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A SHARP TRACE ADAMS INEQUALITY IN $\mathbb{R}^4$ AND EXISTENCE OF THE EXTREMALS

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Let $\Omega \subseteq \mathbb{R}^4$ be a bounded domain with smooth boundary $\partial\Omega$. In this paper, we establish the following sharp form of the trace Adams inequality in $W^{2,2}(\Omega)$ with zero mean value and zero Neumann boundary condition:

$$S(\alpha) = \sup_{\substack{u\in W^{2,2}(\Omega)\setminus\{0\},\ \|\Delta u\|_2\le 1 \\ \int_\Omega u\,dx=0,\ \frac{\partial u}{\partial \nu}|_{\partial\Omega}=0}} \int_{\partial\Omega} e^{\alpha u^2}\, d\sigma < \infty$$

holds if and only if $\alpha \le 12\pi^2$.

Moreover, we prove a classification theorem for the solutions of a class of nonlinear boundary value problem of biharmonic equations on the half-space $\mathbb{R}^4_+$. With this classification result, we can show that $S(12\pi^2)$ is attained by using the blow-up analysis and capacitary estimate. As an application, we prove a sharp trace Adams–Onofri-type inequality in general four-dimensional bounded domains with smooth boundary.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$ and $W^{m,p}(\Omega)$ denote the usual Sobolev space: the completion of $C^\infty(\overline{\Omega})$ under the norm

$$\|\cdot\| = \left(\sum_{|\alpha|\le m} \int_\Omega |D^\alpha u|^p\, dx\right)^{\frac{1}{p}}.$$

If $1 < p < n/m$, the classical Sobolev embedding asserts that $W^{m,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ for $p^* = np/(n-mp)$, and $W^{m,p}(\Omega) \hookrightarrow L^{p^\star}(\partial\Omega)$ for $p^\star = (n-1)p/(n-mp)$. However, when $p = n/m$, it is known that both $W^{m,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ and $W^{m,p}(\Omega) \hookrightarrow L^\infty(\partial\Omega)$ fail.

It is known that the analogue of optimal Sobolev embedding for $W_0^{m,n/m}(\Omega)$ (the Sobolev space consisting of functions vanishing on the boundary $\partial\Omega$) is given by the famous Trudinger–Moser inequality ($m = 1$) [Moser 1970/71; Trudinger 1967] and Adams inequality ($m > 1$) [Adams 1988], which can be

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stated in the form

$$\sup_{\substack{u\in W_0^{m,\frac{n}{m}}(\Omega)\setminus\{0\}\\ \|\Delta^{m/2}u\|_{\frac{n}{m}}\le 1}} \int_\Omega \exp(\alpha|u(x)|^{\frac{n}{n-m}})\,dx \begin{cases} \le c|\Omega| & \text{if } \alpha\le\alpha(n,m),\\ =+\infty & \text{if } \alpha>\alpha(n,m),\end{cases} \tag{1-1}$$

where

$$\alpha(n,m)=\begin{cases} \dfrac{n}{\omega_{n-1}}\Bigg[\dfrac{\pi^{n/2}2^m\Gamma\big(\frac12(m+1)\big)}{\Gamma\big(\frac12(n-m+1)\big)}\Bigg]^{\frac{n}{n-m}} & \text{when } m \text{ is odd},\\ \dfrac{n}{\omega_{n-1}}\Bigg[\dfrac{\pi^{n/2}2^m\Gamma\big(\frac{m}{2}\big)}{\Gamma\big(\frac12(n-m)\big)}\Bigg]^{\frac{n}{n-m}} & \text{when } m \text{ is even}.\end{cases}$$

Here ω_{n-1} denotes the $(n-1)$-dimensional surface measure of the unit ball in $\mathbb{R}^n$. So far, the Trudinger–Moser–Adams inequalities (1-1) have been generalized in many other directions such as the Trudinger–Moser inequalities on the unbounded domains, compact or complete and noncompact Riemannian manifolds, CR spheres, hyperbolic spaces, Heisenberg groups, Hardy–Adams-type inequalities on hyperbolic spaces, etc., to just name a few from a long list of extensive works we refer the interested readers to [Adachi and Tanaka 2000; Cohn and Lu 2004; Chen et al. 2020; 2021; 2022b; 2023a; 2023b; Chen 1990; Li 2005; Cohn and Lu 2001; do Ó 1997; 2024; Fontana 1993; Lam and Lu 2012; 2013; Lam et al. 2014; Li and Lu 2021; Li et al. 2018a; 2018b; 2021; Li and Ruf 2008; Li and Zhu 2022; Liang et al. 2020; Lu and Tang 2013; Lu and Yang 2017; Lu et al. 2024; Ma et al. 2021; Mancini et al. 2013; Nguyen 2018; 2024 Ruf and Sani 2013; Wang 2025; Xue et al. 2025; Yang 2007; Zhang and Zhu 2024; Zhang et al. 2025].

An interesting problem related to the Trudinger–Moser–Adams inequalities lies in investigating the existence of extremal functions. Carleson and Chang [1986] first established the existence of extremals for Trudinger–Moser inequalities on the unit ball through a symmetrization rearrangement inequality combined with the ODE technique. After that, the existence of extremals was proved for any bounded domains in $\mathbb{R}^n$ (see [Flucher 1992; Lin 1996; Adimurthi and Druet 2004]). One can also see [Li 2001; 2005; Li and Ruf 2008; Zhu 2014] for existence of extremals for the Trudinger–Moser inequalities on unbounded domains and compact Riemannian manifolds, and see [Lu and Yang 2009a; DelaTorre and Mancini 2021; Chen et al. 2020] for the existence of extremals for Adams inequalities in bounded and unbounded domains. We note that the Trudinger–Moser–Adams inequalities on the Sobolev spaces $W^{m,\frac{n}{m}}(\Omega)$ without the Dirichlet boundary condition have also been established; the interested readers can refer to [Chang and Yang 1988; Leckband 2005; Cianchi 2005; Lu and Yang 2009b; Tarsi 2012].

In this paper, we are interested in the borderline case of Sobolev trace inequality in $W^{m,\frac{n}{m}}(\Omega)$. As mentioned above, from the Sobolev embedding we know that $W^{m,\frac{n}{m}}(\Omega)\hookrightarrow L^q(\partial\Omega)$ for any $q\in[1,\infty)$, but not for $q=\infty$. (See, e.g., Maz'ja's book [1985]). Adams [1973] showed that $W^{m,\frac{n}{m}}(\Omega)$ can be embedded into the Orlicz space $L_\phi(\partial\Omega)$, with $\phi(t)=\exp(|t|^{n/(n-m)}-1)$ (see also [Maz'ja 1985]). The first optimal trace inequality of Moser type on $\partial\Omega$ was obtained in [Chang and Marshall 1985] in a two-dimensional disk D for functions with zero boundary mean value. Namely,

$$\sup_{\substack{u\in W^{1,2}(D)\setminus\{0\}\\ \int_D|\nabla u|^2\,dx=1,\ \int_{\partial D}u\,d\sigma=0}} \int_{\partial D} e^{\alpha u^2}\,d\sigma<+\infty \quad \text{if and only if } \alpha\le\pi. \tag{1-2}$$

Using the technique of blow-up analysis, Li and Liu [2005] extended the result of [Chang and Marshall 1985] to general bounded domains and obtained the existence of corresponding extremals. Yang [2006] obtained another sharp form of trace Trudinger–Moser inequality for functions with zero mean value in $\Omega \subset \mathbb{R}^2$. Furthermore, Cianchi [2008] formulated a unified framework for high-dimensional Trudinger–Moser-type inequalities on a boundary $\partial\Omega$ or smooth submanifold of arbitrary dimension in $\overline{\Omega}$. In particular, this includes the trace Trudinger–Moser inequality with the zero mean value in $W^{1,n}(\Omega)$ for $n \geq 3$.

The main purpose of this paper is to study the second-order trace Adams inequality on a smooth bounded domain Ω with zero mean value. Set

$$\mathcal{H} = \left\{ u \in W^{2,2}(\Omega) : \|\Delta u\|_2 \leq 1,\ \int_\Omega u\,dx = 0,\ \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0 \right\}.$$

Our main results read as follows.

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}^4$ be a bounded smooth domain with smooth boundary $\partial\Omega$. Then if $\alpha \leq 12\pi^2$, we have*

$$S(\alpha) := \sup_{u\in\mathcal{H}\setminus\{0\}} \int_{\partial\Omega} e^{\alpha u^2}\,d\sigma < \infty. \tag{1-3}$$

The constant $12\pi^2$ is sharp in the sense that if $\alpha > 12\pi^2$, then the supremum $S(\alpha)$ is infinity. Moreover, the supremum is attained if $\alpha \leq 12\pi^2$.

As an immediate consequence of Theorem 1.1, we have the following result when Ω is a four-dimensional ball $\mathbb{B}^4$ and $\mathbb{S}^3$ is its boundary.

Corollary 1.2. *If $\alpha \leq 12\pi^2$, we have*

$$S(\alpha, \mathbb{B}^4) := \sup_{u\in\mathcal{H}\setminus\{0\}} \int_{\mathbb{S}^3} e^{\alpha u^2}\,d\sigma < \infty. \tag{1-4}$$

The constant $12\pi^2$ is sharp in the sense that if $\alpha > 12\pi^2$, then the supremum $S(\alpha, \mathbb{B}^4)$ is infinity. Moreover, the supremum $S(\alpha, \mathbb{B}^4)$ is attained if $\alpha \leq 12\pi^2$.

As an application of Theorem 1.1, we can derive the following trace Adams–Onofri-type inequality on any four-dimensional bounded domains.

Theorem 1.3. *Assume that $\Omega \subseteq \mathbb{R}^4$ is a bounded smooth domain with smooth boundary $\partial\Omega$. For any $u \in W^{2,2}(\Omega)$ with $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$, there exists a constant C such that*

$$\frac{1}{48\pi^2}\int_\Omega |\Delta u|^2\,dx + \frac{1}{|\Omega|}\int_\Omega u\,dx - \log\left(\int_{\partial\Omega} e^u\,d\sigma\right) \geq C.$$

Remark 1.4. The Moser–Onofri and Adams–Onofri inequalities on the sphere can be obtained by using the endpoint differentiation argument (see Beckner's work [1993]). However, this method cannot be used to establish the sharp trace Adams–Onofri inequality on general four-dimensional bounded domains due to its absence of conformal invariance. Our Theorem 1.3 offers a weak version of such a Moser–Onofri-type inequality on general bounded domains in $\mathbb{R}^4$ from the sharp Adams trace inequality obtained in Theorem 1.1.

Remark 1.5. The first-order Sobolev trace inequality was due to Escobar [1988] and Beckner [1993]. The second-order and higher-order Sobolev trace inequalities were established by Ache and Chang [2017], and subsequently by Ngô, Nguyen, and Phan [Ngô et al. 2020], and Q. Yang [2019], where they established the sharp trace Sobolev inequality of higher-order on the real unit ball $\mathbb{B}^{n+1} \subset \mathbb{R}^{n+1}$. Case [2020] establishes a family of sharp Sobolev trace inequalities involving the $W^{k,2}(\mathbb{R}^n_+, y^\alpha)$-norm, which leads to the well-known embedding

$$W^{k,2}(\mathbb{R}^{n+1}_+) \hookrightarrow \bigoplus_{j=0}^{k-1} W^{k-j-\frac{1}{2},2}(\mathbb{R}^n).$$

More recently, Flynn, the second author, and Q. Yang [Flynn et al. 2023] introduced conformally covariant boundary operators for Poincaré–Einstein manifolds satisfying a mild spectral assumption. Using these boundary operators the authors set up related higher-order trace Sobolev inequalities on these manifolds. They later [Flynn et al. 2025] introduced an appropriate family of conformally covariant boundary operators associated to the Siegel domain $\mathcal{U}^{n+1}$ with the Heisenberg group $\mathbb{H}^n$ as its boundary and the complex ball $\mathbb{B}^{n+1}_{\mathbb{C}}$ with the complex sphere $\mathbb{S}^{2n+1}$ as its boundary and prove all higher-order CR Sobolev trace inequalities for the Siegel domain $\mathcal{U}^{n+1}$ and the complex ball $\mathbb{B}^{n+1}_{\mathbb{C}}$. This generalizes the Sobolev trace inequality in the CR setting by Frank, González, Monticelli, and Tan [Frank et al. 2015] in the case $\gamma \in (0, 1)$ to the general case for all $\gamma \in (0, n+1) \setminus \mathbb{N}$.

Remark 1.6. Besides the trace Trudinger–Moser inequalities on bounded domains as discussed earlier, we also refer to the recent article [Chen et al. 2022a] by the authors and Yang, which studies trace Trudinger–Moser and Adams inequalities under various constraints on the upper half-spaces $\mathbb{R}^{2m}_+$ by the Fourier rearrangement and the polyharmonic extension.

The general strategy we use in this paper is exploiting the blow-up analysis. We first prove the subcritical trace Adams inequalities and the existence of extremals by using the sharp subcritical Adams inequalities involved with zero mean value. Then, we take a sequence $\alpha_k \to 12\pi^2$ and find a maximizing sequence $\{u_k\}_k \subset W^{2,2}(\Omega)$ for $S(12\pi^2)$. If u_k is bounded in $L^\infty(\Omega)$, i.e., $c_k := \max_{x\in\Omega} |u_k(x)| < \infty$, we can easily show that u_k converges to a function u in $W^{2,2}(\Omega)$ by elliptic estimates. If $c_k \to +\infty$, i.e., the blow-up arises, we apply the blow-up analysis method to analyze the asymptotic behavior of u_k near and far away from the blow-up point $p \in \partial\Omega$, and derive an upper bound for the trace Adams functional,

$$S(12\pi^2) \le |\partial\Omega| + 2\pi^2 e^{12\pi^2 A_p - \frac{3}{4}}, \tag{1-5}$$

where A_p is the value at p of the trace of the regular part of the Green function for the operator $\Delta^2 + 1/|\Omega|$. Finally, we construct a function sequence in $\mathcal{H}$ to show that the upper bound can actually be surpassed, which implies that the concentration phenomenon will not happen.

Neither the blow-up strategy in the study of second-order Adams inequality with Dirichlet boundary condition (see [Lu and Yang 2009a]) nor the blow-up method for the first-order trace Trudinger–Moser inequality and existence of their extremals (see [Li and Liu 2005; Yang 2006]) can be easily generalized to second-order trace Adams inequality case. In the following, we will introduce the main difference between the proof of the second-order trace Adams inequality and those for the second-order Adams inequality with Dirichlet boundary condition and the trace Trudinger–Moser blow-up analysis (see [Lu and Yang 2009a; Yang 2006]). We will also outline the elements of novelty when we carry out the blow-up procedure for our second-order trace Adams inequality.

First of all, the main difference between the blow-up analysis for the Adams inequality and that for the Adams trace inequality blowing-up is the location of the blow-up points. For the former, the blow-up points must be at some interior points, while for the latter the blow-up points must lie on the boundary of Ω, which leads to the situation where the related Euler–Lagrange equations of the maximizer sequence are some biharmonic equations with the Neumann boundary condition and the analysis of asymptotic behavior near and far away from the blow-up points is totally different.

Second, unlike the first-order case, one cannot show that the maximum point x_k of u_k lies on the boundary $\partial\Omega$ due to the absence of maximum principle of the biharmonic operator Δ^2. We stress that without this fact one cannot analyze the asymptotic behavior of u_k near the blow up points $p \in \partial\Omega$ if the concentration phenomenon occurs. To overcome this difficulty, we will make use of the assumption that

$$\frac{\partial u_k}{\partial \nu} = 0 \quad \text{on } \partial\Omega$$

to show that there exists some point $\tilde{x}_k \in \partial\Omega$ such that

$$|u_k(\tilde{x}_k) - u_k(x_k)| = o_k(1).$$

This important observation allows us to choose the maximum point x_k on the boundary $\partial\Omega$.

Third, when we try to analyze the asymptotic behavior of u_k near the blow up point p, a crucial step is to classify the solutions to the Liouville equation

$$\begin{cases} \Delta^2\psi = 0, & x \in \mathbb{R}^4_+, \\ \dfrac{\partial \Delta\psi}{\partial t} = \exp(24\pi^2\psi), & x \in \partial\mathbb{R}^4_+, \\ \psi(0) = \sup\psi = 0, & \\ \dfrac{\partial\psi}{\partial t} = 0, & x \in \partial\mathbb{R}^4_+, \end{cases}$$

where $\mathbb{R}^4_+$ is the half-space $\{x = (x', t) : x' \in \mathbb{R}^3, t > 0\}$. Instead of assuming $\psi \in W^{2,2}(\mathbb{R}^4_+)$ or other global integrality condition for ψ (see [Ache and Chang 2017; Ndiaye and Sun 2024]), we can prove the classification theorem under the finite growth condition $\int_{B_R^+} |\Delta\psi|\, dx \le CR^2$. This finite growth condition can be verified by the technique of harmonic analysis. Indeed, in order to achieve this goal, we need to prove an important local estimate for Δu_k:

$$\int_{B_\rho(x_k)\cap\Omega} |u_k \Delta u_k|\, dx \le C\rho^2, \tag{1-6}$$

when ρ is small. For this, we first rewrite $u_k\Delta u_k$ in terms of the Riesz potential, then by using the Hardy–Littlewood–Sobolev inequality on the compact manifold with the boundary and the boundedness in $L\log^{\frac{1}{2}} L(\Omega)$of $\Delta^2 u_k$, we can show that $u_k \Delta u_k$ is bounded in the Lorentz space $L^{2,\infty}(\Omega)$, which implies (1-6). Applying this local estimate and a careful computation, we can show that the solution ψ must take the form

$$\psi = -\frac{1}{8\pi^2}\log\Big(\big(1+\big(\tfrac{\pi}{2}\big)^{\frac{2}{3}}t\big)^2+\big(\tfrac{\pi}{2}\big)^{\frac{4}{3}}|x'|^2\Big)+\frac{1}{2^{\frac{8}{3}}\pi^{\frac{4}{3}}}\frac{t}{\big(1+\big(\frac{\pi}{2}\big)^{\frac{2}{3}}t\big)^2+\big(\frac{\pi}{2}\big)^{\frac{4}{3}}|x'|^2}.$$

Fourth, when we try to obtain the upper bound for the trace Adams inequality if the concentration phenomenon occur, an important step consists in finding the sharp lower bounds of the integral of $|\Delta u_k|^2$ on some annular regions when we carry out the capacity estimates. In earlier work [Li and Liu 2005; Yang 2006], this could be achieved by comparing the energy of u_k with the quantity

$$\min_{u\in\{u\,:\,u(R_1)=a,\,u(R_2)=b\}}\int_{\{R_1\le|x|\le R_2\}\cap\mathbb{R}^2_+}|\nabla u|^2\,dx, \tag{1-7}$$

whose extremal function is some harmonic function which can be explicitly obtained by solving some equation on the half-space. However, in the second-order case, finding the explicit expression of the corresponding extremal appears to be very hard. In this work, we will compute the upper bound by directly comparing the Dirichlet energy of u_k with some biharmonic function in the half annular region. In our situation the boundary of the upper half annular region involves some part of $\partial\Omega$ where u_k is not vanishing. This will add a lot of trouble in the comparison of the corresponding calculations since the asymptotic behavior of u_k cannot be obtained in this half annular region. In order to avoid the complicated computations on $\partial\Omega$, we will modify the biharmonic function to cancel the integral on the boundary $\partial\Omega$ (see Section 3.2).

Finally, since $\mathcal{H}$ requires that the test functions not only satisfy $\|\Delta u\|^2 \le 1$, but also satisfy $\partial u/\partial\nu = 0$ on $\partial\Omega$, this makes the construction of test functions more complicated when we try to show that the concentration upper bound can be surpassed.

This paper is organized as follows. Section 2 is devoted to proving the sharp subcritical trace Adams inequality, and showing the existence of extremals. In Section 3, we show that the maximizing sequence must concentrate around the blow-up point when the blow up arises. Moreover, we analyze the asymptotic behavior of the maximizing sequence near and far away from the blow-up point, and derive an upper bound for the trace Adams functional. In Section 4, we prove the existence of extremals by constructing a proper test function sequence, and finish the proof of Theorem 1.1.

2. The best constant for the trace Adams inequality

In this section, we prove that the best constant in Theorem 1.1 is $12\pi^2$. First, we recall the following subcritical Adams inequalities for functions with zero mean value proved by Hang [2022]. Throughout this section, we let $\Omega\subseteq\mathbb{R}^4$ be a bounded smooth domain with smooth boundary $\partial\Omega$. We also recall that

$$\mathcal{H}=\Big\{u\in W^{2,2}(\Omega):\|\Delta u\|_2\le 1,\ \int_\Omega u\,dx=0,\ \frac{\partial u}{\partial\nu}\Big|_{\partial\Omega}=0\Big\}.$$

Lemma 2.1 [Hang 2022, Theorem 3.2]. *For any $\varepsilon > 0$, we have*

$$\sup_{u\in\mathcal{H}\setminus\{0\}} \int_\Omega \exp((16\pi^2-\varepsilon)|u|^2)\,dx < \infty. \tag{2-1}$$

Next, we further prove the following.

Lemma 2.2. *Set $\alpha_2 = \sup\{\alpha : \sup_{u\in\mathcal{H}} \int_\Omega e^{\alpha u^2} < +\infty\}$. Then $\alpha_2 = 16\pi^2$.*

Proof. By Lemma 2.1, we know $\alpha_2 \ge 16\pi^2$. In order to prove the lemma, we only need to show $\alpha \le 16\pi^2$. Taking any $p \in \partial\Omega$, for any $0 < \rho < \delta$, we use the notation $B_\rho = B_\rho(p)$ and set

$$u_k(x) := \begin{cases} \sqrt{\frac{1}{16\pi^2}\log(1/R_k)} - \dfrac{|x|^2}{\rho^2\sqrt{4\pi^2 R_k \log(1/R_k)}} + \dfrac{1}{\sqrt{4\pi^2\log(1/R_k)}} & \text{if } x \in B_{\rho\sqrt[4]{R_k}} \cap \Omega, \\ 1/\sqrt{\pi^2\log(1/R_k)}\,\log(\rho/|x|) & \text{if } x \in (B_\rho \setminus B_{\rho\sqrt[4]{R_k}}) \cap \Omega, \\ \eta_k(|x|) & \text{if } x \in (B_\delta \setminus B_\rho) \cap \Omega, \end{cases}$$

where $\{R_k\}_{k\ge 1} \subset \mathbb{R}^+$, $R_k \searrow 0$, and η_k satisfies

$$\frac{\partial \eta_k}{\partial \nu}\Big|_{\partial B_\rho} = -\frac{1}{\rho\sqrt{\pi^2\log(1/R_k)}}, \quad \frac{\partial \eta_k}{\partial \nu}\Big|_{\partial B_\delta} = 0, \quad \eta_k|_{\partial B_\rho} = \eta_k|_{\partial B_\delta} = 0,$$

and η_k, $\Delta\eta_k$ are all $O(1/\sqrt{\log 1/R_k})$. Since u_k is radial, we can choose some function $\varphi_k(x)$ such that

$$\frac{\partial \varphi_k}{\partial \nu} = \frac{\partial u_k}{\partial \nu} = o_\delta(1) \quad \text{for } x \in \partial\Omega,$$

$\varphi_k(x) = o_\delta(1)$ and $\|\Delta\varphi_k\|_2^2 = o_\delta(1)$ as $\delta \to 0$.

For some fixed $r > \delta$, set

$$U_k(x) := \begin{cases} u_k - \varphi_k & \text{if } x \in B_\delta \cap \Omega, \\ t_k\phi_k & \text{if } x \in (B_r \setminus B_\delta) \cap \Omega, \end{cases}$$

where ϕ_k is a smooth function such that $\operatorname{supp}(\phi_k) \subset B_r \setminus B_\delta$, $\frac{\partial \phi_k}{\partial \nu}\big|_{\partial\Omega} = 0$, and t_k is selected such that $\int_\Omega U_k(x)\,dx = 0$. Easy computation directly gives

$$\|U_k\|_2^2 = O_{k,r}(1), \quad \|\Delta U_k\|_2^2 = 1 + O_{k,r}(1).$$

Normalizing U_k by $\widetilde{U}_k = U_k/\|\Delta U_k\|$, we have $\widetilde{U}_k \in \mathcal{H}$. Then it follows that for any fixed $\alpha > 16\pi^2$, there exists some $\varepsilon_0 > 0$, such that

$$\int_\Omega e^{\alpha \widetilde{U}_k^2} \ge \int_{\Omega\cap B_{\rho\sqrt[4]{R_k}}} e^{\alpha\widetilde{U}_k^2} \ge c\rho^4 e^{\varepsilon_0 \log(1/R_k)} \to \infty,$$

as $k \to \infty$, and the proof is finished. □

Based on Lemma 2.2, we can show that the best constant of the inequality (1-3) is $12\pi^2$.

Lemma 2.3. *Set $I_\alpha(u) = \int_{\partial\Omega} e^{\alpha u^2}\,d\sigma$. Then we have*

$$\sup_{u\in\mathcal{H}\setminus\{0\}} I_\alpha(u) < +\infty \quad \textit{for } \alpha < 12\pi^2 \qquad \textit{and} \qquad \sup_{u\in\mathcal{H}\setminus\{0\}} I_\alpha(u) = +\infty \quad \textit{for } \alpha > 12\pi^2.$$

Proof. Take a smooth vector field $\vec{\nu}(x)$ whose restriction on $\partial\Omega$ is the outward unit normal vector field. Using the divergence theorem and Sobolev embedding theorem, we derive that for any $\varepsilon > 0$,

$$\begin{aligned}\int_{\partial\Omega} e^{(12\pi^2-\varepsilon)u^2}\,d\sigma &= \int_\Omega \operatorname{div}(\vec{\nu}(x)e^{(12\pi^2-\varepsilon)u^2})\,dx \\ &= \int_\Omega \big(\operatorname{div}(\vec{\nu}(x)) + 2(12\pi^2-\varepsilon)u\langle\vec{\nu}(x),\nabla u\rangle\big)e^{(12\pi^2-\varepsilon)u^2}\,dx \\ &\le c\left(1+\int_\Omega |\nabla u|\,|u|e^{(12\pi^2-\varepsilon)u^2}\right)dx \\ &\le c\big(1+\|\nabla u\|_{L^4(\Omega)}\|u\|_{L^p(\Omega)}\|e^{(12\pi^2-\varepsilon)u^2}\|_{L^{(16\pi^2-\varepsilon)/(12\pi^2-\varepsilon)}(\Omega)}\big) \\ &\le c + c(\|\Delta u\|^2_{L^2(\Omega)}+\|u\|^2_{L^2(\Omega)})^{\frac{1}{2}}\|u\|_{L^p(\Omega)}\|e^{(12\pi^2-\varepsilon)u^2}\|_{L^{(16\pi^2-\varepsilon)/(12\pi^2-\varepsilon)}(\Omega)},\end{aligned} \tag{2-2}$$

where $1/p + \frac{1}{4} + (12\pi^2-\varepsilon)/(16\pi^2-\varepsilon) = 1$. This together with Lemma 2.2 yields

$$\sup_{u\in\mathcal{H}\setminus\{0\}} I_{12\pi^2-\varepsilon}(u) < +\infty$$

for any $\varepsilon > 0$. Using the test function $\widetilde{U}_k$ constructed in Lemma 2.2 again, one can easily check that for any $\alpha > 12\pi^2$, $I_\alpha(\widetilde{U}_k) \to +\infty$ as $k \to \infty$. □

Let α_k be an increasing sequence converging to $12\pi^2$. Then by the weak compactness of the Banach space $L^{12\pi^2/\alpha_k}$, there exists an extremal function $u_k \in \mathcal{H}\setminus\{0\}$ such that

$$\int_{\partial\Omega} \exp(\alpha_k|u_k|^2)\,d\sigma = \sup_{u\in\mathcal{H}\setminus\{0\}} \int_{\partial\Omega} \exp(\alpha_k|u|^2)\,d\sigma.$$

Furthermore, we can show that the extremal function $u_k \in \mathcal{H}$ is smooth. For this, we first recall the following elliptic regularity result.

Lemma 2.4 [Troianiello 1987, Theorem 3.17]. *Suppose that $f \in L^p(\Omega)$ and $h \in W^{1,p}(\Omega)$ for some $p \ge 2$. Let $u \in W^{1,2}(\Omega)$ be a solution of*

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = h & \text{on } \partial\Omega. \end{cases}$$

Then $u \in W^{2,p}(\Omega)$.

Lemma 2.5. *For any $\alpha_k < 12\pi^2$, the functional $I_{\alpha_k}(u)$ defined in $\mathcal{H}$ admits a smooth maximizer.*

Proof. Obviously, there exists $u_k \in \mathcal{H}$ such that

$$I_{\alpha_k}(u_k) = \sup_{u\in\mathcal{H}\setminus\{0\}} I_{\alpha_k}(u).$$

Hence u_k satisfies the Euler–Lagrange equation

$$\begin{cases} \Delta^2 u_k = \gamma_k & \text{for all } x \in \Omega, \\ \dfrac{\partial \Delta u_k}{\partial \nu} = \dfrac{u_k \exp(\alpha_k u_k^2)}{\lambda_k} & \text{for all } x \in \partial\Omega, \\ \int_\Omega |\Delta u_k|^2\,dx = 1,\ \int_\Omega u_k\,dx = 0,\ \dfrac{\partial u_k}{\partial \nu} = 0 & \text{for all } x \in \partial\Omega, \end{cases} \tag{2-3}$$

where

$$\lambda_k = -\int_{\partial\Omega} u_k^2 \exp(\alpha_k u_k^2)\, d\sigma, \quad \gamma_k = \int_{\partial\Omega} \frac{u_k \exp(\alpha_k u_k^2)}{\lambda_k |\Omega|}\, d\sigma. \tag{2-4}$$

By the Orlicz embedding (see Lemma 3.4 in [Hang 2022]), we obtain $\exp(u_k^2) \in L^p(\Omega)$ for any $p > 1$. Therefore, $u_k \exp(\alpha_k u_k^2)/\lambda_k \in W^{1,q}(\Omega)$ for any $1 < q < 2$. We claim that $u_k \in L^\infty(\Omega)$. Indeed, we can rewrite (2-3) as the systems

$$\begin{cases} \Delta u_k = v_k & \text{in } \Omega, \\ \dfrac{\partial u_k}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2-5}$$

and

$$\begin{cases} \Delta v_k = \gamma_k & \text{in } \Omega, \\ \dfrac{\partial v_k}{\partial \nu} = h_k & \text{on } \partial\Omega, \end{cases} \tag{2-6}$$

where $h_k = u_k \exp(\alpha_k u_k^2)/\lambda_k$. Applying Lemma 2.4 for (2-6), we know $v_k \in W^{2,q}(\Omega)$. By the Sobolev embedding theorem, we get $v_k \in L^{4q/(4-2q)}(\Omega)$. Using Lemma 2.4 again for (2-5), we derive that $u_k \in W^{2,4q/(4-2q)}(\Omega)$. Since $q > 1$, we can immediately obtain the claim by the Sobolev embedding theorem.

From the boundedness of u_k, we know that $h_k \in W^{1,2}(\Omega)$. Thus we have $h_k \in W^{2,2}(\Omega)$ from Lemma 2.4, and hence $h_k \in W^{1,p}(\Omega)$ for any $p > 2$. By Lemma 2.4 again we have $u_k \in W^{2,p}(\Omega)$ for any $p > 2$, which implies that $u_k \in C^{1,\alpha}(\Omega)$ for some $\alpha \in (0, 1)$ by the Sobolev compact embedding. Since on the boundary $\partial\Omega$,

$$\frac{\partial v_k}{\partial \nu} = h_k = \frac{u_k \exp(\alpha_k u_k^2)}{\lambda_k} \in C^{1,\alpha}(\partial\Omega)$$

and v_k satisfies (2-6), the elliptic regularity gives that $v_k \in C^{2,\alpha}(\Omega)$. Since $\Delta u_k = v_k \in C^{2,\alpha}(\Omega)$ and by (2-5), we can furthermore derive that $u_k \in C^{4,\alpha}(\Omega)$ using the elliptic regularity again. Repeating the above procedure, we obtain $u_k \in C^\infty(\Omega)$. □

Now, we give the following important observation.

Lemma 2.6. $-\liminf_{k\to\infty} \lambda_k > 0,$ *and* $|\gamma_k| < c$ *for some* $c > 0$.

Proof. By the elementary inequality $te^t > e^t - 1$ for all $t > 0$, we have

$$|\partial\Omega| < \sup_{u\in\mathcal{H}\setminus\{0\}} \int_{\partial\Omega} e^{12\pi^2 u^2} = \lim_{k\to\infty} \int_{\partial\Omega} e^{\alpha_k u_k^2} \le |\partial\Omega| - \liminf_{k\to\infty} 12\pi^2 \lambda_k. \tag{2-7}$$

This implies $-\liminf_{k\to\infty} \lambda_k > 0$. By (2-4), (2-7) and Hölder's inequality, we derive

$$|\gamma_k| \le \frac{-1}{\lambda_k |\Omega|} \left(\int_{\partial\Omega} u_k^2 \exp(\alpha_k u_k^2)\, d\sigma \right)^{\frac{1}{2}} \left(\int_{\partial\Omega} \exp(\alpha_k u_k^2)\, d\sigma \right)^{\frac{1}{2}} \le c.$$ □

Set $c_k = |u_k(x_k)| = \max_{x\in\Omega} |u_k(x)|$. If $\{c_k\}$ is bounded, then by the elliptic estimates with respect to (2-3), there exists $u \in \mathcal{H} \cap C^\infty(\Omega)$ such that $u_k \to u$ in $C^\infty(\Omega)$ as $k \to \infty$, and Theorem 1.1 follows immediately. In the sequel, we assume $c_k \to +\infty$ as $k \to \infty$. Passing to a subsequence, we may assume that $u_k(x_k) \ge 0$ for all k, for otherwise we consider $-u_k$ instead of u_k.

3. Blow-up analysis

In this section, we consider the blow-up case, that is $u_k(x_k) \to \infty$ as $k \to \infty$. Applying the Adams inequality [1988], we know that passing to a subsequence, $x_k \to p$ for some $p \in \partial\Omega$. Now, we show that the weak limit of u_k in $W^{2,2}(\Omega)$ is zero. Furthermore, u_k must concentrate around the blow-up point p.

Lemma 3.1. *If $c_k \to +\infty$, then $u_k \rightharpoonup 0$ in $W^{2,2}(\Omega)$ and $u_k \to 0$ in $L^p(\Omega)$ for any $1 \le p < \infty$. Moreover,*

(i) $|\Delta u_k|^2\,dx \to \delta_p$ *in the sense of measures*;

(ii) $e^{\alpha_k u_k^2}$ *is bounded in $L^p(\Omega \setminus B_\delta(p))$, for any $p \ge 1$, $\delta > 0$;*

(iii) $u_k \to 0$ *in $C^{3,\gamma}(\Omega \setminus B_\delta(p))$, for any $\gamma \in (0,1)$, $\delta > 0$.*

Proof. Since u_k is bounded in $W^{2,2}(\Omega)$, we assume that $u_k \rightharpoonup u_0$ in $W^{2,2}(\Omega)$ with some $u_0 \in W^{2,2}(\Omega)$. The compactness of the embedding of $W^{2,2}(\Omega)$ into $L^p(\Omega)$ implies $u_k \to u_0$ in $L^p(\Omega)$ for any $p \ge 1$. If $u_0 \neq 0$, then by the concentration compactness principle (see Proposition 3.2 of [Hang 2022]), $e^{16\pi^2 u_k^2}$ is bounded in $L^p(\Omega)$ for some $p > 1$. Similarly to (2-2), we can find some $\varepsilon_0 > 0$ such that $e^{12\pi^2 u_k^2}$ is bounded in $L^{1+\varepsilon_0}(\partial\Omega)$, hence $\partial\Delta u_k/\partial\nu$ is bounded in $L^{1+\varepsilon_0}(\partial\Omega)$. Using the same argument in Lemma 2.5, we get that u_k is bounded in $L^\infty(\Omega)$. This contradicts $c_k \to +\infty$. Hence, we have $u_0 = 0$.

Now, we show that u_k must concentrate around the blow-up point p. Let

$$A = \left\{ q \in \Omega : \lim_{r\to 0} \liminf_{k\to\infty} \int_{B_r(q)} |\Delta u_k|^2\,dx > 0 \right\}.$$

We claim that A contains only one point. Suppose that the claim does not hold. Then, for any $q \in \Omega$, we have

$$\lim_{r\to 0} \liminf_{k\to\infty} \int_{B_r(q)} |\Delta u_k|^2\,dx < 1.$$

Then there exist positive numbers r and δ such that

$$\int_{B_r(q)} |\Delta u_k|^2\,dx \le \delta(q) < 1.$$

Using the same argument as that in (2-2) again, we see that there exists a constant $\alpha(q) > 12\pi^2$ such that

$$\int_{\partial\Omega\cap B_r(q)} e^{\alpha(q)u_k^2}\,d\sigma \le C_q$$

for some constant C_q depending on q. Hence there exists an $\alpha > 12\pi^2$ such that

$$\int_{\partial\Omega} e^{\alpha u_k^2}\,d\sigma \le C,$$

by using the covering argument. Therefore, it follows from the Vitali convergence lemma that

$$\lim_{k\to+\infty} \int_{\partial\Omega} e^{\alpha_k u_k^2}\,d\sigma = |\partial\Omega|,$$

which is impossible by the choice of u_k. Next we show that $A = \{p\}$ and

$$\lim_{r\to 0}\liminf_{k\to\infty}\int_{B_r(p)} |\Delta u_k|^2 = 1.$$

Suppose not: repeating the argument above, we can obtain that u_k is bounded in $L^\infty(B_\delta(p))$ for some $\delta > 0$, which contradicts with $c_k \to +\infty$, and the statement (i) is proved.

The statement (ii) follows from (i) and Lemma 2.1, and the statement (iii) can be proved by the standard regularity argument; we omit the details. □

To understand the asymptotic behavior of u_k near the blow-up point p, we define

$$r_k^3 = -\frac{\lambda_k}{c_k^2}\exp(-\alpha_k c_k^2).$$

Indeed, r_k decays very fast as $k \to \infty$:

Lemma 3.2. *For any $\gamma < 12\pi^2$, it holds that $e^{\gamma c_k^2} r_k^3 \to 0$ as $k \to \infty$.*

Proof. For any $\gamma < 12\pi^2$, we have

$$\begin{aligned} c_k^2 r_k^3 e^{\gamma c_k^2} &= e^{(\gamma-\alpha_k)c_k^2}\int_{\partial\Omega} u_k^2 e^{\alpha_k u_k^2}\,d\sigma \le \int_{\partial\Omega} u_k^2 e^{\alpha_k u_k^2} e^{(\gamma-\alpha_k)u_k^2}\,d\sigma \\ &= \int_{\partial\Omega} u_k^2 e^{\gamma u_k^2}\,d\sigma \le \left(\int_{\partial\Omega} u_k^s\,d\sigma\right)^{2/s}\left(\int_{\partial\Omega} e^{\gamma s/(s-2) u_k^2}\,d\sigma\right)^{(s-2)/s} \\ &\le c, \end{aligned}$$

provided s is large enough, where we have used the subcritical trace Adams inequality. □

The following important observation allows us to choose suitably approximate $\{x_k\}$ by points on the boundary $\partial\Omega$.

Lemma 3.3. *There exists some point $\tilde{x}_k \in \partial\Omega$ such that*

$$|u_k(\tilde{x}_k) - u_k(x_k)| = o_k(1),$$

as $k \to \infty$.

Proof. If $x_k \notin \partial\Omega$, since $d(x_k, \partial\Omega)$ is sufficiently small, then there exists a unique $y_k \in \Omega$ such that $d_k := d(x_k, \partial\Omega) = |y_k - x_k|$ and $x_k = y_k + d_k\nu_k$, where ν_k is the inner normal vector of boundary $\partial\Omega$ at the point y_k. Hence

$$|u_k(x_k) - u_k(y_k)| \le \int_0^1 \left|\frac{d}{dt}(u_k(y_k + t d_k \nu_k)\right| dt = \int_0^1 \left|\frac{\partial u_k}{\partial \nu_k}(y_k + t d_k, \nu_k) d_k\right| dt \to 0$$

when $k \to +\infty$ by the mean value theorem and the fact that $\frac{\partial u_k}{\partial_{\nu_k}}\big|_{y_k} = 0$, and the proof is finished. □

In view of Lemma 3.3, we can take $x_k = \tilde{x}_k \in \partial\Omega$ and then

$$u_k(x_k) = c_k + o_k(1), \tag{3-1}$$

as $k \to \infty$. Define two sequences of functions on $\partial\Omega$, namely,

$$\begin{cases} \phi_k(x) = u_k(x_k + r_k x)/c_k, & x \in \Omega_k = \{x : x_k + r_k x \in \Omega\}, \\ \psi_k(x) = c_k(u_k(x_k + r_k x) - c_k), & x \in \Omega_k. \end{cases}$$

Up to translation and rotation, we can easily obtain $\Omega_k \to \mathbb{R}^4_+$ as $k \to +\infty$.

Lemma 3.4. $$\phi_k(x) \to 1 \quad \textit{in } C^3_{\text{loc}}(\overline{\mathbb{R}^4_+}).$$

Proof. By (2-3), for k large enough we have

$$\begin{cases} \Delta^2 \phi_k = \dfrac{r_k^4}{c_k} \gamma_k & \text{for all } x \in B_R(0) \cap \Omega_k, \\ \dfrac{\partial}{\partial \nu} \Delta \phi_k = \dfrac{r_k^3 u_k \exp(\alpha_k u_k^2)}{c_k \lambda_k} & \text{for all } x \in B_R(0) \cap \partial\Omega_k, \end{cases} \tag{3-2}$$

for any $R > 0$. By the definition of r_k, we have

$$\left| \frac{r_k^4}{c_k} \gamma_k \right| = \frac{\lambda_k}{c_k^2} \exp(-\alpha_k c_k^2) \frac{r_k}{c_k} \int_{\partial\Omega} \frac{u_k \exp(\alpha_k u_k^2)}{\lambda_k |\Omega|} \, d\sigma \le \frac{|\partial\Omega|}{|\Omega|} \frac{r_k}{c_k^2} \to 0$$

and

$$\left| \frac{r_k^3 u_k \exp(\alpha_k u_k^2)}{c_k \lambda_k} \right| \le \frac{1}{c_k^2} \to 0,$$

as $k \to \infty$. Since ϕ_k is bounded in $L^1_{\text{loc}}(\overline{B_R(0) \cap \Omega_k})$ and $\phi_k(x_k) = 1 + o_k(1)$, by the standard elliptic regularity argument, we have $\phi_k \to 1$ in $C^3_{\text{loc}}(\overline{B_{R/2}(0) \cap \Omega_k})$. □

In order to obtain the limit behavior of ψ_k, we need to check the following growth condition:

Lemma 3.5. $$\int_{B_R(0) \cap \Omega_k} |\Delta \psi_k| \, dx \le C R^2.$$

Proof. Direct computation gives that

$$\int_{B_R(0) \cap \Omega_k} |\Delta \psi_k| \, dx = c_k r_k^{-2} \int_{B_{R r_k}(x_k) \cap \Omega} |\Delta u_k| \, dx.$$

Since $u_k(r_k x + x_k)/c_k \to 1$ in $C^3(B_R \cap \Omega_k)$ for any $R > 0$, in order to prove this lemma we only need to show that

$$(R r_k)^{-2} \int_{B_{R r_k}(x_k) \cap \Omega} |u_k \Delta u_k| \, dx \lesssim 1.$$

Applying Hölder's inequality in Lorentz space (see [O'Neil 1963]), we get

$$\begin{aligned} (R r_k)^{-2} \int_{B_{R r_k}(x_k) \cap \Omega} |u_k \Delta u_k| \, dx &\le (R r_k)^{-2} \|\chi_{B_{R r_k}(x_k) \cap \Omega_k}\|_{L^{2,1}(B_{R r_k}(x_k) \cap \Omega)} \|u_k \Delta u_k\|_{L^{2,\infty}(B_{R r_k}(x_k) \cap \Omega)} \\ &\lesssim \|u_k \Delta u_k\|_{L^{2,\infty}(B_{R r_k}(x_k))}. \end{aligned} \tag{3-3}$$

Now, we start to prove that $\|u_k \Delta u_k\|_{L^{2,\infty}(\Omega)} \lesssim 1$. Let G denote the Green function of the Laplacian operator with Neumann boundary condition:

$$\begin{cases} -\Delta G_x(y) = \delta_x(y) - \dfrac{1}{|\Omega|}, & x, y \in \overline{\Omega}, \\ \dfrac{\partial G}{\partial \nu}\Big|_{\partial\Omega} = 0, & \\ \int_\Omega G_x(y)\, dy = 0, & x \in \overline{\Omega}. \end{cases}$$

Obviously $G_x(y)$ satisfies $G_x(y) \lesssim |x-y|^{-2}$ for any $x, y \in \Omega$. By integration by parts together with $\int_\Omega u_k(x)\, dx = 0$ and $\partial u_k/\partial\nu|_{\partial\Omega} = 0$ and using the fact that $\int_\Omega |\Delta u_k|^2\, dx = 1$ (see the Euler–Lagrange equation (2-3)), we derive that

$$|u_k(x)| \lesssim \int_\Omega |\Delta u_k|\, |x-y|^{-2}\, dy$$

and

$$|\Delta u_k(x)| \lesssim \int_{\partial\Omega} |x-y|^{-2} f_k(y)\, d\sigma_y + \int_\Omega |x-y|^{-2} \gamma_k\, dy + \frac{1}{|\Omega|} \int_\Omega |\Delta u_k|\, dy \lesssim 1 + \int_{\partial\Omega} |x-y|^{-2} f_k(y)\, d\sigma_y,$$

where $f_k = u_k \exp(\alpha_k u_k^2)/\lambda_k$. Then it follows that

$$|u_k(x)|\, |\Delta u_k(x)| \lesssim \left(\int_\Omega |(\Delta u_k)(y)|\, |x-y|^{-2}\, dy \right) \left(1 + \int_{\partial\Omega} |x-z|^{-2} f_k(z)\, d\sigma_z \right). \tag{3-4}$$

Now, we claim that

$$\left\| \left(\int_\Omega |\Delta u_k(y)|\, |x-y|^{-2}\, dy \right) \left(1 + \int_{\partial\Omega} |x-z|^{-2} f_k(z)\, d\sigma_z \right) \right\|_{L^{2,\infty}} \lesssim 1.$$

Recall the Hardy–Littlewood–Sobolev inequality in $\mathbb{R}^n$: for any $f \in L^p(\mathbb{R}^n)$,

$$\left\| \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\theta}}\, dy \right\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)},$$

where $p > 1$, $0 < \theta < n$ and $\frac{1}{q} = \frac{1}{p} - \frac{\theta}{n}$. Hence it follows that

$$\begin{aligned} \left\| \left(\int_\Omega |\Delta u_k(y)|\, |x-y|^{-2}\, dy \right) \right\|_{L^{2,\infty}(\Omega)} &\lesssim \left\| \left(\int_\Omega |\Delta u_k(y)|\, |x-y|^{-2}\, dy \right) \right\|_{L^6(\mathbb{R}^4)} \\ &\lesssim \|\Delta u\|_{L^{\frac{3}{2}}(\Omega)} \lesssim \|\Delta u\|_{L^2(\Omega)} \lesssim 1. \end{aligned} \tag{3-5}$$

Hence it suffices to prove that

$$\left\| \left(\int_\Omega |\Delta u_k(y)|\, |x-y|^{-2}\, dy \right) \left(\int_{\partial\Omega} |x-z|^{-2} f_k(z)\, d\sigma_z \right) \right\|_{L^{2,\infty}} \lesssim 1.$$

For any $\varepsilon > 0$ sufficiently small, using the estimate (see [Maalaoui et al. 2016])

$$|x-y|^{-2} |x-z|^{-2} \le |x-y|^{-2-\varepsilon} |x-z|^{-2+\varepsilon} + |z-y|^{-2} |x-z|^{-2},$$

we obtain

$$\begin{aligned}&\left\|\left(\int_\Omega |\Delta u_k(y)|\,|x-y|^{-2}\,dy\right)\left(\int_{\partial\Omega}|x-z|^{-2}f_k(z)\,d\sigma_z\right)\right\|_{L^{2,\infty}}\\&\quad\le\left\|\left(\int_\Omega |\Delta u_k(y)|\,|x-y|^{-2-\varepsilon}\,dy\right)\left(\int_{\partial\Omega}|x-z|^{-2+\varepsilon}f_k(z)\,d\sigma_z\right)\right\|_{L^{2,\infty}}\\&\qquad\qquad+\left\|\int_{\partial\Omega}\left(\int_\Omega|\Delta u_k(y)|\,|z-y|^{-2}\,dy\right)f_k(z)|x-z|^{-2}\,d\sigma_z\right\|_{L^{2,\infty}}\\&:=\mathrm{I}_1+\mathrm{I}_2.\end{aligned}\tag{3-6}$$

Applying the generalized Hölder's inequality involving the Lorentz norm, we derive that

$$\mathrm{I}_1\le\left\|\int_\Omega|\Delta u_k(y)|\,|x-y|^{-2-\varepsilon}\,dy\right\|_{L^{4/\varepsilon}(\Omega)}\left\|\int_{\partial\Omega}|x-z|^{-2+\varepsilon}f_k(z)\,d\sigma_z\right\|_{L^{4/(2-\varepsilon),\infty}(\Omega)}:=\mathrm{I}_{11}\times\mathrm{I}_{12}.$$

For I_{11}, the boundedness of fractional integral operator directly gives $\mathrm{I}_{11}\lesssim\|\Delta u_k\|_{L^2(\Omega)}$. For I_{12}, we claim that it can be dominated by $\|f_k\|_{L^1(\partial\Omega)}$. Define the auxiliary integral operators

$$T^1_{\varepsilon,r}(x)=\int_{\{\partial\Omega\cap|x-y|<r\}}\frac{f_k(y)}{|x-y|^{2-\varepsilon}}\,d\sigma_y,\quad T^2_{\varepsilon,r}(x)=\int_{\{\partial\Omega\cap|x-y|\ge r\}}\frac{f_k(y)}{|x-y|^{2-\varepsilon}}\,d\sigma_y.$$

Obviously,

$$\int_\Omega|T^1_{\varepsilon,r}(x)|\,dx\le\left(\sup_{y\in\partial\Omega}\int_{\{|x-y|<r\}}\frac{1}{|x-y|^{2-\varepsilon}}\,dx\right)\|f_k\|_{L^1(\partial\Omega)}\lesssim r^{2+\varepsilon}\|f_k\|_{L^1(\partial\Omega)}$$

and

$$\|T^2_{\varepsilon,r}\|_{L^\infty(\Omega)}\le\frac{1}{r^{2-\varepsilon}}\|f_k\|_{L^1(\partial\Omega)}.$$

For any $\lambda>0$, we can write

$$|\{x:T^1_{\varepsilon,r}(x)+T^2_{\varepsilon,r}(x)>2\lambda\}|\le|\{x:T^1_{\varepsilon,r}(x)>\lambda\}|+|\{x:T^2_{\varepsilon,r}(x)>\lambda\}|.$$

Choosing r such that $1/r^{2-\varepsilon}\|f_k\|_{L^1(\partial\Omega)}=\lambda$, then $|\{x:T^2_{\varepsilon,r}(x)>\lambda\}|=0$. Hence, we deduce that

$$|\{x:T^1_{\varepsilon,r}(x)+T^2_{\varepsilon,r}(x)>2\lambda\}|\lesssim\frac{r^{2+\varepsilon}}{\lambda}\|f_k\|_{L^1(\partial\Omega)}=\frac{1}{\lambda^{4/(2-\varepsilon)}}\|f_k\|^{4/(2-\varepsilon)}_{L^1(\partial\Omega)},$$

which gives that $\mathrm{I}_{12}\lesssim\|f_k\|_{L^1(\partial\Omega)}$, and the claim is proved.

Gathering the estimates of I_{11} and I_{12}, we derive that $\mathrm{I}_1\lesssim\|\Delta u_k\|_{L^2(\Omega)}\|f_k\|_{L^1(\partial\Omega)}$. For I_2, obviously

$$\mathrm{I}_2\lesssim\left\|\int_\Omega|\Delta u_k(y)|\,|z-y|^{-2}f_k(z)\,dy\right\|_{L^1(\partial\Omega)}\lesssim\|\Delta u_k\|_{L^2(\Omega)}\left\|\int_{\partial\Omega}|z-y|^{-2}f_k(z)\,d\sigma_z\right\|_{L^2(\Omega)}.$$

According to Corollary 6.16 in [Bennett and Sharpley 1988], we derive that

$$\left\|\int_{\partial\Omega}|z-y|^{-2}f_k(z)\,d\sigma_z\right\|_{L^2(\Omega)}\lesssim\int_{\partial\Omega}f_k(z)\log^{\frac12}(1+f_k(z))\,d\sigma_z.$$

Since $f_k = u_k \exp(\alpha_k u_k^2)/\lambda_k$, it is easy to check that $\int_{\partial\Omega} f_k(z) \log^{\frac{1}{2}}(1+f_k(z))\, d\sigma_z \lesssim 1$.Combining the estimates of I_1 and I_2, we find that

$$\left\| \left(\int_\Omega |\Delta u_k(y)| |x-y|^{-2}\, dy \right) \left(\int_{\partial\Omega} |x-z|^{-2} f_k(z)\, d\sigma_z \right) \right\|_{L^{2,\infty}} \lesssim 1,$$

which accomplishes the proof of Lemma 3.5. □

Lemma 3.6. *We have* $\psi_k(x) \to \psi(x', t)$ *in* $C^3_{\mathrm{loc}}(\overline{B_R^+(0)})$ $(x' \in \partial\mathbb{R}^4_+,\ t \in \mathbb{R}^+)$, *where* $\psi(x', t)$ *satisfies the equations*

$$\begin{cases} \Delta^2 \psi = 0, & x \in \mathbb{R}^4_+, \\ \dfrac{\partial \Delta \psi}{\partial t} = \exp(24\pi^2 \psi), & x \in \partial\mathbb{R}^4_+, \\ \psi(0) = \sup \psi = 0, & \\ \dfrac{\partial \psi}{\partial t} = 0, & x \in \partial\mathbb{R}^4_+. \end{cases}$$

Furthermore, ψ *must take the form*

$$\psi = -\frac{1}{8\pi^2} \log\Big(\big(1 + \big(\tfrac{\pi}{2}\big)^{\frac{2}{3}} t\big)^2 + \big(\tfrac{\pi}{2}\big)^{\frac{4}{3}} |x'|^2 \Big) + \frac{1}{2^{\frac{8}{3}} \pi^{\frac{4}{3}}} \frac{t}{\big(1 + \big(\frac{\pi}{2}\big)^{\frac{2}{3}} t\big)^2 + \big(\frac{\pi}{2}\big)^{\frac{4}{3}} |x'|^2}.$$

Proof. By (2-3), we can easily obtain

$$\begin{cases} \Delta^2 \psi_k = c_k r_k^4 \gamma_k & \text{for all } x \in B_R(0) \cap \Omega_k, \\ \dfrac{\partial \psi_k}{\partial t} = 0 & \text{for all } x \in B_R(0) \cap \partial\Omega_k, \\ \dfrac{\partial \Delta \psi_k}{\partial t} = \dfrac{u_k \exp(\alpha_k \psi_k (1 + u_k/c_k))}{c_k} & \text{for all } x \in B_R(0) \cap \partial\Omega_k, \end{cases} \tag{3-7}$$

for any $R > 0$. Let $-\Delta \psi_k = v_k$; then ψ_k and v_k respectively satisfy the equations

$$\begin{cases} -\Delta \psi_k = v_k & \text{for all } x \in B_R(0) \cap \Omega_k, \\ \dfrac{\partial \psi_k}{\partial t} = 0 & \text{for all } x \in B_R(0) \cap \partial\Omega_k, \end{cases} \tag{3-8}$$

and

$$\begin{cases} -\Delta v_k = c_k r_k^4 \gamma_k & \text{for all } x \in B_R(0) \cap \Omega_k, \\ \dfrac{\partial v_k}{\partial t} = \dfrac{u_k \exp(\alpha_k \psi_k (1 + u_k/c_k))}{c_k} & \text{for all } x \in B_R(0) \cap \partial\Omega_k. \end{cases} \tag{3-9}$$

Noticing

$$\frac{\partial v_k}{\partial t} = \frac{u_k \exp(\alpha_k \psi_k (1 + u_k/c_k))}{c_k} \in L^\infty(B_R(0) \cap \partial\Omega_k),$$

applying Lemma 3.5 and the standard elliptic regularity, we deduce that

$$\|v_k\|_{C^{1,\alpha}(\overline{B_{R/2}(0) \cap \Omega_k})} \lesssim 1.$$

Then there exists some $v \in C^{1,\alpha}(\overline{B_{R/2}(0) \cap \Omega_k})$ such that $v_k \to v$ in $C^{1,\beta}(\overline{B_{R/2}(0) \cap \Omega_k})$ for any $\beta < \alpha$. Let $\widetilde{\psi_k(x)}$ be the even extension of ψ_k with respect to the boundary $\partial B_R^+(0) \cap \partial\mathbb{R}^4_+$; then we have $-\Delta \widetilde{\psi_k} \in C^{1,\alpha}(B_R(0) \cap \Omega_k)$, $\widetilde{\psi_k(x)} \le \psi_k(0) = 0$. Using the Harnack inequality and elliptic regularity estimates, we get $\|\widetilde{\psi_k}\|_{C^{3,\alpha}(B_R(0)\cap\Omega_k)} \lesssim C$. Hence there exists $\psi \in C^{3,\beta}(\overline{B_R(0)\cap\Omega_k})$ such that $\psi_k \to \psi$ in $C^{3,\beta}(\overline{B_R(0)\cap\Omega_k})$ for any $\beta < \alpha$, where ψ satisfies the equation

$$\begin{cases} \Delta^2\psi = 0 & \text{in } \mathbb{R}^4_+, \\ \dfrac{\partial\Delta\psi}{\partial t} = \exp(24\pi^2\psi) & \text{on } \partial\mathbb{R}^4_+, \\ \psi(0) = \sup\psi = 0, & \\ \dfrac{\partial\psi}{\partial t} = 0, & \text{on } \partial\mathbb{R}^4_+. \end{cases}$$

From (3-1), it is not difficult to see that

$$\int_{B_R\cap\partial\mathbb{R}^4_+} \exp(24\pi^2\psi) \le -\!\!\!\!\!\!\int_{B_{Rr_k}\cap\partial\Omega} \frac{u_k^2\exp(\alpha_k u_k^2)}{\lambda_k} \le 1. \tag{3-10}$$

Next, we will prove that ψ must take the form

$$\psi = -\frac{1}{8\pi^2}\log\Big(\big(1+\big(\tfrac{\pi}{2}\big)^{\frac{2}{3}}t\big)^2 + \big(\tfrac{\pi}{2}\big)^{\frac{4}{3}}|x'|^2\Big) + \frac{1}{2^{\frac{8}{3}}\pi^{\frac{4}{3}}}\frac{t}{\big(1+\big(\frac{\pi}{2}\big)^{\frac{2}{3}}t\big)^2 + \big(\frac{\pi}{2}\big)^{\frac{4}{3}}|x'|^2}.$$

Indeed, let $\phi(x) = \int_{\partial\mathbb{R}_4^+} P(x,y')\psi(y',0)\,dy'$, where $x = (x',t),\ y = (y',t)$ and

$$P(x,y') = \frac{4}{\pi^2}\frac{t^3}{|x-y'|^6}$$

is the Poisson kernel for the bi-Laplace operator on the upper half-space. It is not difficult to check that ϕ satisfies the equations

$$\begin{cases} (-\Delta)^2\phi = 0, & x \in \mathbb{R}^4_+, \\ \phi = \psi(x), & x \in \partial\mathbb{R}^4_+, \\ \dfrac{\partial\phi}{\partial t} = 0, & x \in \partial\mathbb{R}^4_+, \end{cases} \tag{3-11}$$

and $\int_{B_R^+(0)} |\Delta\phi|\,dx \le CR^2$.

Let $w = \psi - \phi$. Then w satisfies

$$\begin{cases} (-\Delta)^2 w = 0, & x \in \mathbb{R}^4_+, \\ w = 0, & x \in \partial\mathbb{R}^4_+, \\ \dfrac{\partial w}{\partial t} = 0, & x \in \partial\mathbb{R}^4_+. \end{cases} \tag{3-12}$$

Noticing $\int_{B_R^+(0)} |\Delta w|\,dx \le \int_{B_R^+(0)} |\Delta\psi|\,dx + \int_{B_R^+(0)} |\Delta\phi|\,dx \le CR^2$, one can deduce that w must be equal to zero. Hence $\psi(x) = \int_{\partial\mathbb{R}_4^+} P(x,y')\psi(y',0)\,d\xi$. Set $\psi_0(x') = \psi(x',0)$. Then we know that $\frac{1}{2}\partial\Delta\psi/\partial t|_{\partial\mathbb{R}^4_+} = (-\Delta)^{\frac{3}{2}}\psi_0$ and $\psi_0(x')$ satisfies the following equation in the distributional sense:

$$\begin{cases} (-\Delta)^{\frac{3}{2}}\psi_0 = \frac{1}{2}e^{24\pi^2\psi_0}, & x' \in \partial\mathbb{R}^3, \\ \int_{\mathbb{R}^3} e^{24\pi^2\psi_0(x')}\,dx' \le \frac{1}{2}. \end{cases} \tag{3-13}$$

Let $\eta_0(x') = 8\pi^2\psi_0(x') + \frac{1}{3}\log(2\pi^2)$. Then η_0 satisfies

$$\begin{cases} (-\Delta)^{\frac{3}{2}}\eta_0 = 2e^{3\eta_0}, & x' \in \mathbb{R}^3, \\ \int_{\mathbb{R}^3} e^{3\eta_0}\,dx' \le \pi^2. \end{cases} \tag{3-14}$$

From the result of Hyder [2019], we known that $\eta_0(x')$ can be decomposed as $\eta_0 = v + p$, where p is a polynomial of degree at most 2 and $v(x') = -\alpha \log|x'| + o(\log|x'|)$ as $|x'| \to +\infty$. Furthermore, $\eta_0(x') = \log(2\lambda/(1+\lambda^2|x'-x_0'|^2))$ if and only if p is a constant. Noticing that $\psi(x',t)$ is a Poisson extension of ψ_0 on $\mathbb{R}^4_+$ and $\int_{B_R^+}|\Delta\psi|\,dx \le CR^2$, we deduce that p must be equal to constant. This proves

$$\eta_0(x') = \log\frac{2\lambda}{1+\lambda^2|x'-x_0'|^2}.$$

Since $\psi(x) \le \psi(0) = \sup_{x\in\mathbb{R}^4_+}\psi(x) = 0$, it follows that ψ has the form

$$\psi(x',t) = -\frac{1}{8\pi^2}\log\Big(\big(1+\big(\tfrac{\pi}{2}\big)^{\frac{2}{3}}t\big)^2 + \big(\tfrac{\pi}{2}\big)^{\frac{4}{3}}|x'|^2\Big) + \frac{1}{2^{\frac{8}{3}}\pi^{\frac{4}{3}}}\frac{t}{\big(1+\big(\frac{\pi}{2}\big)^{\frac{2}{3}}t\big)^2+\big(\frac{\pi}{2}\big)^{\frac{4}{3}}|x'|^2},$$

where the second term ensures $\frac{\partial\psi}{\partial t}\big|_{\partial\mathbb{R}^4_+} = 0$. By an easy computation, one can see that

$$\int_{\partial\mathbb{R}^4_+}\exp(24\pi^2\psi(x'))\,dx' = 1. \qquad \square$$

3.1. ***Polyharmonic truncation functions.*** We first introduce some notation. If $x_0 \in \partial\Omega$, for small $\delta > 0$, let $M_{\delta,x_0} = B_\delta(x_0)\cap\overline{\Omega}$. We can choose a Fermi coordinate (see [Manasse and Misner 1963]) for M_{δ,x_0} by the map $\theta : M_\delta \to \overline{B^3_\delta(0)}\times[0,\delta]$, where $\theta(x_0) = 0$. We will identify M_{δ,x_0} with $\overline{B^3_\delta(0)}\times[0,\delta]$ through the map θ. Under the Fermi coordinate, we can write the metric on the $M_{\delta,p}$ as

$$g = g_{ij}\,dx_i\otimes dx_j + dt\otimes dt \quad (i,j\in\{1,2,3\}), \tag{3-15}$$

where $(1-\varepsilon)\delta_{i,j} \le g_{ij} \le (1+\varepsilon)\delta_{i,j}$ for small $\varepsilon > 0$.

We choose a Fermi coordinate system (U_k,θ_k) near the point x_k such that $\theta_k(x_k) = 0$, and $\theta_k(U_k\cap\Omega) \subseteq \mathbb{R}^4_+ = \{x = (x',t)\in\mathbb{R}^4 : t > 0\}$, and $\theta_k(U_k\cap\partial\Omega) \subseteq \partial\mathbb{R}^4_+$. In the following, we make an even extension for $u_k\circ\theta_k^{-1}$ in the direction of t under the Fermi coordinate system (U_k,θ_k):

$$\begin{cases}\tilde u_k(x) = u_k\circ\theta_k^{-1}(x',t) & \text{if } t\ge 0,\\ \tilde u_k(x) = u_k\circ\theta_k^{-1}(x',-t) & \text{if } t<0.\end{cases}$$

Then $\tilde u_k(x)\in W^{2,2}(B_r(0))$ with $\|\Delta\tilde u_k\|_{L^2(B_r(0))} = 2\|\Delta\tilde u_k\|_{L^2(B_r^+(0))}$ for small $r > 0$.

Now, we need some biharmonic truncation functions $\tilde u_k^M$ which was studied in [DelaTorre and Mancini 2021]. Roughly speaking, the value of the truncations functions $\tilde u_k^M$ are close to c_k/M in a small neighborhood of 0, and coincide with $\tilde u_k$ outside the same neighborhood.

Lemma 3.7 [DelaTorre and Mancini 2021, Lemma 4.20]. *For any $M > 1$ and $k\in\mathbb{N}$, there exists a radius $\tilde\rho_k^M > 0$ and a constant $c = c(M)$ such that*

(1) $\tilde u_k \ge \dfrac{c_k}{M}$ *in* $B_{\tilde\rho_k^M}(0)$;

(2) $\left|\tilde u_k - \dfrac{c_k}{M}\right| \le \dfrac{c}{c_k}$ *on* $\partial B_{\tilde\rho_k^M}(0)$;

(3) $|\nabla^l\tilde u_k| \le \dfrac{c}{c_k(\tilde\rho_k^M)^l}$ *on* $\partial B_{\tilde\rho_k^M}(0)$ *for any* $1\le l\le 3$;

(4) $\tilde\rho_k^M \to 0$, *and* $\dfrac{\tilde\rho_k^M}{r_k} \to +\infty$, *as* $k\to\infty$.

Let $\tilde{v}_k^M \in C^4(\overline{B_{\tilde{\rho}_k^M}(0)})$ be the unique solution of

$$\begin{cases} \Delta^2(\tilde{v}_k^M) = 0 & \text{in } B_{\tilde{\rho}_k^M}(0), \\ \partial_\nu^i(\tilde{v}_k^M) = \partial_\nu^i(\tilde{u}_k) & \text{on } \partial B_{\tilde{\rho}_k^M}(0),\ i = 0, 1. \end{cases} \tag{3-16}$$

We consider the function

$$u_k^M = \begin{cases} \tilde{v}_k^M \circ \theta_k & \text{in } \theta_k^{-1}(B_{\tilde{\rho}_k^M}(0)), \\ u_k & \text{in } \Omega \setminus \theta_k^{-1}(B_{\tilde{\rho}_k^M}(0)). \end{cases} \tag{3-17}$$

Lemma 3.8 [DelaTorre and Mancini 2021, Lemma 4.21]. *For any $M > 1$, we have*

$$u_k^M = \frac{c_k}{M} + O(c_k^{-1}),$$

uniformly on $\theta_k^{-1}(\overline{B_{\tilde{\rho}_k^M}(0)})$.

Remark 3.9. Using the explicit form of the Green function of Δ^2 on balls, namely Boggio's formula [1905], and the representation formula of solutions for (3-16), one can see that $\partial u_k^M / \partial \nu = 0$ for any $x \in \theta_k^{-1}(B_{\tilde{\rho}_k^M}(0)) \cap \partial\Omega$.

