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LOWER BOUNDS ON FIBERED YANG–MILLS FUNCTIONALS: GENERIC NEFNESS AND SEMISTABILITY OF DIRECT IMAGES

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The main goal of this paper is to generalize a part of the relationship between mean curvature and Harder–Narasimhan filtrations of holomorphic vector bundles to arbitrary polarized fibrations. More precisely, for a polarized family of complex projective manifolds, we establish lower bounds on a fibered version of Yang–Mills functionals in terms of the Harder–Narasimhan slopes of direct image sheaves associated with high tensor powers of the polarization. We discuss the optimality of these lower bounds and, as an application, provide an analytic characterisation of a fibered version of generic nefness. As another application, we refine the existent obstructions for finding metrics with constant horizontal mean curvature. The study of the semiclassical limit of Hermitian Yang–Mills functionals lies at the heart of our approach.

1. Introduction

Consider a holomorphic submersion $\pi : X \to B$ between compact complex manifolds X and B of dimensions $n+m$ and m, respectively, $n, m \in \mathbb{N}$. Let L be a holomorphic line bundle over X, which is relatively ample with respect to π. We fix a *Gauduchon Hermitian form* ω_B on B, i.e., a positive $(1,1)$-form, such that $\partial\bar{\partial}\omega_B^{m-1} = 0$; see [Gauduchon 1977]. The main goal of this paper is to study the relationship between the so-called horizontal mean curvature of the fibration, which is a certain differential-geometric invariant of the family defined using ω_B, and Harder–Narasimhan ω_B-slopes of direct images $E_k := R^0\pi_* L^k$, which are algebraic invariants.

More precisely, consider a Hermitian metric h^L on L, which is positive along the fibers of π. We denote by

$$\omega(h^L) := \frac{\sqrt{-1}}{2\pi} R^L$$

the first Chern form of (L, h^L), where R^L is the curvature of the Chern connection. When h^L is clear from the context, we omit it from the above notation.

As ω is positive along the fibers, it provides a (smooth) decomposition of the tangent space TX of X into the vertical component $T^V X$, corresponding to the tangent space of the fibers, and the horizontal component $T^H X$, corresponding to the orthogonal complement of $T^V X$ with respect to ω. The form ω then decomposes as $\omega = \omega_V + \omega_H$, $\omega_V \in \mathscr{C}^\infty\big(X, \bigwedge^{1,1} T^{V*} X\big)$, $\omega_H \in \mathscr{C}^\infty\big(X, \bigwedge^{1,1} T^{H*} X\big)$. Upon the natural identification of $T^H X$ with $\pi^* TB$, we may view ω_H as an element from $\mathscr{C}^\infty\big(X, \bigwedge^{1,1} \pi^* T^* B\big)$.

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The triple $(\pi, \omega, T^H X)$ then defines a *Kähler fibration* in the sense of [Bismut et al. 1988, Definition 1.4]. We define the *horizontal mean curvature*, $\bigwedge_{\omega_B} \omega_H(h^L) \in \mathscr{C}^\infty(X)$, as

$$\bigwedge_{\omega_B} \omega_H(h^L) := \frac{\omega_H(h^L) \wedge \omega_B^{m-1}}{\omega_B^m}. \tag{1-1}$$

We say that h^L is *fibered Einstein* if $\bigwedge_{\omega_B} \omega_H(h^L)$ is a constant. By decomposition into horizontal and vertical components, it is easy to see that this condition is equivalent to

$$\omega(h^L)^{n+1} \wedge \pi^* \omega_B^{m-1} = c \cdot \omega(h^L)^n \wedge \pi^* \omega_B^m, \tag{1-2}$$

where c is a constant. By integrating (1-2), we see that the constant c is independent of h^L, since ω_B is Gauduchon; see Section 4 for details. Remark the similarity of (1-2) with the J-equation if $m = 1$, and with optimal symplectic connection equation of Dervan and Sektnan [2021, Proposition 2.7] if the fibers are Fano. For families of manifolds given by projectivizations of vector bundles, the condition (1-2) was introduced by Kobayashi [1996], who called such metrics *Finsler–Einstein* metrics; then Feng, Liu, and Wan [Feng et al. 2019] generalized it for general Kähler submersions, and called such metrics *geodesic Einstein* metrics. If instead of a closed manifold B, one considers manifolds with boundary, equation (1-2) was studied extensively in the past: If B is a 1-dimensional annuli and $c = 0$, this is a geodesic equation in Mabuchi space [1987]; see [Semmes 1992] or [Donaldson 1999]. If B is a bounded smooth strongly pseudoconvex domain in $\mathbb{C}^n$, $c = 0$ and ω_B is the standard Kähler form, (1-2) was called Wess–Zumino–Witten equation in [Donaldson 1999] due to its connection with [Witten 1983].

Remark that in the important case when $X := \mathbb{P}(E^*)$ for some holomorphic vector bundle E over B, $L := \mathscr{O}(1)$, and h^L is induced by a Hermitian metric h^E on E, (L, h^L) is fibered Einstein if and only if (E, h^E) is Hermite–Einstein; see Remark 3.4 for details.

The first main observation of this paper is that the correspondence between fibered Einstein and Hermite Einstein equations is much tighter. Indeed, as we shall see, relying on the work of Ma and Zhang [2023], see Theorem 2.2, the fibered Einstein equation for L is *mutatis mutandis* the semiclassical limit (i.e., $k \to \infty$) of the Hermite–Einstein equation for $E_k := R^0 \pi_* L^k$. Let us now explain the first manifestation of this correspondence.

Recall that a *slope* (or ω_B-slope) of a coherent sheaf $\mathscr{E}$ over B is defined as $\mu(\mathscr{E}) := \deg(\mathscr{E})/\mathrm{rk}(\mathscr{E})$, where the degree, $\deg(\mathscr{E})$, is given by $\deg(\mathscr{E}) := \int_B [c_1(\det \mathscr{E})] \cdot [\omega_B^{m-1}]$; here and after the intersection product is for Bott–Chern and Aeppli cohomology classes, ω_B^{m-1} represents an Aeppli cohomology class since ω_B is Gauduchon, see Section 4 for details, and $\det \mathscr{E}$ is the Knudsen–Mumford determinant of $\mathscr{E}$, see [Knudsen and Mumford 1976]. A torsion-free coherent sheaf $\mathscr{E}$ is called *semistable* or *ω_B-semistable* if for every coherent subsheaf $\mathcal{F}$ of $\mathscr{E}$, verifying $\mathrm{rk}(\mathcal{F}) > 0$, we have $\mu(\mathcal{F}) \leq \mu(\mathscr{E})$. When $\dim B = 1$, these notions clearly do not depend on ω_B.

Theorem 1.1. *Assume that L admits an approximate fibered Einstein metric, i.e., there is $c \in \mathbb{R}$ such that, for any $\epsilon > 0$, there is a relatively positive metric h^L_ϵ on L, verifying the bound*

$$\left| \bigwedge_{\omega_B} \omega_H(h^L_\epsilon) - c \right| < \epsilon. \tag{1-3}$$

Then the vector bundles $E_k := R^0\pi_* L^k$ are ***asymptotically semistable***, *i.e., for any quotient sheaves $\mathcal{Q}_k$ of E_k, $\mathrm{rk}(\mathcal{Q}_k) > 0$, and any $\epsilon > 0$ for k big enough, we have $\mu(\mathcal{Q}_k) \geq \mu(E_k) - \epsilon k$.*

Remark 1.2. (a) It is likely that there is an even closer relationship between fibered Einstein metrics on L and Hermite Einstein metrics on E_k, paralleling the known correspondence between constant scalar curvature and balanced metrics; see [Donaldson 2001].

(b) We conjecture that the converse of Theorem 1.1 also holds. In fact, this will follow as a special case of a more general conjecture, which we discuss after Theorem 1.7.

When Theorem 1.1 is applied to $\pi : \mathbb{P}(E^*) \to B$ for some vector bundle E over B, and $L := \mathscr{O}(1)$, due to a precise relation between the slopes of $\mathrm{Sym}^k E = R^0\pi_* L^k$, $k \in \mathbb{N}$, and E, see [Chen 2011, §3.2], we recover the well-known fact, see [Kobayashi 2014, Theorem 6.10.13], that if E admits approximate Hermite–Einstein metrics, then E is semistable.

The asymptotic semistability condition from Theorem 1.1 seems rather difficult to verify at first sight. We will now discuss some numerical obstructions for it. More precisely, for an irreducible complex analytic subspace $Y \subset X$ of dimension $k + m$, $k \geq 0$, such that the restriction of π to Y, $\pi|_Y : Y \to B$, is a surjection, we define the ω_B-slope, $\mu(Y)$, as

$$\mu(Y) = \frac{1}{k+1} \cdot \frac{\int_Y [c_1(L)^{k+1}] \cdot [\omega_B^{m-1}]}{\int_Y [c_1(L)^k] \cdot [\omega_B^m]}. \tag{1-4}$$

By the Serre vanishing theorem, for $k \in \mathbb{N}^*$ big enough, the sheaf $\mathcal{Q}_k := R^0\pi|_{Y,*} L|_Y^k$ can be realized as a quotient of E_k through the restriction map; see the proof of Proposition 4.7 for details. By the asymptotic version of the Riemann–Roch–Grothendieck theorem, see Theorem 4.1, which we establish in our singular setting, we can calculate the asymptotics of the slopes of $\mathcal{Q}_k$ and E_k, as $k \to \infty$. By comparing the asymptotics of these slopes, we obtain in Section 4 the following result.

Theorem 1.3. *If the vector bundles E_k are asymptotically semistable, then X is* ***numerically semistable***, *i.e., for any Y as above, we have $\mu(Y) \geq \mu(X)$. Moreover, if $\dim B = 1$, then asymptotic semistability of E_k is equivalent to numerical semistability of X.*

Remark 1.4. A combination of Theorems 1.1 and 1.3 shows that existence of approximate fibered Einstein metrics on L implies $\mu(Y) \geq \mu(X)$ for Y above. Feng, Liu, and Wan [Feng et al. 2019, Theorem 2.2], see also [Wan and Wang 2020], established this by different means under an assumption, requiring among others that the projection of the singular locus of Y to B has codimension at least 2.

