t,x), \partial_t\phi_k(v,t,x))\|_{H^1_x\times L^2_x} < v^{2k-1/2},$$

while

$$|t - T_{0,k}| < \frac{\ln(1/v)^{2-\theta/2}}{v}.$$

Also, Theorems 7 and 8 implies that if $v \ll 1$ and

$$-4\frac{\ln(1/v)^{2-\theta}}{v} \leq t \leq -\frac{\ln(1/v)^{2-\theta}}{v},$$

then there exist $e_{k,v}$ satisfying

$$\left|e_{v,k} - \frac{1}{\sqrt{2}}\ln\left(\frac{8}{v^2}\right)\right| \ll 1$$

such that

$$\left\|\phi_k(v,t,x) - H_{0,1}\left(\frac{x - e_{k,v} + vt}{\sqrt{1-v^2}}\right) - H_{-1,0}\left(\frac{x+e_{k,v} - vt}{\sqrt{1-v^2}}\right)\right\|_{H^1_x} + \left\|\partial_t\phi_k(v,t,x) - \frac{v}{\sqrt{1-v^2}}H'_{0,1}\left(\frac{x - e_{k,v} + vt}{\sqrt{1-v^2}}\right) + \frac{v}{\sqrt{1-v^2}}H'_{-1,0}\left(\frac{x+e_{k,v} - vt}{\sqrt{1-v^2}}\right)\right\|_{H^1_x} \ll v^{2k-1/2}. \tag{143}$$

In conclusion, the second item of Theorem 2 follows from the observation above and Remark 22.

Step 2 (proof of first item of Theorem 2). From Step 1, for

$$t_0 = -\frac{\ln(1/v)^{2-\theta}}{v},$$

we obtained that $\phi(t_0, x)$ satisfies (143). Next, we will study the behavior of $\phi(t,x)$ for $t \leq t_0$, which is equivalent to studying the function $\phi_1(t,x) = \phi(-(t+t_0), x)$ for $t \geq 0$.

However, from the estimate (143), we can verify that $(\phi_1(0,x), \partial_t\phi_1(0,x))$ satisfies the hypotheses of Theorem 4, if we consider $y_0 = e_{k,v} - vt_0$ and $0 < v <\ll 1$. Therefore, using the result of Theorem 4 and the identity $\phi_1(t,x) = \phi(-(t+t_0), x)$, we obtain the first item of Theorem 2. □

Appendix A: Auxiliary estimates

In this appendix, we complement our article by demonstrating complementary estimates.

Lemma 30. *For*

$$\mathcal{G}(x) = e^{-\sqrt{2}x} - \frac{e^{-\sqrt{2}x}}{(1+e^{2\sqrt{2}x})^{3/2}} + x\frac{e^{\sqrt{2}x}}{(1+e^{2\sqrt{2}x})^{3/2}} + k_1\frac{e^{\sqrt{2}x}}{(1+e^{2\sqrt{2}x})^{3/2}},$$

we have that

$$\int_{\mathbb{R}} U^{(3)}(H_{0,1}(x))H'_{0,1}(x)^2\mathcal{G}(x)\,dx = \int_{\mathbb{R}} U^{(3)}(H_{0,1}(x))H'_{0,1}(x)^2 e^{-\sqrt{2}x}\,dx - \sqrt{2}\int_{\mathbb{R}}[U''(H_{0,1}(x))-2]H'_{0,1}(x)e^{-\sqrt{2}x}\,dx.$$

Remark 31. Indeed, the value k_1 in Lemma 30 can be replaced by zero, since

$$\int_{\mathbb{R}} U^{(3)}(H_{0,1}(x))H'_{0,1}(x)^3\,dx = 0.$$

Proof of Lemma 30. First, from identity $H''_{0,1}(x) = U'(H_{0,1}(x))$ and integration by parts, we can verify the identity

$$\int_{\mathbb{R}} U^{(3)}(H_{0,1}(x))H'_{0,1}(x)^2\mathcal{G}(x)\,dx = \int_{\mathbb{R}} U'(H_{0,1}(x))[\mathcal{G}''(x) - U''(H_{0,1})\mathcal{G}(x)]\,dx.$$

Also, since $-\mathcal{G}''(x)+U''(H_{0,1}(x))\mathcal{G}(x)=[U''(H_{0,1}(x))-2]e^{-\sqrt{2}x}+8\sqrt{2}H'_{0,1}(x)$ and $\langle H'_{0,1},\, U'(H_{0,1})\rangle=0$, we conclude using integration by parts that

$$\begin{aligned}\int_{\mathbb{R}} U^{(3)}(H_{0,1}(x))H'_{0,1}(x)^2\mathcal{G}(x)\,dx &= -\int_{\mathbb{R}} U'(H_{0,1}(x))[U''(H_{0,1}(x))-2]e^{-\sqrt{2}x}\,dx\\ &= -\int_{\mathbb{R}} H''_{0,1}(x)[U''(H_{0,1}(x))-2]e^{-\sqrt{2}x}\,dx,\\ &= \int_{\mathbb{R}} U^{(3)}(H_{0,1}(x))H'_{0,1}(x)^2e^{-\sqrt{2}x}\,dx - \sqrt{2}\int_{\mathbb{R}}[U''(H_{0,1}(x))-2]H'_{0,1}(x)e^{-\sqrt{2}x}\,dx. \quad \square\end{aligned}$$

Now, using integration by parts and identity (27) of [Moutinho 2023], we have that

$$-\int_{\mathbb{R}}[U''(H_{0,1}(x))-2]e^{-\sqrt{2}x}H'_{0,1}(x)\,dx = -\sqrt{2}\int_{\mathbb{R}}[6H_{0,1}(x)^5 - 8H_{0,1}(x)^3]e^{-\sqrt{2}x}\,dx = 4, \tag{144}$$

from which we deduce the following lemma.

Lemma 32. $\displaystyle\int_{\mathbb{R}} U^{(3)}(H_{0,1}(x))H'_{0,1}(x)^2\mathcal{G}(x)\,dx - \int_{\mathbb{R}} U^{(3)}(H_{0,1}(x))H'_{0,1}(x)^2e^{-\sqrt{2}x}\,dx = 4\sqrt{2}.$

Lemma 33. *There is* $\delta > 0$, $c > 0$ *such that if*

$$0 < v < \delta, \quad d(t) = \frac{1}{\sqrt{2}}\ln\left(\frac{8}{v^2}\cosh(\sqrt{2}vt)^2\right),$$

then for

$$H^{+}_{0,1}(x,t) = H_{0,1}\left(\frac{x-\frac{1}{2}d(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}\right), \quad H^{-}_{0,1}(x,t) = H_{-1,0}\left(\frac{x+\frac{1}{2}d(t)}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}\right),$$

and any $g \in H^1_x(\mathbb{R})$ *such that*

$$\langle g(x),\, \partial_x H^{+}_{0,1}(x,t)\rangle = 0, \quad \langle g(x),\, \partial_x H^{-}_{0,1}(x,t)\rangle = 0,$$

we have

$$c\|g\|^2_{H^1_x} \le \left\langle -\partial_x^2 g(x) + U''(H^{+}_{0,1}(x,t) + H^{-}_{0,1}(x,t))g(x),\, g(x)\right\rangle. \tag{145}$$

Proof of Lemma 33. First, to simplify our computations we let

$$\gamma_{d(t)} = \frac{1}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}.$$

Next, we can verify using a change of variables that

$$\langle U''(H^{+}_{0,1}(x,t))g(x), g(x)\rangle = \sqrt{1-\tfrac{1}{4}\dot{d}(t)^2}\int_{\mathbb{R}} U''(H_{0,1}(y))\big[g\big(\big(y+\tfrac{1}{2}d(t)\gamma_{d(t)}\big)\gamma^{-1}_{d(t)}\big)\big]^2\,dy,$$

and

$$\int_{\mathbb{R}} \frac{dg(x)}{dx}^2\,dx = \frac{1}{\sqrt{1-\frac{1}{4}\dot{d}(t)^2}}\int_{\mathbb{R}}\left[\frac{d}{dy}[g(y\gamma^{-1}_{d(t)})]\right]^2 dy. \tag{146}$$

We now let

$$g_1(t,y) = g\big(y\sqrt{1-\tfrac{1}{4}\dot{d}(t)^2}\big) = g(y\gamma^{-1}_{d(t)}).$$

Moreover, $L = -\partial_x^2 + U''(H_{0,1}(x))$ is a positive operator in $L^2(\mathbb{R})$ when it is restricted to the orthogonal complement of $H'_{0,1}(x)$ in $L^2_x(\mathbb{R})$; see [Jendrej et al. 2022] or [Moutinho 2023] for the proof. In conclusion, we deduce that there is a constant $C > 0$ independent of $v > 0$ such that

$$\left\langle -\frac{d^2}{dx^2}g(x) + U''(H^{+}_{0,1}(x,t))g(x), g(x)\right\rangle \geq C\sqrt{1-\tfrac{1}{4}\dot{d}(t)^2}\,\|g_1(t,y)\|^2_{H^1_y(\mathbb{R})}, \tag{147}$$

so, from $\dot{d}(t) = v\tanh(\sqrt{2}vt)$ and identity (146), we deduce that there is a constant $C_1 > 0$ such that if $v \ll 1$, then

$$\left\langle -\frac{d^2}{dx^2}g(x) + U''(H^{+}_{0,1}(x,t))g(x), g(x)\right\rangle \geq C_1\|g(x)\|^2_{H^1(\mathbb{R})}. \tag{148}$$

Similarly, we can verify for the same constant $C_1 > 0$ that if $\langle g(x), \partial_x H^{-}_{-1,0}(x,t)\rangle_{L^2_x} = 0$ and $v \ll 1$, then

$$\left\langle -\frac{d^2}{dx^2}g(x) + U''(H^{-}_{0,1}(x,t))g(x), g(x)\right\rangle \geq C_1\|g(x)\|^2_{H^1(\mathbb{R})}. \tag{149}$$

The remaining part of the proof proceeds exactly as the proof of Lemma 2.6 of [Moutinho 2023]. □

Lemma 34. *There exist $C > 1$, $c > 0\delta > 0$ such that if $0 < v < \delta$, then for any $(\varphi_1, \varphi_2) \in H^1_x(\mathbb{R}) \times L^2_x(\mathbb{R})$ we have that*

$$\int_{\mathbb{R}} \varphi_2^2 + \partial_x\varphi_1^2 + U''\left(H_{0,1}\left(\frac{x}{\sqrt{1-v^2}}\right)\right)\varphi_1(x)^2\,dx \geq c\|(\varphi_1,\varphi_2)\|^2_{H^1_x\times L^2_x} - C\langle(\varphi_1,\varphi_2), JD_{v,0}(x)\rangle^2.$$

Proof. The proof is completely analogous to that of property (2) of [Kowalczyk et al. 2021, Lemma 2.8]. □

Appendix B: Proof of Theorem 29

We start by letting

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

and we consider for $x \in \mathbb{R}$ and $-1 < v < 1$ the functions

$$\psi^{0}_{-1,0}(x,v) = J\begin{bmatrix} H'_{-1,0}\Big(\frac{x}{\sqrt{1-v^2}}\Big) \\ \frac{v}{1-v^2} H^{(2)}_{-1,0}\Big(\frac{x}{\sqrt{1-v^2}}\Big)\end{bmatrix}, \tag{150}$$

$$\psi^{1}_{-1,0}(x,v) = J\begin{bmatrix} vxH'_{-1,0}\Big(\frac{x}{\sqrt{1-v^2}}\Big) \\ \frac{1}{\sqrt{1-v^2}} H'_{-1,0}\Big(\frac{x}{\sqrt{1-v^2}}\Big) + \frac{v^2x}{1-v^2} H^{(2)}_{-1,0}\Big(\frac{x}{\sqrt{1-v^2}}\Big)\end{bmatrix}, \tag{151}$$

and we write, for $j \in \{0,1\}$, $\psi^{j}_{0,1}(x,v) = \psi^{j}_{-1,0}(-x,-v)$.

Next, we will use Lemma 2.6 of [Chen and Jendrej 2022].

Lemma 35. *The functions*

$$Y^{0}_{-1,0}(v;x,t) = -J\psi^{0}_{-1,0}(x+vt,v), \tag{152}$$

$$Y^{1}_{-1,0}(v;x,t) = -J\psi^{1}_{-1,0}(x+vt,v) + t\sqrt{1-v^2}\,Y^{0}_{-1,0}(v;x+vt,t) \tag{153}$$

are solutions of the linear differential system

$$\frac{d}{dt}\begin{bmatrix} w_1(t) \\ w_2(t)\end{bmatrix} = J\begin{bmatrix} -\frac{\partial^2}{\partial x^2} + U''\Big(H_{-1,0}\Big(\frac{x+vt}{\sqrt{1-v^2}}\Big)\Big) & 0 \\ 0 & 1\end{bmatrix}\begin{bmatrix} w_1(t) \\ w_2(t)\end{bmatrix}, \tag{154}$$

and the functions

$$Y^{0}_{0,1}(v;x,t) = -J\psi^{0}_{0,1}(x-vt,v), \tag{155}$$

$$Y^{1}_{0,1}(v;x,t) = -J\psi^{1}_{0,1}(x-vt,v) + t\sqrt{1-v^2}\,Y^{0}_{0,1}(v;x-vt,t) \tag{156}$$

are solutions of the linear differential system

$$\frac{d}{dt}\begin{bmatrix} w_1(t) \\ w_2(t)\end{bmatrix} = J\begin{bmatrix} -\frac{\partial^2}{\partial x^2} + U''\Big(H_{0,1}\Big(\frac{x-vt}{\sqrt{1-v^2}}\Big)\Big) & 0 \\ 0 & 1\end{bmatrix}\begin{bmatrix} w_1(t) \\ w_2(t)\end{bmatrix}. \tag{157}$$

Now, similarly to [Chen and Jendrej 2022], we consider the linear operator $L_{+,-}(v,t)$ defined by

$$L_{+,-}(v,t) = \begin{bmatrix} -\frac{\partial^2}{\partial x^2} + U''\Big(H_{0,1}\Big(\frac{x-vt}{\sqrt{1-v^2}}\Big) + H_{-1,0}\Big(\frac{x+vt}{\sqrt{1-v^2}}\Big)\Big) & 0 \\ 0 & 1\end{bmatrix}. \tag{158}$$

We recall that

$$H_{0,1}(x) = \frac{e^{\sqrt{2}x}}{\sqrt{1+e^{2\sqrt{2}x}}}, \quad \text{and} \quad \Big|\frac{d^l}{dx^l}H_{0,1}(x)\Big| \lesssim \min(e^{\sqrt{2}x}, e^{-2\sqrt{2}x}) \quad \text{for any } l \in \mathbb{N}.$$

From now on, we let $\psi^{j}_{-1,0}(v;t,x) = \psi^{j}_{-1,0}(x+vt,v)$ and $\psi^{j}_{0,1}(v;t,x) = \psi^{j}_{-1,0}(x-vt,v)$ for any $j \in \{0,1\}$. Furthermore, using Lemma 11, we can verify similarly to the proof of Proposition 2.8 of [Chen and Jendrej 2022] the following result.

Lemma 36. *There exists $C > 0$, such that for any $0 < v < 1$, we have for all $t \in \mathbb{R}_{\geq 1}$ that*

$$\left\| \frac{\partial}{\partial t}\psi^0_{0,1}(v;t,x) - L_{+,-}J\psi^0_{0,1}(v;t,x)\right\|_{L^2_x} \leq C\exp\left(\frac{-2\sqrt{2}v|t|}{\sqrt{1-v^2}}\right),$$

$$\left\| \frac{\partial}{\partial t}\psi^0_{-1,0}(v;t,x) - L_{+,-}J\psi^0_{-1,0}(v;t,x)\right\|_{L^2_x} \leq C\exp\left(\frac{-2\sqrt{2}v|t|}{\sqrt{1-v^2}}\right),$$

$$\left\| \frac{\partial}{\partial t}\psi^1_{0,1}(v;t,x) - L_{+,-}J\psi^1_{0,1}(v;t,x) + \sqrt{1-v^2}\psi^0_{0,1}(v;t,x)\right\|_{L^2_x} \leq C(|t|v+1)v\exp\left(\frac{-2\sqrt{2}v|t|}{\sqrt{1-v^2}}\right),$$

$$\left\| \frac{\partial}{\partial t}\psi^1_{-1,0}(v;t,x) - L_{+,-}J\psi^1_{-1,0}(v;t,x) + \sqrt{1-v^2}\psi^0_{-1,0}(v;t,x)\right\|_{L^2_x} \leq C(|t|v+1)v\exp\left(\frac{-2\sqrt{2}v|t|}{\sqrt{1-v^2}}\right).$$

Next, we consider a smooth cut function $0 \leq \chi(x) \leq 1$ that satisfies

$$\chi(x) = \begin{cases} 1 & \text{if } x \leq 2(1-10^{-3}), \\ 0 & \text{if } x \geq 2. \end{cases}$$

From now on, for each $0 < v < 1$, we consider $p(v) = \frac{1}{2}v(1-10^{-3})$ and we also let

$$\chi_1(v;t,x) = \chi\left(\frac{x+vt}{p(v)t}\right), \quad \chi_2(v;t,x) = 1-\chi\left(\frac{x+vt}{p(v)t}\right).$$

Lemma 37. *There is $c, \delta_0 > 0$ such that if $0 < v < \delta_0$, then*

$$Q(t,r) = \frac{1}{2}\left[\int_{\mathbb{R}} \partial_t r(t,x)^2 + \partial_x r(t,x)^2 + U''\left(H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right) + H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right) r(t,x)^2\,dx\right] + \sum_{j=1}^{2} v\int_{\mathbb{R}} \chi_j(v;t,x)(-1)^j \partial_t r(t,x)\partial_x r(t,x)\,dx$$

satisfies, for any $t \geq \frac{\ln(1/v)}{v}$,

$$Q(t,r) \geq c\|\vec{r}(t)\|^2_{H^1_x\times L^2_x} - \frac{1}{c}\left[\sum_{j=0}^{1}\langle \vec{r}(t), \psi^j_{-1,0}(v;t)\rangle^2 + \langle \vec{r}(t), \psi^j_{0,1}(v;t)\rangle^2\right].$$

Proof. From the definitions of $\psi^1_{-1,0}$ and $\psi^1_{0,1}$, we can verify that there is a constant $C > 0$ such that if $v \ll 1$, then

$$\left|\left\langle r(t), H'_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right)\right\rangle^2\right| \leq C\left[\langle (r(t), \partial_t r(t)), \psi^1_{0,1}(v;t)\rangle^2 + v^2\|(r(t), \partial_t r(t))\|^2_{H^1_x\times L^2_x}\right], \tag{159}$$

$$\left|\left\langle r(t), \dot{H}_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right\rangle^2\right| \leq C\left[\langle (r(t), \partial_t r(t)), \psi^1_{-1,0}(v;t)\rangle^2 + v^2\|(r(t), \partial_t r(t))\|^2_{H^1_x\times L^2_x}\right]. \tag{160}$$

Then, using the estimates (159) and (160), the proof of Lemma 37 is analogous to the demonstration of Lemma 2.3 of [Jendrej et al. 2022] or the proof of Lemma 2.5 in [Moutinho 2023] or the demonstration of Lemma 33 in Appendix A □

Remark 38. Proposition 2.10 of [Chen and Jendrej 2022] implies that for any $0 < v < 1$, there is T_v and c_v such that Lemma 37 holds with c_v in the place of c for all $t \geq T_v$.

Lemma 39. *There exists $C > 0$ such that, for any $0 < v < 1$, if $f(t,x) \in L^\infty_t(\mathbb{R}; H^1_x(\mathbb{R}))$ and $h(t,x) \in L^\infty_t(\mathbb{R}_{\geq 1}; H^1_x(\mathbb{R})) \cap C^1_t(\mathbb{R}_{\geq 1}; L^2_x(\mathbb{R}))$ is a solution of the integral equation associated to the partial differential equation*

$$\partial_t^2 h(t,x) - \partial_x^2 h(t,x) + U''\left(H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right) + H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right)h(t,x) = f(t,x),$$

for some boundary condition $(h(t_0), \partial_t h(t_0)) \in H^1_x(\mathbb{R}) \times L^2_x(\mathbb{R})$, then

$$Q(t,h) = \frac{1}{2}\left[\int_{\mathbb{R}} \partial_t h(t,x)^2 + \partial_x h(t,x)^2 + U''\left(H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right) + H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right)h(t,x)^2\,dx\right] + \sum_{j=1}^{2} v \int_{\mathbb{R}} \chi_j(v;t,x)(-1)^j \partial_t h(t,x)\partial_x h(t,x)\,dx$$

satisfies

$$\left|\frac{\partial}{\partial t} Q(t,h)\right| \leq C\left[\|f(t)\|_{L^2_x}\|(h(t),\partial_t h(t))\|_{H^1_x\times L^2_x} + \|(h(t),\partial_t h(t))\|^2_{H^1_x\times L^2_x}\left(v\exp\left(\frac{-\sqrt{2}vt(1-10^{-3})^2}{\sqrt{1-v^2}}\right) + \frac{1}{t}\right)\right]$$

for all $t \geq 1$.

Proof. First, from the equation satisfied by $h(t,x)$, we obtain that

$$\int_{\mathbb{R}}\left[\partial_t^2 h(t,x) - \partial_x^2 h(t,x) + U''\left(H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right) + H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right)h(t,x)^2\right]\partial_t h(t,x)\,dx = \int_{\mathbb{R}} f(t,x)\partial_t h(t,x)\,dx. \quad (161)$$

As a consequence, we deduce by integration by parts that

$$\begin{aligned}\frac{d}{dt}&\left[\int_{\mathbb{R}} \partial_t h(t)^2 + \partial_x h(t)^2 + U''\left(H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right) + H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right)h(t)^2\,dx\right] \\ &= -\frac{v}{\sqrt{1-v^2}}\int_{\mathbb{R}} U^{(3)}\left(H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right) + H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right)H'_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right)h(t)^2\,dx \\ &\quad + \frac{v}{\sqrt{1-v^2}}\int_{\mathbb{R}} U^{(3)}\left(H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right) + H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right)H'_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)h(t)^2\,dx \\ &\quad + 2\int_{\mathbb{R}} f(t,x)h(t,x)\,dx. \quad (162)\end{aligned}$$

Next, from the definition of $\chi_1(v;t,x)$ and $\chi_2(v;t,x)$, we can verify for each $j\in\{1,2\}$ that

$$\begin{aligned}&\frac{d}{dt}\left[v\int_{\mathbb{R}}\chi_j(v;t,x)(-1)^j\partial_t h(t,x)\partial_x h(t,x)\,dx\right]\\&\quad=v\int_{\mathbb{R}}\chi_j(v;t,x)(-1)^j\partial_t^2 h(t,x)\partial_x h(t,x)\,dx+v\int_{\mathbb{R}}\chi_j(v;t,x)(-1)^j\partial_t h(t,x)\partial_{t,x}^2 h(t,x)\,dx\\&\qquad+O\left(\|\dot\chi\|_{L^\infty_x(\mathbb{R})}\frac{v}{t}\|(h(t),\partial_t h(t))\|^2_{H^1_x\times L^2_x}\right),\end{aligned}$$

from which we deduce using integration by parts that

$$\begin{aligned}&\frac{d}{dt}\left[v\int_{\mathbb{R}}\chi_j(v;t,x)(-1)^j\partial_t h(t,x)\partial_x h(t,x)\,dx\right]\\&\quad=v\int_{\mathbb{R}}\chi_j(v;t,x)(-1)^j\partial_t^2 h(t,x)\partial_x r(t,x)\,dx+O\left(\|\dot\chi\|_{L^\infty_x(\mathbb{R})}\frac{1}{t}\|(h(t),\partial_t h(t))\|^2_{H^1_x\times L^2_x}\right).\end{aligned}\tag{163}$$

From the equation satisfied by $h(t,x)$, we have that

$$\begin{aligned}&v\int_{\mathbb{R}}\chi_j(v;t,x)(-1)^j\partial_t^2 h(t,x)\partial_x h(t,x)\,dx\\&\quad=v\int_{\mathbb{R}}\chi_j(v;t,x)(-1)^j f(t,x)\partial_x h(t,x)\,dx+v\int_{\mathbb{R}}\chi_j(v;t,x)(-1)^j\partial_x^2 h(t,x)\partial_x h(t,x)\,dx\\&\qquad-v\int_{\mathbb{R}}\chi_j(v;t,x)(-1)^j U''\left(H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right)+H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right)h(t,x)\partial_x h(t,x)\,dx.\end{aligned}$$

So, using integration by parts, we obtain for any $j\in\{1,2\}$ that

$$\begin{aligned}&2\sqrt{1-v^2}\int_{\mathbb{R}}\chi_j(v;t,x)\partial_t^2 h(t,x)\partial_x h(t,x)\,dx\\&\quad=\int_{\mathbb{R}}\chi_j(v;t,x)U^{(3)}\left(H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right)+H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right)H_{0,1}'\left(\frac{x-vt}{\sqrt{1-v^2}}\right)h(t,x)^2\,dx\\&\qquad+\int_{\mathbb{R}}\chi_j(v;t,x)U^{(3)}\left(H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right)+H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right)H_{-1,0}'\left(\frac{x+vt}{\sqrt{1-v^2}}\right)h(t,x)^2\,dx\\&\qquad+O\left(\|\chi'\|_{L^\infty_x(\mathbb{R})}\frac{1}{vt}\|(h(t),\partial_t h(t))\|^2_{H^1_x\times L^2_x}+\|f(t)\|_{L^2_x}\|(h(t),\partial_t h(t))\|_{H^1_x\times L^2_x}\right).\end{aligned}$$

From the definitions of $\chi_1(v;t,x)$ and $\chi_2(v;t,x)$, we can verify for all $t>1$ that

$$\begin{aligned}H_{0,1}'\left(\frac{x-vt}{\sqrt{1-v^2}}\right)\chi_1(v;t,x)&<\sqrt{2}\exp\left(-\frac{\sqrt{2}vt(1+2\times10^{-3})}{\sqrt{1-v^2}}\right),\\\dot H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\chi_2(v;t,x)&<\sqrt{2}\exp\left(-\frac{\sqrt{2}vt(1-10^{-3})^2}{\sqrt{1-v^2}}\right).\end{aligned}$$

In conclusion, we obtain that

$$\begin{aligned}\sum_{j=1}^{2} v\int_{\mathbb{R}} \chi_j(v;t,x)(-1)^j\partial_t^2 h(t,x)\partial_x h(t,x)\,dx \\ = \frac{v}{2\sqrt{1-v^2}}\int_{\mathbb{R}} U^{(3)}\Big(H_{0,1}\Big(\frac{x-vt}{\sqrt{1-v^2}}\Big)+H_{-1,0}\Big(\frac{x+vt}{\sqrt{1-v^2}}\Big)\Big)H_{0,1}'\Big(\frac{x-vt}{\sqrt{1-v^2}}\Big)h(t,x)^2\,dx \\ -\frac{v}{2\sqrt{1-v^2}}\int_{\mathbb{R}} U^{(3)}\Big(H_{0,1}\Big(\frac{x-vt}{\sqrt{1-v^2}}\Big)+H_{-1,0}\Big(\frac{x+vt}{\sqrt{1-v^2}}\Big)\Big)H_{-1,0}'\Big(\frac{x+vt}{\sqrt{1-v^2}}\Big)h(t,x)^2\,dx \\ +O\Big(\|\dot\chi\|_{L^\infty_x(\mathbb{R})}\frac{1}{t}\|(h(t),\partial_t h(t))\|^2_{H^1_x\times L^2_x}+v\|f(t)\|_{L^2_x}\|(h(t),\partial_t h(t))\|_{H^1_x\times L^2_x}\Big) \\ +O\Big(v\exp\Big(-\frac{\sqrt{2}vt(1-10^{-3})^2}{(1-v^2)^{1/2}}\Big)\|h(t,x)\|^2_{H^1_x(\mathbb{R})}\Big). \quad (164)\end{aligned}$$

So, using estimate (164), Lemma 39 will follow from the sum of (162) and (163). □

Lemma 40. *There is $C>0$, such that, for any $0<v<1$, if $f(t,x)\in L^\infty_t(\mathbb{R};H^1_x(\mathbb{R}))$ and $h(t,x)\in L^\infty_t(\mathbb{R}_{\geq 1};H^1_x(\mathbb{R}))\cap C^1_t(\mathbb{R}_{\geq 1};L^2_x(\mathbb{R}))$ is a solution of the integral equation associated to the partial differential equation*

$$\partial_t^2 h(t,x)-\partial_x^2 h(t,x)+U''\Big(H_{0,1}\Big(\frac{x-vt}{\sqrt{1-v^2}}\Big)+H_{-1,0}\Big(\frac{x+vt}{\sqrt{1-v^2}}\Big)\Big)h(t,x)=f(t,x)$$

for some boundary condition $(h(t_0),\partial_t h(t_0))\in H^1_x(\mathbb{R})\times L^2_x(\mathbb{R})$, then for $\vec{h}(t)=(h(t,x),\partial_t h(t,x))$ we have

$$\Big|\frac{d}{dt}\langle\vec{h}(t),\psi^0_{-1,0}(v;t)\rangle\Big|\leq C\Big[\|f(t)\|_{L^2_x(\mathbb{R})}+\|\overrightarrow{h(t)}\|_{H^1_x(\mathbb{R})\times L^2_x(\mathbb{R})}\exp\Big(\frac{-2\sqrt{2}vt}{(1-v^2)^{1/2}}\Big)\Big],$$

$$\Big|\frac{d}{dt}\langle\vec{h}(t),\psi^0_{0,1}(v;t)\rangle\Big|\leq C\Big[\|f(t)\|_{L^2_x(\mathbb{R})}+\|\overrightarrow{h(t)}\|_{H^1_x(\mathbb{R})\times L^2_x(\mathbb{R})}\exp\Big(\frac{-2\sqrt{2}vt}{(1-v^2)^{1/2}}\Big)\Big],$$

$$\begin{aligned}\Big|\frac{d}{dt}\langle\vec{h}(t),\psi^1_{-1,0}(v;t)\rangle+(1-v^2)^{1/2}\langle\vec{h}(t),\psi^0_{-1,0}(v;t)\rangle\Big| \\ \leq C\Big[\|f(t)\|_{L^2_x}+\|\overrightarrow{h(t)}\|_{H^1_x\times L^2_x}(|t|v+1)\exp\Big(\frac{-2\sqrt{2}vt}{(1-v^2)^{1/2}}\Big)\Big],\end{aligned}$$

$$\begin{aligned}\Big|\frac{d}{dt}\langle\vec{h}(t),\psi^1_{0,1}(v;t)\rangle+(1-v^2)^{1/2}\langle\vec{h}(t),\psi^0_{0,1}(v;t)\rangle\Big| \\ \leq C\Big[\|f(t)\|_{L^2_x}+\|\overrightarrow{h(t)}\|_{H^1_x\times L^2_x}(|t|v+1)\exp\Big(\frac{-2\sqrt{2}vt}{(1-v^2)^{1/2}}\Big)\Big].\end{aligned}$$

Proof of Lemma 40. This follows directly from the identity

$$\frac{d}{dt}\vec{h}(t)=JL_{+,-}\vec{h}(t)+\begin{bmatrix}0\\ f(t,x)\end{bmatrix}, \quad (165)$$

and from Lemma 36. □

Proof of Theorem 29. For

$$T_0 \geq \frac{4\ln(1/v)}{v},$$

we consider similarly to [Chen and Jendrej 2022] the norms

$$\|u\|_{L^2_{v,T_0}} = \sup_{t\geq T_0} e^{vt}\|u(t,x)\|_{L^2_x(\mathbb{R})}, \quad \|u\|_{H^1_{v,T_0}} = \sup_{t\geq T_0} e^{vt}[\|u(t,x)\|^2_{H^1_x(\mathbb{R})} + \|\partial_t u(t,x)\|^2_{L^2_x(\mathbb{R})}]^{1/2}.$$

Next, from Lemma 40, we can verify using the fundamental theorem of calculus that there is a constant $C > 1$ such that if $v \ll 1$, then for any $t \geq T_0$ we have that

$$|\langle \vec{h}(t), \psi^0_{-1,0}(v;t)\rangle| \leq C\left[\|f\|_{L^2_{v,T_0}}\frac{e^{-vt}}{v} + \|h\|_{H^1_{v,T_0}}\frac{e^{-(2\sqrt{2}+1)vt}}{v}\right], \tag{166}$$

$$|\langle \vec{h}(t), \psi^1_{-1,0}(v;t)\rangle| \leq C\left[\|f\|_{L^2_{v,T_0}}\frac{e^{-vt}}{v^2} + \|h\|_{H^1_{v,T_0}} te^{-(2\sqrt{2}+1)vt} + \|h\|_{H^1_{v,T_0}}\frac{e^{-(2\sqrt{2}+1)vt}}{v^2}\right], \tag{167}$$

$$|\langle \vec{h}(t), \psi^0_{0,1}(v;t)\rangle| \leq C\left[\|f\|_{L^2_{v,T_0}}\frac{e^{-vt}}{v} + \|h\|_{H^1_{v,T_0}}\frac{e^{-(2\sqrt{2}+1)vt}}{v}\right], \tag{168}$$

$$|\langle \vec{h}(t), \psi^1_{0,1}(v;t)\rangle| \leq C\left[\|f\|_{L^2_{v,T_0}}\frac{e^{-vt}}{v^2} + \|h\|_{H^1_{v,T_0}} te^{-(2\sqrt{2}+1)vt} + \|h\|_{H^1_{v,T_0}}\frac{e^{-(2\sqrt{2}+1)vt}}{v^2}\right]. \tag{169}$$

Also, from Lemma 39, we can verify using the fundamental theorem of calculus for any $t \geq T_0$ that there is a constant $K \geq 1$ such that if $v \ll 1$, then

$$\int_t^{+\infty}\left|\frac{d}{ds}Q(s,h)\right| ds \leq K\left[\frac{e^{-2vt}}{v}\|f\|_{L^2_{v,T_0}}\|h\|_{H^1_{v,T_0}} + \|h\|^2_{H^1_{v,T_0}}\left(\frac{e^{-2vt}}{vt} + e^{-t(2v+\sqrt{2}v(1-10^{-3})^2)}\right)\right]. \tag{170}$$

In conclusion, similarly to Step 1 in the proof of Lemma 3.1 of [Chen and Jendrej 2022], we deduce using the estimates (166)–(170) with Lemma 37 that there exists a new constant $C > 1$ such that for any $t \geq T_0$ and $v \ll 1$ we have

$$\|h\|^2_{H^1_{v,T_0}} \leq \frac{C}{v^4}\|f\|^2_{L^2_{v,T_0}}. \tag{171}$$

The fact that the constant C in (171) is independent of v follows from

$$T_0 \geq \frac{4\ln(1/v)}{v},$$

which implies that

$$\frac{e^{-2vt}}{v^4} + \frac{e^{-2vt}}{vt} \ll v^4.$$

We also observe that if $(g_1(t,x), \partial_t g_1(t,x))$ and $(g_2(t,x), \partial_t g_2(t,x))$ are in the space $(g(t), \partial_t g(t)) \in H^1_x(\mathbb{R}) \times L^2_x(\mathbb{R})$ such that

$$\|(g(t), \partial_t g(t))\|_{L^\infty([T_0,+\infty], H^1_x \times L^2_x)} \leq 1, \tag{172}$$

then, since $U \in C^\infty$, we can verify that the function

$$\begin{aligned} N(v,\vec{g})(t,x) = U'\left(H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right) + H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right) + g(t,x)\right) - U'\left(H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right) \\ - U'\left(H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right)\right) - U''\left(H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right) + H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right)\right)g(t,x) \end{aligned} \tag{173}$$

satisfies, for some new constant $C \geq 1$ and any $v \ll 1$,

$$\|N(v, \overrightarrow{g_1(t)}) - N(v, \overrightarrow{g_2(t)})\|_{H^1_x} \leq C[\|g_1(t)\|_{H^1_x} + \|g_2(t)\|_{H^1_x}]\|g_1(t) - g_2(t)\|_{H^1_x},$$

which implies

$$\|N(v, \overrightarrow{g_1(t)}) - N(v, \overrightarrow{g_2(t)})\|_{H^1_{v,T_0}} \leq Ce^{-vt}[\|g_1\|_{H^1_{v,T_0}} + \|g_2\|_{H^1_{v,T_0}}]\|g_1 - g_2\|_{H^1_{v,T_0}}. \tag{174}$$

In conclusion, by repeating the argument of the proof of Proposition 3.6 of [Chen and Jendrej 2022], we can verify using the Lipschitz estimate of (174) and estimate (171) that if

$$T_0 \geq \frac{4\ln(1/v)}{v} \quad \text{and} \quad v \ll 1,$$

then there exists a map

$$S : \{u \in H^1_{v,T_0} \mid \|u\|_{H^1_{v,T_0}} \leq 1\} \to \{u \in H^1_{v,T_0} \mid \|u\|_{H^1_{v,T_0}} \leq 1\} \tag{175}$$

such that $\mu(t, x) = S(u)(t, x)$ is the unique solution of the equation

$$\partial_t^2 \mu(t, x) - \partial_x^2 \mu(t, x) + U''\left(H_{-1,0}\left(\frac{x + vt}{\sqrt{1 - v^2}}\right) + H_{0,1}\left(\frac{x - vt}{\sqrt{1 - v^2}}\right)\right)\mu(t, x) = N(v, \vec{\mu})(t, x), \tag{176}$$

such that $\mu \in H^1_{v,T_0}$. Indeed, the uniqueness is guaranteed by estimate (171) and from estimates (171) and (174) we have that the map S is a contraction in the set

$$B = \{u \in H^1_{v,T_0} \mid \|u\|_{H^1_{v,T_0}} \leq 1\},$$

and so Theorem 29 follows similarly to the proof of Proposition 3.6 of [Chen and Jendrej 2022] by using Banach's fixed point theorem. □

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THE σ_k-LOEWNER–NIRENBERG PROBLEM ON RIEMANNIAN MANIFOLDS FOR $k < \frac{n}{2}$

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Let (M^n, g_0) be a smooth compact Riemannian manifold of dimension $n \geq 3$ with nonempty boundary ∂M. Let $\Gamma \subset \mathbb{R}^n$ be a symmetric convex cone and f a symmetric defining function for Γ satisfying standard assumptions. Under an algebraic condition on Γ, which is satisfied for example by the Gårding cones Γ_k^+ when $k < \frac{1}{2}n$, we prove the existence of a locally Lipschitz viscosity solution $g_u = e^{2u} g_0$ to the fully nonlinear Loewner–Nirenberg problem associated to (f, Γ),

$$\begin{cases} f(\lambda(-g_u^{-1} A_{g_u})) = 1, \ \ \lambda(-g_u^{-1} A_{g_u}) \in \Gamma & \text{on } M \setminus \partial M, \\ u(x) \to +\infty & \text{as } \operatorname{dist}_{g_0}(x, \partial M) \to 0, \end{cases}$$

where A_{g_u} is the Schouten tensor of g_u. Previous results on Euclidean domains show that, in general, u is not differentiable. The solution u is obtained as the limit of smooth solutions to a sequence of fully nonlinear Loewner–Nirenberg problems on approximating cones containing $(1, 0, \ldots, 0)$, for which we also have uniqueness. In the process, we obtain an existence and uniqueness result for the corresponding Dirichlet boundary value problem with finite boundary data, which is also of independent interest. An important feature of our paper is that the existence of a conformal metric g satisfying $\lambda(-g^{-1} A_g) \in \Gamma$ on M is a *consequence* of our results, rather than an assumption.

1. Introduction

A pertinent theme in conformal geometry is to establish the existence of conformal metrics satisfying some notion of constant curvature. For example, given a compact Riemannian manifold (M^n, g_0) of dimension $n \geq 3$ with nonempty boundary ∂M, a natural question is whether there exists a conformal metric which is complete on $M \setminus \partial M$ and has constant negative scalar curvature on $M \setminus \partial M$. In the seminal work of Loewner and Nirenberg [1974], the authors proved among other results the existence and

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uniqueness of such a metric when $M \setminus \partial M$ is a bounded Euclidean domain with smooth boundary[1] and g_0 is the flat metric. Aviles and McOwen [1988] later extended this result to the Riemannian setting; for further related results we refer, e.g., to [Allen et al. 2018; Andersson et al. 1992; Aviles 1982; Finn 1998; Gover and Waldron 2017; Graham 2017; Han and Shen 2020; Han et al. 2024; Jiang 2021; Li 2022b; Mazzeo 1991; Véron 1981]. We note that the related problem of finding conformal metrics with constant scalar curvature on closed manifolds, known as the Yamabe problem, was solved in [Aubin 1970; Schoen 1984; Trudinger 1968; Yamabe 1960].

Since the works of Viaclovsky [2000] and Chang, Gursky and Yang [Chang et al. 2002], there has been significant interest in fully nonlinear generalisations of Yamabe-type problems, including on manifolds with boundary. Suppose that

$$\Gamma \subset \mathbb{R}^n \text{ is an open, convex, connected symmetric cone with vertex at } 0, \tag{1-1}$$

$$\Gamma_n^+ = \{\lambda \in \mathbb{R}^n : \lambda_i > 0 \; \forall 1 \le i \le n\} \subseteq \Gamma \subseteq \Gamma_1^+ = \{\lambda \in \mathbb{R}^n : \lambda_1 + \cdots + \lambda_n > 0\}, \tag{1-2}$$

$$f \in C^\infty(\Gamma) \cap C^0(\overline{\Gamma}) \text{ is concave, 1-homogeneous and symmetric in the } \lambda_i, \tag{1-3}$$

$$f > 0 \text{ in } \Gamma, \quad f = 0 \text{ on } \partial\Gamma, \quad f_{\lambda_i} > 0 \text{ in } \Gamma \text{ for } 1 \le i \le n. \tag{1-4}$$

In this paper, we study the natural generalisation of the Loewner–Nirenberg problem to the fully nonlinear setting on Riemannian manifolds. That is, for (f, Γ) satisfying (1-1)–(1-4) and a compact Riemannian manifold (M, g_0) with nonempty boundary ∂M, we study the existence and uniqueness of a conformal metric $g_u = e^{2u} g_0$ satisfying

$$\begin{cases} f(\lambda(-g_u^{-1} A_{g_u})) = 1, \quad \lambda(-g_u^{-1} A_{g_u}) \in \Gamma & \text{on } M \setminus \partial M, \\ u(x) \to +\infty & \text{as } \mathrm{d}(x, \partial M) \to 0. \end{cases} \tag{1-5}$$

Here,

$$A_g = \frac{1}{n-2}\left(\mathrm{Ric}_g - \frac{R_g}{2(n-1)} g\right)$$

denotes the $(0, 2)$-Schouten tensor of a Riemannian metric g, Ric_g and R_g denote the Ricci curvature tensor and scalar curvature of g, respectively, $\lambda(T)$ denotes the vector of eigenvalues of a $(1, 1)$-tensor T, and $\mathrm{d}(x, \partial M)$ is the distance from $x \in M$ to ∂M with respect to g_0. Typical examples of (f, Γ) satisfying (1-1)–(1-4) are given by $(\sigma_k^{1/k}, \Gamma_k^+)$ for $1 \le k \le n$, where σ_k is the k-th elementary symmetric polynomial and $\Gamma_k^+ = \{\lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0 \; \forall 1 \le j \le k\}$. When $f = \sigma_1$, (1-5) reduces to the original Loewner–Nirenberg problem on Riemannian manifolds discussed above.

Much of the motivation to study (1-5) stems from the fact that, as a consequence of the Ricci decomposition, the Schouten tensor fully determines the conformal transformation properties of the full Riemann curvature tensor. We note that, for $g_u = e^{2u} g_0$, one has the transformation law

$$A_{g_u} = -\nabla_{g_0}^2 u - \tfrac{1}{2} |\nabla_{g_0} u|_{g_0}^2 g_0 + du \otimes du + A_{g_0}, \tag{1-6}$$

which demonstrates the fully nonlinear nature of (1-5) when $f \ne c\sigma_1$. Moreover, (1-5) is nonuniformly elliptic when $f \ne c\sigma_1$.

[1] Loewner and Nirenberg [1974] also considered the problem on a class of nonsmooth Euclidean domains, but we will not be concerned with such generalisations in this paper.

By the 1-homogeneity of f, without loss of generality we may assume

$$f\left(\tfrac{1}{2}, \dots, \tfrac{1}{2}\right) = 1. \tag{1-7}$$

As in [Li and Nguyen 2014], we define μ_Γ^+ to be the number satisfying

$$(-\mu_\Gamma^+, 1, \dots, 1) \in \partial\Gamma.$$

We note that μ_Γ^+ is uniquely determined by Γ and is easily seen to satisfy $\mu_\Gamma^+ \in [0, n-1]$. When $\Gamma = \Gamma_k^+$, one has $\mu_{\Gamma_k^+}^+ = (n-k)/k$.

Our first main result concerns the solution to the Loewner–Nirenberg problem (1-5) under the assumption

$$\mu_\Gamma^+ > 1. \tag{1-8}$$

Observe that, for $\Gamma = \Gamma_k^+$, (1-8) holds if and only if $k < \frac{1}{2}n$. The role of condition (1-8) will be discussed later in the introduction.

Theorem 1.1. *Let (M, g_0) be a smooth compact Riemannian manifold of dimension $n \geq 3$ with nonempty boundary ∂M, and suppose (f, Γ) satisfies (1-1)–(1-4), (1-7) and (1-8). Then there exists a locally Lipschitz viscosity solution to (1-5) satisfying*

$$\lim_{\mathrm{d}(x,\partial M)\to 0} (u(x) + \ln \mathrm{d}(x, \partial M)) = 0, \tag{1-9}$$

which is maximal in the sense that if $\tilde{u}$ is any continuous viscosity solution to (1-5), then $\tilde{u} \leq u$ on $M \setminus \partial M$. Moreover, when $(1, 0, \dots, 0) \in \Gamma$, u is smooth and is the unique continuous viscosity solution to (1-5).

We recall that a continuous function u on $M\setminus\partial M$ is a viscosity subsolution (resp. viscosity supersolution) to the equation in (1-5) if, for any $x_0 \in M \setminus \partial M$ and $\varphi \in C^2(M \setminus \partial M)$ satisfying $u(x_0) = \varphi(x_0)$ and $u(x) \leq \varphi(x)$ near x_0 (resp. $u(x) \geq \varphi(x)$ near x_0), we have $\lambda(-g_\varphi^{-1} A_{g_\varphi})(x_0) \in \{\lambda \in \Gamma : f(\lambda) \geq 1\}$ (resp. $\lambda(-g_\varphi^{-1} A_{g_\varphi})(x_0) \in \mathbb{R}^n\setminus\{\lambda \in \Gamma : f(\lambda) > 1\}$). We say that u is a viscosity solution to the equation in (1-5) if it is both a viscosity subsolution and a viscosity supersolution.

Remark 1.2. In previous work studying equations of the form $f(\lambda(-g_u^{-1} A_{g_u})) = 1$, it has been typical to assume that the background metric g_0 satisfies $\lambda(-g_0^{-1} A_{g_0}) \in \Gamma$ on M (a notable exception is a result of Gursky, Streets and Warren [Gursky et al. 2011], which will be discussed later in the introduction). In contrast, one of the key points of this paper is that we do not assume the existence of such a metric in Theorem 1.1. Rather, the existence of such a metric is established as a by-product of the proof of Theorem 1.1 (see Theorem 1.6), and our proof of Theorem 1.1 would not be substantially simpler even if we were to assume the existence of such a metric from the outset. We note that after our work was submitted, Professor Rirong Yuan [2024] brought to our attention his work, where a conformal metric satisfying $\lambda(-g^{-1} A_g) \in \Gamma$ is constructed under the assumption (1-8) by an entirely different method. See also [Yuan 2022], which considers the existence problem for (1-5) assuming $\lambda(-g_0^{-1} A_{g_0}) \in \Gamma$ and $(1, 0, \dots, 0) \in \Gamma$, and addresses (1-9) and uniqueness of solutions under an even stronger assumption on Γ (see condition (1.20) therein).

Remark 1.3. In the case that $M \setminus \partial M$ is a Euclidean domain, the existence of a Lipschitz viscosity solution to (1-5) was established by Gonzáles, Li and Nguyen [González et al. 2018]. It was also shown in their work that this solution is unique among continuous viscosity solutions. We note that the uniqueness of the viscosity solution obtained in Theorem 1.1 remains an open problem when $M \setminus \partial M$ is not a Euclidean domain and $(1, 0, \dots, 0) \in \partial\Gamma$.

Remark 1.4. In [Li and Nguyen 2021; Li et al. 2023] it was shown that if $M \setminus \partial M$ is a Euclidean domain with disconnected boundary and $\Gamma \subset \Gamma_2^+$ (in particular, this implies $(1, 0, \dots, 0) \in \partial\Gamma$), then the Lipschitz viscosity solution to (1-5) is not differentiable. Thus, in general, the Lipschitz regularity of the solution in Theorem 1.1 cannot be improved to C^1 regularity when $(1, 0, \dots, 0) \in \partial\Gamma$. On the other hand, the existence of a unique smooth solution to (1-5) satisfying (1-9) when $(1, 0, \dots, 0) \in \Gamma$ is new even when $M \setminus \partial M$ is a Euclidean domain. This smoothness result can be viewed as an analogue of the result in [Gursky and Viaclovsky 2003] on the existence of a smooth solution to the σ_k-Yamabe problem for the trace-modified Schouten tensor on closed manifolds.

To describe the proof of Theorem 1.1, we first introduce some notation and an equivalent formulation of the result. For $\tau \in [0, 1]$, $\lambda \in \mathbb{R}^n$ and $e = (1, \dots, 1) \in \mathbb{R}^n$, we define

$$\lambda^\tau := \tau\lambda + (1-\tau)\sigma_1(\lambda)e, \quad f^\tau(\lambda) := \frac{f(\lambda^\tau)}{\tau + n(1-\tau)} \quad \text{and} \quad \Gamma^\tau := \{\lambda : \lambda^\tau \in \Gamma\}.$$

As shown in [Duncan and Nguyen 2023, Appendix A], Γ satisfies (1-1), (1-2) and $(1, 0, \dots, 0) \in \Gamma$ if and only if there exists $\widetilde{\Gamma}$ satisfying (1-1), (1-2) and a number $\tau < 1$ for which $\Gamma = (\widetilde{\Gamma})^\tau$. Note that (1-7) implies $f^\tau\big(\frac{1}{2}, \dots, \frac{1}{2}\big) = 1$. An equivalent formulation of Theorem 1.1 is then as follows.

Theorem 1.1′. *Let (M, g_0) be a smooth compact Riemannian manifold of dimension $n \geq 3$ with nonempty boundary ∂M, and suppose (f, Γ) satisfies (1-1)–(1-4), (1-7) and (1-8). Then, for each $\tau < 1$, there exists a smooth solution u to*

$$\begin{cases} f^\tau(\lambda(-g_u^{-1}A_{g_u})) = 1, \ \ \lambda(-g_u^{-1}A_{g_u}) \in \Gamma^\tau & \text{on } M \setminus \partial M, \\ u(x) \to +\infty & \text{as } d(x, \partial M) \to 0, \end{cases} \tag{1-10}$$

and moreover u satisfies (1-9) and is the unique continuous viscosity solution to (1-10). When $\tau = 1$, there exists a Lipschitz viscosity solution u to (1-10) satisfying (1-9), which is maximal in the sense that if $\tilde{u}$ is any continuous viscosity solution to (1-10), then $\tilde{u} \leq u$ on $M \setminus \partial M$.

Remark 1.5. If we label the solution to (1-10) in Theorem 1.1′ as u^τ for each $\tau \leq 1$, then we will show that, for each compact set $K \subset M \setminus \partial M$, there exists a constant C which is independent of τ but dependent on M, g_0, f, Γ and K such that

$$\|u^\tau\|_{C^{0,1}(K)} \leq C \quad \text{for all } \tau \in [0, 1].$$

In the proof of Theorem 1.1′, we will first prove the existence of a unique smooth solution to (1-10) when $\tau < 1$. The Lipschitz viscosity solution in the case $\tau = 1$ is then obtained in the limit as $\tau \to 1$. In turn, for each $\tau < 1$, the existence of a smooth solution to (1-10) is obtained as the limit of smooth solutions to Dirichlet boundary value problems with finite boundary data. Although we only need to consider constant boundary data in the proof of Theorem 1.1′, we will prove the following more general result.

Theorem 1.6. *Let (M, g_0) be a smooth compact Riemannian manifold of dimension $n \geq 3$ with nonempty boundary ∂M, and suppose (f, Γ) satisfies (1-1)–(1-4) and (1-8). Let $\psi \in C^\infty(M)$ be positive and $\xi \in C^\infty(\partial M)$. Then, for each $\tau < 1$, there exists a smooth solution u to*

$$\begin{cases} f^\tau(\lambda(-g_u^{-1}A_{g_u})) = \psi, \ \ \lambda(-g_u^{-1}A_{g_u}) \in \Gamma^\tau & \text{on } M \setminus \partial M, \\ u = \xi & \text{on } \partial M, \end{cases} \tag{1-11}$$

and moreover u is the unique continuous viscosity solution to (1-11). When $\tau = 1$, there exists a Lipschitz viscosity solution to (1-11).

Remark 1.7. If we label the solution to (1-11) in Theorem 1.6 as u^τ for each $\tau \leq 1$, then we will show that there exists a constant C which is independent of τ but dependent on $M,\ g_0,\ f,\ \Gamma,\ \psi$ and ξ such that

$$\|u^\tau\|_{C^{0,1}(M)} \leq C \quad \text{for all } \tau \in [0, 1].$$

The existence of a smooth solution to (1-11) when $\tau < 1$ is achieved using the continuity method, which relies on obtaining a priori estimates. To keep the introduction concise, we only discuss the C^0 estimates here and postpone the discussion of the other estimates to the main body of the paper. Now, if one assumes $\lambda(-g_0^{-1}A_{g_0}) \in \Gamma$ on M, then it is straightforward to obtain both the a priori upper and lower bounds on solutions to (1-11). Since we do not make such an assumption on g_0, a large portion of our work involves proving the lower bound. The a priori lower bound is obtained in two independent stages, which can be summarised as follows:

(1) First, in Section 2, we prove a local interior gradient estimate on solutions to (1-11) of the form

$$|\nabla_{g_0} u|_{g_0}(x) \leq C(r^{-1} + e^{\sup_{B_r} u}) \quad \text{for } x \in B_{r/2}, \tag{1-12}$$

where B_r is a geodesic ball contained in the interior of M. An important feature is that the estimate (1-12) does not depend on a lower bound for u.

(2) Second, in Section 3.2, we construct suitable barrier functions to prove a lower bound for u in a *uniform neighbourhood of* ∂M — this is one of the key new ideas in this paper.

We note that the assumption $\mu_\Gamma^+ > 1$ is used in both stages above. Once the lower bound in a uniform neighbourhood of ∂M is established in the second step, the local interior gradient estimate from the first step and a trivial global upper bound in Proposition 3.1 then allows one to propagate the lower bound to all of M — see the proof of Proposition 3.2 for the details. As indicated in Remark 1.7 above, it is important that all estimates in the two steps above (as well as the boundary gradient estimates obtained in the main body of the paper — see Proposition 3.8) are independent of τ.

In fact, the proof of Step (2) provides a purely local lower bound: if $x_0 \in \partial M$ and u solves (1-11) in $M \cap B_r(x_0)$, then $u \geq C$ in $M \cap B_{r/2}(x_0)$. In our subsequent work [Duncan and Nguyen 2025], we show that this local lower bound cannot hold when (1-8) fails, that is when $\mu_\Gamma^+ \leq 1$.

We now discuss the two steps above in more detail. Our local interior gradient estimate, which is also of independent interest, is as follows:

Theorem 1.8. *Let (M, g_0) be a smooth Riemannian manifold of dimension $n \geq 3$, possibly with nonempty boundary, and suppose (f, Γ) satisfies (1-1)–(1-4) and (1-8). Fix $\tau \in (0, 1]$, fix a positive function $\psi \in C^\infty(M)$ and suppose that $u \in C^3(B_r)$ satisfies*

$$f^\tau(\lambda(-g_u^{-1}A_{g_u})) = \psi, \quad \lambda(-g_u^{-1}A_{g_u}) \in \Gamma^\tau \tag{1-13}$$

in a geodesic ball B_r contained in the interior of M. Then

$$|\nabla_{g_0} u|_{g_0}(x) \leq C(r^{-1} + e^{\sup_{B_r} u}) \quad \text{for } x \in B_{r/2}, \tag{1-14}$$

where C is a constant depending on n, f, Γ, $\|g_0\|_{C^3(B_r)}$ and $\|\psi\|_{C^1(B_r)}$ but independent of τ and $\inf_{B_r} \psi$.

We note that Theorem 1.8 was previously obtained for $(f, \Gamma) = (\sigma_k^{1/k}, \Gamma_k^+)$ when $k < \frac{1}{2}n$ and $\tau = 1$ in the thesis of Khomrutai [2009].[2] Roughly speaking, one important observation in the thesis is as follows: if $\rho|\nabla_{g_0} u|_{g_0}^2$ attains its maximum at x_0 (here ρ is a cutoff function satisfying standard assumptions), then in a "worst case scenario" (i.e., in a situation where the gradient estimate cannot be obtained somewhat directly), the ordered eigenvalues $\lambda_1(x_0) \geq \cdots \geq \lambda_n(x_0)$ of $(-g_0^{-1}A_{g_u})(x_0)$ are greater than or equal to a perturbation of $(1, \ldots, 1, -1)\frac{1}{2}|\nabla u|^2(x_0)$. But when $k < \frac{1}{2}n$, the vector $(1, \ldots, 1, -1)$ belongs to Γ_k^+, and so by (1-13) and homogeneity of $\sigma_k^{1/k}$, the gradient estimate follows. In our proof of Theorem 1.8, we show that this phenomenon persists for general cones satisfying $\mu_\Gamma^+ > 1$. In order to circumvent certain arguments of Khomrutai that rely on algebraic properties of the σ_k operators, we appeal to some general cone properties recently observed by Yuan [2022].