Lemma 3.10. *For any $M > 1$,*

$$\limsup_{k\to\infty} \int_\Omega |\Delta u_k^M|^2 \, dx \le \frac{1}{M}.$$

Proof. Testing (3-2) with $(u_k - u_k^M)$, by Lemmas 3.7, 2.6 and Remark 3.9, for any $R > 0$, we have

$$\begin{aligned}
&\int_{\theta_k^{-1}(B_{\tilde{\rho}_k^M}(0))\cap\Omega} \Delta u_k \Delta(u_k - u_k^M)\, dx \\
&= \int_{\theta_k^{-1}(B_{\tilde{\rho}_k^M}(0))\cap\Omega} \gamma_k (u_k - u_k^M)\, dx - \int_{\partial(\theta_k^{-1}(B_{\tilde{\rho}_k^M}(0))\cap\Omega)} (u_k - u_k^M) \frac{\partial}{\partial\nu} \Delta u_k \, d\sigma \\
&\qquad\qquad + \int_{\partial(\theta_k^{-1}(B_{\tilde{\rho}_k^M}(0))\cap\Omega)} \frac{\partial}{\partial\nu}(u_k - u_k^M) \Delta u_k \, d\sigma \\
&= \int_{\theta_k^{-1}(B_{\tilde{\rho}_k^M}(0))\cap\Omega} \gamma_k (u_k - u_k^M)\, dx - \int_{\theta_k^{-1}(B_{\tilde{\rho}_k^M}(0))\cap\partial\Omega} (u_k - u_k^M) \frac{u_k \exp(\alpha_k u_k^2)}{\lambda_k} \, d\sigma \\
&\ge - \int_{\theta_k^{-1}(B_{\tilde{\rho}_k^M}(0))\cap\partial\Omega} \lambda_k^{-1} u_k \exp\{\alpha_k u_k^2\} (u_k - u_k^M)\, d\sigma + o_k(1) \\
&\ge - \int_{B_{Rr_k(x_k)}\cap\partial\Omega} \lambda_k^{-1} c_k \exp\{\alpha_k u_k^2\} \Big(c_k - \frac{c_k}{M}\Big)\, d\sigma + o_k(1) \\
&= \int_{B_R^+(0)\cap\partial\mathbb{R}_+^4} \Big(1 - \frac{1}{M}\Big) \exp\Big\{ \frac{u_k(x_k + r_k x) + c_k}{c_k} \alpha_k \psi_k(x) \Big\}\, d\sigma + o_k(1) \\
&\ge \Big(1 - \frac{1}{M}\Big) \int_{B_R^+(0)\cap\partial\mathbb{R}_+^4} \exp\{24\pi^2 \psi(x)\}\, d\sigma + o_k(1).
\end{aligned}$$

Letting $R \to \infty$, we get

$$\int_{\theta_k^{-1}(B_{\tilde{\rho}_k^M}(0))\cap\Omega} \Delta u_k \Delta(u_k - u_k^M)\,dx \geq 1 - \frac{1}{M} + o_k(1). \tag{3-18}$$

Observing that

$$\begin{aligned}\int_\Omega |\Delta u_k^M|^2\,dx &= \int_{\theta_k^{-1}(B_{\tilde{\rho}_k^M}(0))\cap\Omega} |\Delta v_k^M|^2\,dx + \int_{\Omega\setminus\theta_k^{-1}(B_{\tilde{\rho}_k^M}(0))} |\Delta u_k|^2\,dx \\ &= \int_{\theta_k^{-1}(B_{\tilde{\rho}_k^M}(0))\cap\Omega} |\Delta v_k^M|^2\,dx + 1 - \int_{\theta_k^{-1}(B_{\tilde{\rho}_k^M}(0))\cap\Omega} |\Delta u_k|^2\,dx \\ &= \int_{\theta_k^{-1}(B_{\tilde{\rho}_k^M}(0))\cap\Omega} |\Delta v_k^M|^2\,dx + 1 - \int_{\theta_k^{-1}(B_{\tilde{\rho}_k^M}(0))\cap\Omega} \Delta u_k \Delta(u_k - u_k^M)\,dx \\ &\qquad - \int_{\theta_k^{-1}(B_{\tilde{\rho}_k^M}(0))\cap\Omega} \Delta u_k \Delta u_k^M\,dx,\end{aligned}$$

by (3-18) and (3-17), we have

$$\begin{aligned}\int_\Omega |\Delta u_k^M|^2\,dx &\leq \frac{1}{M} + \frac{1}{2}\int_{\theta_k^{-1}(B_{\tilde{\rho}_k^M}(0))\cap\Omega} |\Delta v_k^M|^2\,dx - \frac{1}{2}\int_{\theta_k^{-1}(B_{\tilde{\rho}_k^M}(0))\cap\Omega} \Delta u_k \Delta v_k^M\,dx + o_k(1) \\ &\leq \frac{1}{M} + \frac{1}{2}\int_{\theta_k^{-1}(B_{\tilde{\rho}_k^M}(0))\cap\Omega} \Delta v_k^M \Delta(v_k^M - u_k)\,dx + o_k(1) \\ &= \frac{1}{M} + o_k(1).\end{aligned}$$

□

Lemma 3.11. *We have*

$$\lim_{k\to\infty}\int_{\partial\Omega} \exp(\alpha_k u_k^2)\,d\sigma = \lim_{L\to\infty}\lim_{k\to\infty}\int_{B_{Lr_k(x_k)}\cap\partial\Omega} \exp(\alpha_k u_k^2)\,d\sigma = \lim_{k\to\infty}\frac{-\lambda_k}{c_k^2} + |\partial\Omega|,$$

and consequently,

$$\frac{-\lambda_k}{c_k} \to \infty \quad \textit{and} \quad \sup_k \frac{-c_k^2}{\lambda_k} < \infty.$$

Proof. By Lemmas 3.7 and 3.1, we have

$$\begin{aligned}\int_{\partial\Omega} \exp(\alpha_k u_k^2)\,d\sigma &= \int_{\partial\Omega\cap\theta_k^{-1}(B_{\tilde{\rho}_k^M}(0))} \exp(\alpha_k u_k^2)\,d\sigma + \int_{\partial\Omega\setminus\theta_k^{-1}(B_{\tilde{\rho}_k^M}(0))} \exp(\alpha_k (u_k^M)^2)\,d\sigma \\ &\leq \frac{-M^2\lambda_k(1+o_k(1))}{c_k^2}\int_{\theta_k^{-1}(B_{\tilde{\rho}_k^M}(0))\cap\partial\Omega} \frac{u_k^2}{-\lambda_k}\exp(\alpha_k u_k^2)\,d\sigma + |\partial\Omega| \\ &\leq -(1+o_k(1))M^2\frac{\lambda_k}{c_k^2} + |\partial\Omega|,\end{aligned}$$

Let $k \to +\infty$ and $M \to 1$; we find that

$$\lim_{k\to+\infty}\int_{\partial\Omega} \exp(\alpha_k u_k^2)\,d\sigma \leq -\lim_{k\to\infty}\frac{\lambda_k}{c_k^2} + |\partial\Omega|.$$

On the other hand, we also have

$$\int_{\partial\Omega}\exp(\alpha_k u_k^2)\,d\sigma=\left(\int_{\partial\Omega\setminus B_{Rr_k}(x_k)}+\int_{B_{Rr_k}(x_k)\cap\partial\Omega}\right)\exp(\alpha_k u_k^2)\,d\sigma$$
$$\geq|\partial\Omega|-|B_{Rr_k}\cap\partial\Omega|-\frac{\lambda_k}{c_k^2}\int_{B_R(0)\cap\partial\mathbb{R}^4_+}\exp(\psi_k+o_k(1))\,d\sigma.$$

Letting $k\to+\infty$ and $R\to+\infty$, we get that

$$\lim_{k\to+\infty}\int_{\partial\Omega}\exp(\alpha_k u_k^2)\,d\sigma\geq-\lim_{k\to\infty}\frac{\lambda_k}{c_k^2}+|\partial\Omega|.$$

Combining the above estimates, we accomplish the proof of Lemma 3.11. □

Lemma 3.12. *For any $\varphi\in C^\infty(\partial\Omega)$, one has*

$$-\lim_{k\to\infty}\int_{\partial\Omega}\varphi(x)\frac{c_k u_k}{\lambda_k}\exp(\alpha_k u_k^2)\,d\sigma=\varphi(p).\tag{3-19}$$

Proof. For any fixed $M>1$, and k large enough, we divide $\partial\Omega$ into three parts,

$$\Omega_1=\big(\theta_k^{-1}(B_{\tilde\rho_k^M}(0))\setminus B_{Rr_k}(x_k)\big)\cap\partial\Omega,\quad \Omega_2=\partial\Omega\setminus\theta_k^{-1}(B_{\tilde\rho_k^M}(0)),\quad \Omega_3=B_{Rr_k}(x_k)\cap\partial\Omega,$$

and split the integral as

$$\int_{\partial\Omega}\varphi(x)\frac{c_k u_k}{-\lambda_k}\exp(\alpha_k u_k^2)\,d\sigma=\left(\int_{\Omega_1}+\int_{\Omega_2}+\int_{\Omega_2}\right)\varphi(x)\frac{c_k u_k}{-\lambda_k}\exp(\alpha_k u_k^2)\,d\sigma$$
$$=\mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3.\tag{3-20}$$

For I_1, we have

$$\begin{aligned}|\mathrm{I}_1|&\leq M\sup_{\partial\Omega}|\varphi|\int_{\Omega_1}\frac{u_k^2}{-\lambda_k}\exp(\alpha_k u_k^2)\,d\sigma\\&\leq M\sup_{\partial\Omega}|\varphi|(1+o_k(1))\left(1-\int_{B_{Rr_k}\cap\partial\Omega}\frac{u_k^2}{-\lambda_k}\exp(\alpha_k u_k^2)\,d\sigma\right)\\&\leq M\sup_{\partial\Omega}|\varphi|\left(1-\int_{B_R^+\cap\partial\mathbb{R}^4_+}\exp(24\pi^2\psi)\,d\sigma+o_k(1)\right)\\&\to 0\quad\text{as } k,\,R\to\infty.\end{aligned}\tag{3-21}$$

Next, by Lemma 3.10, Hölder's inequality, Sobolev embedding theorem and Lemma 3.11, we have

$$|\mathrm{I}_2|\leq\sup_{\partial\Omega}|\varphi|\frac{c_k}{-\lambda_k}\int_{\partial\Omega}|u_k|e^{\alpha_k(u_k^M)^2}\,d\sigma\leq\sup_{\partial\Omega}|\varphi|\frac{c_k}{-\lambda_k}\|u_k\|_{L^{p'}(\partial\Omega)}\|e^{\alpha_k(u_k^M)^2}\|_{L^p(\partial\Omega)}$$
$$\leq c\sup_{\partial\Omega}|\varphi|\left|\frac{c_k}{\lambda_k}\right|\to 0\quad\text{as } k\to\infty,\tag{3-22}$$

for some $p>1$ and p' with $\frac{1}{p}+\frac{1}{p'}=1$.

Finally, we have

$$\begin{aligned}
\mathrm{I}_3 &= \int_{B_{Rr_k}\cap\partial\Omega} \varphi(x)\frac{c_k u_k}{-\lambda_k}\exp(\alpha_k u_k^2)\,d\sigma \\
&= \int_{B_R^+\cap\partial\mathbb{R}_+^4} \varphi(r_k x + x_k)\exp\{(\phi_k+1)\alpha_k\psi_k(x)\}\,dx + o_k(1) \\
&= \varphi(p)\int_{B_R^+\cap\partial\mathbb{R}_+^4} \exp\{24\pi^2\psi(x)\}\,d\sigma + o_k(1) \\
&= \varphi(p) + o_{k,R}(1). \qquad (3\text{-}23)
\end{aligned}$$

Combining (3-21), (3-22) and (3-23), we obtain (3-19) and the proof is finished. □

Lemma 3.13. *For any $1 < q < 2$, $c_k u_k \rightharpoonup G$ weakly in $W^{2,q}(\Omega)$. Furthermore, for any $\Omega' \Subset \bar{\Omega}\setminus p$, we have $c_k u_k \to G$ in $C^\infty(\bar{\Omega}')$, where G satisfies*

$$\begin{cases} \Delta^2 G = \delta_p - \frac{1}{|\Omega|} & \text{in } \Omega, \\ \int_\Omega G = 0,\ \dfrac{\partial G}{\partial \nu} = 0,\ \dfrac{\partial \Delta G}{\partial \nu}\Big|_{\partial\Omega\setminus\{p\}} = 0. \end{cases} \qquad (3\text{-}24)$$

Moreover, we have

$$G = -\frac{1}{4\pi^2}\ln|x-p| + A_p + \varphi(x), \qquad (3\text{-}25)$$

where A_p is some constant depending on p, $\varphi(x) \in C^3(\Omega)\cap C^1(\bar{\Omega})$ and $\varphi(p) = 0$.

Proof. From (2-3), we have

$$\begin{cases} \Delta^2(c_k u_k) = c_k\gamma_k & \text{for all } x \in \Omega, \\ \dfrac{\partial}{\partial\nu}\Delta(c_k u_k) = c_k u_k \exp(\alpha_k u_k^2)/\lambda_k & \text{for all } x \in \partial\Omega. \end{cases} \qquad (3\text{-}26)$$

Integrating both sides on Ω, one has

$$\int_\Omega c_k\gamma_k\,dx = \int_\Omega \Delta^2(c_k u_k)\,dx = \int_{\partial\Omega}\frac{\partial\Delta(c_k u_k)}{\partial\nu}\,d\sigma = \int_{\partial\Omega}\frac{c_k u_k\exp(\alpha_k u_k^2)}{\lambda_k}\,d\sigma, \qquad (3\text{-}27)$$

which together with Lemma 3.12 gives $c_k\gamma_k \to -\dfrac{1}{|\Omega|}$ as $k\to\infty$. For any $q \in (1,2)$, we have

$$\int_\Omega |\Delta c_k u_k|^q\,dx = \sup\left\{\int_\Omega \Delta(c_k u_k)\Delta\varphi\,dx : \|\varphi\|_{W^{2,q'}} = 1\right\},$$

where $\frac{1}{q}+\frac{1}{q'} = 1$. By the Sobolev embedding theorem, we have $\sup_{x\in\Omega}|\varphi(x)| < \infty$. Using Lemma 3.12, we have

$$\begin{aligned}
\int_\Omega \Delta(c_k u_k)\Delta\varphi\,dx &= \int_\Omega \Delta^2(c_k u_k)\varphi\,dx - \int_{\partial\Omega}\frac{\partial\Delta(c_k u_k)}{\partial\nu}\varphi\,d\sigma \\
&= \int_\Omega c_k\gamma_k\varphi(x)\,dx - \int_{\partial\Omega}\frac{c_k u_k\varphi\exp(\alpha_k u_k^2)}{\lambda_k}\,d\sigma \\
&= -\frac{1}{|\Omega|}\int_\Omega \varphi(x)\,dx + \varphi(p) + o_k(1) \\
&\le c\sup_{x\in\Omega}|\varphi(x)| < c, \qquad (3\text{-}28)
\end{aligned}$$

which implies that

$$\int_\Omega |\Delta c_k u_k|^q \, dx < c.$$

Combining this and the condition $\int_\Omega c_k u_k \, dx = 0$, $\int_\Omega u_k \, dx = 0$, we derive that $c_k u_k$ is bounded in $W^{2,q}(\Omega)$ for any $1 \le q < 2$. Thus, there exists some $G \in W^{2,q}(\Omega)$ such that $c_k u_k \rightharpoonup G$ in $W^{2,q}(\Omega)$ as $k \to \infty$. Now, letting $k \to \infty$ in (3-28), we have

$$\int_\Omega \Delta G \Delta \varphi \, dx = -\frac{1}{|\Omega|} \int_\Omega \varphi(x) \, dx + \varphi(p).$$

Combining the assumptions on u_k, (3-24) is proved.

For any $\Omega' \Subset \bar{\Omega} \setminus p$, we can choose some function $\phi \in C^\infty(\mathbb{R}^4)$ such that $\phi(x) = 1$ for $x \in \Omega'$ and $\phi(x) = 0$ for x belonging to a small neighborhood of p. By Lemma 3.1, we know that $\phi u_k \to 0$ in $L^2(\Omega')$ as $k \to +\infty$. This together with the convergence $\Delta u_k \to 0$ in $L^2(\Omega')$ as $k \to \infty$ implies that $e^{\alpha_k u_k^2}$ is uniformly bounded in $L^s(\bar{\Omega}')$ for any $s > 1$. Standard elliptic regularity gives that $c_k u_k \to G$ in $C^k(\bar{\Omega}')$ for any positive integer k.

Next, we prove (3-25). Fix $r > 0$, without loss of generality, we assume $p = 0$, and choose some cutoff function $\phi \in C_0^\infty(B_{2r}(0))$ such that $\phi = 1$ in $B_r(0)$. Let

$$g(x) = G(x) + \frac{1}{4\pi^2} \phi(x) \ln|x|.$$

Then we have

$$\Delta^2 g(x) = f \quad \text{in } \Omega,$$

where

$$f(x) = \frac{1}{4\pi^2} \big(\Delta^2 \phi \cdot \ln|x| + 2\nabla\Delta\phi \cdot \nabla \ln|x| + 2\Delta(\nabla\phi \cdot \nabla \ln|x|) + 2\nabla\phi \cdot \nabla\Delta \ln|x| + \phi \cdot \Delta^2 \ln|x| \big) + \delta(x) - \frac{1}{|\Omega|}.$$

Since $1/(4\pi^2)\phi \cdot \Delta^2 \ln|x| = \delta(x)$ in $\mathbb{R}^4_+$, a careful computation yields

$$f(x) = \frac{1}{4\pi^2} \big(\Delta^2 \phi \cdot \ln|x| + 2\nabla\Delta\phi \cdot \nabla \ln|x| + 2\Delta(\nabla\phi \cdot \nabla \ln|x|) + 2\nabla\phi \cdot \nabla\Delta \ln|x| \big) - \frac{1}{|\Omega|}.$$

Observing $G \in W^{2,s}(\Omega)$ for any $1 < s < 2$, we obtain $f(x) \in L^p(\Omega)$ for any $p > 2$. By the standard regularity theory, we get $g(x) \in C^3_{\mathrm{loc}}(\Omega) \cap C^1(\bar{\Omega})$. Let $A_p = g(0)$ and

$$\varphi(x) = g(x) - g(0) + \frac{1}{4\pi^2}(1 - \phi) \ln|x|.$$

Then we have

$$G = -\frac{1}{4\pi^2} \ln|x| + A_p + \varphi(x), \tag{3-29}$$

where A_p is some constant depending on p, $\varphi(x) \in C^3(\Omega) \cap C^1(\bar{\Omega})$ and $\varphi(0) = 0$, and the proof is finished. □

3.2. ***Neck analysis.*** In this subsection, we will use the capacity technique to derive the upper bound of $I_{12\pi^2}(u_k)$ when $c_k \to \infty$. The capacity technique applied to the existence of extremals for Adams inequalities was first used by Lu and Yang in [2009a], and was improved by DelaTorre and Mancini [2021] by comparing the Dirichlet energy of maximizing sequence with the energy of a suitable polyharmonic function.

Based on Lemma 3.11, we only need to give the sharp upper bound of $\lim_{k\to\infty} -\lambda_k/c_k^2$. Let us fix a large $R > 0$ and a small $\delta > 0$ and consider the annular region

$$A_k(R,\delta) := \{x \in \Omega : r_k R \le |x - x_k| \le \delta\}.$$

Our strategy is to compare the Dirichlet energy of u_k on $A_k(R,\delta)$ with the energy of the function

$$\mathcal{W}_k(x) := -\frac{1}{4\pi^2 c_k}(\log|x - x_k| + \rho_k(x)),$$

where $\rho_k(x) \in C^\infty(\overline{\Omega})$ is chosen such that

$$\frac{\partial \mathcal{W}_k(x)}{\partial \nu} = \frac{\partial \Delta \mathcal{W}_k(x)}{\partial \nu} = 0 \quad \text{for } x \in \partial\Omega$$

and $\|\rho_k(x)\|_{C^3} = O(\delta)$.

As a consequence of Lemma 3.6, on $\partial B_{Rr_k}(x_k) \cap \Omega$, we have that

$$u_k(x) = c_k + \frac{\psi((x - x_k)/r_k)}{c_k} + o(c_k^{-1}) = c_k - \frac{1}{4\pi^2 c_k}\log R - \frac{1}{6\pi^2 c_k}\log\frac{\pi}{2} + \frac{O(R^{-1})}{c_k} + o(c_k^{-1}),$$

provided k is large enough. Similarly, a direct computation also gives

$$\Delta^{\frac{j}{2}} u_k = \frac{(\Delta^{j/2}\psi)((x - x_k)/r_k)}{r_k^j c_k} + o(r_k^{-j} c_k^{-1}) = -\frac{K_{2,j/2}}{4\pi^2 r_k^j c_k R^j} e_j(x - x_k) + \frac{O(R^{-j-1})}{r_k^j c_k} + o(r_k^{-j} c_k^{-1})$$

for any $1 \le j \le 3$, where

$$K_{2,j/2} = \begin{cases} 1 & \text{if } j = 1, \\ 2 & \text{if } j = 2, \\ -4 & \text{if } j = 3, \end{cases} \quad \text{and} \quad e_j(x) := \begin{cases} 1 & \text{if } j \text{ is even}, \\ \dfrac{x}{|x|} & \text{if } j \text{ is odd}. \end{cases}$$

Recalling the definition of $\mathcal{W}_k$, we have on $\partial B_{Rr_k}(x_k) \cap \Omega$ that

$$\mathcal{W}_k = \frac{\alpha_k}{12\pi^2} c_k - \frac{1}{12\pi^2 c_k}\log\frac{-\lambda_k}{c_k^2} - \frac{1}{4\pi^2 c_k}\log R + \frac{O(\delta)}{c_k} \tag{3-30}$$

and

$$\Delta^{j/2}\mathcal{W}_k = -\frac{K_{2,j/2}}{4\pi^2 c_k r_k^j R^j} e_j(x - x_k) + \frac{O(\delta)}{c_k} \quad \text{for any } 1 \le j \le 3. \tag{3-31}$$

Hence, we conclude that on $\partial B_{Rr_k}(x_k) \cap \Omega$,

$$u_k(x) - \mathcal{W}_k = \frac{1}{12\pi^2 c_k}\log\frac{-\lambda_k}{c_k^2} - \frac{1}{6\pi^2 c_k}\log\frac{\pi}{2} + \frac{O(R^{-1})}{c_k} + \frac{O(\delta)}{c_k} + o(c_k^{-1}) + \left(1 - \frac{\alpha_k}{12\pi^2}\right)c_k$$

and

$$\Delta^{j/2}(u_k - \mathcal{W}_k) = \frac{O(R^{-j-1})}{r_k^j c_k} + o(r_k^{-j} c_k^{-1}) \quad \text{for any } 1 \le j \le 3.$$

Similarly, in view of Lemma 3.13, we also derive that on $\partial B_\delta(x_k) \cap \Omega$,

$$u_k(x) - \mathcal{W}_k = \frac{A_p}{c_k} + \frac{O(\delta)}{c_k} + o(c_k^{-1})$$

and

$$\Delta^{j/2}(u_k(x) - \mathcal{W}_k) = \frac{O(1)}{c_k} + o(c_k^{-1}) \quad \text{for any } 1 \le j \le 3,$$

where we have also used that $|x - x_k|/|x - p| \to 1$ uniformly on $\partial B_\delta(x_k)$.

Now, we compare $\|\Delta u_k\|_{L^2(A_k(R,\delta))}$ and $\|\Delta \mathcal{W}_k\|_{L^2(A_k(R,\delta))}$. Obviously,

$$\|\Delta u_k\|^2_{L^2(A_k(R,\delta))} - \|\Delta \mathcal{W}_k\|^2_{L^2(A_k(R,\delta))} \ge 2\int_{A_k(R,\delta)} \Delta(u_k - \mathcal{W}_k) \cdot \Delta \mathcal{W}_k \, dx. \tag{3-32}$$

Step 1. Estimates for the right-hand side of (3-32).

Integrating by parts, the integral in the right-hand side equals to

$$\begin{aligned} 2\int_{A_k(R,\delta)} &\Delta(u_k - \mathcal{W}_k) \cdot \Delta \mathcal{W}_k \, dx \\ &= -2\int_{\partial A_k(R,\delta)\setminus\partial\Omega} \nu \cdot ((u_k - \mathcal{W}_k)\Delta^{\frac{3}{2}} \mathcal{W}_k) \, d\sigma + 2\int_{\partial A_k(R,\delta)\setminus\partial\Omega} \nu \cdot (\Delta^{\frac{1}{2}}(u_k - \mathcal{W}_k)\Delta \mathcal{W}_k) \, d\sigma, \end{aligned}$$

where we have used the fact that $\dfrac{\partial \mathcal{W}_k}{\partial \nu} = \dfrac{\partial \Delta \mathcal{W}_k}{\partial \nu} = 0$ on $\partial A_k(R,\delta) \cap \partial\Omega$.

On $\partial B_{Rr_k}(x_k) \cap \Omega$, we have

$$\begin{aligned} (u_k - \mathcal{W}_k)&\Delta^{\frac{3}{2}} \mathcal{W}_k \cdot \nu \\ &= \frac{1}{4\pi^2}\left(\frac{1}{12\pi^2 c_k^2}\log\frac{-\lambda_k}{c_k^2} - \frac{1}{6\pi^2 c_k^2}\log\frac{\pi}{2} + \frac{O(R^{-1})}{c_k^2} + \left(1 - \frac{\alpha_k}{12\pi^2}\right) + \frac{O(\delta)}{c_k^2} + o(c_k^{-2})\right)\frac{K_{2,\frac{3}{2}}}{(r_k R)^3} \\ &= -\frac{1}{\pi^2 (r_k R)^3}\left(\frac{1}{12\pi^2 c_k^2}\log\frac{-\lambda_k}{c_k^2} - \frac{1}{6\pi^2 c_k^2}\log\frac{\pi}{2} + \frac{O(R^{-1})}{c_k^2} + \left(1 - \frac{\alpha_k}{12\pi^2}\right) + \frac{O(\delta)}{c_k^2} + o(c_k^{-2})\right) \end{aligned}$$

and

$$\Delta^{\frac{1}{2}}(u_k - \mathcal{W}_k)\Delta \mathcal{W}_k \cdot \nu = \left(\frac{O(R^{-1})}{c_k^2} + o(c_k^{-2})\right) O(r_k R)^{-3}.$$

Similarly, on $\partial B_\delta(x_k) \cap \Omega$, we have

$$(u_k - \mathcal{W}_k)\Delta^{\frac{3}{2}} \mathcal{W}_k \cdot \nu = \frac{1}{\pi^2 \delta^3}\left(\frac{A_p}{c_k^2} + \frac{O(\delta)}{c_k^2} + o(c_k^{-2})\right)$$

and

$$\Delta^{\frac{1}{2}}(u_k - \mathcal{W}_k)\Delta \mathcal{W}_k \cdot \nu = \left(\frac{O(1)}{c_k^2} + o(c_k^{-2})\right) O(\delta^{-2}).$$

Then we can obtain

$$\int_{A_k(R,\delta)} \Delta(u_k-\mathcal{W}_k)\cdot\Delta\mathcal{W}_k\,dx = \frac{1}{12\pi^2c_k^2}\log\frac{-\lambda_k}{c_k^2}-\frac{1}{6\pi^2c_k^2}\log\frac{\pi}{2}-\frac{A_p}{c_k^2}+\frac{O(R^{-1})}{c_k^2}+\frac{O(\delta)}{c_k^2}+\Big(1-\frac{\alpha_k}{12\pi^2}\Big)+o(c_k^{-2}).$$

Combining the above estimates, we derive that

$$\|\Delta u_k\|^2_{L^2(A_k(R,\delta))}-\|\Delta\mathcal{W}_k\|^2_{L^2(A_k(R,\delta))} \geq \frac{1}{6\pi^2c_k^2}\log\frac{-\lambda_k}{c_k^2}-\frac{1}{3\pi^2c_k^2}\log\frac{\pi}{2}-\frac{2A_p}{c_k^2}+\frac{O(R^{-1})}{c_k^2}+\frac{O(\delta)}{c_k^2}+\Big(2-\frac{\alpha_k}{6\pi^2}\Big)+o(c_k^{-2}). \quad (3\text{-}33)$$

Step 2. Estimates for $\|\Delta u_k\|^2_{L^2(A_k(R,\delta))}$.

We rewrite $\|\Delta u_k\|^2_{L^2(A_k(R,\delta))}$ as

$$\|\Delta u_k\|^2_{L^2(A_k(R,\delta))} = 1-\int_{\Omega\setminus B_\delta(x_k)}|\Delta u_k|^2\,dx-\int_{\Omega\cap B_{Rr_k}(x_k)}|\Delta u_k|^2\,dx. \quad (3\text{-}34)$$

Since

$$\Delta^{\frac12}(\log|x|)=\frac{x}{|x|^2},\quad \Delta(\log|x|)=\frac{2}{|x|^2},\quad \Delta^{1+\frac12}(\log|x|)=-4\frac{x}{|x|^4},$$

we have

$$\begin{aligned}\nu\cdot G(\delta)\Delta^{\frac32}G(\delta) &= -\Big(-\frac{1}{4\pi^2}\ln|\delta|+A_p+o_\delta(1)\Big)\Big(\frac{1}{\pi^2}\cdot\frac{1}{\delta^3}+O(1)\Big)\\ &= -\frac{1}{\pi^2}\frac{1}{\delta^3}\Big(-\frac{1}{4\pi^2}\ln\delta+A_p+o_\delta(1)\Big)\end{aligned} \quad (3\text{-}35)$$

and

$$\nu\cdot\Delta^{\frac12}G(\delta)\Delta G(\delta) = -\Big(-\frac{1}{4\pi^2}\frac{1}{\delta}+O(1)\Big)\Big(-\frac{1}{4\pi^2}\frac{2}{\delta^2}+O(1)\Big) = -\frac{1}{8\pi^4}\frac{1}{\delta^3}(1+o_\delta(1)). \quad (3\text{-}36)$$

Since

$$\int_{\Omega\setminus B_\delta(x_k)}|\Delta G|^2\,dx=\int_{\Omega\cap\partial B_\delta(x_k)}\nu(-G\Delta^{\frac32}G+\Delta^{\frac12}G\Delta G)\,d\sigma,$$

we have by Lemma 3.13,

$$\int_{\Omega\setminus B_\delta(x_k)}|\Delta u_k|^2\,dx=\frac{1}{c_k^2}\Big(-\frac{1}{4\pi^2}\log\delta-\frac{1}{8\pi^2}+A_p+o_\delta(1)+o_k(1)\Big). \quad (3\text{-}37)$$

By Lemma 3.6, we derive that

$$\begin{aligned}\int_{\Omega\cap B_{Rr_k}(x_k)}|\Delta u_k|^2\,dx &= \frac{1}{c_k^2}\int_{B_R^+}|\Delta\psi|^2\,dx+o\Big(\frac{1}{c_k^2}\Big)=\frac{1}{c_k^2}\Big(\int_{\partial B_R^+}\nu(\Delta^{\frac12}\psi\Delta\psi-\psi\Delta^{\frac32}\psi)\,d\sigma\Big)+o\Big(\frac{1}{c_k^2}\Big)\\ &:= \frac{1}{c_k^2}(\mathrm{I}-\mathrm{II})+o\Big(\frac{1}{c_k^2}\Big).\end{aligned} \quad (3\text{-}38)$$

Observe that on $\partial B_R^+\cap\mathbb{R}^4_+$, we also have

$$\psi(x)=-\frac{1}{6\pi^2}\log\frac{\pi}{2}-\frac{1}{4\pi^2}\log R+O\Big(\frac{1}{R}\Big),\quad \nu\Delta^{\frac12}\psi(x)=-\frac{1}{4\pi^2}\frac{1}{R}+O\Big(\frac{1}{R^2}\Big)$$

and

$$
\begin{aligned}
\nu\Delta^{\frac{3}{2}}\psi &= -\frac{1}{4\pi^2}\left(\frac{-4}{\left(\left(t+\left(\frac{2}{\pi}\right)^{\frac{2}{3}}\right)^2+|x'|^2\right)^2}\right)\frac{\left(x',t+\left(\frac{2}{\pi}\right)^{\frac{2}{3}}\right)\cdot(x',t)}{R}+O\left(\frac{1}{R^4}\right)\\
&= \frac{1}{\pi^2}\left(\frac{1}{\left(\left(t+\left(\frac{2}{\pi}\right)^{\frac{2}{3}}\right)^2+|x'|^2\right)^2}\right)\frac{\left(x',t+\left(\frac{2}{\pi}\right)^{\frac{2}{3}}\right)\cdot(x',t)}{R}+O\left(\frac{1}{R^4}\right)\\
&= \frac{1}{\pi^2}\left(\frac{1}{R^4}+O\left(\frac{1}{R^5}\right)\right)(R+O(1))+O\left(\frac{1}{R^4}\right)\\
&= \frac{1}{\pi^2}\frac{1}{R^3}+O\left(\frac{1}{R^4}\right).
\end{aligned}
$$

Hence we can write

$$
\mathrm{II}=\int_{\partial B_R^+\cap\mathbb{R}^4_+}\nu\psi\,\Delta^{\frac{3}{2}}\psi\,d\sigma+\int_{\partial B_R^+\cap\partial\mathbb{R}^4_+}\nu\psi\,\Delta^{\frac{3}{2}}\psi\,d\sigma:=\mathrm{II}_1+\mathrm{II}_2,
$$

where

$$
\begin{aligned}
\mathrm{II}_1 &= \int_{\partial B_R^+\cap\mathbb{R}^4_+}\nu\psi\,\Delta^{\frac{3}{2}}\psi\,d\sigma\\
&= \pi^2R^3\left(-\frac{1}{6\pi^2}\log\frac{\pi}{2}-\frac{1}{4\pi^2}\log R+O\left(\frac{1}{R}\right)\right)\cdot\left(\frac{1}{\pi^2}\frac{1}{R^3}+O\left(\frac{1}{R^4}\right)\right)\\
&= -\frac{1}{4\pi^2}\log R-\frac{1}{6\pi^2}\log\frac{\pi}{2}+O\left(\frac{\log R}{R}\right).
\end{aligned}
$$

Since $\frac{\partial}{\partial t}\Delta\psi=\exp(24\pi^2\psi)$ for $x=(x',0)\in\partial\mathbb{R}^4_+$, set $\psi_0(x')=\psi(x',0)$. We have

$$
\begin{aligned}
\mathrm{II}_2 &= \int_{\partial B_R^+\cap\partial\mathbb{R}^4_+}\nu\psi\,\Delta^{\frac{3}{2}}\psi\,d\sigma\\
&= \int_{B_R^3}-\exp(24\pi^2\psi_0(x'))\psi_0(x')\,dx'\\
&= -\int_{\mathbb{R}^3}\frac{\left(\left(\frac{2}{\pi}\right)^{\frac{2}{3}}\right)^3}{\left(\frac{\pi}{2}\right)^2\left(|x'|^2+\left(\left(\frac{2}{\pi}\right)^{\frac{2}{3}}\right)^2\right)^3}\psi_0(x')\,dx'+O\left(\frac{1}{R}\right)\\
&= -\psi\left(0,\left(\frac{2}{\pi}\right)^{\frac{2}{3}}\right)+O\left(\frac{1}{R}\right)\\
&= \frac{1}{4\pi^2}\log 2-\frac{1}{16\pi^2}+O\left(\frac{1}{R}\right),
\end{aligned}
$$

where B_R^3 denotes the three-dimensional balls with radius R.

So, we have

$$
\begin{aligned}
\mathrm{II} &= -\frac{1}{4\pi^2}\log R-\frac{1}{6\pi^2}\log\frac{\pi}{2}+\frac{1}{4\pi^2}\log 2-\frac{1}{16\pi^2}+O\left(\frac{\log R}{R}\right)\\
&= -\frac{1}{4\pi^2}\log\frac{R}{2}-\frac{1}{6\pi^2}\log\frac{\pi}{2}-m\frac{1}{16\pi^2}+O\left(\frac{\log R}{R}\right).
\end{aligned}
\tag{3-39}
$$

Now, we estimate I, and rewrite it as

$$\mathrm{I}=\int_{\partial B_R^+}\nu\Delta^{\frac12}\psi\,\Delta\psi\,d\sigma=\int_{\partial B_R^+\cap\mathbb{R}^4_+}\nu\Delta^{\frac12}\psi\,\Delta\psi\,d\sigma+\int_{\partial B_R^+\cap\partial\mathbb{R}^4_+}\nu\Delta^{\frac12}\psi\,\Delta\psi\,d\sigma:=\mathrm{I}_1+\mathrm{I}_2.$$

Since on $\partial B_R^+\cap\mathbb{R}^4_+$, we have

$$\nu\Delta^{\frac12}\psi(x)=-\frac{1}{4\pi^2}\frac{1}{R}+O\Big(\frac{1}{R^2}\Big)$$

and $\Delta\psi=-\frac{1}{2\pi^2}\frac{1}{R^2}$, we therefore get

$$\mathrm{I}_1=\int_{\partial B_R^+\cap\mathbb{R}^4_+}\nu\Delta^{\frac12}\psi\,\Delta\psi\,d\sigma=\pi^2R^3\Big(-\frac{1}{4\pi^2}\frac{1}{R}+O\Big(\frac{1}{R^2}\Big)\Big)\Big(-\frac{1}{4\pi^2}\Big(\frac{2}{R^2}\Big)\Big)=\frac{1}{8\pi^2}+O\Big(\frac{1}{R}\Big).$$

Using the fact that $\frac{\partial\psi}{\partial t}=0$ on $\partial\mathbb{R}^4_+$, we obtain

$$\mathrm{I}_2=\int_{\partial B_R^+\cap\partial\mathbb{R}^4_+}\nu\Delta^{\frac12}\psi\,\Delta\psi\,d\sigma=0.$$

Hence we have

$$\mathrm{I}=\frac{1}{8\pi^2}+O\Big(\frac{1}{R}\Big). \tag{3-40}$$

By (3-38), (3-39) and (3-40), we get

$$\begin{aligned}\int_{\Omega\cap B_{Rr_k(x_k)}}|\Delta u_k|^2\,dx&=\frac{1}{c_k^2}\Big(\frac{1}{8\pi^2}+O\Big(\frac{1}{R}\Big)-\Big(-\frac{1}{4\pi^2}\log\frac{R}{2}-\frac{1}{6\pi^2}\log\frac{\pi}{2}-\frac{1}{16\pi^2}+O\Big(\frac{\log R}{R}\Big)\Big)\Big)+o\Big(\frac{1}{c_k^2}\Big)\\&=\frac{1}{c_k^2}\Big(\frac{3}{16\pi^2}+\frac{1}{4\pi^2}\log\frac{R}{2}+\frac{1}{6\pi^2}\log\frac{\pi}{2}\Big)+\frac{1}{c_k^2}O\Big(\frac{\log R}{R}\Big).\end{aligned} \tag{3-41}$$

Combining (3-37) and (3-34), we derive that

$$\begin{aligned}\|\Delta u_k\|^2_{A_k(R,\delta)}&=1-\frac{1}{c_k^2}\Big(-\frac{1}{4\pi^2}\log\delta-\frac{1}{8\pi^2}+A_p+o_{\delta,k}(1)\Big)\\&\qquad-\frac{1}{c_k^2}\Big(\frac{3}{16\pi^2}+\frac{1}{4\pi^2}\log\frac{R}{2}+\frac{1}{6\pi^2}\log\frac{\pi}{2}\Big)+\frac{1}{c_k^2}O\Big(\frac{\log R}{R}\Big)\\&=1-\frac{1}{c_k^2}\Big(\frac{1}{16\pi^2}+\frac{1}{4\pi^2}\log\frac{R}{2\delta}+\frac{1}{6\pi^2}\log\frac{\pi}{2}+A_p+O\Big(\frac{\log R}{R}\Big)+o_{\delta,k}(1)\Big).\end{aligned} \tag{3-42}$$

Step 3. Estimates for $\|\Delta\mathcal{W}_k\|^2_{L^2(A_k(R,\delta))}$.

Since

$$\begin{aligned}\int_{A_k(R,\delta)}|\Delta\mathcal{W}_k|^2\,dx&=-\int_{\partial A_k(R,\delta)}\nu(\mathcal{W}_k\Delta^{\frac32}\mathcal{W}_k-\Delta^{\frac12}\mathcal{W}_k\Delta\mathcal{W}_k)\,d\sigma\\&=-\int_{\Omega\cap\partial B_\delta(x_k)}\nu(\mathcal{W}_k\Delta^{\frac32}\mathcal{W}_k-\Delta^{\frac12}\mathcal{W}_k\Delta\mathcal{W}_k)\,d\sigma\\&\qquad+\int_{\Omega\cap\partial B_{Rr_k(x_k))}}\nu(\mathcal{W}_k\Delta^{\frac32}\mathcal{W}_k-\Delta^{\frac12}\mathcal{W}_k\Delta\mathcal{W}_k)\,d\sigma\\&:=-\mathrm{III}_1+\mathrm{III}_2.\end{aligned} \tag{3-43}$$

From (3-30) and (3-31), we have

$$\begin{aligned}\mathrm{III}_2 &= \Bigg(\Bigg(\frac{\alpha_k}{12\pi^2}c_k - \frac{1}{12\pi^2 c_k}\log\frac{-\lambda_k}{c_k^2} - \frac{1}{4\pi^2 c_k}\log R\Bigg) \\ &\qquad \cdot\Bigg(-\frac{K_{2,\frac{3}{2}}}{4\pi^2 c_k R^3 r_k^3}\Bigg) - \frac{K_{2,\frac{1}{2}}}{4\pi^2 c_k R r_k}\frac{K_{2,1}}{4\pi^2 c_k R^2 r_k^2} + O(\delta)\Bigg)\pi^2 R^3 r_k^3 \\ &= 1 - \frac{1}{c_k^2}\Bigg(\frac{1}{12\pi^2}\log\frac{-\lambda_k}{c_k^2} + \frac{1}{4\pi^2}\log R + \frac{1}{8\pi^2} + O(\delta)\Bigg). \qquad (3\text{-}44)\end{aligned}$$

Similarly, we can also obtain

$$\begin{aligned}\mathrm{III}_1 &= \int_{\Omega\cap\partial B_\delta} \nu(\mathcal{W}_k\Delta^{\frac{3}{2}}\mathcal{W}_k - \Delta^{\frac{1}{2}}\mathcal{W}_k\Delta\mathcal{W}_k)\,d\sigma \\ &= \frac{\nu}{c_k^2}\Bigg(\Bigg(\frac{-1}{4\pi^2}\log\delta + O(\delta)\Bigg)\Bigg(\frac{-K_{2,\frac{3}{2}}}{4\pi^2\delta^3}e_3(x-x_k) + O(\delta)\Bigg) - \frac{-K_{2,\frac{1}{2}}e_1(x-x_k)}{4\pi^2\delta}\frac{-K_{2,\frac{2}{2}}}{4\pi^2\delta^2}\Bigg)\pi^2\delta^3 + \frac{O(\delta)}{c_k^2} \\ &= \frac{1}{c_k^2}\Bigg(\frac{-1}{4\pi^2}\log\delta - \frac{1}{8\pi^2} + O(\delta)\Bigg).\end{aligned}$$

Combining (3-43) and (3-44), we derive that

$$\begin{aligned}\int_{A_k(R,\delta)} |\Delta\mathcal{W}_k|^2\,dx &= 1 - \frac{1}{c_k^2}\Bigg(\frac{-1}{4\pi^2}\log\delta - \frac{1}{8\pi^2} + \frac{1}{12\pi^2}\log\frac{-\lambda_k}{c_k^2} + \frac{1}{4\pi^2}\log R + \frac{1}{8\pi^2} + O(\delta)\Bigg) \\ &= 1 - \frac{1}{c_k^2}\Bigg(\frac{-1}{4\pi^2}\log\frac{\delta}{R} + \frac{1}{12\pi^2}\log\frac{-\lambda_k}{c_k^2} + O(\delta)\Bigg). \qquad (3\text{-}45)\end{aligned}$$

Now, we are in position to give the sharp upper bound for $\lim_{k\to\infty}\frac{-\lambda_k}{c_k^2}$. Indeed, from (3-42), (3-45) and (3-33), we can get

$$\begin{aligned}&\|\Delta u_k\|^2_{A_k(R,\delta)} - \int_{A_k(R,\delta)} |\Delta\mathcal{W}_k|^2\,dx \\ &\quad = 1 - \frac{1}{c_k^2}\Bigg(\frac{1}{16\pi^2} + \frac{1}{4\pi^2}\log\frac{R}{2\delta} + \frac{1}{6\pi^2}\log\frac{\pi}{2} + A_p\Bigg) - 1 + \frac{1}{c_k^2}\Bigg(\frac{-1}{4\pi^2}\log\frac{\delta}{R} + \frac{1}{12\pi^2}\log\frac{-\lambda_k}{c_k^2} + O(\delta)\Bigg) \\ &\quad = \frac{1}{c_k^2}\Bigg(\frac{-1}{4\pi^2}\log\frac{\delta}{R} + \frac{1}{12\pi^2}\log\frac{-\lambda_k}{c_k^2} - \frac{1}{16\pi^2} - \frac{1}{4\pi^2}\log\frac{R}{2\delta} - \frac{1}{6\pi^2}\log\frac{\pi}{2} - A_p + O(\delta)\Bigg) \\ &\quad = \frac{1}{c_k^2}\Bigg(\frac{-1}{4\pi^2}\log\frac{\delta}{R} + \frac{1}{12\pi^2}\log\frac{-\lambda_k}{c_k^2} - \frac{1}{16\pi^2} - \frac{1}{4\pi^2}\log\frac{R}{2\delta} - \frac{1}{6\pi^2}\log\frac{\pi}{2} - A_p + O(\delta)\Bigg) \\ &\quad \ge \frac{1}{c_k^2}\Bigg(\frac{1}{6\pi^2}\log\frac{-\lambda_k}{c_k^2} - 2A_p - \frac{1}{3\pi^2}\log\frac{\pi}{2} + o_{\delta,k}(1) + o(R^{-1})\Bigg) + 2 - \frac{\alpha_k}{6\pi^2},\end{aligned}$$

which implies

$$\lim_{k\to\infty}\frac{-\lambda_k}{c_k^2} \le 2\pi^2\exp\big(-\tfrac{3}{4} + 12\pi^2 A_p\big).$$

Therefore, we can conclude with the following.

Proposition 3.14. *If $c_k \to \infty$, then*

$$\sup_{u\in W^{2,2}(\Omega),\ \|\Delta u\|_2\leqslant 1}\int_{\partial\Omega} e^{12\pi^2u^2}\,dx \leqslant |\partial\Omega| + 2\pi^2 e^{12\pi^2 A_p-\frac{3}{4}}.$$

4. A test functions argument and the proof of Theorem 1.1

In this section, we assume $A_p = \max_{p\in\partial\Omega} A_p$ for some $p\in\partial\Omega$. Now we construct a blowing up sequence ϕ_ε with $\int_\Omega |\Delta\phi_\varepsilon|^2 = 1$, and

$$\int_{\partial\Omega} e^{12\pi^2(\phi_\varepsilon-\bar{\phi}_\varepsilon)^2}\,d\sigma > |\partial\Omega| + 2\pi^2 e^{12\pi^2 A_p-\frac{3}{4}}, \quad \text{where } \bar{\phi}_\varepsilon = \frac{1}{|\Omega|}\int_\Omega \phi_\varepsilon\,dx. \tag{4-1}$$

Take a Fermi coordinate system (U,θ) around p such that $\theta(p)=(0,0)$, θ maps $\partial\Omega\cap U$ inside $\partial\mathbb{R}^4_+$, and for any $\varepsilon>0$ and $x\in\partial\Omega$, there exists $\delta>0$ such that

$$(1-\varepsilon)\theta \le g = g_{ij}\,dx_i\otimes dx_j + dt\otimes dt \le (1+\varepsilon)\theta \quad \text{in } M_\delta,$$

where $M_\delta = \{x\in\Omega_\delta : \operatorname{dist}(\pi(x),p)\le\delta\}$.

Set

$$\widetilde{\phi}_\varepsilon(x',t) = C + \frac{-1/(8\pi^2)\log\Bigl(\bigl(\frac{\pi}{2}\bigr)^{4/3}|x'|^2/\varepsilon^2 + \bigl(\bigl(\frac{\pi}{2}\bigr)^{2/3}t/\varepsilon+1\bigr)^2\Bigr) + B + g_\varepsilon(x',t)}{C}$$

for some constants B, C, where

$$g_\varepsilon(x',t) = \frac{1}{2^{\frac{8}{3}}\pi^{\frac{4}{3}}}\frac{t/\varepsilon}{(1+(\pi/2)^{\frac{2}{3}}t/\varepsilon)^2 + (\pi/2)^{\frac{4}{3}}|x'|^2/\varepsilon^2}.$$

Let $B_r^+ = B_r(p)\cap\Omega$ and R be a function of ε such that $R\to+\infty$ and $R\varepsilon\to 0$ as $\varepsilon\to 0$. Set

$$\phi_\varepsilon = \begin{cases} \widetilde{\phi}_\varepsilon\circ\theta(x) & \text{if } x\in B^+_{R\varepsilon},\\ (G-\eta\beta)/C & \text{if } x\in B^+_{2R\varepsilon}\setminus B^+_{R\varepsilon},\\ G/C & \text{if } x\in\Omega\setminus B^+_{2R\varepsilon},\end{cases}$$

where $\beta = G - C\widetilde{\phi}_\varepsilon\circ\theta(x)$, η is some radial function in $C_0^\infty(B_{2R\varepsilon}(p))$ with $\eta\equiv 1$ on $B_{R\varepsilon}(p)$, and $|\nabla\eta| = O(1/R\varepsilon)$, $|\Delta\eta| = O(1/(R\varepsilon)^2)$. One can easily verify that $\partial\phi_\varepsilon(x)/\partial\nu = 0$ for any $x\in\partial\Omega$.

Now, we estimate $\int_\Omega |\Delta\phi_\varepsilon|^2\,dx$; rewrite it as

$$\int_\Omega |\Delta\phi_\varepsilon|^2\,dx = \biggl(\int_{B^+_{R\varepsilon}} + \int_{\Omega\setminus B^+_{R\varepsilon}}\biggr)|\Delta\phi_\varepsilon|^2\,dx := \mathrm{I}_1 + \mathrm{I}_2. \tag{4-2}$$

Since

$$\begin{aligned}\mathrm{I}_2 &= \int_{\Omega\setminus B^+_{R\varepsilon}} |\Delta\phi_\varepsilon|^2\\ &= \int_{\Omega\setminus B^+_{R\varepsilon}}\frac{|\Delta G|^2}{C^2} + \int_{B^+_{2R\varepsilon}\setminus B^+_{R\varepsilon}}\frac{\bigl|\Delta\bigl(\eta(G-C\widetilde{\phi}_\varepsilon\circ\theta(x))\bigr)\bigr|^2}{C^2} - \frac{2}{C^2}\int_{B^+_{2R\varepsilon}\setminus B^+_{R\varepsilon}}|\nabla G\nabla(G-C\widetilde{\phi}_\varepsilon\circ\theta(x))|^2\\ &:= \mathrm{II}_1 + \mathrm{II}_2 + \mathrm{II}_3.\end{aligned} \tag{4-3}$$

Let C satisfy

$$C+\frac{-\frac{1}{8\pi^2}\log\bigl(\bigl(\frac{\pi}{2}\bigr)^{4/3}R^2\bigr)+B}{C}=\frac{-\frac{1}{4\pi^2}\log R\varepsilon+A_p}{C}, \tag{4-4}$$

by direct computing, one can easily verify that

$$|\mathrm{II}_2|,\,|\mathrm{II}_3|=\frac{1}{C^2}(O(R\varepsilon)+O(R^{-1})). \tag{4-5}$$

Similar as (3-37) and (3-41), we can obtain

$$\begin{aligned}\mathrm{II}_1=\int_{\Omega\setminus B^+_{R\varepsilon}}\frac{|\Delta G|^2}{C^2}&=\int_{\partial(\Omega\setminus B^+_{R\varepsilon})}\nu(-G\Delta^{\frac{3}{2}}G+\Delta^{\frac{1}{2}}G\Delta G)\,d\sigma\\&=\frac{1}{C^2}\Bigl(-\frac{1}{4\pi^2}\log R\varepsilon-\frac{1}{8\pi^2}+A_p+O(R\varepsilon)\Bigr)\end{aligned} \tag{4-6}$$

and

$$\mathrm{I}_1=\int_{B^+_{R\varepsilon}}|\Delta\phi_\varepsilon|^2=\frac{1}{C^2}\biggl(\frac{3}{16\pi^2}+\frac{1}{4\pi^2}\log\frac{R}{2}+\frac{1}{6\pi^2}\log\frac{\pi}{2}+O\Bigl(\frac{\log R}{R}\Bigr)\biggr). \tag{4-7}$$

Combining (4-2), (4-3), (4-5), (4-6) and (4-7), we have

$$\begin{aligned}\int_\Omega|\Delta\phi_\varepsilon|^2&=\frac{1}{C^2}\biggl(\frac{3}{16\pi^2}+\frac{1}{4\pi^2}\log\frac{R}{2}+\frac{1}{6\pi^2}\log\frac{\pi}{2}+O\Bigl(\frac{\log R}{R}\Bigr)\biggr)\\&\qquad+\frac{1}{C^2}\Bigl(-\frac{1}{4\pi^2}\log R\varepsilon-\frac{1}{8\pi^2}+A_p+O(R\varepsilon)+O\Bigl(\frac{1}{R}\Bigr)\Bigr)\\&=\frac{1}{C^2}\biggl(\frac{1}{16\pi^2}+\frac{1}{6\pi^2}\log\frac{\pi}{2}+\frac{1}{4\pi^2}\log\frac{1}{2\varepsilon}+A_p+O(R\varepsilon)+O\Bigl(\frac{\log R}{R}\Bigr)\biggr).\end{aligned}$$

To ensure that $\int_\Omega|\Delta\phi_\varepsilon|^2=1$, we set

$$C^2=\frac{1}{16\pi^2}+\frac{1}{6\pi^2}\log\frac{\pi}{2}+\frac{1}{4\pi^2}\log\frac{1}{2\varepsilon}+A_p+O(R\varepsilon)+O\Bigl(\frac{\log R}{R}\Bigr). \tag{4-8}$$

On the other hand, from (4-4), we have

$$C^2=-\frac{1}{4\pi^2}\log\varepsilon+A_p-B+\frac{1}{6\pi^2}\log\frac{\pi}{2}.$$

Therefore,

$$B=\frac{1}{4\pi^2}\log 2-\frac{1}{16\pi^2}+O(R\varepsilon)+O\Bigl(\frac{\log R}{R}\Bigr). \tag{4-9}$$

A straightforward computation gives

$$\begin{aligned}\bar{\phi}_\varepsilon=\frac{1}{|\Omega|}\int_\Omega\phi_\varepsilon&=\frac{1}{C}\bigl(O((R\varepsilon)^4\log R)+O((R\varepsilon)^4\log\varepsilon)+O((R\varepsilon)^4\log R\varepsilon)\bigr)\\&=\frac{1}{C}\bigl(O((R\varepsilon)^4\log R)+O((R\varepsilon)^4\log\varepsilon)\bigr).\end{aligned}$$

Then

$$\int_{\partial\Omega} \exp(12\pi^2(\phi_\varepsilon - \overline{\phi}_\varepsilon)^2)\, d\sigma$$
$$\geq \int_{\partial B^+_{R\varepsilon}\cap\partial\mathbb{R}^+_4} \exp\bigl(12\pi^2(\widetilde{\phi}_\varepsilon - \overline{\phi}_\varepsilon)^2(x',t)\bigr)\, dx'\, dt$$
$$\geq \int_{\partial B^+_{R\varepsilon}\cap\partial\mathbb{R}^+_4} \exp\biggl(12\pi^2 C^2 - 3\log\biggl(\Bigl(\frac{\pi}{2}\Bigr)^{4/3}\frac{|x'|^2}{\varepsilon^2}+1\biggr) + 24\pi^2 B - 24\pi^2 C\overline{\phi}_\varepsilon\biggr)\, dx'$$
$$= \exp\bigl(12\pi^2C^2 + 24\pi^2 B + O((R\varepsilon)^4\log R) + O((R\varepsilon)^4\log\varepsilon)\bigr)\int_{B^3_{R\varepsilon}} \frac{1}{\bigl((\frac{\pi}{2})^{4/3}|x'|^2/\varepsilon^2+1\bigr)^3}\, dx'.$$

Let $\left(\frac{\pi}{2}\right)^{\frac{2}{3}}\frac{x'}{\varepsilon} = \tilde{x}$. Then

$$\int_{B^3_{R\varepsilon}} \frac{1}{\bigl((\frac{\pi}{2})^{4/3}|x'|^2/\varepsilon^2+1\bigr)^3}\, dx' = \Bigl(\frac{2}{\pi}\Bigr)^2\varepsilon^3\int_{B^3_{(\pi/2)^{2/3}R}} \frac{1}{(\tilde{x}^2+1)^3}\, d\tilde{x} = \Bigl(\frac{2}{\pi}\Bigr)^2\varepsilon^3\int_0^{(\pi/2)^{2/3}R}\frac{4\pi r^2}{(r^2+1)^3}\, dr$$
$$= \Bigl(\frac{2}{\pi}\Bigr)^2\varepsilon^3 4\pi\int_0^{(\pi/2)^{2/3}R}\frac{r^2}{(r^2+1)^3}\, dr$$
$$= \varepsilon^3\Bigl(1+O\Bigl(\frac{1}{R}\Bigr)\Bigr),$$

where we have used the fact that

$$\int_0^\infty \frac{r^2}{(r^2+1)^3}\, dr = \tfrac{1}{16}\pi.$$

Hence, it follows from (4-8) and (4-9) that

$$\int_{\partial B^+_{R\varepsilon}\cap\partial\mathbb{R}^+_4} \exp\bigl(12\pi^2(\phi_\varepsilon - \overline{\phi}_\varepsilon)^2(x',t)\bigr)\, dx'\, dt$$
$$\geq \varepsilon^3\Bigl(1+O\Bigl(\frac{1}{R}\Bigr)\Bigr)\exp\bigl(12\pi^2C^2 + 24\pi^2B + O((R\varepsilon)^4\log R) + O((R\varepsilon)^4\log\varepsilon)\bigr)$$
$$= \varepsilon^3\exp\biggl(12\pi^2\Bigl(\frac{1}{16\pi^2} + \frac{1}{6\pi^2}\log\frac{\pi}{2} + \frac{1}{4\pi^2}\log\frac{1}{2\varepsilon} + A_p\Bigr) + 24\pi^2\Bigl(\frac{1}{4\pi^2}\log 2 - \frac{1}{16\pi^2}\Bigr)\biggr)$$
$$+ O(R\varepsilon) + O\biggl(\frac{\log R}{R} + O((R\varepsilon)^4\log R) + O((R\varepsilon)^4\log\varepsilon)\biggr)$$
$$= \exp\bigl(-\tfrac{3}{4} + 2\log\pi + \log 2 + 12\pi^2A_p\bigr) + O((R\varepsilon)^4\log R) + O((R\varepsilon)^4\log\varepsilon) + O(R\varepsilon) + O\Bigl(\frac{\log R}{R}\Bigr)$$
$$= 2\pi^2\exp\bigl(-\tfrac{3}{4} + 12\pi^2A_p\bigr) + O((R\varepsilon)^4\log R) + O((R\varepsilon)^4\log\varepsilon) + O(R\varepsilon) + O\Bigl(\frac{\log R}{R}\Bigr).$$

Moreover, we have

$$\int_{\Omega\setminus\partial B^+_{R\varepsilon}} \exp(12\pi^2(\phi_\varepsilon - \overline{\phi}_\varepsilon)^2)\, d\sigma \geq \int_{\partial\Omega\setminus\partial B^+_{R\varepsilon}} (1+12\pi^2(\phi_\varepsilon - \overline{\phi}_\varepsilon)^2)\, d\sigma$$
$$\geq |\partial\Omega\setminus\partial B^+_{R\varepsilon}| + \frac{12\pi^2}{C^2}\int_{\partial\Omega\setminus\partial B^+_{2R\varepsilon}} (G - C\overline{\phi}_\varepsilon)^2\, d\sigma.$$

Therefore,

$$\begin{aligned}
&\int_{\partial\Omega} \exp(12\pi^2(\phi_\varepsilon - \overline{\phi}_\varepsilon)^2)\, d\sigma \\
&\geq |\partial\Omega| - O((R\varepsilon)^3) + \frac{12\pi^2}{C^2} \int_{\partial\Omega\setminus\partial B^+_{2R\varepsilon}} \big(G - C(O((R\varepsilon)^4 \log R) + O((R\varepsilon)^4 \log \varepsilon))\big)^2\, d\sigma \\
&\qquad + 2\pi^2 \exp\big(-\tfrac{3}{4} + 12\pi^2 A_p\big) + O((R\varepsilon)^4 \log R) + O((R\varepsilon)^4 \log \varepsilon) + O\Big(\frac{\log R}{R}\Big) + O(R\varepsilon) \\
&= |\partial\Omega| + 2\pi^2 \exp\big(-\tfrac{3}{4} + 12\pi^2 A_p\big) + \frac{12\pi^2}{C^2} \int_{\partial\Omega} G^2 + O((R\varepsilon)^4 \log R) \\
&\qquad + O((R\varepsilon)^4 \log \varepsilon) + O(R\varepsilon) + O\Big(\frac{\log R}{R}\Big).
\end{aligned}$$

Let $R = \log^2 \varepsilon$. Then we have $R \to \infty$ and $R\varepsilon \to 0$, and

$$(R\varepsilon)^4 \log R + (R\varepsilon)^4 \log \frac{1}{\varepsilon} + O\Big(\frac{\log R}{R}\Big) + O(R\varepsilon) = o\Big(\frac{1}{C^2}\Big).$$

Hence

$$\int_{\partial\Omega} \exp(12\pi^2(\phi_\varepsilon - \overline{\phi}_\varepsilon)^2)\, d\sigma > |\partial\Omega| + 2\pi^2 \exp\big(-\tfrac{3}{4} + 12\pi^2 A_p\big),$$

as ε is small enough.

Proof of Theorem 1.1. In the subcritical case $\alpha < 12\pi^2$, the inequality (1-3) and the sharpness of the constant $12\pi^2$ can be obtained from Lemma 2.3. In the critical case, that is $\alpha = 12\pi^2$, we will address the problem by dividing it into two cases. If $c_k = \max_{x\in\Omega} |u_k(x)|$ is bounded, then the inequality (1-3) is obvious, and by the elliptic estimates with respect to (2-3), there exists $u \in \mathcal{H} \cap C^\infty(\Omega)$ such that $u_k \to u$ in $C^\infty(\Omega)$ as $k \to \infty$, and Theorem 1.1 follows immediately. While if we assume that $c_k \to +\infty$ as $k \to \infty$, one can find a contradiction between Proposition 3.14 and the arguments of the test functions for (4-1) in Section 4, this means that c_k must be bounded, and the proof is finished. □

Proof of Theorem 1.3. For any $u \in W^{2,2}(\Omega)$ with $\dfrac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$, define $\bar{u} = \dfrac{1}{|\Omega|} \displaystyle\int_\Omega u\, dx$. Then we can write

$$\begin{aligned}
\int_{\partial\Omega} e^{u-\bar{u}}\, d\sigma &= \int_{\partial\Omega} \exp\Big(\frac{u-\bar{u}}{\|\Delta(u-\bar{u})\|_2} \|\Delta(u-\bar{u})\|_2\Big)\, d\sigma \\
&\leq \int_{\partial\Omega} \exp\Big(12\pi^2 \frac{|u-\bar{u}|^2}{\|\Delta(u-\bar{u})\|_2^2}\Big) \exp\Big(\frac{1}{48\pi^2} \|\Delta(u-\bar{u})\|_2^2\Big)\, d\sigma \leq C_0 e^{1/(48\pi^2)\|\Delta u\|_2^2}, \quad (4\text{-}10)
\end{aligned}$$

where we have used the elementary inequality $ab \leq 12\pi^2 a^2 + \dfrac{1}{48\pi^2} b^2$ and the trace Adams inequality in Theorem 1.1. Then we have

$$\log\Big(\int_{\partial\Omega} e^{u-\bar{u}}\, d\sigma\Big) \leq \log C_0 + \frac{1}{48\pi^2} \|\Delta u\|_2^2.$$

That is

$$\log\Big(\int_{\partial\Omega} e^{u}\, d\sigma\Big) \leq \frac{1}{48\pi^2} \int_\Omega |\Delta u|^2\, dx + \frac{1}{|\Omega|} \int_\Omega u\, dx + \log C_0. \qquad □$$

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SINGULARITIES OF THE CHERN–RICCI FLOW

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We study the nature of finite time singularities for the Chern–Ricci flow, partially answering a question posed by Tosatti and Weinkove. We show that a solution of degenerate parabolic complex Monge–Ampère equations, starting from arbitrarily positive (1,1)-currents, is smooth outside some analytic subset, generalizing works by Di Nezza and Lu. Moreover, we extend Guedj and Lu's recent approach to establish uniform a priori estimates for degenerate complex Monge–Ampère equations on compact Hermitian manifolds. We apply these results to study the Chern–Ricci flow on log terminal varieties starting from a current with mild singularities.

1. Introduction

Finding canonical metrics on complex varieties has been a central problem in complex geometry over the last few decades. Since Yau's solution to Calabi's conjecture, significant progress has been made in this direction. Cao [1985] introduced a parabolic approach to provide an alternative proof of the existence of Kähler–Einstein metrics on manifolds with numerically trivial or ample canonical line bundle via the Kähler–Ricci flow. This flow is only Hamilton's Ricci flow evolving Kähler metrics. Motivated by the classification of complex varieties, Song and Tian [2012; 2017] have proposed an *analytic minimal model program* to classify algebraic varieties with mild singularities using the Kähler–Ricci flow. This approach necessitates a theory of weak solutions for degenerate parabolic complex Monge–Ampère equations starting from rough initial data. Since then, various results have been achieved in this direction. Song and Tian initiated the study of the Kähler–Ricci flow starting from an initial current with continuous potentials. While Guedj and Zeriahi [2017b] (also [Tô 2017]) showed that the Kähler–Ricci flow could be continued from an initial current with zero Lelong numbers. To the author's knowledge, the best results so far have been obtained by Di Nezza and Lu [2017], who successfully ran the Kähler–Ricci flow from an initial

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current with positive Lelong numbers. There have been several related works in such singular settings from a pluripotential theoretical point of view. For further details, we refer to the recent works [Guedj et al. 2020; Dang 2022].

Beyond the Kähler setting, there more recently has been interest in the study of geometric flows in the context of non-Kähler manifolds. Unlike the Kähler case, Hamilton's Ricci flow does not, in general, preserve the special Hermitian condition. It is thus natural to look for another geometric flow of Hermitian metrics, which somehow specializes in the Ricci flow in the Kähler context. Several parabolic flows on complex manifolds that preserve the Hermitian property have been proposed by Streets and Tian [2010; 2011] and Liu and Yang [Liu and Yang 2012]. Additionally, the anomaly flow of $(n-1, n-1)$-forms has been extensively studied by Phong, Picard, and Zhang [Phong et al. 2018a; 2018b].

This paper is devoted to the Chern–Ricci flow, which is an evolution equation of Hermitian metrics on a complex manifold by their Chern–Ricci form, first introduced by Gill [2011] in the setting of manifolds with vanishing first Bott–Chern class. Let (X, ω_0) be a compact n-dimensional Hermitian manifold. The *Chern–Ricci flow* $\omega = \omega(t)$ starting at ω_0 is an evolution equation of Hermitian metrics,

$$\frac{\partial \omega}{\partial t} = -\mathrm{Ric}(\omega), \quad \omega|_{t=0} = \omega_0, \tag{1-1}$$

where $\mathrm{Ric}(\omega)$ is the *Chern–Ricci form* of ω associated to the Hermitian metric $g = (g_{i\bar{j}})$, which in local coordinates is given by

$$\mathrm{Ric}(\omega) = -dd^c \log \det(g).$$

Here $d = \partial + \bar{\partial}$ and $d^c = i(\bar{\partial} - \partial)/2$ are both real operators, so that $dd^c = i\partial\bar{\partial}$. In the Kähler setting, $\mathrm{Ric}(\omega) = iR_{j\bar{k}}\, dz_j \wedge d\bar{z}_k$, where $R_{j\bar{k}}$ is the usual Ricci curvature of ω. Thus, if ω_0 is Kähler, i.e., $d\omega_0 = 0$, (1-1) coincides with the Kähler–Ricci flow. For complex manifolds with $c_1^{\mathrm{BC}}(X) = 0$, Gill [2011] proved the longtime existence of the flow and smooth convergence of the flow to the unique Chern–Ricci flat metric in the $\partial\bar{\partial}$-class of the initial metric. For general complex manifolds, Tosatti and Weinkove [2015, Theorem 1.3] characterized the maximal existence time $T_{\max}$ of the flow as

$$T_{\max} := \sup\bigl\{t > 0 : \exists \psi \in \mathcal{C}^\infty(X) \text{ with } \omega_0 - t\,\mathrm{Ric}(\omega_0) + dd^c\psi > 0\bigr\}.$$

Finite time singularities. Suppose that the flow (1-1) exists on a maximal interval $[0, T_{\max})$ with $T_{\max} < \infty$, so the flow develops a finite time singularity. We say that the Chern–Ricci flow does not develop a singularity at a point $x \in X$ if there exist an open neighborhood $U \ni x$ and a smooth metric $\omega_{T_{\max}}$ on U such that $\omega(t)$ converges to $\omega_{T_{\max}}$ in $\mathcal{C}^\infty_{\mathrm{loc}}(U)$ as $t \to T_{\max}^-$.

The following question was asked by Feldman, Ilmanen, and Knopf [Feldman et al. 2003, Question 2, page 204] for the Kähler–Ricci flow and by Tosatti and Weinkove [2022, Question 6.1] for the Chern–Ricci flow.

Question 1.1. Do singularities of the Chern–Ricci flow form a union of all analytic subvarieties of X for which the volume shrinks to zero as $t \to T_{\max}$?

In the Kähler setting, this question was affirmatively answered by Collins and Tosatti [2015]. When X is a compact complex surface and ω_0 is Gauduchon, i.e., $dd^c\omega_0 = 0$, the Chern–Ricci flow preserves the Gauduchon (pluriclosed) condition, in particular, the limiting form $\alpha_{T_{\max}} = \omega_0 - T_{\max}\,\mathrm{Ric}(\omega_0)$ is Gauduchon. The answer is thus affirmative in this case, due to Gill and Smith [2015] (see also [Tosatti and Weinkove 2013]), where they proved that singularities of the Chern–Ricci flow form a finite union of disjoint (-1)-curves.

We partially answer Question 1.1 under two additional assumptions. First, we assume that the limiting form $\alpha_{T_{\max}}$ is *uniformly noncollapsing*:

$$\int_X (\alpha_{T_{\max}} + dd^c\psi)^n \geq c_0 > 0 \quad \text{for all } \psi \in \mathcal{C}^\infty(X),\ \alpha_{T_{\max}} + dd^c\psi > 0. \tag{1-2}$$

We mention that when $\dim X = 2$ and ω_0 is a Gauduchon metric on X, the latter condition is equivalent to $\int_X \alpha_{T_{\max}}^2 > 0$ (by Stokes' theorem). A simple example (see [Tosatti and Weinkove 2013, Remark 3.1]) where this condition appears is the following. Let Y be a compact Hermitian manifold and $\pi : X \to Y$ be the blowup of a point with exceptional divisor E. Let ω_X and ω_Y be Gauduchon metrics on X and Y respectively, and fix $T_{\max} > 0$. It is known that there is a metric h on the line bundle $\mathcal{O}(E)$ with curvature R_h such that for $C > 0$ large enough, $\omega' = C\pi^*\omega_Y - T_{\max} R_h + dd^c f$ is a Hermitian metric for some $f \in \mathcal{C}^\infty(X)$. By the adjunction formula, we can choose

$$\omega_0 := (C+1)\pi^*\omega_Y + T_{\max}\,\mathrm{Ric}(\omega_X) + dd^c f$$

which is a Gauduchon metric. Hence $\alpha_{T_{\max}} = \pi^*\widetilde{\omega}_Y + dd^c\tilde{f}$ for some Gauduchon metric $\widetilde{\omega}_Y$ and $\tilde{f} \in \mathcal{C}^\infty(X)$; see [Tosatti and Weinkove 2013, Lemma 3.2] or [Buchdahl 2000]. For any $\psi \in \mathcal{C}^\infty(X)$,

$$\int_X (\alpha_{T_{\max}} + dd^c\psi)^2 = \int_X \alpha_{T_{\max}}^2 = \int_Y \widetilde{\omega}_Y^2 > 0.$$

The second assumption is that X has *the bounded mass property*, that is, there exists a Hermitian metric ω_X such that $v_+(\omega_X) < +\infty$ (see Definition 2.4). This condition is automatically satisfied for compact complex surfaces (see [Guedj and Lu 2022]). For further examples of non-Kähler manifolds in higher dimensions, we refer the reader to [Angella et al. 2023]. Our main theorem is the following.

Theorem A. *Let (X, ω_0) be an n-dimensional compact Hermitian manifold with bounded mass property, i.e., $v_+(\omega_0) < +\infty$. Assume that the Chern–Ricci flow* (1-1) *starting at ω_0 exists on the maximal interval* $[0, T_{\max})$ *with $T_{\max} < \infty$ and that the limiting form $\alpha_{T_{\max}}$ is uniformly noncollapsing*:

$$\int_X (\alpha_{T_{\max}} + dd^c\psi)^n \geq c_0 > 0 \quad \text{for all } \psi \in \mathcal{C}^\infty(X) \text{ such that } \alpha_{T_{\max}} + dd^c\psi > 0. \tag{1-3}$$

Then as $t \to T^-$ the metrics $\omega(t)$ converge to $\omega_{T_{\max}}$ in $\mathcal{C}^\infty_{\mathrm{loc}}(\Omega)$ for some Zariski open set $\Omega \subset X$.

The strategy of the proof is as follows. Using the uniformly noncollapsing condition of $\alpha_{T_{\max}}$, we show that there exists a quasiplurisubharmonic function ρ with analytic singularities such that $\alpha_{T_{\max}} + dd^c\rho$ dominates a Hermitian metric. This form is called *big* (see Definition 2.6). Then Ω is the set in which ρ is smooth. In particular, it is Zariski open. Next, we establish several uniform local estimates for ω near the maximal time $T_{\max}$, adapting techniques from [Collins and Tosatti 2015; Gill 2011]. The convergence result follows directly from these estimates.

Degenerate parabolic complex Monge–Ampère equations. In the previous paragraph, we studied the behavior of the Chern–Ricci flow at finite singularity time. It is natural to ask whether the flow can pass through this singularity. To do this, we need to define weak solutions of the Chern–Ricci flows starting from degenerate initial currents on a compact complex variety with mild singularities. This leads us to consider several geometric settings arising in the minimal model program, particularly the case of complex varieties with Kawamata log terminal (klt) singularities. From an analytic point of view, this situation naturally involves densities that may blow up but still belong to L^p spaces for some exponent $p > 1$ whose size depends on the algebraic nature of the singularities.