As we explain later, Theorem 1.1 is a direct consequence of a more refined result concerning lower bounds on fibered Yang–Mills functionals. More precisely, for a relatively Kähler $(1, 1)$-form ω on X and any $c \in \mathbb{R}$, $p \in [1, +\infty[$, we define the fibered Yang–Mills functional as

$$\begin{aligned} \mathrm{FYM}_{p,c}(\pi, \omega) &:= \int_X \big|\textstyle\bigwedge_{\omega_B} \omega_H(x) - c\big|^p \omega^n \wedge \pi^* \omega_B^m(x), \\ \mathrm{FYM}_{+\infty,c}(\pi, \omega) &:= \sup_{x \in X} \big|\textstyle\bigwedge_{\omega_B} \omega_H(x) - c\big|. \end{aligned} \tag{1-5}$$

We also let $\mathrm{FYM}_{p,c}(\pi, h^L) := \mathrm{FYM}_{p,c}(\pi, c_1(L, h^L))$ for a relatively positive metric h^L on L. We will now show that asymptotic Harder–Narasimhan slopes of E_k, as $k \to \infty$, yield lower bounds for these functionals. To readers familiar with Hermitian Yang–Mills theory, see [Atiyah and Bott 1983, Proposition 8.20; Donaldson 1985, Proposition 5; Daskalopoulos and Wentworth 2004, §§2.3, 2.4], this will sound very natural. Indeed, again from the work of Ma and Zhang [2023], one can view the horizontal mean curvature of L as the semiclassical limit (i.e., $k \to \infty$) of the mean curvature of E_k. From this, we establish the lower bounds on the fibered Yang–Mills functionals through the limits of the lower bounds on the Hermitian Yang–Mills functionals of E_k.

To explain this in detail, recall first that any torsion-free coherent sheaf $\mathscr{E}$ on $(B, [\omega_B])$ admits a unique filtration by subsheaves $\mathscr{E}_i$, $i = 1, \dots, s$, also called a *Harder–Narasimhan filtration*,

$$\mathscr{E} = \mathscr{E}_s \supset \mathscr{E}_{s-1} \supset \cdots \supset \mathscr{E}_1 \supset \mathscr{E}_0 = \{0\}, \tag{1-6}$$

such that, for any $1 \leq i \leq s-1$, the quotient sheaf $\mathscr{E}_i/\mathscr{E}_{i-1}$ is the maximal semistable (torsion-free coherent) subsheaf of $\mathscr{E}/\mathscr{E}_{i-1}$, i.e., for any subsheaf of $\mathcal{F}$ of a (torsion-free coherent) sheaf $\mathscr{E}/\mathscr{E}_{i-1}$, we have $\mu(\mathcal{F}) \leq \mu(\mathscr{E}_i/\mathscr{E}_{i-1})$ and $\mathrm{rk}(\mathcal{F}) \leq \mathrm{rk}(\mathscr{E}_i/\mathscr{E}_{i-1})$ if $\mu(\mathcal{F}) = \mu(\mathscr{E}_i/\mathscr{E}_{i-1})$. For the proof of this result in the setting of the Gauduchon form ω_B, see either [Bruasse 2001] or [Greb et al. 2016, Corollary 2.27]. We define the *Harder–Narasimhan slopes*, $\mu_1, \dots, \mu_{\mathrm{rk}(\mathscr{E})}$ of $\mathscr{E}$, such that $\mu(\mathscr{E}_i/\mathscr{E}_{i-1})$ appears among $\mu_1, \dots, \mu_{\mathrm{rk}(\mathscr{E})}$ exactly $\mathrm{rk}(\mathscr{E}_i/\mathscr{E}_{i-1})$ times, and the sequence $\mu_1, \dots, \mu_{\mathrm{rk}(\mathscr{E})}$ is nonincreasing. We let $\mu_{\min} := \mu_{\mathrm{rk}(\mathscr{E})}$, $\mu_{\max} := \mu_1$.

We let $N_k := \mathrm{rk}(E_k)$, and let $\mu_1^k, \dots, \mu_{N_k}^k$ be the Harder–Narasimhan slopes of E_k, and $\mu_{\min}^k$, $\mu_{\max}^k$ be the minimal and the maximal slopes. Define the probability measure η_k^{HN} on $\mathbb{R}$ as

$$\eta_k^{\mathrm{HN}} := \frac{1}{N_k} \sum_{i=1}^{N_k} \delta\left[\frac{\mu_i^k}{k}\right], \tag{1-7}$$

where $\delta[x]$ is the Dirac mass at $x \in \mathbb{R}$. Our lower bounds for the fibered Yang–Mills functionals will build upon the following result.

Theorem 1.5. *The sequence of measures η_k^{HN} converges weakly, as $k \to \infty$, to a probability measure η^{HN} on $\mathbb{R}$, and the limits below exist and relate with η^{HN} as follows*:

$$\eta_{\min}^{\mathrm{HN}} := \lim_{k\to\infty} \frac{\mu_{\min}^k}{k} \leq \operatorname{ess\,inf} \eta^{\mathrm{HN}}, \quad \eta_{\max}^{\mathrm{HN}} := \lim_{k\to\infty} \frac{\mu_{\max}^k}{k} = \operatorname{ess\,sup} \eta^{\mathrm{HN}}. \tag{1-8}$$

Remark 1.6. The proof of Theorem 1.5 follows the arguments from [Chen 2010; Finski 2024b, Theorem 1.1], establishing Theorem 1.5 in the projective setting for flat maps $\pi : X \to B$, for $\dim B = 1$ and $\dim B \geq 1$, respectively. The only difference is that due to the lack of algebraicity, the proofs from [Chen 2010; Finski 2024b] of the linear upper bound on $\mu_{\max}^k$ in $k \in \mathbb{N}^*$, crucial for Theorem 1.5, do not work. Here this bound is obtained by a differential-geometric argument; see Proposition 2.3.

We are finally ready to state our lower bounds for the fibered Yang–Mills functionals.

Theorem 1.7. *For any relatively positive metric h^L on L, we have*

$$\inf_{x\in X} \wedge_{\omega_B}\omega_H(x) \leq \eta^{\mathrm{HN}}_{\min}, \quad \sup_{x\in X} \wedge_{\omega_B}\omega_H(x) \geq \eta^{\mathrm{HN}}_{\max}. \tag{1-9}$$

If, moreover, ω_B is Kähler, then for any $c \in \mathbb{R}$, $p \in [1, +\infty[$, we have

$$\mathrm{FYM}_{p,c}(\pi, h^L) \geq \int_{\mathbb{R}} |x-c|^p d\eta^{\mathrm{HN}}(x) \cdot \int_X [\omega^n]\cdot \pi^*[\omega_B^m]. \tag{1-10}$$

Remark 1.8. (a) As we shall establish in Proposition 4.5, $\eta^{\mathrm{HN}}_{\min} = \eta^{\mathrm{HN}}_{\max}$ if and only if E_k, $k \in \mathbb{N}$, are asymptotically semistable. Hence, (1-9) refines Theorem 1.1.

(b) The left hand-side of (1-10) depends on h^L, but the right-hand side doesn't.

(c) When $X := \mathbb{P}(E^*)$ for some holomorphic vector bundle E over B, $L := \mathscr{O}(1)$, and h^L is induced by a Hermitian metric h^E on E, the result can be deduced from the lower bounds on the Hermitian Yang–Mills functionals due to Atiyah and Bott [1983] and Daskalopoulos and Wentworth [2004].

Note that similar lower bounds in the context of constant scalar curvature metrics were obtained by Donaldson [2005] for the Calabi functional. Here, as in [Donaldson 2005], we expect the bounds from Theorem 1.7 to be tight. In other words, it seems likely that the following conjecture holds.

Conjecture. *In the notation of Theorem 1.7, for any $p \in [1, +\infty[$, $c \in \mathbb{R}$, we have*

$$\begin{aligned} \inf_{h^L} \mathrm{FYM}_{p,c}(\pi, h^L) &= \int_{\mathbb{R}} \left|x-c\right|^p d\eta^{\mathrm{HN}}(x) \cdot \int_X [\omega^n]\cdot \pi^*[\omega_B^m], \\ \inf_{h^L} \mathrm{FYM}_{+\infty,c}(\pi, h^L) &= \max\{|\eta^{\mathrm{HN}}_{\min} - c|, |\eta^{\mathrm{HN}}_{\max} - c|\}, \end{aligned} \tag{1-11}$$

where the infimum is taken among all relatively positive metrics h^L on L.

Remark 1.9. In the recent paper [Finski 2024a], the author established the Conjecture for $p = 1$.

By Remark 1.8(a), if the Conjecture holds for $p = +\infty$, then the converse implication of Theorem 1.1 also follows, upon taking $c := \eta^{\mathrm{HN}}_{\min} = \eta^{\mathrm{HN}}_{\max}$.

For $\pi : \mathbb{P}(E^*) \to B$, $L = \mathscr{O}(1)$, where E is some holomorphic vector bundle over a complex compact manifold B, one can show that the Conjecture holds by the existence of the approximate critical hermitian structures on vector bundles, see [Daskalopoulos and Wentworth 2004, Definition 3.9], and the calculation of the asymptotic slopes of $\mathrm{Sym}^k E = R^0\pi_* L^k$, due to Chen [2011, Theorem 1.2]. The following result is another partial justification of the Conjecture.

Theorem 1.10. *The identity* $\sup_{h^L} \inf_{x\in X} \wedge_{\omega_B}\omega_H(x) = \eta^{\mathrm{HN}}_{\min}$ *holds, where the supremum is taken among all relatively positive metrics h^L on L.*

Remark 1.11. By [Li et al. 2022, Theorem 1.5], when $X := \mathbb{P}(E^*)$ for some holomorphic vector bundle E over B, and $L := \mathscr{O}(1)$, the above theorem remains valid even if one restricts attention to metrics h^L that are induced by a Hermitian metric h^E on E.

