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COMPLEX HESSIAN MEASURES WITH RESPECT TO A BACKGROUND HERMITIAN FORM

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To the memory of Jean-Pierre Demailly

We develop potential theory for m-subharmonic functions with respect to a Hermitian metric on a Hermitian manifold. First, we show that the complex Hessian operator is well defined for bounded functions in this class. This allows us to define the m-capacity and then show the quasicontinuity of m-subharmonic functions. Thanks to this we derive other results parallel to those in pluripotential theory such as the equivalence between polar sets and negligible sets. The theory is then used to study the complex Hessian equation on compact Hermitian manifold with boundary, with the right-hand side of the equation admitting a bounded subsolution. This is an extension of a recent result of Collins and Picard dealing with classical solutions.

1. Introduction

The m-Hessian operator is defined in terms of elementary symmetric polynomials of degree m of eigenvalues of the Hessian matrix of the given function. If the degree is equal to the dimension of the space then one deals with the most important case of the Monge–Ampère operator. One can also consider more general symmetric functions of eigenvalues. The nonlinear equations involving such operators will be called in this article *Hessian-type* equations. They do appear in geometry in problems involving curvatures, like the prescribed Gauss curvature equation or the Lagrangian mean curvature equation. The m-Hessian equations in $\mathbb{R}^n$ were first solved by Caffarelli, Nirenberg and Spruck [Caffarelli et al. 1985] for smooth, nondegenerate data. The study of weak solutions for measures on the right-hand side was initiated by Trudinger and Wang [1997; 1999; 2002] (see also [Wang 2009]).

Here we are interested in the complex setting and weak solutions. For smooth data the first solutions in complex variables were obtained by Vinacua [1988] and S.-Y. Li [2004] who followed the method of [Caffarelli et al. 1985]. Błocki [2005] adopted the methods of pluripotential theory (initiated by Bedford and Taylor [1976; 1982] in relation to the complex Monge–Ampère equation) to define the action of the m-Hessian operator on nonsmooth functions and study weak solutions of the associated equation.

Let $\Omega \subset \mathbb{C}^n$ be an open set and ω be a positive Hermitian $(1,1)$-form on Ω. Let $1 \le m \le n$ be an integer and consider a function $u \in C^2(\Omega, \mathbb{R})$. The complex Hessian operator with respect to ω acts on u by

$$H_m(u) = (dd^c u)^m \wedge \omega^{n-m}.$$

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The operator is elliptic if we restrict ourselves to functions u whose eigenvectors $\lambda = (\lambda_1, \dots, \lambda_n)$ of the complex Hessian matrix $[u_{i\bar{j}}]_{1\le i,j\le n}$, with respect to ω, belong to the Gårding cone

$$\Gamma_m = \{\lambda \in \mathbb{R}^n : S_1(\lambda) > 0, \dots, S_m(\lambda) > 0\},$$

where $S_k(\lambda)$ is the k-th elementary symmetric polynomial on λ. Such a function is called m-ω-subharmonic (or m-ω-sh for short).

For $\omega = dd^c|z|^2$ the standard Kähler form on $\mathbb{C}^n$ Błocki defined nonsmooth m-subharmonic functions. He showed that the Hessian operator acting on a bounded m-subharmonic function is a well-defined positive Radon measure, that the operator is stable under decreasing sequences, and that the homogeneous Dirichlet problem is solvable. The nonhomogeneous one with the right-hand side in L^p, $p > n/m$, was solved by Dinew and Kołodziej in [2014].

On a compact Hermitian manifold (X, ω) the right m-Hessian operator to consider is

$$H_m(u) = (dd^c u + \omega)^m \wedge \omega^{n-m},$$

or more generally

$$H_{m,\alpha}(u) = (dd^c u + \alpha)^m \wedge \omega^{n-m},$$

where α is another (1, 1)-form.

For ω Kähler the counterpart of the Calabi–Yau theorem was shown by Dinew and Kołodziej [2017], with a use of earlier C^2 estimates of Hou, Ma and Wu [Hou et al. 2010]. Having this result an analogue of pluripotential theory yields weak solutions (see [Dinew and Kołodziej 2014]). We refer the readers to [Dinew and Lu 2015; Lu 2013a; 2013b; 2015; Lu and Nguyen 2015] for results in potential theory for m-ω-sh functions on a compact Kähler manifold.

Our first goal here is to develop potential theory for m-subharmonic functions (with respect to a Hermitian metric) on a Hermitian manifold. The results often parallel those of pluripotential theory.

Now we assume that ω is a general Hermitian metric. The complex m-Hessian equation on compact manifolds was solved independently by Székelyhidi [2018] and Zhang [2017]. The current authors obtained weak continuous solutions for the right-hand side in L^p, $p > n/m$ [Kołodziej and Nguyen 2016]. This partially motivates the development of potential theory for m-ω-sh functions on Hermitian manifolds. Unlike in the Kähler case, we have to deal with the nonzero torsion terms $dd^c\omega$ and $d\omega \wedge d^c\omega$. A direct computation shows that for a smooth m-ω-sh function u, $0 \le p \le n-m-1$ and $k \ge 1$, the form $(dd^c u)^k \wedge \omega^p$ may not be positive. Those terms appear when we perform integration by parts. This makes the proofs of basic potential estimates in the Hermitian setting substantially more difficult. For example, we need to fully exploit the properties of the positive cone Γ_m, and show new inequalities on elementary symmetric polynomials to prove the Chern–Levine–Nirenberg (CLN) inequality [Kołodziej and Nguyen 2016] and a variant of the Cauchy–Schwarz inequality in this paper. This coupled with the uniform convergence allows us to define the complex Hessian measure of a continuous m-ω-sh function u as the weak limit of

$$H_m(u) := \lim_{\delta\to 0} H_m(u^\delta) = \lim_{\delta\to 0} (dd^c u^\delta)^m \wedge \omega^{n-m}, \tag{1-1}$$

where $\{u^\delta\}$ is a sequence of smooth m-ω-sh functions converging uniformly to u.

However, if u is a bounded m-ω-sh function, up until now we have not been able to define its complex Hessian measure. Thanks to a basic observation (Lemma 3.1) that $dd^c u$ is a current of order zero we found a very natural way to extend the classical approach of Bedford–Taylor in this case. The first main result of the paper (Theorem 3.3) shows that the measure on the left-hand side of (1-1) can be defined by taking the weak limit inductively. Thus, for bounded m-ω-subharmonic functions u and $u^\delta \downarrow u$ pointwise as $\delta \to 0$,

$$H_m(u) := \lim_{\delta_m \to 0} \cdots \lim_{\delta_1 \to 0} dd^c u^{\delta_m} \wedge \cdots \wedge dd^c u^{\delta_1} \wedge \omega^{n-m}$$

exists in the sense of currents of order zero. The proof is based on the CLN inequality in [Kołodziej and Nguyen 2016]. This is the starting point for proving analogues of Bedford–Taylor results presented in Chapter 1 of [Kołodziej 2005]. The main difference is that we no longer have a nice integration by parts formula for closed positive currents. Instead we need to work with the (nonclosed) currents of order zero in general. This makes our proof of weak convergence of currents under decreasing limits of m-ω-sh bounded functions in Proposition 3.20 more complicated than the one in [Bedford and Taylor 1982]. Next, we obtain (Theorem 4.9) the quasicontinuity of m-ω-sh functions with respect to a suitable m-capacity: for a Borel set $E \subset \Omega$,

$$\mathrm{cap}_m(E) = \sup\left\{ \int_E H_m(u) : u \text{ is } m\text{-}\omega\text{-sh in } \Omega,\ -1 \le u \le 0 \right\}. \tag{1-2}$$

To define this capacity we needed Theorem 3.3. Once quasicontinuity is proven, one obtains weak convergence of mixed wedge products of the forms $dd^c u_j$ for m-ω-sh functions u_j under monotone convergence (Lemmas 5.1 and 5.4) following the classical arguments in [Bedford and Taylor 1982; 1987]. Next we study the polar sets and negligible sets of m-ω-sh functions. In this setting it seems impossible to obtain nice formulae for the capacity of compact or open sets in terms of Hessian measures of extremal functions as it is the case for Monge–Ampère measures. Exploiting further the properties of Γ_m in Section 2.4, especially Proposition 2.15 we are able to compare the outer capacity and the Hessian measures of relative extremal functions in Lemma 7.5. This suffices to give a characterization of a polar set by $\mathrm{cap}_m^*(E) = 0$ in Proposition 7.7. Consequently, we conclude the equivalence of polar sets and negligible sets (Theorem 7.8).

In the last sections we apply the above results to the complex m-Hessian equation. Recently, there is a lot of active research on fully nonlinear elliptic equations on compact Hermitian manifolds with or without boundary (see [Chu and McCleerey 2021; Collins and Picard 2022; Dong 2021; Dong and Li 2021; Guan 1998; Guan and Li 2010; Guedj and Lu 2025; Guo and Phong 2024; Li and Shen 2020; Phong and Tô 2021; Székelyhidi 2018; Székelyhidi et al. 2017; Tosatti and Weinkove 2010; Yuan 2021]) in various geometric contexts.

In particular Collins and Picard [2022] solved the Dirichlet problem for the m-Hessian equation in an open subset of a Hermitian manifold under the hypothesis of existence of a subsolution and smooth data. We extend it in Sections 8 and 9 showing that the existence of a bounded subsolution implies the existence of a bounded solution for both bounded domains (Theorem 8.7) and compact complex manifolds with boundary (Theorem 9.1). In the proof the above equivalence of polar and negligible sets will play a crucial role (see Lemma 8.3). The homogeneous m-Hessian equation on a (Kähler) manifold with boundary was recently solved in a particular case in [Wu 2023] in relation to the Wess–Zumino–Witten-type equation proposed by Donaldson [1999].

2. Generalized *m*-subharmonic functions

2.1. ***Elementary symmetric positive cones.*** In this section we prove important pointwise estimates for elementary symmetric polynomials. Let $1 \le m \le n$ be two integers. The positive cone Γ_m is given by

$$\Gamma_m = \{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n : S_1(\lambda) > 0, \dots, S_m(\lambda) > 0\}, \tag{2-1}$$

where $S_k(\lambda) = \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k}$; and conventionally, $S_0(\lambda) = 1$ and $S_k(\lambda) = 0$ for $k < 0$ or $k > n$. Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \Gamma_m$ be arranged in the decreasing manner, i.e.,

$$\lambda_1 \ge \cdots \ge \lambda_m \ge \cdots \ge \lambda_n.$$

Then, we know from [Ivochkina 1983, Lemma 8] that $\lambda_m > 0$ which is a consequence of a characterization of the cone Γ_m. Namely, for $\{i_1, \dots, i_t\} \subset \{1, \dots, n\}$ such that $k + t \le m$, one has

$$S_{k;i_1 \cdots i_t}(\lambda) > 0, \tag{2-2}$$

where $S_{k;i_1 \cdots i_t}(\lambda) = S_k|_{\lambda_{i_1} = \cdots = \lambda_{i_t} = 0}$. This implies also that for $1 \le k \le m$,

$$S_{k-1}(\lambda) \ge \lambda_1 \cdots \lambda_{k-1}. \tag{2-3}$$

Lemma 2.1. *There exists $\theta = \theta(n, m) > 0$ such that the following statements hold.*

(a) *For $1 \le j \le m$, $\lambda_j S_{m-1;j} \ge \theta S_m$.*

(b) *For $1 \le i \le m-1$, $\lambda_i S_{m-2;im} \ge \theta S_{m-1;m}$.*

Proof. The item (a) follows from [Kołodziej and Nguyen 2016, equation (2.7)], while (b) follows from (a) if we replace n, m and S_m by $n-1$, $m-1$ and $S_{m-1;m}$, respectively. □

Lemma 2.2. *There exists a uniform constant C, depending on n, m, such that the following inequalities are satisfied.*

(a) *For $1 \le i \le m-1$ and $\lambda \in \Gamma_m$,*

$$\frac{\lambda_1 \cdots \lambda_m}{\lambda_i} \le C (S_{m-1;i} S_{m-1})^{\frac{1}{2}}.$$

(b) *Generally, for $1 \le \ell \le n$ and increasing multi-indices $(i_1, \dots, i_{m-1})$,*

$$\prod_{\substack{s=1 \\ i_s \ne \ell}}^{m-1} |\lambda_{i_s}| \le C (S_{m-1;\ell} S_{m-1})^{\frac{1}{2}}.$$

Proof. (a) The inequality is equivalent to saying that there exist uniform constants $c_1, c_2 > 0$ such that for every positive number a,

$$a \frac{\lambda_1 \cdots \lambda_m}{\lambda_i} \le c_1 a^2 S_{m-1;i} + c_2 S_{m-1}, \tag{2-4}$$

where $1 \le i \le m-1$. In fact as we will see in the proof c_1, c_2 are explicitly given constants. We observe that if $a \le 1$, then we can easily get the claim as

$$\frac{\lambda_1 \cdots \lambda_m}{\lambda_i} \le \lambda_1 \cdots \lambda_{m-1} \le S_{m-1}.$$

Now we consider $a > 1$. We prove (a) for the case $i = 1$, the other cases $1 \le i \le m-1$ follow in the same way. The basic identities/inequalities are

$$\begin{aligned} S_{m;1m} + \lambda_1 S_{m-1;1m} = S_{m;m} &= S_m - \lambda_m S_{m-1;m} \\ &\ge -\lambda_m S_{m-1;m}, \end{aligned} \tag{2-5}$$

and

$$S_{m-1;1m} + \lambda_1 S_{m-2;1m} = S_{m-1;m}. \tag{2-6}$$

Multiplying (2-5) by $S_{m-2;1m}$ and (2-6) by $S_{m-1;1m}$ to eliminate λ_1 we get that

$$\begin{aligned} S^2_{m-1;1m} - S_{m;1m} S_{m-2;1m} &\le S_{m-1;m}(S_{m-1;1m} + \lambda_m S_{m-2;1m}) \\ &= S_{m-1;m} S_{m-1;1}. \end{aligned} \tag{2-7}$$

The Newton inequality holds for every $\lambda \in \mathbb{R}^n$ and tells us

$$S_{m;1m} S_{m-2;1m} \le \frac{(m-1)(n-m+1)}{m(n-m+2)} [S_{m-1;1m}]^2 =: c_m [S_{m-1;1m}]^2.$$

Notice that $0 < c_m < 1$. Hence, we derive from the above and (2-7) that

$$S^2_{m-1;1m} - c_m [S_{m-1;1m}]^2 \le S_{m-1;m} S_{m-1;1}.$$

Therefore,

$$S^2_{m-1;1m} \le \frac{1}{1-c_m} S_{m-1;m} S_{m-1;1}. \tag{2-8}$$

Using the Cauchy–Schwarz inequality, (2-8) and

$$S_{m-1;1m} = S_{m-1;1} - \lambda_m S_{m-2;1m},$$

we get

$$\begin{aligned} a^2 S_{m-1;1} + \frac{1}{4(1-c_m)} S_{m-1;m} &\ge a \left[\frac{S_{m-1;1} S_{m-1;m}}{1-c_m} \right]^{\frac{1}{2}} \\ &\ge a |S_{m-1;1m}| \\ &\ge a(-S_{m-1;1} + \lambda_m S_{m-2;1m}). \end{aligned}$$

This implies

$$(a^2 + a) S_{m-1;1} + \frac{1}{4(1-c_m)} S_{m-1;m} \ge a \lambda_m S_{m-2;1m}. \tag{2-9}$$

Since $a \geq 1$, it follows that $2a^2 S_{m-1;1} + C S_{m-1} \geq a\lambda_m S_{m-2;1m}$. So, using Lemma 2.1 and (2-3) we have

$$\begin{aligned}(2a^2 S_{m-1;1} + C S_{m-1})\lambda_1 &\geq a\lambda_m \lambda_1 S_{m-2;1m}\\ &\geq a\theta^2 \lambda_m S_{m-1}\\ &\geq a\theta^2 \lambda_1 \cdots \lambda_m. \end{aligned} \tag{2-10}$$

The proof of the lemma is completed with $c_1 = 2,\ c_2 = 1/(4(1-c_m))$ and $C = \sqrt{2/(1-c_m)}$.

(b) The characterization (2-2) implies that a sum of any $n-m+1$ entries of λ is positive. Hence, we have for $\lambda_{i_s} \leq 0$,

$$|\lambda_{i_s}| \leq (n-m)\lambda_m.$$

So, for $1 \leq \ell \leq m-1$,

$$\prod_{\substack{i_s \neq \ell\\ s=1}}^{m-1} |\lambda_{i_s}| \leq (n-m)^{m-1} \frac{\lambda_1 \cdots \lambda_m}{\lambda_\ell} \leq C[S_{m-1;\ell} S_{m-1}]^{\frac{1}{2}}, \tag{2-11}$$

where we used (a) for the second inequality.

Now we treat the remaining range $m \leq \ell \leq n$. By a result of Lin and Trudinger [1994, Theorem 1.1], we know that $S_{m-1;\ell} \geq \theta S_{m-1}$ for a constant $\theta = \theta(n, m)$ depending only on n, m. This implies

$$S_{m-1} \leq \frac{(S_{m-1;\ell} S_{m-1})^{\frac{1}{2}}}{\sqrt{\theta}}.$$

Thus, the desired inequality easily follows from this and the bound

$$\prod_{\substack{i_s \neq \ell\\ s=1}}^{m-1} |\lambda_{i_s}| \leq (n-m)^{m-1} \lambda_1 \cdots \lambda_{m-1} \tag{2-12}$$

for $m \leq \ell \leq n$. □

2.2. ***The Cauchy–Schwarz inequality.*** Let ω be a Hermitian metric on $\mathbb{C}^n$ and let Ω be a bounded open set in $\mathbb{C}^n$. The positive cone $\Gamma_m(\omega)$, associated to ω, of real (1, 1)-forms is defined as follows. A real form γ is said to belong to $\Gamma_m(\omega)$ if at any point $z \in \Omega$,

$$\gamma^k \wedge \omega^{n-k}(z) > 0 \quad \text{for } k = 1, \ldots, m.$$

Equivalently, in the normal coordinate system with respect to ω at z, diagonalizing $\gamma = \sqrt{-1}\sum_i \lambda_i dz_i \wedge d\bar{z}_i$, we have $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Gamma_m$. Now we will translate the estimates in Section 2.1 into the integral forms.

We can state the following versions of the Cauchy–Schwarz inequality in this setting. Let h be a smooth real-valued function and ϕ, ψ be Borel functions. Let T be a positive current of bidegree $(n-2, n-2)$.

Lemma 2.3. *There exists a uniform constant C depending on ω such that*

$$\left|\int \phi\psi\, dh \wedge d^c\omega \wedge T\right|^2 \leq C \int |\phi|^2\, dh \wedge d^c h \wedge \omega \wedge T \int |\psi|^2\, \omega^2 \wedge T.$$

Proof. The proof of [Nguyen 2016, Proposition 1.4] can be easily adapted. □

The above lemma can be applied for the case $T = \gamma^s \wedge \omega^{n-m+\ell}$, where $\gamma \in \Gamma_m(\omega)$ and $0 \le s, \ell \le m-1$ and $s+\ell = m-1$. Next, we also need to deal with possible nonpositive forms $T' = \gamma^{m-1} \wedge \omega^{n-m-1}$ where the classical Cauchy–Schwarz is not immediately applicable. However, we still have:

Lemma 2.4. *There exists a uniform constant C depending on ω, n, m such for every $\gamma \in \Gamma_m(\omega)$,*

$$\left|\int \phi\psi\, dh \wedge d^c\omega \wedge \gamma^{m-1} \wedge \omega^{n-m-1}\right|^2 \le C \int |\phi|^2\, dh \wedge d^c h \wedge \gamma^{m-1} \wedge \omega^{n-m} \times \int |\psi|^2\, \gamma^{m-1} \wedge \omega^{n-m+1}.$$

Proof. We express the integrands of both sides as follows.

$$\begin{aligned} dh \wedge d^c\omega \wedge \gamma^{m-1} \wedge \omega^{n-m-1} &= f_1(z)\omega^n, \\ dh \wedge d^c h \wedge \gamma^{m-1} \wedge \omega^{n-m} &= [f_2(z)]^2\omega^n, \\ \gamma^{m-1} \wedge \omega^{n-m+1} &= [f_3(z)]^2\omega^n. \end{aligned}$$

Thus, the inequality will follow from the classical Cauchy–Schwarz inequality if we have pointwise $|f_1(z)| \le C f_2(z) f_3(z)$ for every $z \in \Omega$. This is proved by using the normal coordinate system at a given point z with respect to ω which diagonalizes also γ, i.e.,

$$\omega = \sqrt{-1}\sum_{i=1}^n dz_i \wedge d\bar{z}_i, \quad \gamma = \sqrt{-1}\sum_i \lambda_i\, dz_i \wedge d\bar{z}_i,$$

where $\lambda = (\lambda_1, \dots, \lambda_n) \in \Gamma_m$. Define $h_i = \partial h/\partial z_i$. Then, at the point z,

$$\binom{n}{m-1}(f_2)^2 = \sum_{i=1}^n |h_i|^2 S_{m-1;i}, \quad \binom{n}{m-1}(f_3)^2 = S_{m-1}. \tag{2-13}$$

Now, observe that $\gamma^{m-1} \wedge \omega^{n-m-1}$ is an $(n-2, n-2)$-form, so after taking the wedge product with $dh \wedge d^c\omega$ the nonzero contribution give only $\sqrt{-1}\partial h \wedge \bar\partial\omega$ and $\sqrt{-1}\,\bar\partial h \wedge \partial\omega$. As h is a real-valued function, these two forms are mutually conjugate. Let us write

$$\bar\partial\omega = \sum \omega_{i\bar{j}\bar{k}}\, d\bar{z}_k \wedge dz_i \wedge d\bar{z}_j.$$

Define $dV = \left(\sqrt{-1}\right)^{n^2} dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{\wedge} z_1 \wedge \cdots \wedge d\bar{z}_n$. Let $J = (j_1, \dots, j_{n-m-1})$ be an increasing multi-index. Then,

$$\frac{1}{(m-1)!}\partial h \wedge \bar\partial\omega \wedge \gamma^{m-1} \wedge dz_J \wedge d\bar{z}_J/dV = \sum_{\substack{j,\ell\notin J\\ j\neq\ell}} c_{j\bar{j}\bar{\ell}}\, h_\ell \prod_{i_s\notin J\cup\{j,\ell\}} \lambda_{i_s},$$

where $c_{i\bar{i}\bar{\ell}}$ is $\omega_{i\bar{i}\bar{\ell}}$ or $\omega_{i\bar{\ell}\bar{i}}$, and $i_s \in I = (i_1, \dots, i_{m-1})$ which is an increasing multi-index satisfying $I \cap J = \varnothing$. Then, at the point z,

$$|f_1(z)| \le c_0 \sum_{|I|=m-1} \sum_{\ell=1}^n |h_\ell| \prod_{\substack{i_s\neq\ell\\ s=1}}^{m-1} |\lambda_{i_s}|, \tag{2-14}$$

where c_0 is a uniform constant depending only on ω.

By (2-13) and (2-14) we have reduced $|f_1| \le C f_2 f_3$ to the one for symmetric polynomials. To show the latter, by Lemma 2.2(b) for $1 \le \ell \le n$, we have

$$|h_\ell| \prod_{\substack{i_s \neq \ell \\ s=1}}^{m-1} |\lambda_{i_s}| \le C[|h_\ell|^2 S_{m-1;\ell} S_{m-1}]^{\frac{1}{2}} \le C\left(\sum_{i=1}^{n} |h_i|^2 S_{m-1;i}\right)^{\frac{1}{2}} [S_{m-1}]^{\frac{1}{2}}.$$

Taking the sum of (finitely many) terms on the left-hand side of (2-14) the proof of the theorem follows. □

We also need this inequality for wedge products of two forms and more. This is done by solving a linear system of inequalities as in [Kołodziej and Nguyen 2016, page 2226].