On a compact Hermitian n-manifold (X, ω_X), we consider the following degenerate parabolic complex Monge–Ampère equation,

$$\frac{\partial \varphi_t}{\partial t} = \log\left[\frac{(\theta_t + dd^c\varphi_t)^n}{\mu}\right] \tag{1-4}$$

for $t \in (0, T_{\max})$, where $T_{\max} < \infty$ and

- $\theta_t = \theta + t\chi$ is an affine family of smooth semipositive forms, where χ is a smooth (1,1)-form and θ is a smooth, *big* (1,1)-form, that is there is a quasiplurisubharmonic function ρ with analytic singularities such that
$$\theta + dd^c\rho \geq \delta\omega_X \quad \text{for some } \delta > 0;$$
- μ is a positive measure on X of the form
$$\mu = e^{\psi^+ - \psi^-}\, dV_X$$
with $\psi^\pm$ quasiplurisubharmonic functions, being smooth on a given Zariski open subset $U \subset \{\rho > -\infty\}$ and $e^{-\psi^-} \in L^p$ for some $p > 1$ and dV_X a smooth volume form;
- $\varphi : [0, T_{\max}] \times X \to \mathbb{R}$ is the unknown function, with $\varphi_t := \varphi(t, \cdot)$.

We first define the weak solution of the Chern–Ricci flow:

Definition 1.2. A family of functions $\varphi_t : X \to \mathbb{R}$ for $t \in (0, T_{\max})$ is said to be a weak solution of equation (1-4) starting with φ_0 if the following hold:

(1) For each t, φ_t is θ_t-plurisubharmonic on X.

(2) $\varphi_t \to \varphi_0$ in $L^1(X)$ as $t \to 0^+$.

(3) For each $\varepsilon > 0$ there exists a Zariski open set $\Omega_\varepsilon \subset X$ such that the function $(t, x) \mapsto \varphi(t, x) \in \mathcal{C}^\infty([\varepsilon, T_{\max} - \varepsilon] \times \Omega_\varepsilon)$. Furthermore, equation (1-4) satisfies in the classical sense on $[\varepsilon, T_{\max}) \times \Omega_\varepsilon$.

The following theorem establishes the existence of the complex Monge–Ampère flow starting with an initial function φ_0 with small Lelong numbers.

Theorem B. *Let (X, ω_0) be an n-dimensional compact Hermitian manifold and θ a semipositive and big* $(1, 1)$*-form. Let φ_0 be an θ-plurisubharmonic function satisfying $p^*/(2c(\varphi_0)) < T_{\max}$, where p^* is the conjugate exponent of p. Then, there exists a weak solution φ of the flow* (1-4) *starting with φ_0 for $t \in (0, T_{\max})$.*

Here, $c(\varphi_0)$ denotes the integrability index of φ_0, which is the supremum of positive constants $c > 0$ such that $e^{-2c\varphi_0}$ is locally integrable. Thanks to Skoda's integrability theorem, $c(\varphi_0) = +\infty$ if and only if φ_0 has zero Lelong numbers at all points.

Let us briefly outline the strategy for the proof of Theorem B. We first approximate φ_0 by a decreasing sequence of smooth $(\theta + 2^{-j}\omega_X)$-plurisubharmonic functions $\varphi_{0,j}$ thanks to Demailly's regularization theorem. Similarly, $\psi^{\pm}$ are approximated by smooth quasiplurisubharmonic functions. We consider the corresponding solution $\varphi_{t,j}$ to equation (1-4), with $\theta_{t,j} = \theta_t + 2^{-j}\omega_X$. Our goal is to establish several a priori estimates that allow us to take the limit as $j \to +\infty$. Precisely, we aim to show that for any $\varepsilon > 0$, there is a Zariski open set $\Omega_\varepsilon \subset X$ such that for each fixed $0 < T < T_{\max}$ and any compact subset $K \subset \Omega_\varepsilon$,

- $\|\varphi_{t,j}\|_{\mathcal{C}^0([\varepsilon,T]\times K)} \leq C_{\varepsilon,T,K}$;
- $\partial_t \varphi_{t,j}$ is uniformly bounded on $[\varepsilon, T] \times K$;
- $\Delta_{\omega_X} \varphi_{t,j}$ is uniformly bounded on $[\varepsilon, T] \times K$.

We then apply the parabolic Evans–Krylov–Trudinger theory and Schauder estimates to obtain uniform higher-order local estimates for all derivatives of $\varphi_{t,j}$ (see [Gill 2011] for a recent account in the Chern–Ricci flow context). This allows us to pass to the limit and conclude that

$$\varphi_{t,j} \to \varphi_t \in \mathcal{C}^\infty([\varepsilon, T] \times \Omega_\varepsilon)$$

as $j \to +\infty$. Furthermore, we automatically have the weak convergence $\varphi_t \to \varphi_0$ as $t \to 0^+$. Stronger convergence results are discussed in Section 4.4 when φ_0 has less singularity.

We emphasize here that the mild assumption $p^*/(2c(\varphi_0)) < T_{\max}$ guarantees that the approximating flow is well-defined (i.e., not identically $-\infty$) and is crucial for the smoothing properties of the flow. As noted by Di Nezza and Lu [2017] for the Kähler setting, without this assumption, the Kähler–Ricci flow may still run, but it is likely to lose its regularizing effect due to the presence of positive Lelong numbers. In such cases, they highlighted that the main challenge lies in establishing the a priori $\mathcal{C}^0$-estimate. Their approach relies on Kołodziej's method, which uses generalized Monge–Ampère capacities. In contrast, our approach follows the recent developments of Guedj and Lu [2023; 2025], which have the advantage of being applicable to degenerate (1,1)-forms in the non-Kähler context.

We finally apply the previous analysis to treat the case of mildly singular varieties. This allows us to define a good notion of the weak Chern–Ricci flow on complex compact varieties with log terminal singularities. We will discuss it in Section 6 and prove the following.

Theorem C. *Let Y be a compact complex variety with log terminal singularities. Assume that θ_0 is a Hermitian metric such that*

$$T_{\max} := \sup\{t > 0 : \exists \psi \in \mathcal{C}^\infty(Y) \text{ such that } \theta_0 - t\,\mathrm{Ric}(\theta_0) + dd^c\psi > 0\} > 0.$$

Assume that $S_0 = \theta_0 + dd^c\varphi_0$ is a positive $(1,1)$-current with sufficiently small slopes. Then, there exists a family $(\omega_t)_{t\in[0,T_{\max})}$ of positive $(1,1)$-currents on Y starting with S_0 such that

(1) $\omega_t = \theta_0 - t\,\mathrm{Ric}(\theta_0) + dd^c\varphi_t$ *are positive* (1,1)*-currents*;

(2) $\omega_t \to S_0$ *weakly as* $t \to 0^+$;

(3) *for each* $\varepsilon > 0$ *there exists a Zariski open set* Ω_ε *such that on* $[\varepsilon, T_{\max}) \times \Omega_\varepsilon$, ω *is smooth and*

$$\frac{\partial \omega}{\partial t} = -\mathrm{Ric}(\omega).$$

This generalizes previous results of Song and Tian [2017], Guedj and Zeriahi [2017a], Tô [2017], Di Nezza and Lu [2017], Guedj, Lu, and Zeriahi [Guedj et al. 2020] and the author [Dang 2022] to the non-Kähler case, and of Tô [2018], Nie [2017] and the author [Dang 2024] to more degenerate initial data.

Organization of the paper. We establish a priori estimates in Section 3, which will be used to prove Theorem B in Section 4. Theorem A will be proved in Section 5, studying the behavior of the Chern–Ricci flow at noncollapsing finite time singularities. In Section 6, we apply these tools to prove the existence of the weak Chern–Ricci flow with initial degenerate data on compact complex varieties with log terminal singularities, proving Theorem C.

2. Preliminaries

2.1. *Recap on pluripotential theory.* Let X be a compact complex manifold of dimension n, equipped with a Hermitian metric ω_X. We fix θ a smooth semipositive real $(1, 1)$-form on X.

2.1.1. *Quasiplurisubharmonic functions and Lelong numbers.* A function $u \in L^1(X)$ is quasiplurisubharmonic (quasi-psh for short) if it is locally given as the sum of a smooth function and a plurisubharmonic (psh for short) function.

Definition 2.1. A quasi-psh function $\varphi : X \to [-\infty, +\infty)$ is called *θ-plurisubharmonic* (*θ-psh* for short) if it satisfies $\theta_\varphi := \theta + dd^c\varphi \geq 0$ in the weak sense of currents. We let $\mathrm{PSH}(X, \theta)$ denote the set of all θ-psh functions that are not identically $-\infty$.

The set $\mathrm{PSH}(X, \theta)$ is endowed with the $L^1(X)$-topology. By Hartogs' lemma, the map $\varphi \mapsto \sup_X \varphi$ is continuous with respect to this topology. Since the set of closed positive currents in a fixed dd^c-class is compact (in the weak topology), it follows that the set of $\varphi \in \mathrm{PSH}(X, \theta)$, with $\sup_X \varphi = 0$ is compact. We refer the reader to [Demailly 2012; Guedj and Zeriahi 2017a] for basic properties of θ-psh functions.

Quasi-psh functions are, in general, singular, and a convenient way to measure their singularities is the Lelong numbers.

Definition 2.2. Let $x_0 \in X$. Fixing a holomorphic chart $x_0 \in V_{x_0} \subset X$, the *Lelong number* $\nu(\varphi, x_0)$ of a quasi-psh function φ at $x_0 \in X$ is defined as

$$\nu(\varphi, x_0) := \sup\bigl\{\gamma \geq 0 : \varphi(z) \leq \gamma \log \|z - x_0\| + O(1), \text{ on } V_{x_0}\bigr\}.$$

We remark here that this definition does not depend on the choice of local holomorphic charts. In particular, if $\varphi = \log |f|$ in a neighborhood V_{x_0} of x_0, for some holomorphic function f, then $\nu(\varphi, x_0)$ is equal to the vanishing order $\mathrm{ord}_{x_0}(f) := \sup\{k \in \mathbb{N} : D^\gamma f(x_0) = 0, \ \forall |\gamma| < k\}$.

In some contexts, it is more convenient to work with the integrability index rather than the Lelong numbers. The *integrability index* of a quasi-psh function φ at a point $x \in X$ is defined by

$$c(\varphi, x) := \sup\{c > 0 : e^{-2c\varphi} \in L^1(V_x)\},$$

where V_x is some neighborhood around x. This definition does not depend on the choice of the open neighborhood V_x. We denote by $c(\varphi)$ the infimum of $c(\varphi, x)$ for all $x \in X$. Since X is compact, it follows that $c(\varphi) > 0$.

Skoda's integrability theorem (see [Guedj and Zeriahi 2017a, Chapter 2]) yields the following "optimal" relation between the Lelong number of a quasi-psh function φ at a point $x_0 \in X$ and the local integrability index of φ at x_0:

$$\frac{1}{\nu(\varphi, x_0)} \leq c(\varphi, x_0) \leq \frac{n}{\nu(\varphi, x_0)}. \tag{2-1}$$

In particular, $c(\varphi) = +\infty$ if and only if $\nu(\varphi, x) = 0$ for all $x \in X$.

2.1.2. *Monge–Ampère measures.* The complex Monge–Ampère measure $(\theta + dd^c u)^n$ is well-defined for any θ-psh function u which is bounded, as follows from the Bedford–Taylor theory: if $\beta = dd^c \rho$ is a Kähler form such that $\beta > \theta$ in a local open chart $U \subset X$, then u is β-psh and the positive currents $(\beta + dd^c u)^j$ are well-defined for $1 \leq j \leq n$. Thus, the *complex Monge–Ampère measure*,

$$(\theta + dd^c u)^n := \sum_{j=0}^{n} \binom{n}{j} (\beta + dd^c u)^j \wedge (\theta - \beta)^{n-j},$$

is a positive measure on X. Indeed, by Demailly's regularization theorem, we can approximate u by a decreasing sequence of smooth $(\theta + \varepsilon_j \omega_X)$-psh functions u_j. Consequently, $(\theta + dd^c u)^n$ is the limit of positive measures $(\theta + \varepsilon_j \omega_X + dd^c u_j)^n$, ensuring that $(\theta + dd^c u)^n$ is positive.

This definition does not depend on the choice of β by the same arguments. We refer to [Dinew and Kołodziej 2012] for an adaptation of [Bedford and Taylor 1976; 1982] to the Hermitian context. We recall the following maximum principle.

Lemma 2.3. *Let φ, ψ be bounded θ-psh functions such that $\varphi \leq \psi$. Then*

$$\mathbf{1}_{\{\varphi=\psi\}} (\theta + dd^c \varphi)^n \leq \mathbf{1}_{\{\varphi=\psi\}} (\theta + dd^c \psi)^n.$$

Proof. This is a direct consequence of Bedford–Taylor's maximum principle; see [Guedj and Zeriahi 2017a, Theorem 3.23]. We refer the reader to [Guedj and Lu 2022, Lemma 1.2] for a brief proof. □

2.1.3. *Positivity assumptions.* For our purposes, we need to assume a slightly stronger positivity property of the form θ in the sense of [Guedj and Lu 2023].

Definition 2.4. We consider

$$v_-(\theta) := \inf\left\{ \int_X (\theta + dd^c \varphi)^n : \varphi \in \mathrm{PSH}(X, \theta) \cap L^\infty(X) \right\}$$

and

$$v_+(\theta) := \sup\left\{ \int_X (\theta + dd^c \varphi)^n : \varphi \in \mathrm{PSH}(X, \theta) \cap L^\infty(X) \right\}.$$

We emphasize that when θ is Hermitian, the supremum and infimum in the definition of these quantities can be taken over $\mathrm{PSH}(X,\theta)\cap\mathcal{C}^\infty(X)$ due to Demailly's regularization theorem and Bedford–Taylor's convergence results.

Definition 2.5. A function ρ is said to have *analytic singularities* if there exists a constant $c>0$ such that locally on X,

$$\rho = c\log\sum_{j=1}^{N}|f_j|^2 + O(1),$$

where the f_j are holomorphic functions.

Definition 2.6. We say θ is *big* if there exists a θ-psh function with analytic singularities such that $\theta+dd^c\rho\geq\delta\omega_X$ for some $\delta>0$. We let Ω denote the open Zariski set in which ρ is locally bounded.

Such a form appears in some contexts of complex differential geometry. For example, if Y is a compact complex space endowed with a Hermitian metric ω_Y and $\pi:X\to Y$ is a log resolution of singularities, then the form $\theta:=\pi^*\omega_Y$ is big; see, e.g., [Fino and Tomassini 2009, Proposition 3.2]. Moreover, we can find a θ-psh function ρ with analytic singularities such that $\theta+dd^c\rho\geq\delta\omega_X$, and

$$\Omega=\{\rho>-\infty\}=X\setminus\mathrm{Exc}(\pi)=\pi^{-1}(Y_{\mathrm{reg}})\simeq Y_{\mathrm{reg}}.$$

2.1.4. *Envelopes.* Recall that a Borel set $E\subset X$ is (*locally*) *pluripolar* if for each $x\in X$, there exists an open neighborhood U of x and a psh function u on U such that $E\cap U\subset\{u=-\infty\}$. As follows from [Vu 2019, Theorem 1.1] or [Guedj and Lu 2022, Lemma 2.6], the set E is globally pluripolar; i.e., there exists $u\in\mathrm{PSH}(X,\omega_X)$ such that $E\subset\{u=-\infty\}$. Since θ is big, the function $u':=\delta u+\rho$ is θ-psh and its $-\infty$-locus contains E.

Definition 2.7. Given a measurable function $h:X\to\mathbb{R}$, we define the *θ-psh envelope* of h by

$$P_\theta(h):=(\sup\{u\in\mathrm{PSH}(X,\theta):u\leq h\text{ on }X\})^*,$$

where the star means that we take the upper semicontinuous regularization.

We note that this definition is equivalent to the one given in [Guedj and Lu 2022, Definition 2.2]; see [Guedj and Lu 2022, Corollary 2.7].

We have the following result, established in [Guedj and Lu 2022, Theorem 2.3].

Theorem 2.8. *If h is bounded from below, quasi-lower-semicontinuous, and $P_\theta(h)<+\infty$, then*

(1) *$P_\theta(h)$ is a bounded θ-psh function;*

(2) *$P_\theta(h)\leq h$ in $X\setminus P$, for some pluripolar set P;*

(3) *$(\theta+dd^cP_\theta(h))^n$ is concentrated on the contact set $\{P_\theta(h)=h\}$.*

The following $\mathcal{C}^0$-estimate is crucial in the sequel.

Lemma 2.9. *Let θ be a smooth real semipositive and big (1,1)-form. Assume $\varphi\in\mathrm{PSH}(X,\theta)\cap L^\infty(X)$ satisfies*

$$(\theta+dd^c\varphi)^n\leq e^{A\varphi-g}f\,dV_X,$$

where $A > 0$ and f, g are measurable functions such that $e^{A\psi-g} f \in L^q(X)$ with $q > 1$, for some $\psi \in \mathrm{PSH}(X, \delta\theta)$, with $\delta \in (0, 1)$. Then we have the estimate

$$\varphi \geq \psi - C,$$

where C is a positive constant only depending on n, A, δ, θ, q and an upper bound for $\int_X e^{q(A\psi-g)} f^q \, dV_X$.

Proof. We apply the approach recently developed by Guedj and Lu [2023; 2025]. Subtracting a large constant, we can assume that $\varphi \leq 0$. Set $u := P_{(1-\delta)\theta}(\varphi - \psi)$. Fix $M > 0$ so large that $E := \{\psi > -M\}$ is not empty and hence it is nonpluripolar. We claim that the global extremal function $V^*_{E,(1-\delta)\theta}$ of E is not identically $+\infty$, where

$$V_{E,(1-\delta)\theta}(x) := \sup\big\{\varphi(x) : \varphi \in \mathrm{PSH}(X, (1-\delta)\theta),\ \varphi \leq 0 \text{ on } E\big\}.$$

The proof follows almost verbatim from [Guedj and Zeriahi 2017a, Theorem 9.17]. We suppose by contradiction that $\sup_X V_{E,(1-\delta)\theta} = +\infty$. By a lemma of Choquet (see [Guedj and Zeriahi 2017a, Lemma 4.31]), there exists an increasing sequence $u_j \in \mathrm{PSH}(X, (1-\delta)\theta)$ such that $u_j = 0$ on E, $\sup_X u_j \geq 2^j$, and

$$V_{E,(1-\delta)\theta} = (\lim \nearrow u_j)^*.$$

Set $v_j := u_j - \sup_X u_j$. These functions belong to the compact set of $(1-\delta)\theta$-psh functions normalized by $\sup_X w = 0$. Hence, there exists a uniform constant $C > 0$ such that $\int_X v_j \, dV \geq -C$; see [Dinew and Kołodziej 2012, Proposition 2.1]. Since $(1-\delta)\theta \geq 0$, the function $v := \sum_{j\geq 1} 2^{-j} v_j \in \mathrm{PSH}(X, (1-\delta)\theta)$ is a decreasing limit of functions in $\mathrm{PSH}(X, (1-\delta)\theta)$, with $\int_X v \, dV \geq -C$. Since $v = -\infty$ on E, it follows that E is $\mathrm{PSH}(X, (1-\delta)\theta)$-pluripolar. This gives a contradiction.

Since $u \leq \varphi - \psi \leq M$ on E, hence $u - M$ is a candidate defining $V_{E,(1-\delta)\theta}$. Therefore, $\sup_X u \leq M + \sup_X V^*_{E,(1-\delta)\theta}$ is uniformly bounded from above.

Since $\varphi - \psi$ is bounded from below and quasicontinuous, it follows from Theorem 2.8 that

$$((1-\delta)\theta + dd^c u)^n$$

is supported on the contact set $D := \{u + \psi = \varphi\}$. We observe that $u + \psi$ and φ are both θ-psh functions satisfying $u + \psi \leq \varphi$, it follows from Lemma 2.3 that

$$\mathbf{1}_D(\theta + dd^c(u+\psi))^n \leq \mathbf{1}_D(\theta + dd^c\varphi)^n.$$

From these, we have

$$\begin{aligned}((1-\delta)\theta + dd^c u)^n = \mathbf{1}_D((1-\delta)\theta + dd^c u)^n \leq \mathbf{1}_D(\theta + dd^c(u+\psi))^n &\leq \mathbf{1}_D(\theta + dd^c\varphi)^n \\ &\leq \mathbf{1}_D e^{A\varphi - g} f \, dV_X \\ &= \mathbf{1}_D e^{Au} e^{A\psi-g} f \, dV_X.\end{aligned}$$

By assumption, $F := e^{A\psi-g} f \in L^q(X)$, with $q > 1$. Since $(1-\delta)\theta$ is semipositive and big, it follows from [Guedj and Lu 2023, Lemma 2.1] that there exists a uniform constant $m > 0$ only depending on dV_X,

n, q, θ, δ, and $\|e^{A\psi-g}f\|_{L^q}$, such that we can find $v \in \mathrm{PSH}(X, (1-\delta)\theta) \cap L^\infty(X)$ satisfying $-1 \le v \le 0$ and

$$((1-\delta)\theta + dd^c v)^n \ge mF\,dV_X.$$

Hence

$$e^{-A(v+A^{-1}\ln m)}((1-\delta)\theta + dd^c v)^n \ge F\,dV_X \ge e^{-Au}((1-\delta)\theta + dd^c u)^n.$$

The domination principle (see [Guedj and Lu 2023, Proposition 1.14]) yields $u \ge v + A^{-1}\ln m$. This completes the proof. □

2.2. *Equisingular approximation.* Fix φ a θ-psh function on X. We aim at approximating φ by a decreasing sequence of quasi-psh functions which are less singular than φ and such that their singularities are somehow comparable to those of φ. This leads us to apply Demailly's equisingular approximation theorem. For each $c > 0$, we define the *Lelong superlevel sets*

$$E_c(\varphi) := \{x \in X : \nu(\varphi, x) \ge c\}.$$

We also use the notation $E_c(T)$ for a closed positive $(1,1)$-current T. A well-known result of Siu [1974] asserts that the Lelong superlevel sets $E_c(\varphi)$ are analytic subsets of X. We refer the reader to [Demailly 1992, Remark 3.2] for an alternative proof.

The following result of Demailly on the equisingular approximation of a quasi-psh function by quasi-psh functions with analytic singularities is crucial.

Theorem 2.10 (Demailly's equisingular approximation). *Let φ be a θ-psh function on X. There exists a decreasing sequence of quasi-psh functions $(\varphi_m)_{m\in\mathbb{N}}$ such that*

(1) *(φ_m) converges pointwise and in $L^1(X)$ to φ as $m \to +\infty$,*

(2) *φ_m has the same singularities as $1/(2m)$ times a logarithm of a sum of squares of holomorphic functions,*

(3) *$dd^c\varphi_m \ge -\theta - \varepsilon_m\omega_X$, where $\varepsilon_m > 0$ decreases to 0 as $m \to +\infty$,*

(4) *$\int_X e^{2m(\varphi_m-\varphi)}\,dV < +\infty$,*

(5) *φ_m is smooth outside the analytic subset $E_{1/m}(\varphi)$.*

Proof. We briefly outline the idea for the reader's convenience, as it is likely already known to experts. We follow the proof of [Demailly 1992] by applying with the current $T = dd^c\varphi$ and the smooth real (1,1)-form $\gamma = -\theta$. We also borrow notation from there.

For $\delta > 0$ small, let us cover X by $N = N(\delta)$ geodesic balls $B_{2r}(a_j)$ with respect to ω_X such that $X = \bigcup_j B_r(a_j)$ and in terms of coordinates $z^j = (z_1^j, \dots, z_n^j)$,

$$\sum_{\ell=1}^n \lambda_\ell^j i\,dz_\ell^j \wedge d\bar{z}_\ell^j \le \gamma|_{B_{2r}(a_j)} \le \sum_{\ell=1}^n (\lambda_\ell^j + \delta) i\,dz_\ell^j \wedge d\bar{z}_\ell^j,$$

where we have diagonalized $\gamma(a_j)$ at the center a_j. Here, N and r are taken to be uniform. Set $\varphi^j := \varphi|_{B_{2r}(a_j)} - \sum_{\ell=1}^n \lambda_\ell^j |z_\ell^j|^2$. On each $B_{2r}(a_j)$, we define

$$\varphi_{j,\delta,m} := \frac{1}{2m} \log \sum_{k\in\mathbb{N}} |f_{j,m,k}|^2,$$

where $(f_{j,m,k})_{k\in\mathbb{N}}$ is an orthogonal basis of the Hilbert space $\mathcal{H}_{B_{2r}(a_j)}(m\varphi^j)$ of holomorphic functions on $B_{2r}(a_j)$ with finite L^2-norm $\|u\|^2 = \int_{B_{2r}(a_j)} |u|^2 e^{-2m\varphi^j}\, dV(z^j)$. Note that since $dd^c\varphi \geq \gamma$ it follows that $\varphi - \sum_{\ell=1}^n \lambda_\ell^j |z_\ell^j|^2$ is psh on $B_{2r}(a_j)$. The Bergman kernel process applied on each ball $B_{2r}(a_j)$ has provided approximations $\varphi_{j,\delta,m}$ of $\varphi^j = \varphi|_{B_{2r}(a_j)} - \sum_{\ell=1}^n \lambda_\ell^j |z_\ell^j|^2$; it thus remains to glue these functions into a function $\varphi_{\delta,m}$ globally defined on X. For this, we set

$$\varphi_{\delta,m}(x) = \frac{1}{2m} \log\biggl(\sum_j \theta_j(x)^2 \exp\biggl(2m\biggl(\varphi_{j,\delta,m} + \sum_\ell (\lambda_\ell^j - \delta)|z_\ell^j|^2\biggr)\biggr)\biggr),$$

where $(\theta_j)_{1\leq j\leq N}$ is the partition of unity subordinate to the $B_r(a_j)$. Now we take $\delta = \delta_m \searrow 0$ slowly and $\varphi_m = \varphi_{\delta_m,m}$ the same computations as in [Demailly 1992, page 16] ensure that

$$dd^c\varphi_m \geq \gamma - \varepsilon(\delta_m)\omega_X$$

for $m \geq m_0$ sufficiently large and $\varepsilon_m = \varepsilon(\delta_m) \searrow 0$ as $m \to +\infty$. By construction, the properties (1), (2), (3), and (5) are satisfied.

Property (4) is crucial for later use; its proof should be provided. The argument originates from [Demailly et al. 2001, Theorem 2.3, Step 2], using local uniform convergence and the strong Noetherian property. By the properties of the functions φ_m, it suffices to show that on each ball $B_j = B_r(a_j)$,

$$\int_{B_j} e^{2m\varphi_m - 2m\varphi}\, dV = \int_{B_j} \biggl(\sum_{k\in\mathbb{N}} |f_{j,m,k}|^2\biggr) e^{-2m\varphi}\, dV(z^j) < +\infty.$$

We let $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_k \subset \cdots \subset \mathcal{O}(B_{2r}(a_j) \times B_{2r}(a_j))$ denote the sequence of ideal coherent sheaves generated by the holomorphic functions $(f_{j,m,\ell}(z)\overline{f_{j,m,\ell}(\overline{w})})_{\ell\leq k}$ on $B_{2r}(a_j) \times B_{2r}(a_j)$. By the strong Noetherian property (see, e.g., [Demailly 2012, C. II, 3.22]), the sequence $(\mathcal{F}_k)$ is stationary on a compact subset $B_j \times B_j \Subset B_{2r}(a_j) \times B_{2r}(a_j)$ at an index k_0 large enough. Using the Cauchy–Schwarz inequality we have that the sum of the series $U(z,w) = \sum_{k\in\mathbb{N}} f_{j,m,k}(z)\overline{f_{j,m,k}(\overline{w})}$ is bounded from above by

$$\biggl(\sum_{k\in\mathbb{N}} |f_{j,m,k}(z)|^2 \sum_{k\in\mathbb{N}} |f_{j,m,k}(\overline{w})|^2\biggr)^{\frac{1}{2}}$$

hence uniformly convergent on every compact subset of $B_{2r}(a_j) \times B_{2r}(a_j)$. Since the space of sections of a coherent ideal sheaf is closed under the topology of uniform convergence on compact subsets, the Noetherian property guarantees $U(z,w) \in \mathcal{F}_{k_0}(B_j \times B_j)$. Restricting to the conjugate diagonal $w = \bar{z}$, we obtain

$$\sum_{k\in\mathbb{N}} |f_{j,m,k}(z)|^2 \leq C_0\biggl(\sum_{k\leq k_0} |f_{j,m,k}(z)|^2\biggr)$$

on B_j. Since all terms $f_{j,m,k}$ have the L^2-norm equal to 1 with respect to the weight $e^{-2m\varphi}$, this completes the proof. □

Using this, one obtains the following lemma, which is slightly more general than the one in [Di Nezza and Lu 2017].

Lemma 2.11. *Let* θ *be a big* (1,1)*-form. Assume* $\varphi \in \mathrm{PSH}(X, \theta)$*. Then for each* $\varepsilon > 0$ *there exist* $c(\varepsilon) > 0$ *and* $\psi_\varepsilon \in \mathrm{PSH}(X, \theta) \cap \mathcal{C}^\infty\big(X \setminus (\{\rho = -\infty\} \cup E_{c(\varepsilon)}(\varphi))\big)$ *such that*

$$\int_X e^{\frac{2}{\varepsilon}(\psi_\varepsilon - \varphi)}\, dV_X < +\infty. \tag{2-2}$$

Proof. The proof is quite close to that of [Di Nezza and Lu 2017, Lemma 2.7]. Recall that the bigness of θ implies that there exists ρ an θ-psh function with analytic singularities and $\sup_X \rho = 0$ such that

$$\theta + dd^c \rho \geq 3\delta_0 \omega_X \quad \text{for a fixed constant } \delta_0 > 0.$$

Let $c(\varphi)$ be the integrability index of φ. We can assume that $c(\varphi) < +\infty$; otherwise we are done. By Theorem 2.10, we can find (φ_m) a Demailly's equisingular approximant of φ. We have that φ_m is smooth in the complement of the analytic subset $E_{1/m}(\varphi)$ and

$$\theta + dd^c \varphi_m \geq -\varepsilon_m \delta_0 \omega_X$$

for $\varepsilon_m > 0$ decreasing to zero as m goes to $+\infty$. We notice that the errors $\varepsilon_m > 0$ appear in the gluing process; see Theorem 2.10. We choose $m = m(\varepsilon)$ to be the smallest positive integer such that

$$m > \frac{2}{\varepsilon(1+\varepsilon_m)}, \quad \frac{2\varepsilon_m}{\varepsilon(1+\varepsilon_m)} < c(\varphi).$$

We now set

$$\psi_\varepsilon := \frac{\varphi_m}{1+\varepsilon_m} + \frac{\varepsilon_m}{1+\varepsilon_m}\rho. \tag{2-3}$$

Thus, we have

$$\theta + dd^c \psi_\varepsilon \geq \frac{\varepsilon_m}{1+\varepsilon_m} 2\delta_0 \omega_X := 2\kappa\omega_X.$$

Holder's inequality ensures that (2-2) holds, noticing that $\rho \leq 0$. We easily see that ψ_ε is smooth in the complement of $\{\rho = -\infty\} \cup E_{c(\varepsilon)}(\varphi)$ with $c(\varepsilon) = m(\varepsilon)^{-1}$. □

3. A priori estimates

3.1. ***Notation.*** We use some notation from [Di Nezza and Lu 2017, Section 3.1]. Until further notice, X denotes a compact complex manifold of dimension n, endowed with a reference Hermitian form ω_X. Following the strategy outlined in the introductory section, we assume, in this part, that θ and $\theta_t = \theta + t\chi$, $t \in (0, T_{\max})$, are Hermitian metrics, and φ_0 is a smooth strictly θ-psh function. We denote by $\mu := f\, dV_X$ a positive measure with density $\|f\|_{L^p} \leq C$ uniformly, for some $p > 1$. For higher-order estimates, we assume moreover that

$$f = e^{\psi^+ - \psi^-},$$

where $\psi^\pm$ are smooth quasi-psh functions. Recall that ρ is a θ-psh function with analytic singularities such that $\theta + dd^c \rho$ dominates a Hermitian form. We may assume that $\sup_X \rho = 0$. We remark that as follows from [Guedj and Lu 2023], a priori bounds below remain valid when θ is semipositive and big, $f \in L^p(X, \omega_X)$, and φ_0 is merely bounded and θ-psh.

We consider φ_t a smooth solution of the parabolic complex Monge–Ampère equation

$$\frac{\partial \varphi_t}{\partial t} = \log\left[\frac{(\theta_t + dd^c\varphi_t)^n}{\mu}\right], \quad \varphi|_{t=0} = \varphi_0 \tag{3-1}$$

on $[0, T_{\max})$; see, e.g., [Tosatti and Weinkove 2015]. We should keep in mind that φ_t plays the role of its approximants $\varphi_{t,j}$ in establishing a priori estimates. For brevity, we will suppress the index j.

We fix T and S such that

$$\frac{p^*}{2c(\varphi_0)} < T < S < T_{\max},$$

where p^* is the conjugate exponent of p, i.e., $\dfrac{1}{p} + \dfrac{1}{p^*} = 1$. Since we are interested in the behavior of the flow (3-1) near zero, we can assume that

$$\theta_S \geq (1-a)\theta \quad \text{for } a \in \big[0, \tfrac{1}{2}\big).$$

It is truly natural in several geometric contexts; for example, θ_t are the pullback of Hermitian forms. Thus, for each $t \in [0, S]$, we have

$$\theta_t = \frac{t\theta_S}{S} + \frac{S-t}{S}\theta \geq \left(1 - \frac{at}{S}\right)\theta.$$

During the proof, we use the notation $\omega_t := \theta_t + dd^c\varphi_t$ for the smooth path of Hermitian forms and denote $\Delta_t = \mathrm{tr}_{\omega_t} dd^c$ the corresponding time-dependent Laplacian operator on functions.

We fix $\varepsilon_0 > 0$ small and let $\psi_0 := \psi_{\varepsilon_0}$ as established in Lemma 2.11. By construction, ψ_0 is smooth outside an analytic subset $\{\rho = -\infty\} \cup E_{c(\varepsilon)}(\varphi_0)$ and satisfies

$$\theta + dd^c\psi_0 \geq 2\kappa\omega_X. \tag{3-2}$$

We let E_1, E_2 denote the quantities

$$E_1 := \int_X e^{2(\psi_0 - \varphi_0)/\varepsilon_0}\, dV_X < +\infty, \quad E_2 := \int_X e^{-p^*\psi_0/T}\, dV_X < +\infty.$$

Observe that E_1 is finite by Lemma 2.11, while E_2 is finite since $p^*/(2c(\varphi_0)) < T$ and that ψ_0 is less singular than φ_0. We should emphasize that φ_0 in this a priori estimate section plays a role in its approximating sequence $\varphi_{0,j}$ (which are smooth strictly θ-psh functions decreasing to φ_0). The corresponding sequence E_1^j is uniformly bounded from above in j. Hence we can pass the limit.

In what follows, we use C to denote a positive constant whose value may change from line to line but is uniformly controlled.

3.2. ***Uniform estimate.*** We first look for an upper a priori bound for φ_t. We recall that

$$\tfrac{1}{2}\theta \leq \theta_t \leq A\omega_X \quad \text{for all } t \in [0, T]$$

for $A > 0$ sufficiently large. It follows from [Tosatti and Weinkove 2010] that there exist a constant c and a smooth $A\omega_X$-psh function Φ normalized by $\inf_X \Phi = 0$ such that

$$(A\omega_X + dd^c\Phi)^n = e^c f\, dV_X.$$

Proposition 3.1. *For any $(t, x) \in [0, T] \times X$, there exists a uniform constant $C > 0$ such that*

$$\varphi_t(x) \le C.$$

Proof. For any $(t, x) \in [0, T] \times X$, we set $v(t, x) = \Phi(x) + ct + \sup_X \varphi_0$. Then, we can check that

$$\frac{\partial v}{\partial t} = \log\left[\frac{(A\omega_X + dd^c v_t)^n}{\mu}\right], \quad \text{while } \frac{\partial \varphi}{\partial t} \le \log\left[\frac{(A\omega_X + dd^c\varphi_t)^n}{\mu}\right],$$

and $v_0 \ge \varphi_0$. Hence, by the classical maximum principle, we have $v(t, x) \ge \varphi(t, x)$ for $(t, x) \in [0, T] \times X$. Consequently, this provides an upper bound for $\varphi(t, x)$:

$$\sup_X |\Phi| + \max(c, 0)T + \sup_X \varphi_0. \qquad \square$$

We fix two positive constants α, β such that

$$\frac{p^*}{2c(\varphi_0)} < \frac{1}{\alpha} < \frac{1}{\alpha - \beta} < T_{\max} \quad \text{and} \quad \theta + (\alpha - \beta)\chi \ge 0.$$

We observe that the density $e^{-\alpha\varphi_0} f$ belongs to L^q for $q > 1$. Indeed, we choose $\delta > 0$ so small ($\alpha(p^* + \delta) < 2c(\varphi_0)$) that

$$\frac{1}{q} = \frac{1}{p} + \frac{1}{p^* + \delta} \quad \text{with } q > 1.$$

Applying Hölder's inequality and Skoda's theorem, we have

$$\int_X e^{-\alpha q\varphi_0} f^q\, dV \le \|f\|^q_{L^p} \left(\int_X e^{-\alpha(p^*+\delta)\varphi_0}\, dV\right)^{q/p^*+\delta} < +\infty.$$

Thus, by [Tosatti and Weinkove 2010], there exists a smooth θ-psh function u such that

$$\beta^n(\theta + dd^c u)^n = e^{\beta u - \alpha\varphi_0} f\, dV.$$

Proposition 3.2. *For $t \in (0, \alpha^{-1})$,*

$$(1 - \alpha t)\varphi_0 + \beta t u + n(t \log t - t) \le \varphi_t.$$

In particular, there exists a uniform constant $C > 0$ such that

$$\varphi_0 - C(t - t\log t) \le \varphi_t \quad \textit{for all } t \in (1, \alpha^{-1}).$$

Proof. The proof is identical to that of [Guedj and Zeriahi 2017b, Lemma 2.9]. $\square$

Before establishing a lower bound for the solution φ_t, we first prove an upper bound for its time derivative, $\dot{\varphi}_t := \dfrac{\partial \varphi}{\partial t}$.

Proposition 3.3. *For all* $(t, x) \in (0, T] \times X$,

$$\dot{\varphi}_t(x) \leq \frac{\varphi_t(x) - \varphi_0(x)}{t} + n. \tag{3-3}$$

Proof. The proof is identical to that of [Guedj and Zeriahi 2017b, Proposition 3.1] (also in [Guedj et al. 2020]). ☐

We follow the approach in [Di Nezza and Lu 2017] to derive the following uniform estimate for the complex parabolic Monge–Ampère equation.

Theorem 3.4. *Fix* $\varepsilon > p^*\varepsilon_0$. *For* $t \in [\varepsilon, T]$, *we obtain the estimate*

$$\varphi_t \geq \left(1 - \frac{bt}{T}\right)\psi_0 - C,$$

where $b \in \left(a, \frac{1}{2}\right)$ *and* $C > 0$ *is a uniform constant.*

Proof. Fixing $t \in [\varepsilon, T]$, it follows from Proposition 3.3 that

$$(\theta_t + dd^c\varphi_t)^n = e^{\dot{\varphi}_t} \leq e^{n+(\varphi_t - \varphi_0)/t} f\, dV.$$

We set

$$\psi_t := \left(1 - \frac{bt}{T}\right)\psi_0$$

for $b \in \left(a, \frac{1}{2}\right)$ close to a. We recall that

$$\theta_t \geq \left(1 - \frac{at}{S}\right)\theta.$$

It then follows that ψ_t is $\delta\theta_t$-psh with $\delta \in (0, 1)$ only depending on $\varepsilon_0, a, b, T, S$. More precisely,

$$\delta = \frac{TS - bS\varepsilon_0}{TS - aT\varepsilon_0}.$$

Using similar arguments as in the proof of [Di Nezza and Lu 2017, Theorem 3.2], we can bound the quantity

$$\int_X e^{q(\psi_t - \varphi_0)/t} f^q\, dV < +\infty \tag{3-4}$$

for some $q > 1$, in terms of $\|f\|_{L^p}$, E_1 and E_2. To establish this, we fix $\gamma > 0$ small enough and choose $q > 1$ such that

$$\frac{1}{q} = \frac{1}{p} + \frac{1}{2p^* + \gamma} + \frac{1}{2p^* + \gamma}.$$

By Hölder's inequality, we obtain

$$\int_X e^{q(\psi_t - \varphi_0)/t} f^q\, dV \leq \|f\|_{L^p}^q \left(\int_X e^{(2p^*+\gamma)(\psi_0 - \varphi_0)/t}\, dV\right)^{q/(2p^*+\gamma)} \left(\int_X e^{-(2p^*+\gamma)b\psi_0/T}\, dV\right)^{q/(2p^*+\gamma)}.$$

The second term on the right-hand side is finite due to the construction of ψ_0 in Lemma 2.11. Also, since ψ_0 is less singular than φ_0, the third term is finite.

From (3-4), we apply Lemma 2.9 with $A = 1/t$ and $g = \varphi_0/t - n$ to obtain the desired estimate. It is important to note that our $\mathcal{C}^0$-estimate depends only on n, θ, q, the fixed parameters ε_0, ε, T, S, and an upper bound for E_1 and E_2. □

Remark 3.5. When φ_0 is bounded or, more generally, has zero Lelong numbers, it was shown in [Tô 2018] (generalizing the result of [Guedj and Zeriahi 2017b] in the Kähler context) that the estimate (3-3) ensures a lower bound for φ_t using the Kolodziej–Nguyen theorem [2015]. Unfortunately, this method cannot be applied in more general cases, such as when φ_0 is more singular, for example, when it has a positive Lelong number. To analyze the singularities of the initial potential φ_0 in such cases, Guedj and Lu's approach [2023] could help.

3.3. ***Laplacian estimate.*** We recall the following classical inequality.

Lemma 3.6. *Let α, β be two positive* (1,1)*-forms. Then*

$$n\left(\frac{\alpha^n}{\beta^n}\right)^{\frac{1}{n}} \leq \operatorname{tr}_\beta(\alpha) \leq n\left(\frac{\alpha^n}{\beta^n}\right)(\operatorname{tr}_\alpha(\beta))^{n-1}.$$

We define

$$\Psi_t := \left(1 - \frac{bt}{S}\right)\psi_0,$$

where ψ_0 is defined in Lemma 2.11 with $\varepsilon_0 > 0$ fixed.

To establish the $\mathcal{C}^2$-estimate, it is necessary to derive a lower bound for $\dot{\varphi}_t = \dfrac{\partial \varphi}{\partial t}$.

Proposition 3.7. *Fix $\varepsilon > p^*\varepsilon_0$. For $(t, x) \in (\varepsilon, T] \times X$,*

$$\dot{\varphi}_t(x) \geq n \log(t - \varepsilon) + A(\Psi_t - \varphi_t) - C,$$

where $A, C > 0$ are positive constants only depending on ε, T, $\|f\|_{L^p}$, and an upper bound for E_1 and E_2.

Proof. The proof is almost identical to that of [Di Nezza and Lu 2017, Proposition 3.5]. The only difference is that we use Theorem 3.4 instead of the corresponding result in [Di Nezza and Lu 2017, Theorem 3.2]. We include the proof for the reader's convenience.

Since $\mu = f\, dV$ is a smooth volume form, the main result of Tosatti and Weinkove [2010] ensures that there exists a constant c_1 and $\phi_1 \in \operatorname{PSH}(X, \theta) \cap \mathcal{C}^\infty(X)$ such that

$$(\theta + dd^c\phi_1)^n = e^{c_1}\mu, \quad \sup_X \phi_1 = 0.$$

From [Guedj and Lu 2023, Theorems 2.2, 3.4], it follows that $|c_1| + \|\phi_1\|_{L^\infty} \leq C$, where $C > 0$ depends only on the semipositivity and bigness of θ, n, dV_X, p and $\|f\|_p$.

We define

$$G(t, x) := \dot{\varphi}_t(x) + A(\varphi_t - \Psi_t) - \phi_1 - n \log(t - \varepsilon)$$

for a constant $A > 0$ to be determined hereafter. Observe that G achieves its minimum on $[\varepsilon, T] \times X$ at some point $(t_0, x_0) \in (\varepsilon, T] \times (X \setminus \{\psi_0 = -\infty\})$. In the following, all computations will be performed at this point. We compute

$$\Big(\frac{\partial}{\partial t} - \Delta_t\Big) G = A\dot{\varphi}_t - \frac{n}{t-\varepsilon} + A\frac{b\psi_0}{S} - nA + A\,\mathrm{tr}_{\omega_t}(\theta_t + dd^c\Psi_t) + \mathrm{tr}_{\omega_t}(\chi + dd^c\phi_1).$$

We observe that

$$\theta_t + dd^c\Psi_t = \frac{t(b-a)}{S}\theta + \Big(1 - \frac{bt}{S}\Big)(\theta + dd^c\psi_0) \geq \frac{\varepsilon(b-a)}{S}\theta + \tfrac{1}{2}2\kappa\omega_X.$$

We now choose $A > 0$ so big that

$$A(\theta_t + dd^c\Psi_t) + \chi \geq \theta.$$

Therefore

$$\Big(\frac{\partial}{\partial t} - \Delta_t\Big) G \geq A\dot{\varphi}_t - \frac{n}{t-\varepsilon} + A\frac{b\psi_0}{S} - nA + \mathrm{tr}_{\omega_t}(\theta + dd^c\phi_1). \tag{3-5}$$

On the other hand, Lemma 3.6 ensures that

$$\mathrm{tr}_{\omega_t}(\theta + dd^c\phi_1) \geq n\bigg(\frac{(\theta + dd^c\phi_1)^n}{\omega_t^n}\bigg)^{\frac{1}{n}} = ne^{-\dot{\varphi}_t + c_1/(n)}.$$

Using the elementary inequality $\gamma y - \log y \geq -C_\gamma$ for any small constant $\gamma > 0$ and $y > 0$, we observe that

$$A\dot{\varphi}_t + ne^{(-\dot{\varphi}_t + c_1)/n} \geq e^{-\dot{\varphi}_t/n - C_1} - C_2.$$

Substituting this into (3-5), it follows from the minimum principle that at (t_0, x_0),

$$\dot{\varphi}_t \geq -n\log\bigg(C_2 + \frac{n}{t-\varepsilon} - \frac{Ab\psi_0}{S} + nA\bigg) - nC_1,$$

and hence

$$G(t_0, x_0) \geq -C_3 - n\log\bigg(C_2(t_0 - \varepsilon) + n - \frac{Ab(t_0-\varepsilon)\psi_0}{S}\bigg) - \frac{Abt_0(S-T)}{ST}\psi_0,$$

where we have used Theorem 3.4. Thus, we obtain a uniform lower bound for $G(t_0, x_0)$, and the desired lower bound follows. □

We are now in a position to establish the $\mathcal{C}^2$-estimate. We follow the computations of [Tosatti and Weinkove 2015, Lemma 4.1] (see also [Tô 2018, Lemma 6.4]), where they use the technical trick introduced by Phong and Sturm [2010]. Recall that the measure μ is of the form

$$\mu = e^{\psi^+ - \psi^-}\, dV_X,$$

where $\psi^\pm$ are smooth $K\omega_X$-psh functions on X for uniform constant $K > 0$. For simplicity, we assume $K = 1$ and normalize $\sup_X \psi^\pm = 0$.

Theorem 3.8. *Fix $\varepsilon > p^*\varepsilon_0$. For $(t, x) \in [\varepsilon, T] \times X$ we have the bound*

$$(t-\varepsilon)\log \mathrm{tr}_{\omega_X}(\omega_t) \leq -B\psi_0 - C\psi^- + C,$$

where B, C are positive constants depending only on ε, T, $\|e^{-\psi^-}\|_{L^p}$, and an upper bound for E_1 and E_2.

Proof. We follow the computations of [Gill 2011; Tô 2018] (which are due to the trick of Phong and Sturm [2010]) with modification to deal with unbounded functions. The constant C denotes various uniform constants, which may differ throughout the argument.

Consider

$$H := (t-\varepsilon)\log \operatorname{tr}_{\omega_X}(\omega_t) - \gamma(u), \quad (t,x)\in[\varepsilon,T]\times X,$$

where $\gamma:\mathbb{R}\to\mathbb{R}$ is a smooth, concave, increasing function such that $\lim_{t\to+\infty}\gamma(t)=+\infty$, and

$$u(t,x) := \varphi_t(x) - \Psi_t(x) - \kappa\psi^- + 1 \geq 1,$$

as follows from Theorem 3.4, and ψ_0, $\psi^- \leq 0$. We will show that H is uniformly bounded from above for an appropriate choice of γ.

We let g denote the Riemann metric associated with ω_X and $\tilde{g}$ the one associated with $\omega_t := \theta_t + dd^c\varphi_t$. Since H goes to $-\infty$ on the boundary of $X_0 := \{x\in X : \psi_0(x) > -\infty\}$, H achieves its maximum on $[\varepsilon,T]\times X$ at some point $(t_0,x_0)\in(\varepsilon,T]\times X_0$.

At this maximum point, we use the following local coordinate systems due to Guan and Li [2010, Lemma 2.1, (2.19)]:

$$g_{i\bar{j}} = \delta_{ij}, \quad \frac{\partial g_{i\bar{i}}}{\partial z_j} = 0 \quad \text{and} \quad \tilde{g}_{i\bar{j}} \text{ is diagonal.}$$

Following the computations in [Tô 2018, equation (3.20)], we have

$$\Delta_t \operatorname{tr}_{\omega_X}(\omega_t) \geq \sum_{i,j} \tilde{g}^{i\bar{i}}\tilde{g}^{j\bar{j}}\tilde{g}_{i\bar{j}j}\tilde{g}_{j\bar{i}\bar{j}} - \operatorname{tr}_{\omega_X}\operatorname{Ric}(\omega_t) - C_1\operatorname{tr}_{\omega_X}(\omega_t)\operatorname{tr}_{\omega_t}(\omega_X). \tag{3-6}$$

From standard arguments, as in [Guedj and Lu 2023, equation (4.5)], we obtain

$$\frac{|\partial \operatorname{tr}_{\omega_X}(\omega_t)|^2_{\tilde{\omega}_t}}{(\operatorname{tr}_{\omega_X}(\omega_t))^2} \leq \frac{1}{\operatorname{tr}_{\omega_X}(\omega_t)}\Bigl(\sum_{i,j}\tilde{g}^{i\bar{i}}\tilde{g}^{j\bar{j}}\tilde{g}_{i\bar{j}j}\tilde{g}_{j\bar{i}\bar{j}}\Bigr) + C\frac{\operatorname{tr}_{\omega_t}(\omega_X)}{(\operatorname{tr}_{\omega_X}(\omega_t))^2} + \frac{2}{(\operatorname{tr}_{\omega_X}(\omega_t))^2}\operatorname{Re}\sum_{i,j,k}\tilde{g}^{i\bar{i}}T_{ij\bar{j}}\tilde{g}_{k\bar{i}\bar{k}}, \tag{3-7}$$

where $T_{ij\bar{j}} := \tilde{g}_{j\bar{j}i} - \tilde{g}_{i\bar{j}j}$ is the torsion term corresponding to θ_t which is controlled: $|T_{ij\bar{j}}|\leq C$. Now at the point (t_0,x_0), we have $\partial_{\bar{i}}H=0$; hence

$$(t-\varepsilon)\sum_k \tilde{g}_{k\bar{k}\bar{i}} = \operatorname{tr}_{\omega_X}(\omega_t)\gamma'(u)u_{\bar{i}}.$$

The Cauchy–Schwarz inequality yields

$$\left|\frac{2}{(\operatorname{tr}_{\omega_X}(\omega_t))^2}\operatorname{Re}\sum_{i,j,k}\tilde{g}^{i\bar{i}}T_{ij\bar{j}}\tilde{g}_{k\bar{k}\bar{i}}\right| \leq C\frac{\gamma'(u)(t_0-\varepsilon)}{-\gamma''(u)}\frac{\operatorname{tr}_{\omega_t}(\omega_X)}{(\operatorname{tr}_{\omega_X}(\omega_t))^2} + \frac{-\gamma''(u)}{t_0-\varepsilon}|\partial u|^2_{\omega_t},$$

and hence

$$\left|\frac{2}{(\operatorname{tr}_{\omega_X}(\omega_t))^2}\operatorname{Re}\sum_{i,j,k}\tilde{g}^{i\bar{i}}T_{ij\bar{j}}\tilde{g}_{k\bar{i}\bar{k}}\right| \leq C\Bigl(\frac{\gamma'(u)T}{-\gamma''(u)}+1\Bigr)\frac{\operatorname{tr}_{\omega_t}(\omega_X)}{(\operatorname{tr}_{\omega_X}(\omega_t))^2} + \frac{-\gamma''(u)}{t_0-\varepsilon}|\partial u|^2_{\omega_t},$$

using that $|\tilde g_{k\bar k i} - \tilde g_{k i \bar k}| \le C$. From this, the inequality (3-7) becomes

$$\frac{|\partial \operatorname{tr}_{\omega_X}(\omega_t)|^2_{\omega_t}}{(\operatorname{tr}_{\omega_X}(\omega_t))^2} \le \frac{1}{\operatorname{tr}_{\omega_X}(\omega_t)}\Bigl(\sum_{i,j} \tilde g^{i\bar i}\tilde g^{j\bar j}\tilde g_{i\bar j j}\tilde g_{j\bar i \bar j}\Bigr) + C\Bigl(\frac{\gamma'(u)T}{-\gamma''(u)}+2\Bigr)\frac{\operatorname{tr}_{\omega_t}(\omega_X)}{(\operatorname{tr}_{\omega_X}(\omega_t))^2} + \frac{-\gamma''(u)}{t_0-\varepsilon}|\partial u|^2_{\omega_t}. \tag{3-8}$$

Set $\alpha := \operatorname{tr}_{\omega_X}(\omega_t)$. We compute

$$\begin{aligned}\dot\alpha &= \operatorname{tr}_{\omega_X}(\chi) - \operatorname{tr}_{\omega_X}\operatorname{Ric}(\omega_t) - \operatorname{tr}_{\omega_X} dd^c(\psi^+ - \psi^-) + \operatorname{tr}_{\omega_X}(\operatorname{Ric}(\omega_X))\\ &\le \operatorname{tr}_{\omega_X}(C_1\omega_X + dd^c\psi^-) - \operatorname{tr}_{\omega_X}\operatorname{Ric}(\omega_t),\end{aligned}$$

where we have used the fact that $\operatorname{tr}_{\omega_X}(\chi)$ is bounded from above, together with the trivial inequality $n \le \operatorname{tr}_{\omega_X}(\omega_t)\operatorname{tr}_{\omega_t}(\omega_X)$. Combining this with (3-6) and (3-8), we infer that

$$\begin{aligned}\frac{\dot\alpha}{\alpha} - \Delta_t\log\alpha &= \frac{\dot\alpha}{\alpha} - \frac{\Delta_t\alpha}{\alpha} + \frac{|\partial\alpha|^2_{\omega_t}}{\alpha^2}\\ &\le \frac{\operatorname{tr}_{\omega_t}(C_1\omega_X + dd^c\psi^-)}{\alpha} + C\Bigl(\frac{\gamma'(u)T}{-\gamma''(u)}+2\Bigr)\frac{\operatorname{tr}_{\omega_t}(\omega_X)}{\alpha^2} + \frac{-\gamma''(u)}{t_0-\varepsilon}|\partial u|^2_{\omega_t}.\end{aligned} \tag{3-9}$$

From this, at the maximum point (t_0, x_0),

$$\begin{aligned}0 \le \Bigl(\frac{\partial}{\partial t} - \Delta_t\Bigr)H &= \log\alpha + (t-\varepsilon)\Bigl(\frac{\dot\alpha}{\alpha} - \Delta_t\log\alpha\Bigr) - \gamma'(u)\dot u + \gamma'(u)\Delta_t u + \gamma''(u)|\partial u|^2_{\omega_t}\\ &\le \log\alpha + \frac{C_3\operatorname{tr}_{\omega_t}(\omega_X + dd^c\psi^-)}{\alpha} + C_4\Bigl(\frac{\gamma'(u)T}{-\gamma''(u)}+2\Bigr)\frac{\operatorname{tr}_{\omega_t}(\omega_X)}{\alpha^2}\\ &\qquad - \gamma'(u)\dot\varphi_t + \gamma'(u)\dot\Psi_t + \gamma'(u)\Delta_{\omega_t}(\varphi_t - \Psi_t - \kappa\psi^-),\end{aligned} \tag{3-10}$$

with $C_3, C_4 > 0$ under control. Moreover, since $\theta_t \ge (1 - at/S)\theta$,

$$\theta_t + dd^c\Psi_t \ge \Bigl(1 - \frac{bt}{S}\Bigr)2\kappa\omega_X.$$

Thus we obtain

$$\Delta_t(\varphi_t - \Psi_t) \le n - \kappa\operatorname{tr}_{\omega_t}(\omega_X). \tag{3-11}$$

Substituting (3-11) into (3-10), we obtain

$$\begin{aligned}0 \le \log\alpha + \frac{C_3\operatorname{tr}_{\omega_t}(\omega_X + dd^c\psi^-)}{\alpha} - \gamma'(u)(n - \kappa\operatorname{tr}_{\omega_t}(\omega_X + dd^c\psi^-)) - \gamma'(u)\dot\varphi_t - \gamma'(u)\frac{b\psi_0}{S}\\ + C_4\Bigl(\frac{\gamma'(u)T}{-\gamma''(u)}+2\Bigr)\frac{\operatorname{tr}_{\omega_t}(\omega_X)}{(\operatorname{tr}_{\omega_X}(\omega_t))^2} + C_5.\end{aligned}$$

We now choose the function γ to obtain a simplified formulation. We set

$$\gamma(u) := \frac{C_3+3}{\min(\kappa,1)}u + \log(u).$$

Since $u \ge 1$ we have

$$\frac{C_3+3}{\min(\kappa,1)} \le \gamma'(u) \le 1 + \frac{C_3+3}{\min(\kappa,1)}, \qquad \frac{\gamma'(u)T}{-\gamma''(u)} + 2 \le C_5u^2.$$

Using $\operatorname{tr}_{\omega_X}(\omega_X + dd^c\psi^-) \le \operatorname{tr}_{\omega_t}(\omega_X + dd^c\psi^-)\operatorname{tr}_{\omega_X}(\omega_t)$ we obtain

$$0 \le \log\alpha - \gamma'(u)\dot{\varphi}_t - \gamma'(u)\frac{b\psi_0}{S} - 3\operatorname{tr}_{\omega_t}(\omega_X) + C_6(u^2+1)\frac{\operatorname{tr}_{\omega_t}(\omega_X)}{\alpha^2}. \tag{3-12}$$

If at the point (t_0, x_0) we have $\alpha^2 \le C_6(u^2+1)$, then

$$H(t_0, x_0) \le T\log\sqrt{C_6(u^2+1)} - \gamma(u) \le C_7,$$

and we are done. Otherwise, we assume that, at (t_0, x_0), $\alpha^2 \ge C_6(u^2+1)$. Applying Lemma 3.6, we obtain

$$\log\alpha = \log\operatorname{tr}_{\omega_X}(\omega_t) \le (n-1)\log\operatorname{tr}_{\omega_t}(\omega_X) + \log n + \dot{\varphi}_t - \psi^-$$

using that $\sup_X \psi^+ = 0$. Plugging this into (3-12), we obtain

$$0 \le C_5 + (n-1)\log\operatorname{tr}_{\omega_t}(\omega_X) - 2\operatorname{tr}_{\omega_t}(\omega_X) - (\gamma'(u)-1)\dot{\varphi}_t - \gamma'(u)\frac{b\psi_0}{S} - \psi^-,$$

or equivalently,

$$\operatorname{tr}_{\omega_t}(\omega_X) \le C_8 - (\gamma'(u)-1)\dot{\varphi}_t - \gamma'(u)\frac{b\psi_0}{S} - \psi^- \tag{3-13}$$

since $(n-1)\log y - 2y \le -y + O(1)$ for $y > 0$. In particular, we have

$$\dot{\varphi}_t \le \frac{C_5}{\gamma'(u)-1} - \frac{\gamma'(u)}{\gamma'(u)-1}\frac{b\psi_0}{S} \le \frac{C_5}{A-1} - \frac{bA\psi_0}{(A-1)S} - \frac{\psi^-}{A-1} \tag{3-14}$$

at (t_0, x_0), since $\operatorname{tr}_{\omega_t}(\omega_X) \ge 0$ and $A \le \gamma'(u) \le A+1$ with $A =: (C_3+3)/\min(\kappa, 1)$. It follows from Lemma 3.6 that

$$\operatorname{tr}_{\omega_t}(\omega_X) \ge n\exp\left(\frac{-\dot{\varphi}_t + \psi^-}{n}\right).$$

Plugging this into (3-13), we obtain

$$\operatorname{tr}_{\omega_t}(\omega_X) \le C_9 - \gamma'(u)\frac{b\psi_0}{S} - \gamma'(u)\psi^- \le C_9 - \frac{(A+1)b\psi_0}{S} - (A+1)\psi^-$$

with $C_9 > 0$ under control; since $e^y - Dy \ge -C$ for $y \in \mathbb{R}$, $D > 0$, we apply with $y = (-\dot{\varphi}_t + \psi^-)/n$. Again Lemma 3.6 yields

$$\log\alpha \le (n-1)\log\left(C_9 - \frac{b(A+1)\psi_0}{S} - (A+1)\psi^-\right) + \log n + \dot{\varphi}_t - \psi^-.$$

Combining this together with (3-14), we have at (t_0, x_0)

$$\begin{aligned} H \le C_{10} - A\left[\varphi_t - \left(1 - \frac{bt}{S} - \frac{b(t-\varepsilon)}{(A-1)S}\right)\psi_0\right] + \left(A\kappa - 1 - \frac{1}{A-1}\right)\psi^- \\ + (t-\varepsilon)(n-1)\log\left(C_9 - \frac{b(A+1)\psi_0}{S} - (A+1)\psi^-\right). \end{aligned}$$

Up to increasing $A > 0$ if necessary, so that

$$\eta := \frac{b\varepsilon}{T} - \frac{b\varepsilon}{S} - \frac{bT}{(A-1)S} > 0,$$

and since $\psi_0 \le 0$, we obtain, at (t_0, x_0),

$$H \le C_{10} - A\left[\varphi_t - \left(1 - \frac{bt}{T}\right)\psi_0\right] + A\eta\psi_0 + A\kappa/2\psi^- \\ + (t-\varepsilon)(n-1)\log\left(C_9 - \frac{b(A+1)\psi_0}{S} - (A+1)\psi^-\right).$$

The second term is uniformly bounded from above by Theorem 3.4. Since $-\gamma y + \log y$ is bounded from above for $y > 0$, we conclude that H achieves a uniform bound at (t_0, x_0). This completes the proof. □

3.4. ***Estimates near the zero time.*** Recall that there exists a θ-psh function ρ with analytic singularities such that $\sup_X \rho = 0$ and

$$\theta + dd^c\rho \ge 3\delta_0\omega_X$$

for some $\delta_0 > 0$. The main result of Tosatti and Weinkove [2010] ensures that there exists a constant c_1 and $\phi_1 \in \mathrm{PSH}(X, \theta) \cap \mathcal{C}^\infty(X)$ such that

$$(\theta + dd^c\phi_1)^n = e^{c_1} d\mu, \quad \sup_X \phi_1 = 0.$$

Proposition 3.9. *Assume that ψ_1, ψ_2 are two smooth ω_X-psh functions satisfying*

$$\dot\varphi_0 \ge C_1\psi_1, \quad \varphi_0 \ge \tfrac{1}{2}(\rho + \delta_0\psi_2)$$

for some constants $C_1 > 0$. Fix $T_1 \in (0, T_{\max})$ such that $\theta_t > \frac{1}{2}\theta$ for all $t \in [0, T_1]$. Then there exists a uniform constant $C_2 > 0$ only depending on C_1, δ_0, T_1 and $\sup_X |\phi_1|$ such that

$$\dot\varphi_t \ge C_2(\rho + \delta_0\psi_2 + 1) + C_1\psi_1 \quad \textit{for all } t \in [0, T_1].$$

Proof. The proof is identical to that of Proposition 3.7. We consider

$$H(t,x) := \dot\varphi_t - C_1\psi_1 + A\big(\varphi_t - \tfrac{1}{2}(\rho + \delta_0\psi_2)\big) - \phi_1$$

for $A > 0$ to be chosen later. We observe that H achieves its minimum at some point $(t_0, x_0) \in [0, T_1] \times X$. If $t_0 = 0$, we are done by assumptions. Otherwise, by the minimum principle, we have at (t_0, x_0),

$$0 \ge \left(\frac{\partial}{\partial t} - \Delta_t\right)H \ge -An + A\dot\varphi_t + (-C_1 + A\delta_0)\,\mathrm{tr}_{\omega_t}(\omega_X) + \mathrm{tr}_{\omega_t}(dd^c\phi_1)$$

using $\theta_t + dd^c\frac{1}{2}(\rho + \delta_0\psi_2) \ge \delta_0\omega_X$. Now, we choose $A = \delta_0(C_1 + 1)$, thus

$$\mathrm{tr}_{\omega_t}(\omega_X + dd^c\phi_1) \ge n\left(\frac{(\theta + dd^c\phi_1)^n}{\omega_t^n}\right)^{1/n} = ne^{(-\dot\varphi_t + c_1)/n}$$

using Lemma 3.6. Together with the inequality $e^y \ge By - C_B$, we obtain a uniform lower bound for $\dot\varphi_t$ at (t_0, x_0). On the other hand, by Proposition 3.2 we see that $\varphi_t \ge \varphi_0 - c(t)$, so

$$\varphi_t \ge \tfrac{1}{2}(\rho + \delta_0\psi_2) - c(t),$$

where $c(t) \to 0$ as $t \to 0$. The lower bound for $H(t_0, x_0)$ thus follows, finishing the proof. □

Proposition 3.10. *Assume that ψ_1, ψ_2 are two smooth ω_X-psh functions satisfying*

$$\Delta_{\omega_X}\varphi_0 \le e^{-C_1\psi_1}, \quad \varphi_0 \ge \tfrac{1}{2}(\rho+\delta_0\psi_2)$$

for some constants $C_1>0$. Fix $T_1\in(0,T_{\max})$ such that $\theta_t>\frac{1}{2}\theta$ for all $t\in[0,T_1]$. Then there exist uniform constants $C_2>0$, $C_3>0$ only depending on C_1, δ_0 and T_1 such that

$$\operatorname{tr}_{\omega_X}(\omega_t)\le C_3 e^{-C_1\psi_1-C_2(\rho+\delta_0\psi_2+\delta_0\psi^-)} \quad \textit{for all } t\in[0,T_1].$$

Proof. Consider the function

$$H(t,\cdot)=\log\operatorname{tr}_{\omega_X}(\omega_t)+C_1\psi_1-\gamma(u),$$

where $\gamma:\mathbb{R}\to\mathbb{R}$ is a smooth concave increasing function such that $\lim_{t\to+\infty}\gamma(t)=+\infty$ and

$$u(t,x):=\varphi_t(x)-\tfrac{1}{2}(\rho(x)+\delta_0\psi_2(x))+\delta_0\psi^-(x)+1.$$

We suppose that H achieves its maximum at a point $(t_0,x_0)\in[0,T_1]\times X$, with $x_0\in\{\rho>-\infty\}$. If $t_0=0$, then $H(0,\cdot)\le\log n-\gamma(1)$. Otherwise, assume $t_0>0$. We proceed by computing at this point. By the maximum principle and the arguments in Theorem 3.8, we have

$$\begin{aligned}0\le\Big(\frac{\partial}{\partial t}-\Delta_t\Big)H\le \frac{C\operatorname{tr}_{\omega_t}(\omega_X+dd^c\psi^-)}{\operatorname{tr}_{\omega_X}(\omega_t)}&-\gamma'(u)(n-\delta_0\operatorname{tr}_{\omega_t}(\omega_X+dd^c\psi^-))+C\\ &-C_1\operatorname{tr}_{\omega_t}(dd^c\psi_1)-\gamma'(u)\dot\varphi_t+C\bigg(\frac{\gamma'(u)}{-\gamma''(u)}+2\bigg)\frac{\operatorname{tr}_{\omega_t}(\omega_X)}{(\operatorname{tr}_{\omega_X}(\omega_t))^2}.\end{aligned}\tag{3-15}$$

Here, we use $\theta_t+dd^c\frac{1}{2}(\rho+\delta_0\psi_2)\ge\delta_0\omega_X$. We set

$$\gamma(u):=\frac{C+C_1+3}{\min(\kappa,1)}u+\ln(u).$$

We then proceed in the same way as in the proof of Theorem 3.8 to obtain the uniform upper bound for $H(t_0,x_0)$. This finishes the proof. ☐

4. Degenerate Monge–Ampère flows

4.1. ***Proof of Theorem B.*** By Demailly's regularization theorem (Theorem 2.10), we can find two sequences $\psi_j^\pm\in\mathcal{C}^\infty(X)$ such that

- $\psi_j^\pm$ decreases pointwise to $\psi^\pm$ on X and the convergence is in $\mathcal{C}^\infty_{\mathrm{loc}}(U)$;
- $dd^c\psi^\pm\ge-\omega_X$.

We note that $|\sup_X\psi_j^\pm|$ is uniformly bounded, and for all j,

$$\|e^{-\psi_j^-}\|_{L^p}\le\|e^{-\psi^-}\|_{L^p}.$$

Thanks to Demailly's regularization theorem again, we can find a smooth sequence $(\varphi_{0,j})$ of strictly $(\theta+2^{-j}\omega_X)$-psh functions decreasing towards φ_0. We set $\theta_{t,j}=\theta_t+2^{-j}\omega_X$ and $\mu_j=e^{\psi_j^+-\psi_j^-}$. It follows from [Tosatti and Weinkove 2015, Theorem 1.2] (see also [Tô 2018]) that there exists a unique function $\varphi_j \in \mathcal{C}^\infty([0,T)\times X)$ such that

$$\begin{cases} \dfrac{\partial \varphi_{t,j}}{\partial t} = \log\left[\dfrac{(\theta_{t,j}+dd^c\varphi_{t,j})^n}{\mu_j}\right], \\ \varphi_j|_{t=0} = \varphi_{0,j}. \end{cases} \tag{4-1}$$

It follows from the maximum principle that the sequence $\varphi_{t,j}$ decreases with respect to j. Moreover, Proposition 3.1 ensures that $\sup_X \varphi_{t,j}$ is uniformly bounded from above. By Proposition 3.2, as $j \to +\infty$, the family $\varphi_{t,j}$ decreases to φ_t, which is a well-defined θ_t-psh function on X. Following the same arguments as in [Tô 2018, Section 4.1], we conclude that $\varphi_t \to \varphi_0$ in $L^1(X)$ as $t \to 0^+$.

Next, we study the partial regularity of φ_t for small t. We fix $\varepsilon_0 > 0$ and $\varepsilon > p^*\varepsilon_0$. Let T and S be as defined in Section 3.1. Let ρ be a θ-psh function with analytic singularities along D such that $\theta + dd^c\rho$ dominates a Hermitian form, where $D := \{\rho = -\infty\}$. By Lemma 2.11, there is a function $\psi_0 \in \mathrm{PSH}(X,\theta)\cap\mathcal{C}^\infty\big(X\setminus(D\cup E_c(\varphi_0))\big)$, where $c = c(\varepsilon_0) > 0$, such that

$$\int_X e^{2(\psi_0-\varphi_0)/\varepsilon_0}\,dV_X < +\infty.$$

We assume without loss of generality that $\psi_0 \le 0$. Since $\dfrac{p^*}{2c(\varphi_0)} < T$ and ψ_0 is less singular than φ_0, we also have

$$\int_X e^{-p^*\psi_0/T}\,dV_X < +\infty.$$

We note that since φ_0 is a decreasing limit of a smooth sequence $\varphi_{0,j}$, the corresponding constants for $\varphi_{0,j}$ are uniformly bounded (in j), and we can pass to the limit as $j \to +\infty$.

