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FOCUSING DYNAMICS OF 2D BOSE GASES IN THE INSTABILITY REGIME

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We consider the dynamics of a 2D Bose gas with an interaction potential of the form $N^{2\beta-1}w(N^{\beta}\cdot)$ for $\beta \in \left(0, \frac{3}{2}\right)$. The interaction may be chosen to be negative and large, leading to the instability regime where the corresponding focusing cubic nonlinear Schrödinger equation (NLS) may blow up in finite time. We show that to leading order, the N-body quantum dynamics can be effectively described by the NLS prior to the blow-up time. Moreover, we prove the validity of the Bogoliubov approximation, where the excitations from the condensate are captured in a norm approximation of the many-body dynamics.

1. Introduction

Since the pioneering work of Bose [1924] and Einstein [1925], and especially after the experimental realization of the Bose–Einstein condensation [Anderson et al. 1995; Davis et al. 1995], there has been a remarkable effort to understand the macroscopic behavior of interacting Bose gases from first principles. From the mathematical point of view, the theory of interacting Bose gases goes back to Bogoliubov [1947], who proposed an effective method to transform a weakly interacting Bose gas to a noninteracting one, subject to a modification of the kinetic operator due to the interaction effect. While the original work of Bogoliubov focuses on the spectral property of bosonic systems towards a microscopic explanation for Landau's criteria of superfluidity, his ideas are also applicable to quantum dynamics. In the present paper, we will justify Bogoliubov's approximation in the dynamical setting for a class of Bose gases with attractive interactions.

In the presence of large attractive interaction potentials, blow-up phenomena have been observed in experiments with ultracold Bose gases [Bradley et al. 1995; Cornish et al. 2000; Donley et al. 2001]. In these experimental settings, first a repulsive interaction was used to prepare an initial state, and then the interaction was switched to attractive by means of Feshbach resonances. When the strength of the attractive interaction was increased beyond a critical threshold, a blow-up process happened, where a large fraction of the condensate was lost [Roberts et al. 2001]. Heuristically, this behavior can be explained by describing the condensate by the solution of a focusing cubic nonlinear Schrödinger equation (NLS), which may exhibit a finite-time blow-up.

In the present work, we will focus on the instability regime for dilute Bose gases in two dimensions, where the corresponding NLS is mass-critical. Before the blow-up time, we give a rigorous derivation of Bose–Einstein condensation and Bogoliubov's theory; in particular, we prove that the many-body dynamics are effectively described by the solution of the NLS, and that the kinetic energy of the system

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diverges in finite time. To our knowledge, this is the first result of this kind for dilute Bose gases in the instability regime.

1.1. ***Mathematical setting.*** In the framework of many-body quantum physics, the dynamics of a system of N (spinless) bosons in $\mathbb{R}^2$ can be described by the linear N-body Schrödinger equation

$$\begin{cases} \mathrm{i}\partial_t \Psi_N(t) = H_N \Psi_N(t), \\ \quad \Psi_N(0) = \Psi_{N,0}, \end{cases} \tag{1-1}$$

where the wave function $\Psi_N(t)$ belongs to $L^2_s(\mathbb{R}^{2N})$, the space of square integrable functions of N variables in $\mathbb{R}^2$ satisfying the bosonic symmetry

$$\Psi_N(t, x_1, \dots, x_N) = \Psi_N(t, x_{\sigma(1)}, \dots, x_{\sigma(N)}) \quad \text{for all } \sigma \in S_N \text{ and } x_i \in \mathbb{R}^2, \tag{1-2}$$

where S_N denotes the set of all permutations of $\{1, \dots, N\}$. We will work on a nonrelativistic system with short-range interactions, where the underlying Hamiltonian is typically given by

$$H_N = \sum_{j=1}^{N} (-\Delta_j) + \frac{1}{N-1} \sum_{1 \le j < k \le N} w_N(x_j - x_k), \tag{1-3}$$

where

$$w_N(x) = N^{2\beta} w(N^\beta x), \quad \beta > 0, \tag{1-4}$$

with a real-valued, even and bounded potential w. We do not impose any positivity condition on w; in particular, the attractive case $w \le 0$ is allowed.

When w is bounded, the Hamiltonian H_N is self-adjoint on $L^2_s(\mathbb{R}^{2N})$ with the same domain as the noninteracting Hamiltonian. Therefore, the linear Schrödinger equation (1-1) has a unique global solution $\Psi_N(t) = e^{-\mathrm{i}tH_N}\Psi_N(0)$ with $t \in \mathbb{R}$, for every initial state $\Psi_N(0) \in L^2_s(\mathbb{R}^{2N})$. The major challenge in the analysis of (1-1) is that the relevant dimension grows fast as $N \to \infty$, making it very difficult to extract helpful information about the quantum system. Therefore, in practice, it is desirable to obtain collective descriptions by reasonable approximations, based on suitable assumptions on the initial state. In the present work, we will assume that the initial state exhibits the Bose–Einstein condensation (BEC), and that the particles outside of the BEC have bounded kinetic energy. These assumptions allow a rigorous derivation of effective nonlinear equations describing the BEC and the excitations which are computable by numerical methods.

Roughly speaking, Bose–Einstein condensation (BEC) is the phenomenon where many particles occupy a common quantum state. In particular, this is the case when the N-body wave function is approximately given by a factorized state, namely

$$\Psi_N(t, x_1, x_2, \dots, x_N) \approx \varphi(t, x_1)\varphi(t, x_2)\cdots\varphi(t, x_N) \tag{1-5}$$

in an appropriate sense. Here the normalized function $\varphi(t, \cdot) \in L^2(\mathbb{R}^2)$ describes the condensate, and its evolution is governed by the cubic nonlinear Schrödinger equation (NLS)

$$\begin{cases} \mathrm{i}\partial_t \varphi(t,x) = \big(-\Delta_x + b\,|\varphi(t,x)|^2 - \mu(t)\big)\varphi(t,x), \\ \qquad \varphi(0,x) = \varphi_0(x), \end{cases} \tag{1-6}$$

where

$$b=\int_{\mathbb{R}^2} w,\quad \mu(t)=\tfrac{1}{2}b\int_{\mathbb{R}^2}|\varphi(t,x)|^4\,\mathrm{d}x. \tag{1-7}$$

The equation (1-6) can be formally obtained from (1-1) using the assumption (1-5) and the fact that $w_N(x)=N^{2\beta}w(N^\beta x)\to b\delta(x)$ weakly.

The coupling constant $b=\int w$ plays a crucial role in (1-6). The focusing case $b<0$ and the defocusing case $b>0$ correspond to rather different physical situations. In particular, we are interested in the focusing case where the NLS (1-6) may blow up in finite time, even if the initial datum $\varphi(0)$ is smooth. The possibility of the finite-time blow up is closely related to instability, which we will explain below.

1.2. ***Stability vs. instability.*** Since the 2D cubic NLS (1-6) is mass critical, it is well-known from [Weinstein 1983] that the possibility of the finite-time blow up for H^1-solution depends not only on the sign of the interaction, but also on its strength. To be precise, let us denote the critical interaction strength as the optimal constant $a^*>0$ in the Gagliardo–Nirenberg interpolation inequality

$$\Big(\int_{\mathbb{R}^2}|\nabla f(x)|^2\,\mathrm{d}x\Big)\Big(\int_{\mathbb{R}^2}|f(x)|^2\,\mathrm{d}x\Big)\ge \tfrac{1}{2}a^*\int_{\mathbb{R}^2}|f(x)|^4\,\mathrm{d}x\quad \text{for all } f\in H^1(\mathbb{R}^2). \tag{1-8}$$

Equivalently, $a^*=\|Q\|_{L^2}^2$ where Q is the unique positive solution of

$$-\Delta Q+Q-Q^3=0\quad\text{in } \mathbb{R}^2$$

(see [McLeod and Serrin 1987; Kwong 1989]). From [Weinstein 1983, Theorems 3.1 and 4.2], we have two distinct regimes:

- *NLS stability regime*: $b>-a^*$. The NLS (1-6) has a unique global solution for all initial data $\varphi_0\in H^1(\mathbb{R}^2)$ satisfying $\|\varphi_0\|_{L^2}=1$.
- *NLS instability regime*: $b<-a^*$. A finite-time blow up occurs, for example, if the initial data $\varphi_0\in H^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2;|x|^2\,\mathrm{d}x)$ satisfy $\|\varphi_0\|_{L^2}=1$ and

$$\int_{\mathbb{R}^2}|\nabla\varphi_0(x)|^2\,\mathrm{d}x+\tfrac{1}{2}b\int_{\mathbb{R}^2}|\varphi_0(x)|^4\,\mathrm{d}x<0. \tag{1-9}$$

In the instability regime, we refer to [Merle and Raphael 2004] for the universality of the blow-up profile, [Merle and Raphael 2005] for a precise description of the solutions near the blow-up time, and [Merle and Raphael 2003; 2006; Raphael 2005] for works on the blow-up rate. We also refer to [Merle 1993] for a complete characterization of the minimal-mass blow-up solutions in the special case $b=-a^*$.

Unlike the NLS (1-6), for the N-body quantum dynamics (1-1), the solution $\Psi_N(t)$ exists globally for every L^2-initial datum. Nevertheless, we can still discuss stability and instability regimes by considering the boundedness of the energy per particle.

- *Many-body stability regime*: The system is stable of the second kind if

$$H_N\ge -CN \tag{1-10}$$

for some constant $C>0$ independent of N (see [Lieb and Seiringer 2010]). In principle, (1-10) is stronger than the NLS stability. By testing (1-10) against factorized states, we see that $\int w\ge -a^*$. However, the condition $\int w \geq -a^*$, or even $\int |w_-| < a^*$ with $w_- = \min\{w, 0\}$ the negative part of w, does not imply the many-body stability (1-10), except if $\beta \leq \frac{1}{2}$ [Lewin 2015]. The range of β guaranteeing (1-10) can be improved for trapped systems; see [Lewin et al. 2016; 2017; 2018; Nam and Rougerie 2020].

- *Many-body instability regime*: If $\int w < -a^*$, then (1-10) fails to hold. More precisely, we have

$$H_N \geq -CN^{1+2\beta}, \tag{1-11}$$

and the optimality of (1-11) can be seen by testing against factorized states and using $\|w_N\|_{L^\infty} \sim CN^{2\beta}$. In particular, (1-11) allows the energy per particle to diverge to $-\infty$ as $N \to \infty$, which is consistent with blow up of the NLS (1-6).

Our goal is to make a rigorous connection from the many-body Schrödinger equation (1-1) to the NLS (1-6) in the instability regime.

1.3. ***Derivation of NLS from many-body dynamics.*** The rigorous derivation of the NLS from the many-body Schrödinger equation (1-1) has been studied since the 1970s, initiated by Hepp [1974], Ginibre and Velo [1979] and Spohn [1980], and has gained renewed interest since the 2000s with important developments including the derivation of the Gross–Pitaevskii equation in 3D by Erdős, Schlein and Yau [Erdős et al. 2009; 2010]. We refer to [Benedikter et al. 2016] for a pedagogical introduction to the subject and a detailed discussion of the literature. In particular, in the defocusing case ($w \geq 0$), we refer to [Kirkpatrick et al. 2011; Jeblick et al. 2019] for the derivation of the 2D NLS (1-6) and [Chen and Holmer 2017; Boßmann 2020] for the derivation of the effectively 2D dynamics of strongly confined 3D systems.

In the focusing case ($w \leq 0$) in 2D, most of the existing works in the literature are based on the stability condition $\int |w_-| < a^*$. In this case, the focusing NLS (1-6) is globally well-posed, and its derivation from the many-body equation (1-1) was given by Chen and Holmer [2017] and Jeblick and Pickl [2018] under the technical addition of a trapping potential like $V(x) = |x|^s$, enabling them to use the many-body stability (1-10) for $0 < \beta < (s+1)/(s+2)$ by [Lewin et al. 2017]. Since the stability (1-10) was later extended to trapped systems for $0 < \beta < 1$ [Nam and Rougerie 2020], the approaches in [Chen and Holmer 2017; Jeblick and Pickl 2018] are conceptually applicable for that range of β. After that, Nam and Napiórkowski [2019] removed the trapping potential and derived (1-6) for all $0 < \beta < 1$, still under the crucial assumption $\int |w_-| < a^*$.

In the present paper, we will give a novel derivation of the focusing NLS (1-6) which covers arbitrarily negative potentials w and all $\beta \in \left(0, \frac{3}{2}\right)$. Without the stability condition $\int |w_-| < a^*$, one only has the very weak bound (1-11) instead of (1-10), and to our knowledge, the derivation of the NLS (1-6) prior to the blow-up time is only available for $\beta < \frac{1}{2}$, following the methods in [Pickl 2010; Chen and Holmer 2017; Nam and Napiórkowski 2017a; 2019; Jeblick and Pickl 2018; Chong 2021]. Although we do not expect our extended range $\beta \in \left(0, \frac{3}{2}\right)$ to be optimal, it is sufficiently large and in particular covers the physical setting of dilute Bose gases where $\beta > \frac{1}{2}$. Actually, we will derive (1-6) from a stronger result, namely a norm approximation of the many-body quantum dynamics also describing the fluctuations around the condensate in the spirit of Bogoliubov's theory. That result requires further notation and explanation, which we defer to the next section.

We conjecture that our results hold for all $\beta \in (0, \infty)$, and also for $\beta = \beta_N \to 0$ slowly such that $\lim_{N\to\infty} \log(N^{\beta})/N = 0$. The latter scaling regime was considered in [Caraci 2021] for the repulsive case $w \geq 0$. For the repulsive potential $w \geq 0$, it is also possible to consider the critical scaling regime with $w_N(x) = e^{2N} w(N^{\beta} x)$. In this so-called Gross–Pitaevskii regime, it was proved in [Jeblick et al. 2019] that the correlations at short distance leads to a subtle correction to the leading order where the coupling constant b in (1-6) must be replaced by the zero-scattering energy of w. It is natural to expect a similar result for the attractive case $w \leq 0$, but this remains an open problem.

It is also interesting to consider the derivation of the focusing NLS in one and in three dimensions. In three dimensions, the focusing cubic NLS always has finite-time blow-up for all strength of the interaction, and the asymptotic behavior of the many-body quantum dynamics has been established for $0 < \beta < \frac{1}{3}$ in [Nam and Napiórkowski 2017a; Chong 2021]. It remains open to understand the case $\beta > \frac{1}{3}$. In one dimension, the focusing cubic NLS is globally well-posed (it is mass subcritical), and the norm approximation of the many-body quantum dynamics for all $\beta > 0$ has been derived in [Nam and Napiórkowski 2019]. It is possible to obtain the finite-time blow-up in one dimension by considering a quantum system with three-body interactions, and we expect that the techniques introduced in the present paper is also helpful for the corresponding problem (see, e.g., [Nguyen and Ricaud 2024] for a related model in the stationary setting).

Finally, we refer to [Michelangeli and Schlein 2012] for a pioneering study on the rigorous understanding of the many-body dynamical instability for bosons. This work is based on a different setting where the particles have a relativistic dispersion law and an attractive potential of the form $|x|^{-1}$, ensuring that the corresponding Hartree theory has finite-time blow-up for a sufficiently large interaction coupling constant. On one hand, the general idea of proving the instability by Fock space from [Michelangeli and Schlein 2012] is very helpful for us (see, in particular, Corollary 4). On the other hand, on the technical side, the analysis in [Michelangeli and Schlein 2012] does not extend to our case. More precisely, while the N-independent interaction potential considered in [Michelangeli and Schlein 2012] places the system in a mean-field regime, controlling the N-dependent potential will be the main task of our approach.

2. Main results

Recall that we consider the Schrödinger equation (1-1) with the Hamiltonian H_N given in (1-3), where $w_N(x) = N^{2\beta} w(N^{\beta} x)$ as in (1-4). We will give rigorous descriptions of the macroscopic behavior of the many-body dynamics $\Psi_N(t) = e^{-itH_N} \Psi_{N,0}$ when $N \to \infty$, including the NLS (1-6) as the leading-order approximation, and a norm approximation in $L^2_s(\mathbb{R}^{2N})$ as the second-order approximation.

We always impose the following condition on the interaction potential:

Assumption 1. Let $w \in L^{\infty}(\mathbb{R}^2)$ be compactly supported and $w(x) = w(-x) \in \mathbb{R}$.

We do not put any assumption on the sign and the size of w.