Remark 1.9. For gradient estimates on solutions to equations of the form (1-13) which depend on two-sided C^0 bounds, see for instance [Guan 2008; Gursky and Viaclovsky 2003]. For gradient estimates for the related positive cone equation, see e.g., [Chen 2005; Guan and Wang 2003; Jin et al. 2007; Li 2009; Li and Li 2003; Viaclovsky 2002; Wang 2006].

Remark 1.10. We have been informed that in an upcoming work of Baozhi Chu, YanYan Li and Zongyuan Li [Chu et al. 2023], a Liouville-type theorem for a fully nonlinear, degenerate elliptic Yamabe-type equation on negative cones is proved for all $\mu_\Gamma^+ \neq 1$. As an application of this Liouville-type theorem and the method in [Li 2009] (which dealt with local gradient estimates for equations on positive cones), the authors obtain local interior gradient estimates for solutions to (1-13) depending only on one-sided C^0 bounds for all $\mu_\Gamma^+ \neq 1$ without assuming concavity of f. Counterexamples to both results are also given when $\mu_\Gamma^+ = 1$. This proof is entirely different from our proof of Theorem 1.8.

We now turn to the second step mentioned above, namely the lower bound in a neighbourhood of ∂M. This is achieved through constructing suitable comparison functions on small annuli; the main step here is to prove the following proposition (see Proposition 3.4 for a more precise version).

Proposition 1.11. *Suppose (f, Γ) satisfies (1-1)–(1-4) and (1-8), let g_0 be a Riemannian metric defined on a neighbourhood Ω of the origin in $\mathbb{R}^n$, let $m \in \mathbb{R}$ and define $A_{r_-, r_+} := \{x : r_- < \mathrm{d}_{g_0}(x, 0) < r_+\}$.*

[2] We would like to thank Baozhi Chu, YanYan Li and Zongyuan Li for bringing [Khomrutai 2009] to our attention.

Then there exist constants $S > 1$ and $0 < R < 1$ depending on g_0, f, Γ and m such that, whenever $1 < r_+/r_- < S$ and $r_+ < R$, there exists a solution to

$$\begin{cases} f(\lambda(-g_w^{-1}A_{g_w})) \geq 1, \quad \lambda(-g_w^{-1}A_{g_w}) \in \Gamma & \text{on } A_{r_-,r_+}, \\ w(x) = m & \text{for } x \in \mathbb{S}_{r_-}, \\ w(x) \to -\infty & \text{as } \mathrm{d}_{g_0}(x, \mathbb{S}_{r_+}) \to 0. \end{cases}$$

Our construction of w in Proposition 1.11 is modelled on the radial solutions of Chang, Han and Yang [Chang et al. 2005] to the σ_k-Yamabe equation on annular domains in $\mathbb{R}^n$ when $k < \frac{1}{2}n$. To apply Proposition 1.11 to complete the second step, we attach a collar neighbourhood N to ∂M and cover a neighbourhood of ∂M in M by sufficiently small annuli whose centres lie in N and whose inner boundaries touch ∂M. On each of these annuli, the solutions constructed in Proposition 1.11 then serve as the desired lower bound by the comparison principle. See the proof of Proposition 3.3 for details.

Remark 1.12. The assumption $\mu_\Gamma^+ > 1$ plays an important role in our proof of Proposition 1.11, and in fact a similar construction is not possible when $\mu_\Gamma^+ \leq 1$. More precisely, given a smooth metric g_0 defined on an annulus $A_{r,R}$ and given a cone Γ satisfying (1-1), (1-2) and $\mu_\Gamma^+ \leq 1$, there is no smooth metric $g_w = e^{2w}g_0$ satisfying $\lambda(-g_w^{-1}A_{g_w}) \in \Gamma$ on $A_{r,R}$ and for which $w \to -\infty$ at either boundary component of $A_{r,R}$. The proof of this nonexistence result uses arguments different in nature to those considered in this paper and appears in our more recent work [Duncan and Nguyen 2025].

For the remainder of the introduction, we discuss in more detail how our results and methods compare to previous work on fully nonlinear problems of Loewner–Nirenberg type. As mentioned before, when $M \setminus \partial M$ is a Euclidean domain, the existence of a Lipschitz viscosity solution to (1-5), as well as uniqueness of this solution among continuous viscosity solutions, was established in [González et al. 2018]. Moreover, counterexamples to C^1 regularity were given in [Li and Nguyen 2021; Li et al. 2023]. The proof in [González et al. 2018] uses Perron's method, which in turn uses canonical solutions on interior/exterior balls and a comparison principle on Euclidean domains established in [Li et al. 2018]. Since one cannot use exterior balls in the Riemannian setting and since it is not currently known whether the comparison principle in [Li et al. 2018] extends to the Riemannian setting, a different approach to that in [González et al. 2018] is required to prove Theorem 1.1′.

On the other hand, for $(f, \Gamma) = (\sigma_k^{1/k}, \Gamma_k^+)$, $2 \leq k \leq n$, Gursky, Streets and Warren [Gursky et al. 2011] proved the existence of a unique smooth solution to (1-5) with the Ricci tensor in place of the Schouten tensor (see Remark 1.13 below for the relation between this result and Theorem 1.1′, and see also [Wang 2021; Li 2022a] for some further related results). As in the present paper, the solution of Gursky, Streets and Warren is constructed as a limit of solutions with finite boundary data, and these solutions are in turn obtained using the continuity method. Their method for obtaining an a priori lower bound on solutions is different to ours and is instead based on the explicit construction of a global subsolution. Roughly speaking, the subsolution construction in [Gursky et al. 2011] uses the fact that, in the analogous formula to (1-6) for the Ricci tensor, the gradient terms are collectively nonnegative definite and so can be neglected in certain computations. In our case, the gradient terms do not have an overall sign, thus leading to our new approach for the lower bound discussed above.

Remark 1.13. Since $\mu^+_{\Gamma^+_k} = (n-k)/k$, it is easy to see that $\mu^+_{(\Gamma^+_k)^\tau} = (n-k)/k + (n-1)(1-\tau)$. Thus

$$\mu^+_{(\Gamma^+_k)^\tau} > 1 \quad \text{if and only if} \quad \tau < a_{n,k} := \frac{n-k+k(n-2)}{k(n-1)}.$$

On the other hand, for $\tau = (n-2)/(n-1)$, we have

$$(\sigma_k^{1/k})^\tau(\lambda(-g_u^{-1}A_{g_u})) = \frac{1}{n-1}\cdot\sigma_k^{1/k}(\lambda(-g_u^{-1}\operatorname{Ric}_{g_u})).$$

Since $(n-2)/(n-1) < a_{n,k}$ if and only if $k < n$, we therefore see that Theorem 1.1′ recovers the result of [Gursky et al. 2011] for $k < n$.

The plan of the paper is as follows: In Section 2 we prove the local interior gradient estimate stated in Theorem 1.8. In Section 3 we consider the Dirichlet boundary value problem (1-11), proving Theorem 1.6. Finally, in Section 4 we turn to the fully nonlinear Loewner–Nirenberg problem (1-10), proving Theorem 1.1′ (and hence Theorem 1.1).

Notation. Throughout the rest of the paper, if X is a $(1,1)$-tensor satisfying $\lambda(X) \in \Gamma$ then we frequently write $f(X) := f(\lambda(X))$.

2. Proof of Theorem 1.8: the local interior gradient estimate

In this section we prove the local interior gradient estimate stated in Theorem 1.8. Throughout the section, unless otherwise stated all derivatives and norms are taken with respect to g_0. Moreover, C will denote a constant that may change from line to line and depends only on n, f, Γ, $\|g_0\|_{C^3(B_r)}$ and $\|\psi\|_{C^1(B_r)}$.

2.1. ***Set-up and main ideas of the proof.*** Our set-up for the proof of Theorem 1.8 is similar to that in the related works [Chen 2005; Guan and Wang 2003; Jin et al. 2007; Khomrutai 2009; Li 2009; Li and Li 2003; Wang 2006] on local gradient estimates. Throughout this section we write $S = A_{g_0}$ and

$$W = \nabla^2 u + \tfrac{1}{2}|\nabla u|^2 g_0 - du\otimes du - S.$$

By a standard argument, it suffices to consider the case $r = 1$ in the proof of Theorem 1.8. Suppose $\rho \in C_c^\infty(B_1)$ is a cutoff function in B_1 with $\rho = 1$ on $B_{1/2}$, $|\nabla\rho| \le C\rho^{1/2}$ and $|\nabla^2\rho| \le C$. Set $H = \rho|\nabla u|^2$ and suppose H attains a maximum at x_0. We may assume that $|\nabla u| \ge 1$ at x_0, otherwise we are done. Choosing suitable normal coordinates centred at x_0, we may also assume $W = (w_{ij})$ is diagonal at x_0 with $w_{11} \ge \cdots \ge w_{nn}$, and hence at x_0 we have

$$\begin{cases} w_{ii} = u_{ii} - u_i^2 + \frac{1}{2}|\nabla u|^2 - S_{ii} & \text{for all } 1 \le i \le n, \\ u_{ij} = u_i u_j + S_{ij} & \text{for } i \ne j. \end{cases} \tag{2-1}$$

Using the fact that $H_i(x_0) = 0$ for each i, we obtain at x_0

$$\sum_{l=1}^n u_{il}u_l = -\frac{\rho_i}{2\rho}|\nabla u|^2, \tag{2-2}$$

and hence

$$\left|\sum_{l=1}^{n} u_{il}u_l\right| \le C\rho^{-1/2}|\nabla u|^2. \tag{2-3}$$

For A_0 a large number to be fixed later, we may assume at x_0 that

$$\rho^{-1/2} \le C\frac{|\nabla u|}{A_0} \quad \text{and} \quad |S| \le \frac{|\nabla u|^2}{A_0}, \tag{2-4}$$

otherwise we are done. Note that, by combining (2-3) with the first estimate in (2-4), we have

$$\left|\sum_{l=1}^{n} u_{il}u_l\right| \le C\frac{|\nabla u|^3}{A_0}. \tag{2-5}$$

Denote by F_τ^{ij} the coefficients of the linearised operator at $(g_0^{-1}W)(x_0)$, that is,

$$F_\tau^{ij} = \frac{\partial f^\tau}{\partial A_{ij}}\bigg|_{A=(g_0^{-1}W)(x_0)}.$$

Then (F_τ^{ij}) is a positive definite, diagonal matrix. Also define

$$\mathcal{F}_\tau = \sum_{i=1}^{n} F_\tau^{ii} \quad \text{and} \quad \tilde{u}_{ij} := u_{ij} - S_{ij}.$$

By homogeneity and concavity of f, it is easy to see that $\mathcal{F}_\tau \ge 1/C > 0$: indeed, writing $\lambda = \lambda(g_0^{-1}W)(x_0)$, we have

$$\mathcal{F}_\tau = \sum_{i=1}^{n} \frac{\partial f^\tau}{\partial \lambda_i}(\lambda) = f^\tau(\lambda) + \sum_{i=1}^{n} \frac{\partial f^\tau}{\partial \lambda_i}(\lambda)(1-\lambda_i) \ge f^\tau(1,\dots,1). \tag{2-6}$$

With our set-up and notation established, we now briefly discuss the main ideas in the proof of Theorem 1.8. The first step is to obtain the following lemma.

Lemma 2.1. *Under the same hypotheses as Theorem 1.8 but without the restriction $\mu_\Gamma^+ > 1$, there exists a constant C such that*

$$0 \ge -C\mathcal{F}_\tau(1+e^{2u})|\nabla u|^2 - C\rho\mathcal{F}_\tau\frac{|\nabla u|^4}{A_0} + \rho\sum_{i,l} F_\tau^{ii}\tilde{u}_{il}^2 \quad \textit{at } x_0. \tag{2-7}$$

The proof of Lemma 2.1 is by now standard and will be given in Section 2.2.

Now, in the case that the positive term on the right-hand side of (2-7) dominates $|\nabla u|^4\mathcal{F}_\tau$, in the sense that

$$\sum_{i,l} F_\tau^{ii}\tilde{u}_{il}^2 \ge \varepsilon|\nabla u|^4\mathcal{F}_\tau \quad \text{at } x_0 \tag{2-8}$$

for a suitably chosen small constant $\varepsilon > 0$, then the desired gradient estimate is routine (the details will be given later). On the other hand, if (2-8) fails for our suitably chosen small constant $\varepsilon > 0$, we will see that the ordered eigenvalues $w_{11} \ge \cdots \ge w_{nn}$ of W at x_0 are greater than or equal to a perturbation of $(1,\dots,1,-1)\frac{1}{2}|\nabla u|^2$. As mentioned in the introduction, this phenomenon was previously observed in the case $(f,\Gamma)=(\sigma_k^{1/k},\Gamma_k^+)$ when $k<\frac{1}{2}n$ in the thesis of Khomrutai [2009]. Using the fact that $(1,\dots,1,-1)\in\Gamma$ (this is the only place in the proof of Theorem 1.8 where the assumption $\mu_\Gamma^+>1$ is used), the gradient estimate again follows. The details will be given in Section 2.3.

2.2. *Proof of Lemma 2.1.* We follow closely the proof in [Guan and Wang 2003]. In what follows, all computations are implicitly carried out at x_0. First observe that, by (2-2),

$$H_{ij}=\biggl(\rho_{ij}-\frac{2\rho_i\rho_j}{\rho}\biggr)|\nabla u|^2+2\rho\sum_{l=1}^n u_{lij}u_l+2\rho\sum_{l=1}^n u_{il}u_{jl},$$

and hence, by positivity of (F_τ^{ij}) and nonpositivity of (H_{ij}),

$$\begin{aligned}0\ge\sum_{i=1}^n F_\tau^{ii}H_{ii}&=\sum_{i=1}^n F_\tau^{ii}\biggl[\biggl(\rho_{ii}-\frac{2\rho_i^2}{\rho}\biggr)|\nabla u|^2+2\rho\sum_{l=1}^n u_{lii}u_l+2\rho\sum_{l=1}^n u_{il}^2\biggr]\\&=-C|\nabla u|^2\mathcal{F}_\tau+2\rho\sum_{i,l}F_\tau^{ii}u_{lii}u_l+2\rho\sum_{i,l}F_\tau^{ii}u_{il}^2.\end{aligned}\tag{2-9}$$

Now, commuting derivatives yields

$$\begin{aligned}\sum_{i,l}F_\tau^{ii}u_{lii}u_l&\ge\sum_{i,l}F_\tau^{ii}u_{iil}u_l-C|\nabla u|^2\mathcal{F}_\tau\\&=\sum_{i,l}F_\tau^{ii}\bigl[(w_{ii})_l-\bigl(\tfrac{1}{2}|\nabla u|^2-u_i^2\bigr)_l+(S_{ii})_l\bigr]u_l-C|\nabla u|^2\mathcal{F}_\tau\\&=\sum_{l=1}^n(\psi e^{2u})_lu_l-\mathcal{F}_\tau\sum_{k,l}u_{kl}u_ku_l+2\sum_{i,l}F_\tau^{ii}u_{il}u_iu_l+\sum_{i,l}F_\tau^{ii}(S_{ii})_lu_l-C|\nabla u|^2\mathcal{F}_\tau,\end{aligned}\tag{2-10}$$

where to reach the last line we have used the fact that f^τ is homogeneous of degree 1 to assert that $\sum_i F_\tau^{ii}(w_{ii})_l=(f^\tau(g_0^{-1}W))_l=(\psi e^{2u})_l$. Also, since $|\nabla u|\ge 1$, we can bound the penultimate term in (2-10) from below by $-C|\nabla u|^2\mathcal{F}_\tau$, and also observe that

$$\sum_{l=1}^n(\psi e^{2u})_lu_l=\sum_{l=1}^n e^{2u}\psi_lu_l+2e^{2u}\psi|\nabla u|^2\ge -Ce^{2u}|\nabla u|^2.\tag{2-11}$$

Also, by (2-5) we have

$$-\mathcal{F}_\tau\sum_{k,l}u_{kl}u_ku_l\ge -C\frac{|\nabla u|^4}{A_0}\mathcal{F}_\tau,\tag{2-12}$$

and likewise

$$2\sum_{i,l}F_\tau^{ii}u_{il}u_iu_l=2\sum_i\biggl(F_\tau^{ii}u_i\sum_l u_{il}u_l\biggr)\ge -2\sum_i\biggl(|F_\tau^{ii}u_i|\biggl|\sum_l u_{il}u_l\biggr|\biggr)\ge -C\frac{|\nabla u|^4}{A_0}\mathcal{F}_\tau.\tag{2-13}$$

Substituting (2-11)–(2-13) back into (2-10) and recalling $\mathcal{F}_\tau\ge 1/C$, we get

$$\sum_{i,l}F_\tau^{ii}u_{lii}u_l\ge -C(1+e^{2u})|\nabla u|^2\mathcal{F}_\tau-C\mathcal{F}_\tau\frac{|\nabla u|^4}{A_0},$$

and, substituting this back into (2-9), we see

$$0 \geq -C\mathcal{F}_\tau(1+e^{2u})|\nabla u|^2 - C\rho\mathcal{F}_\tau \frac{|\nabla u|^4}{A_0} + 2\rho \sum_{i,l} F_\tau^{ii} u_{il}^2. \tag{2-14}$$

The desired estimate (2-7) then follows from (2-14) and the following inequality, which is a consequence of the Cauchy–Schwarz inequality and the second inequality in (2-4):

$$\sum_{i,l} F_\tau^{ii} u_{il}^2 \geq \frac{1}{2}\sum_{i,l} F_\tau^{ii} \tilde{u}_{il}^2 - \frac{1}{A_0}\mathcal{F}_\tau|\nabla u|^4. \qquad \square$$

2.3. ***Proof of Theorem 1.8.*** We begin this section by stating a central result in our argument, namely Proposition 2.2. The proof of Theorem 1.8 is then given assuming the validity of Proposition 2.2 — this should serve to elucidate the ideas outlined at the end of Section 2.1. The proof of Proposition 2.2 will be given later in the section and consists of a series of technical lemmas.

To this end, for $1 > \delta_0 \geq A_0^{-1/10}$ a small number to be fixed later, define the set

$$\mathcal{I} = \Big\{ i \in \{1, \dots, n\} : \big| w_{jj} + \tfrac{1}{2}|\nabla u|^2 \big| < 2\delta_0^2 |\nabla u|^2 \Big\}.$$

We remind the reader that all computations are implicitly carried out at x_0, and that we have the ordering $w_{11} \geq \cdots \geq w_{nn}$. We will prove:

Proposition 2.2. *There exists a constant $\widetilde{C} > 1$ depending only on n, f, Γ, $\|g_0\|_{C^3(B_r)}$ and $\|\psi\|_{C^1(B_r)}$ such that if $A_0^{-1/10} \leq \delta_0 \leq \widetilde{C}^{-1}$ and*

$$\sum_{i,l} F_\tau^{ii} \tilde{u}_{il}^2 < \widetilde{C}^{-1}\delta_0^4 |\nabla u|^4 \mathcal{F}_\tau, \tag{2-15}$$

then:

(1) $\mathcal{I} = \{n\}$, *and*

(2) $\big| w_{n-1,n-1} - \frac{1}{2}|\nabla u|^2 \big| < 2\delta_0 |\nabla u|^2$.

Assuming the validity of Proposition 2.2 for now, let us complete the proof of Theorem 1.8.

Proof of Theorem 1.8. We start by fixing $\widetilde{C}$ sufficiently large so that Proposition 2.2 applies. Then, for $A_0 > \widetilde{C}^{10}$ to be fixed later, if $A_0^{-1/10} \leq \delta_0 \leq \widetilde{C}^{-1}$ and (2-15) is satisfied,

$$w_{n-1,n-1} = (1+a_{n-1})\frac{|\nabla u|^2}{2} \quad \text{and} \quad w_{nn} = -(1+a_n)\frac{|\nabla u|^2}{2}$$

for some $|a_{n-1}|, |a_n| \leq 4\delta_0$. On the other hand, since $w_{11} \geq \cdots \geq w_{nn}$ for each $\alpha = 1, \dots, n-2$, we can write $w_{\alpha\alpha} = w_{n-1,n-1} + X_\alpha$ for some $X_\alpha \geq 0$. Therefore

$$\begin{pmatrix} w_{11} \\ \vdots \\ w_{n-2,n-2} \\ w_{n-1,n-1} \\ w_{nn} \end{pmatrix} = \begin{pmatrix} X_1 \\ \vdots \\ X_{n-2} \\ 0 \\ 0 \end{pmatrix} + \frac{|\nabla u|^2}{2} \underbrace{\begin{pmatrix} 1+a_{n-1} \\ \vdots \\ 1+a_{n-1} \\ 1+a_{n-1} \\ -(1+a_n) \end{pmatrix}}_{\mathcal{B}}, \tag{2-16}$$

with the first vector on the right-hand side of (2-16) clearly belonging to $\overline{\Gamma^\tau}$ for each $\tau \le 1$ since each entry is nonnegative. We also observe that $\mathcal{B}$ is a perturbation of $\mathcal{B}_0 := (1, \dots, 1, -1)$ and that $\mathcal{B}_0 \in \Gamma^\tau$ for any $\tau \le 1$ since we assume $\mu_\Gamma^+ > 1$. Therefore, since $|a_{n-1}|, |a_n| \le 4\delta_0$, for $\widetilde{C}$ sufficiently large we will have $\mathcal{B} \in \Gamma$ with $f^\tau(\mathcal{B}) \ge \frac{1}{2} f^\tau(\mathcal{B}_0)$. Monotonicity of f then implies

$$\psi e^{2u} = f^\tau(w_{11}, \dots, w_{nn}) \ge \tfrac{1}{2}|\nabla u|^2 f^\tau(\mathcal{B}) \ge \tfrac{1}{4}|\nabla u|^2 f^\tau(\mathcal{B}_0),$$

which implies the desired gradient estimate.

It remains to address the case that, for the value of $\widetilde{C}$ fixed in the foregoing argument, (2-15) is not satisfied. Then

$$\sum_{i,l} F_\tau^{ii} \tilde{u}_{il}^2 \ge \widetilde{C}^{-1} A_0^{-2/5} |\nabla u|^4 \mathcal{F}_\tau, \tag{2-17}$$

and substituting (2-17) into (2-7) we therefore have

$$0 \ge -C\mathcal{F}_\tau(1+e^{2u})|\nabla u|^2 - C\rho\mathcal{F}_\tau \frac{|\nabla u|^4}{A_0} + \widetilde{C}^{-1} A_0^{-2/5} \rho |\nabla u|^4 \mathcal{F}_\tau.$$

Multiplying through by $\widetilde{C} A_0^{2/5} \rho$ then yields the estimate

$$0 \ge -\widetilde{C} C A_0^{2/5} \rho (1+e^{2u})|\nabla u|^2 - \frac{\widetilde{C} C}{A_0^{3/5}} \rho^2 |\nabla u|^4 + \rho^2 |\nabla u|^4. \tag{2-18}$$

It follows that if we choose $A_0 \ge \max\{(2\widetilde{C} C)^{5/3}, \widetilde{C}^{10}\}$ (where C and $\widetilde{C}$ are the constants in (2-18)), then we have (for a possibly different constant C)

$$0 \ge -C\rho(1+e^{2u})|\nabla u|^2 + \tfrac{1}{2}\rho^2 |\nabla u|^4, \tag{2-19}$$

and therefore

$$H^2 = \rho^2 |\nabla u|^4 \le C(1+e^{2u})H. \tag{2-20}$$

After dividing through by H we again arrive at the desired gradient estimate. □

The rest of the section is devoted to the proof of Proposition 2.2, which we obtain through a series of three lemmas. In the first of these lemmas we show that if $A_0^{-1/10} \le \delta_0 \le \widetilde{C}^{-1}$ for $\widetilde{C}$ sufficiently large, then $\mathcal{I} \ne \varnothing$.

Lemma 2.3. *There exists a constant $\widetilde{C} > 1$ depending only on n, f, Γ, $\|g_0\|_{C^3(B_r)}$ and $\|\psi\|_{C^1(B_r)}$ such that if $A_0^{-1/10} \le \delta_0 \le \widetilde{C}^{-1}$, then $\mathcal{I} \ne \varnothing$.*

Proof. It is clear that, for $\delta_0 \le \sqrt{1/n}$, there is at least one index $j \in \{1, \dots, n\}$ such that $u_j^2 \ge \delta_0^2 |\nabla u|^2$. We claim that, for such an index j, we have $j \in \mathcal{I}$. We follow the method in [Guan and Wang 2003]. We know that, for $l \ne j$, we have $u_{jl} = u_j u_l + S_{jl}$ and therefore

$$\sum_{l \ne j} u_{jl} u_l = \sum_{l \ne j} u_j u_l^2 + \sum_{l \ne j} S_{jl} u_l.$$

It follows that

$$\sum_{l=1}^{n} u_{jl}u_l = \sum_{l\neq j} u_j u_l^2 + \sum_{l\neq j} S_{jl}u_l + u_{jj}u_j$$
$$= u_j|\nabla u|^2 + \sum_{l\neq j} S_{jl}u_l + u_{jj}u_j - u_j^3$$
$$= \sum_{l\neq j} S_{jl}u_l - u_j((u_j^2 - |\nabla u|^2) - u_{jj}).$$

Hence

$$\left|u_j((u_j^2 - |\nabla u|^2) - u_{jj}) - \sum_{l\neq j} S_{jl}u_l\right| = \left|\sum_{l=1}^{n} u_{jl}u_l\right| \overset{(2\text{-}5)}{\leq} C\frac{|\nabla u|^3}{A_0}.$$

It follows that

$$|u_j||(u_j^2 - |\nabla u|^2) - u_{jj}| \leq C\frac{|\nabla u|^3}{A_0} + \left|\sum_{l\neq j} S_{jl}u_l\right| \overset{(2\text{-}4)}{\leq} C\frac{|\nabla u|^3}{A_0} \leq C\delta_0^{10}|\nabla u|^3, \tag{2-21}$$

where to reach the last inequality we have used $A_0^{-1/10} \leq \delta_0$. Substituting $|u_j| \geq \delta_0|\nabla u|$ back into (2-21) yields

$$|(u_j^2 - |\nabla u|^2) - u_{jj}| \leq C\delta_0^9|\nabla u|^2. \tag{2-22}$$

Next, substituting $u_{jj} = w_{jj} + u_j^2 - \frac{1}{2}|\nabla u|^2 + S_{jj}$ into (2-22) and again applying (2-4), we obtain

$$\left|w_{jj} + \tfrac{1}{2}|\nabla u|^2\right| \leq C\delta_0^9|\nabla u|^2 + \frac{|\nabla u|^2}{A_0} = C\delta_0^9|\nabla u|^2 + \delta_0^{10}|\nabla u|^2. \tag{2-23}$$

It is clear that one can then choose $\widetilde{C}$ sufficiently large so that the right-hand side of (2-23) is less than $2\delta_0^2|\nabla u|^2$ for $\delta_0 \leq \widetilde{C}^{-1}$. Once such a choice is made, we see that (2-23) implies $j \in \mathcal{I}$, which proves the claim and therefore the lemma. □

In our subsequent arguments we will use the following proposition, which is essentially a consequence of [Yuan 2022, Theorem 1.4] — see Appendix A for a summary of the proof.

Proposition 2.4. *Suppose Γ satisfies (1-1) and (1-2) with $\Gamma \neq \Gamma_n^+$ (equivalently, $\mu_\Gamma^+ > 0$). Then there exists a constant $\theta = \theta(n, \Gamma) > 0$ such that, for any $\lambda \in \Gamma$ with $\lambda_1 \geq \cdots \geq \lambda_n$,*

$$\frac{\partial f}{\partial \lambda_i}(\lambda) \geq \theta \sum_{j=1}^{n} \frac{\partial f}{\partial \lambda_j}(\lambda) \quad \text{if } i \in \{n-1, n\} \text{ or } \lambda_i \leq 0. \tag{2-24}$$

We are now in a position to show that if one additionally assumes (2-15) holds for $\widetilde{C}$ sufficiently large, then $|\mathcal{I}| = \{n\}$ (recall once again the ordering $w_{11} \geq \cdots \geq w_{nn}$).

Lemma 2.5. *There exists a constant $\widetilde{C} > 1$ depending only on n, f, Γ, $\|g_0\|_{C^3(B_r)}$ and $\|\psi\|_{C^1(B_r)}$ such that if $A_0^{-1/10} \leq \delta_0 \leq \widetilde{C}^{-1}$ and (2-15) is satisfied, then $|\mathcal{I}| = \{n\}$.*

Proof. We first claim that if $\widetilde{C}$ is sufficiently large and (2-15) holds, then $u_{jj} > -2\delta_0^2|\nabla u|^2$ for $j \in \mathcal{I}$. Indeed, suppose for a contradiction that this is not the case. Then we would have

$$\sum_{i,l} F_\tau^{ii}\tilde{u}_{il}^2 \geq F_\tau^{jj}\tilde{u}_{jj}^2 \overset{(2\text{-}4)}{\geq} \frac{1}{2}F_\tau^{jj}u_{jj}^2 - F_\tau^{jj}\frac{|\nabla u|^4}{A_0^2} \geq 2F_\tau^{jj}\delta_0^4|\nabla u|^4 - F_\tau^{jj}\delta_0^{20}|\nabla u|^4 \geq F_\tau^{jj}\delta_0^4|\nabla u|^4 \geq \theta\delta_0^4|\nabla u|^4\mathcal{F}_\tau, \tag{2-25}$$

with the last inequality following from Proposition 2.4 — note that Proposition 2.4 applies in this case since $w_{jj} < 0$ by virtue of $j \in \mathcal{I}$ if $\widetilde{C}$ is sufficiently large. But this contradicts (2-15) if $\widetilde{C}$ is sufficiently large, proving the claim.

By the claim, we may therefore suppose that $\widetilde{C}$ is large enough so that $u_{jj} > -2\delta_0^2|\nabla u|^2$ whenever $j \in \mathcal{I}$. Then, for $j \in \mathcal{I}$, we therefore have

$$-2\delta_0^2|\nabla u|^2 - u_j^2 + \tfrac{1}{2}|\nabla u|^2 - S_{jj} < u_{jj} - u_j^2 + \tfrac{1}{2}|\nabla u|^2 - S_{jj} = w_{jj} < -\tfrac{1}{2}|\nabla u|^2 + 2\delta_0^2|\nabla u|^2,$$

with the last inequality following from the definition of $\mathcal{I}$. That is,

$$-u_j^2 < (-1+4\delta_0^2)|\nabla u|^2 + S_{jj} \overset{(2\text{-}4)}{<} (-1+4\delta_0^2)|\nabla u|^2 + \delta_0^{10}|\nabla u|^2 < (-1+5\delta_0^2)|\nabla u|^2. \tag{2-26}$$

Clearly (2-26) cannot hold for more than one index if $10\delta_0^2 < 1$. Hence $|\mathcal{I}| \leq 1$ for $\widetilde{C}$ sufficiently large, and after increasing $\widetilde{C}$ further if necessary so that $\mathcal{I} \neq \varnothing$ (recall that this is possible by Lemma 2.3), it must be the case that $|\mathcal{I}| = 1$, i.e., $\mathcal{I} = \{n\}$. □

To finish the proof of Proposition 2.2 it remains to show (after taking $\widetilde{C}$ larger if necessary) that $\left|w_{n-1,n-1} - \frac{1}{2}|\nabla u|^2\right| < 2\delta_0|\nabla u|^2$. This is the focus of the next lemma.

Lemma 2.6. *There exists a constant $\widetilde{C} > 1$ depending only on n, f, Γ, $\|g_0\|_{C^3(B_r)}$ and $\|\psi\|_{C^1(B_r)}$ such that if $A_0^{-1/10} \leq \delta_0 \leq \widetilde{C}^{-1}$ and (2-15) is satisfied, then*

$$\left|w_{n-1,n-1} - \tfrac{1}{2}|\nabla u|^2\right| < 2\delta_0|\nabla u|^2.$$

Proof. **Step 1:** In this first step we show

$$w_{n-1,n-1} > \left(\tfrac{1}{2} - 2\delta_0\right)|\nabla u|^2. \tag{2-27}$$

Suppose for a contradiction that $w_{n-1,n-1} \leq \left(\frac{1}{2} - 2\delta_0\right)|\nabla u|^2$, i.e.,

$$u_{n-1,n-1} - u_{n-1}^2 - S_{n-1,n-1} \leq -2\delta_0|\nabla u|^2. \tag{2-28}$$

Either $u_{n-1}^2 < \delta_0|\nabla u|^2$ or $u_{n-1}^2 \geq \delta_0|\nabla u|^2$. In the former case, (2-28) then implies

$$u_{n-1,n-1} < -\delta_0|\nabla u|^2 + S_{n-1,n-1} \overset{(2\text{-}4)}{<} -\delta_0|\nabla u|^2 + \delta_0^{10}|\nabla u|^2 < -\tfrac{1}{2}\delta_0|\nabla u|^2 \tag{2-29}$$

if $\delta_0 < \frac{1}{2}$, and one obtains a contradiction as in (2-25) if $\widetilde{C}$ is sufficiently large — note that Proposition 2.4 is again justified since $w_{n-1,n-1}$ is the second lowest eigenvalue. If instead $u_{n-1}^2 \geq \delta_0|\nabla u|^2$, the proof of Lemma 2.3 shows that $n-1 \in \mathcal{I}$. This contradicts the conclusion $|\mathcal{I}| = \{n\}$ of Lemma 2.5 if $\widetilde{C}$ is sufficiently large. Thus (2-27) is established, which completes the proof of Step 1.

Step 2: In this second step we show

$$w_{n-1,n-1} < \bigl(\tfrac{1}{2} + 2\delta_0\bigr)|\nabla u|^2. \tag{2-30}$$

Indeed, we have

$$w_{n-1,n-1} = u_{n-1,n-1} - u_{n-1}^2 + \tfrac{1}{2}|\nabla u|^2 - S_{n-1,n-1} \overset{(2\text{-}4)}{\le} |u_{n-1,n-1}| + \tfrac{1}{2}|\nabla u|^2 + \delta_0^{10}|\nabla u|^2.$$

But $|u_{n-1,n-1}| \le \delta_0|\nabla u|^2$, else one would obtain a contradiction as in (2-25) if $\widetilde{C}$ is sufficiently large (again we are using the fact $w_{n-1,n-1}$ is the second lowest eigenvalue, so Proposition 2.4 applies). The estimate (2-30) thus follows, which completes the proof of Step 2.

With (2-27) and (2-30) established, the proof of Lemma 2.6 is complete. □

Proof of Proposition 2.2. This is an immediate consequence of Lemmas 2.3, 2.5 and 2.6. □

3. Proof of Theorem 1.6: the Dirichlet boundary value problem

As discussed in the introduction, in the proof of Theorem 1.1′, we will first address the corresponding Dirichlet boundary value problem with finite boundary data. To this end, in this section we prove Theorem 1.6. Our proof uses the continuity method, and we proceed according to the following steps:

(1) In Section 3.1 we give a routine proof of the global upper bound on solutions for $\tau \le 1$, independent of whether or not $\mu_\Gamma^+ > 1$.

(2) In Section 3.2 we prove the global lower bound on solutions for $\tau \le 1$ when $\mu_\Gamma^+ > 1$. As outlined in the introduction, we use two main ingredients: our local interior gradient estimate obtained in Theorem 1.8 and a lower bound in a uniform neighbourhood of ∂M, which is obtained by constructing suitable comparison functions on small annuli (see Propositions 3.3 and 3.4).

(3) In Section 3.3 we prove the global gradient estimate for $\tau \le 1$ when $\mu_\Gamma^+ > 1$. To obtain the lower bound for the normal derivative on ∂M, we use our comparison functions on small annuli constructed in Section 3.2, and to obtain the upper bound for the normal derivative on ∂M, we use comparison functions similar to that of [Guan 2008] (this latter argument does not use $\mu_\Gamma^+ > 1$). For the interior estimates we use Theorem 1.8, and for estimates near ∂M we appeal to the proof of Theorem 1.8.

(4) In Section 3.4 we prove the global Hessian estimate for $\tau < 1$ following arguments of [Guan 2008]. These estimates apply whether or not $\mu_\Gamma^+ > 1$.

(5) In Section 3.5 we complete the proof of Theorem 1.6: we first prove the existence of a unique smooth solution when $\tau < 1$ using the continuity method, and we then obtain a Lipschitz viscosity solution in the case $\tau = 1$ in the limit as $\tau \to 1$.

We point out that, in order to obtain a Lipschitz viscosity solution in the limit $\tau \to 1$ in Section 3.5, it is important that our a priori C^1 estimates obtained in Sections 3.1–3.3 are uniform in $\tau \in [0, 1]$. On the other hand, the global Hessian estimate in Section 3.4 deteriorates as $\tau \to 1$; this is to be expected in view of the work in [Li and Nguyen 2021; Li et al. 2023], where the nonexistence of C^2 solutions is established for all Euclidean domains with disconnected smooth boundary when $\tau = 1$.

3.1. *Upper bound.* The global upper bound on solutions to (1-11) is routine and does not require the assumption $\mu_\Gamma^+ > 1$.

Proposition 3.1. *Suppose (f, Γ) satisfies (1-1)–(1-4), and let $\tau \le 1$. Let $\psi \in C^\infty(M)$ be positive and $\xi \in C^\infty(\partial M)$. Then there exists a constant C which is independent of τ but dependent on g_0, f, Γ, a lower bound for $\inf_M \psi$ and an upper bound for $\sup_{\partial M} \xi$ such that any C^2 solution to (1-11) satisfies $u \le C$ on M.*

Proof. Suppose the maximum of u occurs at $x_0 \in M$. If $x_0 \in \partial M$, then $u(x_0) \le \xi(x_0)$. If $x_0 \in M \setminus \partial M$, then $\nabla_{g_0}^2 u(x_0) \le 0$ and $du(x_0) = 0$, and hence

$$\psi(x_0) e^{2u(x_0)} \le f^\tau(-g_0^{-1} A_{g_0})(x_0),$$

which yields

$$u(x_0) \le \frac{1}{2} \ln\left(\frac{f^\tau(-g_0^{-1} A_{g_0})}{\psi}\right)(x_0). \qquad \square$$

3.2. *Lower bound.* In this section we obtain the global lower bound on solutions to (1-11).

Proposition 3.2. *Suppose (f, Γ) satisfies (1-1)–(1-4) and (1-8), and let $\tau \le 1$. Let $\psi \in C^\infty(M)$ be positive and $\xi \in C^\infty(\partial M)$. Then there exists a constant C which is independent of τ but dependent on g_0, f, Γ, an upper bound for $\|\psi\|_{C^1(M)}$ and a lower bound for $\inf_{\partial M} \xi$ such that any C^3 solution to (1-11) satisfies $u \ge C$ on M.*

There are two main ingredients in our proof of Proposition 3.2: our local interior gradient estimate from Theorem 1.8 and a lower bound in a uniform neighbourhood of ∂M; the assumption $\mu_\Gamma^+ > 1$ plays a role at both stages. As pointed out before, a delicate point is that we do not assume that the background metric satisfies $\lambda(-g_0^{-1} A_{g_0}) \in \Gamma$ on M — if such an assumption is made, then the proof of the lower bound is as straightforward as the proof of Proposition 3.1. In our case, the global lower bound requires more work and is one of the key steps in this paper.

To state our result concerning the lower bound near ∂M, for $\delta > 0$, we define

$$M_\delta = \{x \in M : \mathrm{d}(x, \partial M) < \delta\},$$

where $\mathrm{d}(x, \partial M)$ is the distance from x to ∂M with respect to g_0. It is well known that, for $\delta > 0$ sufficiently small, M_δ is a tubular neighbourhood of ∂M. We show the following.

Proposition 3.3. *Under the same hypotheses as Proposition 3.2, there exists a constant $\delta > 0$ which is independent of τ but dependent on g_0, f, Γ, an upper bound for $\sup_M \psi$ and a lower bound for $\inf_{\partial M} \xi$ such that any C^3 solution u to (1-11) satisfies $u \ge \inf_{\partial M} \xi - 1$ in M_δ.*

Assuming the validity of Proposition 3.3 for now, we give the proof of Proposition 3.2.

Proof of Proposition 3.2. Let $\delta > 0$ be as in the statement of Proposition 3.3, so that u satisfies the lower bound $u \ge \inf_{\partial M} \xi - 1$ in M_δ. It follows that

$$u \ge \inf_{\partial M} \xi - 1 - \mathrm{diam}(M, g_0) \sup_{M \setminus M_\delta} |\nabla_{g_0} u|_{g_0} \quad \text{in } M. \tag{3-1}$$

On the other hand, by Theorem 1.8 and the uniform upper bound for u obtained in Proposition 3.1, we have

$$|\nabla_{g_0} u|_{g_0} \leq C(\delta^{-1} + 1) \quad \text{in } M \setminus M_\delta. \tag{3-2}$$

Substituting (3-2) into (3-1), the proof of Proposition 3.2 is complete. □

Roughly speaking, to prove Proposition 3.3 we cover a neighbourhood of ∂M by small annuli on which we construct suitable comparison functions. The construction of such comparison functions is given in the following proposition (which is a more precise version of Proposition 1.11 stated in the introduction). For a Riemannian metric g_0 defined on a neighbourhood of the origin in $\mathbb{R}^n$, let $r(x) = \mathrm{d}_{g_0}(0, x)$, let $\mathbb{S}_r = \partial\mathbb{B}_r$ denote the geodesic sphere of radius r centred at the origin, and denote by A_{r_1,r_2} the annulus $\mathbb{B}_{r_2} \setminus \overline{\mathbb{B}}_{r_1}$. We also write

$$\beta = \frac{2}{\mu_\Gamma^+ - 1}$$

and recall the convention $g_w = e^{2w} g_0$.

Proposition 3.4. *Suppose (f, Γ) satisfies (1-1)–(1-4) and (1-8), and let g_0 be a Riemannian metric defined on a neighbourhood Ω of the origin in $\mathbb{R}^n$. Fix a constant $\varepsilon > 0$. Then there exists a constant $C > 1$ depending only on g_0, f and Γ, and a constant $0 < R < 1$ depending additionally on ε, such that, for each $m \in \mathbb{R}$,*

$$w(r) := (\beta + \varepsilon) \ln\left(\frac{r_+ - r}{r_+ - r_-}\right) + m \tag{3-3}$$

satisfies

$$\begin{cases} f(\lambda(-g_w^{-1} A_{g_w})) \geq \dfrac{f(-\mu_\Gamma^+ + C^{-1}\varepsilon, 1, \dots, 1)}{Ce^{2m}(r_+ - r_-)^2} > 0, \quad \lambda(-g_w^{-1} A_{g_w}) \in \Gamma & \text{on } A_{r_-,r_+}, \\ w(x) = m & \text{for } x \in \mathbb{S}_{r_-}, \\ w(x) \to -\infty & \text{as } \mathrm{d}(x, \mathbb{S}_{r_+}) \to 0, \end{cases} \tag{3-4}$$

whenever $1 < r_+/r_- < 1 + \varepsilon/(2(\beta + 2))$ and $r_+ < R$.

Remark 3.5. Our choice of w in (3-3) is motivated by the work of Chang, Han and Yang [Chang et al. 2005] on radial solutions to the σ_k-Yamabe equation on annular domains in $\mathbb{R}^n$. Indeed, when $\varepsilon = 0$ and $\mu_\Gamma^+ = (n-k)/k$, (3-8) corresponds to the leading order term in the solution to the σ_k-Yamabe equation in Γ_k^- on annular domains in $\mathbb{R}^n$ for $k < \frac{1}{2}n$.

Remark 3.6. We reiterate that Proposition 3.4 relies crucially on the assumption $\mu_\Gamma^+ > 1$ and that a similar construction is not possible when $\mu_\Gamma^+ \leq 1$ — see Remark 1.12 in the introduction.

Assuming the validity of Proposition 3.4 for now, we first give the proof of Proposition 3.3 — the reader may wish to refer to Figure 1 in the following argument.

Proof of Proposition 3.3. We attach a collar neighbourhood N to ∂M such that g_0 extends smoothly to $M \cup N$; we denote this extension also by g_0. Let

$$D = \inf_{x \in \partial M} \mathrm{d}_{g_0}(x, \partial(M \cup N))$$

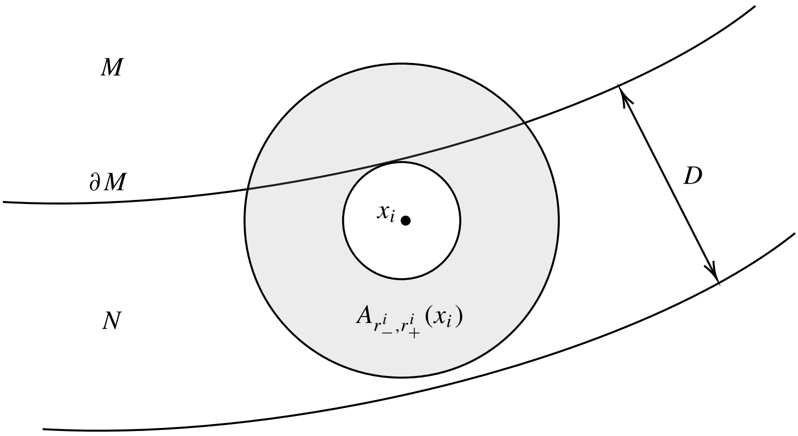

Figure 1. An annulus in the covering of a neighbourhood of ∂M in M in the proof of Proposition 3.3.

denote the thickness of N. Fix $\varepsilon > 0$ and let $m = \inf_{\partial M} \xi$, and cover a neighbourhood of ∂M in M by a finite collection of annuli $\{A_{r^i_-,r^i_+}(x_i)\}_{1\leq i\leq K}$ centred at x_i such that the collection $\{A_{r^i_-,(r^i_-+r^i_+)/2}(x_i)\}$ still covers a neighbourhood of ∂M in M, and such that, for each i,

(1) $x_i \in N$,

(2) $r^i_- + r^i_+ < D$,

(3) $r^i_- = \mathrm{d}_{g_0}(x_i, \partial M)$,

(4) the closed ball $\overline{B_{r^i_+}(x_i)}$ is contained in a single normal coordinate chart (U_i, ζ_i) mapping x_i to the origin,

(5) $$\frac{r^i_+}{r^i_-} \leq 1 + \frac{\varepsilon}{2(\beta+2)},$$

(6) $r_i^+ < R$ is sufficiently small so that
$$\frac{f(-\mu_\Gamma^+ + C^{-1}\varepsilon, 1, \ldots, 1)}{Ce^{2m}(r^i_+ - r^i_-)^2} \geq \sup_M \psi$$
(here C and R are as in the statement of Proposition 3.4, where we are implicitly identifying the annulus $A_{r^i_-,r^i_+}(x_i)$ with its image under ζ_i, which is possible by property (4)).

In what follows, we continue to implicitly make the identification between $A_{r^i_-,r^i_+}(x_i)$ and its image under ζ_i.

Let w_i denote the solution obtained in Proposition 3.4 on $A_{r^i_-,r^i_+}(x_i)$ with $\varepsilon > 0$ and $m = \inf_{\partial M} \xi$ as fixed above. Since w_i is radially decreasing and $w_i(x) = \inf_{\partial M} \xi$ for $x \in \mathbb{S}_{r^i_-}(x_i)$, we have $w_i \leq \inf_{\partial M} \xi$ on $A_{r^i_-,r^i_+}(x_i) \cap \partial M$. On the other hand, $w_i = -\infty < u$ on $\mathbb{S}_{r^i_+}(x_i)$. Therefore, the comparison principle (see Proposition 3.7 below) yields $u \geq w_i$ on $A_{r^i_-,r^i_+}(x_i) \cap M$ for each i. This yields a finite lower bound for u on $A_{r^i_-,(r^i_-+r^i_+)/2}(x_i)$. Since we assume the collection $\{A_{r^i_-,(r^i_-+r^i_+)/2}(x_i)\}$ still covers a neighbourhood of ∂M in M, we may piece together the estimates for u on each annulus $A_{r^i_-,(r^i_-+r^i_+)/2}(x_i)$ to obtain the desired estimate for u on a uniform neighbourhood of ∂M in M. □

In the above proof we made use of the following comparison principle.

Proposition 3.7 (comparison principle). *Let $\alpha > 0$ be a positive constant and (M, g) a compact Riemannian manifold with nonempty boundary ∂M. Suppose $u, v \in C^0(M)$ with at least one of u or v belonging to $C^2(M \setminus \partial M)$. If $f(-g_u^{-1}A_{g_u}) \geq f(-g_v^{-1}A_{g_v}) \geq \alpha > 0$ in the viscosity sense on $M \setminus \partial M$ and $u \leq v$ on ∂M, then $u \leq v$ in M.*

In the proof of Proposition 3.3, we only needed Proposition 3.7 in the case that both $u, v \in C^2(M \setminus \partial M)$. In this case, the proof of Proposition 3.7 is standard in light of the fact that if $f(-g_v^{-1}A_{g_v}) > 0$, c is a positive constant and $w = v + c$, then $f(-g_w^{-1}A_{g_w}) < f(-g_v^{-1}A_{g_v})$. The case when $u \in C^0(M)$ in Proposition 3.7 will be needed later in the paper. When $u \in C^2(M \setminus \partial M)$, Proposition 3.7 follows from [Caffarelli et al. 2013, Theorem 2.1], since the proof on page 130 therein applies also on Riemannian manifolds with boundary. When $v \in C^2(M \setminus \partial M)$, Proposition 3.7 again follows from [Caffarelli et al. 2013, Theorem 2.1], therein considering $\widetilde{F}(x, s, p, M) := -F(x, -s, -p, -M)$ in place of F.

We now give the proof of Proposition 3.4.

Proof of Proposition 3.4. It will be more convenient to write our conformal metrics in the form $g^v = v^{-2}g_0$, so that $g_w = g^v$ for $e^{2w} = v^{-2}$. Then the $(0, 2)$-Schouten tensor of g^v is given by

$$(A_{g^v})_{ij} = v^{-1}(\nabla^2_{g_0} v)_{ij} - \tfrac{1}{2}v^{-2}|\nabla_{g_0} v|^2_{g_0}(g_0)_{ij} + (A_{g_0})_{ij}.$$

In a fixed normal coordinate system based at the origin, it follows that if $v = v(r)$ then

$$((g^v)^{-1}A_{g^v})^p_j = v^2\left(\lambda\delta^p_j + \chi\frac{x^p x_j}{r^2}\right) + O(r^2)v|v_{rr}| + O(r)v|v_r| + O(1)v^2 \quad \text{as } r \to 0, \tag{3-5}$$

where

$$\lambda = \frac{v_r}{rv}\left(1 - \frac{rv_r}{2v}\right) \quad \text{and} \quad \chi = \frac{v_{rr}}{v} - \frac{v_r}{vr}; \tag{3-6}$$

we refer the reader to Appendix B for the derivation of (3-5). Therefore

$$(-(g^v)^{-1}A_{g^v})^p_j \geq -v^2\left(\lambda\delta^p_j + \chi\frac{x^p x_j}{r^2}\right) - |\Psi|\delta^p_j \tag{3-7}$$

in the sense of matrices, where $|\Psi| = O(r^2)v|v_{rr}| + O(r)v|v_r| + O(1)v^2$ as $r \to 0$.

Step 1: In this first step we compute and estimate the quantities on the right-hand side of (3-7) for our particular choice of w in (3-3), i.e., for

$$v(r) = e^{-\Lambda}(r_+ - r)^{-\beta-\varepsilon}, \tag{3-8}$$

where we have written $\Lambda = m - (\beta + \varepsilon)\ln(r_+ - r_-)$. For shorthand we write $\varphi(r) = r_+ - r$. Then

$$v_r = e^{-\Lambda}(\beta + \varepsilon)\varphi^{-\beta-\varepsilon-1} \quad \text{and} \quad v_{rr} = e^{-\Lambda}(\beta + \varepsilon)(\beta + \varepsilon + 1)\varphi^{-\beta-\varepsilon-2}, \tag{3-9}$$

from which it follows that

$$\frac{v_r}{rv} = (\beta + \varepsilon)r^{-1}\varphi^{-1}, \quad \frac{rv_r}{2v} = \frac{\beta + \varepsilon}{2}r\varphi^{-1} \quad \text{and} \quad \frac{v_{rr}}{v} = (\beta + \varepsilon)(\beta + \varepsilon + 1)\varphi^{-2}.$$

Therefore

$$\lambda = \frac{v_r}{rv}\left(1 - \frac{rv_r}{2v}\right) = (\beta+\varepsilon)r^{-1}\varphi^{-1}\left(1 - \frac{\beta+\varepsilon}{2}r\varphi^{-1}\right) \tag{3-10}$$

and

$$\chi = \frac{v_{rr}}{v} - \frac{v_r}{vr} = -(\beta+\varepsilon)r^{-1}\varphi^{-1}(1-(\beta+\varepsilon+1)r\varphi^{-1}). \tag{3-11}$$

For Ψ we estimate using (3-9) to get

$$\begin{aligned} |\Psi| &\le Cr^2 v|v_{rr}| + Crv|v_r| + Cv^2 \\ &\le Cre^{-2\Lambda}\varphi^{-2\beta-2\varepsilon-2}(r+\varphi+r^{-1}\varphi^2) \le C_1 re^{-2\Lambda}\varphi^{-2\beta-2\varepsilon-2} =: \eta, \end{aligned} \tag{3-12}$$

where to obtain the final estimate in (3-12) we have used the fact that $r, \varphi \le 1$ and

$$r^{-1}\varphi^2 \le \frac{r_+^2}{r_-} \le r_+\left(1 + \frac{\varepsilon}{2(\beta+2)}\right) \le C.$$

Step 2: We now use the computations from Step 1 to analyse the eigenvalues of the matrix on the right-hand side of (3-7), or more precisely the eigenvalues of

$$-v^2\left(\lambda\delta_j^p + \chi\frac{x^p x_j}{r^2}\right) - \eta\delta_j^p,$$

which are given by

$$-(\chi v^2 + \lambda v^2 + \eta, \lambda v^2 + \eta, \dots, \lambda v^2 + \eta).$$

We write this vector of eigenvalues more conveniently as

$$(-\lambda v^2 - \eta)\left(\frac{\chi v^2}{\lambda v^2 + \eta} + 1, 1, \dots, 1\right).$$

We make the following two claims:

Claim 1: There exist constants $c_1 > 0$ and $0 < R_1 < 1$ depending only on g_0, f and Γ such that

$$-\lambda v^2 - \eta > c_1 e^{-2\Lambda}\varphi^{-2\beta-2\varepsilon-2} \quad \text{in } \{r_- < r < r_+\} \tag{3-13}$$

whenever $1 < r_+/r_- < 1 + \varepsilon/(2(\beta+2))$ and $r_+ < R_1$.

Claim 2: There exists a constant $c_2 > 0$ depending only on g_0, f and Γ, and a constant $0 < R_2 < 1$ depending additionally on ε such that

$$\frac{\chi v^2}{\lambda v^2 + \eta} + 1 > -\mu_\Gamma^+ + c_2\varepsilon \quad \text{in } \{r_- < r < r_+\} \tag{3-14}$$

whenever $1 < r_+/r_- < 1 + \varepsilon/(2(\beta+2))$ and $r_+ < R_2$.

Once the claims are proved, Proposition 3.4 is obtained as follows. First fix r_+ and r_- such that $1 < r_+/r_- < 1 + \varepsilon/(2(\beta+2))$ and $r_+ < \min\{R_1, R_2\}$. By Claim 2 and the definition of μ_Γ^+,

$$f\left(\frac{\chi v^2}{\lambda v^2 + \eta} + 1, 1, \dots, 1\right) > f(-\mu_\Gamma^+ + c_2\varepsilon, 1, \dots, 1) > 0 \quad \text{in } \{r_- < r < r_+\}.$$

Then, by Claim 1, it follows that

$$f\Big((-\lambda v^2-\eta)\Big(\frac{\chi v^2}{\lambda v^2+\eta}+1, 1, \dots, 1\Big)\Big) > c_1 e^{-2\Lambda}\varphi^{-2\beta-2\varepsilon-2} f(-\mu_\Gamma^+ + c_2\varepsilon, 1, \dots, 1) \quad \text{in } \{r_- < r < r_+\},$$

from which (3-4) follows. To complete the proof of Proposition 3.4, it therefore remains to prove Claims 1 and 2.