Corollary 2.5. *There exists a uniform constant C, depending on ω, n, m, such that the following inequalities hold.*

(a) *For $\eta, \gamma \in \Gamma_m(\omega)$,*

$$\left|\int \phi\psi \, dh \wedge d^c\omega \wedge \eta^k \wedge \gamma^{m-k-1} \wedge \omega^{n-m-1}\right|^2 \le C \int |\phi|^2 \, dh \wedge d^c h \wedge (\eta+\gamma)^{m-1} \wedge \omega^{n-m} \int |\psi|^2 \, (\eta+\gamma)^{m-1} \wedge \omega^{n-m+1}.$$

(b) *Generally, for $\gamma_1, \dots, \gamma_{m-1} \in \Gamma_m(\omega)$,*

$$\left|\int \phi\psi \, dh \wedge d^c\omega \wedge \gamma_1 \wedge \cdots \wedge \gamma_{m-1} \wedge \omega^{n-m-1}\right|^2 \le C \int |\phi|^2 \, dh \wedge d^c h \wedge \left(\sum_{i=1}^{m-1} \gamma_i\right)^{m-1} \wedge \omega^{n-m} \times \int |\psi|^2 \left(\sum_{i=1}^{m-1} \gamma_i\right)^{m-1} \wedge \omega^{n-m+1}.$$

2.3. *m-subharmonic functions on Hermitian manifolds.* Let us recall the definition of generalized m-subharmonic function in the Hermitian setting (see [Błocki 2005; Kołodziej and Nguyen 2016; Gu and Nguyen 2018]). Let Ω be a bounded open set in $\mathbb{C}^n$ and let ω be a Hermitian metric on Ω.

Definition 2.6. An upper semicontinuous function $u : \Omega \to [-\infty, +\infty)$ is called m-ω-subharmonic if $u \in L^1_{\text{loc}}(\Omega)$ and for any collection $\gamma_1, \dots, \gamma_{m-1} \in \Gamma_m(\omega)$,

$$dd^c u \wedge \gamma_1 \wedge \cdots \wedge \gamma_{m-1} \wedge \omega^{n-m} \ge 0$$

in the sense of currents.

Remark 2.7. By [Gårding 1959] a function $u \in C^2(\Omega)$ is m-ω-sh if and only if $dd^c u \in \overline{\Gamma_m(\omega)}$, that is $dd^c u \wedge \gamma_1 \wedge \cdots \wedge \gamma_{m-1} \ge 0$ pointwise in Ω. Thus, the estimates for forms in $\Gamma_m(\omega)$ are applicable to $dd^c u$ if u is a smooth m-ω-sh function.

It follows from [Michelsohn 1982] that for $\gamma_1, \dots, \gamma_{m-1} \in \Gamma_m(\omega)$ there is a unique $(1, 1)$ positive form α such that

$$\alpha^{n-1} = \gamma_1 \wedge \cdots \wedge \gamma_{m-1} \wedge \omega^{n-m}.$$

The above definition of m-ω-sh function can be expressed more familiarly, in terms of potential theory, using the notion of α-subharmonicity (see, e.g., [Gu and Nguyen 2018, Definition 2.1, Lemma 9.10]). Thanks to this, many potential-theoretic properties of m-ω-sh functions can be derived from those of α-sh functions. For example, if two m-ω-sh functions are equal almost everywhere (with respect to the Lebesgue measure), then they are equal everywhere [Gu and Nguyen 2018, Corollary 9.7]. One also has the "gluing property" that allows one to modify the function outside a compact subset. This is an immediate consequence of [Gu and Nguyen 2018, Lemma 9.5].

Lemma 2.8. *Let $U \subset \Omega$ be two open sets in $\mathbb{C}^n$. Let u be m-ω-sh in U and v be m-ω-sh in Ω. Assume that* $\limsup_{z\to x} u(z) \le v(x)$ *for every $x \in \partial U \cap \Omega$. Then,*

$$\widetilde{u} = \begin{cases} \max\{u, v\} & \text{on } U, \\ v & \text{on } \Omega \setminus U, \end{cases}$$

is m-ω-sh in Ω.

Because of this we have the following way of reducing a proof to a simpler case (see [Kołodziej 2005, page 7] for the proof).

Localization principle. *If we are to prove the weak convergence or local estimate for a family of locally uniformly bounded m-ω-sh functions, it is no loss of generality to assume that the functions are defined in a ball and are all equal on some neighborhood of the boundary.*

For a bounded psh function its convolution with a radial smoothing kernel provides locally a smooth, decreasing approximation of this function. It is no longer the case for generalized m-ω-sh. However, in a strictly m-pseudoconvex domain Ω, that is for $\Omega = \{\rho < 0\}$, where $\rho \in C^2(\overline{\Omega})$ is strictly m-ω-sh and $d\rho \neq 0$ on $\partial\Omega$, we still have (nonexplicit) way of approximation by smooth m-ω-sh functions.

Proposition 2.9. *Let $\Omega \Subset \mathbb{C}^n$ be strictly m-pseudoconvex domain. Let u be a bounded m-ω-sh function in a neighborhood of $\overline{\Omega}$. Then, there exists a sequence of smooth m-ω-sh functions $u_j \in C^\infty(\overline{\Omega})$ that decreases to u pointwise in $\overline{\Omega}$.*

Proof. The proof is the same as the one of [Gu and Nguyen 2018, Theorem 3.18] if we replace the ball by a strictly m-pseudoconvex domain and invoke [Collins and Picard 2022, Theorem 1.1] for the smooth solution of the Dirichlet problem on a strictly m-pseudoconvex domain. □

2.4. ***Integral estimates for smooth functions.*** Let Ω be a bounded open set in $\mathbb{C}^n$. Let $-1 \le v \le w \le 0$ be smooth m-ω-sh functions in Ω such that $v = w$ in a neighborhood of $\partial\Omega$. Let ρ be a smooth m-ω-sh function such that $-1 \le \rho \le 0$. Using the notation $\gamma := dd^c\rho$, $h = w - v$ we consider

$$e_{(q,k,s)} = \int h^{q+1}\gamma^k \wedge (dd^c v)^s \wedge \omega^{n-k-s}, \tag{2-15}$$

where $q \ge 0$, the integers $0 \le k \le m$ and $0 \le s \le m - k$. We wish to bound

$$e_{(q,m,0)} = \int h^{q+1}\gamma^m \wedge \omega^{n-m},$$

by the integrals

$$e_{(r,0,i)} = \int h^{r+1} (dd^c v)^i \wedge \omega^{n-i},$$

where $i = 0, \dots, m$ and $1 \le r < q$. This is done via repeated use of the integration by parts to replace γ by $dd^c v$. However, there are three different cases depending on the total degree $k+s$ of the form $\gamma^k \wedge (dd^c v)^s$ that we need to deal with separately.

- Case 1: $k+s = m$.
- Case 2: $k+s = m-1$.
- Case 3: $k+s \le m-2$.

Let us start with an auxiliary inequality that we use frequently below.

Lemma 2.10. *Let $p \ge 1$ and $0 \le k \le m-1$. There exists a constant C depending on ω, n, m such that*:

(a) *For $0 \le s \le m-1-k$,*

$$\int h^{p-1}\, dh \wedge d^c h \wedge \gamma^k \wedge (dd^c v)^s \wedge \omega^{n-k-s-1} \le \int h^p \gamma^k \wedge (dd^c v)^{s+1} \wedge \omega^{n-k-s-1} + C \int h^{p+1} (\gamma + dd^c v)^{k+s} \wedge \omega^{n-k-s}.$$

(b) *For $0 \le s \le m-3-k$,*

$$\int h^{p-1}\, dh \wedge d^c h \wedge \gamma^k \wedge (dd^c v)^s \wedge \omega^{n-k-s-1} \le \int h^p \gamma^k \wedge (dd^c v)^{s+1} \wedge \omega^{n-k-s-1} + C \int h^{p+1} \gamma^k \wedge (dd^c v)^s \wedge \omega^{n-k-s}.$$

Proof. (a) Note first that $0 \le h \le 1$ and also $T := \gamma^k \wedge (dd^c v)^s \wedge \omega^{n-k-s-1}$ and $dd^c w \wedge T$ are positive forms for $n-s-k-1 \ge n-m$. Therefore,

$$\begin{aligned} p(p+1) h^{p-1} dh \wedge d^c h \wedge T &= [dd^c h^{p+1} - (p+1) h^p dd^c h] \wedge T \\ &\le [dd^c h^{p+1} + (p+1) h^p dd^c v] \wedge T. \end{aligned}$$

Hence,

$$\int h^{p-1}\, dh \wedge d^c h \wedge \gamma^k \wedge (dd^c v)^s \wedge \omega^{n-s-k-1} \le \int (dd^c h^{p+1} + h^p dd^c v) \wedge \gamma^k \wedge (dd^c v)^s \wedge \omega^{n-s-k-1}. \tag{2-16}$$

It remains to estimate the product involving the first term in the bracket. By integration by parts and [Kołodziej and Nguyen 2016, Lemma 2.3],

$$\begin{aligned} \int dd^c h^{p+1} \wedge \omega^{n-s-k-1} \wedge \gamma^k \wedge (dd^c v)^s &= \int h^{p+1} dd^c (\omega^{n-s-k-1}) \wedge \gamma^k \wedge (dd^c v)^s \\ &\le C \int h^{p+1} (\gamma + dd^c v)^{k+s} \wedge \omega^{n-m+1}. \end{aligned} \tag{2-17}$$

Combining the last two inequalities the proof of the lemma follows.

(b) The proof is very similar, we first have (2-16). Then, in the middle integral of (2-17) one can express $dd^c(\omega^{n-s-k-1}) = \eta \wedge \omega^{n-m}$ for a smooth $(m-s-k, m-s-k)$-form η. Then, since $\gamma, dd^c v \in \Gamma_m(\omega)$, in this case the inequality has a better form

$$\left| \int h^{p+1} \eta \wedge \gamma^k \wedge (dd^c v)^s \wedge \omega^{n-m} \right| \le C \int h^{p+1} \gamma^k \wedge (dd^c v)^s \wedge \omega^{n-k-s}. \qquad \square$$

Let us start with the simplest situation in Case 1. We are going to show that

$$e_{(q,m,0)} \le C(e_{(q-1,m-1,1)} + e_{(q-1,m-1,0)}), \tag{2-18}$$

where C is a uniform constant depending on ω, m, n. Equivalently:

Lemma 2.11. *Let $q \ge 1$. Then,*

$$\int_\Omega (w-v)^{q+1} \gamma^m \wedge \omega^{n-m} \le C \int_\Omega (w-v)^q \gamma^{m-1} \wedge dd^c v \wedge \omega^{n-m} + C \int_\Omega (w-v)^q \gamma^{m-1} \wedge \omega^{n-m+1}.$$

Proof. Recall that $h := w - v \ge 0$ and $h = 0$ near $\partial\Omega$. A direct computation gives

$$\begin{aligned} dd^c(h^{q+1}\omega^{n-m}) &= q(q+1)h^{q-1} dh \wedge d^c h \wedge \omega^{n-m} \\ &\quad + (q+1)h^q dd^c h \wedge \omega^{n-m} \\ &\quad + (q+1)(n-m)h^q d\omega \wedge d^c h \wedge \omega^{n-m-1} \\ &\quad + (q+1)(n-m)h^q dh \wedge d^c \omega \wedge \omega^{n-m-1} \\ &\quad + h^{q+1} dd^c \omega^{n-m} \\ &=: T_0 + T_1 + T_2 + T_3 + T_4. \end{aligned} \tag{2-19}$$

By integration by parts,

$$\begin{aligned} \int h^{q+1} dd^c \rho \wedge \gamma^{m-1} \wedge \omega^{n-m} &= \int \rho\, dd^c(h^{q+1}\omega^{n-m}) \wedge \gamma^{m-1} \\ &= \int \rho (T_0 + T_1 + T_2 + T_3 + T_4) \wedge \gamma^{m-1}. \end{aligned} \tag{2-20}$$

Case 1a: *estimate of* T_0. Since $-1 \le \rho \le 0$ and T_0 is a positive current,

$$\rho T_0 \wedge \gamma^{m-1} \le 0. \tag{2-21}$$

Case 1b: *estimate of* T_1. Similarly, because v is a m-ω-sh function,

$$\begin{aligned} \rho T_1 \wedge \gamma^{m-1} &= (q+1)\rho h^q (dd^c w - dd^c v) \wedge \omega^{n-m} \wedge \gamma^{m-1} \\ &\le (q+1) h^q dd^c v \wedge \gamma^{m-1} \wedge \omega^{n-m}. \end{aligned} \tag{2-22}$$

Case 1c: *estimate of* T_4. Using the inequality [Kołodziej and Nguyen 2016, Lemma 2.3]

$$\gamma^{m-1} \wedge dd^c \omega^{n-m} \le C_{m,n} \gamma^{m-1} \wedge \omega^{n-m+1}, \tag{2-23}$$

we can estimate the last term T_4,

$$\left|\int \rho h^{q+1}\gamma^{m-1}\wedge dd^c\omega^{n-m}\right| \le C\int |\rho| h^{q+1}\gamma^{m-1}\wedge\omega^{n-m+1} \le Ce_{(q,m-1,0)}, \tag{2-24}$$

where we used again the fact that $|\rho|\le 1$.

Case 1d: *estimate of* T_2 *and* T_3. We use the Cauchy–Schwarz inequality in Lemma 2.4, where an extra uniform constant appears. Let us estimate T_2 (for T_3 the estimate is completely the same). By the Cauchy–Schwarz inequality,

$$\begin{aligned}\left|\int \rho h^q\, d\omega\wedge d^c h\wedge\omega^{n-m-1}\wedge\gamma^{m-1}\right|^2 &\le C\int |\rho| h^{q-1}\, dh\wedge d^c h\wedge\omega^{n-m}\wedge\gamma^{m-1}\int |\rho| h^{q+1}\omega^{n-m+1}\wedge\gamma^{m-1}\\ &\le C\Big(\int h^{q-1}\, dh\wedge d^c h\wedge\gamma^{m-1}\wedge\omega^{n-m}+\int h^{q+1}\gamma^{m-1}\wedge\omega^{n-m+1}\Big)^2\\ &\le C(e_{(q-1,m-1,1)}+e_{(q,m-1,0)})^2,\end{aligned} \tag{2-25}$$

where we used Lemma 2.10 with $s=0$, $k=m-1$ and $p=q$ for the first integral in the last inequality, namely,

$$\int h^{q-1}\, dh\wedge d^c h\wedge\omega^{n-m}\wedge\gamma^{m-1}\le C(e_{(q-1,m-1,1)}+e_{(q,m-1,0)}).$$

Combining the estimates (2-21), (2-22), (2-24) and (2-25) and the fact that $e_{(q,\bullet,\bullet)}\le e_{(q-1,\bullet,\bullet)}\le e_{(q-2,\bullet,\bullet)}$ we conclude the proof of the lemma. □

We can now state the general inequality in Case 1.

Corollary 2.12. *For* $1\le k\le m$ *and* $s=m-k\ge 0$ *and* $q\ge 2$,

$$e_{(q,k,s)}\le c_k\sum_{i=0}^{k-1} e_{(q-1,i,m-i)}+C\sum_{i=0}^{m-1} e_{(q-2,i,m-1-i)}.$$

Proof. The proof is by induction in k but "downward". For $k=m$ it is Lemma 2.11. Assume that it is true for every $k+1\le \ell\le m$, i.e., we have

$$e_{(q,\ell,m-\ell)}\le c_\ell\sum_{i=0}^{k} e_{(q-1,i,m-i)}+c_\ell\sum_{i=0}^{m-1} e_{(q-2,i,m-1-i)}. \tag{2-26}$$

We proceed to prove the conclusion holds for k. The strategy is the same as in the proof of the last lemma, however we need to estimate T_2 and T_3 more carefully. We repeat the steps of the proof of Lemma 2.11 replacing $dd^c\rho\wedge\Gamma$ by $dd^c\rho\wedge\Gamma^{(s)}$, where

$$\Gamma=\gamma^{m-1}\wedge\omega^{n-m}\quad\text{and}\quad \Gamma^{(s)}=\gamma^{m-1-s}\wedge(dd^c v)^s\wedge\omega^{n-m},$$

in the integrand on the left-hand side. The corresponding estimates for T_0, T_1 are similar. Namely,

$$\rho T_0 \wedge \gamma^{m-1-s} \wedge (dd^c v)^s \le 0, \tag{2-21$'$}$$

and

$$\rho T_1 \wedge \gamma^{m-s-1} \le (q+1) h^q (dd^c v)^{s+1} \wedge \gamma^{m-1-s} \wedge \omega^{n-m}. \tag{2-22$'$}$$

The one for T_4 is

$$\begin{aligned}\gamma^{m-1-s} \wedge (dd^c v)^s \wedge dd^c \omega^{n-m} &\le C_{m,n} (\gamma + dd^c v)^{m-1} \wedge \omega^{n-m+1} \\ &\le C_{m,n} [\gamma^{m-1} + \gamma^{m-2} \wedge dd^c v + \cdots + (dd^c v)^{m-1}] \wedge \omega^{n-m+1}.\end{aligned} \tag{2-23$'$}$$

Hence, integrating both sides and using $0 \le h \le 1$, leads to

$$\left| \int h^{q+1} \gamma^{m-1-s} \wedge (dd^c v)^s \wedge dd^c \omega^{n-m} \right| \le C \sum_{i=0}^{m-1} e_{(q,i,m-1-i)}. \tag{2-24$'$}$$

Lastly for T_2 and T_3, we need to use Corollary 2.5,

$$\begin{aligned} I^2 := &\left| \int \rho h^q d\omega \wedge d^c h \wedge \omega^{n-m-1} \wedge \gamma^{m-1-s} \wedge (dd^c v)^s \right|^2 \\ &\le C \int h^{q-1} (\gamma + dd^c v)^{m-1} \wedge \omega^{n-m+1} \times \int h^{q+1} dh \wedge d^c h \wedge (\gamma + dd^c v)^{m-1} \wedge \omega^{n-m}. \end{aligned}$$

The standard Cauchy–Schwarz inequality (Lemma 2.3) gives, for $\varepsilon > 0$ to be determined later,

$$I \le \frac{C}{\varepsilon} \int h^{q-1} (\gamma + dd^c v)^{m-1} \wedge \omega^{n-m+1} + \varepsilon \int h^{q+1}\, dh \wedge d^c h \wedge (\gamma + dd^c v)^{m-1} \wedge \omega^{n-m}.$$

By the last inequality in (2-23$'$) the first integral is bounded by

$$\frac{C}{\varepsilon} \sum_{i=0}^{m-1} e_{(q-2,i,m-1-i)}.$$

To bound the second integral we use

$$(\gamma + dd^c v)^{m-1} \wedge \omega^{n-m} \le C_{m,n} \sum_{i=0}^{m-1} \gamma^i \wedge (dd^c v)^{m-1-i} \wedge \omega^{n-m},$$

and then Lemma 2.10(a) for $k = i$, $s = m-1-i$ and $p = q+2$. This gives a bound for the second integral by

$$C\varepsilon \sum_{i=0}^{m-1} e_{(q+1,i,m-i)} + C\varepsilon \sum_{i=0}^{m-1} e_{(q+2,i,m-1-i)}.$$

Let us consider the first sum above:

$$\varepsilon \sum_{i=0}^{m-1} e_{(q+1,i,m-i)} = \varepsilon \sum_{i \ge k+1} e_{(q+1,i,m-i)} + \varepsilon \sum_{i=0}^{k-1} e_{(q+1,i,m-i)} + \varepsilon e_{(q+1,k,m-k)}.$$

Applying the induction hypothesis (2-26) to the first term on the right, we derive

$$\varepsilon \sum_{i=0}^{m-1} e_{(q+1,i,m-i)} \le \varepsilon b_k \biggl(e_{(q,k,m-k)} + \sum_{i=0}^{k-1} e_{(q,i,m-i)} \biggr) + \varepsilon b_k \sum_{i=0}^{m-1} e_{(q-1,i,m-1-i)} + \varepsilon \sum_{i=0}^{k-1} e_{(q+1,i,m-i)} + \varepsilon e_{(q+1,k,m-k)},$$

where $b_k = c_m + \cdots + c_{k+1}$.

Since $e_{(q+2,\bullet,\bullet)} \le e_{(q+1,\bullet,\bullet)} \le e_{(q,\bullet,\bullet)} \le e_{(q-1,\bullet,\bullet)}$, it follows from the above estimates that

$$I \le \varepsilon(1+b_k) e_{(q,k,m-k)} + \varepsilon \sum_{i=0}^{k-1} e_{(q-1,i,m-i)} + \biggl[\varepsilon(b_k+1) + \frac{C}{\varepsilon} + C\varepsilon \biggr] \sum_{i=0}^{m-1} e_{(q-2,i,m-1-i)}. \tag{2-25$'$}$$

Thus, combining (2-21′), (2-22′), (2-24′) and (2-25′) we have

$$e_{(q,k,s)} \le C e_{(q-1,k-1,s+1)} + C \sum_{i=0}^{m-1} e_{(q,i,m-1-i)} + \varepsilon(1+b_k) e_{(q,k,s)} + \varepsilon \sum_{i=0}^{k-1} e_{(q-1,i,m-i)} + \biggl[\varepsilon(b_k+1) + \frac{C}{\varepsilon} + C\varepsilon \biggr] \sum_{i=0}^{m-1} e_{(q-2,i,m-1-i)}.$$

Now we can choose ε so that $\varepsilon(1+b_k) = \frac{1}{2}$. Since $s = m-k$, regrouping terms on the right-hand side (decreasing the first parameter in $e_{(q,i,m-1-i)}$ if necessary) we get for possibly larger $C > 0$ that

$$\tfrac{1}{2} e_{(q,k,m-k)} \le (C+\varepsilon) \sum_{i=0}^{k-1} e_{(q-1,i,m-i)} + \biggl(C + \frac{C}{\varepsilon} \biggr) \sum_{i=0}^{m-1} e_{(q-2,i,m-1-i)}. \qquad \square$$

Next, we consider Case 2.

Lemma 2.13. *For* $1 \le k \le m-1$ *and* $s = m-1-k \ge 0$ *and* $q \ge 1$, *we have*

$$e_{(q,k,s)} \le C \biggl(e_{(q-1,k-1,s+1)} + \sum_{i=0}^{m-2} e_{(q-1,i,m-2-i)} \biggr).$$

Proof. The basic computation using integration by parts that corresponds to (2-19) starts with

$$dd^c(h^{q+1}\omega^{n-m+1}) := T_0 + T_1 + T_2 + T_3 + T_4, \tag{2-19$''$}$$

where each term has higher exponent of ω. The estimates for T_0, T_1 are the same as the ones in (2-21′) and (2-22′). Precisely,

$$\rho T_0 \wedge \gamma^{m-2-s} \wedge (dd^c v)^s \le 0, \tag{2-21$''$}$$

and

$$\rho T_1 \wedge \gamma^{m-2-s} \le (q+1) h^q \gamma^{m-2-s} \wedge (dd^c v)^{s+1} \wedge \omega^{n-m+1}. \tag{2-22$''$}$$

The one for T_4 is

$$\gamma^{m-2-s}\wedge(dd^c v)^s\wedge dd^c(\omega^{n-m+1})\le C_{m,n}(\gamma+dd^c v)^{m-2}\wedge\omega^{n-m+2}$$
$$\le C_{m,n}[\gamma^{m-2}+\gamma^{m-3}\wedge dd^c v+\cdots+(dd^c v)^{m-2}]\wedge\omega^{n-m+2}. \quad (2\text{-}23'')$$

Integrating both sides and using the fact that $0\le h\le 1$ yield

$$\left|\int\rho h^{q+1}\gamma^{m-2-s}\wedge(dd^c v)^s\wedge dd^c(\omega^{n-m+1})\right|\le C\sum_{i=0}^{m-2}e_{(q,i,m-2-i)}. \quad (2\text{-}24'')$$

Next, the corresponding inequalities for T_2 and T_3 are easier. This is due to the fact that

$$T_2\wedge\gamma^{m-2-s}\wedge(dd^c v)^s=C_0h^q d\omega\wedge d^c h\wedge\omega^{n-m}\wedge\gamma^{m-2-s}\wedge(dd^c v)^s.$$

Therefore, the classical Cauchy–Schwarz inequality (Lemma 2.3) is sufficient. Namely,

$$I^2:=\left|\int\rho h^q d\omega\wedge d^c h\wedge\omega^{n-m}\wedge\gamma^{m-2-s}\wedge(dd^c v)^s\right|^2$$
$$\le C\int h^q\gamma^{m-2-s}\wedge(dd^c v)^s\wedge\omega^{n-m+2}\times\int h^q\,dh\wedge d^c h\wedge\gamma^{m-2-s}\wedge(dd^c v)^s\wedge\omega^{n-m+1}.$$

The Cauchy–Schwarz inequality gives

$$I\le C\int h^q\gamma^{m-2-s}\wedge(dd^c v)^s\wedge\omega^{n-m+2}+C\int h^q\,dh\wedge d^c h\wedge\gamma^{m-2-s}\wedge(dd^c v)^s\wedge\omega^{n-m+1}.$$

Here, the first integral in the sum is $e_{(q-1,k-1,s)}$. By applying Lemma 2.10(a) for $k-1$, s and $p=q+1$ one gets a bound for the second integral by

$$\int h^{q+1}\gamma^{m-2-s}\wedge(dd^c v)^{s+1}\wedge\omega^{n-m+1}+C\int h^{q+2}(\gamma+dd^c v)^{m-2}\wedge\omega^{n-m+2}$$
$$\le e_{(q,k-1,s+1)}+C\sum_{i=0}^{m-2}e_{(q+1,i,m-2-i)}.$$

Combining this with the decreasing property in the first parameter of $e_{(q,\bullet,\bullet)}$ we get

$$I\le C\left[e_{(q-1,k-1,s)}+e_{(q,k-1,s+1)}+\sum_{i=0}^{m-2}e_{(q+1,i,m-2-i)}\right]$$
$$\le C\left[e_{(q-1,k-1,s+1)}+\sum_{i=0}^{m-2}e_{(q-1,i,m-2-i)}\right]. \quad (2\text{-}25'')$$

Finally, combining (2-21″), (2-22″), (2-24″) and (2-25″) one completes the proof of the lemma. □

Lastly, we consider Case 3.