2.1. ***Derivation of the NLS.*** Let us recall the following well-known result concerning the NLS (1-6) (see, e.g., [Cazenave 2003, Theorem 4.10.1]):

Lemma 2. *For every $b \in \mathbb{R}$ and $\varphi_0 \in H^1(\mathbb{R}^2)$ with $\|\varphi_0\|_{L^2} = 1$, there exists a unique solution*

$$\varphi \in C([0, T_{\max}), H^1(\mathbb{R}^2))$$

of (1-6) *with a unique maximal time $T_{\max} \in (0, \infty]$. Moreover, if $T_{\max} < \infty$, then*

$$\lim_{t \nearrow T_{\max}} \|\varphi(t)\|_{H^1} = \infty. \tag{2-1}$$

For nontrivial interactions w, the many-body quantum state $\Psi_N(t)$ is not expected to be close to the factorized state $\varphi(t)^{\otimes N}$ in norm (see Theorem 5 below). Therefore, the leading-order approximation (1-5) has to be understood in an average sense, which can be formulated properly in terms of reduced density matrices. For every normalized vector $\Psi_N \in L^2_s(\mathbb{R}^{2N})$, its one-body density matrix $\gamma^{(1)}_{\Psi_N}$ is a nonnegative operator on $L^2(\mathbb{R}^2)$ with kernel

$$\gamma^{(1)}_{\Psi_N}(x; y) = \int_{\mathbb{R}^{2(N-1)}} \Psi_N(x, x_2, \ldots, x_N) \overline{\Psi_N(y, x_2, \ldots, x_N)} \, \mathrm{d}x_2 \cdots \mathrm{d}x_N. \tag{2-2}$$

Equivalently, it can be obtained by taking the partial trace

$$\gamma^{(1)}_{\Psi_N} = \mathrm{Tr}_{2 \to N} |\Psi_N\rangle\langle\Psi_N|. \tag{2-3}$$

Clearly, if $\Psi_N = \varphi^{\otimes N}$, then $\gamma^{(1)}_{\Psi_N} = |\varphi\rangle\langle\varphi|$ (the rank-one projection onto $\varphi \in L^2(\mathbb{R}^2)$). In general, the approximation

$$\gamma^{(1)}_{\Psi_N} \approx |\varphi\rangle\langle\varphi| \tag{2-4}$$

with respect to the trace norm is an appropriate interpretation of (1-5). Our first main result is a rigorous derivation of the NLS (1-6) from (1-1).

Theorem 3 (NLS evolution of the condensate). *Let $\beta \in \left(0, \frac{3}{2}\right)$, $0 < \alpha_1 < \min\left(\beta, \frac{1}{8}, \frac{1}{16}(3 - 2\beta)\right)$ and let w satisfy Assumption 1. Let $\varphi(t)$ be the solution of* (1-6) *on the maximal time interval* $[0, T_{\max})$ *as in Lemma 2 with initial datum $\varphi_0 \in H^4(\mathbb{R}^2)$, $\|\varphi_0\|_{L^2} = 1$. Let $\Psi_N(t)$ be the solution of* (1-1) *with a normalized initial state $\Psi_{N,0} \in L^2_s(\mathbb{R}^{2N})$ satisfying*

$$N \,\mathrm{Tr}((1 - \Delta) q \gamma^{(1)}_{\Psi_{N,0}} q) \le C, \quad q = 1 - |\varphi_0\rangle\langle\varphi_0|, \tag{2-5}$$

for some constant $C > 0$. Then for every $t \in [0, T_{\max})$, we have Bose–Einstein condensation in the state $\varphi(t)$, i.e.,

$$\mathrm{Tr}\left|\gamma^{(1)}_{\Psi_N(t)} - |\varphi(t)\rangle\langle\varphi(t)|\right| \le C_t N^{-\alpha_1} \tag{2-6}$$

for sufficiently large N, where C_t is independent of N and continuous on $[0, T_{\max})$.

The initial condition (2-5) means that at the time $t = 0$, the total kinetic energy of all excited particles outside the condensate φ_0 is bounded. Thus, there are only few excitations, which is a key assumption allowing us to control the fluctuations around the condensate $\varphi(t)$ for all $t \in [0, T_{\max})$ by using an energy method. The kinetic bound (2-5) has been proven for the ground state or low-lying excited states of trapped systems with suitable repulsive interactions; see, e.g., [Seiringer 2011; Lewin et al. 2015b].

In Theorem 3 we do not make any assumption on the sign of the potential w, but our result is mostly interesting in the focusing case $w \le 0$. It substantially extends the result in [Nam and Napiórkowski 2019] where the NLS (1-6) was derived under the stability condition $\int |w_-| < a^*$ and the smaller range $\beta \in (0,1)$ (see also [Chen and Holmer 2017; Jeblick and Pickl 2018] for earlier related results). Without the stability condition, the derivation of the NLS (1-6) prior to the blow-up time is only available for $\beta < \frac{1}{2}$, for instance by following the methods in Pickl 2010; Chen and Holmer 2017; Nam and Napiórkowski 2017a; 2019; Jeblick and Pickl 2018; Chong 2021] and using the uniform-in-N bounds on the Hartree equation which we prove in Lemma 10.

The speed of the divergence of C_t as $t \to T_{\max}$ depends on the solution of the NLS in Lemma 2, but it is beyond the scope of this paper to investigate the quantitative behavior of this nonlinear problem.

The following statement is a direct consequence of Theorem 3 and the definition of $T_{\max}$ in Lemma 2.

Corollary 4 (Many-body blow up). *We keep the same assumptions as in Theorem 3, and assume additionally that $T_{\max} < \infty$. Then there is a sequence $N(t) \in \mathbb{N}$ such that $N(t) \to \infty$ as $t \nearrow T_{\max}$ and such that*

$$\lim_{t \nearrow T_{\max}} \frac{1}{N(t)} \Big\langle \Psi_{N(t)}(t), \sum_{j=1}^{N(t)} (-\Delta_j) \Psi_{N(t)}(t) \Big\rangle = \lim_{t \nearrow T_{\max}} \operatorname{Tr}(-\Delta \gamma^{(1)}_{\Psi_N(t)}) = \infty. \tag{2-7}$$

The implication of Corollary 4 follows from a well-known argument (see [Michelangeli and Schlein 2012, Remark 2]): for every $t \in [0, T_{\max})$, the trace convergence in Theorem 3 and Fatou's lemma imply that

$$\liminf_{N \to \infty} \operatorname{Tr}((1-\Delta)\gamma^{(1)}_{\Psi_N}(t)) \ge \operatorname{Tr}\big((1-\Delta)|\varphi(t)\rangle\langle\varphi(t)|\big) = \|\varphi(t)\|^2_{H^1(\mathbb{R}^2)}. \tag{2-8}$$

Therefore, if $T_{\max} < \infty$, then the one-body blow-up condition (2-1) implies the many-body blow-up result (2-7). Note that (2-8) is only an inequality, hence the reverse direction, which would imply that the many-body blow-up phenomenon does not occur at any fixed time $t \in [0, T_{\max})$, cannot be deduced from Theorem 3. We expect that this holds true, but a proof would require some additional analysis, which we will not pursue in the present work. Moreover, it is an open question whether the number of the excitations blows up as $t \to T_{\max}$.

We will derive Theorem 3 from a stronger result, namely the norm convergence of the many-body dynamics (see Theorem 5 below). On the technical side, it would be interesting if one could prove Theorem 3 directly using an analysis at the level of density matrices, but we do not see how to achieve this (without going to the norm approximation).

2.2. ***Norm approximation.*** Let us now discuss the fluctuations around the condensate. For this purpose, we first introduce the Hartree-type equation

$$\begin{cases} i\partial_t u_N(t,x) = \big(-\Delta_x + (w_N * |u_N(t,\cdot)|^2)(x) - \mu_N(t)\big) u_N(t,x) =: h(t) u_N(t,x), \\ \qquad\qquad u_N(0,x) = \varphi_0(x), \end{cases} \tag{2-9}$$

with

$$\mu_N(t) = \frac{1}{2} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} |u_N(t,x)|^2 w_N(x-y) |u_N(t,y)|^2 \, dx \, dy. \tag{2-10}$$

The Hartree dynamics (2-9) plays the same role as the NLS dynamics (1-6) in the leading-order description, but using the former is slightly more natural for the second-order approximation (see Lewin et al. 2015a;

Nam and Napiórkowski 2017a; 2017b; 2019; Brennecke et al. 2019] for a similar choice). In particular, (2-9) has a unique global solution, and $\|u_N(t)\|_{H^1}$ is bounded uniformly in N and locally in time when $t \in [0, T_{\max})$, with $T_{\max}$ given in Lemma 2. Moreover, since $u_N(t) \to \varphi(t)$ in $L^2(\mathbb{R}^2)$ as $N \to \infty$, the convergence (2-6) remains true if $\varphi(t)$ is replaced by $u_N(t)$ (see Lemma 10 for the details).

To describe the excitations around the condensate, it is convenient to switch to a Fock space setting where the number of particles is not fixed. Let us introduce the one-body excited space

$$\mathfrak{H}_\perp(t) = \{u_N(t)\}^\perp \subset \mathfrak{H} = L^2(\mathbb{R}^2) \tag{2-11}$$

and the (bosonic) Fock spaces over $\mathfrak{H}_\perp$

$$\mathcal{F}_\perp^{\leq N}(t) = \bigoplus_{k=0}^{N} \bigotimes_{\text{sym}}^{k} \mathfrak{H}_\perp \subset \mathcal{F}_\perp(t) = \bigoplus_{k\geq 0} \bigotimes_{\text{sym}}^{k} \mathfrak{H}_\perp \subset \mathcal{F} = \bigoplus_{k\geq 0} \bigotimes_{\text{sym}}^{k} \mathfrak{H}. \tag{2-12}$$

Note that $\mathcal{F}_\perp(t)$ and its subspace $\mathcal{F}_\perp^{\leq N}(t)$ are time-dependent via $u_N(t)$, and they are naturally embedded in the full Fock space $\mathcal{F}$ over $\mathfrak{H}$.

Let us recall the standard second quantization formalism, where the creation and annihilation operators on $\mathcal{F}$, $a^\dagger(f)$ and $a(f)$, are defined by

$$(a^\dagger(f)\boldsymbol{\chi})^{(k)}(x_1, \dots, x_k) = \frac{1}{\sqrt{k}} \sum_{j=1}^{k} f(x_j)\chi^{(k-1)}(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_k) \quad \text{for all } k \geq 1, \tag{2-13a}$$

$$(a(f)\boldsymbol{\chi})^{(k)}(x_1, \dots, x_k) = \sqrt{k+1} \int_{\mathbb{R}^2} \mathrm{d}x\, \overline{f(x)}\chi^{(k+1)}(x_1, \dots, x_k, x) \quad \text{for all } k \geq 0 \tag{2-13b}$$

for all $f \in L^2(\mathbb{R}^2)$ and $\boldsymbol{\chi} = (\chi^{(k)})_{k=0}^{\infty} \in \mathcal{F}$. It is also convenient to introduce the operator-valued distributions $a_x^\dagger$, a_x by

$$a^\dagger(f) = \int \mathrm{d}x\, f(x) a_x^\dagger, \quad a(f) = \int \mathrm{d}x\, \overline{f(x)} a_x, \tag{2-14}$$

which satisfy the canonical commutation relations

$$[a_x, a_y^\dagger] = \delta(x-y), \quad [a_x, a_y] = [a_x^\dagger, a_y^\dagger] = 0. \tag{2-15}$$

Using this language, we define the second quantization of one- and two-body operators as

$$\begin{aligned} \mathrm{d}\Gamma_1(T) &= 0 \oplus \bigoplus_{k\geq 1} \sum_{j=1}^{k} T_j = \iint T(x; x') a_x^\dagger a_{x'}\, \mathrm{d}x\, \mathrm{d}x', \\ \mathrm{d}\Gamma_2(S) &= 0 \oplus 0 \oplus \bigoplus_{k\geq 2} \sum_{1\leq i<j\leq k} S_{ij} = \tfrac{1}{2} \iiiint S(x, y; x', y') a_x^\dagger a_y^\dagger a_{x'} a_{y'}\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}x'\, \mathrm{d}y', \end{aligned} \tag{2-16}$$

for $T(x, x')$ and $S(x, y; x, y')$ the kernels of the operators T on $\mathfrak{H}$ and S on $\mathfrak{H}^2$ (see, e.g., [Solovej 2014, Section 7]). In this language, the Hamiltonian (1-3) can be expressed equivalently as

$$H_N = \mathrm{d}\Gamma_1(-\Delta) + \frac{1}{N-1}\mathrm{d}\Gamma_2(w_N) \tag{2-17}$$

on $\mathfrak{H}^N$. We also introduce the number operator on Fock space $\mathcal{F}$,

$$\mathcal{N} = \mathrm{d}\Gamma_1(\mathbb{1}), \tag{2-18}$$

where $\mathbb{1}$ this is the identity operator on $\mathfrak{H}$, and define the cut-off functions

$$\mathbb{1}^{\le m} = \mathbb{1}(\mathcal{N} \le m), \quad \mathbb{1}^{> m} = \mathbb{1}(\mathcal{N} > m) \qquad \text{for all } m \in (0,\infty). \tag{2-19}$$

Following the approach in [Lewin et al. 2015a; 2015b], the N-body dynamics $\Psi_N(t) \in L_s^2(\mathbb{R}^{2N})$ can be decomposed as

$$\Psi_N(t) = \sum_{k=0}^{N} u_N(t)^{\otimes(N-k)} \otimes_s \phi_N^{(k)}(t) = \sum_{k=0}^{N} \frac{a^\dagger(u_N(t))^{\otimes(N-k)}}{\sqrt{(N-k)!}} \phi_N^{(k)}(t) \tag{2-20}$$

for $\otimes_s$ the symmetric tensor product and where the vector

$$\Phi_N(t) = (\phi_N^{(k)}(t))_{k=0}^{N} \in \mathcal{F}_\perp^{\le N}(t) \subset \mathcal{F}_\perp(t) \tag{2-21}$$

describes the excitations around the condensate $u_N(t)$ (see Section 3 for details).

Our goal is to approximate the N-body dynamics $\Phi_N(t)$ by the solution $\Phi(t)$ of the simpler evolution equation

$$\begin{cases} \mathrm{i}\partial_t \Phi(t) = \mathbb{H}(t)\Phi(t), \\ \quad \Phi(0) = \Phi_0, \end{cases} \tag{2-22}$$

where $\mathbb{H}(t)$ denotes the Bogoliubov Hamiltonian

$$\mathbb{H}(t) = \mathrm{d}\Gamma_1(h(t) + K_1(t)) + \tfrac{1}{2}\left(\iint K_2(t,x,y) a_x^\dagger a_y^\dagger \,\mathrm{d}x\,\mathrm{d}y + \text{h.c.}\right). \tag{2-23}$$

In (2-23), $h(t)$ is defined in the Hartree equation (2-9), and

$$K_1(t) = q(t)\widetilde{K}_1(t)q(t), \quad K_2(t) = q(t) \otimes q(t)\widetilde{K}_2(t), \tag{2-24}$$

where

$$q(t) = 1 - p(t) = 1 - |u_N(t)\rangle\langle u_N(t)| \tag{2-25}$$

and the kernel of the operator $\widetilde{K}_1(t)$ and the function $\widetilde{K}_2(t) \in \mathfrak{H}^2$ are given by

$$\begin{aligned} \widetilde{K}_1(t,x,y) &= u_N(t,x) w_N(x-y)\overline{u_N(y)}, \\ \widetilde{K}_2(t,x,y) &= u_N(t,x) w_N(x-y) u_N(y). \end{aligned} \tag{2-26}$$

The effective generator $\mathbb{H}(t)$ emerges from the Bogoliubov approximation when we write H_N in the second quantization formalism, then implement the c-number substitution $a(u_N), a^\dagger(u_N) \mapsto \sqrt{N}$, and finally keep only the terms that are quadratic in creation and annihilation operators. Note that $\mathbb{H}(t)$ is an operator on the full Fock space $\mathcal{F}$ since $h(t)$ does not leave $\mathfrak{H}_\perp(t)$ invariant, but it does not contradict the fact that $\Phi(t) \in \mathcal{F}_\perp(t)$ (see, e.g., [Lewin et al. 2015a] for a detailed explanation). Moreover, $\mathbb{H}(t)$ is N-dependent, although we do not make this explicit in the notation. The Bogoliubov equation (2-22) is globally well-posed (see Lemma 7).

Now we are ready to state our second main result.

Theorem 5 (Bogoliubov excitations from the condensate). *Let $\beta \in \left(0, \frac{3}{2}\right)$, $0 < \alpha_2 < \min\left(\frac{1}{8}, \frac{1}{16}(3-2\beta)\right)$ and let w satisfy Assumption 1. Let $u_N(t)$ be the solution of the Hartree equation* (2-9) *with initial datum $\varphi_0 \in H^4(\mathbb{R}^2)$, $\|\varphi_0\| = 1$. Let $\Phi(t) = (\phi^{(k)}(t))_{k=0}^{\infty} \in \mathcal{F}_\perp(t)$ be the solution of the Bogoliubov equation* (2-22) *with initial datum $\Phi_0 = (\phi_0^{(k)})_{k=0}^{\infty} \in \mathcal{F}_\perp(0)$ satisfying $\|\Phi_0\| = 1$ and*

$$\langle \Phi_0, \mathrm{d}\Gamma_1(1-\Delta)\Phi_0 \rangle \le C \tag{2-27}$$

for some constant $C \ge 0$. Let $\Psi_N(t)$ the solution of the Schrödinger equation (1-1) *with initial datum*

$$\Psi_{N,0} = \sum_{k=0}^{N} {\varphi_0}^{\otimes(N-k)} \otimes_s \phi_0^{(k)} = \sum_{k=0}^{N} \frac{a^\dagger(\varphi_0)^{\otimes(N-k)}}{\sqrt{(N-k)!}} \phi_0^{(k)}. \tag{2-28}$$

Then, for all $t \in [0, T_{\max})$, we have the norm approximation

$$\left\| \Psi_N(t) - \sum_{k=0}^{N} u_N(t)^{\otimes(N-k)} \otimes_s \phi^{(k)}(t) \right\| \le C_t N^{-\alpha_2}, \tag{2-29}$$

where the constant C_t is independent of N and continuous in $t \in [0, T_{\max})$.