Note: We will use at various stages the fact that

$$1 < \frac{r_+}{r_-} < 1 + \frac{\varepsilon}{2(\beta+2)} \iff 0 < \varphi r^{-1} < \frac{\varepsilon}{2(\beta+2)} \quad \text{in } \{r_- < r < r_+\}. \tag{3-15}$$

Proof of Claim 1. Suppose $1 < r_+/r_- < 1 + \varepsilon/(2(\beta+2))$ and $r_+ < 1$. We start by computing

$$-\lambda v^2 = e^{-2\Lambda}\varphi^{-2\beta-2\varepsilon-2}(\beta+\varepsilon)\Big(\frac{\beta+\varepsilon}{2} - \varphi r^{-1}\Big). \tag{3-16}$$

By (3-15) and (3-16), it follows that

$$-\lambda v^2 \geq \frac{1}{C} e^{-2\Lambda}\varphi^{-2\beta-2\varepsilon-2} \quad \text{in } \{r_- < r < r_+\}. \tag{3-17}$$

Recalling also that

$$\eta = C_1 r e^{-2\Lambda}\varphi^{-2\beta-2\varepsilon-2}, \tag{3-18}$$

we see that (3-17) and (3-18) imply

$$-\lambda v^2 - \eta \geq (C^{-1} - C_1 r) e^{-2\Lambda}\varphi^{-2\beta-2\varepsilon-2} \quad \text{in } \{r_- < r < r_+\}. \tag{3-19}$$

The inequality (3-13) then follows from (3-19) after taking r_+ sufficiently small. This completes the proof of Claim 1. □

Proof of Claim 2. Suppose $1 < r_+/r_- < 1 + \varepsilon/(2(\beta+2))$ and $r_+ < 1$. By (3-16) and the fact that $\mu_\Gamma^+ = (2+\beta)/\beta$, we have

$$-\lambda v^2 - \mu_\Gamma^+\lambda v^2 = -\frac{2+2\beta}{\beta}\lambda v^2 = \frac{2+2\beta}{\beta} e^{-2\Lambda}\varphi^{-2\beta-2\varepsilon-2}(\beta+\varepsilon)\Big(\frac{\beta+\varepsilon}{2} - \varphi r^{-1}\Big), \tag{3-20}$$

and, by the formula for χ in (3-11), we have

$$-\chi v^2 = e^{-2\Lambda}(\beta+\varepsilon)\varphi^{-2\beta-2\varepsilon-2}(\varphi r^{-1} - (\beta+\varepsilon+1)). \tag{3-21}$$

It follows from (3-20) and (3-21) that

$$-\chi v^2 - \lambda v^2 - \mu_\Gamma^+\lambda v^2 = e^{-2\Lambda}\varphi^{-2\beta-2\varepsilon-2}\frac{\beta+\varepsilon}{\beta}(\varepsilon - (\beta+2)r^{-1}\varphi). \tag{3-22}$$

On the other hand, by (3-15), we have

$$\frac{\beta+\varepsilon}{\beta}(\varepsilon - (\beta+2)r^{-1}\varphi) > \frac{\varepsilon}{2} \quad \text{in } \{r_- < r < r_+\},$$

which when substituted into (3-22) yields

$$-\chi v^2 - \lambda v^2 - \mu_\Gamma^+ \lambda v^2 > \frac{\varepsilon}{2} e^{-2\Lambda} \varphi^{-2\beta-2\varepsilon-2} \quad \text{in } \{r_- < r < r_+\}. \tag{3-23}$$

Recalling (3-18), the estimate (3-23) therefore implies

$$-\chi v^2 - \lambda v^2 - \mu_\Gamma^+ \lambda v^2 - \eta - \mu_\Gamma^+ \eta \geq \left(\frac{\varepsilon}{2} - Cr\right) e^{-2\Lambda} \varphi^{-2\beta-2\varepsilon-2} \quad \text{in } \{r_- < r < r_+\}. \tag{3-24}$$

After taking r_+ smaller if necessary (but in a way that only depends on ε and the constant C in (3-24)), we therefore have

$$-\chi v^2 - \lambda v^2 - \mu_\Gamma^+ \lambda v^2 - \eta - \mu_\Gamma^+ \eta \geq \frac{\varepsilon}{4} e^{-2\Lambda} \varphi^{-2\beta-2\varepsilon-2} \quad \text{in } \{r_- < r < r_+\},$$

or equivalently

$$\frac{\chi v^2}{\lambda v^2 + \eta} + 1 \geq -\mu_\Gamma^+ + \frac{\frac{\varepsilon}{4} e^{-2\Lambda} \varphi^{-2\beta-2\varepsilon-2}}{-\lambda v^2 - \eta}. \tag{3-25}$$

On the other hand, by (3-16), we have

$$0 < -\lambda v^2 - \eta \leq -\lambda v^2 \leq C e^{-2\Lambda} \varphi^{-2\beta-2\varepsilon-2} \quad \text{in } \{r_- < r < r_+\}.$$

Thus, if r_+ is chosen sufficiently small (but depending only on g_0, f, Γ and ε), we see

$$\frac{\chi v^2}{\lambda v^2 + \eta} + 1 \geq -\mu_\Gamma^+ + c\varepsilon \quad \text{in } \{r_- < r < r_+\},$$

as required. This completes the proof of Claim 2. ☐

As explained above, with Claims 1 and 2 established, the proof of Proposition 3.4 is complete. ☐

3.3. *Gradient estimate.* In this section we prove the global gradient estimate.

Proposition 3.8. *Suppose (f, Γ) satisfies (1-1)–(1-4) and (1-8), and let $\tau \leq 1$. Let $\psi \in C^\infty(M)$ be positive and $\xi \in C^\infty(\partial M)$. Then there exists a constant C which is independent of τ but dependent on g_0, f, Γ and upper bounds for $\|\psi\|_{C^1(M)}$, $\|\xi\|_{C^2(\partial M)}$ and $\|u\|_{C^0(M)}$ such that any C^3 solution to (1-11) satisfies $|\nabla_{g_0} u|_{g_0} \leq C$ on M.*

Proof. By a conformal change of background metric, we may assume without loss of generality that $\xi \equiv 0$.

By our interior local gradient estimate in Theorem 1.8, we only need to prove the gradient estimate near the boundary, say in $B_{1/2}(y_0) \cap M$, where $y_0 \in \partial M$ is arbitrary. Consider $H = \rho |\nabla_{g_0} u|_{g_0}^2$, where ρ is a smooth cutoff function satisfying $\rho = 1$ on $B_{1/2}(y_0)$, $\rho = 0$ outside $B_1(y_0)$, $|\nabla_{g_0} \rho|_{g_0} \leq C \rho^{1/2}$ and $|\nabla_{g_0}^2 \rho|_{g_0} \leq C$. Suppose that H attains its maximum at $x_0 \in M$. If $x_0 \notin B_1(y_0) \cap M$, then $\nabla_{g_0} u = 0$ in $B_{1/2}(y_0) \cap M$ and we are done. If $x_0 \in B_1(y_0) \cap (M \setminus \partial M)$, then our proof of Theorem 1.8 applies and we again obtain the desired estimate. It remains to consider the case that $x_0 \in B_1(y_0) \cap \partial M$.

We first observe that, since $\xi \equiv 0$ on ∂M, the tangential derivatives of u on ∂M vanish. Therefore, we only need to bound the normal derivative $\nabla_\nu u(x_0)$, where ν denotes the inward pointing unit normal to ∂M at x_0. We first consider the lower bound for $\nabla_\nu u(x_0)$. With the same setup and notation as in the proof of Proposition 3.3, except now with $m = u(x_0) = 0$, consider an annulus $A_{r_-,r_+}(y)$ satisfying $\mathbb{S}_{r_-}(y) \cap \partial M = \{x_0\}$ and conditions (1)–(6) in the proof of Proposition 3.3. Then the function w on $A_{r_-,r_+}(y)$, as defined in (3-3), satisfies $w \le u$ on $A_{r_-,r_+}(y) \cap \partial M$ since $w(x_0) = u(x_0)$ and w is radially decreasing. By the comparison principle stated in Proposition 3.7, it follows that $w \le u$ on $A_{r_-,r_+}(y) \cap M$. Thus, for $x \in A_{r_-,r_+}(y) \cap M$, we have

$$\frac{u(x) - u(x_0)}{\mathrm{d}(x, x_0)} = \frac{u(x) - w(x_0)}{\mathrm{d}(x, x_0)} \ge \frac{w(x) - w(x_0)}{\mathrm{d}(x, x_0)},$$

which implies

$$\nabla_\nu u(x_0) \ge \nabla_\nu w(x_0).$$

For the upper bound for $\nabla_\nu u(x_0)$, we use a barrier function constructed in [Guan 2008]. First observe that, since $\Gamma \subset \Gamma_1^+$, we have

$$0 < \sigma_1(-g_0^{-1} A_{g_u}) = \Delta_{g_0} u + \frac{n-2}{2} |\nabla_{g_0} u|_{g_0}^2 - \sigma_1(g_0^{-1} A_{g_0}).$$

Now let $d(x) = \mathrm{d}(x, \partial M)$ and recall $M_\delta = \{x \in M : d(x) < \delta\}$. It is well known that, for sufficiently small $\delta > 0$, d is smooth in M_δ with $|\nabla_{g_0} d|_{g_0} = 1$. To obtain an upper bound for $\nabla_\nu u(x_0)$, it suffices to find a function $\bar{u} \in C^3(M_\delta)$ satisfying

$$\begin{cases} \sigma_1(-g_0^{-1} A_{g_{\bar{u}}}) \le 0 & \text{in } M_\delta, \\ \bar{u} = u & \text{on } \partial M, \\ \bar{u} \ge u & \text{on } \partial M_\delta \setminus \partial M. \end{cases} \tag{3-26}$$

Indeed, once such a function $\bar{u}$ is obtained, the maximum principle implies $\bar{u} \ge u$ on M_δ, and it follows that, for any $x \in M_\delta$, we have

$$\frac{u(x) - u(x_0)}{\mathrm{d}(x, x_0)} = \frac{u(x) - \bar{u}(x_0)}{\mathrm{d}(x, x_0)} \le \frac{\bar{u}(x) - \bar{u}(x_0)}{\mathrm{d}(x, x_0)},$$

which implies $\nabla_\nu u(x_0) \le \nabla_\nu \bar{u}(x_0)$.

To this end, we define as in [Guan 2008]

$$\bar{u}(x) = \frac{1}{n-2} \ln \frac{d(x) + \delta^2}{\delta^2}.$$

We first observe that $\bar{u}|_{\partial M} = 0 = u|_{\partial M}$. Next we calculate $\sigma_1(-g_0^{-1} A_{g_{\bar{u}}})$. In what follows, we denote by ∇d the differential of d (whereas $\nabla_{g_0} d$ will continue to denote the gradient of d with respect to g_0). Routine computations yield

$$\nabla_{g_0} \bar{u}(x) = \frac{1}{n-2} \frac{\nabla_{g_0} d(x)}{d(x) + \delta^2}$$

and

$$\nabla_{g_0}^2 \bar{u}(x) = \frac{1}{n-2} \left(\frac{\nabla_{g_0}^2 d(x)}{d(x) + \delta^2} - \frac{\nabla d(x) \otimes \nabla d(x)}{(d(x) + \delta^2)^2} \right),$$

from which it follows that

$$\sigma_1(-g_0^{-1}A_{g_{\bar u}}) = \Delta_{g_0}\bar u + \frac{n-2}{2}|\nabla_{g_0}\bar u|^2_{g_0} - \sigma_1(g_0^{-1}A_{g_0}) \le C - \frac{1}{2(n-2)}\frac{1}{(d(x)+\delta^2)^2} + \frac{C}{d(x)+\delta^2}, \tag{3-27}$$

where we have used the fact that $|\nabla_{g_0}d|_{g_0} = 1$ and $|\Delta_{g_0}d| \le C$ in M_δ for δ sufficiently small. We then see that the negative term on the last line of (3-27) dominates the remaining terms for $\delta > 0$ sufficiently small. Therefore, for $\delta > 0$ sufficiently small, we have $\sigma_1(-g_0^{-1}A_{g_{\bar u}}) \le 0$ in M_δ.

Finally, we observe that on $\partial M_\delta \setminus \partial M$ we have

$$\bar u = \frac{1}{n-2}\ln\Big(\frac{\delta+\delta^2}{\delta^2}\Big) \ge \frac{1}{n-2}\ln(1/\delta).$$

Choosing δ smaller if necessary so that

$$\frac{1}{n-2}\ln(1/\delta) \ge \max_M u \quad \text{on } \partial M_\delta \setminus \partial M,$$

the construction of $\bar u$ is complete. This completes the proof of Proposition 3.8. □

3.4. ***Hessian estimate.*** In this section we give the global Hessian estimate assuming $\tau < 1$.

Proposition 3.9. *Suppose (f, Γ) satisfies (1-1)–(1-4), and let $\tau < 1$. Let $\psi \in C^\infty(M)$ be positive and $\xi \in C^\infty(\partial M)$. Then there exists a constant C depending on g_0, f, Γ, $(1-\tau)^{-1}$ and upper bounds for $\|\psi\|_{C^2(M)}$, $\|\xi\|_{C^2(M)}$ and $\|u\|_{C^1(M)}$ such that any solution to (1-11) satisfies $|\nabla^2_{g_0}u|_{g_0} \le C$ on M.*

We point out that we do not require $\mu^+_\Gamma > 1$ in Proposition 3.9.

Proof. If the maximum of $|\nabla^2_{g_0}u|_{g_0}$ occurs in $M \setminus \partial M$, then one can appeal to the proof of the global estimate in [Gursky and Viaclovsky 2003] if $f = \sigma_k^{1/k}$, or the proof of the global estimate in [Guan 2008] for general (f, Γ) satisfying (1-1)–(1-4). So we suppose that the maximum occurs at a point $x_0 \in \partial M$. Let e_n denote the interior unit normal vector field on ∂M, and fix an orthonormal frame $\{e_1, \dots, e_{n-1}\}$ for the tangent bundle of ∂M near x_0. By parallel transporting along geodesics normal to ∂M, we may extend this to an orthonormal frame $\{e_1, \dots, e_n\}$ for the tangent bundle of M near x_0. Since $(\nabla^2_{g_0}u)_{ij}(x_0) = (\nabla^2_{g_0}\xi)_{ij}(x_0)$ for $i, j \ne n$, we only need to estimate $(\nabla^2_{g_0}u)_{ij}(x_0)$ when at least one of i or j are equal to n. The proof is almost identical to that in [Guan 2008], but for the convenience of the reader we summarise the argument here. In what follows, all computations are carried out in a neighbourhood of x_0 on which the frame $\{e_1, \dots, e_n\}$ is defined.

Still with the convention $g_u = e^{2u}g_0$, it will be convenient to write the equation in (1-11) in the equivalent form

$$f(\lambda(-g_0^{-1}A^\tau_{g_u})) = \psi e^{2u}, \quad \lambda(-g_u^{-1}A^\tau_{g_u}) \in \Gamma \quad \text{on } M \setminus \partial M, \tag{3-28}$$

where

$$A^\tau_{g_u} = \tau A_{g_u} + (1-\tau)\sigma_1(-g_u^{-1}A_{g_u})g_u = -\tau\nabla^2_{g_0}u - (1-\tau)\Delta_{g_0}u g_0 - b_{n,\tau}|\nabla_{g_0}u|^2_{g_0}g_0 + \tau du\otimes du + A^\tau_{g_0}$$

and $b_{n,\tau} = \frac{1}{2}(n-2-(n-3)\tau)$. Writing $F[u] = f(\lambda(-g_0^{-1}A^\tau_{g_u}))$ and

$$F^{ij} = \frac{\partial f}{\partial A_{ij}}\bigg|_{A=-g_0^{-1}A^\tau_{g_u}},$$

the linearisation of F at u in the direction η (excluding zero-order terms) is given by

$$\begin{aligned}\mathcal{L}\eta &= F^{ij}(\tau(\nabla^2_{g_0}\eta)_{ij} + (1-\tau)\Delta_{g_0}\eta(g_0)_{ij} + 2b_{n,\tau}\langle\nabla_{g_0}u, \nabla_{g_0}\eta\rangle_{g_0}(g_0)_{ij} - 2\tau\,\partial_i u, \partial_j\eta)\\ &= F^{ij}(\tau(\nabla^2_{g_0}\eta)_{ij} - 2\tau\,\partial_i u\,\partial_j\eta) + ((1-\tau)\Delta_{g_0}\eta + 2b_{n,\tau}\langle\nabla_{g_0}u, \nabla_{g_0}\eta\rangle_{g_0})\sum_i F^{ii}. \qquad (3\text{-}29)\end{aligned}$$

Now suppose $\delta > 0$ is sufficiently small so that $d(x) = \mathrm{d}(x, \partial M)$ is smooth in $M_\delta = \{x \in M : d(x) < \delta\}$. For a positive constant N to be determined later, define

$$v = \frac{N}{2}d^2 - d. \qquad (3\text{-}30)$$

A routine computation shows that, for $\delta > 0$ sufficiently small,

$$|\mathcal{L}d| \le C_0 \sum_i F^{ii} \quad \text{in } M_\delta, \qquad (3\text{-}31)$$

where C_0 is a constant independent of τ but depending on g_0 and an upper bound for $\|u\|_{C^1(M)}$. It follows that

$$\begin{aligned}\mathcal{L}d^2 &= 2d\mathcal{L}d + 2(1-\tau)|\nabla_{g_0}d|^2_{g_0}\sum_i F^{ii} + 2F^{ij}\,\partial_i d\,\partial_j d\\ &\ge 2d\mathcal{L}d + 2(1-\tau)\sum_i F^{ii}\\ &\ge 2((1-\tau) - C_0 d)\sum_i F^{ii} \quad \text{in } M_\delta. \qquad (3\text{-}32)\end{aligned}$$

Choosing $N \ge 4(1+C_0)/(1-\tau)$ and subsequently $\delta \le \min\{N^{-1}, C_0^{-1}\}$, one sees from (3-31) and (3-32) that the function v defined in (3-30) satisfies

$$\mathcal{L}v \ge \sum_i F^{ii} \quad \text{and} \quad v \le -\frac{d}{2} \quad \text{in } M_\delta. \qquad (3\text{-}33)$$

With (3-33) in hand, one can then show the following.

Lemma 3.10. *Fix $\delta > 0$ sufficiently small as in the foregoing argument. If $h \in C^2(\overline{M}_\delta)$ satisfies $h \le 0$ on ∂M, $h(z_0) = 0$ for some $z_0 \in \partial M$ and*

$$-\mathcal{L}h \le C_1 \sum_i F^{ii} \quad \textit{in } M_\delta \qquad (3\text{-}34)$$

for some constant C_1, then

$$(\nabla_{g_0}h)_n(z_0) \le C, \qquad (3\text{-}35)$$

where C is a constant depending on g_0, C_1, $(1-\tau)^{-1}$ and upper bounds for $\|h\|_{C^0(\overline{M}_\delta)}$ and $\|u\|_{C^1(M)}$.

Proof. It is clear from the definition of v that we can choose $A > 0$ large (depending on $\|h\|_{C^0(\overline{M}_\delta)}$) such that $-Av - h \geq 0$ on ∂M_δ. On the other hand, using (3-33) and (3-34), we have

$$\mathcal{L}(-Av - h) \leq (-A + C_1)\sum_i F^{ii} \quad \text{in } M_\delta,$$

and hence $\mathcal{L}(-Av - h) \leq 0$ in M_δ for A sufficiently large. Thus, for A sufficiently large the maximum principle yields $-Av-h \geq 0$ in M_δ, and since $(-Av-h)(z_0) = 0$, it follows that $(\nabla_{g_0}(-Av-h))_n(z_0) \geq 0$, i.e., $(\nabla_{g_0}h)_n(z_0) \leq -A(\nabla_{g_0}v)_n(z_0)$. The estimate (3-35) then follows. □

We now continue the proof of Proposition 3.9. Suppose $i \in \{1, \dots, n-1\}$ and define $h = \pm(\nabla_{g_0}(u-\bar{\xi}))_i$, where (as in the proof of Proposition 3.8) $\bar{\xi}$ denotes the extension of ξ to M_δ such that $\bar{\xi}$ is constant along geodesics normal to ∂M. By differentiating (3-28), one can show directly that $|\mathcal{L}(\nabla_{g_0}u)_i| \leq C\sum_i F^{ii}$, and by (2-6) we also have $|\mathcal{L}\bar{\xi}| \leq C \leq C\sum_i F^{ii}$. Therefore h satisfies the assumptions of Lemma 3.10, and it follows from Lemma 3.10 that

$$|(\nabla^2_{g_0}u)_{in}(x_0)| \leq C.$$

It remains to estimate the double normal derivative $(\nabla^2_{g_0}u)_{nn}(x_0)$. Note that, since $\{e_1, \dots, e_n\}$ is an orthonormal frame and $(\nabla^2_{g_0}u)_{ii}(x_0) = (\nabla^2_{g_0}\xi)_{ii}(x_0)$ for $i \in \{1, \dots, n-1\}$, obtaining an upper (resp. lower) bound for $(\nabla^2_{g_0}u)_{nn}(x_0)$ is equivalent to obtaining an upper (resp. lower) bound for $\Delta_{g_0}u(x_0)$. Now, since $\Gamma \subseteq \Gamma_1^+$, the lower bound $\Delta_{g_0}u \geq -C$ in M is immediate. To obtain the upper bound for $(\nabla^2_{g_0}u)_{nn}(x_0)$, we may assume $(\nabla^2_{g_0}u)_{nn}(x_0) \geq 1$, otherwise we are done. We may also assume that, with respect to the frame $\{e_1, \dots, e_n\}$, the Hessian of u at x_0 is given by $\nabla^2_{g_0}u(x_0) = \operatorname{diag}((\nabla^2_{g_0}u)_{11}(x_0), \dots, (\nabla^2_{g_0}u)_{nn}(x_0))$. Then, by (3-28), monotonicity of f and our estimates for $(\nabla^2_{g_0}u)_{ij}(x_0)$ when i and j are not both equal to n, we have

$$\psi(x_0)e^{2u(x_0)} = f(-g_0^{-1}A^\tau_{g_u}(x_0)) \geq f((1-\tau)(\nabla^2_{g_0}u)_{nn}(x_0)g_0 + B), \tag{3-36}$$

where B is a symmetric matrix bounded in terms of $\|u\|_{C^1(M)}$. Observing that, by homogeneity of f,

$$\frac{1}{t}f(tg_0 + B) = f(g_0 + t^{-1}B) \to f(g_0) \quad \text{as } t \to \infty,$$

we see that (3-36) implies an upper bound for $(\nabla^2_{g_0}u)_{nn}(x_0)$. □

3.5. *Proof of Theorem 1.6.* We first prove the existence of a smooth solution to (1-11) when $\tau < 1$. Fix $\varepsilon > 0$, and let $S_\varepsilon = \{\tau \in [0, 1-\varepsilon]$: (1-11) admits a solution in $C^{2,\alpha}(M)\}$. Since (1-11) admits a unique smooth solution when $\tau = 0$, S_ε is nonempty. A computation as in (3-29) (but now including zero-order terms) shows that the linearised operator is invertible as a mapping from $C^{2,\alpha}(M)$ to $C^\alpha(M)$, from which openness of S_ε follows. By Propositions 3.1 and 3.2, solutions to (1-11) admit a global C^0 estimate. By Proposition 3.8, solutions to (1-11) therefore admit a global C^1 estimate. Note that, at this point, the estimates are independent of ε. By Proposition 3.9, one then obtains the global C^2 estimate on solutions to (1-11), which do now depend on ε. With the C^2 estimate established, (1-11) becomes uniformly elliptic, and the regularity theory of Evans and Kyrlov [Evans 1982; Krylov 1982; 1983] then implies a $C^{2,\alpha}$ estimate. Thus S_ε is also closed, and so $S_\varepsilon = [0, 1-\varepsilon]$. Since $\varepsilon > 0$ was arbitrary, existence of a $C^{2,\alpha}$

solution to (1-11) for any $\tau < 1$ then follows. Higher regularity then follows from classical Schauder theory, and uniqueness is a consequence of the comparison principle in Proposition 3.7.

Now, since the solutions obtained to (1-11) are uniformly bounded in $C^1(M)$ as $\tau \to 1$, along a sequence $\tau_i \to 1$ these solutions converge uniformly to some $u \in C^{0,1}(M)$. The proof that u is a viscosity solution to (1-11) when $\tau = 1$ is exactly the same as in the proof of Theorem 1.3 in [Li and Nguyen 2021] and is omitted here. □

4. Proof of Theorem 1.1′: the fully nonlinear Loewner–Nirenberg problem

In this section we prove Theorem 1.1′. Our proof proceeds according to the following steps:

(1) In Section 4.1 we construct a smooth solution to (1-10) when $\tau < 1$. The solution is obtained as the limit of solutions with constant finite boundary data $m \in \mathbb{R}$ (which we know to exist by Theorem 1.6) as $m \to \infty$.

(2) In Section 4.2 we prove that there exists a smooth solution u to (1-10) when $\tau < 1$ satisfying the asymptotics stated in (1-9).

(3) In Section 4.3 we prove that *any* smooth solution to (1-10) must satisfy (1-9) when $\tau < 1$. When combined with the maximum principle, this will imply that the solution u obtained to (1-10) is unique when $\tau < 1$.

(4) In Section 4.4 we complete the proof of Theorem 1.1′.

4.1. ***Existence of a smooth solution to (1-10) when $\tau < 1$.*** Fix $\tau < 1$, and suppose that (f, Γ) satisfies (1-1)–(1-4), (1-7) and (1-8). By Theorem 1.6, we know that, for each $m \in \mathbb{R}$, there exists a unique smooth solution u_m to

$$\begin{cases} f^\tau(\lambda(-g_{u_m}^{-1}A_{g_{u_m}})) = 1, \quad \lambda(-g_{u_m}^{-1}A_{g_{u_m}}) \in \Gamma^\tau & \text{on } M \setminus \partial M, \\ u_m = m & \text{on } \partial M. \end{cases} \tag{4-1}$$

In this section we show that, in the limit $m \to \infty$, one obtains a smooth solution u to (1-10).

Proposition 4.1. *Fix $\tau < 1$, and suppose that (f, Γ) satisfies (1-1)–(1-4), (1-7) and (1-8). Let u_m denote the unique smooth solution to (4-1). Then a subsequence of $\{u_m\}_m$ converges locally uniformly as $m \to \infty$ to a solution $u \in C^\infty(M \setminus \partial M)$ of (1-10). Moreover, given any constant $\alpha > 0$, there exists a constant $\delta > 0$ independent of τ but dependent on g_0, α, f and Γ such that $u \geq \alpha$ in $M_\delta \setminus \partial M$.*

Proof. Since the comparison principle in Proposition 3.7 implies $u_{m+1} \geq u_m$, to prove the existence of a limit $u \in C^\infty(M \setminus \partial M)$ solving (1-10), it suffices to show that, for each compact set $K \subset M \setminus \partial M$, there exists a constant C independent of m such that $\|u_m\|_{C^2(K)} \leq C$; higher order estimates then follow from the work of Evans and Krylov [Evans 1982; Krylov 1982] and classical Schauder theory.

The lower bound is trivial (and in fact global) since $u_m \geq u_1$ for all m. Next we address the local upper bound — note that whilst we obtained a global upper bound in Proposition 3.1, the bound therein depends on m, which is insufficient for our current purposes. Recalling the normalisation $f\left(\frac{1}{2}, \ldots, \frac{1}{2}\right) = 1$, we have by concavity and homogeneity of f

$$f(\lambda) \leq f\left(\frac{\sigma_1(\lambda)}{n}e\right) + \nabla f\left(\frac{\sigma_1(\lambda)}{n}e\right) \cdot \left(\lambda - \frac{\sigma_1(\lambda)}{n}e\right) = \frac{f(e)}{n}\sigma_1(\lambda) = \frac{2}{n}\sigma_1(\lambda) \tag{4-2}$$

for $\lambda \in \Gamma$, and thus any solution to the equation in (4-1) satisfies $R_{g_{u_m}} \leq -n(n-1)$. On the other hand, by [Aviles and McOwen 1988], there exists a smooth metric $g_w = e^{2w}g_0$ satisfying

$$\begin{cases} R_{g_w} = -n(n-1) & \text{on } M \setminus \partial M, \\ w(y) \to +\infty & \text{as } \mathrm{d}(y, \partial M) \to 0. \end{cases} \tag{4-3}$$

By the comparison principle for the semilinear equation (4-3), $u_m \leq w$ in $M \setminus \partial M$ for each m, which yields a finite upper bound for u_m on any compact subset of $M \setminus \partial M$ which is independent of m. The local gradient estimate then follows from Theorem 1.8, or alternatively one can appeal to [Guan 2008, Theorem 2.1] since we have the two-sided C^0 bound at this point. For the local Hessian estimate, we appeal to [Guan 2008, Theorem 3.1]. We therefore obtain the full C^2 estimate $\|u_m\|_{C^2(K)} \leq C(K)$ on any compact set $K \subset M \setminus \partial M$, as required.

It remains to prove the second assertion in the statement of Proposition 4.1. Fix $\alpha > 0$ and consider the solution $u_{\alpha+1}$ to (4-1) with $m = \alpha + 1$. Since $u_{\alpha+1}$ admits a global C^0 estimate depending only g_0, α, f and Γ, there exists a constant $\delta > 0$ depending only on g_0, α, f and Γ such that $u_{\alpha+1} \geq \alpha$ in M_δ. By the comparison principle in Proposition 3.7, $u \geq u_{\alpha+1}$ in $M \setminus \partial M$, and in particular $u \geq \alpha$ in $M_\delta \setminus \partial M$, as required. □

4.2. ***Asymptotics.*** Fix $\tau < 1$ and suppose that (f, Γ) satisfies (1-1)–(1-4), (1-7) and (1-8). In this section we show that there exists a smooth solution u to (1-10) satisfying (1-9), that is

$$\lim_{\mathrm{d}(x,\partial M)\to 0} (u(x) + \ln \mathrm{d}(x, \partial M)) = 0. \tag{4-4}$$

Remark 4.2. At this point of the argument, we do not know that this constructed solution coincides with the one obtained in Section 4.1, although we will later see in Section 4.3 that this is the case.

We start by proving an upper bound on the growth of any smooth solution to the equation in (1-10), irrespective of the boundary data or whether $\tau < 1$ or $\mu_\Gamma^+ > 1$.

Proposition 4.3. *Let (M, g_0) be a smooth Riemannian manifold with nonempty boundary and suppose that (f, Γ) satisfies (1-1)–(1-4) and (1-7). Then there exist constants $\delta > 0$ and $C > 0$ depending only on g_0 such that any continuous metric $g_u = e^{2u}g_0$ satisfying*

$$f(\lambda(-g_u^{-1}A_{g_u})) \geq 1, \quad \lambda(-g_u^{-1}A_{g_u}) \in \Gamma \quad \textit{in the viscosity sense on } M \setminus \partial M \tag{4-5}$$

satisfies

$$u(x) + \ln \mathrm{d}(x, \partial M) \leq C\mathrm{d}(x, \partial M)^{1/2} \quad \textit{in } M_\delta \setminus \partial M. \tag{4-6}$$

In particular, any continuous metric $g_u = e^{2u}g_0$ satisfying (4-5) satisfies

$$\limsup_{\mathrm{d}(x,\partial M)\to 0} (u(x) + \ln \mathrm{d}(x, \partial M)) \leq 0. \tag{4-7}$$

Proof. By (4-2), the comparison principle for viscosity sub- and supersolutions to uniformly elliptic equations implies that if $g_w = e^{2w} g_0$ satisfies

$$\begin{cases} \sigma_1(-g_w^{-1} A_{g_w}) \le \frac{1}{2} n & \text{in } \Omega \Subset M \setminus \partial M, \\ w(x) \to +\infty & \text{as } \mathrm{d}(x, \partial\Omega) \to 0, \end{cases} \tag{4-8}$$

then $u \le w$ in Ω. Since $\sigma_1(-g_w^{-1} A_{g_w}) = -(2(n-1))^{-1} R_{g_w}$, the transformation law for scalar curvature implies that the equation in (4-8) is equivalent to

$$-\frac{S_{g_0}}{n-1} + 2\Delta_{g_0} w + (n-2)|\nabla_{g_0} w|^2_{g_0} \le n e^{2w}. \tag{4-9}$$

We follow an argument of Gursky, Streets and Warren [Gursky et al. 2011], in turn based on the original argument of Loewner and Nirenberg [1974], to construct such local supersolutions near ∂M. For a point x_0 a distance d from ∂M, consider a point z_0 a distance $R > d$ from ∂M, which lies along the shortest path geodesic from x_0 to ∂M. We may assume R is small enough so that $\Delta_{g_0} d^2(z_0, \cdot) \ge 1$ on $B_R(z_0)$, and so that there exists a function h defined on $[0, R^2]$ satisfying

$$(n-2)(h')^2 + 2h'' \le 0, \quad h' > \max_M |S_{g_0}| + \widetilde{C}(g_0), \quad h(0) = 0, \tag{4-10}$$

where $\widetilde{C}(g_0)$ is a sufficiently large constant to be fixed in the proof. Indeed, once $\widetilde{C}(g_0)$ is fixed, the function $h(t) = \sqrt{t + \varepsilon^2} - \varepsilon$ satisfies (4-10) for ε sufficiently small and t in a sufficiently small interval $[0, R^2]$.

Let r denote the distance from z_0, and define on $B_R(z_0)$ the radial function

$$w(r) = -\ln(R^2 - r^2) + h(R^2 - r^2) + \ln\alpha,$$

where $\alpha > 0$ is to be determined. Exactly as in the proof of Lemma 5.2 in [Gursky et al. 2011], a direct computation shows that, for R sufficiently small and $\widetilde{C}(g_0)$ sufficiently large, the left-hand side of (4-9) satisfies

$$-\frac{S_{g_0}}{n-1} + 2\Delta_{g_0} w + (n-2)|\nabla_{g_0} w|^2_{g_0} \le \frac{4nR^2}{(R^2 - r^2)^2} e^{2h} = \frac{4nR^2}{\alpha^2} e^{2w}. \tag{4-11}$$

Therefore, if we take $\alpha = 2R$, we see w indeed satisfies (4-9). We then obtain

$$\begin{aligned} u(x_0) \le w(x_0) &= -\ln(R^2 - (R-d)^2) + h(R^2 - (R-d)^2) + \ln(2R) \\ &= -\ln(d(2R-d)) + h(d(2R-d)) + \ln(2R) \\ &= -\ln d - \ln\Big(1 - \frac{d}{2R}\Big) + h(d(2R-d)). \end{aligned}$$

But $h(d(2R-d)) = \sqrt{d(2R-d) + \varepsilon^2} - \varepsilon \le \sqrt{d(2R-d)} \le C\sqrt{d}$ and

$$\ln\Big(1 - \frac{d}{2R}\Big) \ge -\frac{d}{2R} \ge -C\sqrt{d}$$

for sufficiently small d, and thus (4-6) follows. The inequality (4-7) is a clear consequence of (4-6). □

We are now in a position to prove the existence of a smooth solution to (1-10) when $\tau < 1$ with the desired asymptotic behaviour in (4-4).

Proposition 4.4. *Fix $\tau < 1$, and suppose that (f, Γ) satisfies (1-1)–(1-4), (1-7) and (1-8). Then there exists a smooth solution $g_v = e^{2v} g_0$ to (1-10) and a constant C independent of τ but dependent on g_0, f and Γ such that the following holds: for each $\varepsilon > 0$ sufficiently small, there exists a constant $a \gg 0$ independent of τ but dependent on g_0, ε, C, f and Γ such that*

$$v(x) + \ln \mathrm{d}(x, \partial M) \geq \ln \sqrt{1 - 2\varepsilon} - \ln(1 + a\mathrm{d}(x, \partial M)) \quad \text{in } A_\varepsilon^a \subset M, \tag{4-12}$$

where

$$A_\varepsilon^a = \Big\{ x \in M \setminus \partial M : d(x) + ad(x)^2 \leq \frac{\varepsilon}{C} \Big\}.$$

In particular,

$$\lim_{\mathrm{d}(x, \partial M) \to 0} (v(x) + \ln \mathrm{d}(x, \partial M)) = 0. \tag{4-13}$$

Proof. Consider an exhaustion of M by smooth compact manifolds with boundary defined by

$$M_{(j)} = \{ x \in M : \mathrm{d}(x, \partial M) \geq j^{-1} \}.$$

By Proposition 4.1, for each j, there exists a smooth solution $g_{v_{(j)}} = e^{2v_{(j)}} g_0$ to

$$\begin{cases} f^\tau(-g_{v_{(j)}}^{-1} A_{g_{v_{(j)}}}) = 1, \quad \lambda(-g_{v_{(j)}}^{-1} A_{g_{v_{(j)}}}) \in \Gamma^\tau & \text{on } M_{(j)} \setminus \partial M_{(j)}, \\ v_{(j)}(x) \to +\infty & \text{as } \mathrm{d}(x, \partial M_{(j)}) \to 0. \end{cases}$$

(Note that we put parentheses around the index j to avoid confusion with the solutions u_m to (4-1)). Since $v_{(j)}(x) \to +\infty$ as $\mathrm{d}(x, \partial M_{(j)}) \to 0$, the comparison principle in Proposition 3.7 implies that if $j < m$, then

$$v_{(m)}|_{M_{(j)}} < v_{(j)}. \tag{4-14}$$

Now, as justified in the proof of Proposition 4.1, a subsequence of $\{v_{(j)}\}_j$ converges locally uniformly to some $v \in C^\infty(M \setminus \partial M)$. We claim that v is our desired function. It is clear that v solves the equation in (1-10). We now establish (4-12), which we split into two steps: in the first step we show $v(x) \to +\infty$ as $\mathrm{d}(x, \partial M) \to 0$, and in the second step we prove (4-12).

Step 1: In this first step we show that $v(x) \to +\infty$ as $\mathrm{d}(x, \partial M) \to 0$. To this end, let $d(x) = \mathrm{d}(x, \partial M)$, and define $\varphi = -\ln(B(d + ad^2))$, $g_\varphi = e^{2\varphi} g_0$, where a and B are positive constants to be determined. Writing $e^{2\varphi} = \psi^{-2}$, so that $\psi = B(d + ad^2)$, we compute near ∂M

$$|\nabla_{g_0} \psi|_{g_0}^2 = B^2(1 + 2ad)^2 \quad \text{and} \quad \nabla_{g_0}^2 \psi = B(1 + 2ad)\nabla_{g_0}^2 d + 2aB \nabla d \otimes \nabla d,$$

where ∇d denotes the differential of d. It follows that, near ∂M,

$$\begin{aligned} -g_\varphi^{-1} A_{g_\varphi} &= g_0^{-1}\big(-\psi \nabla_{g_0}^2 \psi + \tfrac{1}{2} |\nabla_{g_0} \psi|_{g_0}^2 g_0 - \psi^2 A_{g_0}\big) \\ &= B^2 g_0^{-1}\big(\tfrac{1}{2} g_0 + 2a^2 d^2 [g_0 - \nabla d \otimes \nabla d - d \nabla_{g_0}^2 d] - d(1 + 3ad)\nabla_{g_0}^2 d \\ &\qquad + 2ad[g_0 - \nabla d \otimes \nabla d] - d^2(1 + ad)^2 A_{g_0}\big). \end{aligned} \tag{4-15}$$

Taking for instance $a = 1$, we then see that, for δ fixed sufficiently small and B fixed sufficiently large,

$$f^\tau(-g_\varphi^{-1} A_{g_\varphi}) \geq 1 \quad \text{in } M_\delta \setminus \partial M. \tag{4-16}$$

To use (4-16) to show $v(x) \to +\infty$ as $\mathrm{d}(x, \partial M) \to 0$, we follow the proof of [Loewner and Nirenberg 1974, Theorem 5]. For $m \gg 1$, denote by S_m the set where $\varphi(x) = -\ln(B(d + d^2)) \geq m$. We may assume (by taking m sufficiently large) that S_m is a tubular neighbourhood of ∂M contained in M_δ. Let $\Sigma_m = \partial S_m \setminus \partial M$ and $D_m = \min_{\Sigma_m} v$, and suppose J is sufficiently large so that $\Sigma_m \subset M_{(j)}$ for all $j \geq J$. Then $\varphi = m$ and $v \geq D_m$ on Σ_m, and, by the monotonicity in (4-14), we also have $v_{(j)} \geq D_m$ on Σ_m for each $j \geq J$. Therefore

$$v_{(j)} + \max\{0, m - D_m\} \geq m = \varphi \quad \text{on } \Sigma_m \tag{4-17}$$

and

$$v_{(j)} + \max\{0, m - D_m\} = \infty > \varphi \quad \text{on } \partial M_{(j)}. \tag{4-18}$$

In light of (4-16)–(4-18), the comparison principle in Proposition 3.7 implies $v_{(j)} + \max\{0, m - D_m\} \geq \varphi$ on $M_{(j)} \cap S_m$. Sending $j \to \infty$, it follows that $v + \max\{0, m - D_m\} \geq \varphi$ in S_m, and in particular $v(x) \to +\infty$ as $\mathrm{d}(x, \partial M) \to 0$.

Step 2: In this second step we show that v satisfies (4-12). The method is essentially a quantitative version of Step 1, requiring a more careful choice of parameters a and B in the definition of φ.

We first claim that the two quantities in the square parentheses in (4-15) are nonnegative definite for sufficiently small d. Indeed, observe that $g_0(x) - \nabla d(x) \otimes \nabla d(x)$ is the induced metric on $\partial M_{d(x)} \setminus \partial M$ and is therefore nonnegative definite. Moreover, $\nabla^2_{g_0} d$ is a bounded tensor near ∂M whose kernel contains ∇d. Hence $\nabla^2_{g_0} d$ is bounded from above by $C(g_0 - \nabla d \otimes \nabla d)$ for some constant C depending only on (M, g_0). Therefore, $g_0 - \nabla d \otimes \nabla d - d\nabla^2_{g_0} d$ is nonnegative definite for d sufficiently small, as claimed.

In light of (4-15) and the above claim, we see that, for δ chosen sufficiently small independently of a (but depending on (M, g_0)) and $\widehat{C} \geq 1$ a constant such that $|A_{g_0}|_{g_0}, |\nabla^2_{g_0} d|_{g_0} \leq \widehat{C}$ on M_δ, we have

$$\begin{aligned} -g_\varphi^{-1} A_{g_\varphi} &\geq B^2 g_0^{-1}\big(\tfrac{1}{2} g_0 - d(1+3ad)\nabla^2_{g_0} d - d^2(1+ad)^2 A_{g_0}\big) \\ &\geq B^2 g_0^{-1}\big(\tfrac{1}{2} - \widehat{C}d - \widehat{C}(1+3a)d^2 - 2\widehat{C}ad^3 - \widehat{C}a^2 d^4\big) g_0 \end{aligned} \tag{4-19}$$

in $M_\delta \setminus \partial M$. Since we will eventually take a large, we may assume $a \geq 1$, in which case (4-19) implies

$$-g_\varphi^{-1} A_{g_\varphi} \geq B^2\big(\tfrac{1}{2} - \widehat{C}[d + 4ad^2 + 2ad^3 + a^2 d^4]\big)\,\mathrm{Id} \quad \text{in } M_\delta \setminus \partial M. \tag{4-20}$$

Now fix $\varepsilon > 0$ small, define $B = 1/\sqrt{1 - 2\varepsilon}$ and denote by $\hat{A}^a_\varepsilon$ the set

$$\hat{A}^a_\varepsilon = \left\{x \in M \setminus \partial M : \varphi(x) = -\ln(B(d + ad^2)) \geq -\ln \frac{\varepsilon}{100\widehat{C}}\right\} = \left\{x \in M \setminus \partial M : d + ad^2 \leq \frac{\varepsilon\sqrt{1 - 2\varepsilon}}{100\widehat{C}}\right\},$$

where $\widehat{C}$ is the constant in (4-20). It is easily verified that, in $\hat{A}^a_\varepsilon$, we have $\widehat{C}(d + 4ad^2 + 2ad^3 + a^2 d^4) \leq \varepsilon$. Moreover, if we define

$$\Sigma^a_\varepsilon = \partial \hat{A}^a_\varepsilon \setminus \partial M,$$

then Σ^a_ε converges to ∂M as a increases. It follows from these two facts and (4-20) that, for a sufficiently large (depending only on (M, g_0)),

$$-g_\varphi^{-1} A_{g_\varphi} \geq B^2 \operatorname{diag}\big(\tfrac{1}{2} - \varepsilon, \ldots, \tfrac{1}{2} - \varepsilon\big) = \operatorname{diag}\big(\tfrac{1}{2}, \ldots, \tfrac{1}{2}\big) \quad \text{in } \hat{A}^a_\varepsilon.$$

It then follows from our normalisation $f\left(\frac{1}{2},\dots,\frac{1}{2}\right)=1$ that

$$f^\tau(-g_\varphi^{-1}A_{g_\varphi})\geq 1 \quad \text{in } \hat{A}^a_\varepsilon. \tag{4-21}$$

We now let

$$C^a_\varepsilon=\min_{\Sigma^a_\varepsilon} v.$$

Since $v(x)\to+\infty$ as $\mathrm{d}(x,\partial M)\to 0$ (by Step 1) and since Σ^a_ε converges to ∂M as a increases, we can choose a large enough so that $C^a_\varepsilon\geq -\ln(\varepsilon/(100\widehat{C}))$. Moreover, this choice of a depends only on g_0, ε, $\widehat{C}$, f and Γ: since each $v_{(j)}$ was constructed according to the procedure in the proof of Proposition 4.1, we know from the second statement in Proposition 4.1 that there exists $\delta=\delta(g_0,\varepsilon,\widehat{C},f,\Gamma)>0$ such that $v_{(j)}\geq -\ln(\varepsilon/(100\widehat{C}))$ in $(M_{(j)})_\delta\setminus\partial M_{(j)}$ for each j. Taking $j\to\infty$, we see $v\geq -\ln(\varepsilon/(100\widehat{C}))$ in $M_\delta\setminus\partial M$. Therefore, to ensure $C^a_\varepsilon\geq -\ln(\varepsilon/(100\widehat{C}))$, one only needs to pick a large depending on $\delta=\delta(g_0,\varepsilon,\widehat{C},f,\Gamma)$.

We now fix such a value of a and suppose J is sufficiently large so that $\Sigma^a_\varepsilon\subset M_{(j)}$ for all $j\geq J$. Then $\varphi=-\ln(\varepsilon/(100\widehat{C}))$ and $v\geq C^a_\varepsilon$ on Σ^a_ε, and, by the monotonicity in (4-14), we also have $v_{(j)}\geq C^a_\varepsilon$ on Σ^a_ε for each $j\geq J$. Therefore,

$$v_{(j)}\geq -\ln\frac{\varepsilon}{100\widehat{C}}=\varphi \quad \text{on } \Sigma^a_\varepsilon \tag{4-22}$$

and

$$v_{(j)}=\infty>\varphi \quad \text{on } \partial M_{(j)}. \tag{4-23}$$

In light of (4-21)–(4-23), the comparison principle in Proposition 3.7 then yields

$$v_{(j)}\geq\varphi \quad \text{in } \hat{A}^a_\varepsilon\cap M_{(j)}.$$

Sending $j\to\infty$, it follows that $v\geq\varphi$ in $\hat{A}^a_\varepsilon$, i.e.,

$$v\geq\varphi=-\ln(B(d+ad^2))=\ln\sqrt{1-2\varepsilon}-\ln d-\ln(1+ad) \quad \text{in } \hat{A}^a_\varepsilon.$$

This is precisely (4-12) after relabelling constants, and thus the second step is complete.

To complete the proof of the proposition, we observe that (4-12) implies

$$\liminf_{\mathrm{d}(x,\partial M)\to 0}(v(x)+\ln\mathrm{d}(x,\partial M))\geq\ln\sqrt{1-2\varepsilon},$$

and, since $\varepsilon>0$ is arbitrary, it follows that

$$\liminf_{\mathrm{d}(x,\partial M)\to 0}(v(x)+\ln\mathrm{d}(x,\partial M))\geq 0. \tag{4-24}$$

By (4-24) and Proposition 4.3, we therefore see that v satisfies (4-13). □

4.3. ***Uniqueness.*** Having just established the existence of a smooth solution to (1-10) satisfying (4-4) when $\tau<1$ and $\mu^+_\Gamma>1$, we now turn to uniqueness of solutions. We start with the following.

Proposition 4.5. *Fix $\tau<1$, and suppose that (f,Γ) satisfies (1-1)–(1-4), (1-7) and (1-8). Then any continuous viscosity solution $g_u=e^{2u}g_0$ to (1-10) satisfies (4-4).*

Proof. Let u be a continuous viscosity solution to (1-10). By Proposition 4.3, we know that u satisfies $\limsup_{\mathrm{d}(x,\partial M)\to 0}(u(x)+\ln \mathrm{d}(x,\partial M)) \leq 0$, so it remains to show

$$\liminf_{\mathrm{d}(x,\partial M)\to 0}(u(x)+\ln \mathrm{d}(x,\partial M)) \geq 0. \tag{4-25}$$

To prove (4-25), we attach a collar neighbourhood N to ∂M, extend g_0 smoothly to $M\cup N$ and consider the sequence $\{M^{(j)}\}_j$ of smooth compact manifolds with boundary given by

$$M^{(j)} = \{x \in M\cup N : \mathrm{d}(x, M) \leq j^{-1}\}.$$

Note that for $x\in M$ and j sufficiently large, $\mathrm{d}(x,\partial M^{(j)}) = \mathrm{d}(x,\partial M)+j^{-1}$. Fix $\varepsilon>0$. By Proposition 4.4, there exist constants $\delta>0$ and $a>0$ depending on g_0, ε, f, Γ but independent of j, and a smooth metric $g_{u^{(j)}} = e^{2u^{(j)}}g_0$ for each j such that

$$f^\tau(-g_{u^{(j)}}^{-1}A_{g_{u^{(j)}}}) = 1, \quad \lambda(-g_{u^{(j)}}A_{g_{u^{(j)}}}) \in \Gamma^\tau \quad \text{on } M^{(j)}\setminus\partial M^{(j)}$$

and

$$u^{(j)}(x)+\ln \mathrm{d}(x,\partial M^{(j)}) \geq \ln\sqrt{1-2\varepsilon} - \ln(1+a\mathrm{d}(x,\partial M^{(j)})) \quad \text{in } (M^{(j)})_\delta\setminus\partial M^{(j)}.$$

In particular, for j sufficiently large so that $(M^{(j)})_\delta \cap M \neq \varnothing$, we have

$$u^{(j)}(x)+\ln\left(\mathrm{d}(x,\partial M)+\frac{1}{j}\right) \geq \ln\sqrt{1-2\varepsilon} - \ln\left(1+a\mathrm{d}(x,\partial M)+\frac{a}{j}\right) \quad \text{in } M_{\delta-1/j}. \tag{4-26}$$

Now, by the comparison principle in Proposition 3.7, $u^{(j)}|_M \leq u$ for each j, and thus (4-26) implies

$$u(x)+\ln\left(\mathrm{d}(x,\partial M)+\frac{1}{j}\right) \geq \ln\sqrt{1-2\varepsilon} - \ln\left(1+a\mathrm{d}(x,\partial M)+\frac{a}{j}\right) \quad \text{in } M_{\delta-1/j}\setminus\partial M. \tag{4-27}$$

After taking $j\to\infty$ in (4-27), it follows that

$$\liminf_{\mathrm{d}(x,\partial M)\to 0}(u(x)+\ln \mathrm{d}(x,\partial M)) \geq \ln\sqrt{1-2\varepsilon},$$

and, since $\varepsilon>0$ is arbitrary, we obtain (4-25). ☐

Finally we prove uniqueness of solutions to (1-10) when $\tau<1$.

Proposition 4.6. *Fix $\tau<1$, suppose that (f,Γ) satisfies (1-1)–(1-4), (1-7) and (1-8), and let v denote the smooth solution to (1-10) obtained in Proposition 4.4. Then v is the unique continuous viscosity solution to (1-10).*

Proof. Suppose that w is a continuous viscosity solution to (1-10). By Proposition 4.5, both v and w satisfy (4-4). For $\delta\geq 0$, define $\Sigma_\delta = \{d=\delta\}$. Then, for each $\varepsilon>0$, there exists a minimal $\delta_\varepsilon>0$ such that $w\leq v+\varepsilon$ on $\Sigma_{\delta_\varepsilon}$. Writing $v_\varepsilon = v+\varepsilon$, we have

$$f^\tau(-g_0^{-1}A_{g_{v_\varepsilon}}) = f^\tau(-g_0^{-1}A_{g_v}) = e^{2v} < e^{2v_\varepsilon},$$

and thus v_ε is a supersolution of the equation in (1-10). By the comparison principle in Proposition 3.7, it follows that $w\leq v+\varepsilon$ on $M\setminus M_{\delta_\varepsilon}$. By minimality of δ_ε, we have $\delta_\varepsilon\to 0$ as $\varepsilon\to 0$, and thus $w\leq v$ on $M\setminus\partial M$. Reversing the roles of w and v, we see that $w\geq v$ on $M\setminus\partial M$, and therefore $w=v$. ☐

4.4. ***Proof of Theorem 1.1′.*** The existence of a smooth solution to (1-10) for each $\tau < 1$, the asymptotic behaviour stated in (1-9) and uniqueness in the class of continuous viscosity solutions follow from Propositions 4.4 and 4.6. Let us denote these solutions by u^τ. As observed previously, these solutions u^τ satisfy a locally uniform C^1 estimate which is independent of τ; i.e., for each compact set $K \subset M \setminus \partial M$, there exists a constant C independent of τ but dependent on g_0, f, Γ and K such that

$$\|u^\tau\|_{C^1(K)} \leq C.$$

It follows that a subsequence of $\{u^\tau\}$ converges locally uniformly in $C^{0,\alpha}$ to some $u \in C^{0,1}_{\mathrm{loc}}(M, g_0)$ for each $\alpha \in (0,1)$. As noted in the proof of Theorem 1.6 in Section 3.5, the fact that u is a viscosity solution to (1-10) when $\tau = 1$ follows from exactly the same argument as in the proof of [Li and Nguyen 2021, Theorem 1.4]. It remains to show that u satisfies the asymptotics in (1-9) and is maximal.

To this end, first note that, since we only require u to be a viscosity subsolution in Proposition 4.3,

$$\limsup_{\mathrm{d}(x,\partial M)\to 0} (u(x) + \ln \mathrm{d}(x,\partial M)) \leq 0. \tag{4-28}$$

To show that

$$\liminf_{\mathrm{d}(x,\partial M)\to 0} (u(x) + \ln \mathrm{d}(x,\partial M)) \geq 0, \tag{4-29}$$

we first recall that u is the $C^{0,\alpha}$ limit of the solutions u^τ as $\tau \to 1$. By Proposition 4.4, for each $\varepsilon > 0$ sufficiently small, there exist constants $\delta > 0$ and $a > 0$ independent of τ (but dependent on g_0, ε, f and Γ) such that

$$u^\tau(x) + \ln \mathrm{d}(x,\partial M) \geq \ln\sqrt{1-2\varepsilon} - \ln(1 + a\mathrm{d}(x,\partial M)) \quad \text{in } M_\delta \setminus \partial M. \tag{4-30}$$

Taking $\tau \to 1$ in (4-30), we obtain

$$u(x) + \ln \mathrm{d}(x,\partial M) \geq \ln\sqrt{1-2\varepsilon} - \ln(1 + a\mathrm{d}(x,\partial M)) \quad \text{in } M_\delta \setminus \partial M,$$

and (4-29) then follows exactly as in the proof of Proposition 4.4.

Finally, to see that u is maximal, suppose that $\tilde{u}$ is another continuous viscosity solution to (1-10). By Proposition 4.3, (4-28) holds with $\tilde{u}$ in place of u, and we also know that (1-9) is satisfied with u^τ in place of u for each $\tau \leq 1$. Combining these facts, it follows that, for each $\tau \leq 1$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\tilde{u} \leq u^\tau_\varepsilon := u^\tau + \varepsilon \quad \text{in } M_\delta \setminus \partial M.$$

On the other hand, $f^\tau(-g^{-1}_{u^\tau_\varepsilon} A_{g_{u^\tau_\varepsilon}}) = e^{-2\varepsilon} f^\tau(-g^{-1}_{u^\tau} A_{g_{u^\tau}}) < 1$ on $M \setminus \partial M$ and $f^\tau(-g^{-1}_{\tilde{u}} A_{g_{\tilde{u}}}) \geq 1$ in the viscosity sense on $M \setminus \partial M$; to see this latter inequality, observe

$$\begin{aligned} f^\tau(\lambda) = \frac{1}{\tau + n(1-\tau)} f(\tau\lambda + (1-\tau)\sigma_1(\lambda)e) &\geq \frac{1}{\tau + n(1-\tau)}(\tau f(\lambda) + (1-\tau)\sigma_1(\lambda) f(e)) \\ &\overset{(4\text{-}2)}{\geq} \frac{1}{\tau + n(1-\tau)}\left(\tau f(\lambda) + (1-\tau)\frac{nf(\lambda)}{f(e)} f(e)\right) = f(\lambda). \end{aligned}$$

By the comparison principle in Proposition 3.7, it follows that $\tilde{u} \leq u^\tau_\varepsilon$ in $M \setminus M_\delta$, and therefore $\tilde{u} \leq u^\tau_\varepsilon$ in $M \setminus \partial M$. Taking $\varepsilon \to 0$ and then $\tau \to 1$, it follows that $\tilde{u} \leq u$ in $M \setminus \partial M$, as claimed. □

Appendix A: Proof of Proposition 2.4: a cone property

Proposition 2.4 is essentially a consequence of [Yuan 2022, Theorem 1.4]. We summarise the details here for the convenience of the reader. Let Γ be any cone satisfying (1-1) and (1-2), and define

$$\kappa_\Gamma = \max\{k : (\underbrace{0,\dots,0}_{k},\underbrace{1,\dots,1}_{n-k}) \in \Gamma\}.$$

Assume for now that there exists a constant $\theta = \theta(n,\Gamma) > 0$ such that, whenever $\lambda \in \Gamma$ with $\lambda_1 \geq \cdots \geq \lambda_n$,

$$\frac{\partial f}{\partial \lambda_i}(\lambda) \geq \theta \sum_{j=1}^{n} \frac{\partial f}{\partial \lambda_j}(\lambda) \quad \text{for } i \geq n - \kappa_\Gamma. \tag{A-1}$$

Since $\kappa_\Gamma = 0$ if and only if $\Gamma = \Gamma_n^+$, we see that $\kappa_\Gamma \geq 1$ whenever $\Gamma \neq \Gamma_n^+$, and thus (2-24) holds for $i \in \{n-1, n\}$. Also, it is easy to see that κ_Γ is equal to the maximum number of negative entries a vector in Γ can have; i.e.,

$$\kappa_\Gamma = \max\{k : (-\alpha_1,\dots,-\alpha_k,\alpha_{k+1},\dots,\alpha_n) \in \Gamma,\ \alpha_j > 0 \text{ for all } 1 \leq j \leq n\}.$$

Thus (2-24) also holds if $\lambda_i \leq 0$.