Lemma 2.14. *For $1\le k\le m-2$ and $0\le s\le m-2-k$ and $q\ge 1$ we have*

$$e_{(q,k,s)}\le C[e_{(q-1,k-1,s+1)}+e_{(q,k-1,s)}].$$

Proof. We need to estimate $dd^c\rho\wedge\gamma^{k-1}\wedge(dd^cv)^s\wedge\omega^{n-k-s}$, where $n-k-s\geq n-m+2$. Then, there is a significant change in basic computation of

$$dd^c(h^{q+1}\omega^{n-k-s}) = T_0+T_1+T_2+T_3+T_4, \tag{2-19'''}$$

where all forms T_i contain powers of ω with the exponent at least $n-m$. The estimates for T_0, T_1 are the same as in Corollary 2.12 and improved estimates for T_2, T_3 are as in Lemma 2.13. Moreover the bound for T_4 is easier. Namely, since $\gamma^{k-1}\wedge(dd^cv)^s\wedge\omega^{n-m}$ is a positive form, one obtains

$$\gamma^{k-1}\wedge(dd^cv)^s\wedge dd^c(\omega^{n-k-s})\leq C\gamma^{k-1}\wedge(dd^cv)^s\wedge\omega^{n-k-s+1}. \tag{2-23'''}$$

Hence, multiplying both sides by ρh^{q+1} and then integrating we get

$$\left|\int\rho h^{q+1}\gamma^{k-1}\wedge(dd^cv)^s\wedge dd^c(\omega^{n-k-s})\right|\leq Ce_{(q,k-1,s)}, \tag{2-24'''}$$

where we used the fact that $-1\leq\rho\leq 0$.

Next, the estimates for T_2 and T_3 are

$$\begin{aligned} I &:= \left|\int\rho h^q\, d\omega\wedge d^ch\wedge\omega^{n-k-s-1}\wedge\gamma^{k-1}\wedge(dd^cv)^s\right| \\ &\leq C\int h^{q+1}\gamma^{k-1}\wedge(dd^cv)^s\wedge\omega^{n-k-s}+C\int h^{q-1}\,dh\wedge d^ch\wedge\gamma^{k-1}\wedge(dd^cv)^s\wedge\omega^{n-k-s}. \end{aligned}$$

Using Lemma 2.10(b) for $k-1$, s and $p=q-1$ in the last integral above yields

$$\begin{aligned} I &\leq Ce_{(q,k-1,s)}+C[e_{(q-1,k-1,s+1)}+e_{(q,k-1,s)}] \\ &\leq C[e_{(q-1,k-1,s+1)}+e_{(q,k-1,s)}]. \end{aligned} \tag{2-25'''}$$

Combining the estimates for T_0, T_1 in (2-19‴), (2-24‴) and (2-25‴) we complete the proof of lemma. □

We are ready to state the main inequality.

Proposition 2.15. *Let $e_{(q,k,s)}$ be the numbers defined by* (2-15). *Then, for $q=3m$,*

$$e_{(q,m,0)}\leq C\sum_{s=0}^{m}e_{(0,0,s)},$$

where $C=C(\omega,n,m)$ is a uniform constant.

Proof. We start with Lemma 2.11 which gives

$$e_{(q,m,0)}\leq C[e_{(q-1,m-1,1)}+e_{(q-1,m-1,0)}]. \tag{2-27}$$

Then, the first term in the bracket is estimated via Corollary 2.12. Applying this corollary $m-1$ times and using decreasing property of $e_{(p,k,s)}$ in the first parameter, we get

$$e_{(q-1,m-1,1)}\leq Ce_{(q-m,0,m)}+C\sum_{i=0}^{m-1}e_{(q-m-2,i,m-1-i)}. \tag{2-28}$$

The second term in the bracket in (2-27) satisfies $e_{(q-1,m-1,0)} \le e_{(q-m-2,m-1,0)}$. Next, we use Lemma 2.13 for each term $e_{(q',\ell,m-1-\ell)}$ with $q' = q-m-2$ in the sum above. Namely, applying the lemma ℓ times and using the decreasing property again, we get

$$e_{(q',\ell,m-1-\ell)} \le Ce_{(q'-\ell,0,m-1)} + C\sum_{i=0}^{m-2} e_{(q'-\ell-1,i,m-2-i)}.$$

The smallest value of the first parameter in the last sum is $q'-m$ for $\ell = m-1$. Hence,

$$\sum_{i=0}^{m-1} e_{(q',i,m-1-i)} \le Ce_{(q'-\ell,0,m-1)} + C\sum_{i=0}^{m-2} e_{(q'-m,i,m-2-i)}. \tag{2-29}$$

It remains to apply Lemma 2.14 for each term $e_{(q'',\ell,m-2-\ell)}$ in the sum on the right-hand side, where $q'' = q'-m = q-2m-2$. Again, we have

$$\begin{aligned} e_{(q'',\ell,m-2-\ell)} &\le Ce_{(q''-1,\ell-1,m-\ell-1)} + Ce_{(q'',\ell-1,m-\ell-2)} \\ &\le \cdots \\ &\le Ce_{(q''-\ell,0,m-2)} + \sum_{i=0}^{\ell-1} e_{(q''-i,\ell-1-i,m-\ell-2+i)}. \end{aligned}$$

Therefore, an easy induction argument gives us

$$e_{(q'',\ell,m-2-\ell)} \le C\sum_{s=0}^{m-2} e_{(q''-\ell,0,s)}. \tag{2-30}$$

Combining (2-27), (2-28), (2-29) and (2-30) we arrive at

$$e_{(q,m,0)} \le C\sum_{s=0}^{m} e_{(q''-m+2,0,s)} = C\sum_{s=0}^{m} e_{(0,0,s)}$$

as $q''-m+2 = q-3m = 0$. □

So far all considered functions were smooth, however, by [Kołodziej and Nguyen 2016, Proposition 2.11] we know that the integrands on both sides of the above statements (Lemmas 2.10–2.14, Corollary 2.12 and Proposition 2.15) are well defined for continuous m-ω-sh functions. Let us record the following observation.

Remark 2.16. Let Ω be a strictly m-pseudoconvex domain. The statements above are still valid for continuous m-ω-sh functions v, w and $-1 \le \rho \le 0$ satisfying $-1 \le v \le w \le 0$ and $v = w$ in a neighborhood of $\partial\Omega$.

In fact, there exist decreasing sequences of m-ω-sh functions v_j, w_j and ρ_j belonging to $C^\infty(\overline{\Omega})$ such that $v_j \downarrow v$, $w_j \downarrow w$ and $\rho_j \downarrow \rho$ (uniformly) in $\overline{\Omega}$, and moreover,

$$-1 \le v_j \le w_j \le 0, \quad -1 \le \rho_j \le 0.$$

For plurisubharmonic functions the usual convolution with standard kernels produces the approximating sequence and hence, the property $v_j = w_j$ near the boundary is preserved. Then, the integration by parts is not affected and passing to the limit as $j \to +\infty$ gives the desired inequalities. However, in this new setting we used a different way to obtain the approximating sequence. The property that $v_j = w_j$ near the boundary of Ω needs to be verified, which is possible via the stability estimates for complex Hessian equations. However, we can get around this by showing the uniform convergence to zero of the sequence $h_j = w_j - v_j$ near the boundary.

Let $\Omega' \Subset \Omega$ be a smooth domain such that $v = w$ outside Ω'. Let $T_j = (dd^c \rho_j)^k \wedge (dd^c v_j)^s$, where $k + s \le m$. Then, it follows from the weak convergence [Kołodziej and Nguyen 2016, Proposition 2.11] and the CLN inequality [Kołodziej and Nguyen 2016, Proposition 2.9] that for $p \ge 1$,

$$\lim_{j\to\infty} \int_{\Omega'} h_j^p \, T_j \wedge \omega^{n-k-s} = \int_{\Omega} h^p \, T \wedge \omega^{k-s}.$$

Therefore, we reduce the required inequality to the case of smooth functions v_j, w_j and ρ_j. However the integration by parts in (2-20) will contain the extra boundary terms:

$$\int_{\Omega'} h_j^p \, dd^c \rho_j \wedge T_j \wedge \omega^{n-k-s} = \int_{\Omega'} \rho_j \, dd^c (h_j^p \omega^{n-k-s}) \wedge T_j + E_1 + E_2,$$

where

$$\begin{aligned} E_1 &= \int_{\partial\Omega'} h_j^p \, d^c \rho_j \wedge \omega^{n-k-s} \wedge T_j; \\ E_2 &= -\int_{\partial\Omega'} \rho_j \, d^c (h_j^p \omega^{n-k-s}) \wedge T_j \\ &= -\int_{\partial\Omega'} \rho_j h_j^{p-1} \, (p d^c h_j \wedge \omega^{n-k-s} + h_j d^c \omega^{n-k-s}) \wedge T_j. \end{aligned}$$

By the CLN inequality and $h_j \to 0$ uniformly on a neighborhood of $\partial\Omega'$ as $j \to \infty$, the two boundary terms go to zero when we pass to the limit.

Remark 2.17. We will see later that the above statements also hold for bounded m-ω-sh functions once we define the wedge product for currents related to such functions and prove the weak convergence under decreasing sequences.

3. Wedge product for bounded functions

In this section we prove the existence of the wedge product of currents where the dd^c operator is applied to bounded m-ω-sh functions. Then, we introduce a weighted Sobolev space associated to these resulting positive currents. This is a crucial technical tool that allows us to establish the integration by parts formulae for (nonclosed) currents of order zero.

3.1. ***The wedge product for bounded m-ω-sh functions.*** Let $\Omega \subset \mathbb{C}^n$ be a bounded open set. We are going to show that the wedge product of bounded m-ω-sh functions can be defined inductively by

$$dd^c v \wedge dd^c u = dd^c (v \, dd^c u)$$

and similarly for more terms. Let us start with the following observation.

Lemma 3.1. *Let u be a bounded m-ω-sh function. Then, $dd^c u$ is a real $(1, 1)$-current of order zero whose coefficients are signed Radon measures.*

Proof. Without loss of generality we may assume $-1 \le u \le 0$. We will show that there is a signed measure μ such that

$$dd^c u(\phi) = \int \phi \, d\mu$$

for every (smooth) test $(n-1, n-1)$-form. By a theorem in distribution theory in Federer's book [1969, Section 4.15] it is enough to show that for all $K \subset \Omega$ compact there exists $C = C(K, \Omega)$ such that

$$|dd^c u(\phi)| \le C \|\phi\|_K$$

for any smooth test form ϕ with $\operatorname{supp} \phi \subset K$. Here we define

$$\|\phi\|_K = \sum_{i,j} \sup_K |\phi_{i\bar{j}}|,$$

where the $\phi_{i\bar{j}}$ are coefficients of the form ϕ. By the definition of dd^c-operator for currents

$$dd^c u(\phi) = u(dd^c \phi) = \int u \wedge dd^c \phi.$$

Thus, for a smooth sequence of m-ω-sh functions $\{u_s\}$ that decrease to u we have

$$dd^c u(\phi) = \lim_{s\to\infty} (dd^c u_s)(\phi) = \lim_{s\to\infty} \int dd^c u_s \wedge \phi.$$

Since $dd^c u_s \in \Gamma_m(\omega)$, it follows from [Kołodziej and Nguyen 2016, Corollary 2.4] that

$$|dd^c u_s \wedge \phi| \le C \|\phi\|_K dd^c u_s \wedge \omega^{n-1} \tag{3-1}$$

for some universal constant depending on n, m. It follows that

$$\left| \int dd^c u_s \wedge \phi \right| \le C \|\phi\|_K \int_K dd^c u_s \wedge \omega^{n-1}.$$

Thanks to the CLN inequality [Kołodziej and Nguyen 2016, Proposition 2.9] we know that the last integral on the right-hand side is bounded by a constant $C(K, \Omega)\|u_s\|_{L^\infty}$. This finishes the proof.

Furthermore, for any test function $\chi \ge 0$,

$$0 \le \lim_{s\to\infty} \int \chi \, \omega^{n-1} \wedge dd^c u_s = dd^c u(\chi \omega^{n-1}).$$

In other words, $dd^c u \wedge \omega^{n-1}$ is a (positive) Radon measure. Moreover, it follows from (3-1) that the total variation of coefficients of $dd^c u$ is dominated by $dd^c u \wedge \omega^{n-1}$. Therefore, the coefficients of $T = dd^c u$ are signed Radon measures. □

It follows from Lemma 3.1 that we can multiply $dd^c u$ by a bounded function v. In fact, we can continue to define inductively the wedge product in this way and obtain a real current of order zero for up to $m - 1$ bounded functions.

Lemma 3.2. *Let $u_1, \dots, u_p$ with $1 \le p \le m-1$ be bounded m-ω-sh functions. Assume $T_0 = 1$ and $T_{p-1} = dd^c u_{p-1} \wedge \dots \wedge dd^c u_1$. Then, the current*

$$dd^c u_p \wedge \dots \wedge dd^c u_1 := dd^c[u_p T_{p-1}]$$

is a well-defined real (p, p)-current of order zero whose coefficients are signed Radon measures. Moreover,

$$\mathscr{L}_p(u_p, \dots, u_1) := dd^c u_p \wedge \dots \wedge dd^c u_1 \wedge \omega^{n-m}$$

is a positive $(n-m+p, n-m+p)$-current.

Proof. We argue by induction in p. If $p=1$, then this is Lemma 3.1. Assume that it holds for $1 \le p \le m-2$, we need to show that the lemma holds for $p+1$. For the simplicity of notation we present the argument for $p+1=2$ and note that the proof in the general case is completely the same. Let us write $u_1 = u$ and $u_2 = v$ and assume that $-1 \le u, v \le 0$. Since $dd^c u$ has signed measure coefficients, the currents

$$S = v dd^c u \quad \text{and} \quad dd^c v \wedge dd^c u := dd^c S = dd^c(v dd^c u)$$

are well defined. Our next goal is to show that the real (closed) current $dd^c S$ of bidegree (2, 2) has order zero, thus its coefficients are signed Radon measures.

As above we will show that for each compact set $K \subset \Omega$, there is a uniform constant C such that

$$|dd^c S(\phi)| \le C \|\phi\|_K \tag{3-2}$$

for every smooth test form with $\operatorname{supp} \phi \subset K$, where $\|\phi\|_K$ denotes the C^0-norm of the form ϕ. To this end, let $\{v_\ell\}_{\ell \ge 1}$ and $\{u_s\}_{s \ge 1}$ be sequences of smooth m-ω-sh functions such that

$$v_\ell \downarrow v \quad \text{and} \quad u_s \downarrow u.$$

We may also assume that $-1 \le u_s, v_\ell \le 0$. Fix ϕ a test form. By definition we have

$$dd^c S(\phi) = S(dd^c \phi) = \int v \, dd^c u \wedge dd^c \phi. \tag{3-3}$$

Since $dd^c u \wedge dd^c \phi$ has signed Radon measure coefficients, it follows from the dominated convergence theorem that

$$\int v \, dd^c u \wedge dd^c \phi = \lim_{\ell \to \infty} \int v_\ell \, dd^c u \wedge dd^c \phi = \lim_{\ell \to \infty} \lim_{s \to \infty} \int v_\ell \, dd^c u_s \wedge dd^c \phi.$$

By the integration by parts formula,

$$\int v_\ell \, dd^c u_s \wedge dd^c \phi = \int dd^c v_\ell \wedge dd^c u_s \wedge \phi.$$

Notice that $m-1 \ge 2$. Applying [Kołodziej and Nguyen 2016, Corollary 2.4] for $\gamma_1 = dd^c u_s$, $\gamma_2 = dd^c v_\ell$ belonging to $\Gamma_m(\omega)$, we get

$$|dd^c v_\ell \wedge dd^c u_s \wedge \phi| \le C \|\phi\|_K [dd^c(u_s + v_\ell)]^2 \wedge \omega^{n-2}. \tag{3-4}$$

Integrating both sides and then applying the CLN inequality for the right-hand side we get

$$\left|\int dd^c v_\ell \wedge dd^c u_s \wedge \phi\right| \le C(K,\Omega)\|\phi\|_K.$$

Letting $s \to \infty$ and then $\ell \to \infty$ in this order, we conclude from (3-2) the desired estimate (3-3). Hence, $dd^c S$ is a (2, 2) current of order zero.

Furthermore, for a test function $\chi \ge 0$,

$$dd^c S \wedge \omega^{n-2}(\chi) = \lim_{\ell\to\infty}\lim_{s\to\infty}\int \chi\, dd^c v_\ell \wedge dd^c u_s \wedge \omega^{n-2} \ge 0.$$

Thus, $dd^c S \wedge \omega^{n-2}$ is a positive Radon measure. It follows from (3-4) (by passing to the limit) that the coefficients of $dd^c S$ are signed Radon measures.

Finally, for smooth m-ω-sh functions $u_1^{j_1}, \dots, u_p^{j_p}$, the form

$$\mathscr{L}_p(u_p^{j_p}, \dots, u_1^{j_1}) = dd^c u_p^{j_p} \wedge \cdots \wedge dd^c u_1^{j_1} \wedge \omega^{n-m}$$

is positive by Gårding's inequality [1959]. Letting $j_1 \to \infty, \dots, j_p \to \infty$ in this order, for decreasing sequences approximating the functions $u_p, \dots, u_1$ we obtain that $\mathscr{L}_p(u_p, \dots, u_1)$ is a positive current of bidegree $(n-m+p, n-m+p)$. □

Thus, we defined inductively for bounded m-ω-sh functions, $u_1, \dots, u_{m-1}$, a real closed current

$$S := dd^c u_{m-1} \wedge \cdots \wedge dd^c u_1 \tag{3-5}$$

of order zero whose coefficients are signed Radon measures. Furthermore,

$$S \wedge \omega^{n-m+1} \tag{3-6}$$

is a positive Radon measure. In the last step we multiply S by a bounded function $v = u_m$.

Theorem 3.3. *Let S be as in* (3-5) *and let v be a bounded m-ω-sh function. Then,*

$$T := dd^c(vS)$$

is a real closed (m, m)-current and $T \wedge \omega^{n-m}$ is a positive Radon measure.

Proof. Let $\chi \ge 0$ be a test function. By definition

$$T \wedge \omega^{n-m}(\chi) = T(\chi\omega^{n-m}) = \int vS \wedge dd^c(\chi\omega^{n-m}).$$

Since S is of order zero, it follows from the dominated convergence theorem that for a sequence of smooth m-ω-sh functions $v_\ell \downarrow v$,

$$\int vS \wedge dd^c(\chi\omega^{n-m}) = \lim_{\ell\to\infty}\int v_\ell S \wedge dd^c(\chi\omega^{n-m}).$$

Using the inductive definition of S and the integration by parts, we get

$$\int v_\ell S \wedge dd^c(\chi\omega^{n-m}) = \lim_{j_{m-1}\to\infty} \cdots \lim_{j_1\to\infty} \int dd^c v_\ell \wedge dd^c u^{j_{m-1}} \wedge \cdots \wedge u^{j_1} \wedge dd^c(\chi\omega^{n-m})$$
$$= \lim_{j_{m-1}\to\infty} \cdots \lim_{j_1\to\infty} \int dd^c v_\ell \wedge dd^c u^{j_{m-1}} \wedge \cdots \wedge dd^c u^{j_1} \wedge (\chi\omega^{n-m}).$$

Since $v_\ell, u^{j_1}, \ldots, u^{j_{m-1}}$ are smooth m-ω-sh functions, the last integrand is positive and the proof follows. □

Consequently the following definition is justified.

Definition 3.4. Let $u_1, \ldots, u_p$ with $1 \le p \le m$ be bounded m-ω-sh functions in Ω. The wedge product is given inductively by

$$dd^c u_p \wedge \cdots \wedge dd^c u_1 := dd^c[u_p T_{p-1}],$$

where $T_0 = 1$ and $T_{p-1} = dd^c u_{p-1} \wedge \cdots \wedge dd^c u_1$.

We are most interested in the following positive Radon measures.

Definition 3.5. Let u be a bounded m-ω-sh function. Then, the Hessian operator $H_m(u)$ is defined by

$$H_m(u) := (dd^c u)^m \wedge \omega^{n-m} = \mathscr{L}_m(u, \ldots, u).$$

Moreover, for $1 \le s \le m$, we also write

$$H_s(u) = (dd^c u)^s \wedge \omega^{n-s}.$$

For convenience, we summarize the results in Lemma 3.2 and Theorem 3.3 as follows. Let $\{u_s^j\}_{j\ge1}$ be decreasing sequences such that $u_s^j \downarrow u_s$ for each $1 \le s \le m$. Define

$$T^{j_p\cdots j_1} = dd^c u_p^{j_p} \wedge \cdots \wedge dd^c u_1^{j_1}. \tag{3-7}$$

Then, we have $T^{j_p\cdots j_2} \wedge dd^c u_1 = \lim_{j_1\to\infty} T^{j_p\cdots j_1}$, and consequently

$$dd^c u_p \wedge \cdots \wedge dd^c u_1 = \lim_{j_p\to\infty} \cdots \lim_{j_1\to\infty} T^{j_p\cdots j_1}. \tag{3-8}$$

For $1 \le p \le m-1$, each limit in (3-8) is in the sense of currents of order zero. For $p = m$, we have

$$dd^c u_p \wedge \cdots \wedge dd^c u_1 \wedge \omega^{n-m} = \lim_{j_p\to\infty} \cdots \lim_{j_1\to\infty} T^{j_p\cdots j_1} \wedge \omega^{n-m}, \tag{3-9}$$

where each limit is also in the sense of currents of order zero.

Remark 3.6. At this point the definition of the wedge product in Definition 3.4 is not symmetric with respect to $u_1, \ldots, u_p$. Furthermore, if the metric ω is Kähler or just the standard Kähler form $\beta = dd^c|z|^2$ in $\mathbb{C}^n$, then by (3-9) it follows that Definition 3.4 coincides with the ones given before in [Błocki 2005; Dinew and Kołodziej 2014].

By taking the limit inductively as in (3-8) and (3-9) we can state now the CLN inequality for bounded functions.

Proposition 3.7 (CLN inequality). *Let $K \Subset U \Subset \Omega$, where K is compact and U is open. Let $u, u_1, \dots, u_p$, be bounded m-ω-sh functions in Ω, where $1 \le p \le m$. Then, there exists a constant C depending on K, U, ω such that*

(a) $\int_K (dd^c u)^p \wedge \omega^{n-p} \le C(1 + \|u\|_{L^\infty(U)})^p$;

(b) $\int_K dd^c u_1 \wedge \dots \wedge dd^c u_p \wedge \omega^{n-p} \le C\big(1 + \sum_{s=1}^p \|u_s\|_{L^\infty(U)}\big)^p$.