We assume now that B (and, hence, X) is projective. Recall that a line bundle L on X is called *nef* if for any irreducible curve C in X, $\int_C c_1(L) \geq 0$. It is well-known that this condition is equivalent to the existence of metrics with an arbitrarily small negative part of the curvature, as stated precisely in [Demailly 1992, Proposition 4.2]. One of the central concerns in complex geometry is the study of variations of this result, which provides a "dictionary" between the algebraic and analytic definitions of positivity. Let us now explain an application of Theorem 1.10 in this context.

We fix a very ample integral multipolarization $[\omega_{B,1}], \dots, [\omega_{B,m-1}]$ on B, which is a collection of very ample classes from $H^{1,1}(B, \mathbb{C}) \cap H^2(B, \mathbb{Z})$. We say that a $\mathbb{Q}$-line bundle L on X is $([\omega_{B,1}], \dots, [\omega_{B,m-1}])$-*generically fibered nef with respect to* π if there is $l_0 \in \mathbb{N}^*$ such that, for any regular curve $C = C(l) \subset B$, $l = (l_1, \dots, l_{m-1}) \in \mathbb{N}^{*(m-1)}$, $l_i \geq l_0$, $i = 1, \dots, m-1$, given by a complete intersection of *generic* divisors from classes $l_1[\omega_{B,1}], \dots, l_{m-1}[\omega_{B,m-1}]$, the restriction of $c_1(L)$ to $\pi^{-1}(C)$ is nef. When π is the projectivization of a vector bundle, an equivalent definition was given by Miyaoka [1987], see also [Peternell 2012]. The general case was introduced in [Finski 2024b]. We say L is *stably* $([\omega_{B,1}], \dots, [\omega_{B,m-1}])$-*generically fibered nef with respect to* π if for some (or any) ample line bundle L_0 on X, for any $\epsilon > 0$, $\epsilon \in \mathbb{Q}$, the $\mathbb{Q}$-line bundle $L \otimes L_0^{\epsilon}$ is $([\omega_{B,1}], \dots, [\omega_{B,m-1}])$-generically fibered nef with respect to π. Recall that L is called *relatively nef with respect to* π if its restriction to every fiber is nef. As we explain in Section 3, from the previously obtained algebraic description of $\eta^{\mathrm{HN}}_{\min}$ from [Finski 2024b, Corollary 1.4], Theorem 1.10 can be used to prove the following result.

Theorem 1.12. *Consider a holomorphic submersion* $\pi : X \to B$ *between projective manifolds* B, X. *A relatively nef line bundle* L *on* X *is stably* $([\omega_{B,1}], \dots, [\omega_{B,m-1}])$-*generically fibered nef if and only if for any* (*or some*) *Kähler forms* $\omega_{B,1}, \dots, \omega_{B,m-1}$ *on* B *in* $[\omega_{B,1}], \dots, [\omega_{B,m-1}]$, *and any* (*or some*) *Kähler form* ω_X *on* X, *for any* $\epsilon > 0$, *there is a Hermitian metric* h^L_ϵ *on* L, *such that*

$$\omega(h^L_\epsilon) \wedge \pi^* \omega_{B,1} \wedge \cdots \wedge \pi^* \omega_{B,m-1} \geq -\epsilon \cdot \omega_X^m, \tag{1-12}$$

where by this we mean that the volume forms obtained by the restriction to every m-*dimensional complex hyperplane of each side of* (1-12) *compares as required in* (1-12); *see* [Demailly 2012, (III.1.6)].

Remark 1.13. Curiously, even though the forms $\omega_{B,1}, \dots, \omega_{B,m-1}$ are Kähler, if these forms are different, then in the proof of Theorem 1.12, we need to apply Theorem 1.10 for a non-Kähler Gauduchon form ω_B, constructed from $\omega_{B,1}, \dots, \omega_{B,m-1}$. This was our main motivation to write this article in the current generality. However, Theorem 1.12 is new even if the forms are equal.

In conclusion, it seems for us that a proof of the Conjecture might rely either on the techniques of geometric flows or continuity method as in [Donaldson 1985; Uhlenbeck and Yau 1986]. In this vein, the recent a priori bounds for Monge–Ampère and Hessian equations established by Guo, Phong, and Tong [2023] and Guo, Phong, Tong, and Wang [2021] will likely play an important role.

Note that for a Hermitian Yang–Mills functional, the analogous conjecture holds due to results of Atiyah and Bott [1983], Daskalopoulos and Wentworth [2004], Sibley [2015], Jacob [2016] and Li, Zhang, and Zhang [Li et al. 2022]; see Theorem 2.5 for a precise statement. We also mention the recent works of Xia [2021], Hisamoto [2023] and Dervan and Székelyhidi [2020], see also [Collins et al. 2022], proving versions of the Conjecture in the context of constant scalar curvature metrics.

In a different direction, when B is a bounded smooth strongly pseudoconvex domain in $\mathbb{C}^n$, $c = 0$ and ω_B is the standard Kähler form on $B \subset \mathbb{C}^n$, Donaldson [1992] and Coifman and Semmes [1993] established that Dirichlet problem associated with the Hermite–Einstein equation has solutions for any vector bundle over B (in particular for E_k, $k \in \mathbb{N}$). Wu [2023] showed that the Dirichlet problem associated with (1-2) always has weak solutions, and these solutions can be obtained as the semiclassical limit of the solutions of the Hermite–Einstein equations on E_k. See also [Phong and Sturm 2006; Rubinstein and Zelditch 2010; Song and Zelditch 2010] for earlier results in this direction. In other words, a phenomenon similar to Theorem 1.7 is present: there is a relation between the Hermite–Einstein and the Wess–Zumino–Witten equations. The major difference between these developments and our paper is that in our boundaryless setting, neither Hermite–Einstein equations nor fibered Einstein equations have solutions in general, and the methods of [Donaldson 1992; Coifman and Semmes 1993; Wu 2023] do not apply.

This article will be organized as follows. In Section 2, we will establish Theorems 1.5 and 1.7. We discuss how horizontal curvature behaves with respect to a restriction to a subfamily in Section 3, and using this, we establish Theorems 1.10 and 1.12. Finally, in Section 4, we establish a numerical obstruction for asymptotic semistability of direct images from Theorem 1.3.

2. Fibered Yang–Mills functionals through the semiclassical limit

The main goal of this section is to prove Theorems 1.5 and 1.7. The theory of Toeplitz operators and Hermitian Yang–Mills theory, which we recall below, will be particularly useful for that.

We begin by recalling some facts about Toeplitz operators. Let Y be a complex projective manifold of dimension n with an ample line bundle L. We fix a positive Hermitian metric h^L on L. We denote by ω its first Chern form, $c_1(L, h^L)$. For smooth sections f, f' of L^k, $k \in \mathbb{N}$, over Y, we define the L^2-scalar product using the pointwise scalar product $\langle \cdot, \cdot \rangle_{h^L}$ induced by h^L as

$$\langle f, f' \rangle_{L^2(Y)} := \int_Y \langle f(x), f'(x) \rangle_{h^{L^{\otimes k}}} \cdot \omega^n(x). \tag{2-1}$$

Recall that the *Bergman projector* B_k is given by the orthogonal projection (with respect to the scalar product (2-1)) from the space of L^2-sections of L^k to $H^0(Y, L^k)$. For any bounded function f on Y, we then define the Toeplitz operator, $T_k(f) : H^0(Y, L^k) \to H^0(Y, L^k)$, as

$$T_k(f)(s) = B_k(f \cdot s), \quad s \in H^0(Y, L^k). \tag{2-2}$$

Proposition 2.1. *For any bounded function $f : Y \to \mathbb{R}$, we have the inequalities*

$$\inf f \cdot \mathrm{Id}_{H^0(Y,L^k)} \leq T_k(f) \leq \sup f \cdot \mathrm{Id}_{H^0(Y,L^k)}, \tag{2-3}$$

where by $A \leq B$ we mean that the difference $B - A$ is positive definite. Moreover, if f is smooth, then for any continuous function $\phi : \mathbb{R} \to \mathbb{R}$, we have

$$\begin{aligned} \lim_{k\to\infty} \frac{\operatorname{Tr}[\phi(T_k(f))]}{\dim H^0(Y, L^k)} &= \frac{\int_{x\in Y} \phi(f(x))\omega^n(x)}{\int_Y [\omega^n]}, \\ \lim_{k\to\infty} \|\phi(T_k(f))\| &= \max\{|\sup \phi(f)|, |\inf \phi(f)|\}, \end{aligned} \tag{2-4}$$

where $\|\cdot\|$ is the operator norm.

Proof. The statement (2-3) follows from the trivial fact that if f is a positive function, then the operator $T_k(f)$ is positive-definite. The statement (2-4) is a restatement of the weak convergence of spectral measures of Toeplitz operators due to Boutet de Monvel and Guillemin [1981, Theorem 13.13]. For an alternative proof through Bergman kernel expansion, see [Ma and Marinescu 2007, Theorem 7.4.1], [Barron et al. 2014, Theorem 3.8] or [Finski 2022, Appendix A]. See also [Ma and Marinescu 2012; 2013] for generalizations and more refined results. □

Now, the reason why Toeplitz operators are relevant to this paper is because they appear as the principal term in the asymptotic expansion of the curvature of L^2-metrics on direct images of a polarized fibrations. More precisely, consider a proper holomorphic submersion $\pi : X \to B$ between complex manifolds X and B of dimensions $n+m$ and m, respectively, $n, m \in \mathbb{N}$. Let L be a holomorphic line bundle over X, which is relatively ample with respect to π. Endow L with a relatively positive Hermitian metric h^L. Let $k \in \mathbb{N}$ be large enough that

$$E_k := R^0\pi_* L^k \tag{2-5}$$

is locally free. The L^2-product (2-1) then defines a smooth Hermitian metric h^{E_k} on E_k. We denote by $R^{E_k} \in \mathscr{C}^\infty\big(B, \bigwedge^2 T^*B \otimes \mathrm{End}(E_k)\big)$ the curvature of its Chern connection.