It remains to justify (A-1), for which we follow [Yuan 2022]. By concavity, $f_i(\lambda) \geq f_j(\lambda)$ whenever $\lambda_i \leq \lambda_j$. In particular, our ordering implies

$$\frac{\partial f}{\partial \lambda_n}(\lambda) \geq \frac{1}{n} \sum_{j=1}^{n} \frac{\partial f}{\partial \lambda_j}(\lambda),$$

which establishes (A-1) for $\Gamma = \Gamma_n^+$.

On the other hand, for a general cone Γ satisfying (1-1) and (1-2), we have

$$\sum_{i=1}^{n} f_i(\lambda)\mu_i > 0 \quad \text{whenever } \lambda, \mu \in \Gamma. \tag{A-2}$$

Suppose $\Gamma \neq \Gamma_n^+$, in which case it is clear that $\kappa_\Gamma > 0$, and fix any $\alpha_1,\dots,\alpha_n > 0$ such that

$$(-\alpha_1,\dots,-\alpha_{\kappa_\Gamma},\alpha_{\kappa_\Gamma+1},\dots,\alpha_n) \in \Gamma.$$

Then (A-2) implies

$$\sum_{i=\kappa_\Gamma+1}^{n} \alpha_i f_{n-i+1}(\lambda) - \sum_{i=1}^{\kappa_\Gamma} \alpha_i f_{n-i+1}(\lambda) > 0. \tag{A-3}$$

We may assume $\alpha_1 \geq \cdots \geq \alpha_{\kappa_\Gamma}$, in which case (A-3) implies

$$f_{n-\kappa_\Gamma}(\lambda) > \frac{\alpha_1}{\sum_{i=\kappa_\Gamma+1}^{n} \alpha_i} f_n(\lambda).$$

The desired estimate then follows for all $i \geq n - \kappa_\Gamma$, again by our ordering. □

Appendix B: The Schouten tensor for a radial conformal factor

In this appendix we prove the formula (3-5). In normal coordinates, $r = \sqrt{x_1^2 + \cdots + x_n^2}$, and therefore $\partial_i v(r) = (x_i/r)v_r$. It follows that

$$|\nabla_{g_0} v|_{g_0}^2 = g_0^{ij}\partial_i v \partial_j v = \frac{g_0^{ij} x_i x_j}{r^2} v_r^2 = v_r^2,$$

where we have used the fact that
$$\frac{\partial}{\partial r} = \frac{x_i}{\sqrt{x_1^2+\cdots+x_n^2}}\frac{\partial}{\partial x_i}$$
has unit magnitude. Moreover,
$$(\nabla^2_{g_0} v)_{ij} = \partial_i\partial_j v - \Gamma^k_{ij}\partial_k v = \frac{\delta_{ij}}{r}v_r + \frac{x_i x_j}{r}\left(\frac{v_{rr}}{r} - \frac{v_r}{r^2}\right) - \Gamma^k_{ij}\partial_k v.$$
Combining the above, we therefore see that
$$\begin{aligned}(g_v^{-1}A_{g_v})^p_j &= v^2(g_0^{-1}A_{g_v})^p_j = v^2 g_0^{pi}(A_{g_v})_{ij} \\ &= v^2\left[\frac{g_0^{pi}\delta_{ij}}{vr}v_r + g_0^{pi}\frac{x_i x_j}{vr}\left(\frac{v_{rr}}{r} - \frac{v_r}{r^2}\right) - g_0^{pi}\frac{\Gamma^k_{ij}x_k v_r}{vr} - \frac{v_r^2}{2v^2}\delta^p_j + (g_0^{-1}A_{g_0})^p_j\right].\end{aligned}$$
Now write $g_0^{pi} = \delta^{pi} + \chi^{pi}$, where $\chi = O(r^2)$ as $r \to 0$. Then
$$\begin{aligned}(g_v^{-1}A_{g_v})^p_j &= v^2\left[\frac{\delta^p_j}{vr}v_r + \frac{x^p x_j}{vr}\left(\frac{v_{rr}}{r} - \frac{v_r}{r^2}\right) - \frac{v_r^2}{2v^2}\delta^p_j\right] \\ &\qquad + \underbrace{v^2\left[\frac{\chi^{pi}\delta_{ij}}{vr}v_r + \chi^{pi}\frac{x_i x_j}{vr}\left(\frac{v_{rr}}{r} - \frac{v_r}{r^2}\right) - g_0^{pi}\frac{\Gamma^k_{ij}x_k v_r}{vr} + (g_0^{-1}A_{g_0})^p_j\right]}_{=\Psi^p_j} \\ &= v^2\left(\lambda\delta^p_j + \chi\frac{x^p x_j}{r^2}\right) + \Psi^p_j,\end{aligned}$$
where λ and χ are as in (3-6). Now, since $\chi = O(r^2)$, we have
$$v^2\frac{\chi^{pi}\delta_{ij}}{vr}v_r = O(r)v|v_r|, \qquad v^2\chi^{pi}\frac{x_i x_j}{vr}\left(\frac{v_{rr}}{r} - \frac{v_r}{r^2}\right) = O(r^2)v|v_{rr}| + O(r)v|v_r|,$$
and, since $\Gamma^k_{ij} = O(r)$ and $(g_0^{-1}A_{g_0})^p_j = O(1)$, we also have
$$v^2 g_0^{pi}\frac{\Gamma^k_{ij}x_k v_r}{vr} = O(r)v|v_r| \quad\text{and}\quad v^2(g_0^{-1}A_{g_0})^p_j = O(1)v^2.$$
The claim (3-5) then follows.

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ENTROPY SOLUTIONS TO THE MACROSCOPIC INCOMPRESSIBLE POROUS MEDIA EQUATION

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We investigate maximal potential energy dissipation as a selection criterion for subsolutions (coarse-grained solutions) in the setting of the unstable Muskat problem. We show that both (a) imposing this criterion on the level of convex integration subsolutions and (b) the strategy of Otto based on a relaxation via minimizing movements lead to the same nonlocal conservation law. Our main result shows that this equation admits an entropy solution for unstable initial data with an analytic interface.

1. Introduction

An outstanding open problem in hydrodynamics is the description of unstable interface configurations quickly leading to turbulent regimes. Examples are the thoroughly studied Saffman–Taylor [Saffman and Taylor 1958], Rayleigh–Taylor [Rayleigh 1882; Taylor 1950] and Kelvin–Helmholtz [Thomson 1871] instabilities. In these unstable regimes, Eulerian quantities — such as the velocity field — are very irregular, and the Lagrangian trajectories typically fail to be uniquely defined. Hence uniqueness is not to be expected at the microscopic level, a phenomenon that in the physics literature is known as spontaneous stochasticity [Thalabard et al. 2020], and instead it will be desirable to have a well-defined deterministic evolution at the macroscopic level. The current paper provides such a macroscopic evolution in the context of the incompressible porous medium equation derived from maximal potential energy dissipation.

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Keywords: incompressible porous media, Saffman–Taylor instability, relaxation, subsolutions, nonlocal hyperbolic conservation law, mixing zone, level sets.

1.1. ***IPM and interfaces.*** Throughout the article, we will consider the incompressible porous media (IPM) equation, given by

$$\begin{aligned} \partial_t \rho + \operatorname{div}(\rho v) &= 0, \\ \operatorname{div} v &= 0, \\ v &= -\nabla p - \rho e_2 \end{aligned} \tag{1-1}$$

on the two-dimensional periodic strip $\mathbb{T} \times \mathbb{R}$, where $\mathbb{T}$ denotes the flat 1-torus of length 2π, and over a time interval $[0, T)$, $T > 0$. Here the (normalized) fluid density $\rho : [0, T) \times \mathbb{T} \times \mathbb{R} \to \mathbb{R}$, the velocity $v : [0, T) \times \mathbb{T} \times \mathbb{R} \to \mathbb{R}^2$ and the pressure $p : [0, T) \times \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ are the unknowns, and

$$-e_2 := (0, -1)^T \in \mathbb{R}^2$$

is the direction of gravity.

The model describes the evolution of a two-dimensional density-dependent incompressible fluid in an overdamped scenario (the porous medium) and under the influence of gravity. It consists of the law for mass conservation, the incompressibility condition for the velocity field and Darcy's law (see [Allaire 1989; Darcy 1856; Muskat 1934; Saffman and Taylor 1958; Sánchez-Palencia 1980] for more physical background). Constants such as mobilities (viscosities), permeability of the medium, and gravity have been set to 1. System (1-1) also models the motion of an incompressible and viscous fluid in a Hele-Shaw cell [Saffman and Taylor 1958], a different physical scenario with the same mathematical formulation.

Concerning initial conditions, we are interested in the unstable interface case, i.e.,

$$\rho_0(x) = \begin{cases} +1, & x_2 > \gamma_0(x_1), \\ -1, & x_2 < \gamma_0(x_1), \end{cases} \tag{1-2}$$

for a graph $\gamma_0 : \mathbb{T} \to \mathbb{R}$.

Generally speaking, if the initial data ρ_0 is sufficiently regular it is well known that the IPM equation has a unique regular local-in-time solution; see [Castro et al. 2009; Córdoba et al. 2007]. However, the problem of formation of singularities versus global existence is still open and only partial results are known. For example, the existence of solutions with Sobolev norms unbounded in time has recently been proven in [Kiselev and Yao 2023].

In the case of discontinuous initial data of the type (1-2), the situation is even more subtle as the following dichotomy shows: If the denser fluid is below the lighter one, then the problem is stable and the existence of solutions is well known (see Section 2.1). However, if the lighter fluid is below the heavier one, the problem is ill-posed (at least in the Muskat sense, see Section 2.1, and in the sense of bounded weak solutions, see Section 2.2).

1.2. ***Macroscopic IPM.*** In spite of this difficulty, there have been several attempts to understand the evolution of such an initial configuration at least in the coarse-grained picture. Namely, on the one hand, Felix Otto [1999] discovered that, in the Lagrangian formulation, IPM is a gradient flow, and he suggested in the unstable situation a relaxation based on the corresponding minimizing movements scheme in the Wasserstein setting (JKO scheme). On the other hand, [Córdoba et al. 2011b] showed that

(nonstrict) IPM subsolutions
Definition 2.4 (nonstrict: with "$\leq$" instead of "$<$" in (2-4))
existence: [Arnaiz et al. 2021; Castro et al. 2021; Förster and Székelyhidi 2018; Noisette and Székelyhidi 2021] and boxes below
uniqueness: no

macroscopic IPM solutions
Definition 3.1
existence: boxes below
uniqueness: no

IPM solutions
Definition 2.1
existence: [Siegel et al. 2004] and induced by subsolutions via [Székelyhidi 2012]
uniqueness: no

macroscopic IPM entropy solutions
Definition 3.4
existence: Theorem 3.2
uniqueness: open

Figure 1. Relation of (sub)solutions in the unstable nonflat interface case: note that each IPM solution is indeed also a macroscopic IPM solution due to the fact that IPM solutions satisfy $\rho(t,x)^2 = 1$ for almost every (t,x). Concerning the strictness of the stated inclusions, the listed references [Arnaiz et al. 2021; Castro et al. 2021; Förster and Székelyhidi 2018; Noisette and Székelyhidi 2021] provide subsolutions different from macroscopic IPM solutions. We also believe that the other two inclusions are strict, e.g., by methods similar to the ones used in the present paper, it should be possible to construct a nonflat two-shock solution to macroscopic IPM that is neither an entropy solution nor an IPM solution.

IPM can be recast as a differential inclusion in the Tartar framework and therefore fits the adaptation of convex integration in hydrodynamics by De Lellis and Székelyhidi [2009; 2010]. Subsequently, the full relaxation of the differential inclusion has been computed in [Székelyhidi 2012] leading to a concept of coarse-grained solutions (subsolutions in the convex integration jargon). In Section 2 we present precise definitions and review the historical landmarks of the theory. As an overview the reader can also consult two diagrams: one concerning the various notions of (sub)solutions occurring in the paper and their relations, see Figure 1, and another concerning the steps of the relaxations, see Figure 2.

Let us remark that [Székelyhidi 2012] proved in the case of a flat interface that Otto's relaxation selects a convex integration subsolution, which turns out to be the global-in-time entropy solution to a

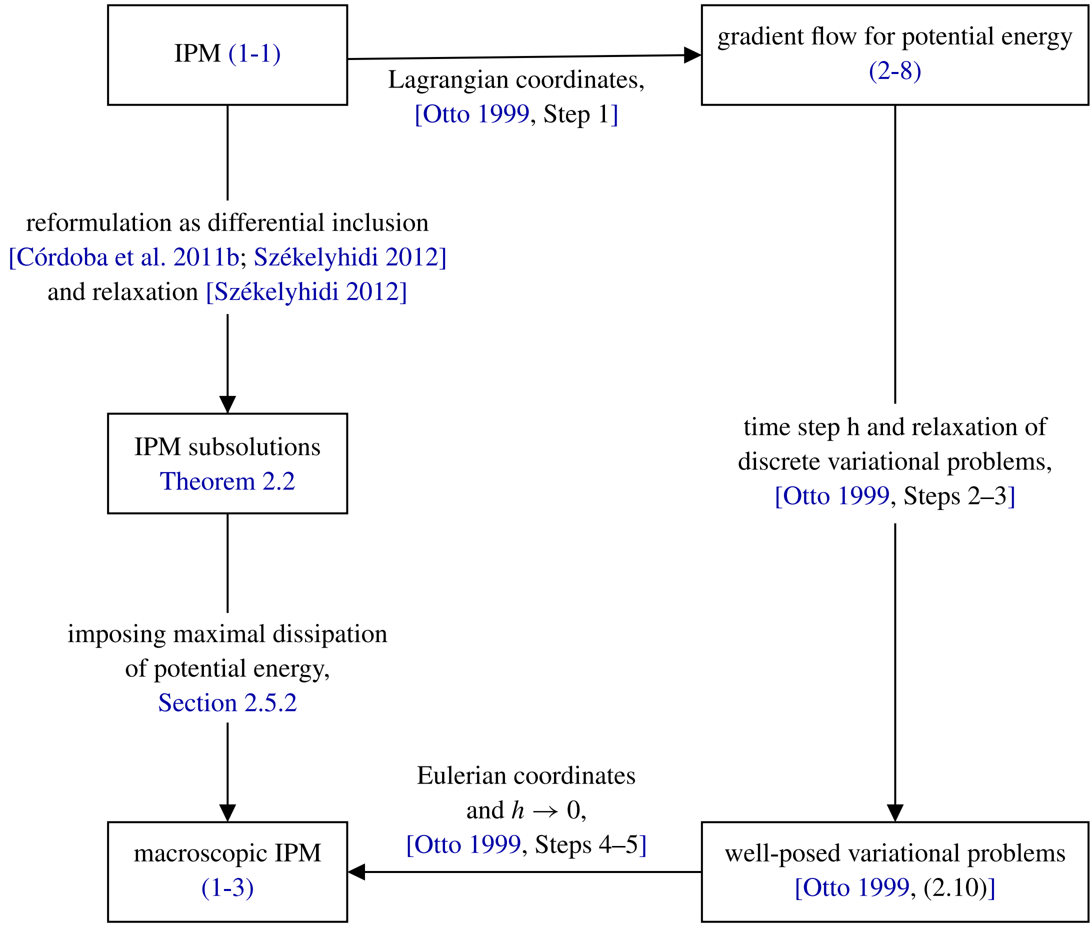

Figure 2. Relaxation of IPM in Eulerian coordinates via subsolutions on the left and in Lagrangian coordinates via minimizing movements on the right.

one-dimensional Burgers equation, reconciling both relaxation theories. In the case of a nonflat interface, the theory of convex integration starting from [Castro et al. 2021] has provided a number of subsolutions [Arnaiz et al. 2021; Castro et al. 2022; Förster and Székelyhidi 2018; Noisette and Székelyhidi 2021]. In all these situations, the starting point is an ansatz for the coarse-grained density $\bar{\rho}$ and for the growth of the mixing zone motivated in analogy to the flat case. These subsolutions show that also on a macroscopic level plenty of different evolutions are possible, such that a selection, which so far has not been available, has to be made for an attempt to claim uniqueness.

The aim of this paper is to use maximal potential energy dissipation as a selection criterion. Since, as discovered by Otto, in Lagrangian coordinates IPM is a gradient flow with respect to potential energy, this seems a natural approach. In any case, we first revisit the strategy proposed by Otto [1999] in the case of nonflat interfaces (the scheme is explained in Section 2.5.1 and Appendix B). We then reconcile it by selecting the subsolution in the convex integration terminology which at each time instant dissipates the most potential energy.

It can be shown that both (a) the relaxed minimizing movements scheme provided in [Otto 1999] (at least formally) and (b) imposing maximal potential energy dissipation among convex integration subsolutions (rigorously) lead to the equation

$$\begin{aligned} \partial_t \rho + \operatorname{div}(\rho v) + \partial_{x_2}(\rho^2) &= 0, \\ \operatorname{div} v &= 0, \\ v &= -\nabla p - \rho e_2, \end{aligned} \tag{1-3}$$

which will be referred to as *macroscopic IPM*. In Section 2 we explain in detail how the Muskat problem, the theory of subsolutions, convex integration for IPM and Otto's relaxation are connected. The derivation of (1-3) from the JKO scheme is known to experts, but as far as we are aware the arguments around maximal potential energy dissipation for subsolutions are new. In particular, it will be explained in which way (1-3) can offer a selection criterion for IPM subsolutions based on a natural extension of the gradient flow structure of IPM.

(Entropy) solutions to macroscopic IPM are subsolutions to IPM as long as they exist. By introducing a parameter $0 < \mu < 1$ in the first equation,

$$\partial_t \rho + \operatorname{div}(\rho v + \mu \rho^2 e_2) = 0,$$

macroscopic IPM produces strict subsolutions. Hence, by a suitable h-principle, see Theorem 2.2, the time of existence of microscopic solutions to IPM will be dictated by the time of existence of (1-3). This is in stark contrast to [Castro et al. 2021], where a rarefaction-like ansatz with a prescribed speed of opening of the mixing zone is made and a resulting time- and space-dependent parameter $\mu(t, x)$ is derived which is smaller than 1 just for short times.

In general we emphasize that, contrary to the procedure of [Castro et al. 2021] (and also of [Arnaiz et al. 2021; Castro et al. 2022; Förster and Székelyhidi 2018; Noisette and Székelyhidi 2021]), i.e., deriving a macroscopic equation from an ansatz, we here follow the reversed process, i.e., we consider based on a selection a fixed equation for the macroscopic evolution and derive properties of its solutions, such as the speed of opening of the mixing zone. We believe that this is a necessity when it comes to potential applications addressing for instance the prediction of a unique mixing zone evolution.

1.3. ***Existence result and idea of proof.*** The bulk of the paper is devoted to proving the existence of an entropy solution for (1-3) with (1-2) as initial data. System (1-3) can be written as a single scalar nonlocal hyperbolic conservation law,

$$\partial_t \rho + \operatorname{div}(\rho T[\rho]) + \partial_{x_2}(\rho^2) = 0, \tag{1-4}$$

where $v = T[\rho]$ is a zeroth-order singular integral operator. Contrary to other nonlocal conservation laws with a more regular nonlocal feedback — see [Amadori and Shen 2012; Amorim 2012; Betancourt et al. 2011; Blandin and Goatin 2016; Colombo et al. 2012] for examples and [Keimer and Pflug 2023] for a recent overview — a general existence and uniqueness theory for nonlocal terms as in (1-4) is not available. We bypass this by using the structure of the two-phase initial data (1-2). This approach, born out of necessity, not only provides us with the existence of a solution but in addition allows us to learn about certain properties of it. More precisely, by showing that the Burgers' term $\partial_{x_2}(\rho^2)$ is able to tear up the initial discontinuity of the density even in the presence of the incompressible velocity v, we will prove the existence of a local-in-time solution which is Lipschitz for $t > 0$. This fact is highly nontrivial and presents many technical difficulties that will be tackled in Sections 4–6, which together form the proof of our main theorem and will be described below. A careful statement of our main theorem itself, containing further properties of the solution, can be found in Section 3. We have preferred to state the existence theorem for (1-3) after the reader is hopefully convinced by Section 2 that (1-3) renders a macroscopic description for the unstable Muskat problem consistent with maximal potential energy dissipation.

One main ingredient of our proof is to look at the evolution of level sets of the density ρ in suitably scaled coordinates and to adjust properly to leading-order terms of this evolution. These steps, carried out in Section 4, reduce the initial value problem (1-2), (1-3) to a fixed-point problem of the type

$$\eta(t, y) = \frac{1}{t^{1+\alpha}} \int_0^t \int_{-2}^2 \int_{\mathbb{T}} (K_s[\eta(s, \cdot)])(y, z)(h_s[\partial_{y_1}\eta(s, \cdot)])(y, z)\, dz_1\, dz_2\, ds - \frac{1}{t^\alpha} h_0(y) \tag{1-5}$$

for functions $\eta : [0, T) \times \mathbb{T} \times (-2, 2) \to \mathbb{R}$ describing the evolution of the level sets in superlinear order with respect to $t > 0$ small. The constants ± 2 for the domain of η are coming from the rarefaction speed of Burgers' equation. Moreover, here $h_0(y)$ is one of the mentioned leading-order terms — in fact the first-order term — depending on the initial graph γ_0 and $\alpha \in (0, 1)$. Moreover, for each $s > 0$, $y \in \mathbb{T} \times (-2, 2)$ and $\xi : \mathbb{T} \times (-2, 2) \to \mathbb{R}$ fixed, the function $z \mapsto (K_s[\xi])(y, z)$ is a convolution kernel of order -1 induced by the Biot–Savart law. The dependence on ξ involves both $\xi(z)$ and $\xi(y)$ in the form of the difference $\xi(y) - \xi(z)$. Similarly, the function $(y, z) \mapsto h_s[\partial_{y_1}\xi](y, z)$, again considered for a fixed s and ξ, depends on the difference $\partial_{y_1}\xi(y) - \partial_{y_1}\xi(z)$. Thus, after integration in z, the regularity of the right-hand side of (1-5) with respect to y is the regularity of $\partial_{y_1}\eta$, i.e., the right-hand side when seen as an operator looses one derivative in y_1.

In addition, as one of the main difficulties — also for potential equivalent reformulations of (1-5) where the above loss of a derivative might be avoided — we would like to point out that the kernels $K_s[\xi]$ degenerate as $s \to 0$ to a one-dimensional kernel with singularity $\sim 1/(y_1 - z_1)$, i.e., to an integral kernel of order 0. Thus, estimates for $K_s[\xi](y, \cdot)$ as a kernel of order -1 cannot be obtained uniformly in s.

Regardless, in order to keep the paper enjoyable, we deal with (1-5) and its loss of derivative by considering real analytic initial interfaces. This allows us to use an adaptation of the Nirenberg–Nishida abstract Cauchy–Kovalevskaya theorem. Still, the application of it — even when we continue to ignore the so far not mentioned factor $t^{-(1+\alpha)}$ on the right-hand side — takes quite a lot of effort. It is the second main part of our proof and can be found in Section 5.

Finally, Section 6 puts everything together to give a solution to the macroscopic IPM equation. In Appendices A, B and C, we give a proof of a version of the abstract Cauchy–Kovalevskaya theorem needed for our situation, and we give some more details regarding the derivation of the macroscopic IPM equation.

1.4. ***On the entropy condition.*** We emphasize that the solution we find is an entropy solution of (1-3), or rather (1-4). The notion of an entropy solution is stated in Definition 3.4. This is consistent with the flat case $\gamma_0 = 0$ where, as said earlier, the relaxed minimizing movements scheme of Otto [1999] converges to the entropy solution of (1-4) which in that case reduces to Burgers' equation. For an extended discussion concerning the selection of the entropy solution by the minimizing movements scheme (including other gradient flows as counterexamples where a corresponding selection fails), we refer to [Gigli and Otto 2013], where the IPM relaxation is revisited in the flat setting of Otto's original work [1999]. In addition see also [Otto 2001] for a stability result in the flat case. Concerning the general, nonflat case, it was also conjectured by Otto (personal communication) that the convergence of the minimizing movements scheme to an entropy solution remains true.

Moreover, we point out that some sort of choice among solutions of (1-3) is critical in order to have a selection criterion. Indeed, already in the flat case solutions are clearly not unique, and also in the general case nonentropic solutions for (1-3) can be obtained in an easier way, for instance via (2-1) below; see Remark 3.3. We believe that the requirement of being an entropy solution leads to uniqueness for the initial value problem (1-2), (1-3), but, since the velocity v depends on ρ in a comparably singular nonlocal way, standard methods do not seem to work and uniqueness of entropy solutions to macroscopic IPM stands as an interesting open question. In any case, we emphasize that for the scheme we present there is a unique solution, and therefore our maximal dissipating subsolution is amenable to numerical calculations.

1.5. ***Further questions.*** Besides the question of uniqueness of the found entropy solution, our work opens the door to many other questions with various levels of difficulty, such as improving the regularity of the solutions, considering initial interfaces (not being analytical or not being a graph, as for example in [Castro et al. 2022]) or other densities as initial data as well. It would be interesting to see whether the JKO scheme does converge rigorously or what happens in the case of different mobilities [Mengual 2022; Otto 1999]. On a more general level, there might be other selection criteria for IPM, for example, based on surface tension [Jacobs et al. 2021] or on vanishing diffusion [Menon and Otto 2005; 2006]. Finally, we emphasize that our selection criterion ultimately is tailored to the gradient flow structure of IPM, and for other equations the reasoning necessarily must be different. In any case, we hope our work encourages the research on finding a deterministic coarse-grained evolution in the presence of instabilities.

2. Ill-posedness and relaxation

The unstable interface initial value problem considered here is highly ill-posed. In this section we explain in which sense this ill-posedness holds, as well as a strategy based on convex integration and the relaxation of [Otto 1999] to overcome it. This section, having the purpose to fully motivate equation (1-3), is mostly a review of existing results. Except for the derivation in Section 2.5.2 showing that maximal potential energy dissipating subsolutions coincide with Otto's relaxation, we do not claim any novelty. However, we are not aware that the computations in Section 2.5.1 can be found in the literature. A reader only interested in solving system (1-3) can go directly to Section 3.

2.1. ***The Muskat problem.*** If one assumes that

$$\rho(x,t)=\begin{cases}\rho_{\mathrm{up}}, & x_2>f(x_1,t),\\ \rho_{\mathrm{down}}, & x_2<f(x_1,t),\end{cases}$$

a closed equation from (1-1) can be obtained for the interface $(x_1, f(x_1,t))$. Indeed,

$$\partial_t f(x,t)=\frac{\rho_{\mathrm{down}}-\rho_{\mathrm{up}}}{4\pi}\int_{\mathbb{T}}\frac{\sin(y)(f_x(x,t)-f_x(x-y,t))}{\cosh(f(x,t)-f(x-y,t))-\cos(y)}\,dy. \tag{2-1}$$

This equation is usually known in the literature as the Muskat equation honoring M. Muskat [1934].

In the case $\rho_{\mathrm{down}}>\rho_{\mathrm{up}}$, the problem is stable and local existence and regularity of solutions can be proven in different functional settings and situations [Abels and Matioc 2022; Agrawal et al. 2023; Alazard and Lazar 2020; Alazard and Nguyen 2021b; 2021a; 2023; 2022; Cameron 2019; Chen et al. 2022; Cheng et al. 2016; Choi et al. 2007; Córdoba and Gancedo 2007; Córdoba et al. 2011a; 2013; 2014; Deng et al. 2017; Escher and Matioc 2011; García-Juárez et al. 2022; 2024; Matioc 2019; Nguyen and Pausader 2020; Shi 2023], as well as global for small and medium size initial data [Alonso-Orán and Granero-Belinchón 2022; Constantin et al. 2013; 2016; 2017; Córdoba and Lazar 2021; Dong et al. 2023; Gancedo and Lazar 2022; Granero-Belinchón and Lazar 2020]. The existence of singularities for large initial data is shown in [Castro et al. 2012a; 2013] and also in [Córdoba et al. 2015; 2017].

However, if $\rho_{\mathrm{down}}<\rho_{\mathrm{up}}$, the Muskat equation is ill-posed [Córdoba and Gancedo 2007; Siegel et al. 2004]. Surprisingly, convex integration has allowed us to construct solutions to IPM starting in these kinds of unstable situations. They have been called mixing solutions and, in them, the initial interface between the two different densities disappears and a strip arises in which the two densities mix. We elaborate on these mixing solutions in the next sections. For a general picture of convex integration in the context of fluid dynamics, we refer to the surveys [Buckmaster and Vicol 2021; De Lellis and Székelyhidi 2019; 2022].

2.2. ***IPM as differential inclusion.*** The first examples of nonuniqueness of weak solutions for (1-1) using convex integration were given in [Córdoba et al. 2011b] by Córdoba, Gancedo and the second author for the initial value $\rho_0=0$. Their method bypasses the computation of the relaxation by means of so-called T_4 configurations. After this, Székelyhidi [2012] established the explicit relaxation of (1-1) for initial data of two-phase type, enabling a systematic investigation of interface problems in IPM. While the results in [Córdoba et al. 2011b; Székelyhidi 2012] established ill-posedness of IPM in the class of essentially bounded solutions, Isett and Vicol [2015] could also show the existence of compactly supported $C^{\alpha}_{t,x}$-solutions for $\alpha<\frac{1}{9}$. The starting point of our investigation is the relaxation of [Székelyhidi 2012], which we will describe in this subsection.

In the following we consider initial data with $|\rho_0|=1$ almost everywhere. The corresponding notion of weak solutions is fixed in Definition 2.1 below. Note that, for such initial data, the last condition in the definition is an additional consistency requirement coming from the continuity, or rather transport, equation in (1-1).

Definition 2.1. A pair $\rho \in L^\infty((0,T)\times\mathbb{T}\times\mathbb{R})$, $v \in L^\infty(0,T; L^2(\mathbb{T}\times\mathbb{R};\mathbb{R}^2))$ is a solution of (1-1), (1-2) provided, for any $\varphi \in \mathcal{C}_c^\infty([0,T)\times\mathbb{T}\times\mathbb{R})$, we have

$$\int_0^T\int_{\mathbb{T}\times\mathbb{R}} \rho\partial_t\varphi + \rho v\cdot\nabla\varphi\,dx\,dt + \int_{\mathbb{T}\times\mathbb{R}} \rho_0\varphi(0,\cdot)\,dx = 0,$$
$$\int_0^T\int_{\mathbb{T}\times\mathbb{R}} v\cdot\nabla\varphi\,dx\,dt = 0,$$
$$\int_0^T\int_{\mathbb{T}\times\mathbb{R}} (v+\rho e_2)\cdot\nabla^\perp\varphi\,dx\,dt = 0,$$

and $|\rho(t,x)| = 1$ for almost every $(t,x)\in(0,T)\times\mathbb{T}\times\mathbb{R}$.

A key step in [Córdoba et al. 2011b; Székelyhidi 2012] is to recast weak solutions as defined above as solutions to a differential inclusion, to be able to use the Murat–Tartar compensated compactness formalism [Tartar 1979].

A pair (ρ, v) is a weak solution if and only if the triple

$$(\rho, v, m) \in L^\infty((0,T)\times\mathbb{T}\times\mathbb{R}) \times (L^\infty(0,T; L^2(\mathbb{T}\times\mathbb{R})))^2$$

satisfies the linear system

$$\begin{aligned} \partial_t\rho + \operatorname{div} m &= 0,\\ \operatorname{div} v &= 0,\\ v &= -\nabla p - \rho e_2,\\ \rho(0,\cdot) &= \rho_0 \end{aligned} \tag{2-2}$$

distributionally, i.e., in analogy to Definition 2.1, together with

$$(\rho(t,x), v(t,x), m(t,x)) \in K := \{(\rho, v, m)\in\mathbb{R}^5 : |\rho| = 1,\ m = \rho v\} \tag{2-3}$$

for almost every $(t,x)\in(0,T)\times\mathbb{T}\times\mathbb{R}$.

Then the relaxation of the incompressible porous media equation is understood as the relaxation of the corresponding differential inclusion; i.e., in the pointwise nonlinear constraint (2-3), the set K is replaced by its convex (or more generally Λ-convex) hull. Up to technicalities, one can recover highly oscillatory solutions from this set, as the main theorem of [Székelyhidi 2012] shows.

Theorem 2.2 [Székelyhidi 2012]. *Let $\bar\rho \in L^\infty((0,T)\times\mathbb{T}\times\mathbb{R})$ and $\bar v, \bar m \in L^\infty(0,T; L^2(\mathbb{T}\times\mathbb{R}))$ satisfy (2-2) in the sense of distributions. Suppose that there exists a bounded and open set $\mathscr{U}\subset(0,T)\times\mathbb{T}\times\mathbb{R}$ such that (2-3) holds for almost every $(t,x)\notin\mathscr{U}$, while $(\bar\rho,\bar v,\bar m)$ are continuous on $\mathscr{U}$ with*

$$(\bar\rho(t,x), \bar v(t,x), \bar m(t,x)) \in \{(\rho, v, m)\in\mathbb{R}^5 : |\rho| < 1,\ |2(m-\rho v) + (1-\rho^2)e_2| < (1-\rho^2)\} \tag{2-4}$$

for every $(t,x)\in\mathscr{U}$. Then there exist infinitely many weak solutions (ρ, v) of (1-1), (1-2) that coincide with $(\bar\rho,\bar v)$ outside of $\mathscr{U}$ and are arbitrarily close to $(\bar\rho,\bar v)$ in the weak $L^2(\mathscr{U})$-topology.

In the case of the IPM system, the set on the right-hand side of (2-4) is indeed only the interior of the Λ-convex hull of K, see [Székelyhidi 2012] for a precise definition, which does not coincide with the full convex hull as opposed to the Euler equations. Still, (2-4) describes all possible weak limits of solutions to the IPM system, see [Székelyhidi 2012]. In view of that, one can therefore truly speak about the full relaxation of IPM in the context of two-phase mixtures.

This fact has been quantified in [Castro et al. 2019], where the relation between solutions and subsolutions has been made precise through an adapted h-principle. In particular, this leads to additional properties of the solutions like a degraded macroscopic behavior or the turbulent mixing at every time-slice property. The latter means that the solutions (ρ, v) induced by $(\bar{\rho}, \bar{v}, \bar{m})$ satisfy $\rho \in \mathcal{C}^0([0, T); L^2_{\mathrm{weak}}(\mathbb{T} \times (-R, R)))$, where R is some positive number with $\mathscr{U} \subset (0, T) \times \mathbb{T} \times (-R, R)$, and

$$\left(\int_B (1 - \rho(t, x))\, dx\right)\left(\int_B (1 + \rho(t, x))\, dx\right) > 0 \tag{2-5}$$

for any $t \in (0, T)$ and any ball B fully contained in $\mathscr{U}_t := \{x \in \mathbb{T} \times \mathbb{R} : (t, x) \in \mathscr{U}\}$.

For later purposes, we also point out the following possible upgrade of Theorem 2.2, which is obtained by using convex integration as in [Castro et al. 2019; De Lellis and Székelyhidi 2010].

Lemma 2.3. *Let $(\bar{\rho}, \bar{v}, \bar{m})$ be as in Theorem 2.2 and $\delta : [0, T) \to \mathbb{R}$ continuous with $\delta(0) = 0$, $\delta(t) > 0$, $t > 0$. Then there exist infinitely many solutions (ρ, v) as in Theorem 2.2 with the additional property that*

$$\left|\int_{\mathbb{T}\times\mathbb{R}} (\bar{\rho}(t, x) - \rho(t, x)) x_2\, dx\right| \leq \delta(t)$$

for almost every $t \in [0, T)$.

Definition 2.4. Any triple $(\bar{\rho}, \bar{v}, \bar{m})$ satisfying the conditions of Theorem 2.2 is called a subsolution of (1-1), (1-2). The set $\mathscr{U}$, in other papers frequently also denoted by Ω_{mix}, is called the mixing zone of the subsolution.

Theorem 2.2 shifts the focus from a single solution to the investigation of subsolutions which are understood as possible coarse-grained or averaged solutions. As subsolutions play the central role also in the present investigation, we will frequently omit the bars in notation and instead mark solutions by $(\rho_{\mathrm{sol}}, v_{\mathrm{sol}})$ in case there is a chance of confusion.

2.3. ***Examples of subsolutions.*** The first examples of nonconstant subsolutions have been given in the same paper of Székelyhidi [2012] for the perfectly flat initial interface, $\rho_0(x) = \mathrm{sign}(x_2)$. Keeping the one-dimensional structure of the initial data, one sees that $v = 0$, $m = -\alpha(1 - \rho^2)e_2$, $\alpha \in (0, 1)$ reduces (2-2), (2-3) to the one-dimensional conservation law

$$\partial_t \rho + \alpha \partial_{x_2}(\rho^2) = 0,$$

which has a unique entropy solution given by

$$\rho(t, x) = \begin{cases} 1, & x_2 > 2\alpha t, \\ x_2/(2\alpha t), & -2\alpha t < x_2 < 2\alpha t, \\ -1, & x_2 < -2\alpha t. \end{cases}$$

It also has been mentioned in [Székelyhidi 2012] that the limiting case $\alpha = 1$ is in agreement with the relaxation of Otto [1999]. It coincides with (1-3) in the flat situation, see Section 4.1. In addition, this case gives an upper bound for the mixing zone. More precisely, it has been shown in [Székelyhidi 2012] that the mixing zone at time $t > 0$, $\mathscr{U}_t$, of any one-dimensional subsolution emanating from $\rho_0(x) = \mathrm{sign}(x_2)$ is contained in the strip $[-1, 1] \times (-2t, 2t)$. A similar subsolution in the harder case of different viscosities was studied in [Mengual 2022]. Actually, the Λ-hull of IPM with different viscosities and densities is computed in that paper.

In the context of IPM and differential inclusions, we would also like to mention [Hitruhin and Lindberg 2021] which addresses the stationary, i.e., time-independent, IPM system. In that paper the lamination convex hull of that system is computed, and in addition a rigidity result for its subsolutions and an application for long-term limits of (1-1) is given.

The first examples of subsolutions giving rise to mixing solutions, i.e., solutions obtained from the subsolution via convex integration with property (2-5), for IPM starting in a nonflat interface $(x_1, f_0(x_1))$ were provided in [Castro et al. 2021]. In this paper the density ρ of the subsolution is Lipschitz and the prescribed speed of opening of the mixing zone $c(x_1)$ ($= 2\alpha$ in the flat case above) satisfies $1 \le c < 2$ and, as indicated, might depend on x_1. The result of [Castro et al. 2021] holds for initial data $f_0 \in H^5(\mathbb{R})$, i.e., in a regime where the Muskat problem cannot be solved. A numerical analysis of these subsolutions can be found in [Castro 2017], where the formation of fingers can be observed. In [Arnaiz et al. 2021], the semiclassical viewpoint developed in [Castro et al. 2021] is taken one step further (using semiclassical Sobolev spaces for example), providing an alternative proof to the main result of [Castro et al. 2021]. Indeed this later approach improves the subsolutions with respect to their regularity, as the boundary of the mixing zone is in $H^{5-1/c(x_1)}$, where $c(x_1)$ is the local speed of opening of the mixing zone, instead of merely in H^4.

Förster and Székelyhidi [2018] constructed mixing solutions with an initial interface $f_0 \in \mathcal{C}^{3+\alpha}$ relaxing the initial regularity needed in [Castro et al. 2021] but relying on subsolutions with piecewise constant density instead of Lipschitz. In this case the speed of opening of the mixing zone is $0 < c < 2$ with c uniform in x_1. Thereafter the same kind of subsolutions have been constructed in [Noisette and Székelyhidi 2021] with variable speed of opening.

As mentioned before, mixing solutions obtained via convex integration are not unique. There are two reasons for this fact: (a) different subsolutions can be found, (b) infinitely many solutions, corresponding to different distributions of the density, emanate from every fixed subsolution. In order to deal with point (b), in [Castro et al. 2019] it has been shown that all the solutions obtained from a fixed subsolution can be chosen in such a way that they share averages over large sets, i.e., they are the same as the subsolution at a macroscopic level. One of the main points of the present paper is to deal with point (a). A particular instance of this multiplicity will be illustrated in Section 2.4 below.

The constructions of the subsolutions above seem to rely on the Saffman–Taylor instability (heavy fluid on top of a lighter fluid). In [Castro et al. 2021] it was observed that there also exist mixing solutions in the stable regime (see also [Förster and Székelyhidi 2018]) which build on Kelvin–Helmholtz-type instabilities (discontinuity of the velocity field, see [Mengual 2022] for a thorough discussion of this phenomena at the level of the hulls). Actually, the analysis in [Castro et al. 2021] indicates that the mixing can be created around any point of the interface which is not both flat (with zero slope) and stable. We call points having zero slope in the stable regime fully stable points. It happens that, in an initially overhanging interface, there must be always a fully stable point. Partially unstable situations therefore require one to find compatibility between the Muskat solution and mixing solutions, see [Castro et al. 2022]. Remarkably, the construction in that paper allows one to answer the question on how to prolongate in time the singular solutions to the Muskat problem found in [Castro et al. 2012a; 2013], namely as mixing solutions.

As a last remark we would like to point out that the subsolutions constructed in [Castro et al. 2021; 2022; Förster and Székelyhidi 2018; Noisette and Székelyhidi 2021] are local in time in the sense that, although the involved functions exist over a potentially larger time interval, a small time interval has to be chosen in order to guarantee that they take values inside the convex hull, i.e., that (2-4) holds. This is in contrast to the flat cases [Mengual 2022; Székelyhidi 2012] and to the subsolution constructed in the present paper. Although here we will only prove a local-in-time existence result, the involved functions take values in (the closure of) the convex hull as long as they exist.

2.4. *The subsolution selection problem.* As described, the constructions from the previous subsection contain ansatzes for certain properties of the subsolution and hence for the induced mixing solutions of (1-1). To illustrate this freedom in the simplest case, let us discuss the flat interface with $\gamma_0(x_1)=0$ in slightly more detail. As in [Székelyhidi 2012], setting $v\equiv 0,\ m=m_2(t,x_2)e_2,\ \rho=\rho(t,x_2)$, one sees that $(\rho,0,m)$ is a subsolution if and only if

$$\partial_t\rho+\partial_{x_2}m_2=0,\qquad \rho(0,x)=\operatorname{sign}(x_2),\qquad |\rho|\le 1,$$
$$|2m_2+1-\rho^2|<1-\rho^2\ \text{ when } |\rho|<1,\qquad m_2=0\ \text{ when } |\rho|=1,$$

and the required continuity conditions hold. Thus one could make the ansatz

$$m_2=-\frac{1-\rho^2}{2}+\frac{1-\rho^2}{2}\xi_2 \tag{2-6}$$

with $\xi_2:[0,T)\times\mathbb{R}\to\mathbb{R}$ satisfying $|\xi_2|<1$ and for any such ξ_2 solve the conservation law

$$\partial_t\rho+\partial_{x_2}\left((\xi_2(t,x_2)-1)\frac{1-\rho^2}{2}\right)=0$$

with initial data $\rho_0(x_2)=\operatorname{sign}(x_2)$ to get plenty of subsolutions with different mixing zones and density profiles. Note that in this sense ξ_2, or rather the whole relation (2-6), plays the role of a constitutive law.

Summarizing once more, these examples show that not only does each subsolution induce infinitely many solutions of the incompressible porous media equation sharing a common coarse-grained, or averaged, behavior, but there are also infinitely many possibilities for this averaged evolution via the vast amount of possible subsolutions. This is a common problem in the construction of turbulent solutions emanating from unstable interface initial data, as for instance also for the Kelvin–Helmholtz instability

[Gebhard and Kolumbán 2022a; Mengual and Székelyhidi 2023; Székelyhidi 2011] and the Rayleigh-Taylor instability in the context of the Euler equations [Gebhard and Kolumbán 2022b; Gebhard et al. 2021; 2024]. We emphasize that, however, our criteria builds on the gradient flow structure of IPM, and therefore different ideas should be used in the case of the Euler equations, see Section 2.5.3 for a short overview of strategies used so far.

2.5. ***A selection criterion.*** We now focus in the general, not necessarily flat, case on the selection of subsolutions in terms of choosing an appropriate relation between m, ρ and v such that (2-4) holds provided $|\rho| \leq 1$.

First we will review the strategy proposed by F. Otto [1999] to relax system (1-1) based on its gradient flow structure in Lagrangian coordinates, and we will formally obtain (1-3) from this relaxation. The strategy of Otto does not rely on the notion of a subsolution in the context of differential inclusions as in Section 2. However, the solution of (1-3) will be a (nonstrict) subsolution with

$$m = \rho v - (1-\rho^2)e_2.$$

Thereafter, we will also give an argument to derive (1-3) in Eulerian coordinates directly based on subsolutions. Also, here the starting point will be the gradient flow structure of (1-1). This second argument shows that the relaxation of Otto selects among all subsolutions precisely those that maximize the dissipation of potential energy at every time instant.

The relations are summarized in Figure 2 on page 2244.

2.5.1. *Otto's relaxation.* In this section we give a very brief summary of Otto's five-step strategy leading to the macroscopic IPM equation (1-3). The discussion is not rigorous and even then we have put most of the explicit calculations in Appendix B. We adapt our notation to that of [Otto 1999], which, due to a different normalization, studies the evolution of

$$s(x,t) = \frac{1-\rho\big(x, \tfrac{1}{2}t\big)}{2}$$

instead of $\rho(x,t)$, i.e., contrary to other sections the density s is now taking values in $[0,1]$. In these coordinates the IPM system (1-1) reads

$$\begin{aligned} \partial_t s + u\cdot\nabla s &= 0,\\ \operatorname{div} u &= 0, \\ u &= -\nabla\Pi + s e_2; \end{aligned} \tag{2-7}$$

see Appendix B.

The starting point (Step 1) of Otto's relaxation is the vital fact that, when formulated in Lagrangian coordinates, IPM can be seen as a gradient flow with respect to the potential energy

$$E[\Phi] = -\int s(x,0)\Phi(x)\cdot e_2$$

on the manifold

$$M_0 = \{\Phi \text{ one-to-one and onto, smooth, volume-preserving maps}\}.$$

More precisely, if (s, u, Π) is a solution of (2-7), then the flow $\Phi(x,t)$ induced by u satisfies

$$\int \partial_t \Phi(\,\cdot\,,t)\cdot w = -dE[\Phi(\,\cdot\,,t)]w \quad \text{for all } w \in T_{\Phi(\,\cdot\,,t)}M_0, \tag{2-8}$$

where $dE[\Phi]w$ is the Fréchet derivative of the functional E at the point $\Phi \in M_0$ in the direction

$$w \in T_\Phi M_0 = \{w \text{ smooth and such that } \nabla\cdot(w\circ\Phi^{-1}) = 0\}.$$

Fast-forwarding a bit, the next steps of Otto consist of the introduction of a time discretization with step size $h > 0$ in the form of a minimizing movements scheme (Step 2), the extension of the underlying manifold M_0 to its L^2-closure in order to turn the potentially ill-posed discrete variational problems emanating from Step 2 to well-posed ones (Step 3), and a translation of the now existing sequence of minimizers back to Eulerian coordinates (Step 4). At this point there exists a sequence of functions $\theta^{(k)}$ corresponding to $s(\,\cdot\,,t)$ at time $t = kh$, but of course potentially on a coarse-grained or "locally averaged" level, which is characterized by the following JKO scheme: $\theta^{(0)} = s(\,\cdot\,,0)$ and, given $\theta^{(k)}$, $\theta^{(k+1)}$ is the minimizer in K of

$$\tfrac{1}{2}\operatorname{dist}^2(\theta^{(k)},\theta) + \tfrac{1}{2}\operatorname{dist}^2(1-\theta^{(k)},1-\theta) - h\int \theta(x)x_2, \tag{2-9}$$

where the set K consists of measurable θ taking values in $[0,1]$ and such that $\int\theta = \int s(x,0)$, and $\operatorname{dist}^2(\theta_0,\theta_1)$ for $\theta_0,\theta_1 \in K$ is the L^2-Wasserstein distance

$$\operatorname{dist}^2(\theta_0,\theta_1) = \inf_{\Phi\in I(\theta_0,\theta_1)} \int \theta_0(x)|\Phi(x)-x|^2\,dx$$

with

$$I(\theta_0,\theta_1) = \left\{\Phi : \int \theta_1(y)\zeta(y)\,dy = \int \theta_0(x)\zeta(\Phi(x))\,dx \ \ \forall \zeta \in \mathcal{C}_0^0\right\}.$$

Notice that this indeed is a relaxation of the original problem since the densities are no longer taking values in $\{0,1\}$ and the transport maps are not necessarily injective.

The fifth and last step consists of passing to the limit $h \to 0$ whenever this is possible. Otto [1999] proved that this is the case for the unstable flat situation

$$s(x,0) = \begin{cases} 0, & x_2 > 0, \\ 1, & x_2 < 0, \end{cases}$$

and that the limit of θ_h defined by

$$\theta_h(x,t) := \theta^{(k)}(x), \qquad t \in [kh,(k+1)h)$$

is the unique entropy solution of the conservation law

$$\partial_t\theta + \partial_{x_2}(\theta(1-\theta)) = 0.$$

For a different proof of this statement we refer to the work of Gigli and Otto [2013], which in particular also contains a further examination of the relation between the minimizing movements scheme and the entropy condition.

In fact, it was conjectured by Otto (personal communication) that the described scheme, if it converges, should also lead to an entropy solution of the macroscopic IPM equation in the general, nonflat case. We refer to Section 3 for the definition of entropy solutions.

In the rest of this section we sketch how at least formally system (1-3), or rather its equivalent reformulation in terms of $s(x,t)$, arises from the JKO-characterization (2-9) of the discrete functions $\theta^{(k)}$ when *assuming* suitable convergence. Our presentation here, as well as in Appendix B which contains some more details, is devoted to conveying that the scheme indeed leads to the macroscopic IPM equation rather than to providing a rigorous proof which we defer to future work. A similar computation was derived by Otto (personal communication).

Fix t and write for simplicity $\theta^0 := \theta_h(t)$, $\theta^1 := \theta_h(t+h)$. Furthermore, let Φ^h denote the transport map corresponding to $\mathrm{dist}^2(\theta^0,\theta^1)$ and $\bar{\Phi}^h$ the transport map corresponding to $\mathrm{dist}^2(1-\theta^0, 1-\theta^1)$. Then it can be shown that there are functions a^h, $\bar{a}^h$ such that

$$\Phi^h(x) = x + (\nabla a^h \circ \Phi^h)(x),$$
$$\bar{\Phi}^h(x) = x + (\nabla \bar{a}^h \circ \bar{\Phi}^h)(x).$$

This in fact is a consequence of Brenier's theorem [1991]; still an argument is also provided in Appendix B.

Moreover, it can be deduced from first variations of the functional (2-9) that

$$a^h - \bar{a}^h = hx_2. \tag{2-10}$$

Now, we write $a^h = hp^h$, $\bar{a}^h = h\bar{p}^h$ and make the strong assumption that the introduced functions p^h, $\bar{p}^h$ have a well defined $\mathcal{C}^2$ limit denoted by p, $\bar{p}$. Moreover, we also assume that $\theta_h(t,x)$ is converging in a strong enough sense and denote the limit function by $\theta(t,x)$.

If this is the case we can pass to the limit $h \to 0$ and obtain, see Appendix B,

$$\partial_t \theta = -\mathrm{div}(\theta \nabla p), \tag{2-11}$$
$$\partial_t \theta = \Delta \bar{p} - \mathrm{div}(\theta \nabla \bar{p}). \tag{2-12}$$

Now (2-10) yields $p = \bar{p} + x_2$. Thus (2-11), (2-12) imply that

$$\Delta \bar{p} = \mathrm{div}((\nabla \bar{p} - \nabla p)\theta) = -\partial_{x_2}\theta. \tag{2-13}$$

Therefore, from (2-12) and (2-13), we deduce

$$\partial_t \theta = -\partial_{x_2}\theta - \mathrm{div}(\nabla \bar{p}\theta) = -\partial_{x_2}\theta - \mathrm{div}((\nabla \bar{p} + \theta e_2)\theta) + \mathrm{div}(\theta^2 e_2).$$

To finish we define $u = \nabla \bar{p} + \theta e_2$, which clearly satisfies $\mathrm{div}\, u = 0$, to get

$$\partial_t \theta + u \cdot \nabla \theta + \partial_{x_2}\theta - 2\theta \partial_{x_2}\theta = 0,$$
$$u = \nabla \bar{p} + \theta e_2,$$
$$\mathrm{div}\, u = 0.$$

Undoing the change of coordinates from the beginning, i.e., considering

$$\rho(t,x) = 1 - 2s(x, 2t),$$

one obtains (1-3). As said, more details can be found in Appendix B.

2.5.2. *Transfer to subsolutions.* Now we give an alternative derivation of the macroscopic system (1-3), taking a different route after Step 1 of Otto's relaxation; i.e., the starting point is again the gradient flow structure of IPM saying that solutions of (1-1) seek to maximize the dissipation of potential energy at every time instance. However, at this point we do not care in which precise sense the dissipation is maximized (in Lagrangian coordinates with respect to the L^2-metric on the manifold of area preserving diffeomorphisms). We instead simply extend the principle of maximal energy dissipation for solutions of (1-1) to its relaxation given in Theorem 2.2; i.e., we seek to investigate also subsolutions that decrease the potential energy at every time instant as much as possible.

Suppose that (ρ, v, m) is a subsolution in the sense of Definition 2.4. We define its associated relative potential energy

$$E_{\mathrm{rel}}(t) := \int_{\mathbb{T}\times\mathbb{R}} (\rho(t,x) - \rho_0(x))x_2\,dx \tag{2-14}$$

and, for now formally, compute

$$\partial_t E_{\mathrm{rel}}(t) = -\int_{\mathbb{T}\times\mathbb{R}} x_2 \operatorname{div} m(t,x)\,dx = \int_{\mathbb{T}\times\mathbb{R}} m_2(t,x)\,dx. \tag{2-15}$$

Moreover, similar to (2-6), condition (2-4) implies

$$m = \rho v - \frac{1-\rho^2}{2}e_2 + \frac{1-\rho^2}{2}\xi$$

almost everywhere for some $\xi : [0,T)\times\mathbb{T}\times\mathbb{R} \to \mathbb{R}^2$ satisfying $|\xi| < 1$. Plugging this into (2-15), one deduces

$$\partial_t E_{\mathrm{rel}}(t) = \int_{\mathbb{T}\times\mathbb{R}} \rho v_2 - (1-\rho^2)\frac{1-\xi_2}{2}\,dx.$$

Hence considering $\rho(t,\cdot)$, and therefore also $v(t,\cdot)$, see Section 4.2 below, to be given, one easily sees that the energy dissipation at time t is maximized in the closure of all admissible ξ with the choice $\xi(t,x) = -e_2$.

Hence choosing constantly $\xi = -e_2$, and therefore

$$m = \rho v - (1-\rho^2)e_2, \tag{2-16}$$

we deduce that (nonstrict) subsolutions that maximize at each time instant the dissipation of potential energy are characterized as solutions of

$$\begin{aligned} \partial_t\rho + \operatorname{div}(\rho v - (1-\rho^2)e_2) &= 0, \\ \operatorname{div} v &= 0, \\ v &= -\nabla p - \rho e_2. \end{aligned} \tag{2-17}$$

The above formal computation in (2-15) can be made rigorous under mild decay assumptions, as for instance shown in Appendix C. Here, however, we would like to state some further remarks.

First of all we emphasize that, by choosing m as in (2-16), we do not obtain a subsolution in the sense of Definition 2.4, since (2-4) holds only in a nonstrict sense; thus we speak about a nonstrict subsolution.

By considering instead

$$m = \rho v - \mu(1-\rho^2)e_2, \tag{2-18}$$

i.e., $\xi = (1-2\mu)e_2$ with μ arbitrarily close to 1 but $\mu < 1$, one obtains strict subsolutions and hence actual mixing solutions via Theorem 2.2, arbitrarily close to the nonstrict ones with maximal energy dissipation. However, in the remainder of the paper we will solve (2-17) as the outstanding case and remark that a similar analysis leads to a subsolution corresponding to the system with m given by (2-18); see also Remark 3.3.

Moreover, we would like to point out that, in the flat case, where $v = 0$, system (2-17) is exactly the hyperbolic conservation law found in [Székelyhidi 2012], whose entropy solution corresponds to the maximum speed of expansion of the mixing zone; see Section 2.3.

Furthermore, we remark that, given a strict subsolution (ρ, v, m) with relative potential energy $E_{\mathrm{rel}}(t)$ defined in (2-14), one obtains infinitely many mixing solutions $(\rho_{\mathrm{sol}}, v_{\mathrm{sol}})$ as in Theorem 2.2 with the additional property that their relative potential energy at almost every time t is arbitrarily close to $E_{\mathrm{rel}}(t)$; see Lemma 2.3. In this sense there also exist actual mixing solutions with potential energy decay arbitrarily close to the maximal decay for subsolutions characterized by (2-17).