Under the decomposition (2-28), the kinetic condition (2-5) is equivalent to condition (2-27) in Theorem 5 (see Remark 6 below). Strictly speaking, the state $\Phi_{N,0}$ in (2-28) is not normalized in $L_s^2(\mathbb{R}^{2N})$, but the condition (2-27) ensures that

$$1 \ge \|\Psi_{N,0}\|^2 = 1 - \|\mathbb{1}_{\{\mathcal{N}>N\}}\Phi_0\|^2 \ge 1 - \langle \Phi_0, (\mathcal{N}/N)\Phi_0 \rangle \ge 1 - CN^{-1}. \tag{2-30}$$

In the mean-field regime $\beta = 0$, the norm approximation in the form (2-29) was first given in [Lewin et al. 2015a], and higher-order corrections to Bogoliubov's theory were recently derived in [Boßmann et al. 2021]. For repulsive interaction $w \ge 0$, the validity of Bogoliubov's theory in 3D was extended to $0 < \beta < 1$ in [Brennecke et al. 2019] and $\beta = 1$ in [Caraci et al. 2025] (see also [Nam and Napiórkowski 2017a; 2017b] for earlier results). We expect that the ideas from these works in 3D apply to handle the repulsive case in 2D as well, possibly allowing a larger value of β in 2D. Our result is mainly interesting in the attractive case $w \le 0$ in 2D, where the validity of Bogoliubov's theory was known only for $0 < \beta < 1$ in the stability regime $\int |w_-| > -a^*$ [Nam and Napiórkowski 2019].

There have been many works devoted to the dynamics around the coherent states in Fock space, initiated in [Hepp 1974; Ginibre and Velo 1979; Grillakis et al. 2010; 2011; Boccato et al. 2017]. Our method is also applicable to this setting, but we skip the details. We refer to [Lewin et al. 2015a] for a detailed comparison between the N-body setting and the Fock space situation, and [Benedikter et al. 2016] for further results.

The ideas of our proof are explained in the next section, and the full technical details are provided afterwards.

Notation. We will use $C > 0$ for a general constant which may depend on w and φ_0 and which may vary from line to line. We also use the notation C_t to highlight the time dependence. When it is unambiguous, we abbreviate the L^2-norm and the corresponding inner product by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$, respectively.

3. Proof strategy

In this section we explain the main ingredients of the proof. We will focus on Theorem 5, which implies Theorem 3. Our approach is based on Bogoliubov's approximation where the fluctuations around the condensate are effectively described by an evolution equation with a quadratic generator in Fock space. The main mathematical challenge is to justify this approximation by rigorous estimates. Let us first give an overview of the proof strategy, and then we come to the detailed setting.

As an important input of Bogoliubov's theory [1947], we expect that most particles are in the condensate $u_N(t)$, which is governed by the Hartree equation (2-9). The first step in our analysis is to establish several uniform-in-N bounds for the Hartree dynamics, which is nontrivial due to the instability issue. These one-body estimates require a careful adaptation of the analysis of the NLS (1-6) in [Cazenave 2003], which will be discussed in Section 4. In the following, we will focus on the many-body aspects of the proof.

In order to extract the excitations, namely the particles outside the condensate, from the N-body wave function $\Psi_N(t)$, we use the unitary transformation $U_N(t)$ introduced in [Lewin et al. 2015b]. This is a mathematical tool to implement Bogoliubov's c-number substitution [1947], resulting in the evolution $\Phi_N(t) = U_N(t)\Psi_N(t)$ on the excited Fock space $\mathcal{F}_\perp^{\le N}(t)$ where the generator $\mathcal{G}_N(t)$ was computed explicitly in [Lewin et al. 2015a]. Thus, we can rewrite (2-29) in terms of excitations as

$$\|\Phi_N(t) - \Phi(t)\|^2 \le C_t N^{-\alpha_2} \tag{3-1}$$

for all $t \in [0, T_{\max})$, where $\Phi(t)$ is the solution to the Bogoliubov equation (2-22).

The main difficulty in proving (3-1) is the lack of the stability of the second kind (1-10). More precisely, with an arbitrarily negative potential w, we do not expect to have a good lower bound for the generator $\mathcal{G}_N(t)$ of $\Phi_N(t)$, which in turn prevents us from obtaining a good kinetic bound for $\Phi_N(t)$. A key observation in [Nam and Napiórkowski 2019] is that a weaker version of the stability (1-10) holds if we restrict to a space of few excitations. Rigorously, for the truncated dynamics $\Phi_{N,M}(t) \in \mathcal{F}_\perp^{\le M}(t)$ which is associated to the generator $\mathbb{1}^{\le M}\mathcal{G}_N(t)\mathbb{1}^{\le M}$ with a parameter $M = N^{1-\delta}$, $\delta \in (0,1)$, it was proved in [Nam and Napiórkowski 2019] that $\Phi_{N,M}$ satisfies an essentially uniform kinetic bound, and hence $\|\Phi_{N,M}(t) - \Phi(t)\|$ can be controlled efficiently (see Lemma 7 below).

Thus, by the triangle inequality, the main missing ingredient for (3-1) is a good estimate for the norm $\|\Phi_N(t) - \Phi_{N,M}(t)\|$. For this term, we cannot use the analysis in [Nam and Napiórkowski 2019], which crucially relies on the stability condition $\int |w_-| < a^*$. The main novelty of the present paper is the introduction of a new method which does not require any information about the full dynamics Φ_N. This kind of idea was previously used in [Nam and Napiórkowski 2017a], where various propagation bounds were established by Cauchy–Schwarz inequalities of the form

$$|\langle \Phi_N, A\Phi_{N,M}\rangle| \le \|\Phi_N\| \, \|A\Phi_{N,M}\|. \tag{3-2}$$

However, this approach is insufficient to handle the dilute regime where $\beta > \frac{1}{2}$ because the Hilbert–Schmidt norm of the operator with integral kernel $w_N(x-y)u_N(x)u_N(y)$ diverges as N^β. An alternative would be to follow the approach in [Nam and Napiórkowski 2019], which avoids the Cauchy–Schwarz argument described above and uses a different strategy, relying on the a priori estimate $\langle \Phi_N, \mathrm{d}\Gamma_1(1-\Delta)\Phi_N\rangle \leq C_{t,\varepsilon}(N+N^{2\beta})$ ([Nam and Napiórkowski 2019, Lemma 10]. However, in the instability regime, we do not have the a priori bound (1-10) but only (1-11). This leads to the much weaker kinetic estimate $\langle \Phi_N, \mathrm{d}\Gamma_1(1-\Delta)\Phi_N\rangle \leq CN^{1+2\beta}$, which is not sufficient to close the argument in [Nam and Napiórkowski 2019] for $\beta > \frac{1}{2}$.

To improve the Cauchy–Schwarz argument, we write $1 = \mathcal{W}^{-1}\mathcal{W}$ with a suitable weight $\mathcal{W} > 0$—eventually we choose $\mathcal{W} = \big(1+\mathrm{d}\Gamma_2(|w_N|)\big)^{1/2}$; see (3-22)—and split (3-2) into

$$|\langle \Phi_N, A\Phi_{N,M}\rangle| \leq |\langle \Phi_N, \mathcal{W}^{-1}A\mathcal{W}\Phi_{N,M}\rangle| + \big|\langle \Phi_N, \mathcal{W}^{-1}[\mathcal{W}, A]\Phi_{N,M}\rangle\big|. \tag{3-3}$$

The first term on the right-hand side of (3-3) looks similar to $|\langle \Phi_N, A\Phi_{N,M}\rangle|$ but it is easier to bound by the Cauchy–Schwarz inequality provided that we can bound $\|A^*\mathcal{W}^{-1}\Phi_N\|$ in terms of $\|\Phi_N\|$ in an average sense. For the second term on the right-hand side, we gain some cancelation due to the commutator $[\mathcal{W}, A]$, which eventually ensures that $\|\mathcal{W}^{-1}[\mathcal{W}, A]\Phi_{N,M}\|$ is much smaller than $\|A\Phi_{N,M}\|$.

Now let us provide further details of the above ingredients.

3.1. ***Reformulation of the Schrödinger equation.*** Our starting point is a reformulation of the Schrödinger equation (1-1), following the method in [Lewin et al. 2015a; 2015b].

Let $u_N(t)$ be the Hartree evolution in (2-9). To factor out the contribution of the condensate, we use the excitation map $U_N(t) : \mathfrak{H}^N(t) \to \mathcal{F}_\perp^{\leq N}(t)$ defined by

$$U_N(t) = \bigoplus_{k=0}^{N} q(t)^{\otimes k}\left(\frac{a(u_N(t))^{N-k}}{\sqrt{(N-k)!}}\right), \tag{3-4}$$

where $q(t) = 1 - |u_N(t)\rangle\langle u_N(t)|$ as in (2-25). It was proven in [Lewin et al. 2015b] that $U_N(t)$ is a unitary transformation and its inverse is given by (2-20), namely

$$U_N(t)^*\Phi = \sum_{k=0}^{N} u_N(t)^{\otimes(N-k)} \otimes_s \phi^{(k)} = \sum_{k=0}^{N} \frac{a^\dagger(u_N(t))^{\otimes(N-k)}}{\sqrt{(N-k)!}}\phi^{(k)}$$

for all $\Phi = (\phi_k)_{k=0}^N \in \mathcal{F}_\perp^{\leq N}(t)$. Heuristically, the mapping U_N provides an efficient way of focusing on the fluctuations around the Hartree state $u_N(t)^{\otimes N}$; in particular, $U_N(t)u_N(t)^{\otimes N} = \Omega$ is the vacuum of $\mathcal{F}_\perp(t)$.

It was also proven in [Lewin et al. 2015b] that, for $f, g \in \mathfrak{H}_\perp(t)$, we have these identities on $\mathcal{F}_\perp^{\leq N}(t)$:

$$\begin{aligned} U_N a^\dagger(u_N)a(u_N)U_N^* &= N - \mathcal{N}, \\ U_N a^\dagger(f)a(u_N)U_N^* &= a^\dagger(f)\sqrt{N-\mathcal{N}}, \\ U_N a^\dagger(u_N)a(g)U_N^* &= \sqrt{N-\mathcal{N}}a(g), \\ U_N a^\dagger(f)a(g)U_N^* &= a^\dagger(f)a(g). \end{aligned} \tag{3-5}$$

If Bose–Einstein condensation holds, then, in an average sense, $\mathcal{N} \ll N$ in $\mathcal{F}_\perp^{\leq N}(t)$. Therefore, (3-5) can be interpreted as a rigorous implementation of Bogoliubov's c-number substitution [1947], where $a(u_N)$ and $a^\dagger(u_N)$ are formally replaced by the scalar number $\sqrt{N}$.

Remark 6. From (3-5) we have $U_N^* \,\mathrm{d}\Gamma_1(qAq)U_N = \mathrm{d}\Gamma_1(qAq)$ for any operator A on $\mathfrak{H}$. Consequently, (2-5) is equivalent to (2-27).

Now we consider the transformed dynamics

$$\Phi_N(t) = U_N(t)\Psi_N(t). \tag{3-6}$$

The Schrödinger equation (1-1) can be written in the equivalent form

$$\begin{cases} \mathrm{i}\partial_t \Phi_N(t) = \mathcal{G}_N(t)\Phi_N(t), \\ \Phi_N(0) = U_N(0)^*\Psi_{N,0}. \end{cases} \tag{3-7}$$

Here, the generator $\mathcal{G}_N(t)$ can be computed explicitly, using the second-quantized form (2-17) and the rules (3-5) (see [Lewin et al. 2015a, Appendix B]), as

$$\mathcal{G}_N(t) = (\mathrm{i}\partial_t U_N(t))U_N^*(t) + U_N(t)H_N U_N^*(t) = \tfrac{1}{2}\sum_{j=0}^{4} \mathbb{1}^{\le N}(\mathbb{G}_j + \mathbb{G}_j^*)\mathbb{1}^{\le N} \tag{3-8}$$

with

$$\mathbb{G}_0 = \mathrm{d}\Gamma_1(h) + \mathrm{d}\Gamma_1(K_1)\frac{N-\mathcal{N}}{N-1} + \mathrm{d}\Gamma_1(q(t)(h+\Delta)q(t))\frac{1-\mathcal{N}}{N-1}, \tag{3-9a}$$

$$\mathbb{G}_1 = -2a^\dagger\big(q(t)(w_N * |u_N(t)|^2)u_N(t)\big)\frac{\mathcal{N}\sqrt{N-\mathcal{N}}}{N-1}, \tag{3-9b}$$

$$\mathbb{G}_2 = \iint K_2(t,x,y)a_x^\dagger a_y^\dagger \,\mathrm{d}x\,\mathrm{d}y \frac{\sqrt{(N-\mathcal{N})(N-\mathcal{N}-1)}}{N-1}, \tag{3-9c}$$

$$\mathbb{G}_3 = \iiiint (q(t)\otimes q(t) w_N \mathbb{1}\otimes q(t))(x,y;x',y')u_N(t,x)a_x^\dagger a_y^\dagger a_{y'}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}x'\,\mathrm{d}y' \,\frac{\sqrt{N-\mathcal{N}}}{N-1}, \tag{3-9d}$$

$$\mathbb{G}_4 = \frac{1}{N-1}\,\mathrm{d}\Gamma_2(q(t)\otimes q(t) w_N q(t)\otimes q(t)). \tag{3-9e}$$

Recall that $h(t)$ is given in (2-9), and $K_1(t)$ and $K_2(t)$ are given in (2-24). In the above notation, w_N denotes the function $w_N : \mathbb{R}^2 \to \mathbb{R}$ in $\mathbb{G}_1(t)$, and the two-body multiplication operator $w_N(x-y)$ in $\mathbb{G}_3(t)$ and $\mathbb{G}_4(t)$.

3.2. ***Simplified equations.*** Following Bogoliubov's heuristic ideas [1947], we consider a simplification of (3-7), where only the quadratic terms $\mathbb{G}_0$ and $\mathbb{G}_2$ in the generator are kept. This leads to the Bogoliubov equation (2-22), whose well-posedness is well-known; see, e.g., [Nam and Napiórkowski 2019, Lemma 5].

Lemma 7 (Bogoliubov dynamics). *Let w satisfy Assumption 1, let $u_N(t)$ be the Hartree evolution in* (2-9) *with initial state $\varphi_0 \in H^4(\mathbb{R}^2)$, and let $\Phi_0 \in \mathcal{F}_\perp(0)$ be a normalized vector satisfying* (2-27). *Then the Bogoliubov equation* (2-22) *with initial condition Φ_0 has a unique global solution*

$$\Phi \in C([0,\infty),\mathcal{F}) \cap L^\infty_{\mathrm{loc}}\big((0,\infty),\mathcal{Q}(\mathrm{d}\Gamma_1(1-\Delta))\big)$$

and $\Phi(t) \in \mathcal{F}_\perp(t)$ for all $t > 0$. Moreover, for every $t \in [0, T_{\max})$ and $\varepsilon > 0$, we have

$$\langle \Phi(t), \mathrm{d}\Gamma_1(1-\Delta)\Phi(t)\rangle \le C_{t,\varepsilon} N^\varepsilon, \tag{3-10}$$

where $T_{\max}$ is given in Lemma 2 and the constant $C_{t,\varepsilon}$ is independent of N.

Proof. The global well-posedness of $\Phi(t)$ is shown in [Lewin et al. 2015a, Theorem 7]. The kinetic bound (3-10) follows from the analysis in [Nam and Napiórkowski 2019, Lemma 5] and the uniform bounds of $u_N(t)$, which will be given later in Lemma 10. □

In order to estimate the difference $\|\Phi_N(t) - \Phi(t)\|$, we follow [Nam and Napiórkowski 2019] and introduce the truncated dynamics $\Phi_{N,M}(t) \in \mathcal{F}_\perp^{\leq M}(t)$, which solve the equation

$$\begin{cases} \mathrm{i}\partial_t \Phi_{N,M}(t) = \mathbb{1}^{\leq M} \mathcal{G}_N(t) \mathbb{1}^{\leq M} \Phi_{N,M}(t), \\ \Phi_{N,M}(0) = \mathbb{1}^{\leq M} \Phi_0. \end{cases} \tag{3-11}$$

As explained in [Nam and Napiórkowski 2019], the main advantage of (3-11) is that the truncated generator is stable, namely

$$\mathbb{1}^{\leq M} \mathcal{G}_N(t) \mathbb{1}^{\leq M} \geq \tfrac{1}{2}\, \mathrm{d}\Gamma_1(-\Delta) - C_{t,\varepsilon} N^{\varepsilon} \tag{3-12}$$

for all $t \in [0, T_{\max})$ and $M \ll N$. This allows us to establish an efficient kinetic bound for $\Phi_{N,M}(t)$, which is not available for Φ_N. Consequently, it is much easier to compare $\Phi_{N,M}(t)$ with the Bogoliubov dynamics. We collect some known properties of $\Phi_{N,M}(t)$ in the following lemma.