Recall that $\psi^\pm$ are smooth (merely locally bounded) in a Zariski open set $U \subset X\setminus D$. We will show that φ_t is smooth on $U \setminus E_c(\varphi_0)$ for each $t > \varepsilon$. Let K be an arbitrarily compact subset of $U \setminus E_c(\varphi_0)$. It follows from Proposition 3.1, Theorem 3.4, and the remark above that

$$\sup_{[\varepsilon,T]\times K} |\varphi_j| \le C(\varepsilon, T, K).$$

Next, Proposition 3.7 yields

$$\sup_{[\varepsilon,T]\times K} |\dot{\varphi}_j| \le C(\varepsilon, T, K).$$

Moreover, by Theorem 3.8, we also obtain a uniform bound for $\Delta\varphi_t^j$:

$$\sup_{[\varepsilon,T]\times K} |\Delta\varphi_j| \le C(\varepsilon, T, K).$$

Using the complex parabolic Evans–Krylov–Trudinger theory, together with parabolic Schauder estimates (see, e.g., [Boucksom and Guedj 2013, Theorem 4.1.4]), we derive higher-order estimates for φ_j on $[\varepsilon, T]\times K$:

$$\|\varphi_j\|_{\mathcal{C}^k([\varepsilon,T]\times K)} \le C(\varepsilon, T, K, k).$$

This ensures that φ_j is relatively compact in $\mathcal{C}^\infty([\varepsilon, T] \times (U \setminus E_c(\varphi_0))$ since K was taken arbitrarily. By passing to the limit in (4-1), we deduce that φ satisfies (1-4) in the classical sense on $[\varepsilon, T] \times \Omega_\varepsilon$ with $\Omega_\varepsilon = U \setminus E_{c(\varepsilon)}(\varphi_0)$.

4.2. ***Uniqueness.*** We now follow the argument in [Guedj and Zeriahi 2017b] to prove that the solution φ to equation (1-4) constructed in the previous part is the unique maximal solution in the following sense:

Proposition 4.1. *Let ψ_t be a weak solution to equation* (1-4) *with initial data φ_0. Then $\psi_t \le \varphi_t$ for all $t \in (0, T_{\max})$.*

Proof. By construction in the previous paragraph, $\varphi_{t,j}$ are smooth $(\theta_t + 2^{-j}\omega_X)$-psh functions decreasing pointwise to φ_t. It thus suffices to show that $\psi_t \le \varphi_{t,j}$ for all fixed j.

Fix $0 < T < T_{\max}$ and $2^{-j} > \varepsilon > \delta > 0$. We let $U_\varepsilon \subset X$ denote the Zariski open set in which $\psi_{t+\varepsilon}$ is smooth. We can find a ω_X-psh function ϕ with analytic singularities along $X \setminus U_\varepsilon$; see, e.g., [Demailly and Paun 2004]. We apply the maximum principle to the function $H := \psi_{t+\varepsilon} - \varphi_{t+\varepsilon,j} + \delta\phi$. Suppose that H achieves its maximum on $[0, T-\varepsilon] \times X$ at $(t_\varepsilon, x_\varepsilon)$ with $t_\varepsilon > 0$. Note that $x_\varepsilon \in U_\varepsilon$. We thus have

$$0 \le \frac{\partial}{\partial t} H \le \log\left[\frac{(\theta_{t+\varepsilon} + dd^c\varphi_{t+\varepsilon,j} - \delta dd^c\phi)^n}{(\theta_{t+\varepsilon} + 2^{-j}\omega_X + dd^c\varphi_{t+\varepsilon,j})^n}\right] < 0$$

using that $-dd^c\phi \le \omega_X$, which is a contradiction. Letting $\delta \searrow 0$, we obtain

$$\psi_{t+\varepsilon}(x) - \varphi_{t+\varepsilon,j}(x) \le \sup_X(\psi_\varepsilon - \varphi_{\varepsilon,j}).$$

Moreover, since $(\varepsilon, x) \mapsto \varphi_{\varepsilon,j}(x)$ is continuous, it follows from Hartogs' lemma (see [Guedj and Zeriahi 2017a, Proposition 8.4]) that

$$\sup_X(\psi_\varepsilon - \varphi_{\varepsilon,j}) \xrightarrow{\varepsilon \to 0} \sup_X(\varphi_0 - \varphi_{0,j}) \le 0.$$

Letting $\varepsilon \to 0$, the desired inequality follows. □

The uniqueness we have just shown is referred to as "maximally stretched" by P. Topping [2010, Remark 1.9] in the context of Riemann surfaces.

4.3. ***Short time behavior.*** In this subsection, we study the behavior of the solution to the degenerate Monge–Ampère flow in a short time. We show that the flow φ_t starting from a current with positive Lelong numbers also has positive Lelong numbers for a sufficiently short time. This result follows almost verbatim from the Kähler case, as discussed in [Di Nezza and Lu 2017, Section 4.2].

Theorem 4.2. *If φ_0 has positive Lelong numbers, then*

$$E_c(\varphi_0) \subset E_{c(t)}(\varphi_t), \quad c(t) = c - 2nt.$$

In particular, the maximal solution φ_t has positive Lelong numbers for any $t < \dfrac{1}{2nc(\varphi_0)}$.

Proof. The proof is identical to that of [Di Nezza and Lu 2017, Theorem 4.5]. We give a sketch of the proof here. Fix $x_0 \in E_c(\varphi_0)$. We can find a cutoff function $\chi \in \mathcal{C}^\infty(X)$ with support near x_0 and $\chi = 1$

on a neighborhood of x_0. Define $\phi := \chi(x)c \log \|x - x_0\|$, which is $B\omega_X$-psh, and $e^{2\phi/c} \in \mathcal{C}^\infty(X)$. Since $x_0 \in E_c(\varphi_0)$ we can choose ϕ so that $\phi \geq \varphi_0$ by adding a positive constant. By Lemma 4.3, we obtain

$$\varphi_t \leq (1 - 2nt/c)\phi + Ct,$$

which implies $\nu(\varphi_t, x_0) \geq c - 2nt$. If $t < 1/(2nc(\varphi_0))$, then by Skoda's integrability theorem, $e^{-2\varphi_0/c}$ is not integrable for $2nt < c < 1/c(\varphi_0)$. Therefore, $E_c(\varphi_0)$ is not empty, neither is $E_{c(t)}(\varphi_t)$ for sufficiently small $t > 0$. □

Lemma 4.3. *Assume that $\phi \in \mathrm{PSH}(X, \omega_X)$ satisfies $e^{\gamma\phi} \in \mathcal{C}^\infty(X)$ for some constant $\gamma > 0$, and $0 \geq \psi^\pm \geq \phi \geq \varphi_0$. Then, there exists a positive constant C depending on an upper bound for $dd^c e^{\gamma\phi}$ such that*

$$\varphi(t) \leq (1 - (n\gamma + 1)t)\phi + Ct \quad \textit{for all } t \in [0, 1/n\gamma].$$

Proof. Assume that $\theta_t \leq \omega_X$ for $t \in [0, 1/(n\gamma + 1)]$. As argued in [Di Nezza and Lu 2017, Lemma 4.4], we can assume that ϕ is smooth and work with the approximants $\varphi_{t,j}$ instead. We choose $C > 0$ depending only on an upper bound for $dd^c e^{\gamma\phi}$, such that $dd^c\phi \leq Ce^{-\gamma\phi}\omega_X$. Consider the function

$$\phi_t := (1 - (n\gamma + 1)t)\phi + t \log(2^n C^n).$$

We observe that

$$0 \leq \omega_X + dd^c\phi \leq 2Ce^{-\gamma\phi}\omega_X,$$

and hence

$$(\omega_X + dd^c\phi_t)^n \leq (2C)^n e^{-n\gamma\phi}\omega_X^n \leq e^{\dot{\phi}_t + \psi^+ - \psi^-}\omega_X^n.$$

Therefore, ϕ_t is a supersolution to the parabolic equation

$$(\omega_X + dd^c u_t)^n = e^{\dot{u}_t + \psi^+ - \psi^-}\omega_X^n,$$

while $\varphi_{t,j}$ is a subsolution. By the classical maximum principle, it follows that $\varphi_{t,j} \leq \phi_t$ for any fixed j. This completes the proof. □

4.4. *Convergence at time zero.* We study in this part the convergence at zero of the degenerate complex Monge–Ampère flow.

We recall the quasimonotone convergence in the sense of Guedj and Trusiani [2023]: φ_j converges quasimonotonically to φ if $P_\theta(\inf_{\ell \geq j} \varphi_\ell)$ is a sequence of θ-psh functions that increases to φ.

Theorem 4.4. *The flow φ_t converges quasimonotonically to φ_0 as $t \to 0^+$.*

Proof. By Proposition 3.2, we have that for small $t > 0$,

$$\varphi_t \geq \varphi_0 - C(t - t \log t).$$

It follows that

$$P_\theta\Big(\inf_{0<s\leq t} \varphi_s\Big) \geq \varphi_0 - C(t - t \log t),$$

which completes the proof. □

Theorem 4.5. *Assume that φ_0 is continuous in an open set $U \subset X$. Then φ_t converges to φ_0 in $L^\infty_{\mathrm{loc}}(U)$.*

Proof. The proof closely follows the arguments in the Kähler case [Di Nezza and Lu 2017]. Without loss of generality, we assume that $\varphi_t \le 0$. By Proposition 3.2, there exists a uniform constant $C > 0$ and a small time t_0 such that for $0 \le s < t \le t_0$,

$$\varphi_s - C(t-s)\log(t-s) - C(t-s) \le \varphi_t.$$

Set $u_t := \varphi_t + (C+\log 4)t - Ct\log t$. Substituting $s = t/2$, we deduce that $u_t \ge u_{t/2}$, hence the sequence $u_{t_0 2^{-j}}$ decreases to $u_0 = \varphi_0$. The conclusion follows from Dini's theorem. □

We also have the same result as in the Kähler case [Di Nezza and Lu 2017, Theorem 6.3]. We assume that θ is a big form and that $f = e^{\psi^+ - \psi^-} \in L^p$, for some $p > 1$, where $\psi^\pm$ are quasi-psh functions. Assume moreover that $\psi^- \in L^\infty_{\mathrm{loc}}(X \setminus D)$ for some closed set $D \subset X$. It follows from [Guedj and Lu 2023, Theorem 4.1] that there exists a bounded θ-psh function φ_0 such that $\sup_X \varphi_0 = 0$ and

$$(\theta + dd^c \varphi_0)^n = cf\, dV.$$

We recall that there is $\rho \in \mathrm{PSH}(X, \theta)$ with analytic singularities along a closed subset E such that $\theta + dd^c\rho \ge 2\delta\omega_X$ for some $\delta > 0$. Set $U := X \setminus (D \cup E)$.

Theorem 4.6. *Assume φ_0 is as above. Let φ_t be the weak solution of equation* (1-4) *with the initial data φ_0. Then φ_t converges to φ_0 in $\mathcal{C}^\infty_{\mathrm{loc}}(U)$.*

Proof. The proof is quite close to [Di Nezza and Lu 2017, Theorem 6.3]. We sketch the key steps for the reader's convenience. First, we approximate $\psi^\pm$ by their smooth approximants $\psi^\pm_j$, thanks to [Demailly 1992]. We then apply the Tosatti–Weinkove theorem [2010] to obtain smooth $(\theta + 2^{-j}\omega_X)$-psh functions $\varphi_{0,j}$ such that $\sup_X \varphi_{0,j} = 0$ and

$$(\theta + 2^{-j}\omega_X + dd^c\varphi_{0,j})^n = c_j e^{\psi_j^+ - \psi_j^-}\, dV.$$

Note here that the $f_j = e^{\psi_j^+ - \psi_j^-}$ have uniform L^p-norms. The same arguments as in [Guedj and Lu 2023, Theorem 4.2] show that

- $c_j \to c > 0$;
- for any $\varepsilon > 0$, $\varphi_{0,j} \ge \varepsilon(\rho + \delta\psi^-) - C(\varepsilon)$;
- $\Delta_{\omega_X}\varphi_{0,j} \le e^{-C(\varepsilon)(\rho+\delta\psi^-)}$.

Let $\varphi_{t,j}$ be a smooth solution to equation (1-4) with initial data $\varphi_{0,j}$. The sequence $\varphi_{t,j}$ converges to the unique weak solution φ_t. We apply Propositions 3.9 and 3.10, together with bootstrapping arguments, to obtain locally uniform estimates for all derivatives of $\varphi_{t,j}$. This leads to convergence in $\mathcal{C}^\infty_{\mathrm{loc}}(U)$. □

5. Finite time singularities

In this section, we study the finite time singularities of the Chern–Ricci flow and provide a proof of Theorem A.

We consider a family of Hermitian metrics $\omega(t)$ evolving under the Chern–Ricci flow (1-1) with the initial Hermitian metric ω_0. Suppose that the maximal existence time of the flow is finite, i.e., $T_{\max} < \infty$.

The form $\alpha_{T_{\max}} := \omega_0 - T_{\max}\operatorname{Ric}(\omega_0)$ is nef in the sense of [Guedj and Lu 2022]; i.e., for each $\varepsilon > 0$ there exists $\psi_\varepsilon \in \mathcal{C}^\infty(X)$ such that $\alpha_{T_{\max}} + dd^c\psi_\varepsilon \geq -\varepsilon\omega_0$. Indeed, for $\varepsilon > 0$,

$$\alpha_{T_{\max}} + \varepsilon\omega_0 = (1+\varepsilon)\left(\omega_0 - \frac{T_{\max}}{1+\varepsilon}\operatorname{Ric}(\omega_0)\right),$$

and since $T_{\max}/(1+\varepsilon) < T_{\max}$, we have $\omega_0 - T_{\max}/(1+\varepsilon)\operatorname{Ric}(\omega_0) + dd^c\psi > 0$ for some smooth function ψ. We assume that $\alpha_{T_{\max}}$ is *uniformly noncollapsing*, i.e.,

$$\int_X (\alpha_{T_{\max}} + dd^c\psi)^n \geq c_0 > 0 \quad \text{for all } \psi \in \operatorname{PSH}(X, \alpha_{T_{\max}}) \cap \mathcal{C}^\infty(X). \tag{5-1}$$

This condition implies that the volume of $(X, \omega(t))$ does not collapse to zero as $t \to T_{\max}^-$.

Theorem 5.1. *Let α be a nef* (1,1)*-form satisfying the uniformly noncollapsing condition* (5-1). *If X admits a Hermitian metric ω_X such that $v_+(\omega_X) < +\infty$ then α is big.*

Conversely, if α is big and $v_-(\omega_X) > 0$ then α is uniformly noncollapsing.

When α is semipositive or closed the result was proved by Guedj and Lu [2022, Theorems 4.6, 4.9], answering the transcendental Grauert–Riemenschneider conjecture [Demailly and Paun 2004, Conjecture 0.8]. For our purposes, we would like to extend it when α is no longer closed.

Proof. The proof of this theorem follows the same lines as in [Guedj and Lu 2022, Theorem 4.6], which is based on ideas from Chiose [2016], so we omit it here. □

Remark 5.2. When ω_0 is closed, or more generally, is a Guan–Li metric, i.e., $dd^c\omega_0 = dd^c\omega_0^2 = 0$, the condition (5-1) is simply written as $\int_X \alpha_{T_{\max}}^n > 0$. The assumption $v_+(\omega_X) < \infty$ or $v_-(\omega_X) > 0$ is independent of the choice of the Hermitian ω_X, as shown in [Guedj and Lu 2022, Proposition 3.2]. For additional examples of manifolds where such conditions hold, we refer the reader to [Angella et al. 2023].

This result is a slight generalization of [Nguyen 2016, Theorem 4.3], where α is closed semipositive, and X admits a pluriclosed metric, i.e., $dd^c\omega_X = 0$.

As a consequence of Theorem 5.1, we give a slight improvement of the main result in [Tosatti and Weinkove 2012] (see also [Nguyen 2016, Theorem 4.1]) which extends the one of Demailly [1993] to the non-Kähler setting.

Theorem 5.3. *Let X be a compact complex n-manifold equipped with a Hermitian metric ω_X satisfying $v_+(\omega_X) < \infty$. Let α be a nef* (1,1)*-form. Assume that $x_1, \dots, x_N \in X$ are fixed points and $\tau_1, \dots, \tau_N$ are positive constants such that*

$$0 < \sum_{j=1}^N \tau_j^n < \int_X (\alpha + dd^c\psi)^n \quad \textit{for all } \psi \in \operatorname{PSH}(X, \alpha) \cap \mathcal{C}^\infty(X).$$

Then, there exists an α-psh function φ with logarithmic poles at $x_1, \dots, x_N \in X$,

$$\varphi(z - x_j) \leq \tau_j \log\|z - x_j\| + O(1)$$

in local coordinates near x_j, for all $j = 1, \dots, N$.

Proof. By Theorem 5.1, we know that α is big. The rest of the proof follows in the same manner as in [Tosatti 2016, Theorem 1.3]. □

We go back to the Chern–Ricci flow. Observe that one can deduce the Chern–Ricci flow (1-1) to a parabolic complex Monge–Ampère equation

$$\frac{\partial \varphi_t}{\partial t} = \log\left[\frac{(\alpha_t + dd^c\varphi_t)^n}{\omega_0^n}\right], \quad \alpha_t + dd^c\varphi > 0, \quad \varphi(0) = 0,$$

where $\alpha_t := \omega_0 - t\,\mathrm{Ric}(\omega_0)$. We assume that the form $\alpha_{T_{\max}}$ is uniformly noncollapsing. By Theorem 5.1, there exists a function ρ with analytic singularities such that

$$\alpha_{T_{\max}} + dd^c\rho \geq 2\delta_0\omega_0$$

for some $\delta_0 > 0$. We observe that

$$\alpha_t + dd^c\rho = \frac{1}{T_{\max}}\big((T_{\max} - t)(\omega_0 + dd^c\rho) + t(\alpha_{T_{\max}} + dd^c\rho)\big) \geq \delta_0\omega_0 \tag{5-2}$$

for $t \in [T_{\max} - \varepsilon, T_{\max}]$ with sufficiently small $\varepsilon > 0$. Set

$$\Omega := X \setminus \{\rho = -\infty\}.$$

We establish uniform $\mathcal{C}^\infty_{\mathrm{loc}}$ estimates on Ω.

Lemma 5.4. *There is a uniform constant* $C_0 > 0$ *such that on* $[0, T_{\max}) \times X$ *we have*

(i) $\varphi \leq C_0$;

(ii) $\dot\varphi \leq C_0$;

(iii) $\varphi \geq \rho - C_0$;

(iv) $\dot\varphi \geq C_0\rho - C_0$.

Proof. The proofs of (i) and (ii) follow directly from the classical maximum principle; see, e.g., [Tosatti and Weinkove 2015, Lemma 4.1] or [Tian and Zhang 2006].

For (iii), we set $\psi := \varphi - \rho$. Note that the function $\psi + At \geq -C$ holds on $[0, T_{\max} - \varepsilon]$ with ε as above. Fix $T_{\max} - \varepsilon < T' < T_{\max}$, and assume that $\psi + At$ achieves its minimum at $(t_0, x_0) \in [0, T'] \times X$ with $t_0 \in (T_{\max} - \varepsilon, T']$. Note that $x_0 \in \Omega$. We compute at (t_0, x_0),

$$0 \geq \frac{\partial\psi}{\partial t} + A = \log\frac{(\alpha_t + dd^c\rho + dd^c\psi)^n}{\omega_0^n} + A \geq \log\frac{(\delta_0\omega_0)^n}{\omega_0^n} + A \geq -C + A,$$

where we have used the estimate (5-2). If we choose $A > C$, then we get a contradiction. Thus, we obtain the lower bound for ψ, completing the proof.

For (iv), we apply the minimum principle to

$$Q = \dot\varphi + A\psi + Bt,$$

where A and B are large constants that will be chosen later. Our goal is to show that $Q \geq -C$ on $X \times [0, T_{\max})$. As above, we observe that $Q \geq -C$ on $[0, T_{\max} - \varepsilon] \times X$. Given any $T_{\max} - \varepsilon < T' < T_{\max}$, suppose that Q achieves its minimum on $[0, T'] \times X$ at some point (t_0, x_0) with $t_0 \in (T_{\max} - \varepsilon, T']$. Note that $x_0 \in \Omega$. At this point, we have

$$0 \geq \Big(\frac{\partial}{\partial t} - \Delta_\omega\Big) Q = -\mathrm{tr}_\omega \mathrm{Ric}(\omega_0) + A\dot{\varphi} - An + A\,\mathrm{tr}_\omega(\alpha_t + dd^c\rho) + B$$
$$\geq \delta_0 \,\mathrm{tr}_\omega\, \omega_0 + A \log \frac{\omega^n}{\omega_0^n} + \mathrm{tr}_\omega\, \omega_0 - An + B,$$

where A is chosen so large that

$$(A-1)(\alpha_t + dd^c\rho) - \mathrm{Ric}(\omega_0) \geq \omega_0$$

for $t \in [T_{\max} - \varepsilon, T_{\max}]$. But since $A \log y - \delta_0 y^{1/n}$ is bounded from above for $y > 0$, the arithmetic-geometric inequality yields

$$\delta_0 \,\mathrm{tr}_\omega\, \omega_0 + A \log \frac{\omega^n}{\omega_0^n} \geq \delta_0 \bigg(\frac{\omega_0^n}{\omega^n}\bigg)^{1/n} + A \log \frac{\omega^n}{\omega_0^n} \geq -C_1$$

for a uniform constant $C_1 > 0$. If we choose $B = C_1 + An$, we obtain

$$0 \geq \Big(\frac{\partial}{\partial t} - \Delta_\omega\Big) Q \geq \mathrm{tr}_\omega\, \omega_0 > 0$$

which leads to a contradiction. Thus, the desired estimate follows. □

Lemma 5.5. *There is a uniform constant $C > 0$ such that on* $[0, T_{\max}) \times X$ *we have*

$$\mathrm{tr}_{\omega_0}\, \omega(t) \leq C e^{-C\rho}.$$

Proof. Set $\psi = \varphi - \rho + C_0 \geq 0$. We apply the maximum principle to

$$Q = \log \mathrm{tr}_{\omega_0}\, \omega - A\psi + e^{-\psi},$$

where $A > 0$ will be determined later. The idea of using the last term in Q is due to Phong and Sturm [2010] and was used in the context of the Chern–Ricci flow in [Tosatti and Weinkove 2013; 2015; Tô 2018]. Note that $e^{-\psi} \in (0, 1]$.

It suffices to show that Q is uniformly bounded from above. Again, it follows from the definition of Q that $Q \leq C$ on $[0, T_{\max} - \varepsilon] \times X$ for a uniform $C > 0$. Fix $T_{\max} - \varepsilon < T' < T_{\max}$, and suppose that Q achieves its maximum at some point $(t_0, x_0) \in [0, T'] \times X$ with $t \in (T_{\max} - \varepsilon, T']$. In what follows, we compute at this point. From [Tosatti and Weinkove 2015, Proposition 3.1, also (4.2)] we have

$$\Big(\frac{\partial}{\partial t} - \Delta_\omega\Big) \log \mathrm{tr}_{\omega_0}\, \omega \leq \frac{2}{(\mathrm{tr}_{\omega_0}\, \omega)^2} \mathrm{Re}(g^{\bar{q}k} (T_0)_{kp}^p \partial_{\bar{q}}\, \mathrm{tr}_{\omega_0}\, \omega) + C\,\mathrm{tr}_\omega\, \omega_0,$$

where $(T_0)_{kp}^p$ denotes the torsion terms corresponding to ω_0. At the maximum point (x_0, t_0) of Q, we have $\partial_i Q = 0$; hence

$$\frac{1}{\mathrm{tr}_{\omega_0}\, \omega} \partial_i\, \mathrm{tr}_{\omega_0}\, \omega - A\partial_i \psi - e^{-\psi} \partial_i \psi = 0.$$

Thus, the Cauchy–Schwarz inequality yields

$$\begin{aligned}\left|\frac{2}{(\operatorname{tr}_{\omega_0}\omega)^2}\operatorname{Re}(g^{\bar{q}k}(T_0)^p_{kp}\partial_{\bar{q}}\operatorname{tr}_{\omega_0}\omega)\right| &\le \left|\frac{2}{(\operatorname{tr}_{\omega_0}\omega)^2}\operatorname{Re}\big((A+e^{-\psi})g^{\bar{q}k}(T_0)^p_{kp}\partial_{\bar{q}}\psi\big)\right| \\ &\le e^{-\psi}|\partial\psi|^2_\omega + C(A+1)^2e^{\psi}\frac{\operatorname{tr}_\omega\omega_0}{(\operatorname{tr}_{\omega_0}\omega)^2}\end{aligned}$$

for uniform $C>0$ only depending on the torsion term. It thus follows that, at the point (t_0, x_0),

$$\begin{aligned}0\le\Big(\frac{\partial}{\partial t}-\Delta_\omega\Big)Q &\le C(A+1)^2e^{\psi}\frac{\operatorname{tr}_\omega\omega_0}{(\operatorname{tr}_{\omega_0}\omega)^2}+C\operatorname{tr}_\omega\omega_0-(A+e^{-\psi})\dot{\varphi}+(A+e^{-\psi})\operatorname{tr}_\omega(\omega-(\alpha_t+dd^c\rho)) \\ &\le C(A+1)^2e^{\psi}\frac{\operatorname{tr}_\omega\omega_0}{(\operatorname{tr}_{\omega_0}\omega)^2}+(C-A\delta_0)\operatorname{tr}_\omega\omega_0+(A+1)\log\frac{\omega_0^n}{\omega^n},\end{aligned}\tag{5-3}$$

where we have used $\alpha_t+dd^c\rho\ge\delta_0\omega_0$. If at (x_0,t_0), $(\operatorname{tr}_{\omega_0}\omega)^2\le e^{\psi}C(A+1)^2$ then at the same point we obtain

$$Q\le C+\tfrac{1}{2}\psi-A\psi+e^{-\psi}\le C+1.$$

Since $\psi\ge 0$, we are done. Otherwise, we choose $A=\delta_0^{-1}(C+2)$. Hence, from (5-3) one gets

$$\operatorname{tr}_\omega\omega_0\le C\log\frac{\omega_0^n}{\omega^n}+C.$$

By Lemma 3.6, we obtain

$$\operatorname{tr}_{\omega_0}\omega\le n(\operatorname{tr}_\omega\omega_0)^{n-1}\frac{\omega^n}{\omega_0^n}\le C\frac{\omega^n}{\omega_0^n}\bigg(\log\frac{\omega_0^n}{\omega^n}\bigg)^{n-1}+C\le C'$$

since $\omega^n/\omega_0^n\le C_0$ by Lemma 5.4, and $y\mapsto y|\log y|^{n-1}$ is bounded from above as $y\to 0$. Thanks to Lemma 5.4(iii), Q is bounded from above at its maximum, finishing the proof. ☐

Proof of Theorem A. Let $K\subset\Omega$ be any compact set. It follows from Lemmas 5.4 and 5.5 that there exists a constant $C_K>0$ such that on $[0,T_{\max})\times K$,

$$C_K^{-1}\omega_0\le\omega(t)\le C_K\omega_0.$$

Applying the local higher-order estimates of Gill [2011, Section 4], we obtain uniform $\mathcal{C}^\infty$ estimates for $\omega(t)$ on compact subsets of Ω. Consequently, there exists a constant c_K such that

$$\frac{\partial}{\partial t}\omega=-\operatorname{Ric}(\omega)\le c_K\omega\quad\text{on }[0,T_{\max})\times K.$$

This implies that $e^{-c_Kt}\omega(t)$ decreases in t and is bounded from below. Hence $\omega(t)$ converges to $\omega_{T_{\max}}$ as $t\to T_{\max}$, and since we have uniform estimates on compact subsets of Ω, we see that the convergence is in $\mathcal{C}^\infty_{\text{loc}}(\Omega)$. This finishes the proof. ☐

6. The Chern–Ricci flow on varieties with log terminal singularities

In this section, we extend our previous analysis to the case of compact complex varieties with *mild singularities*. We refer the reader to [Eyssidieux et al. 2009, Section 5] for a brief introduction to the complex analysis on mildly singular varieties.

We assume here that Y is a $\mathbb{Q}$-Gorenstein variety, i.e., Y is a normal complex space such that its canonical divisor K_Y is $\mathbb{Q}$-Cartier. We denote the singular set of Y by Y_{sing} and let $Y_{\text{reg}} := Y \setminus Y_{\text{sing}}$. Given a log resolution of singularities $\pi : X \to Y$ (which may and will always be chosen to be an isomorphism over Y_{reg}), there exists a unique (exceptional) $\mathbb{Q}$-divisor $\sum a_i E_i$ with simple normal crossings (snc for short) such that

$$K_X = \pi^* K_Y + \sum_i a_i E_i .$$

The coefficients $a_i \in \mathbb{Q}$ are called the *discrepancies* of Y along E_i.

Definition 6.1. We say that Y has *log terminal* (*lt* for short) singularities if and only if $a_i > -1$ for all i.

The following definition of *adapted measure* is introduced in [Eyssidieux et al. 2009, Section 6]:

Definition 6.2. Let h be a smooth Hermitian metric on the $\mathbb{Q}$-line bundle $\mathcal{O}_Y(K_Y)$. The corresponding adapted measure $\mu_{Y,h}$ on Y_{reg} is locally defined by choosing a nowhere vanishing section σ of mK_Y over a small open set U and setting

$$\mu_{Y,h} := \frac{(i^{mn^2} \sigma \wedge \bar{\sigma})^{1/m}}{|\sigma|_{h^m}^{2/m}} .$$

The point of the definition is that the measure $\mu_{Y,h}$ does not depend on the choice of σ, so it is globally defined. The arguments above show that Y has log terminal singularities if and only if $\mu_{Y,h}$ has a finite total mass on Y, which can be considered as a Radon measure on the whole of Y. Then $\chi = dd^c \log \mu_{Y,h}$ is a well-defined smooth closed (1, 1)-form on Y.

Given a Hermitian form ω_Y on Y, there exists a unique Hermitian metric $h = h(\omega_Y)$ of K_Y such that

$$\omega_Y^n = \mu_{Y,h} .$$

We have the following definition.

Definition 6.3. The *Ricci curvature form* of ω_Y is $\operatorname{Ric}(\omega_Y) := -dd^c \log h$.

We recall the *slope* of a quasi-psh function ϕ at y in the sense of [Berman et al. 2019]. Choosing local generators (f_j) of the maximal ideal $\mathfrak{m}_y$ of $\mathcal{O}_{Y,y}$, we define

$$s(\phi, y) = \sup\left\{ s \geq 0 : \varphi \leq s \log \sum |f_j| + O(1) \right\}.$$

Note that this definition is independent of the choice of (f_j). By [Berman et al. 2019, Theorem A.2] there is $C > 0$ such that for any log resolution $\pi : X \to Y$,

$$\nu(\phi \circ \pi, E) \leq C s(\phi, y),$$

with E a prime divisor lying above y. In particular, the Lelong numbers of $\phi \circ \pi$ are sufficiently small if the $s(\phi, y)$ is also sufficiently small at all points $y \in Y$. We refer to [Pan 2025] for related results.

Applying the analysis in the previous section, we obtain the existence of the Chern–Ricci flow on log terminal singularities. This generalizes the result in [Dang 2024, Theorem E].

Theorem 6.4. *Let Y be a compact complex variety with log terminal singularities. Assume that θ_0 is a Hermitian metric such that*

$$T_{\max} := \sup\big\{t > 0 : \exists \psi \in \mathcal{C}^\infty(Y) \text{ such that } \theta_0 - t\,\mathrm{Ric}(\theta_0) + dd^c\psi > 0\big\} > 0.$$

Assume that $S_0 = \theta_0 + dd^c\phi_0$ is a positive (1,1)-current with small slopes. Then, there exists a family $(\omega_t)_{t\in[0,T_{\max})}$ of positive (1,1)-currents on Y starting with S_0 such that

(1) $\omega_t = \theta_0 - t\,\mathrm{Ric}(\theta_0) + dd^c\varphi_t$ *are positive* (1,1)*-currents*;

(2) $\omega_t \to S_0$ *weakly as* $t \to 0^+$;

(3) *for each $\varepsilon > 0$, there exists a Zariski open set Ω_ε such that on $[\varepsilon, T_{\max}) \times \Omega_\varepsilon$, ω is smooth and*

$$\frac{\partial \omega}{\partial t} = -\mathrm{Ric}(\omega).$$

Proof. It is classical that solving the (weak) Chern–Ricci flow is equivalent to solving a complex Monge–Ampère equation flow. Let χ be a closed smooth (1,1)-form that represents $c_1^{\mathrm{BC}}(K_Y)$. Given $T \in (0, T_{\max})$, there is a function $\psi_T \in \mathcal{C}^\infty(Y)$ such that $\theta_0 - t\,\mathrm{Ric}(\theta_0) + dd^c\psi_T > 0$. We set for $t \in [0, T]$,

$$\hat{\theta}_t := \theta_0 + t\chi, \quad \text{with } \chi = -\mathrm{Ric}(\theta_0) + dd^c\frac{\psi_T}{T},$$

which defines an affine path of Hermitian forms. Since χ is a smooth representative of $c_1^{\mathrm{BC}}(K_Y)$, one can find a smooth metric h on the $\mathbb{Q}$-line bundle $\mathcal{O}_Y(K_Y)$ with curvature form χ. We obtain $\mu_{Y,h}$, the adapted measure corresponding to h. The Chern–Ricci flow is equivalent to the following complex Monge–Ampère flow:

$$(\hat{\theta}_t + dd^c\phi_t)^n = e^{\partial_t\phi}\mu_{Y,h}. \tag{6-1}$$

Let $\pi : X \to Y$ be a log resolution of singularities. We have seen that the measure

$$\mu := \pi^*\mu_{Y,h} = f\,dV \quad \text{where } f = \prod_i |s_i|^{2a_i}$$

has poles (corresponding to $a_i < 0$) or zeroes (corresponding to $a_i > 0$) along the exceptional divisors $E_i = (s_i = 0)$, and dV is a smooth volume form. Passing to the resolution, the flow (6-1) becomes

$$\frac{\partial\varphi}{\partial t} = \log\left[\frac{(\theta_t + dd^c\varphi_t)^n}{\mu}\right], \tag{6-2}$$

where $\theta_t := \pi^*\hat{\theta}_t$ and $\varphi := \pi^*\phi$. Since $(\hat{\theta}_t)_{t\in[0,T]}$ is a smooth family of Hermitian forms, it follows that the family of semipositive forms $[0, T] \ni t \mapsto \theta_t$ satisfies all our requirements. We also have that $\theta := \pi^*\theta_0$, the latter is smooth, semipositive, and big but no longer Hermitian. We fix a θ-psh function ρ with analytic singularities along a divisor $E = \pi^{-1}(Y_{\mathrm{sing}})$ such that $\theta + dd^c\rho \geq 2\delta\omega_X$ with $\delta > 0$. We observe that since the Lelong numbers $\nu(\varphi_0, x)$ are sufficiently small, we have the assumption $p^*/(2c(\varphi_0)) < T_{\max}$ by Skoda's integrability theorem. The result therefore follows from Theorem B. □

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THE EXISTENCE OF TOPOLOGICAL SOLUTIONS TO THE CHERN–SIMONS MODEL ON LATTICE GRAPHS

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We prove the existence of topological solutions to the self-dual Chern–Simons model and the abelian Higgs system on the lattice graphs $\mathbb{Z}^n$ for $n \geqslant 2$. This extends results of Huang, Lin and Yau (2020) from finite graphs to lattice graphs.

1. Introduction

Various vortex problems have been extensively studied in recent decades, which play important roles in quantum physics, solid state physics and so on. The existence of topological and nontopological solutions in these models has been rigorously proven in mathematics. In $\mathbb{R}^2$, we consider the self-dual Chern–Simons vortex equation

$$\Delta u = \lambda e^u (e^u - 1) + 4\pi \sum_{j=1}^{M} n_j \delta_{p_j} \tag{1}$$

and the abelian Higgs equation

$$\Delta u = \lambda (e^u - 1) + 4\pi \sum_{j=1}^{M} n_j \delta_{p_j} \tag{2}$$

with positive integers $n_1, \dots, n_M$ and distinct vortices $p_1, \dots, p_M \in \mathbb{R}^2$. Here $\lambda > 0$, and δ_{p_j} is the Dirac mass at p_j. A solution of (1) or (2) is called topological if $u(x) \to 0$ as $|x| \to +\infty$, and called nontopological if $u(x) \to -\infty$ as $|x| \to +\infty$.

For the abelian Higgs system (2), Jaffe and Taubes [1980] proved the existence and uniqueness of general finite energy multivortex solutions to the Bogomol'nyi equations, and there have been many studies on this model since then, such as [Jacobs and Rebbi 1979; Jaffe and Taubes 1980; Wang and Yang 1992]. The self-dual Chern–Simons system (1) is the minimal self-dual model containing the Chern–Simons term. The Chern–Simons vortices were discovered in [Jackiw and Weinberg 1990; Hong et al. 1990], which attracted people to investigate the existence problem. The existence of topological solutions in $\mathbb{R}^2$ was established in [Wang 1991; Spruck and Yang 1995] by the variational method and iteration argument, and the existence of self-dual doubly periodic vortex solutions was proved in [Caffarelli and Yang 1995]. Later, nontopological solutions were studied in the literature, e.g., [Chen et al. 1994; Chae and Imanuvilov 2000; Chan et al. 2002; Choe et al. 2011], and see [Dunne 1995; Han 2014; Struwe and Tarantello 1998; Chae and Kim 1997] for other related results.

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Recently, people have paid attention to the elliptic equations on graphs. Grigor'yan, Lin and Yang first studied nonlinear elliptic equations on graphs; see, e.g., [Grigor'yan et al. 2016b; 2017]. In a seminal paper [Huang et al. 2020], Huang, Lin and Yau proved the existence result for solutions to (1) on finite graphs. Furthermore, on a finite graph, the existence of solutions to the generalized self-dual Chern–Simons equation was proved in [Lü and Zhong 2021; Hou and Sun 2022], and the existence of solutions to the Chern–Simons Higgs model has been recently proved in [Li et al. 2024] using topological degree methods. See [Grigor'yan et al. 2016a; Huang et al. 2021; Chao and Hou 2023] for other related results. For infinite graphs, existence results for the Kazdan–Warner equation were proved in [Ge and Jiang 2018; Keller and Schwarz 2018] on graphs with positive spectrum and canonically compactifiable graphs, while lattice graphs $\mathbb{Z}^n$, i.e., discrete analogs of Euclidean spaces $\mathbb{R}^n$, are excluded. In this paper, our main contribution is to extend the results in [Huang et al. 2020] from finite graphs to lattice graphs. We study the Chern–Simons equation (1) on $\mathbb{Z}^n$ for $n \geqslant 2$, and prove the existence of topological solutions. Furthermore, using topological solutions of (1), we prove the existence of topological solutions to the abelian Higgs equation (2) on $\mathbb{Z}^n$ for $n \geqslant 2$.

We consider the infinite integer lattice $\mathbb{Z}^n$ for $n \geqslant 2$; see Section 2 for details. We define the distance on $\mathbb{Z}^n$ by

$$d(x, y) = \sum_{i=1}^{n} |x_i - y_i|, \quad x, y \in \mathbb{Z}^n,$$

and write $d(x) = d(x, 0)$. For any function $u : \mathbb{Z}^n \to \mathbb{R}$, the l^p-norm of u is defined as

$$\|u\|_{l^p(\mathbb{Z}^n)} = \begin{cases} \left(\sum_{x\in\mathbb{Z}^n} |u(x)|^p\right)^{1/p} & \text{for } 1 \leqslant p < \infty, \\ \sup_{x\in\mathbb{Z}^n} |u(x)| & \text{for } p = \infty. \end{cases}$$

The Laplacian is defined as

$$\Delta u(x) = \sum_{d(x,y)=1} (u(y) - u(x)).$$

In the following we mainly consider topological solutions to the self-dual Chern–Simons vortex equation on $\mathbb{Z}^n$

$$\begin{cases} \Delta u = \lambda e^u (e^u - 1) + 4\pi \sum_{j=1}^{M} n_j \delta_{p_j} & \text{on } \mathbb{Z}^n, \\ \lim_{d(x)\to+\infty} u(x) = 0, \end{cases}$$

and construct a topological solution to the above equation, which is maximal among all possible solutions. Our main result is as follows.

Theorem 1.1. *Equation* (1) *has a topological solution $u \in l^p(\mathbb{Z}^n)$ on $\mathbb{Z}^n$ for $1 \leqslant p \leqslant \infty$ and $n \geqslant 2$, which is maximal among all possible solutions. Furthermore*, *we have the decay estimate*

$$u = O(e^{-m(1-\epsilon)d(x)}),$$

where $m = \ln(1 + \lambda/(2n)),\ 0 < \epsilon < 1$.

In this paper, we provide two proofs of Theorem 1.1. In Proof A, we adopt the exhaustion method and the discrete isoperimetric inequality. Proof A is novel, which relies on the discrete nature of graphs in an essential way. First, by a contradiction argument, we prove the existence of solutions on a finite subset Ω with Dirichlet boundary condition in Lemma 3.2. In order to prove the existence result on $\mathbb{Z}^n$, we apply the exhaustion method. This approach was first introduced by Lin and Yang [2022] to the analysis on graphs. Considering a sequence of finite subsets

$$\Omega_0 \subset \Omega_1 \subset \cdots \subset \Omega_k \subset \cdots, \quad \bigcup_{i=1}^{\infty} \Omega_i = \mathbb{Z}^n,$$

and corresponding monotone solution sequence $\{u_{\Omega_i}\}$ obtained by Lemma 3.2, we denote by $\tilde{u}^i(x)$ the null extension to $\mathbb{Z}^n$ of u_{Ω_i}; see Section 2.1. By passing to the limit, to avoid the triviality of the limit, one needs to prove a uniform bound for all $\tilde{u}^i$. Suppose that it is not true. Then there exists $\lim_{i\to+\infty} \tilde{u}^i(x_i) = -\infty$ for a vertex sequence $\{x_i\}$. Set

$$\begin{aligned} A_1^i &= \{x \in \Omega_i : -C \leqslant \tilde{u}^i(x) \leqslant 0\}, \\ A_2^i &= \{x \in \Omega_i : \tilde{u}^i(x) < -(2n+1)C - \lambda\}, \\ A_3^i &= \{x \in \Omega_i : -(2n+1)C - \lambda \leqslant \tilde{u}^i(x) < -C\}, \end{aligned}$$

where $C = 4\pi \sum_{j=1}^{M} n_j$. Since $|\Delta \tilde{u}^i(x)|$ is bounded on Ω_i, we may prove that

$$A_3^i \neq \varnothing \quad \text{and} \quad \lim_{i\to+\infty} |A_2^i| = +\infty.$$

These imply that $\lim_{i\to+\infty} |A_3^i| = +\infty$ by the isoperimetric inequality on $\mathbb{Z}^n$. However, summing over Ω_i in the equation, we know that $\sum_{x\in\Omega_i} e^{\tilde{u}^i}(1-e^{\tilde{u}^i})$ is uniformly bounded, which yields a contradiction. Hence we prove the l^∞-convergence $\tilde{u}^i \to u$ on $\mathbb{Z}^n$, and this limit is a topological solution. Applying the maximum principle and Lemma 3.3, we finally get the decay estimate and the maximality of the constructed solution.

In Proof B, we follow the methods in [Spruck and Yang 1995]. We prove a key lemma, Lemma 3.4, which provides the uniform l^2-norm estimate of the solution on a finite subset Ω with Dirichlet boundary condition. To prove this lemma, let $F(u)$ be the natural functional associated to the equation (1) on Ω. By Green's identities on graphs, we prove that $F(u_k)$ decreases with respect to k and has a upper bound which only depends on n, λ and $\sum_{j=1}^{M} n_j$. Applying the discrete Gagliardo–Nirenberg–Sobolev inequality proved in [Porretta 2020], we have

$$\|u_k\|_{l^2(\Omega)} \leqslant C_3(F(u_k)+1) \leqslant C_4,$$

where C_3, C_4 only depend on n, λ and $\sum_{j=1}^{M} n_j$. Thanks to this lemma, we may pass to the limit and get the solution $u_\Omega \in l^2(\Omega)$ on Ω. Since its l^2-norm is uniformly bounded, we construct the solution u of equation (1) on $\mathbb{Z}^n$ by the exhaustion method. As in Proof A, we prove the decay estimate and the maximality of the solution.

With the help of Theorem 1.1, we prove the existence of topological solutions to the abelian Higgs model.

Theorem 1.2. *Equation* (2) *has a unique topological solution $u' \in l^p(\mathbb{Z}^n)$ on $\mathbb{Z}^n$ for $1 \leqslant p \leqslant \infty$ and $n \geqslant 2$, satisfying $u \leqslant u' \leqslant 0$, where u is constructed in Theorem 1.1. Furthermore*, *there holds the decay estimate*

$$u' = O(e^{-m(1-\epsilon)d(x)}),$$

where $m = \ln(1+\lambda/(2n)),\ 0 < \epsilon < 1$.

To prove Theorem 1.2, we apply the subsupersolution method. By choosing $\omega_1 = 0$ as a supersolution and $\omega_2 = u$ as a subsolution, where u is constructed in Theorem 1.1, we obtain a solution to (2) by the monotone iteration argument.

The paper is organized as follows: In next section, we introduce the setting of graphs. In Section 3, we give two proofs of Theorem 1.1. In Section 4, we prove Theorem 1.2.

2. Preliminaries

2.1. ***The setting of*** $\mathbb{Z}^n$**.** Consider the infinite integer lattice graph $\mathbb{Z}^n$, $n \geqslant 2$, consisting of the set of vertices

$$V = \mathbb{Z}^n = \big\{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_i \in \mathbb{Z},\ \forall 1 \leqslant i \leqslant n\big\}$$

and the set of edges

$$E = \left\{\{x, y\} : x, y \in \mathbb{Z}^n, \sum_{i=1}^{n} |x_i - y_i| = 1\right\},$$

and we write $x \sim y$ if $\{x, y\} \in E$. We denote by $C(\mathbb{Z}^n) = \{u : \mathbb{Z}^n \to \mathbb{R}\}$ the set of functions on $\mathbb{Z}^n$, by $\operatorname{supp}(u) = \{x \in \mathbb{Z}^n : u(x) \neq 0\}$ the support of u, and by $C_0(\mathbb{Z}^n)$ the set of functions with finite support. For a finite subset $\Omega \subset \mathbb{Z}^n$, we define the boundary of Ω as

$$\delta\Omega := \big\{y \in \mathbb{Z}^n \setminus \Omega : \exists x \in \Omega \text{ such that } y \sim x\big\},$$

and write $\overline{\Omega} = \Omega \cup \delta\Omega$. For $u \in C(\Omega)$, the null extension to $\mathbb{Z}^n$ of u is defined as

$$\tilde{u}(x) = \begin{cases} u(x) & \text{on } \Omega, \\ 0 & \text{on } \Omega^c. \end{cases}$$

We define the difference operator as

$$\nabla_{xy} u = u(y) - u(x),\ u \in C(\mathbb{Z}^n), \quad x, y \in \mathbb{Z}^n.$$

For $f, g \in C(\overline{\Omega})$, we introduce a bilinear form,

$$D_\Omega(f, g) := \tfrac{1}{2} \sum_{\substack{x, y \in \Omega \\ x \sim y}} \nabla_{xy} f \nabla_{xy} g + \sum_{\substack{x \in \Omega,\ y \in \delta\Omega \\ x \sim y}} \nabla_{xy} f \nabla_{xy} g,$$

and we write $D_\Omega(f) = D_\Omega(f, f)$ for the Dirichlet energy of f on Ω. For $f \in C(\overline{\Omega})$, the directional derivative operator $\partial f / \partial \vec{n}$ at $x \in \delta\Omega$ is defined as

$$\frac{\partial f}{\partial \vec{n}}(x) := \sum_{\substack{y \in \Omega \\ x \sim y}} (f(x) - f(y)).$$

The following are Green's identities on graphs; see, e.g., [Grigor'yan 2018].

Lemma 2.1. *Let $f, g \in C(\mathbb{Z}^n)$ and Ω be a finite subset of $\mathbb{Z}^n$.*

(a) *If $f \in C_0(\mathbb{Z}^n)$, we have*

$$\frac{1}{2}\sum_{\substack{x,y\in\mathbb{Z}^n\\ x\sim y}} \nabla_{xy} f \nabla_{xy} g = -\sum_{x\in\mathbb{Z}^n} f(x)\Delta g(x).$$

(b)

$$D_\Omega(f, g) = -\sum_{x\in\Omega} f(x)\Delta g(x) + \sum_{x\in\delta\Omega} f(x)\frac{\partial g}{\partial \vec{n}}(x).$$

2.2. ***Maximum principle and discrete functional inequalities.*** In this subsection we introduce a maximum principle, the isoperimetric inequality, and the discrete Gagliardo–Nirenberg–Sobolev inequality on $\mathbb{Z}^n$, which play key roles in the proofs of the main results. The following maximum principle is well-known.

Lemma 2.2. *Let Ω be a finite subset of $\mathbb{Z}^n$. For any positive $f \in C(\bar{\Omega})$, suppose that a function $v \in C(\bar{\Omega})$ satisfies*

$$\begin{cases} (\Delta - f)v \geqslant 0 & \text{on } \Omega, \\ v \leqslant 0 & \text{on } \delta\Omega. \end{cases}$$

We have $v \leqslant 0$ on $\bar{\Omega}$.

Proof. We prove the result by contradiction. Suppose that there exists $x \in \Omega$ such that $v(x) = \sup_{y\in\bar{\Omega}} v(y) = c > 0$. By the equation,

$$\Delta v(x) \geqslant f(x)v(x) > 0.$$

This implies that there exists $x_0 \sim x,\ x_0 \in \bar{\Omega}$, such that $v(x_0) > v(x) = c$, which yields a contradiction. □

By the above lemma, we have the following corollary.

Corollary 2.3. *For any positive $f \in C(\mathbb{Z}^n)$, suppose that a function $v \in l^2(\mathbb{Z}^n)$ satisfies*

$$\begin{cases} (\Delta - f)v \geqslant 0 & \text{on } \mathbb{Z}^n, \\ \lim_{d(x)\to+\infty} v(x) \leqslant 0. \end{cases}$$

Then $v \leqslant 0$ on $\mathbb{Z}^n$.

The isoperimetric inequality is well-known on $\mathbb{Z}^n$, see, e.g., [Barlow 2017], which is needed for our Proof A. For $K \subset \mathbb{Z}^n$, we denote by $|K|$ the cardinality of the set K.

Lemma 2.4. *There exists a constant C_n, only depending on the dimension n, such that for any finite $\Omega \subset \mathbb{Z}^n$,*

$$|\delta\Omega| \geqslant C_n |\Omega|^{\frac{n-1}{n}}.$$

For $p \geqslant 1$, we define the $D^{1,p}$-norm as

$$\|u\|_{D^{1,p}(\mathbb{Z}^n)} := \left(\sum_{x\in\mathbb{Z}^n}\sum_{y\sim x} |u(y) - u(x)|^p\right)^{\frac{1}{p}}.$$

In Proof B, we need the discrete Gagliardo–Nirenberg–Sobolev inequality on $\mathbb{Z}^n$. Since $\mathbb{Z}^n$ is a discrete regular mesh, the proof of Theorem 4.1 in [Porretta 2020] yields the following discrete Gagliardo–Nirenberg–Sobolev inequality.

Lemma 2.5 [Porretta 2020]. *Let $n \geqslant 2$, $p > 1$, $\gamma \geqslant p$ and $p' = p/(p-1)$. Then for any $u \in l^p(\mathbb{Z}^n)$, we have*

$$\|u\|^{\gamma}_{l^{\gamma n/(n-1)}(\mathbb{Z}^n)} \leqslant C(p, n, \gamma)\|u\|_{D^{1,p}(\mathbb{Z}^n)}\|u\|^{\gamma-1}_{l^{(\gamma-1)p'}(\mathbb{Z}^n)}.$$

Remark 2.6. Although Theorem 4.1 in [Porretta 2020] requires $p > n$ and $\gamma > p$, the above inequality in fact holds for any $p > 1$ and $\gamma \geqslant p$ by the same argument in [Porretta 2020]. For $n \geqslant 2$, choose

$$p = \gamma = 2, \quad p' = 2.$$

With a well-known fact that for any $q \geqslant p$,

$$\|u\|_{l^q(\mathbb{Z}^n)} \leqslant \|u\|_{l^p(\mathbb{Z}^n)},$$

see, e.g., [Huang et al. 2015], we get for $u \in l^2(\mathbb{Z}^n)$,

$$\|u\|_{l^4(\mathbb{Z}^n)} \leqslant \|u\|_{l^{2n/(n-1)}(\mathbb{Z}^n)} \leqslant C'_n \|u\|^{\frac{1}{2}}_{D^{1,2}(\mathbb{Z}^n)}\|u\|^{\frac{1}{2}}_{l^2(\mathbb{Z}^n)}.$$

3. Existence theorems for the Chern–Simons equation

In this section we consider the existence of topological solutions to the Chern–Simons equation, and we give two proofs of Theorem 1.1. To prove this theorem, we first consider an iterative sequence on a finite subset of $\mathbb{Z}^n$.

Let Ω_0 be a finite subset of $\mathbb{Z}^n$, satisfying $\Omega_0 \supset \{p_j\}_{j=1}^M$, and Ω be an arbitrary connected finite subset such that $\Omega_0 \subset \Omega \subset \mathbb{Z}^n$. We write

$$g = 4\pi \sum_{j=1}^{M} n_j \delta_{p_j}, \quad C = 4\pi \sum_{j=1}^{M} n_j,$$

and it is obvious that $g \in l^p(\mathbb{Z}^n)$ for any $p \geqslant 1$. Choose a constant $K > 2\lambda > 0$. Let $u_0 = 0$ and consider the iterative equations

$$\begin{cases} (\Delta - K)u_k = \lambda e^{u_{k-1}}(e^{u_{k-1}} - 1) + g - K u_{k-1} & \text{on } \Omega, \\ u_k = 0 & \text{on } \delta\Omega. \end{cases} \tag{3}$$

Lemma 3.1. *Let the sequence $\{u_k\}$ be given in* (3). *Then for each k, u_k is uniquely defined and*

$$0 = u_0 \geqslant u_1 \geqslant u_2 \geqslant \cdots.$$

Proof. First we have

$$\begin{cases} (\Delta - K)u_1 = g & \text{on } \Omega, \\ u_1 = 0 & \text{on } \delta\Omega. \end{cases} \tag{4}$$

One easily sees the existence and uniqueness of the solution u_1 on Ω. Using Lemma 2.2, we obtain that $u_1 \leqslant 0$.

Suppose that $0 = u_0 \geqslant u_1 \geqslant u_2 \geqslant \cdots \geqslant u_i$. Since

$$\lambda e^{u_i}(e^{u_i} - 1) + g - K u_i \in l^2(\Omega),$$

we have the existence and uniqueness of the solution u_{i+1}. From the equations (3) we get

$$\begin{aligned}(\Delta - K)(u_{i+1}-u_i) &= \lambda(e^{2u_i}-e^{2u_{i-1}}) - \lambda(e^{u_i}-e^{u_{i-1}}) - K(u_i-u_{i-1})\\ &\geqslant 2\lambda e^{2\omega}(u_i-u_{i-1}) - K(u_i-u_{i-1}) \geqslant K(e^{2\omega}-1)(u_i-u_{i-1}) \geqslant 0,\end{aligned}$$

where ω is a function satisfying $u_i \leqslant \omega \leqslant u_{i-1}$. This implies that $u_{i+1} \leqslant u_i$ by Lemma 2.2 and proves this lemma. □

Proof A of Theorem 1.1. By Lemma 3.1, we prove the convergence of the monotone sequence $\{u_k\}$.

Lemma 3.2. *Let $\{u_k\}$ be the sequence defined by* (3). *Then there exists $u_\Omega \in C(\overline{\Omega})$ such that*

$$u_k \to u_\Omega \text{ on } \overline{\Omega},$$

which satisfies

$$\begin{cases}\Delta u_\Omega = \lambda e^{u_\Omega}(e^{u_\Omega}-1) + g & \text{on } \Omega,\\ u_\Omega = 0 & \text{on } \delta\Omega.\end{cases} \tag{5}$$

Proof. Since Ω is finite and the sequence is monotone, the pointwise limit u_Ω of u_k exists. It suffices to show that u_Ω is bounded. We first consider the set

$$B(\Omega) = \{x \in \Omega : \exists y \in \delta\Omega \text{ such that } y \sim x\}.$$

Summing over Ω in (3), by Lemma 2.1 we obtain

$$\sum_{x\in\delta\Omega} \frac{\partial u_k}{\partial \vec{n}}(x) + \lambda \sum_{x\in\Omega} e^{u_{k-1}}(1-e^{u_{k-1}}) = \sum_{x\in\Omega} g(x) + K\sum_{x\in\Omega}(u_k(x)-u_{k-1}(x)) \leqslant 4\pi \sum_{j=1}^{M} n_j = C.$$

This yields that

$$\sum_{x\in B(\Omega)} |u_k(x)| \leqslant C.$$

In particular, for any $x \in B(\Omega)$, the sequence $\{u_k(x)\}$ is uniformly bounded.

If $x_1 \sim x_0,\ x_1 \in \Omega$ and $x_0 \in B(\Omega)$, we claim that $\{u_k(x_1)\}$ is uniformly bounded. Equation (3) at x_0 shows that

$$\begin{aligned}|\Delta u_k(x_0)| &\leqslant K|u_k(x_0) - u_{k-1}(x_0)| + \lambda|e^{u_{k-1}(x_0)}(e^{u_{k-1}(x_0)}-1)| + |g(x_0)|\\ &\leqslant K|u_k(x_0)| + \tfrac{1}{4}\lambda + C \leqslant (K+1)C + \tfrac{1}{4}\lambda.\end{aligned}$$

Note that

$$\Delta u_k(x_0) = \sum_{y\sim x_0}(u_k(y)-u_k(x_0)) \leqslant u_k(x_1) - 2nu_k(x_0) \leqslant u_k(x_1) + 2nC$$

and $u_k(x_1) < 0$. We obtain that $\{u_k(x_1)\}$ is uniformly bounded.

Since Ω is connected, we repeat the above process, and get that $\{u_k(x)\}$ is uniformly bounded on Ω, which completes the proof of this lemma. □

Let $\Omega_i,\ 1 \leqslant i < \infty$, be finite and connected subsets satisfying

$$\Omega_0 \subset \Omega_1 \subset \cdots \subset \Omega_k \subset \cdots, \quad \bigcup_{i=1}^{\infty} \Omega_i = \mathbb{Z}^n.$$

We write $u^i = u_{\Omega_i}$ for simplicity. To prove Theorem 1.1, we need the following lemma.

Lemma 3.3. *Let Ω be a finite subset of $\mathbb{Z}^n$ and $\{u_k\}$ be the sequence defined by* (3). *For any function $V \in C(\overline{\Omega})$ satisfying*

$$\begin{cases} \Delta V \geqslant \lambda e^V(e^V - 1) + g & \text{on } \Omega, \\ V(x) \leqslant 0 & \text{on } \delta\Omega, \end{cases}$$

we have

$$0 = u_0 \geqslant u_1 \geqslant \cdots \geqslant u_k \geqslant \cdots \geqslant u_\Omega \geqslant V.$$

Proof. First, one has

$$\Delta V \geqslant \lambda e^V(e^V - 1) + g \geqslant \lambda e^V(e^V - 1).$$

We claim that $\sup_{x\in\Omega} V(x) \leqslant 0$. If not, choose $V(x_0) = \sup_{x\in\Omega} V(x) > 0$ for some $x_0 \in \Omega$. Then

$$0 \geqslant \Delta V(x_0) \geqslant \lambda e^{V(x_0)}(e^{V(x_0)} - 1) > 0,$$

which yields a contradiction and proves the claim.

Suppose that $V \leqslant u_k$. Then

$$(\Delta - K)(u_{k+1} - V) \leqslant \lambda(e^{2u_k} - e^{2V}) - \lambda(e^{u_k} - e^V) - K(u_k - V) \leqslant K(e^{2\omega} - 1)(u_k - V) \leqslant 0,$$

where the function ω satisfies $V \leqslant \omega \leqslant u_k \leqslant 0$. This implies that $V \leqslant u_{k+1}$ by Lemma 2.2 and proves this lemma by the induction. □

Finally, we use these lemmas to prove Theorem 1.1.

Proof of Theorem 1.1. For any integers $1 \leqslant j \leqslant k$, we have $\Omega_j \subset \Omega_k$. On $\overline{\Omega}_j$, since $u^k \leqslant 0$, one easily sees that u^k satisfies the conditions in Lemma 3.3, and we obtain

$$u^k \leqslant u^j \quad \text{on } \overline{\Omega}_j.$$

Let $\tilde{u}^i$ be the null extension to $\mathbb{Z}^n$ of u^i on Ω_i. Note that

$$0 \geqslant \tilde{u}^1 \geqslant \tilde{u}^2 \geqslant \cdots \geqslant \tilde{u}^k \geqslant \cdots$$

on $\mathbb{Z}^n$.

In order to prove the convergence of the sequence $\{\tilde{u}^i\}$ to a nontrivial limit, we need to give a uniform lower bound of $\{\tilde{u}^i\}$. We argue by contradiction. Suppose that

$$\lim_{i\to+\infty} \|\tilde{u}^i\|_{l^\infty} = +\infty.$$

We write

$$\begin{aligned} A_1^i &= \{x \in \Omega_i : -C \leqslant \tilde{u}^i(x) \leqslant 0\}, \\ A_2^i &= \{x \in \Omega_i : \tilde{u}^i(x) < -(2n+1)C - \lambda\}, \\ A_3^i &= \{x \in \Omega_i : -(2n+1)C - \lambda \leqslant \tilde{u}^i(x) < -C\}. \end{aligned}$$

By the conditions we assume, we get $A_2^i \neq \varnothing$ when $i \geqslant i_0$ for some i_0. In the following, we only consider $i \geqslant i_0$. Following the proof of Lemma 3.2, we have

$$\sum_{x\in B(\Omega_i)} |\tilde{u}^i(x)| \leqslant C,$$

which yields $B(\Omega_i) \subset A_1^i$ so that $A_1^i \neq \varnothing$. To obtain the uniform l^∞-norm, we show the contradiction in three steps.

(i) We claim that

$$A_3^i \neq \varnothing.$$

Suppose that $A_3^i = \varnothing$, which is equivalent to $A_1^i \cup A_2^i = \Omega_i$, and there exist two vertices $x, y \in \Omega_i$, satisfying $x \sim y,\ x \in A_1^i,\ y \in A_2^i$. Note that

$$\Delta \tilde{u}^i(x) = \sum_{z \sim x} (\tilde{u}^i(z) - \tilde{u}^i(x)) \leqslant \tilde{u}^i(y) - 2n\tilde{u}^i(x) < -(2n+1)C - \lambda + 2nC = -C - \lambda$$

and

$$|\Delta \tilde{u}^i(x)| \leqslant |g(x)| + \lambda |e^{\tilde{u}^i}(1 - e^{\tilde{u}^i})| < C + \lambda.$$

This yields a contradiction. Thus we have $A_3^i \neq \varnothing$.

(ii) We claim that

$$\lim_{i \to +\infty} |A_2^i| = +\infty.$$

By

$$\lim_{i \to +\infty} \|\tilde{u}^i\|_{l^\infty} = +\infty,$$

we choose a sequence $\{x_i\}$, where $x_i \in \Omega_i$, satisfying

$$\lim_{i \to +\infty} \tilde{u}^i(x_i) = -\infty.$$

Consider the function

$$w_i(x) = \tilde{u}^i(x - x_1 + x_i)$$

and

$$\Omega_i' = \{x \in \mathbb{Z}^n : x - x_1 + x_i \in \Omega_i\}.$$

We have

$$\lim_{i \to +\infty} w_i(x_1) = -\infty.$$

Let i be sufficiently large. From the proof in (i) we also have the estimate that for any $x \in \Omega_i'$,

$$|\Delta w_i(x)| < C + \lambda.$$

Consider an arbitrary vertex $y_1 \in \Omega_i',\ y_1 \sim x_1$, and by the above facts,

$$\Delta w_i(y_1) = \sum_{z \sim y_1} (w_i(z) - w_i(y_1)) \leqslant -2n w_i(y_1) + w_i(x_1).$$

This implies that

$$w_i(y_1) < \frac{1}{2n} w_i(x_1) + \frac{C + \lambda}{2n}.$$

Since Ω_i' is connected, $x_1 \in \Omega_i'$ and

$$\lim_{i \to +\infty} |\Omega_i'| = +\infty,$$

there exist $y_1 \sim x_1$ and a subsequence, still denoted by $\{\Omega_i'\}$, satisfying $y_1 \in \Omega_i'$. One obtains that

$$\liminf_{i\to+\infty} w_i(y_1) = -\infty.$$

Repeating the above process,

$$\limsup_{i\to+\infty} \big|\{y \in \Omega_i' : \liminf_{i\to+\infty} w_i(y) = -\infty\}\big| = +\infty.$$

The monotonically decreasing sequence $\{\tilde{u}^i\}$ guarantees that

$$A_2^i \subset A_2^{i+1}.$$

By letting $i \to +\infty$, one easily sees that

$$\lim_{i\to+\infty} |A_2^i| = \limsup_{i\to+\infty} |A_2^i| = +\infty.$$

(iii) From (i) and (ii) we want to prove that

$$\limsup_{i\to+\infty} |A_3^i| = +\infty.$$

We argue by contradiction.

Suppose that

$$\limsup_{i\to+\infty} |A_3^i| = N < +\infty.$$

We focus on the set $\Omega_i \setminus A_3^i = A_1^i \cup A_2^i$, which can be divided into a union of several disjoint connected subsets, i.e.,

$$\Omega_i \setminus A_3^i = \bigcup_{j=1}^{l} O_j,$$

and we have

$$\delta O_j \subset \delta\Omega_i \cup A_3^i.$$

From the proof of (i), we have

$$O_j \subset A_1^i \quad \text{or} \quad O_j \subset A_2^i.$$

For some $1 \leqslant l_1 \leqslant l-1$, without loss of generality, we may assume that

$$A_1^i = \bigcup_{j=1}^{l_1} O_j \quad \text{and} \quad A_2^i = \bigcup_{j=l_1+1}^{l} O_j.$$

Since $B(\Omega_i) \subset A_1^i$, we get

$$\delta\Omega_i \subset \bigcup_{j=1}^{l_1} \delta O_j.$$

Thus for $l_1 + 1 \leqslant j \leqslant l$,

$$\delta O_j \subset A_3^i.$$

For any $x \in \Omega_i$, since $O_1, \dots, O_l$ are disjoint connected sets and $|\delta\{x\}| = 2n$, there are no more than $2n$ sets from the family $\{O_j\}_{j=1}^{l}$ satisfying $x \in \delta O_j$.

By the isoperimetric inequality in Lemma 2.4, we have the estimate

$$|A_2^i| = \sum_{j=l_1+1}^{l} |O_j| \leqslant \Big(\frac{1}{C_n}\Big)^{\frac{n}{n-1}} \sum_{j=l_1+1}^{l} |\delta O_j|^{\frac{n}{n-1}} \leqslant \Big(\frac{1}{C_n}\Big)^{\frac{n}{n-1}} \Big(\sum_{j=l_1+1}^{l} |\delta O_j|\Big)^{\frac{n}{n-1}} \leqslant \Big(\frac{2n}{C_n}\Big)^{\frac{n}{n-1}} |A_3^i|^{\frac{n}{n-1}} \leqslant \Big(\frac{2nN}{C_n}\Big)^{\frac{n}{n-1}},$$

which contradicts the claim proved in (ii) by letting $i \to +\infty$.

With this fact, we choose a small constant ϵ satisfying

$$0 < \epsilon < \inf_{x \in [-(n+1)C-\lambda, -C]} e^x(1-e^x).$$

From (5), we get

$$\sum_{x \in \delta\Omega_i} \frac{\partial \tilde{u}^i}{\partial \vec{n}}(x) + \lambda \sum_{x \in \Omega_i} e^{\tilde{u}^i}(1-e^{\tilde{u}^i}) = \sum_{x \in \Omega_i} g(x) = 4\pi \sum_{j=1}^{M} n_j = C,$$

which implies that

$$\sum_{x \in \Omega_i} e^{\tilde{u}^i}(1-e^{\tilde{u}^i}) \leqslant \frac{C}{\lambda} \quad \text{and} \quad \sum_{x \in A_3^i} \epsilon \leqslant \frac{C}{\lambda}.$$

This yields a contradiction to (iii). Thus $\{\tilde{u}^i\}$ has a uniform bound in $l^\infty(\mathbb{Z}^n)$, and we have the pointwise convergence

$$\lim_{i \to +\infty} \tilde{u}^i(x) = u(x) \quad \text{for all } x \in \mathbb{Z}^n,$$

where $u \in l^\infty(\mathbb{Z}^n)$ and u satisfies the self-dual Chern–Simons vortex equation (1). From the above inequality and the fact that u has a lower bound, we pass to the limit, and get that for any $i \geqslant 1$,

$$e^{\inf_{x \in \mathbb{Z}^n} u(x)} \sum_{x \in \Omega_i} (1-e^u) \leqslant \sum_{x \in \Omega_i} e^u(1-e^u) \leqslant \frac{C}{\lambda},$$

which yields that u is a topological solution.

That the solution is maximal follows from Lemma 3.3. On any finite subset Ω, by Lemma 3.3, we obtain that the solution u_Ω is maximal. On $\mathbb{Z}^n$, we suppose that there exists another topological solution f of the self-dual Chern–Simons vortex equation. From the proof of Lemma 3.3, we observe that $f \leqslant 0$ on $\mathbb{Z}^n$. Applying Lemma 3.3 on Ω_i, we have

$$f \leqslant u^i.$$

For a fixed integer $k \geqslant 1$ and for $i \geqslant k$ we have

$$f(x) \leqslant \underline{\lim}_{i \to \infty} u^i(x) = u(x) \quad \text{on } \Omega_k.$$

For any $x \in \mathbb{Z}^n$, there exists a sufficiently large integer k satisfying $x \in \Omega_k$ such that $f(x) \leqslant u(x)$. Thus we obtain $f \leqslant u$ on $\mathbb{Z}^n$, and the solution u is maximal among all possible solutions.

The last part is to prove the decay estimate

$$u = O(e^{-m(1-\epsilon)d(x)}),$$

where $m = \ln(1+\lambda/(2n))$. Note that the solution u satisfies

$$\Delta u = \lambda e^u(e^u - 1) \quad \text{on } \overline{\Omega}_0^c.$$

Since

$$\lim_{d(x)\to+\infty} u(x) = 0,$$

for any $0 < \epsilon < 1$, we can choose $R \geqslant 1$ sufficiently large that

$$\lambda e^{2u} \geqslant 2n\Big[\Big(1+\frac{\lambda}{2n}\Big)^{1-\epsilon} - 1\Big], \quad d(x) \geqslant R.$$

Then for $d(x) \geqslant R$,

$$\Delta u = \lambda e^u(e^u - 1) = \lambda e^{u+\omega} u \leqslant \lambda e^{2u} u \leqslant c_3 u,$$

where the function ω satisfies $u \leqslant \omega \leqslant 0$ and $c_3 = 2n[(1+\lambda/(2n))^{1-\epsilon} - 1]$.