2.5.3. *Comparison to selection criteria in related problems.* As mentioned in Section 2.4, the selection of a meaningful subsolution is a general problem when studying hydrodynamic instabilities via differential inclusions. We briefly give an overview of previously applied selection criteria.

In the case of a perfectly flat interface, the selection typically is done by reducing the subsolution system to a one-dimensional hyperbolic conservation law and picking the unique entropy solution as a natural candidate. This has been done in the context of the Kelvin–Helmholtz instability for the Euler equations [Székelyhidi 2011], the Rayleigh–Taylor instability for the inhomogeneous Euler equations [Gebhard et al. 2021], and as discussed in all detail above for the flat unstable Muskat problem in IPM [Székelyhidi 2012].

Another approach, selecting the subsolution that at initial time maximizes the total energy dissipation, has been applied in the context of the nonflat Kelvin–Helmholtz instability [Mengual and Székelyhidi 2023] within the class of all subsolutions with vorticity concentrated on a finite number of sheets, and thereafter in the class of one-dimensional self-similar subsolutions emanating from the flat Rayleigh–Taylor instability modeled by the Euler equations in Boussinesq approximation [Gebhard and Kolumbán 2022b]. This strategy has been motivated by the entropy rate admissibility criterion of Dafermos [1973], which has also been investigated in [Chiodaroli and Kreml 2014; Feireisl 2014] for convex integration solutions of the compressible Euler equations. In view of Section 2.5.2, also the selection criterion considered in the present paper falls into that category. However, in contrast to [Gebhard and Kolumbán 2022b; Mengual and Székelyhidi 2023], the selection applies among all possible subsolutions (with certain natural decay at infinity) and not only within a special subclass, and it applies at all times instead of only the initial time.

Another way to select subsolutions globally in time has been studied in [Gebhard et al. 2024] in the context of the flat Rayleigh–Taylor instability for the Euler equations in Boussinesq approximation.

Similar to Section 2.5.2 above, the underlying geometric principle of the equation, in that case the least action principle, has been imposed on the level of subsolutions leading to a degenerate elliptic variational problem that turns out to be formally equivalent to the direct relaxation of the least action principle by Brenier [1989]. However, solutions obtained from this relaxation conserve the total energy, which is inconsistent with anomalous energy dissipation present in turbulent regimes. In view of that, in [Gebhard et al. 2024] an additional term, responsible for energy dissipation but subject to certain choices, has been added in the variational problem. In contrast, the relaxation of IPM considered here is not relying on any comparable choices.

3. The main result

According to the previous section, we consider on $\mathbb{T}\times\mathbb{R}$ the system

$$\begin{aligned}\partial_t\rho+\operatorname{div}(\rho v+\rho^2e_2)&=0,\\ \operatorname{div} v&=0,\\ v&=-\nabla p-\rho e_2\end{aligned}\tag{3-1}$$

with initial data (1-2), i.e.,

$$\rho_0(x)=\begin{cases}+1, & x_2>\gamma_0(x_1),\\ -1, & x_2<\gamma_0(x_1),\end{cases}$$

for a sufficiently regular function $\gamma_0:\mathbb{T}\to\mathbb{R}$. In fact we here consider the case of a real analytic initial interface. For completeness we also state the notion of a general weak solution to system (3-1).

Definition 3.1. A pair $\rho\in L^\infty((0,T)\times\mathbb{T}\times\mathbb{R})$, $v\in L^\infty(0,T;L^2(\mathbb{T}\times\mathbb{R};\mathbb{R}^2))$ is a solution of (3-1), (1-2) provided, for any $\varphi\in\mathcal{C}_c^\infty([0,T)\times\mathbb{T}\times\mathbb{R})$, we have

$$\begin{aligned}\int_0^T\int_{\mathbb{T}\times\mathbb{R}}\rho\partial_t\varphi+(\rho v+\rho^2e_2)\cdot\nabla\varphi\,dx\,dt+\int_{\mathbb{T}\times\mathbb{R}}\rho_0\varphi(0,\cdot\,)\,dx&=0,\\ \int_0^T\int_{\mathbb{T}\times\mathbb{R}}v\cdot\nabla\varphi\,dx\,dt&=0,\\ \int_0^T\int_{\mathbb{T}\times\mathbb{R}}(v+\rho e_2)\cdot\nabla^\perp\varphi\,dx\,dt&=0.\end{aligned}$$

Theorem 3.2. *Let $\gamma_0:\mathbb{T}\to\mathbb{R}$ be real analytic. Then the initial value problem (3-1), (1-2) has a local-in-time solution with the following properties*:

(i) *ρ and v are continuous on $[0,T)\times\mathbb{T}\times\mathbb{R}\setminus\{(0,x_1,\gamma_0(x_1)):x_1\in\mathbb{T}\}$.*

(ii) *$\rho(t,\cdot\,)$ is Lipschitz continuous at positive times and $v(t,\cdot\,)$ is log-Lipschitz continuous, with*

$$\begin{aligned}\|\nabla\rho(t,\cdot\,)\|_{L^\infty(\mathbb{T}\times\mathbb{R})}&\le C_0t^{-1},\\ |v(t,x)-v(t,x')|&\le C_0t^{-1}|(x-x')\log|x-x'||\end{aligned}\tag{3-2}$$

for $t\in(0,T)$, $x,x'\in\mathbb{T}\times\mathbb{R}$, $|x-x'|\le\frac12$ and a constant $C_0>0$ depending on γ_0.

(iii) *For $t \in (0,T)$, there exist two real analytic curves $\gamma_t(\,\cdot\,,\pm 1) : \mathbb{T} \to \mathbb{R}$ such that $\rho(t,x) = 1$ whenever $x_2 \geq \gamma_t(x_1, 1)$ and $\rho(t,x) = -1$ whenever $x_2 \leq \gamma_t(x_1,-1)$. Moreover, $\rho(t,\cdot)$ maps the remaining set into $(-1,1)$. Also there, the level sets $\Gamma_t(h) := \{x \in \mathbb{T}\times\mathbb{R} : \rho(t,x) = h\}$, $h \in (-1,1)$, are given by graphs of real analytic functions $\gamma_t(\,\cdot\,,h) : \mathbb{T} \to \mathbb{R}$. Furthermore, the joint map $[0,T)\times\mathbb{T}\times[-1,1] \to \mathbb{R}$, $(t,x_1,h) \mapsto \gamma_t(x_1,h)$ belongs to the space $\mathcal{C}^1([0,T);\mathcal{C}^1(\mathbb{T}\times[-1,1]))$, and there exists a real analytic function $s_0 : \mathbb{T} \to \mathbb{R}$ such that*

$$\gamma_t(x_1,h) = \gamma_0(x_1) + t(2h + s_0(x_1)) + o(t) \tag{3-3}$$

with respect to $\|\cdot\|_{\mathcal{C}^1(\mathbb{T}\times[-1,1])}$ as $t \to 0$.

(iv) *For any locally Lipschitz continuous $\eta : \mathbb{R} \to \mathbb{R}$, we have the balance*

$$\partial_t(\eta(\rho)) + \operatorname{div}(\eta(\rho)v + Q(\rho)e_2) = 0, \tag{3-4}$$

with initial data $\eta(\rho)(0,\cdot) = \eta(\rho_0)$ and flux $Q(\rho) := \int_0^\rho 2\eta'(s)s\,ds$.

Remark 3.3. (a) In fact the function $s_0 : \mathbb{T} \to \mathbb{R}$ appearing in (3-3) is precisely the normal part of the initial velocity when evaluated in $(x_1, \gamma_0(x_1))$. See Section 4.2, in particular equation (4-9), for the definition and further discussion.

(b) Note that (iii) implies that ρ is piecewise $\mathcal{C}^1$ with the exceptional set given by

$$\{(t, x_1, \gamma_t(x_1,\pm 1) : t \in [0,T),\, x_1 \in \mathbb{T}\}.$$

(c) Equation (3-4) is a priori understood in analogy to Definition 3.1, i.e., in a distributional sense. However, given the regularity of ρ and v, it in fact holds pointwise almost everywhere on $(0,T)\times\mathbb{T}\times\mathbb{R}$; see Section 6.

(d) Since convex functions are locally Lipschitz, the balance (3-4) in particular states that ρ is an entropy solution for the conservation law $\partial_t\rho + \operatorname{div}(\rho v +^2 e_2) = 0$, see Definition 3.4 below.

(e) We notice that, for an analytic initial interface, the Muskat equation (2-1) can be solved for short time in order to find a solution to the macroscopic IPM system (3-1), which at the same time is also a solution for IPM (see [Castro et al. 2012a] and in the case of the vortex-sheet problem [Castro et al. 2012b]). However, this solution is not an entropy solution. Moreover, piecewise constant solutions of (3-1) also could be constructed but again they would not be entropy solutions.

(f) As discussed earlier in Section 2.5.2, the solution (ρ, v) given by Theorem 3.2 induces only a nonstrict subsolution by setting $m := \rho v - (1-\rho^2)e_2$. However, an analogous existence statement remains true when replacing the first equation of (3-1) by

$$\partial_t\rho + \operatorname{div}(\rho v + \mu\rho^2 e_2) = 0$$

corresponding to a choice of m as in (2-18) and thus to strict subsolutions when $\mu < 1$. This can be seen for instance by rescaling time and considering the nonlocal velocity field $\mu^{-1}v$ in Sections 4 and 5.

(g) Notice that (iii) describes precisely the mixing zone $\mathscr{U}$ of the subsolution, see Definition 2.4, where the corresponding solutions develop a mixing behavior. In particular, from (3-3) one can deduce the initial growth of the mixing zone, which is linear in time. When combined with [Castro et al. 2019], it also implies the observed degraded mixing property of solutions (the closer to the upper boundary, the bigger the volume fraction of the heavier fluid). In particular, by letting μ tend to 1, our method predicts a unique mixing zone selected by maximal potential energy dissipation which can be compared with experiments, as opposed to subsolutions where the mixing zone depends on an a priori ansatz.

(h) The time of existence $T > 0$ of the found solution depends on how well γ_0 can be extended holomorphically onto a complex strip, see, e.g., Lemma 5.6. In addition T is capped by 1. While the latter is an artificial bound making our proof of existence at some points slightly less technical, the former dependence is naturally appearing in proofs relying on Cauchy–Kovalevskaya theorems. The question regarding a global-in-time solution, may it be as a general entropy solution or as a solution of the level set formulation introduced in Section 4, is open.

(i) The choice of the periodic infinite strip $\mathbb{T} \times \mathbb{R}$ as our spatial domain seemed to us to be the least technical choice. Compared to the whole plane $\mathbb{R}^2$, one does not need to speak about decay/flatness at $x_1 \to \pm\infty$, still we believe that our approach can be adapted to that setting. The same is true for the bounded periodic domain $\mathbb{T} \times (0, 1)$, where the necessary estimates for the Biot–Savart kernel, see Lemma 5.5, have to be derived on a more abstract level. However, the situation in a bounded domain with vertical boundaries is more delicate and not within the scope of this paper.

For completeness we include in the following the notion of an entropy solution for equation (3-1). Note that (3-1) is a nonlocal hyperbolic conservation law. As is common for such equations, see, e.g., [Amadori and Shen 2012; Amorim 2012; Betancourt et al. 2011; Blandin and Goatin 2016; Colombo et al. 2012], the notion of an entropy solution is the one for the corresponding local conservation law where the otherwise nonlocal velocity field is considered as a fixed local one.

Definition 3.4 (entropy solution). A solution (ρ, v) in the sense of Definition 3.1 is called an entropy solution provided, for any $\varphi \in \mathcal{C}_c^\infty([0, T) \times \mathbb{T} \times \mathbb{R})$, $\varphi \geq 0$ and any convex $\eta : \mathbb{R} \to \mathbb{R}$ with induced flux $Q(\rho) := \int_0^\rho 2\eta'(s)s\, ds$, we have

$$\int_0^T \int_{\mathbb{T}\times\mathbb{R}} \eta(\rho)\partial_t\varphi + (\eta(\rho)v + Q(\rho)e_2) \cdot \nabla\varphi\, dx\, dt + \int_{\mathbb{T}\times\mathbb{R}} \eta(\rho_0)\varphi(0, \cdot\,)\, dx \geq 0.$$

We remark that typically the set of η for which the stated imbalance is required to hold is taken to be a strict subset of all convex functions, such as for instance the family $\{r \mapsto |r - c| : c \in \mathbb{R}\}$ of Kružkov [1970]; see also [Dafermos 2016]. Since our solution already satisfies the stronger property (iv), we refrain at this point from restricting the set of entropies.

In any case, due to the nature of the nonlocality of our velocity field — which is a zeroth-order singular integral operator with respect to the density ρ (see Section 4.2) — the uniqueness of the found entropy solution remains open.

4. Level set formulation

We begin our investigation with a look at the illustrative example of a perfectly flat initial interface $\gamma_0(x_1) = 0$ (Section 4.1) and some known facts concerning the nonlocal velocity field v — in particular at initial time — in the nonflat case (Section 4.2). Thereafter, with the beginning of Section 4.3, we will reformulate problem (3-1), (1-2) as a suitable fixed-point problem.

4.1. ***The flat interface.*** In the perfectly flat case, $\gamma_0 = 0$, an x_1-independent solution of (3-1) is obtained by observing that $v = 0$ and solving the Riemann problem for Burgers' equation

$$\partial_t \rho + \partial_{x_2}(\rho^2) = 0, \qquad \rho(0, x_2) = \operatorname{sign}(x_2).$$

The unique entropy solution is Lipschitz continuous at positive times and explicitly given by

$$\rho(t, x) = \begin{cases} 1, & x_2 > 2t, \\ x_2/(2t), & |x_2| \le 2t, \\ -1, & x_2 < -2t. \end{cases}$$

As discussed earlier, see Section 2.3, this solution bounds the mixing zone in the class of all one-dimensional IPM subsolutions.

However, in rescaled coordinates $y \mapsto x$, $x = (y_1, t y_2)$, the solution is given by the stationary profile

$$\rho(t, y_1, t y_2) = \phi_0(y) := \begin{cases} 1, & y_2 > 2, \\ \frac{1}{2} y_2, & |y_2| \le 2, \\ -1, & y_2 < -2, \end{cases} \tag{4-1}$$

or in other words the level sets $\rho(t, \cdot)^{-1}(\{h\})$, $h \in (-1, 1)$, are given by flat lines $\{x : x_2 = 2ht\}$ that as time evolves are pulled apart with speed $2h$.

Of course these are simple reformulations, but a key point in our analysis is an appropriate extension of this principle to the general, nonflat case where the velocity field does not vanish. This will be done by keeping the profile $\phi_0(y)$ on the right-hand side of (4-1) and allowing the transformation $y \mapsto x$ to be of the type $x = (y_1, t y_2 + f(t, y))$, i.e., we keep the "pulling"-term $t y_2$ dealing with the Burgers' term $\partial_{x_2}(\rho^2)$ in the equation and allow the level sets to have a general form reacting to the nonlocal velocity field. The details in terms of induced equations for f are in Sections 4.3–4.5.

4.2. ***Biot–Savart and the initial velocity field.*** The flat case discussed in the previous subsection is a very special case in the sense that $v = 0$ and the resulting equation is local. In the general case a key feature of both systems, IPM and the relaxation, is the nonlocal relation between the density ρ and the velocity field v. More precisely, the last two equations in (1-1), (3-1), respectively, i.e., the incompressibility condition and Darcy's law, can be understood by means of a zeroth-order convolution operator. Indeed, taking the curl of Darcy's law, one sees that, at each time, $v(t, \cdot)$ is an incompressible vector field with vorticity given by

$$\partial_{x_1} v_2(t, x) - \partial_{x_2} v_1(t, x) = -\partial_{x_1} \rho(t, x). \tag{4-2}$$

Thus, when requiring decay as $|x_2| \to \infty$, the velocity field v is, at least in the case of our interest, uniquely determined in terms of the Biot–Savart operator

$$v(t,x) = (K * (-\partial_{x_1}\rho(t,\cdot)))(x) = \int_{\mathbb{T}\times\mathbb{R}} K(x-z)(-\partial_{x_1}\rho(t,z))\,dz. \tag{4-3}$$

On $\mathbb{T}\times\mathbb{R}$ the kernel K is given by

$$K(z) := \frac{1}{4\pi}\frac{(-\sinh(z_2),\sin(z_1))^T}{\cosh(z_2)-\cos(z_1)}, \tag{4-4}$$

and, as usual, K is the orthogonal gradient of the corresponding Green's function

$$G(z) := \frac{1}{4\pi}\log(\cosh(z_2)-\cos(z_1)). \tag{4-5}$$

Relation (4-3) has to be interpreted accordingly at initial time $t=0$ due to the fact that $-\partial_{x_1}\rho_0$ is only a measure supported on the interface

$$\Gamma_0 := \{(x_1,\gamma_0(x_1)) : x_1 \in \mathbb{T}\}.$$

Thus, the initial velocity field $v_0(x)$ is the one of a vortex-sheet and therefore discontinuous across the interface.

Lemma 4.1. *The unique square integrable solution of*

$$v = -\nabla p - \rho_0 e_2, \quad \operatorname{div} v = 0 \quad \text{on } \mathbb{T}\times\mathbb{R} \tag{4-6}$$

is given by

$$v_0(x) = \int_{\mathbb{T}} K\begin{pmatrix} x_1 - z_1 \\ x_2 - \gamma_0(z_1)\end{pmatrix} 2\gamma_0'(z_1)\,dz_1 \tag{4-7}$$

for $x \notin \Gamma_0$, while the one-sided limits at Γ_0 are given by

$$\lim_{\substack{\pm(y_2-\gamma_0(y_1))>0\\ y\to(x_1,\gamma_0(x_1))}} v_0(y) = \text{p.v.}\int_{\mathbb{T}} K\begin{pmatrix} x_1 - z_1 \\ \gamma_0(x_1) - \gamma_0(z_1)\end{pmatrix} 2\gamma_0'(z_1)\,dz_1 \mp \frac{\gamma_0'(x_1)}{1+\gamma_0'(x_1)^2}\begin{pmatrix} 1 \\ \gamma_0'(x_1)\end{pmatrix}. \tag{4-8}$$

Proof. First of all one can check that the right-hand side of (4-7) defines a locally integrable solution of (4-6) with exponential decay as $|x_2| \to \infty$. Thus standard elliptic estimates imply that this is the only solution with these properties.

In order to compute the one-sided limits, we write

$$K(z) = \frac{1}{2\pi}\frac{z^\perp}{|z|^2}\eta(z_1) + K_{\text{reg}}(z),$$

where $\eta : \mathbb{T}\times\mathbb{R}$ is a smooth periodic cutoff function with $\eta(z_1) = 1$ for $|z_1| \le 1$ and $\eta(z_1) = 0$ for $|z_1| \ge 2$, and the regular part $K_{\text{reg}} : \mathbb{T}\times\mathbb{R} \to \mathbb{R}^2$,

$$K_{\text{reg}}(z) := K(z) - \frac{1}{2\pi}\frac{z^\perp}{|z|^2}\eta(z_1),$$

is smooth. In fact K_{reg} is harmonic where $\eta(z_1) = 1$. Furthermore, using complex notation, we write $z^\perp/|z|^2 = (1/(iz))^*$, where z^* denotes complex conjugation.

Then, denoting by $v_{0,\mathrm{reg}}$ the contribution from the regular part K_{reg}, we have

$$v_0(y) - v_{0,\mathrm{reg}}(y) = \left(\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{2\gamma_0'(z_1)}{y-(z_1+i\gamma_0(z_1))}\,dz_1\right)^* = -\left(\frac{1}{2\pi i}\int_{\Gamma_0} \frac{1}{\xi - y}\,\frac{2\gamma_0'(\xi_1)}{1+i\gamma_0'(\xi_1)}\,d\xi\right)^*$$

for $y \notin \Gamma_0$. Now taking one sided limits $y \to x \in \Gamma_0$, expression (4-8) follows from the Sokhotski–Plemelj formula; see [Muskhelishvili 1972]. □

Formulas (4-8) show that the initial velocity field is still continuous across the interface in the normal direction. Therefore the (not normalized) normal velocity at the interface $s_0 : \mathbb{T} \to \mathbb{R}$,

$$s_0(x_1) := v_0(x_1, \gamma_0(x_1)) \cdot \begin{pmatrix} -\gamma_0'(x_1) \\ 1 \end{pmatrix} = \text{p.v.}\int_{\mathbb{T}} K\begin{pmatrix} x_1 - z_1 \\ \gamma_0(x_1) - \gamma_0(z_1) \end{pmatrix} 2\gamma_0'(z_1)\,dz_1 \cdot \begin{pmatrix} -\gamma_0'(x_1) \\ 1 \end{pmatrix}, \tag{4-9}$$

is well-defined. It will play an important role in our further analysis as it dictates the motion of Lagrangian particles at the interface to first order when ignoring the Burgers' term $\partial_{x_2}(\rho^2)$.

4.3. ***Rescaling and level set function.*** We now transform problem (3-1), (1-2) in terms of level sets. The reformulation here is understood on a formal level. We will solve the derived fixed-point problem in Section 5 and a posteriori justify the transformations in Section 6.

The starting point is the following ansatz for ρ capturing the effect of the Burgers' part described in Section 4.1. Assume that there exists $f : [0, T) \times \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ sufficiently regular with

$$f(0, y) = \gamma_0(y_1) \tag{4-10}$$

and such that, for every $t \in (0, T)$, $y_1 \in \mathbb{T}$, the map $\mathbb{R} \to \mathbb{R}$, $y_2 \mapsto ty_2 + f(t, y_1, y_2)$ is a monotone diffeomorphism.

Then each of the transformations $X_t : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$, $t \in (0, T)$,

$$X_t(y) = \begin{pmatrix} y_1 \\ ty_2 + f(t, y) \end{pmatrix},$$

is a diffeomorphism as well.

We now seek to find a solution of (3-1), (1-2) on $[0, T)$ having the property that

$$\rho(t, X_t(y)) = \phi_0(y_2) = \begin{cases} +1, & y_2 \geq 2, \\ \frac{1}{2}y_2, & y_2 \in (-2, +2), \\ -1, & y_2 \leq -2. \end{cases} \tag{4-11}$$

For $t > 0$, we compute

$$DX_t(y) = \begin{pmatrix} 1 & 0 \\ \partial_{y_1} f(t, y) & t + \partial_{y_2} f(t, y) \end{pmatrix}, \tag{4-12}$$

$$DX_t(y)^{-1} = \begin{pmatrix} 1 & 0 \\ \frac{-\partial_{y_1} f(t,y)}{t+\partial_{y_2} f(t,y)} & \frac{1}{t+\partial_{y_2} f(t,y)} \end{pmatrix}, \tag{4-13}$$

$$\nabla\rho(t, X_t(y)) = \frac{1}{2(t + \partial_{y_2} f(t, y))} \begin{pmatrix} -\partial_{y_1} f(t, y) \\ 1 \end{pmatrix} \mathbb{1}_{(-2,2)}(y_2), \tag{4-14}$$

so that the first equation of (3-1) — when written in nondivergence form — under the ansatz (4-11) is equivalent to

$$0=\mathbb{1}_{(-2,2)}(y_2)\begin{pmatrix}-\partial_{y_1}f(t,y)\\1\end{pmatrix}\cdot\left(\begin{pmatrix}0\\y_2+\partial_t f(t,y)-2\phi_0(y_2)\end{pmatrix}-v(t,X_t(y))\right).$$

Since $2\phi_0(y_2)=\mathbb{1}_{(-2,2)}(y_2)=y_2$, expanding the above equation leads to

$$\partial_t f(t,y)=v(t,y_1,ty_2+f(t,y))\cdot\begin{pmatrix}-\partial_{y_1}f(t,y)\\1\end{pmatrix}\tag{4-15}$$

for $(t,y_1,y_2)\in(0,T)\times\mathbb{T}\times(-2,2)$.

Note that, in view of (4-14), the velocity field in (4-15) is always considered in directions normal to the level sets of ρ.

4.4. ***Transformation of the velocity field.*** For $t>0$, we have that $v(t,\cdot)$ (in all reasonable scenarios) is given by the Biot–Savart law (4-3); see Section 4.2.

Applying the transformation $X_t(y)$, we compute the velocity field

$$v(t,y_1,ty_2+f(t,y))=v(t,X_t(y))$$

occurring in (4-15). First of all, formulas (4-12) and (4-14) imply

$$\begin{aligned}v(t,X_t(y))&=-\int_{\mathbb{T}\times\mathbb{R}}K(X_t(y)-z)\partial_{x_1}\rho(t,z)\,dz\\&=-\int_{\mathbb{T}\times\mathbb{R}}K(X_t(y)-X_t(z))\partial_{x_1}\rho(t,X_t(z))\det DX_t(z)\,dz\\&=\frac{1}{2}\int_{-2}^{2}\int_{\mathbb{T}}K(X_t(y)-X_t(z))\partial_{y_1}f(t,z)\,dz_1\,dz_2.\end{aligned}$$

Next we compute the full right-hand side of (4-15) and exploit the fact that the velocity field $v(t,X_t(y))$ is only needed in normal directions. More precisely, for $z\neq y$, we have

$$\begin{aligned}&\partial_{y_1}f(t,z)K(X_t(y)-X_t(z))\cdot\begin{pmatrix}-\partial_{y_1}f(t,y)\\1\end{pmatrix}\\&\quad=\partial_{y_1}f(t,z)\nabla G(X_t(y)-X_t(z))\cdot\begin{pmatrix}1\\\partial_{y_1}f(t,y)\end{pmatrix}\\&\quad=\partial_{y_1}f(t,y)\nabla G(X_t(y)-X_t(z))\cdot\begin{pmatrix}1\\\partial_{y_1}f(t,z)\end{pmatrix}-\partial_1 G(X_t(y)-X_t(z))(\partial_{y_1}f(t,y)-\partial_{y_1}f(t,z))\\&\quad=-\partial_{y_1}f(t,y)\frac{d}{dz_1}(G(X_t(y)-X_t(z)))-K_2(X_t(y)-X_t(z))(\partial_{y_1}f(t,y)-\partial_{y_1}f(t,z)).\end{aligned}$$

Thus after integration we obtain an additional cancelation in the convolution, i.e.,

$$v(t,X_t(y))\cdot\begin{pmatrix}-\partial_{y_1}f(t,y)\\1\end{pmatrix}=-\frac{1}{2}\int_{-2}^{2}\int_{\mathbb{T}}K_2(X_t(y)-X_t(z))(\partial_{y_1}f(t,y)-\partial_{y_1}f(t,z))\,dz_1\,dz_2.\tag{4-16}$$

4.5. ***Equation for f.*** Combining (4-16) with (4-15), we see that (3-1) can — after our ansatz — be written in the closed form

$$\partial_t f(t,y) = -\frac{1}{2}\int_{-2}^{2}\int_{\mathbb{T}} K_2(\tilde{\Delta} X_t(y,z))(\partial_{y_1} f(t,y) - \partial_{y_1} f(t,z))\,dz_1\,dz_2, \tag{4-17}$$

where

$$\tilde{\Delta} X_t(y,z) := X_t(y) - X_t(z) = \begin{pmatrix} y_1 - z_1 \\ t(y_2 - z_2) + f(t,y) - f(t,z) \end{pmatrix}$$

also depends on f. Via translation in z_1, equation (4-17) can also be written as

$$\partial_t f(t,y) = -\frac{1}{2}\int_{-2}^{2}\int_{\mathbb{T}} K_2(\Delta X_t(y,z))\Delta\partial_{y_1} f_t(y,z)\,dz_1\,dz_2, \tag{4-18}$$

where we have used the abbreviation

$$\begin{aligned} \Delta X_t(y,z) &:= \begin{pmatrix} z_1 \\ t(y_2 - z_2) + f(t,y_1,y_2) - f_t(t,y_1 - z_1, z_2) \end{pmatrix}, \\ \Delta\partial_{y_1} f_t(y,z) &:= \partial_{y_1} f(t,y_1,y_2) - \partial_{y_1} f(t,y_1 - z_1, z_2). \end{aligned} \tag{4-19}$$

The latter form turns out to be more convenient to work with.

4.6. ***One more ansatz.*** One important assumption in the above derivation is the invertibility of the maps $(X_t)_{t>0}$. In order to guarantee this, we further make the ansatz

$$f(t,y) = \gamma_0(y_1) + t s_0(y_1) + \tfrac{1}{2} t^{1+\alpha}\eta(t,y), \tag{4-20}$$

where $\alpha \in (0,1)$ and the functions $s_0 : \mathbb{T} \to \mathbb{R}$, $\eta : (0,T) \times \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ are sufficiently regular. In order to avoid potential confusion, we emphasize that the function η has nothing to do with an entropy; compare with the η appearing in Definition 3.4. Furthermore, we remark that the particular choice of $\alpha \in (0,1)$ is not important; see Section 5.7 for further discussion.

By this ansatz f satisfies (4-10), and the desired invertibility can be assumed true for a small time interval (depending on $\|\partial_{y_2}\eta\|_{L^\infty}$). Moreover, since at $t = 0$ we have $\partial_t f = s_0$, $\partial_{y_1} f = \gamma_0'$, passing formally to the limit on the right-hand side of (4-18), one sees that s_0 necessarily is given by

$$-\frac{1}{2}\int_{-2}^{2}\int_{\mathbb{T}} K_2(\Delta X_0(y_1,z_1))\Delta\gamma_0'(y_1,z_1)\,dz_1\,dz_2, \tag{4-21}$$

where

$$\begin{aligned} \Delta X_0(y_1,z_1) &:= \begin{pmatrix} z_1 \\ \gamma_0(y_1) - \gamma_0(y_1 - z_1) \end{pmatrix}, \\ \Delta\gamma_0'(y_1,z_1) &:= \gamma_0'(y_1) - \gamma_0'(y_1 - z_1). \end{aligned}$$

A quick computation similar to the one in Section 4.4 and comparison with (4-9) shows that the above expression is precisely the normal component of the initial velocity evaluated at $(y_1, \gamma_0(y_1))$, i.e., (4-21). This shows that the function $s_0(y_1)$ is indeed forced to be the normal component of $v_0(y_1, \gamma_0(y_1))$.

Finally we integrate (4-18) in time and use (4-20), (4-21) in order to deduce that, for f to be a solution to (4-18), η must be a solution of the fixed-point problem

$$\eta(t,y) = -\frac{1}{t^{1+\alpha}} \int_0^t \int_{-2}^{2} \int_{\mathbb{T}} K_2(\Delta X_s(y,z)) \Delta \partial_{y_1} f_s(y,z) - K_2(\Delta X_0(y_1,z_1)) \Delta \gamma_0'(y_1,z_1)\, dz_1\, dz_2\, ds. \tag{4-22}$$

Note that η and $\partial_{y_1}\eta$ enter the right-hand side through (4-19), (4-20).

5. Existence of a solution for analytic graphs

Our goal is to show that, for a real analytic $\gamma_0 : \mathbb{T} \to \mathbb{R}$, there exists a unique local-in-time solution η of problem (4-22). The proof relies on the following version of the abstract Cauchy–Kovalevskaya theorem based on the formulation of Nishida [1977]; see also [Nirenberg 1972].

In order to avoid confusion we emphasize that, throughout Section 5, every symbol ρ, ρ', $\bar{\rho}$, ρ_0 denotes a positive constant referring to the size of the domain of analyticity. This is done in analogy to [Nishida 1977]. At no time in Section 5 do we mention the density function $\rho(t,x)$, which we seek to construct, or the initial density $\rho_0(x)$.

Theorem 5.1. *Let $(B_\rho)_{\rho\in(0,\rho_0)}$, $\rho_0 > 0$, be a scale of Banach spaces with $\|\cdot\|_{\rho'} \le \|\cdot\|_\rho$ for $0 < \rho' < \rho < \rho_0$, and consider the integral equation*

$$u(t) = \frac{1}{a(t)} \int_0^t F(u(s), s)\, ds \tag{5-1}$$

for a given continuous function $a : [0,\infty) \to \mathbb{R}$ with $a(t) > 0$ for $t > 0$. If F is such that

(i) *there exist $R > 0$, $T > 0$ such that, for every $0 < \rho' < \rho < \rho_0$, the map*

$$\{u \in B_\rho : \|u\|_\rho < R\} \times [0,T) \to B_{\rho'}, \quad (u,t) \mapsto F(u,t),$$

is well-defined and continuous,

(ii) *there exists $b : [0,T) \to [0,\infty)$ continuous such that, for any $0 < \rho' < \rho < \rho_0$ and all $u, v \in B_\rho$, $\|u\|_\rho < R$, $\|v\|_\rho < R$, $t \in [0,T)$, we have*

$$\|F(u,t) - F(v,t)\|_{\rho'} \le \frac{b(t)}{\rho - \rho'} \|u - v\|_\rho,$$

(iii) *$F(0,\cdot) \in L^1(0,T; B_\rho)$ for any $\rho \in (0,\rho_0)$, and there exists $c : [0,T) \to [0,\infty)$ continuously differentiable on $(0,T)$ and continuous on $[0,T)$ with $c(0) = 0$ as well as $c'(t) > 0$ for $t > 0$ such that, for all $\rho \in (0,\rho_0)$, $t \in (0,T)$, we have*

$$\frac{1}{a(t)} \int_0^t \|F(0,s)\|_\rho\, ds \le \frac{c(t)}{\rho_0 - \rho},$$

(iv) *for a constant $K > 0$, the functions $a(t)$, $b(t)$, $c(t)$ appearing in (5-1), (ii), (iii) satisfy the relation*

$$\sup_{s\in(0,t)} \left| \frac{b(s)c(s)}{c'(s)} \right| \le K a(t) c(t), \quad t \in (0,T), \tag{5-2}$$

then there exists a constant $\bar{a} = \bar{a}(K, R) > 0$ *and a unique* $u(t)$ *which, for any* $\rho \in (0, \rho_0)$, *maps the interval* $\{t \in [0, T) : c(t) < \bar{a}(\rho_0 - \rho)\}$ *continuously into the* R*-ball of* B_ρ. *Moreover,* u *satisfies* (5-1) *and*

$$\|u(t)\|_\rho = O\left(\frac{c(t)}{\rho_0 - \rho}\right) \quad \text{as } t \to 0.$$

In particular, $u(0) = 0$.

For the choices $a(t) = 1,\ b(t) = c_1,\ c(t) = c_2 t$ with some constants $c_1, c_2 > 0$, the above theorem is the abstract Cauchy–Kovalevskaya theorem in the formulation of Nishida [1977]. The proof of Theorem 5.1 requires indeed just some minor modifications which are presented in Appendix A. For a related generalization of the abstract Cauchy–Kovalevskaya theorem, see also [Reissig 1987; 1988].

We will apply Theorem 5.1 in the following situation.

Lemma 5.2. *Let* $c_1, c_2 > 0$ *and* $\alpha \in (0, 1)$. *There exist* $T = T(\alpha),\ K = K(c_1, c_2) > 0$ *such that* $a(t) := t^{1+\alpha}$, $b(t) := c_1 t^{1+\alpha}|\log t|,\ c(t) := c_2 t^{1-\alpha}|\log t|$ *satisfy* (5-2).

Proof. Consider $T \in (0, 1)$ such that

$$(1-\alpha)|\log t| \geq 2, \qquad |\log t| t^\alpha \leq 1$$

for all $t \in (0, T)$. Then, for $0 < s < t < T$, we have

$$\frac{b(s)c(s)}{c'(s)} = c_1 \frac{s^{2+\alpha}|\log s|^2}{(1-\alpha)|\log s| - 1} \leq c_1 s^2 |\log s| \leq c_1 t^2 |\log t| = \frac{c_1}{c_2} a(t)c(t).$$

Thus (5-2) holds with $K := c_1 c_2^{-1}$. □

5.1. ***Banach spaces.*** Set

$$\Omega_0 := \mathbb{T} \times (-2, 2)$$

as well as

$$U_\rho := \{z \in \mathbb{C} : |\mathrm{Im}(z)| < \rho\}, \qquad \Omega_\rho := U_\rho \times (-2, 2)$$

for $\rho > 0$.

We define the space B_ρ to consist of all continuous functions $\eta : \Omega_0 \to \mathbb{R},\ y \mapsto \eta(y)$, which satisfy

(i) for every $y_2 \in (-2, 2)$, the function $\eta(\cdot, y_2)$ extends to a holomorphic function $U_\rho \to \mathbb{C}$ which is again denoted by $\eta(\cdot, y_2)$,

(ii) the derivative $\partial_{y_2}\eta : \Omega_\rho \to \mathbb{C}$ exists and is uniformly continuous, and $\partial_{y_2}\eta(\cdot, y_2)$ is holomorphic on U_ρ for every $y_2 \in (-2, 2)$,

(iii) the norm

$$\|\eta\|_\rho := \|\eta\|_{L^\infty(\Omega_\rho)} + \|\partial_{y_1}\eta\|_{L^\infty(\Omega_\rho)} + \|\partial_{y_2}\eta\|_{L^\infty(\Omega_\rho)}$$

is finite.

For clarification, the extension in (i) strictly speaking is the extension of the 2π-periodic function $\eta(\cdot, y_2) : \mathbb{R} \to \mathbb{R}$. The extension $U_\rho \to \mathbb{C},\ y_1 \mapsto \eta(y_1, y_2)$, therefore is periodic in the real part of y_1. Moreover, $\partial_{y_1}\eta$ denotes the complex derivative in the first component, while $\partial_{y_2}\eta$ is the real partial derivative with respect to the second component. Although the two derivatives are of slightly different nature, we still use a gradient notation $\nabla_y \eta := (\partial_{y_1}\eta, \partial_{y_2}\eta)^T$.

Clearly each B_ρ is a Banach space and $B_\rho \subset B_{\rho'}$, $\|\cdot\|_{\rho'} \le \|\cdot\|_\rho$ whenever $\rho' < \rho$. Moreover, for the introduced scale of spaces, we have the following lemma, which is a direct consequence of Cauchy's integral formula for analytic functions.

Lemma 5.3 (Cauchy). *Let $0 < \rho' < \rho$ and $\eta \in B_\rho$. Then, for $j = 1, 2$, we have*

$$\|\partial_{y_1}\partial_{y_j}\eta\|_{L^\infty(\Omega_{\rho'})} \le \frac{C}{\rho - \rho'}\|\eta\|_\rho$$

for $C = (2\pi)^{-1}$.

In particular, $\partial_{y_1}\eta$ is — as is η itself — Lipschitz continuous on Ω_0. This together with the assumed uniform continuity of $\partial_{y_2}\eta$ implies the following.

Lemma 5.4. *Let $\rho > 0$ and $\eta \in B_\rho$. Then $\eta : \Omega_0 \to \mathbb{R}$ extends to $\mathcal{C}^1(\bar{\Omega}_0)$, and $\eta(\,\cdot\,, y_2)$ and $\partial_{y_2}\eta(\,\cdot\,, y_2)$ are real analytic for each $y_2 \in [-2, 2]$.*

Also note that $\partial_{y_2}\partial_{y_1}\eta(y) = \partial_{y_1}\partial_{y_2}\eta(y)$ for $\eta \in B_\rho,\ y \in \Omega_\rho$, for instance, by means of Cauchy's integral formula.

5.2. ***Notation.*** From now on we fix $\alpha \in (0, 1)$ and a real analytic initial datum $\gamma_0 : \mathbb{T} \to \mathbb{R}$. Clearly γ_0 can be extended to a holomorphic function defined on $U_{2\rho_0}$ for some $\rho_0 > 0$ small.

Hence all (complex) derivatives are uniformly bounded on U_{ρ_0}, e.g., there exist a constant $C_0 > 0$ such that

$$\|\gamma_0'\|_{L^\infty(U_{\rho_0})} \le C_0. \tag{5-3}$$

More generally, henceforth, $C_0 > 0$ always denotes a constant depending solely on the $L^\infty(U_{\rho_0})$-norm of a fixed finite amount of derivatives of γ_0. (A detailed look at the proof reveals that the first five derivatives of γ_0 are sufficient. However, the precise number is not important.) In contrast $C > 0$ usually denotes a constant not depending on γ_0. Both constants typically change from line to line. Also we point out that distinguishing C_0 from C is not essential for the proof of Theorem 3.2.

For a pair $a = (a_1, a_2) \in \mathbb{R} \times \mathbb{C}$, we define

$$|a|_* := (|a_1|^2 + |a_2|^2)^{1/2} = (a_1^2 + a_2 a_2^*)^{1/2}. \tag{5-4}$$

Moreover, whenever we write $|z_1|$ for $z_1 \in \mathbb{T}$, we mean the absolute value of the unique representative of z_1 in $[-\pi, \pi)$. In particular, we will also use $|a|_*$ for pairs $a \in \mathbb{T} \times \mathbb{C}$.

For any function $g : \Omega_{\rho_0} \to \mathbb{C}^n$ or $h : U_{\rho_0} \to \mathbb{C}^n$ we use the abbreviation

$$\Delta g(y, z) := g(y) - g(y_1 - z_1, z_2) \quad \text{or} \quad \Delta h(y_1, z_1) := h(y_1) - h(y_1 - z_1) \tag{5-5}$$

for $y = (y_1, y_2) \in \Omega_{\rho_0},\ z = (z_1, z_2) \in \Omega_0$ and $y_1 \in U_{\rho_0},\ z_1 \in \mathbb{T}$, respectively. In the proofs we will usually omit the points (y, z) and simply write Δg and Δh.

Furthermore, for $t \ge 0$ and $\eta \in B_{\rho_0}$, we define $f_t^\eta : \Omega_{\rho_0} \to \mathbb{C},\ X_t^\eta : \Omega_{\rho_0} \to \mathbb{C}^2$,

$$f_t^\eta(y) := \gamma_0(y_1) + ts_0(y_1) + \tfrac{1}{2}t^{1+\alpha}\eta(y), \quad X_t^\eta(y) := \begin{pmatrix} y_1 \\ ty_2 + f_t^\eta(y) \end{pmatrix}. \tag{5-6}$$

The function $s_0 : U_{\rho_0} \to \mathbb{C}$ will be introduced in Lemma 5.6 below. At time $t = 0$, we simply write $X_0(y_1)$ instead of $X_0^\eta(y)$. The second component of $X_t^\eta(y)$ is denoted by $X_{t,2}^\eta(y)$. There is no need to distinguish the first component, since it is just given by y_1.

5.3. *Preliminary lemmas.* In order to define the function F as a complex extension of the functional appearing in (4-22), we need some preparation.

Recall that the second component K_2 of the Biot–Savart kernel on $\mathbb{T} \times \mathbb{R}$ is given by

$$K_2(a) = K_2(a_1, a_2) = \frac{1}{4\pi} \frac{\sin(a_1)}{\cosh(a_2) - \cos(a_1)}.$$

Thus, for fixed $a_1 \in \mathbb{T}$, the canonical extension of $K_2(a_1, \cdot)$ to $a_2 \in \mathbb{C}$ is holomorphic on the open set $\{a_2 \in \mathbb{C} : \cosh(a_2) - \cos(a_1) \neq 0\}$. We define

$$\mathcal{U} := \{a \in \mathbb{T} \times \mathbb{C} : \cosh(a_2) - \cos(a_1) \neq 0\}.$$

Lemma 5.5. *Let $\kappa \in \left(0, \frac{1}{2}\right)$. The sets*

$$\mathcal{U}^\kappa := \left\{(a_1, a_2) \in \mathbb{T} \times \mathbb{C} : |\mathrm{Im}(a_2)| < \kappa(|a_1| + |\mathrm{Re}(a_2)|),\ |\mathrm{Im}(a_2)| < \tfrac{\pi}{2}\right\}$$

are subsets of $\mathcal{U}$ with $\partial\mathcal{U}^\kappa \cap \partial\mathcal{U} = \{0\}$. Moreover, there exists a constant $C > 0$ depending on κ such that, for all $a \in \mathcal{U}^\kappa$, $j = 0, 1, 2$, we have

$$|\partial_{a_2}^j K_2(a)| \leq C|a|_*^{-(1+j)}. \tag{5-7}$$

Proof. Let $a \in \overline{\mathcal{U}}^\kappa \cap \partial\mathcal{U}$, $a_2 = u + iv$. Then

$$0 = \cosh(a_2) - \cos(a_1) = \cosh(u)\cos(v) - \cos(a_1) + i\sinh(u)\sin(v)$$

implies $v = 0$, and thus $\cosh(u) = \cos(a_1)$, which is only possible for $a_1 = u = 0$; or $u = 0$ and $\cos(v) = \cos(a_1)$, which in the closure of $\mathcal{U}^\kappa$ is again only possible for $a_1 = u = 0$. Thus $\mathcal{U}^\kappa \subset \mathcal{U}$ and $\partial\mathcal{U}^\kappa \cap \partial\mathcal{U} = \{0\}$.

For the second part we split the analysis into three regions: $a \in \mathcal{U}^\kappa$, $|a|_*$ close to 0; $a \in \mathcal{U}^\kappa$, $|a|_*$ large; and the remaining subset of $\mathcal{U}^\kappa$.

Let us start with $a \in \mathcal{U}^\kappa$, $|a|_*$ close to 0. Writing again $a = (a_1, u + iv)$ and using that $v^2 \leq 2\kappa^2(a_1^2 + u^2)$, we have

$$\begin{aligned}
|a_1^2 + a_2^2| &= (a_1^4 + 2a_1^2(u^2 - v^2) + (u^2 - v^2)^2 + 4u^2v^2)^{1/2} \\
&\geq ((1 - 4\kappa^2)a_1^4 + 2a_1^2(1 - 2\kappa^2)u^2 + (u^2 + v^2)^2)^{1/2} \\
&\geq (1 - 4\kappa^2)(a_1^4 + |a_2|^4)^{1/2} \geq \tfrac{1}{2}(1 - 4\kappa^2)|a|_*^2.
\end{aligned}$$

Then, for $a \in \mathcal{U}^\kappa$, a small, it follows that

$$|K_2(a)| \leq \frac{1}{2\pi} \frac{|a_1| + O(|a_1|^3)}{|a_1^2 + a_2^2| - O(|a_1|^4) - O(|a_2|^4)} \leq \frac{1}{\pi} \frac{|a|_* + O(|a|_*^3)}{(1 - 4\kappa^2)|a|_*^2 - O(|a|_*^4)} \leq \frac{C}{|a|_*}.$$

Doing the same for higher-order derivatives, it follows that there exists $\varepsilon > 0$ such that (5-7) holds for all $a \in \mathcal{U}^\kappa$ with $|a|_* < \varepsilon$. We fix such an ε.

Next let us consider the opposite regime $a = u + iv \in \mathcal{U}^\kappa$, $|a|_*$ large. Note that this necessarily means that $|u|$ has to be large since $|a_1| \le \pi$, $|v| \le \frac{\pi}{2}$ by definition of $\mathcal{U}^\kappa$. We then estimate

$$\begin{aligned}|\cosh(a_2) - \cos(a_1)| &\ge |\cosh(u)\cos(v) + i \sinh(u)\sin(v)| - 1 \\ &= (\cosh^2(u) - \sin^2(v))^{1/2} - 1 \\ &\ge \left(\frac{u^2}{2} - 1\right)^{1/2} - 1.\end{aligned}$$

Consequently one can find constants $C > 0$ and $R > 0$ such that (5-7) with $j = 0$ holds for all $a \in \mathcal{U}^\kappa$ with $|a|_* > R$. Again this procedure can be extended to higher-order derivatives giving an R as above but with (5-7) valid for $j = 0, 1, 2$ for $|a|_* > R$. Let us also fix such an R.

Let now a be in the remaining set, i.e., $a \in \mathcal{U}^\kappa$ with $\varepsilon \le |a|_* \le R$. The closure of this set is compact and bounded away from $\partial\mathcal{U}$, where the denominator of K_2 vanishes. Therefore the existence of $C > 0$ such that (5-7) holds also on this set follows just by continuity of $\partial^j_{a_2} K_2$, $j = 0, 1, 2$. This finishes the proof of the lemma. □

Lemma 5.6. *Let $\rho_0 > 0$ be chosen such that γ_0 extends holomorphically to $U_{2\rho_0}$ with*

$$4\|\mathrm{Im}(\gamma_0')\|_{L^\infty(U_{\rho_0})} < 1. \tag{5-8}$$

Then the complex extension of the initial normal velocity $s_0 : \Omega_0 \to \mathbb{R}$,

$$s_0(y_1) := -2 \int_{\mathbb{T}} K_2(\Delta X_0(y_1, z_1))\Delta\gamma_0'(y_1, z_1)\,dz_1,$$

is holomorphic on U_{ρ_0}. Moreover, the $L^\infty(U_{\rho_0})$-norm of any finite number of derivatives of s_0 can be bounded by C_0. In particular, all derivatives of s_0 are given by differentiation under the integral.

Proof. By (5-8) one estimates

$$|\mathrm{Im}(\gamma_0(y_1) - \gamma_0(y_1 - z_1))| < \tfrac{1}{4}|z_1|$$

for $z_1 \in [-\pi, \pi)$, $z_1 \ne 0$, $y_1 \in U_{\rho_0}$. Thus, by Lemma 5.5 the composition of K_2 with $y_1 \mapsto \Delta X_0(y_1, z_1)$ is holomorphic for every $z_1 \ne 0$. Moreover, again by Lemma 5.5, for such z_1, we have

$$|\partial_{y_1}(K_2(\Delta X_0(y_1, z_1)))| \le C\left(\frac{|\Delta\gamma_0'(y_1, z_1)|^2}{|\Delta X_0(y_1, z_1)|_*^2} + \frac{|\Delta\gamma_0''(y_1, z_1)|}{|\Delta X_0(y_1, z_1)|_*}\right) \le C_0.$$

It follows that s_0 is holomorphic and that $\|s_0'\|_{L^\infty(U_{\rho_0})} \le C_0$. The same can be shown for higher-order derivatives. □

The following two lemmas provide careful estimates needed for the compensation of various terms appearing in the definition of our nonlinear map F below. We are also careful with the uniform integrability as we need to be able to neglect what happens in some small sets.

Lemma 5.7. *Let $\rho_0 > 0$ be as in Lemma 5.6, and let $R > 0$. There exists $T = T(R, C_0, \alpha) \in (0, 1)$ such that, for all $\eta \in B_\rho$, $\|\eta\|_\rho < R$, $\rho \in (0, \rho_0)$ and $t \in [0, T)$, $y \in \Omega_\rho$, $z \in \Omega_0$, we have $\Delta X^\eta_t(y, z) \in \overline{\mathcal{U}^{3/8}}$ and*

$$t|y_2 - z_2| \le C_0 |\Delta X^\eta_t(y, z)|_*. \tag{5-9}$$

Proof. First of all chose $T \in (0,1)$ with $T^\alpha R \leq 1$. Then, omitting the (y,z) dependence in the notation, see Section 5.2, we have

$$\begin{aligned}\tfrac{1}{2}t|y_2-z_2| &\leq t|y_2-z_2| - \tfrac{1}{2}t^{1+\alpha}R|y_2-z_2|\\ &\leq \left|t(y_2-z_2)+\tfrac{1}{2}t^{1+\alpha}\operatorname{Re}(\eta(y)-\eta(y_1,z_2))\right|\\ &= \left|\operatorname{Re}\big(\Delta X^\eta_{t,2}-\Delta\gamma_0 - t\Delta s_0 - \tfrac{1}{2}t^{1+\alpha}(\eta(y_1,z_2)-\eta(y_1-z_1,z_2))\big)\right|\\ &\leq |\operatorname{Re}(\Delta X^\eta_{t,2})| + (C_0(1+T)+T^{1+\alpha}R)|z_1| \leq |\operatorname{Re}(\Delta X^\eta_{t,2})| + C_0|z_1|.\end{aligned}$$

This implies (5-9).

In order to see that $\Delta X^\eta_t \in \overline{\mathcal{U}^{3/8}}$, we use (5-8) as well as the just shown inequality to deduce

$$\begin{aligned}|\operatorname{Im}(\Delta X^\eta_{t,2})| &= \left|\operatorname{Im}\big(\Delta\gamma_0 + t\Delta s_0 + \tfrac{1}{2}t^{1+\alpha}\Delta\eta\big)\right|\\ &\leq \big(\tfrac{1}{4}+TC_0+T^{1+\alpha}R\big)|z_1| + T^\alpha R t|y_2-z_2|\\ &\leq \big(\tfrac{1}{4}+T(C_0+1)\big)|z_1| + T^\alpha R|\operatorname{Re}(\Delta X^\eta_{t,2})| + T^\alpha RC_0|z_1|.\end{aligned}$$

Thus by choosing $T>0$ even smaller, we have the desired inequality

$$|\operatorname{Im}(\Delta X^\eta_{t,2})| \leq \tfrac{3}{8}(|z_1| + |\Delta\operatorname{Re}(X^\eta_{t,2})|). \qquad \square$$

Lemma 5.8. *Let $\rho_0,\ R,\ T>0$ be as in Lemma 5.7. For $\eta \in B_\rho$, $\|\eta\|_\rho < R$, $\rho \in (0,\rho_0)$ and $y \in \Omega_\rho$, $t \in (0,T)$, we have*

$$\int_{\Omega_0} \frac{1}{|\Delta X^\eta_t(y,z)|_*}\,dz \leq C_0|\log t|. \tag{5-10}$$

The integrability of $|\Delta X^\eta_t(y,\cdot)|_*^{-1}$ is uniform with respect to $y \in \Omega_0$ and with respect to t considered on any interval of the form $[t_0,T)$ with $t_0>0$.

Proof. In view of (5-9), we have

$$\begin{aligned}\int_{\Omega_0}\frac{1}{|\Delta X^\eta_t|_*}\,dz &\leq C_0\int_{\Omega_0}\frac{1}{|z_1|+t|y_2-z_2|}\,dz = C_0\int_{\mathbb{T}}\int_{y_2-2}^{y_2+2}\frac{1}{|z_1|+t|z_2|}\,dz_2\,dz_1\\ &\leq C_0\int_0^\pi\int_0^4\frac{1}{z_1+tz_2}\,dz_2\,dz_1 = C_0\Big(\frac{\pi}{t}\log\Big(1+\frac{4t}{\pi}\Big)+4\log\Big(\frac{\pi}{4t}+1\Big)\Big),\end{aligned}$$

which is of order $|\log t|$. Note here that $t<1$ since T is assumed to be less than 1.

The uniform integrability follows from

$$\frac{1}{|z_1|+t_0|z_2|} \in L^1(\mathbb{T}\times(-4,4)). \qquad \square$$

5.4. *Definition of F*. Let us fix $\rho_0>0$ as in Lemma 5.6. Take $R=1$ and a corresponding $T\in(0,1)$ from Lemma 5.7.

We define the application $(\eta,t)\mapsto F(\eta,t)=F_t(\eta)$ by setting

$$F_t(\eta)(y) := -\int_{\Omega_0} K_2(\Delta X^\eta_t(y,z))\Delta\partial_{y_1} f^\eta_t(y,z) - K_2(\Delta X_0(y_1,z_1))\Delta\gamma_0'(y_1,z_1)\,dz$$

for $t>0$ and $F_0(\eta)(y)=0$.

Lemma 5.9. *F when seen as a map*

$$\{\eta \in B_\rho : \|\eta\|_\rho < 1\} \times [0, T) \to B_{\rho'}$$

is well-defined for all $0 < \rho' < \rho < \rho_0$. Moreover, for $\eta \in B_\rho$, $\|\eta\|_\rho < 1$, the map $[0, T) \to B_{\rho'}$, $t \mapsto F_t(\eta)$, is continuous.

Proof. Let $\eta \in B_\rho$, $\|\eta\|_\rho < 1$ and $t \in (0, T)$. In view of Lemma 5.6, it remains to look at

$$\widetilde{F}_t(\eta)(y) := F_t(\eta)(y) + 2s_0(y_1) = -\int_{\Omega_0} K_2(\Delta X_t^\eta(y, z))\Delta\partial_{y_1} f_t^\eta(y, z)\, dz.$$

For $y \in \Omega_\rho$ and $z \in \Omega_0$ with $z_1 \neq 0$, one computes

$$\partial_{y_1}(K_2(\Delta X_t^\eta)\Delta\partial_{y_1} f_t^\eta) = \partial_{a_2} K_2(\Delta X_t^\eta)(\Delta\partial_{y_1} f_t^\eta)^2 + K_2(\Delta X_t^\eta)\Delta\partial_{y_1}^2 f_t^\eta, \tag{5-11}$$

$$\partial_{y_2}(K_2(\Delta X_t^\eta)\Delta\partial_{y_1} f_t^\eta) = \partial_{a_2} K_2(\Delta X_t^\eta)\big(t + \tfrac{1}{2}t^{1+\alpha}\partial_{y_2}\eta(y)\big)\Delta\partial_{y_1} f_t^\eta + K_2(\Delta X_t^\eta)\tfrac{1}{2}t^{1+\alpha}\partial_{y_2}\partial_{y_1}\eta(y), \tag{5-12}$$

where, as usual, we have omitted the (y, z) dependence in the Δ-notation.