Proof. (a) Without loss of generality we may assume that $\|u\|_{L^\infty(\Omega)} \le 1$. We can cover K by finitely many small balls, hence by the localization principle we can assume that K and U are concentric balls. Let $\{u^j\}_{j\ge1}$ be sequences of smooth m-ω-sh functions such that $u^j \downarrow u$ as $j \to \infty$. Let $0 \le \chi \le 1$ be a cut-off function such that $\chi \equiv 1$ on K and $\operatorname{supp} \chi \subset U$. By the CLN inequality for smooth functions [Kołodziej and Nguyen 2016, Proposition 2.9]

$$\int \chi \mathscr{L}_p(u^{j_p}, \dots, u^{j_1}) \wedge \omega^{m-p} \le C\bigg(1 + \|\sum_{s=1}^{p} u^{j_s}\|^p_{L^\infty(U)}\bigg).$$

By the Hartogs lemma for ω-sh functions [Gu and Nguyen 2018, Lemma 9.14] it follows that

$$\lim_{j\to\infty} \|1 + u^j\|_{L^\infty(U)} = \lim_{j\to\infty} \sup_{\overline{U}} (1 + u^j) = 1 + \sup_{\overline{U}} u \le 1 + \|u\|_{L^\infty(U)}.$$

Letting $j_1 \to \infty, \dots, j_p \to \infty$ in this order, we get

$$\begin{aligned}\int_K \mathscr{L}_p(u) \wedge \omega^{m-p} &\le \lim_{j_p\to\infty} \cdots \lim_{j_1\to\infty} \int \chi \mathscr{L}_p(u^{j_p}, \dots, u^{j_1}) \wedge \omega^{m-p} \\ &\le C \lim_{j_p\to\infty} \cdots \lim_{j_1\to\infty} \bigg(1 + \bigg\|\sum_{s=1}^{p} u^{j_s}\bigg\|_{L^\infty}\bigg)^p \\ &= C(1 + \|u\|_{L^\infty(U)})^p.\end{aligned}$$

(b) Observe that for $v := u_1 + \dots + u_p$ we have $\mathscr{L}_p(v) \ge \mathscr{L}(u_1, \dots, u_p)$ as positive currents. So, (b) is an immediate consequence of (a). □

We also need a simplified version of Lemma 2.3 for particular positive currents. Let $-1 \le u_1, \dots, u_{m-2} \le 0$ be bounded m-ω-sh functions in Ω. Denote by T the current

$$T = dd^c u_1 \wedge \dots \wedge dd^c u_{m-2} \wedge \omega^{n-m}. \tag{3-10}$$

Corollary 3.8. *Let ϕ be a Borel function such that* $\operatorname{supp} \phi = K \Subset \Omega$. *Assume also $|\phi| \le 1$. Let $0 \le h \le 1$ be a smooth m-ω-sh function in Ω. There exists a constant $C = C(\omega, K)$ such that*

$$\left|\int \phi \, dh \wedge d^c \omega \wedge T\right|^2 \le C \int |\phi| \, T \wedge \omega^2.$$

Proof. By Lemma 2.3, it is enough to show that $\int |\phi| \, dh \wedge d^c h \wedge T \wedge \omega \le C$. In fact, since h^2 and h are m-ω-sh, we have

$$2dh \wedge d^c h \wedge T \wedge \omega = [dd^c h^2 - 2h dd^c h] \wedge T \wedge \omega \le dd^c h^2 \wedge T \wedge \omega.$$

Multiplying both sides by $|\phi|$ and integrating them, we get

$$\int |\phi|\, dh \wedge d^c h \wedge T \wedge \omega \leq \int |\phi|\, dd^c h^2 \wedge T \wedge \omega.$$

Using the assumption on ϕ the right-hand side is bound by

$$\int_K dd^c h^2 \wedge T \wedge \omega.$$

The desired estimate follows from the CLN inequality for bounded functions (Proposition 3.7). □

3.2. *Weighted Sobolev space.* Let $K \subset \Omega$ be a compact subset and let ψ be a strictly plurisubharmonic defining function for Ω. Let us denote by $P^* = P^*(\Omega, K, \psi)$ the set of bounded m-ω-sh functions that

$$u(z) = \psi, \quad z \in \Omega \setminus K.$$

Let $u_1, \dots, u_p \in P^*$ with $1 \leq p \leq m-1$ and define

$$\tau = dd^c u_1 \wedge \cdots \wedge dd^c u_p, \tag{3-11}$$

and conventionally $\tau = 1$ for $p = 0$. By definition this is a closed (p, p)-current of order zero. We first make an additional assumption on the current τ. By considering $u_i = u_i + \delta|z|^2$ for some $\delta > 0$ small, we may assume that there is $c_0 > 0$ such that

$$\tau \wedge \omega^{n-p-1} \geq c_0 \omega^{n-1}. \tag{3-12}$$

Hence, the trace measure (with respect to ω) of the positive current $\tau \wedge \omega^{n-p}$ satisfies

$$\mu_p := \tau \wedge \omega^{n-p} \geq c_0 \omega^n. \tag{3-13}$$

Let us introduce a norm associated to τ on the space smooth functions $C^\infty(\Omega, \mathbb{R})$. If v, w are smooth functions in Ω, then we can define

$$\langle v, w\rangle_\tau = \int dv \wedge d^c w \wedge \tau \wedge \omega^{n-p-1} \quad \text{and} \quad \|v\|_\tau^2 = \langle v, v\rangle_\tau = \int dv \wedge d^c v \wedge \tau \wedge \omega^{n-p}.$$

Let us consider the inner product

$$\langle v, w\rangle = \int vw\beta^n + \langle v, w\rangle_\tau,$$

where $\beta = dd^c|z|^2$, and the associated norm

$$\|v\|^2 = \langle v, v\rangle = \|v\|_{L^2}^2 + \|v\|_\tau^2.$$

The assumption makes this inner product positive definite. In fact, it is clear that $\langle \bullet, \bullet\rangle$ satisfies the properties of an inner product, namely

(i) $\langle v, v\rangle \geq 0$ and $\langle v, v\rangle = 0$ if and only if $v = 0$;

(ii) $\langle av_1 + bv_2, w\rangle = a\langle v_1, w\rangle + b\langle v_2, w\rangle$;

(iii) $\langle v, w\rangle = \langle w, v\rangle$.

This implies also that it satisfies the triangle inequality

$$|\langle v, w\rangle| \le \sqrt{\langle v, v\rangle\langle w, w\rangle}.$$

Let us denote by $W^{1,2}(\Omega, \mu_p)$ the real Hilbert space completion of $C^\infty(\Omega, \mathbb{R})$ with respect to this norm. Let us state a property of this space immediately following from the definition.

Lemma 3.9. *$W^{1,2}(\Omega, \mu_p)$ is uniformly convex. Consequently, it is reflexive.*

Our goal is to show that for each $0 \le p \le m-1$,

$$P^* \subset W^{1,2}(\Omega, \mu_p).$$

If we take $u_p = \psi$ the strictly psh defining function for Ω, then clearly

$$W^{1,2}(\Omega, \mu_p) \subset W^{1,2}(\Omega, \mu_{p-1}).$$

Therefore, we only need to prove two special cases for $p = 0$ and $p = m-1$. Let us start with this simpler case $p = 0$, i.e., $\tau = 1$, $\tau \wedge \omega^{n-1} = \omega^{n-1}$ and $\mu_0 = \omega^n$ which is equivalent to the Lebesgue measure $dV = \beta^n$. This is a classical result if $\omega = \beta$ is the standard Kähler metric (see, e.g., [Hörmander 1994, Proposition 3.4.19]).

Lemma 3.10. *All functions in P^* belong to the Sobolev space $W^{1,2}(\Omega)$.*

Proof. Let $u \in P^*$. By the approximation theorem there is $\{u_j\}_{j\ge1} \subset P^*$ a sequence of smooth functions such that $u_j \downarrow u$ pointwise. We may assume that $0 \le u, u_j \le 1$. Observe that the sequence $\{u_j\}_{j\ge1}$ is uniformly bounded in $W^{1,2}(\Omega)$. Indeed, by the ω-subharmonicity

$$\begin{aligned}\int du_j \wedge d^c u_j \wedge \omega^{n-1} &= \int \big(\tfrac{1}{2} dd^c u_j^2 - 2u_j dd^c u_j\big) \wedge \omega^{n-1}\\ &\le \frac{1}{2}\int dd^c u_j^2 \wedge \omega^{n-1}.\end{aligned}$$

The last integral is bounded by a constant $C = C(K, \psi, \Omega)$ by an application of the CLN inequality. Therefore, the sequence of (vector-valued) functions ∇u_j is uniformly bounded in $L^2(\Omega)$. Thus, by the weak compactness theorem, ∇u_j converges weakly to a vector-valued function $\boldsymbol{v} \in L^2(\Omega)$.

To complete the proof we are going to show that $\boldsymbol{v} = \nabla u$. First, note that $u_j \to u$ in $L^2(\Omega)$ by the dominated convergence theorem. So, $(u_j, \nabla u_j) \to (u, \boldsymbol{v})$ weakly in the Hilbert space $L^2(\Omega) \times L^2(\Omega, \mathbb{R}^n)$. The Mazur lemma [Yosida 1995, Chapter V, Theorem 2] implies that there is a sequence of convex combinations

$$\widetilde{u}_\ell = \sum_{j=1}^{\ell} \lambda_{\ell,j}(u_j, \nabla u_j), \quad \lambda_{\ell,j} \ge 0 \text{ and } \sum_{j=1}^{\ell} \lambda_{\ell,j} = 1,$$

which converges in norm to $(u, \boldsymbol{v})$ in $L^2(\Omega) \times L^2(\Omega, \mathbb{R}^n)$. It follows that

$$w_\ell = \sum_{j=1}^{\ell} \lambda_{\ell,j} u_j$$

is a Cauchy sequence in $W^{1,2}(\Omega)$. Hence, there exist a limit function

$$\widetilde{u} = \lim_{\ell\to\infty} w_\ell \in W^{1,2}(\Omega).$$

It follows that $u = \widetilde{u}$ and $\boldsymbol{v} = \nabla\widetilde{u} = \nabla u$. □

We proceed to prove the most general case $1 \le p = m-1$ when

$$\mu = \mu_p := \tau \wedge \omega^{n-m+1}$$

is a positive Radon measure. Equivalently, we wish to show that if $v \in P^*$, then the form dv has coefficients in $L^2(\Omega, d\mu)$. For a smooth function v we have

$$\int |\nabla v|^2 \, dV \le \frac{1}{c_0} \int dv \wedge d^c v \wedge \tau \wedge \omega^{n-m}.$$

This implies that $W^{1,2}(\Omega,\mu) \subset W^{1,2}(\Omega)$. In particular, the gradient ∇u is well defined in $W^{1,2}(\Omega,\mu)$.

Proposition 3.11. *Let $v \in P^*$ and $\{v_\ell\}_{\ell\ge 0} \subset P^*$ be a sequence of smooth m-ω-sh functions such that $v_\ell \downarrow v$ pointwise in Ω. Then v_ℓ converges weakly to v in $W^{1,2}(\Omega,\mu)$, i.e., for every $w \in W^{1,2}(\Omega,\mu)$,*

$$\lim_{\ell\to\infty} \langle v_\ell, w\rangle = \langle v, w\rangle.$$

In particular, $P^* \subset W^{1,2}(\Omega,\mu)$.

Proof. We may assume that $0 \le v \le v_\ell \le 1$. Then, there exists a uniform constant $C = C(\omega, n, m, \psi, \Omega)$ such that

$$\|\nabla v_\ell\|_\tau^2 = \int dv_\ell \wedge d^c v_\ell \wedge \tau \wedge \omega^{n-m} \le C\|v_\ell\|_{L^\infty}^2.$$

In fact, since $dv_\ell \wedge d^c v_\ell = \frac{1}{2} dd^c v_\ell^2 - v_\ell dd^c v_\ell$, and $dd^c v_\ell \wedge \tau \wedge \omega^{n-m}$ is a positive measure and $v_\ell \ge 0$, it follows that

$$\int dv_\ell \wedge d^c v_\ell \wedge \tau \wedge \omega^{n-m} \le \int dd^c v_\ell^2 \wedge \tau \wedge \omega^{n-m}.$$

Since v_ℓ^2 is also m-ω-sh, the desired inequality follows from the CLN inequality.

In the same way as in the previous proof, via Mazur's lemma, one proves that $v \in W^{1,2}(\Omega,\mu)$. □

Remark 3.12. The proofs given in Lemma 3.10 and Proposition 3.11 are inspired by the one in the book by Heinonen, Kilpeläinen and Martio [Heinonen et al. 2006, Theorem 1.30].

3.3. ***Integration by parts for bounded functions.*** In this section we show "integration by parts inequalities" for bounded m-ω-sh functions which are smooth near the boundary. In particular, they hold for functions in the class P^*. First we deal with the case of the wedge product of $m-2$ bounded functions and consider

$$\eta = dd^c u_1 \wedge \cdots \wedge dd^c u_{m-2}, \tag{3-14}$$

where $u_1, \ldots, u_{m-2} \in P^*$ as in Section 3.2. Recall that we also impose the assumption (3-12) for η, namely

$$\eta \wedge \omega^{n-m+1} \ge c_0 \omega^{n-1}, \tag{3-15}$$

and denote the corresponding trace measure by μ:

$$\mu = \eta \wedge \omega^{n-m+2}. \tag{3-16}$$

The following integration by parts formula for currents of order zero is used frequently below. Let ϕ and ρ be smooth functions such that $\phi = 0$ near $\partial\Omega$. Then,

$$\int \phi\, dd^c\rho \wedge \eta \wedge \omega^{n-m+1} = -\int d\phi \wedge d^c\rho \wedge \eta \wedge \omega^{n-m+1} + \int \phi\, d^c\rho \wedge d\omega \wedge \eta \wedge \omega^{n-m}. \tag{3-17}$$

If $v, w \in P^* \subset W^{1,2}(\Omega, \mu)$ are smooth, then

$$\langle v, w\rangle_\eta = \int dv \wedge d^c w \wedge \eta \wedge \omega^{n-m+1}.$$

By abusing the notation for general $v, w \in P^*$ we will write the integral on the right understanding that it is the inner product $\langle v, w\rangle_\eta$. The first version of the "integration by parts inequality" in the class P^* reads as follows.

Lemma 3.13. *Let $\{v_\ell\}_{\ell\geq 1}$ be a sequence of smooth functions from P^* and let $w \in P^*$ be smooth. Assume $v_\ell \leq w$ and $v_\ell \downarrow v \in P^*$. Then,*

$$-\int d(w-v) \wedge d^c v \wedge \eta \wedge \omega^{n-m+1} \leq \int (w-v) dd^c v \wedge \eta \wedge \omega^{n-m+1} + C\biggl(\int (w-v)\eta \wedge \omega^{n-m+2}\biggr)^{\frac{1}{2}}.$$

The constant C depends on ω, K, ψ and the uniform norm of functions, but it is independent of δ in (3-12).

Proof. Without loss of generality we may assume that $0 \leq v, v_\ell, w \leq 1$ and $0 \leq u_i \leq 1$. By the weak convergence in the space $W^{1,2}(\Omega, \mu)$ (Proposition 3.11) and the dominated convergence theorem it is enough to prove that

$$-\int d(w-v_\ell) \wedge d^c v \wedge \eta \wedge \omega^{n-m+1} \leq \int (w-v_\ell) dd^c v \wedge \eta \wedge \omega^{n-m+1} + C\biggl(\int (w-v_\ell)\eta \wedge \omega^{n-m+2}\biggr)^{\frac{1}{2}}.$$

To this end we use once more the weak convergence property when ℓ is fixed. Note that $w - v_\ell$ is a continuous function whose support is compact in Ω. It follows from the weak convergence of currents (of order zero) that

$$\int (w-v_\ell) dd^c v \wedge \eta \wedge \omega^{n-m+1} = \lim_{j\to\infty} \int (w-v_\ell) dd^c v_j \wedge \eta \wedge \omega^{n-m+1}.$$

Now for each ℓ and j, the integration by parts formula (3-17) gives

$$-\int d(w-v_\ell) \wedge d^c v_j \wedge \eta \wedge \omega^{n-m+1} = \int (w-v_\ell) dd^c v_j \wedge \eta \wedge \omega^{n-m+1} + \int (w-v_\ell) d^c v_j \wedge d\omega \wedge \eta \wedge \omega^{n-m}.$$

By Corollary 3.8 applied for $\phi = w - v_\ell$ and CLN inequality (Proposition 3.7) for bounded functions, the second integral on the right-hand side is bounded by

$$C\biggl(\int (w-v_\ell)\eta \wedge \omega^{n-m+2}\biggr)^{\frac{1}{2}}.$$

The desired inequality follows by letting $j \to \infty$. □

Lemma 3.14. *Let v, w and $\{v_\ell\}$ be as in Lemma 3.13. Let ρ be a bounded m-ω-sh in Ω such that $-1 \le \rho \le 0$. Then,*

$$\int_\Omega (w-v)dd^c\rho\wedge\eta\wedge\omega^{n-m+1} \le -\int_\Omega d(w-v)\wedge d^c\rho\wedge\eta\wedge\omega^{n-m+1} + C\biggl(\int_\Omega (w-v)\wedge\eta\wedge\omega^{n-m+2}\biggr)^{\frac{1}{2}}.$$

The dependence of C is the same as in the previous lemma.

Proof. Since the supports of the integrands on both sides are contained in K, we replace ρ by $\max\{\rho, A\psi\}$ for some $A > 0$ large such that $A\psi \le -1$ on a neighborhood of K. Thus, we may assume that $\rho \in P^*(\Omega, K', A\psi)$ for a compact subset $K \subset K' \Subset \Omega$. By the dominated convergence theorem for Radon measures and the weak convergence in Proposition 3.11 it is enough to prove the inequality for v_ℓ in the place of v, that is

$$\int_\Omega (w-v_\ell)dd^c\rho\wedge\eta\wedge\omega^{n-m+1} \le -\int_\Omega d(w-v_\ell)\wedge d^c\rho\wedge\eta\wedge\omega^{n-m+1} + C\biggl(\int_\Omega (w-v_\ell)\eta\wedge\omega^{n-m+2}\biggr)^{\frac{1}{2}}. \tag{3-18}$$

For a fixed ℓ the function $w - v_\ell$ is continuous, with its support in Ω. We can apply the limit argument once more to assume that ρ also is smooth as follows. By the approximation theorem for m-ω-sh functions we can find a decreasing sequence $\{\rho_j\}_{j\ge1} \subset P^*$ of smooth m-ω-sh functions such that $\rho_j \downarrow \rho$. We may assume that $-A \le \rho_j \le 0$. Then,

$$dd^c\rho_j\wedge\eta\wedge\omega^{n-m+1} \to dd^c\rho\wedge\eta\wedge\omega^{n-m+1}$$

weakly as measures. So, if the inequality holds for ρ_j, then so it does for ρ.

Using again the integration by parts formula (3-17) we have

$$\int_\Omega (w-v_\ell)dd^c\rho_j\wedge\eta\wedge\omega^{n-m+1} = -\int_\Omega d(w-v_\ell)\wedge d^c\rho_j\wedge\eta\wedge\omega^{n-m+1} + \int_\Omega (w-v_\ell)d^c\rho_j\wedge d\omega\wedge\eta\wedge\omega^{n-m}.$$

As in the proof of Lemma 3.13, by an application of Cauchy–Schwarz inequality (Corollary 3.8) and then the CLN inequality for the second integral on the right-hand side, we conclude that the inequality holds for ρ_j and so does (3-18). □

Lastly we can state the important estimate for functions in P^* and for $\eta = dd^cu_1\wedge\cdots\wedge dd^cu_{m-2}$, where $u_1, \dots, u_{m-2} \in P^*$ without strict positive assumption on η.

Lemma 3.15. *Let v, w and $\{v_\ell\}$ be as in Lemma 3.13. Let ρ be a bounded m-ω-sh in Ω such that $-1 \le \rho \le 0$. Then,*

$$\int_\Omega (w-v)dd^c\rho\wedge\eta\wedge\omega^{n-m+1}$$
$$\le C\biggl(\int_\Omega (w-v)dd^cv\wedge\eta\wedge\omega^{n-m+1}\biggr)^{\frac{1}{2}} + C\biggl(\int_\Omega (w-v)\eta\wedge\omega^{n-m+2}\biggr)^{\frac{1}{2}} + C\biggl(\int_\Omega (w-v)\eta\wedge\omega^{n-m+2}\biggr)^{\frac{1}{4}},$$

with the same dependence of C as in previous lemmas.

Proof. Observe that it is enough to prove the inequality for $u_i := u_i + \delta|z|^2$ and then let $\delta \to 0^+$. Therefore without loss of generality we assume that η satisfies conditions (3-15) and (3-16). As in the proof of Lemma 3.14 we may assume that $\rho \in P^*$ and it is smooth. It follows from (3-18) that

$$\int_\Omega (w - v_\ell) dd^c \rho \wedge \eta \wedge \omega^{n-m+1} \\ \leq -\int_\Omega d(w - v_\ell) \wedge d^c \rho \wedge \eta \wedge \omega^{n-m+1} + C\left(\int_\Omega (w - v_\ell) \eta \wedge \omega^{n-m+2}\right)^{\frac{1}{2}}. \quad (3\text{-}19)$$

It remains to bound the first integral on the right-hand side. Using the Cauchy–Schwarz inequality (Lemma 2.3) for $T = \eta \wedge \omega^{n-m+1}$ we get

$$\left|\int_\Omega d(w - v_\ell) \wedge d^c \rho \wedge T\right|^2 \leq C_1 \int_\Omega d(w - v_\ell) \wedge d^c(w - v_\ell) \wedge T, \quad (3\text{-}20)$$

where C_1 is the bound for

$$C \int_K d\rho \wedge d^c \rho \wedge T$$

and so the dependence of C_1 is as in the statement which is a simple consequence of the CLN inequality.

Next, we have

$$\int_\Omega d(w - v_\ell) \wedge d^c(w - v_\ell) \wedge T \\ = -\int_\Omega (w - v_\ell) dd^c(w - v_\ell) \wedge T + \int_\Omega (w - v_\ell) d^c(w - v_\ell) \wedge d\omega \wedge \eta \wedge \omega^{n-m}. \quad (3\text{-}21)$$

Let us estimate the last integral. The Cauchy–Schwarz inequality and the CLN inequality imply that

$$\left|\int_\Omega (w - v_\ell) d^c(w - v_\ell) \wedge d\omega \wedge \eta \wedge \omega^{n-m}\right|^2 \leq C_2 \int_\Omega (w - v_\ell) \eta \wedge \omega^{n-m+2}, \quad (3\text{-}22)$$

where C_2 is the bound for

$$\int_\Omega (w - v_\ell) d(w - v_\ell) \wedge d^c(w - v_\ell) \wedge \eta \wedge \omega^{n-m+1}.$$

Notice also that as $w - v_\ell \geq 0$,

$$-(w - v_\ell) dd^c(w - v_\ell) \wedge T \leq (w - v_\ell) \wedge dd^c v_\ell \wedge T. \quad (3\text{-}23)$$

Combining (3-19), (3-20), (3-21), (3-22) and (3-23) we get

$$\int_\Omega (w - v_\ell) dd^c \rho \wedge \eta \wedge \omega^{n-m+1} \\ \leq C\left(\int_\Omega (w - v_\ell) dd^c v_\ell \wedge T\right)^{\frac{1}{2}} + C\left(\int_\Omega (w - v_\ell) \eta \wedge \omega^{n-m+2}\right)^{\frac{1}{2}} + C\left(\int_\Omega (w - v_\ell) \eta \wedge \omega^{n-m+2}\right)^{\frac{1}{4}}. \quad (3\text{-}24)$$

Notice that by smooth sequence $\rho_j \downarrow \rho$ we obtain this inequality for $\rho \in P^*$. To finish the proof we need to know that passing to the limit as $\ell \to \infty$ we get the inequality in the statement. This is known for the limits of all integrals above except for the first integral on the right-hand side. To handle this, we use the fact that $v_\ell \geq v$. Then

$$\int_\Omega (w - v_\ell) dd^c v_\ell \wedge \eta \wedge \omega^{n-m+1} \leq \int_\Omega (w - v) dd^c v_\ell \wedge \eta \wedge \omega^{n-m+1}. \tag{3-25}$$

Applying Lemma 3.14 for the right-hand side, we have

$$\int_\Omega (w - v) dd^c v_\ell \wedge \eta \wedge \omega^{n-m+1} \leq -\int_\Omega d(w - v) \wedge d^c v_\ell \wedge \eta \wedge \omega^{n-m+1} + C\bigg(\int_\Omega (w - v_\ell)\tau \wedge \omega^{n-m+2}\bigg)^{\frac{1}{2}}, \tag{3-26}$$

Combining three inequalities (3-24), (3-25) and (3-26), we obtain

$$\int_\Omega (w - v_\ell) dd^c \rho \wedge \eta \wedge \omega^{n-m+1}$$
$$\leq C\bigg(-\int_\Omega d(w-v)\wedge d^c v_\ell \wedge \eta \wedge \omega^{n-m+1}\bigg)^{\frac{1}{2}} + C\bigg(\int_\Omega (w-v_\ell)\eta\wedge\omega^{n-m+2}\bigg)^{\frac{1}{2}} + C\bigg(\int_\Omega (w-v_\ell)\eta\wedge\omega^{n-m+2}\bigg)^{\frac{1}{4}}.$$

Letting $\ell \to \infty$ and invoking Proposition 3.11 gives

$$\int_\Omega (w - v) dd^c \rho \wedge \eta \wedge \omega^{n-m+1}$$
$$\leq C\bigg(-\int_\Omega d(w-v)\wedge d^c v \wedge \eta \wedge \omega^{n-m+1}\bigg)^{\frac{1}{2}} + C\bigg(\int_\Omega (w-v)\eta\wedge\omega^{n-m+2}\bigg)^{\frac{1}{2}} + C\bigg(\int_\Omega (w-v)\eta\wedge\omega^{n-m+2}\bigg)^{\frac{1}{4}}.$$

Now the desired estimate follows from Lemma 3.13. □

Remark 3.16. In Lemmas 3.13, 3.14, 3.15 we only need to assume that w is continuous after using one more approximation argument. In fact, it is enough to assume $w \in P^*$.