Lemma 8 (Truncated dynamics). *We keep the assumptions of Lemma 7. Let $M = N^{1-\delta}$ for some constant $\delta \in (0,1)$. Then* (3-11) *has a unique global solution $\Phi_{N,M}(t) \in \mathcal{F}_\perp^{\leq M}(t)$ with $t \in [0,\infty)$. Moreover, for every $t \in [0, T_{\max})$ and $\varepsilon = \varepsilon(\delta) > 0$, we have*

$$\langle \Phi_{N,M}(t), \mathrm{d}\Gamma_1(1-\Delta) \Phi_{N,M}(t) \rangle \leq C_{t,\varepsilon} N^{\varepsilon} \tag{3-13}$$

and

$$\|\Phi_{N,M}(t) - \Phi(t)\|^2 \leq C_{t,\varepsilon} N^{\varepsilon} \left(\sqrt{\frac{M}{N}} + \frac{1}{M} \right). \tag{3-14}$$

Proof. The global well-posedness of $\Phi_{N,M}(t)$ follows from the general method in [Lewin et al. 2015a, Theorem 7] (see also [Nam and Napiórkowski 2019, Section 6]). Given the uniform bounds of $u_N(t)$ in Lemma 10, the bounds (3-13) and (3-14) follow from the arguments in Lemmas 11 and 15 in [Nam and Napiórkowski 2019], respectively. □

3.3. *From the truncated to the full dynamics.* Given Lemma 8, the missing piece for the proof of Theorem 5 is an estimate for $\|\Phi_N(t) - \Phi_{N,M}(t)\|$. The main new ingredient of the present paper is the following bound:

Proposition 9. *We keep the assumptions of Lemma 7. Let $M = N^{1-\delta}$ for some constant $\delta \in (0,1)$. Let Φ_N and $\Phi_{N,M}$ be solutions of* (3-7) *and* (3-11), *with initial data $\Phi_N(0) = \mathbb{1}^{\leq N}\Phi_0$, $\Phi_{N,M}(0) = \mathbb{1}^{\leq M}\Phi_0$, respectively. Then for every $t \in [0, T_{\max})$ and every $\varepsilon > 0$, we have*

$$\|\Phi_N(t) - \Phi_{N,M}(t)\|^2 \leq C_{t,\varepsilon} N^{\varepsilon} \left(\frac{1}{\sqrt{M}} + \frac{N^{\beta}}{M^{3/2}} \right). \tag{3-15}$$

Eventually, we will take $\delta > 0$ small, hence the condition $\beta < \frac{3}{2}$ is needed to ensure that the error term $N^{\beta}/M^{3/2}$ on the right-hand side of (3-15) is negligible.

In order to prove Proposition 9, by norm conservation of $\|\Phi_N(t)\|$ and $\|\Phi_{N,M}(t)\|$, it suffices to show that $\langle\Phi_N(t), \Phi_{N,M}(t)\rangle$ is close to 1. For technical reasons, it is more convenient to consider $\langle\Phi_N(t), f_M^2\Phi_{N,M}(t)\rangle$ with f_M a smoothened of version $\mathbb{1}^{\le M}$. To be precise, we fix a smooth function $f:\mathbb{R}\to[0,1]$ such that $f(s)=1$ for $s\le\frac{1}{2}$ and $f(s)=0$ for $s\ge 1$, and define the operator f_M on $\mathcal{F}$ by

$$f_M = f\left(\frac{\mathcal{N}}{M}\right). \tag{3-16}$$

We will deduce Proposition 9 from a Grönwall argument and the estimate

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\langle\Phi_N(t), f_M^2\Phi_{N,M}(t)\rangle\right| \le C_{t,\varepsilon}N^{\varepsilon}\left(\frac{1}{\sqrt{M}}+\frac{N^{\beta}}{M^{3/2}}\right). \tag{3-17}$$

It remains to explain the proof of (3-17). Let us drop the time dependence from the notation where it is unambiguous. From (3-7) and (3-11), we have

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\langle\Phi_N(t), f_M^2\Phi_{N,M}(t)\rangle\right| = \left|\Im\langle\Phi_N, [\mathcal{G}_N, f_M^2]\Phi_{N,M}\rangle\right| \tag{3-18}$$

since $\Phi_{N,M}\in\mathcal{F}_\perp^{\le M}$ and $f_M^2\mathbb{1}^{\le M}=f_M^2$. Then it is straightforward to decompose $\mathcal{G}_N$ into the sum of $\mathbb{G}_j$ as in (3-8). Since f_M is a function of $\mathcal{N}$, only the particle number nonpreserving terms $\mathbb{G}_1$, $\mathbb{G}_2$ and $\mathbb{G}_3$ contribute to the commutator.

One of the most difficult terms is the quadratic one $\left|\langle\Phi_N, [\mathbb{G}_2, f_M^2]\Phi_{N,M}\rangle\right|$, where two annihilation operators hit Φ_N. Since $\mathbb{G}_2$ only changes the number of particles by at most 2, the commutator with f_M^2 allows us to gain a factor M^{-1}. Therefore, estimating (3-18) essentially boils down to proving a bound for

$$\frac{1}{M}\left|\iint \mathrm{d}x\,\mathrm{d}y\, u_N(x)u_N(y)w_N(x-y)\langle\Phi_N, a_x^\dagger a_y^\dagger\Phi_{N,M}\rangle\right|. \tag{3-19}$$

In [Nam and Napiórkowski 2019], a variant of this term was estimated using a kinetic bound for Φ_N based on the method in [Lewin 2015] and the stability condition $\int|w_-|<a^*$. In the present paper, since we are considering a general potential w including the instability regime $\int w_- < -a^*$, we only have

$$\langle\Phi_N, \mathrm{d}\Gamma_1(1-\Delta)\Phi_N\rangle \le CN^{1+2\beta}, \tag{3-20}$$

which can be deduced from a variant of the energy lower bound (1-11). However, the latter bound is too weak, and inserting it in the analysis in [Nam and Napiórkowski 2019] produces a solution only for $\beta<\frac{1}{2}$.

Another idea, which can be extracted from the approach in [Nam and Napiórkowski 2017a], is to handle (3-19) by the Cauchy–Schwarz inequality

$$|\langle\Phi_N, a_x^\dagger a_y^\dagger\Phi_{N,M}\rangle| \le \|\Phi_N\|\,\|a_x^\dagger a_y^\dagger\Phi_{N,M}\|. \tag{3-21}$$

(To be precise, a variant of this argument was used in [Nam and Napiórkowski 2017a] to compare Φ_N directly with the Bogoliubov dynamics Φ.) The advantage of (3-21) is that no information about Φ_N is needed. However, since we have to couple (3-21) with the singular potential $w_N(x-y)$ in (3-19), we eventually obtain a large factor $\|w_N\|_{L^\infty}\sim N^{2\beta}$, and the final bound is only good for $\beta<\frac{1}{2}$.

Thus, to cover the extended range $\beta \in \big(0, \frac{3}{2}\big)$, new ideas are needed to handle (3-19). In the present paper, on the one hand, we will not rely on any information of Φ_N; moreover, instead of using directly (3-21) we will further decompose (3-19) by introducing a weight given by

$$\mathcal{R} := \mathrm{d}\Gamma_2(|w_N|) + 1 = \tfrac{1}{2} \int \mathrm{d}x\,\mathrm{d}y\,|w_N(x-y)| a_x^\dagger a_y^\dagger a_x a_y + 1. \tag{3-22}$$

By inserting $1 = \mathcal{R}^{-1/2}\mathcal{R}^{1/2}$, we can write

$$a_x^\dagger a_y^\dagger = \mathcal{R}^{-1/2} a_x^\dagger a_y^\dagger \mathcal{R}^{1/2} + \mathcal{R}^{-1/2}[\mathcal{R}^{1/2}, a_x^\dagger a_y^\dagger]. \tag{3-23}$$

Then, using the triangle inequality we can bound (3-19) by

$$\begin{aligned} &\frac{1}{M} \iint \mathrm{d}x\,\mathrm{d}y\,|u_N(x)|\,|u_N(y)|\,|w_N(x-y)|\,|\langle \Phi_N, \mathcal{R}^{-1/2} a_x^\dagger a_y^\dagger \mathcal{R}^{1/2} \Phi_{N,M}\rangle| \\ &\qquad + \frac{1}{M} \iint \mathrm{d}x\,\mathrm{d}y\,|u_N(x)|\,|u_N(y)|\,|w_N(x-y)|\,\big|\langle \Phi_N, \mathcal{R}^{-1/2}[\mathcal{R}^{1/2}, a_x^\dagger a_y^\dagger]\Phi_{N,M}\rangle\big|. \end{aligned} \tag{3-24}$$

The key point is that although the first term in (3-24) looks similar to (3-19), it is much easier to control. Indeed, by the Cauchy–Schwarz inequality, we have

$$\begin{aligned} &\frac{1}{M} \iint \mathrm{d}x\,\mathrm{d}y\,|u_N(x)|\,|u_N(y)|\,|w_N(x-y)|\,|\langle \Phi_N, \mathcal{R}^{-1/2} a_x^\dagger a_y^\dagger \mathcal{R}^{1/2} \Phi_{N,M}\rangle| \\ &\quad \le \frac{1}{M} \iint \mathrm{d}x\,\mathrm{d}y\,|u_N(x)|\,|u_N(y)|\,|w_N(x-y)|\,\|a_x a_y \mathcal{R}^{-1/2}\Phi_N\|\,\|\mathcal{R}^{1/2}\Phi_{N,M}\| \\ &\quad \le \frac{1}{M} \bigg(\iint \mathrm{d}x\,\mathrm{d}y\,|w_N(x-y)|\,\|a_x a_y \mathcal{R}^{-1/2}\Phi_N\|^2 \bigg)^{1/2} \\ &\qquad\qquad \times \bigg(\iint \mathrm{d}x\,\mathrm{d}y\,|u_N(x)|^2 |u_N(y)|^2 |w_N(x-y)| \bigg)^{1/2} \|\mathcal{R}^{1/2}\Phi_{N,M}\|. \end{aligned} \tag{3-25}$$

Then, by the definition of $\mathcal{R}$, we can bound

$$\begin{aligned} \iint \mathrm{d}x\,\mathrm{d}y\,|w_N(x-y)|\,\|a_x a_y \mathcal{R}^{-1/2}\Phi_N\|^2 &= \big\langle \Phi_N, \mathcal{R}^{-1/2}\,\mathrm{d}\Gamma_2(|w_N|)\mathcal{R}^{-1/2}\Phi_N \big\rangle \\ &\le \|\Phi_N\|^2, \end{aligned} \tag{3-26}$$

without relying on any information on Φ_N. The other factors in (3-25) can be bounded efficiently using $\|w_N\|_{L^1} \le C$ and good estimates on $\Phi_{N,M}$ and u_N, given that $\mathcal{R}$ can essentially be controlled in terms of the kinetic energy; see (5-3). All this allows us to bound (3-25) by $C_{t,\varepsilon} N^{\varepsilon}/\sqrt{M}\,\|\Phi_N\|$, which appears as the first error term on the right-hand side of (3-17).

We still have to bound the second term in (3-24). This term looks complicated, but in principle, we gain a huge cancelation from the commutator $[\mathcal{R}^{1/2}, a_x^\dagger a_y^\dagger]$ due to the fact that $\mathcal{R}$ is a "local operator". To make it more transparent, we can use

$$\mathcal{R}^{1/2} = \frac{1}{\pi} \int_0^\infty \frac{1}{\sqrt{s}} \frac{\mathcal{R}}{\mathcal{R}+s}\,\mathrm{d}s \tag{3-27}$$

to write, for any operator B,

$$[\mathcal{R}^{1/2}, B] = \frac{1}{\pi}\int_0^\infty \mathrm{d}s\, \frac{1}{\sqrt{s}}\left[\frac{\mathcal{R}}{\mathcal{R}+s}, B\right] = \frac{1}{\pi}\int_0^\infty \mathrm{d}s\, \frac{\sqrt{s}}{\mathcal{R}+s}[\mathcal{R}, B]\frac{1}{\mathcal{R}+s}. \tag{3-28}$$

In particular, a straightforward computation shows that

$$\begin{aligned}[\mathcal{R}, a_x^\dagger a_y^\dagger] &= \big[\mathrm{d}\Gamma_2(|w_N|), a_x^\dagger a_y^\dagger\big] \\ &= |w_N(x-y)| a_x^\dagger a_y^\dagger + \int \mathrm{d}z\, \big(|w_N(z-x)| + |w_N(z-y)|\big) a_x^\dagger a_y^\dagger a_z^\dagger a_z.\end{aligned} \tag{3-29}$$

Let us take $|w_N(x-y)|a_x^\dagger a_y^\dagger$ from (3-29) and insert it in (3-28). The corresponding contribution from the second term in (3-24) can be controlled by

$$\frac{1}{M}\int_0^\infty \mathrm{d}s\,\sqrt{s}\iint \mathrm{d}x\,\mathrm{d}y\,|u_N(x)|\,|u_N(y)|\,|w_N(x-y)|^2 \left\| a_x a_y \frac{\mathcal{R}^{-1/2}}{\mathcal{R}+s}\Phi_N\right\| \left\| \frac{1}{\mathcal{R}+s}\Phi_{N,M}\right\|. \tag{3-30}$$

The resolvents $(\mathcal{R}+s)^{-1}$ are important in two respects: one the one hand, they provide sufficient decay in s via the estimate $(\mathcal{R}+s)^{-1} \le (1+s)^{-1}$. On the other hand, they compensate for the singular interaction, which is similar to the argument in (3-26) although $|w_N|^2$ is way more singular than $|w_N|$.

To combine these two ideas, we apply the Cauchy–Schwarz inequality on $\mathbb{R}^2 \times \mathbb{R}^2$ to (3-30), where we write

$$|w_N|^2 = |w_N|^{1+\varepsilon/2}|w_N|^{1-\varepsilon/2} \tag{3-31}$$

for $\varepsilon > 0$ small. We estimate the term coming from $|w_N|^{1-\varepsilon/2}$ using

$$\frac{\mathcal{R}^{-1/2}}{\mathcal{R}+s}\,\mathrm{d}\Gamma(|w_N|^{2-\varepsilon})\frac{\mathcal{R}^{-1/2}}{\mathcal{R}+s} \le \frac{\mathcal{R}^{1-\varepsilon}}{(\mathcal{R}+s)^2} \le \frac{1}{(1+s)^{1+\varepsilon}}. \tag{3-32}$$

Here, we used that

$$\mathrm{d}\Gamma_2(|w_N|^{2-\varepsilon}) \le \big(\mathrm{d}\Gamma_2(|w_N|)\big)^{2-\varepsilon} \le \mathcal{R}^{2-\varepsilon}, \tag{3-33}$$

which relies heavily on the locality of $\mathcal{R}$, namely $\mathrm{d}\Gamma_2(|w_N|)$ is the second quantization of a two-body multiplication operator (see Lemma 12). For the term coming from $|w_N|^{1+\varepsilon/2}$, by calculating the L^2-norm of $|w_N|^{1+\varepsilon/2}$, which appears in (3-31), we eventually obtain $C_{t,\varepsilon}N^\varepsilon N^\beta/M^{3/2}$, the second error term on the right-hand side of (3-17). The cubic term $\langle \Phi_N, [\mathbb{G}_3, f_M^2]\Phi_{N,M}\rangle$ can be handled similarly. This completes our overview of the main ingredients of the proof.

In our analysis, the restriction to $\beta < \frac{3}{2}$ seems a purely technical limitation. We conjecture that it could be improved. For example, one would be able to cover all $\beta > 0$ if the higher-moment estimate $\langle \Phi_{N,M}, \mathcal{N}^b \Phi_{N,M}\rangle \le C_{b,\varepsilon}N^\varepsilon$ were established for all $b \ge 1$. At present, however, we are not able to show that, and we will only use the first moment bound ($b = 1$) from Lemma 8.

Organization of the paper. In Section 4, we establish uniform-in-N estimates for the Hartree dynamics. The most technical part of the paper is contained in Section 5 where we prove Proposition 9. From this, we conclude the main results in Section 6.

4. Uniform estimates for Hartree evolution

In this section, we consider the Hartree evolution u_N in (2-9). By Assumption 1, it is globally well-posed in H^k, $k \in \{1, 2, \dots\}$, for any fixed N by [Cazenave 2003, Corollary 6.1.2]. However, it is a priori not clear whether $\|u_N(t)\|_{H^k}$ is bounded uniformly in N for fixed $t \in [0, T_{\max})$. In the following lemma, we prove such uniform bounds for all times prior to the NLS blow-up time $T_{\max}$.