Consider the function $h(x) = -e^{-m(1-\epsilon)d(x)}$. Let e_i be the vector whose i-th component is 1 and the others are 0. For $x \in \overline{\Omega}_0^c$ and $d(x) \geqslant R$, suppose that $d(x) = t \geqslant R \geqslant 1$, and we have

$$\Delta h(x) = \sum_{y\sim x}\big(h(y) - h(x)\big) = \sum_{i=1}^{n}\big(h(x+e_i) + h(x-e_i) - 2h(x)\big).$$

If $x_i \neq 0$, then

$$h(x+e_i) + h(x-e_i) - 2h(x) = -e^{-m(1-\epsilon)(t-1)} - e^{-m(1-\epsilon)(t+1)} + 2e^{-m(1-\epsilon)t}.$$

If $x_i = 0$, we have

$$\begin{aligned} h(x+e_i) + h(x-e_i) - 2h(x) &= -2e^{-m(1-\epsilon)(t+1)} + 2e^{-m(1-\epsilon)t} \\ &\geqslant -e^{-m(1-\epsilon)(t-1)} - e^{-m(1-\epsilon)(t+1)} + 2e^{-m(1-\epsilon)t}. \end{aligned}$$

Therefore, we have the inequality

$$\begin{aligned} \Delta h(x) &\geqslant n\big[-e^{-m(1-\epsilon)(t-1)} - e^{-m(1-\epsilon)(t+1)} + 2e^{-m(1-\epsilon)t}\big] = n[e^{-m(1-\epsilon)} + e^{m(1-\epsilon)} - 2]h(x) \\ &= n\Big[\Big(1+\frac{c_3}{2n}\Big) + \frac{1}{1+c_3/(2n)} - 2\Big]h(x) \geqslant n\Big[2\Big(1+\frac{c_3}{2n}\Big) - 2\Big]h(x) = c_3 h(x). \end{aligned}$$

Fix a subset

$$\Omega_0' = \{x \in \mathbb{Z}^n : d(x) \geqslant R_1 \geqslant R\},$$

which satisfies $\Omega_0' \cap \overline{\Omega}_0 = \varnothing$. By choosing a large constant $C(\epsilon)$, we obtain

$$(\Delta - c_3)(C(\epsilon)h - u) \geqslant 0 \quad \text{on } \Omega_0',$$

$$\lim_{|x|\to+\infty}(C(\epsilon)h - u)(x) = 0 \quad \text{and} \quad C(\epsilon)h(x) - u(x) \leqslant 0 \qquad \text{if } d(x) = R_1.$$

These imply that

$$0 \geqslant u(x) \geqslant -C(\epsilon)e^{-m(1-\epsilon)d(x)} \quad \text{on } \Omega_0',$$

completing Theorem 1.1. As a consequence, we obtain $u \in l^p(\mathbb{Z}^n),\ 1 \leqslant p \leqslant \infty$. □

Proof B of Theorem 1.1. Proof B follows the methods in [Spruck and Yang 1995]. We mainly prove the following key lemma.

Lemma 3.4. *Let $n \geqslant 2$, $\lambda > 0$, and Ω_0 be a finite subset of $\mathbb{Z}^n$ containing the distinct points $\{p_j\}_{j=1}^M$. For any finite subset $\Omega \supset \Omega_0$, the boundary value problem*

$$\begin{cases} \Delta u = \lambda e^u(e^u - 1) + 4\pi \sum_{j=1}^M n_j \delta_{p_j} & \text{on } \Omega, \\ u(x) = 0 & \text{on } \delta\Omega, \end{cases}$$

has a solution $u_\Omega : \bar{\Omega} \to \mathbb{R}$. This solution is maximal among all possible solutions and satisfies that $\|u_\Omega\|_{l^2(\Omega)} \leqslant C_0$, where C_0 only depends on n, λ and C.

By Green's identities in Lemma 2.1, we consider the following functional on Ω:

$$F(u) = \tfrac{1}{2} D_\Omega(u) + \sum_{x\in\Omega} \big[\tfrac{1}{2}\lambda(e^{u(x)} - 1)^2 + g(x)u(x)\big].$$

We prove the following lemma which states that $F(u_k)$ deceases with respect to k.

Lemma 3.5. *Let $\{u_k\}$ be the sequence defined by* (3). *Then*

$$c_0 \geqslant F(u_1) \geqslant F(u_2) \geqslant \cdots \geqslant F(u_k) \geqslant \cdots,$$

where the constant c_0 only depends on n, C, λ.

Proof. Multiplying (3) by $u_k - u_{k-1}$ and summing over Ω, we obtain

$$\sum_{x\in\Omega} (\Delta - K)u_k(x)[u_k(x) - u_{k-1}(x)] \\ = \sum_{x\in\Omega} \big[\lambda e^{u_{k-1}}(e^{u_{k-1}} - 1)(u_k - u_{k-1}) - K u_{k-1}(u_k - u_{k-1}) + g(u_k - u_{k-1})\big](x). \quad (6)$$

By Green's identities in Lemma 2.1,

$$\sum_{x\in\Omega} \Delta u_k(x)(u_k(x) - u_{k-1}(x)) = -D_\Omega(u_k - u_{k-1}, u_k) = -D_\Omega(u_k) + D_\Omega(u_{k-1}, u_k).$$

Combining it with equation (6), we get

$$D_\Omega(u_k) - D_\Omega(u_{k-1}, u_k) + \sum_{x\in\Omega} K(u_k(x) - u_{k-1}(x))^2 \\ = -\sum_{x\in\Omega} [\lambda e^{u_{k-1}}(e^{u_{k-1}} - 1)(u_k - u_{k-1}) + g(u_k - u_{k-1})](x).$$

Consider the function

$$\varphi(x) = \frac{\lambda}{2}(e^x - 1)^2 - \frac{K}{2}x^2,$$

which is concave for any $x \leqslant 0$. Hence

$$\frac{\varphi(u_{k-1}) - \varphi(u_k)}{u_{k-1} - u_k} \geqslant \varphi'(u_{k-1}) = \lambda e^{u_{k-1}}(e^{u_{k-1}} - 1) - K u_{k-1}.$$

That is,

$$\frac{\lambda}{2}(e^{u_k} - 1)^2 \leqslant \frac{\lambda}{2}(e^{u_{k-1}} - 1)^2 + \frac{K}{2}(u_k - u_{k-1})^2 + \lambda e^{u_{k-1}}(e^{u_{k-1}} - 1)(u_k - u_{k-1}).$$

By the fact

$$\begin{aligned}|D_\Omega(u_{k-1},u_k)| &\leqslant \frac{1}{2}\sum_{\substack{x,y\in\Omega\\ x\sim y}}|\nabla_{xy}u_{k-1}\nabla_{xy}u_k| + \sum_{\substack{x\in\Omega,y\in\delta\Omega\\ x\sim y}}|\nabla_{xy}u_{k-1}\nabla_{xy}u_k|\\ &\leqslant \frac{1}{4}\sum_{\substack{x,y\in\Omega\\ x\sim y}}(|\nabla_{xy}u_{k-1}|^2+|\nabla_{xy}u_k|^2) + \frac{1}{2}\sum_{\substack{x\in\Omega,y\in\delta\Omega\\ x\sim y}}(|\nabla_{xy}u_{k-1}|^2+|\nabla_{xy}u_k|^2)\\ &= \frac{1}{2}D_\Omega(u_{k-1}) + \frac{1}{2}D_\Omega(u_k),\end{aligned}$$

we obtain that

$$F(u_k) \leqslant F(u_k) + \frac{K}{2}\|u_{k-1}-u_k\|^2_{l^2(\Omega)} \leqslant F(u_{k-1}).$$

Thus we only need to prove $F(u_1) \leqslant c_0$. Note that

$$\begin{aligned}D_\Omega(u_1) &= \frac{1}{2}\sum_{\substack{x,y\in\Omega\\ x\sim y}}|\nabla_{xy}u_1|^2 + \sum_{\substack{x\in\Omega,y\in\delta\Omega\\ x\sim y}}|\nabla_{xy}u_1|^2\\ &\leqslant \sum_{\substack{x,y\in\Omega\\ x\sim y}}(u_1(x)^2+u_1(y)^2) + 2\sum_{\substack{x\in\Omega,y\in\delta\Omega\\ x\sim y}}(u_1(x)^2+u_1(y)^2)\\ &\leqslant 4n\|u_1\|^2_{l^2(\Omega)}\end{aligned}$$

and $|e^{u_1}-1| = 1-e^{u_1} \leqslant -u_1$. Then we have the estimate

$$\begin{aligned}F(u_1) &\leqslant \frac{1}{2}\cdot 4n\|u_1\|^2_{l^2(\Omega)} + \frac{\lambda}{2}\sum_{x\in\Omega}u_1(x)^2 + \frac{1}{2}\sum_{x\in\Omega}[g(x)^2+u_1(x)^2]\\ &= c_1 + c_2\|u_1\|^2_{l^2(\Omega)},\end{aligned}$$

where c_1, c_2 are constants that only depend on n, λ and C. Multiplying (4) by u_1 and summing over Ω, we have

$$D_\Omega(u_1) + K\sum_{x\in\Omega}u_1(x)^2 = -\sum_{x\in\Omega}g(x)u_1(x).$$

This yields

$$K\sum_{x\in\Omega}u_1(x)^2 \leqslant \frac{1}{2K}\sum_{x\in\Omega}g(x)^2 + \frac{K}{2}\sum_{x\in\Omega}u_1(x)^2.$$

Hence,

$$\sum_{x\in\Omega}u_1(x)^2 \leqslant \frac{\|g\|^2_{l^2(\mathbb{Z}^n)}}{K^2},$$

which completes the proof. □

Our aim is a uniform control of the l^2-norm of $\{u_k\}$. By Lemma 3.5, we can use the functional $F(u_k)$ to control the l^2-norm of u_k. In fact, we prove the following lemma, which states that the functional F is coercive.

Lemma 3.6. *Let $v \in l^2(\bar{\Omega})$ and $v(x) = 0$ for all $x \in \delta\Omega$. Then*

$$\|v\|_{l^2(\Omega)} \leqslant C_2(F(v)+1),$$

where C_2 only depends on n, C, λ. In particular, let $\{u_k\}$ be the sequence defined by (3)*. We have for any $k \geqslant 1$,*

$$\|u_k\|_{l^2(\Omega)} \leqslant C_2(F(u_k)+1) \leqslant C_0,$$

where C_0 only depends on n, C, λ.

Proof. For any function $v \in l^2(\bar{\Omega})$ with $v(x) = 0$ for all $x \in \delta\Omega$. Let $\tilde{v}$ be the null extension to $\mathbb{Z}^n$ of v on Ω. Hence $\tilde{v} \in l^2(\mathbb{Z}^n)$. By Lemma 2.5, we have

$$\|\tilde{v}\|^4_{l^4(\mathbb{Z}^n)} \leqslant C'_n \|\tilde{v}\|^2_{D^{1,2}(\mathbb{Z}^n)} \|\tilde{v}\|^2_{l^2(\mathbb{Z}^n)}.$$

Note that

$$\|\tilde{v}\|^4_{l^4(\mathbb{Z}^n)} = \sum_{x\in\Omega} v(x)^4, \quad \|\tilde{v}\|^2_{l^2(\mathbb{Z}^n)} = \sum_{x\in\Omega} v(x)^2, \quad \text{and} \quad \|\tilde{v}\|_{D^{1,2}(\mathbb{Z}^n)} \leqslant (2D_\Omega(v))^{\frac{1}{2}}.$$

This yields that

$$\sum_{x\in\Omega} v(x)^4 \leqslant C_3 D_\Omega(v) \sum_{x\in\Omega} v(x)^2, \tag{7}$$

where $C_3 = 2C'_n$. Since $e^v - 1 \geqslant v$ and $1 - e^{-v} \geqslant v/(1+v)$ for $v \geqslant 0$,

$$|e^v - 1|^2 \geqslant \left(\frac{|v|}{1+|v|}\right)^2.$$

By (7) we obtain

$$\begin{aligned}
F(v) &= \tfrac{1}{2} D_\Omega(v) + \sum_{x\in\Omega} \left[\tfrac{1}{2}\lambda(e^{v(x)}-1)^2 + g(x)v(x)\right] \\
&\geqslant \tfrac{1}{2} D_\Omega(v) + \tfrac{1}{2}\lambda \sum_{x\in\Omega} \left(\frac{|v(x)|}{1+|v(x)|}\right)^2 - \|g\|_{l^{4/3}(\mathbb{Z}^n)} \|v\|_{l^4(\Omega)} \\
&\geqslant \tfrac{1}{2} D_\Omega(v) + \tfrac{1}{2}\lambda \sum_{x\in\Omega} \left(\frac{|v(x)|}{1+|v(x)|}\right)^2 - C_4 (D_\Omega(v))^{\frac{1}{4}} \left(\sum_{x\in\Omega} v(x)^2\right)^{\frac{1}{4}} \\
&\geqslant \tfrac{1}{2} D_\Omega(v) + \tfrac{1}{2}\lambda \sum_{x\in\Omega} \left(\frac{|v(x)|}{1+|v(x)|}\right)^2 - \epsilon \|v\|_{l^2(\Omega)} - \frac{C_4}{\epsilon} (D_\Omega(v))^{\frac{1}{2}} \\
&\geqslant \tfrac{1}{2} D_\Omega(v) + \tfrac{1}{2}\lambda \sum_{x\in\Omega} \left(\frac{|v(x)|}{1+|v(x)|}\right)^2 - \epsilon \|v\|_{l^2(\Omega)} - \tfrac{1}{4} D_\Omega(v) - C_4 \\
&= \tfrac{1}{4} D_\Omega(v) + \tfrac{1}{2}\lambda \sum_{x\in\Omega} \left(\frac{|v(x)|}{1+|v(x)|}\right)^2 - \epsilon \|v\|_{l^2(\Omega)} - C_4,
\end{aligned} \tag{8}$$

where $\epsilon > 0$ is a sufficiently small constant which will be chosen below, and C_4 is a uniform constant only depending on ϵ, C, C'_n which may change its value from line to line.

By the inequality (7), we have the estimate

$$
\begin{aligned}
\Big(\sum_{x\in\Omega} v(x)^2\Big)^2 &= \Big[\sum_{x\in\Omega}\frac{|v(x)|}{1+|v(x)|}(1+|v(x)|)|v(x)|\Big]^2\\
&\leqslant \sum_{x\in\Omega}\Big(\frac{|v(x)|}{1+|v(x)|}\Big)^2\sum_{x\in\Omega}(1+|v(x)|)^2 v(x)^2\\
&\leqslant 2\sum_{x\in\Omega}\Big(\frac{|v(x)|}{1+|v(x)|}\Big)^2\sum_{x\in\Omega}(v(x)^2+v(x)^4)\\
&\leqslant 2\sum_{x\in\Omega}\Big(\frac{|v(x)|}{1+|v(x)|}\Big)^2\sum_{x\in\Omega}v(x)^2+2C_3\sum_{x\in\Omega}\Big(\frac{|v(x)|}{1+|v(x)|}\Big)^2 D_\Omega(v)\sum_{x\in\Omega}v(x)^2\\
&\leqslant \tfrac{1}{2}\Big(\sum_{x\in\Omega}v(x)^2\Big)^2+C_5\Big[\Big(\sum_{x\in\Omega}\Big(\frac{|v(x)|}{1+|v(x)|}\Big)^2\Big)^2+\Big(\sum_{x\in\Omega}\Big(\frac{|v(x)|}{1+|v(x)|}\Big)^2\Big)^2 D_\Omega(v)^2\Big]\\
&\leqslant \tfrac{1}{2}\Big(\sum_{x\in\Omega}v(x)^2\Big)^2+C_5\Big[1+\Big(\sum_{x\in\Omega}\Big(\frac{|v(x)|}{1+|v(x)|}\Big)^2\Big)^4+D_\Omega(v)^4\Big],
\end{aligned}
$$

where C_5, C_6 are uniform constants only depending on C'_n. This yields that

$$
\|v\|_{l^2(\Omega)}\leqslant C_6\Big[1+\sum_{x\in\Omega}\Big(\frac{|v(x)|}{1+|v(x)|}\Big)^2+D_\Omega(v)\Big]. \tag{9}
$$

We choose $\epsilon=\dfrac{\min\{1/8,\lambda/4\}}{C_6}$, and by combining (8) with (9), we obtain

$$
\|v\|_{l^2(\Omega)}\leqslant C_2(F(v)+1).
$$

By Lemma 3.5, we have

$$
\|u_k\|_{l^2(\Omega)}\leqslant C_2(F(u_k)+1)\leqslant C_0,
$$

where C_2, C_0 only depend on n, C, λ. □

Proof of Lemma 3.4. By Lemmas 3.6 and 3.1, we obtain

$$
u_k\to u_\Omega \quad \text{in } l^2(\Omega), \qquad \text{and} \qquad \|u_\Omega\|_{l^2(\Omega)}\leqslant C_0.
$$

Since Δ is a local operator, by the pointwise convergence the function $u_\Omega\in l^2(\Omega)$ is the solution to the equation

$$
\begin{cases}\Delta u=\lambda e^u(e^u-1)+g & \text{on } \Omega,\\ u(x)=0 & \text{on } \delta\Omega.\end{cases}
$$

This finishes the main proof of Lemma 3.4. For the rest, it remains to prove that this solution is maximal, which we argue the same as in Proof A. □

Let Ω_i be finite and connected subsets satisfying

$$\Omega_0 \subset \Omega_1 \subset \cdots \subset \Omega_k \subset \cdots, \quad \bigcup_{i=1}^{\infty} \Omega_i = \mathbb{Z}^n,$$

and we write $u^i = u_{\Omega_i}$. Finally we use these lemmas to prove Theorem 1.1.

Proof of Theorem 1.1. As in Proof A, for any integer $1 \leqslant j \leqslant k$, one easily sees that

$$u^k \leqslant u^j \quad \text{on } \overline{\Omega}_j.$$

Let $\tilde{u}^k$ be the null extension to $\mathbb{Z}^n$ of u_k. Then

$$0 \geqslant \tilde{u}^1 \geqslant \tilde{u}^2 \geqslant \cdots \geqslant \tilde{u}^k \geqslant \cdots$$

on $\mathbb{Z}^n$. Noting that $\|\tilde{u}^k\|_{l^2(\mathbb{Z}^n)} \leqslant C_0$ for any $k \geqslant 1$, we have the pointwise convergence

$$\tilde{u}^k(x) \to \ u(x), \quad \text{for all } x \in \mathbb{Z}^n,$$

and $u \in l^2(\mathbb{Z}^n)$. Hence u satisfies the equations

$$\begin{cases} \Delta u = \lambda e^u(e^u - 1) + 4\pi \sum_{j=1}^{M} \delta_{p_j} & \text{on } \mathbb{Z}^n, \\ \lim_{d(x) \to +\infty} u(x) = 0, \end{cases}$$

which is a topological solution. Analogous to Proof A, one can show that this solution is maximal and satisfies the decay estimate. This implies that $u \in l^p(\mathbb{Z}^n)$ for any $1 \leqslant p \leqslant \infty$, and we finish Proof B. □

4. Existence theorems of the abelian Higgs equation

Note that the topological solution to the Chern–Simons model, obtained in Theorem 1.1, serves as a subsolution of the abelian Higgs equation. In this section we will prove the existence of topological solutions to the abelian Higgs equation (2) on $\mathbb{Z}^n$ for $n \geqslant 2$ using the subsupersolution approach. We prove the existence of topological solutions to (2) by a monotone iteration method.

Definition 4.1. We call a function ω a supersolution or a subsolution of (2) if, on $\mathbb{Z}$,

$$\Delta\omega \leqslant \lambda(e^{\omega} - 1) + g \quad \text{or} \quad \Delta\omega \geqslant \lambda(e^{\omega} - 1) + g, \quad \text{respectively.}$$

Lemma 4.2. *The function $\omega_1 = 0$ is a supersolution of* (2), *and the function $\omega_2 = u$ given by Theorem 1.1 is a subsolution of* (2).

Proof. Since $g \geqslant 0$, we have

$$\Delta\omega_1 = 0 \leqslant \lambda(e^{\omega_1} - 1) + g.$$

Noting that $\omega_2 \leqslant 0$, we have

$$\Delta\omega_2 = \lambda e^{\omega_2}(e^{\omega_2} - 1) + g \geqslant \lambda(e^{\omega_2} - 1) + g. \qquad \square$$

Similar to Section 3, we define an iterative sequence as follows. For $K > \lambda$, let $u'_0 = 0$ and consider the following equations, $k \geqslant 1$,

$$\begin{cases} (\Delta - K)u'_k = \lambda(e^{u'_{k-1}} - 1) + g - K u'_{k-1} & \text{on } \mathbb{Z}^n, \\ \lim\limits_{d(x)\to+\infty} u'_k(x) = 0. \end{cases} \tag{10}$$

We have the following lemma.

Lemma 4.3. *Let* $\{u'_k\}$ *be the sequence defined by* (10). *Then for each* k, u'_k *is uniquely defined and*

$$\omega_1 = 0 = u'_0 \geqslant u'_1 \geqslant u'_2 \geqslant \cdots \geqslant \omega_2.$$

Proof. It is clear that u'_1 is unique and $u'_1 \in l^2(\mathbb{Z}^n)$. Since

$$(\Delta - K)(\omega_2 - u'_1) \geqslant \lambda(e^{\omega_2} - 1) - K\omega_2 \geqslant (\lambda - K)\omega_2 \geqslant 0,$$

with the boundary conditions, we prove that $u'_1 \geqslant \omega_2$ by Corollary 2.3.

Suppose that

$$0 = u'_0 \geqslant u'_1 \geqslant u'_2 \geqslant \cdots \geqslant u'_i \geqslant \omega_2$$

and we have the existence and uniqueness of $u'_{i+1} \in l^2(\mathbb{Z}^n)$. By calculation, we obtain

$$\begin{aligned}(\Delta - K)(u'_{i+1} - u'_i) = \lambda(e^{u'_i} - e^{u'_{i-1}}) - K(u'_i - u'_{i-1}) &\geqslant \lambda e^{\eta_1}(u'_i - u'_{i-1}) - K(u'_i - u'_{i-1}) \\ &\geqslant K(e^{\eta_1} - 1)(u'_i - u'_{i-1}) \geqslant 0\end{aligned}$$

and

$$(\Delta - K)(\omega_2 - u'_{i+1}) \geqslant \lambda(e^{\omega_2} - e^{u'_i}) - K(\omega_2 - u'_i) \geqslant (\lambda e^{\eta_2} - K)(\omega_2 - u'_i) \geqslant 0,$$

where the functions η_1, η_2 satisfy

$$u'_i \leqslant \eta_1 \leqslant u'_{i-1} \leqslant 0 \quad \text{and} \quad \omega_2 \leqslant \eta_2 \leqslant u'_i \leqslant 0.$$

These yield that

$$\omega_2 \leqslant u'_{i+1} \leqslant u'_i.$$

□

Finally, we give a sketch of the proof of Theorem 1.2.

Proof of Theorem 1.2. By Lemma 4.3, the monotone sequence $\{u'_k\}$ is bounded in $l^2(\mathbb{Z}^n)$. Hence, we get the pointwise convergence

$$u'_k(x) \to u'(x) \quad \text{for all } x \in \mathbb{Z}^n,$$

and we obtain that $u' \in l^2(\mathbb{Z}^n)$ and u' is a topological solution to (2). In addition, if there exists another topological solution f, then

$$\Delta(u' - f) = \lambda(e^{u'} - e^f).$$

Hence there exists a function f', satisfying $\min\{u', f\} \leqslant f' \leqslant \max\{u', f\}$, such that

$$(\Delta - \lambda e^{f'})(u' - f) = 0.$$

By the maximum principle, we obtain that this solution is unique.

Furthermore, since $0 \geqslant u' \geqslant \omega_2 = u$ and

$$u = O(e^{-m(1-\epsilon)d(x)}),$$

we have

$$u' = O(e^{-m(1-\epsilon)d(x)})$$

and $u' \in l^p(\mathbb{Z}^n)$ for any $1 \leqslant p \leqslant \infty$. □

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ENTROPY MAXIMIZATION IN THE TWO-DIMENSIONAL EULER EQUATIONS

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We consider a variational problem related to entropy maximization in the two-dimensional Euler equations, in order to investigate the long-time dynamics of solutions with bounded vorticity. Using variations on the classical min-max principle and borrowing ideas from optimal transportation and quantitative rearrangement inequalities, we prove results on the structure of entropy maximizers arising in the investigation of the long-time behavior of vortex patches. We further show that the same techniques apply in the study of stability of the canonical Gibbs measure associated to a system of point vortices.

1. Long-time dynamics in two-dimensional perfect fluids

The Euler equations describing the motion of an inviscid and incompressible fluid in a two-dimensional regular simply connected domain $M \subset \mathbb{R}^2$ read

$$\begin{cases} \partial_t \omega + \boldsymbol{u} \cdot \nabla \omega = 0, \\ \omega|_{t=0} = \omega^{\mathrm{in}}, \end{cases} \tag{E}$$

where $\omega(t,x) : \mathbb{R} \times M \to \mathbb{R}$ is the vorticity and $\boldsymbol{u}(t,x) : \mathbb{R} \times M \to \mathbb{R}^2$ is the divergence-free velocity field, related to ω through the *Biot-Savart* law

$$\boldsymbol{u} = \nabla^\perp \psi = (-\partial_{x_2}\psi, \partial_{x_1}\psi), \qquad \begin{cases} \Delta\psi = \omega & \text{in } M, \\ \psi = 0 & \text{on } \partial M. \end{cases} \tag{1-1}$$

In short, $\boldsymbol{u} = \nabla^\perp \Delta^{-1}\omega$. The transport nature of equations (E) translates into the representation of solutions via the method of characteristics,

$$\omega(t,x) = \omega^{\mathrm{in}} \circ \Phi_t^{-1}(x),$$

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where

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t(x) = \boldsymbol{u}(t, \Phi_t(x)), \quad \Phi_0(x) = x,$$

is the Lagrangian flow. Thanks to Yudovich theory [1963], the Euler equations can be seen as a well-posed, weak-$*$ continuous dynamical system on the compact metric space given by the unit ball in L^∞

$$X := \{\omega \in L^\infty(M) : \|\omega\|_{L^\infty} \le 1\},$$

endowed with the weak-$*$ topology (see [Nguyen 2022] for a recent proof of this fact). It is then natural to ask what is the generic long-time picture of solutions to (E). A central conjecture due to V. Šverák [2012] posits that generic initial data give rise to solutions whose orbits are not precompact in L^2. While this conjecture, in its generality, remains currently out of reach, the recent *inviscid damping* results [Bedrossian and Masmoudi 2015; Masmoudi and Zhao 2024; Ionescu and Jia 2020; 2022; 2023] validate it in certain perturbative regimes.

A related point of view revolves around the idea that the velocity causes a cascade towards high frequencies that averages (i.e., *mixes*) the vorticity in infinite time. The Euler equations (E) preserve physically relevant quantities, hence said frequency cascade is constrained to be consistent with many conservation laws. These are the *kinetic energy*,

$$\mathsf{E}(\omega) = \tfrac{1}{2}\int_M |\nabla^\perp \Delta^{-1}\omega(x)|^2\,\mathrm{d}x = \tfrac{1}{2}\int_M |\boldsymbol{u}(x)|^2\,\mathrm{d}x,$$

the *circulation*,

$$\mathsf{K}(\omega) = \int_{\partial M} \boldsymbol{u}\cdot \mathrm{d}\ell = \int_M \omega(x)\,\mathrm{d}x,$$

and the *Casimirs*,

$$\mathsf{S}_f(\omega) = \frac{1}{|M|}\int_M f(\omega(x))\,\mathrm{d}x, \quad \text{for any continuous } f : \mathbb{R} \to \mathbb{R}.$$

Along any sequence of times tending to infinity, weak compactness implies the existence of subsequential limit points ω_∞ for the dynamics. While E and K are continuous in the weak-$*$ topology, and hence

$$(\mathsf{E}, \mathsf{K})(\omega^{\mathrm{in}}) = (\mathsf{E}, \mathsf{K})(\omega_\infty), \tag{1-2}$$

the Casimirs may lose information at infinite time. In particular, if $\omega(t_j) \overset{*}{\rightharpoonup} \omega_\infty$, along a sequence of time $t_j \to \infty$, then only upper-semicontinuity can be deduced, namely

$$\mathsf{S}_f(\omega^{\mathrm{in}}) = \limsup_{j\to\infty} \mathsf{S}_f(\omega(t_j)) \le \mathsf{S}_f(\omega_\infty), \quad \text{for any continuous concave } f : \mathbb{R} \to \mathbb{R}.$$

A strict inequality above is associated to mixing and is often observed in the long-time limit of the two-dimensional Euler equations.

To give the above observations a robust mathematical framework, one can account for the Euler evolution and its long-time limits by considering the weakly-$*$ closed set

$$\mathcal{O}_{\mathrm{in}} = \overline{\big\{\omega^{\mathrm{in}} \circ \Phi \mid \Phi : M \to M \text{ is an area preserving diffeomorphism}\big\}}^{*}, \tag{1-3}$$

which can be seen as the *orbit* of the natural action of the volume-preserving diffeomorphism group on the vorticity field ω^{in}. Although this set may strictly contain the Ω-limit set of ω^{in}, we can get close to the dynamics of $2d$ Euler by intersecting $\mathcal{O}_{\text{in}}$ with the various conservation laws (1-2). This approach provides the basis of Shnirelman's maximal mixing theory [1993], explored and revisited recently in [Dolce and Drivas 2022].

1.1. ***A statistical mechanics perspective.*** For specific choices of f (e.g., $f(\omega) = -\omega \log \omega$), S_f can be seen as a measure of *entropy*, which is a measure of the number of possible configurations at the microscopic level that leads to the observable macrostate. The *second law of thermodynamics* states that the *entropy* of an isolated system will never decrease, but will instead tend to increase over time until it reaches a maximum value at equilibrium.

L. Onsager argued in his seminal [1949] work that under certain ergodicity assumptions, Euler flows originating from point vortices should relax to vorticities that maximize the Boltzmann entropy

$$\mathsf{S}(\omega) := -\frac{1}{|M|}\int_M \omega(x) \log \omega(x)\, \mathrm{d}x, \tag{1-4}$$

subject to all conservation laws, in analogy with equilibrium statistical mechanics. The field has seen tremendous growth since then, with the development of *statistical hydrodynamics* theories corresponding to variational problems of the type

$$\text{maximize } \mathsf{S}_f(\omega) \text{ subject to } \omega \in \mathcal{O}_{\text{in}} \text{ and } \mathsf{E}(\omega) = \mathsf{E}(\omega^{\text{in}}), \tag{1-5}$$

for suitable choices of f, see the reviews [Robert 1995; Bouchet and Venaille 2012]. A rigorous picture for point vortices has been established in the seminal articles [Caglioti et al. 1992; 1995; Eyink and Spohn 1993; Kiessling 1993]. In this article we center in the choice of the Boltzmann entropy (1-4), but other choices of f are also physically relevant.

In the canonical formalism of statistical mechanics, the maximal entropy functional is computed via a variational problem over a microcanonical ensemble as

$$\mathcal{S}(\mathsf{e}) = \max\{\mathsf{S}_f(\omega) : \omega \in \mathcal{O}_{\text{in}},\ \mathsf{E}(\omega) = \mathsf{e}\}, \tag{1-6}$$

and is expected to be concave with respect to the energy level e. However, even in the simple scenario of a single vortex patch in a disk, this seems to be an open question [Šverák 2012].

One of the aims of this article is to give a general strategy to show the concavity of $\mathcal{S}$, via a variation of the classical min-max principle. The energy constraint in (1-6) can indeed be rewritten as

$$\mathcal{S}(\mathsf{e}) = \max_{\omega \in \mathcal{O}_{\text{in}}} \Big\{ \mathsf{S}_f(\omega) + \inf_{\beta \in \mathbb{R}} \beta(\mathsf{e} - \mathsf{E}(\omega)) \Big\}, \tag{1-7}$$

since

$$\inf_{\beta \in \mathbb{R}} \beta(\mathsf{e} - \mathsf{E}(\omega)) = \begin{cases} 0 & \text{if } \mathsf{E}(\omega) = \mathsf{e}, \\ -\infty & \text{otherwise.} \end{cases}$$

By formally commuting the max and the inf in (1-7), also known as the min-max principle, we are able to rewrite

$$\mathcal{S}(\mathsf{e}) = \inf_{\beta\in\mathbb{R}}\Big\{\beta\mathsf{e} + \max_{\omega\in\mathcal{O}_{\text{in}}}(\mathsf{S}_f(\omega) - \beta\mathsf{E}(\omega))\Big\} = \inf_{\beta\in\mathbb{R}}\Big\{\beta\mathsf{e} + \max_{\omega\in\mathcal{O}_{\text{in}}}\mathsf{F}_\beta(\omega)\Big\}, \tag{1-8}$$

in terms of the associated free energy

$$\mathsf{F}_\beta(\omega) := \mathsf{S}_f(\omega) - \beta\mathsf{E}(\omega). \tag{1-9}$$

If the min-max principle applies, it follows immediately that $\mathcal{S}$ is concave as it is now written as the infimum over affine functions of e. The rigorous justification of the min-max theorem is included in the Appendix, whose main requirement is the uniqueness of maximizers of F_β in $\mathcal{O}_{\text{in}}$ for every $\beta\in\mathbb{R}$. This discussion leads to the following conditional result.

Theorem 1 (uniqueness implies concavity). *Assume that $\omega^{\text{in}} = \mathbb{1}_A$, for some $A\subset M$, and suppose that $f\in C([0,\infty))\cap C^1((0,\infty))$ is strictly concave with $f'(z)\to-\infty$ as $z\to 0^+$. If for any achievable energy level the constrained entropy maximization problem* (1-5) *has a unique maximizer, then the maximal entropy functional $\mathcal{S}$ in* (1-6) *is strictly concave.*

Remark. The uniqueness of the entropy maximization problem (1-5) can be relaxed to allow uniqueness up to transformations that preserve the entropy and the energy, see Remark A.2. For instance, in the case of radially symmetric domains, we can for instance allow for uniqueness up to rotations.

Showing the uniqueness property required in the above Theorem 1 in general settings is an interesting and challenging open question. In statistical physics, nonuniqueness of the equilibrium state is directly related to phase transitions in the associated system [Georgii 2011; Delgadino et al. 2021; 2023]. For interacting particle [Carrillo et al. 2020; Chayes and Panferov 2010] and spin glass systems, several toy models exhibiting discontinuous phase transitions exist. In the case of $2d$ Euler, it could be expected that for nontrivial geometries, a discontinuous phase transitions takes place (specifically, the relevant type are first-order or discontinuous phase transitions).

While in the variational problem it is just a Lagrange multiplier, β plays the role of the inverse temperature in the language of statistical mechanics. More specifically, if there exists a unique $\beta=\beta(\mathsf{e})$ that attains the infimum in (1-8), we obtain the classical statistical mechanics identity (Clausius law)

$$\beta = \frac{\mathrm{d}\mathcal{S}}{\mathrm{de}}.$$

That is to say, the derivative of the maximal entropy with respect to the energy is the inverse temperature of the system. In particular, $\beta(\mathsf{e})$ is strictly decreasing in e, and hence we can write the energy associated to a given inverse temperature level β (see Figure 1).

1.2. ***Main results for vortex patches.*** One of the purposes of this article is to show the uniqueness of maximizers in (1-5) in the case when ω^{in} is a vortex patch on the disk, by exploiting the theory of radial rearrangements. We therefore consider $M=\mathbb{D}$, the unit disk centered at the origin, and ω^{in} the indicator

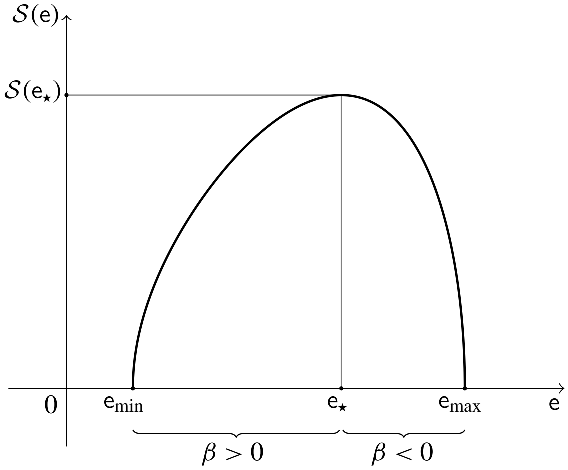

Figure 1. The function $\mathsf{e} \mapsto \mathcal{S}(\mathsf{e})$. The three cases $\beta \to -\infty$, $\beta = 0$ and $\beta \to \infty$ are associated with the energy levels $\mathsf{e}_{\max}$, $\mathsf{e}_\star$ and $\mathsf{e}_{\min}$, respectively.

function of a set $A \subset \mathbb{D}$, i.e.,

$$\omega^{\text{in}}(x) = \mathbb{1}_A(x), \qquad \frac{|A|}{|\mathbb{D}|} = \mathsf{m} \in (0, 1). \tag{1-10}$$

In this context, the corresponding set in (1-3) takes the particularly amenable form [Dolce and Drivas 2022]

$$\mathcal{O} := \left\{ \omega \in L^\infty : 0 \le \omega \le 1,\ \frac{1}{|\mathbb{D}|} \int_{\mathbb{D}} \omega(x)\,\mathrm{d}x = \mathsf{m} \right\}. \tag{1-11}$$

Since $\mathcal{O}$ is weak-$*$ compact, the weak continuity of E implies the existence of a maximum and a minimum, denoted $\mathsf{e}_{\max}$ (resp. $\mathsf{e}_{\min}$), achieved at vorticity $\omega_{\max}$ (resp. $\omega_{\min}$). Also, since $0 \notin \mathcal{O}$ and $\mathsf{E}(\omega) = 0$ if and only if $\omega = 0$, it necessarily holds that $\mathsf{e}_{\min} > 0$. In fact, $\mathsf{e}_{\max}$ and $\mathsf{e}_{\min}$ and corresponding vorticities can be computed explicitly, see Lemma 3.2. Given $\mathsf{e} \in [\mathsf{e}_{\min}, \mathsf{e}_{\max}]$, we are interested in the maximization problem

$$\text{maximize } \mathsf{S}(\omega) \text{ subject to } \omega \in \mathcal{O} \text{ and } \mathsf{E}(\omega) = \mathsf{e}, \tag{1-12}$$

where S denotes the Boltzmann entropy

$$\mathsf{S}(\omega) := -\frac{1}{|\mathbb{D}|} \int_{\mathbb{D}} \omega(x) \log \omega(x)\,\mathrm{d}x. \tag{1-13}$$

Remark (on the choice of entropy). Our analysis does not heavily rely on the specific form (1-13), although for the patch problem, several other entropies can be considered, consistent with classical theories of statistical hydrodynamics. For instance, in the Robert–Sommeria–Miller theory [Robert 1990; 1991; Miller 1990; Robert and Sommeria 1991], the choice of S should be dictated by the form of the initial datum: assume that $\mathbb{D}$ is partitioned into a disjoint union of sets $\{A_\ell\}_{\ell=1}^N$, and $\omega^{\text{in}} = \sum_\ell a_\ell \mathbb{1}_{A_\ell}$ with

$a_\ell \in [0, 1]$. Then one can define the entropy (generated by $\omega^{\rm in}$) as

$$\mathsf{S}_{\mathsf{rsm}}(\omega) := \sup\left\{-\frac{1}{|\mathbb{D}|}\int_{\mathbb{D}}\sum_\ell \rho_\ell(x)\log\rho_\ell(x)\,\mathrm{d}x \,:\, \omega=\sum_\ell a_\ell\rho_\ell,\ 0\le\rho_\ell\le 1,\ \sum_\ell \rho_\ell = 1\right\}. \tag{1-14}$$

The case of a vortex patch (1-10) leads to explicit computations: since $A_1 = A$, $A_2 = \mathbb{D}\setminus A$, $a_1 = 1$ and $a_2 = 0$, the only possible choice in (1-14) is when $\rho_1 = \omega$ and $\rho_2 = 1-\omega$, leading to

$$\mathsf{S}_{\mathsf{rsm}}(\omega) := -\frac{1}{|\mathbb{D}|}\int_{\mathbb{D}}\big[\omega(x)\log\omega(x) + (1-\omega(x))\log(1-\omega(x))\big]\,\mathrm{d}x,$$

Another possibility, introduced by Turkington [1999], is to consider a different maximization problem compared to (1-14), that, in the case of a vortex patch, allows vorticity to mix on the whole range of small scales $a \in [a_2, a_1] = [0, 1]$. This leads to the entropy

$$\mathsf{S}_{\mathsf{t}}(\omega) := \frac{1}{|\mathbb{D}|}\int_{\mathbb{D}} f_{\mathsf{t}}(\omega(x))\,\mathrm{d}x,$$

with

$$f_{\mathsf{t}}(\omega) = \sup\left\{-\int_0^1 \rho(y)\log\rho(y)\,\mathrm{d}y \,:\, \rho\ge 0,\ \int_0^1\rho(y)\,\mathrm{d}y = 1,\ \int_0^1 y\rho(y)\,\mathrm{d}y = \omega\right\}.$$

This amounts to performing an entropy maximization over all probability densities $\rho\,\mathrm{d}y$ in (0, 1) which give ω as their mean value.

The first main result of this article is the following characterization of the maximal entropy functional, in complete analogy with classical statistical mechanics.

Theorem 2. *The function* $\mathcal{S} : [\mathsf{e}_{\min}, \mathsf{e}_{\max}] \to [0, \infty)$ *defined as*

$$\mathcal{S}(\mathsf{e}) = \max\{\mathsf{S}(\omega) \,:\, \omega\in\mathcal{O},\ \mathsf{E}(\omega) = \mathsf{e}\}$$

is strictly concave. Moreover,

(a) $\mathcal{S}$ *has a unique*, *strictly positive global maximum at* $\mathsf{e}_\star = \pi\,\mathsf{m}^2/16$;

(b) $\mathcal{S}(\mathsf{e}_{\min}) = \mathcal{S}(\mathsf{e}_{\max}) = 0$;

(c) $\mathcal{S}$ *is increasing on* $[\mathsf{e}_{\min}, \mathsf{e}_\star]$ *and decreasing on* $[\mathsf{e}_\star, \mathsf{e}_{\max}]$.

Remark. Theorem 2(c) is a consequence of concavity and Theorem 2(a)–(b).

In light of Theorem 1, the rigorous justification of the min-max principle in the Appendix needed for Theorem 2 requires the uniqueness of maximizers of F_β in $\mathcal{O}$ for every $\beta\in\mathbb{R}$. This is the second main result of this article.

Theorem 3. *For any* $\beta\in\mathbb{R}$, *there exists a unique solution* $\omega_\beta\in\mathcal{O}$ *of the maximization problem*

$$\textit{maximize } \mathsf{F}_\beta(\omega) \textit{ subject to } \omega\in\mathcal{O}.$$

Moreover,

- *if* $\beta\ge 0$, *then* ω_β *is radially increasing*,
- *if* $\beta\le 0$, *then* ω_β *is radially decreasing.*

In particular, ω_β *is constant when* $\beta = 0$.

Our uniqueness proof follows closely the developments in uniqueness of steady states for the standard Keller–Segel model [Calvez and Carrillo 2012], and their homogeneous variants [Calvez et al. 2021; Carrillo et al. 2015]. In broad terms, the strategy is to find suitable interpolation curves between two competitor states, such that the free energy over the curve is convex or monotone. The seminal paper of McCann [1997] implements this idea with the interpolation curves given by the geodesics of the optimal transportation distance. We also mention the novel interpolation curve in [Delgadino et al. 2022] for radially decreasing states, which unfortunately is not directly applicable to our setting. Inspired by [Calvez and Carrillo 2012; Calvez et al. 2021; Carrillo et al. 2015], we first use rearrangement theory [Talenti 1976; Kesavan 2006] to reduce the problem to one-dimensional radially decreasing profiles. Then we consider the optimal transportation interpolation between a maximizer and a competitor. Finally, employing Jensen's inequality and the Euler–Lagrange equation for the maximizer, we can show the strict monotonicity of the free energy along the interpolation curve. One of the main differences with [Calvez et al. 2021; Carrillo et al. 2015] is that we need to deal with the L^∞ constraint, imposed by (1-11).

1.3. ***On the conservation of angular momentum.*** The constrained maximization problem (1-12) takes into account the conservation laws (1-2), but neglects the additional symmetries of the disk that give rise to conservation of *angular momentum*

$$\mathsf{A}(\omega) = -\int_{\mathbb{D}} \tfrac{1}{2}(1-|x|^2)\omega(x)\,\mathrm{d}x = \int_{\mathbb{D}} x^\perp \cdot \boldsymbol{u}(x)\,\mathrm{d}x, \tag{1-15}$$

which is weak-$*$ continuous, like the energy E.

The inclusion of this extra constraint changes the picture dramatically. Indeed, entropy maximizers are no longer necessarily radially symmetric, as the following heuristics illustrate. Ignoring for the moment the L^∞ constraint that the set $\mathcal{O}$ in (1-11) imposes, the limiting case of maximal angular momentum $\mathsf{a}_{\max} = 0$ is achieved only for states ω which are supported on the boundary $\partial\mathbb{D}$. In which case, if ω_{rad} is radial then $\omega_{\mathrm{rad}} = c\delta_{\partial\mathbb{D}}$ and $\mathsf{E}(\omega_{\mathrm{rad}}) = 0$. On the other hand, the nonradial state $\omega_{x_0} = \delta_{x_0}$ with $x_0 \in \partial\mathbb{D}$ also has zero angular momentum, and formally has unbounded energy $\mathsf{E}(\omega_{x_0}) = +\infty$. This extreme situation hints that constraining the angular momentum implies there are energy levels that are not achievable by radial vorticities. A rigorous statement describing this situation is contained in the following result.

Theorem 4. *For* $\mathsf{m} \in (0, 1)$ *and* $L \geq 1$, *consider the set*

$$\mathcal{O}_L := \left\{\omega \in L^\infty : 0 \leq \omega \leq L,\ \frac{1}{|\mathbb{D}|}\int_{\mathbb{D}} \omega(x)\,\mathrm{d}x = \mathsf{m}\right\}.$$

For the radial optimization problem, there exists $C > 1$ *independent of* L, a *and* m *such that*

$$\sup\bigl\{\mathsf{E}(\omega) : \omega \in \mathcal{O}_L,\ \mathsf{A}(\omega) = \mathsf{a},\ \omega \text{ is radial}\bigr\} \leq C\left(\mathsf{m}|\mathsf{a}| + |\mathsf{a}|^2 \log\left(\frac{L}{|\mathsf{a}|}\right)\right). \tag{1-16}$$

For the nonradial case, if $L \geq 4\pi^2\mathsf{m}^3/|\mathsf{a}|^2$, *we have the lower bound*

$$\sup\{\mathsf{E}(\omega) : \omega \in \mathcal{O}_L,\ \mathsf{A}(\omega) = \mathsf{a}\} \geq \frac{\pi\,\mathsf{m}^2}{4}\log\left(\frac{L|\mathsf{a}|^2}{64\pi^2\mathsf{m}^3}\right). \tag{1-17}$$

In particular, there exist $\mathsf{a}_* \in \left(-\frac{1}{2}, 0\right)$ *and* $Q > 2\pi$ *depending on* m, *such that if* $\mathsf{a} \in (\mathsf{a}_*, 0)$ *and* $L = Q^2 m^3/|\mathsf{a}|^2$, *then*

$$\sup\left\{\mathsf{E}(\omega) : \omega \in \mathcal{O}_L,\ \mathsf{A}(\omega) = \mathsf{a},\ \omega \text{ is radial}\right\} < \sup\left\{\mathsf{E}(\omega) : \omega \in \mathcal{O}_L,\ \mathsf{A}(\omega) = \mathsf{a}\right\}.$$

The radial bound (1-16) follows by utilizing the formula

$$\mathsf{E}(\omega) = \frac{1}{4\pi}\int_0^1 \frac{1}{r}\left|\int_{B_r} \omega(x)\,\mathrm{d}x\right|^2 \mathrm{d}r,$$

which is valid only for radial functions. The bound follows by estimating the amount of vorticity near the origin, using the L^∞ bound and the angular momentum. The lower bound (1-17) follows by calculating the energy of a vortex patch of the form

$$\omega_{x_0,L} = L\mathbb{1}_{B_{\sqrt{\mathsf{m}/L}}(x_0)}.$$

The complete proof of Theorem 4 is postponed to Section 5.

We note that the proof of Theorem 4 is similar in spirit to [Dolce and Drivas 2022, Theorem 2]. The main difference is that Dolce and Drivas consider the case of a periodic channel (hence not simply connected) instead of the disk. In their case, the conserved quantity of interest is the linear momentum instead of the angular momentum.

1.4. ***On the stability of Onsager solutions.*** The Euler–Lagrange equations associated to the variational problem (1-12) resemble those appearing in the context of mean-field limits of point-vortices studied in [Caglioti et al. 1992; 1995; Eyink and Spohn 1993; Kiessling 1993]. Specifically, in the setting of the unit disk and for $\beta \in (-8\pi, \infty)$, there exists a unique radial solution to the mean field equation

$$\omega_\beta = \frac{\mathrm{e}^{\beta\psi_\beta}}{Z_\beta} \qquad \text{and} \qquad \begin{cases} \Delta\psi_\beta = \omega_\beta & \text{in } \mathbb{D}, \\ \psi_\beta = 0 & \text{on } \partial\mathbb{D}, \end{cases} \tag{1-18}$$

where

$$Z_\beta = \int_{\mathbb{D}} \mathrm{e}^{\beta\psi_\beta(x)}\,\mathrm{d}x,$$

which is given in radial variables by

$$\omega_\beta(r) = \frac{1 - A(\beta)}{\pi}\frac{1}{(1 - A(\beta)r^2)^2}, \quad \text{with } A(\beta) = \frac{\beta}{8\pi + \beta}. \tag{1-19}$$

We call such steady Euler flows *Onsager solutions*, as they appeared first in [Onsager 1949]. A result of [Caglioti et al. 1995], rephrased with the terminology of this article, states that such solutions arise as maximizers of the same free energy F_β in (1-9) over the set

$$\mathcal{P} = \left\{\omega \in L^1 : \omega \geq 0,\ \int_{\mathbb{D}} \omega(x)\,\mathrm{d}x = 1,\ \int_{\mathbb{D}} \omega(x)\log\omega(x)\,\mathrm{d}x < \infty\right\}.$$

Moreover, we notice by (1-19) that as $\beta \to -8\pi$ we have $\omega_\beta \rightharpoonup \delta_0$, see Theorem 6.1 below for a precise statement. This restriction of $\beta > -8\pi$ did not apply to the previous results as we considered the vortex patch problem, which has the conservation of the L^∞ norm which prevents blow-up.

Quantitative stability of these solutions in L^1 has been addressed in [Lemou 2016], using the stability of the Hardy–Littlewood inequality applied to angular momentum. The techniques developed in the proof of Theorem 3 allow us to prove the following qualitative L^2-stability result.

Theorem 5. *For any $\beta \in (-8\pi, \infty)$, the solution ω_β to* (1-18) *is L^2 stable with respect to L^∞ perturbations. That is, for any $\varepsilon > 0$ and any positive $\omega^{\mathrm{in}} \in L^\infty$, there exists $\delta = \delta(\varepsilon, \|\omega^{\mathrm{in}}\|_{L^\infty}) > 0$ such that if $\|\omega^{\mathrm{in}} - \omega_\beta\|_{L^2} < \delta$, the corresponding Euler solution $\omega = \omega(t)$ is such that $\|\omega(t) - \omega_\beta\|_{L^2} < \varepsilon$ for any $t > 0$.*

The case $\beta \geq 0$ follows from the classical method of Arnold [1966; Arnold and Khesin 1998], as the right-hand side of (1-18) is an increasing function (see also [Gallay and Šverák 2024] for a recent revisitation of the method). However, the case $\beta < 0$ is nontrivial, and it will be our main focus. Indeed, Arnold criteria for stability of steady Euler solutions satisfying the semilinear elliptic equation $\Delta\psi = F(\psi)$ require that

$$-\lambda_1 < F' < 0 \quad \text{or} \quad 0 < F' < \infty,$$

where $\lambda_1 > 0$ is the first eigenvalue of the Dirichlet Laplacian. Such a condition is clearly violated by (1-18).

To obtain this Lyapunov stability result we make use of a quantitative Jensen's inequality and an adaptation of Talenti's original argument [1976]. In particular, we borrow ideas from [Amato et al. 2024], which combine the arguments of the quantitative versions of Polya–Szegő [Cianchi et al. 2008] and Hardy–Littlewood [Cianchi and Ferone 2008] inequalities, specialized to the solutions of the Poisson equation to obtain a quantitative version of Talenti's inequality.

We mention that the uniqueness of Onsager-type solutions in the sphere $\mathbb{S}^2$ was recently addressed in [Gui and Moradifam 2018] by studying Onofri's inequality [1982], which settled a conjecture in conformal geometry [Chang and Yang 1987]. The stability of Theorem 5 is related to the dual formulation of Onofri's inequality, which was exploited recently in [Carlen 2025] to obtain stability of the log-HLS inequality.

2. Uniqueness implies concavity

The purpose of this section is to prove Theorem 1, for vortex patches of the form (1-10) in a general two-dimensional simply connected domain $M \subset \mathbb{R}^2$. In which case, we will make use of the characterization (1-11) for the orbit of the patch, with the disk $\mathbb{D}$ replaced by M. As mentioned in Section 1.2, the concavity of $\mathcal{S}$ is a consequence of the min-max principle stated in Proposition A.1. The only nontrivial requirement is stated in Proposition A.1(e), which requires the uniqueness of maximizers $\omega_\beta \in \mathcal{O}$ of the functional

$$\mathsf{L}(\omega, \beta) = \beta\mathsf{e} + \mathsf{S}_f(\omega) - \beta\mathsf{E}(\omega)$$

for a given fixed $\beta \in \mathbb{R}$.

In the generality of Section 1.1, the energy functional E still achieves a maximum and minimum values $\mathsf{e}_{\max} \geq \mathsf{e}_{\min} \geq 0$ on $\mathcal{O}$. Theorem 1 is then a consequence of Proposition A.1 and the following result.

Proposition 2.1. *Assume that the free energy F_β has a unique maximizer $\omega_\beta \in \mathcal{O}$ for each $\beta \in \mathbb{R}$. Then the function $\mathsf{e} \mapsto \mathcal{S}(\mathsf{e})$ is strictly concave on $[\mathsf{e}_{\min}, \mathsf{e}_{\max}]$.*

We start by deriving an Euler–Lagrange equation, which we will use in the proof of Proposition 2.1.

Lemma 2.2. *Assume* $f \in C([0,\infty)) \cap C^1((0,\infty))$ *is concave and* $\lim_{z\to 0^+} f'(z) = -\infty$. *Any maximizer* $\bar\omega$ *of* F_β *over* $\mathcal{O}$ *satisfies* $\inf \bar\omega > 0$, *and there exists* $\bar\lambda = \bar\lambda(\bar\omega)$ *such that*

$$\frac{1}{|M|} f'(\bar\omega) + \beta\bar\psi = \bar\lambda \quad \text{a.e. on } \{\bar\omega < 1\}, \tag{2-1}$$

where

$$\begin{cases} \Delta\bar\psi = \bar\omega & \text{in } M, \\ \bar\psi = 0 & \text{on } \partial M. \end{cases}$$

Proof. To prove (2-1), we consider a positive smooth function φ, such that

$$\frac{1}{|M|}\int_M (1-\bar\omega)\varphi = 1. \tag{2-2}$$

We take the perturbation

$$\bar\omega_\varepsilon = \frac{\mathsf{m}}{\mathsf{m}+\varepsilon}(\bar\omega + \varepsilon(1-\bar\omega)\varphi),$$

with $\varepsilon > 0$ small enough to satisfy $\bar\omega_\varepsilon \in \mathcal{O}$. Taking a variation of the entropy, and using concavity, we know

$$\liminf_{\varepsilon\to 0^+} \frac{\mathsf{S}_f(\bar\omega_\varepsilon) - \mathsf{S}_f(\bar\omega)}{\varepsilon} \geq -\frac{1}{|M|}\int_M f'(\bar\omega)(1-\bar\omega)\varphi - \frac{1}{\mathsf{m}}\mathsf{S}_f(\bar\omega).$$

Similarly, taking a variation of the Energy we get

$$\lim_{\varepsilon\to 0^+} \frac{\mathsf{E}(\bar\omega_\varepsilon) - \mathsf{E}(\bar\omega)}{\varepsilon} = -\int_M \bar\psi(1-\bar\omega)\varphi - \frac{2}{\mathsf{m}}\mathsf{E}(\bar\omega).$$

Using the maximality property of $\bar\omega$, we know that

$$\lim_{\varepsilon\to 0^+} \frac{\mathsf{F}_\beta(\bar\omega_\varepsilon) - \mathsf{F}_\beta(\bar\omega)}{\varepsilon} \leq 0,$$

which immediately implies that

$$\int_M \left(\frac{f'(\bar\omega)}{|M|} + \beta\bar\psi\right)(1-\bar\omega)\varphi \geq \bar\lambda(\bar\omega) := \frac{1}{\mathsf{m}}(\mathsf{S}(\bar\omega) - 2\beta\mathsf{E}(\bar\omega))$$

for any positive and smooth test function φ which satisfies (2-2). Therefore,

$$\frac{f'(\bar\omega)}{|M|} + \beta\bar\psi \geq \bar\lambda \quad \text{a.e. on } \{\bar\omega < 1\}.$$

Using that $f'(0) = -\infty$ and that $\bar\psi$ is uniformly bounded, we have that $\bar\omega$ is uniformly bounded below in M. Hence, we have that the perturbation

$$\bar\omega_\varepsilon = \frac{\mathsf{m}}{\mathsf{m}-\varepsilon}(\bar\omega - \varepsilon\varphi) \in \mathcal{O}$$

for any positive smooth φ satisfying

$$\frac{1}{|M|}\int_M \varphi = 1$$

and $\varepsilon > 0$ small enough. Following the same arguments as above, we obtain the reverse inequality

$$\frac{f'(\bar\omega)}{|M|} + \beta\bar\psi \leq \bar\lambda \quad \text{a.e. on } M,$$

and (2-1) follows. □

Proof of Proposition 2.1. Applying the min-max principle in Proposition A.1, the maximal entropy function can be written as an infimum over affine functions in e, namely

$$\mathcal{S}(\mathsf{e}) = \inf_{\beta\in\mathbb{R}} \Big\{ \beta\mathsf{e} + \max_{\omega\in\mathcal{O}_{\text{in}}} \mathsf{F}_\beta(\omega) \Big\}, \tag{2-3}$$

and hence it is concave in e. We now verify its strict concavity. For any $\beta \in \mathbb{R}$, let

$$g(\beta) := \max_{\omega\in\mathcal{O}_{\text{in}}} \mathsf{F}_\beta(\omega).$$

Being the supremum of affine functions of β, g is a proper lower semicontinuous convex function. Therefore, for any $\beta \in \mathbb{R}$, the subdifferential $\partial g(\beta)$ is nonempty and monotone. We define the set

$$\beta(\mathsf{e}) := \{\beta \in \mathbb{R} : \mathcal{S}(\mathsf{e}) = \beta\mathsf{e} + g(\beta)\}.$$

We claim that $\beta(\mathsf{e})$ is nonempty in $(\mathsf{e}_{\min}, \mathsf{e}_{\max})$, single-valued and monotone as a function of e.

• $\beta(\mathsf{e})$ **is nonempty.** For $\mathsf{e} \in (\mathsf{e}_{\min}, \mathsf{e}_{\max})$, we first show that

$$\lim_{\beta\to\pm\infty} (\beta\mathsf{e} + g(\beta)) = +\infty. \tag{2-4}$$

The expression for the energy optimizers $\omega_{\max}$ and $\omega_{\min}$ can be computed explicitly (see (3-6) and (3-7)). In particular, $\omega_{\max}, \omega_{\min} \in \{0, 1\}$, and hence $\mathsf{S}_f(\omega_{\min})$, $\mathsf{S}_f(\omega_{\max})$ are finite.

For $\beta > 0$, we have

$$-\beta\mathsf{e}_{\min} + \mathsf{S}_f(\omega_{\min}) = \mathsf{F}_\beta(\omega_{\min}) \le \max_{\omega\in\mathcal{O}_{\text{in}}} \mathsf{F}_\beta(\omega) = g(\beta),$$

which implies the bound

$$\beta(\mathsf{e} - \mathsf{e}_{\min}) + \mathsf{S}_f(\omega_{\min}) \le \beta\mathsf{e} + g(\beta),$$

and shows that $\lim_{\beta\to\infty} (\beta\mathsf{e} + g(\beta)) = +\infty$. Similarly, for $\beta < 0$, we have the bound

$$-\beta\mathsf{e}_{\max} + \mathsf{S}_f(\omega_{\max}) = \mathsf{F}_\beta(\omega_{\max}) \le g(\beta),$$

which implies the bound

$$-\beta(\mathsf{e}_{\max} - \mathsf{e}) + \mathsf{S}_f(\omega_{\max}) \le \beta\mathsf{e} + g(\beta),$$

and the claim (2-4) follows. If $\{\beta_n\} \subset \mathbb{R}$ is a minimizing sequence such that

$$\mathcal{S}(\mathsf{e}) = \lim_{n\to\infty} (\beta_n\mathsf{e} + g(\beta_n)),$$

then by (2-4) we must have that $\{\beta_n\}$ is bounded and therefore has a limit point $\bar\beta$. By the lower semicontinuity of f we find from (2-3) that $\bar\beta\mathsf{e} + g(\bar\beta) = \mathcal{S}(\mathsf{e})$, proving the claim.

• **Characterization.** If $\beta \in \beta(\mathsf{e})$, then

$$-\mathsf{e} \in \partial g(\beta), \tag{2-5}$$

because β is a minimizer. Next we show that the subdifferential of f is given by

$$\partial g(\beta) = \{-\mathsf{E}(\omega_\beta)\}, \tag{2-6}$$

where ω_β is the unique maximizer of F_β over $\mathcal{O}_{\rm in}$. Since g is the pointwise supremum of affine functions $\mathsf{G}_\omega(\beta) := \mathsf{S}_f(\omega) - \beta\mathsf{E}(\omega)$ over $\omega \in \mathcal{O}_{\rm in}$, the subdifferential $\partial g(\beta)$ is in general [Zălinescu 2002, Theorem 2.4.18] given by

$$\partial g(\beta) = \mathrm{co}\biggl(\bigcup_{\omega\in\mathcal{O}_{\rm in}} \{\partial \mathsf{G}_\omega(\beta) \,:\, \mathsf{G}_\omega(\beta) = g(\beta)\}\biggr),$$

where $\mathrm{co}(B)$ denotes the closed convex hull of the set B. Due to our uniqueness assumption and the fact that G_ω is differentiable, in our case the above identity reduces to

$$\partial g(\beta) = \Bigl\{\frac{\mathrm{d}}{\mathrm{d}\beta} G_\omega(\beta)\big|_{\omega=\omega_\beta}\Bigr\} = \{-\mathsf{E}(\omega_\beta)\},$$

which is (2-6).

• **Strict monotonicity.** Thanks to (2-5)–(2-6), the function $\beta(\mathsf{e})$ is given implicitly by the equation

$$\mathsf{E}(\omega_{\beta(\mathsf{e})}) = \mathsf{e}.$$

Hence $\beta(\mathsf{e})$ is well-defined and monotone if we can prove that the mapping $\beta \to \mathsf{E}(\omega_\beta)$ is strictly monotone.

Given $\beta_1 < \beta_2$, we want to show that $\mathsf{E}(\omega_{\beta_1}) > \mathsf{E}(\omega_{\beta_2})$. By the Euler–Lagrange condition (2-1), we know that $\omega_{\beta_1} \neq \omega_{\beta_2}$. Hence, by uniqueness of maximizers, we have

$$\mathsf{F}_{\beta_1}(\omega_{\beta_1}) > \mathsf{F}_{\beta_1}(\omega_{\beta_2}), \quad \mathsf{F}_{\beta_2}(\omega_{\beta_2}) > \mathsf{F}_{\beta_2}(\omega_{\beta_1}),$$

which implies

$$(\beta_2-\beta_1)(\mathsf{E}(\omega_{\beta_2}) - \mathsf{E}(\omega_{\beta_1})) = (\mathsf{F}_{\beta_1}(\omega_{\beta_2}) - \mathsf{F}_{\beta_1}(\omega_{\beta_1})) + (\mathsf{F}_{\beta_2}(\omega_{\beta_1}) - \mathsf{F}_{\beta_2}(\omega_{\beta_2})) < 0.$$

This shows the desired strict monotonicity. □

Remark. The assumption that $\omega^{\rm in} = \mathbb{1}_A$ for some $A \subset M$ is only used to show that for any $\beta_1 \neq \beta_2$ we have that $\omega_{\beta_1} \neq \omega_{\beta_2}$, which stems from the Euler–Lagrange condition (2-1).

3. Free energy and entropy maximizers

In this section we analyze various aspects of entropy maximization related to the free energy F_β. We specialize from now on in setting of Section 1.2, hence considering the vortex patch problem in the unit disk $M = \mathbb{D}$, with entropy given by (1-13). Since our analysis is based on rearrangements inequality, we first take some time to review the basic concepts of radial rearrangements. The assumptions on f are those of Theorem 1.

3.1. ***Rearrangements and radial symmetry of maximizers.*** A standard technique for studying maximizers is utilizing rearrangements of mass. Given a set $B \subset \mathbb{D}$, its symmetric rearrangement $B^\sharp$ is the open centered ball whose volume agrees with B, namely

$$B^\sharp = \{x \in \mathbb{R}^2 \,:\, \pi|x|^2 < |B|\}.$$

Given a function $\omega \in \mathcal{O}$, its *symmetric decreasing rearrangement* is defined by

$$\omega^\sharp(x) = \int_0^1 \mathbb{1}_{[\omega>t]^\sharp}(x)\,\mathrm{d}t.$$

Notice that $\omega^\sharp$ is radial. The *symmetric increasing rearrangement* of ω is

$$\omega_\sharp(x) = \omega^\sharp(\sqrt{1-|x|^2}\,).$$

It follows directly from a theorem of Talenti [1976, Theorem 1(v)] that

$$\mathsf{E}(\omega) = \tfrac{1}{2}\|\nabla\Delta^{-1}\omega\|_{L^2}^2 \le \tfrac{1}{2}\|\nabla\Delta^{-1}\omega^\sharp\|_{L^2}^2 = \mathsf{E}(\omega^\sharp) \tag{3-1}$$

for any $\omega \in \mathcal{O}$, with equality if and only if $\omega = \omega^\sharp$; see [Carlen and Loss 1992; Kesavan 2006]. We are also interested in comparing the kinetic energy among *radial* vorticities. If ω is radial, then the corresponding stream function can be explicitly derived from (1-1) as

$$\psi(r) = -\int_r^1 \frac{1}{s}\int_0^s \omega(\bar{s})\bar{s}\,\mathrm{d}\bar{s}\,\mathrm{d}s. \tag{3-2}$$

For any radial function $g \in \mathcal{O}$, define

$$M_g(r) = 2\pi\int_0^r g(s)s\,\mathrm{d}s = \int_{B_r} g(x)\,\mathrm{d}x, \quad r \in (0,1], \tag{3-3}$$

where B_r is the ball of radius r centered at the origin. Thanks to (3-2), for any radial $\omega \in \mathcal{O}$, it is not hard to see that

$$\mathsf{E}(\omega) = \pi\int_0^1 |\partial_r\psi(r)|^2 r\,\mathrm{d}r = \frac{1}{4\pi}\int_0^1 \frac{1}{r}|M_\omega(r)|^2\,\mathrm{d}r. \tag{3-4}$$

Moreover, we have the following comparison principle in the radial case.

Lemma 3.1. *Let $\omega \in \mathcal{O}$ be a radial function. Then*

$$\mathsf{E}(\omega_\sharp) \le \mathsf{E}(\omega) \le \mathsf{E}(\omega^\sharp).$$

Proof. Thanks to the Hardy–Littlewood inequality[1] and the fact that $(\mathbb{1}_{B_r})^\sharp = \mathbb{1}_{B_r}$, we have

$$\int_{\mathbb{D}} \mathbb{1}_{B_r}(x)\omega_\sharp(x)\,\mathrm{d}x \le \int_{\mathbb{D}} \mathbb{1}_{B_r}(x)\omega(x)\,\mathrm{d}x \le \int_{\mathbb{D}} \mathbb{1}_{B_r}(x)\omega^\sharp(x)\,\mathrm{d}x,$$

implying that $M_{\omega_\sharp} \le M_\omega \le M_{\omega^\sharp}$. The claim follows from (3-4). □

The representation (3-4) of E is also useful to compute explicitly the energy for specific vorticities. For instance,

$$\omega_\star \equiv \mathsf{m} \quad\Longrightarrow\quad \mathsf{e}_\star := \mathsf{E}(\omega_\star) = \frac{\pi\mathsf{m}^2}{16}. \tag{3-5}$$

[1] For any two measurable functions $f, g : \mathbb{D} \to [0,\infty)$, it holds that $\int_{\mathbb{D}} f^\sharp g_\sharp \le \int_{\mathbb{D}} fg \le \int_{\mathbb{D}} f^\sharp g^\sharp$.

Now, for any radial $\omega \in \mathcal{O}$, we have that $M_\omega(r) \le \pi \min\{r^2, \mathsf{m}\}$. Defining

$$\omega_{\max}(r) = \begin{cases} 1, & r \in (0, \sqrt{\mathsf{m}}), \\ 0, & r \in (\sqrt{\mathsf{m}}, 1), \end{cases} \tag{3-6}$$

we then have that $M_\omega \le M_{\omega_{\max}}$. Similarly,

$$M_\omega(r) = M_\omega(1) - 2\pi \int_r^1 \omega(s) s \, \mathrm{d}s \ge \pi(\mathsf{m} - 1 + r^2),$$

so that $M_\omega \ge M_{\omega_{\min}}$, where

$$\omega_{\min}(r) = \begin{cases} 0, & r \in (0, \sqrt{1-\mathsf{m}}), \\ 1, & r \in (\sqrt{1-\mathsf{m}}, 1). \end{cases} \tag{3-7}$$

As a consequence, by a direct computation of $\mathsf{E}(\omega_{\min})$ and $\mathsf{E}(\omega_{\max})$ we have the following result, which we state without proof.

Lemma 3.2. *For any radial $\omega \in \mathcal{O}$ we have*

$$\mathsf{e}_{\min} \le \mathsf{E}(\omega) \le \mathsf{e}_{\max},$$

where

$$\mathsf{e}_{\min} := \mathsf{E}(\omega_{\min}) = \frac{\pi \mathsf{m}^2}{16} - \frac{\pi}{8}(1-\mathsf{m})\big(\mathsf{m} + (1-\mathsf{m})\log(1-\mathsf{m})\big)$$

and

$$\mathsf{e}_{\max} := \mathsf{E}(\omega_{\max}) = \frac{\pi \mathsf{m}^2}{16} + \frac{\pi \mathsf{m}^2}{8} |\log \mathsf{m}|.$$

In fact, the above functions satisfy the stronger property below.

Lemma 3.3. *The functions $\omega_{\min}$ and $\omega_{\max}$ are the unique functions that achieve their energy levels, that is*

$$\begin{aligned} \mathsf{E}(\omega) = \mathsf{e}_{\min} \quad &\Longrightarrow \quad \omega = \omega_{\min}, \\ \mathsf{E}(\omega) = \mathsf{e}_{\max} \quad &\Longrightarrow \quad \omega = \omega_{\max}. \end{aligned} \tag{3-8}$$

Proof. Since $\omega \mapsto \mathsf{E}(\omega)$ is a convex function, it has a unique global minimizer in $\mathcal{O}$. Moreover, such minimizer is radially symmetric, since E is invariant under rotation. Thus, Lemma 3.2 implies that $\omega_{\min}$ is the unique minimizer. Turning to (3-8), we know by (3-1) that any energy maximizer is necessarily radially decreasing, and by Lemma 3.2 that $\omega_{\max}$ is one of them. If $\bar{\omega}$ is another radially decreasing maximizer, it is not hard to see that $\bar{\omega} = \omega_{\max}$ if and only if $M_{\bar{\omega}} = M_{\omega_{\max}}$, i.e., if and only if $\mathsf{E}(\bar{\omega}) = \mathsf{E}(\omega_{\max})$. □

3.2. ***Relaxed maximization problems.*** The strategy to prove Theorem 2 is to study (relaxed versions of) the maximization problem (1-12) and apply a min-max principle. In turn, we will see how this is reduced to study uniqueness of maximizers for the free energy F_β in (1-9), as $\beta \in \mathbb{R}$ varies. We begin with the following observations.