For further applications we need the following lemma.

Lemma 3.17. *Let u be a bounded m-ω-sh. Suppose $\{u_j\}_{j\geq 1}$ is a decreasing sequence of smooth m-ω-sh functions with $\|u_j\|_{L^\infty} \leq 1$ such that $u_j \downarrow u$ pointwise. Assume also that all $u_j = u$ on a neighborhood of $\partial\Omega$. Let $-1 \leq \rho \leq 0$ be m-ω-sh functions. Then, for every $0 \leq s \leq m-1$,*

$$\lim_{j\to+\infty} \sup_\rho \bigg\{ \int_\Omega (u_j - u)(dd^c \rho)^s \wedge \omega^{n-s} \bigg\} = 0.$$

Proof. Let $K \Subset \Omega$ be the compact set such that $u_j = u$ on $\Omega \setminus K$ for all j. First we can modify ρ outside K by $A\psi$ without changing the integral value. Hence, we may assume that $\rho \in P^*$. Then, applying repeatedly Lemma 3.15 to replace $dd^c\rho$ by $dd^c u$ we get

$$\int_\Omega (u_j - u)(dd^c \rho)^{m-1} \wedge \omega^{n-m+1} \leq C \sum_{s=1}^{m-1} [e_{(0,0,s)}]^{1/2^s},$$

where

$$e_{(0,0,s)} = \int_\Omega (u_j - u)(dd^c u)^s \wedge \omega^{n-s}.$$

Thus, the proof of the lemma follows by the dominated convergence theorem applied to each term on the right-hand side. □

We get immediate consequences by the classical arguments. Let us define the $(m-1)$-capacity in the class of m-ω-sh functions. For a Borel set $E \subset \Omega$,

$$\mathrm{cap}_{m-1}(E) = \sup\left\{ \int_E H_{m-1}(\rho) : -1 \le \rho \le 0,\ \rho \text{ is } m-\omega\text{-sh in } \Omega \right\}.$$

Corollary 3.18. *If $\{u_j\}_{j\ge 1}$ is a uniformly bounded decreasing sequence of smooth m-ω-sh functions such that $u_j \downarrow u$ (m-ω-sh function), then $u_j \to u$ in* $\mathrm{cap}_{m-1}(\bullet)$.

Corollary 3.19. *Every bounded m-ω-sh function is quasicontinuous with respect to* $\mathrm{cap}_{m-1}(\bullet)$.

The quasicontinuity in $(m-1)$-capacity allows us to obtain the weak convergence for the wedge product of m-ω-sh bounded functions (compare [Bedford and Taylor 1987, Proposition 3.2]).

Proposition 3.20. *Let $0 \le p \le m-1$. Let $v, u_1, \dots, u_p$ be uniformly bounded m-ω-sh functions in Ω. Let $\{v_\ell\}_{\ell\ge 1}$, $\{u_s^j\}_{j\ge 1}$ be decreasing sequences of smooth m-ω-sh functions such that $v_\ell \downarrow v$ and $u_s^j \downarrow u_s$ for each $1 \le s \le p$. Then*

$$v_j dd^c u_1^j \wedge \cdots \wedge dd^c u_p^j \to v dd^c u_1 \wedge \cdots \wedge dd^c u_p$$

as $j \to \infty$ in the sense of currents of order zero. Consequently,

$$dd^c v_j \wedge dd^c u_1^j \wedge \cdots \wedge dd^c u_p^j \wedge \omega^{n-p} \to dd^c v \wedge dd^c u_1 \wedge \cdots \wedge dd^c u_p \wedge \omega^{n-p}$$

in the sense of currents.

Proof. By the localization principle we may assume Ω is a ball and all functions belong to P^*. Without loss of generality we may assume that $-1 \le v_j, u_s^j, v, u \le 0$. Define $S^j = dd^c u_1^j \wedge \cdots \wedge dd^c u_p^j$ and $S = dd^c u_1 \wedge \cdots \wedge dd^c u_p$. Both of them are currents of order zero. Let χ be a test form with $\operatorname{supp}\chi = K$. We need to show that

$$\lim_{j\to\infty} \left| \int (v_j S^j - vS) \wedge \chi \right| = 0.$$

We argue by induction over p. It is obvious for $p = 0$. Assume that it is true for $p-1 \le m-2$. This means that

$$u_1^j dd^c u_2^j \wedge \cdots \wedge dd^c u_p^j \to u_1 dd^c u_2 \wedge \cdots \wedge dd^c u_p$$

in the sense of currents of order zero. Hence,

$$S^j = dd^c[u_1^j dd^c u_2^j \wedge \cdots \wedge dd^c u_p^j] \to dd^c[u_1 dd^c u_2 \wedge \cdots \wedge dd^c u_p] = S.$$

Furthermore, by [Kołodziej and Nguyen 2016, Corollary 2.4], for a test form ϕ, we have

$$|S^j \wedge \phi| \le C\|\phi\|_K \widetilde{S}^j \wedge \omega^{n-p}, \tag{3-27}$$

where $\widetilde{S}^j = \big(dd^c \sum_{s=1}^p u_s^j\big)^p$ and $\|\phi\|_K$ is the C^0-norm of ϕ on K. Note that $\widetilde{S}^j \to \widetilde{S} = \big(dd^c \sum_{s=1}^p u_s\big)^p$ as the weak convergence holds for the wedge product of currents associated to p bounded functions. Letting $j \to \infty$ one obtains

$$|S \wedge \phi| \le C\|\phi\|_K \widetilde{S} \wedge \omega^{n-p}. \tag{3-28}$$

We are ready to show that the conclusion holds for $p \le m-1$. The uniform constants C below depend additionally on the uniform norm of coefficients of χ.

Let $\varepsilon > 0$. By the quasicontinuity with respect to $(m-1)$-capacity in Corollary 3.19, we can find an open set G such that $\operatorname{cap}_{m-1}(G) < \varepsilon$ and u_s^j, u_s, v are continuous on $\Omega \setminus G$. Note that $\Omega \setminus G$ is compact in Ω. It follows from Dini's theorem that $u_s^j \to u_s$ uniformly as $j \to \infty$ on that set. We first write

$$\int (v_j S^j - vS) \wedge \chi = \int (v_j - v) S^j \wedge \chi + \int v(S^j - S) \wedge \chi. \tag{3-29}$$

For the first term on the right, it follows from (3-27) that

$$\left|\int (v_j - v) S^j \wedge \chi\right| \le C\bigg(\int_{\Omega\setminus G} (v_j - v)\widetilde{S}^j \wedge \omega^{n-p} + n^p \operatorname{cap}_{m-1}(G)\bigg), \tag{3-30}$$

where we used the fact that

$$\int_G (v_j - v)\widetilde{S}^j \wedge \omega^{n-p} \le n^p \operatorname{cap}_{m-1}(G).$$

Next, we estimate the second integral in (3-29). Let $-1 \le \widetilde{v} \le 0$ be a continuous extension of v from $\Omega \setminus G$ to Ω. By induction hypothesis we know that $S^j \to S$ weakly, as currents. So,

$$\lim_{j\to\infty} \int \widetilde{v}(S^j - S) \wedge \chi = 0. \tag{3-31}$$

Moreover, similarly as above

$$\left|\int_G \widetilde{v} S^j \wedge \chi\right| \le Cn^p \operatorname{cap}_{m-1}(G),$$

and by (3-28)

$$\left|\int_G \widetilde{v} S \wedge \chi\right| \le C\left|\int_G \widetilde{S} \wedge \omega^{n-p}\right| \le Cn^p \operatorname{cap}_{m-1}(G).$$

Then,

$$\left|\int v(S^j - S) \wedge \chi\right| \le \left|\int \widetilde{v}(S^j - S) \wedge \chi\right| + 4Cn^p \operatorname{cap}_{m-1}(G). \tag{3-32}$$

Combining (3-29), (3-30), (3-31) and (3-32) we conclude that

$$\lim_{j\to\infty} \left|\int (v_j S^j - vS) \wedge \chi\right| \le C\varepsilon.$$

This holds for arbitrary ε, so the proof of the proposition follows. □

Remark 3.21. The proposition showed that the wedge product of currents associated to continuous m-ω-sh functions given in [Kołodziej and Nguyen 2016, Proposition 2.11] coincides with the one in Definition 3.4.

Corollary 3.22. *The wedge products*

$$dd^c u_1 \wedge \cdots \wedge dd^c u_{m-1}, \quad dd^c u_1 \wedge \cdots \wedge dd^c u_m \wedge \omega^{n-m}$$

are symmetric with respect to bounded m-ω-sh functions $u_1, \ldots, u_{m-1}, u_m$.

We can now extend the generalized Cauchy–Schwarz inequality in Lemma 2.4 and Corollary 2.5 to the case of bounded functions. We state here a simple version which is essential to prove the quasicontinuity later on. This is a stronger version of Corollary 3.8. Let $-1 \le u_1, \ldots, u_{m-1} \le 0$ be bounded m-ω-sh functions in Ω. Define

$$\tau = dd^c u_1 \wedge \cdots \wedge dd^c u_{m-1}. \tag{3-33}$$

Corollary 3.23. *Let $0 \le \phi \le 1$ be a continuous function such that* $\operatorname{supp}\phi = K \Subset \Omega$. *Let $0 \le h \le 1$ be a smooth m-ω-sh function. There exists a constant $C = C(\omega, K)$ such that*

$$\left| \int \phi \, dh \wedge d^c\omega \wedge \tau \wedge \omega^{n-m-1} \right|^2 \le C \int \phi \, \widetilde{\tau} \wedge \omega^{n-m+1},$$

where $\widetilde{\tau} = \big(dd^c \sum_{s=1}^{m-1} u_s\big)^{m-1}$.

Proof. The left-hand side is well defined because the coefficients of S are signed Radon measures. Let $\{u_s^j\}_{j\ge1}$ be a sequence of smooth m-ω-sh functions such that $u_s^j \downarrow u_s$ for each $1 \le s \le m-1$. We may assume $-1 \le u_s^j \le 0$. Define

$$S^j = dd^c u_1^j \wedge \cdots \wedge dd^c u_{m-1}^j, \quad \widetilde{S}^j = (dd^c u_1^j + \cdots + dd^c u_{m-1}^k)^{m-1}.$$

Applying Corollary 2.5 for $\gamma_s = dd^c u_s^j$, $s = 1, \ldots, p$, we get

$$\left| \int \phi \, dh \wedge d^c\omega \wedge S^j \wedge \omega^{n-m-1} \right|^2 \le C \int \phi \, dh \wedge d^c h \wedge \left(\sum_{s=1}^{m-1} \gamma_s \right)^{m-1} \wedge \omega^{n-m} \times \int \phi \, \widetilde{S}^j \wedge \omega^{n-m+1}.$$

As in the proof of Corollary 3.8, the CLN inequality implies that the first integral inequality on the right-hand side is bounded by a constant $C = C(\omega, K)$. By Proposition 3.20 the last integral converges to $\int \phi \, \widetilde{\tau} \wedge \omega^{n-m+1}$. □

4. Quasicontinuity

Having the Hessian measure defined for bounded m-ω-sh functions, we can introduce the m-capacity (see [Bedford and Taylor 1982]): for a Borel subset $E \subset \Omega$,

$$\operatorname{cap}_m(E) = \sup\left\{ \int_E (dd^c\rho)^m \wedge \omega^{n-m} : \rho \text{ is } m-\omega\text{-sh in } \Omega, -1 \le \rho \le 0 \right\}. \tag{4-1}$$

Here in fact $\mathrm{cap}_m(E) = \mathrm{cap}_m(E, \Omega)$ but we shall often suppress Ω in the notation if the domain is fixed. Then, this is an inner capacity, namely,

$$\mathrm{cap}_m(E) = \sup\{\mathrm{cap}_m(K) : K \text{ is compact subset of } E\}.$$

Proposition 4.1. *Let Ω be a open set in $\mathbb{C}^n$ and* $\mathrm{cap}_m(E) = \mathrm{cap}_m(E, \Omega)$.

(a) *If $E_1 \subset E_2$, then* $\mathrm{cap}_m(E_1) \le \mathrm{cap}_m(E_2)$.

(b) *If $E \subset \Omega_1 \subset \Omega_2$, then* $\mathrm{cap}_m(E, \Omega_2) \le \mathrm{cap}_m(E, \Omega_1)$.

(c) $\mathrm{cap}_m\big(\bigcup_j E_j\big) \le \sum_j \mathrm{cap}_m(E_j)$.

(d) *If $E_1 \subset E_2 \subset \cdots$ are Borel sets in Ω and $E := \bigcup_j E_j$, then* $\mathrm{cap}_m(E) = \lim_j \mathrm{cap}_m(E_j)$.

Definition 4.2 (convergence in capacity). A sequence of Borel functions u_j in Ω is said to converge in capacity (or in $\mathrm{cap}_m(\bullet)$) to u if for any $\delta > 0$ and $K \Subset \Omega$,

$$\lim_{j\to\infty} \mathrm{cap}_m(K \cap |u_j - u| \ge \delta) = 0.$$

Proposition 4.3. *If $\{u_j\}_{j\ge1}$ is a uniformly bounded sequence of continuous m-ω-sh functions that decreases to a bounded m-ω-sh function u in Ω, then u_j converges to u in* $\mathrm{cap}_m(\bullet)$.

The proof of this proposition will need the improved versions of Lemma 3.13, 3.14, 3.15 in which $\eta \wedge \omega$ is replaced by

$$\tau = dd^c u_1 \wedge \cdots \wedge dd^c u_{m-1},$$

where $u_1, \dots, u_{m-1} \in P^*$ in (3-11) and the positivity assumption (3-12) is satisfied. This is done thanks to Corollary 3.23. Since the proofs follow line by line those from the previous section after replacing $\eta \wedge \omega$ by τ and applying Corollary 3.23 instead of Corollary 3.8, we only state the lemmas.

Lemma 4.4. *Let $\{v_\ell\}_{\ell\ge1}$ be a sequence of smooth functions from P^* and let $w \in P^*$ be smooth. Assume $v_\ell \le w$ and $v_\ell \downarrow v \in P^*$. Then,*

$$-\int d(w-v) \wedge d^c v \wedge \tau \wedge \omega^{n-m} \le \int (w-v) dd^c v \wedge \tau \wedge \omega^{n-m} + C\bigg(\int (w-v)\tau \wedge \omega^{n-m+1}\bigg)^{\frac{1}{2}},$$

where the constant C depends on ω, K, ψ and the uniform norm of functions, but it is independent of δ in (3-12).

Lemma 4.5. *Let v, w and $\{v_\ell\}$ be as in Lemma 4.4. Let ρ be a bounded m-ω-sh in Ω such that $-1 \le \rho \le 0$. Then,*

$$\int_\Omega (w-v) dd^c \rho \wedge \tau \wedge \omega^{n-m} \le -\int_\Omega d(w-v) \wedge d^c \rho \wedge \tau \wedge \omega^{n-m} + C\bigg(\int_\Omega (w-v) \wedge \tau \wedge \omega^{n-m+1}\bigg)^{\frac{1}{2}}.$$

The dependence of C is the same as in the previous lemma.

As in Lemma 3.15 the strict positivity assumption (3-12) is not needed in the following statement.

Lemma 4.6. *Let v, w and $\{v_\ell\}$ be as in Lemma 4.4. Let ρ be a bounded m-ω-sh in Ω such that $-1 \le \rho \le 0$. Then,*

$$\int_\Omega (w-v)dd^c\rho \wedge \tau \wedge \omega^{n-m} \le C\Big(\int_\Omega (w-v)dd^c v \wedge \tau \wedge \omega^{n-m}\Big)^{\frac{1}{2}} + C\Big(\int_\Omega (w-v)\tau \wedge \omega^{n-m+1}\Big)^{\frac{1}{2}} + C\Big(\int_\Omega (w-v)\tau \wedge \omega^{n-m+1}\Big)^{\frac{1}{4}}$$

with the same dependence of C as in previous lemmas.

Remark 4.7. In Lemmas 4.4, 4.5, 4.6 it is enough to assume that $w \in P^*$ after using one more approximation argument.

Thanks to Lemma 4.6, we can easily prove the last case $s = m$ of Lemma 3.17.

Lemma 4.8. *Let u be a bounded m-ω-sh. Suppose $\{u_j\}_{j\ge 1}$ is a decreasing sequence of smooth m-ω-sh functions with $\|u_j\|_{L^\infty} \le 1$ such that $u_j \downarrow u$ pointwise. Assume also that all $u_j = u$ on a neighborhood of $\partial\Omega$. Let $-1 \le \rho \le 0$ be m-ω-sh functions. Then,*

$$\lim_{j\to+\infty} \sup_\rho \Big\{\int_\Omega (u_j - u)(dd^c\rho)^m \wedge \omega^{n-m}\Big\} = 0. \tag{4-2}$$

End of proof of Proposition 4.3. Because of (b) and (c) in Proposition 4.1 we can assume that Ω is a ball and all functions are equal near the boundary. Let $\delta > 0$. We wish to show that

$$\lim_{j\to\infty} \mathrm{cap}_m(\{u_j - u > \delta\}) = 0.$$

Argument by contradiction. Suppose that the statement were not true. Then, there would exist $\varepsilon > 0$, a subsequence $\{u_{j_s}\} \subset \{u_j\}$ and a sequence of m-ω-sh functions ρ_{j_s} with $-1 \le \rho_{j_s} \le 0$ such that

$$\limsup_{j_s\to+\infty} \int_{\{u_{j_s}-u>\delta\}} H_m(\rho_{j_s}) \ge \varepsilon.$$

On the other hand, by Markov's inequality,

$$\int_{\{u_{j_s}-u>\delta\}} H_m(\rho_{j_s}) \le \frac{1}{\delta}\int_\Omega (u_{j_s} - u) H_m(\rho_{j_s}).$$

Lemma 4.8 shows that the right-hand side converges to zero. This leads to a contradiction. □

Theorem 4.9. *Let Ω be a bounded open set in $\mathbb{C}^n$. Let u be a m-ω-sh function in Ω. Then, for every $\varepsilon > 0$, there exists an open subset $U \subset \Omega$ with $\mathrm{cap}_m(U, \Omega) < \varepsilon$ such that u restricted to $\Omega \setminus U$ is continuous.*

Proof. The result is local and, moreover, it can be reduced to the case of bounded functions by the property (7-5) whose proof will use only bounded functions. We can use the classical argument in [Bedford and Taylor 1982, Theorem 3.5] since there exists a sequence of smooth m-ω-sh functions that decrease to u pointwise (Proposition 2.9) and our capacity is subadditive by Proposition 4.1. □

We obtain the convergence in capacity for monotone sequences of uniformly bounded functions. In particular, the smoothness assumption in Proposition 4.3 can be relaxed by yet another approximation. We recall first a classical result in measure theory, giving a short proof for the reader's convenience.

Lemma 4.10. *Let μ_j be a sequence of positive Radon measures with compact support in Ω. Assume that μ_j converges weakly to μ. Let $\Omega \Supset F_1 \supset F_2 \supset \cdots$ be a sequence of decreasing closed subsets in Ω satisfying*

$$\lim_{j\to\infty} \mu(F_j) = 0.$$

Then, $\lim_{j\to+\infty} \mu_j(F_j) = 0$.

Proof. Fix $\varepsilon > 0$. By the assumption there exists $j_0 > 0$ such that $\mu(F_{j_0}) < \varepsilon$. Using the inclusions we have $\mu_j(F_j) \le \mu_j(F_{j_0})$, for $j > j_0$. The weak convergence implies

$$\limsup_{j\to\infty} \mu_j(F_j) \le \limsup_{j\to\infty} \mu_j(F_{j_0}) \le \mu(F_{j_0}) < \varepsilon.$$

It follows that

$$0 \le \liminf_{j\to\infty} \mu_j(F_j) \le \limsup_{j\to\infty} \mu_j(F_j) \le \varepsilon.$$

This holds for every $\varepsilon > 0$. □

Corollary 4.11. *Let Ω be a bounded open set in $\mathbb{C}^n$. Let $\{u_j\}_{j\ge1}$ be a uniformly bounded and monotone sequence of m-ω-sh functions such that either $u_j \downarrow u$ pointwise or $u_j \uparrow u$ almost everywhere for a bounded m-ω-sh function u in Ω. Then, u_j converges to u in m-capacity.*

Proof. The localization principle applies in both cases. We can assume that Ω is a ball and the u_j are equal to a fixed smooth psh function in a neighborhood of the boundary. Hence, $u - u_j = 0$ on $\Omega \setminus K$ for a fixed compact subset K. Let $\delta > 0$. We wish to show that

$$\lim_{j\to\infty} \mathrm{cap}_m(|u-u_j| \ge \delta) = 0.$$

Arguing by contradiction, suppose that this were not true. Then, there would exist $\varepsilon > 0$ and a sequence of m-ω-sh functions ρ_j with $-1 \le \rho_j \le 0$ such that

$$\limsup_{j\to\infty} \int_{\{|u-u_j|\ge\delta\}} H_m(\rho_j) \ge \varepsilon.$$

Fix a cut-off function χ with compact support in Ω and equal 1 on K. By the CLN inequality for bounded functions (Proposition 3.7) we may assume that $\mu_j := \chi H_m(\rho_j)$ converges weakly to a positive Radon measure μ.

We use the quasicontinuity. Find an open set $G \subset \Omega$ such that $\mathrm{cap}_m(G) \le \varepsilon/2$ and the restrictions of u_j, u to $\Omega \setminus G$ are continuous functions. Since u_j is a monotone sequence, it follows from Dini's lemma that it converges uniformly on $\Omega \setminus G$. In particular, the sets $F_j := \{|u-u_j| \ge \delta\} \setminus G$ are closed in Ω and satisfy $\lim_{j\to\infty} \mu(F_j) = 0$. Hence,

$$\int_{\{|u-u_j|\ge\delta\}} H_m(\rho_j) \le \int_{F_j} \chi H_m(\rho_j) + \mathrm{cap}_m(G) \le \mu_j(F_j) + \tfrac{1}{2}\varepsilon.$$

Letting $j \to \infty$ and using Lemma 4.10 this leads to a contradiction. □

5. Weak convergence

5.1. ***Convergence theorems for decreasing sequences.*** Let Ω be a bounded open set in $\mathbb{C}^n$. We have continuity of wedge products of m-ω-sh functions under decreasing sequences of bounded functions.

Lemma 5.1. *Let $v, u_1, \dots, u_p$, $1 \le p \le m$, be a bounded m-ω-sh functions in Ω. Let $\{v^j\}_{j\ge 1}$ and $\{u_s^j\}_{j\ge 1}$ be uniformly bounded sequences of m-ω-sh such that $v^j \downarrow v$ and $u_s^j \downarrow u_s$ as $j \to +\infty$, for each $s = 1, \dots, p$. Then,*

(a) $\lim_{j\to\infty} \mathscr{L}(u_1^j, \dots, u_p^j) = \mathscr{L}(u_1, \dots, u_p)$;

(b) $\lim_{j\to+\infty} v^j \mathscr{L}(u_1^j, \dots, u_p^j) = v\mathscr{L}(u_1, \dots, u_p)$;

where the convergence is understood in the sense of currents of order zero.

Proof. (a) By the localization principle we may assume that all functions are defined in a ball Ω and they are equal to a fixed smooth psh function ψ outside $\Omega' \Subset \Omega$. Let $\{u_s^{j,\delta}\}_{\delta>0}$ be decreasing sequences of smooth m-ω-sh functions such that $u_s^{j,\delta} \downarrow u_s^j$ as $\delta \to 0$. Similarly, let $\{u_s^\delta\}_{\delta>0}$ be approximating sequences for u_s. We may assume that all involved functions are negative and of uniform norm less than one.