Theorem 2.2 [Ma and Zhang 2023, Theorem 0.4]. *There are $C > 0$, $k_0 \in \mathbb{N}$, such that, for any $k \geq k_0$,*

$$\left\| \frac{\sqrt{-1}}{2\pi} R^{E_k} - k \cdot T_k(\omega_H) \right\| \leq C, \tag{2-6}$$

where $\|\cdot\|$ is the operator norm, and we naturally extended the definition of Toeplitz operators from functions to bounded sections of $\pi^*\bigwedge^2 T^*B$ as follows: for a decomposition $\omega_H = \sum f_{ij} dz_i d\bar{z}_j$, where $z_1, \dots, z_n$ are local coordinates on B, we let $T_k(\omega_H) := \sum T_k(f_{ij}) dz_i d\bar{z}_j$.

As we shall explain below, Theorem 2.2 is the crucial ingredient connecting fibered Yang–Mills functionals with Hermitian Yang–Mills functionals. But before this, let us mention another application of Theorem 2.2 to the study of Harder–Narasimhan slopes of direct images.

We fix now a *Gauduchon Hermitian form ω_B* on B. As before Theorem 1.5, we denote by $\mu^k_{\max}$ the maximal Harder–Narasimhan ω_B-slope of E_k.

Proposition 2.3. *There is $C > 0$, such that $\mu^k_{\max} \leq Ck$ for any $k \in \mathbb{N}^*$.*

Proof. For $p = 1, \dots, \mathrm{rk}(E_k)$, we denote by $R^{\bigwedge^p E_k}$ the curvature of the Chern connection on $\bigwedge^p E_k$, induced by the metric $h^{\bigwedge^p E_k}$ induced by h^{E_k}. By Theorem 2.2 and the very definition of $\bigwedge_{\omega_B}$ from (1-1), we conclude that there is $C > 0$ such that, for any $k \in \mathbb{N}^*$, we have

$$\frac{\sqrt{-1}}{2\pi} \textstyle\bigwedge_{\omega_B} R^{\bigwedge^p E_k} \leq Cpk \cdot \mathrm{Id}_{\bigwedge^p E_k}. \tag{2-7}$$

Let F be a line subbundle of $\bigwedge^p E_k$. We denote by h^F the Hermitian metric on F, induced by the metric h^{E_k}. By (2-7) and the well-known principle that curvature decreases in holomorphic subbundles, see [Demailly 2012, (V.14.6)], we deduce

$$\textstyle\bigwedge_{\omega_B} c_1(F, h^F) \leq Cpk. \tag{2-8}$$

However, it is classical, see [Kobayashi 2014, proofs of Lemma 5.7.16 and Theorem 5.8.3], that we have

$$\mu_{\max}^k \leq \max_{p=1,\dots,\mathrm{rk}(E_k)} \sup_{F\subset \bigwedge^p E_k} \frac{1}{p}\int_B c_1(F,h^F)\wedge\omega_B^{m-1}, \tag{2-9}$$

where the second supremum is taken over line subbundles F. We conclude by (2-8) and (2-9). □

Proof of Theorem 1.5. Taking into account the linear bound from Proposition 2.3, the proof of Theorem 1.5 is the same as in [Chen 2010; Finski 2024b, Theorem 1.1]. Let us briefly recall the main steps for completeness. We introduce the (nonincreasing) filtrations $\mathcal{F}_k(\lambda)$, $\lambda\in\mathbb{R}$, of E_k by coherent (torsion-free) subsheaves (defined over B), so that $\mathcal{F}_k(\lambda)$ is the maximal subsheaf of E_k such that all of its Harder–Narasimhan slopes are bigger than λ. The filtration $\mathcal{F}_k$ is just a "renaming" of the Harder–Narasimhan filtration of E_k. Now, for any $b\in B$, we denote by $\mathcal{F}_b$ the filtration induced by $\mathcal{F}_k(\lambda)$ on $R(X_b,L_b)=\bigoplus_{k=0}^{\infty}H^0(X_b,L_b^k)$ of the fiber $X_b=\pi^{-1}(b)$, $L_b=L|_{X_b}$, $b\in B$. It was established in [Chen 2010] for $\dim B=1$ and in [Finski 2024b, Proposition 2.5] for any projective B, that for generic $b\in B$, the above filtration is submultiplicative, i.e., for any $t,s\in\mathbb{R}$, $k,l\in\mathbb{N}$, we have

$$\mathcal{F}_b^t H^0(X_b,L_b^k)\cdot\mathcal{F}_b^s H^0(X_b,L_b^l)\subset\mathcal{F}_b^{t+s}H^0(X_b,L_b^{k+l}). \tag{2-10}$$

Observe, however, that the projectivity assumption was never used in [Finski 2024b, Proposition 2.5], and so submultiplicativity holds for general complex manifolds B. Theorem 1.5 is then a formal consequence of the submultiplicativity and Proposition 2.3, saying that the above filtration is bounded in the terminology of [Boucksom and Chen 2011]. For the (different) proofs of this last result, see [Chen 2010, théorème 3.4.3; Boucksom and Chen 2011, Theorem A; Finski 2025, Theorem 1.9]. □

Let us now recall some crucial facts from Hermitian Yang–Mills theory, following the pioneering work of Atiyah and Bott [1983] and later developments by Donaldson [1985], Daskalopoulos and Wentworth [2004], and others. We fix a compact complex manifold B of dimension m with a Gauduchon Hermitian form ω_B on B. Let E be a holomorphic vector bundle of rank r over B. For a Hermitian metric h^E on E, we denote by R^E its curvature. For any $p\in[1,+\infty[$, $c\in\mathbb{R}$, we define the *Hermitian Yang–Mills functional* as

$$\begin{aligned}\mathrm{HYM}_{p,c}(E,h^E)&:=\int_B \mathrm{Tr}\Big[\Big|\frac{\sqrt{-1}}{2\pi}\textstyle\bigwedge_{\omega_B}R_x^E-c\cdot\mathrm{Id}_E\Big|^p\Big]\omega_B^m(x),\\ \mathrm{HYM}_{+\infty,c}(E,h^E)&:=\sup_{x\in X}\Big\|\frac{\sqrt{-1}}{2\pi}\textstyle\bigwedge_{\omega_B}R_x^E-c\cdot\mathrm{Id}_E\Big\|,\end{aligned} \tag{2-11}$$

where $\|\cdot\|$ means the operator norm, and $|A|:=\sqrt{AA^*}$ for $A\in\mathrm{End}(V)$ on a Hermitian vector space (V,H).

As in (1-7), we denote the Harder–Narasimhan ω_B-slopes of E by $\mu_1,\dots,\mu_r$. Let $\mu_{\min}:=\mu_r$, $\mu_{\max}:=\mu_1$, be the minimal and the maximal slopes. We define the probability measure

$$\mu_E:=\frac{1}{r}\sum_{i=1}^r\delta[\mu_i]. \tag{2-12}$$

The following result lies at the heart of this paper.

Theorem 2.4. *For any $c \in \mathbb{R}$ and a Hermitian metric h^E on E, we have*

$$\mathrm{HYM}_{+\infty,c}(E, h^E) \geq \max\{|\mu_{\min} - c|, |\mu_{\max} - c|\}. \tag{2-13}$$

If, moreover, ω_B is Kähler, then for any $p \in [1, +\infty[$, we have

$$\mathrm{HYM}_{p,c}(E, h^E) \geq r \cdot \int_{\mathbb{R}} |x - c|^p d\mu_E(x) \tag{2-14}$$

For the proof of Theorem 2.4 for $p \in [1, +\infty[$, see [Atiyah and Bott 1983, Proposition 8.20] if $\dim B = 1$, [Daskalopoulos and Wentworth 2004, Lemma 2.17, Corollary 2.22, Proposition 2.25] if B is Kähler of any dimension (even though the article [Daskalopoulos and Wentworth 2004] is written for surfaces, see [Sibley 2015, §3.1]). For the proof of the first part, consult [Li et al. 2022, Theorem 1.5]. It is remarkable that the bounds from Theorem 2.4 are actually tight. Although we will not use this result in what follows, we state it for the reader's convenience, as it clarifies our motivation for the Conjecture.

Theorem 2.5. *In the notation Theorem 2.4, assume that ω_B is Kähler. Then for any $c \in \mathbb{R}$, $p \in [1, +\infty[$, we have*

$$\inf_{h^E} \mathrm{HYM}_{p,c}(E, h^E) = r \cdot \int_{\mathbb{R}} |x - c|^p d\mu_E(x) \cdot \int_B [\omega_B^m]. \tag{2-15}$$

where the infimum is taken over all Hermitian metrics h^E on E.

For the proof of Theorem 2.5 for $p \in [1, +\infty[$, see [Atiyah and Bott 1983, Proposition 8.20] if $\dim B = 1$. For higher dimensions, this is a direct consequence of the existence of L^p-approximate critical hermitian structure on E, see [Daskalopoulos and Wentworth 2004, Definition 3.9] for the definition, and [Daskalopoulos and Wentworth 2004, Theorem 3.11; Sibley 2015, Theorem 1.3] for the proofs if B is Kähler of dimension 2 and any dimension, respectively. See also [Jacob 2016, Theorems 2, 3].