In order to get uniform integrability, we use (5-9) to estimate

$$\begin{aligned} |\Delta\partial_{y_1}^j f_t^\eta| &\le C_0|z_1| + \|\partial_{y_1}^{j+1}\eta\|_{L^\infty(\Omega_{\rho'})}|z_1| + \|\partial_{y_2}\partial_{y_1}^j\eta\|_{L^\infty(\Omega_{\rho'})} t|y_2 - z_2| \\ &\le C_0(1 + \|\partial_{y_1}^j\nabla_y\eta\|_{L^\infty(\Omega_{\rho'})})|\Delta X_t^\eta|_* \end{aligned} \tag{5-13}$$

for $y \in \Omega_{\rho'}$, $j = 0, 1, 2$. Now (5-13) and Lemmas 5.5 and 5.7 imply

$$|K_2(\Delta X_t^\eta)\Delta\partial_{y_1} f_t^\eta| \le C_0(1 + \|\partial_{y_1}\nabla_y\eta\|_{L^\infty(\Omega_{\rho'})}). \tag{5-14}$$

As a consequence $\widetilde{F}_t$, and thus F_t, maps at least into $L^\infty(\Omega_{\rho'})$. Moreover, combining similarly (5-7), (5-13) for $j = 1, 2$, and (5-9) to estimate (5-11), (5-12), one sees that

$$|\partial_{y_1}(K_2(\Delta X_t^\eta)\Delta\partial_{y_1} f_t^\eta)| \le C_0(1 + \|\partial_{y_1}\nabla_y\eta\|^2_{L^\infty(\Omega_{\rho'})} + \|\partial_{y_1}^2\nabla_y\eta\|_{L^\infty(\Omega_{\rho'})}) \tag{5-15}$$

and, recalling $t < 1$, $|\partial_{y_2}\eta(y)| < 1$, that

$$|\partial_{y_2}(K_2(\Delta X_t^\eta)\Delta\partial_{y_1} f_t^\eta)| \le C_0\frac{t}{|\Delta X_t^\eta|_*}(1 + \|\partial_{y_1}\nabla_y\eta\|_{L^\infty(\Omega_{\rho'})}). \tag{5-16}$$

It follows that the complex derivative $\partial_{y_1} F_t(\eta)$ exists and is bounded on $\Omega_{\rho'}$. Moreover, in view of Lemma 5.8, the same is true for the (real) derivative $\partial_{y_2} F_t(\eta)$.

Next we turn to the required uniform continuity of $\partial_{y_2} F_t(\eta)$ on $\Omega_{\rho'}$. First of all observe that the corresponding integrant (5-12) as a function of $(z, y) \in \Omega_0 \times \Omega_{\rho'}$ is uniformly continuous on subsets which have their z_1-component bounded away from 0. Here one uses the Cauchy integral formula and the assumed uniform continuity of $\partial_{y_2}\eta$ on the larger set Ω_ρ in order to conclude the uniform continuity of $\partial_{y_2}\partial_{y_1}\eta(y) = \partial_{y_1}\partial_{y_2}\eta(y)$. This together with the uniform integrability of the majorant given in (5-16) via Lemma 5.8 implies that $\partial_{y_2} F_t(\eta)$ is uniformly continuous on $\Omega_{\rho'}$; see also the argument below for continuity in time.

Moreover, in a similar way as above for (5-14)–(5-16), one can check that, for any $y \in \Omega_{\rho'}$, $z_1 \neq 0$,

$$|\partial_{y_1}\partial_{y_2}(K_2(\Delta X_t^\eta)\Delta\partial_{y_1} f_t^\eta)| \leq \frac{C_0 t}{|\Delta X_t^\eta|_*}(1 + \|\partial_{y_1}\nabla_y \eta\|^2_{L^\infty(\Omega_{\rho'})} + \|\partial^2_{y_1}\nabla_y \eta\|_{L^\infty(\Omega_{\rho'})}),$$

which implies that also $\partial_{y_2} F_t(\eta)$ is complex-differentiable in y_1.

In order to conclude $F_t(\eta) \in B_{\rho'}$, it therefore only remains to observe that $F_t(y) \in \mathbb{R}$ for $y \in \Omega_0$.

It remains to prove the continuity of $[0, T) \ni t \mapsto F_t(\eta) \in B_{\rho'}$. Let $t, t_0 \in (0, T)$ and take $\delta > 0$ sufficiently small. For $z \in \Omega_0$ with $|z_1| > \delta$ as well as $y \in \Omega_{\rho'}$, we have

$$|K_2(\Delta X_t^\eta)\Delta\partial_{y_1} f_t^\eta - K_2(\Delta X_{t_0}^\eta)\Delta\partial_{y_1} f_{t_0}^\eta| \leq \frac{C_0}{\delta^2}|t - t_0|$$

due to Lemmas 5.5 and 5.7. On the set $\{z \in \Omega_0 : |z_1| < \delta\}$, one uses the uniform majorant given in (5-14) to conclude the continuity of $(0, T) \ni t \mapsto F_t(\eta)$ with respect to $\|\cdot\|_{L^\infty(\Omega_{\rho'})}$.

For the corresponding continuity of $\partial_{y_1} F_t(\eta)$, $\partial_{y_2} F_t(\eta)$ with respect to $\|\cdot\|_{L^\infty(\Omega_{\rho'})}$, one uses a similar combination of Lipschitz continuity on $|z_1| > \delta$ and uniform integrability on the strip $|z_1| < \delta$ induced by (5-15), (5-16) and Lemma 5.8.

Finally, continuity at $t_0 = 0$ can be shown in the exact same way by noting that, compared to Lemma 5.8, the additional factor t in (5-16) for $\partial_{y_2} F_t(\eta)$ causes $t|\Delta X_t^\eta|_*^{-1}$ to be uniformly integrable with respect to t taken from the open interval $t \in (0, T)$. □

Remark 5.10. The continuity of F as stated in (i) of Theorem 5.1 will follow from Lemma 5.9 when combined with the Lipschitz property of Lemma 5.11 below.

5.5. *Contraction property.* Next we will verify (ii) of Theorem 5.1, with $b(t) = C_0 t^{1+\alpha}|\log t|$. Let ρ_0, R, $T > 0$ be as in Section 5.4. Recall that $R = 1$ and $T = T(R, C_0, \alpha) < 1$. Without loss of generality we also assume $\rho_0 < 1$.

Lemma 5.11. *For all* $0 < \rho' < \rho < \rho_0$, $\eta, \zeta \in B_\rho$, $\|\eta\|_\rho < 1$, $\|\zeta\|_\rho < 1$ *and* $t \in [0, T)$, *we have*

$$\|F_t(\eta) - F_t(\zeta)\|_{\rho'} \leq \frac{C_0 t^{1+\alpha}|\log t|}{\rho - \rho'}\|\eta - \zeta\|_\rho.$$

For the proof of Lemma 5.11 we first of all state some estimates implied by the lemmas in Section 5.3.

Lemma 5.12. *Let* $0 < \rho' < \rho < \rho_0$ *and* $\eta, \xi, \zeta \in B_\rho$ *with* $\|\eta\|_\rho$, $\|\xi\|_\rho$, $\|\zeta\|_\rho < 1$. *For* $y \in \Omega_{\rho'}$, $z \in \Omega_0$, $t \in [0, T)$, *we have*

$$t|\Delta\zeta(y, z)| \leq C_0\|\zeta\|_{\rho'}|\Delta X_t^\xi(y, z)|_*, \tag{5-17}$$

$$t|\Delta\partial_{y_1}\zeta(y, z)| \leq \frac{C_0}{\rho - \rho'}\|\zeta\|_\rho|\Delta X_t^\xi(y, z)|_*, \tag{5-18}$$

$$|\Delta\partial_{y_1} f_t^\eta(y, z)| \leq \frac{C_0}{\rho - \rho'}|\Delta X_t^\xi(y, z)|_*, \tag{5-19}$$

$$|\Delta\zeta(y, z)\Delta\partial_{y_1} f_t^\eta(y, z)| \leq C_0\|\zeta\|_{\rho'}|\Delta X_t^\xi(y, z)|_*, \tag{5-20}$$

$$|\Delta\zeta(y, z)\Delta\partial^2_{y_1} f_t^\eta(y, z)| \leq \frac{C_0}{\rho - \rho'}\|\zeta\|_{\rho'}|\Delta X_t^\xi(y, z)|_*. \tag{5-21}$$

Proof. By (5-9) in Lemma 5.7, one deduces

$$t|\Delta\zeta| \le \|\nabla_y \zeta\|_{L^\infty(\Omega_{\rho'})}(|z_1| + t|y_2 - z_2|) \le C_0\|\zeta\|_{\rho'}|\Delta X_t^{\xi}(y,z)|_*.$$

This shows (5-17). Inequality (5-18) is obtained in the same way by additionally applying Cauchy's Lemma 5.3.

Next, (5-19) follows from

$$|\Delta\partial_{y_1} f_t^\eta| \le C_0|z_1| + t^{1+\alpha}|\Delta\partial_{y_1}\eta|$$

and (5-18), while (5-20) is a consequence of

$$|\Delta\zeta\,\Delta\partial_{y_1} f_t^\eta| \le C_0\|\zeta\|_{L^\infty(\Omega_{\rho'})}|z_1| + t^{1+\alpha}|\Delta\zeta|\|\partial_{y_1}\eta\|_{L^\infty(\Omega_{\rho'})}$$

and (5-17).

Finally, (5-21) is achieved in the same way as (5-20) but with an additional use of Lemma 5.3. □

Proof of Lemma 5.11. Let $0 < \rho' < \rho < \rho_0$ and $\eta, \zeta \in B_\rho$ be as stated. For $\lambda \in [0,1]$, define

$$\xi_\lambda := \lambda\eta + (1-\lambda)\zeta.$$

Then $\xi_\lambda \in B_\rho$ and $\|\xi_\lambda\|_\rho < 1$.

Now for $y \in \Omega_{\rho'}$ we write

$$|F_t(\eta)(y) - F_t(\zeta)(y)| \le \int_{\Omega_0} |(K_2(\Delta X_t^\eta) - K_2(\Delta X_t^\zeta))\Delta\partial_{y_1} f_t^\eta|\,dz + \int_{\Omega_0} |K_2(\Delta X_t^\zeta)(\Delta\partial_{y_1} f_t^\eta - \Delta\partial_{y_1} f_t^\zeta)|\,dz. \quad (5\text{-}22)$$

In order to estimate the first term, we first use the fundamental theorem of calculus to write

$$\int_{\Omega_0} |K_2(\Delta X_t^\eta) - K_2(\Delta X_t^\zeta)||\Delta\partial_{y_1} f_t^\eta|\,dz = \int_{\Omega_0}\left|\int_0^1 \partial_{a_2}K_2(\Delta X_t^{\xi_\lambda})\tfrac{1}{2}t^{1+\alpha}(\Delta\eta - \Delta\zeta)\,d\lambda\right||\Delta\partial_{y_1} f_t^\eta|\,dz.$$

Now, $|\Delta\partial_{y_1} f_t^\eta|$ is dealt with by (5-19) with $\xi = \xi_\lambda$, Lemma 5.5 and its equation (5-7) are used to deal with $\partial_{a_2}K_2(\Delta X_t^{\xi_\lambda})$ and by definition of $\|\cdot\|_{\rho'}$ we arrive at the estimate

$$\begin{aligned}\int_{\Omega_0} |K_2(\Delta X_t^\eta) - K_2(\Delta X_t^\zeta)||\Delta\partial_{y_1} f_t^\eta|\,dz &= \int_{\Omega_0}\left|\int_0^1 \partial_{a_2}K_2(\Delta X_t^{\xi_\lambda})\tfrac{1}{2}t^{1+\alpha}(\Delta\eta - \Delta\zeta)\,d\lambda\right||\Delta\partial_{y_1} f_t^\eta|\,dz \\ &\le C_0 t^{1+\alpha}\|\eta-\zeta\|_{\rho'}\int_{\Omega_0}\int_0^1 \frac{1}{|\Delta X_t^{\xi_\lambda}|_*^2}\frac{|\Delta X_t^{\xi_\lambda}|_*}{\rho-\rho'}\,d\lambda\,dz \le \frac{C_0 t^{1+\alpha}|\log t|}{\rho-\rho'}\|\eta-\zeta\|_\rho,\end{aligned}$$

where the last inequality is a direct application of Lemma 5.8. Again by Lemmas 5.5 and 5.8, the second term in (5-22) is bounded by

$$\int_{\Omega_0} |K_2(\Delta X_t^\zeta)(\Delta\partial_{y_1} f_t^\eta - \Delta\partial_{y_1} f_t^\zeta)|\,dz \le C\int_{\Omega_0}\frac{1}{|\Delta X_t^\zeta|_*}t^{1+\alpha}|\Delta\partial_{y_1}\eta - \Delta\partial_{y_1}\zeta|\,dz \le C_0 t^{1+\alpha}|\log t|\|\eta-\zeta\|_\rho.$$

Thus,

$$\|F_t(\eta)-F_t(\zeta)\|_{L^\infty(\Omega_{\rho'})}\le \frac{C_0 t^{1+\alpha}|\log t|}{\rho-\rho'}\|\eta-\zeta\|_\rho.$$

Let us now turn to the corresponding inequality with ∂_{y_1}. In a similar way as before we write the decomposition

$$\partial_{y_1}F_t(\eta)(y)-\partial_{y_1}F_t(\zeta)(y)=-\int_{\Omega_0}A_1+A_2+A_3+A_4\,dz,$$

where

$$\begin{aligned}
A_1&:=(\partial_{a_2}K_2(\Delta X_t^\eta)-\partial_{a_2}K_2(\Delta X_t^\zeta))(\Delta\partial_{y_1}f_t^\eta)^2,\\
A_2&:=\partial_{a_2}K_2(\Delta X_t^\zeta)((\Delta\partial_{y_1}f_t^\eta)^2-(\Delta\partial_{y_1}f_t^\zeta)^2),\\
A_3&:=(K_2(\Delta X_t^\eta)-K_2(\Delta X_t^\zeta))\Delta\partial_{y_1}^2 f_t^\eta,\\
A_4&:=K_2(\Delta X_t^\zeta)(\Delta\partial_{y_1}^2 f_t^\eta-\Delta\partial_{y_1}^2 f_t^\zeta),
\end{aligned}$$

see (5-11). Regarding A_1, we use (5-19) and (5-20) to deduce

$$\begin{aligned}
\int_{\Omega_0}|A_1|\,dz&\le C\int_{\Omega_0}\int_0^1\frac{1}{|\Delta X_t^{\xi_\lambda}|_*^3}t^{1+\alpha}|\Delta(\eta-\zeta)\Delta\partial_{y_1}f_t^\eta||\Delta\partial_{y_1}f_t^\eta|\,dz\,d\lambda\\
&\le\frac{C_0t^{1+\alpha}}{\rho-\rho'}\|\eta-\zeta\|_{\rho'}\int_0^1\int_{\Omega_0}\frac{1}{|\Delta X_t^{\xi_\lambda}|_*}\,dz\,d\lambda\le\frac{C_0t^{1+\alpha}|\log t|}{\rho-\rho'}\|\eta-\zeta\|_\rho.
\end{aligned}$$

By making use of (5-21) instead of (5-20), one can bound $\int_{\Omega_0}|A_3|\,dz$ in a similar way. We omit the details.

Next for A_2, inequality (5-19) implies

$$\int_{\Omega_0}|A_2|\,dz\le C\int_{\Omega_0}\frac{1}{|\Delta X_t^\zeta|_*^2}|\Delta\partial_{y_1}f_t^\eta+\Delta\partial_{y_1}f_t^\zeta|t^{1+\alpha}|\Delta\partial_{y_1}\eta-\Delta\partial_{y_1}\zeta|\,dz\le\frac{C_0t^{1+\alpha}|\log t|}{\rho-\rho'}\|\eta-\zeta\|_\rho.$$

Finally, the estimate for $\int_{\Omega_0}|A_4|\,dz$ is a straightforward consequence of Lemmas 5.5, 5.7, 5.8 and Cauchy's Lemma 5.3.

Summarizing, we have shown

$$\|\partial_{y_1}F_t(\eta)-\partial_{y_1}F_t(\zeta)\|_{L^\infty(\Omega_{\rho'})}\le\frac{C_0t^{1+\alpha}|\log t|}{\rho-\rho'}\|\eta-\zeta\|_\rho.$$

It therefore remains to check ∂_{y_2}. Again we write the decomposition

$$\partial_{y_2}F_t(\eta)(y)-\partial_{y_2}F_t(\zeta)(y)=-\int_{\Omega_0}B_1+B_2+B_3+B_4\,dz,$$

where, see (5-12),

$$\begin{aligned}
B_1&:=(\partial_{a_2}K_2(\Delta X_t^\eta)-\partial_{a_2}K_2(\Delta X_t^\zeta))\big(t+\tfrac12t^{1+\alpha}\partial_{y_2}\eta(y)\big)\Delta\partial_{y_1}f_t^\eta,\\
B_2&:=\partial_{a_2}K_2(\Delta X_t^\zeta)\big[\big(t+\tfrac12t^{1+\alpha}\partial_{y_2}\eta(y)\big)\Delta\partial_{y_1}f_t^\eta-\big(t+\tfrac12t^{1+\alpha}\partial_{y_2}\zeta(y)\big)\Delta\partial_{y_1}f_t^\zeta\big],\\
B_3&:=(K_2(\Delta X_t^\eta)-K_2(\Delta X_t^\zeta))\tfrac12t^{1+\alpha}\partial_{y_2}\partial_{y_1}\eta(y),\\
B_4&:=K_2(\Delta X_t^\zeta)\tfrac12t^{1+\alpha}(\partial_{y_2}\partial_{y_1}\eta(y)-\partial_{y_2}\partial_{y_1}\zeta(y)).
\end{aligned}$$

Since $t < 1$ and $|\partial_{y_2}\eta(y)| < 1$, we get

$$\int_{\Omega_0} |B_1|\,dz \le Ct^{1+\alpha} \int_0^1\!\!\int_{\Omega_0} \frac{1}{|\Delta X_t^{\xi_\lambda}|_*^3} t|\Delta\eta - \Delta\zeta||\Delta\partial_{y_1} f_t^\eta|\,dz\,d\lambda \\ \le \frac{C_0 t^{1+\alpha}|\log t|}{\rho - \rho'} \|\eta - \zeta\|_\rho$$

by (5-17) and (5-19).

Moreover,

$$\int_{\Omega_0} |B_2|\,dz \le C \int_{\Omega_0} \frac{1}{|\Delta X_t^\zeta|_*^2} \big[t^{1+\alpha}|\partial_{y_2}\eta(y) - \partial_{y_2}\zeta(y)||\Delta\partial_{y_1} f_t^\eta| \\ + (t + t^{1+\alpha}|\partial_{y_2}\zeta(y)|)t^{1+\alpha}|\Delta\partial_{y_1}\eta - \Delta\partial_{y_1}\zeta|\big]\,dz \\ \le \frac{C_0 t^{1+\alpha}|\log t|}{\rho - \rho'} \|\eta - \zeta\|_\rho$$

by use of (5-19) in the first term as well as (5-18) in the second.

The estimate for $\int_{\Omega_0} |B_3|\,dz$ follows in analogy to $\int_{\Omega_0} |B_1|\,dz$ utilizing (5-17) and Cauchy's Lemma 5.3, whereas the estimate for $\int_{\Omega_0} |B_4|\,dz$ relies solely on Lemma 5.3.

This finishes the proof of Lemma 5.11. □

5.6. *The affine term.* In order to complete the list of ingredients of Theorem 5.1, we investigate $F_t(0)$. As usual, we consider $\rho_0 \in (0, 1)$ to be fixed according to Lemma 5.6 and $R = 1$, $T = T(R, C_0, \alpha) \in (0, 1)$ given by Lemma 5.7.

Lemma 5.13. *For any $\rho \in (0, \rho_0)$, $t \in (0, T)$, we have*

$$\|F_t(0)\|_\rho \le C_0 t|\log t|.$$

Proof. Let $y \in \Omega_\rho$. Recall that

$$\Delta X_t^0 = \begin{pmatrix} z_1 \\ \Delta\gamma_0 + t\Delta s_0 + t(y_2 - z_2) \end{pmatrix} = \Delta X_0 + \begin{pmatrix} 0 \\ t\Delta s_0 + t(y_2 - z_2) \end{pmatrix}, \quad z \in \Omega_0.$$

In view of Lemmas 5.5, 5.7, 5.8 and the boundedness of Ω_0, we have

$$|F_t(0)(y)| \le \int_{\Omega_0} |K_2(\Delta X_t^0) - K_2(\Delta X_0)||\Delta\gamma_0'| + t|K_2(\Delta X_t^0)|A|\Delta s_0'|\,dz \\ \le \int_0^1\!\!\int_{\Omega_0} |\partial_{a_2} K_2(\Delta X_{\lambda t}^0)t(y_2 - z_2 + \Delta s_0)||\Delta\gamma_0'|\,dz\,d\lambda + C_0 t \\ \le C_0 t\left(1 + \int_0^1\!\!\int_{\Omega_0} \frac{1}{|\Delta X_{\lambda t}^0|_*}\,dz\,d\lambda\right) \\ \le C_0 t\left(1 + \int_0^1 |\log(\lambda t)|\,d\lambda\right) \\ \le C_0 t|\log t|.$$

Next we estimate $\partial_{y_1} F_t(0)$. One has

$$|\partial_{y_1} F_t(0)(y)| \le \int_{\Omega_0} |\partial_{a_2} K_2(\Delta X_t^0) - \partial_{a_2} K_2(\Delta X_0)||\Delta\gamma_0'|^2 + |K_2(\Delta X_t^0)|t|\Delta s_0''| \\ + |K_2(\Delta X_t^0) - K_2(\Delta X_0)||\Delta\gamma_0''| + |\partial_{a_2} K_2(\Delta X_t^0)|(2t\Delta\gamma_0'\Delta s_0' + t^2|\Delta s_0'|^2)\,dz.$$

The terms appearing on the right-hand side can be dealt with in a similar way as above.

Finally, we also state

$$|\partial_{y_2} F_t(0)(y)| \le \int_{\Omega_0} |\partial_{a_2} K_2(\Delta X_t^0)|t|\Delta\gamma_0' + t\Delta s_0'|\,dz \le C_0 t|\log t|.$$

This finishes the proof of Lemma 5.13. □

Remark 5.14. Note that Lemma 5.13 implies

$$\frac{1}{t^{1+\alpha}} \int_0^t \|F_s(0)\|_\rho\,ds \le C_0 t^{1-\alpha}|\log t| \le \frac{C_0}{\rho_0-\rho} t^{1-\alpha}|\log t|;$$

i.e., Theorem 5.1 (iii) holds with $a(t) = t^{1+\alpha},\ c(t) = C_0 t^{1-\alpha}|\log t|$.

5.7. *Conclusion and additional remarks.* In Sections 5.1–5.6 we have verified all the conditions of Theorem 5.1. As a consequence we deduce the following statement.

Proposition 5.15. *Let $\rho_0 > 0$ be as in Lemma 5.6. There exists $\bar a = \bar a(C_0) > 0,\ T = T(C_0,\alpha) > 0$ and a unique function $t \mapsto \eta_t$ with the properties that, for every $\rho \in (0,\rho_0)$, the map*

$$I_\rho := \{t \in [0,T) : C_0 t^{1-\alpha}|\log t| < \bar a(\rho_0-\rho)\} \ni t \mapsto \eta_t \in B_\rho$$

is continuous with $\|\eta_t\|_\rho < 1,\ t \in I_\rho$, and such that, for all $y \in \Omega_\rho,\ t \in I_\rho$, we have

$$\eta_0(y) = 0, \qquad \eta_t(y) = \frac{1}{t^{1+\alpha}} \int_0^t F_s(\eta_s)(y)\,ds. \tag{5-23}$$

We finish the investigation of the fixed-point problem (4-22) with some accompanying remarks concerning properties of the solution η_t given by Proposition 5.15.

The first addresses regularity. In contrast to the analyticity of η_t in y_1, we only know that η_t is continuously differentiable in y_2. Using (5-23) it seems possible to upgrade the regularity with respect to y_2. However, since $F_s(\eta_s)(y)$ involves the integration over the finite interval $(-2,2)$ with respect to z_2, in contrast to $\mathbb{T}$ for the integration in z_1, the maximal regularity for $\eta_t(y_1,\cdot) : [-2,2] \to \mathbb{C}$ is expected to be finite. In any case, since a higher regularity of η_t with respect to y_2 would only improve the regularity of our subsolution inside the mixing zone and not across its boundary, we have not pursued this topic any further.

Next we turn to the role of the parameter α. Suppose that we set up problem (4-22) with respect to two different choices $0 < \alpha < \beta < 1$ leading to two different right-hand sides involving $F_t^\alpha(\eta),\ F_t^\beta(\eta)$. Our previous analysis gives two solutions $\eta_t^\alpha,\ \eta_t^\beta$ with corresponding intervals $I_\rho^\alpha \subset [0,T^\alpha),\ I_\rho^\beta \subset [0,T^\beta)$, $\rho \in (0,\rho_0)$. Note that the intervals $I_\rho^\alpha,\ I_\rho^\beta$ are defined with the same $\bar a$ and recall that $T^\alpha, T^\beta \in (0,1)$.

Lemma 5.16. *We have* $t^{\beta-\alpha}\eta_t^\beta = \eta_t^\alpha$ *on* $[0, \min\{T^\alpha, T^\beta\})$.

Proof. Define $J_\rho^\alpha := I_\rho^\alpha \cap [0, T^\beta)$ and $J_\rho^\beta := I_\rho^\beta \cap [0, T^\alpha)$. Then $J_\rho^\beta \subset J_\rho^\alpha$ due to the fact that $\beta > \alpha$ and $t < 1$. Both functions $t^{\beta-\alpha}\eta_t^\beta$, η_t^α are continuous maps from J_ρ^β into the unit ball of B_ρ, $\rho \in (0, \rho_0)$, and they both vanish at $t = 0$. Moreover, it is easy to check that

$$t^{\beta-\alpha}\eta_t^\beta(y) = t^{\beta-\alpha}\frac{1}{t^{1+\beta}}\int_0^t F_s^\beta(\eta_s^\beta)(y)\,ds = \frac{1}{t^{1+\alpha}}\int_0^t F_s^\alpha(s^{\beta-\alpha}\eta_s^\beta)(y)\,ds.$$

Thus Proposition 5.15 implies $t^{\beta-\alpha}\eta_t^\beta = \eta_t^\alpha$ as long as both are defined. □

Both solutions $t^{\beta-\alpha}\eta_t^\beta$, η_t^α of (5-23) then extend uniquely to a common maximal solution of (5-23) enjoying the properties of Proposition 5.15. Moreover, Lemma 5.16 shows that $t^{1+\alpha}\eta_t^\alpha$ is independent of the considered $\alpha \in (0, 1)$. Hence the induced function $f(t, y)$, defined in Section 6 below, is independent of $\alpha \in (0, 1)$.

Finally we remark that, for the choice $\alpha = 1$ in ansatz (4-20), a more careful analysis would have been required. In that case the initial value $\eta_0(y)$ is not expected to be given by 0 and the estimate given in Lemma 5.13 does not even lead to boundedness of $t^{-2}\int_0^t \|F_s(0)\|_\rho\,ds$. However, since this analysis has not been needed in order to prove existence of a Lipschitz solution of (3-1), we leave the case $\alpha = 1$ as a possible future improvement.

6. Justification of ansatzes

We will now verify that η provided by Proposition 5.15 indeed induces — when undoing the transformations stated in Section 4 — a solution of the macroscopic IPM system (3-1).

Given η from Proposition 5.15, we first of all define $f : [0, T) \times \bar{\Omega}_0 \to \mathbb{R}$,

$$f(t, y) := f_t^{\eta_t}(y) = \gamma_0(y_1) + ts_0(y_1) + \tfrac{1}{2}t^{1+\alpha}\eta_t(y),$$

where $T = T(C_0, \alpha) > 0$ can be taken as the endpoint of the interval $I_{\rho_0/2}$ for instance. Also recall Lemma 5.4 if needed for the extension to the closure of Ω_0.

Lemma 6.1. *We have* $f \in \mathcal{C}^1([0, T); \mathcal{C}^1(\bar{\Omega}_0))$, *with*

$$\|\partial_{y_2} f(t, \cdot)\|_{L^\infty(\Omega_0)} \le \tfrac{1}{2}t^{1+\alpha}, \quad t \in (0, T). \tag{6-1}$$

Moreover, the functions $f(t, \cdot, y_2)$, $\partial_t f(t, \cdot, y_2)$, $\partial_{y_2} f(t, \cdot, y_2)$, $t \in [0, T)$, $y_2 \in [-2, 2]$, *are real analytic, and* f *satisfies the initial value problem* $f(0, y) = \gamma_0(y_1)$,

$$\partial_t f(t, y) = -\frac{1}{2}\int_{\Omega_0} K_2(\Delta X_t^{\eta_t}(y, z))(\partial_{y_1} f(t, y) - \partial_{y_1} f(t, y_1 - z_1, z_2))\,dz \tag{6-2}$$

for $t \in [0, T)$, $y \in \bar{\Omega}_0$.

Proof. As a direct consequence of Lemma 5.9 and Proposition 5.15, one sees that f belongs to $\mathcal{C}^1([0, T); B_{\rho_0/2})$ and satisfies (6-1), (6-2) for $t \in [0, T)$, $y \in \Omega_{\rho_0/2}$. The statement follows from the definition of the spaces B_ρ and Lemma 5.4. □

We are now able to prove our main result.

Proof of Theorem 3.2. Let f be as in Lemma 6.1. Define the open space-time set

$$\mathscr{U} := \{(t,x) \in (0,T)\times\mathbb{T}\times\mathbb{R} : -2t + f(t,x_1,-2) < x_2 < 2t + f(t,x_1,2)\}$$

as well as the slices

$$\mathscr{U}_t := \{x \in \mathbb{T}\times\mathbb{R} : (t,x)\in\mathscr{U}\}, \quad t\in(0,T).$$

As a consequence of (6-1), the maps $X_t : \Omega_0 \to \mathscr{U}_t$,

$$X_t(y) := \begin{pmatrix} y_1 \\ ty_2 + f(t,y) \end{pmatrix}, \quad t \in (0,T),$$

are $\mathcal{C}^1$ diffeomorphisms with the property that the joint maps $(0,T)\times\Omega_0 \to \mathbb{T}\times\mathbb{R}$, $(t,y)\mapsto X_t(y)$, and $\mathscr{U}\to\mathbb{T}\times\mathbb{R}$, $(t,x)\mapsto X_t^{-1}(x)$, are also of class $\mathcal{C}^1$.

In view of (4-11), we thus can indeed define the density $\rho : [0,T)\times\mathbb{T}\times\mathbb{R}\to\mathbb{R}$ by setting

$$\rho(t,x) := \begin{cases} 1, & x_2 \ge 2t + f(t,x_1,2), \\ \frac{1}{2}(X_t^{-1}(x))_2, & x \in \mathscr{U}_t, \\ -1, & x_2 \le -2t + f(t,x_1,-2), \end{cases}$$

for $t\in(0,T)$ and $\rho(0,x) := \rho_0(x)$. Here $(X_t^{-1}(x))_2$ denotes the second component of $X_t^{-1}(x)$. Observe that ρ is continuous except at points $(0,x_1,\gamma_0(x_1))$, $x_1\in\mathbb{T}$, and piecewise $\mathcal{C}^1$ with the exceptional set being $\partial\mathscr{U}\subset[0,T)\times\mathbb{T}\times\mathbb{R}$. Moreover, as long as t is positive, $\rho(t,\cdot)$ is Lipschitz continuous and there exists a constant $C_0>0$ depending on the initial data such that

$$|\nabla\rho(t,x)| \le \frac{C_0}{t}\mathbb{1}_{\mathscr{U}_t}(x) \tag{6-3}$$

for all $(t,x)\notin\partial\mathscr{U}$.

Moreover, standard elliptic estimates show that v defined through (4-3), (4-7) and (4-8) is the unique L^2 solution of the last two equations of (3-1).

The stated log-Lipschitz continuity of $v(t,\cdot)$, $t>0$, is a consequence of the Biot–Savart operator acting on a compactly supported L^∞-vorticity; see [Marchioro and Pulvirenti 1994]. In addition, it is also easy to see that $v:[0,T)\times\mathbb{T}\times\mathbb{R}\to\mathbb{R}^2$ is continuous except at the one-dimensional set $\{(0,x_1,\gamma_0(x_1)) : x_1\in\mathbb{T}\}$.

Hence we have shown properties (i), (ii) of Theorem 3.2. Moreover, observe that property (iii) holds by construction, with γ_t given by

$$\gamma_t(x_1,h) := t2h + f(t,x_1,2h), \quad x_1\in\mathbb{T}, \quad h\in[-1,1].$$

It thus remains to show that the first equation of (3-1) and the entropy balances (3-4) are satisfied.

The regularity of ρ implies

$$\int_{\mathbb{T}\times\mathbb{R}} \rho(t,\cdot)v(t,\cdot)\cdot\nabla\varphi\,dx = -\int_{\mathbb{T}\times\mathbb{R}} v(t,\cdot)\cdot\nabla\rho(t,\cdot)\varphi\,dx$$

for all $t \in (0, T)$, $\varphi \in \mathcal{C}^\infty(\mathbb{T} \times \mathbb{R})$. It follows that (ρ, v) is a solution in the sense of Definition 3.1 if and only if the transport form

$$\partial_t \rho + v \cdot \nabla \rho + 2\rho \partial_{x_2} \rho = 0 \tag{6-4}$$

of the equation is satisfied pointwise in $(0, T) \times \mathbb{T} \times \mathbb{R} \setminus \partial \mathscr{U}$.

At points $(t, x) \notin \overline{\mathscr{U}}$, equation (6-4) trivially holds. Inside $\mathscr{U}$ one can check that the computations in Section 4.3 are possible showing that (6-4) is equivalent to (4-15). Note that in Section 4.3 we have formally assumed that the X_t are global diffeomorphisms mapping $\mathbb{T} \times \mathbb{R}$ to itself, but as the reader can easily see, it is enough to have transformations from Ω_0 to the corresponding $\mathscr{U}_t$.

Observing also that the computations in Section 4.4 are legal in our scenario, one sees that (6-4) on $\mathscr{U}$ is indeed equivalent to (6-2).

Finally, let $\eta : \mathbb{R} \to \mathbb{R}$ be an arbitrary Lipschitz continuous function, and define the function $Q : \mathbb{R} \to \mathbb{R}$,

$$Q(u) := \int_0^u 2\eta'(s) s \, ds,$$

which is also Lipschitz continuous when restricted to any compact interval of $\mathbb{R}$. Consequently we have enough regularity to deduce (3-4) by multiplying (6-4) with $\eta'(\rho(t, x))$ and applying the chain rule. This finishes the proof of Theorem 3.2. □

Appendix A: The abstract Cauchy–Kovalevskaya theorem

Proof of Theorem 5.1. As indicated in Section 5, the proof of Theorem 5.1 is a slight modification of the original proof in [Nishida 1977].

Let $a_0 > 0$ and set $a_{k+1} := a_k(1 - (k+2)^{-2})$, $k = 0, 1, \dots$. Then

$$a := \lim_{k \to \infty} a_k > 0.$$

For $\rho \in (0, \rho_0)$ and $k = 0, 1, \dots,$ we define the intervals

$$I_{k,\rho} := \{t \in [0, T) : c(t) < a_k(\rho_0 - \rho)\}.$$

We also define for a function u with $u : I_{k,\rho} \to B_\rho$ continuous for any $\rho \in (0, \rho_0)$ the norm

$$M_k[u] := \sup\left\{ \|u(t)\|_\rho \left(\frac{a_k(\rho_0 - \rho)}{c(t)} - 1 \right) : \rho \in (0, \rho_0),\, t \in I_{k,\rho} \right\}.$$

Note that, for $c(t) = t$, one recovers Nishida's setup. Now one recursively constructs the sequence

$$u_0(t) := 0, \qquad u_{k+1}(t) := \begin{cases} \frac{1}{a(t)} \int_0^t F(u_k(s), s) \, ds, & t \in (0, T), \\ 0, & t = 0. \end{cases}$$

We claim that, for a_0 chosen sufficiently small, the recursion is well-defined, that each $u_k : I_{k,\rho} \to B_\rho$ is continuous with $\|u_k(t)\|_\rho < \frac{1}{2} R$ for $t \in I_{k,\rho}$, $\rho \in (0, \rho_0)$, and that

$$\lambda_{k-1} := M_k[u_k - u_{k-1}] \le (4K a_0)^{k-1} a_0, \tag{A-1}$$

where $K > 0$ is the constant appearing in (5-2).

We first of all note that $u_1(t)$ exists and satisfies the stated continuity condition due to assumptions (i) and (iii). Moreover, for $t \in I_{0,\rho}$, we have $\|u_1(t)\|_\rho < a_0$. Thus, we pick a_0 at least as small as $\frac{1}{2}R$. One also easily checks that $\lambda_0 \le a_0$.

From now on we proceed by induction. Assume that the recursion with the above stated properties is possible up to some $k \ge 1$. Then it is also clear that $u_{k+1} : I_{k+1,\rho} \to B_\rho$ is well-defined as well as continuous on the open interval $I_{k+1,\rho} \setminus \{0\}$ for any $\rho \in (0, \rho_0)$.

If we assume for now that (A-1) also holds for λ_k, then, for $t \in I_{k+1,\rho}$, we obtain in analogy to [Nishida 1977] the estimate

$$\begin{aligned}\|u_{k+1}(t)\|_\rho \le \sum_{j=0}^{k} \lambda_j \left(\frac{a_j(\rho_0-\rho)}{c(t)} - 1 \right)^{-1} &\le \sum_{j=0}^{k} \lambda_j \left(\frac{a_j}{a_{j+1}} - 1 \right)^{-1} \\ &\le a_0 \sum_{j=0}^{k} (4Ka_0)^j (j+2)^2 < \tfrac{1}{2}R\end{aligned}$$

by choice of a_0 independent of k. Moreover, the first inequality in the above line of estimates applied at times $t > 0$ with $c(t) < \frac{1}{2}a(\rho_0-\rho)$ also gives

$$\|u_{k+1}(t)\|_\rho \le c(t) \sum_{j=0}^{k} \frac{\lambda_j}{a_j(\rho_0-\rho)-c(t)} \le \frac{2a_0 c(t)}{a(\rho_0-\rho)} \sum_{j=0}^{k} (4Ka_0)^j,$$

which shows that u_{k+1} is also continuous with respect to $\|\cdot\|_\rho$ at $t = 0$.

To finish the induction it thus remains to show (A-1) for λ_k. The clever move is to use the contraction property with a different Banach space at each time τ inside the integral. Namely, exactly as in [Nishida 1977, p. 631], the contraction property of F (Theorem 5.1 (ii)) with

$$\rho(\tau) := \frac{1}{2}\left(\rho_0 - \frac{c(\tau)}{a_k} + \rho \right)$$

and the definition of λ_{k-1} lead to

$$\|u_{k+1}(t) - u_k(t)\|_\rho \le \frac{4\lambda_{k-1}a_k}{a(t)} \int_0^t \frac{b(\tau)c(\tau)}{(a_k(\rho_0-\rho)-c(\tau))^2}\, d\tau$$

for $t \in I_{k,\rho}$. At this point we use (5-2) and a change of variables to obtain

$$\|u_{k+1}(t) - u_k(t)\|_\rho \le 4\lambda_{k-1}a_k K c(t) \int_0^{c(t)} \frac{1}{(a_k(\rho_0-\rho)-\xi)^2}\, d\xi$$

from where one can conclude $\lambda_k \le 4K\lambda_{k-1}a_0$ by following [Nishida 1977] again.

Now Theorem 5.1 follows as in [Nirenberg 1972; Nishida 1977]. □

Appendix B: More on Otto's relaxation

We here add some more details regarding the fifth step of Otto's relaxation [1999] in the general nonflat case, which has only been sketched in Section 2.5.1.

Before doing that we will quickly convince ourselves that the setting in [Otto 1999] is indeed equivalent to the formulation of IPM considered in our paper. Otto considers the equations

$$\begin{aligned} \partial_t s + u \cdot \nabla s &= 0, \\ \nabla \cdot u &= 0, \\ u &= -\nabla p + s e_2, \end{aligned} \tag{B-1}$$

which correspond with [Otto 1999, (1.1)–(1.2)] and the first equation on page 875 of [Otto 1999] with $\lambda = 1$. The parameter λ in that paper is the quotient of the mobilities. In our case, we have taken both mobilities equal to one and then $\lambda = 1$. More importantly, in that paper,

$$s = \{0, 1\}, \tag{B-2}$$

however

$$\rho = \{-1, 1\} \tag{B-3}$$

in our case.

Let us see how we can go from (1-1), (B-3) to (B-1)–(B-2). Firstly we define

$$\bar{s} = \tfrac{1}{2}(1 - \rho), \qquad \rho = 1 - 2\bar{s},$$

and thus

$$\begin{aligned} \partial_t \bar{s} + v \cdot \nabla \bar{s} &= 0, \\ \nabla \cdot v &= 0, \\ v &= -\nabla(p + x_2) + 2\bar{s} e_2, \end{aligned}$$

with

$$\bar{s} = \{0, 1\}.$$

We define $\bar{u} = \frac{1}{2} v$ and $\overline{\Pi} = \frac{1}{2}(p + x_2)$, which yields

$$\begin{aligned} \partial_t \bar{s} + 2\bar{u} \cdot \nabla \bar{s} &= 0, \\ \nabla \cdot \bar{u} &= 0, \\ \bar{u} &= -\nabla \overline{\Pi} + \bar{s} e_2. \end{aligned}$$

Finally we take $s(x, t) = \bar{s}\big(x, \frac{1}{2}t\big)$, $u\big(x, \frac{1}{2}t\big) = \bar{u}\big(x, \frac{1}{2}t\big)$ and $\Pi(x, t) = \overline{\Pi}\big(x, \frac{1}{2}t\big)$; thus

$$\begin{aligned} \partial_t s + u \cdot \nabla s &= 0, \\ \nabla \cdot u &= 0, \\ u &= -\nabla \Pi + s e_2, \end{aligned}$$

with $s = \{0, 1\}$, which agrees with (B-1)–(B-2) (up to a relabeling of the pressure). Therefore, if we show that (B-1)–(B-2) relaxes to

$$\begin{aligned} \partial_t s + u \cdot \nabla s + \partial_{x_2} s - 2s\partial_{x_2} s &= 0, \\ \nabla \cdot u &= 0, \\ u &= -\nabla \Pi + s e_2, \end{aligned} \tag{B-4}$$

with $s \in [0,1]$, by undoing the previous transformations, we see that (1-1), (B-3) relaxes to

$$\begin{aligned} \partial_t \rho + v \cdot \nabla \rho + 2\rho \partial_{x_2} \rho &= 0, \\ \nabla \cdot v &= 0, \\ v &= -\nabla p - \rho e_2, \end{aligned} \tag{B-5}$$

with $\rho \in [-1,1]$.

Next, we begin our formal discussion with the outcome of the fourth step of Otto, after which there exists for each $h > 0$ a sequence of "coarse-grained" functions $\{\theta^k\}_{k=0}^{N(h)}$ that are characterized by the following JKO scheme (which we understand as a minimizing movements scheme with respect to the Wasserstein distance):

$\theta^{(k+1)}$ is the minimizer in K of

$$\tfrac{1}{2}\,\mathrm{dist}^2(\theta^{(k)}, \theta) + \tfrac{1}{2}\,\mathrm{dist}^2(1-\theta^{(k)}, 1-\theta) - h \int \theta(x) x_2, \tag{B-6}$$

where the set K consists of measurable θ taking values in $[0,1]$ and such that $\int \theta = \int s(x,0)$, and $\mathrm{dist}^2(\theta_0, \theta_1)$ is the L^2-Wasserstein distance

$$\mathrm{dist}^2(\theta_0, \theta_1) = \inf_{\Phi \in I(\theta_0,\theta_1)} \int \theta_0(x) |\Phi(x) - x|^2 \, dx,$$

with

$$I(\theta_0, \theta_1) = \left\{ \Phi : \int \theta_1(y) \zeta(y)\, dy = \int \theta_0(x) \zeta(\Phi(x))\, dx \ \ \forall \zeta \in \mathcal{C}_0^0 \right\}.$$

In the definition of $I(\theta_0, \theta_1)$, we have been deliberately imprecise and defer the reader to [Otto 1999] for the proper definition. Even more, in order to make the exposition clearer, in the following we will assume that the minimizer exists, that it is smooth and that it satisfies pointwise the corresponding Monge–Ampere equation; i.e.,

$$I(\theta_0, \theta_1) = \{\Phi \text{ diffeomorphism} : (\theta_1 \circ \Phi)(x) J_\Phi(x) = \theta_0(x)\}.$$

Here J_Φ denotes the Jacobian determinant $\det D\Phi$.

As explained in Section 2.5.1, our goal is to show, on a formal level, that the limit as $h \to 0$ — we will assume that it exists in the first place — of the functions

$$\theta_h(x,t) := \theta^{(k)}(x), \quad t \in [kh, (k+1)h)),$$

is characterized by system (B-4).

We begin with the Euler–Lagrange equation of (B-6). For a given $\theta_0 \in K$, let θ_1 be the minimizer in K of

$$F[\theta] \equiv \tfrac{1}{2}\,\mathrm{dist}^2(\theta_0, \theta) + \tfrac{1}{2}\,\mathrm{dist}^2(1-\theta_0, 1-\theta) - h \int \theta(x) x_2.$$

Then we have that

$$D_\theta F[\theta_1]\psi = \frac{d}{d\tau} F[\theta_1 + \tau\psi]\Big|_{\tau=0} = 0,$$

where we simply assume that $\theta_1+\tau\psi\in K$; i.e., we in particular consider ψ with $\int\psi=0$. In order to compute $D_\theta F[\theta_1]\psi$, we first look at $D_\theta \operatorname{dist}^2(\theta_0,\theta_1)\psi$. Let $\Phi_0^\tau\in I(\theta_0,\theta_1+\tau\psi)$ be such that

$$\operatorname{dist}^2(\theta_0,\theta_1+\tau\psi)=\inf_{\Phi\in I(\theta_0,\theta_1+\tau\psi)}\int\theta_0(x)|\Phi(x)-x|^2\,dx=\int\theta_0(x)|\Phi_0^\tau(x)-x|^2\,dx.$$

We define

$$w\circ\Phi_0^0=\frac{d\Phi_0^\tau}{d\tau}\bigg|_{\tau=0},$$

and thus

$$\tfrac{1}{2}D_\theta\operatorname{dist}^2(\theta_0,\theta_1)\psi=\int\theta_0(x)(\Phi_0^0(x)-x)\cdot(w\circ\Phi_0^0)(x)\,dx. \tag{B-7}$$

We next compute for which w we have $\Phi_0^\tau\in I(\theta_0,\theta_1+\tau\psi)$. We have

$$J_{\Phi_0^\tau}(x)((\theta_1+\tau\psi)\circ\Phi_0^\tau)(x)=\theta_0(x). \tag{B-8}$$

Taking a τ-derivative in (B-8) and evaluating at $\tau=0$ yields

$$J_{\Phi_0^0}\operatorname{div}w\circ\Phi_0^0\theta_1\circ\Phi_0^0+J_{\Phi_0^0}w\circ\Phi_0^0\cdot\nabla\theta_1\circ\Phi_0^0+J_{\Phi_0^0}\psi\circ\Phi_0^0=0,$$

which reduces to

$$\operatorname{div}(w\theta_1)+\psi=0. \tag{B-9}$$

In addition, Φ_0^0 minimizes

$$\int\theta_0(x)|\Phi(x)-x|^2\,dx$$

in $I(\theta_0,\theta_1)$. So, for every family of flows $(\Phi_\delta^0)\in I(\theta_0,\theta_1)$, we have that

$$\frac{d}{d\delta}\int\theta_0(x)|\Phi_\delta^0(x)-x|^2\,dx\bigg|_{\delta=0}=0.$$

That is,

$$\int\theta_0(x)(\Phi_0^0(x)-x)\cdot(\overline{w}\circ\Phi_0^0)(x)\,dx=0,$$

where if Φ_δ is the flow of a vector field $\overline{w}$,

$$\overline{w}\circ\Phi_0^0=\frac{d\Phi_\delta^0}{d\delta}\bigg|_{\delta=0}, \tag{B-10}$$

$$\operatorname{div}(\theta_1\overline{w})=0. \tag{B-11}$$

The condition (B-11), equivalent to $\Phi_\delta\in I(\theta_0,\theta_1)$, is deduced by differentiating

$$J_{\Phi_\delta^0}(x)(\theta_1\circ\Phi_\delta^0)(x)=\theta_0(x) \tag{B-12}$$

with respect to δ.

Therefore, we have

$$0=\int\theta_0(x)(\Phi_0^0(x)-x)\cdot(\overline{w}\circ\Phi_0^0)(x)\,dx=\int\theta_1(x)\overline{w}(x)\cdot(x-(\Phi_0^0)^{-1}(x))\,dx,$$

where in the last equality we have used the definition of $I(\theta_0, \theta_1)$. Since $\overline{w}$ is an arbitrary vector field, Hodge decomposition implies that

$$x - (\Phi_0^0)^{-1}(x) = \nabla a(x) \tag{B-13}$$

for some function a. In order to avoid technicalities, we here have implicitly assumed that θ_1 does not vanish.

From (B-7), (B-9) and (B-13), we see that

$$\tfrac{1}{2} D_\theta \operatorname{dist}^2(\theta_0, \theta_1)\psi = \int \theta_1(x) w(x) \cdot \nabla a(x)\, dx = -\int \nabla \cdot (\theta_1 w)(x) a(x)\, dx = \int \psi(x) a(x).$$

We have obtained that

$$\tfrac{1}{2} D_\theta \operatorname{dist}^2(\theta_0, \theta_1)\psi = \int \psi(x) a(x) \Phi_0^0(x) = x + (\nabla a \circ \Phi_0^0)(x).$$

Similar computations yield

$$\tfrac{1}{2} D_\theta \operatorname{dist}^2(1-\theta_0, 1-\theta_1)\psi = -\int \psi(x) \bar{a}(x) \overline{\Phi}_0^0(x) = x + (\nabla \bar{a} \circ \overline{\Phi}_0^0)(x),$$

and putting everything together we arrive at

$$D_\theta F[\theta_1]\psi = \int (a(x) - \bar{a}(x) - h x_2)\psi(x)\, dx = 0 \tag{B-14}$$

for all ψ with $\int \psi = 0$. Moreover, since $\Phi_0^0 \in I(\theta_0, \theta_1)$ and $\overline{\Phi}_0^0 \in I(1-\theta_0, 1-\theta_1)$,

$$\theta_1(x) = J_{(\Phi_0^0)^{-1}}(x)\theta_0(x - \nabla a(x)), \tag{B-15}$$

$$(1-\theta_1)(x) = J_{(\overline{\Phi}_0^0)^{-1}}(x)(1-\theta_0)(x - \nabla \bar{a}(x)). \tag{B-16}$$

Note that so far we have omitted the h-dependence of the functions a, $\bar{a}$, Φ_0^0, $\overline{\Phi}_0^0$ in our notation. We continue doing so when introducing $p = a/h$, $\bar{p} = \bar{a}/h$ which, up to a constant, satisfy

$$p - \bar{p} = x_2$$

by (B-14). Note that the constant is irrelevant since only derivatives of p and $\bar{p}$ will play a role. To obtain a formal limit as $h \to 0$, we will assume in the following that the h-dependent functions p and $\bar{p}$ have a well-defined $\mathcal{C}^2$ limit, which will again be denoted by p and $\bar{p}$.

Now we take said limit. On one hand we have from (B-15) that

$$\begin{aligned}\frac{\theta_1(x) - \theta_0(x)}{h} &= \frac{J_{(\Phi_0^0)^{-1}}(x)\theta_0(x - h\nabla p(x)) - \theta_0(x)}{h} \\ &= \frac{(J_{(\Phi_0^0)^{-1}}(x) - 1)\theta_0(x - h\nabla p(x)) + \theta_0(x - h\nabla p(x)) - \theta_0(x)}{h}.\end{aligned}$$

Recall that Φ_0^0 is linked to p via (B-13), and thus, since $\Phi_0^0(x) \to x$, we have

$$J_{(\Phi_0^0)^{-1}}(x) - 1 = -h\Delta p(x)$$

at first order in h. Thus, when letting $h \to 0$ in the difference quotient, we arrive at

$$\partial_t \theta = -\Delta p \theta - \nabla p \cdot \nabla \theta. \tag{B-17}$$

On the other hand, we have from (B-16) that

$$\frac{\theta_1(x) - \theta_0(x)}{h} = \frac{1 - J_{(\bar{\Phi}_0^0)^{-1}}(x) + J_{(\bar{\Phi}_0^0)^{-1}}(x)\theta_0(x - h\nabla \bar{p}(x)) - \theta_0(x)}{h}.$$

Passing to the limit yields

$$\partial_t \theta = \Delta \bar{p} - \Delta \bar{p}\theta - \nabla \theta \cdot \nabla \bar{p}. \tag{B-18}$$

In order for (B-17) and (B-18) to agree, we have

$$\Delta \bar{p} - \nabla \cdot (\nabla \bar{p} \theta) = -\nabla \cdot (\nabla p \theta),$$

and since $p = \bar{p} + x_2$ we have

$$\Delta \bar{p} = -\nabla \cdot (\nabla x_2 \theta) = -\partial_{x_2} \theta. \tag{B-19}$$

Therefore, from (B-18) and (B-19),

$$\partial_t \theta = -\partial_{x_2}\theta - \nabla \cdot (\nabla \bar{p}\theta) = -\partial_{x_2}\theta - \nabla \cdot ((\nabla \bar{p} + \theta e_2)\theta) + \nabla \cdot (\theta^2 e_2).$$

To finish we define $u = \nabla \bar{p} + \theta e_2$, which clearly satisfies $\nabla \cdot u = 0$, to get

$$\begin{aligned} \partial_t \theta + u \cdot \nabla \theta + \partial_{x_2}\theta - 2\theta \partial_{x_2}\theta &= 0, \\ u &= \nabla \bar{p} + \theta e_2, \\ \nabla \cdot u &= 0, \end{aligned}$$

which agrees with (B-4).

Appendix C: Rigorous energy dissipation

In Section 2.5.2 equation (2-15), we have formally computed the decay rate of the total potential energy. For completeness we give sufficient conditions when this computation is justified. Also for completeness, we show that the subsolution given by Theorem 3.2 indeed satisfies the sufficient conditions.

Lemma C.1. *Let $\rho_0 \in L^1_{\mathrm{loc}}(\mathbb{T} \times \mathbb{R})$ be some initial data, and further suppose that the pair of functions $(\rho, m) \in L^1_{\mathrm{loc}}((0, T) \times \mathbb{T} \times \mathbb{R}; \mathbb{R} \times \mathbb{R}^2)$ satisfies*

$$\partial_t \rho + \operatorname{div} m = 0, \qquad \rho(0, \cdot\,) = \rho_0$$

on $(0, T) \times \mathbb{T} \times \mathbb{R}$ in the sense of distributions. If there exists $\alpha > 0$ such that

$$m_2,\ (\rho - \rho_0)x_2,\ (\rho - \rho_0)|x_2|^{1+\alpha} \in \mathcal{C}^0([0, T); L^1(\mathbb{T} \times \mathbb{R})), \qquad m_2|x_2|^\alpha \in L^1((0, T) \times \mathbb{T} \times \mathbb{R}),$$

then the relative potential energy defined in (2-14) belongs to $\mathcal{C}^1([0, T))$, and we have

$$\frac{d}{dt} E_{\mathrm{rel}}(t) = \int_{\mathbb{T} \times \mathbb{R}} m_2(t, x)\, dx.$$

Proof. Let $R > 0$ and $\varphi_R : \mathbb{R} \to [0, 1]$ be a cutoff function with $\varphi_R(x_2) = 1$ for $|x_2| \leq R$, $\varphi_R(x_2) = 0$ for $|x_2| \geq 2R$ and $|\varphi_R'(x_2)| \leq 2R^{-1}$, $x_2 \in \mathbb{R}$.

We use the abbreviation $E(t) = E_{\text{rel}}(t)$ and define

$$E_R(t) := \int_{\mathbb{T}\times\mathbb{R}} (\rho(t, x) - \rho_0(x)) x_2 \varphi_R(x_2)\, dx.$$

Note that $E(t)$, $E_R(t)$ are well-defined at every time $t \in [0, T)$, and we have

$$|E(t) - E_R(t)| \leq \|(\rho(t, \cdot) - \rho_0)|x_2|^{1+\alpha}\|_{L^1(\mathbb{T}\times\mathbb{R})} \frac{1}{R^\alpha} = O(R^{-\alpha})$$

uniformly in time as $R \to \infty$. Thus

$$h^{-1}(E(t+h) - E(t)) = h^{-1}(E_R(t+h) - E_R(t)) + h^{-1} O(R^{-\alpha}).$$

Moreover, the assumed continuity conditions and approximation of the indicator function of $[t, t+h]$ imply

$$\begin{aligned} E_R(t+h) - E_R(t) &= \int_t^{t+h} \int_{\mathbb{T}\times\mathbb{R}} m \cdot \nabla(x_2 \varphi_R(x_2))\, dx\, ds \\ &= \int_t^{t+h} \int_{\mathbb{T}\times\mathbb{R}} m_2\, dx\, ds + \int_t^{t+h} \int_{\mathbb{T}\times\mathbb{R}} m_2(\varphi_R(x_2) - 1 + x_2 \varphi_R'(x_2))\, dx\, ds. \end{aligned}$$

Now the latter term can be bounded by $5R^{-\alpha}\|m_2|x_2|^\alpha\|_{L^1((0,T)\times\mathbb{T}\times\mathbb{R})}$, implying that

$$h^{-1}(E(t+h) - E(t)) = h^{-1} \int_t^{t+h} \int_{\mathbb{T}\times\mathbb{R}} m_2\, dx\, ds + h^{-1} O(R^{-\alpha}).$$

The statement follows. □

Let (ρ, v) be the solution constructed in Theorem 3.2 and set

$$m = \rho v - (1 - \rho^2) e_2.$$

Lemma C.2. *In addition to the properties stated in Theorem 3.2, the velocity field v satisfies*

$$|v(t, x)| \leq C e^{-|x_2|}$$

whenever $|x_2| \geq R$ for constants $C, R > 0$ independent of t. The pair (ρ, m) in particular satisfies the conditions of Lemma C.1.