Let χ be a test form such that $\operatorname{supp}\chi = K \Subset \Omega$. We consider the difference

$$M_{(j,\delta)} = \int \chi[\mathscr{L}(u_1^{j,\delta}, \dots, u_p^{j,\delta}) - \mathscr{L}(u_1^\delta, \dots, u_p^\delta)].$$

By Proposition 3.20 $\lim_{\delta\to 0} \mathscr{L}(u_1^\delta, \dots, u_p^\delta) = \mathscr{L}(u_1, \dots, u_p)$. So, the proof is completed as soon as we show that

$$\lim_{j\to\infty} \lim_{\delta\to 0} |M_{(j,\delta)}| = 0.$$

We claim that

$$|M_{(j,\delta)}| \le C \sum_{s=1}^{p} \int_K |u_s^{j,\delta} - u_s^\delta| (dd^c \rho^{j,\delta})^{p-1} \wedge \omega^{n-p+1}, \tag{5-1}$$

where $\rho^{j,\delta} = \frac{1}{2p} \sum_{s=1}^{p} (u_s^{j,\delta} + u_s^\delta)$. In fact,

$$[\mathscr{L}(u_1^{j,\delta}, \dots, u_p^{j,\delta}) - \mathscr{L}(u_1^\delta, \dots, u_p^\delta)] = \sum_{s=1}^{p} dd^c(u_s^{j,\delta} - u_s^\delta) \wedge T_s \wedge \omega^{n-m},$$

where

$$T_s(j,\delta) = dd^c u_1^{j,\delta} \wedge \dots \wedge dd^c u_{s-1}^{j,\delta} \wedge dd^c u_{s+1}^\delta \wedge \dots \wedge dd^c u_p^\delta.$$

(Here one should use the obvious modifications for $s = 1$ and $s = p$.) Notice that T_s is a smooth closed $(p-1, p-1)$-form. By integration by parts,

$$\int \chi\omega^{n-m} \wedge dd^c(u_s^{j,\delta} - u_s^\delta) \wedge T_p = \int (u_s^{j,\delta} - u_s^\delta) dd^c(\chi\omega^{n-m}) \wedge T_s.$$

Since $0 \le p-1 \le m-1$, it follows from [Kołodziej and Nguyen 2016, Corollary 2.4] that

$$|dd^c(\chi\omega^{n-m}) \wedge T_s| \le 2^m C[dd^c(u_1^{j,\delta} + \dots + u_p^\delta)]^{p-1} \wedge \omega^{n-p+1}$$

for a uniform constant C depending only on ω and χ. This proves (5-1).

At this point we no longer have the continuity of u_s^j and u_s, however, we can make use of the quasicontinuity. Let $\varepsilon > 0$. Find an open set G such that all $u_s^{j,\delta}$, u_s^j and also u_s^j, u_s are continuous on $\Omega \setminus G$ and $\mathrm{cap}_m(G) < \varepsilon$. We know that $\Omega \setminus G$ is compact in Ω and by Dini's theorem for $s = 1, \ldots, p$ we have $u_s^{j,\delta} \to u_s^j$ and $u_s^\delta \to u_s$ uniformly as $\delta \to 0$ on that set. Therefore,

$$\int_K |u_s^{j,\delta} - u_s^\delta|(dd^c\rho^{j,\delta})^{p-1} \wedge \omega^{n-p+1} \le \int_{\Omega\setminus G} |u_s^{j,\delta} - u_s^\delta|(dd^c\rho^{j,\delta})^{p-1} \wedge \omega^{n-p+1} + \mathrm{cap}_m(G).$$

To estimate the integral on the right-hand side we use

$$\begin{aligned}\int_{\Omega\setminus G} |u_s^{j,\delta} - u_s^\delta|(dd^c\rho^{j,\delta})^{p-1} \wedge \omega^{n-p+1} \le \int_{\Omega\setminus G} &(u_s^{j,\delta} - u_s^j)(dd^c\rho^{j,\delta})^{p-1} \wedge \omega^{n-p+1}\\ &+ \int_{\Omega\setminus G} (u_s^\delta - u_s)(dd^c\rho^{j,\delta})^{p-1} \wedge \omega^{n-p+1}\\ &+ \int_{\Omega\setminus G} (u_s^j - u_s)(dd^c\rho^{j,\delta})^{p-1} \wedge \omega^{n-p+1}.\end{aligned}$$

By the uniform convergence and then the CLN inequality (Proposition 3.7) the first two terms go to zero as $\delta \to 0$. Moreover, $\rho^{j,\delta}$ is a sequence of smooth m-ω-sh functions decreasing to $\rho^j = \frac{1}{2p}\sum_{s=1}^p (u_s^j + u_s)$ as $\delta \to 0$. This implies by Proposition 3.20 that $\mathscr{L}_{p-1}(\rho^{j,\delta})$ converges weakly to $\mathscr{L}_{p-1}(\rho^j)$ as $\delta \to 0$. Combining this with the continuity on $\Omega \setminus G$ we get

$$\lim_{\delta\to 0} \int_{\Omega\setminus G} (u_s^j - u_s)\mathscr{L}_{p-1}(\rho^{j,\delta}) = \int_{\Omega\setminus G} (u_s^j - u_s)\mathscr{L}_{p-1}(\rho^j).$$

It follows that

$$\lim_{\delta\to 0} |M_{(j,\delta)}| \le \int_{\Omega\setminus G} (u_s^j - u_s)\mathscr{L}_{p-1}(\rho^j) + \varepsilon.$$

Letting $j \to +\infty$ we have by the uniform convergence,

$$\lim_{j\to\infty} \lim_{\delta\to 0} |M_{(j,\delta)}| \le \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the proof is completed.

(b) For simplicity we assume that $u_1 = \cdots = u_p = u$ and also $\{u_s^j\} = \{u^j\}$. The general case follows in the same way. Then we write

$$\mathscr{L}_p(u) = \mathscr{L}_p(u, \ldots, u).$$

Since v^j decreases to v and $\mathscr{L}_p(u^j)$ converges weakly to $\mathscr{L}_p(u)$ thanks to (a), any weak limit Θ of the sequence $v^j\mathscr{L}_p(u^j)$ satisfies

$$\Theta \le v\mathscr{L}_p(u).$$

Hence, $v\mathscr{L}_p(u) - \Theta$ is a positive current. In particular,

$$v\mathscr{L}_p(u) \wedge \omega^{m-p} - \Theta \wedge \omega^{m-p} = vH_p(u) - \Theta \wedge \omega^{m-p}$$

is a positive Radon measure. To show the converse let $\chi \geq 0$ be a test function in Ω. We will show that

$$\int \chi \Theta \wedge \omega^{m-p} \geq \int \chi v H_p(u).$$

In fact, since $v^j H_p(u^j)$ converges weakly to $\Theta \wedge \omega^{m-p}$, it is enough to show

$$\lim_{j\to+\infty} \int \chi v^j H_p(u^j) \geq \int \chi v H_p(u). \tag{5-2}$$

Let $\varepsilon > 0$ and choose an open set $G \subset \Omega$ such that $\mathrm{cap}_m(G, \Omega) \leq \varepsilon$ and v, v_s are all continuous on $F = \Omega \setminus G$. Since v is continuous on F, there is a continuous extension g to Ω such that $v = g$ on F with the same uniform norm. Without loss of generality we also assume that $0 \leq v, v_s, g \leq 1$. Hence,

$$\begin{aligned}
\int \chi v H_p(u) &\leq \int_F \chi v H_p(u) + \mathrm{cap}_m(G) \\
&= \int_F \chi g H_p(u) + \mathrm{cap}_m(G) \\
&\leq \int \chi g H_p(u) + \mathrm{cap}_m(G) \\
&= \lim_{j\to\infty} \int \chi g H_p(u^j) + \mathrm{cap}_m(G) \\
&\leq \lim_{j\to\infty} \int_F \chi g H_p(u^j) + 2\,\mathrm{cap}_m(G).
\end{aligned}$$

By Dini's theorem v^j converges to $v = g$ uniformly on F, so the last integral does not exceed

$$\lim_{j\to\infty} \int_F \chi v^j H_p(u^j) + \varepsilon \leq \lim_{j\to\infty} \int \chi v^j H_p(u^j) + \varepsilon.$$

Therefore, we have proved that

$$\int \chi v H_p(u) \leq \lim_{j\to\infty} \int \chi v^j H_p(u^j) + 2\,\mathrm{cap}_m(G) + \varepsilon.$$

Since $\mathrm{cap}_m(G) \leq \varepsilon$ and $\varepsilon > 0$ is arbitrary, the inequality (5-2) follows. □

Corollary 5.2. *Let u, v be bounded m-ω-sh functions and $T = dd^c v_1 \wedge \cdots \wedge dd^c v_{m-p} \wedge \omega^{n-m}$ for bounded m-ω-sh functions $v_1, \ldots, v_{m-p}$, where $1 \leq p \leq m$. Then,*

$$\mathbb{1}_{\{u<v\}} (dd^c \max\{u, v\})^p \wedge T = \mathbb{1}_{\{u<v\}} (dd^c v)^p \wedge T.$$

Consequently,

$$(dd^c \max\{u, v\})^p \wedge T \geq \mathbb{1}_{\{u\geq v\}} (dd^c u)^p \wedge T + \mathbb{1}_{\{u<v\}} (dd^c v)^p \wedge T.$$

Proof. Given the weak convergence results under decreasing sequences in the above lemma, the proof of [Guedj and Zeriahi 2017, Theorem 3.27] can be easily adapted to the current case. □

We conclude this section by going back to the extensions of results in Section 2.4, noted in Remark 2.17. We give here only the most important statement that will be used later.

Corollary 5.3. *Let $\Omega \Subset \mathbb{C}^n$ be strictly m-pseudoconvex. Let $-1 \le v \le w \le 0$ be bounded m-ω-sh functions such that* $\lim_{z\to\partial\Omega}(w-v) = 0$. *Let ρ be a bounded m-ω-sh function such that $-1 \le \rho \le 0$. There is a constant $C = C(\omega, n, m)$ such that*

$$\int_\Omega (w-v)^{3m}(dd^c\rho)^m \wedge \omega^{n-m} \le C \sum_{s=0}^{m} \int_\Omega (w-v)(dd^c v)^s \wedge \omega^{n-s}.$$

Proof. Let us replace w by $w_\varepsilon = \max\{w-\varepsilon, v\}$ for $\varepsilon > 0$, so that $w_\varepsilon = v$ in a neighborhood of $\partial\Omega$. If we could prove the inequality for w_ε and v, then by letting $\varepsilon \to 0$, the domination convergence theorem would imply the required inequality. Let $\Omega' \Subset \Omega$ be a smooth subdomain such that $w = v$ on $\Omega \setminus \Omega'$. Then the integrals on both sides will not change if we modify v, w outside Ω'. Hence, we may further assume that $w = v = \psi$ on $\Omega\setminus\Omega'$ with ψ a smooth m-ω-sh defining function for Ω.

Using the quasicontinuity it is easy to see from Lemma 5.1(b) that for smooth decreasing sequences $w_j \downarrow w$, $v_j \downarrow v$ and $\rho_j \downarrow \rho$ we have

$$\lim_{j\to\infty} \int_\Omega (w_j - v_j)^{3m} H_m(\rho_j) = \int_\Omega (w-v)^{3m} H_m(\rho),$$

and for $0 \le s \le m$,

$$\lim_{j\to\infty} \int_\Omega (w_j - v_j) H_s(v_j) = \int_\Omega (w-v) H_s(v).$$

Therefore, it is enough to prove the inequality for smooth functions $v_j \le w_j \le 0$ and $-1 \le \rho \le 0$. Notice that $w_j \to w$ and $v_j \to v$ uniformly on $\Omega\setminus\Omega'$ (this is the reason why we modify w, v near the boundary). Thus, we can follow the argument in Remark 2.16 and conclude that the extra terms will vanish after passing to the limit as $j \to +\infty$. Hence, the proof for the bounded functions case follows. □

5.2. ***Convergence theorems for increasing sequences.*** With a similar proof as that of Lemma 5.1 we get:

Lemma 5.4. *Let $1 \le p \le m$ and $v, u_1, \dots, u_p$ be bounded m-ω-sh functions. Suppose that $\{v^j\}_{j\ge1}$, $\{u_s^j\}_{j\ge1}$ are uniformly bounded increasing sequences of m-ω-sh functions such that $v^j \uparrow v$ and $u_s^j \uparrow u_s$ (almost everywhere) as $j \to \infty$ for $s = 1, \dots, p$. Then,*

$$\lim_{j\to+\infty} v^j \mathscr{L}(u_1^j, \dots, u_p^j) = v\mathscr{L}(u_1, \dots, u_p) \tag{5-3}$$

in the sense of currents of order zero.

Corollary 5.5. *Let Ω be a bounded open set. Let $\mathcal{U}_m$ be a uniformly bounded family of m-ω-sh functions in Ω. Define $v(x) = \sup\{v_\alpha(x) : v_\alpha \in \mathcal{U}_m\}$. Then, the set*

$$N := \{v < v^*\}$$

has zero measure with respect to any measure $\mathscr{L}(u_1, \dots, u_m) = dd^c u_1 \wedge \dots \wedge dd^c u_m \wedge \omega^{n-m}$, where the u_i are bounded m-ω-sh functions. In particular, $\mathrm{cap}_m(N, \Omega) = 0$.

Proof. By Choquet's lemma we can reduce the argument to the case when $\mathcal{U}_m$ is an increasing sequence $\{v_j\}_{j\ge1}$ with $w = \sup_j v_j$ and $N = \{w < v^*\}$. It follows from the proof of [Gu and Nguyen 2018, Corollary 9.9] that $w = v^*$ almost everywhere. Therefore, Lemma 5.4 implies $v_j \mathscr{L}(u_1, \dots, u_m)$ converges weakly to $v^* \mathscr{L}(u_1, \dots, u_m)$. Then, the positive currents $(v^* - v_j)\mathscr{L}(u_1, \dots, u_m)$ converge weakly to zero and hence, for any compact set $K \subset \Omega$,

$$\lim_{j\to\infty} \int_K (v^* - v_j)\mathscr{L}(u_1, \dots, u_m) = 0.$$

By the monotone convergence theorem, $\lim_{j\to\infty} \int_K (w - v_j)\mathscr{L}(u_1, \dots, u_m) = 0$. Therefore,

$$\int_K (v^* - w)\mathscr{L}(u_1, \dots, u_m) = 0.$$

In other words, $v^* = w$ a.e. on K with respect to $\mathscr{L}(u_1, \dots, u_m)$. The last conclusion follows from the inner regularity of capacity. □

6. Comparison principle

Let Ω be a bounded open set which is relatively compact in a strictly m-pseudoconvex bounded domain D in $\mathbb{C}^n$. Fix a constant $\boldsymbol{B}$ such that on $\overline{\Omega}$,

$$-\boldsymbol{B}\omega^2 \le dd^c\omega \le \boldsymbol{B}\omega^2, \quad -\boldsymbol{B}\omega^3 \le d\omega \wedge d^c\omega \le \boldsymbol{B}\omega^3.$$

Let ρ be a strictly psh function satisfying $\rho \le 0$ and $dd^c\rho \ge \omega$ in D. In this section we assume all functions are defined in D which means that they can be approximated by a decreasing sequence of smooth m-ω-sh functions in a neighborhood of $\overline{\Omega}$.

Theorem 6.1. *Let u, v be bounded m-ω-sh functions in Ω such that $d = \sup_\Omega(v - u) > 0$, and $\liminf_{z\to\partial\Omega}(u - v)(z) \ge 0$. Fix $0 < \varepsilon < \min\{\frac{1}{2}, d/(2\|\rho\|_\infty)\}$. Let us denote for $0 < s < \varepsilon_0 := \varepsilon^3/16\boldsymbol{B}$,*

$$U(\varepsilon, s) := \{u < (v + \varepsilon\rho) + S(\varepsilon) + s\}, \quad \text{where } S(\varepsilon) = \inf_\Omega[u - (v + \varepsilon\rho)].$$

Then,

$$\int_{U(\varepsilon,s)} H_m(v + \varepsilon\rho) \le \left(1 + \frac{Cs}{\varepsilon^m}\right) \int_{U(\varepsilon,s)} H_m(u),$$

where C is a uniform constant depending on m, n, ω.

Proof. If u, v are smooth, then the proof follows from [Gu and Nguyen 2018, Lemmas 3.8, 3.9 and 3.10]. To pass from the smooth case to the bounded case we use the quasicontinuity of m-ω-sh functions and the argument as the one in [Bedford and Taylor 1982, Theorem 4.1] (see also [Kołodziej 2005, Theorem 1.16]). The proof is readily adaptable with obvious changes of notation. Here we only indicate the points of difference that we need to take care of. Firstly, replacing u by $u + \delta$ with $\delta > 0$ and then letting $\delta \downarrow 0$ we may assume that $\{u < v\} \Subset \Omega' \Subset \Omega$ and $u \ge v + \delta$ on $\Omega \setminus \Omega'$. By restricting u, v to a smaller domain we may assume that u, v are defined in a neighborhood of $\overline{\Omega}$.

Let $\{u_k\}_{k\ge1}$, $\{v_j\}_{j\ge1}$ be sequences of smooth m-ω-sh functions in a neighborhood of $\overline{\Omega}$ (Proposition 2.9) such that $u_k \downarrow u$ and $v_j \downarrow v$ pointwise in $\overline{\Omega}$. Define $d_{jk} = \sup_{\overline{\Omega}}(v_j - u_k)$. Then, for $j \ge k > 0$ large we have

$$d_{jk} \ge \tfrac{1}{2}d > 0.$$

In fact, for small $\epsilon > 0$ there exists $x \in \Omega$ such that $d - \epsilon \le v(x) - u(x)$. So, for $k > k_0$ large enough,

$$d - 2\epsilon \le v(x) - u_k(x) \le v_j - u_k \le d_{jk}.$$

We get the desired inequality by letting $j \to \infty$ and then $\epsilon \to 0$. Next, since $u \ge v + \delta$ on a compact set $K = \overline{\Omega} \setminus \Omega'$, we have $u_k \ge v + \delta$ for every $k \ge 1$. Since u_k is continuous, by Hartogs' lemma for ω-sh functions [Gu and Nguyen 2018, Lemma 9.14], there is $j_k \ge k > 0$ large enough such that for $j \ge j_k$,

$$v_j + \delta \le u_k \quad \text{on } K.$$

Thus, there exist subsequences of $\{u_k\}$ and $\{v_j\}$, which can be used in the argument from [Bedford and Taylor 1982, Theorem 4.1]. □

Corollary 6.2. *Let u, v be bounded m-ω-sh functions in a neighborhood of $\overline{\Omega}$ such that*

$$\liminf_{z \to \partial\Omega} (u - v)(z) \ge 0.$$

Assume that $H_m(v) \ge H_m(u)$ in Ω. Then, $u \ge v$ on Ω.

Proof. Arguing by contradiction, suppose that $\sup_\Omega(v - u) = d > 0$. Hence, there exist $\delta, a > 0$ so small that $\sup_\Omega[(1 + a)v - (u + \delta)] > d/2$ and $\liminf_{z \to \partial\Omega}[(u + \delta) - (1 + a)v](z) \ge 0$. Applying Theorem 6.1 for $\widetilde{u} = u + \delta$ and $\widetilde{v} = (1 + a)v$, we have for $0 < s < \varepsilon_0$,

$$\int_{U(\varepsilon,s)} H_m(\widetilde{v} + \varepsilon\rho) \le \left(1 + \frac{Cs}{\varepsilon^m}\right) \int_{U(\varepsilon,s)} H_m(u).$$

Observe that

$$H_m(\widetilde{v} + \varepsilon\rho) \ge (1 + a)^m H_m(v) + \varepsilon^m H_m(\rho) \ge (1 + a)^m H_m(u) + \varepsilon^m H_m(\rho).$$

Hence, we derive from the above inequality that

$$\varepsilon^m \int_{U(\varepsilon,s)} H_m(\rho) \le 0$$

for $s > 0$ so small that $(1 + a)^m \ge 1 + Cs/\varepsilon^m$. Therefore, the Lebesgue measure of $U(\varepsilon, s)$ is zero. This is impossible as it is a nonempty quasiopen set for $0 < s < \varepsilon_0$. □

The above argument also gives:

Corollary 6.3 (domination principle). *Let u, v be bounded m-ω-sh such that* $\limsup_{z \to \partial\Omega} |u(z) - v(z)| = 0$ *and $\int_{\{u<v\}} H_m(u) = 0$. Then, $u \ge v$ in Ω.*

7. Polar sets and negligible sets

In this section we study the polar sets and negligible sets of m-ω-sh functions. We obtain here results analogous to those in pluripotential theory from [Bedford and Taylor 1982]. Let us first give the definitions.

Definition 7.1 (m-polar sets). A set E in $\mathbb{C}^n$ is m-polar if for each $z \in E$ there is an open set $z \in U$ and a m-ω-sh function u in U such that $E \cap U \subset \{u = -\infty\}$.

Let $\{u_\alpha\}$ be a family of m-ω-sh functions in Ω which is locally bounded from above. Then, the function

$$u(z) = \sup_\alpha u_\alpha(z)$$

need not be m-ω-sh, but its upper semicontinuous regularization

$$u^*(z) = \limsup_{x \to z} u(x) \geq u(z)$$

is m-ω-sh (see [Gu and Nguyen 2018, Proposition 2.6-(c)]). A set of the form

$$N = \{z \in \Omega : u(z) < u^*(z)\} \tag{7-1}$$

is called *m-negligible*.

An n-polar/n-negligible set is pluripolar/negligible. Clearly a pluripolar set is a m-polar (or m-negligible sets) for every $1 \leq m \leq n$. More generally, a m-polar (resp. m-negligible) is a $(m-1)$-polar (resp. $(m-1)$-negligible) set. An effective way to study these sets is by extremal functions.

Definition 7.2. Let E be a subset of a bounded open set $\Omega \subset \mathbb{C}^n$. We define

$$u_E = u_{E,\Omega} = \sup\{v(x) : v \text{ is } m\text{-}\omega\text{-sh in } \Omega,\ u \leq 0,\ u \leq -1 \text{ on } E\}$$

By Choquet's lemma u_E is the limit of an increasing sequence of m-ω-sh functions. It follows from [Gu and Nguyen 2018, Corollary 9.9] that u_E^* is m-ω-sh and $u_E = u_E^*$ almost everywhere. Moreover, $u_E^* \equiv 0$ if and only if there exists an increasing sequence of m-ω-sh functions $\{v_j\}_{j \geq 1}$ satisfying

$$v_j \leq 0, \quad v_j \leq -1 \text{ on } E, \quad \int_\Omega |v_j|\, dV_{2n} \leq 2^{-j}. \tag{7-2}$$

Lemma 7.3. *Let Ω be a bounded open set in $\mathbb{C}^n$.*

(i) *If $E_1 \subset E_2$, then $u_{E_2} \leq u_{E_2}$.*

(ii) *If $E \subset \Omega_1 \subset \Omega_2$, then $u_{E,\Omega_2} \leq u_{E,\Omega_1}$.*

(iii) *Let K_j be nonincreasing sequence of compact subset in Ω and $K = \bigcap_j K_j$. Then, $u_{K_j}^*$ increases almost everywhere to u_K^*.*

(iv) *If $u_{E_j}^* \equiv 0$ and $E = \bigcup_{j=1}^\infty E_j$, then $u_E^* \equiv 0$.*

Suppose moreover that Ω is strictly m-pseudoconvex.

(v) *If $E \Subset \Omega$, then* $\lim_{z \to \partial\Omega} u_E^* = 0$.

(vi) *For every set $E \subset \Omega$, $H_m(u_E^*) \equiv 0$ on $\Omega \setminus \overline{E}$.*

Proof. The properties (i) and (ii) are obvious from the definition, and also $\lim_j u_{K_j} \leq u_K^*$ in (iii). To prove the reverse inequality let v be a m-ω-sh with $v \leq 0$ and $u \leq -1$ on K. For $\varepsilon > 0$, the open set $U_\varepsilon = \{u < -1 + \varepsilon\}$ contains K. Hence, $K_j \subset U_\varepsilon$ for j large enough. So, $v - \varepsilon \leq u_{K_j}^*$. Taking the supremum over all such functions v we get $u_K - \varepsilon \leq u := \lim_j u_{K_j}$. Letting $\varepsilon \to 0$ we obtain the conclusion. The statement that $u = u^*$ almost everywhere follows from [Gu and Nguyen 2018, Corollary 9.9].

(iv) Let $\varepsilon > 0$. By (7-2) we can choose a sequence $v_j \le 0$, $v_j \le -1$ on E_j and $\int_\Omega |v_j|\, dV_{2n} \le \varepsilon 2^{-j}$. Then, $v = \sum_j v_j$ is a m-ω-sh function satisfying $v \le 0$, $v \le -1$ on E and $\int_\Omega |v|\, dV_{2n} \le \varepsilon$. Hence, $u_E^* \equiv 0$.

(v) Let ψ be a strictly m-ω-sh defining function of Ω. Then, for $A > 1$ large enough, $A\psi \le u_E^*$. This finishes the proof.