Lemma 10. *Let w satisfy Assumption 1. Let $\varphi_0 \in H^4(\mathbb{R}^2)$ and $T_{\max}$ be as in Lemma 2. Then, for every $T \in [0, T_{\max})$, there exists a constant $C = C(T, \varphi_0) > 0$ such that, for all $t \in [0, T]$ and all N sufficiently large,*

$$\|u_N(t)\|_{L^\infty} \le C\,\|u_N(t)\|_{H^2(\mathbb{R}^2)} \le C, \quad \|\partial_t u_N(t)\|_{H^2(\mathbb{R}^2)} \le C. \tag{4-1}$$

Moreover, for $\varphi(t)$ the solution of the NLS (1-6),

$$\|u_N(t) - \varphi(t)\|_{L^2(\mathbb{R}^2)} \le C N^{-\beta}. \tag{4-2}$$

For interactions satisfying the stability condition $\int_{\mathbb{R}^2} |w_-| < a^*$, (4-1) has been shown in [Nam and Napiórkowski 2019, Lemma 4]. As explained in [Nam and Napiórkowski 2019], the key point is to get the uniform bound $\|u_N(t)\|_{H^1(\mathbb{R}^2)} \le C$, and the rest follows from a rather general argument. In the stability regime considered in [Nam and Napiórkowski 2019], this bound follows immediately from energy conservation and (1-8), namely

$$\begin{aligned}
\mathcal{E}[u_N(t)] &= \|\nabla u_N(t)\| + \tfrac{1}{2} \iint \mathrm{d}x\,\mathrm{d}y\, |u_N(t,x)|^2 w_N(x-y) |u_N(t,y)|^2 \\
&\ge \|\nabla u_N(t)\| - \tfrac{1}{2} \|(w_N)_-\|_{L^1} \|u_N(t)\|_{L^4}^4 \\
&\ge \|\nabla u_N(t)\|^2 \left(1 - \frac{\int_{\mathbb{R}^2} |w_-|}{a^*}\right).
\end{aligned} \tag{4-3}$$

For a general w in Lemma 10, the estimate (4-3) is not available, hence the estimate of $\|u_N(t)\|_{H^1(\mathbb{R}^2)}$ is more complicated. Instead of studying u_N directly as in [Nam and Napiórkowski 2019], we will focus on the difference

$$\theta_N(t) = u_N(t) - \varphi(t). \tag{4-4}$$

We will bound $\theta_N(t)$ by a bootstrap argument consisting of two steps:

(1) If $\|\theta_N(t)\| \le \delta = \sqrt{a^*/(32\,\|w\|_{L^1})}$, then $\|\nabla \theta_N(t)\| \le C$. This follows from energy conservation and the a priori bound $\|\varphi(t)\|_{H^1(\mathbb{R}^2)} \le C$ on $[0, T]$.

(2) If $\|\theta_N(s)\| \le \delta$ for all $s \in [0, t]$, then $\|\theta_N(t)\| \le N^{-\beta} \ll \delta$ for sufficiently large N. To prove this, we use Step 1 and Gronwall's lemma.

The conclusion thus follows from the continuity of the map $t \mapsto \|\theta_N(t)\|$ and the initial condition $\theta_N(0) = 0$. Now let us go to the details.

Proof of Lemma 10. For simplicity, we consider the solutions $u_N(t)$ and $\varphi(t)$ of (2-9) and (1-6) without the phases $\mu_N(t)$ and $\mu(t)$, respectively. Due to the gauge transformation $u_N \mapsto \exp\{-\mathrm{i}\int_0^t \mu(s)\,\mathrm{d}s\} u_N$, this does not change the N-dependence of the estimates (4-1). Let $\delta = \sqrt{a^*/(32\,\|w\|_{L^1})}$ with a^* as in (1-8).

Step 1. Assume that $\|\theta_N(t)\| \le \delta$ for some $t \in [0, T]$. Using the energy conservation of (2-9) and the fact $\|w_N\|_{L^1} = \|w\|_{L^1}$, we can bound

$$\begin{aligned}\mathcal{E}[u_N(t)] = \mathcal{E}[\varphi_0] &= \|\nabla\varphi_0\|^2 + \tfrac{1}{2}\iint |\varphi_0(x)|^2 w_N(x-y)|\varphi_0(y)|^2\,\mathrm{d}x\,\mathrm{d}y \\ &\le \|\nabla\varphi_0\|^2 + \tfrac{1}{2}\|w\|_{L^1}\|\varphi_0\|_{L^4}^4 \le C. \end{aligned} \tag{4-5}$$

On the other hand, using (1-8) and the assumption $\|\theta_N(t)\| \le \delta$, we can bound

$$\|\theta_N(t)\|_{L^4}^4 \le \frac{2\delta^2}{a^*}\|\nabla\theta_N(t)\|^2 = \frac{1}{16\,\|w\|_{L^1}}\|\nabla\theta_N(t)\|^2.$$

Combining with $\frac{1}{2}\|\nabla\theta_N(t)\|^2 \le \|\nabla u_N(t)\|^2 + \|\nabla\varphi(t)\|^2$, we find that

$$\begin{aligned}\mathcal{E}_N[u_N(t)] &\ge \|\nabla u_N(t)\|^2 - \tfrac{1}{2}\|w\|_{L^1}\|\theta_N(t)+\varphi(t)\|_{L^4}^4 \\ &\ge \big(\tfrac{1}{2}\|\nabla\theta_N(t)\|^2 - \|\nabla\varphi(t)\|^2\big) - 4\|w\|_{L^1}\big(\|\theta_N(t)\|_{L^4}^4 + \|\varphi(t)\|_{L^4}^4\big) \\ &\ge \tfrac{1}{4}\|\nabla\theta_N(t)\|^2 - C, \end{aligned} \tag{4-6}$$

where we used that $\|\nabla\varphi(t)\| \le C$ on $[0, T]$. Consequently, (4-5) and (4-6) imply that $\|\nabla\theta_N(t)\|^2 \le C$.

Step 2. Let $s \in [0, t]$ and assume that $\|\theta_N(s)\| \le \delta$. Then, dropping the time dependence from the notation, we find that

$$\begin{aligned}\tfrac{1}{2}\big|\partial_s\|\theta_N(s)\|^2\big| &= \big|\Im\big\langle\theta_N, (w_N * |u_N|^2)\theta_N + \big(w_N * (|u_N|^2 - |\varphi|^2)\big)\varphi + (w_N * |\varphi|^2 - b\,|\varphi|^2)\varphi\big\rangle\big| \\ &\le \|\theta_N\|\,\|\varphi\|_{L^\infty}\big(\big\|w_N * (|u_N|^2 - |\varphi|^2)\big\| + \big\|w_N * |\varphi|^2 - b\,|\varphi|^2\big\|\big) \\ &=: \|\theta_N\|\,\|\varphi\|_{L^\infty}(A_1 + A_2). \end{aligned} \tag{4-7}$$

On the right-hand side of (4-7), we have $\|\theta_N(s)\| \le \delta$ by our assumption, and $\|\varphi(s)\|_{L^\infty} \le C$ since $\varphi(s) \in H^2(\mathbb{R}^2)$ [Cazenave 2003, Theorems 5.3.1 and 5.4.1] and Sobolev's embedding $H^2(\mathbb{R}^2) \subset L^\infty(\mathbb{R}^2)$ [Lieb and Loss 2001, Theorem 8.8 (iii)].

Estimate of A_1. Using

$$\big||u_N|^2 - |\varphi|^2\big| = \big|(|u_N| - |\varphi|)(|u_N| + |\varphi|)\big| \le |\theta_N|^2 + 2\,|\varphi|\,|\theta_N|, \tag{4-8}$$

we can bound

$$A_1 \le \|w_N\|_{L^1}(\|\theta_N\|_{L^4}^2 + 2\,\|\varphi\|_{L^\infty}\|\theta_N\|) \le C\,\|\theta_N\|. \tag{4-9}$$

In the last estimate, we used (1-8) and the bound $\|\nabla\theta_N(t)\|^2 \le C$ from Step 1.

Estimate of A_2. Observing that $b = \widehat{w}_N(0)$ and $\widehat{w}_N(\xi) = \widehat{w}(\xi/N^\beta)$, Plancherel's theorem yields

$$A_2 \le \left\|\frac{\widehat{w}(\,\cdot\,/N^\beta) - \widehat{w}(0)}{|\cdot|}\right\|_{L^\infty}\big\||\cdot|\,\widehat{|\varphi|^2}\big\|_{L^2} \le CN^{-\beta}. \tag{4-10}$$

Here, we used $\|\nabla\varphi\| \le C$ and the fact that $\widehat{w}$ is Lipschitz.

In summary, inserting (4-9) and (4-10) in (4-7), we arrive at

$$\partial_s \|\theta_N(s)\|^2 \le C\big(\|\theta_N(s)\|^2 + N^{-2\beta}\big). \tag{4-11}$$

Consequently, we obtain $\|\theta_N(t)\| \le CN^{-\beta}$ by Gronwall's lemma since $\theta_N(0) = 0$.

Conclusion. Define

$$t_N^{\max} = \sup\{t \in [0,T] : \|\theta_N(t)\| \le \delta\}. \tag{4-12}$$

Assume that $t_N^{\max} < T$. By [Cazenave 2003, Theorem 4.10.1], the map $[0,T] \ni t \mapsto \|\theta_N(t)\|$ is continuous, hence $\|\theta_N(s)\| \le \delta$ for $s \in [0, t_N^{\max}]$. By Step 2, this implies that

$$\|\theta_N(t_N^{\max})\| \le CN^{-\beta} < \delta \tag{4-13}$$

for sufficiently large N, which contradicts $t_N^{\max} < T$. Hence, $t_N^{\max} \ge T$, and consequently

$$\|\theta_N(t)\| \le \delta \quad \text{for all } t \in [0,T].$$

By Step 1, we get $\|\nabla\theta_N(t)\|^2 \le C$. Therefore,

$$\|u_N(t)\|_{H^1} \le \|\theta_N(t)\|_{H^1} + \|\varphi(t)\|_{H^1} \le C \quad \text{for all } t \in [0,T]. \tag{4-14}$$

The remaining estimates in (4-1) can be deduced from the H^1-bound as in [Nam and Napiórkowski 2019, Lemma 4], using Duhamel's formula. The bound (4-2) also follows from the above argument, where the error term $N^{-\beta}$ comes from (4-10). □

5. From the truncated to the full dynamics

In this section we prove Proposition 9. As explained in Section 3.3, the key step is to prove the propagation bound (3-17). We use (3-18) and (3-8) to decompose

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\langle \Phi_N(t), f_M^2 \Phi_{N,M}(t)\rangle\right| \le \sum_{j=1}^{3} \big|\langle \Phi_N, [(\mathbb{G}_j + \mathbb{G}_j^*), f_M^2]\Phi_{N,M}\rangle\big| \tag{5-1}$$

with $\mathbb{G}_j$ given in (3-8). In the next subsections, we will handle the cases $j = 1, 2, 3$ separately, and then conclude (3-17) as well as Proposition 9.

As a preparation, let us collect here two auxiliary estimates which will be used repeatedly in this section. The first one is a simple Sobolev-type estimate.

Lemma 11. *Let $W \in L^s(\mathbb{R}^2)$ with $s \in (1,2]$ and denote by $W(x-y)$ the corresponding two-body multiplication operator. Then*

$$\mathrm{d}\Gamma_2\big(|W(x-y)|\big) \le C_s \|W\|_{L^s(\mathbb{R}^2)} \mathcal{N}\, \mathrm{d}\Gamma_1(1-\Delta) \tag{5-2}$$

as operators on $\mathcal{F}$.

In particular, Assumption 1 guarantees that $w \in L^{1+\varepsilon}(\mathbb{R}^2)$ for every $\varepsilon > 0$, hence Lemma 11 implies that

$$\mathrm{d}\Gamma_2\big(|w_N(x-y)|\big) \le C_\varepsilon N^{\varepsilon} \mathcal{N}\, \mathrm{d}\Gamma_1(1-\Delta). \tag{5-3}$$

Here we used the fact that

$$\int_{\mathbb{R}^2} |w_N(x)|^\alpha \, \mathrm{d}x = N^{2\beta(\alpha-1)} \int |w(x)|^\alpha \, \mathrm{d}x \quad \text{for all } \alpha > 0. \tag{5-4}$$

Proof. Using Sobolev's embedding $L^{s'}(\mathbb{R}^2) \supset H^1(\mathbb{R}^2)$ with $1/s' + 1/s = 1$ [Lieb and Loss 2001, Theorem 8.8], we have

$$\begin{aligned}\iint \mathrm{d}x\, \mathrm{d}y\, |W(x-y)|\, |f(x,y)|^2 &\le C_s \int \mathrm{d}x\, \|W(x-\cdot\,)\|_{L^s(\mathbb{R}^2)} \|f(x,\cdot\,)\|^2_{H^1(\mathbb{R}^2)} \\ &= C_s \|W\|_{L^s(\mathbb{R}^2)} \langle f, (1-\Delta_y) f\rangle_{L^2(\mathbb{R}^2\times\mathbb{R}^2)}\end{aligned} \tag{5-5}$$

for all $f \in H^1(\mathbb{R}^2 \times \mathbb{R}^2)$. Therefore, we have the two-body inequality

$$|W(x-y)| \le C_s \|W\|_{L^s(\mathbb{R}^2)} (1-\Delta_y) \tag{5-6}$$

for each $y \in \mathbb{R}^2$, which implies the second-quantized form (5-2). □

The second estimate is concerned with the second quantization of a two-body multiplication operator:

Lemma 12. *Let $A \ge 0$ be a multiplication operator on $\mathfrak{H}^2$ such that $A(x,y) = A(y,x)$ and let $s \in [1,\infty)$. Then*

$$\mathrm{d}\Gamma_2(A^s) \le [\mathrm{d}\Gamma_2(A)]^s. \tag{5-7}$$

Proof. On every k-particle sector $\mathfrak{H}^k$, $k \ge 2$, we have

$$\mathrm{d}\Gamma_2(A^s) = \sum_{1\le i<j\le k} A_{ij}^s \le \bigg(\sum_{1\le i<j\le k} A_{ij}\bigg)^s = [\mathrm{d}\Gamma_2(A)]^s. \tag{5-8}$$

This concludes the proof since $\mathrm{d}\Gamma_2(A)$ preserves the particle number. □

5.1. *Estimate of the linear terms.* We consider first the linear terms in (5-1).

Lemma 13. *For every $t \in [0, T_{\max})$ and $\varepsilon > 0$, we have*

$$\big|\langle \Phi_N, [(\mathbb{G}_1 + \mathbb{G}_1^*), f_M^2]\Phi_{N,M}\rangle\big| \le C_{t,\varepsilon} \frac{N^\varepsilon}{\sqrt{N}}. \tag{5-9}$$

Proof. From the definition of $\mathbb{G}_1$ in (3-8), we obtain

$$\big|\langle \Phi_N, [\mathbb{G}_1, f_M^2]\Phi_{N,M}\rangle\big| = 2\,\big|\langle \Phi_N, a^\dagger\big(q(t)(w_N * |u_N|^2)u_N\big)\omega_1 \Phi_{N,M}\rangle\big| \tag{5-10}$$

with

$$\omega_1 = \frac{\mathcal{N}\sqrt{N-\mathcal{N}}}{N-1}\bigg(f^2\Big(\frac{\mathcal{N}}{M}\Big) - f^2\Big(\frac{\mathcal{N}+1}{M}\Big)\bigg), \tag{5-11}$$

where we used that $g(\mathcal{N})a_x^\dagger = a_x^\dagger g(\mathcal{N}+1)$. For f as in (3-16), we have

$$|\omega_1| \le \frac{C\mathcal{N}}{M\sqrt{N}} \mathbb{1}^{\le M} \tag{5-12}$$

in the sense of operators on $\mathcal{F}_\perp^{\leq N}(t)$. We will use the simple bound

$$a(v)a^\dagger(v) = a^\dagger(v)a(v) + \|v\|^2 \leq (\mathcal{N}+1)\|v\|^2, \tag{5-13}$$

where $v = q(t)(w_N * |u_N|^2)u_N$ satisfies

$$\|v\|_{L^2} \leq \big\|(w_N * |u_N|^2)u_N\big\|_{L^2} \leq \|w_N\|_{L^1}\|u_N\|_{L^2}^2\|u_N\|_{L^\infty} \leq C_t \tag{5-14}$$

by Lemma 10. Therefore, by the Cauchy–Schwarz inequality, we deduce from (5-10) and (5-13) that

$$\begin{aligned}\big|\langle \Phi_N, [\mathbb{G}_1, f_M^2]\Phi_{N,M}\rangle\big| &\leq 2\,\|\Phi_N\|\,\big\|a^\dagger\big(q(t)(w_N * |u_N|^2)u_N\big)(\mathcal{N}+1)^{-1/2}\big\|_{\mathrm{op}}\|(\mathcal{N}+1)^{1/2}\omega_1\Phi_{N,M}\| \\ &\leq C_t\left\|\frac{(\mathcal{N}+1)^{3/2}}{M\sqrt{N}}\mathbb{1}^{\leq M}\Phi_{N,M}\right\| \leq C_{t,\varepsilon}\frac{N^\varepsilon}{\sqrt{N}}. \end{aligned}\tag{5-15}$$

In the last estimate, we used that

$$\|\mathcal{N}^{1/2}\Phi_{N,M}\|^2 = \langle \Phi_{N,M}, \mathcal{N}\Phi_{N,M}\rangle \leq C_{t,\varepsilon}N^\varepsilon, \tag{5-16}$$

which follows from the kinetic estimate (3-13) in Lemma 8. Similarly, we also get

$$\big|\langle \Phi_N, [\mathbb{G}_1^*, f_M^2]\Phi_{N,M}\rangle\big| \leq C_{t,\varepsilon}\frac{N^\varepsilon}{\sqrt{N}} \tag{5-17}$$

since

$$\big|\langle \Phi_N, [\mathbb{G}_1^*, f_M^2]\Phi_{N,M}\rangle\big| = 2\,\big|\big\langle \Phi_N, a\big(q(t)(w_N * |u_N|^2)u_N\big)\widetilde{\omega}_1\Phi_{N,M}\big\rangle\big|, \tag{5-18}$$

where

$$\widetilde{\omega}_1 = \frac{\mathcal{N}\sqrt{N-\mathcal{N}+1}}{N-1}\left(f^2\left(\frac{\mathcal{N}}{M}\right) - f^2\left(\frac{\mathcal{N}-1}{M}\right)\right) \tag{5-19}$$

as an operator on $\mathcal{F}_\perp^{\leq N}$. From (5-15) and (5-17), we obtain (5-9). □

5.2. *Estimate of the quadratic terms.* We turn to the quadratic terms in (5-1).