Proof. Regarding the second component, one easily sees that

$$|v_2(t, x)| \leq |\mathscr{U}_t| \|\partial_{x_1} \rho(t, \cdot)\|_{L^\infty(\mathbb{T}\times\mathbb{R})} \|K_2(x - \cdot)\|_{L^\infty(\mathscr{U}_t)},$$

which can be bounded by $Ce^{-|x_2|}$ for $|x_2| \geq R$ with constants $C, R > 0$ independent of time.

Regarding the first component, we cannot exploit the decay of the kernel, since $K_1(z) \to \mp 1$ as $z_2 \to \pm\infty$. Still by subtracting vanishing horizontal averages, we deduce

$$\begin{aligned} |v_1(t,x)| &= \left| \int_{\mathbb{T}\times\mathbb{R}} \partial_{x_1}\rho(t,y)(K_1(x-y) - K_1(x-(0,y_2))\,dy \right| \\ &\le |\mathscr{U}_t| \|\partial_{x_1}\rho(t,\cdot)\|_{L^\infty(\mathbb{T}\times\mathbb{R})} \|\partial_{z_1}K_1(x-\cdot)\|_{L^\infty(\mathscr{U}_t^*)}\pi, \end{aligned}$$

where $\mathscr{U}_t^*$ is the set of points obtained by taking all segments between $y \in \mathscr{U}_t$ and $(0, y_2)$. It is only important that those sets are bounded uniformly in time, which allows us to argue as above for v_2, since $\partial_{z_1}K_1$ now has the required decay. □

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UNIFORM BOUNDS FOR BILINEAR SYMBOLS WITH LINEAR K-QUASICONFORMALLY EMBEDDED SINGULARITY

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We prove bounds in the strict local $L^2(\mathbb{R}^d)$ range for trilinear Fourier multiplier forms with a d-dimensional singular subspace. Given a fixed parameter $K \geq 1$, we treat multipliers with nondegenerate singularity that are push-forwards by K-quasiconformal matrices of suitable symbols. As particular applications, our result recovers the uniform bounds for the one-dimensional bilinear Hilbert transforms in the strict local L^2 range, and it implies the uniform bounds for two-dimensional bilinear Beurling transforms, which are new, in the same range.

1. Introduction

Let $d \geq 1$, and let Γ_0 be the linear subspace of $\mathbb{R}^{3\times d}$ consisting of all vectors (ξ_1, ξ_2, ξ_3) with $\xi_1+\xi_2+\xi_3=0$. Trilinear Fourier multiplier forms on Γ_0 are studied in order to understand mapping properties of bilinear Fourier multiplier operators on $\mathbb{R}^d$. In the present paper, we prove bounds in the strict local L^2 range for multipliers whose singular set can be written as an image of the d-dimensional diagonal of $\mathbb{R}^{3\times d}$ under a block K-quasiconformal matrix. Our bounds depend on the matrix through the parameter K alone; in this sense we prove bounds uniform in isotropic dilations and rotations. We comment more on the motivation for such bounds after stating the main result.

We normalize the Fourier transform of a Schwartz function as

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i x\cdot\xi}\,\mathrm{d}x.$$

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Let $1 < p < \infty$. We denote the L^p norm of a measurable function by

$$\|f\|_p^p := \int_{\mathbb{R}^d} |f(x)|^p \,\mathrm{d}x.$$

Let $K \geq 1$. A linear map

$$L = L_1 \oplus L_2 \oplus L_3$$

mapping $\mathbb{R}^{3\times d}$ to itself is said to be block K-quasiconformal if, for all $n \in \{1, 2, 3\}$, we have $L_n : \mathbb{R}^d \to \mathbb{R}^d$ and

$$\|L_n\|_{\mathrm{op}}^d \leq K \det L_n.$$

We say that L is nontrivial if additionally

$$L_1 + L_2 + L_3 = 0.$$

Theorem 1.1. *Let $d \geq 1$, $K \geq 1$ and*

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1, \quad 2 < p_1, p_2, p_3 < \infty.$$

There exists a constant $C = C(d, K, p_1, p_2, p_3)$ such that the following holds.

Let $m : \mathbb{R}^{3\times d} \to \mathbb{C}$ satisfy

$$|\partial_1^{\gamma_1} \partial_2^{\gamma_2} \partial_3^{\gamma_3} m(\xi)| \leq \sup\{|\xi - (\tau, \tau, \tau)|^{-|\gamma|} : \tau \in \mathbb{R}^d\} \tag{1-1}$$

for all $\gamma \in \mathbb{N}^{3\times d}$ with $|\gamma| \leq 100d$. Let L be a nontrivial block K-quasiconformal matrix. Define

$$\Lambda_m(f_1, f_2, f_3) = \int_{\mathbb{R}^{3\times d}} \delta_0(\xi_1 + \xi_2 + \xi_3) \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) \hat{f}_3(\xi_3) m(L^{-1}\xi) \,\mathrm{d}\xi,$$

where δ_0 is the Dirac mass at the origin.

Then, for all triples of Schwartz functions f_1, f_2 and f_3 on $\mathbb{R}^d$,

$$|\Lambda_m(f_1, f_2, f_3)| \leq C \prod_{n=1}^{3} \|f_n\|_{p_n}.$$

We use a symbol m defined on all of $\mathbb{R}^{3\times d}$ for convenience, but instead of that, a symbol only defined on Γ_0 with conditions stated using directional differential operators within the space Γ_0 could be used as well. Similarly, the use of the mapping L in the definition of the form is a compact way to express a set of certain anisotropic symbol estimates on m through the simple condition (1-1). We point out that the restriction of Theorem 1.1 to the strict local L^2 range is likely not to be sharp. Moreover, we do not see any obvious obstruction for an analogy of our result for higher orders of multilinearity. The only missing ingredient for the latter seems to be a suitable generalization of the uniform paraproduct estimate as in [Muscalu et al. 2002b]. However, we did not attempt any of these extensions in order to keep the technicalities in this paper more limited and have better focus on some of the key ideas of our approach. For related work in $d = 1$ extending the range of exponents of the bilinear Hilbert transform, see [Di Plinio and Thiele 2016; Li 2006; Oberlin and Thiele 2011; Thiele 2002; Uraltsev and Warchalski 2022].

The simplest interesting special case of Theorem 1.1 is $d = K = 1$, when $L = (L_1, L_2, L_3)$ is a vector of nonzero real numbers adding up to 0 and

$$m(\xi_1, \xi_2, \xi_3) = \frac{L_1\xi_1 + L_2\xi_2 + L_3\xi_3}{\sqrt{(L_1\xi_1 + L_2\xi_2 + L_3\xi_3)^2 + (\xi_1 + \xi_2 + \xi_3)^2}},$$

which restricted to the hyperplane $\xi_1 + \xi_2 + \xi_3 = 0$ reads as

$$m(\xi_1, \xi_2, \xi_3) = \operatorname{sgn}(L_1\xi_1 + L_2\xi_2 + L_3\xi_3).$$

In this case, Λ_m is a scalar multiple of the trilinear form dual to the bilinear Hilbert transform, which can be written on the spatial side as

$$\text{p.v.} \iint_{\mathbb{R}^2} f_1(x + M_1 t) f_2(x + M_2 t) f_3(x + M_3 t) \frac{\mathrm{d}x\, \mathrm{d}t}{t}, \tag{1-2}$$

where $M = (M_1, M_2, M_3)$ is a unit vector perpendicular to both $(1, 1, 1)$ and L. No two components of M are equal, because no component of L is zero. This condition is referred to as nondegeneracy of M. The case of (1-2) with two components of the unit vector M equal is called degenerate. If for example $M_3 = M_1$, we have

$$\Lambda_m(f_1, f_2, f_3) = \int_{\mathbb{R}} f_1(x) f_3(x) \left[\text{p.v.} \int_{\mathbb{R}} f_2(x + t) \frac{\mathrm{d}t}{t}\right] \mathrm{d}x.$$

One obtains L^p bounds for this form by Hölder's inequality and bounds for the linear Hilbert transform. Bounds for the nondegenerate case of the bilinear Hilbert transform require a different argument and were shown in the exponent range of Theorem 1.1 in [Lacey and Thiele 1997], albeit with constants blowing up as M tends to a degenerate value. Bounds uniform in M were later proven in [Grafakos and Li 2004] for the first time. These results are covered by Theorem 1.1.

The simplest example of our main theorem which is new is the case where $d = 2$, $K = 1$ and (L_1, L_2, L_3) is a triple of conformal matrices adding up to zero. In this case, we identify $\mathbb{R}^2$ with $\mathbb{C}$ and view the application of the matrices L_n as multiplication by complex numbers. Moreover, we set

$$m(\zeta_1, \zeta_2, \zeta_3) = \frac{(\overline{L_1\zeta_1 + L_2\zeta_2 + L_3\zeta_3})^2}{|L_1\zeta_1 + L_2\zeta_2 + L_3\zeta_3|^2 + |\zeta_1 + \zeta_2 + \zeta_3|^2}.$$

Similar computations as for the bilinear Hilbert transform identify Λ_m as a scalar multiple of what one might call the bilinear Beurling transform

$$\text{p.v.} \iint_{\mathbb{C}^2} f_1(z + M_1\zeta) f_2(z + M_2\zeta) f_3(z + M_3\zeta) \frac{\mathrm{d}A(z)\, \mathrm{d}A(\zeta)}{\zeta^2},$$

where A denotes the area measure. Thus our main theorem implies L^p bounds in the strictly locally L^2 range for the bilinear Beurling transform uniformly in M. The Beurling kernel ζ^{-2} can be replaced by any standard Calderón–Zygmund kernel arising from a Mikhlin multiplier.

In dimension $d = 1$, the cases for L allowed in Theorem 1.1 together with a small number of easily understood degenerate cases provide an exhaustive picture of all cases of L. The situation in higher dimensions is more complicated. There are completely nondegenerate cases, completely degenerate cases in the sense that $L_n = 0$ for some n, and further there is a zoo of distinct cases that one may call partially degenerate. For fixed K, our main theorem proves uniform bounds for the nondegenerate cases as one approaches the completely degenerate cases inside a cone that stays away from the partially degenerate cases. Within the conformal context, our theorem covers all cases including the degenerate ones. In this respect, we show that the setting of one complex dimension is quite analogous to the setting of one real dimension.

Concerning the general case, a list, not exhaustive, of five partially degenerate cases for $d = 2$ was described in [Demeter and Thiele 2010], and four of the cases were shown to be bounded, albeit without any attempt to prove uniform bounds. The remaining case, called the twisted paraproduct, was later treated in [Kovač 2012] (see also [Bernicot 2012] for preliminary results and [Durcik 2015; 2017] for further work). A further partially degenerate case is the triangular Hilbert transform described in [Kovač et al. 2015], where one dimension of the kernel is integrated out because it projects to zero in the arguments of all functions. The triangular Hilbert transform is not known to satisfy any L^p bounds, and it is well understood that presently known techniques are insufficient to obtain such bounds. A version of Theorem 1.1 with uniformity in K, as opposed to our assumption on K being fixed, would imply bounds for the triangular Hilbert transform. Bounds for the triangular Hilbert transform as well as some of the known bounds for other partially degenerate cases in $d = 2$ would, in turn, imply bounds for the so-called Carleson operator in the corresponding L^p spaces, see [Carleson 1966; Fefferman 1973; Hunt 1968]. A more systematic classification of the partially degenerate cases appears in [Warchalski 2019], where also some uniform bounds are proven in a discrete model.

The main technical novelty of the current work is the application of our previous work [Fraccaroli et al. 2022], where we improved and extended the method of phase plane projections, previously studied in [Muscalu et al. 2002a] in dimension 1, to higher dimensions. In order to apply the set-up introduced in [Fraccaroli et al. 2022], we have to reformulate the standard phase space decomposition of the form Λ_m in a new way. Unlike the existing literature using either stopping times and outer measures, see [Do and Thiele 2015], or a tree-selection algorithm with various size functionals acting on families of multitiles, see [Grafakos and Li 2004; Lacey and Thiele 1997; Thiele 2002], our proof arranges the tree-selection in a different way. In particular, unlike our main inspiration [Muscalu et al. 2002a], we put emphasis on choosing the top intervals and top frequencies and let them define regions in phase space, the trees. Each tree, a region in the phase space, is then divided into a boundary and a core. The treatise of these two parts can be separated into two independent modules. The estimation of the boundaries is completely independent of paraproduct theory of any kind, just invoking Hölder's inequality. The estimation of the cores in turn relies on two real analysis lemmas, one on paraproduct estimates and one on phase space localization, which are stand-alone results that do not make any explicit reference to the notion of a tree. Clarifying the roles of the core part and the boundary part of a tree is the main insight we are communicating. Later, at the level of tree selection, we further notice that almost all nontrivial phase space interaction of the selected trees is encoded in their boundary parts. Summing up, while the paraproduct theory of boundaries is very simple and that of cores more complicated, the orders of complexity are swapped when carrying out the tree selection.

We close the introduction commenting a bit more on the background context of the study uniform bounds for multilinear operators. On one hand, one may use uniform bounds over parametrized families of singular operators to conclude bounds for superpositions of these operators as the parameter varies. While integrable rather than uniform dependence on the parameter may suffice for this purpose in some applications, even integrable dependence may need more work than the basic nonuniform bounds. We refer to [Muscalu 2014a; 2014b; 2014c] for a discussion about connections to Calderón commutators and the Cauchy integral over Lipschitz curves as the original motivation for studying the bilinear Hilbert transforms. Secondly, multilinear forms whose multipliers are characteristic functions of convex sets E are closely related to uniform bounds for multipliers which are characteristic functions of half-planes relative to tangent lines of E. This connection appears in [Demeter and Gautam 2012; Grafakos and Li 2006; Li 2008; Lie 2015; Muscalu 2000; Saari and Thiele 2023].

Finally, we describe the structure of the present paper. Section 2 contains the outline of the proof of Theorem 1.1, which is organized into four propositions. These principal propositions are proved in Sections 3, 4, 5 and 6, one proposition in each section. Theorem 1.1 is deduced from the contents of the outline Section 2 in Section 7. Sections 3–6 are independent of each other and only make reference to Section 2. Section 7 depends on arguments in Sections 3–6 only through the propositions stated in Section 2. Section 5 is slightly longer than its siblings, and it is divided further into an outline part and five further numbered subsections, which only refer to Section 2 and the overview part of Section 5.

2. Outline of the proof

We fix the dimension $d \geq 1$, dilation parameters $k_2 > k_1 > k_0 \geq 3$ with $k_i - k_j > 100d$ for $0 \leq j < i \leq 2$, and the triple of exponents (p_1, p_2, p_3) satisfying

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1, \qquad 2 < p_1, p_2, p_3 < \infty.$$

Let

$$\varepsilon = \min\{p_1 - 2, p_2 - 2, p_3 - 2\}.$$

In addition, we fix a number $\alpha > 2d$, $\alpha < 8d$. We further fix linear maps L_1, L_2 and L_3 as in Theorem 1.1. For $n \in \{1, 2, 3\}$, we choose $v_n \in \mathbb{Z}$ such that

$$2^{v_n - 1} < \|L_n\|_{\mathrm{op}} \leq 2^{v_n}.$$

Fix an index $n_* \in \{1, 2, 3\}$ such that

$$v_{n_*} = \min\{v_1, v_2, v_3\}. \tag{2-1}$$

As the condition (1-1) is invariant under scaling $\xi \mapsto \lambda\xi$, we may assume that $v_{n_*} = 0$.

Denote by $B(x, r)$ the open ball centered at $x \in \mathbb{R}^d$ and with radius r. For $\xi \in \mathbb{R}^{3\times d}$, $r > 0$, and $n \in \{1, 2, 3\}$, define $Q_n(\xi, r) \subset \mathbb{R}^d$ to be the minimal open rectangular box with sides parallel to the coordinate axes containing $B(\xi_n, 2^{v_n} r)$. Let

$$Q(\xi, r) = Q_1(\xi, r) \times Q_2(\xi, r) \times Q_3(\xi, r).$$

Let

$$\Gamma = \{(L_1\tau, L_2\tau, L_3\tau) : \tau \in \mathbb{R}^d\}.$$

Let $\mathcal{W}$ be a maximal set of pairwise disjoint rectangles of the form $Q(\xi, 2^{-j})$ with $\xi \in \mathbb{R}^{3\times d}$ and $j \in \mathbb{Z}$ with

$$\Gamma \cap Q(\xi, 2^{k_0-j}) = \varnothing$$

and

$$\Gamma \cap Q(\xi, 2^{k_0+1-j}) \neq \varnothing.$$

For all $N > 0$, let $\mathcal{W}_N$ be the finite subset of $\mathcal{W}$ defined by

$$\mathcal{W}_N = \{Q(\xi, 2^{-j}) \in \mathcal{W} : |\xi|, |j| \le N\}.$$

For a cube with sides parallel to the coordinate axes $I \subset \mathbb{R}^d$, define the mollified distance ρ_I by

$$\rho_I(x) = \inf\{r > 1 : x \in (2r-1)I\},$$

where aI denotes the cube with the same center as I and a times the side-length. Moreover, for a Borel set $F \subset \mathbb{R}^d$, define

$$\rho_I(F) = \inf\{\rho_I(x) : x \in F\}.$$

Definition 2.1 (frequency cut-offs). Let $E \subset \mathbb{R}^{3\times d}$ be bounded with open interior. Define $\Phi_n^{\alpha}(E)$ to be the set of continuous complex-valued functions ϕ on $\mathbb{R}^d$ with

$$|\phi(x)| \le 2^{(v_n-j)d} \rho^{-\alpha}_{[0,2^{j-v_n})^d}(x)$$

for all $x \in \mathbb{R}^d$ and

$$\operatorname{supp}\hat{\phi} \subset \{\xi_n : \xi \in E\},$$

where $j \in \mathbb{Z}$ is maximal such that there exists $\xi \in \mathbb{R}^{3\times d}$ with $E \subset Q(\xi, 2^{-j})$.

In Section 7, Theorem 1.1 is reduced to Proposition 2.2 below, where the multiplier is replaced by a sum of tensor multipliers.

Proposition 2.2 (weak estimate for tensor model). *Let*

$$\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = 1, \quad 2 < q_1, q_2, q_3 < \infty.$$

There exists a constant $C = C(d, \alpha, k_0, q_1, q_2, q_3)$ such that the following holds.

For $Q \in \mathcal{W}$ and $n \in \{1, 2, 3\}$, let $\phi_{Q,n} \in \Phi_n^{4\alpha}(Q)$. For each $n \in \{1, 2, 3\}$, let $f_n \in L^2(\mathbb{R}^d)$ be a function such that

$$\|f_n\|_\infty \le 2.$$

Then, for all $N > 0$,

$$\left| \sum_{Q \in \mathcal{W}_N} \int_{\mathbb{R}^d} \prod_{n=1}^{3} [\phi_{Q,n} * f_n(x)] \,\mathrm{d}x \right| \le C \prod_{n=1}^{3} \|f_n\|_2^{2/q_n}. \tag{2-2}$$

The proof of Proposition 2.2 can be found in Section 6. It requires several intermediate results, which we state next. The following frequency-localized version of Proposition 2.2 will play a role inside the proof. While the singularity of the bilinear multiplier in Proposition 2.2 can still be truly d-dimensional, Proposition 2.3 only deals with a point singularity in the spirit of more classical Coifman–Meyer multilinear multipliers. Proposition 2.3 will be proven in Section 3.

Proposition 2.3 (frequency-localized estimate). *Let k be a positive integer and*

$$\frac{1}{q_1}+\frac{1}{q_2}+\frac{1}{q_3}=1, \quad 2<q_1,q_2,q_3<\infty.$$

There exists a constant $C=C(d,\alpha,k_0,k,q_1,q_2,q_3)$ such that the following holds.

Let $\eta\in\Gamma$. For $Q\in\mathcal{W}$ and $n\in\{1,2,3\}$, let $\phi_{Q,n}\in\Phi_n^{4\alpha}(Q)$. For each $n\in\{1,2,3\}$, let $f_n\in L^{q_n}(\mathbb{R}^d)$. Then, for all $N>0$,

$$\left|\sum_{\substack{Q\in\mathcal{W}_N\\ \eta\in 2^k Q}}\int_{\mathbb{R}^d}\prod_{n=1}^{3}[\phi_{Q,n}*f_n(x)]\,\mathrm{d}x\right|\le C\prod_{n=1}^{3}\|f_n\|_{q_n}.$$

The reduction of Proposition 2.2 to Proposition 2.3 features a stopping-time argument, which introduces spatial truncations in addition to the mere frequency localization discussed so far and utilizes the notion of trees defined below.

For $k\in\mathbb{Z}$, let $\mathcal{D}_k=\{2^k([0,1)^d+l):l\in\mathbb{Z}^d\}$ and $\mathcal{D}=\bigcup_{k\in\mathbb{Z}}\mathcal{D}_k$. An element of $\mathcal{D}$ is called a dyadic cube.

Definition 2.4 (multitile, n-tile). A product $I\times Q$ is called a multitile if $I\in\mathcal{D}$ and $Q\in\mathcal{W}$ and $|Q_{n_*}|^{-1}=|I|$. For a multitile $I\times Q$ and $n\in\{1,2,3\}$, we call the product $I\times Q_n$ an n-tile. If $P=I\times Q$ is a multitile, we write I_P for I and Q_P for Q.

Definition 2.5 (tree). Let $\mathcal{V}$ be a finite subset of multitiles, let $\xi\in\Gamma$, and let $I_0\in\mathcal{D}$. Assume there exists at least one $P\in\mathcal{V}$ with $I_P=I_0$ and $\xi\in 2^{k_2+1}Q_P$. Then the triple $(\xi,I_0,\mathcal{V})$ defines a tree T. We write ξ_T for ξ, I_T for I_0, $\mathcal{V}_T$ for $\mathcal{V}$, and j_T for the top scale $\log_2|I_0|^{1/d}$. Attached to the tree T are the following objects:

- The family $\mathcal{P}_T$ of multitiles in $\mathcal{V}$ with $I_P\subset I_T$ and
$$\xi_T\in 2^{k_2+1}Q_P.$$
- The family $\mathcal{B}_T$ of multitiles $P\in\mathcal{P}_T$ with
$$\xi_T\in 2^{k_2+1}Q_P\setminus 2^{k_1+1}Q_P.$$
- The family $\mathcal{I}_T$ of dyadic cubes $I\in\mathcal{D}$ such that there exist P and P' in $\mathcal{P}_T\setminus\mathcal{B}_T$ with
$$I_P\subset I\subset I_{P'}.$$

The following definition gives a gauge to the size of a function near a tree.

Definition 2.6 (main sizes). Let $1 \le p \le \infty$, $n \in \{1, 2, 3\}$, and $f \in L^p(\mathbb{R}^d)$. Let T be a tree. We define

$$\Sigma^{\text{bdr}}_{n,p,f}(T) = \sup_{P \in \mathcal{B}_T} \sup_{\phi \in \Phi^{4\alpha}_n(Q_P)} \frac{\|\rho^{-\alpha}_{I_P}[\phi * f]\|_p}{|I_P|^{1/p}},$$

$$\Sigma^{\text{sum}}_{n,f}(T) = \left(\frac{1}{|I_T|} \sum_{P \in \mathcal{B}_T} \sup_{\phi \in \Phi^{4\alpha}_n(Q_P)} \|1_{I_P}[\phi * f]\|_2^2\right)^{1/2},$$

$$\Sigma^{\text{cor}}_{n,p,f}(T) = \sup_{i \in \mathbb{Z}} \sup_{I \in \mathcal{D}_i \cap \mathcal{I}_T} \sup_{\phi \in \Phi^{4\alpha}_n(Q_{T,i})} \frac{\|\rho^{-\alpha}_{I}[\phi * f]\|_p}{|I|^{1/p}},$$

where $Q_{T,i} = Q(\xi_T, 2^{k_1+5d-i})$ and $1/\infty$ is understood to be 0.

Heuristically, the core size is large enough to control a phase space paraproduct, but it is slightly too imprecise in terms of phase space localization. In order to maintain the information about frequency localization of a tree, the frequencies seen as peripheral with respect to the top frequency must be measured with a different kind of size, the sum size. The pair of sum size and core size are together strong enough to control the paraproduct and maintain the phase space localization, but in order to sum together the trees of different amplitudes, this couple still fails by a logarithmic blowup. To adjust this last piece, a multiplicative fraction of the sum size is replaced by the boundary size, which is a sup size again, but of nature lacunary with respect to the top frequency. After this last adjustment, the triple of sizes succeeds in the task of controlling the paraproduct, maintaining phase space localization and recovering summability over amplitudes. In the following proposition, we control the phase space paraproduct by the sizes. The proof can be found in Section 4.

Proposition 2.7 (phase space-localized estimate). *Let*

$$\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = 1, \qquad 2 < q_1, q_2, q_3 < \infty.$$

There exists a constant $C = C(d, \alpha, k_0, k, q_1, q_2, q_3)$ such that the following holds.

Let T be a tree. For each $P \in \mathcal{P}_T$ and $n \in \{1, 2, 3\}$, let $\phi_{P,n} \in \Phi^{4\alpha}_n(Q_P)$. Then, for any $n' \in \{1, 2, 3\}$,

$$\left|\sum_{P \in \mathcal{B}_T} \int_{\mathbb{R}^d} 1_{I_P}(x) \prod_{n=1}^{3} [\phi_{P,n} * f_n(x)]\, dx\right| \le C|I_T| \Sigma^{\text{bdr}}_{n',\infty,f_{n'}}(T) \prod_{n \ne n'} \Sigma^{\text{sum}}_{n,f_n}(T), \tag{2-3}$$

$$\left|\sum_{P \in \mathcal{P}_T \setminus \mathcal{B}_T} \int_{\mathbb{R}^d} 1_{I_P}(x) \prod_{n=1}^{3} [\phi_{P,n} * f_n(x)]\, dx\right| \le C|I_T| \prod_{n=1}^{3} \Sigma^{\text{cor}}_{n,q_n,f}(T). \tag{2-4}$$

The remaining ingredient of the proof of Proposition 2.2 is a partition of the set of all multitiles into trees, to which Proposition 2.7 can be applied. This last proposition will be proved in Section 5.

Proposition 2.8 (decomposition of the phase space). *There exists a constant $C = C(d, \alpha, k_0, k_1, k_2)$ such that the following holds.*

Let $N, N' > 0$. Let $\mathcal{V}$ be the finite subset of multitiles defined by

$$\mathcal{V} = \{P : Q_P \in \mathcal{W}_N,\ I_P \subset [-N'2^N, N'2^N]^{3\times d}\},$$

with $\mathcal{W}$ as in Proposition 2.2. For each $M \in \mathbb{Z} \cup \{-\infty\}$, there exists a family of trees $\mathcal{T}_M$ such that

$$\mathcal{V} = \bigcup_{M \in \mathbb{Z} \cup \{-\infty\}} \bigcup_{T \in \mathcal{T}_M} \mathcal{P}_T,$$

and the following hold for each $n \in \{1, 2, 3\}$:

- *For each tree $T \in \mathcal{T}_M$ for which there exists $P \in \mathcal{P}_T$ with $2^{k_1+1} Q_P \ni \xi_T$, we have*

$$\Sigma^{\mathrm{cor}}_{n,2,f_n}(T) \le 2^{M/2} \| f_n \|_2.$$

- *For every tree $T \in \mathcal{T}_M$, we have*

$$\Sigma^{\mathrm{bdr}}_{n,2,f_n}(T) + \Sigma^{\mathrm{sum}}_{n,f_n}(T) \le 2^{M/2} \| f_n \|_2.$$

- *For every tree T with $\mathcal{V}_T \subset \mathcal{V}$, we have*

$$\Sigma^{\mathrm{cor}}_{n,2,f_n}(T) + \Sigma^{\mathrm{bdr}}_{n,2,f_n}(T) + \Sigma^{\mathrm{sum}}_{n,f_n}(T) \le C \| f_n \|_\infty. \tag{2-5}$$

- *We have*

$$\sum_{T \in \mathcal{T}_M} 2^M |I_T| \le C. \tag{2-6}$$

Complementary notation. We conclude the section outlining the proof by listing some notational conventions that we intentionally omitted when describing the strategy of the proof but which will be helpful for understanding the proofs. In what follows, a constant C will depend on d, α, ε, k_0, k_1, and k_2. The exact dependence will be implicit in our arguments. We occasionally use the shorthand notation $A \lesssim B$ when $A \le CB$ for such a constant C.

Concerning the frequency cut-offs, see Definition 2.1, we use the following shorthand notations:

- Given $\xi \in \mathbb{R}^{3 \times d}$ and $j \in \mathbb{Z}$, we define

$$\Phi^{\alpha}_{n,j}(\xi) = \Phi^{\alpha}_n(Q(\xi, 2^{-j})).$$

- Given $\xi \in \mathbb{R}^{3 \times d}$ and $j \in \mathbb{Z}$, we define

$$\Psi^{\alpha}_{n,j}(\xi) = \Phi^{\alpha}_n(Q(\xi, 2^{-j}) \setminus Q(\xi, 2^{-j-2})).$$

- Given $\xi \in \mathbb{R}^{3 \times d}$, we denote by $M_n(\xi, E)$ the set of ϕ such that

$$\sup_{\tau \in \mathbb{R}^d \setminus \{\xi_n\}} |(\tau - \xi_n)^\beta \partial^\beta \hat{\phi}(\tau)| \le 2^{-\nu_n |\beta|}, \quad \operatorname{supp} \hat{\phi} \subset E$$

  for all $\beta \in \mathbb{N}^d$ with $|\beta| \le 100d$. We call such a ϕ a normalized n-Mikhlin cut-off to E at ξ.

3. Proof of Proposition 2.3: paraproduct

Let η and $\phi_{Q,n}$ be given as in Proposition 2.3. By a translation on the Fourier transform side we may assume $\eta = 0$. By definition of $\mathcal{W}$, for each $Q \in \mathcal{W}$ we have $0 \notin 2Q$. Hence there exists $n \in \{1, 2, 3\}$ such that $0 \notin 2Q_n$. By splitting into three cases and estimating (2-2) in each case separately, we may assume without loss of generality that $0 \notin 2Q_1$ for all $Q \in \mathcal{W}$. Further, for each $j \in \mathbb{Z}$, there exists at most $C(d,k)$ distinct elements $Q \in \mathcal{W}$ with $2^k Q \ni 0$ and $|Q_{n_*}| = 2^{-jd}$. By splitting into $C(d,k)$ further subcases, we may assume there exists at most one such Q. Even further, for each Q with $|Q_{n_*}| = 2^{-jd}$, there exist $C(d,k)$, $\widetilde{C}(d,k)$, and $\{c_{j',n} : |c_{j',n}| \le \widetilde{C}(d,k),\ |j'-j| \le C(d,k)\}$ such that

$$\phi_{Q,n} = \sum_{j':|j'-j|\le C(d,k)} c_{j',n}\phi_{j',n}, \quad \phi_{j',n} \in \Phi^{4\alpha}_{n,j'}(0).$$

Hence we may further reduce the study to the case where $\phi_{Q,n}$ is replaced by $\phi_{j,n}$ as above and v_n is replaced by v_n' with $|v_n - v_n'| \le C(d,k)$. Hence we aim at bounding

$$\left|\sum_{j\in\mathcal{N}} \int_{\mathbb{R}^d} c_j \prod_{n=1}^{3} [\phi_{j,n} * f_n(x)]\,\mathrm{d}x\right|,$$

where $\mathcal{N} \subset \mathbb{Z}$ is finite, $\phi_{j,1} \in \Psi^{4\alpha}_{1,j}(0)$, and $\phi_{j,n} \in \Phi^{4\alpha}_{n,j}(0)$ for $n \in \{2,3\}$.

Let χ be a Schwartz function on $\mathbb{R}^d$ such that $\hat{\chi}(\tau) = 0$ for $|\tau| \ge 2$ and $\hat{\chi}(\tau) = 1$ for $|\tau| \le 1$. Define, for $l \in \mathbb{Z}$,

$$\chi_l(x) = 2^{-ld}\chi(2^{-l}x),$$

and, for each $j \in \mathcal{N}$ and $n \in \{2,3\}$, define

$$\hat{\rho}_{j,n} = \hat{\phi}_{j,n} - \hat{\phi}_{j,n}(0)\hat{\chi}_{j-v_n}.$$

By the triangle inequality, it suffices to prove, for any collection

$$\{c_j : |c_j| \le 1,\ j \in \mathcal{N}\},$$

bounds for the tree expressions

$$\mathrm{I} = \left|\sum_{j\in\mathcal{N}} \int_{\mathbb{R}^d} [c_j\phi_{j,1} * f_1(x)][\rho_{j,2} * f_2(x)][\phi_{j,3} * f_3(x)]\,\mathrm{d}x\right|, \tag{3-1}$$

$$\mathrm{II} = \left|\sum_{j\in\mathcal{N}} \int_{\mathbb{R}^d} [c_j\phi_{j,1} * f_1(x)][\chi_{j-v_2} * f_2(x)][\rho_{j,3} * f_3(x)]\,\mathrm{d}x\right|, \tag{3-2}$$

$$\mathrm{III} = \left|\sum_{j\in\mathcal{N}} \int_{\mathbb{R}^d} [c_j\phi_{j,1} * f_1(x)][\chi_{j-v_2} * f_2(x)][\chi_{j-v_3} * f_3(x)]\,\mathrm{d}x\right| \tag{3-3}$$

separately.

We begin with (3-1). We estimate it with Cauchy–Schwartz in $\mathcal{N}$ and Hölder in $\mathbb{R}^d$ by

$$\left\|\left(\sum_{j\in\mathcal{N}} |\phi_{j,1} * f_1|^2\right)^{1/2}\right\|_{q_1} \left\|\left(\sum_{j\in\mathcal{N}} |\rho_{j,2} * f_2|^2\right)^{1/2}\right\|_{q_2} \left\|\sup_{j\in\mathcal{N}} |\phi_{j,3} * f_3|\right\|_{q_3}.$$

The term (3-2) is estimated similarly by

$$\left\|\left(\sum_{j\in\mathcal{N}} |\phi_{j,1} * f_1|^2\right)^{1/2}\right\|_{q_1} \left\|\sup_{j\in\mathcal{N}} |\chi_{j-v_2} * f_2|\right\|_{q_2} \left\|\left(\sum_{j\in\mathcal{N}} |\rho_{j,3} * f_3|^2\right)^{1/2}\right\|_{q_3}.$$

In both cases, we can apply the standard square function estimate (see Theorem 5.1.2 in [Grafakos 2008]) and maximal function estimates to obtain the desired bound. This completes the proof for I and II.

It remains to estimate (3-3). We telescope χ_{j-v_2} and χ_{j-v_3} into functions $\psi_l := \chi_{l-1} - \chi_l$ and thus write

$$\mathrm{III} \lesssim \sum_{m_1=-v_1-10}^{-v_1+10} \biggl| \sum_{\substack{m_2 \geq -v_2 \\ m_3 \geq -v_3}} \sum_{j \in \mathcal{N}} \int_{\mathbb{R}^d} [\phi_{m_1+j} * f_1(x)] \prod_{n=2}^{3} [\psi_{m_n+j} * f_n(x)] \, \mathrm{d}x \biggr|, \tag{3-4}$$

where $\phi_{m_1+j} = \phi_{j,1} * \psi_{m_1+j}$.

We fix a triple $(\kappa_1, \kappa_2, \kappa_3) \in \mathbb{Z}^3$ and restrict the sums to $m_n \in \kappa_n + 1000d\mathbb{Z}$ for $n \in \{2, 3\}$ and $j \in \widetilde{\mathcal{N}} = \kappa_1 + 1000d\mathbb{Z}$. By the triangle inequality and summation over the $(1000d)^3$ values of $(\kappa_1, \kappa_2, \kappa_3)$, it suffices to bound the restricted sum. Consider then a fixed term in the sum (3-4). Such a term is nonzero only if

$$0 \in (\operatorname{supp} \hat{\phi}_{m_1+j} + \operatorname{supp} \hat{\psi}_{m_2+j} + \operatorname{supp} \hat{\psi}_{m_3+j}).$$

Recalling that we work with indices modulo $1000d$, this happens only if two of the numbers in $\{m_1, m_2, m_3\}$ are equal and the remaining one is larger.

We first assume $m_1 = m_n \leq m_{n'}$ for fixed $n, n' \in \{2, 3\}$. Then, for $\mathfrak{m} = \max(m_1, -v_{n'})$, we bound (3-4) by

$$\begin{aligned}
&\biggl| \sum_{m_{n'} \geq \mathfrak{m}} \sum_{j \in \widetilde{\mathcal{N}}} \int_{\mathbb{R}^d} [\phi_{m_1+j} * f_1(x)][\psi_{m_1+j} * f_n(x)][\psi_{m_{n'}+j} * f_{n'}(x)] \, \mathrm{d}x \biggr| \\
&\qquad \leq \int_{\mathbb{R}^d} \sum_{j \in \widetilde{\mathcal{N}}} |\phi_{m_1+j} * f_1(x)| \, |\psi_{m_1+j} * f_n(x)| \biggl| \sum_{m_{n'} \geq \mathfrak{m}} \psi_{m_{n'}+j} * f_{n'}(x) \biggr| \, \mathrm{d}x \\
&\qquad \leq \biggl\| \biggl(\sum_{j \in \widetilde{\mathcal{N}}} |\phi_{m_1+j} * f_1|^2 \biggr)^{1/2} \biggr\|_{q_1} \biggl\| \biggl(\sum_{j \in \widetilde{\mathcal{N}}} |\psi_{m_1+j} * f_n|^2 \biggr)^{1/2} \biggr\|_{q_n} \biggl\| \sup_{j \in \widetilde{\mathcal{N}}} \biggl| \sum_{m_{n'} \geq \mathfrak{m}+j} \psi_{m_{n'}} * f_{n'} \biggr| \biggr\|_{q_{n'}}.
\end{aligned}$$

These factors are bounded by the square function estimate and maximally truncated singular integral estimate, which completes the proof in this case.

Assume then that $m_2 = m_3 \leq m_1$. Now, for $\mathfrak{m} = \max(-v_2, -v_3)$, we bound (3-4) by

$$\begin{aligned}
&\biggl| \sum_{j \in \widetilde{\mathcal{N}}} \int_{\mathbb{R}^d} [\phi_{m_1+j} * f_1(x)] \sum_{k=j+\mathfrak{m}}^{j+m_1} [\psi_k * f_2(x)][\psi_k * f_3(x)] \, \mathrm{d}x \biggr| \\
&\qquad = \biggl| \int_{\mathbb{R}^d} \sum_{k \in \mathbb{Z}} [\psi_k * f_2(x)][\psi_k * f_3(x)] \sum_{j \in \widetilde{\mathcal{N}} \cap \{k-m_1, \dots, k-\mathfrak{m}\}} [\phi_{m_1+j} * f_1(x)] \biggr| \, \mathrm{d}x \\
&\qquad \leq \biggl\| \biggl(\sum_{k \in \mathbb{Z}} |\psi_k * f_2|^2 \biggr)^{1/2} \biggr\|_{q_2} \biggl\| \biggl(\sum_{k \in \mathbb{Z}} |\psi_k * f_3|^2 \biggr)^{1/2} \biggr\|_{q_3} \biggl\| \sup_{k \in \mathbb{Z}} \biggl| \sum_{j \in \widetilde{\mathcal{N}} \cap \{k-m_1, \dots, k-\mathfrak{m}\}} \phi_{m_1+j} * f_1 \biggr| \biggr\|_{q_1}.
\end{aligned}$$

Again, the bound follows by the square function estimate and maximally truncated singular integral estimate, and the proof is complete. □

4. Proof of Proposition 2.7: tree estimate

Boundary part. Given any family of multitiles $\mathcal{F} \subset \mathcal{P}_T$, we define

$$\Lambda_{\mathcal{F}}(f_1, f_2, f_3) = \sum_{P \in \mathcal{F}} \int_{\mathbb{R}^d} 1_{I_P}(x) \prod_{n=1}^{3} [\phi_{P,n} * f_n(x)] \, dx.$$

We start with the easier bound (2-3).

Proposition 4.1. *There exists a constant C such that, for any $n' \in \{1, 2, 3\}$,*

$$|\Lambda_{\mathcal{B}_T}(f_1, f_2, f_3)| \le C |I_T| \Sigma^{\mathrm{bdr}}_{n,\infty,f_{n'}}(T) \prod_{n \neq n'} \Sigma^{\mathrm{sum}}_{n,f_n}(T).$$

Proof. By Hölder's inequality in $\mathbb{R}^d$ and the Cauchy–Schwartz inequality in $\mathcal{B}_T$,

$$|\Lambda_{\mathcal{B}_T}(f_1, f_2, f_3)| \le \sup_{P \in \mathcal{B}_T} \|1_P[\phi_{P,n'} * f_{n'}]\|_\infty \prod_{n \neq n'} \biggl(\sum_{P \in \mathcal{B}_T} \|1_P[\phi_{P,n} * f_n]\|_2^2 \biggr)^{1/2}. \qquad \square$$

We turn to estimating the form $\Lambda_{\mathcal{B}_T \setminus \mathcal{P}_T}$, which is the main source of difficulty in the proof. Here we will need several auxiliary tools, including Proposition 2.3 and some results from [Fraccaroli et al. 2022].

Phase space projections. Define, for $j \in \mathbb{Z}$,

$$\mathcal{I}_{T,j} := \{I_P : P \in \mathcal{P}_T \setminus \mathcal{B}_T\} \cap \mathcal{D}_j, \qquad E_j^0 := \bigcup \mathcal{I}_{T,j}.$$

Define further, for each integer $k \ge 1$,

$$\mathcal{I}^k_{T,j} := \{I \in \mathcal{D}_j : \rho_I(E_j^0) \le k\}, \qquad E_j^k := \bigcup \mathcal{I}^k_{T,j}.$$

Finally, for $\xi \in \mathbb{R}^{3\times d}$ and $n \in \{1, 2, 3\}$, we let $\mathrm{Mod}_{n,\xi}$ be the mapping such that

$$\mathrm{FT}(\mathrm{Mod}_{n,\xi} f)(\tau) = \hat{f}(\tau + \xi_n),$$

where FT is the Fourier transform. We define the phase space localization by using the construction from [Fraccaroli et al. 2022].

Definition 4.2 (phase plane projection). Let $v \ge 0$ be an integer, $n \in \{1, 2, 3\}$, and T be a tree. Let h be a Schwartz function. We define $\Pi_{T,n} h = \mathrm{Mod}_{n,-\xi}\, g$, where g is the output of Theorem 1.1 in [Fraccaroli et al. 2022] based on the input parameter $m = v_n$, input function $f = \mathrm{Mod}_{n,\xi}\, h$, input cube $U = I_T$, and the input M being the family of minimal cubes in $\bigcup_{j \in \mathbb{Z}} \mathcal{I}_{T,j}$.

By scaling, we can now quote the following result from [Fraccaroli et al. 2022].

Theorem 4.3 [Fraccaroli et al. 2022, Theorem 1.1]. *Let $1 \le p \le \infty$ and $1/p + 1/p' = 1$. Let $\alpha > d$ and $0 \le k \le k_1 + 4d$. There exists a constant $C = C(d, \alpha, p, k_0, k_1)$ such that the following holds.*

Let T be a tree and fix $n \in \{1, 2, 3\}$. Then, for every $j \le j_T$ and $J \in \mathcal{D}_j$,

$$\|\Pi_{T,n} f\|_p \le C \Sigma^{\mathrm{cor}}_{n,p,f}(T) |I_T|^{1/p} \tag{4-1}$$

and

$$\sum_{i\le j_T}\sum_{\substack{I\in\mathcal{I}_{T,i}\\ I\subset J}}\sup_{\phi\in\Phi^{4\alpha}_{n,i-k}(\xi)}|I|^{1/p'}\|\rho_I^{-3\alpha}[\phi*(f-\Pi_{T,n}f)]\|_p\le C\Sigma^{\mathrm{cor}}_{n,p,f}(T)|J|. \tag{4-2}$$

For every $j\le j_T$ and $J\in\mathcal{D}_j$ such that $I\not\subset 3J$ for any $I\in\mathcal{I}_{T,j}$,

$$\sum_{i\le j_T}\sup_{\substack{I\in\mathcal{D}_i\setminus\mathcal{I}_T\\ I\subset J}}\sup_{\psi\in\Psi^{4\alpha}_{n,i-k}(\xi)}|I|^{-1/p}\|\rho_I^{-3\alpha}[\psi*\Pi_{T,n}f]\|_p\le C\Sigma^{\mathrm{cor}}_{n,p,f}(T)\|1_{7I_T}\rho_J^{-\alpha}\|_\infty. \tag{4-3}$$

Proof of Proposition 2.7. It remains to prove

$$|\Lambda_{\mathcal{P}_T\setminus\mathcal{B}_T}(f_1,f_2,f_3)|\le C|I_T|\prod_{n=1}^{3}\Sigma^{\mathrm{cor}}_{n,q_n,f_n}(T),$$

as by Proposition 4.1 we already know (2-3) to hold.

Core part. By decomposing $\Lambda_{\mathcal{P}_T\setminus\mathcal{B}_T}$ into $C(d,k_0,k_1)$ many distinct sums, we can assume that, for each $j\in\mathbb{Z}$, there is at most one $Q\in\mathcal{W}$ such that $Q_P=Q$ and $|I_P|=2^{jd}$ for some $P\in\mathcal{P}_T\setminus\mathcal{B}_T$. We pick a sequence of functions

$$\phi_{j,n}\in\Phi^{4\alpha}_n(Q)$$

such that

$$\begin{aligned}|\Lambda_{\mathcal{P}_T\setminus\mathcal{B}_T}(f_1,f_2,f_3)|&\le\sum_{j\in\mathbb{Z}}\max_{P:Q_P=Q}\left|\int_{\mathbb{R}^d}1_{E_j^1}(x)\prod_{n=1}^{3}[\phi_{P,n}*f_n(x)]\,\mathrm{d}x\right|\\&\le C\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^d}1_{E_j^1}(x)\prod_{n=1}^{3}[\phi_{j,n}*f_n(x)]\,\mathrm{d}x.\end{aligned}$$

We define

$$\begin{aligned}\Lambda_{\mathcal{C}_T}(f_1,f_2,f_3)&:=\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^d}1_{E_j^1}(x)\prod_{n=1}^{3}[\phi_{j,n}*f_n(x)]\,\mathrm{d}x,\\ \Lambda_{\mathcal{C}_T,c}(f_1,f_2,f_3)&:=\sum_{\substack{j\in\mathbb{Z}\\ E_j^0\neq\varnothing}}\int_{\mathbb{R}^d}1_{(E_j^1)^c}(x)\prod_{n=1}^{3}[\phi_{j,n}*f_n(x)]\,\mathrm{d}x.\end{aligned}$$

We compute

$$\begin{aligned}|\Lambda_{\mathcal{C}_T}(f_1,f_2,f_3)|&\le|\Lambda_{\mathcal{C}_T,c}(\Pi_{T,1}f_1,\Pi_{T,2}f_2,\Pi_{T,3}f_3)|\\&\quad+|\Lambda_{\mathcal{C}_T}(\Pi_{T,1}f_1,\Pi_{T,2}f_2,\Pi_{T,3}f_3)+\Lambda_{\mathcal{C}_T,c}(\Pi_{T,1}f_1,\Pi_{T,2}f_2,\Pi_{T,3}f_3)|\\&\quad+|\Lambda_{\mathcal{C}_T}(f_1-\Pi_{T,1}f_1,\Pi_{T,2}f_2,\Pi_{T,3}f_3)|\\&\quad+|\Lambda_{\mathcal{C}_T}(f_1,f_2-\Pi_{T,2}f_2,\Pi_{T,3}f_3)|\\&\quad+|\Lambda_{\mathcal{C}_T}(f_1,f_2,f_3-\Pi_{T,3}f_3)|\\&=:\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V}.\end{aligned} \tag{4-4}$$

For clarity, we state three auxiliary facts before estimating the five terms above.

Lemma 4.4. *For each $j \in \mathbb{Z}$, there is $n_j \in \{1,2,3\}$ and coefficients $c_{i,j}$ and functions $\phi_{i,j} \in \Psi^{4\alpha}_{n,j+i}(\xi_T)$ such that*

$$\phi_{P,n_j} = \sum_{i=-k_1-3d}^{-k_0+1} c_{i,j}\phi_{i,j}, \qquad \sum_{i=-k_1-3d}^{-k_0+1} |c_{i,j}| \le C(d).$$

Proof. For each $j \in \mathbb{Z}$ and $P \in \mathcal{P}_T \setminus \mathcal{B}_T$ with $|I_P| = 2^{jd}$, we know that $\xi_T \notin 2^{k_0} Q_P$. Hence there exists at least one $n_j \in \{1,2,3\}$ such that $(\xi_T)_{n_j} \notin 2^{k_0} Q_{n_j}$. The claim follows from this. □

Lemma 4.5. *Let $\mathcal{A}$ be the set of dyadic cubes I maximal with $|I| \le |I_T|$ and $J \subset 3I$ for no $J \in \mathcal{I}_T$ with $|J| \le |I|$. Then*

$$\mathcal{A}_j = \{J \in \mathcal{A} : |J| \ge 2^{jd}\}$$

is a partition of $\mathbb{R}^d \setminus E^1_j$.

Proof. Disjointness follows from maximality. If $x \in \mathbb{R}^d \setminus \bigcup \mathcal{A}_j$, then $J \in \mathcal{D}_j$ with $x \in J$ satisfies $3J \supset I$ for some $I \in \mathcal{I}_T$ with $|I| \le |J|$. Then $\hat{I} \in \mathcal{D}_j$ with $\hat{I} \supset I$ satisfies $\hat{I} \in \mathcal{I}_{T,j}$ and $J \subset 3\hat{I}$. Hence $J \subset E^1_j$. The inclusion $\bigcup \mathcal{A}_j \subset \mathbb{R}^d \setminus E^1_j$ follows by definition. □

Lemma 4.6. *Let $j \in \mathbb{Z}$ and $J \in \mathcal{D}_j$ be such that $5J \supset I$ for some $I \in \mathcal{I}_{T,j}$. Then*

$$\|1_J[\phi_{j,n} * \Pi_{T,n} f_n]\|_{q_n} \le C|J|^{1/q_n} \Sigma^{\mathrm{cor}}_{n,q_n,f_n}(T).$$

Proof. This follows by applying (4-2) to J and restricting the sum on the left-hand side to a single term as

$$\begin{aligned}\|1_J[\phi_{j,n} * \Pi_{T,n} f_n]\|_{q_n} &\le \|1_J[\phi_{j,n} * (\Pi_{T,n} f_n - f_n)]\|_{q_n} + \|1_J[\phi_{j,n} * f_n]\|_{q_n} \\ &\le C|J|^{1/q_n} \Sigma^{\mathrm{cor}}_{n,q_n,f_n}(T).\end{aligned}$$ □

Now we can estimate the five terms in (4-4). To estimate I, we recall that, for each $j \in \mathbb{Z}$, there exists $n_j \in \{1,2,3\}$ as in Lemma 4.4. We fix n_j to be one of them so that the three sets

$$\mathcal{N}_n = \{j \in \mathbb{Z} : E^1_j \ne \varnothing,\ n_j = n\}$$

partition the subset of $\mathbb{Z}$ appearing in the definition of I. Then

$$\begin{aligned}\mathrm{I} = |\Lambda_{\mathcal{C}_T,c}(\Pi_{T,1} f_1, \Pi_{T,2} f_2, \Pi_{T,3} f_3)| &\le \sum_{\nu=1}^{3} \int_{\mathbb{R}^d} \sum_{j \in \mathcal{N}_\nu} 1_{(E^1_j)^c} \prod_{n=1}^{3} |\phi_{j,n} * \Pi_{T,n} f_n(x)|\,\mathrm{d}x \\ &\le \sum_{\nu=1}^{3} \Bigl(\prod_{n \ne \nu} \|M_{\mathrm{HL}} \Pi_{T,n} f_n\|_{q_n}\Bigr) \Bigl\| \sum_{j \in \mathcal{N}_\nu} 1_{(E^1_j)^c} |\phi_{j,\nu} * \Pi_{T,\nu} f_\nu| \Bigr\|_{q_\nu},\end{aligned}$$

where M_{HL} is the Hardy–Littlewood maximal function. By the maximal function theorem and (4-1) from Theorem 4.3,

$$\|M_{\mathrm{HL}} \Pi_{T,n} f_n\|_{q_n} \le C|I_T|^{1/q_n} \Sigma^{\mathrm{cor}}_{n,q_n,f_n}(T).$$

By Lemma 4.5 and Minkowski's inequality,

$$\Big\|\sum_{j\in\mathcal{N}_\nu} 1_{(E_j^1)^c}|\phi_{j,\nu}*\Pi_{T,\nu}f_\nu|\Big\|_{q_\nu}^{q_\nu} = \sum_{J\in A}\Big\|1_J\sum_{j\in\mathcal{N}_\nu}\sum_{I\in\mathcal{D}_j\setminus\mathcal{I}_{T,j}^1} 1_I|\phi_{j,\nu}*\Pi_{T,\nu}f_\nu|\Big\|_{q_\nu}^{q_\nu}$$
$$\le \sum_{J\in A}\bigg(\sum_{j\in\mathcal{N}_\nu}\bigg(\sum_{\substack{I\in\mathcal{D}_j\setminus\mathcal{I}_{T,j}^1\\ I\subset J}}\|1_I[\phi_{j,\nu}*\Pi_{T,\nu}f_\nu]\|_{q_\nu}^{q_\nu}\bigg)^{1/q_\nu}\bigg)^{q_\nu}.$$

By Lemma 4.4 and (4-3) from Theorem 4.3,

$$\sum_{j\in\mathcal{N}_\nu}\bigg(\sum_{\substack{I\in\mathcal{D}_j\setminus\mathcal{I}_{T,j}^1\\ I\subset J}}\|1_I[\phi_{j,\nu}*\Pi_{T,\nu}f_\nu]\|_{q_\nu}^{q_\nu}\bigg)^{1/q_\nu} \le |J|^{1/q_n}\sum_{i=-k_1-3d}^{-k_0+1}\sum_{j\in\mathcal{N}_\nu}\sup_{\substack{I\in\mathcal{D}_j\setminus\mathcal{I}_{T,j}^1\\ I\subset J}}\sup_{\psi\in\Psi_{\nu,j+i}^{4\alpha}(\xi_T)}\frac{\|1_I[\psi*\Pi_{T,\nu}f_\nu]\|_{q_\nu}}{|I|^{1/q_\nu}}$$
$$\le C|J|^{1/q_\nu}\Sigma_{\nu,q_\nu,f_\nu}^{\mathrm{cor}}(T)\|1_{7I_T}\rho_J^{-\alpha}\|_\infty.$$

Summing the q_ν-th power over J concludes the proof.

To estimate

$$\mathrm{II} = |\Lambda_{\mathcal{C}_T}(\Pi_{T,1}f_1,\Pi_{T,2}f_2,\Pi_{T,3}f_3)+\Lambda_{\mathcal{C}_T,c}(\Pi_{T,1}f_1,\Pi_{T,2}f_2,\Pi_{T,3}f_3)|,$$

it suffices to apply Proposition 2.3 (the global paraproduct estimate) and (4-1) in Theorem 4.3 (the L^p estimate for the phase space projection). The desired bound follows.

We move to estimate III + IV + V. Note that, for $n\in\{1,2,3\}$ and $J\in\mathcal{I}_{T,j}^1$, by definition of $\Sigma_{n,p,f}^{\mathrm{cor}}(T)$,

$$\|1_J[\phi_{j,n}*f_n]\|_{q_n}\le |J|^{1/q_n}\Sigma_{n,q_n,f_n}^{\mathrm{cor}}(T),$$

and further, by Lemma 4.6,

$$\|1_J[\phi_{j,n}*\Pi_{T,n}f_n]\|_{q_n}\le C|J|^{1/q_n}\Sigma_{n,q_n,f_n}^{\mathrm{cor}}(T).$$

By these estimates and Hölder's inequality,

$$\mathrm{III}+\mathrm{IV}+\mathrm{V}\le C\max_{n\in\{1,2,3\}}\bigg\{\bigg(\prod_{n'\neq n}\Sigma_{n',q_{n'},f_{n'}}^{\mathrm{cor}}(T)\bigg)\sum_{j\in\mathbb{Z}}\sum_{J\in\mathcal{I}_{T,j}^1}|J|^{1-1/q_n}\|1_J[\phi_{j,n}*(f_n-\Pi_{T,n}f_n)]\|_{q_n}\bigg\},$$

from which the claim follows by (4-2) of Theorem 4.3. □

5. Proof of Proposition 2.8: tree selection

We start by defining two auxiliary sizes that are needed to complement those in Definition 2.6.

Definition 5.1. Under the set-up of Definition 2.6, define

$$\Sigma_{n,p,f}^{\mathrm{bdr,top}}(T)=\sup_{\substack{P\in\mathcal{B}_T\\ I_P=I_T}}\sup_{\phi\in\Phi_n^{4\alpha}(Q_P)}\frac{\|\rho_{I_T}^{-\alpha}[\phi*f]\|_p}{|I_T|^{1/p}},$$
$$\Sigma_{n,p,f}^{\mathrm{cor,top}}(T)=\sup_{\phi\in\Phi_{n,j_T-k_1-5d}^{4\alpha}(\xi_T)}\frac{\|\rho_I^{-\alpha}[\phi*f]\|_p}{|I_T|^{1/p}}.$$

We formalize the idea of greedy selection by stating the following definition.