(vi) Given the unique continuous solution of the Dirichlet problem for the homogeneous Hessian equation [Gu and Nguyen 2018, Theorem 3.15] in small balls, the result follows from a classical balayage argument. □

The outer capacity $\mathrm{cap}_m^*(\bullet)$ is defined as

$$\mathrm{cap}_m^*(E) = \inf\{\mathrm{cap}_m(U) : E \subset U,\ U \subset \Omega \text{ is open}\}. \tag{7-3}$$

Then, we have basic properties which follow easily from the corresponding ones of the capacity cap_m.

Proposition 7.4. *Let Ω be a bounded open set in $\mathbb{C}^n$. Then,*

(i) $\mathrm{cap}_m^*(E_1) \le \mathrm{cap}_m^*(E_2)$ *if* $E_1 \subset E_2 \subset \Omega$;

(ii) $\mathrm{cap}_m^*(E_1, \Omega_1) \ge \mathrm{cap}_m^*(E, \Omega_2)$ *if* $E \subset \Omega_1 \subset \Omega_2$;

(iii) $\mathrm{cap}_m^*\big(\bigcup_j E_j\big) \le \sum_j \mathrm{cap}_m^*(E_j)$.

Lemma 7.5. *Let $\Omega \Subset \mathbb{C}^n$ be a strictly m-pseudoconvex domain. Let $E \Subset \Omega$ a Borel subset. Then*

$$\int_\Omega H_m(u_E^*) \le \mathrm{cap}_m^*(E) \le C \sum_{s=0}^m \int_\Omega (-u_E^*) H_s(u_E^*).$$

Proof. We prove first the left-hand side inequality. Assume that $E = \overline{E}$ is compact. Lemma 7.3(vi) implies

$$\int_\Omega H_m(u_K^*) = \int_K H_m(u_K^*) \le \mathrm{cap}_m(K) \le \mathrm{cap}_m^*(K).$$

Assume $E = G$ is an open subset. We can find an increasing sequence of compact sets K_j such that $\bigcup_j K_j = G$. It is easy to see that $u_{K_j}^*$ decreases to $u_G = u_G^*$ on Ω. Hence, by the weak convergence theorem for decreasing sequences, $H_m(u_{K_j}^*) \to H_m(u_G)$ weakly. This implies

$$\int_\Omega H_m(u_G) = \lim_{j\to\infty} \int_\Omega H_m(u_{K_j}^*) \le \lim_{j\to\infty} \mathrm{cap}_m(K_j) \le \mathrm{cap}_m(G).$$

Since $\mathrm{cap}_m(G) = \mathrm{cap}_m^*(G)$, the conclusion follows.

Now let E be a Borel subset. By definition there exists a sequence of open sets $\{O_j\}$ in Ω containing E such that $\mathrm{cap}_m^*(E) = \lim_j \mathrm{cap}_m(O_j)$. Replacing O_j by $\bigcap_{1\le s\le j} O_j$ we may assume that $\{O_j\}_{j\ge 1}$ is decreasing. Moreover, by Choquet's lemma there exists an increasing sequence $\{v_j\}$ of negative m-ω-sh functions in Ω such that $v_j = -1$ on E and $\lim_j v_j = u_E$ almost everywhere on Ω. Set $G_j = O_j \cap \{v_j < -1 + 1/j\}$. Then, $E \subset G_j \subset O_j$ and

$$v_j - \frac{1}{j} \le u_{G_j} \le u_E.$$

So, $\lim_{j\to\infty} \mathrm{cap}_m(G_j) = \mathrm{cap}_m^*(E)$ and u_{G_j} increases to u_E almost everywhere on Ω. Therefore, by the weak convergence for increasing sequences (Lemma 5.4),

$$\int_\Omega H_m(u_E^*) = \lim_{j\to\infty} \int_\Omega H_m(u_{G_j}) \le \lim_{j\to\infty} \mathrm{cap}_m(G_j) = \mathrm{cap}_m^*(E).$$

Thus, the proof of left-hand side inequality is completed.

Next we prove the other one. Let $E \Subset \Omega$ be a Borel subset and consider the sets G_j defined as above. Then, $\lim_j \mathrm{cap}_m(G_j) = \mathrm{cap}_m^*(E)$. We also have for $0 \le s \le m$,

$$\lim_{j\to\infty} \int_\Omega (-u_{G_j}) H_s(u_{G_j}) = \int_\Omega (-u_E^*) H_s(u_E^*)$$

by the weak convergence for increasing sequence again. Thus, it is enough to prove the inequality for $E = G \Subset \Omega$ an open subset.

To this end let $-1 \le \rho \le 0$ be a m-ω-sh function in Ω. Since $G \subset \{u_G = -1\}$ and $u_G = u_G^*$ it follows that for $q \ge 1$,

$$\int_G H_m(\rho) \le \int (-u_G)^q H_m(\rho).$$

Applying Corollary 5.3 for $v = 0$ and $u = u_G$ we get

$$\int_\Omega (-u_G)^{3m} H_m(\rho) \le C \sum_{s=0}^{m} \int_\Omega (-u_G) H_s(u_G).$$

Taking the supremum over all such functions ρ, we get the desired inequality. □

Remark 7.6. For a compact set K in a strictly m-pseudoconvex domain Ω, $\mathrm{cap}(K, \Omega) = 0$ if and only if $\mathrm{cap}_m^*(K, \Omega) = 0$.

Proposition 7.7. *In a strictly m-pseudoconvex domain Ω the following are equivalent*:

(a) $u_{E,\Omega}^* = 0$.

(b) $E \subset \{u = -\infty\}$ *for an m-ω-sh function $u < 0$ in Ω.*

(c) $\mathrm{cap}_m^*(E, \Omega) = 0$.

Proof. The implication (a) ⇒ (b) follows from (7-2) by setting $u = \sum_{j\ge1} v_j$. Conversely, $E \subset \{v = -\infty\}$, where $v < 0$ and m-ω-sh, implies $u_E \ge v/j$ for $j = 1, 2 \ldots.$ So $u_E = 0$ outside $\{v = -\infty\}$ whose Lebesgue measure is zero. Hence, $u_E^* = 0$ by [Gu and Nguyen 2018, Corollary 9.7]. The implication (c) ⇒ (a) follows from the fact that $H_m(u_E^*) \equiv 0$ and the domination principle (Corollary 6.2) if E is relatively compact in Ω. The general case follows from the countable subadditivity of cap_m^* and the corresponding property of u_E^* above.

To prove (b) ⇒ (c) let us fix an open subset $V \Subset \Omega$ and define $\mathcal{O}_j = \{u < -j\} \cap V$. Let $\varepsilon > 0$. We wish to find an open subset $E \subset G \subset \Omega$ with $\mathrm{cap}_m(G) < \varepsilon$. Indeed, we have $0 \ge u_{\mathcal{O}_j} \ge \max\{u/j, -1\}$, where $u_{\mathcal{O}_j}$ is the relative extremal function. Then, $u_{\mathcal{O}_j} \uparrow 0$ a.e. on Ω by using ω-subharmonicity. Now the right-hand side inequality in Lemma 7.5 gives

$$\mathrm{cap}_m(\mathcal{O}_j) \le C \sum_{s=0}^{m} e_{(0,0,s)},$$

where $e_{(0,0,s)} = \int_\Omega (-u_{\mathcal{O}_j}) H_s(u_{\mathcal{O}_j})$. Applying the weak convergence theorem for increasing convergence sequences we get that $H_s(u_{\mathcal{O}_j}) \to 0$ weakly in Ω, $1 \le s \le m$. Furthermore, for $s = m$ and $s = 0$,

$$\lim_{j\to\infty} \int_\Omega (-u_{\mathcal{O}_j}) H_m(u_{\mathcal{O}_j}) = 0 = \lim_{j\to\infty} \int_\Omega (-u_{\mathcal{O}_j}) \omega^n.$$

Now we claim that for $1 \le s \le m-1$,

$$\lim_{j\to\infty} \int_\Omega (-u_{\mathcal{O}_j}) H_s(u_{\mathcal{O}_j}) = 0. \tag{7-4}$$

Assume this is true for a moment and let us finish the proof. The above facts imply that

$$\lim_{j\to\infty} \mathrm{cap}_m(\mathcal{O}_j) = 0. \tag{7-5}$$

Take a sequence of open sets V_s exhausting Ω. Choose $\mathcal{O}_{j_s} = \{u < -j_s\} \cap V_s$ such that $\mathrm{cap}_m(\mathcal{O}_{j_s}) < \varepsilon/2^s$. Define $G = \bigcup_{s\ge 1} \mathcal{O}_{j_s}$ which is an open set containing E and which has capacity less than ε.

Finally, let us verify (7-4). Let ρ be strictly m-ω-sh defining function for Ω. Since $\mathcal{O}_j \subset V \Subset \Omega$, we have $u_{\mathcal{O}_j} \ge u_V \ge A\rho$ for a constant $A > 0$ depending only on V, Ω by the proof of Lemma 7.3(v). Hence, the $u_{\mathcal{O}_j}$ can be extended to a neighborhood $\widetilde{\Omega}$ of $\overline{\Omega}$ by $A\rho$ (see, e.g., (8-9)). By the CLN inequality there is a uniform constant $C = C(\Omega, \widetilde{\Omega})$ such that for every $j \ge 1$,

$$\int_\Omega H_s(u_{\mathcal{O}_j}) \le C.$$

Then, for a fixed $\varepsilon > 0$,

$$\begin{aligned}\int_\Omega (-u_{\mathcal{O}_j}) H_s(u_{\mathcal{O}_j}) &\le A \int_\Omega |\rho| H_s(u_{\mathcal{O}_j}) \\ &\le A\varepsilon \int_{\{|\rho|<\varepsilon\}} H_s(u_{\mathcal{O}_j}) + A \int_{\{|\rho|\ge\varepsilon\}} H_s(u_{\mathcal{O}_j}) \\ &\le AC\varepsilon + A \int_{\{|\rho|\ge\varepsilon\}} H_s(u_{\mathcal{O}_j}).\end{aligned}$$

Since $H_s(u_{\mathcal{O}_j}) \to 0$ weakly as $j \to \infty$ and $\{|\rho| \ge \varepsilon\} \subset \Omega$ is compact, letting $j \to \infty$ we get

$$\lim_{j\to\infty} \int_\Omega (-u_{\mathcal{O}_j}) H_s(u_{\mathcal{O}_j}) \le AC\varepsilon.$$

This holds for arbitrary $\varepsilon > 0$, where A, C are uniform constants independent of ε. □

Theorem 7.8. *m-negligible sets are m-polar.*

Proof. The result is local, so we may assume that all functions are defined on a bounded strictly m-pseudoconvex domain Ω. Thanks to the characterization in Proposition 7.7 it is enough to show that a negligible set E has outer capacity zero. Let $\{u_j\}$ be the sequence in the definition of the negligible set and put $u = \sup_j u_j$. By Choquet's lemma we may assume this is an increasing sequence. Let $\varepsilon > 0$. By quasi-continuity we can find an open set $G \subset \Omega$ such that $\mathrm{cap}_m(G) < \varepsilon$ and u, the u_j are continuous on $F := \Omega \setminus G$.

Since $\mathrm{cap}_m^*(\bullet)$ is countably subadditive, it is enough to show that

$$\mathrm{cap}_m^*(E \cap K) = 0$$

for a fixed compact subset $K \subset \Omega$. Observe that for all rational numbers $r < t$, the sets

$$K_{rt} = K \cap F \cap \{u \le r < t \le u^*\}$$

are compact, because u is lower semicontinuous in $K \cap F$ and u^* is upper-semicontinuous. Also, $(K \cap E) \setminus G$ is contained in the countable union of such compact sets. Thus, by countable subadditivity, it remains to verify $\mathrm{cap}_m^*(K_{rt}) = 0$.

Since u is lower semicontinuous on K, there exists a constant c such that $u \ge c$ on K. Define $u_c = \max\{u, c\}$ and notice that $K_{rt} \subset K'_{rt}$, where

$$K'_{rt} = K \cap F \cap \{u_c \le r < t \le u_c^*\}$$

are compact sets. Since $\{u_c < u_c^*\}$ has inner capacity cap_m zero, we have $\mathrm{cap}_m(K'_{rt}) = 0$. This implies that $\mathrm{cap}_m^*(K'_{rt}) = 0$ by Remark 7.6. □

We have the following analogue of Josefson's theorem whose proof is the same as the one of [Kołodziej 2005, Theorem 1.23].

Theorem 7.9. *For any m-polar subset E of $\mathbb{C}^n$, there exists a m-ω-sh function h on $\mathbb{C}^n$ such that $E \subset \{h = -\infty\}$.*

8. Dirichlet problem in domains in $\mathbb{C}^n$

Let Ω be a bounded strictly m-pseudoconvex domain in $\mathbb{C}^n$. The comparison principle in Corollary 6.2 coupled with the proof of [Gu and Nguyen 2018, Lemma 3.13] gives the following stability estimate for the complex Hessian equation:

Proposition 8.1. *Let $u, v \in C^0(\overline{\Omega})$ be m-ω-sh in Ω and satisfy*

$$H_m(u) = f\,dV_{2n}, \quad H_m(v) = g\,dV_{2n}$$

with $0 \le f, g \in L^p(\Omega)$ and $p > n/m$. Then

$$\|u - v\|_{L^\infty} \le \sup_{\partial\Omega} |u - v| + C\|f - g\|_{L^p(\Omega)}^{\frac{1}{m}},$$

where $C = C(m, n, p, \Omega)$.

Let $\psi \in C^\infty(\partial\Omega)$. Given a smooth positive function $f \in C^\infty(\overline{\Omega}, \mathbb{R})$, there is always a smooth m-ω-sh subsolution $\underline{u} \in C^\infty(\overline{\Omega})$, that is

$$H_m(\underline{u}) \ge f(z), \qquad \underline{u} = \psi \quad \text{on } \partial\Omega. \tag{8-1}$$

The stability estimate and an easy approximation argument imply that we can solve the Dirichlet problem when the right-hand side is in L^p, $p > n/m$, after invoking the solution for the smooth data due to Collins and Picard [2022].

Theorem 8.2. *Let $0 \le f \in L^p(\Omega)$ for some $p > n/m$. Suppose $\varphi \in C^0(\partial\Omega)$. Then there exists a unique continuous m-ω-sh function $u \in C^0(\overline{\Omega})$ solving the Dirichlet problem*

$$H_m(u) = f\omega^n, \qquad u = \varphi \quad \text{on } \partial\Omega.$$

Now we wish to solve the equation with the right-hand side just being a positive Radon measure assuming the existence of a subsolution. We use ideas from [Kołodziej and Nguyen 2023a], however several steps require very different proofs.

Assume $\widetilde{\Omega}$ is a neighborhood of $\overline{\Omega}$. Let us define a slightly modified Cegrell class

$$\widetilde{\mathcal{E}}_0(\Omega) = \Big\{u \text{ is bounded and } m\text{-}\omega\text{-sh in } \widetilde{\Omega} : \lim_{z\to\partial\Omega} u(z) = 0\Big\}. \tag{8-2}$$

The set $\widetilde{\Omega}$ is suppressed in this notation. By the CLN inequality for $u \in \widetilde{\mathcal{E}}_0(\Omega)$ we have $\int_\Omega H_m(u) < +\infty$. We introduce this modified class to control the integrals of the wedge products of currents associated to bounded m-ω-sh functions.

Now we follow the steps in [Kołodziej and Nguyen 2023a]. The first one corresponds to [Kołodziej and Nguyen 2023a, Lemma 2.1] which in turn was inspired by [Cegrell 1998, Lemma 5.2].

Lemma 8.3. *Let λ be a finite positive Radon measure on Ω which vanishes on m-polar sets. Let $\{u_j\}_{j\ge1} \subset \widetilde{\mathcal{E}}_0(\Omega)$ be a uniformly bounded in $\widetilde{\Omega}$ sequence that converges dV-a.e. to $u \in \widetilde{\mathcal{E}}_0(\Omega)$. Then, there exists a subsequence u_{j_s} such that*

$$\lim_{j_s\to\infty} \int_\Omega u_{j_s}\, d\lambda = \int_\Omega u\, d\lambda.$$

Proof. By the comparison principle, all functions are negative on Ω. The proof of [Kołodziej and Nguyen 2023a, Lemma 2.1] is applicable provided that the m-negligible sets are m-polar and this is the content of Theorem 7.8. □

Applying the lemma twice for the sequences $\{u_j\}$, $\max\{u_j, u\}$ and combining with the identity $2\max\{u_j, u\} = u_j + u + |u_j - u|$ we easily get a corollary.

Corollary 8.4. *Let λ and $\{u_j\}_{j\ge1}$ be as in Lemma 8.3. Then, there is a subsequence, still denoted by $\{u_j\}$, such that $\lim_{j\to\infty} \int_\Omega |u_j - u|\, d\lambda = 0$.*

The following result is crucial for proving the weak convergence later.

Lemma 8.5. *Let $d\lambda$ and $\{u_j\}_{j\ge1}$ be as in Lemma 8.3. Let $\{w_j\}_{j\ge1} \subset \widetilde{\mathcal{E}}_0(\Omega)$ be uniformly bounded in $\widetilde{\Omega}$. Suppose that w_j converges in capacity — $\mathrm{cap}_m(\bullet, \Omega)$ — to $w \in \widetilde{\mathcal{E}}_0(\Omega)$. Then,*

$$\lim_{j\to\infty} \int_\Omega |u - u_j| H_m(w_j) = 0.$$

Remark 8.6. Since all functions are uniformly bounded on the fixed neighborhood $\widetilde{\Omega}$ of $\overline{\Omega}$, with $\|u_j\|_{L^\infty}, \|w_j\|_{L^\infty} \le A$, by Proposition 3.7 there exist two positive constants C_1, C_2 depending only on sup-norm of the u_j and the w_j (and the domains $\Omega, \widetilde{\Omega}$), such that

$$\sup_j \int_\Omega H_m(u_j) \le C_1, \quad \sup_j \int_\Omega H_m(w_j) \le C_2.$$

Proof. Note that $|u-u_j| = (\max\{u, u_j\}-u_j)+(\max\{u, u_j\}-u)$. Observe first that by the Hartogs lemma and quasicontinuity of u ([Gu and Nguyen 2018, Lemma 9.14] and Theorem 4.9) $\phi_j := \max\{u, u_j\} \to u$ in capacity. Fix $\varepsilon > 0$. We have for j large,

$$\begin{aligned}\int_\Omega (\max\{u, u_j\} - u) H_m(w_j) &\le \int_{\{|\phi_j - u| > \varepsilon\}} H_m(w_j) + \varepsilon \int_\Omega H_m(w_j)\\ &\le A^m \operatorname{cap}_m(|\phi_j - u| > \varepsilon) + C_2 \varepsilon.\end{aligned}$$

Therefore, $\lim_{j\to\infty} \int (\phi_j - u) H_m(w_j) = 0$. Here and in what follows we drop the domain Ω in the integrals if no confusion arises.

Next, we consider for $j > k$,

$$\int (\phi_j - u_j) H_m(w_j) - \int (\phi_j - u_j) H_m(w_k) = \int (\phi_j - u_j) dd^c(w_j - w_k) \wedge T \wedge \omega^{n-m},$$

where $T = T(j,k) = \sum_{s=1}^{n-1} (dd^c w_j)^s \wedge (dd^c w_k)^{m-1-s}$. Let us write $h_j = \phi_j - u_j$. Now the proof gets more complicated than the one in [Kołodziej and Nguyen 2023a, Lemma 2.3] as the integration by parts produces more terms involving the torsion of ω.

By the integration by parts

$$\int h_j dd^c(w_j - w_k) \wedge T \wedge \omega^{n-m} = \int (w_j - w_k) dd^c(h_j \omega^{n-m}) \wedge T. \tag{8-3}$$

This integration by parts formula is justified by an approximation argument as follows. All functions are continuous on the boundary $\partial\Omega$ with zero value there, so the approximating sequences of smooth functions converge to zero uniformly on $\partial\Omega$. Moreover, the functions are defined and uniformly bounded on a neighborhood $\widetilde{\Omega}$ so the total masses of wedge products are uniformly bounded by the CLN inequality. Hence, the boundary terms vanish after passing to the limit (see Remark 2.16 and also [Guedj and Zeriahi 2017, Proposition 3.7]).

By a direct calculation,

$$dd^c(h_j\omega^{n-m}) = dd^c h_j \wedge \omega^{n-m} + h_j dd^c \omega^{n-m} + (n-m)[dh_j \wedge d^c\omega + d\omega \wedge d^c h_j] \wedge \omega^{n-m-1}. \tag{8-4}$$

For the first term we obtain a bound

$$\int (w_j - w_k) dd^c h_j \wedge \omega^{n-m} \wedge T \le \int |w_j - w_k| dd^c(\phi_j + u_j) \wedge T \wedge \omega^{n-m}, \tag{8-5}$$

and using inequality [Kołodziej and Nguyen 2016, Lemma 2.3] for the second term

$$\int (w_j - w_k) h_j dd^c \omega^{n-m} \wedge T \le C \int |w_j - w_k| h_j [dd^c(w_j + w_k)]^{m-1} \wedge \omega^{n-m+1}. \tag{8-6}$$

Next, since two terms in the bracket of (8-4) are mutually conjugate, we only estimate the first one. To this end we will use the Cauchy–Schwarz inequality (Corollary 2.5):

$$\left|\int (w_j - w_k) dh_j \wedge d^c\omega \wedge \omega^{n-m-1} \wedge T\right|^2$$
$$\leq C \int |w_j - w_k| dh_j \wedge d^c h_j \wedge [dd^c(w_j + w_k)]^{m-1} \wedge \omega^{n-m} \times \int |w_j - w_k| [dd^c(w_j + w_k)]^{m-1} \wedge \omega^{n-m+1}.$$

Let us consider the second factor in the product. Since $\|w_j\|_{L^\infty}, \|u_j\|_{L^\infty} \leq A$ in Ω, it follows that

$$\begin{aligned}
&\int_\Omega |w_j - w_k| [dd^c(\phi_j + u_j)]^{m-1} \wedge \omega^{n-m+1} \\
&\qquad \leq A \int_{\{|w_j - w_k| > \varepsilon\}} [dd^c(\phi_j + u_j)]^{m-1} \wedge \omega^{n-m+1} + \varepsilon \int_{\{|w_j - w_k| \leq \varepsilon\}} [dd^c(\phi_j + u_j)]^{m-1} \wedge \omega^{n-m+1} \\
&\qquad \leq (2A)^m \operatorname{cap}_m(|w_j - w_k| > \varepsilon) + C\varepsilon, \qquad (8\text{-}7)
\end{aligned}$$

where the uniform bound for the integral in the second line follows from the CLN inequality as all functions are uniformly bounded in $\widetilde{\Omega}$. It means that the left-hand side of (8-7) is less than $2C\varepsilon$ for some k_0 and every $j > k \geq k_0$.

As for the first factor in the product we observe that

$$dh_j \wedge d^c h_j \leq 2du_j \wedge d^c u_j + 2d\phi_j \wedge d^c \phi_j,$$

and $2d\phi_j \wedge d^c\phi_j = dd^c\phi_j^2 - 2\phi_j dd^c \phi_j$ and similarly for u_j. Therefore, we can apply the estimate as in (8-7) for this integral. The same is true for the integrals on the right-hand sides of (8-5) and (8-6).

Thus,

$$\begin{aligned}
\int (\phi_j - u_j) H_m(w_j) &\leq \int (\phi_j - u_j) H_m(w_k) + \left| \int (\phi_j - u_j) H_m(w_j) - \int (\phi_j - u_j) H_m(w_k) \right| \\
&\leq \int (\phi_j - u_j) H_m(w_k) + 8C\varepsilon \\
&\leq \int |u - u_j| H_m(w_k) + 8C\varepsilon.
\end{aligned}$$

Fix $k = k_0$ and apply Corollary 8.4 for $d\lambda = H_m(w_{k_0})$ to get that, for $j \geq k_1 \geq k_0$,

$$\int (\phi_j - u_j) H_m(w_j) \leq (8C + 1)\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the proof of the lemma is completed. □

Let μ be a positive Radon measure on a bounded strictly m-pseudoconvex domain Ω. Assume that there exists a bounded m-ω-sh function $\underline{u}$ in Ω such that

$$H_m(\underline{u}) \geq \mu, \qquad \lim_{x \to z \in \partial\Omega} \underline{u}(x) = 0. \qquad (8\text{-}8)$$

This function $\underline{u}$ is called a subsolution for $d\mu$. Our goal is to prove the following.

Theorem 8.7. *Let $\varphi \in C^0(\partial\Omega)$ and let μ be a positive Radon measure in Ω. Assume that μ admits a bounded subsolution $\underline{u}$ as in* (8-8). *Then, there exists a unique bounded m-ω-sh function u solving* $\lim_{z\to x} u(z) = \varphi(x)$ *for $x \in \partial\Omega$,*

$$H_m(u) = \mu \quad \text{in } \Omega.$$

We first make some reduction steps. In fact it is enough to prove the statement under the following additional assumptions on the measure $d\mu$, the boundary data φ and the subsolution $\underline{u}$.