Lemma 14. *For every $t \in [0, T_{\max})$ and $\varepsilon > 0$, we have*

$$\big|\langle \Phi_N, [\mathbb{G}_2, f_M^2]\Phi_{N,M}\rangle\big| \leq C_{t,\varepsilon}\left(\frac{1}{\sqrt{M}} + \frac{N^\beta}{M^{3/2}}\right)N^\varepsilon, \tag{5-20}$$

$$\big|\langle \Phi_N, [\mathbb{G}_2^*, f_M^2]\Phi_{N,M}\rangle\big| \leq C_{t,\varepsilon}\frac{N^\varepsilon}{M}. \tag{5-21}$$

Proof. The bound (5-20) is one of the most difficult estimates in this section. We use the strategy explained in Section 3.3.

Step 1. Let us abbreviate

$$\omega_2 = \frac{\sqrt{(N-\mathcal{N})(N-\mathcal{N}-1)}}{N-1}\left(f^2\left(\frac{\mathcal{N}}{M}\right) - f^2\left(\frac{\mathcal{N}+2}{M}\right)\right) \tag{5-22}$$

as an operator on $\mathcal{F}_{\perp}^{\leq N}$. For $N \geq 2$, we have

$$|\omega_2| \leq \frac{C}{M} \mathbb{1}^{>M/2}. \tag{5-23}$$

We also observe that in the relevant estimate for $\mathbb{G}_2$, $K_2 = q \otimes q \widetilde{K}_2$ in (2-24) can be replaced by $\widetilde{K}_2$ as for any $\chi, \chi' \in \mathcal{F}_{\perp}(t)$ we have

$$\left\langle \chi, \iint \mathrm{d}x\, \mathrm{d}y\, K_2(x, y) u_N(x) u_N(y) a_x^\dagger a_y^\dagger \chi' \right\rangle = \left\langle \chi, \iint \mathrm{d}x\, \mathrm{d}y\, \widetilde{K}_2(x, y) u_N(x) u_N(y) a_x^\dagger a_y^\dagger \chi' \right\rangle. \tag{5-24}$$

Hence, we can write

$$\langle \Phi_N, [\mathbb{G}_2, f_M^2] \Phi_{N,M} \rangle = \iint \mathrm{d}x\, \mathrm{d}y\, w_N(x-y) u_N(x) u_N(y) \langle \Phi_N, a_x^\dagger a_y^\dagger \omega_2 \Phi_{N,M} \rangle. \tag{5-25}$$

By writing

$$a_x^\dagger a_y^\dagger = \mathcal{R}^{-1/2} a_x^\dagger a_y^\dagger \mathcal{R}^{1/2} + \mathcal{R}^{-1/2} [\mathcal{R}^{1/2}, a_x^\dagger a_y^\dagger] \tag{5-26}$$

with $\mathcal{R} = \mathrm{d}\Gamma_2(|w_N|) + 1$ as in (3-22) and using the triangle inequality, we find that

$$\left| \langle \Phi_N, [\mathbb{G}_2, f_M^2] \Phi_{N,M} \rangle \right| \leq \mathcal{E}_1 + \mathcal{E}_2 \tag{5-27}$$

where

$$\mathcal{E}_1 = \iint \mathrm{d}x\, \mathrm{d}y\, |w_N(x-y)|\, |u_N(x)|\, |u_N(y)|\, |\langle \Phi_N, \mathcal{R}^{-1/2} a_x^\dagger a_y^\dagger \mathcal{R}^{1/2} \omega_2 \Phi_{N,M} \rangle|, \tag{5-28}$$

$$\mathcal{E}_2 = \iint \mathrm{d}x\, \mathrm{d}y\, |w_N(x-y)|\, |u_N(x)|\, |u_N(y)|\, \left| \langle \Phi_N, \mathcal{R}^{-1/2} [\mathcal{R}^{1/2}, a_x^\dagger a_y^\dagger] \omega_2 \Phi_{N,M} \rangle \right|. \tag{5-29}$$

Step 2. Now let us estimate $\mathcal{E}_1$. By the Cauchy–Schwarz inequality,

$$\begin{aligned} \mathcal{E}_1 &\leq \iint \mathrm{d}x\, \mathrm{d}y\, |w_N(x-y)|\, |u_N(x)|\, |u_N(y)|\, \|a_x a_y \mathcal{R}^{-1/2} \Phi_N\|\, \|\mathcal{R}^{1/2} \omega_2 \Phi_{N,M}\| \\ &\leq \left(\iint \mathrm{d}x\, \mathrm{d}y\, |w_N(x-y)|\, |u_N(x)|^2 |u_N(y)|^2 \right)^{1/2} \\ &\qquad \times \left(\iint \mathrm{d}x\, \mathrm{d}y\, |w_N(x-y)|\, \|a_x a_y \mathcal{R}^{-1/2} \Phi_N\|^2 \right)^{1/2} \|\mathcal{R}^{1/2} \omega_2 \Phi_{N,M}\|. \end{aligned} \tag{5-30}$$

From Lemma 10, we can bound

$$\iint \mathrm{d}x\, \mathrm{d}y\, |w_N(x-y)|\, |u_N(x)|^2 |u_N(y)|^2 \leq \|u_N\|_{L^\infty}^2 \|u_N\|_{L^2}^2 \|w_N\|_{L^1} \leq C_t. \tag{5-31}$$

Moreover, (3-26) yields

$$\iint \mathrm{d}x\, \mathrm{d}y\, |w_N(x-y)|\, \|a_x a_y \mathcal{R}^{-1/2} \Phi_N\|^2 \leq \|\Phi_N\|^2 \leq 1. \tag{5-32}$$

From Lemma 11 and the kinetic estimate in Lemma 8, we get

$$\left\langle \Phi_{N,M}, \mathrm{d}\Gamma_2(|w_N|) \Phi_{N,M} \right\rangle \leq C_\varepsilon N^\varepsilon \langle \Phi_{N,M}, M\, \mathrm{d}\Gamma_1(1-\Delta) \Phi_{N,M} \rangle \leq C_{t,\varepsilon} M N^{2\varepsilon}. \tag{5-33}$$

Combining this with (5-23) and the fact that $\mathcal{R}$ commutes with ω_2, we find that

$$\|\mathcal{R}^{1/2}\omega_2\Phi_{N,M}\|^2 \le \frac{C}{M^2}\langle\Phi_{N,M},\mathcal{R}\Phi_{N,M}\rangle \le \frac{C_{t,\varepsilon}N^{\varepsilon}}{M}. \tag{5-34}$$

Inserting (5-31), (5-32) and (5-34) in (5-30), we conclude that

$$\mathcal{E}_1 \le \frac{C_{t,\varepsilon}N^{\varepsilon}}{\sqrt{M}}\|\Phi_N\| \tag{5-35}$$

for every constant $\varepsilon > 0$.

Step 3. We turn to estimate the second term $\mathcal{E}_2$, which is more involved. Using (3-29) and (3-28), we get

$$\begin{aligned}[\mathcal{R}^{1/2}, a_x^\dagger a_y^\dagger] = \frac{1}{\pi}\int_0^\infty \mathrm{d}s\, \frac{\sqrt{s}}{\mathcal{R}+s}|w_N(x-y)|a_x^\dagger a_y^\dagger\frac{1}{\mathcal{R}+s} \\ + \frac{1}{\pi}\int_0^\infty \mathrm{d}s\int \mathrm{d}z\,\frac{\sqrt{s}}{\mathcal{R}+s}\big(|w_N(x-z)|+|w_N(y-z)|\big)a_x^\dagger a_y^\dagger a_z^\dagger a_z\frac{1}{\mathcal{R}+s}.\end{aligned} \tag{5-36}$$

This allows us to write

$$\begin{aligned}\mathcal{E}_2 &= \iint \mathrm{d}x\,\mathrm{d}y\,|w_N(x-y)|\,|u_N(x)|\,|u_N(y)|\,\big|\langle\Phi_N,\mathcal{R}^{-1/2}[\mathcal{R}^{1/2},a_x^\dagger a_y^\dagger]\omega_2\Phi_{N,M}\rangle\big| \\ &\le C(\mathcal{E}_{2,1}+\mathcal{E}_{2,2}),\end{aligned} \tag{5-37}$$

where

$$\mathcal{E}_{2,1} = \int_0^\infty \mathrm{d}s\,\sqrt{s}\iint \mathrm{d}x\,\mathrm{d}y\,|w_N(x-y)|^2|u_N(x)|\,|u_N(y)|\left|\langle\Phi_N,\frac{\mathcal{R}^{-1/2}}{\mathcal{R}+s}a_x^\dagger a_y^\dagger\frac{1}{\mathcal{R}+s}\omega_2\Phi_{N,M}\rangle\right|, \tag{5-38}$$

$$\begin{aligned}\mathcal{E}_{2,2} = \int_0^\infty \mathrm{d}s\,\sqrt{s}\iiint \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,|w_N(x-y)|\,|w_N(x-z)|\,|u_N(x)|\,|u_N(y)| \\ \times\left|\langle\Phi_N,\frac{\mathcal{R}^{-1/2}}{\mathcal{R}+s}a_x^\dagger a_y^\dagger a_z^\dagger a_z\frac{1}{\mathcal{R}+s}\omega_2\Phi_{N,M}\rangle\right|.\end{aligned} \tag{5-39}$$

Estimate of $\mathcal{E}_{2,1}$. By the Cauchy–Schwarz inequality, we find for any constant $\varepsilon \in (0,1)$ that

$$\begin{aligned}\mathcal{E}_{2,1} &\le \int_0^\infty \mathrm{d}s\sqrt{s}\iint \mathrm{d}x\,\mathrm{d}y\,|w_N(x-y)|^2|u_N(x)|\,|u_N(y)|\left\|a_x a_y\frac{\mathcal{R}^{-1/2}}{\mathcal{R}+s}\Phi_N\right\|\left\|\frac{1}{\mathcal{R}+s}\omega_2\Phi_{N,M}\right\| \\ &\le \int_0^\infty \mathrm{d}s\,\sqrt{s}\bigg(\iint \mathrm{d}x\,\mathrm{d}y\,|w_N(x-y)|^{2+\varepsilon}|u_N(x)|^2|u_N(y)|^2\bigg)^{1/2} \\ &\qquad\times\bigg(\iint \mathrm{d}x\,\mathrm{d}y\,|w_N(x-y)|^{2-\varepsilon}\left\|a_x a_y\frac{\mathcal{R}^{-1/2}}{\mathcal{R}+s}\Phi_N\right\|^2\bigg)^{1/2}\left\|\frac{1}{\mathcal{R}+s}\omega_2\Phi_{N,M}\right\|.\end{aligned} \tag{5-40}$$

By Lemma 10, we obtain

$$\begin{aligned}\iint \mathrm{d}x\,\mathrm{d}y\,|w_N(x-y)|^{2+\varepsilon}|u_N(x)|^2|u_N(y)|^2 &\le \|u_N\|_{L^\infty}^2\|u_N\|_{L^2}^2\||w_N|^{2+\varepsilon}\|_{L^1} \\ &\le C_t N^{2\beta(1+\varepsilon)}.\end{aligned} \tag{5-41}$$

Moreover, using that

$$\iint \mathrm{d}x\,\mathrm{d}y\,|w_N(x-y)|^{2-\varepsilon}a_x^\dagger a_y^\dagger a_x a_y = \mathrm{d}\Gamma_2(|w_N|^{2-\varepsilon}) \le \big(\mathrm{d}\Gamma_2(|w_N|)\big)^{2-\varepsilon} \le \mathcal{R}^{2-\varepsilon} \tag{5-42}$$

by Lemma 12, we can bound

$$\begin{aligned}\iint \mathrm{d}x\,\mathrm{d}y\,|w_N(x-y)|^{2-\varepsilon}\left\|a_x a_y \frac{\mathcal{R}^{-1/2}}{\mathcal{R}+s}\Phi_N\right\|^2 &= \left\langle \frac{\mathcal{R}^{-1/2}}{\mathcal{R}+s}\Phi_N, \mathrm{d}\Gamma_2(|w_N|^{2-\varepsilon})\frac{\mathcal{R}^{-1/2}}{\mathcal{R}+s}\Phi_N\right\rangle \\ &\le \left\langle \Phi_N, \frac{\mathcal{R}^{1-\varepsilon}}{(\mathcal{R}+s)^2}\Phi_N\right\rangle \le \frac{1}{(1+s)^{1+\varepsilon}}.\end{aligned} \tag{5-43}$$

In the last estimate we used that $\mathcal{R} \ge 1$. Moreover, using again the fact that $\mathcal{R}$ commutes with ω_2, we find with (5-23) that

$$\begin{aligned}\left\|\frac{1}{\mathcal{R}+s}\omega_2\Phi_{N,M}\right\|^2 &\le \frac{C}{M^2(1+s)^2}\langle \Phi_{N,M}, \mathbb{1}^{>M/2}\Phi_{N,M}\rangle \\ &\le \frac{C}{M^2(1+s)^2}\left\langle \Phi_{N,M}, \frac{2\mathcal{N}}{M}\Phi_{N,M}\right\rangle \le \frac{C_{t,\varepsilon}}{M^3(1+s)^2}N^\varepsilon.\end{aligned} \tag{5-44}$$

Here in the last estimate, we used the kinetic bound in Lemma 8. Inserting (5-41), (5-43) and (5-44) in (5-40) we find that, for every constant $\varepsilon \in (0,1)$,

$$\begin{aligned}\mathcal{E}_{2,1} &\le C_{t,\varepsilon}\int_0^\infty \mathrm{d}s\,\sqrt{s}\sqrt{N^{2\beta(1+\varepsilon)}}\sqrt{\frac{1}{(1+s)^{1+\varepsilon}}}\sqrt{\frac{1}{M^3(1+s)^2}N^\varepsilon} \\ &\le C_{t,\varepsilon}\frac{N^{(1+\varepsilon)\beta+\varepsilon/2}}{M^{3/2}}\int_0^\infty \frac{\mathrm{d}s}{(1+s)^{1+\varepsilon/2}} \le C_{t,\varepsilon}\frac{N^{(1+\varepsilon)\beta+\varepsilon/2}}{M^{3/2}}.\end{aligned} \tag{5-45}$$

Estimate of $\mathcal{E}_{2,2}$. Similarly, for every constant $\varepsilon > 0$ small, by the Cauchy–Schwarz inequality,

$$\begin{aligned}\mathcal{E}_{2,2} &= \int_0^\infty \mathrm{d}s\,\sqrt{s}\iiint \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,|w_N(x-y)|\,|w_N(x-z)|\,|u_N(x)|\,|u_N(y)| \\ &\qquad\qquad \times \left|\left\langle \Phi_N, \frac{\mathcal{R}^{-1/2}(\mathcal{N}+1)^{-1/2}}{\mathcal{R}+s}a_x^\dagger a_y^\dagger a_z^\dagger a_z \frac{(\mathcal{N}+3)^{1/2}}{\mathcal{R}+s}\omega_2\Phi_{N,M}\right\rangle\right| \\ &\le \|u_N\|_{L^\infty}^2 \int_0^\infty \mathrm{d}s\,\sqrt{s}\iiint \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,|w_N(x-y)|\,|w_N(x-z)| \\ &\qquad\qquad \times \left\|a_x a_y a_z \frac{\mathcal{R}^{-1/2}(\mathcal{N}+1)^{-1/2}}{\mathcal{R}+s}\Phi_N\right\| \left\|a_z\frac{(\mathcal{N}+3)^{1/2}}{\mathcal{R}+s}\omega_2\Phi_{N,M}\right\| \\ &\le C_t\int_0^\infty \mathrm{d}s\,\sqrt{s}\left(\iiint \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,|w_N(x-y)|^{2-\varepsilon}\left\|a_x a_y a_z\frac{\mathcal{R}^{-1/2}(\mathcal{N}+1)^{-1/2}}{\mathcal{R}+s}\Phi_N\right\|^2\right)^{1/2} \\ &\qquad \times \left(\int \mathrm{d}z\left\|a_z\frac{(\mathcal{N}+3)^{1/2}}{\mathcal{R}+s}\omega_2\Phi_{N,M}\right\|^2\int \mathrm{d}x\,|w_N(x-z)|^2\int \mathrm{d}y\,|w_N(x-y)|^\varepsilon\right)^{1/2}.\end{aligned} \tag{5-46}$$