Lemma 3.4. *For every $\mathsf{e} \in [\mathsf{e}_{\min}, \mathsf{e}_{\max}]$, the constrained maximization problem* (1-12) *admits at least one solution.*

Proof. Notice that S_f is strictly concave and upper semicontinuous. Let ω_n be a maximizing sequence. Since $\mathcal{O}$ is weak-$*$ compact, there exists $\bar\omega \in \mathcal{O}$ such that $\omega_n \overset{*}{\rightharpoonup} \bar\omega$ up to subsequences, and since $\mathsf{E}(\omega_n) \to \mathsf{E}(\bar\omega)$ by continuity, $\mathsf{E}(\bar\omega) = \mathsf{e}$. Moreover,

$$\sup\bigl\{\mathsf{S}_f(\omega) \,:\, \omega \in \mathcal{O},\ \mathsf{E}(\omega) = \mathsf{e}\bigr\} = \limsup_{n\to\infty} \mathsf{S}_f(\omega_n) \le \mathsf{S}_f(\bar\omega),$$

as S_f is upper semicontinuous. Hence $\bar\omega$ is a maximizer. □

It is also important to identify the global entropy maximizer, regardless of the energy constraint.

Lemma 3.5. *The unconstrained maximization problem*

$$\textit{maximize } \mathsf{S}_f(\omega) \textit{ subject to } \omega \in \mathcal{O}$$

has a unique maximizer given by the constant solution $\omega_\star \equiv \mathsf{m}$.

Proof. The proof follows from the previous Lemma 3.4, with the uniqueness stemming from the strict concavity of the entropy functional. Assume now that $\bar\omega$ is the unique maximizer, and let $\Phi : \mathbb{D} \to \mathbb{D}$ be a volume-preserving diffeomorphism. Then $\bar\omega \circ \Phi \in \mathcal{O}$, and $\mathsf{S}_f(\bar\omega \circ \Phi) = \mathsf{S}_f(\bar\omega)$. Thus, by uniqueness, $\bar\omega \circ \Phi = \bar\omega$, and since Φ is arbitrary, this implies $\bar\omega$ is constant, hence equal to its average. □

Thanks to the computation in (3-5), Theorem 2(a) is proved. The role of $\mathsf{e}_\star = \mathsf{E}(\omega_\star)$ in (3-5) is quite interesting. Indeed, for a fixed energy level $\mathsf{e} \le \mathsf{e}_\star$, the maximization problem (1-12) can be relaxed into a convex one.

Lemma 3.6. *If* $\mathsf{e}_{\min} \le \mathsf{e} \le \mathsf{e}_\star$, *then the constrained maximization problem* (1-12) *is equivalent to the relaxed problem*

$$\textit{maximize } \mathsf{S}_f(\omega) \textit{ subject to } \omega \in \mathcal{O},\ \mathsf{E}(\omega) \le \mathsf{e}. \tag{3-9}$$

In particular, *problem* (3-9), *and hence* (1-12), *have a unique solution.*

If $\mathsf{e}_\star \le \mathsf{e} \le \mathsf{e}_{\max}$, *then the constrained maximization problem* (1-12) *is equivalent to the relaxed problem*

$$\textit{maximize } \mathsf{S}_f(\omega) \textit{ subject to } \omega \in \mathcal{O},\ \mathsf{E}(\omega) \ge \mathsf{e}. \tag{3-10}$$

Proof. We first look at the case $\mathsf{e} \le \mathsf{e}_\star$ and argue by contradiction. Fix any maximizer $\bar\omega \in \mathcal{O}$ with $\mathsf{E}(\bar\omega) < \mathsf{e}$. For any $t \in [0, 1]$, the convex combination $\omega_t = (1-t)\bar\omega + t\omega_\star$ is in $\mathcal{O}$. Moreover, using the continuity of E, we have that for t small enough $\mathsf{E}(\omega_t) < \mathsf{e}$. Since $\bar\omega$ is a maximizer,

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathsf{S}_f(\omega_t)\Big|_{t=0} \le 0.$$

However, a direct computation shows

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathsf{S}_f(\omega_t)\Big|_{t=0} = \frac{1}{|\mathbb{D}|}\int_{\mathbb{D}} (\omega_\star - \bar\omega) f'(\bar\omega) = \frac{1}{|\mathbb{D}|}\int_{\mathbb{D}} (\omega_\star - \bar\omega)(f'(\bar\omega) - f'(\omega_\star)) > 0,$$

hence reaching a contradiction. Since the condition $\mathsf{E}(\omega) \le \mathsf{e}$ in (3-9) is convex and the entropy functional is strictly concave, uniqueness of solution follows immediately. The case $\mathsf{e} \ge \mathsf{e}_\star$ is completely analogous, however (3-10) is not a convex problem and therefore uniqueness of solutions is not as immediate as the previous case. □

4. Uniqueness of maximizers at negative temperature

In light of Proposition 2.1, we have reduced the problem to the investigation of uniqueness of maximizers for the free energy F_β. This is the main result of this section.

Theorem 4.1. *For any $\beta \in \mathbb{R}$, there exists a unique maximizer ω_β of F_β over $\mathcal{O}$.*

For $\beta \geq 0$, F_β is strictly concave and therefore admits a unique maximizer, which is necessarily radial due to the invariance of the functionals under rotations. Hence, this section is devoted to the study of the case of negative inverse temperature $\beta < 0$.

4.1. *The Euler–Lagrange equations.* Despite the loss of concavity of F_β for $\beta < 0$, Talenti's inequality (3-1) implies that if ω is not radially symmetric and decreasing then

$$\mathsf{F}_\beta(\omega) = \mathsf{S}(\omega) - \beta\mathsf{E}(\omega) = \mathsf{S}(\omega^\sharp) - \beta\mathsf{E}(\omega) < \mathsf{S}(\omega^\sharp) - \beta\mathsf{E}(\omega^\sharp) = \mathsf{F}_\beta(\omega^\sharp). \tag{4-1}$$

Thus, any maximizer is radially symmetric and decreasing. First, we derive the Euler–Lagrange equations, which we refine from Lemma 2.2.

Lemma 4.2. *Let $\beta < 0$. Any maximizer $\bar\omega$ of F_β over $\mathcal{O}$ is radially decreasing and satisfies the following: there exist $r_{\bar\omega} \in [0, 1)$ and $\bar\lambda = \bar\lambda(\bar\omega) \in [0, \infty)$ such that*

$$\begin{aligned} \bar\omega(r) &= 1, \quad \textit{on } [0, r_{\bar\omega}), \\ -\log\bar\omega(r) - \frac{\beta}{2}\int_r^1 \frac{1}{s} M_{\bar\omega}(s)\,\mathrm{d}s &= \bar\lambda, \quad \textit{on } (r_{\bar\omega}, 1). \end{aligned} \tag{4-2}$$

As a consequence, $\bar\omega(r) \geq \mathrm{e}^{-\bar\lambda} > 0$ on $[0, 1]$.

Proof. By (4-1) we know that any maximizer must be radially symmetric and decreasing. Hence, the first property that there exists a $r_{\bar\omega} \in [0, 1)$ such that $\{\bar\omega = 1\} = B_{r_{\bar\omega}}$ follows immediately.

To prove (4-2), we simply rely on Lemma 2.2. In the case of Boltzmann entropy, $f'(z) = -1 - \log(z)$. Moreover, we can make use of the expression (3-2) for radial stream functions with (3-3) and conclude the proof. □

We use the nondegeneracy that stems from (4-2), to show that out of the set of possible maximizers $\bar\omega$, there exists at least one maximizer with the smallest possible radius $r_{\bar\omega}$.

Lemma 4.3. *Let $\beta < 0$. There exists a maximizer ω_β of F_β over $\mathcal{O}$ such that*

(a) $\{\omega_\beta = 1\} = [0, r_\beta)$;

(b) *For $\lambda_\beta = \bar\lambda(\omega_\beta)$ in (4-2), it holds that*

$$-\log\omega_\beta(r) - \frac{\beta}{2}\int_r^1 \frac{1}{s} M_{\omega_\beta}(s)\,\mathrm{d}s = \lambda_\beta \quad \textit{on } (r_\beta, 1); \tag{4-3}$$

(c) $r_\beta \leq r_{\bar\omega}$ *for any maximizer $\bar\omega$.*

Proof. By the upper-semicontinuity of the entropy and the continuity of the energy, it follows that the set of maximizers

$$\mathcal{M} = \left\{\bar{\omega} \in \mathcal{O} \,:\, \mathsf{F}_\beta(\bar{\omega}) = \max_{\omega\in\mathcal{O}} \mathsf{F}_\beta(\omega)\right\}$$

is weak-$*$ compact. As (c) suggests, we want r_β to be defined by

$$r_\beta = \inf_{\bar{\omega}\in\mathcal{M}} r_{\bar{\omega}}.$$

We consider $\{\bar{\omega}_n\}_{n\in\mathbb{N}} \subset \mathcal{M}$ a minimizing sequence such that

$$\lim_{n\to\infty} r_{\bar{\omega}_n} = r_\beta.$$

Using compactness, we know that, up to subsequence, there exist an accumulation point $\bar{\omega}_\beta \in \mathcal{M}$ such that $\bar{\omega}_n \overset{*}{\rightharpoonup} \omega_\beta$. The fact that ω_β satisfies the Euler–Lagrange condition (4-3) follows from the maximality, so that to complete the proof we just need to show (a), namely

$$\{\omega_\beta = 1\} = [0, r_\beta).$$

We notice that $\{\bar{\omega}_n\}_{n\in\mathbb{N}}$ is a sequence of radially decreasing and bounded functions, hence they are of bounded variation, which implies the convergences is strong in $L^p(\mathbb{D})$ for any $p \in [1, \infty)$. Using that $r_\beta \le r_{\bar{\omega}_n}$, we know that $\bar{\omega}_n = 1$ on $r \in [0, r_\beta)$, which implies the inclusion $[0, r_\beta) \subset \{\omega_\beta = 1\}$, for instance by passing to a further subsequence that converges pointwise. We will show the reverse inclusion by using the Euler–Lagrange equation for $\bar{\omega}_n$. Indeed, differentiating in r the Euler–Lagrange condition (4-2) for $\bar{\omega}_n$, we get the equation

$$-\frac{\partial_r \bar{\omega}_n(r)}{\bar{\omega}_n(r)} + \frac{\beta}{2}\frac{1}{r} M_{\omega_\beta}(r) = 0, \quad \text{for } r > r_n.$$

Using the strong convergence of $\bar{\omega}_n \to \omega_\beta$, and the distributional convergence of the derivatives $\partial_r \bar{\omega}_n \rightharpoonup \partial_r \omega_\beta$, we obtain that

$$\partial_r \omega_\beta(r) = \frac{\beta}{2}\omega_\beta(r)\frac{1}{r} M_{\omega_\beta}(r) < 0 \quad \text{for } r > r_\beta,$$

which implies that

$$\omega_\beta < 1 \quad \text{for } r > r_\beta. \qquad \square$$

Written for the corresponding radial stream function ψ_β, the Euler–Lagrange equation (4-3) reads

$$\Delta\psi_\beta = \mathrm{e}^{-\lambda_\beta}\mathrm{e}^{\pi\beta\psi_\beta}. \tag{4-4}$$

4.2. ***Uniqueness of maximizers.*** We will show that ω_β is in fact unique maximizer of the free energy F_β. We start by considering $\bar{\omega}$ a general radial competitor. Below we write an expression for $\mathsf{F}_\beta(\bar{\omega})$ in terms of ω_β and the Brenier [1991] map (or optimal transport map) between them. As both functions are radial (and decreasing), we know that the optimal mapping is also radial and increasing. Just like the one-dimensional case it can be represented implicitly by the cumulative distribution functions. To avoid some pathological regularity situations, from now on we assume that both the source and target measure are bounded above and below, which is satisfied by maximizers; see Equation (4-3). Namely, there exists a unique strictly increasing map $T : [0, 1] \to [0, 1]$ such that $T(0) = 0,\ T(1) = 1$ and

$$\int_0^r \omega_\beta(s)s\,\mathrm{d}s = \int_0^{T(r)} \bar{\omega}(s)s\,\mathrm{d}s \quad \text{for any } r \in [0, 1]. \tag{4-5}$$

In particular, from (3-3) we have

$$M_{\omega_\beta}(r) = 2\pi \int_0^r \omega_\beta(s)s\,\mathrm{d}s = 2\pi \int_0^{T(r)} \bar{\omega}(s)s\,\mathrm{d}s = M_{\bar{\omega}}(T(r)). \tag{4-6}$$

Using the Monge–Ampère equation associated to the change of variable we obtain the relationship

$$\frac{r}{T(r)T'(r)}\omega_\beta(r) = \bar{\omega}(T(r)) \quad \text{for any } r \in [0, 1]. \tag{4-7}$$

To simplify the notation we define the function

$$\phi(r) := \frac{T(r)T'(r)}{r}. \tag{4-8}$$

The next results is a comparison between the energy and entropy of ω_β and $\bar{\omega}$.

Lemma 4.4. *Let ω_β and $\bar{\omega}$ be two radial functions in $\mathcal{O}$ that are bounded below away from zero. Then*

$$\mathsf{S}(\bar{\omega}) - 2\int_0^1 \omega_\beta(r) \log \phi(r) r\,\mathrm{d}r = \mathsf{S}(\omega_\beta), \tag{4-9}$$

$$\mathsf{E}(\bar{\omega}) + \int_0^1 \left(\int_r^1 \frac{1}{s}\omega_\beta(s) M_{\omega_\beta}(s)\,\mathrm{d}s \right) \log \phi(r) r\,\mathrm{d}r \le \mathsf{E}(\omega_\beta), \tag{4-10}$$

where $\phi(r)$ is defined in (4-8).

Remark. We note that in the previous result we do not use the optimality in of $\bar{\omega}$ or ω_β in any strong way, it only requires that both vorticities are radial.

Proof of Lemma 4.4. Using the change of variable $r = T(s)$ given by the Brenier map (4-5), we can rewrite the entropy as

$$\mathsf{S}(\bar{\omega}) = -2\int_0^1 \bar{\omega}(r) \log(\bar{\omega}(r)) r\,\mathrm{d}r = -2\int_0^1 \bar{\omega}(T(s)) \log\bigl(\bar{\omega}(T(s))\bigr) T(s)T'(s)\,\mathrm{d}s.$$

Using (4-7), we obtain

$$\mathsf{S}(\bar{\omega}) = -2\int_0^1 \omega_\beta(s) \log\left(\frac{\omega_\beta(s)}{\phi(s)}\right) s\,\mathrm{d}s,$$

which coincides with the desired (4-9).

For the energy, we first integrate by parts to obtain the different representation

$$\mathsf{E}(\bar{\omega}) = \frac{1}{4\pi}\int_0^1 \frac{|M_{\bar{\omega}}(r)|^2}{r}\,\mathrm{d}r = -\int_0^1 M_{\bar{\omega}}(r)\bar{\omega}(r)\log(r) r\,\mathrm{d}r.$$

Next, we perform the change of variables $r = T(s)$ and use (4-6) and (4-7) to obtain

$$\mathsf{E}(\bar{\omega}) = -\int_0^1 M_{\bar{\omega}}(T(s))\bar{\omega}(T(s))\log(T(s))T(s)T'(s)\,\mathrm{d}s = -\int_0^1 M_{\omega_\beta}(s)\omega_\beta(s)\log(T(s)) s\,\mathrm{d}s. \tag{4-11}$$

We use that $\phi(r) = [T^2]'/2r$ to rewrite

$$\log(T(s)) = \tfrac{1}{2}\log(T^2(s)) = \tfrac{1}{2}\log\left(2\int_0^s \phi(a)a\,\mathrm{d}a\right).$$

Then, we normalize the integral to be able to apply Jensen's inequality and we obtain

$$\log(T(s)) = \tfrac{1}{2}\log(s^2) + \tfrac{1}{2}\log\left(\frac{2}{s^2}\int_0^s \phi(a)a\,\mathrm{d}a\right) \geq \log(s) + \frac{1}{s^2}\int_0^s \log(\phi(a))a\,\mathrm{d}a.$$

Using this inequality on (4-11) we have

$$\mathsf{E}(\bar{\omega}) \leq \mathsf{E}(\omega_\beta) - \int_0^1 \left(\frac{1}{s^2}\int_0^s \log(\phi(a))a\,\mathrm{d}a\right) M_{\omega_\beta}(s)\omega_\beta(s)s\,\mathrm{d}s.$$

The result (4-10) follows by applying Fubini's theorem. □

Finally, we show that ω_β is the unique maximizer of F_β.

Lemma 4.5. *Let $\beta < 0$. Let r_β and ω_β be given by Lemma 4.3. Assume that $\bar{\omega} \in \mathcal{O}$ is radial, bounded below away from* 0, *and satisfies* $(0, r_\beta) \subset \{\bar{\omega} = 1\}$. *Then we have the inequality*

$$\mathsf{F}_\beta(\bar{\omega}) - 2\omega_\beta(1)\int_0^1 \log\phi(r)r\,\mathrm{d}r \leq \mathsf{F}_\beta(\omega_\beta), \tag{4-12}$$

where $\phi(r)$ is defined in (4-8). *Moreover, if $\bar{\omega} \neq \omega_\beta$, then the energy difference is positive, namely*

$$-2\omega_\beta(1)\int_0^1 \log\phi(r)r\,\mathrm{d}r > 0. \tag{4-13}$$

Proof. Applying Lemma 4.4, we have

$$\mathsf{F}_\beta(\bar{\omega}) + \int_0^1 \left(-2\omega_\beta(r) - \beta\int_r^1 \frac{1}{s}\omega_\beta(s)M_{\omega_\beta}(s)\,\mathrm{d}s\right)\log(\phi(r))r\,\mathrm{d}r \leq \mathsf{F}_\beta(\omega_\beta). \tag{4-14}$$

Notice that by the hypothesis $(0, r_\beta) \subset \{\bar{\omega} = 1\}$, Brenier's map is trivial on $[0, r_\beta)$. That is to say

$$\phi(r) = 1 \quad \text{on } [0, r_\beta).$$

For $r \in (r_\beta, 1)$, we can use the Euler–Lagrange equation (4-3) to simplify the remainder. More specifically, taking a derivative of (4-3) we obtain

$$\partial_r\omega_\beta(r) = \frac{\beta}{2}\frac{\omega_\beta(r)M_{\omega_\beta}(r)}{r}.$$

Integrating back on $(r, 1)$, we deduce that

$$\omega_\beta(1) = \omega_\beta(r) + \frac{\beta}{2}\int_r^1 \frac{1}{s}\omega_\beta(s)M_{\omega_\beta}(s)\,\mathrm{d}s \quad \text{for any } r \in (r_\beta, 1). \tag{4-15}$$

Replacing back (4-15) into (4-14), we obtain the desired (4-12).

To show (4-13), we apply Jensen's inequality

$$2\int_0^1 \log\phi(r)r\,\mathrm{d}r \leq \log\left(2\int_0^1 \phi(r)r\,\mathrm{d}r\right) = \log\left(2\int_0^1 T(r)T'(r)\,\mathrm{d}r\right) = \log(T^2(1) - T^2(0)) = 0. \tag{4-16}$$

The equality in Jensen's inequality can only occur if $\phi(r) = C$ is constant, which implies that Brenier's map $T(r) = r$ is the identity. The conclusion that the defect is positive if $\bar\omega \neq \omega_\beta$ follows directly from the previous argument, and the fact that $\omega_\beta(1) > 0$ by Lemma 4.2. □

We conclude this section with the proof of Theorem 4.1, which is a consequence of the results above.

Proof of Theorem 4.1. We will show that ω_β given in Lemma 4.3 is the unique maximizer of F_β. Assume $\bar\omega$ is also maximizer of F_β, then by Lemmas 4.2 and 4.3 it satisfies the hypothesis of Lemma 4.5. Applying Lemma 4.5 and $\mathsf{F}_\beta(\bar\omega) = \mathsf{F}_\beta(\omega_\beta)$, we obtain $\omega_\beta = \bar\omega$. □

5. Nonradial energy maximizers at fixed angular momentum

This section is dedicated to the proof of Theorem 4. Section 5.1 contains the upper bound on the kinetic energy E for radial functions on $\mathcal{O}_L$, and Section 5.2 contains the lower bound by computing the energy of an explicit vortex patch. The conclusion follows then by choosing L in terms of the angular momentum. As in the statement of Theorem 4, we let $\mathsf{m} \in (0, 1)$ and $L \geq 1$ and set

$$\mathcal{O}_L := \left\{\omega \in L^\infty : 0 \leq \omega \leq L,\ \frac{1}{|\mathbb{D}|}\int_{\mathbb{D}} \omega(x)\,\mathrm{d}x = \mathsf{m}\right\}.$$

The proof is carried out in the next sections.

5.1. ***Upper bounds on the kinetic energy for radial functions.*** We claim that there exists a constant $C \geq 1$, independent of L, a, and m, such that

$$\sup\left\{\mathsf{E}(\omega) : \omega \in \mathcal{O}_L,\ \mathsf{A}(\omega) = \mathsf{a},\ \omega \text{ is radial}\right\} \leq C(\mathsf{m}|\mathsf{a}| + |\mathsf{a}|^2 \log(L/|\mathsf{a}|)). \tag{5-1}$$

Now, if $\omega \in \mathcal{O}_L$ is a radial function, from (1-15) we deduce that

$$\mathsf{A}(\omega) = -\frac{1}{2}\left[2\pi\int_0^1 \omega(r)(1-r^2)r\,\mathrm{d}r\right] = -\frac{1}{2}\left[\pi\mathsf{m} - \int_0^1 \partial_r M_\omega(r) r^2\,\mathrm{d}r\right] = -\int_0^1 M_\omega(r) r\,\mathrm{d}r.$$

Thus, for every $r \in [0, 1]$, we have the pointwise identity

$$\tfrac{1}{2}M_\omega(r)(1-r^2) = \tfrac{1}{2}\int_0^r \partial_s[M_\omega(s)(1-s^2)]\,\mathrm{d}s = \pi\int_0^r \omega(s)(1-s^2)s\,\mathrm{d}s - \int_0^r M_\omega(s)s\,\mathrm{d}s.$$

In particular,

$$\tfrac{1}{2}M_\omega(r)(1-r^2) \leq \pi\int_0^r \omega(s)(1-s^2)s\,\mathrm{d}s \leq -\mathsf{A}(\omega),$$

so that any radial function $\omega \in \mathcal{O}_L$ with $\mathsf{A}(\omega) = \mathsf{a}$ satisfies

$$M_\omega(r) \leq \min\left\{\pi L r^2,\ \frac{2|\mathsf{a}|}{1-r^2},\ \pi\mathsf{m}\right\}.$$

This implies

$$M_\omega(r) \leq \begin{cases} \pi L r^2, & r^2 \leq 1 - \sqrt{1 - \dfrac{8|\mathsf{a}|}{\pi L}}, \\ \dfrac{2|\mathsf{a}|}{1-r^2}, & 1 - \sqrt{1 - \dfrac{8|\mathsf{a}|}{\pi L}} < r^2 \leq 1 - \dfrac{2|\mathsf{a}|}{\pi\mathsf{m}}, \\ \pi\mathsf{m}, & 1 - \dfrac{2|\mathsf{a}|}{\pi\mathsf{m}} < r^2 \leq 1. \end{cases}$$

Thanks to (3-4), we then have $\mathsf{E}(\omega) \le \mathsf{E}_1 + \mathsf{E}_2 + \mathsf{E}_3$, where

$$\mathsf{E}_1 = \frac{1}{4\pi}\int_0^{r^2=1-\sqrt{1-8|\mathsf{a}|/(\pi L)}} \frac{1}{r}|\pi L r^2|^2\,\mathrm{d}r = \frac{\pi L^2}{16}\left(1-\sqrt{1-\frac{8|\mathsf{a}|}{\pi L}}\right)^2 \lesssim |\mathsf{a}|^2,$$

$$\begin{aligned}\mathsf{E}_2 &= \frac{1}{4\pi}\int_{r^2=1-\sqrt{1-8|\mathsf{a}|/(\pi L)}}^{r^2=1-2|\mathsf{a}|/(\pi \mathsf{m})} \frac{1}{r}\left|\frac{2|\mathsf{a}|}{1-r^2}\right|^2\mathrm{d}r \\ &= \frac{|\mathsf{a}|^2}{2\pi}\left[\frac{\pi\mathsf{m}}{2|\mathsf{a}|} + \log\left(\frac{\pi\mathsf{m}}{2|\mathsf{a}|}-1\right) - \frac{1}{\sqrt{1-8|\mathsf{a}|/(\pi L)}} - \log\left(\frac{1-\sqrt{1-8|\mathsf{a}|/(\pi L)}}{\sqrt{1-8|\mathsf{a}|/(\pi L)}}\right)\right] \\ &\lesssim \mathsf{m}|\mathsf{a}| + |\mathsf{a}|^2\log(L/|\mathsf{a}|),\end{aligned}$$

$$\mathsf{E}_3 = \frac{1}{4\pi}\int_{r^2=1-2|\mathsf{a}|/(\pi\mathsf{m})}^{1} \frac{1}{r}|\mathsf{m}|^2\,\mathrm{d}r = -\frac{\pi\mathsf{m}^2}{8}\log\left(1-\frac{2|\mathsf{a}|}{\pi\mathsf{m}}\right) \lesssim \mathsf{m}|\mathsf{a}|.$$

Thus (5-1) follows by collecting the above three bounds.

5.2. ***Kinetic energy of a vortex patch near the boundary.*** Next, we compute the energy of a vortex approximation near the boundary. We consider the vortex patch of height $L > 0$ around $x_0 \in \mathbb{D}$, given by

$$\omega_{x_0,L} = L\mathbb{1}_{B_{\sqrt{\mathsf{m}/L}}(x_0)},$$

where we impose that L satisfies

$$L \ge \frac{4\mathsf{m}}{(1-|x_0|)^2}, \tag{5-2}$$

so that $\omega_{x_0,L} \in \mathcal{O}_L$. To estimate the kinetic energy of $\omega_{x_0,L}$, we use the explicit Green's function of the Laplace operator on the unit disk, so that

$$\psi_{x_0,L}(x) = \frac{1}{2\pi}\int_{\mathbb{D}} \log\frac{|x-y|}{|y|\,|x-y_*|}\omega_{x_0,L}(y)\,\mathrm{d}y = \frac{L}{2\pi}\int_{B_{\sqrt{\mathsf{m}/L}}(x_0)} \log\frac{|x-y|}{|y|\,|x-y_*|}\,\mathrm{d}y,$$

where $y_* = y/|y|^2$. Thus

$$\mathsf{E}(\omega_{x_0,L}) = -\tfrac{1}{2}\int_{\mathbb{D}} \psi_{x_0,L}(x)\omega_{x_0,M}(x)\,\mathrm{d}x = \frac{L^2}{2\pi}\int_{B_{\sqrt{\mathsf{m}/L}}(x_0)}\int_{B_{\sqrt{\mathsf{m}/L}}(x_0)} \log\frac{|y|\,|x-y_*|}{|x-y|}\,\mathrm{d}y\,\mathrm{d}x. \tag{5-3}$$

For $x, y \in B_{\sqrt{\mathsf{m}/L}}(x_0)$, we have the bound

$$|x-y| \le 2\sqrt{\frac{\mathsf{m}}{L}}. \tag{5-4}$$

Using that $|y| > \frac{1}{2}$ and $y_* \notin \mathbb{D}$, from (5-2) we deduce the bound

$$|y|\,|x-y_*| \ge \frac{1}{2}\left[1-|x_0|-\sqrt{\frac{\mathsf{m}}{L}}\right] = \frac{1}{2}\left[1-|x_0|-\frac{1-|x_0|}{2}\right] \ge \tfrac{1}{4}(1-|x_0|). \tag{5-5}$$

Plugging (5-4)–(5-5) into (5-3) we obtain the bound

$$\mathsf{E}(\omega_{x_0,L}) \ge \frac{\pi\mathsf{m}^2}{2}\log\left(\sqrt{\frac{L}{\mathsf{m}}}\frac{(1-|x_0|)}{8}\right). \tag{5-6}$$

Computing explicitly the angular momentum, we find

$$\mathsf{A}(\omega_{x_0,L}) = -\tfrac{1}{2}\int_{\mathbb{D}} (1-|x|^2)\omega_{x_0,L}(x)\,\mathrm{d}x = -\frac{\pi \mathsf{m}}{2}\Big(1-|x_0|^2-\frac{\mathsf{m}}{2L}\Big) \geq -\pi\mathsf{m}(1-|x_0|).$$

Hence, from (5-6) we have the bound

$$\mathsf{E}(\omega_{x_0,L}) \geq \frac{\pi \mathsf{m}^2}{2}\log\bigg(\frac{\sqrt{L}\,|\mathsf{A}(\omega_{x_0,L})|}{8\pi\sqrt{m^3}}\bigg), \tag{5-7}$$

as long as

$$L \geq \frac{4\pi^2\mathsf{m}^3}{|\mathsf{A}(\omega_{x_0,L})|^2}$$

to satisfy (5-2).

5.3. ***Proof of Theorem 4.*** The bounds (1-16) and (1-17) are included in the sections above. Given an angular momentum a, we consider the height

$$L = Q^2\pi^2\frac{\mathsf{m}^3}{|\mathsf{a}|^2}, \tag{5-8}$$

with $Q > 2$ to be chosen below. By (5-7) and (5-8), we have the bound

$$\sup\{\mathsf{E}(\omega) : \omega \in \mathcal{O}_L,\ \mathsf{A}(\omega) = \mathsf{a}\} \geq \frac{\pi \mathsf{m}^2}{2}\log\big(\tfrac{1}{8}Q\big)$$

which is independent of a. We pick

$$Q = 8\mathrm{e}^{\frac{2}{\pi\mathsf{m}^2}},$$

which implies

$$\sup\{\mathsf{E}(\omega) : \omega \in \mathcal{O}_L,\ \mathsf{A}(\omega) = \mathsf{a}\} \geq 1.$$

For radial functions, we use (5-1) to get the bound

$$\sup\{\mathsf{E}(\omega) : \omega \in \mathcal{O}_L,\ \mathsf{A}(\omega) = \mathsf{a},\ \omega \text{ is radial}\} \leq C\bigg(\mathsf{m}|\mathsf{a}| + |\mathsf{a}|^2\log\bigg(Q^2\pi^2\frac{\mathsf{m}^3}{|\mathsf{a}|^3}\bigg)\bigg).$$

So to finish the proof we need to pick $\mathsf{a}_* < 0$ close enough to zero depending only on m such that for any $\mathsf{a} \in (\mathsf{a}_*, 0)$, we have

$$C\bigg(\mathsf{m}|\mathsf{a}| + |\mathsf{a}|^2\log\bigg(Q^2\pi^2\frac{\mathsf{m}^3}{|\mathsf{a}|^3}\bigg)\bigg) < 1,$$

which implies the desired inequality.

6. Stability of Onsager solutions with negative inverse temperature

Equations of Liouville-type such as (4-4) arise in the classical setting of mean-field limits of the canonical Gibbs measure associated to a system of point vortices. We state below an important result from [Caglioti et al. 1995] for Onsager solutions, namely solutions to the mean-field equation (1-18).

Theorem 6.1 [Caglioti et al. 1995, Section 5]. *Let $\beta \in (-8\pi, \infty)$. Onsager solutions*

$$\omega_\beta(r) = \frac{1 - A(\beta)}{\pi} \frac{1}{(1 - A(\beta) r^2)^2} \quad \textit{with} \quad A(\beta) = \frac{\beta}{8\pi + \beta}$$

are the unique maximizer of

$$\mathsf{F}_\beta(\omega) = \mathsf{S}(\omega) - \beta \mathsf{E}(\omega) = -\int_{\mathbb{D}} \omega \log \omega \, \mathrm{d}x - \frac{\beta}{2} \int_{\mathbb{D}} |\nabla \Delta^{-1} \omega|^2 \, \mathrm{d}x$$

over the set

$$\mathcal{P} = \left\{ \omega \in L^1 \,:\, \omega \geq 0,\ \int_{\mathbb{D}} \omega(x) \, \mathrm{d}x = 1,\ \int_{\mathbb{D}} \omega(x) \log \omega(x) \, \mathrm{d}x < \infty \right\}.$$

Moreover, we have the convergence

$$\lim_{\beta \to -8\pi^-} \omega_\beta \to \delta_0,$$

weakly in the sense of measures.

The purpose of this section is to prove Theorem 5. As mentioned already, the ideas related to (quantitative) rearrangement inequalities and elliptic equations from [Talenti 1976; Amato et al. 2024; Cianchi et al. 2008; Cianchi and Ferone 2008] are here revisited in the case of the disk $\mathbb{D}$ and vorticities ω satisfying an L^∞ bound. The key result for us is the following stability result with respect to the H^{-1} norm and its radially decreasing rearrangement.

Lemma 6.2. *Consider a positive vorticity distribution $\omega \in L^\infty$ such that $\int_{\mathbb{D}} \omega \, \mathrm{d}x = \mathsf{m} > 0$. Then, for every $\varepsilon > 0$, there exists $\delta > 0$ such that if*

$$\mathsf{E}(\omega^\sharp) - \mathsf{E}(\omega) < \delta,$$

then there exists $x_* \in \mathbb{R}^2$ such that $|x_*| \leq \varepsilon$ and

$$\|\omega - \omega^\sharp(\,\cdot\, - x_*)\|_{L^1(\mathbb{R}^2)} < \varepsilon,$$

where ω and $\omega^\sharp$ are extended by zero outside the disk.

The proof is deferred until after the proof of Arnold's stability.

6.1. ***Proof of Theorem 5.*** We now proceed with the proof of the main result in this section.

Proof of Theorem 5. Throughout the proof we take $\|\omega^{\mathrm{in}} - \omega_\beta\|_{L^2} < \delta$ progressively smaller, and we keep changing ε accordingly and without renaming it. We will also omit the dependence on t of the solution $\omega = \omega(t)$, as the proof is carried for any arbitrary $t \geq 0$. We proceed in several steps. To simplify the notation, we first assume that

$$\int_{\mathbb{D}} \omega_{\mathrm{in}}(x) \, \mathrm{d}x = \mathsf{m} = 1 \tag{6-1}$$

so that we have unit mass, and in the last step we generalize to the case $\mathsf{m} \neq 1$.

Step 0. We show that for any $\varepsilon > 0$ small enough, we can pick $\|\omega_\beta - \omega^{\text{in}}\|_{L^2} < \delta$ small enough that the corresponding Euler solution $\omega = \omega(t)$ is such that

$$0 \le \mathsf{F}_\beta[\omega_\beta] - \mathsf{F}_\beta(\omega(t)) < \varepsilon \quad \text{for all } t \in [0, \infty). \tag{6-2}$$

Proof of Step 0. We notice that F_β is continuous with respect to the L^2 norm. For what concerns the kinetic energy part, by the triangle and Poincaré inequalities there exists $C_0 = C_0(\|\omega'\|_{L^2}, \|\omega\|_{L^2}) > 0$ such that

$$|\mathsf{E}(\omega) - \mathsf{E}(\omega')| = \tfrac{1}{2}\big|\|\omega\|_{H^{-1}} - \|\omega'\|_{H^{-1}}\big|(\|\omega\|_{H^{-1}} + \|\omega'\|_{H^{-1}}) \le C_0\|\omega - \omega'\|_{L^2} \quad \text{for all } \omega, \omega' \in L^\infty \cap \mathcal{P}.$$

For the Boltzmann entropy part, we notice that

$$|\omega \log \omega| \lesssim 1 + |\omega|^2. \tag{6-3}$$

Thus if $\omega_n \to \omega$ in L^2, then up to subsequences $\omega_n \log \omega_n \to \omega \log \omega$ almost everywhere and (6-3) implies uniform integrability. Thus, the Vitali convergence theorem implies that $\mathsf{S}(\omega_n) \to \mathsf{S}(\omega)$.

Hence, given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, \beta)$ such that if $\|\omega_\beta - \omega^{\text{in}}\|_{L^2} < \delta$, then

$$|\mathsf{F}_\beta(\omega_\beta) - \mathsf{F}_\beta(\omega^{\text{in}})| < \varepsilon.$$

Using assumption (6-1), that ω_β is the unique maximizer over $\mathcal{P}$ of F_β, and that the mass and free energy F_β are conserved along the evolution of the Euler equations, we obtain the desired inequality (6-2).

Step 1. We can choose $\|\omega^{\text{in}} - \omega_\beta\|_{L^2} < \delta$, such that

$$\|\omega(t, \cdot) - \omega^\sharp(t, \cdot - x_*)\|_{L^2(\mathbb{R}^2)} < \varepsilon \quad \text{for all } t \in [0, \infty), \tag{6-4}$$

where $x_* = x_*(t) \in \mathbb{R}$ satisfies $|x_*| < \varepsilon$, and both functions are extended to $\mathbb{R}^2$ by zero outside the disk.

Proof of Step 1. Using (6-2), we have

$$0 \le \underbrace{\mathsf{F}_\beta(\omega_\beta) - \mathsf{F}_\beta(\omega^\sharp(t))}_{\ge 0} + \underbrace{\mathsf{F}_\beta(\omega^\sharp(t)) - \mathsf{F}_\beta(\omega(t))}_{\ge 0} < \varepsilon \quad \text{for all } t \in [0, \infty), \tag{6-5}$$

where the positivity of each term follows from the fact that ω_β is the optimizer of F_β over $\mathcal{P}$, and that $\mathsf{F}_\beta(\omega^\sharp(t)) \ge \mathsf{F}_\beta(\omega(t))$ for $\beta < 0$, in view of (4-1). Hence, using that $\mathsf{S}(\omega^\sharp) = \mathsf{S}(\omega)$ we can conclude that

$$0 \le \mathsf{E}(\omega^\sharp(t)) - \mathsf{E}(\omega(t)) < \varepsilon.$$

Up to notation, the conclusion of (6-4) follows from the quantitative Talenti's inequality in Lemma 6.2. To get the stability in L^2 we just need to interpolate the above bound with the bounds in L^∞.

Step 2. We consider the Brenier map $T : [0, 1] \to [0, 1]$ such that $T(0) = 0$, $T(1) = 1$ and

$$\int_0^r \omega_\beta(s) s \,\mathrm{d}s = \int_0^{T(r)} \omega^\sharp(s) s \,\mathrm{d}s \quad \text{for any } r \in [0, 1]. \tag{6-6}$$

For every $\varepsilon > 0$, we can pick $\|\omega_\beta - \omega^{\text{in}}\|_{L^2} < \delta$ small enough such that

$$\int_0^1 |\omega_\beta(r) - \omega^\sharp(T(r))|^2 \, r \,\mathrm{d}r < \varepsilon \quad \text{and} \quad \int_0^1 H\Big(\frac{TT'}{r}\Big) \, r \,\mathrm{d}r < \varepsilon, \tag{6-7}$$

where $H(u) = -\log u + u - 1$.

Proof of Step 2. By (6-5) we have that

$$0 \le \mathsf{F}_\beta(\omega_\beta) - \mathsf{F}_\beta(\omega^\sharp(t)) < \varepsilon.$$

Applying Lemma 4.5, we have that

$$0 \le -2\omega_\beta(1)\int_0^1 \log\Big(\frac{TT'}{r}\Big) r\,\mathrm{d}r \le \mathsf{F}_\beta(\omega_\beta) - \mathsf{F}_\beta(\omega^\sharp) < \varepsilon.$$

The first inequality follows from an application of Jensen's inequality as in (4-16).

Next, we apply a quantitative version of Jensen's inequality. Given $G(\,\cdot\,) = -\log(\,\cdot\,)$, considering the random variable $X = TT'/r$ and the probability measure $2r\,\mathrm{d}r$ in [0, 1], we have

$$\mathbb{E}X = 2\int_0^1 TT'\,\mathrm{d}r = T^2(1) - T^2(0) = 1.$$

Hence, for the function $G(\,\cdot\,) = -\log(\,\cdot\,)$, we have

$$\mathbb{E}G(X) = \mathbb{E}\big[G(X) - G(\mathbb{E}X) - G'(\mathbb{E}X)(X - \mathbb{E}X)\big].$$

We define the convex function

$$H(x) := -\log x + x - 1,$$

which satisfies the inequality

$$\tfrac{1}{4}\min\{|x-1|^2, |x-1|\} \le H(x) \quad \text{for all } x > 0. \tag{6-8}$$

Using the observations above, we have

$$2\int_0^1 H\Big(\frac{TT'}{r}\Big) r\,\mathrm{d}r \le -2\int_0^1 \log\Big(\frac{TT'}{r}\Big) r\,\mathrm{d}r < \varepsilon.$$

Differentiating (6-6), we have

$$\frac{T(r)T'(r)}{r} = \frac{\omega_\beta(r)}{\omega^\sharp(T(r))}.$$

Applying (6-8) and using that $\omega^\sharp, \omega_\beta \in L^\infty$, we can pick $\tilde{a} = \tilde{a}(\|\omega^\sharp\|_{L^\infty}, \|\omega_\beta\|_{L^\infty}) > 0$ small enough so that

$$\tilde{a}|\omega_\beta(r) - \omega^\sharp(T(r))|^2 \le \tfrac{1}{4}\min\bigg\{\bigg|\frac{\omega_\beta(r)}{\omega^\sharp(T(r))} - 1\bigg|^2, \bigg|\frac{\omega_\beta(r)}{\omega^\sharp(T(r))} - 1\bigg|\bigg\} \le H\bigg(\frac{\omega_\beta(r)}{\omega^\sharp(T(r))}\bigg).$$

Therefore, up to relabelling ε we have

$$\int_0^1 |\omega_\beta(r) - \omega^\sharp(T(r))|^2 r\,\mathrm{d}r \le \varepsilon.$$

Step 3. We exhibit a control on how far the Brenier map is from the identity. That is to say, we can choose $\|\omega_\beta - \omega^{\mathrm{in}}\|_{L^2} < \delta$ small enough so that

$$|T(s) - s| < \varepsilon \quad \text{for all } s \in [0, 1]. \tag{6-9}$$

Proof of Step 3. By (6-7), for δ small enough we have the inequality

$$\int_0^s H\Big(\frac{TT'}{r}\Big) r \,\mathrm{d}r \le \int_0^1 H\Big(\frac{TT'}{r}\Big) r \,\mathrm{d}r < \varepsilon.$$

Applying Jensen's inequality, we have

$$\frac{s^2}{2} H\Big(\frac{T^2(s)}{s^2}\Big) = \frac{s^2}{2} H\Big(\frac{2}{s^2}\int_0^s \frac{TT'}{r} r \,\mathrm{d}r\Big) \le \int_0^s H\Big(\frac{TT'}{r}\Big) r \,\mathrm{d}r < \varepsilon.$$

Using (6-8), we have

$$\tfrac{1}{4} s^2 \min\bigg\{\bigg|\frac{T^2(s)}{s^2} - 1\bigg|^2, \bigg|\frac{T^2(s)}{s^2} - 1\bigg|\bigg\} \le \frac{s^2}{2} H\Big(\frac{T^2(s)}{s^2}\Big).$$

Using the mass constraint and the L^∞ bound on ω we can take $r > 0$ small enough depending on $\|\omega^{\mathrm{in}}\|_{L^\infty}$ such that

$$\omega^\sharp(x) \ge \frac{1 - \|\omega^{\mathrm{in}}\|_{L^\infty} |x|^2}{1 - |x|^2} \ge \tfrac{1}{2} \quad \text{for all } |x| < r.$$

Hence, using (6-6), we have that for r small enough

$$C^{-1} T(r) \le \int_0^{T(r)} \omega^\sharp(s) s \,\mathrm{d}s = \int_0^r \omega_\beta(s) s \,\mathrm{d}s \le Cr,$$

which implies that uniformly

$$\sup_{r\in[0,1]} \bigg|\frac{T^2(r)}{r^2} - 1\bigg| < C(\|\omega^\sharp\|_{L^\infty}).$$

Hence, up to relabeling ε, we have

$$|T(s) - s|^2 \bigg|\frac{T(s)}{s} + 1\bigg|^2 = s^2 \bigg|\frac{T^2(s)}{s^2} - 1\bigg|^2 < \varepsilon,$$

and the conclusion follows.

Step 4. For every $\varepsilon > 0$, we can pick $\|\omega_\beta - \omega^{\mathrm{in}}\|_{L^2} < \delta$ small enough so that

$$\int_0^1 |\omega_\beta(r) - \omega^\sharp(r)| r \,\mathrm{d}r < \varepsilon. \tag{6-10}$$

Proof of Step 4. Changing variables and applying the triangle inequality, we notice that

$$\begin{aligned}\int_0^1 |\omega_\beta(r) - \omega^\sharp(r)| r \,\mathrm{d}r &= \int_0^1 |\omega_\beta(T(r)) - \omega^\sharp(T(r))|\, T(r) T'(r) \,\mathrm{d}r \\ &\le \underbrace{\int_0^1 |\omega_\beta(T(r)) - \omega_\beta(r)| T(r) T'(r) \,\mathrm{d}r}_{\mathrm{I}} + \underbrace{\int_0^1 |\omega_\beta(r) - \omega^\sharp(T(r))| T(r) T'(r) \,\mathrm{d}r}_{\mathrm{II}}.\end{aligned}$$

Applying (6-9) and the smoothness of ω_β, we get the bound

$$\mathrm{I} \le \|\partial_r \omega_\beta\|_{L^\infty} \|T - r\|_{L^\infty} \int_0^1 T(r) T'(r) \,\mathrm{d}r < \|\partial_r \omega_\beta\|_{L^\infty} \varepsilon.$$

We manipulate the second term to get

$$\mathrm{II} \le \int_0^1 |\omega_\beta(r) - \omega^\sharp(T(r))| r \,\mathrm{d}r + \int_0^1 |\omega_\beta(r) - \omega^\sharp(T(r))|(TT' - r)\,\mathrm{d}r \le \sqrt{\frac{\varepsilon}{2}} + \varepsilon,$$

where we have used Cauchy–Schwarz and (6-7). Up to renaming ε, (6-10) now follows.

Step 5. We now conclude the proof of the theorem, under the unit mass assumption (6-1).

Proof of Step 5. We combine (6-4) and (6-10) to obtain that

$$\|\omega_\beta(\cdot - x_*) - \omega(t)\|_{L^1(\mathbb{R}^2)} < 2\varepsilon, \quad \text{for all } t \ge 0,$$

for some $x_* \in \mathbb{R}^2$ such that $|x_*| \le \varepsilon$, where we have extended the functions by zero outside the disk $\mathbb{D}$. Using the continuity of the L^1 norm over translations for ω_β, we can conclude that, up to renaming ε,

$$\|\omega_\beta - \omega(t)\|_{L^1(\mathbb{R}^2)} < \varepsilon.$$

To get the stability in L^2 we just need to interpolate the above bound with the bounds in L^∞.

Step 6. We conclude the proof of the theorem, without the unit mass assumption (6-1).

Proof of Step 6. We start by showing that the maximizer of the problem

$$\max_{\omega \in \mathcal{P}_\mathsf{m}} \mathsf{F}_\beta(\omega) = \mathsf{S}(\omega) - \beta \mathsf{E}(\omega) = -\int_\mathbb{D} \omega \log \omega \,\mathrm{d}x - \frac{\beta}{2}\int_\mathbb{D} |\nabla \Delta^{-1}\omega|^2 \,\mathrm{d}x, \tag{6-11}$$

where

$$\mathcal{P}_\mathsf{m} = \left\{ \omega \in L^1 : \omega \ge 0, \int_\mathbb{D} \omega(x)\,\mathrm{d}x = \mathsf{m}, \int_\mathbb{D} \omega(x) \log \omega(x)\,\mathrm{d}x < \infty \right\},$$

is continuous with respect to the mass parameter m in any L^p with $p \in [1, \infty)$, as the long the parameter $\mathsf{m}\beta > -8\pi$. By rescaling, we find

$$\max_{\omega \in \mathcal{P}_\mathsf{m}} -\int_\mathbb{D} \omega \log \omega \,\mathrm{d}x - \frac{\beta}{2}\int_\mathbb{D} |\nabla\Delta^{-1}\omega|^2\,\mathrm{d}x = \max_{\omega \in \mathcal{P}_1} -\mathsf{m}\int_\mathbb{D} \omega\log\omega\,\mathrm{d}x - \frac{\beta \mathsf{m}^2}{2}\int_\mathbb{D} |\nabla\Delta^{-1}\omega|^2\,\mathrm{d}x - \mathsf{m}\log\mathsf{m}.$$

Hence, for general m, we get that the maximizer of (6-11) is given by

$$\omega_* = \mathsf{m}\omega_{\mathsf{m}\beta},$$

which by (1-19) is continuous in any topology as long as it does not blow up $\mathsf{m}\beta > -8\pi$.

Using the continuity of the mass with respect to the L^2 norm, we can pick δ small enough so that $\|\omega_\beta - \omega_{\mathrm{in}}\|_{L^2} < \delta$ implies

$$\|\mathsf{m}\omega_{\mathsf{m}\beta} - \omega_\beta\|_{L^2} < \varepsilon \quad \text{and} \quad \|\mathsf{m}\omega_{\mathsf{m}\beta} - \omega_{\mathrm{in}}\|_{L^2} < \varepsilon, \tag{6-12}$$

where

$$\mathsf{m} = \int_\mathbb{D} \omega_{\mathrm{in}}(x)\,\mathrm{d}x.$$

To conclude the proof, we need to repeat *Steps 0–5* replacing the role of ω_β by $\mathsf{m}\omega_{\mathsf{m}\beta}$. We conclude that we can pick δ small enough that $\|\omega_\beta - \omega_{\mathrm{in}}\|_{L^2} < \delta$ implies $\|\mathsf{m}\omega_{\mathsf{m}\beta} - \omega(t)\|_{L^2} < \varepsilon$ for all $t > 0$, and the result follows, up to relabeling ε, by (6-12). □

6.2. ***Stability of rearrangements.*** We now proceed to prove Lemma 6.2, which constitutes the crucial step in the proof of Theorem 5.

Proof of Lemma 6.2. We consider (up to signs!) the associated stream functions to ω and its rearrangement $\omega^\#$

$$\begin{cases} -\Delta\phi = \omega & \text{in } \mathbb{D}, \\ \phi = 0 & \text{on } \partial\mathbb{D}, \end{cases} \qquad \begin{cases} -\Delta\bar{\phi} = \omega^\sharp & \text{in } \mathbb{D}, \\ \bar{\phi} = 0 & \text{on } \partial\mathbb{D}. \end{cases} \tag{6-13}$$

A celebrated theorem of Talenti [1976, Theorem 1] states that

$$\phi^\sharp(x) \le \bar{\phi}(x) \quad \text{for all } x \in \mathbb{D},$$

and

$$\mathsf{E}(\omega) = \tfrac{1}{2}\|\nabla\phi\|_{L^2}^2 \le \tfrac{1}{2}\|\nabla\bar{\phi}\|_{L^2}^2 = \mathsf{E}(\omega^\sharp).$$

To prove the lemma, we again proceed in steps.

Step 1. We show that under our hypothesis, there exists $C(\|\omega\|_{L^\infty}, \mathsf{m}) > 0$ such that

$$\|\bar{\phi} - \phi^\sharp\|_{L^\infty} \le C(\mathsf{E}(\omega^\sharp) - \mathsf{E}(\omega)). \tag{6-14}$$

Proof of Step 1. For $h \ge 0$, we consider the distribution function

$$u(h) = |\{x \in \mathbb{D} : \phi(x) > h\}|,$$

whose derivative is

$$u'(h) = -\int_{\partial[\phi>h]} \frac{1}{|\nabla\phi|}\, \mathrm{d}\mathcal{H}^1.$$

Considering the perimeter of the level sets and the isoperimetric inequality we obtain

$$2\pi^{\frac{1}{2}} u(h)^{\frac{1}{2}} \le \mathrm{Per}([\phi > h]) = \int_{\partial[\phi>h]} \mathrm{d}\mathcal{H}^1 \le \left(-u'(h)\int_{\partial[\phi>h]} |\nabla\phi|\, \mathrm{d}\mathcal{H}^1\right)^{\frac{1}{2}}.$$

Next, we use the first equation in (6-13) to compute the last integral,

$$\int_{\partial[\phi>h]} |\nabla\phi|\, \mathrm{d}\mathcal{H}^1 = \int_{[\phi>h]} -\Delta\phi\, \mathrm{d}x = \int_{[\phi>h]} \omega\, \mathrm{d}x \le \int_0^{(u(h)/\pi)^{1/2}} \omega^\sharp(s) s\, \mathrm{d}s.$$

Putting the last two equations together, we get the inequality

$$4\pi u(h) \le -u'(h) \int_0^{(u(h)/\pi)^{1/2}} \omega^\sharp(s) s\, \mathrm{d}s. \tag{6-15}$$

Noting that for the rearranged vorticity $\omega^\sharp$ all the inequalities are in fact equalities, we obtain that the distribution $v(h) = |\{x \in \mathbb{D} : \bar{\phi}(x) > h\}|$ function satisfies

$$4\pi v(h) = -v'(h) \int_0^{(v(h)/\pi)^{1/2}} \omega^\sharp(s) s\, \mathrm{d}s. \tag{6-16}$$

Using the boundary condition we have that

$$u(0) = v(0) = \pi,$$

and hence we can use the derivative equations (6-15) and (6-16) to conclude that

$$u(h) \le v(h) \quad \text{for all } h \ge 0.$$

Using the inequality above with

$$u(\phi^\sharp(x)) = \pi |x|^2 = v(\bar{\phi}(x)) \quad \text{and} \quad v'(h) < 0, \tag{6-17}$$

we get Talenti's inequality

$$\phi^\sharp(x) \le \bar{\phi}(x). \tag{6-18}$$

Using (6-17), with (6-15) and (6-16), we obtain that

$$\partial_r \phi^\sharp(r) \ge \partial_r \bar{\phi}(r) \quad \text{for all } r \in [0, 1],$$

where we have abused notation and considered $\phi^\sharp$ and $\bar{\phi}$ with respect to the radial variable. This implies

$$\max_{x \in \mathbb{D}} |\bar{\phi}(x) - \phi^\sharp(x)| = \bar{\phi}(0) - \phi^\sharp(0).$$

Since $\omega,\ \omega^\sharp \in L^\infty$, the corresponding stream functions $\phi,\ \bar{\phi}$ are Lipschitz-continuous. As radial rearrangements are contractive in the Lipschitz norm, we have

$$\|\phi^\sharp\|_{W^{1,\infty}} \le \|\phi\|_{W^{1,\infty}} \lesssim \|\omega\|_{L^\infty}.$$

Hence, there exists $r > 0$ small enough depending only on $\|\omega\|_{L^\infty}$ such that

$$|\bar{\phi}(x) - \phi^\sharp(x)| \ge \tfrac{1}{2} \|\bar{\phi} - \phi^\sharp\|_{L^\infty} \quad \text{for all } |x| < r. \tag{6-19}$$

Using the mass constraint and the L^∞ bound on ω, we can take $r > 0$ small enough depending on $\|\omega\|_{L^\infty}$ and m such that

$$\omega^\sharp(x) \ge \frac{\mathsf{m} - \|\omega\|_{L^\infty} |x|^2}{1 - |x|^2} \ge \frac{\mathsf{m}}{2} \quad \text{for all } |x| < r. \tag{6-20}$$

Now we are ready to show (6-14). By the Hardy–Littlewood inequality,

$$\begin{aligned} \mathsf{E}(\omega^\sharp) - \mathsf{E}(\omega) = \tfrac{1}{2} \int_{\mathbb{D}} \bar{\phi}\omega^\sharp - \phi\omega \, \mathrm{d}x = \tfrac{1}{2} \int_{\mathbb{D}} \phi^\sharp \omega^\sharp - \phi\omega + (\bar{\phi} - \phi^\sharp)\omega^\sharp \, \mathrm{d}x &\ge \tfrac{1}{2} \int_{\mathbb{D}} (\bar{\phi} - \phi^\sharp)\omega^\sharp \, \mathrm{d}x \\ &\ge \tfrac{1}{2} \int_{B_r} (\bar{\phi} - \phi^\sharp)\omega^\sharp \, \mathrm{d}x \\ &\ge c(\|\omega\|_{L^\infty}, \mathsf{m}) \|\bar{\phi} - \phi^\sharp\|_{L^\infty}, \end{aligned}$$

where we used (6-18), (6-19) and (6-20).

Step 2. For any $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|\bar{\phi} - \phi^\sharp\|_{L^\infty} < \delta$, there exists $x_* \in \mathbb{R}^2$ such that $|x_*| < \varepsilon$ and

$$\|\omega - \omega^\sharp(\,\cdot\, - x_*)\|_{L^1(\mathbb{R}^2)} < \varepsilon,$$

where the functions ω and $\omega^\sharp$ are extended by zero outside the disk.

Proof of Step 2. We choose x_* to be the optimizer of

$$\|\phi - \phi^\sharp(\cdot - x_*)\|_{L^2(\mathbb{R}^2)} = \inf_{x_0 \in \mathbb{R}^2} \|\phi - \phi^\sharp(\cdot - x_0)\|_{L^2(\mathbb{R}^2)}.$$

Applying [Amato et al. 2024, Section 5, equation (82)], we have that there exists a $C > 0$ such that

$$C^{-1} \min\bigl(|x_*|, \tfrac{1}{2}\bigr) \leq |\mathbb{D} \triangle (\mathbb{D} + x_*)| \leq C \|\bar{\phi} - \phi^\sharp\|_{L^\infty}^{1/4}.$$

The proof of [Amato et al. 2024, Theorem 1.4] shows exactly that we can pick $\|\bar{\phi} - \phi^\sharp\|_{L^\infty} < \delta$ small enough that

$$\inf_{x_0 \in \mathbb{R}^2} \|\omega - \omega^\sharp(\cdot - x_0)\|_{L^1} \leq \|\omega - \omega^\sharp(\cdot - x_*)\|_{L^1} < \varepsilon. \qquad \square$$

Appendix: A min-max principle

We show a variation of the classical min-max principle, which can be found in [Ekeland and Témam 1999, Chapter VI].

Proposition A.1. *Let A and B be closed convex sets of a Banach space $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$, and consider a proper functional $\mathsf{L} : A \times B \to \mathbb{R}$. Assume the following*:

(a) *For every $\beta \in A$, the function $\omega \mapsto \mathsf{L}(\beta, \omega)$ is weakly upper semicontinuous.*

(b) *For every $\omega \in B$, the function $\beta \mapsto \mathsf{L}(\beta, \omega)$ is convex and lower semicontinuous.*

(c) *The functional L is coercive in β. More specifically, there exists a function $g : [0, \infty) \to [0, \infty)$ such that $\lim_{u\to\infty} g(u) = \infty$ and for any $\beta \in A$ there exists $\omega \in B$ such that*

$$\mathsf{L}(\beta, \omega) \geq g(\|\beta\|_1).$$

(d) *The set B is bounded, and hence weakly compact.*

(e) *For every $\beta \in A$, the function $\omega \mapsto \mathsf{L}(\beta, \omega)$ has a unique maximizer ω_β.*

Then

$$\inf_{\beta \in A} \sup_{\omega \in B} \mathsf{L}(\beta, \omega) = \sup_{\omega \in B} \inf_{\beta \in A} \mathsf{L}(\beta, \omega).$$

Remark A.2. The (e) can be weakened to the following: for any $\beta_* \in A$, any two maximizers ω_1^* and ω_2^* satisfy

$$\mathsf{L}(\beta, \omega_1^*) = \mathsf{L}(\beta, \omega_2^*) \quad \text{for any other } \beta \in A.$$

In the case of the Euler equation in a radial domain this means that the min-max principle applies if we know that the maximizers are unique up to rigid rotations, which preserves the energy and the entropy.

Proof of Proposition A.1. First of all, we observe that

$$\mathsf{L}(\beta, \omega) \geq \inf_{\beta \in A} \mathsf{L}(\beta, \omega) \quad \text{for all } \omega \in B,$$

so that

$$\sup_{\omega \in B} \mathsf{L}(\beta, \omega) \geq \sup_{\omega \in B} \inf_{\beta \in A} \mathsf{L}(\beta, \omega),$$

and thus

$$\inf_{\beta\in A}\sup_{\omega\in B}\mathsf{L}(\beta,\omega)\geq\sup_{\omega\in B}\inf_{\beta\in A}\mathsf{L}(\beta,\omega),$$

so we only need to prove the reverse inequality. Define

$$f(\beta):=\mathsf{L}(\beta,\omega_\beta)=\sup_{\omega\in B}\mathsf{L}(\beta,\omega),$$

where $\omega_\beta\in B$ is assumed to be the unique maximizer from (e). The function $\beta\mapsto f(\beta)$ is convex and lower semicontinuous, being the envelope of convex lower semicontinuous functions by (b). Therefore by convexity and coercivity (c) it attains its lower bound at some $\bar\beta\in A$, so that

$$f(\bar\beta)=\min_{\beta\in A}f(\beta)=\min_{\beta\in A}\max_{\omega\in B}\mathsf{L}(\beta,\omega)$$

and

$$f(\bar\beta)\geq\mathsf{L}(\bar\beta,\omega)\quad\text{for all }\omega\in B.$$

Now, by convexity (b), for every $\beta\in A,\ \omega\in B$ and $t\in(0,1)$, we have

$$\mathsf{L}((1-t)\bar\beta+t\beta,\omega)\leq(1-t)\mathsf{L}(\bar\beta,\omega)+t\mathsf{L}(\beta,\omega).$$

In particular, taking $\beta_t=(1-t)\bar\beta+t\beta$ we consider $\omega=\omega_{\beta_t}$ given by (e) and we find

$$f(\bar\beta)\leq f(\beta_t)=\mathsf{L}(\beta_t,\omega_{\beta_t})\leq(1-t)\mathsf{L}(\bar\beta,\omega_{\beta_t})+t\mathsf{L}(\beta,\omega_{\beta_t})\leq(1-t)f(\bar\beta)+t\mathsf{L}(\beta,\omega_{\beta_t}),$$

implying

$$f(\bar\beta)\leq\mathsf{L}(\beta,\omega_{\beta_t})\quad\text{for all }\beta\in A.\tag{A-1}$$

Now, by compactness (d), as $t\to0$, ω_{β_t} converges weakly to some $\bar\omega\in B$, up to subsequences. Next, we claim that $\bar\omega=\omega_{\bar\beta}$. Indeed,

$$\mathsf{L}(\beta_t,\omega_{\beta_t})\geq\mathsf{L}(\beta_t,\omega)\quad\text{for all }\omega\in B,$$

and from convexity (b) we have

$$(1-t)\mathsf{L}(\bar\beta,\omega_{\beta_t})+t\mathsf{L}(\beta,\omega_{\beta_t})\geq\mathsf{L}(\beta_t,\omega)\quad\text{for all }\omega\in B.$$

Since $\mathsf{L}(\beta,\omega_{\beta_t})\leq f(\beta)<\infty$, we can use the semicontinuity (a) and (b) to pass to the limit as $t\to0$ and obtain

$$\mathsf{L}(\bar\beta,\bar\omega)\geq\limsup_{t\to0}(1-t)\mathsf{L}(\bar\beta,\omega_{\beta_t})+t\mathsf{L}(\beta,\omega_{\beta_t})\geq\liminf_{t\to0}\mathsf{L}(\beta_t,\omega)\geq\mathsf{L}(\bar\beta,\omega)\quad\text{for all }\omega\in B,$$

proving the claim that $\bar\omega=\omega_{\bar\beta}$, the unique maximizer by (e). We can now pass to the limit in (A-1), using weak upper semicontinuity (a), to get

$$f(\bar\beta)\leq\mathsf{L}(\beta,\omega_{\bar\beta})\quad\text{for all }\beta\in A.$$

Thus

$$\min_{\beta\in A}\max_{\omega\in B}\mathsf{L}(\beta,\omega)\leq\min_{\beta\in A}\mathsf{L}(\beta,\omega_{\bar\beta})\leq\max_{\omega\in B}\min_{\beta\in A}\mathsf{L}(\beta,\omega),$$

as we needed, and the proof is complete. □

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NORM-VARIATION OF TRIPLE ERGODIC AVERAGES FOR COMMUTING TRANSFORMATIONS

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We prove an r-variation estimate, $r > 4$, in the norm for ergodic averages with respect to three commuting transformations. It is not known whether such estimates hold for all $r \geq 2$ as in the analogous cases for one or two commuting transformations, or whether such estimates hold for any $r < \infty$ for more than three commuting transformations.

1. Introduction

We prove the following norm variation bound for three commuting transformations.

Theorem 1.1. *For all $r > 4$, there exists a constant $C > 0$ such that the following holds. Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space, $T_0, T_1, T_2 \colon X \to X$ mutually commuting measure preserving transformations, and let J and $n_0 < n_1 < \cdots < n_J$ be positive integers. For any $f_0, f_1 \in L^8(X)$ and $f_2 \in L^4(X)$, each of respective norm* 1, *we have the bound*

$$\sum_{j=1}^{J} \left\| M_{n_j}(f_0, f_1, f_2) - M_{n_{j-1}}(f_0, f_1, f_2) \right\|_{L^2(X)}^{r} \leq C,$$

where we have defined for almost every $x \in X$

$$M_n(f_0, f_1, f_2)(x) := \frac{1}{n} \sum_{i=0}^{n-1} f_0(T_0^i x) f_1(T_1^i x) f_2(T_2^i x).$$

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Norm variation bounds with $r \geq 2$ for one transformation were proven in [Jones et al. 1996] and for two commuting transformations in [Durcik et al. 2019a], following earlier work [Kovač 2016] in the finite group setting. Norm variation bounds with any $r < \infty$ for any number of commuting transformations were stated as an open problem in the closing section of [Avigad and Rute 2015]. Any such bounds remain unknown for more than three commuting transformations. It is natural to conjecture norm variation bounds for $r \geq 2$ for any number of commuting transformations. The passage from two to three commuting transformations is a critical transition as present techniques very clearly fail to address the sharp variation threshold $r \geq 2$.

Norm variation bounds for any r are strong quantitative forms of norm convergence. Qualitative norm convergence for three or more commuting transformations was proven by Tao [2008] by finitary methods. The case for two commuting transformations had been shown before using the tools from ergodic theory. Ergodic theoretic proofs of Tao's result were given in [Host 2009; Austin 2010], and a generalization to transformations generating a nilpotent group was proven in [Walsh 2012].

Norm convergence should be compared with the more difficult question of pointwise convergence almost everywhere. Such pointwise convergence is known by the classical Birkhoff theorem for a single transformation [Birkhoff 1931], with pointwise variational estimates proven in [Bourgain 1989]. Pointwise convergence almost everywhere remains a widely recognized open problem even in the case of two general commuting transformations. This contrasts with recent developments in the area concerning multiple ergodic averages with actions of polynomial powers $T^{p(n)}$, including a number of pointwise almost everywhere convergence results under the umbrella of the Furstenberg–Bergelson–Leibman conjecture such as the bilinear but not completely linear polynomial averages in [Krause et al. 2022] or the multiparameter polynomial averages in [Bourgain et al. 2023]. For further history on the ergodic means discussed in the present paper, we refer to the paper on two commuting transformations [Durcik et al. 2019a].

By a variant of the well-known Calderón transference principle, Theorem 1.1 follows from Theorem 1.2 below. We do not elaborate on the transference principle in the present paper but refer to the case of two commuting transformations in [Durcik et al. 2019a]. It reduces quantitative convergence results to analogous results on individual orbits of the action of the group spanned by the commuting transformations and parametrized by $\mathbb{Z}^3$. The further transfer from $\mathbb{Z}^3$ to $\mathbb{R}^3$ as in Theorem 1.2 is harmless and it can be made a part of the transference principle in our setting, unlike in the setting of actions $T^{p(n)}$ with polynomials of higher degree which face number theoretic complications.

Theorem 1.2. *For all $r > 4$, there exists a constant $C > 0$ such that the following holds. For any positive integer J and positive real numbers $t_0 < t_1 < \cdots < t_J$, any $f_0, f_1 \in L^8(\mathbb{R}^3)$ and $f_2 \in L^4(\mathbb{R}^3)$ with respective norm* 1, *we have*

$$\sum_{j=1}^{J} \left\| M_{t_j}(f_0, f_1, f_2) - M_{t_{j-1}}(f_0, f_1, f_2) \right\|_{L^2(\mathbb{R}^3)}^r \leq C, \tag{1-1}$$

where, with e_0, e_1, e_2 the standard unit vectors in $\mathbb{R}^3$, we have defined for almost every $x \in \mathbb{R}^3$

$$M_t(f_0, f_1, f_2)(x) := \frac{1}{t} \int_0^t f_0(x + \tau e_0) f_1(x + \tau e_1) f_2(x + \tau e_2)\, d\tau. \tag{1-2}$$

Only the choice of tuple of exponents $(8, 8, 4)$ breaks the symmetry between the three functions in the above theorems. One therefore concludes the analogous estimates for permutations of these exponents. Interpolation gives further tuples of exponents, for example the symmetric tuple $(6, 6, 6)$.

Theorem 1.2 is proven using the theory of singular Brascamp–Lieb forms. A singular Brascamp–Lieb datum $D = (n, S, \Pi, (\Pi_s)_{s\in S})$ is a tuple containing the dimension $n \geq 1$ of the domain of integration, the finite set S parametrizing the tuple of input functions, and linear maps Π and Π_s for $s \in S$ on the domain $\mathbb{R}^n$, where Π_s maps onto the domain of the input function with parameter s, typically of smaller dimension than n. Together with some singular integral kernel K on the range of Π, the singular Brascamp–Lieb form $\Lambda_{D,K}$ is defined as

$$\Lambda_{D,K}((f_s)_{s\in S}) = \int_{\mathbb{R}^n} K(\Pi x) \prod_{s\in S} f_s(\Pi_s x)\, dx,$$

where the integral is defined in some principal value sense or, if the kernel has additional qualitative regularity as is mostly the case in the present paper, in the Lebesgue integral sense. We also often talk about the multiplier m of the form, which is the Fourier transform of the kernel K. A singular Brascamp–Lieb inequality estimates this form by a constant times the product of Lebesgue norms $\prod_{s\in S} \|f_s\|_{p_s}$ for some tuple of exponents p_s.

Singular Brascamp–Lieb inequalities with the kind of data appearing in this paper are studied in [Durcik et al. 2019b; 2022; Durcik and Thiele 2021; Muscalu and Zhai 2022] when K is a classical Calderón–Zygmund kernel. Compared with this work, the novelty in the present paper is that the kernels K do not satisfy uniform Calderón–Zygmund bounds but rather multiparameter symbol estimates arising naturally in the investigation of variation norms. These symbol estimates no longer synchronize with a Whitney decomposition of frequency space but rather involve regions determined by an arbitrary sequence of jumps between scales, such as the red regions with arbitrary eccentricity in Figure 1.

Multiparameter singular Brascamp–Lieb forms of this type appear in more basic form already in the case of two commuting transformations [Durcik et al. 2019a]. Compared to two transformations, novel challenges for three commuting transformations arise from the absence of the cubical structure of the main singular Brascamp–Lieb form relevant to Theorem 1.2. For two commuting transformations the set S of the Brascamp–Lieb datum can be naturally identified with the corners of a square, but for three commuting transformations it cannot be identified with corners of a cube, but rather with vertices of a triangular prism. Cubical structure is important to allow a loss-free symmetrization of the form along the reflection symmetries of the cube. Lacking such cubical structure, the techniques available to us lead to an unavoidable loss analogous to the work on cancellation for the simplex Hilbert transform [Durcik et al. 2019b] and also [Durcik and Kovač 2022; Durcik and Stipčić 2025]. One novelty in the present paper is that this loss needs to be absorbed by a relaxation of the variation norm parameter r towards $r > 4$. In other words, we cannot allow a loss in difference between largest and smallest scale involved, i.e., in the total number of intermediate scales involved, but only a loss in the much smaller number of jumps between the scales. Thus we need to develop an analysis that carefully uses and preserves the particular structure of multipliers as depicted in Figure 1 throughout the argument.