Lemma 8.8. *We may assume additionally that*

(a) *φ is smooth on $\partial\Omega$;*

(b) *μ has compact support in Ω;*

(c) *the support of $\nu = (dd^c\underline{u})^m \wedge \omega^{n-m}$ is compact in Ω;*

(d) *$\underline{u}$ can be extended as a m-ω-sh function to a neighborhood $\widetilde{\Omega}$ of $\overline{\Omega}$.*

Proof. *Step* 1: (c) $\Rightarrow$ (d). Define $K := \operatorname{supp}\nu$. Let ρ be a strictly m-ω-sh defining function for Ω. In particular, ρ is defined in a neighborhood $\widetilde{\Omega}$ of $\overline{\Omega}$. For $A > 0$ (to be chosen), we define

$$v = \begin{cases} \max\{\underline{u}, A\rho\} & \text{in } \Omega, \\ \quad A\rho & \text{on } \widetilde{\Omega}\setminus\Omega. \end{cases} \tag{8-9}$$

Notice that $\lim_{x\to\partial\Omega}(\underline{u} - A\rho) = 0$, so the function v is well defined. We claim that for a sufficiently large A we have $\underline{u} \ge A\rho$ on Ω, and then v is a required extension. Indeed, let $U \Subset \Omega$ be a neighborhood of K. Since $\sup_{\overline{U}} \rho \le -\delta$ for some $\delta > 0$ and $\underline{u}$ is bounded, we can choose $A > 0$ large enough so that $v \ge A\rho$ on $\overline{U}$. Furthermore, $(dd^c v)^m \wedge \omega^{n-m} \equiv 0$ on $\Omega \setminus \overline{U}$ and $v \ge A\rho$ on the boundary $\partial(\Omega\setminus\overline{U})$. The domination principle implies that $v \ge A\rho$ on $\Omega\setminus\overline{U}$.

Remark 8.9. The constant $A > 0$ chosen in this argument depends only on the defining function ρ, the support of ν and the sup-norm of $\underline{u}$.

Step 2: (b) $\Rightarrow$ (c). This follows from the classical balayage argument. However, several ingredients are only available recently. We give a detailed argument. Assume that $\operatorname{supp}\mu \subset U$ which is an open subset relatively compact in Ω. We define an envelope

$$v = \sup\{w : w \text{ is } m\text{-}\omega\text{-sh in } \Omega,\ w \le \underline{u} \text{ on } U,\ w \le 0\}.$$

It follows from [Gu and Nguyen 2018, Proposition 2.6] that the upper semicontinuous regularization $v^* \ge v$ is also m-ω-sh function in Ω. Hence, $v^* = v$ belongs to the family in the definition of the envelope. So $\underline{u} \le v$. Thus, $\underline{u} = v$ on U containing $\operatorname{supp}\mu$. Therefore,

$$H_m(v) \ge \mu \quad \text{in } \Omega.$$

Now we verify that

$$H_m(v) \equiv 0 \quad \text{on } \Omega\setminus\overline{U}.$$

Let $B(a,r) \Subset \Omega \setminus \overline{U}$ be a small ball. By using the solution to the homogeneous Hessian equation (Theorem 8.2) one can find a continuous m-ω-sh function $h \geq v$ in Ω which is maximal in $B(a,r)$ in the sense that $H_m(h) \equiv 0$ in $B(a,r)$ and $h = v$ on $\Omega \setminus B(a,r)$.

Observe also $U \subset \Omega \setminus B(a,r)$ and the function h is a candidate in the envelope. So, $h \leq v$ in Ω. Hence, $h = v$ everywhere. We have $H_m(v) \equiv 0$ on $B(a,r)$. The ball is arbitrary so we get the desired property for v.

Step 3: (b). If the problem is solvable for measures with compact support, then it is solvable for a general measure. In fact, let $\eta_j \uparrow 1$ be a sequence of cut-off functions. Then, $\eta_j \mu$ admits $\underline{u}$ as a bounded subsolution. Solve $H_m(w) = \eta_j \mu$ to obtain a bounded m-ω-sh function u_j with $u_j = \varphi$ on $\partial\Omega$. Denote by $h \in C^0(\overline{\Omega})$ a unique m-ω-sh solution to $H_m(h) \equiv 0$ in Ω with $h = \varphi$ on $\partial\Omega$. Then, $H_m(\underline{u} + h) \geq H_m(u_j)$. The domination principle gives

$$\underline{u} + h \leq u_j \leq h,$$

and the sequence u_j is decreasing. Define $u = \lim_j u_j$. Then, it is a solution to $H_m(u) = \mu$ with the boundary data φ.

Step 4: (a). If we can solve the Dirichlet problem for smooth boundary data, then it is solvable for the continuous one. Indeed, let φ_j be a sequence of smooth functions decreasing to φ. Find a bounded m-ω-sh function u_j in Ω such that

$$H_m(u_j) = \mu, \qquad \lim_{z \to x} u_j(z) = \varphi_j(x) \quad \text{for every } x \in \partial\Omega.$$

Let h_j be the solution the homogeneous equation $H_m(h_j) \equiv 0$ with the boundary data φ_j. By the domination principle, $u_j \geq h_j + \underline{u}$ and u_j is a decreasing sequence. Since φ_j is uniformly bounded, so is h_j. Therefore, the function $u = \lim_j u_j$ is a required solution to the equation. □

We proceed to prove the theorem under the assumptions (a)–(d).

Proof. Recall that we assumed that subsolution $\underline{u}$ is defined in a neighborhood $\widetilde{\Omega}$ of $\overline{\Omega}$. Hence, $\underline{u} \in \widetilde{\mathcal{E}}_0(\Omega)$ in the sense of (8-2). By Proposition 2.9 we can find a decreasing sequence of smooth m-ω-sh functions v_j defined in $\widetilde{\Omega}$ and such that $v_j \downarrow \underline{u}$ pointwise. Since $\underline{u}$ is bounded, $\{v_j\}$ is uniformly bounded. Next, let us write

$$H_m(v_j) = f_j dV \quad \text{in } \widetilde{\Omega},$$

where dV is the Euclidean volume form on $\mathbb{C}^n$. Observe that v_j is no longer zero on the boundary of Ω, however we can modify it by solving the Dirichlet problem to find $\bar{v}_j \in C^0(\overline{\Omega})$, m-ω-sh satisfying

$$H_m(\bar{v}_j) = f_j dV \quad \text{in } \Omega, \qquad \bar{v}_j = 0 \quad \text{on } \partial\Omega.$$

Since $\underline{u}$ continuous on $\partial\Omega$, by Dini's theorem, v_j converges uniformly to $\underline{u}$ on $\partial\Omega$. As a consequence of the stability estimate for the right-hand side in L^p, we have

$$\|\bar{v}_j - v_j\|_{L^\infty(\Omega)} \leq \sup_{\partial\Omega} |\bar{v}_j - v_j| = \sup_{\partial\Omega} |v_j|.$$

Hence, $\{\bar{v}_j\}$ is also uniformly bounded on $\overline{\Omega}$. Also by Dini's theorem $v_j \to \underline{u}$ uniformly on compact sets where $\underline{u}$ is continuous. By the stability estimate above $\bar{v}_j \to \underline{u}$ uniformly on such compact sets. Combining this with the quasicontinuity of $\underline{u}$, we get also that

$$\bar{v}_j \to \underline{u} \quad \text{in capacity as } \; j \to \infty.$$

Thus the $\bar{v}_j$ have zero boundary values, but in general those are only continuous functions. Also $H_m(\bar{v}_j)$ converges weakly to ν whose support is compact in Ω. Thus,

$$\sup_j \int_\Omega H_m(\bar{v}_j) \le C.$$

Moreover all $\bar{v}_j$ can be extended so that they form a uniformly bounded sequence in $\widetilde{\mathcal{E}}_0(\Omega)$ as follows. Set $\check{v}_j := \max\{\bar{v}_j, A\rho\}$ where A is the same as in (8-2). We verify that

$$\check{v}_j \to \underline{u} \quad \text{in capacity as } \; j \to \infty. \tag{8-10}$$

In fact, we fix an $\varepsilon > 0$. Then,

$$\{z \in \Omega : |\check{v}_j - \underline{u}| > \varepsilon\} = \{z \in \Omega : \check{v}_j - \underline{u} > \varepsilon\} \cup \{z \in \Omega : \check{v}_j - \underline{u} < -\varepsilon\}.$$

Note that $A\rho \le \underline{u}$ in Ω, which implies $\{\check{v}_j - \underline{u} > \varepsilon\} = \{\bar{v}_j - \underline{u} > \varepsilon\}$. So,

$$\mathrm{cap}_m(\check{v}_j - \underline{u} > \varepsilon) \to 0 \quad \text{as } \; j \to \infty.$$

On the other, since $\bar{v}_j \le \check{v}_j$ we have

$$\{\check{v}_j - \underline{u} < -\varepsilon\} \subset \{\bar{v}_j - \underline{u} < -\varepsilon\}.$$

The capacity of the latter set tends to zero, so the same is true for the former. This finishes the proof of the claim $\check{v}_j \to \underline{u}$ in the capacity cap_m.

We are now ready to produce a sequence of functions whose limit point will be the desired solution. By the Radon–Nikodym theorem we can write $\mu = g\nu$ for a Borel measurable function $0 \le g \le 1$. Assume first that g is continuous (we will relax the assumption on g at the end of the proof) and solve for $u_j \in \mathrm{SH}_m(\omega) \cap C^0(\overline{\Omega})$, the equation

$$H_m(u_j) = g\chi f_j dV, \qquad u_j = \varphi \quad \text{on } \; \partial\Omega,$$

where $0 \le \chi \le 1$ is the cut-off function in Ω such that $\chi \equiv 1$ on a neighborhood of $\operatorname{supp}\nu$ and $\operatorname{supp}\chi \Subset \Omega$. Set

$$u = \bigl(\limsup_{j\to\infty} u_j\bigr)^*. \tag{8-11}$$

Let us show first that $\{u_j\}$ is uniformly bounded. Indeed, let ψ be a smooth m-ω-sh solution to $(dd^c\psi)^m \wedge \omega^{n-m} = 1$ in $\overline{\Omega}$ with $\psi = \varphi$ on $\partial\Omega$. Thus, we may assume that ψ is m-ω-sh on $\widetilde{\Omega}$. At this point we used the smoothness of φ. Furthermore, let $h \in C^0(\overline{\Omega})$ be a m-ω-sh solution to

$$H_m(h) = 0 \quad \text{in } \; \Omega, \qquad h = \varphi \quad \text{on } \; \partial\Omega.$$

It follows from the domination principle that

$$\psi + \check{v}_j \le u_j \le h.$$

(Here we may increase A so that $\check{v}_j = \max\{\bar{v}_j, A\rho\} = \bar{v}_j$ in a neighborhood of supp ν and supp χ.) The above lower and upper bounds of u_j are continuous on the boundary $\partial\Omega$ and equal to φ there. So u_j and u have the same property. Thus passing to a subsequence we may assume that

$$u_j \to u \quad \text{in } L^1(\Omega), \qquad u_j \to u \quad \text{a.e. in } dV. \tag{8-12}$$

The next step is to prove that $H_m(u) = \mu$. Observe that $\psi + \check{v}_j$ is defined in the neighborhood $\widetilde{\Omega}$ of $\overline{\Omega}$. This combined with $\psi + \check{v}_j \le u_j$ allows us to extend u_j to $\widetilde{\Omega}$ by setting

$$\widetilde{u}_j = \begin{cases} \max\{u_j, \psi + \check{v}_j\} & \text{on } \Omega, \\ \psi + \check{v}_j & \text{on } \widetilde{\Omega} \setminus \Omega. \end{cases}$$

Using again the smoothness of φ, we can find ψ' a strictly m-sh function on a (possibly smaller) neighborhood $\widetilde{\Omega}$ of $\overline{\Omega}$ satisfying

$$\psi' = -\varphi \quad \text{on } \partial\Omega.$$

Then, we have clearly

$$\widehat{u}_j := u_j + \psi' \in \widetilde{\mathcal{E}}_0(\Omega) \tag{8-13}$$

for all $j \ge 1$. Consequently, we get the important uniform bound for the total mass of mixed Hessian operators.

Lemma 8.10. *Let $T_{j,k} = (dd^c \widehat{u}_j)^s \wedge (dd^c \check{v}_k)^\ell \wedge \omega^{n-s-\ell}$, where $0 \le s + \ell \le m$. There exists a uniform constant C independent of j, k such that*

$$\int_\Omega T_{j,k} \le C.$$

Proof. All functions belong to $\widetilde{\mathcal{E}}_0(\Omega)$ defined on the fixed neighborhood $\widetilde{\Omega}$ of $\overline{\Omega}$. Thanks to Remark 8.9 we know that the extensions of $\check{v}_i$ are uniformly bounded on $\widetilde{\Omega}$. Hence, the extensions of u_j above are uniformly bounded as well. The proof follows from the CLN inequality. □

The next result corresponds to [Kołodziej and Nguyen 2023a, Lemma 3.5].

Lemma 8.11. *There exists a subsequence $\{u_{j_s}\}$ such that for*

$$w_s := \max\left\{u_{j_s}, u - \frac{1}{s}\right\}$$

the following claims hold:

(a) $\lim_{s\to\infty} \int_\Omega |u_{j_s} - u| H_m(u) = 0$.

(b) $\lim_{s\to\infty} \int_\Omega |u_{j_s} - u| H_m(w_s) = 0$.

(c) $\lim_{s\to\infty} \int_\Omega |u_{j_s} - u| H_m(u_{j_s}) = 0$.

Proof. (a) Since u is bounded in Ω by Proposition 7.7 the measure $H_m(u)$ vanishes on m-polar sets. Define $\widehat{u} = u + \psi'$. Then $\widehat{u}$ is the limit (a.e.-dV) of the sequence $\{\widehat{u}_j\}_{j\geq 1} \subset \widetilde{\mathcal{E}}_0(\Omega)$ from (8-13). Thus, $\widehat{u} \in \widetilde{\mathcal{E}}_0(\Omega)$. Notice that $\widehat{u}_j - u = u_j - u$, so the assumptions of Corollary 8.4 are satisfied.

(b) Clearly, $\widehat{w}_s = w_s + \psi' \in \widetilde{\mathcal{E}}_0(\Omega)$. Since $w_s \to u$ in capacity as $s \to \infty$, the same convergence holds for $\widehat{w}_s \to \widehat{u}$. It follows from Lemma 8.5 that

$$0 = \lim_{s\to\infty} \int_\Omega |\widehat{w}_s - \widehat{u}| H_m(\widehat{w}_s) \geq \lim_{s\to\infty} \int_\Omega |u_{j_s} - u| H_m(w_s).$$

(c) We use

$$H_m(u_{j_s}) = g\chi f_{j_s} dV \leq \chi H_m(\bar{v}_{j_s}) \leq H_m(\check{v}_{j_s}),$$

where the first inequality follows from $0 \leq g \leq 1$ and the last one is by $\bar{v}_j = \check{v}_j$ in a neighborhood of $\operatorname{supp} \nu$ which can be taken to be the neighborhood of $\operatorname{supp} \chi$. Taking into account the convergence in capacity of $\check{v}_{j_s}$ to $\underline{u}$, as $j_s \to \infty$, the proof of (c) follows again from Lemma 8.5. □

We are in the position to conclude that u from (8-11) is indeed the solution.

The argument of [Kołodziej and Nguyen 2023a, Lemma 3.6] is readily applicable to conclude that there exists a subsequence $\{u_{j_s}\}_{s\geq 1}$ of $\{u_j\}_{j\geq 1}$ such that

$$H_m(u_{j_s}) \to H_m(u) \quad \text{weakly}.$$

Hence, if $0 \leq g \leq 1$ is a continuous function whose support is compact in Ω, then there exists a unique bounded m-ω-sh function with $u = \varphi$ on $\partial\Omega$ and $H_m(u) = g H_m(\underline{u})$. The general case of a Borel function $0 \leq g \leq 1$ follows from the argument in [Kołodziej and Nguyen 2023a, page 11] at the end of the proof of Theorem 3.1. □

9. Hessian equations on Hermitian manifolds with boundary

Let $(\overline{M}, \omega)$ be a smooth compact Hermitian manifold of dimension n with nonempty boundary ∂M. Then, $\overline{M} = M \cup \partial M$, where M is a complex manifold of dimension n. Let $1 \leq m \leq n$ be an integer and $\alpha \in \Gamma_m(\omega)$ be a real (1, 1)-form.

Recently Collins and Picard [2022] solved the Dirichlet problem in M for the Hessian equation $(\alpha + dd^c u)^m \wedge \omega^{n-m} = f\omega^n$, for smooth data, assuming the existence of a subsolution. The goal of this section is to extend this result to the case of bounded functions. The special case of the Monge–Ampère equation was treated in [Kołodziej and Nguyen 2023a, Theorem 1.2]. The theorem below is also a significant improvement of [Gu and Nguyen 2018, Theorem 1.3].

Recall from [Gu and Nguyen 2018, Definition 2.4, Lemma 9.10] that a function $u : M \to [-\infty, +\infty)$ is called (α, m)-ω-subharmonic if it can be written locally as a sum of a smooth function and a ω-sh function, and globally for any collection $\gamma_1, \dots, \gamma_{m-1} \in \Gamma_m(M, \omega)$,

$$(\alpha + dd^c u) \wedge \gamma_1 \wedge \cdots \wedge \gamma_{m-1} \wedge \omega^{n-m} \geq 0 \quad \text{on } M \tag{9-1}$$

in the weak sense of currents. Denote by $\mathrm{SH}_{\alpha,m}(M, \omega)$ or $\mathrm{SH}_{\alpha,m}(\omega)$ the set of all (α, m)-ω-sh functions on M.

If Ω is a local coordinate chart on M and ρ is a strictly psh function on Ω such that

$$dd^c \rho \geq \alpha \quad \text{on } \Omega,$$

then $u + \rho$ is a m-ω-sh function on Ω. Using this fact, we can easily extend the definition of the wedge product for currents associated to bounded (α, m)-ω-sh functions by using partition of unity and the local one (Definition 3.4). Namely, write $\tau = \alpha - dd^c \rho$ which is a smooth $(1, 1)$-form. Then, $\alpha + dd^c u = dd^c(u + \rho) + \tau$. We define

$$(\alpha + dd^c u)^m \wedge \omega^{n-m} := \sum_{k=0}^{m} \binom{m}{k} [dd^c(u+\rho)]^k \wedge \tau^{m-k} \wedge \omega^{n-m} = \sum_{k=0}^{m} \binom{m}{k} \mathcal{L}_k(u+\rho) \wedge \tau^{m-k}.$$

This gives a positive Radon measure on M by the weak convergence theorem. Similarly, the wedge product for bounded (α, m)-ω-sh functions $u_1, \dots, u_m$,

$$(\alpha + dd^c u_1) \wedge \cdots \wedge (\alpha + dd^c u_m) \wedge \omega^{n-m},$$

is a well-defined positive Radon measure. Since the definition is local, all local results for m-ω-sh functions in a local coordinate chart transfer to (α, m)-ω-sh functions on the manifold M. For simplicity we define $\alpha_u := \alpha + dd^c u$ and

$$H_{m,\alpha}(u) = (\alpha + dd^c u)^m \wedge \omega^{n-m}.$$

Now, given a positive Radon measure μ on M and a continuous boundary data $\varphi \in C^0(\partial M, \mathbb{R})$ we wish to solve the Dirichlet problem

$$\begin{cases} u \in \mathrm{SH}_{\alpha,m}(\omega) \cap L^\infty(\overline{M}), \\ H_{m,\alpha}(u) = \mu, \\ \lim_{z\to x} u(x) = \varphi(x) & \text{for } x \in \partial M. \end{cases} \tag{9-2}$$

Let us state a general existence result.

Theorem 9.1. *Assume there exists a bounded (α, m)-ω-sh function $\underline{u}$ on M such that* $\lim_{z\to x} \underline{u}(z) = \varphi(x)$ *for $x \in \partial M$ and $H_{m,\alpha}(\underline{u}) \geq \mu$ on M. Then, there is a solution to the Dirichlet problem* (9-2).

Remark 9.2. In the general setting the uniqueness is not known, unlike in a bounded strictly m-pseudoconvex domain (Theorem 8.7). On the other hand, if we assume further that either the manifold M is Stein, or both ω and α are closed forms, then the solution will be unique.

As we use the Perron envelope method to show the theorem the most important ingredient is the proof of the special case $M \equiv \Omega$ a ball in $\mathbb{C}^n$.

Lemma 9.3. *Let $\varphi \in C^0(\partial\Omega, \mathbb{R})$. Suppose $\mu \leq H_m(v)$ for some bounded m-ω-sh function v in Ω with* $\lim_{z\to x} v(z) = 0$ *for $x \in \partial\Omega$. Then, there exists a unique (α, m)-ω-sh function u in Ω solving*

$$H_{m,\alpha}(u) = \mu \quad \text{in } \Omega, \qquad \lim_{z\to x} u(z) = \varphi(x) \quad \text{for } x \in \partial\Omega. \tag{9-3}$$

The proof of this lemma is a straightforward extension of Theorem 8.7 so we omit the proof.

Proof of Theorem 9.1. Let us proceed with the proof of the bounded subsolution theorem on $\overline{M}$. Consider the set of functions

$$\mathcal{B}(\varphi, \mu) := \{w \in \mathrm{SH}_{\alpha,m}(M, \omega) \cap L^\infty(\overline{M}) : H_{m,\alpha}(w) \geq \mu,\ w^*_{|\partial M} \leq \varphi\}, \tag{9-4}$$

where $w^*(x) = \limsup_{M \ni z \to x} w(z)$ for every $x \in \partial M$. Clearly, $\underline{u} \in \mathcal{B}(\varphi, \mu)$. Let us solve the linear PDE finding $h_1 \in C^0(\overline{M}, \mathbb{R})$ such that

$$(\alpha + dd^c h_1) \wedge \omega^{n-1} = 0, \qquad h_1 = \varphi \quad \text{on } \partial M. \tag{9-5}$$

Since $(\alpha + dd^c w) \wedge \omega^{n-1} \geq 0$ for $w \in \mathrm{SH}_{\alpha,m}(M, \omega)$, the maximum principle for the Laplace operator with respect to ω gives

$$w \leq h_1 \quad \text{for all } w \in \mathcal{B}(\varphi, \mu).$$

Set

$$u(z) = \sup_{w \in \mathcal{B}(\varphi,\mu)} w(z) \quad \text{for every } z \in M. \tag{9-6}$$

Then, by Choquet's lemma and the fact that $\mathcal{B}(\varphi, \mu)$ satisfies the lattice property, $u = u^* \in \mathcal{B}(\varphi, \mu)$. Again by the definition of u, we have $\underline{u} \leq u \leq h_1$. It follows that

$$\lim_{z \to x} u(z) = \varphi(x) \quad \text{for every } x \in \partial M. \tag{9-7}$$

Lemma 9.4 (lift). *Let $v \in \mathcal{B}(\varphi, \mu)$. Let $B \Subset M$ be a small coordinate ball (a chart biholomorphic to a ball in $\mathbb{C}^n$). Then, there exists $\widetilde{v} \in \mathcal{B}(\varphi, \mu)$ such that $v \leq \widetilde{v}$ and $H_{m,\alpha}(\widetilde{v}) = \mu$ on B.*

Proof. Given the solution in a small coordinate ball in Lemma 9.3, the proof from [Kołodziej and Nguyen 2023a, Lemma 3.7] is readily adaptable here. □

By (9-7) it remains to show that the function u above satisfies $H_{m,\alpha}(u) = \mu$. Let $B \Subset M$ be a small coordinate ball. It is enough to check $H_{m,\alpha}(u) = \mu$ on B. Let $\widetilde{u}$ be the lift of u as in Lemma 9.4. It follows that $\widetilde{u} \geq u$ and $H_{m,\alpha}(\widetilde{u}) = \mu$ on B. However, by the definition $\widetilde{u} \leq u$ on M. Thus, $\widetilde{u} = u$ on B, in particular on B we have $H_{m,\alpha}(\widetilde{u}) = H_{m,\alpha}(u) = \mu$. □

Remark 9.5. We can also study the continuity of the solution to the Dirichlet problem (9-2) for a measure that is well dominated by capacity as in [Kołodziej and Nguyen 2023a, Section 4] and the weak solution to the complex Hessian-type equations such as a generalization of the Monge–Ampère equation in [Kołodziej and Nguyen 2023b]. We leave these to future projects.

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