Proof of Theorem 1.7. Preserving the notation introduced in (1-7), we define the probability measure, $\eta^{\mathrm{HN}}_{k,0}$, $k \in \mathbb{N}$, on $\mathbb{R}$ as

$$\eta^{\mathrm{HN}}_{k,0} := \frac{1}{N_k} \sum_{i=1}^{N_k} \delta[\mu_i^k], \tag{2-16}$$

We apply Theorem 2.4 for (E_k, h^{E_k}), $k \in \mathbb{N}$, $c \in \mathbb{R}$, to get

$$\mathrm{HYM}_{+\infty,ck}(E_k, h^{E_k}) \geq \max\{|\mu^k_{\min} - ck|, |\mu^k_{\max} - ck|\}. \tag{2-17}$$

If, moreover, ω_B is Kähler, then for any $p \in [1, +\infty[$, we have

$$\mathrm{HYM}_{p,ck}(E_k, h^{E_k}) \geq N_k \cdot \int_{\mathbb{R}} |x - ck|^p d\eta^{\mathrm{HN}}_{k,0}(x) \cdot \int_B [\omega_B^m]. \tag{2-18}$$

Directly from Theorem 2.2 and Proposition 2.1, under the respective assumptions, for any $p \in [1, +\infty[$, $c \in \mathbb{R}$, we have

$$\begin{aligned} \lim_{k \to \infty} \frac{\mathrm{HYM}_{p,ck}(E_k, h^{E_k})}{k^p \cdot N_k} &= \frac{\mathrm{FYM}_{p,c}(\pi, h^L)}{\int_{X_b} [\omega^n]}, \\ \lim_{k \to \infty} \frac{\mathrm{HYM}_{+\infty,ck}(E_k, h^{E_k})}{k} &= \mathrm{FYM}_{+\infty,c}(\pi, h^L), \end{aligned} \tag{2-19}$$

where $b \in B$ is an arbitrary point, and X_b is the fiber of π at b.

We now divide both sides of the first inequality of (2-17) by $k^p \cdot N_k$, take the limit $k \to \infty$, and apply Theorem 1.5 and (2-19) to deduce

$$\mathrm{FYM}_{p,c}(\pi, h^L) \geq \int_{\mathbb{R}} |x - c|^p d\eta^{\mathrm{HN}}(x) \cdot \int_{X_b} [\omega^n] \cdot \int_B [\omega_B^m]. \tag{2-20}$$

This establishes Theorem 1.7 for $p \in [1, +\infty[$, as $\int_{X_b} [\omega^n] \cdot \int_B [\omega_B^m] = \int_X [\omega^n] \cdot \pi^* [\omega_B^m]$. To get Theorem 1.7 for $p = +\infty$, we divide both sides of the second inequality of (2-17) by k, take the limit $k \to \infty$, and apply Theorem 1.5 and (2-19). □

3. Horizontal curvature on subfamilies and generic fibered nefness

The main goal of this section is to establish Theorems 1.10 and 1.12. For this, we construct a sequence of metrics on the polarizing line bundle from a sequence of Hermitian metrics on direct images. To show that horizontal mean curvature behaves well under this procedure, we rely on a fibered analogue of the principle that "a curvature of a vector bundle increases under taking quotients".

More precisely, consider a holomorphic submersion $\pi : X \to B$ between compact complex manifolds X and B. We let $m := \dim B$. Consider an embedding $\iota : Y \hookrightarrow X$ of a smooth complex manifold Y, such that restriction of π, $\pi|_Y : Y \to B$, is a submersion. We fix a $(1, 1)$-form ω_X on X, which is positive along the fibers of π, and let $\omega_Y := \iota^* \omega_X$. We fix a Hermitian $(1, 1)$-form ω_B on B, and denote by $\bigwedge_{\omega_B} \omega_{Y,H} \in \mathscr{C}^\infty(Y)$, $\bigwedge_{\omega_B} \omega_{X,H} \in \mathscr{C}^\infty(X)$, the horizontal mean curvatures of ω_Y and ω_X, respectively.

Lemma 3.1. *For any $y \in Y$, we have $\bigwedge_{\omega_B} \omega_{Y,H}(y) \geq \bigwedge_{\omega_B} \omega_{X,H}(\iota(y))$.*

Remark 3.2. An equivalent result was established in [Feng et al. 2019, (2.2)] by a slightly different method.

Proof. Let us fix $y \in Y$, and denote by $b := \pi|_Y(y)$, $x := \iota(y)$, and by $e_1, \dots, e_m$ an orthonormal basis of $T_b^{1,0} B$ with respect to ω_B. We denote by $e_1^X, \dots, e_m^X \in T_x^{1,0} X$ the horizontal lifts of $e_1, \dots, e_m$, defined with respect to ω_X, i.e., $d\pi(e_i^X) = e_i$, $i = 1, \dots, m$ and $e_1^X, \dots, e_m^X$ are orthogonal (with respect to ω_X) to the tangent space of the fibers, $T^V X$, of π. Similarly, we denote by $e_1^Y, \dots, e_m^Y \in T_y^{1,0} Y$ the horizontal lifts of $e_1, \dots, e_m$, defined with respect to ω_Y. Clearly, using implicitly the embedding of $T_y Y$ in $T_x X$ through ι, we can write $e_i^Y = e_i^X + v_i$, where $v_i \in T_x^V X$. But then, since e_i^X and v_i are orthogonal with respect to ω_X, and ω_X is positive in the vertical directions, we obtain $\sqrt{-1}\omega_Y(e_i^Y, \bar{e}_i^Y) = \sqrt{-1}\omega_X(e_i^X, \bar{e}_i^X) + \sqrt{-1}\omega_X(v_i, \bar{v}_i) \geq \sqrt{-1}\omega_X(e_i^X, \bar{e}_i^X)$. By taking a sum of the above inequality over all $i = 1, \dots, m$, we establish the needed inequality. □

Another ingredient we need is the calculation of the horizontal mean curvature for projectivizations of vector bundles. More precisely, let (F, h^F) be a Hermitian vector bundle over B of rank r. Let $\mathcal{O}(1)$ be the hyperplane bundle over $\mathbb{P}(F^*)$, $\pi : \mathbb{P}(F^*) \to X$. We endow $\mathcal{O}(1)$ with the metric $h^{\mathcal{O}(1)}$ induced by h^F. We denote by R^F the curvature of the Chern connection on (F, h^F), by ω the first Chern class of $(\mathcal{O}(1), h^{\mathcal{O}(1)})$, and by ω_H its horizontal component.

Lemma 3.3. *In the above notation, for any $x \in X$, we have*

$$\inf_{y \in \mathbb{P}(F_x^*)} \bigwedge\nolimits_{\omega_B} \omega_H(y) = \inf_{\substack{f \in F_x \\ \|f\|_{h^F}=1}} \left\langle \frac{\sqrt{-1}}{2\pi} \bigwedge\nolimits_{\omega_B} R^F f, f \right\rangle_{h^F}. \tag{3-1}$$

Proof. We fix some local coordinates $z := (z_1, \dots, z_n)$ on B, centered at $x \in B$, and a local normal frame $f_1, \dots, f_r$ of F at x, defined in a neighborhood U of x. By a *normal frame* we mean a holomorphic frame satisfying $\langle f_i, f_j \rangle_{h^F} = \delta_{ij} - \sum_{\lambda\mu} d_{\lambda\mu ij} z_\lambda \bar{z}_\mu + O(|z|^3)$ for some constants $d_{\lambda\mu ij}$. We denote by $f_1^*, \dots, f_r^*$ the dual frame of F^*. The above data defines a trivialization of $U \times \mathbb{P}(\mathbb{C}^r) \to \mathbb{P}(F^*)$ near $\pi^{-1}(x)$ as follows. For $a := (a_1, \dots, a_r)$, where $a_i \in \mathbb{C},\ 1 \le i \le r$, and not all a_i are equal to zero, the trivialization is given by the map $(z, [a]) \to \left[\sum_{i=1}^r a_i f_i^*(z)\right] \in \mathbb{P}(F^*)$. Now we take $a_1 = 1$ and let $b_i := a_i,\ 2 \le i \le r,\ b := (b_i)$. Then (z, b) gives a chart for $\mathbb{P}(F^*)$. The well-known formula, see [Demailly 2012, Formula (V.15.15)], shows that at the point $(x, [f_1^*]) \in \mathbb{P}(F^*)$, the curvature, $R^{\mathcal{O}(1)}$, of the hyperplane bundle $(\mathcal{O}(1), h^{\mathcal{O}(1)})$, equals

$$R^{\mathcal{O}(1)}_{(x,[f_1^*])} = \sum_{2 \le j \le r} db_j \wedge d\bar{b}_j + \langle R^F f_1, f_1 \rangle_{h^F}. \tag{3-2}$$

In particular, we see that the vertical part of the form $\omega = c_1(\mathcal{O}(1), h^{\mathcal{O}(1)})$ is the Fubini–Study form induced by h^F, and the horizontal part of ω, ω_H, evaluated at $(x, [f_1^*]) \in \mathbb{P}(E^*)$, coincides with $\sqrt{-1}/(2\pi)\langle R^F f_1, f_1 \rangle_{h^F}$. The result follows directly from this. □

Remark 3.4. From the proof of the above lemma, we see that $\bigwedge_{\omega_B} \omega_H$ is constant if and only if $\bigwedge_{\omega_B} R^F$ is the identity endomorphism up to a constant, which means that ω is fibered Einstein if and only if (F, h^F) is Hermite Einstein.

Recall also the following result.