In the last estimate, we used the uniform bound $\|u_N\|_{L^\infty} \le C_t$ from Lemma 10. Using again (5-42) and $\mathcal{R} \ge 1$ we find that

$$\iiint \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,|w_N(x-y)|^{2-\varepsilon}\left\|a_x a_y a_z \frac{\mathcal{R}^{-1/2}(\mathcal{N}+1)^{-1/2}}{\mathcal{R}+s}\Phi_N\right\|^2$$
$$= \left\langle \frac{\mathcal{R}^{-1/2}(\mathcal{N}+1)^{-1/2}}{\mathcal{R}+s}\Phi_N, \mathrm{d}\Gamma_2(|w_N|^{2-\varepsilon})(\mathcal{N}-2)\frac{\mathcal{R}^{-1/2}(\mathcal{N}+1)^{-1/2}}{\mathcal{R}+s}\Phi_N\right\rangle$$
$$\le \left\langle \Phi_N, \frac{\mathcal{R}^{1-\varepsilon}}{(\mathcal{R}+s)^2}\Phi_N\right\rangle \le \frac{1}{(1+s)^{1+\varepsilon}}. \tag{5-47}$$

Since w is bounded and compactly supported, we get

$$\int \mathrm{d}x\,|w_N(x-z)|^2 \int \mathrm{d}y\,|w_N(x-y)|^\varepsilon \le CN^{2\beta\varepsilon}. \tag{5-48}$$

Moreover, using (5-23) together with $\mathcal{N}^2 \le M\,\mathrm{d}\Gamma_1(1-\Delta)$ on $\mathcal{F}^{\le M}$ and Lemma 8, we have

$$\int \mathrm{d}z\,\left\|a_z \frac{(\mathcal{N}+3)^{1/2}}{\mathcal{R}+s}\omega_2\Phi_{N,M}\right\|^2 = \frac{C}{M^2}\left\langle \Phi_{N,M}, \frac{\mathcal{N}(\mathcal{N}+3)}{(\mathcal{R}+s)^2}\Phi_{N,M}\right\rangle \tag{5-49}$$
$$\le \frac{C}{M^2(1+s)^2}\langle \Phi_{N,M}, \mathcal{N}^2\Phi_{N,M}\rangle \le \frac{C_{t,\varepsilon}N^\varepsilon}{M(1+s)^2}. \tag{5-50}$$

Therefore, we deduce from (5-46) that

$$\mathcal{E}_{2,2} \le C_{t,\varepsilon}\int_0^\infty \mathrm{d}s\,\sqrt{s}\sqrt{\frac{1}{(1+s)^{1+\varepsilon}}}\sqrt{\frac{N^\varepsilon}{M(1+s)^2}N^{2\beta\varepsilon}} \le C_{t,\varepsilon}\frac{N^{(\beta+1/2)\varepsilon}}{\sqrt{M}}. \tag{5-51}$$

Putting (5-45) and (5-51) together, we conclude from (5-37) that

$$\mathcal{E}_2 \le C_{t,\varepsilon}\left(\frac{N^{(1+\varepsilon)\beta+\varepsilon/2}}{M^{3/2}} + \frac{N^{(\beta+1/2)\varepsilon}}{\sqrt{M}}\right). \tag{5-52}$$

Conclusion of (5-20). Inserting (5-35) and (5-52) in (5-27), we obtain (5-20).

Step 4. It remains to prove (5-21). Similarly to (5-25), we can write

$$\langle \Phi_N, [\mathbb{G}_2^*, f_M^2]\Phi_{N,M}\rangle = \iint \mathrm{d}x\,\mathrm{d}y\,w_N(x-y)\overline{u_N(x)u_N(y)}\langle \Phi_N, a_x a_y \widetilde{\omega}_2\Phi_{N,M}\rangle \tag{5-53}$$

with

$$\widetilde{\omega}_2 = \frac{\sqrt{(N-\mathcal{N}+2)(N-\mathcal{N}+1)}}{N-1}\left(f^2\left(\frac{\mathcal{N}-2}{M}\right) - f^2\left(\frac{\mathcal{N}}{M}\right)\right) \tag{5-54}$$

as an operator on $\mathcal{F}_\perp^{\le N}$. This term is much easier to estimate than (5-25) since now two annihilators hit $\Phi_{N,M}$. To be precise, we have

$$|\widetilde{\omega}_2| \le \frac{C}{M} \tag{5-55}$$

similarly to (5-23). Therefore, by the Cauchy–Schwarz inequality,

$$\begin{aligned}\big|\langle\Phi_N,[\mathbb{G}_2^*,f_M^2]\Phi_{N,M}\rangle\big| &\le \iint \mathrm{d}x\,\mathrm{d}y\,|w_N(x-y)|\,|u_N(x)|\,|u_N(y)|\,\|\Phi_N\|\,\|a_xa_y\widetilde{\omega}_2\Phi_{N,M}\| \\ &\le \|\Phi_N\|\bigg(\iint \mathrm{d}x\,\mathrm{d}y\,|w_N(x-y)|\,|u_N(x)|^2|u_N(y)|^2\bigg)^{1/2} \\ &\qquad\qquad \times\bigg(\iint \mathrm{d}x\,\mathrm{d}y\,|w_N(x-y)|\,\|a_xa_y\widetilde{\omega}_2\Phi_{N,M}\|^2\bigg)^{1/2} \\ &\le C_t\big\langle\Phi_{N,M},\mathrm{d}\Gamma_2(|w_N|)|\widetilde{\omega}_2|^2\Phi_{N,M}\big\rangle^{1/2} \\ &\le \frac{C_{t,\varepsilon}}{M^2}\langle\Phi_{N,M},N^\varepsilon\mathcal{N}\,\mathrm{d}\Gamma_1(1-\Delta)\Phi_{N,M}\rangle^{1/2}\le C_{t,\varepsilon}\frac{N^\varepsilon}{M}. \end{aligned} \tag{5-56}$$

Here we used again (5-31), Lemma 11 and the kinetic estimate in Lemma 8. Thus, (5-21) holds true. This completes the proof of Lemma 14. □

5.3. ***Estimate of the cubic terms.*** Concerning the cubic terms in (5-1), we have the following bounds:

Lemma 15. *Let $\Phi_N\in\mathcal{F}_\perp(t)$, $t\in[0,T_{\max})$ and $\varepsilon>0$. Then*

$$\big|\langle\Phi_N,[\mathbb{G}_3,f_M^2]\Phi_{N,M}\rangle\big|\le C_{t,\varepsilon}\bigg(\frac{1}{\sqrt{N}}+\frac{N^\beta}{M\sqrt{N}}\bigg)N^\varepsilon, \tag{5-57}$$

$$\big|\langle\Phi_N,[\mathbb{G}_3^*,f_M^2]\Phi_{N,M}\rangle\big|\le C_{t,\varepsilon}\frac{N^\varepsilon}{\sqrt{N}}. \tag{5-58}$$

Proof. Again, (5-57) is much more difficult than (5-58). We will proceed similarly to the quadratic terms.

Step 1. Analogously to (5-23), we define

$$\omega_3=\sqrt{1-\frac{\mathcal{N}-1}{N-1}}\bigg(f^2\Big(\frac{\mathcal{N}}{M}\Big)-f^2\Big(\frac{\mathcal{N}+1}{M}\Big)\bigg) \tag{5-59}$$

as an operator on $\mathcal{F}_\perp^{\le N}$, which satisfies

$$|\omega_3|\le\frac{C}{M}\mathbb{1}^{\le M}. \tag{5-60}$$

Moreover, similarly to (5-25) we can write

$$\langle\Phi_N,[\mathbb{G}_3,f_M^2]\Phi_{N,M}\rangle=\frac{1}{\sqrt{N}}\iint \mathrm{d}x\,\mathrm{d}y\,w_N(x-y)u_N(x)\langle\Phi_N,a_x^\dagger a_y^\dagger a_y\omega_3\Phi_{N,M}\rangle. \tag{5-61}$$

By decomposing $a_x^\dagger a_y^\dagger a_y$ as

$$a_x^\dagger a_y^\dagger a_y=\mathcal{R}^{-1/2}a_x^\dagger a_y^\dagger a_y\mathcal{R}^{1/2}+\mathcal{R}^{-1/2}[a_x^\dagger a_y^\dagger a_y,\mathcal{R}^{1/2}], \tag{5-62}$$

we obtain

$$\big|\langle\Phi_N,[\mathbb{G}_3,f_M^2]\Phi_{N,M}\rangle\big|\le\mathcal{E}_3+\mathcal{E}_4, \tag{5-63}$$

where

$$\mathcal{E}_3 = \frac{1}{\sqrt{N}} \iint \mathrm{d}x\,\mathrm{d}y\, |w_N(x-y)|\, |u_N(x)|\, |\langle \Phi_N, \mathcal{R}^{-1/2} a_x^\dagger a_y^\dagger a_y \mathcal{R}^{1/2} \omega_3 \Phi_{N,M}\rangle|, \tag{5-64}$$

$$\mathcal{E}_4 = \frac{1}{\sqrt{N}} \iint \mathrm{d}x\,\mathrm{d}y\, |w_N(x-y)|\, |u_N(x)|\, \big|\langle \Phi_N, \mathcal{R}^{-1/2} [a_x^\dagger a_y^\dagger a_y, \mathcal{R}^{1/2}] \omega_3 \Phi_{N,M}\rangle\big|. \tag{5-65}$$

Step 2. Let us first estimate $\mathcal{E}_3$. By the Cauchy–Schwarz inequality,

$$\begin{aligned}\mathcal{E}_3 &\le \frac{1}{\sqrt{N}} \iint \mathrm{d}x\,\mathrm{d}y\, |w_N(x-y)|\, |u_N(x)|\, \|a_x a_y \mathcal{R}^{-1/2} \Phi_N\|\, \|a_y \mathcal{R}^{1/2} \omega_3 \Phi_{N,M}\| \\ &\le \frac{\|u_N\|_{L^\infty}}{\sqrt{N}} \bigg(\iint \mathrm{d}x\,\mathrm{d}y\, |w_N(x-y)|\, \|a_x a_y \mathcal{R}^{-1/2} \Phi_N\|^2 \bigg)^{1/2} \\ &\qquad\qquad \times \bigg(\iint \mathrm{d}x\,\mathrm{d}y\, |w_N(x-y)|\, \|a_y \mathcal{R}^{1/2} \omega_3 \Phi_{N,M}\|^2 \bigg)^{1/2}. \end{aligned} \tag{5-66}$$

We can simplify the right-hand side using (3-26) and Lemma 10. Moreover, by (5-60) and Lemma 11,

$$|\omega_3|^2 \mathcal{N} \mathcal{R} \le \frac{C_\varepsilon N^\varepsilon}{M^2} \mathcal{N}^2\, \mathrm{d}\Gamma_1(1-\Delta) \le C_\varepsilon N^\varepsilon\, \mathrm{d}\Gamma_1(1-\Delta) \tag{5-67}$$

on $\mathcal{F}^{\le M}$. Combining this with the kinetic bound in Lemma 8, we find that

$$\begin{aligned}\iint \mathrm{d}x\,\mathrm{d}y\, |w_N(x-y)|\, \|a_y \mathcal{R}^{1/2} \omega_3 \Phi_{N,M}\|^2 &= \|w_N\|_{L^1} \langle \Phi_{N,M}, |\omega_3|^2 \mathcal{N} \mathcal{R} \Phi_{N,M}\rangle \\ &\le C_\varepsilon N^{2\varepsilon} \end{aligned} \tag{5-68}$$

for every constant $\varepsilon > 0$. Therefore, we deduce from (5-66) that

$$\mathcal{E}_3 \le \frac{C_{t,\varepsilon} N^\varepsilon}{\sqrt{N}}. \tag{5-69}$$

Step 3. Now we turn to the complicated error term $\mathcal{E}_4$. A direct computation shows that

$$\big[\mathrm{d}\Gamma_2(|w_N|), a_x^\dagger a_y^\dagger a_y\big] = |w_N(x-y)| a_x^\dagger a_y^\dagger a_y + \int \mathrm{d}z\, |w_N(x-z)| a_x^\dagger a_y^\dagger a_z^\dagger a_z a_y, \tag{5-70}$$

and with (3-28) this yields

$$\begin{aligned}[\mathcal{R}^{1/2}, a_x^\dagger a_y^\dagger a_y] &= \frac{1}{\pi} \int_0^\infty \mathrm{d}s\, \frac{\sqrt{s}}{\mathcal{R}+s} |w_N(x-y)| a_x^\dagger a_y^\dagger a_y \frac{1}{\mathcal{R}+s} \\ &\quad + \frac{1}{\pi} \int_0^\infty \mathrm{d}s \int \mathrm{d}z\, \frac{\sqrt{s}}{\mathcal{R}+s} |w_N(x-z)| a_x^\dagger a_y^\dagger a_z^\dagger a_z a_y \frac{1}{\mathcal{R}+s}. \end{aligned} \tag{5-71}$$

Thus, by the triangle inequality and the bound $\|u_N\|_{L^\infty} \le C_t$ from Lemma 10, we can split

$$\begin{aligned}\mathcal{E}_4 &= \frac{1}{\sqrt{N}} \iint \mathrm{d}x\,\mathrm{d}y\, |w_N(x-y)|\, |u_N(x)|\, \big|\langle \Phi_N, \mathcal{R}^{-1/2} [a_x^\dagger a_y^\dagger a_y, \mathcal{R}^{1/2}] \omega_3 \Phi_{N,M}\rangle\big| \\ &\le C_t(\mathcal{E}_{4,1} + \mathcal{E}_{4,2}), \end{aligned} \tag{5-72}$$

where

$$\mathcal{E}_{4,1} = \frac{1}{\sqrt{N}} \int_0^\infty \mathrm{d}s \, \sqrt{s} \iint \mathrm{d}x \, \mathrm{d}y \, |w_N(x-y)|^2 \left| \left\langle \Phi_N, \frac{\mathcal{R}^{-1/2}}{\mathcal{R}+s} a_x^\dagger a_y^\dagger a_y \frac{1}{\mathcal{R}+s} \omega_3 \Phi_{N,M} \right\rangle \right|, \tag{5-73}$$

$$\mathcal{E}_{4,2} = \frac{1}{\sqrt{N}} \int_0^\infty \mathrm{d}s \, \sqrt{s} \iiint \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \, |w_N(x-y)| \, |w_N(x-z)| \times \left| \left\langle \Phi_N, \frac{\mathcal{R}^{-1/2}}{\mathcal{R}+s} a_x^\dagger a_y^\dagger a_z^\dagger a_z a_y \frac{1}{\mathcal{R}+s} \omega_3 \Phi_{N,M} \right\rangle \right|. \tag{5-74}$$

Estimate of $\mathcal{E}_{4,1}$. By the Cauchy–Schwarz inequality we have

$$\begin{aligned} \mathcal{E}_{4,1} &\le \frac{1}{\sqrt{N}} \int_0^\infty \mathrm{d}s \, \sqrt{s} \iint \mathrm{d}x \, \mathrm{d}y \, |w_N(x-y)|^2 \left\| a_x a_y \frac{\mathcal{R}^{-1/2}}{\mathcal{R}+s} \Phi_N \right\| \left\| a_y \frac{1}{\mathcal{R}+s} \omega_3 \Phi_{N,M} \right\| \\ &\le \frac{1}{\sqrt{N}} \int_0^\infty \mathrm{d}s \, \sqrt{s} \left(\iint \mathrm{d}x \, \mathrm{d}y \, |w_N(x-y)|^{2-\varepsilon} \left\| a_x a_y \frac{\mathcal{R}^{-1/2}}{\mathcal{R}+s} \Phi_N \right\|^2 \right)^{1/2} \\ &\qquad \times \left(\iint \mathrm{d}x \, \mathrm{d}y \, |w_N(x-y)|^{2+\varepsilon} \left\| a_y \frac{1}{\mathcal{R}+s} \omega_3 \Phi_{N,M} \right\|^2 \right)^{1/2}. \end{aligned} \tag{5-75}$$

The right-hand side can be simplified using (5-43) and the estimate

$$\begin{aligned} &\iint \mathrm{d}x \, \mathrm{d}y \, |w_N(x-y)|^{2+\varepsilon} \left\| a_y \frac{1}{\mathcal{R}+s} \omega_3 \Phi_{N,M} \right\|^2 \\ &\qquad = \| |w_N|^{2+\varepsilon} \|_{L^1} \left\langle \Phi_{N,M}, \frac{\mathcal{N} |\omega_3|^2}{(\mathcal{R}+s)^2} \Phi_{N,M} \right\rangle \le C_{t,\varepsilon} N^{(1+\varepsilon)2\beta} \frac{N^\varepsilon}{M^2 (1+s)^2}, \end{aligned} \tag{5-76}$$

which follows from (5-60), $\mathcal{R} \ge 1$, and the kinetic bound in Lemma 8. Altogether, this gives

$$\begin{aligned} \mathcal{E}_{4,1} &\le \frac{C_{t,\varepsilon}}{\sqrt{N}} \int_0^\infty \mathrm{d}s \, \sqrt{s} \sqrt{\frac{1}{(1+s)^{1+\varepsilon}}} \sqrt{N^{(1+\varepsilon)2\beta} \frac{N^\varepsilon}{M^2 (1+s)^2}} \\ &\le C_{t,\varepsilon} \frac{N^{(1+\varepsilon)\beta + \varepsilon/2}}{\sqrt{N} M}. \end{aligned} \tag{5-77}$$