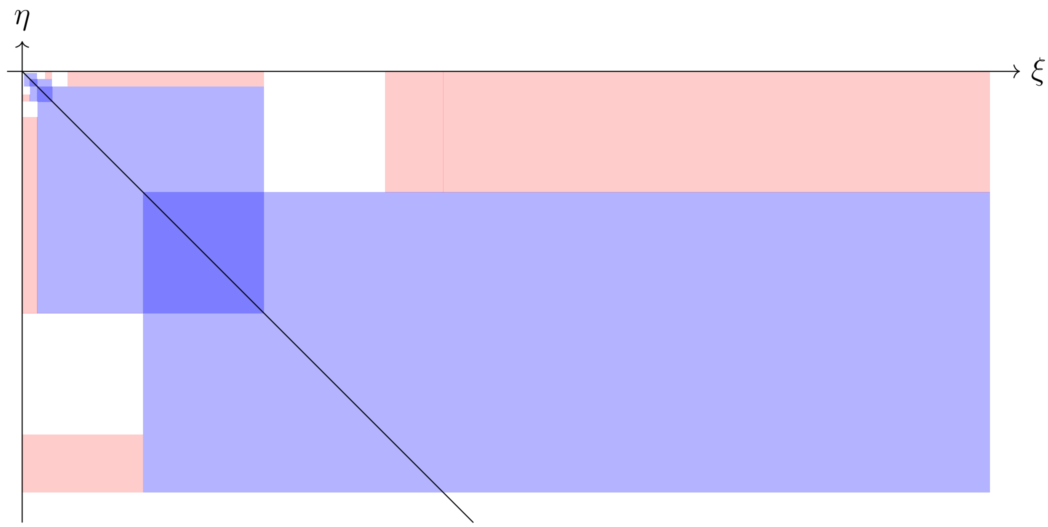

Figure 1. Structure of $m = \widehat{K}$ in Theorem 1.2.

We next provide an overview over the arguments of the present paper, which is structured into intermediate propositions and sections as in Figure 2.

Theorem 1.2 is deduced in Section 3 from (3-1), an estimate in terms of a fixed number J of jumps in the variation, which can be thought of as an endpoint estimate at $r = 4$ for (1-1). This endpoint estimate is reduced to two singular Brascamp–Lieb estimates, both with datum D_1 defined in (2-1), but with different two-dimensional kernels illustrated in Figure 1. For simplicity we focus on one quadrant in our discussion, as the other quadrants do not pose additional difficulties. The first singular Brascamp–Lieb estimate, Proposition 2.1, takes care of the so-called short variation with a multiplier that lives near the dark blue overlap regions of the light blue squares in Figure 1. The size of each dark blue square is comparable to its distance to the origin, a property we call Whitney. Proposition 2.3 takes care of the so-called long variation with multipliers living at the light blue squares themselves. Each light blue square includes potentially many scales and therefore is not in general Whitney. However, the piece of the multiplier associated with each light blue square has elementary tensor structure and telescopes into the difference between its largest and its smallest scale. For both of these propositions, it is important that the number of squares involved is controlled by J.

Proposition 2.3 is proved in Section 5. Multipliers vanishing on the diagonal in Figure 1 play a role as auxiliary objects. We use a positivity of multipliers symmetric across the diagonal to pass to a similar multiplier m_1 associated with light blue squares but constant on the diagonal. We then define two further multipliers m_2 and m_3 so that $m_1 + m_2 + m_3$ is constant in the entire plane. This constant multiplier allows a trivial bound, reducing the estimate for m_1 to estimates for m_2 and m_3. Multiplier m_2 is addressed in Proposition 2.6. It is supported near the red sticks in Figure 1. Each stick is away from the diagonal and has possibly many scales and is therefore not in general Whitney. However, the multiplier associated with each stick is an elementary tensor and as such telescopes into a small number of scales. The multiplier

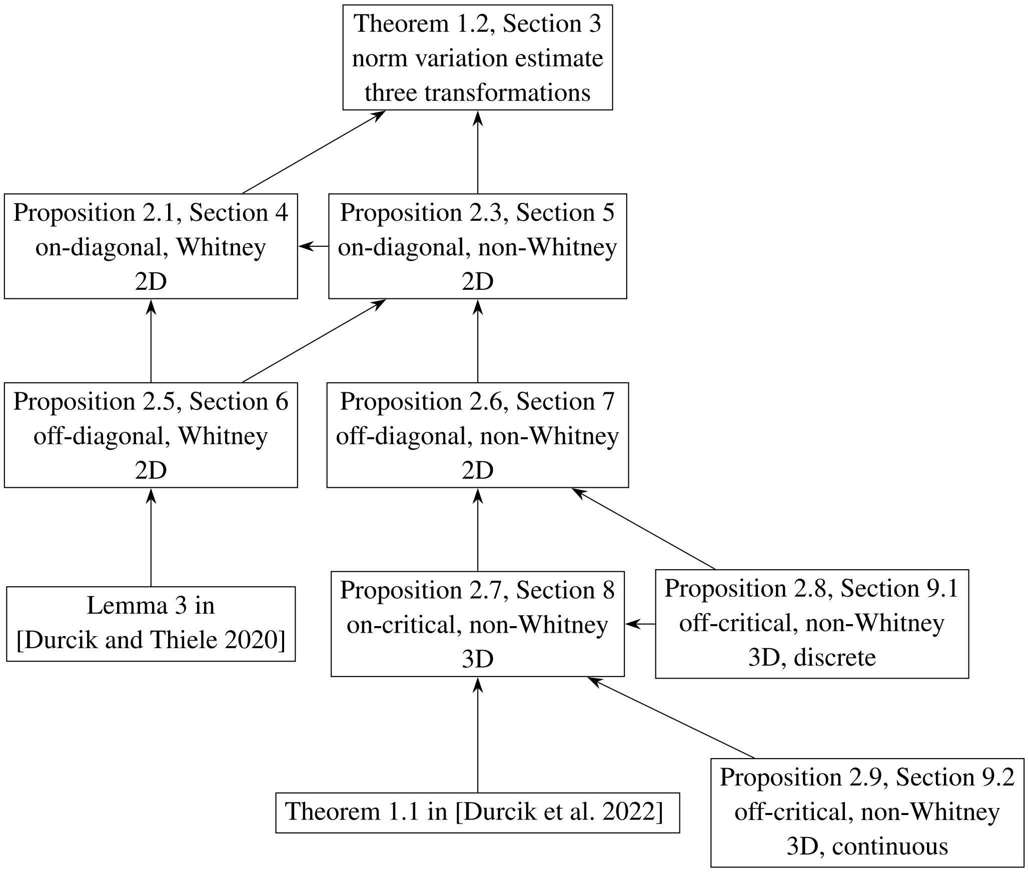

Figure 2. Structure of the proof of the main theorem.

$m_1 + m_2$ is constant both on the diagonal as well as on the white L-shaped regions in Figure 3. The multiplier m_3 is addressed in Proposition 2.5. It is supported in the at most J purple regions in between the white L-shaped regions and vanishes on the diagonal. Each purple region has a single scale and is Whitney. We decompose m_3 into a lacunary family of cones towards the diagonal shown in Figure 3. Each lacunary piece is estimated with Lemma 3 in [Durcik and Thiele 2020]. Thanks to vanishing on the diagonal, one has a geometric sum for the estimates in this family.

Proposition 2.1 is proved in Section 4. We combine the dark blue squares with a suitable family of light blue squares with tensor structure to obtain a multiplier vanishing on the diagonal. The light blue squares are estimated with Proposition 2.3, while the multiplier vanishing on the diagonal is estimated with Proposition 2.5.

Proposition 2.6 is proved in Section 7. Using the off-diagonal property of the multiplier to preserve crucial cancellation in the innermost integral, we apply the Cauchy–Schwarz inequality in the remaining integrals. We estimate one of the factors on the right-hand side of Cauchy–Schwarz using that the multiplier has J summands, which leads to the loss of $J^{1/2}$. The other factor we estimate loss-free thanks to the above mentioned cancellation. This loss-free estimate takes the form of a singular Brascamp–Lieb

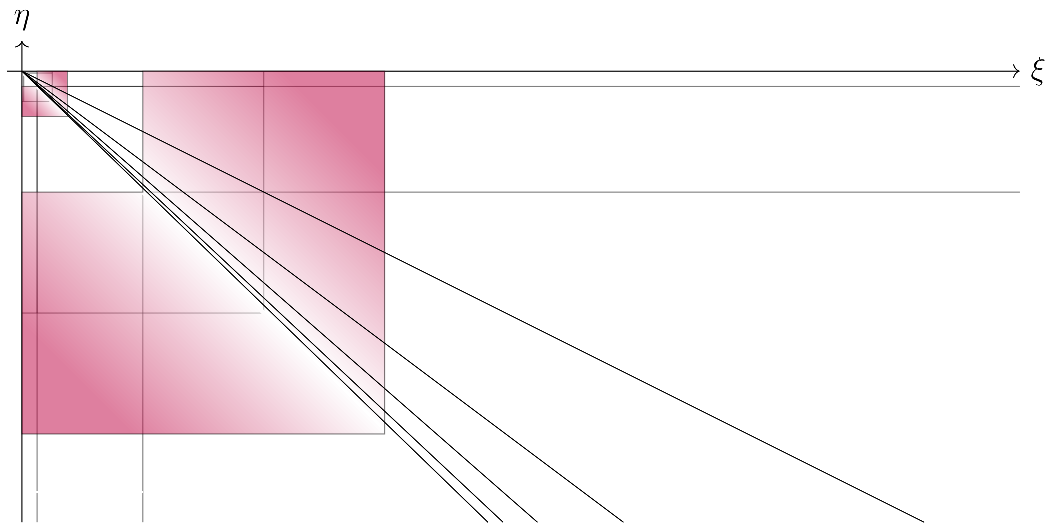

Figure 3. Lacunary cones.

form with datum D_2 defined in (2-10). The multiplier m is now three-dimensional, but consisting of pieces that are naturally of the form

$$\phi_1(\theta \cdot v_1)\phi_2(\theta \cdot v_2), \tag{1-3}$$

with two vectors $v_1 = (0, 0, 1)$ and $v_2 = (1, -1, 0)$ as shown in Figure 4. Typical behavior of the functions ϕ_1 and ϕ_2 is shown in Figure 4 on the planes perpendicular to v_1 and v_2. An important role is played by multipliers vanishing on the critical space spanned by v_1 and v_2. Such multipliers are estimated in Propositions 2.8 and 2.9 in the non-Whitney case and by Theorem 1.1 in [Durcik et al. 2022] in the Whitney case. The multiplier m does not vanish on the critical space. It is first modified using Proposition 2.8 towards a multiplier m' that also consists of pieces as in (1-3) but is more symmetric in the variables ξ, η. The multiplier m' is then estimated by Proposition 2.7.

Proposition 2.7 is proved in Section 8. The key to estimating non-Whitney multipliers such as m' is a new variant of a telescoping identity in this context that concerns three-dimensional multipliers with a two-dimensional flavor as in Figure 4 and (1-3). This identity telescopes a trivial multiplier, the product of the lightest blue squares minus the product of the darkest blue squares, into two sums consisting of products of a square in one plane with a difference of the corresponding square with a consecutive square in the other plane. One sum is arranged to allow a positivity argument using reflection symmetry across the diagonal in the ξ, η plane, the other sum is arranged similarly to allow a positivity argument with symmetry in the shown skew coordinates in the other plane. As the two positive sums add to a trivial multiplier, both are individually bounded. The given multiplier m' can be dominated by one of these constructed multipliers, using various modifications with Propositions 2.8 and 2.9 and Theorem 1.1 in [Durcik et al. 2022].

Propositions 2.8 and 2.9 are proved in Section 9. Vanishing of the multiplier on the critical space allows a lacunary decomposition away from the critical space and a further Cauchy–Schwarz. This leads to singular Brascamp–Lieb estimates with a standard cubical datum and three-fold telescoping identities

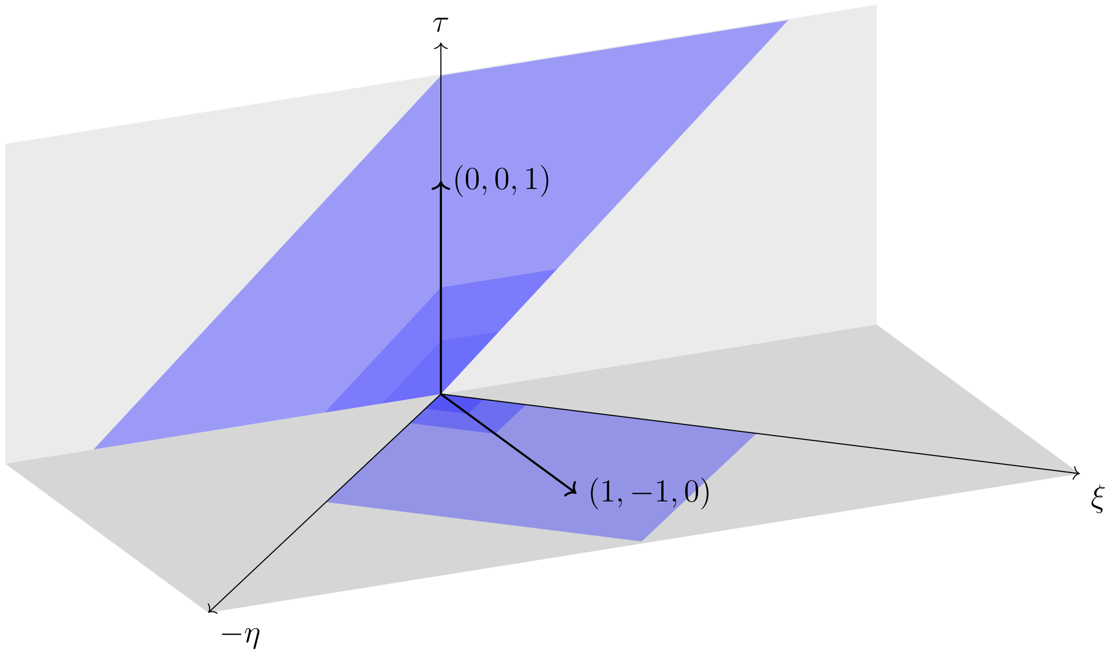

Figure 4. Three-dimensional multiplier.

for three-dimensional kernels. The non-Whitney property requires telescoping along the scales of the variation sequences. There is a mix of discrete telescoping and partial integration, with Proposition 2.8 more discrete and Proposition 2.9 more continuous. Analogous but simpler techniques appear in the Whitney case in [Durcik et al. 2022, Theorem 1.1].

We have kept the sections past Sections 1 and 2 independent of each other; each proves the theorem or one or two propositions and uses some of the other propositions or cited theorems as black boxes.

While it is plausible that our approach can be upgraded to an iterative scheme that handles more than three commuting transformations, we decided to complete and circulate the argument in the case of three transformations. This case has a single transition step with the important new techniques and does not appear to involve all the complications that one expects for the general case.

2. A collection of propositions on singular Brascamp–Lieb forms

This section contains a number of propositions stating cancellation estimates for singular Brascamp–Lieb forms for some data and some class of kernels and with symmetric tuples of test functions.

The first four propositions and two corollaries share a common datum D_1, which, after suitable change of variables, arises directly out of the original problem in Theorem 1.2. Put coordinates

$$x = (x_0, x_1, x_2, x_3^0, x_3^1)$$

on $\mathbb{R}^5$. Define

$$D_1 := (5, S, \Pi, (\Pi_s)_{s\in S}) \tag{2-1}$$

with $S = \{0, 1, 2\} \times \{0, 1\}$, with Π mapping $\mathbb{R}^5$ to $\mathbb{R}^2$ as

$$\Pi(x) = (x_3^0 - x_0 - x_1 - x_2, x_3^1 - x_0 - x_1 - x_2),$$

and with Π_s for $s = (k, j)$ mapping $\mathbb{R}^5$ to $\mathbb{R}^3$ as

$$\Pi_{(k,j)}(x) = (x_0, x_1, x_2) - x_k e_k + x_3^j e_k.$$

Each of the three following propositions will have a constant C, a parameter J and formulate a class of kernels K such that the singular Brascamp–Lieb estimate

$$|\Lambda_{D_1,K}((f_s)_{s\in S})| \le C J^{\frac{1}{2}} \tag{2-2}$$

holds for all tuples of real-valued Schwartz functions $(f_s)_{s\in S}$ such that

$$f_{(k,0)} = f_{(k,1)} \tag{2-3}$$

for each $k \in \{0, 1, 2\}$ and

$$\|f_{(0,j)}\|_8 = \|f_{(1,j)}\|_8 = \|f_{(2,j)}\|_4 = 1 \tag{2-4}$$

for each $j \in \{0, 1\}$. We point out that the symmetry assumption (2-3) arises naturally when reducing Theorem 1.2 to the propositions stated below. While our arguments could be modified in order to prove these propositions without the extra assumption (2-3), we decided not to pursue this line of generalization.

Define for any function ϕ on $\mathbb{R}^d$ the L^1 normalized scaling

$$\phi_{(t)}(x) = t^{-d}\phi(t^{-1}x).$$

Define the Fourier transform $\widehat{\phi}$ of ϕ by integration against the kernel $(x, \xi) \mapsto e^{-2\pi i x\cdot\xi}$.

The kernels in the next proposition satisfy standard two-dimensional symbol estimates with bounds depending on the parameter k. They consist of pieces satisfying a positivity assumption. Such positivity assumption is used in the proof by adding further positive terms so as to achieve better behavior on some frequency diagonal. The complexity of these kernels is bounded by J.

Proposition 2.1 (on-diagonal, Whitney, 2D [proved in Section 4]). *Let $\lambda = \frac{3}{2}$. There exists a constant $C > 0$ such that the following holds for all $k \le 0$. Let J be a positive integer and $(k_j)_{j=1}^J$ a finite strictly monotone increasing sequence of integers. Let*

$$K = \sum_{j=1}^{J} \Phi_j,$$

where for each $1 \le j \le J$ we assume Φ_j is a real-valued function on $\mathbb{R}^2$, with symmetry

$$\Phi_j(u, v) = \Phi_j(v, u)$$

and positivity in the sense

$$\int_{\mathbb{R}^2} f(u)\overline{f(v)}\Phi_j(u, v)\, du\, dv \ge 0 \tag{2-5}$$

for all complex-valued f. We assume further

$$\operatorname{supp}(\widehat{\Phi}_j) \subset \big([-2^{-k_j+20}, -2^{-k_j-20}] \cup [2^{-k_j-20}, 2^{-k_j+20}]\big)^2$$

and, for all $(u, v) \in \mathbb{R}^2$,

$$|(\Phi_j)_{(2^{-k_j})}(u, v)| \leq 2^{\lambda k}(1 + 2^k|u + v|)^{-10}(1 + |u - v|)^{-10}. \tag{2-6}$$

Then estimate (2-2) *holds for any tuple as in* (2-3), (2-4).

We note that the particular value $\lambda = \frac{3}{2}$ is not essential for the proof of Proposition 2.1. Evidently, the analogous statement of the proposition becomes stronger for smaller values of λ. Our proof can be pushed to $\lambda > 1$ at the expense of allowing the constant in (2-2) to depend on λ. On the other hand, the upper bound $\lambda < 2$ is needed to apply Proposition 2.1 to prove Theorem 1.2. There are also constants 10 and 20 chosen in this proposition which need to be large enough but also need to relate to similar other constants in other propositions to follow.

If, in the above proposition, each Φ_j is an elementary tensor of a suitable function ϕ_j with itself, then symmetry and positivity are automatic, and k is naturally chosen as 0. We formulate this as an immediate corollary.

Corollary 2.2. *There exists a constant $C > 0$ such that the following holds. Let J be a positive integer and $(k_j)_{j=1}^J$ a finite strictly monotone increasing sequence of integers. Let*

$$K = \sum_{j=1}^{J} \phi_j \otimes \phi_j,$$

where for each $1 \leq j \leq J$ we assume ϕ_j is a real-valued function on $\mathbb{R}$ with

$$\operatorname{supp}(\widehat{\phi}_j) \subset [-2^{-k_j+20}, -2^{-k_j-20}] \cup [2^{-k_j-20}, 2^{-k_j+20}]$$

and, for all $u \in \mathbb{R}$,

$$|(\phi_j)_{(2^{-k_j})}(u)| \leq (1 + |u|)^{-20}.$$

Then estimate (2-2) *holds for any tuple as in* (2-3), (2-4).

We need the following technical notion of pairs in the next proposition. Let $N = 2^{18}$. This large number is necessitated by a somewhat inefficient referral in the proof of Proposition 2.7 to a theorem in [Durcik et al. 2022]. A more hands-on approach should be able to make this number much more moderate, but this is certainly not important for our argument. A *c-pair* is a pair (ϕ_0, ϕ_1) of two real-valued integrable even functions satisfying the following assumptions. Their Fourier transforms $\widehat{\phi}_0, \widehat{\phi}_1$ map $\mathbb{R}$ to $[0, 1]$, are supported on $[-1, 1]$ and constant 1 on $[-2^{-1}, 2^{-1}]$. They satisfy

$$(\widehat{\phi}_0)^2 + (1 - \widehat{\phi}_1)^2 = 1 \quad \text{and} \quad \|\widehat{\phi}_0^{(N+30)}\|_\infty, \|\widehat{\phi}_1^{(N+30)}\|_\infty \leq c. \tag{2-7}$$

Here and in what follows, $\varphi^{(k)}$ stands for the k-th derivative of φ. Lemma 2.10 below shows that there is c such that a c-pair exists. When c is at most one million times the infimum of all positive numbers c' such that a c'-pair exists, then (ϕ_0, ϕ_1) is called a universal pair. A left window is a function ϕ such that there exists a function ψ such that (ϕ, ψ) is a universal pair. A right window is a function ϕ such that there exists a function ρ such that (ρ, ϕ) is a universal pair. Note that functions ϕ that are both left and right window may exist, but a notion of two sided windows needs caution as the corresponding functions ψ and ρ may not satisfy this notion.

The kernels of the next proposition do not satisfy two-dimensional symbol estimates, at least not uniformly in the choices of sequences k_j and l_j. They still consist of pieces with a positivity assumption and elementary tensor structure with only two different scales in it and have complexity controlled by J.

Proposition 2.3 (on-diagonal, non-Whitney, 2D [proved in Section 5]). *There exists $C > 0$ such that the following holds. Let J be a positive integer and $(k_j)_{j=1}^J$ and $(l_j)_{j=1}^J$ two finite sequences of integers that are interlaced in the sense that $k_j + 10 < l_j$ for $1 \le j \le J$ and $l_j < k_{j+1}$ for $1 \le j < J - 1$. Consider a kernel*

$$K = \sum_{j=1}^{J} (\phi_{0,j} - \phi_{1,j}) \otimes (\phi_{0,j} - \phi_{1,j}),$$

where, for each j, $(\phi_{0,j})_{(2^{-k_j})}$ is a left window and $(\phi_{1,j})_{(2^{-l_j})}$ is a right window.

Then estimate (2-2) *holds for any tuple as in* (2-3), (2-4).

Using Corollary 2.2, we have the following corollary of Proposition 2.3,

Corollary 2.4. *The variant of Proposition 2.3, where the assumption $k_j + 10 < l_j$ is replaced by the assumption $k_j < l_j$, holds.*

To see this corollary, we split the sequence into terms with $k_j + 10 \ge l_j$ and $k_j + 10 < l_j$. The former terms are estimated with Corollary 2.2, while the latter are estimated with Proposition 2.3.

In contrast to the last proposition, the kernel of the next proposition does not oscillate on the critical frequency diagonal $\xi + \eta = 0$. The complexity still is controlled by J. We no longer have the positivity assumptions, but we do satisfy standard symbol estimates, with bounds depending on the parameter k.

Proposition 2.5 (off-diagonal, Whitney, 2D [proved in Section 6]). *Let $\lambda = \frac{3}{2}$. There exists a constant $C > 0$ such that the following holds for all $k \le 0$. Let J be a positive integer and let $(k_j)_{j=1}^J$ be a finite strictly increasing sequence of integers. Let $(\Phi_j)_{j=1}^J$ be a finite sequence of real-valued functions on $\mathbb{R}^2$. Assume that*

$$\operatorname{supp}(\widehat{\Phi}_j) \subseteq \{(\xi, \eta) \in \mathbb{R}^2 : 2^{-k_j-30} \le |(\xi, \eta)| \le 2^{-k_j+30}\}.$$

Assume further that, for all $(u, v) \in \mathbb{R}^2$,

$$|(\Phi_j)_{(2^{-k_j})}(u, v)| \le 2^{\lambda k}(1 + 2^k|u + v|)^{-4}(1 + |u - v|)^{-4} + (1 + |u + v|)^{-4}(1 + |u - v|)^{-4}. \tag{2-8}$$

Let K be defined by

$$K = \sum_{j=1}^{J} \Phi_j,$$

and assume that $\widehat{K}$ vanishes on the diagonal $\{(\xi, \eta) \in \mathbb{R}^2 : \xi + \eta = 0\}$.

Then estimate (2-2) *holds for any tuple as in* (2-3), (2-4).

The kernel of the next proposition also vanishes on the critical diagonal. It does not satisfy standard two-dimensional symbol estimates uniformly in k_j. It has no positivity assumption, but similarly to some of the positive kernels above it is a sum of J tensors with few scales in it.

Proposition 2.6 (off-diagonal, non-Whitney, 2D [proved in Section 7]). *There exists a constant $C > 0$ such that the following holds. Let J be a positive integer and $(k_j)_{j=0}^J$ a finite increasing sequence of integers with $k_{j-1} + 10 \le k_j$ for $1 \le j \le J$. For $1 \le j \le J$, let $\phi_{0,j}$, $\phi_{1,j}$, $\phi_{2,j}$ be functions such that $(\phi_{0,j})_{(2^{-k_{j-1}})}$ is a left window, while $(\phi_{1,j})_{(2^{-k_j})}$ and $(\phi_{2,j})_{(2^{4-k_j})}$ are right windows. Define*

$$K = \sum_{j=1}^{J} (\phi_{0,j} - \phi_{2,j}) \otimes \phi_{1,j}. \tag{2-9}$$

Then estimate (2-2) *holds for any tuple as in* (2-3), (2-4).

The remaining propositions share a singular Brascamp–Lieb datum D_2. The datum D_2 arises as a reduction from D_1 after a Cauchy–Schwarz inequality. Put coordinates $x = (x_0, x_1, x_2^0, x_3^0, x_2^1, x_3^1)$ on $\mathbb{R}^6$. Define

$$D_2 := (6, S, \Pi, (\Pi_s)_{s\in S}) \tag{2-10}$$

with $S = \{0, 1\} \times \mathcal{C}$, where $\mathcal{C}$ is the set of functions $j : \{0, 1\} \to \{0, 1\}$, with Π mapping $\mathbb{R}^6$ to $\mathbb{R}^3$ as

$$\Pi(x) = \big(x_2^0 - x_0 - x_1 - x_3^0,\, x_2^1 - x_0 - x_1 - x_3^0,\, x_3^1 - x_3^0\big),$$

and with Π_s for $s = (k, j)$ mapping $\mathbb{R}^6$ to $\mathbb{R}^3$ as

$$\Pi_{(k,j)}(x) = (x_k, x_2^{j(0)}, x_3^{j(1)}).$$

For this datum D_2 and a kernel K, we are interested in a loss-free estimate

$$|\Lambda_{D_2,K}((f_s)_{s\in S})| \le C \tag{2-11}$$

for any tuple of real-valued Schwartz functions $(f_s)_{s\in S}$ with

$$f_{(k,j)} = f_{(k,j')} \tag{2-12}$$

for all $k \in \{0, 1\}$ and $j, j' \in \mathcal{C}$, and

$$\|f_s\|_8 = 1 \tag{2-13}$$

for all $s \in S$.

The next proposition is a variant of Proposition 2.3, adjusted to the datum D_2. The kernel has some positivity properties and pieces arising from suitable elementary tensor structure. The complexity J here is not relevant, as we obtain estimates independent of J.

We write g for the Gaussian $g(x) = e^{-\pi|x|^2}$, typically in one dimension but occasionally in more than one dimension. We have $\hat{g} = g$. We write h for the derivative of the Gaussian in one dimension, $h(x) = -2\pi x g(x)$. Recall $N = 2^{18}$.

Proposition 2.7 (on-critical, non-Whitney, 3D [proved in Section 8]). *There exists a constant $C > 0$ such that the following holds. Let $\alpha \geq 1$. Let J be a positive integer and $(k_j)_{j=0}^J$ a finite increasing sequence of integers with $k_{j-1} + 10 \leq k_j$ for $1 \leq j \leq J$. Let $(m_j)_{j=1}^J$ be a sequence of real numbers with $k_j - 1 \leq m_j \leq k_j$ for $1 \leq j \leq J$. For $0 \leq j \leq J$, let χ_j be a function such that $(\chi_j)_{(2^{2-k_j})}$ is a left window, and let ϕ_j be such that $\widehat{\phi}_j \geq 0$ and*

$$(\widehat{\phi}_j)^2 = (\widehat{\chi}_{j-1})^2 - (\widehat{\chi}_j)^2.$$

Let

$$K(u, v, z) = \alpha^{-N} \sum_{j=1}^{J} \int_{\mathbb{R}} g_{(\alpha 2^{m_j})}(u+p) g_{(\alpha 2^{m_j})}(v+p) \phi_j(z+p) \phi_j(p)\, dp.$$

Then estimate (2-11) *holds for any tuple as in* (2-12), (2-13).

Proposition 2.7 will be proven using the next two propositions. Both involve the datum D_2. Both exploit a vanishing of the function $\widehat{K}$ on the critical space $\xi + \eta = 0$.

Proposition 2.8 (off-critical, non-Whitney, 3D, discrete [proved in Section 9]). *There is a constant C such that the following holds. Let J be a positive integer. For $1 \leq i \leq 2$, let $(a_{i,j})_{j=1}^J$ be increasing sequences of positive real numbers.*

For $1 \leq j \leq J$, let $\rho_j : \mathbb{R}^4 \to \mathbb{R}$ be a continuous function satisfying

$$\int_{\mathbb{R}^2} |\rho_j|(u_1+p, u_2+p, u_3+r, u_4+r)\, dp\, dr \leq a_{1,j}^{-1}(1+a_{1,j}^{-1}|u_1-u_2|)^{-2} a_{2,j}^{-1}(1+a_{2,j}^{-1}|u_3-u_4|)^{-2} \quad (2\text{-}14)$$

for every $(u_1, u_2, u_3, u_4) \in \mathbb{R}^4$. Let $(c_j)_{j=0}^J$ be an increasing sequence of positive real numbers, well-separated in that $2c_{j-1} \leq c_j$ for $1 \leq j \leq J$. Let χ be a left window. For $1 \leq j \leq J$ let $\phi_j : \mathbb{R} \to \mathbb{R}$ be a continuous function, which exists due to the left window property of χ, satisfying $\widehat{\phi}_j \geq 0$ and

$$(\widehat{\phi}_j)^2 = (\widehat{\chi}_{(c_{j-1})})^2 - (\widehat{\chi}_{(c_j)})^2.$$

Let K be defined by

$$K(u, v, z) = \sum_{j=1}^{J} \int_{\mathbb{R}^3} \phi_j(p) \phi_j(q) \rho_j(u+p+q+r, v+p+q+r, z+r, r)\, dp\, dq\, dr. \quad (2\text{-}15)$$

Then estimate (2-11) *holds for any tuple as in* (2-12), (2-13).

The orthogonal complement $V^\perp$ of the subspace

$$V = \{(\xi, \eta, \tau, -(\xi+\eta+\tau), -(\xi+\eta), -(\xi+\eta)) : \xi, \eta, \tau \in \mathbb{R}\}$$

of $\mathbb{R}^6$ can be parametrized as

$$\{(p+q+r, p+q+r, r, r, p, q) : p, q, r \in \mathbb{R}\}.$$

As (2-15) is an integral over $V^\perp$ of a function F in $\mathbb{R}^6$, its Fourier transform is the restriction to V of the Fourier transform of $\widehat{F}$ to that subspace. Hence, for some universal constant C,

$$\widehat{K}(\xi,\eta,\tau)=C\sum_{j=1}^{J}\widehat{\phi}_j(\xi+\eta)^2\hat{\rho}_j(\xi,\eta,\tau,-\xi-\eta-\tau). \tag{2-16}$$

This expression shows the vanishing of $\widehat{K}(\xi,\eta,\tau)$ on the hyperplane $\xi+\eta=0$.

Also in the following proposition, $\widehat{K}$ vanishes on $\xi+\eta$. It is made up by a very specific part in the variables ξ, η and a rather general part in the variables τ and $\tau+\xi+\eta$.

Proposition 2.9 (off-critical, non-Whitney, 3D, continuous [proved in Section 9]). *There is a constant C such that the following holds. Let J be a positive integer and $(a_j)_{j=0}^J$, $(b_j)_{j=1}^J$ be increasing sequences of positive real numbers. For $1\le j\le J$ let $\phi_j:\mathbb{R}^2\to\mathbb{R}$ be a continuous function satisfying*

$$|\phi_j(u_1,u_2)|\le(b_j)^{-2}(1+b_j^{-1}|(u_1,u_2)|)^{-4}. \tag{2-17}$$

Let K be a kernel such that

$$\widehat{K}(\xi,\eta,\tau)=\sum_{j=1}^{J}\int_{a_{j-1}}^{a_j}t^2(\xi+\eta)^2g(t\xi)g(t\eta)\,\frac{dt}{t}\widehat{\phi}_j(\tau,-\xi-\eta-\tau). \tag{2-18}$$

Then estimate (2-11) *holds for any tuple as in* (2-12), (2-13).

We remark on a symmetry in the datum D_2. We do a change of variables in the kernel using the linear map

$$L(a,b,c)=(a+b-c,a-b,c).$$

Define

$$\widetilde{\Pi}(x):=L\circ\Pi(x)=\big(x_2^0+x_2^1-x_3^0-x_3^1-2(x_0+x_1),x_2^0-x_2^1,x_3^1-x_3^0\big).$$

Define $\widetilde{D}_2$ from D_2 by replacing Π by $\widetilde{\Pi}$, and choose $\widetilde{K}$ so that $\widetilde{K}\circ L=K$. We obtain

$$\Lambda_{D_2,K}((f_s)_{s\in S})=\Lambda_{\widetilde{D}_2,\widetilde{K}}((f_s)_{s\in S}).$$

The map $\widetilde{\Pi}$ has a symmetry under interchanging the last two entries at the same time as precomposing with the involution

$$(x_0,x_1,x_2^0,x_3^0,x_2^1,x_3^1)\mapsto(x_0,x_1,-x_3^0,-x_2^0,-x_3^1,-x_2^1).$$

This involution can be seen as acting on the tuple of functions f_s, and hence we have the following consequence for the associated form. Define $\widetilde{K}^*(a,b,c)=\widetilde{K}(a,c,b)$. For $j\in\mathcal{C}$, define $j^*\in\mathcal{C}$ by $j^*(l)=j(1-l)$ and define $f^*_{(k,j)}(a,b,c)=f_{(k,j^*)}(a,-c,-b)$. Then

$$\Lambda_{\widetilde{D}_2,\widetilde{K}}((f_s)_{s\in S})=\Lambda_{\widetilde{D}_2,\widetilde{K}^*}((f_s^*)_{s\in S}). \tag{2-19}$$

We finally introduce a further datum D_A, which is associated with a regular 3×3 matrix A and has $n=6$. Let S be the set of functions $S:\{0,1,2\}\to\{0,1\}$. We put coordinates $x=(x_1^0,x_2^0,x_3^0,x_1^1,x_2^1,x_3^1)$

on $\mathbb{R}^6$. We define

$$D_A := (6, S, \Pi, (\Pi_s)_{s\in S}), \tag{2-20}$$

where the projection $\Pi : \mathbb{R}^6 \to \mathbb{R}^3$ is given by

$$\Pi(x)^T = (I, A)x^T, \tag{2-21}$$

where I is the 3×3 identity matrix and (I, A) is a 3×6 block matrix. For $s \in S$, $\Pi_s : \mathbb{R}^6 \to \mathbb{R}^3$ is given by

$$\Pi_s(x) = (x_1^{s(0)}, x_2^{s(1)}, x_3^{s(2)}).$$

Note that after the relabelling of the coordinates, this datum has the same components as the datum D_2 except for the choice of the projection Π. We have used transposes in (2-21) as we usually write vectors as rows while the matrix equation (2-21) expects columns. The datum D_A will be used in the proofs of Propositions 2.5, 2.8, and 2.9. In the latter two cases we will only use it with $A = -I$.

We conclude this section with the previously announced existence result.

Lemma 2.10. *There exists a $c > 0$ and a c-pair as defined near* (2-7).

Proof. Let $\psi : [0, \infty) \to \mathbb{R}$ be a smooth monotone decreasing function with

$$\psi(x) = \tfrac{1}{2} \quad \text{for } x \in \left[0, \tfrac{5}{6}\right], \qquad \psi(x) = 0 \quad \text{for } x \in [1, \infty).$$

Let $\rho : [0, \infty) \to \mathbb{R}$ be a smooth monotone increasing function with

$$\rho(x) = 0 \quad \text{for } x \in \left[0, \tfrac{1}{2}\right], \qquad \rho(x) = 3^{\frac{1}{2}} \quad \text{for } x \in \left[\tfrac{4}{6}, \infty\right).$$

There exists a smooth even function ϕ_0 on $\mathbb{R}$ such that its Fourier transform is nonnegative and satisfies on $[0, \infty)$

$$(\widehat{\phi_0})^2 = (4 - \rho^2)\psi^2,$$

because the right-hand side equals ψ^2 on $\left[\tfrac{4}{6}, \infty\right)$ and is bounded below by $\tfrac{1}{4}$ on $\left[0, \tfrac{5}{6}\right]$ and constant 1 on $\left[0, \tfrac{1}{2}\right)$. There exists a smooth even function ϕ_1 on $\mathbb{R}$ such that its Fourier transform is nonnegative and fulfills on the interval $[0, \infty)$

$$(1 - \widehat{\phi_1})^2 = 1 - (4 - \rho^2)\psi^2,$$

because the right-hand side equals $\tfrac{1}{4}\rho^2$ on $\left[0, \tfrac{5}{6}\right]$ and is bounded below by $\tfrac{3}{4}$ on $\left[\tfrac{4}{6}, \infty\right)$ and constant on $\left[0, \tfrac{1}{2}\right]$. The pair (ϕ_0, ϕ_1) then satisfies the assumptions for a c-pair with

$$c = \max(\|\widehat{\phi}_0^{(N+30)}\|_\infty, \|\widehat{\phi}_1^{(N+30)}\|_\infty). \qquad \square$$

We write $A \lesssim B$ if there exists a constant $C > 0$ such that $|A| \le C|B|$ uniformly over all values of parameters appearing in the expressions A and B.

3. Proof of Theorem 1.2 from Proposition 2.1 and Corollaries 2.2 and 2.4

This section follows the corresponding argument in [Durcik et al. 2019a] for two commuting transformations with minor modifications. We summarize and streamline the argument.

Let J be given, without loss of generality we may assume $J > 2$. Let also positive real numbers $t_0 < t_1 < \cdots < t_J$ be given. Let f_0, f_1, f_2 be real-valued measurable functions on $\mathbb{R}^3$, normalized as

$$\|f_0\|_4 = \|f_1\|_8 = \|f_2\|_8 = 1.$$

We will prove a weak-type endpoint estimate at $r = 4$, namely for any $f_0 \in L^4(\mathbb{R}^3)$ and $f_1, f_2 \in L^8(\mathbb{R}^3)$ with respective norm 1,

$$\sum_{j=1}^{J} \|M_{t_j}(f_0, f_1, f_2) - M_{t_{j-1}}(f_0, f_1, f_2)\|_2^2 \lesssim J^{\frac{1}{2}}. \tag{3-1}$$

We call (3-1) an endpoint estimate as it would follow from the hypothetical inequality (1-1) with $r = 4$ by the Cauchy–Schwarz inequality, and conversely (3-1) implies (1-1) for parameters $r > 4$. Namely, (3-1) allows by Chebyshev's inequality to estimate the number of λ-jumps of the norm by $O(\lambda^{-4})$, which then allows to deduce (1-1) by a layer cake representation of the r-variation. Theorem 1.2 will thus follow as soon as we prove (3-1).

We decompose the characteristic function $\mathbb{1}_{[0,1)}$ into smoother functions. Let χ be a left window and define

$$\theta := \chi - \chi_{(2)}.$$

Then $\hat{\theta}$ is supported in $[-1, -2^{-2}] \cup [2^{-2}, 1]$ and, as detailed in [Durcik et al. 2019a, Section 2.4],

$$\mathbb{1}_{[0,1)} = \mathbb{1}_{[0,1)} * \chi + \sum_{k=-\infty}^{-1} \mathbb{1}_{[0,\infty)} * \theta_{(2^k)} - \sum_{k=-\infty}^{-1} \mathbb{1}_{[1,\infty)} * \theta_{(2^k)} =: \varphi + \sum_{k=-\infty}^{-1} \varphi_{0,k} + \sum_{k=-\infty}^{-1} \varphi_{1,k}. \tag{3-2}$$

For $\vartheta \in L^1(\mathbb{R})$ we define in analogy with (1-2) for $x \in \mathbb{R}^3$

$$M_t^{\vartheta}(f_0, f_1, f_2)(x) := \int_{\mathbb{R}} f_0(x + ue_0) f_1(x + ue_1) f_2(x + ue_2) \vartheta_{(t)}(u)\, du.$$

Using (3-2) and the triangle inequality on the sum in k, it suffices to show in place of (1-1) for every $k \le -1$,

$$\sum_{j=1}^{J} \|M_{t_j}^{\varphi}(f_0, f_1, f_2) - M_{t_{j-1}}^{\varphi}(f_0, f_1, f_2)\|_2^2 \lesssim J^{\frac{1}{2}}, \tag{3-3}$$

$$\sum_{j=1}^{J} \|M_{t_j}^{\varphi_{0,k}}(f_0, f_1, f_2) - M_{t_{j-1}}^{\varphi_{0,k}}(f_0, f_1, f_2)\|_2^2 \lesssim 2^{2k} J^{\frac{1}{2}}, \tag{3-4}$$

$$\sum_{j=1}^{J} \|M_{t_j}^{\varphi_{1,k}}(f_0, f_1, f_2) - M_{t_{j-1}}^{\varphi_{1,k}}(f_0, f_1, f_2)\|_2^2 \lesssim 2^{\gamma k} J^{\frac{1}{2}}, \tag{3-5}$$

where $\gamma = \frac{1}{2}$. In fact, it will follow from our argument that inequality (3-5) continues to hold with any $\gamma < 1$, at the expense of allowing the constant in that inequality to depend on γ. The estimate (3-3) is acceptable and the estimates (3-4) and (3-5) give a geometric series over $k \le -1$ and are thus acceptable as well.

We first prove (3-3). We reduce further (3-3) to the analogous estimate but with the bump function φ replaced by one whose Fourier transform is constant near the origin. We write

$$\varphi = \chi + (\varphi - \chi) = \chi + \sum_{l=-2}^{\infty} (\varphi - \chi) * \theta_{(2^l)} =: \chi + \sum_{l=-2}^{\infty} \varphi_{2,l}.$$

It then suffices to show

$$\sum_{j=1}^{J} \| M_{t_j}^{\chi}(f_0, f_1, f_2) - M_{t_{j-1}}^{\chi}(f_0, f_1, f_2) \|_2^2 \lesssim J^{\frac{1}{2}}, \tag{3-6}$$

$$\sum_{j=1}^{J} \| M_{t_j}^{\varphi_{2,l}}(f_0, f_1, f_2) - M_{t_{j-1}}^{\varphi_{2,l}}(f_0, f_1, f_2) \|_2^2 \lesssim 2^{-2l} J^{\frac{1}{2}}. \tag{3-7}$$

We first prove (3-6). We split into long and short variation as in [Jones et al. 2008]. Enlarging the sequence t_j if necessary while at most doubling the number of terms and retaining at least a quarter of the left-hand side of (3-6), we may assume that for each t_j there is a t_i which is an integer power of 2 with $t_i \le t_j < 2t_i$. Let $(k_i)_{i=0}^{I}$ be the increasing sequence of all k_i such that the power 2^{k_i} occurs in the sequence $(t_j)_{j=1}^{J}$. We have $I \le J$. It then suffices to show the short and long variation bounds,

$$\sum_{i=0}^{I} \sum_{j:\, 2^{k_i} < t_j \le 2^{k_i+1}} \| M_{t_j}^{\chi}(f_0, f_1, f_2) - M_{t_{j-1}}^{\chi}(f_0, f_1, f_2) \|_2^2 \lesssim J^{\frac{1}{2}}, \tag{3-8}$$

$$\sum_{i=1}^{I} \| M_{2^{k_i}}^{\chi}(f_0, f_1, f_2) - M_{2^{k_{i-1}}}^{\chi}(f_0, f_1, f_2) \|_2^2 \lesssim J^{\frac{1}{2}}. \tag{3-9}$$

We first discuss the short variation (3-8). We define $T\chi(s) := (s\chi(s))'$, so that

$$(T\chi)_{(t)}(s) = -t\partial_t(\chi_{(t)}(s)),$$

and we will use T throughout the section. By the fundamental theorem of calculus and the Cauchy–Schwarz inequality, we have for $x \in \mathbb{R}^3$ and every $1 \le i \le I$,

$$\sum_{j:\, 2^{k_i} < t_j \le 2^{k_i+1}} \left| M_{t_j}^{\chi}(f_0, f_1, f_2)(x) - M_{t_{j-1}}^{\chi}(f_0, f_1, f_2)(x) \right|^2 \le \int_1^2 \left(M_{2^{k_i}t}^{T\chi}(f_0, f_1, f_2)(x) \right)^2 \frac{dt}{t}.$$

It then suffices to show

$$\sum_{i=0}^{I} \int_{\mathbb{R}^3} \int_1^2 \left(M_{2^{k_i}t}^{T\chi}(f_0, f_1, f_2)(x) \right)^2 \frac{dt}{t}\, dx \lesssim J^{\frac{1}{2}}.$$

Expanding the square and moving the integral in t outside, the left-hand side becomes

$$\int_1^2 \sum_{i=0}^{I} \int_{\mathbb{R}^5} \left[\prod_{n=0}^{2} f_n(x + ue_n) f_n(x + ve_n) \right] (T\chi)_{(2^{k_i}t)}(u) (T\chi)_{(2^{k_i}t)}(v)\, dx\, du\, dv\, \frac{dt}{t}. \tag{3-10}$$

The expression (3-10) takes the form

$$\int_1^2 \Lambda_{D_1,K_t}((f_s)_{s\in S})\,\frac{dt}{t}, \tag{3-11}$$

where for $s=(k,j)\in\{0,1,2\}\times\{0,1\}$ and $y=(y_0,y_1,y_2)$ we have set

$$f_s(y)=f_k\big(y-(y_0+y_1+y_2-y_k)e_k\big) \tag{3-12}$$

and

$$K_t(u,v):=\sum_{i=0}^{I}(T\chi)_{(2^{k_i}t)}(u)(T\chi)_{(2^{k_i}t)}(v). \tag{3-13}$$

Indeed, writing $x=(x_0,x_1,x_2)$ and changing variables

$$u=x_3^0-x_0-x_1-x_2,\quad v=x_3^1-x_0-x_1-x_2, \tag{3-14}$$

we obtain with the projections Π_s of the datum D_1

$$f_{(k,0)}(\Pi_{(k,0)}(x,x_3^0,x_3^1))=f_k(x+ue_k),\quad f_{(k,1)}(\Pi_{(k,1)}(x,x_3^0,x_3^1))=f_k(x+ve_k).$$

It suffices to prove bounds uniformly for fixed $t\in[1,2]$ on the integrand of (3-11). For this we apply Corollary 2.2 with the sequence $(k_i)_{i=0}^I$ and ϕ_i suitable multiples of $(T\chi)_{(2^{k_i}t)}$ and use

$$\operatorname{supp}(\widehat{T}\chi)\subset[-1,-2^{-1}]\cup[2^{-1},1],\quad |T\chi(u)|\lesssim(1+|u|)^{-20}. \tag{3-15}$$

This proves (3-8).

Next, we prove the long variation bound (3-9). Recalling the universal pair (χ,ϕ), by the triangle inequality, it suffices to show

$$\sum_{i=1}^{I}\|M^{\chi}_{2^{k_{i-1}}}(f_0,f_1,f_2)-M^{\phi}_{2^{k_i}}(f_0,f_1,f_2)\|_2^2\lesssim J^{\frac12}, \tag{3-16}$$

$$\sum_{i=1}^{I}\|M^{\chi}_{2^{k_i}}(f_0,f_1,f_2)-M^{\phi}_{2^{k_i}}(f_0,f_1,f_2)\|_2^2\lesssim J^{\frac12}. \tag{3-17}$$

We first prove (3-16). We expand out the square of the L^2 norm to reduce matters to estimating

$$\sum_{i=1}^{I}\int_{\mathbb{R}^5}\Big[\prod_{n=0}^{2}f_n(x+ue_n)f_n(x+ve_n)\Big](\chi_{(2^{k_{i-1}})}-\phi_{(2^{k_i})})(u)(\chi_{(2^{k_{i-1}})}-\phi_{(2^{k_i})})(v)\,dx\,du\,dv. \tag{3-18}$$

Performing the same change of variables in the Brascamp–Lieb datum as in (3-10), we rewrite it as

$$\Lambda_{D_1,K}((f_s)_{s\in S}), \tag{3-19}$$

with

$$K(u,v)=\sum_{i=1}^{I}(\chi_{(2^{k_{i-1}})}-\phi_{(2^{k_i})})(u)(\chi_{(2^{k_{i-1}})}-\phi_{(2^{k_i})})(v).$$

We estimate this with Corollary 2.4 of Propositions 2.1 and 2.3, using that $\widehat{\chi}$ is a left window and $\widehat{\phi}$ is a right window, and after splitting the sum into even and odd indices j to assure spacing of the sequences k_j and l_j. This completes the discussion of (3-16). Similarly, estimating (3-17) reduces to estimating a form (3-19) with kernel

$$K(u,v)=\sum_{i=1}^{I}(\chi_{(2^{k_i})}-\phi_{(2^{k_i})})(u)(\chi_{(2^{k_i})}-\phi_{(2^{k_i})})(v).$$

This is done with Corollary 2.2. This completes the discussion of (3-17) and thus the discussion of (3-6).

Next, we consider the decaying lacunary pieces near the origin (3-7). We define

$$\varphi_{3,l}(x):=2^l(\varphi_{2,l})_{(2^{-l})}(x)$$

and we replace t_j by $2^l t_j$, using that the sequence t_j was arbitrary, to turn (3-7) into

$$\sum_{j=1}^{J}\|M_{t_j}^{\varphi_{3,l}}(f_0,f_1,f_2)-M_{t_{j-1}}^{\varphi_{3,l}}(f_0,f_1,f_2)\|_2^2\lesssim J^{\frac{1}{2}}. \tag{3-20}$$

Analogously to our discussion of (3-6), we pass to short and long variation. The short variation we estimate analogously using in place of (3-15)

$$\operatorname{supp}(\widehat{T}\varphi_{3,l})\subset[-1,-2^{-1}]\cup[2^{-1},1],\quad |T\varphi_{3,l}(u)|\lesssim(1+|u|)^{-20}, \tag{3-21}$$

which follows because $\widehat{\varphi}-\widehat{\chi}$ vanishes at the origin. This completes the estimate for the short variation.

The long variation we expand similarly as (3-18) above into

$$\Lambda(f_0,f_1,f_2):=\sum_{i=1}^{I}\int_{\mathbb{R}^5}\biggl[\prod_{n=0}^{2}f_n(x+ue_n)f_n(x+ve_n)\biggr] \\ \times\bigl((\varphi_{3,l})_{(2^{k_{i-1}})}-(\varphi_{3,l})_{(2^{k_i})}\bigr)(u)\bigl((\varphi_{3,l})_{(2^{k_{i-1}})}-(\varphi_{3,l})_{(2^{k_i})}\bigr)(v)\,dx\,du\,dv. \tag{3-22}$$

By the distributive law, (3-22) is the difference of the two terms of the form

$$\sum_{i=1}^{I}\int_{\mathbb{R}^5}\biggl[\prod_{n=0}^{2}f_n(x+ue_n)f_n(x+ve_n)\biggr](\varphi_{3,l})_{(2^{m_i})}(u)\bigl((\varphi_{3,l})_{(2^{k_{i-1}})}-(\varphi_{3,l})_{(2^{k_i})}\bigr)(v)\,dx\,du\,dv, \tag{3-23}$$

with $m_i=k_i$ and with $m_i=k_{i-1}$, respectively. We write (3-23) as

$$\sum_{i=1}^{I}\int_{\mathbb{R}^3}\biggl[\int_{\mathbb{R}}\prod_{n=0}^{2}f_n(x+ue_n)(\varphi_{3,l})_{(2^{m_i})}(u)\,du\biggr]\biggl[\int_{\mathbb{R}}\prod_{n=0}^{2}f_n(x+ve_n)\bigl((\varphi_{3,l})_{(2^{k_{i-1}})}-(\varphi_{3,l})_{(2^{k_i})}\bigr)(v)\,dv\biggr]dx$$

and apply the Cauchy–Schwarz inequality in x and in the summation. This gives

$$\Lambda(f_0,f_1,f_2)\le\tilde{\Lambda}(f_0,f_1,f_2)^{\frac{1}{2}}\Lambda(f_0,f_1,f_2)^{\frac{1}{2}}$$

with

$$\tilde{\Lambda}(f_0,f_1,f_2)=\biggl[\sum_{i=1}^{I}\int_{\mathbb{R}^5}\biggl[\prod_{n=0}^{2}f_n(x+ue_n)f_n(x+ve_n)\biggr](\varphi_{3,l})_{(2^{m_i})}(u)(\varphi_{3,l})_{(2^{m_i})}(v)\,dx\,du\,dv\biggr]^{\frac{1}{2}}. \tag{3-24}$$

By bootstrapping, it suffices to prove a bound on $\tilde{\Lambda}(f_0,f_1,f_2)$ in place of $\Lambda(f_0,f_1,f_2)$. This should be compared with the integrand in (3-10) for fixed t. By the same change of variables as there, (3-24) equals $\Lambda_{D_1,K}((f_s)_{s\in S})$ with

$$K(u,v)=\sum_{i=1}^{I}(\varphi_{3,l})_{(2^{m_i})}(u)(\varphi_{3,l})_{(2^{m_i})}(v). \tag{3-25}$$

Applying Corollary 2.2 of Proposition 2.1 yields a bound for this term and finishes the proof of (3-7). The assumptions of Corollary 2.2 are satisfied, which can be verified similarly as inequalities (3-21) observed earlier. This completes the proof of the estimate (3-3).

Now we prove (3-4). We write

$$\varphi_{0,k} = \mathbb{1}_{(-\infty,0)} * \theta_{(2^k)} = 2^k \tilde{\theta}_{(2^k)},$$

where $\tilde{\theta} := \mathbb{1}_{(-\infty,0)} * \theta$ is the primitive of θ. It has high-order decay since θ has integral zero. By rescaling, it suffices to show

$$\sum_{j=1}^{J} \| M_{t_j}^{\tilde{\theta}}(f_0, f_1, f_2) - M_{t_{j-1}}^{\tilde{\theta}}(f_0, f_1, f_2)\|_2^2 \lesssim J^{\frac{1}{2}}.$$

This now follows in the same way as (3-20), using

$$\operatorname{supp}(\hat{\tilde{\theta}}) \subset [-1, -2^{-2}] \cup [2^{-2}, 1]$$

and high-order decay of $\tilde{\theta}$. This completes the proof of (3-4).

It remains to prove (3-5). Define

$$\varphi_{4,k}(u) := 2^k \tilde{\theta}(u - 2^{-k}).$$

We have

$$\varphi_{1,k}(u) = (2^k \tilde{\theta}(u - 2^{-k}))_{(2^k)} = (\varphi_{4,k})_{(2^k)}(u).$$

By rescaling, it suffices to show

$$2^{-\gamma k} \sum_{j=1}^{J} \| M_{t_j}^{\varphi_{4,k}}(f_0, f_1, f_2) - M_{t_{j-1}}^{\varphi_{4,k}}(f_0, f_1, f_2)\|_2^2 \lesssim J^{\frac{1}{2}}.$$

We split into long and short variation as in (3-20). To estimate the short variation, we use the fundamental theorem of calculus and the Cauchy–Schwarz inequality, which yields the bound

$$\begin{aligned} &\sum_{i=0}^{I} \sum_{j:\, 2^{k_i} < t_j \le 2^{k_i+1}} 2^{-\gamma k} \| M_{t_j}^{\varphi_{4,k}}(f_0, f_1, f_2) - M_{t_{j-1}}^{\varphi_{4,k}}(f_0, f_1, f_2)\|_2^2 \\ &\lesssim \Bigg[\Bigg[\sum_{i=0}^{I} 2^{-(\gamma+1)k} \int_{\mathbb{R}^3} \int_1^2 \big(M_{2^{k_i} t}^{\varphi_{4,k}}(f_0, f_1, f_2)(x) \big)^2 \frac{dt}{t}\, dx \Bigg] \\ &\qquad \times \Bigg[\sum_{i=0}^{I} 2^{(1-\gamma)k} \int_{\mathbb{R}^3} \int_1^2 \big(M_{2^{k_i} t}^{T\varphi_{4,k}}(f_0, f_1, f_2)(x) \big)^2 \frac{dt}{t}\, dx \Bigg] \Bigg]^{\frac{1}{2}}. \end{aligned} \tag{3-26}$$

We are going to estimate each factor in the square brackets as $\lesssim J^{\frac{1}{2}}$. We begin with the first factor, that we expand as

$$\sum_{i=0}^{I} \int_{\mathbb{R}^5} \Bigg[\prod_{n=0}^{2} f_n(x + u e_n) f_n(x + v e_n) \Bigg] \Bigg[2^{-(\gamma+1)k} \int_1^2 (\varphi_{4,k})_{(2^{k_i} t)}(u) (\varphi_{4,k})_{(2^{k_i} t)}(v) \frac{dt}{t} \Bigg] dx\, du\, dv.$$

Similarly as (3-11) and (3-13), this takes the form

$$\Lambda_{D_1,K}((f_s)_{s\in S})$$

with

$$K(u,v)=\sum_{i=0}^{I}\left[\int_1^2 2^{-(\gamma+1)k}(\varphi_{4,k})_{(t)}(u)(\varphi_{4,k})_{(t)}(v)\,\frac{dt}{t}\right]_{(2^{k_i})}=:\sum_{i=0}^{I}\Phi_{(2^{k_i})}(u,v).$$

We apply Proposition 2.1 with $\lambda=2-\gamma>1$, using that Φ is symmetric and positive as a superposition of positive terms, and using

$$\operatorname{supp}(\widehat{\Phi})\subseteq([-1,-2^{-3}]\cup[2^{-3},1])^2$$

and the bound

$$\begin{aligned}|\Phi(u,v)| &\le 2^{-(\gamma+1)k}\int_1^2 |\varphi_{4,k}(t^{-1}u)\varphi_{4,k}(t^{-1}v)|\,dt\\ &\le 2^{(1-\gamma)k}\int_1^2 |\tilde{\theta}(t^{-1}(u-t2^{-k}))\tilde{\theta}(t^{-1}(v-t2^{-k}))|\,dt\\ &\lesssim 2^{(1-\gamma)k}\int_1^2 (1+|u+v-t2^{1-k}|)^{-10}(1+|u-v|)^{-10}\,dt\\ &\lesssim 2^{(2-\gamma)k}\int_{2^{1-k}}^{2^{2-k}} (1+|u+v-t|)^{-10}(1+|u-v|)^{-10}\,dt\\ &\lesssim 2^{(2-\gamma)k}(1+2^k|u+v|)^{-10}(1+|u-v|)^{-10}.\end{aligned}\tag{3-27}$$

Here we estimated the integral for $|u+v|<2^{3-k}$ by the integral over $\mathbb{R}$ and for $|u+v|>2^{3-k}$ we estimated the integrand by its supremum norm. We used along the way decay estimates of $\tilde{\theta}$ it inherits from the window χ.

We turn to the second factor in (3-26). We proceed as above; in place of (3-27) we compute

$$\begin{aligned}2^{(1-\gamma)k}&\left|\int_1^2 t\partial_t((\varphi_{4,k})_{(t)}(u))t\partial_t((\varphi_{4,k})_{(t)}(v))\,\frac{dt}{t}\right|\\ &=2^{(3-\gamma)k}\left|\int_1^2 t\partial_t\big(t^{-1}\tilde{\theta}(t^{-1}(u-t2^{-k}))\big)t\partial_t\big(t^{-1}\tilde{\theta}(t^{-1}(v-t2^{-k}))\big)\,\frac{dt}{t}\right|.\end{aligned}$$

Applying Leibniz and chain rules, most terms will be analogous to the above. However, when a derivative falls on $t2^{-k}$, we obtain a factor 2^{-k}. The worst term is the one where both derivatives fall on the $t2^{-k}$. Thus we get the estimate

$$\lesssim 2^{(1-\gamma)k}\int_1^2 (1+|u+v-t2^{1-k}|)^{-10}(1+|u-v|)^{-10}\,\frac{dt}{t}.$$

As above, this is estimated by

$$\lesssim 2^{(2-\gamma)k}(1+2^k|u+v|)^{-10}(1+|u-v|)^{-10}.$$

To treat the long variation, we proceed as for (3-22), where after a bootstrapping estimate we are led to estimate, analogously to (3-25), $\Lambda_{D_1,K}((f_s)_{s\in S})$ with

$$K(u,v) = 2^{-\gamma k}\sum_{i=1}^{I}(\varphi_{4,k})_{(2^{m_i})}(u)(\varphi_{4,k})_{(2^{m_i})}(v).$$

Similarly as in (3-27) we estimate

$$\begin{aligned}2^{-\gamma k}|\varphi_{4,k}(u)\varphi_{4,k}(v)| &= 2^{(2-\gamma)k}|\tilde\theta(u-2^{-k})\tilde\theta(v-2^{-k})|\\ &\lesssim 2^{(2-\gamma)k}(1+|u+v-2^{1-k}|)^{-10}(1+|u-v|)^{-10}\\ &\lesssim 2^{(2-\gamma)k}(1+2^k|u+v|)^{-10}(1+|u-v|)^{-10}.\end{aligned}$$

Applying Proposition 2.1 again completes the proof of (3-5).

4. Proof of Proposition 2.1 using Propositions 2.3 and 2.5

Let $\lambda = \frac{3}{2}$. Let $k \le 0$, let J be a positive integer and $(k_j)_{j=1}^J$ a strictly increasing sequence of integers. By splitting into a hundred subsequences, using the triangle inequality to separate these sequences, we may assume $k_j + 100 \le k_{j+1}$ for $1 \le j < J$.

Let Φ_j for $1 \le j \le J$ be as in Proposition 2.1. In particular, $\widehat{\Phi}_j(\xi,-\xi)$ is continuous and even in ξ by the symmetry assumption on the kernel Φ_j. Furthermore, we claim that $\widehat{\Phi}_j(\xi,-\xi)$ is positive for all $\xi\in\mathbb{R}$. To see this, first apply Plancherel to the positivity assumption (2-5) in Proposition 2.1 to conclude

$$0 \le \int_{\mathbb{R}^2}\hat f(-\xi)\overline{\hat f(\eta)}\widehat{\Phi}_j(\xi,\eta)\,d\xi\,d\eta$$

for all Schwartz functions f. Now we see the claim by using testing functions $\hat f$ which approximate the Dirac delta at ξ.

As $\|\Phi_j\|_1$ has a universal bound, for suitable universal constant c we have

$$\widehat{\Phi}_j(\xi,-\xi) \le c(\widehat{\phi}_{0,j}-\widehat{\phi}_{1,j})(\xi)^2$$

with even real functions $\phi_{0,j}$ and $\phi_{1,j}$, such that $(\phi_{0,j})_{2^{-k_j+25}}$ is a left window and $(\phi_{1,j})_{2^{-k_j-25}}$ is a right window. Moreover, there exists a real even function ψ_j such that

$$\widehat{\psi}_j(\xi)^2 := 2c(\widehat{\phi}_{0,j}-\widehat{\phi}_{1,j})(\xi)^2 - \widehat{\Phi}_j(\xi,-\xi). \tag{4-1}$$

Namely, outside the support of $\xi\mapsto\widehat{\Phi}_j(\xi,-\xi)$, the function $\widehat{\psi}_j$ can be chosen to equal $\sqrt{2c}(\widehat{\phi}_{0,j}-\widehat{\phi}_{1,j})$, while on a neighborhood of this support, the function on the right-hand side is at least c and thus has square root. The function $(\widehat{\psi}_j)_{(2^{-k_j+25})}$ has support in $[-1,1]$. To understand derivative bounds for this function, let $F(\xi) = (\widehat{\Phi}_j)_{(2^{-k_j})}(\xi,-\xi)$. Then we have, for $0\le a\le 8$,

$$|F^{(a)}(\xi)| = \left|(-2\pi i)^a\int_{\mathbb{R}^2}(\Phi_j)_{(2^{-k_j})}(u,v)(u-v)^a e^{-2\pi i\xi(u-v)}\,du\,dv\right| \lesssim 2^{(\lambda-1)k},$$

by (2-6). Thus,

$$|((\widehat{\psi}_j)_{(2^{-k_j})})^{(a)}| \lesssim 1, \tag{4-2}$$

as one can see outside the support of $(\widehat{\Phi}_j)_{(2^{-k_j})}$ from bounds for derivatives of the windows and on the support using a lower bound on the right-hand side of (4-1) and upper bounds on the derivative of the right-hand side of (4-1).

To show a bound on $\Lambda_{D_1,K}((f_s)_{s\in S})$ with $K=\sum_{j=1}^J \Phi_j$, which is positive, it suffices to show a bound on $\Lambda_{D_1,K_0}((f_s)_{s\in S})$ with

$$K_0=\sum_{j=1}^J \Phi_j+\psi_j\otimes\psi_j$$

because the form associated with the datum D_1 and the difference K_0-K is positive as well.

By Proposition 2.3, the form $\Lambda_{D_1,K_1}((f_s)_{s\in S})$ is bounded, where

$$K_1=2c\sum_{j=1}^J(\phi_{0,j}-\phi_{1,j})\otimes(\phi_{0,j}-\phi_{1,j}).$$

Hence it suffices to prove a bound on $\Lambda_{D_1,K_3}((f_s)_{s\in S})$, where $K_3=K_0-K_1$.

This is done by an application of Proposition 2.5. Note that we have on the diagonal

$$\widehat{K}_3(\xi,-\xi)=\sum_{j=1}^J \widehat{\Phi}_j(\xi,-\xi)+\widehat{\psi}_j(\xi)^2-2c(\widehat{\phi}_{0,j}-\widehat{\phi}_{1,j})(\xi)^2=0.$$

We verify the remaining assumptions of Proposition 2.5 for

$$\Psi_j:=\Phi_j+\psi_j\otimes\psi_j-2c(\phi_{0,j}-\phi_{1,j})\otimes(\phi_{0,j}-\phi_{1,j}).$$

We have

$$\begin{aligned}\operatorname{supp}(\widehat{\Phi}_j)&\subseteq\big([-2^{-k_j+20},-2^{-k_j-20}]\cup[2^{-k_j-20},2^{-k_j+20}]\big)^2,\\ \operatorname{supp}\big((\widehat{\phi}_{0,j}-\widehat{\phi}_{1,j})\otimes(\widehat{\phi}_{0,j}-\widehat{\phi}_{1,j})\big)&\subseteq\big([-2^{-k_j+25},-2^{-k_j-26}]\cup[2^{-k_j-26},2^{-k_j+25}]\big)^2,\\ \operatorname{supp}(\widehat{\psi}_j\otimes\widehat{\psi}_j)&\subseteq\big([-2^{-k_j+25},-2^{-k_j-26}]\cup[2^{-k_j-26},2^{-k_j+25}]\big)^2.\end{aligned}$$

Thus,

$$\operatorname{supp}(\widehat{\Psi}_j)\subseteq\big\{(\xi,\eta)\in\mathbb{R}^2:2^{-k_j-30}<|(\xi,\eta)|\le 2^{-k_j+30}\big\}.$$

Note also that, using in particular (4-2),

$$\begin{aligned}|(\Phi_j)_{(2^{-k_j})}(u,v)|&\lesssim 2^{\lambda k}(1+2^k|u+v|)^{-10}(1+|u-v|)^{-10},\\ \big|((\phi_{0,j}-\phi_{1,j})\otimes(\phi_{0,j}-\phi_{1,j}))_{(2^{-k_j})}(u,v)\big|&\lesssim(1+|u+v|)^{-4}(1+|u-v|)^{-4},\\ |(\psi_j\otimes\psi_j)_{(2^{-k_j})}(u,v)|&\lesssim(1+|u+v|)^{-4}(1+|u-v|)^{-4}.\end{aligned}$$

Hence

$$|(\Psi_j)_{(2^{-k_j})}(u,v)|\lesssim 2^{\lambda k}(1+2^k|u+v|)^{-4}(1+|u-v|)^{-4}+(1+|u+v|)^{-4}(1+|u-v|)^{-4}.$$

The final claim now follows from Proposition 2.5.

5. Proof of Proposition 2.3 using Propositions 2.5 and 2.6

Let J be a positive integer and $(k_j)_{j=1}^J$ and $(l_j)_{j=1}^J$ two finite sequences of integers with $k_j + 10 < l_j$ for $1 \le j \le J$ and $l_j < k_{j+1}$ for $1 \le j < J - 1$. By splitting the sequence into subsequences of even and odd j if necessary, we may assume without loss of generality that $l_j + 10 < k_{j+1}$ for each $1 \le j < J$. Assume a tuple $(f_s)_{s\in S}$ as in (2-3) and (2-4) is given.

Assume we are given $\phi_{0,j}$ and $\phi_{1,j}$ for each j such that $(\phi_{0,j})_{(2^{-k_j})}$ is a left window and $(\phi_{1,j})_{(2^{-l_j})}$ is a right window. Pick corresponding functions $\psi_{0,j}$ and $\psi_{1,j}$ so that the rescaled functions give universal pairs, and hence

$$(1-\widehat{\phi}_{1,j})^2 + (\widehat{\psi}_{0,j})^2 = 1, \tag{5-1}$$

$$(\widehat{\phi}_{0,j})^2 + (1-\widehat{\psi}_{1,j})^2 = 1. \tag{5-2}$$

Then

$$(1-\widehat{\psi}_{1,1})^2 + \sum_{j=1}^{J}(\widehat{\phi}_{0,j}-\widehat{\phi}_{1,j})^2 + \sum_{j=1}^{J-1}(\widehat{\psi}_{0,j}-\widehat{\psi}_{1,j+1})^2 + (\widehat{\psi}_{0,J})^2 = 1. \tag{5-3}$$

To see this, note that at every point at most one of the functions $\widehat{\phi}_{0,j}, \widehat{\phi}_{1,j},\ 1 \le j \le J$, is neither 0 nor 1, and the functions $\widehat{\psi}_{0,j}, \widehat{\psi}_{1,j}$ are neither 0 nor 1 precisely when the respective function $\widehat{\phi}_{1,j}, \widehat{\phi}_{0,j}$ is not 0 or 1. Therefore, at any point at most one pair $(\widehat{\psi}_{0,j}, \widehat{\phi}_{1,j})$ or $(\widehat{\psi}_{1,j}, \widehat{\phi}_{0,j})$ takes values other than 0 and 1, and we can apply (5-1) or (5-2), respectively.