Proposition 3.5 [Li et al. 2022, Theorem 1.5]. *For any $\epsilon > 0$, there is a Hermitian metric h^E_ϵ on E, such that the associated curvature, R^E_ϵ, for any $b \in B$, $e \in E_b$, verifies*

$$\frac{\sqrt{-1}}{2\pi} \bigwedge\nolimits_{\omega_B} R^E_\epsilon \ge (\mu_{\min} - \epsilon) \cdot \mathrm{Id}_E .$$

Proof of Theorem 1.10. First of all, for a given $\epsilon > 0$, let $k \in \mathbb{N}$ be such that $\mu^k_{\min}/k > \eta^{\mathrm{HN}}_{\min} - \frac{1}{2}\epsilon$. By Proposition 3.5, there is a metric h^ϵ_k on E_k such that, for the associated curvature, $R^{E_k}_\epsilon$, we have

$$\frac{\sqrt{-1}}{2\pi} \bigwedge\nolimits_{\omega_B} R^{E_k}_\epsilon \ge \left(\mu^k_{\min} - \tfrac{1}{2}\epsilon\right) \cdot \mathrm{Id}_{E_k} . \tag{3-3}$$

We denote by ω_k the $(1, 1)$-form on $\mathbb{P}(E_k^*)$, given by the first Chern class of the curvature of the hyperplane line bundle induced by the metric h^ϵ_k. We denote by $\omega_{H,k}$ the horizontal part of this curvature. From Lemma 3.3, the choice of $k \in \mathbb{N}$ and (3-3), we deduce

$$\inf_{x \in \mathbb{P}(E_k^*)} \bigwedge\nolimits_{\omega_B} \omega_{H,k}(x) \ge k \cdot (\eta^{\mathrm{HN}}_{\min} - \epsilon). \tag{3-4}$$

We will now assume that k was chosen big enough so that L^k is relatively ample. Consider now the Kodaira embedding $\iota_k : X \hookrightarrow \mathbb{P}(E_k^*)$. It is well-known that there is a canonical isomorphism between $\iota_k^* \mathscr{O}(1)$ and L^k. We denote by h_ϵ^L the metric induced on L by the pull-back; then $\omega(h_\epsilon^L) = \frac{1}{k}\iota_k^* \omega_{H,k}$. By Lemma 3.1 and (3-4), we conclude that $\inf_{x \in X} \bigwedge_{\omega_B} \omega_H(h_\epsilon^L) \geq \eta_{\min}^{\mathrm{HN}} - \epsilon$. Since $\epsilon > 0$ can be taken arbitrarily small, we deduce $\sup_{h^L} \inf_{x \in X} \bigwedge_{\omega_B} \omega_H(h^L) \geq \eta_{\min}^{\mathrm{HN}}$. In combination with the upper bound from (1-9), this finishes the proof. □

Now, let us establish Theorem 1.12. As in the statement of Theorem 1.12, we fix any Kähler forms $\omega_{B,1}, \dots, \omega_{B,m-1}, \omega_X$. Then the form $\omega_{B,1} \wedge \cdots \wedge \omega_{B,m-1}$ is positive in the sense of [Demailly 2012, (III.1.1)]. By [Michelsohn 1982, (4.8)], there is a Hermitian form ω_B on B, verifying

$$\omega_B^{m-1} = \omega_{B,1} \wedge \cdots \wedge \omega_{B,m-1}. \tag{3-5}$$

Note that ω_B is automatically Gauduchon, but not necessarily Kähler. Since the form ω_B is Gauduchon, it makes sense to define the ω_B-degree and study the Harder–Narasimhan ω_B-slopes, as we did before Theorem 1.1. We remark that due to a relation (3-5), this ω_B-degree coincides with the degree associated with a multipolarization $([\omega_{B,1}], \dots, [\omega_{B,m-1}])$, as defined in [Finski 2024b, before (1.1)]. Below, the invariant $\eta_{\min}^{\mathrm{HN}}$ and other quantities are calculated with respect to ω_B. The following result, together with Theorem 1.10, lie at the core of the proof of Theorem 1.12.

Proposition 3.6 [Finski 2024b, Proposition 5.2]. *A relatively ample line bundle L over X is stably $([\omega_{B,1}], \dots, [\omega_{B,m-1}])$-generically fibered nef with respect to π if and only if $\eta_{\min}^{\mathrm{HN}} \geq 0$.*

Proof of Theorem 1.12. Let us first establish Theorem 1.12 under an additional assumption that L is relatively ample. We assume first that L is stably $([\omega_{B,1}], \dots, [\omega_{B,m-1}])$-generically fibered nef with respect to π. By Theorem 1.10 and Proposition 3.6, we establish that for any $\epsilon > 0$, there is a relatively positive Hermitian metric h_ϵ^L on L, such that $\bigwedge_{\omega_B} \omega_H(h_\epsilon^L) > -\epsilon$. From the definition of $\bigwedge_{\omega_B}$ and the trivial fact that there is $C > 0$, such that $\omega_B^m < C\omega_X^m$, we establish

$$\omega_H(h_\epsilon^L) \wedge \pi^* \omega_{B,1} \wedge \cdots \wedge \pi^* \omega_{B,m-1} \geq -C\epsilon \cdot \omega_X^m. \tag{3-6}$$

But the form $\omega(h_\epsilon^L)$ is relatively positive, so (3-6) implies (1-12) for $\epsilon := C\epsilon$, which finishes the proof of one direction of Theorem 1.12 under an additional assumption that L is relatively ample.

To prove the opposite direction under the same additional assumption that L is relatively ample, assume that we have a sequence of metrics h_ϵ^L, verifying (1-12). We will now show that one can cook up a sequence of relatively positive metrics, $h_{\epsilon,0}^L$, verifying similar bounds. Indeed, let us fix an arbitrary relatively positive metric h_0^L on L. By (1-12), it is easy to see that there is $c > 0$ such that, for any $\epsilon > 0$, over the fibers, the following inequality is satisfied: $c_1(L, h_\epsilon^L) \geq -\epsilon c \cdot c_1(L, h_0^L)$. Then an easy calculation shows that the sequence of metrics $h_{\epsilon,0}^L := (h_\epsilon^L)^{1-2c\epsilon} \cdot (h_0^L)^{2c\epsilon}$ is positive along the fibers and verifies the inequality (1-12) with $C\epsilon$ in place of ϵ, for some $C > 0$. Then as in (3-6), there is $C > 0$ such that, for any $\epsilon > 0$, we have $\bigwedge_{\omega_B} \omega_H(h_{\epsilon,0}^L) > -C\epsilon$. By Theorem 1.10, we then conclude that $\eta_{\min}^{\mathrm{HN}} \geq 0$, which implies that L is stably $([\omega_{B,1}], \dots, [\omega_{B,m-1}])$-generically fibered nef by Proposition 3.6.

We now only assume that L is relatively nef. We assume first that L is stably $([\omega_{B,1}], \dots, [\omega_{B,m-1}])$-generically fibered nef with respect to π. Now, for any $\delta \in \mathbb{Q}$, $\delta > 0$, consider the $\mathbb{Q}$-line bundle $L_\delta := L \otimes L_0^\delta$, where L_0 is some ample line bundle on X. Clearly, L_δ is relatively ample, and it is also stably $([\omega_{B,1}], \dots, [\omega_{B,m-1}])$-generically fibered nef with respect to π. The already established relatively ample case of Theorem 1.12 says that L_δ is $([\omega_{B,1}], \dots, [\omega_{B,m-1}])$-generically fibered nef if and only if for any $\epsilon > 0$, there is a Hermitian metric $h_\epsilon^{L_\delta}$ on L_δ, such that the analogue of (1-12) holds. Let h_0^L be now an arbitrary positive metric on L_0. It is easy to see that if δ and ϵ are sufficiently small, then the metric h_ϵ^L on L, which is constructed as the only metric verifying $h_\epsilon^{L_\delta} = h_\epsilon^L \cdot (h_0^L)^\delta$, will satisfy the analogue of (1-12) (for $C\epsilon$ instead of ϵ for some $C > 0$). This shows one direction of Theorem 1.12.

Inversely, if for any $\epsilon > 0$ there is a metric h_ϵ^L as in (1-12), then the metrics $h_\epsilon^{L_\delta}$, defined by the above formula, will also satisfy a similar inequality. Hence, by the already established case of Theorem 1.12, L_δ is then stably $([\omega_{B,1}], \dots, [\omega_{B,m-1}])$-generically fibered nef for any $\delta \in \mathbb{Q}$, $\delta > 0$. In particular, for any $\delta \in \mathbb{Q}$, $\delta > 0$, the line bundle $L_{2\delta} = L_\delta \otimes L_0^\delta$ is $([\omega_{B,1}], \dots, [\omega_{B,m-1}])$-generically fibered nef for any $\delta \in \mathbb{Q}$, $\delta > 0$, which means that L is stably $([\omega_{B,1}], \dots, [\omega_{B,m-1}])$-generically fibered nef, as $L_{2\delta} = L \otimes L_0^{2\delta}$. This finishes the proof. □

4. Asymptotic Riemann–Roch–Grothendieck and semistability

The main goal of this section is to prove a numerical obstruction for asymptotic semistability of direct images from Theorem 1.3. This will be based on an asymptotic version of Riemann–Roch–Grothendieck theorem, which we establish here in the singular setting.

To begin, let us recall some basic facts about Bott–Chern and Aeppli cohomologies. Let Y be a compact complex manifold of dimension n. We denote by $\Omega^{(p,q)}(Y)$ the vector space of (p,q)-differential forms on Y, $p, q \in \mathbb{N}$, and define $\partial : \Omega^{(p,q)}(Y) \to \Omega^{(p+1,q)}(Y)$, $\bar{\partial} : \Omega^{(p,q)}(Y) \to \Omega^{(p,q+1)}(Y)$, as usual. Recall that Bott–Chern cohomology, $H_{BC}^{p,q}(Y)$, is defined as

$$H_{BC}^{p,q}(Y) := \frac{\big(\ker \partial : \Omega^{(p,q)}(Y) \to \Omega^{(p+1,q)}(Y)\big) \cap \big(\ker \bar{\partial} : \Omega^{(p,q)}(Y) \to \Omega^{(p,q+1)}(Y)\big)}{\operatorname{im} \partial\bar{\partial} : \Omega^{(p-1,q-1)}(Y) \to \Omega^{(p,q)}(Y)}. \tag{4-1}$$

Recall that Aeppli cohomology, $H_A^{p,q}(Y)$, is defined as

$$H_A^{p,q}(Y) := \frac{\ker \partial\bar{\partial} : \Omega^{(p,q)}(Y) \to \Omega^{(p+1,q+1)}(Y)}{\big(\operatorname{im} \partial : \Omega^{(p-1,q)}(Y) \to \Omega^{(p,q)}(Y)\big) + \big(\operatorname{im} \bar{\partial} : \Omega^{(p,q-1)}(Y) \to \Omega^{(p,q)}(Y)\big)}. \tag{4-2}$$

It is standard that for compact Kähler manifolds, the two cohomologies coincide. For $p, q = 0, \dots, n$, we have the natural pairing

$$\wedge : H_{BC}^{p,q}(Y) \times H_A^{n-p,n-q}(Y) \to \mathbb{C}, \tag{4-3}$$

given by the wedge product and integration. If $p : Y \to B$ is a holomorphic map between compact complex manifolds, then for $s := \dim Y - \dim B$, we have a natural map $p_* : H_{BC}^{p,q}(Y) \to H_{BC}^{p-s,q-s}(B)$, defined by the pairing (4-3) and the pull-back p^*.