Definition 5.2 (selection). Let $\mathcal{V}$ be a finite set of multitiles. Let $\mathcal{T}$ be the family of all trees in any of the subsets of $\mathcal{V}$. Let S be a positive integer. A selection is a mapping $\sigma : \{1, \ldots, S\} \to \mathcal{T}$ such that

- $\sigma(1)$ is a tree in $\mathcal{V}_{\sigma(1)} = \mathcal{V}$,
- $\sigma(i+1)$ is a tree in $\mathcal{V}_{\sigma(i+1)} = \mathcal{V}_{\sigma(i)} \setminus \mathcal{P}_{\sigma(i)}$ for all $i \in \{1, \ldots, S-1\}$.

To prove Proposition 2.8, we will construct several selections over the initial set of multitiles. We first show that selections based on top size defined above have good orthogonality properties and as a second step we show that convexity properties allow us to infer estimates for main sizes of Definition 2.6 from those for the auxiliary top sizes of Definition 5.1. There will be three different selection processes. The first selection serves to identify the trees with large core size. The following proposition shows that they have controlled overlap.

Proposition 5.3. *There exists a constant C such that the following holds.*

Let $D > 1$. Let $f \in L^2(\mathbb{R}^d)$. Let $\mathcal{V}$ be a finite set of multitiles and let σ be a selection in $\mathcal{V}$. Let $M > 0$. Assume the following properties of the selection:

- *If I_i is the top cube of $\sigma(i)$ and if I_{i+1} is the top cube of $\sigma(i+1)$, then $|I_{i+1}| \le |I_i|$ for all $i \in \{1, \ldots, S-1\}$.*
- *For each $i \in \{1, \ldots, S\}$, there exists $A_i \in \mathcal{P}_{\sigma(i)}$ with $2^{k_1+1} Q_{A_i} \ni \xi_{\sigma(i)}$.*
- *For each $i \in \{1, \ldots, S\}$, we have $M \le (\Sigma^{\mathrm{cor,top}}_{n,2,f} \circ \sigma)(i) \le DM$.*

Then

$$\left(\sum_{i=1}^{S} M^2 |I_{\sigma(i)}| \right)^{1/2} \le CD \|f\|_2.$$

The next selection serves to remove the trees that contain a lacunary multitile, not treated by the core size, that however happens to give a large contribution.

Proposition 5.4. *There exists a constant C such that the following holds.*

Let $D > 1$. Let $f \in L^2(\mathbb{R}^d)$. Let $\mathcal{V}$ be a finite set of multitiles and let σ be a selection in $\mathcal{V}$. Let $M > 0$. Assume the following properties of the selection:

- *If I_i is the top cube of $\sigma(i)$ and if I_{i+1} is the top cube of $\sigma(i+1)$, then $|I_{i+1}| \le |I_i|$ for all $i \in \{1, \ldots, S-1\}$.*
- *For each $i \in \{1, \ldots, S\}$, we have $M \le (\Sigma^{\mathrm{bdr,top}}_{n,2,f} \circ \sigma)(i) \le DM$.*

Then

$$\left(\sum_{i=1}^{S} M^2 |I_{\sigma(i)}| \right)^{1/2} \le CD \|f\|_2.$$

The third selection removes the trees whose boundaries are contributing a lot to the right-hand side of Proposition 2.7. While the choice order of the previous selections was based on metric geometry, only using the size of the top cube, the treatise of the lacunary parts of the trees requires us to carry out a cone decomposition and consider an order of selection based on that.

Proposition 5.5. *There exists a constant C such that the following holds.*

Let $D > 1$, and let e be a unit vector orthogonal to $d-1$ coordinate axes. Let $f \in L^2(\mathbb{R}^d)$. Let $\mathcal{V}$ be a finite set of multitiles, and let σ be a selection in $\mathcal{V}$. Let $M > 0$. For each $i \in \{1, \dots, S\}$, define

$$C_i = \{\xi \in \mathbb{R}^d : |\xi - \xi_{\sigma(i)}| \le 2(\xi - \xi_{\sigma(i)}) \cdot e\},$$

and let $\mu_i \in M_n(\xi_{\sigma(i)}, C_i)$. Assume the following properties of the selection:

- *For all $i \in \{1, \dots, S-1\}$, assume that $\xi_{\sigma(i)} \cdot e \ge \xi_{\sigma(i+1)} \cdot e$.*
- *For each $i \in \{1, \dots, S\}$, we have $M \le (\Sigma^{\mathrm{sum}}_{n,\mu_i * f} \circ \sigma)(i)$ and $(\Sigma^{\mathrm{bdr}}_{n,2,\mu_i * f} \circ \sigma)(i) \le DM$.*

Then

$$\left(\sum_{i=1}^{S} M^2 |I_{\sigma(i)}| \right)^{1/2} \le CD\|f\|_2.$$

To apply the propositions stated above, we still have to solve the discrepancy between the definitions of sizes in Definitions 2.6 and 5.1. This is the content of the last proposition of this section. We need one more definition.

Definition 5.6 (convex collection). A finite family of multitiles $\mathcal{V}$ is a convex collection if, for any tree T on $\mathcal{V}$ and

$$j_{min} = \min_{P \in \mathcal{P}_T} \log_2 |I_P|^{1/d},$$

the condition $j \in \mathbb{Z} \cap \{i : j_{min} \le i \le j_T\}$ implies that there exist $P \in \mathcal{P}_T$ with $|I_P| = 2^{jd}$ and the condition that $2^{k_1+1} Q_P \ni \xi_T$ for some $P \in \mathcal{P}_T$ implies $2^{k_1+1} Q_{P'}$ for a $P' \in \mathcal{P}_T$ with $I_{P'} = I_T$.

For the purpose of the proof of our main theorem, the convex collections are the only ones that matter. The importance of the convex collections lies in the fact that every tree on a convex collection has a subtree whose size is attained by one of its top multitiles.

Moreover, for a tree T, we set

$$\Theta(T) = \begin{cases} 1 & \text{if there exists } P \in \mathcal{P}_T \text{ with } 2^{k_1+1} Q_P \ni \xi_T. \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 5.7. *Let $\mathcal{V}$ be a convex family. Let $\{e_\delta : 1 \le \delta \le 2d\}$ be the unit vectors orthogonal to the $(d-1)$-dimensional coordinate hyperplanes. Let*

$$C_e = \{\xi \in \mathbb{R}^d : |\xi - \xi_{\sigma(i)}| \le 2(\xi - \xi_{\sigma(i)}) \cdot e\}.$$

Let $\mu^{\delta,n} \in M_n(C_{e_\delta})$ with

$$\sum_{\delta=1}^{2d} \hat{\mu}^{\delta,n}(\xi) = 1, \quad \xi \ne 0.$$

For a tree T on $\mathcal{V}$, we set

$$\hat{\mu}^{\delta}_{T,n}(\xi) = \hat{\mu}^{\delta,n}(\xi - (\xi_T)_n).$$

Let $M \in \mathbb{Z}$ be such that, for all trees T on $\mathcal{V}$,

$$\max_{\substack{1\le\delta\le 2d\\ n\in\{1,2,3\}}} \max\{\Sigma^{\mathrm{bdr}}_{n,2,f_n}(T),\ \Sigma^{\mathrm{sum}}_{n,\mu_T^{\delta,n}*f_n}(T),\ \Theta(T)\Sigma^{\mathrm{cor}}_{n,2,f_n}(T)\} \le 2^{M/2}\|f_n\|_2.$$

Then there exists a selection σ on $\mathcal{V}$ such that

$$\widetilde{\mathcal{V}}_{M-1} = \mathcal{V} \setminus \bigcup_{i=1}^{S} \mathcal{P}_{\sigma(i)}$$

is a convex family such that, for all trees on $\widetilde{\mathcal{V}}_{M-1}$,

$$\max_{\substack{1\le\delta\le 2d\\ n\in\{1,2,3\}}} \max\{\Sigma^{\mathrm{bdr}}_{n,2,f_n}(T),\ \Sigma^{\mathrm{sum}}_{n,\mu_T^{\delta,n}*f_n}(T),\ \Theta(T)\Sigma^{\mathrm{cor}}_{n,2,f_n}(T)\} \le 2^{(M-10d)/2}\|f_n\|_2 \tag{5-1}$$

and

$$\sum_{i=1}^{S} 2^M |I_{\sigma(i)}| \lesssim 1. \tag{5-2}$$

Auxiliary propositions for almost orthogonality. In this subsection, we prove two additional estimates that are needed in the proofs of Propositions 5.3, 5.4 and 5.5.

Proposition 5.8. *Let $\alpha > 2d$. There exists a constant C such that the following holds.*

Let $j \in \mathbb{Z}$, $k \ge 0$, $\xi \in \mathbb{R}^d$, and $f \in L^\infty(\mathbb{R}^d)$. Let $\varphi \in \Phi^{4\alpha}_{n,j-k}(\xi)$ and I be a cube with $|I| = 2^{jd}$. Denote by $\mathcal{M}_{\mathrm{HL}}$ the Hardy–Littlewood maximal function. Then, for all $x \in \mathbb{R}^d$,

$$|\varphi * (\rho_I^{-\alpha} f)(x)| \le C\rho_I(x)^{-\alpha}\mathcal{M}_{\mathrm{HL}} f(x).$$

Proof. As for $j' \le j$ we have $\rho_{[0,2^{j'})^d} \ge \rho_{[0,2^j)^d}$, then, for any $\varphi \in \Phi^{4\alpha}_{n,j-k}(\xi)$ and $x \in \mathbb{R}^d$, we have

$$|\rho^{\alpha}_{[0,2^{j-k})^d}(x)\varphi(x)| \le 2^{(j-k+v_n)d}\rho^{-3\alpha}_{[0,2^{j-k-v_n})^d}(x).$$

We also have

$$\rho^{-\alpha}_{[0,2^j)^d}(x-y)\rho_I^{-\alpha}(y) \le C\rho_I^{-\alpha}(x)$$

for all $x, y \in \mathbb{R}^d$. Indeed, if $2\rho_I(y) \ge \rho_I(x)$, this is clear, and if $2\rho_I(y) \le \rho_I(x)$, then

$$\rho_{[0,2^j)^d}(x-y) \ge \rho_I(x) - \rho_I(y) \ge \frac{\rho_I(x)}{2}.$$

In conclusion,

$$\begin{aligned}|\varphi * (\rho_I^{-\alpha} f)(x)| &= \left|\int_{\mathbb{R}^d} \rho^{\alpha}_{[0,2^j)^d}(x-y)\varphi(x-y)\rho^{-\alpha}_{[0,2^j)^d}(x-y)\rho_I^{-\alpha}(y)f(y)\,\mathrm{d}y\right| \\ &\le C\rho_I(x)^{-\alpha}[2^{(j-k+v_n)d}\rho^{-3\alpha}_{[0,2^{j-k-v_n})^d} * (\rho_I^{-\alpha}|f|)](x).\end{aligned}$$

□

The second auxiliary proposition is essentially a restatement of Lemmata 5.1, 5.2 and 5.3 in [Muscalu et al. 2002a]. Also this estimate is needed in the proofs of Propositions 5.3, 5.4 and 5.5.

Proposition 5.9. *Let A_1 be a positive constant, and let $\alpha > d$. Then there exists a constant A_2 such that the following holds.*

Let $J \in \mathcal{D}$. Let $\mathcal{I} \subset \mathcal{D}$ be a family of cubes satisfying

$$\sum_{\substack{I \in \mathcal{I} \\ I \subset I'}} |I| \le A_1 |I'|$$

for all cubes I' and $|I| \le |J|$ for all $I \in \mathcal{I}$. For each $I \in \mathcal{I}$, let $g_I \in L^2(\mathbb{R})$ be given. Then

$$\Big\| \rho_J^{-\alpha} \sum_{I \in \mathcal{I}} |I|^{1/2} g_I \rho_I^{-\alpha} \Big\|_2 \le A_2 |J|^{1/2} \sup_{I \in \mathcal{I}} \|g_I\|_2. \tag{5-3}$$

Proof. We first prove the reminiscent inequality

$$\Big\| \sum_{I \in \mathcal{I}} |I|^{1/2} g_I 1_{DI} \Big\|_2 \le 2D^d \sqrt{5A_1} \sup_{I \in \mathcal{I}} \|g_I\|_2 \Big(\sum_{I \in \mathcal{I}} |I| \Big)^{1/2} \tag{5-4}$$

for all odd numbers $D \ge 3$. Here the nonlocal cut-off functions are replaced by sharp cut-off functions.

Fix a family $\mathcal{I}$ and the corresponding functions g_I. Let $\mathcal{I}' \subset \mathcal{I}$ be finite. Let A be the sharp constant for the inequality (5-4) when considered over all finite subfamilies of $\mathcal{I}'$. Then

$$\begin{aligned}\Big\| \sum_{I \in \mathcal{I}'} |I|^{1/2} g_I 1_{DI} \Big\|_2^2 &\le 2 \sum_{I \in \mathcal{I}'} \sum_{\substack{J \in \mathcal{I}' \\ DJ \subset 5DI}} \langle |I|^{1/2} g_I 1_{DI}, |J|^{1/2} g_J 1_{DJ} \rangle \le 2 \sum_{I \in \mathcal{I}'} |I|^{1/2} \|g_I\|_2 \Big\| \sum_{\substack{J \in \mathcal{I}' \\ DJ \subset 5DI}} |J|^{1/2} g_J 1_{DJ} \Big\|_2 \\ &\le 2AD^{d/2} \sup_{I \in \mathcal{I}'} \|g_I\|_2^2 \sum_{I \in \mathcal{I}'} |I|^{1/2} \Big(\sum_{\substack{J \in \mathcal{I}' \\ J \subset 5DI}} |J| \Big)^{1/2} \le 2\sqrt{5A_1} AD^d \sup_{I \in \mathcal{I}'} \|g_I\|_2^2 \Big(\sum_{I \in \mathcal{I}'} |I| \Big).\end{aligned}$$

Consequently, $A \le 2D^d \sqrt{5A_1}$. As this constant is independent of $\mathcal{I}'$ and the functions g_I, the proof of (5-4) is complete.

To prove (5-3), we write

$$\rho_I^{-\alpha} \le \sum_{k=1}^{\infty} k^{-\alpha} 1_{(2k-1)I} \quad \text{and} \quad \rho_J^{-\alpha} \le 1_J + \sum_{l=1}^{\infty} l^{-\alpha} 1_{(2l+1)J \setminus (2l-1)J}.$$

Set $\mathcal{I}_{k,l} = \{I \in \mathcal{I} : (2k-1)I \cap (2l+1)J \ne \varnothing\}$. Then

$$\begin{aligned}\Big\| \rho_J^{-\alpha} \sum_{I \in \mathcal{I}} |I|^{1/2} g_I \rho_I^{-\alpha} \Big\|_2 &\le \sum_{k,l=1}^{\infty} k^{-\alpha} l^{-\alpha} \Big\| 1_{(2l+1)J \setminus (2l-1)J} \sum_{I \in \mathcal{I}_{k,l}} |I|^{1/2} g_I 1_{(2k-1)I} \Big\|_2 \\ &\le \sum_{k,l=1}^{\infty} k^{-\alpha} l^{-\alpha} \Big\| 1_{(2\max\{l,k\}+2)J} \sum_{I \in \mathcal{I}_{k,l}} |I|^{1/2} g_I 1_{(2k-1)I} \Big\|_2 \\ &\lesssim \sum_{k,l=1}^{\infty} 2^{-d/2} k^{-\alpha-d/2} l^{-\alpha} |(2\max\{l,k\}+2)J|^{1/2} \lesssim |J|^{1/2}.\end{aligned}$$

□

Finally, we state as a separate proposition the obvious fact that elements of $\mathcal{W}$ with overlapping projections are close to each other in the product space too, something that is a direct consequence of the defining inequalities of $\mathcal{W}$.

Proposition 5.10. *Let $a \geq 0$. Let Q, $Q' \in \mathcal{W}$ satisfy $|Q| \geq |Q'|$. Assume that, for some $n \in \{1, 2, 3\}$, there exist*

$$\xi \in 2^a Q_n \cap 2^a Q'_n, \quad \eta \in 2^a Q \cap \Gamma, \quad \zeta \in 2^a Q' \cap \Gamma.$$

Then $2^{a+4} Q \supset Q'$.

Proof. First we note that the projection $\mathfrak{P} : \Gamma \to \mathbb{R}^d$ defined through $\mathfrak{P}\xi = \xi_n$ is a bijection. This follows from the regularity of L and the fact $\Gamma = \{L(\tau, \tau, \tau) : \tau \in \mathbb{R}^d\}$. Consider the metrics

$$\operatorname{dist}_{\text{full}}(\xi, \eta) = \inf\{r : \eta \in Q(\xi, r)\},$$
$$\operatorname{dist}_n(\xi_n, \eta_n) = \inf\{r : \eta_n \in Q_n(\xi_n, r)\}.$$

The left inverse $\mathfrak{P}^{-1}$ is a 2-Lipschitz mapping $(\mathbb{R}^d, \operatorname{dist}_n) \to (\Gamma, \operatorname{dist}_{\text{full}})$ following directly from the definition of the metrics. We infer that

$$\operatorname{dist}_{\text{full}}(\mathfrak{P}^{-1}\xi, \mathfrak{P}^{-1}\mathfrak{P}\eta) \leq 2 \operatorname{dist}_n(\xi, \mathfrak{P}\eta),$$
$$\operatorname{dist}_{\text{full}}(\mathfrak{P}^{-1}\xi, \mathfrak{P}^{-1}\mathfrak{P}\zeta) \leq 2 \operatorname{dist}_n(\xi, \mathfrak{P}\zeta),$$

so that

$$\operatorname{dist}_{\text{full}}(\mathfrak{P}^{-1}\mathfrak{P}\zeta, \mathfrak{P}^{-1}\mathfrak{P}\eta) \leq 2^{a+2} \operatorname{diam} Q_n,$$

where the diameter diam is computed with respect to d_n. As $|Q| \geq |Q'|$, we conclude $2^{a+3} Q \cap Q' \neq \varnothing$, and the claim follows. □

As an immediate corollary of Proposition 5.10, we conclude that the multitiles $P \in \mathcal{B}_T$ have all their frequency projections supported far from the projections of the top frequency. This will imply important L^2 orthogonality properties for the sum size.

Proposition 5.11. *Given $P \in \mathcal{B}_T$ with $|I_P| = 2^{jd}$ for some $j \in \mathbb{Z}$, we have, for all $n \in \{1, 2, 3\}$,*

$$(Q_P)_n \subset \mathbb{R}^d \setminus Q_n(\xi_T, 2^{-j-k_2}).$$

Proof. Define $Q = Q_P$. By construction, $2^{k_0+1} Q \cap \Gamma \neq \varnothing$. On the other hand, as $P \in \mathcal{B}_T$, we know that $\xi_T \notin 2^{k_1+1} Q$. Set $Q' = Q(\xi_T, 2^{-j+k_1/50})$. Then $Q' \cap Q = \varnothing$. It follows by Proposition 5.10 that $Q'_n \cap Q_n = \varnothing$. □

5.1. *Proof of Proposition 5.3: core size.*

For each $i \in \{1, \dots, S\}$, we find

$$\phi_i \in \Phi^{4\alpha}_{n, j_{\sigma(i)} - k_1 - 5d}(\xi_{\sigma(i)})$$

such that

$$c_i = \|\rho^{-\alpha}_{I_{\sigma(i)}}[\phi_i * f]\|_2, \quad M\sqrt{|I_{\sigma(i)}|} \leq c_i \leq DM\sqrt{|I_{\sigma(i)}|}.$$

Let

$$g_i = \frac{\rho^{-\alpha}_{I_{\sigma(i)}}[\phi_i * f]}{\|\rho^{-\alpha}_{I_{\sigma(i)}}[\phi_i * f]\|_2}.$$

Now

$$\sum_{i=1}^{S} M^2|I_{\sigma(i)}| \lesssim \sum_{i=1}^{S} c_i|I_{\sigma(i)}|^{1/2}\langle f, \bar{\phi}_i * (\rho_{I_{\sigma(i)}}^{-\alpha} g_i)\rangle \le \|f\|_2 \left\| \sum_{i=1}^{S} c_i|I_{\sigma(i)}|^{1/2}[\phi_i * (\rho_{I_{\sigma(i)}}^{-\alpha} g_i)] \right\|_2 .$$

Expanding the square and using the symmetry, we obtain

$$\left\| \sum_{i=1}^{S} |I_{\sigma(i)}|^{1/2} c_i[\phi_i * (\rho_{I_T}^{-\alpha} g_i)] \right\|_2^2 \lesssim D^2M^2 \sum_{i=1}^{S} |I_{\sigma(i)}|^{1/2} \sum_{l=i}^{S} |I_{\sigma(l)}|^{1/2} \langle \rho_{I_{\sigma(l)}}^{-\alpha} g_l, \phi_l * [\phi_i * (\rho_{I_{\sigma(i)}}^{-\alpha} g_i)]\rangle. \tag{5-5}$$

Let

$$\mathcal{A}_i = \{l \in \{i, \dots, S\} : \operatorname{supp} \hat{\phi}_l \cap \operatorname{supp} \hat{\phi}_i \ne \varnothing\}.$$

By Proposition 5.8, for all $l \in \mathcal{A}_i$,

$$\phi_l * [\phi_i * (\rho_{I_{\sigma(i)}}^{-\alpha} g_i)] \le C\rho_{I_{\sigma(i)}}^{-\alpha} \mathcal{M}_{\mathrm{HL}} g_i,$$

where $\mathcal{M}_{\mathrm{HL}}$ is the Hardy–Littlewood maximal function. Using this estimate, the Cauchy–Schwarz inequality and the Hardy–Littlewood maximal function theorem, we bound the right-hand side of (5-5) by

$$CD^2M^2 \sum_{i=1}^{S} |I_{\sigma(i)}| \left\| \frac{1}{|I_{\sigma(i)}|^{1/2}} \sum_{l\in\mathcal{A}_i} |I_{\sigma(l)}|^{1/2} g_l \rho_{I_{\sigma(l)}}^{-\alpha} \rho_{I_{\sigma(i)}}^{-\alpha} \right\|_2 .$$

By hypothesis, for each $l \in \{1, \dots, S\}$, there exists a top multitile $A_l \in \mathcal{P}_{\sigma(l)}$ with $\xi_{\sigma(l)} \in 2^{k_1+1} Q_{A_l}$. Hence, given $l, j \in \mathcal{A}_i$ with $l > j$, we have

$$(2 \operatorname{supp} \hat{\phi}_l) \cap (2 \operatorname{supp} \hat{\phi}_j) \ne \varnothing.$$

Therefore, we have

$$(2 \operatorname{supp} \hat{\phi}_l) \cap (2 \operatorname{supp} \hat{\phi}_j) \ne \varnothing \quad \text{and} \quad I_{\sigma(l)} \cap I_{\sigma(j)} \ne \varnothing$$

only if $|I_{\sigma(l)}| \gtrsim |I_{\sigma(j)}|$, as otherwise Proposition 5.10 would imply

$$2^{k_2+1} Q_{A_l} \supset 2^{k_1+5} Q_{A_l} \supset 2^{k_1+1} Q_{A_j} \ni \xi_{\sigma(j)},$$

which in turn would contradict $A_l \in \mathcal{V}_{\sigma(l)}$. By the definition of the selection, $|I_{\sigma(l)}| \le |I_{\sigma(j)}|$. Moreover, for every fixed $I_{\sigma(j)}$, there are only up to $C(d, k_0)$ elements $l \in \mathcal{A}_i$ such that $I_{\sigma(j)} = I_{\sigma(l)}$, so we can conclude that, for any $i \in \{1, \dots, S\}$,

$$\left\| \sum_{l\in\mathcal{A}_i} 1_{I_{\sigma(l)}} \right\|_\infty \lesssim 1;$$

hence $\{I_{\sigma(l)} : l \in \mathcal{A}_i\}$ is a Carleson family. By Proposition 5.9,

$$\left\| \frac{1}{|I_{\sigma(i)}|^{1/2}} \sum_{l\in\mathcal{A}_i} |I_{\sigma(l)}|^{1/2} g_l \rho_{I_{\sigma(l)}}^{-\alpha} \rho_{I_{\sigma(i)}}^{-\alpha} \right\|_2 \lesssim 1,$$

and we have shown the claim for the sum over all $i \in \{1, \dots, S\}$. ∎

5.2. ***Proof of Proposition 5.4: boundary size.*** For each $i \in \{1, \dots, S\}$, we find

$$\phi_i \in \Phi_n^{4\alpha}(Q_i),$$

where $Q_i = Q_{P_i}$ and P_i is a top multitile of $\sigma(i)$ such that

$$c_i = \|\rho_{I_{\sigma(i)}}^{-\alpha}[\phi_i * f]\|_2, \quad M\sqrt{|I_{\sigma(i)}|} \le c_i \le DM\sqrt{|I_{\sigma(i)}|}.$$

Let

$$g_i = \frac{\rho_{I_T}^{-\alpha}[\phi_i * f]}{\|\rho_{I_T}^{-\alpha}[\phi_i * f]\|_2}.$$

Now

$$\sum_{i=1}^{S} M^2|I_{\sigma(i)}| \lesssim \sum_{i=1}^{S} c_i|I_{\sigma(i)}|^{1/2}\langle f, \bar{\phi}_i * (\rho_{I_{\sigma(i)}}^{-\alpha} g_i)\rangle \le \|f\|_2 \Big\| \sum_{i=1}^{S} c_i|I_{\sigma(i)}|^{1/2}[\phi_i * (\rho_{I_{\sigma(i)}}^{-\alpha} g_i)]\Big\|_2.$$

Fix $\kappa \in \{0, \dots, 99\}$. Write $\mathcal{L} = \{i \in \{1, \dots, S\} : \log_2|I_i|^{1/d} \in \kappa + 100\mathbb{Z}\}$. Expanding the square and using the symmetry, we obtain

$$\Big\| \sum_{i\in\mathcal{L}} |I_{\sigma(i)}|^{1/2} c_i[\phi_i * (\rho_{I_T}^{-\alpha} g_i)]\Big\|_2^2 \lesssim D^2M^2 \sum_{i\in\mathcal{L}} |I_{\sigma(i)}|^{1/2} \sum_{\substack{l\in\mathcal{L} \\ |I_l|\le|I_i|}} |I_{\sigma(l)}|^{1/2}\langle g_l\rho_{I_{\sigma(l)}}^{-\alpha}, \phi_l * [\phi_i * (\rho_{I_{\sigma(i)}}^{-\alpha} g_i)]\rangle.$$

Let

$$\mathcal{A}_i = \{l \in \mathcal{L} : l \in \{i, \dots, S\}, \operatorname{supp}\hat{\phi}_l \cap \operatorname{supp}\hat{\phi}_i \ne \varnothing\}.$$

By Proposition 5.8, the Cauchy–Schwarz inequality and the estimates for the Hardy–Littlewood maximal function as above, it suffices to prove a bound by constant of

$$\Big\| \frac{1}{|I_{\sigma(i)}|^{1/2}} \sum_{l\in\mathcal{A}_i} |I_{\sigma(l)}|^{1/2} g_l \rho_{I_{\sigma(l)}}^{-\alpha} \rho_{I_{\sigma(i)}}^{-\alpha}\Big\|_2.$$

Indeed, by the triangle inequality we can then sum over $\kappa \in \{0, \dots, 99\}$ to conclude the proof. By Proposition 5.9, it hence remains to show that $\{I_{\sigma(l)} : l \in \mathcal{A}_i\}$ is a Carleson family.

Given $l, j \in \mathcal{A}_i$ with $l > j \ge i$ and hence $|I_{\sigma(l)}| \le |I_{\sigma(j)}|$, we have

$$(Q_j)_n \cap (Q_i)_n \ne \varnothing, \quad (Q_l)_n \cap (Q_i)_n \ne \varnothing.$$

Therefore, we have

$$I_{\sigma(l)} \cap I_{\sigma(j)} \ne \varnothing$$

only if $|I_{\sigma(l)}| = |I_{\sigma(j)}|$, as otherwise Proposition 5.10 would imply

$$2^{k_2+1}Q_l \supset 2^{k_2+1}Q_j \ni \xi_{\sigma(j)},$$

which in turn would contradict $P_l \in \mathcal{V}_{\sigma(l)}$. Therefore $I_{\sigma(j)}$ and $I_{\sigma(l)}$ are pairwise disjoint unless $I_{\sigma(j)} = I_{\sigma(l)}$. However, as above, for every fixed $I_{\sigma(j)}$, there are only up to $C(d, k_0)$ elements $l \in \mathcal{A}_i$ such that $I_{\sigma(j)} = I_{\sigma(l)}$. Hence $\{I_{\sigma(l)} : l \in \mathcal{A}_i\}$ is a Carleson family, and the proof is complete. □

5.3. *Proof of Proposition 5.5: sum size.* Consider an index $i \in \{1, \dots, S\}$. For $\theta \geq 1$, we define

$$C_i(\theta) = \{\xi \in \mathbb{R}^d : |\xi - \xi_{\sigma(i)}| \leq \theta(\xi - \xi_{\sigma(i)}) \cdot e\}.$$

Write

$$a_i(j, \theta) = C_i(\theta) \cap (Q_n(\xi_{\sigma(i)}, 2^{-j+1}) \setminus Q_n(\xi_{\sigma(i)}, 2^{-j})).$$

We let

$$\mathcal{B}^i_j = \{P \in \mathcal{B}_{\sigma(i)} : (Q_P)_n \cap a_i(j, 2) \neq \varnothing\}.$$

We note that if $P \in \mathcal{B}^i_j$, then by Proposition 5.11,

$$(Q_P)_n \subset \bigcup_{k=j-50k_2}^{j+50k_2} a_i(k, 10).$$

For each $P \in \mathcal{B}^i_j$, we find $\phi_P \in M_n(\xi_{\sigma(i)}, (Q_P)_n)$ with $\hat{\phi}_P \subset (Q_P)_n$ such that, for

$$c_P = \|\rho^{-\alpha}_{I_P}[\phi_P * f]\|_2,$$

we have

$$\sum_{j\in\mathbb{Z}} \sum_{P\in\mathcal{B}^i_j} c_P^2 \gtrsim M^2 |I_{\sigma(i)}|, \quad c_P \leq DM\sqrt{|I_P|}.$$

Let

$$g_P = \frac{\rho^{-\alpha}_{I_P}[\phi_P * f]}{\|\rho^{-\alpha}_{I_P}[\phi_P * f]\|_2}$$

if $P \in \mathcal{B}^i_j$ for some i and j, and $g_P = 0$ otherwise.

Now

$$\begin{aligned}\sum_{i=1}^S M^2 |I_{\sigma(i)}| &\lesssim \sum_{i=1}^S \sum_{j\in\mathbb{Z}} \sum_{P\in\mathcal{B}^i_j} c_P \langle f, \bar{\phi}_P * (\rho^{-\alpha}_{I_P} g_P)\rangle \\ &\leq \|f\|_2 \left\| \sum_{i=1}^S \sum_{j\in\mathbb{Z}} \sum_{P\in\mathcal{B}^i_j} c_P[\phi_P * (\rho^{-\alpha}_{I_P} g_P)] \right\|_2.\end{aligned}$$

By the triangle inequality, we may restrict the sum over $j \in \mathbb{Z}$ to a sum over $j \in \kappa + 1000k_2\mathbb{Z}$ and integer κ. For fixed κ and every $i \in \{1, \dots, S\}$ we define

$$\mathcal{E}^i_\kappa = \bigcup_{j\in\kappa+1000k_2\mathbb{Z}} \mathcal{B}^i_j.$$

Squaring the second factor and using symmetry, we compute

$$\begin{aligned}&\left\| \sum_{i=1}^S \sum_{P\in\mathcal{E}^i_\kappa} c_P[\phi_P * (\rho^{-\alpha}_{I_P} g_P)] \right\|_2^2 \\ &\quad\lesssim \left(\sup_{P\in\bigcup_{i=1}^S \mathcal{E}^i_\kappa} \frac{c_P^2}{|I_P|} \right) \sum_{s=1}^S \sum_{P\in\mathcal{E}^s_\kappa} |I_P|^{1/2} \sum_{l=1}^S \sum_{\substack{P'\in\mathcal{E}^l_\kappa \\ |I_{P'}|\leq|I_P|}} |I_{P'}|^{1/2} \langle \rho^{-\alpha}_{I_{P'}} g_{P'}, \phi_{P'} * [\phi_P * (\rho^{-\alpha}_{I_P} g_P)]\rangle. \quad (5\text{-}6)\end{aligned}$$

Fix s and $P \in \mathcal{P}_{\sigma(s)}$, and let

$$\mathcal{A}_P = \Big\{ P' \in \bigcup_{j \le 0} \mathcal{B}^s_{j_{\sigma(s)}+1000j} : \operatorname{supp} \hat{\phi}_P \cap \operatorname{supp} \hat{\phi}_{P'} \neq \varnothing \Big\}.$$

By Proposition 5.11, we may apply Proposition 5.8 to bound

$$\phi_{P'} * [\phi_P * (\rho_{I_P}^{-\alpha} g_P)] \lesssim \rho_{I_P}^{-\alpha} \mathcal{M}_{\mathrm{HL}} g_P.$$

By the Cauchy–Schwarz inequality and the Hardy–Littlewood maximal function theorem as above, we hence obtain

$$\begin{aligned} &\sum_{P' \in \mathcal{A}_P} |I_{P'}|^{1/2} \langle \rho_{I_{P'}}^{-\alpha} g_{P'}, \phi_{P'} * [\phi_P * (\rho_{I_P}^{-\alpha} g_P)] \rangle \\ &\lesssim \sum_{P' \in \mathcal{A}_P} |I_{P'}|^{1/2} \langle \rho_{I_P}^{-\alpha} \rho_{I_{P'}}^{-\alpha} g_{P'}, \mathcal{M}_{\mathrm{HL}} g_P \rangle \lesssim \Big\| \sum_{P' \in \mathcal{A}_P} |I_{P'}|^{1/2} \rho_{I_P}^{-\alpha} \rho_{I_{P'}}^{-\alpha} g_{P'} \Big\|_2 \lesssim |I_P|^{1/2} \rho_{I_P}(\mathbb{R}^d \setminus I_{\sigma(s)}), \end{aligned} \tag{5-7}$$

where the last inequality follows by Proposition 5.12 below, the fact that, for every fixed I_P, there are only up to $C(d, k_0)$ elements $P' \in \mathcal{A}_P$ such that $I_P = I_{P'}$, and Proposition 5.9.

Proposition 5.12. *Assume that $L \in \sigma(l)$, $H \in \sigma(h)$, and that $L, H \in \mathcal{A}_P$. Assume additionally that $|I_H| < |I_L| < |I_P|$. Then*

$$I_L \cap I_H = \varnothing, \quad (I_L \cup I_H) \cap I_P = \varnothing.$$

Proof. Because $L, H \in \mathcal{A}_P$,

$$(Q_L)_n \cap (Q_P)_n \neq \varnothing, \quad (Q_H)_n \cap (Q_P)_n \neq \varnothing.$$

As $2^{2000k_2}|I_H| \le 2^{1000k_2}|I_L| \le |I_P|$, this implies, by Proposition 5.10,

$$\xi_{\sigma(s)} \in 2^{k_2+1} Q_P \subset 2^{k_2+1} Q_L \subset 2^{k_2+1} Q_H. \tag{5-8}$$

Further,

$$a_l(j, 10) \cap a_h(j', 10) \neq \varnothing,$$

with $j' \le j - 10k_2$ only if $\xi_{\sigma(l)} \cdot e > \xi_{\sigma(h)} \cdot e$. Hence $s < l < h$, and the claim follows by (5-8), as otherwise it would contradict $L \in \mathcal{V}_{\sigma(l)}$ and $H \in \mathcal{V}_{\sigma(h)}$. □

Applying the estimate (5-7) to the second factor on the right-hand side of (5-6), we obtain

$$\begin{aligned} \sum_{s=1}^{S} \sum_{P \in \mathcal{E}^s_\kappa} |I_P|^{1/2} \Big\| \sum_{P' \in \mathcal{A}_P} |I_{P'}|^{1/2} \rho_{I_P}^{-\alpha} \rho_{I_{P'}}^{-\alpha} g_{P'} \Big\|_2 &\lesssim \sum_{s=1}^{S} \sum_{P \in \mathcal{P}_{\sigma(s)}} |I_P| \rho_{I_P}^{-\alpha}(\mathbb{R}^d \setminus I_{\sigma(s)}) \\ &\lesssim \sum_{s=1}^{S} |I_{\sigma(s)}| \sum_{k=0}^{\infty} k 2^{-k} \lesssim \sum_{s=1}^{S} |I_{\sigma(s)}|. \end{aligned}$$

This concludes the proof of Proposition 5.5. □

5.4. ***Proof of Proposition 5.7: recursion.*** To streamline the language, we introduce the following definition.

Definition 5.13 (admissible tree). Let $\mathcal{V}$ be a finite subset of multitiles. A tree $T = T(\xi, I_0, \mathcal{V})$ is said to be n-admissible with respect to boundary size if

$$\Sigma^{\mathrm{bdr}}_{n,2,f_n}(T) \leq \Sigma^{\mathrm{bdr,top}}_{n,2,f_n}(T).$$

It is said to be n-admissible with respect to core size if

$$\Sigma^{\mathrm{cor}}_{n,2,f_n}(T) \leq \Sigma^{\mathrm{cor,top}}_{n,2,f_n}(T).$$

Proposition 5.14. *Let $N, N' > 0$. The family of multitiles*

$$\mathcal{V} = \{P : Q_P \in \mathcal{W}_N,\ I_P \subset [-N'2^N, N'2^N]^{3\times d}\},$$

with $\mathcal{W}$ in Proposition 2.2, is a convex collection. If σ is a selection on a convex collection, then $\mathcal{V}_{\sigma(i)}$ is a convex collection for all $i \in \{1, \dots, S\}$.

Proof. The proof is clear. □

Now we can proceed to the actual proof. We define the selection on $\mathcal{V}$ as follows. For notational purposes, we set $I_{\sigma(0)} = \mathbb{R}^d$, $\mathcal{P}_{\sigma(0)} = \varnothing$, and $\mathcal{V}_{\sigma(0)} = \mathcal{V}$. Finally, without loss of generality and only for notational convenience, assume $\|f_n\|_2 = 1$ for all n.

Suppose $\sigma(i-1)$ has been defined. A tree T is called an X-tree if $X(T) \geq 2^{(M-10d)/2}$ for some size X. We first define the selection by choosing repeatedly $\Sigma^{\mathrm{cor,top}}_{n,2,f_n}$-trees with $n = 1$ such that trees with larger top cubes are chosen first, only admissible trees are chosen, and only trees T with $\Theta(T) \neq 0$ are chosen. We denote by s_n the number of steps at which we reach the last tree chosen. This number is finite as there are only finitely many multitiles in the original collection $\mathcal{V}$.

We replace $\mathcal{V}$ with $\mathcal{V}_{\sigma(s_n)} \setminus \mathcal{P}_{\sigma(s_n)}$. We repeat the same process with X replaced by $\Sigma^{\mathrm{cor,top}}_{n,2,f_n}$, first with $n = 2$ and then with $n = 3$. This way, we create three selections. The first is σ_1 on $\mathcal{V}$. The second is σ_2 on $\mathcal{V}_{\sigma(s_1)} \setminus \mathcal{P}_{\sigma(s_1)}$, and the third is σ_3 on $\mathcal{V}_{\sigma(s_2)} \setminus \mathcal{P}_{\sigma(s_2)}$. By Proposition 5.3, each of these selections satisfies (5-2). Set

$$\mathcal{V}_1 = \mathcal{V} \setminus \bigcup_{n=1}^{3} \bigcup_{i=s_{n-1}+1}^{s_n} \mathcal{P}_{\sigma(i)}.$$

A tree T on $\mathcal{V}_1$ is either inadmissible or satisfies

$$\Theta(T)\Sigma^{\mathrm{cor,top}}_{n,2,f_n}(T) \leq 2^{(M-10d)/2}.$$

For admissible trees, the latter condition is the desired size bound. On the other hand, if an inadmissible tree violates the size bound, then, by convexity of the reference family, it contains an admissible subtree violating the size bound. But this possibility was just ruled out. This concludes the treatise with respect to the core size.

We repeat the same selection process with $\Sigma^{\mathrm{bdr,top}}_{n,2,f_n}$ instead of $\Theta\Sigma^{\mathrm{cor,top}}_{n,2,f_n}$, and this gives us three more selections σ_4, σ_5, and σ_6 and a family $\mathcal{V}_2$ such that the trees in the selections satisfy (5-2) by Proposition 5.4 and, by the argument as in the case of the core size, the size bound is also valid.

It remains to treat the sum size. Let $\{e_\delta : 1 \le \delta \le 2d\}$ be the collection of unit vectors orthogonal to the $(d-1)$-dimensional hyperplanes. Let $\mu^{\delta,n} \in M_n(C_{e_\delta})$, with

$$\sum_{\delta=1}^{2d} \hat{\mu}^{\delta,n}(\xi) = 1, \quad \xi \neq 0.$$

For a tree T on $\mathcal{V}_2$ or any of its subfamilies, we set $\hat{\mu}^{\delta}_{T,n}(\xi) = \hat{\mu}^{\delta,n}(\xi - (\xi_T)_n)$. We run the selection choosing trees T such that

$$\Sigma^{\mathrm{sum}}_{n,2,\mu^{\delta,n}*f_n}(T) \ge 2^{(M-10d)/2},$$

so that those with maximal $e_\delta \cdot (\xi_T)_n$ are chosen first. Again, we repeat this process for each n and each δ. Each of the $6d$ selections satisfies the hypotheses of Proposition 5.5, and the trees not chosen satisfy (5-1). Collecting the trees in all of the selections constructed so far, we obtain the family $\mathcal{T}_M$, and the proof is complete. □

5.5. *Conclusion of the proof of Proposition 2.8.* Because the family of multitiles $\mathcal{V}$ in the hypothesis of the proposition is finite, there exists M such that the hypothesis of Proposition 5.7 holds. The claim except for (2-5) follows by induction.

To prove (2-5), we first note that, for any tree T and any $p \in [1, \infty]$,

$$\Sigma^{\mathrm{cor}}_{n,p,f_n}(T) + \Sigma^{\mathrm{bdr}}_{n,p,f_n}(T) \le C(d)\|f_n\|_\infty$$

is obvious. It remains to bound the sum size.

Let η be a smooth function with $\eta \gtrsim 1_{I_T}$ and $\operatorname{supp}\hat{\eta} \subset Q_{n_*}(0, 2^{-j_T})$. Let $\{\varphi_P \in \Phi^{4\alpha}_n(Q_P) : P \in \mathcal{B}_T\}$ be functions that almost achieve the supremum in the definition of the sum size. First, we note

$$\sum_{P\in\mathcal{B}_T} \|1_{I_P}[\varphi_P * (1_{\mathbb{R}^d\setminus 3I_T} f_n)]\|_2^2 \lesssim \|f_n\|_\infty \sum_{P\in\mathcal{B}_T} |I_{I_P}|\rho^{-\alpha}_{I_P}(\mathbb{R}^d \setminus 3I_T) \lesssim |I_T|\|f_n\|_\infty.$$

Second, we note

$$\sum_{P\in\mathcal{B}_T} \|1_{I_P}[\varphi_P * (1_{3I_T} f_n)]\|_2^2 \lesssim \sum_{k=k_1}^{k_2} \sum_{j\le j_T} \|1_T[\varphi_{j,k} * (1_{3I_T} f_n)]\|_2^2 \tag{5-9}$$

for a family of sequences $\{\{\varphi_{j,k} \in \Psi^{4\alpha}_{n,j-k}(\xi_T) : j \le j_T\} : k \in \{k_1, \dots, k_2\}\}$. Without loss of generality, we fix k and we drop it from notation. For terms with $j_T - j \le 100$, we estimate

$$\|1_T[\varphi_j * (1_{3I_T} f_n)]\|_2^2 \lesssim |I_T|\|f_n\|_\infty$$

as in the cases of core and boundary sizes. For terms with $j_T - j > 100$, we note that

$$[\mathbb{R}^d \setminus Q_n(\xi_T, 2^{-j+k-100})] \supset (\operatorname{supp}\hat{\eta} + \operatorname{supp}\hat{\varphi}_j) \supset \operatorname{supp}(\hat{\eta} * \hat{\varphi}_j).$$

Consequently, we have $\operatorname{supp}(\hat{\eta}*\hat{\varphi}_j)\cap\operatorname{supp}(\hat{\eta}*\hat{\varphi}_{j'})=\varnothing$ whenever $|j-j'|\geq 100$, and $\sum_{j<j_T-100}\hat{\eta}*\hat{\varphi}_j$ is a Mikhlin multiplier with bounds only depending on the dimension.

We bound each inner sum in (5-9) by

$$\sum_{j<j_T-100}\|\eta[\varphi_j*(1_{3I_T}f_n)]\|_2^2 \lesssim \sum_{l=1}^{100}\sum_{\substack{j\in l+100\mathbb{Z}\\ j<j_T-100}}\|\eta[\varphi_j*(1_{3I_T}f_n)]\|_2^2 \lesssim \sum_{l=1}^{100}\Big\|\eta\sum_{\substack{j\in l+100\mathbb{Z}\\ j<j_T-100}}[\varphi_j*(1_{3I_T}f_n)]\Big\|_2^2 \lesssim \|1_{3I_T}f_n\|_2^2,$$

where the last step followed by the Mikhlin multiplier theorem. The claim then follows. □

6. Proof of Proposition 2.2: tensorized model form

By dominated convergence, there exists $N'=N'(d,\alpha,\varepsilon,k_0,f_n,N)>0$ such that, for the finite subset of multitiles $\mathcal{V}=\{P: Q_P\in\mathcal{W}_N,\ I_P\subset[-N'2^N,N'2^N]^{3\times d}\}$, we have

$$\Big|\sum_{Q\in\mathcal{W}_N}\int_{\mathbb{R}^d}\prod_{n=1}^{3}[\phi_{Q,n}*f_n(x)]\,\mathrm{d}x\Big| \leq 2\Big|\sum_{P\in\mathcal{V}}\int_{\mathbb{R}^d}1_{I_P}(x)\prod_{n=1}^{3}[\phi_{P,n}*f_n(x)]\,\mathrm{d}x\Big|.$$

Consider the families of trees $\mathcal{T}_M$ as in Proposition 2.8. By the triangle inequality, we get the upper bound

$$\sum_{M\in\mathbb{N}\cup\{-\infty\}}\sum_{T\in\mathcal{T}_M}\Big|\sum_{P\in\mathcal{P}_T}\int_{\mathbb{R}^d}1_{I_P}(x)\prod_{n=1}^{3}[\phi_{P,n}*f_n(x)]\,\mathrm{d}x\Big|.$$

By Proposition 2.7, we get the upper bound

$$\sum_{M\in\mathbb{N}\cup\{-\infty\}}\sum_{T\in\mathcal{T}_M}|I_T|\Big(\Sigma^{\mathrm{bdr}}_{n_*,\infty,f_{n_*}}(T)\prod_{n\neq n_*}\Sigma^{\mathrm{sum}}_{n,f_n}(T)+\prod_{n=1}^{3}\Sigma^{\mathrm{cor}}_{n,q_n,f_n}(T)\Big), \tag{6-1}$$

where the second summand in brackets appears if and only if $\mathcal{P}_T\setminus\mathcal{B}_T\neq\varnothing$, i.e., if there exists $P\in\mathcal{P}_T$ with $2^{k_1+1}Q_P\ni\xi_T$.

By log-convexity and (2-5) from Proposition 2.8, we have

$$\Sigma^{\mathrm{cor}}_{n,q_n,f_n}(T)\lesssim\Sigma^{\mathrm{cor}}_{n,2,f_n}(T)^{2/q_n},\qquad \Sigma^{\mathrm{sum}}_{n,f_n}(T)\lesssim\Sigma^{\mathrm{sum}}_{n,f_n}(T)^{2/q_n}.$$

By the local Bernstein's inequality (see, e.g., Proposition 1.2 in [Fraccaroli et al. 2022]),

$$\Sigma^{\mathrm{bdr}}_{n,\infty,f_n}(T)\lesssim 2^{dv_n/2}\Sigma^{\mathrm{bdr}}_{n,2,f_n}(T),\qquad \Sigma^{\mathrm{cor}}_{n,q_n,f_n}(T)\lesssim 2^{dv_n(1/2-1/q_n)}\Sigma^{\mathrm{cor}}_{n,2,f_n}(T).$$

In addition, we know by Proposition 2.8 that, for all $n\in\{1,2,3\}$ and $T\in\mathcal{T}_M$,

$$\Sigma^{\mathrm{bdr}}_{n,2,f_n}(T)\lesssim\min\{1,2^{M/2}\|f_n\|_2\},\qquad \Sigma^{\mathrm{sum}}_{n,f_n}(T)\leq 2^{M/2}\|f_n\|_2,\qquad \Sigma^{\mathrm{cor}}_{n,2,f_n}(T)\lesssim 1.$$

Moreover, if there exists $P\in\mathcal{P}_T$ with $2^{k_1+1}Q_P\ni\xi_T$, we also have

$$\Sigma^{\mathrm{cor}}_{n,2,f_n}(T)\leq 2^{M/2}\|f_n\|_2.$$

Recalling that there is n_* such that $v_{n_*}=0$, we hence bound (6-1) by

$$\sum_{M\in\mathbb{N}\cup\{-\infty\}}\sum_{T\in\mathcal{T}_M}|I_T|\min\{1,2^{M/2}\|f_{n_*}\|_2\}\prod_{n\neq n_*}2^{M/q_n}\|f_n\|_2^{2/q_n}$$
$$\lesssim\sum_{M\in\mathbb{N}\cup\{-\infty\}}\min\{2^{-M/q_{n*}},2^{M(1/2-1/q_{n*})}\|f_{n_*}\|_2\}\prod_{n\neq n_*}\|f_n\|_2^{2/q_n}\lesssim\prod_{n=1}^{3}\|f_n\|_2^{2/q_n},$$

where the first inequality used equation (2-6) from Proposition 2.8. This concludes the proof of Proposition 2.2. □

7. Proof of Theorem 1.1

The integral

$$\int_{\mathbb{R}^{3\times d}}\delta_0(\xi_1+\xi_2+\xi_3)\hat{f}_1(\xi_1)\hat{f}_2(\xi_2)\hat{f}_3(\xi_3)m(L^{-1}\xi)\,d\xi \tag{7-1}$$

is absolutely convergent for Schwartz functions f_1, f_2, and f_3. Approximating m with a symbol supported in a compact set not meeting $\{(\tau,\tau,\tau):\tau\in\mathbb{R}^d\}$, we conclude by the Lebesgue dominated convergence theorem, boundedness of m and absolute convergence of the integral that it suffices to assume m is compactly supported.

By multilinear interpolation [Janson 1988], it suffices to prove a bound for the integral (7-1) by

$$C\prod_{n=1}^{3}\|f_n\|_{q_n}$$

when $f_n=1_{E_n}$ for measurable sets E_n of finite measure and where C is a constant independent of all E_n and m. Because m is assumed to be compactly supported, the integral (7-1) is absolutely convergent even with $f_n=1_{E_n}$. By standard convolution approximation and the dominated convergence theorem, we see that it suffices to bound the integral (7-1) by

$$C\prod_{n=1}^{3}|E_n|^{1/q_n}$$

whenever f_n is a smooth function with

$$\|f_n\|_\infty\le 2,\qquad \|f_n\|_2\le 2|E_n|^{1/2}.$$

Indeed, the convolution mollification converges in all L^p norms with p finite, in particular with $p\in\{2,q_n\}$. We see that, for each $n\in\{1,2,3\}$, the function f_n satisfies the assumptions on Proposition 2.2.

Next we form a Whitney-type decomposition of $\mathbb{R}^{3\times d}\setminus\Gamma$. For each $\xi\in\mathbb{R}^{3\times d}\setminus\Gamma$, set

$$r_\xi=\tfrac{3}{4}\inf\{r>0:Q(\xi,r)\cap\Gamma\neq\varnothing\},$$

where $Q(\xi,r)\subset\mathbb{R}^{3\times d}$ is the open rectangular box defined in Section 2. Let

$$\mathcal{A}=\{Q(\xi,r):r=2^{-k_0}r_\xi\}.$$

We let $\mathcal{W}$ be a maximal pairwise disjoint family of $Q \in \mathcal{A}$ such that $Q \cap [-2^N, 2^N]^{3\times d} \neq \varnothing$, where $N \geq 100$ is an integer such that $\operatorname{supp} m \subset L^{-1}([-2^N, 2^N]^{3\times d})$. It is clear that, for each fixed $k \leq k_0$,

$$\{2^k Q : Q \in \mathcal{W}\}$$

has bounded overlap. Note that

$$\operatorname{supp} m \subset L^{-1}\Big(\bigcup_{Q\in\mathcal{W}} 5Q\Big).$$

For $Q \in \mathcal{W}$, we define $Q_n = \{\xi_n \in \mathbb{R}^d : \xi \in Q\}$. Let $\{\eta_Q : Q \in \mathcal{W}\}$ form a partition of unity adapted to $\mathcal{W}$, meaning that, for each $Q \in \mathcal{W}$, the smooth function $\eta_Q \geq 0$ is supported in $6Q$ and satisfies the bounds

$$|\partial_1^{\gamma_1}\partial_2^{\gamma_2}\partial_3^{\gamma_3}\eta_Q(\xi)| \leq C_\gamma |Q_1|^{-|\gamma_1|/d}|Q_2|^{-|\gamma_2|/d}|Q_3|^{-|\gamma_3|/d}$$

for constants C_γ only depending on $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{N}^{3\times d}$.

Let $\chi_{Q,n}$ be a smooth function with

$$1_{7Q_n} \leq \chi_{Q,n} \leq 1_{8Q_n}, \quad |\partial^\gamma \chi_{Q,n}(\tau)| \leq C_\gamma |Q_n|^{-|\gamma|/d}$$

for all $\gamma \in \mathbb{N}^d$ and $|\gamma| \leq 100d$. Let $\chi_Q(\xi) = \chi_{Q,1}(\xi_1)\chi_{Q,2}(\xi_2)\chi_{Q,3}(\xi_3)$ for $\xi \in \mathbb{R}^{3\times d}$.

Let A_Q be a linear mapping sending $7Q - \mathfrak{P}(\xi_Q)$ into $[0, 2\pi)^{3\times d}$ and such that

$$A_Q([8Q - \mathfrak{P}(\xi_Q)]) \setminus (-2\pi, 2\pi)^{3\times d} \neq \varnothing,$$

where ξ_Q is such that $Q = Q(\xi_Q, r)$ and $\mathfrak{P}$ is the orthogonal projection of $\mathbb{R}^{3\times d}$ onto Γ. Such a matrix is of block form: $A_Q = A_{Q,1} \oplus A_{Q,2} \oplus A_{Q,3}$. We expand as a Fourier series

$$m_Q(L^{-1}\xi) := \eta_Q(\xi) m(L^{-1}\xi) = \chi_Q(\xi) \sum_{k\in\mathbb{Z}^{3\times d}} a_{Q,k} \prod_{n=1}^{3} e^{2\pi i k_n\cdot A_n \xi_n},$$

so that $m = \sum_{Q\in\mathcal{W}} m_Q$.

Write

$$m_{Q,k,n}(L_n^{-1}\xi_n) = \chi_{Q,n}(\xi_n) e^{2\pi i k_n\cdot A_n\xi_n} \quad \text{and} \quad a_k = \sup_{Q\in\mathcal{W}} |a_{Q,k}|.$$

For the function $\phi_{Q,k,n}$ defined by

$$\hat{\phi}_{Q,k,n}(\tau) = (1 + |k_n|)^{-4\alpha} m_{Q,k,n}(L_n^{-1}\tau),$$

we have $c\phi_{Q,k,n} \in \Phi_n^{4\alpha}(Q)$ up to a bounded multiplicative constant c independent of Q, k, and n.

Now we can write

$$\begin{aligned}
&\left|\int_{\mathbb{R}^{3\times d}} \delta_0(\xi_1+\xi_2+\xi_3)\hat{f}_1(\xi_1)\hat{f}_2(\xi_2)\hat{f}_3(\xi_3) m(L^{-1}\xi)\,\mathrm{d}\xi\right| \\
&\qquad \leq \sum_{k\in\mathbb{Z}^{3\times d}} |a_k| \left|\sum_{Q\in\mathcal{W}} \int_{\mathbb{R}^{3\times d}} \delta_0(\xi_1+\xi_2+\xi_3) \prod_{n=1}^{3} m_{Q,k,n}(L_n^{-1}\xi_n)\hat{f}_n(\xi_n)\,\mathrm{d}\xi\right| \\
&\qquad \lesssim \sum_{k\in\mathbb{Z}^{3\times d}} |a_k|(1+|k|)^{12\alpha} \left|\sum_{Q\in\mathcal{W}} \int_{\mathbb{R}^d} \prod_{n=1}^{3} [\phi_{Q,k,n} * f_n(x)]\,\mathrm{d}x\right|.
\end{aligned}$$

By Proposition 2.2, this is bounded by

$$C\prod_{n=1}^{3}|E_n|^{1/q_n}\sum_{k\in\mathbb{Z}^{3\times d}}|a_k|(1+|k|)^{12\alpha}.$$

By smoothness of the symbol m and the upper bound on α, we know $|a_k|\le C|k|^{-12\alpha-3d-1}$, and hence the proof is complete. □

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