Estimate of $\mathcal{E}_{4,2}$. By the Cauchy–Schwarz inequality,

$$\begin{aligned} \mathcal{E}_{4,2} &= \frac{1}{\sqrt{N}} \int_0^\infty \mathrm{d}s \, \sqrt{s} \iiint \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \, |w_N(x-y)| \, |w_N(x-z)| \\ &\qquad \times \left| \left\langle \Phi_N, \frac{\mathcal{R}^{-1/2} (\mathcal{N}+2)^{-1/2}}{\mathcal{R}+s} a_x^\dagger a_y^\dagger a_z^\dagger a_z a_y \frac{(\mathcal{N}+3)^{1/2}}{\mathcal{R}+s} \omega_3 \Phi_{N,M} \right\rangle \right| \\ &\le \frac{1}{\sqrt{N}} \int_0^\infty \mathrm{d}s \, \sqrt{s} \iiint \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \, |w_N(x-y)| \, |w_N(x-z)| \\ &\qquad \times \left\| a_x a_y a_z \frac{\mathcal{R}^{-1/2} (\mathcal{N}+2)^{-1/2}}{\mathcal{R}+s} \Phi_N \right\| \left\| a_y a_z \frac{(\mathcal{N}+3)^{1/2}}{\mathcal{R}+s} \omega_3 \Phi_{N,M} \right\| \\ &\le \frac{1}{\sqrt{N}} \int_0^\infty \mathrm{d}s \, \sqrt{s} \left(\iiint \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \, |w_N(x-y)|^{2-\varepsilon} \left\| a_x a_y a_z \frac{\mathcal{R}^{-1/2} (\mathcal{N}+2)^{-1/2}}{\mathcal{R}+s} \Phi_N \right\|^2 \right)^{1/2} \\ &\qquad \times \left(\iiint \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \, |w_N(x-y)|^{\varepsilon} |w_N(x-z)|^2 \left\| a_y a_z \frac{(\mathcal{N}+3)^{1/2}}{\mathcal{R}+s} \omega_3 \Phi_{N,M} \right\|^2 \right)^{1/2}. \end{aligned} \tag{5-78}$$

We can bound

$$\iiint \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,|w_N(x-y)|^{2-\varepsilon}\left\| a_x a_y a_z \frac{\mathcal{R}^{-1/2}(\mathcal{N}+2)^{-1/2}}{\mathcal{R}+s}\Phi_N \right\|^2 = \iint \mathrm{d}x\,\mathrm{d}y\,|w_N(x-y)|^{2-\varepsilon}\left\| a_x a_y \frac{\mathcal{R}^{-1/2}}{\mathcal{R}+s}\Phi_N \right\|^2 \le \frac{1}{(1+s)^{1+\varepsilon}} \tag{5-79}$$

as in (5-43). Since w is bounded and compactly supported, we have the pointwise estimate

$$\begin{aligned} |w_N(x-y)|^{\varepsilon}|w_N(x-z)|^2 &= |w_N(x-y)|^{\varepsilon}|w_N(x-z)|^2 \mathbb{1}_{\{|y-z|\le CN^{-\beta}\}} \\ &\le CN^{4\beta}|w_N(x-y)|^{\varepsilon}\mathbb{1}_{\{|y-z|\le CN^{-\beta}\}}. \end{aligned} \tag{5-80}$$

Moreover, the operators $\mathrm{d}\Gamma_2(\mathbb{1}_{\{|y-z|\le CN^{-\beta}\}})$, $\mathcal{R}$, $\mathcal{N}$ and ω_3 all commute. Consequently, using $\mathcal{R}\ge 1$ and (5-60), we can bound

$$\begin{aligned} \iint \mathrm{d}y\,\mathrm{d}z\,\mathbb{1}_{\{|y-z|\le CN^{-\beta}\}}\left\| a_y a_z \frac{(\mathcal{N}+3)^{1/2}}{\mathcal{R}+s}\omega_3\Phi_{N,M}\right\|^2 &= \left\langle \Phi_{N,M}, \mathrm{d}\Gamma_2(\mathbb{1}_{\{|y-z|\le CN^{-\beta}\}})\frac{\mathcal{N}+3}{(\mathcal{R}+s)^2}|\omega_3|^2\Phi_{N,M}\right\rangle \\ &\le \frac{C}{M(1+s)^2}\langle \Phi_{N,M}, \mathrm{d}\Gamma_2(\mathbb{1}_{\{|y-z|\le CN^{-\beta}\}})\Phi_{N,M}\rangle. \end{aligned} \tag{5-81}$$

Using Lemma 11 with $s=2\beta/(2\beta-\varepsilon)$, we obtain

$$\mathrm{d}\Gamma_2(\mathbb{1}_{\{|y-z|\le CN^{-\beta}\}}) \le C_\varepsilon N^{\varepsilon}N^{-2\beta}\mathcal{N}\,\mathrm{d}\Gamma_1(1-\Delta) \tag{5-82}$$

for every $\varepsilon>0$. Therefore, together with Lemma 8, we deduce that

$$\begin{aligned} \iiint \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,|w_N(x-y)|^{\varepsilon}|w_N(x-z)|^2\left\| a_y a_z \frac{(\mathcal{N}+3)^{1/2}}{\mathcal{R}+s}\omega_3\Phi_{N,M}\right\|^2 &\le \frac{CN^{4\beta}}{M(1+s)^2}\big\||w_N|^{\varepsilon}\big\|_{L^1}\langle \Phi_{N,M}, \mathrm{d}\Gamma_2(\mathbb{1}_{\{|y-z|\le CN^{-\beta}\}})\Phi_{N,M}\rangle \\ &\le \frac{C_{t,\varepsilon}N^{(2\beta+2)\varepsilon}}{(1+s)^2}. \end{aligned} \tag{5-83}$$

Inserting (5-79) and (5-83) in (5-78) we find that

$$\mathcal{E}_{4,2} = \frac{C_{t,\varepsilon}}{\sqrt{N}}\int_0^\infty \mathrm{d}s\,\sqrt{s}\sqrt{\frac{1}{(1+s)^{1+\varepsilon}}}\sqrt{\frac{N^{(2\beta+2)\varepsilon}}{(1+s)^2}} \le \frac{C_{t,\varepsilon}N^{(\beta+1)\varepsilon}}{\sqrt{N}} \tag{5-84}$$

for every constant $\varepsilon>0$. From (5-77) and (5-84) we get

$$\mathcal{E}_4 \le C_{t,\varepsilon}\left(\frac{N^{\beta}}{\sqrt{N}M}+\frac{1}{\sqrt{N}}\right)N^{\varepsilon}. \tag{5-85}$$

Conclusion of (5-57). Given the decomposition (5-63), the desired bound (5-57) follows immediately from (5-69) and (5-85).

Step 4. It remains to prove (5-58). Similarly to (5-61), we can write

$$\langle \Phi_N, [\mathbb{G}_3^*, f_M^2]\Phi_{N,M}\rangle = \frac{1}{\sqrt{N}} \iint \mathrm{d}x\, \mathrm{d}y\, w_N(x-y)\overline{u_N(x)}\langle \Phi_N, a_y^\dagger a_x a_y \widetilde{\omega}_3 \Phi_{N,M}\rangle \tag{5-86}$$

with

$$\widetilde{\omega}_3 = \sqrt{1-\frac{\mathcal{N}}{N-1}}\Big(f^2\Big(\frac{\mathcal{N}-1}{M}\Big) - f^2\Big(\frac{\mathcal{N}}{M}\Big)\Big) \tag{5-87}$$

as an operator on $\mathcal{F}_\perp^{\le N}$, which satisfies

$$|\widetilde{\omega}_3| \le \frac{C}{M}\mathbb{1}^{\le M}. \tag{5-88}$$

By the Cauchy–Schwarz inequality,

$$\begin{aligned}
&\big|\langle \Phi_N, [\mathbb{G}_3^*, f_M^2]\Phi_{N,M}\rangle\big| \\
&\quad= \frac{1}{\sqrt{N}}\bigg|\iint \mathrm{d}x\, \mathrm{d}y\, w_N(x-y)\overline{u_N(x)}\langle (\mathcal{N}+1)^{-1/2}\Phi_N, a_y^\dagger a_x a_y \mathcal{N}^{1/2}\widetilde{\omega}_3 \Phi_{N,M}\rangle\bigg| \\
&\quad\le \frac{\|u_N\|_{L^\infty}}{\sqrt{N}} \iint \mathrm{d}x\, \mathrm{d}y\, |w_N(x-y)|\, \|a_y(\mathcal{N}+1)^{-1/2}\Phi_N\|\, \|a_x a_y \mathcal{N}^{1/2}\widetilde{\omega}_3\Phi_{N,M}\| \\
&\quad\le \frac{C_t}{\sqrt{N}}\Big(\iint \mathrm{d}x\, \mathrm{d}y\, |w_N(x-y)|\, \|a_y(\mathcal{N}+1)^{-1/2}\Phi_N\|^2\Big)^{1/2} \\
&\qquad\qquad\qquad \times \Big(\iint \mathrm{d}x\, \mathrm{d}y\, |w_N(x-y)|\, \|a_x a_y \mathcal{N}^{1/2}\widetilde{\omega}_3\Phi_{N,M}\|^2\Big)^{1/2} \\
&\quad= \frac{C_t}{\sqrt{N}}\langle \Phi_N, \|w_N\|_{L^1}\mathcal{N}(\mathcal{N}+1)^{-1}\Phi_N\rangle^{1/2}\big\langle \Phi_{N,M}, \mathrm{d}\Gamma_2(|w_N|)\mathcal{N}|\widetilde{\omega}_3|^2\Phi_{N,M}\big\rangle^{1/2} \\
&\quad\le \frac{C_{t,\varepsilon}N^\varepsilon}{\sqrt{N}}\|\Phi_N\|\langle \Phi_{N,M}, \mathrm{d}\Gamma_1(1-\Delta)\mathcal{N}^2\mathbb{1}^{\le M}M^{-2}\Phi_{N,M}\rangle^{1/2} \le \frac{C_{t,\varepsilon}}{\sqrt{N}}N^{2\varepsilon},
\end{aligned} \tag{5-89}$$

where we used Lemma 11 and the kinetic bound in Lemma 8. This concludes the proof of (5-58) and thus of Lemma 15. □

5.4. ***Conclusion of Proposition 9.*** First, inserting the bounds from Lemmas 13, 14 and 15 in (5-1), and using $M \le N$ to simplify some error terms, we find that the desired propagation bound (3-17) holds true, namely that

$$\Big|\frac{\mathrm{d}}{\mathrm{d}t}\langle \Phi_N(t), f_M^2\Phi_{N,M}(t)\rangle\Big| \le C_{t,\varepsilon}N^\varepsilon\Big(\frac{1}{\sqrt{M}} + \frac{N^\beta}{M^{3/2}}\Big).$$

Proof of Proposition 9. Define

$$\mathcal{B}(t) := 1 - \Re\langle \Phi_N(t), f_M^2\Phi_{N,M}(t)\rangle. \tag{5-90}$$

By (2-27) and by definition (3-16) of f_M, we obtain

$$\mathcal{B}(0) = \langle \Phi_0, (1-f_M^2)\Phi_0\rangle \le \langle \Phi_0, \mathbb{1}^{>M/2}\Phi_0\rangle \le \frac{2}{M}\langle \Phi_0, \mathcal{N}\Phi_0\rangle \le \frac{C}{M}. \tag{5-91}$$

Combining this with (3-17), we can therefore bound

$$\mathcal{B}(t) \le C_{t,\varepsilon}\left(\frac{1}{\sqrt{M}} + \frac{N^{\beta}}{M^{3/2}}\right) \tag{5-92}$$

for all $t \in [0, T_{\max})$ and $\varepsilon > 0$.

To conclude Proposition 9, we prove that

$$\|\Phi_N(t) - \Phi_{N,M}(t)\|^2 \le 4\mathcal{B}(t). \tag{5-93}$$

Let us drop the time dependence from the notation for simplicity and write

$$\begin{aligned}\|\Phi_N - \Phi_{N,M}\|^2 &= \|\Phi_N\|^2 + \|\Phi_{N,M}\|^2 - 2\,\Re\langle\Phi_N, \Phi_{N,M}\rangle \\ &\le 2 - 2\,\Re\langle\Phi_N, f_M^2\Phi_{N,M}\rangle - 2\,\Re\langle\Phi_N, g_M^2\Phi_{N,M}\rangle.\end{aligned} \tag{5-94}$$

Here we defined $g_M^2 = 1 - f_M^2$ and used that $\|\Phi_N\| \le 1$, $\|\Phi_{N,M}\| \le 1$. Moreover, by the Cauchy–Schwarz inequality,

$$\begin{aligned}2\,|\langle\Phi_N, g_M^2\Phi_{N,M}\rangle| &\le \|g_M\Phi_N\|^2 + \|g_M\Phi_{N,M}\|^2 \\ &\le 2 - \|f_M\Phi_N\|^2 - \|f_M\Phi_{N,M}\|^2 \le 2 - 2\,|\langle\Phi_N, f_M^2\Phi_{N,M}\rangle|.\end{aligned} \tag{5-95}$$

Thus, (5-93) follows immediately. □

6. Conclusion of the main theorems

6.1. ***Proof of Theorem 5.*** Let $M = N^{1-\delta}$ with $\delta \in (0,1)$. Recall that $\Phi_N(t)$ and $\Phi_{N,M}(t)$ are defined in (3-6) and (3-11), respectively. Since $U_N : \mathfrak{H}^N \to \mathcal{F}_\perp^{\le N}(t)$ is a unitary transformation, the desired norm approximation (2-29) is equivalent to

$$\|\Phi_N(t) - \Phi(t)\|^2 \le C_t N^{-\alpha_2}. \tag{6-1}$$

By Lemma 8 and Proposition 9, we can bound

$$\begin{aligned}\|\Phi_N(t) - \Phi(t)\|^2 &\le 2\,\|\Phi_N(t) - \Phi_{N,M}(t)\|^2 + 2\,\|\Phi_{N,M}(t) - \Phi(t)\|^2 \\ &\le C_{t,\varepsilon}N^{\varepsilon}\left(\frac{1}{\sqrt{M}} + \frac{N^{\beta}}{M^{3/2}} + \sqrt{\frac{M}{N}}\right) \\ &= C_{t,\varepsilon}N^{\varepsilon}(N^{(\delta-1)/2} + N^{3\delta/2+\beta-3/2} + N^{-\delta/2})\end{aligned} \tag{6-2}$$

for all $t \in [0, T_{\max})$ and $\varepsilon > 0$. Here we have put back $M = N^{1-\delta}$ at the end. The optimal choice for δ is

$$\delta = \begin{cases}\frac{1}{4}(3-2\beta) & \text{if } \beta \ge \frac{1}{2}, \\ \frac{1}{2} & \text{if } \beta \le \frac{1}{2},\end{cases} \tag{6-3}$$

which implies (6-1) with every $0 < \alpha_2 < \min\left(\frac{1}{8}, \frac{1}{16}(3-2\beta)\right)$. □

6.2. ***Proof of Theorem 3.*** The implication of the convergence of density matrices from the norm convergence is well-known; see, e.g., [Lewin et al. 2015a, Corollary 2]. Here we recall a quick derivation for the reader's convenience. We will again drop the time dependence from the notation. Let $q = 1 - p = 1 - |u_N\rangle\langle u_N|$ as in (2-25). By using the rules (3-5) (see also Remark 6), Theorem 5 and Lemma 7, it follows that

$$N\,\mathrm{Tr}(q\gamma^{(1)}_{\Psi_N}q) = \|\sqrt{\mathcal{N}}\Phi_N\|^2 \le 2\,\|\sqrt{\mathcal{N}}\mathbb{1}^{\le N}(\Phi_N - \Phi)\|^2 + 2\,\|\sqrt{\mathcal{N}}\Phi\|^2 \le C_{t,\varepsilon}(N^{1-2\alpha_2} + N^{\varepsilon}).$$

Then by the triangle and Cauchy–Schwarz inequalities, we conclude that

$$\begin{aligned}\mathrm{Tr}\left|\gamma^{(1)}_{\Psi_N} - |\varphi\rangle\langle\varphi|\right| &\le \mathrm{Tr}\left|p - |\varphi\rangle\langle\varphi|\right| + \mathrm{Tr}\,|p(\gamma^{(1)}_{\Psi_1} - 1)p| + \mathrm{Tr}\,|q\gamma^{(1)}_{\Psi_1}q| + 2\,\mathrm{Tr}\,|p\gamma^{(1)}_{\Psi_1}q| \\ &\le 2\,\|u_N - \varphi\|_{L^2} + 2\,\mathrm{Tr}(q\gamma^{(1)}_{\Psi_1}q) + 2\sqrt{\mathrm{Tr}\,|q\gamma^{(1)}_{\Psi_1}q|}\sqrt{\mathrm{Tr}\,|p\gamma^{(1)}_{\Psi_1}p|} \\ &\le C_{t,\varepsilon}(N^{-\beta} + N^{-\alpha_2}N^{(\varepsilon-1)/2}).\end{aligned}$$

Here we used $\mathrm{Tr}(p) = \mathrm{Tr}\,\gamma^{(1)}_{\Psi_N} = 1$ and Lemma 10. Thus (2-6) holds for $\alpha_1 = \min(\beta, \alpha_2)$. ☐

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