The Bott–Chern cohomology can be generally defined for arbitrary complex analytic spaces Y in the sense of [Demailly 2012, Definition II.5.2], where one considers differential forms on Y obtained by pullbacks of smooth differential forms through local embeddings of the space into complex vector spaces. Due to a theorem of Lelong, see [Demailly 2012, Theorem III.2.7], the intersection pairing (4-3) can still be defined in this setting, and so the slope (1-4) is well-defined. By resolving the singularities, we can extend the definition of the pushforward for maps between a compact complex analytic spaces Y and a compact manifold B.

Now, let E be a holomorphic vector bundle over Y. Using Chern–Weil theory, one can construct for any Hermitian metric h^E on E a corresponding Chern character form, $\mathrm{ch}(E, h^E)$, which is a d-closed form in $\bigoplus_{p=0}^{+\infty} \Omega^{(p,p)}(Y)$. Bott and Chern [1965] showed that the resulting class in Bott–Chern cohomology doesn't depend on the choice of the metric. This gives a definition of the Chern character, $\mathrm{ch}(E)$ of a vector bundle E with values in Bott–Chern cohomology.

Bismut, Shu, and Wei [Bismut et al. 2023] generalized the definition of the Chern character with values in Bott–Chern cohomology for any coherent sheaf $\mathscr{E}$ on Y. If $\mathscr{E}$ has a finite locally free projective resolution (which is always the case if Y is projective), this construction corresponds to the one given by the alternating sum of Chern characters of the resolution. By [Bismut et al. 2023, §8.6], the (k, k)-component of the Chern character, $\mathrm{ch}_k(\mathscr{E})$, we have

$$\mathrm{ch}_0(\mathscr{E}) = \mathrm{rk}(\mathscr{E}), \quad \mathrm{ch}_1(\mathscr{E}) = c_1(\det \mathscr{E}), \tag{4-4}$$

where $\det \mathscr{E}$ is the Knudsen–Mumford determinant [1976]. Without entering into details of the construction, we mention that the absence of finite locally free projective resolutions of coherent sheaves for general complex manifolds is circumvented in [Bismut et al. 2023] by the use of so-called antiholomorphic superconnections; see [Bismut et al. 2023, Theorem 6.7].

Now, recall that for any proper holomorphic map $p : Y \to B$, and any coherent sheaf $\mathscr{E}$, the Grauert theorem tells that the direct image sheaves $R^q p_* \mathscr{E}$, $q \in \mathbb{N}$, are coherent. The main result of this section goes as follows.

Theorem 4.1. *Let Y be an irreducible compact complex analytic space, L an arbitrary line bundle on Y and $\mathscr{E}$ a coherent sheaf on Y. Let $p : Y \to B$ be a holomorphic map to a compact complex manifold B. Then for any $r \in \mathbb{N}$, and $s := \dim Y - \dim B + r$, in the Bott–Chern cohomology*,

$$\lim_{k\to\infty} \frac{1}{k^s} \sum_{t=0}^{\dim Y} (-1)^t \mathrm{ch}_r(R^t p_*(\mathscr{E} \otimes L^k)) = \frac{\mathrm{rk}(\mathscr{E})}{s!} \cdot p_*(c_1(L)^s). \tag{4-5}$$

Remark 4.2. Despite a huge amount of literature, we were not able to find the proof of this result under the stated hypotheses (even for projective Y, B). For flat maps p and relatively ample L, this result can be alternatively established using Knudsen–Mumford expansions; see [Knudsen and Mumford 1976; Phong et al. 2008, Theorem 3]. Note, however, that flatness doesn't pass through subfamilies, and there is no flatness assumption in Theorem 1.3.

The proof of this result relies on the recent result of [Bismut et al. 2023] establishing the Riemann–Roch–Grothendieck theorem in Bott–Chern cohomology for arbitrary holomorphic maps between (*smooth*!) complex manifolds. More precisely, the main result of [Bismut et al. 2023] says the following.

Theorem 4.3. *Let $p : Y \to B$ be a holomorphic map between compact complex manifolds. Then for any coherent sheaf $\mathscr{E}$ on Y, the identity*

$$\mathrm{Td}(TB) \cdot \sum_{t=0}^{\dim Y} (-1)^t \mathrm{ch}(R^t p_*(\mathscr{E})) = p_*(\mathrm{Td}(TY) \cdot \mathrm{ch}(\mathscr{E})) \tag{4-6}$$

holds in Bott–Chern cohomology, where $\mathrm{Td}(TB)$, $\mathrm{Td}(TY)$ *are the Todd classes.*

Proof of Theorem 4.1. Note first that for smooth manifolds Y, the result follows directly from Theorem 4.3 by (4-4) and the fact that the 0-degree part of the Todd class is identity. If $\dim Y = 0$, then Y is automatically smooth, and, hence, Theorem 4.1 holds as stated.

We will argue by induction on the dimension of Y. For this, we consider a resolution of singularities $f : \hat{Y} \to Y$ of Y. For any $q = 1, \dots, \dim Y$, we define the sheaves $\mathscr{Q}_q$ on Y as $\mathscr{Q}_q := R^q f_* f^* \mathscr{E}$, where $f^*\mathscr{E} := f^{-1}\mathscr{E} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_{\hat{Y}}$. We define the sheaf $\mathscr{Q}_0$ on Y by the short exact sequence

$$0 \to \mathscr{E} \to R^0 f_* f^* \mathscr{E} \to \mathscr{Q}_0 \to 0. \tag{4-7}$$

By the Grauert theorem, the sheaves $\mathscr{Q}_q$, $q = 0, \dots, \dim Y$, are coherent. Since the resolution of singularities is biholomorphic away from a subset of singular points of Y, and over the locally free locus of $\mathscr{E}$, by the projection formula, see [Hartshorne 1977, Exercise II.5.1d)], we have $R^0 f_* f^* \mathscr{E} = \mathscr{E}$, the supports of the sheaves $\mathscr{Q}_q$, $q = 0, \dots, \dim Y$, are *proper* analytic subsets of Y, which, by irreducibility of Y, have strictly smaller dimension than Y; see [Demailly 2012, Proposition II.4.2.6]. By this and the usual *devissage* techniques (see [EGA III_1 1961, théorème 3.1.2], cf. [Hartshorne 1977, Proposition I.7.4]), for any $q = 0, \dots, \dim Y$, there is $r(q) \in \mathbb{N}$, and complex analytic subspaces $\iota_{i,q} : Z_{i,q} \hookrightarrow Y$, with some ideal sheaves $\mathscr{J}_{i,q}$ on $Z_{i,q}$ and a filtration $\mathscr{F}_{i,q}$ of $\mathscr{Q}_q$, $i = 0, \dots, r(q)$, $\mathscr{F}_{0,q} = \{0\}$, $\mathscr{F}_{r(q),q} = \mathscr{Q}_q$, $\mathscr{F}_{i-1,q} \subset \mathscr{F}_{i,q}$, $i = 1, \dots, r_q$, such that, for any $i = 1, \dots, r(q)$, we have $\mathscr{F}_{i,q}/\mathscr{F}_{i-1,q} = \iota_{i,q,*}(\mathscr{J}_{i,q})$. We let $p_{i,q} := p \circ \iota_{i,q}$, $i = 1, \dots, r(q)$, $\hat{p} := p \circ f$. We argue that for any $k \in \mathbb{N}$, we have

$$\begin{aligned}\sum_{t=0}^{\dim Y} & (-1)^t \mathrm{ch}(R^t p_*(\mathscr{E} \otimes L^k)) \\ &= \sum_{t=0}^{\dim Y} (-1)^t \mathrm{ch}(R^t \hat{p}_*(f^*\mathscr{E} \otimes f^* L^k)) - \sum_{t,u=0}^{\dim Y} (-1)^{t+u} \sum_{i=1}^{r(q)} \mathrm{ch}(R^t (p_{i,u})_*(\mathscr{J}_{i,u} \otimes \iota_{i,u}^* L^k)).\end{aligned} \tag{4-8}$$

Once (4-8) is established, Theorem 4.1 would follow by induction, as the space $\hat{Y}$ is smooth, and so for the first summand on the right-hand side of (4-8), the smooth version of Theorem 4.1 applies, and the second summand doesn't contribute to the asymptotics by induction hypothesis, as all $Z_{i,q}$ have strictly smaller dimensions than Y.

Now, let us establish (4-8). First of all, since in the derived category, there is a canonical isomorphism between the functors $R\hat{p}_*$ and $Rp_* Rf_*$, see [Bismut et al. 2023, (3.13)], and the construction of Chern character, defined using derived category of coherent sheaves, factors through the K-theory of the derived category of coherent sheaves [Bismut et al. 2023, Theorem 8.11 and §8.9], we have

$$\sum_{t=0}^{\dim Y} (-1)^t \mathrm{ch}(R^t \hat{p}_*(f^*\mathscr{E} \otimes f^*L^k)) = \sum_{t,u=0}^{\dim Y} (-1)^{t+u} \mathrm{ch}\big(R^t p_*(R^u f_*(f^*\mathscr{E} \otimes f^*L^k))\big). \tag{4-9}$$

Now, from the exact sequence (4-7), using again the fact that the construction of the Chern character passes through the formation of K-theory, we obtain

$$\sum_{t=0}^{\dim Y} (-1)^t \mathrm{ch}\big(R^t p_*(R^0 f_*(f^*\mathscr{E} \otimes f^*L^k))\big) = \sum_{t=0}^{\dim Y} (-1)^t \mathrm{ch}(R^t p_*(\mathscr{E} \otimes L^k)) + \sum_{t=0}^{\dim Y} (-1)^t \mathrm{ch}(R^t p_*(\mathscr{Q}_0 \otimes L^k)). \tag{4-10}$$

Similarly, for any $q = 0, \dots, \dim Y$, we obtain

$$\sum_{t=0}^{\dim Y} (-1)^t \mathrm{ch}(R^t p_*(\mathscr{Q}_q \otimes L^k)) = \sum_{t=0}^{\dim Y} (-1)^t \sum_{i=0}^{r(q)} \mathrm{ch}\big(R^t p_*(\iota_{i,q,*}(\mathscr{J}_{i,q}) \otimes L^k)\big). \tag{4-11}$$

Remark, however, that since $\iota_{i,q}$ is a closed embedding, $\iota_{i,q,*}$ is an exact functor, so we have $R^v \iota_{i,q,*} = 0$ for $v = 1, \dots, \dim Y$, and $R^0 \iota_{i,q,*} = \iota_{i,q,*}$. Moreover, by the projection formula, see [Hartshorne 1977, Exercise II.5.1d)], we have $R^0 \iota_{i,q,*}(\mathscr{J}_{i,q}) \otimes L^k = R^0 \iota_{i,q,*}(\mathscr{J}_{i,q} \otimes \iota_{i,q}^* L^k)$. In particular, for any $t, q = 0, \dots, \dim Y$, $i = 1, \dots, r(q)$, we can write

$$\mathrm{ch}(R^t p_*(\iota_{i,q,*}(\mathscr{J}_{i,q}) \otimes L^k)) = \sum_{v=0}^{\dim Y} (-1)^v \mathrm{ch}\big(R^t p_*(R^v \iota_{i,q,*}(\mathscr{J}_{i,q} \otimes \iota_{i,q}^* L^k))\big). \tag{4-12}$$

But using the same argument as in (4-9), we have

$$\sum_{t,v=0}^{\dim Y} (-1)^{t+v} \mathrm{ch}\big(R^t p_*(R^v \iota_{i,q,*}(\mathscr{J}_{i,q} \otimes \iota_{i,q}^* L^k))\big) = \sum_{t=0}^{\dim Y} (-1)^t \mathrm{ch}\big(R^t (p_{i,q})_*(\mathscr{J}_{i,q} \otimes \iota_{i,q}^* L^k)\big). \tag{4-13}$$

Now, a combination of (4-9), (4-10), (4-11), (4-12) and (4-13), gives us (4-8). □

Now, let us finally establish an application of Theorem 4.1 towards the study of Harder–Narasimhan slopes of direct images. We fix an irreducible compact complex analytic space Y of dimension $k+m$, $k \geq 0$, with a surjective holomorphic map $\pi : Y \to B$. Let ω_B be a Gauduchon Hermitian form on B. Let L be a relatively ample line bundle on Y. Recall that in (1-4), we defined the slope of Y, and before Theorem 1.1, we defined the slopes of coherent sheaves.

Lemma 4.4.
$$\mu(Y) \cdot \int_B [\omega_B^m] = \lim_{k \to \infty} \mu(R^0 \pi_* L^k)/k.$$

Proof. By the Serre vanishing theorem, the higher direct images $R^v \pi_* L^k$, $v \geq 1$, vanish. By (4-4), we conclude that

$$\mu(R^0 \pi_* L^k) = \frac{\int_Y [\mathrm{ch}_1(R^0 \pi_* L^k)] \cdot [\omega_B^{m-1}]}{\mathrm{ch}_0(R^0 \pi_* L^k)}. \tag{4-14}$$

The result now follows directly from Theorem 4.1. □

Proposition 4.5. *The vector bundles E_k are asymptotically semistable if and only if $\eta^{\mathrm{HN}}_{\min} = \eta^{\mathrm{HN}}_{\max}$. Moreover, if E_k are asymptotically semistable, then for any subsheaves $\mathcal{F}_k$ of E_k, $\mathrm{rk}(\mathcal{F}_k) > 0$, and any $\epsilon > 0$, for k big enough, we have $\mu(\mathcal{F}_k) \leq \mu(E_k) + \epsilon k$.*

Proof. The maximal and the minimal slopes satisfy

$$\begin{aligned} \mu^k_{\max} &= \sup\big\{\mu(\mathcal{F}_k) : \mathcal{F}_k \text{ is a subsheaf of } E_k\big\}, \\ \mu^k_{\min} &= \inf\big\{\mu(\mathcal{Q}_k) : \mathcal{Q}_k \text{ is a quotient sheaf of } E_k\big\}. \end{aligned} \tag{4-15}$$

Now, by Theorem 1.5, as $k \to \infty$, we have

$$\lim_{k\to\infty} \frac{\mu(E_k)}{k} = \int_{\mathbb{R}} x d\eta^{\mathrm{HN}}(x). \tag{4-16}$$

From (4-15) and (4-16), we see that E_k are asymptotically semistable if and only if $\eta^{\mathrm{HN}}_{\min} = \operatorname{ess\,sup} \eta^{\mathrm{HN}}$. However, by Theorem 1.5, $\operatorname{ess\,sup} \eta^{\mathrm{HN}}$ coincides with $\eta^{\mathrm{HN}}_{\max}$, which finishes the proof of the first part of the theorem. The proof of the second statement of Proposition 4.5 follows directly by (4-15) and the first part. □

Remark 4.6. From [Finski 2024b, Proposition 5.1], see also [Xu and Zhuang 2020], we know that if $\dim B = 1$, then $\operatorname{ess\,inf} \eta^{\mathrm{HN}}$ coincides with $\eta^{\mathrm{HN}}_{\min}$. The above proof shows that for $\dim B = 1$, the condition on the subsheaves from Proposition 4.5 is equivalent to asymptotic semistability.

Proof of Theorem 1.1. This follows immediately from Theorem 1.7 and Proposition 4.5. □

Proposition 4.7. *For any complex analytic subspace Y of X as in Theorem 1.3, the bound*

$$\mu(Y) \cdot \int_B [\omega_B^m] \geq \eta^{\mathrm{HN}}_{\min}$$

holds.

Proof. Consider the following short exact sequence of sheaves associated with Y

$$0 \to \mathscr{J}_Y \to \mathscr{O}_X \to \iota_* \mathscr{O}_Y \to 0, \tag{4-17}$$

where $\mathscr{J}_Y$ is the ideal sheaf of Y, consisting of local holomorphic functions on X, vanishing along Y, and $\mathscr{O}_Y$ is the structure sheaf of Y associated with the reduced scheme structure of Y, i.e., defined by (4-17). By considering a long exact sequence of direct images associated with (4-17) and the map π, and using the Serre vanishing theorem, we conclude that the restriction map $R^0\pi_* L^k \to R^0\pi|_{Y,*} L|_Y^k$ is surjective. Then, in the notation of (4-15), we have $\mu^k_{\min} \leq \mu(R^0\pi|_{Y,*} L|_Y^k)$. We deduce Proposition 4.7 from Lemma 4.4 by diving by k and passing to the limit $k \to \infty$. □

Proof of Theorem 1.3. First of all, by Lemma 4.4, we conclude that

$$\lim_{k\to\infty} \mu(E_k)/k = \mu(X) \cdot \int_B [\omega_B^m] \geq \eta^{\mathrm{HN}}_{\min}. \tag{4-18}$$

Moreover, since $\operatorname{ess\,sup} \eta^{\mathrm{HN}}$ coincides with $\eta^{\mathrm{HN}}_{\max}$ by [Finski 2024b, Theorem 1.1], by (4-16), we conclude that the equality in the above inequality holds if and only if $\eta^{\mathrm{HN}}_{\min} = \eta^{\mathrm{HN}}_{\max}$, i.e., when E_k is asymptotically semistable by Proposition 4.5. In particular, if E_k is asymptotically semistable, then by $\eta^{\mathrm{HN}}_{\min} = \eta^{\mathrm{HN}}_{\max}$, Proposition 4.7 and (4-18), we establish the first part of Theorem 1.3.

Let us now establish the second part. By a reformulation of the result of Xu and Zhuang [2020, Lemma 2.26 and Proposition 2.28] from [Finski 2024b, (1.5)], we have

$$\eta^{\mathrm{HN}}_{\min} = \inf_{C \subset X} \mu(C) \cdot \int_B [\omega_B^m], \tag{4-19}$$

where C runs over all irreducible curves in X, with project surjectively to B. In particular, from (4-19), we conclude that

$$\eta^{\mathrm{HN}}_{\min} \geq \inf_{Y \subset X} \mu(Y) \cdot \int_B [\omega_B^m], \tag{4-20}$$

where Y are as in Theorem 1.3. A combination of (4-18) and (4-20) shows that if $\inf_{Y \subset X} \mu(Y) = \mu(X)$, then both (4-18) and (4-20) are actually equalities. By the remark after (4-18), this implies that E_k is asymptotically semistable, which finishes the proof. □

Remark 4.8. It is interesting to know if the second part of Theorem 1.3 continues to hold for $\dim B > 1$. There are several potential pitfalls for that. First, since B is not necessarily projective, there might be very few analytic subspaces $Y \subset X$, projecting surjectively to B. Second, even for projective B, the main result of [Finski 2024b, Theorem 1.4] implies that the analogue of the bound (4-19) becomes tight if one considers among Y all subcurves in X projecting to generic curves over the base. But it seems that there are many more curves like that than analytic subspaces projecting surjectively to the base.

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