As $\Lambda_{D_1,K}((f_s)_{s\in S})$ in Proposition 2.3 is positive, it suffices to estimate its sum with another positive term, and thus it suffices to estimate $\Lambda_{D_1,K_1}((f_s)_{s\in S})$ with

$$\widehat{K}_1(\xi,\eta) = (1-\widehat{\psi}_{1,1})(\xi)(1-\widehat{\psi}_{1,1})(\eta) + \sum_{j=1}^{J}(\widehat{\phi}_{0,j}-\widehat{\phi}_{1,j})(\xi)(\widehat{\phi}_{0,j}-\widehat{\phi}_{1,j})(\eta) + \sum_{j=1}^{J-1}(\widehat{\psi}_{0,j}-\widehat{\psi}_{1,j+1})(\xi)(\widehat{\psi}_{0,j}-\widehat{\psi}_{1,j+1})(\eta) + (\widehat{\psi}_{0,J})(\xi)(\widehat{\psi}_{0,J})(\eta).$$

This can be rewritten in a more compressed form as

$$\widehat{K}_1 = \sum_{j=0}^{2J}(\widehat{\varphi}_{0,j}-\widehat{\varphi}_{1,j}) \otimes (\widehat{\varphi}_{0,j}-\widehat{\varphi}_{1,j}),$$

where $\widehat{\varphi}_{0,0} = 1$, for $1 \le j \le J$,

$$\varphi_{0,2j-1} = \phi_{0,j}, \quad \varphi_{0,2j} = \psi_{0,j}, \quad \varphi_{1,2j-1} = \phi_{1,j}, \quad \varphi_{1,2j-2} = \psi_{1,j},$$

and $\varphi_{1,2J} = 0$. Define for $1 \le j \le J$

$$m_{2j-2} = k_j \quad \text{and} \quad m_{2j-1} = l_j.$$

Observe that for each $1 \le j \le 2J-1$ we have that $(\varphi_{0,j})_{(2^{-m_{j-1}})}$ is a left window and $(\varphi_{1,j})_{(2^{-m_j})}$ is a right window.

In order to apply Proposition 2.6, we introduce for $0 \le j \le 2J$ the functions

$$\varphi_{2,j} = (\varphi_{1,j})_{(2^{-4})}.$$

Observe that $(\varphi_{2,j})_{2^{4-m_j}}$ is a right window whenever $0 \le j \le 2J-1$. We write for $\widehat{K}_1$

$$-\sum_{j=0}^{2J} (\widehat{\varphi}_{0,j} - \widehat{\varphi}_{2,j}) \otimes \widehat{\varphi}_{1,j} + \widehat{\varphi}_{1,j} \otimes (\widehat{\varphi}_{0,j} - \widehat{\varphi}_{2,j}) \tag{5-4}$$

$$-\sum_{j=0}^{2J} (\widehat{\varphi}_{2,j} - \widehat{\varphi}_{1,j}) \otimes \widehat{\varphi}_{1,j} + \widehat{\varphi}_{1,j} \otimes (\widehat{\varphi}_{2,j} - \widehat{\varphi}_{1,j}) \tag{5-5}$$

$$+\sum_{j=0}^{2J} \widehat{\varphi}_{0,j} \otimes \widehat{\varphi}_{0,j} - \widehat{\varphi}_{1,j} \otimes \widehat{\varphi}_{1,j}. \tag{5-6}$$

In (5-4), the bound for the sum of these terms over $1 \le j \le 2J-1$ follows from Proposition 2.6, applied to the sequence $(m_j)_{j=0}^{2J-1}$ and the rescaled windows $\varphi_{0,j}$, $\varphi_{1,j}$, $\varphi_{2,j}$ for $1 \le j \le 2J-1$. The term for $j = 2J$ in (5-4) vanishes. To deal with the term for $j = 0$ in (5-4), we use $\widehat{\varphi}_{0,0} = 1$ and rewrite this term as

$$-\widehat{\varphi}_{1,0}(\eta) + \widehat{\varphi}_{2,0}(\xi)\widehat{\varphi}_{1,0}(\eta) + \widehat{\varphi}_{1,0}(\xi) - \widehat{\varphi}_{1,0}(\xi)\widehat{\varphi}_{2,0}(\eta).$$

Denoting by f_k, $k = 0, 1, 2$, the functions defined via (3-12) and using the change of variables as in (3-14), we estimate

$$\left|\Lambda_{D_1,\varphi_{1,0}\otimes\delta}((f_s)_{s\in S})\right| = \left| \int_{\mathbb{R}^4} \left[\prod_{k=0}^{2} f_k(x + ue_k) f_k(x) \right] \varphi_{1,0}(u)\, dx\, du \right| \le \|\varphi_{1,0}\|_1 \lesssim 1,$$

where for a fixed u we used Hölder's inequality in x and δ denotes the Dirac delta at the origin. Similarly,

$$\begin{aligned} \left|\Lambda_{D_1,\varphi_{1,0}\otimes\varphi_{2,0}}((f_s)_{s\in S})\right| &= \left| \int_{\mathbb{R}^5} \left[\prod_{k=0}^{2} f_k(x + ue_k) f_k(x + ve_k) \right] \varphi_{1,0}(u)\varphi_{2,0}(v)\, dx\, du\, dv \right| \\ &\le \|\varphi_{1,0}\|_1 \|\varphi_{2,0}\|_1 \lesssim 1. \end{aligned}$$

By symmetry, this bounds the form associated with the $j = 0$ summand in (5-4).

It remains to estimate the form associated with K_2 where $\widehat{K}_2$ is the sum of (5-5), (5-6). As $\widehat{K}_1$ is constant 1 on the diagonal $\xi + \eta = 0$ by (5-3) and the stick terms (5-4) vanish on this diagonal, the function $\widehat{K}_2$ is still constant 1 on this diagonal.

We define K_3 by $\widehat{K}_3 := \widehat{K}_2 - 1$. It suffices to prove bounds for the form associated with K_3, because $K_2 - K_3$ is the Dirac delta and

$$\left|\Lambda_{D_1,K_2-K_3}((f_s)_{s\in S})\right| = \left| \int_{\mathbb{R}^3} \prod_{k=0}^{2} f_k^2(x)\, dx \right| \le 1,$$

where the functions f_k are as in (3-12). We rewrite $\widehat{K}_3$ as

$$-\sum_{j=0}^{2J}(\widehat{\varphi}_{2,j}-\widehat{\varphi}_{1,j})\otimes\widehat{\varphi}_{1,j}+\widehat{\varphi}_{1,j}\otimes(\widehat{\varphi}_{2,j}-\widehat{\varphi}_{1,j}) \tag{5-7}$$

$$+\sum_{j=1}^{2J}\widehat{\varphi}_{0,j}\otimes\widehat{\varphi}_{0,j}-\widehat{\varphi}_{1,j-1}\otimes\widehat{\varphi}_{1,j-1}, \tag{5-8}$$

where we have reshuffled (5-6) and used $\widehat{\phi}_{0,0}=1$ and $\widehat{\phi}_{1,2J}=0$. Bounds for the sum of (5-7) and (5-8) follow from Proposition 2.5. Indeed, for each $0\le j\le 2J$,

$$\operatorname{supp}((\widehat{\varphi}_{2,j}-\widehat{\varphi}_{1,j})\otimes\widehat{\varphi}_{1,j})\subseteq\big([-2^{-m_j+4},-2^{-m_j-1}]\cup[2^{-m_j-1},2^{-m_j+4}]\big)\times[-2^{-m_j},2^{-m_j}].$$

By symmetry, the j-th summand in (5-7) is supported in

$$\big\{(\xi,\eta)\in\mathbb{R}^2:2^{-m_j-30}<|(\xi,\eta)|\le 2^{-m_j+30}\big\}=:A.$$

The j-th summand also satisfies a bound by

$$\big|(\varphi_{2,j}-\varphi_{1,j})\otimes\varphi_{1,j}+\varphi_{1,j}\otimes(\varphi_{2,j}-\varphi_{1,j})\big|_{(2^{-m_j})}(u,v)\lesssim(1+|u+v|)^{-4}(1+|u-v|)^{-4}$$

due to the functions being windows.

Similarly, for $0\le j\le 2J-1$ we have

$$\operatorname{supp}\big(\widehat{\varphi}_{0,j+1}\otimes\widehat{\varphi}_{0,j+1}-\widehat{\varphi}_{1,j}\otimes\widehat{\varphi}_{1,j}\big)\subseteq[-2^{-m_j},2^{-m_j}]^2\setminus[-2^{-m_j-1},2^{-m_j-1}]^2\subseteq A$$

and the decay

$$\big|\varphi_{0,j+1}\otimes\varphi_{0,j+1}-\varphi_{1,j}\otimes\varphi_{1,j}\big|_{(2^{-m_j})}(u,v)\lesssim(1+|u+v|)^{-4}(1+|u-v|)^{-4}.$$

Thus, bounds for Λ_{D_1,K_3} follow from Proposition 2.5.

6. Proof of Proposition 2.5 using Lemma 3 in [Durcik and Thiele 2020]

Given a regular 3×3 matrix A, let D_A be the datum defined in (2-20). We recall the following lemma, which is a special instance of a more general result proved in [Durcik and Thiele 2020].

Lemma 6.1 [Durcik and Thiele 2020, Lemma 3]. *For all $0<\varepsilon<1$, there exists a constant C such that the following holds.*

Let A be a regular 3×3 matrix which differs from $-I$ by at most one row and satisfies

$$|\det A|>\varepsilon \quad \textit{and} \quad \|A\|_{\mathrm{HS}}\le\varepsilon^{-1}, \tag{6-1}$$

where $\|A\|_{\mathrm{HS}}$ stands for the Hilbert–Schmidt norm of A. With S as in the datum D_A, let $(f_s)_{s\in S}$ be a tuple of real-valued Schwartz functions such that $\|f_s\|_8=1$ for all $s\in S$. Let $i=1,2,3$, and let K be the kernel satisfying

$$K(\Pi x)=\int_0^\infty\int_{\mathbb{R}^3}(\partial_i\partial_{i+3}g)_{(t)}\big(x+((-Ap^T)^T,p)\big)\,dp\,\frac{dt}{t}. \tag{6-2}$$

Then

$$|\Lambda_{D_A,K}((f_s)_{s\in S})|\le C.$$

Proof of Proposition 2.5. Let $\lambda = \frac{3}{2}$. Let $k \le 0$ be given. Let an integer $J \ge 1$ and a strictly increasing sequence $(k_j)_{j=1}^J$ of integers be given. Let $(\Phi_j)_{j=1}^J$ and K be given as in the proposition. Let $(f_s)_{s\in S}$ be given as in (2-3) and (2-4). Set $f_k := f_{(k,0)} = f_{(k,1)}$ for each $k = 0, 1, 2$.

Let $\theta : \mathbb{R} \to \mathbb{R}$ be a function whose Fourier transform is supported in $\left[-2, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, 2\right]$ and whose derivatives up to order 8 are $\lesssim 1$. Assume further that

$$\int_0^\infty \hat{\theta}(r\xi)\,\frac{dr}{r} = 1$$

for all $\xi \ne 0$. We do the two-parameter lacunary decomposition of $\widehat{K}$ in directions $\xi+\eta$ and $\xi-\eta$ and collect these pieces into lacunary cones away from the line $\xi+\eta=0$ centered at the origin. In detail, we write

$$\widehat{K}(\xi,\eta) = \int_0^\infty \widehat{K}^{(z)}(\xi,\eta)\,\frac{dz}{z} \tag{6-3}$$

with

$$\widehat{K}^{(z)}(\xi,\eta) = \int_0^\infty \widehat{K}(\xi,\eta)\hat{\theta}(t(\xi-\eta))\hat{\theta}(z^{-1}t(\xi+\eta))\,\frac{dt}{t}. \tag{6-4}$$

We break the integral in (6-3) into the integrals over the domains $(0,1)$ and $(1,\infty)$ and do the estimates for these integrals separately. We begin with the case $z \in (0,1)$. Here we do an estimate for each z separately and show for all $z < 1$ that

$$|\Lambda_{D_1,K^{(z)}}((f_s)_{s\in S})| \lesssim z^{(\lambda-1)^2/(2\lambda)} J^{\frac{1}{2}}, \tag{6-5}$$

which is an integrable upper bound with respect to the measure dz/z. Fix $z \in (0,1)$.

Let g be the one-dimensional Gaussian and let $h = g'$. Set $\widehat{\omega} = (\hat{h})^{-1}\hat{\theta}$. The function $\widehat{\omega}$ satisfies similar support and derivative estimates as $\hat{\theta}$ since $\hat{h}$ and its derivatives are essentially constant on the support of $\hat{\theta}$. In addition, let $\widehat{\phi}$ be a function supported in the annulus $\frac{1}{16} \le |(\xi,\eta)| \le 16$ such that its derivatives up to order 8 are $\lesssim 1$ and $\widehat{\phi}(\xi,\eta)\hat{g}(\xi)\hat{g}(\eta) = 1$ if $\frac{1}{8} \le |(\xi,\eta)| \le 8$. Then, for all $\xi,\eta \in \mathbb{R}$,

$$\hat{\theta}(\xi-\eta)\hat{\theta}(z^{-1}(\xi+\eta)) = \hat{\theta}(\xi-\eta)\widehat{\omega}(z^{-1}(\xi+\eta))\hat{h}(z^{-1}(\xi+\eta))\widehat{\phi}(\xi,\eta)\hat{g}(\xi)\hat{g}(\eta). \tag{6-6}$$

Note that this equality holds since the left-hand side is supported in the set where

$$\tfrac{1}{8} \le |(\xi,\eta)| \le 8. \tag{6-7}$$

Indeed, on the support of the left-hand side of (6-6) we have $|\xi+\eta| \le 2z \le 2$ and $\frac{1}{2} \le |\xi-\eta| \le 2$. This yields (6-7).

For $z \in (0,1)$ and $t > 0$ we define the function $w^{z,t}$ via

$$\widehat{w}^{z,t}(\xi,\eta) = \widehat{K}(t^{-1}(\xi,\eta))\widehat{\phi}(\xi,\eta)\hat{\theta}(\xi-\eta)\widehat{\omega}(z^{-1}(\xi+\eta)). \tag{6-8}$$

Let Π be the projection associated with the datum D_1. Using the Fourier inversion formula and equations (6-4), (6-6) and (6-8), we write $K^{(z)}(\Pi x)$ as

$$\int_0^\infty\int_{\mathbb{R}^2} \widehat{w}^{z,t}(t(\xi,\eta))\hat{h}(z^{-1}t(\xi+\eta))\hat{g}(t\xi)\hat{g}(t\eta)\exp\bigl(2\pi i\bigl(\xi(x_3^0-x_0-x_1-x_2)+\eta(x_3^1-x_0-x_1-x_2)\bigr)\bigr)\,d\xi\,d\eta\,\frac{dt}{t}. \tag{6-9}$$

Since the Fourier transform of $w^{z,t}$ is supported in the set where $\frac{1}{8} \le |(\xi, \eta)| \le 8$, we observe that $w^{z,t}$ vanishes unless t is in the set

$$M := \bigcup_{j=1}^{J} [2^{k_j-33}, 2^{k_j+33}].$$

We may thus restrict the region of t-integration in (6-9) to M. Further, we may interpret the inner integral in (6-9) as the integral of the Fourier transform of the function

$$(y_0, y_1, y_2, y_3, y_4) \mapsto w^{z,t}_{(t)}(y_0 + x_0 + x_1, y_1 + x_0 + x_1) h_{(z^{-1}t)}(y_2 + x_2) g_{(t)}(y_3 + x_3^0) g_{(t)}(y_4 + x_3^1)$$

over the hyperplane

$$\{(-\xi, -\eta, -\xi - \eta, \xi, \eta) : \xi \in \mathbb{R},\ \eta \in \mathbb{R}\}.$$

It is therefore up to universal multiplicative constant equal to the integral of the function itself over the orthogonal complement

$$\{(p + q - r, q - r, r, p + q, q) : p, q, r \in \mathbb{R}\}.$$

The form $\Lambda_{D_1, K^{(z)}}((f_s)_{s \in S})$ can then be rewritten as

$$\int_M \int_{\mathbb{R}^8} \Big[\prod_{s \in S} f_s(\Pi_s x)\Big] w^{z,t}_{(t)}(x_0 + x_1 + p + q - r, x_0 + x_1 + q - r) \\ \times h_{(z^{-1}t)}(x_2 + r) g_{(t)}(x_3^0 + p + q) g_{(t)}(x_3^1 + q)\, dx\, dp\, dq\, dr\, \frac{dt}{t}. \tag{6-10}$$

We write the integral in x_2 as the innermost and use the Cauchy–Schwarz inequality in the remaining variables. This bounds (6-10) by the geometric mean of

$$\int_M \int_{\mathbb{R}^7} \Big[\prod_{i=0,1} |f_2(x_0, x_1, x_3^i)|^2\Big] \big|w^{z,t}_{(t)}(x_0 + x_1 + p + q - r, x_0 + x_1 + q - r)\big| \\ \times g_{(t)}(x_3^0 + p + q) g_{(t)}(x_3^1 + q)\, dx_0\, dx_1\, dx_3^0\, dx_3^1\, dp\, dq\, dr\, \frac{dt}{t} \tag{6-11}$$

and

$$\int_0^\infty \int_{\mathbb{R}^7} \Big[\int_{\mathbb{R}} \Big[\prod_{i=0,1} f_0(x_3^i, x_1, x_2) f_1(x_0, x_3^i, x_2)\Big] h_{(z^{-1}t)}(x_2 + r)\, dx_2\Big]^2 \\ \times \big|w^{z,t}_{(t)}(x_0 + x_1 + p + q - r, x_0 + x_1 + q - r)\big| \\ \times g_{(t)}(x_3^0 + p + q) g_{(t)}(x_3^1 + q)\, dx_0\, dx_1\, dx_3^0\, dx_3^1\, dp\, dq\, dr\, \frac{dt}{t}. \tag{6-12}$$

In order to bound (6-11) and (6-12), we prove a pointwise estimate for $w^{z,t}$. We first claim

$$|w^{z,t}(u, v)| \lesssim z^\lambda. \tag{6-13}$$

To verify the claim, we observe that since $\widehat{K}$ vanishes on the diagonal $\xi+\eta=0$, the function $\widehat{K}_{(t^{-1})}*\phi$ has the same property. Therefore

$$\begin{aligned}|\widehat{K}_{(t^{-1})}*\phi(\xi,\eta)| &= \big|\widehat{K}_{(t^{-1})}*\phi(\xi,\eta)-\widehat{K}_{(t^{-1})}*\phi\big((\xi-\eta)/2,-(\xi-\eta)/2\big)\big| \\ &= \left|\int_{\mathbb{R}^2} K_{(t^{-1})}*\phi(u,v)e^{-\pi i(\xi-\eta)(u-v)}(e^{-\pi i(\xi+\eta)(u+v)}-1)\,du\,dv\right| \\ &\lesssim \int_{\mathbb{R}^2}\big|K_{(t^{-1})}*\phi(u,v)\big|\min\{|\xi+\eta|\,|u+v|,1\}\,du\,dv \\ &\le |\xi+\eta|^{\lambda-1}\int_{\mathbb{R}^2}\big|K_{(t^{-1})}*\phi(u,v)\big||u+v|^{\lambda-1}\,du\,dv, \end{aligned} \tag{6-14}$$

as $\lambda-1\in(0,1)$. We observe that

$$|K_{(t^{-1})}*\phi(u,v)|\lesssim 2^{\lambda k}(1+2^k|u+v|)^{-4}(1+|u-v|)^{-4}+(1+|u+v|)^{-4}(1+|u-v|)^{-4}, \tag{6-15}$$

thanks to the derivative estimates on ϕ, to the support properties of $\widehat{\phi}$ and $\widehat{\Phi}_j$ and to (2-8). Therefore,

$$\begin{aligned}&\int_{\mathbb{R}^2}\big|K_{(t^{-1})}*\phi(u,v)\big||u+v|^{\lambda-1}\,du\,dv \\ &\qquad\lesssim 2^k\int_{\mathbb{R}^2}(1+2^k|u+v|)^{\lambda-5}(1+|u-v|)^{-4}\,du\,dv+\int_{\mathbb{R}^2}(1+|u+v|)^{\lambda-5}(1+|u-v|)^{-4}\,du\,dv\lesssim 1.\end{aligned}$$

Combining this with (6-14) and passing to $w^{z,t}$, we thus obtain

$$|\widehat{w}^{z,t}(\xi,\eta)|\lesssim z^{\lambda-1}.$$

Estimating the Fourier inversion formula by $L^1\to L^\infty$ bounds, inequality (6-13) follows.

We note that the right-hand side of (6-13) has the desired decay as z tends to 0, however, it does not have a good behavior with respect to (u,v). We therefore derive a yet another estimate for $w^{z,t}$ in which the right-hand side possesses merely L^1 scaling in z but decays sufficiently fast as $|(u,v)|$ tends to infinity. We set

$$F(u,v)=\omega_{(z^{-1})}((u+v)/2)\theta((u-v)/2).$$

By (6-8), we have $w^{z,t}=K_{(t^{-1})}*\phi*F$. Recall that the functions $\widehat{\omega}$ and $\hat{\theta}$ are supported in $[-2,2]$ and have derivatives up to order 8 bounded by $\lesssim 1$. Using (6-15), we therefore obtain

$$|w^{z,t}(u,v)|\lesssim 2^{\lambda k}(1+2^k|u+v|)^{-4}(1+|u-v|)^{-4}+z(1+z|u+v|)^{-4}(1+|u-v|)^{-4}\quad\text{if } 2^k\le z \tag{6-16}$$

and

$$|w^{z,t}(u,v)|\lesssim z(1+z|u+v|)^{-4}(1+|u-v|)^{-4}\quad\text{if } z\le 2^k. \tag{6-17}$$

Finally, we write $|w^{z,t}|=|w^{z,t}|^{(\lambda-1)/(2\lambda)}|w^{z,t}|^{(\lambda+1)/(2\lambda)}$ and use the estimate (6-13) for the first factor and the estimates (6-16) and (6-17) for the second factor. This yields the desired bounds

$$|w^{z,t}(u,v)|\lesssim z^{\frac{(\lambda-1)^2}{2\lambda}}[z(1+z|u+v|)^{-2-\frac{2}{\lambda}}+2^k(1+2^k|u+v|)^{-2-\frac{2}{\lambda}}](1+|u-v|)^{-2-\frac{2}{\lambda}} \tag{6-18}$$

if $2^k \le z$, and

$$|w^{z,t}(u,v)| \lesssim z^{\frac{(\lambda-1)^2}{2\lambda}} z(1+z|u+v|)^{-2-\frac{2}{\lambda}}(1+|u-v|)^{-2-\frac{2}{\lambda}} \quad \text{if } z \le 2^k. \tag{6-19}$$

Having inequalities (6-18) and (6-19) at our disposal, we proceed to bound the term (6-11). We observe that this term can be written as

$$\int_M \int_{\mathbb{R}^5} \Big[\prod_{i=0,1} |f_2(x_0,x_1,x_3^i)|^2 \Big] \times [|w^{z,t}| * (g\otimes g)]_{(t)}\big(x_3^0 - x_0 - x_1 + r, x_3^1 - x_0 - x_1 + r\big)\, dx_0\, dx_1\, dx_3^0\, dx_3^1\, dr\, \frac{dt}{t}. \tag{6-20}$$

Applying the Cauchy–Schwarz inequality, we bound (6-20) with

$$v_{z,t} := [|w^{z,t}| * (g\otimes g)]_{(t)}$$

by

$$\int_M \prod_{i=0,1} \Big[\int_{\mathbb{R}^5} |f_2(x_0,x_1,x_3^i)|^4 v_{z,t}(x_3^0 - x_0 - x_1 + r, x_3^1 - x_0 - x_1 + r)\, dx_0\, dx_1\, dx_3^0\, dx_3^1\, dr \Big]^{\frac{1}{2}} \frac{dt}{t}.$$

The product of the square roots of the integrals for $i = 0, 1$ equals

$$\|f_2\|_4^4 \|v_{z,t}\|_1 \lesssim z^{\frac{(\lambda-1)^2}{2\lambda}}.$$

The last identity can be seen by integrating first in x_3^{1-i} and then in r to obtain the L^1 norm of $v_{z,t}$. What remains is then the L^4 norm of f_2 raised to the fourth power. Using that $\int_M dt/t \lesssim J$, we deduce that (6-11) is bounded by a multiple of

$$z^{\frac{(\lambda-1)^2}{2\lambda}} J.$$

We next focus on the term (6-12). Using the estimates (6-18) and (6-19), bounding the form (6-12) reduces to estimating

$$\int_0^\infty \int_{\mathbb{R}^7} \Big[\int_{\mathbb{R}} \Big[\prod_{i=0,1} f_0(x_3^i, x_1, x_2) f_1(x_0, x_3^i, x_2) \Big] h_{(z^{-1}t)}(x_2 + r)\, dx_2 \Big]^2 \times t^{-1}\gamma\big(1 + t^{-1}\gamma|x_0 + x_1 + p/2 + q - r|\big)^{-2-\frac{2}{\lambda}} t^{-1}(1 + t^{-1}|p|)^{-2-\frac{2}{\lambda}} \times g_{(t)}(x_3^0 + p + q) g_{(t)}(x_3^1 + q)\, dx_0\, dx_1\, dx_3^0\, dx_3^1\, dp\, dq\, dr\, \frac{dt}{t},$$

where $\gamma = z$, or $\gamma = 2^k$ if $2^k \le z$. We will prove a bound independent of z and k, which will bound (6-12) by $\lesssim z^{(\lambda-1)^2/(2\lambda)}$ thanks to the extra factor $z^{(\lambda-1)^2/(2\lambda)}$ in (6-18) and (6-19).

We dominate

$$\begin{aligned} t^{-1}\gamma\big(1 + t^{-1}\gamma|x_0 + x_1 + p/2 + q - r|\big)^{-2-\frac{2}{\lambda}} t^{-1}(1 + t^{-1}|p|)^{-2-\frac{2}{\lambda}} & \\ \lesssim t^{-2}\gamma\big(1 + t^{-1}\big|(\gamma(x_0 + x_1 + p/2 + q - r), 2p)\big|\big)^{-2-\frac{2}{\lambda}} & \\ \lesssim \int_2^\infty g_{(\alpha\gamma^{-1}t)}(x_0 + x_1 + p/2 + q - r) g_{(\alpha t)}(2p)\, \frac{d\alpha}{\alpha^{1+2/\lambda}}. & \end{aligned}$$

It thus suffices to estimate the form

$$\int_0^\infty\int_{\mathbb{R}^7}\bigg[\int_{\mathbb{R}}\bigg[\prod_{i=0,1} f_0(x_3^i,x_1,x_2)f_1(x_0,x_3^i,x_2)\bigg]h_{(z^{-1}t)}(x_2+r)\,dx_2\bigg]^2$$
$$\times g_{(\alpha\gamma^{-1}t)}(x_0+x_1+p/2+q-r)g_{(\alpha t)}(2p)g_{(t)}(x_3^0+p+q)g_{(t)}(x_3^1+q)\,dx_0\,dx_1\,dx_3^0\,dx_3^1\,dp\,dq\,dr\,\frac{dt}{t} \tag{6-21}$$

with a bound $\lesssim \alpha$ and then integrate over α, using that $2/\lambda > 1$. We claim that

$$\int_{\mathbb{R}} g_{(\alpha\gamma^{-1}t)}(x_0+x_1+p/2+q-r)g_{(\alpha t)}(2p)g_{(t)}(x_3^0+p+q)\,dp$$
$$\lesssim g_{(2^{1/2}\alpha\gamma^{-1}t)}(x_0+x_1+q-r)g_{(\alpha t)}(x_3^0+q). \tag{6-22}$$

Indeed, we have

$$g_{(\alpha t)}(2p) \lesssim e^{-\pi\gamma^2\alpha^{-2}t^{-2}(p/2)^2}g_{(2^{-1/2}\alpha t)}(p).$$

The elementary inequality $e^{-2(a+b)^2}e^{-2b^2} \le e^{-a^2}$ yields

$$g_{(\alpha\gamma^{-1}t)}(x_0+x_1+p/2+q-r)e^{-\pi\gamma^2\alpha^{-2}t^{-2}(p/2)^2} \lesssim g_{(2^{1/2}\alpha\gamma^{-1}t)}(x_0+x_1+q-r).$$

Thus, the left-hand side of (6-22) is bounded by

$$g_{(2^{1/2}\alpha\gamma^{-1}t)}(x_0+x_1+q-r)(g_{(2^{-1/2}\alpha t)}*g_{(t)})(x_3^0+q) \lesssim g_{(2^{1/2}\alpha\gamma^{-1}t)}(x_0+x_1+q-r)g_{(\alpha t)}(x_3^0+q),$$

as desired.

Expressing further $g_{(2^{1/2}\alpha\gamma^{-1}t)}(x_0+x_1+q-r)$ as a convolution of two Gaussians and using the evenness of the Gaussian, (6-21) is bounded by

$$\alpha\int_0^\infty\int_{\mathbb{R}^7}\bigg[\int_{\mathbb{R}}\bigg[\prod_{i=0,1} f_0(x_3^i,x_1,x_2)f_1(x_0,x_3^i,x_2)\bigg]h_{(z^{-1}t)}(x_2+r)\,dx_2\bigg]^2$$
$$\times g_{(\alpha\gamma^{-1}t)}(x_0+p)g_{(\alpha\gamma^{-1}t)}(x_1-p+q-r)g_{(\alpha t)}(x_3^0+q)g_{(\alpha t)}(x_3^1+q)\,dx_0\,dx_1\,dx_3^0\,dx_3^1\,dp\,dq\,dr\,\frac{dt}{t}. \tag{6-23}$$

After renaming of variables, naming the variable x_2 that is twice an integration variable once as x_2^0 and once as x_2^1, then renaming the variables $x_0, x_1, x_2^0, x_2^1, x_3^0, x_3^1$ in this order as $x_1^1, x_1^0, x_3^0, x_3^1, x_2^0, x_2^1$, and finally introducing functions $\tilde{f}_0(a,b,c) = f_0(b,a,c)$ and $\tilde{f}_1 = f_1$, we write (6-23) as

$$\alpha\int_0^\infty\int_{\mathbb{R}^7}\bigg[\prod_{i=0,1}\int_{\mathbb{R}} \tilde{f}_0(x_1^0,x_2^0,x_3^i)\tilde{f}_1(x_1^1,x_2^0,x_3^i)\tilde{f}_0(x_1^0,x_2^1,x_3^i)\tilde{f}_1(x_1^1,x_2^1,x_3^i)h_{(z^{-1}t)}(x_3^i+r)\,dx_3^i\bigg]$$
$$\times g_{(\alpha\gamma^{-1}t)}(x_1^0-p+q-r)g_{(\alpha\gamma^{-1}t)}(x_1^1+p)g_{(\alpha t)}(x_2^0+q)g_{(\alpha t)}(x_2^1+q)\,dx_1^0\,dx_1^1\,dx_2^0\,dx_2^1\,dp\,dq\,dr\,\frac{dt}{t}.$$

Let S and $(\Pi_s)_{s\in S}$ be as in the datum D_A. Introducing $f_s = \tilde{f}_{s(1)}$ for $s \in S$, we may write the last display as

$$\alpha\int_0^\infty\int_{\mathbb{R}^9}\bigg[\prod_{s\in S} f_s(\Pi_s x)\bigg]g_{(\alpha\gamma^{-1}t)}(x_1^0-p+q-r)g_{(\alpha\gamma^{-1}t)}(x_1^1+p)$$
$$\times g_{(\alpha t)}(x_2^0+q)g_{(\alpha t)}(x_2^1+q)h_{(z^{-1}t)}(x_3^0+r)h_{(z^{-1}t)}(x_3^1+r)\,dx\,dp\,dq\,dr\,\frac{dt}{t}.$$

Let $V:\mathbb{R}^3\to\mathbb{R}^3$ be a mapping given by $V(v_0,v_1,v_2)=(\alpha\gamma^{-1}v_0,\alpha v_1,z^{-1}v_2)$. We perform the change of variables with respect to this mapping for each of the triples (p,q,r), (x_1^0,x_2^0,x_3^0) and (x_1^1,x_2^1,x_3^1). After this transformation, the above form becomes

$$\alpha\int_0^\infty\int_{\mathbb{R}^9}\Bigl[\prod_{s\in S}\alpha^{\frac14}\gamma^{-\frac18}z^{-\frac18}f_s(V\Pi_s x)\Bigr]g_{(t)}(x_1^0-p+\gamma q-\gamma z^{-1}\alpha^{-1}r)g_{(t)}(x_1^1+p)$$
$$\times g_{(t)}(x_2^0+q)g_{(t)}(x_2^1+q)h_{(t)}(x_3^0+r)h_{(t)}(x_3^1+r)\,dx\,dp\,dq\,dr\,\frac{dt}{t}. \tag{6-24}$$

This can be recognized as an α multiple of

$$\Lambda_{D_A,K}((\alpha^{\frac14}\gamma^{-\frac18}z^{-\frac18}f_s\circ V)_{s\in S}),$$

where K has the form (6-2) with $i=3$ and

$$A=\begin{pmatrix}1 & -\gamma & \gamma z^{-1}\alpha^{-1}\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}.$$

Since $0<\gamma\le z\le 1\le\alpha$, the matrix A satisfies the assumption (6-1) with $\varepsilon=5^{-1/2}$. Observing further that the function $\alpha^{1/4}\gamma^{-1/8}z^{-1/8}f_s\circ V$ has the same L^8 norm as f_s, we deduce from Lemma 6.1 that (6-24) is bounded by $\lesssim\alpha$. This yields the desired bound for (6-12).

Combining the estimates for (6-11) and (6-12), we obtain (6-5).

It remains to consider the part of the integral in (6-3) where $z\in(1,\infty)$. Let φ be the function defined via its Fourier transform by

$$\widehat{\varphi}(\xi)=\int_1^\infty\hat{\theta}(z\xi)\,\frac{dz}{z}.$$

Then we can write

$$\int_1^\infty\widehat{K}^{(z)}(\xi,\eta)\,\frac{dz}{z}=\int_0^\infty\widehat{K}(\xi,\eta)\widehat{\varphi}(t(\xi-\eta))\hat{\theta}(t(\xi+\eta))\,\frac{dt}{t}. \tag{6-25}$$

Formally, this expression has the same form as (6-4) when $z=1$, except that the function $\hat{\theta}$ is at one occurrence replaced by $\widehat{\varphi}$. Due to this similarity, we will denote (6-25) by $\widehat{K}^{(1)}(\xi,\eta)$. Note that $\hat{\theta}$ is supported in $\bigl[-2,-\frac12\bigr]\cup\bigl[\frac12,2\bigr]$, $\widehat{\varphi}$ is supported in $[-2,2]$ and the support properties of $\hat{\theta}$ and $\widehat{\varphi}$ ensure that $\widehat{\varphi}(\xi-\eta)\hat{\theta}(\xi+\eta)$ is supported in the set where $\frac18\le|(\xi,\eta)|\le 8$. We may therefore apply an argument analogous to the case $z\in(0,1)$, arriving at the estimate

$$|\Lambda_{D_1,K^{(1)}}((f_s)_{s\in S})|\lesssim J^{\frac12}.$$

Combining this with (6-5) yields the conclusion of the proposition. □

7. Proof of Proposition 2.6 using Propositions 2.7 and 2.8

Let $(k_j)_{j=0}^J$ be a finite increasing sequence of integers with $k_{j-1}+10\le k_j$. For $1\le j\le J$, let $\phi_{0,j}$, $\phi_{1,j}$, $\phi_{2,j}$ be rescaled respective left or right windows as in the proposition, and define K as in (2-9). Let $(f_s)_{s\in S}$ be a tuple of functions as in (2-3) and (2-4). Set $f_k:=f_{(k,0)}=f_{(k,1)}$ for each $k=0,1,2$.

Let (χ, ϕ) be a universal pair and define $\chi_j := \chi_{(2^{k_j-2})}$ and $\phi_j := \phi_{(2^{k_j-2})}$. Define

$$\phi_{3,j} := \chi_{j-1} - \phi_j \tag{7-1}$$

and consequently,

$$(\widehat{\phi}_{3,j})^2 = (\widehat{\chi}_{j-1})^2 - (\widehat{\chi}_j)^2.$$

If (ξ, η) is in the support of $(\widehat{\phi}_{0,j} - \widehat{\phi}_{2,j}) \otimes \widehat{\phi}_{1,j}$, then

$$2^{-k_j+3} - 2^{-k_j} \le |\xi + \eta| \le 2^{-k_{j-1}} + 2^{-k_j}.$$

In this range, $\widehat{\chi}_{j-1}(\xi + \eta)$ is constant 1 and $\widehat{\chi}_j(\xi + \eta)$ is constant zero. We can therefore introduce artificial factors $\widehat{\phi}_{3,j}$ in $\widehat{K}$ as follows:

$$\widehat{K}(\xi, \eta) = \sum_{j=1}^{J} (\widehat{\phi}_{0,j} - \widehat{\phi}_{2,j})(\xi)\widehat{\phi}_{1,j}(\eta) = \sum_{j=1}^{J} (\widehat{\phi}_{0,j} - \widehat{\phi}_{2,j})(\xi)\widehat{\phi}_{1,j}(\eta)\widehat{\phi}_{3,j}(-\xi - \eta).$$

Taking the Fourier transform, we obtain for some universal constant C,

$$\begin{aligned} K(u, v) &= \int_{\mathbb{R}^2} \sum_{j=1}^{J} (\widehat{\phi}_{0,j} - \widehat{\phi}_{2,j})(\xi) e^{2\pi i u\xi} \widehat{\phi}_{1,j}(\eta) e^{2\pi i v\eta} \widehat{\phi}_{3,j}(-\xi - \eta)\, d\xi\, d\eta \\ &= C \sum_{j=1}^{J} \int_{\mathbb{R}} (\phi_{0,j} - \phi_{2,j})(u + p)\phi_{1,j}(v + p)\phi_{3,j}(p)\, dp, \end{aligned} \tag{7-2}$$

where we used that the integral of a function in $\mathbb{R}^3$ over the diagonal $\{(p, p, p) : p \in \mathbb{R}\}$ equals the integral of its Fourier transform over the orthogonal complement of the diagonal, suitably normalized.

Therefore, with S and Π_s as in the datum D_1, and doing a variable transformation $p \to x_2 + p$,

$$\begin{aligned} \Lambda_{D_1,K}((f_s)_{s\in S}) = \sum_{j=1}^{J} \int_{\mathbb{R}^6} \Big[\prod_{s\in S} f_s(\Pi_s x)\Big] \\ \times (\phi_{0,j} - \phi_{2,j})(x_3^0 - x_0 - x_1 + p)\phi_{1,j}(x_3^1 - x_0 - x_1 + p)\phi_{3,j}(x_2 + p)\, dx\, dp. \end{aligned} \tag{7-3}$$

We apply Fubini in (7-3) to have the integral in x_2 as the innermost and then apply the Cauchy–Schwarz inequality in $x_0, x_1, x_3^0, x_3^1, p$, which bounds $|\Lambda_{D_1,K}((f_s)_{s\in S})|$ up to a constant by the geometric mean of

$$\sum_{j=1}^{J} \int_{\mathbb{R}^5} \Big[\prod_{i=0,1} |f_2(x_0, x_1, x_3^i)|^2\Big] \mu_j(x_3^0 - x_0 - x_1 + p, x_3^1 - x_0 - x_1 + p)\, dx_0\, dx_1\, dx_3^0\, dx_3^1\, dp \tag{7-4}$$

and

$$\begin{aligned} \sum_{j=1}^{J} \int_{\mathbb{R}^5} \Big[\int_{\mathbb{R}} \Big[\prod_{i=0,1} f_0(x_3^i, x_1, x_2) f_1(x_0, x_3^i, x_2)\Big] \phi_{3,j}(x_2 + p)\, dx_2\Big]^2 \\ \times \mu_j\big(x_3^0 - x_0 - x_1 + p, x_3^1 - x_0 - x_1 + p\big)\, dx_0\, dx_1\, dx_3^0\, dx_3^1\, dp, \end{aligned} \tag{7-5}$$

where we have introduced the weight μ_j defined by

$$\mu_j(u,v) = |\phi_{0,j} - \phi_{2,j}|(u)|\phi_{1,j}|(v).$$

We will estimate (7-4) as $\lesssim J$ and (7-5) as $\lesssim 1$, thereby proving Proposition 2.6.

We begin with (7-4). Applying the Cauchy–Schwarz inequality in the remaining integration variables, we bound (7-4) by

$$\sum_{j=1}^{J} \prod_{i=0,1} \Big[\int_{\mathbb{R}^5} |f_2(x_0,x_1,x_3^i)|^4 \mu_j\big(x_3^0 - x_0 - x_1 + p, x_3^1 - x_0 - x_1 + p\big)\, dx_0\, dx_1\, dx_3^0\, dx_3^1\, dp \Big]^{\frac{1}{2}} = \sum_{j=1}^{J} \|f_2\|_4^4 \|\mu_j\|_1 \lesssim J.$$

Here the identity is seen by integrating first in x_3^{1-i} then in p to obtain the L^1 norm of μ_j.

It remains to estimate (7-5). We use decay of μ_j thanks to control of derivatives of Fourier transform of windows and the superposition estimate

$$(1+|(u,v)|)^{-N-20} \lesssim \int_1^\infty g_{(\alpha)}(u) g_{(\alpha)}(v)\, \frac{d\alpha}{\alpha^{N+10}},$$

which we scale isotropically and anisotropically, to dominate

$$\mu_j(u,v) \lesssim \int_1^\infty g_{(\alpha 2^{k_j})}(u) g_{(\alpha 2^{k_j})}(v)\, \frac{d\alpha}{\alpha^{N+10}} + \int_1^\infty g_{(\alpha 2^{k_{j-1}})}(u) g_{(\alpha 2^{k_j})}(v)\, \frac{d\alpha}{\alpha^{N+10}}.$$

By superposition of positive terms, it suffices to estimate as $\lesssim \alpha^N$ the variant of (7-5) with $\mu_j(u,v)$ replaced by

$$g_{(\alpha 2^{l_j})}(u) g_{(\alpha 2^{k_j})}(v)$$

and for each of the sequences $l_j = k_j$ and $l_j = k_{j-1}$. Define the sequence of real numbers $(m_j)_{j=1}^J$ by

$$2^{2m_j} + 2^{2m_j} = 2^{2k_j} + 2^{2l_j}.$$

Note that $k_j - 1 \le m_j \le k_j$, because $l_j \le k_j$. Adding and subtracting terms, it suffices to estimate as $\lesssim \alpha^N$ the variants of (7-5) with $\mu_j(u,v)$ replaced by

$$g_{(\alpha 2^{m_j})}(u) g_{(\alpha 2^{m_j})}(v) \tag{7-6}$$

and by $(\nu_j)_{(\alpha)}(u,v)$, where

$$\nu_j(u,v) := g_{(2^{l_j})}(u) g_{(2^{k_j})}(v) - g_{(2^{m_j})}(u) g_{(2^{m_j})}(v). \tag{7-7}$$

We begin with (7-6). We need to estimate

$$\sum_{j=1}^{J} \int_{\mathbb{R}^5} \Big[\int_{\mathbb{R}} \Big[\prod_{i=0,1} f_0(x_3^i, x_1, x_2) f_1(x_0, x_3^i, x_2) \Big] \phi_{3,j}(x_2+p)\, dx_2 \Big]^2 \times g_{(\alpha 2^{m_j})}(x_3^0 - x_0 - x_1 + p) g_{(\alpha 2^{m_j})}(x_3^1 - x_0 - x_1 + p)\, dx_0\, dx_1\, dx_3^0\, dx_3^1\, dp. \tag{7-8}$$

A renaming of variables, naming the variable x_2 that is twice an integration variable once as x_2^0 and once as x_2^1, then renaming the variables $x_0, x_1, x_2^0, x_2^1, x_3^0, x_3^1$ in this order as $x_1, x_0, x_3^0, x_3^1, x_2^0, x_2^1$, and finally introducing functions $\tilde{f}_0(a, b, c) = f_0(b, a, c)$ and $\tilde{f}_1 = f_1$, we write (7-8) as

$$\sum_{j=1}^{J} \int_{\mathbb{R}^5} \left[\prod_{i=0}^{1} \int_{\mathbb{R}} \tilde{f}_0(x_0, x_2^0, x_3^i) \tilde{f}_1(x_1, x_2^0, x_3^i) \tilde{f}_0(x_0, x_2^1, x_3^i) \tilde{f}_1(x_1, x_2^1, x_3^i) \phi_{3,j}(x_3^i + p)\, dx_3^i \right] \times g_{(\alpha 2^{m_j})}(x_2^0 - x_0 - x_1 + p) g_{(\alpha 2^{m_j})}(x_2^1 - x_0 - x_1 + p)\, dx_0\, dx_1\, dx_2^0\, dx_2^1\, dp. \quad (7\text{-}9)$$

Introducing for the datum D_2 the tuple $f_{(k,j)} = \tilde{f}_k$ for $k = 0, 1$ and $j \in \mathcal{C}$, we may write (7-9) as $\Lambda_{D_2, K_1}((f_s)_{s \in S})$, with

$$K_1(u, v, z) = \sum_{j=1}^{J} \int_{\mathbb{R}} g_{(\alpha 2^{m_j})}(u + p) g_{(\alpha 2^{m_j})}(v + p) \phi_{3,j}(z + p) \phi_{3,j}(p)\, dp.$$

Proposition 2.7 implies $\Lambda_{D_2, K_1}((f_s)_{s \in S}) \lesssim \alpha^N$.

It remains to estimate the term with (7-7). We may assume $l_j = k_{j-1}$, because (7-7) vanishes in the case $k_j = l_j$. With similar transformations as for term (7-6), we write the form associated with (7-7) as $\Lambda_{D_2, K_2}((f_s)_{s \in S})$ with

$$K_2(u, v, z) = \sum_{j=1}^{J} \int_{\mathbb{R}} (\nu_j)_{(\alpha)}(u + p, v + p) \phi_{3,j}(z + p) \phi_{3,j}(p)\, dp.$$

We decompose $\nu_j = \sum_{n \in \mathbb{Z}} \nu_{j,n}$, where

$$\hat{\nu}_{j,0}(\xi, \eta) = \hat{\nu}_j(\xi, \eta)((\widehat{\chi}_{(2^{k_{j-1}})})^2 - (\widehat{\chi}_{(2^{k_j})})^2)(\xi + \eta),$$

and for $n < 0$,

$$\hat{\nu}_{j,n}(\xi, \eta) = \hat{\nu}_j(\xi, \eta)((\widehat{\chi}_{(2^{k_{j-1}+n})})^2 - (\widehat{\chi}_{(2^{k_{j-1}+n+1})})^2)(\xi + \eta) \quad (7\text{-}10)$$

and for $n > 0$,

$$\hat{\nu}_{j,n}(\xi, \eta) = \hat{\nu}_j(\xi, \eta)((\widehat{\chi}_{(2^{k_j+n-1})})^2 - (\widehat{\chi}_{(2^{k_j+n})})^2)(\xi + \eta). \quad (7\text{-}11)$$

We split $K_2 = \sum_{n \in \mathbb{Z}} K_{2,n}$ accordingly and estimate for each n

$$\Lambda_{D_2, K_{2,n}}((f_s)_{s \in S}) \lesssim 2^{-|n|}.$$

Upon summing over n, we obtain the desired bound for (7-7).

We begin with $n = 0$. We have, similarly as in (7-2), for some universal constant C,

$$K_2(u, v, z) = C \sum_{j=1}^{J} \int_{\mathbb{R}^3} (\hat{\nu}_j)_{(\alpha)}(\xi, \eta) e^{2\pi i (u\xi + v\eta)} \widehat{\phi}_{3,j}(\tau) e^{2\pi i z\tau} \widehat{\phi}_{3,j}(-\tau - \xi - \eta)\, d\xi\, d\eta\, d\tau,$$

and thus

$$\widehat{K}_{2,0}(\xi, \eta, \tau) = C \sum_{j=1}^{J} ((\widehat{\chi}_{(\alpha 2^{k_{j-1}})})^2 - (\widehat{\chi}_{(\alpha 2^{k_j})})^2)(\xi + \eta) (\hat{\nu}_j)_{(\alpha)}(\xi, \eta) \widehat{\phi}_{3,j}(\tau) \widehat{\phi}_{3,j}(-\tau - \xi - \eta).$$

Preparing to apply Proposition 2.8, we note that $K_{2,0}$ is of the form (2-15) with ρ_j defined by

$$\rho_j(u_1, u_2, u_3, u_4) := (\nu_j)_{(\alpha)}(u_1, u_2)\phi_{3,j}(u_3)\phi_{3,j}(u_4),$$

as can be seen from the Fourier transform side (2-16). We do not attempt to show that ρ_j itself satisfies the assumptions of Proposition 2.8, but we split into eight pieces by the distributive law, splitting ν_j into two pieces as in its definition (7-7) and each $\phi_{3,j}$ into two as in its definition (7-1). A typical piece is

$$g_{(\alpha 2^{l_j})}(u_1)g_{(\alpha 2^{k_j})}(u_2)\chi_{j-1}(u_3)\phi_j(u_4),$$

which satisfies the assumptions of Proposition 2.8, because

$$\int_{\mathbb{R}^2} g_{(\alpha 2^{l_j})}(u_1+p)g_{(\alpha 2^{k_j})}(u_2+p)\chi_{j-1}(u_3+r)\phi_j(u_4+r)\,dp\,dr \\ \lesssim (g*g)_{(\alpha 2^{1+k_j})}(u_1-u_2)2^{-k_j}(1+2^{-k_j}|u_3-u_4|)^{-2}.$$

This along with similar estimates for the other seven pieces completes the bound for $\Lambda_{D_2,K_{2,0}}((f_s)_{s\in S})$ by Proposition 2.8.

We turn to $n > 0$. We introduce artificial factors that are constant 1 where relevant, using that the sequence k_j is well separated, and write

$$\widehat{K}_{2,n}(\xi,\eta,\tau) = C\sum_{j=1}^{J}((\widehat{\chi}_{(\alpha 2^{k_{j-1}+n+1})})^2 - (\widehat{\chi}_{(\alpha 2^{k_j+n+1})})^2)(\xi+\eta) \\ \times((\widehat{\chi}_{(\alpha 2^{k_j+n-1})})^2 - (\widehat{\chi}_{(\alpha 2^{k_j+n})})^2)(\xi+\eta)(\hat{\nu}_j)_{(\alpha)}(\xi,\eta)\widehat{\phi}_{3,j}(\tau)\widehat{\phi}_{3,j}(-\tau-\xi-\eta).$$

This kernel is of the form (2-15) with

$$\hat{\rho}_j(\xi,\eta,\tau,\sigma) = (\hat{\nu}_{j,n})_{(\alpha)}(\xi,\eta)\widehat{\phi}_{3,j}(\tau)\widehat{\phi}_{3,j}(\sigma),$$

with $\nu_{j,n}$ defined in (7-11). We break both functions $\widehat{\phi}_{3,j}$ into pieces as above. All pieces are done similarly, we discuss a typical piece of ρ_j given by

$$\hat{\varrho}_j(\xi,\eta,\tau,\sigma) = (\hat{\nu}_{j,n})_{(\alpha)}(\xi,\eta)\widehat{\chi}_{j-1}(\tau)\widehat{\phi}_j(\sigma).$$

Using that χ_j and ϕ_j are even, we have

$$\int_{\mathbb{R}}\chi_{j-1}(u_3+r)\phi_j(u_4+r)\,dr = (\chi_{j-1}*\phi_j)(u_3-u_4) \lesssim 2^{-k_j}(1+2^{-k_j}|u_3-u_4|)^{-2}.$$

With Lemma 7.1 below, we obtain

$$\int_{\mathbb{R}^2}|\varrho_j|(u_1+p, u_2+p, u_3+r, u_4+r)\,dp\,dr \\ \lesssim 2^{-n}\alpha^{-1}2^{-k_j}(1+\alpha^{-1}2^{-k_j}|u_1-u_2|)^{-2}2^{-k_j}(1+2^{-k_j}|u_3-u_4|)^{-2}.$$

Proposition 2.8 gives $\Lambda_{D_2,K_{2,n}}((f_s)_{s\in S}) \lesssim 2^{-n}$, as desired.

Lemma 7.1. *We have for every* $1 \le j \le J$ *and every* $x, y \in \mathbb{R}$ *the estimate*

$$|\nu_{j,n}(x,y)| \lesssim 2^{-n}2^{-k_j}(1+2^{-k_j}|x-y|)^{-4}2^{-n-k_j}(1+2^{-n-k_j}|x+y|)^{-4}.$$

Proof of Lemma 7.1. Scaling by a factor 2^{k_j} allows us to assume $k_j = 0$ and $-1 \le m_j \le 0$ and $l_j \le 0$. We fix j and omit the index j. We thus have to prove

$$|\nu_n(x,y)| \lesssim 2^{-2n}(1+|x-y|)^{-4}(1+2^{-n}|x+y|)^{-4}, \tag{7-12}$$

where

$$\hat{\nu}_n(\xi,\eta) = ((\widehat{\chi}_{(2^{-1})})^2 - (\widehat{\chi})^2)(2^n(\xi+\eta))\hat{\nu}(\xi,\eta) \tag{7-13}$$

with

$$\hat{\nu}(\xi,\eta) = g(2^l\xi)g(\eta) - g(2^m\xi)g(2^m\eta). \tag{7-14}$$

We claim that for $0 \le \alpha \le 4$ and $0 \le \beta \le 4$ and $|\xi+\eta| \le 1$

$$|\partial^{\alpha}_{(1,-1)}\partial^{\beta}_{(1,1)}\hat{\nu}(\xi,\eta)| \lesssim |\xi+\eta|^{\max\{1-\beta,0\}}(1+|\xi-\eta|)^{-2}.$$

For $\beta > 0$, this follows by deriving (7-14) and using the decay of Gaussians and their derivatives for those Gaussians whose argument contains m or k, because $-1 \le m \le 0$ and $k = 0$. Here we also use the fact that whenever $|\xi+\eta| \le 1$ and $|\xi-\eta| \ge 1$, then the three quantities $|\xi-\eta|$, $|\xi|$ and $|\eta|$ are comparable.

We next estimate the term with $\beta = 0$. By the choice of m, the function $\hat{\nu}$ vanishes on the diagonal $\xi+\eta = 0$, and thus the same property holds also for $\partial^{\alpha}_{(1,-1)}\hat{\nu}$. Therefore,

$$\begin{aligned}
|\partial^{\alpha}_{(1,-1)}\hat{\nu}(\xi,\eta)| &= \big|\partial^{\alpha}_{(1,-1)}\hat{\nu}(\xi,\eta) - \partial^{\alpha}_{(1,-1)}\hat{\nu}\big((\xi-\eta)/2, -(\xi-\eta)/2\big)\big| \\
&\lesssim |\xi+\eta|\left|\int_0^1 \partial^{\alpha}_{(1,-1)}\partial_{(1,1)}\hat{\nu}\big((\xi-\eta)/2 + r(\xi+\eta)/2, -(\xi-\eta)/2 + r(\xi+\eta)/2\big)\,dr\right| \\
&\lesssim |\xi+\eta| \sup_{0\le r\le 1}\big|\partial^{\alpha}_{(1,-1)}\partial_{(1,1)}\hat{\nu}\big((\xi-\eta)/2 + r(\xi+\eta)/2, -(\xi-\eta)/2 + r(\xi+\eta)/2\big)\big| \\
&\lesssim |\xi+\eta|(1+|\xi-\eta|)^{-2}.
\end{aligned}$$

Turning to $\hat{\nu}_n$ as in (7-13), using that $\widehat{\chi}^2_{(2^{-1})} - \widehat{\chi}^2$ is supported in $[-2,2]$, we obtain by differentiating

$$\begin{aligned}
|\hat{\nu}_n(\xi,\eta)| &\lesssim 2^{-n}1_{|2^n(\xi+\eta)|<1}(1+|\xi-\eta|)^{-2}, \\
|\partial^4_{(1,-1)}\hat{\nu}_n(\xi,\eta)| &\lesssim 2^{-n}1_{|2^n(\xi+\eta)|<1}(1+|\xi-\eta|)^{-2}, \\
|\partial^4_{(1,1)}\hat{\nu}_n(\xi,\eta)| &\lesssim 2^{3n}1_{|2^n(\xi+\eta)|<1}(1+|\xi-\eta|)^{-2}, \\
|\partial^4_{(1,-1)}\partial^4_{(1,1)}\hat{\nu}_n(\xi,\eta)| &\lesssim 2^{3n}1_{|2^n(\xi+\eta)|<1}(1+|\xi-\eta|)^{-2}.
\end{aligned}$$

Hence, estimating the Fourier inversion formula crudely by $L^1 \to L^\infty$ bounds,

$$|\nu_n(x,y)| \lesssim 2^{-2n}, \quad |x-y|^4|\nu_n(x,y)| \lesssim 2^{-2n}, \quad |x+y|^4|\nu_n(x,y)| \lesssim 2^{2n}, \quad |x-y|^4|x+y|^4|\nu_n(x,y)| \lesssim 2^{2n}.$$

We can summarize these findings into (7-12), as can be seen by splitting into four cases depending on whether $2^n \le |x+y|$ or $2^n > |x+y|$ and depending on whether $1 \le |x-y|$ or $1 > |x-y|$. This proves the lemma. □

We finally turn to $n < 0$. As in the previous case, we introduce an artificial factor and write

$$\widehat{K}_{2,n}(\xi,\eta,\tau) = C\sum_{j=1}^{J}((\widehat{\chi}_{(\alpha 2^{k_{j-1}+n-1})})^2 - (\widehat{\chi}_{(\alpha 2^{k_j+n-1})})^2)(\xi+\eta) \times ((\widehat{\chi}_{(\alpha 2^{k_{j-1}+n})})^2 - (\widehat{\chi}_{(\alpha 2^{k_{j-1}+n+1})})^2)(\xi+\eta)(\hat{\nu}_j)_{(\alpha)}(\xi,\eta)\widehat{\phi}_{3,j}(\tau)\widehat{\phi}_{3,j}(-\tau-\xi-\eta).$$

This kernel is of the form (2-15) with

$$\hat{\rho}_j(\xi,\eta,\tau,\sigma) = (\hat{\nu}_{j,n})_{(\alpha)}(\xi,\eta)\widehat{\phi}_{3,j}(\tau)\widehat{\phi}_{3,j}(\sigma)$$

with $\nu_{j,n}$ as in (7-10).

We break both functions $\widehat{\phi}_{3,j}$ into pieces as above. All pieces are done similarly, we discuss a typical piece of ρ_j given by

$$\hat{\varrho}_j(\xi,\eta,\tau,\sigma) = (\hat{\nu}_{j,n})_{(\alpha)}(\xi,\eta)\widehat{\chi}_{j-1}(\tau)\widehat{\phi}_j(\sigma).$$

With Lemma 7.2, we obtain

$$\int_{\mathbb{R}^2} |\varrho_j|(u_1+p, u_2+p, u_3+r, u_4+r)\,dp\,dr \lesssim 2^n\alpha^{-1}2^{-k_j}(1+\alpha^{-1}2^{-k_j}|u_1-u_2|)^{-4}2^{-k_j}(1+2^{-k_j}|u_3-u_4|)^{-2}.$$

Proposition 2.8 gives $\Lambda_{D_2,K_{2,n}}((f_s)_{s\in S}) \lesssim 2^n$, as desired.

Lemma 7.2. *We have for every $1 \le j \le J$ and every $x, y \in \mathbb{R}$ the estimate*

$$|\nu_{j,n}(x,y)| \lesssim 2^n 2^{-k_{j-1}}(1+2^{-k_{j-1}}|x|)^{-4}2^{-k_j}(1+2^{-k_j}|y|)^{-4}.$$

Proof. We split the function

$$\hat{\nu}_j(\xi,\eta) = g(2^{l_j}\xi)g(2^{k_j}\eta) - g(2^{m_j}\xi)g(2^{m_j}\eta)$$

into its two summands and consider the summands separately. Consider the term

$$g(2^{l_j}\xi)g(2^{k_j}\eta).$$

Scaling by the factor 2^{l_j} in ξ and 2^{k_j} in η reduces the matter to proving

$$|\mu_n(x,y)| \lesssim 2^n(1+|x|)^{-4}(1+|y|)^{-4}, \tag{7-15}$$

where

$$\widehat{\mu}_n(\xi,\eta) = (\widehat{\chi}^2 - \widehat{\chi}^2_{(2)})(2^n\xi + 2^{n+l}\eta)g(\xi)g(\eta)$$

with $l \le 0$. On the support of the function

$$(\widehat{\chi}^2 - \widehat{\chi}^2_{(2)})(2^n\xi + 2^{n+l}\eta),$$

we have

$$|\partial_\xi^\alpha\partial_\eta^\beta g(\xi)g(\eta)| \lesssim 2^n(1+|\xi|)^{-4}(1+|\eta|)^{-4}$$

for all $0 \le \alpha, \beta \le 4$. By the Leibniz rule, analogous bounds hold for $\widehat{\mu}_n$. The function μ_n then satisfies the bound (7-15). This is the desired estimate for the term $g(2^{l_j}\xi)g(2^{k_j}\eta)$.

To estimate the term $g(2^{m_j}\xi)g(2^{m_j}\eta)$, we rescale by 2^{m_j} in both variables and claim

$$|\mu_n(x,y)| \lesssim 2^n 2^{5l}(1+|x|)^{-4}(1+|y|)^{-4},$$

where

$$\widehat{\mu}_n(\xi,\eta) = (\widehat{\chi}^2 - \widehat{\chi}^2_{(2)})(2^{n+l}(\xi+\eta))g(\xi)g(\eta)$$

and $l = k_{j-1} - m_j \le 0$. This follows similarly as before, using the decay of the Gaussians. As

$$2^{5l}(1+|x|)^{-4} \lesssim 2^{-l}(1+2^{-l}|x|)^{-4},$$

this completes the proof of the lemma. □

8. Proof of Proposition 2.7 using Propositions 2.8, 2.9, and Theorem 1.1 in [Durcik et al. 2022]

Let $\alpha \ge 1$. Let J be a positive integer and $(k_j)_{j=0}^J$ a finite increasing sequence of integers with $k_{j-1}+10 \le k_j$ for $1 \le j \le J$, let $(m_j)_{j=1}^J$ be a sequence of real numbers with $k_j - 1 \le m_j \le k_j$. For $0 \le j \le J$, let χ_j be a function such that $(\chi_j)_{(2^{2-k_j})}$ is a left window and let ϕ_j be as in the statement of the proposition, i.e.,

$$(\widehat{\phi}_j)^2 = (\widehat{\chi}_{j-1})^2 - (\widehat{\chi}_j)^2.$$

Let a tuple $(f_s)_{s\in S}$ be given as in (2-12), (2-13) and write $f_{(0,j)} = f_0,\ f_{(1,j)} = f_1$ for any $j \in \mathcal{C}$.

Taking the Fourier transform, the kernel K of the proposition reads as

$$\widehat{K}(\xi,\eta,\tau) = \alpha^{-N}\sum_{j=1}^{J} \hat{g}_{(\alpha 2^{m_j})}(\xi)\hat{g}_{(\alpha 2^{m_j})}(\eta)\widehat{\phi}_j(\tau)\widehat{\phi}_j(-\xi-\eta-\tau).$$

Define the kernel K_1 by

$$\widehat{K}_1(\xi,\eta,\tau) := \alpha^{-N}\sum_{j=1}^{J} \hat{g}_{(\alpha 2^{m_j})}(\xi)\hat{g}_{(\alpha 2^{m_j})}(\eta)\big(\widehat{\chi}_{j-1}(\tau)\widehat{\chi}_{j-1}(-\tau-\xi-\eta) - \widehat{\chi}_j(\tau)\widehat{\chi}_j(-\tau-\xi-\eta)\big).$$

Therefore, on the critical space $\xi+\eta=0$, the kernels are equal, i.e., for all ξ,τ we have

$$\widehat{K}(\xi,-\xi,\tau) = \widehat{K}_1(\xi,-\xi,\tau).$$

By the triangle inequality, it suffices to estimate $\Lambda_{D_2,K-K_1}$ and Λ_{D_2,K_1}.

We begin with the latter. Since $\alpha \ge 1$, we observe that it in fact suffices to prove the (stronger) bound $|\Lambda_{D_2,\alpha^N K_1}((f_s)_{s\in S})| \lesssim 1$. Define the kernel K_2 by

$$\widehat{K}_2(\xi,\eta,\tau) := \sum_{j=1}^{J}\big(\hat{g}_{(\alpha 2^{m_{j-1}})}(\xi)\hat{g}_{(\alpha 2^{m_{j-1}})}(\eta) - \hat{g}_{(\alpha 2^{m_j})}(\xi)\hat{g}_{(\alpha 2^{m_j})}(\eta)\big)\widehat{\chi}_{j-1}(\tau)\widehat{\chi}_{j-1}(-\tau-\xi-\eta)$$

and define

$$\widehat{\sigma}_j(\xi,\eta,\tau) := \hat{g}_{(\alpha 2^{m_j})}(\xi)\hat{g}_{(\alpha 2^{m_j})}(\eta)\widehat{\chi}_j(\tau)\widehat{\chi}_j(-\tau-\xi-\eta).$$

Here, we formally set $m_0 = k_0$. By telescoping, we have

$$\alpha^N K_1 + K_2 = \sigma_0 - \sigma_J.$$

For each j, $\Lambda_{D_2,\sigma_j}((f_s)_{s\in S})$ equals

$$\int_{\mathbb{R}^7}\Big[\prod_{s\in S} f_s(\Pi_s x)\Big] g_{(\alpha 2^{m_j})}(x_2^0-x_0-x_1+p)g_{(\alpha 2^{m_j})}(x_2^1-x_0-x_1+p)\chi_j(x_3^0+p)\chi_j(x_3^1+p)\,dx\,dp,$$

where $x=(x_0,x_1,x_2^0,x_2^1,x_3^0,x_3^1)$. This can be estimated using a classical Brascamp–Lieb inequality as

$$|\Lambda_{D_2,\sigma_j}((f_s)_{s\in S})| \lesssim \|g_{(\alpha 2^{m_j})}\|_1^2 \|\chi_j\|_1^2 \prod_{s\in S}\|f_s\|_8 \lesssim 1. \tag{8-1}$$

One can verify this Brascamp–Lieb inequality by interpolation between estimates that put one of the functions f_s in L^1 and all others in L^∞.

The estimate of $\Lambda_{D_2,\alpha^N K_1}$ is thus reduced to an estimate of Λ_{D_2,K_2}, which we now proceed to do. We use the fundamental theorem of calculus to split up a difference of Gaussians with parameters a, b as

$$\begin{aligned} g(a\xi)g(a\eta)-g(b\xi)g(b\eta) &= \int_a^b -t\partial_t(g(t\xi)g(t\eta))\,\frac{dt}{t} = 2\pi\int_a^b t^2(\xi^2+\eta^2)g(t\xi)g(t\eta)\,\frac{dt}{t} \\ &= 2\pi\int_a^b t^2(\xi+\eta)^2 g(t\xi)g(t\eta)\,\frac{dt}{t} - 4\pi\int_a^b t^2\xi\eta g(t\xi)g(t\eta)\,\frac{dt}{t}. \end{aligned} \tag{8-2}$$

Using this splitting, in place of Λ_{D_2,K_2} we may estimate Λ_{D_2,K_3} and Λ_{D_2,K_4} with

$$\widehat{K}_3(\xi,\eta,\tau) := \sum_{j=1}^{J}\int_{\alpha 2^{m_{j-1}}}^{\alpha 2^{m_j}} t^2(\xi+\eta)^2 g(t\xi)g(t\eta)\,\frac{dt}{t}\,\widehat{\chi}_{j-1}(\tau)\widehat{\chi}_{j-1}(-\tau-\xi-\eta)$$

and, using $h=g'$ and that $\hat{h}(\xi)$ is a constant multiple of $\xi\hat{g}(\xi)$,

$$\widehat{K}_4(\xi,\eta,\tau) := \sum_{j=1}^{J}\int_{\alpha 2^{m_{j-1}}}^{\alpha 2^{m_j}} \hat{h}(t\xi)\hat{h}(t\eta)\,\frac{dt}{t}\,\widehat{\chi}_{j-1}(\tau)\widehat{\chi}_{j-1}(-\tau-\xi-\eta).$$

Proposition 2.9 gives

$$|\Lambda_{D_2,K_3}((f_s)_{s\in S})| \lesssim 1.$$

We turn to Λ_{D_2,K_4}, which we write on the spatial side as

$$\begin{aligned} \sum_{j=1}^{J}\int_{\alpha 2^{m_{j-1}}}^{\alpha 2^{m_j}}\int_{\mathbb{R}^5}\Big[\int_{\mathbb{R}}\Big[\prod_{i=0,1} f_0(x_0,x_2^i,x_3)f_1(x_1,x_2^i,x_3)\Big]h_{(t)}(x_3+p)\,dx_3\Big]^2 \\ \times \chi_{j-1}(x_2^0-x_0-x_1+p)\chi_{j-1}(x_2^1-x_0-x_1+p)\,dx_0\,dx_1\,dx_2^0\,dx_2^1\,dp\,\frac{dt}{t}. \end{aligned}$$

Using positivity of the square in this expression, we may dominate

$$|\chi_{j-1}(u)\chi_{j-1}(v)| \le \int_1^\infty g_{(\beta 2^{m_{j-1}})}(u)g_{(\beta 2^{m_{j-1}})}(v)\beta^{-N}\,\frac{d\beta}{\beta}.$$

Then it suffices to estimate for fixed $\beta\ge 1$ the form Λ_{D_2,K_5}, where

$$\widehat{K}_5(\xi,\eta,\tau) := \sum_{j=1}^{J}\int_{\alpha 2^{m_{j-1}}}^{\alpha 2^{m_j}} \hat{h}_{(t)}(\xi)\hat{h}_{(t)}(\eta)\,\frac{dt}{t}\,\hat{g}_{(\beta 2^{m_{j-1}})}(\tau)\hat{g}_{(\beta 2^{m_{j-1}})}(-\tau-\xi-\eta).$$

We introduce a new kernel

$$\widehat{K}_6(\xi,\eta,\tau)=\sum_{j=1}^{J}\hat{g}_{(\alpha 2^{m_j})}(\xi)\hat{g}_{(\alpha 2^{m_j})}(\eta)\int_{\beta 2^{m_{j-1}}}^{\beta 2^{m_j}}\hat{h}_{(t)}(\tau)\hat{h}_{(t)}(-\tau-\xi-\eta)\,\frac{dt}{t}.$$

The kernel K_6 is symmetric to K_5 under the symmetry (2-19). We note that, for some M, which is even in all variables and symmetric under switching the first two variables or switching the second two variables,

$$K_5(x,y,z)=\int_{\mathbb{R}}M(x+p,y+p,z+p,p)\,dp.$$

With $\widetilde{K}_5$ as defined near (2-19), we have

$$\begin{aligned}\widetilde{K}_5(x,y,z)&=\int_{\mathbb{R}}M\bigl(\tfrac{1}{2}(x+y+z)+p,\tfrac{1}{2}(x-y+z)+p,z+p,p\bigr)\,dp\\&=\int_{\mathbb{R}}M\bigl(-p,-y-p,-\tfrac{1}{2}(x-z+y)-p,-\tfrac{1}{2}(x+z+y)-p\bigr)\,dp,\end{aligned}$$

where we obtained the last identity by the substitution of p by $-p-\frac{1}{2}(x+y+z)$. For $\widetilde{K}_5^*$ as defined near (2-19), we obtain

$$\widetilde{K}_5^*(x,y,z)=\int_{\mathbb{R}}M\bigl(-p,-z-p,-\tfrac{1}{2}(x-y+z)-p,-\tfrac{1}{2}(x+y+z)-p\bigr)\,dp.$$

Using that M is an even function and that it is invariant under interchanging the first two entries or the second two entries, we obtain

$$\widetilde{K}_5^*(x,y,z)=\int_{\mathbb{R}}M\bigl(z+p,p,\tfrac{1}{2}(x+y+z)+p,\tfrac{1}{2}(x-y+z)+p\bigr)\,dp.$$

Inverting the tilde operation, we identify the kernel

$$K_6^*(x,y,z)=\int_{\mathbb{R}}M(z+p,p,x+p,y+p)\,dp.$$

Hence, the star symmetry acts on M by interchanging the first two variables with the second two variables in M.

As $\Lambda_{D_2,K_5}((f_s)_{s\in S})$ is positive by the above construction, it follows by symmetry that Λ_{D_2,K_6} is positive as well and it suffices to estimate the sum Λ_{D_2,K_5+K_6}.

We reverse the arguments leading from K_2 to K_4, with a Gaussian in place of χ_{j-1}, and apply these arguments both to K_5 and symmetrically to K_6.

In place of Λ_{D_2,K_3}, we obtain the corresponding forms Λ_{D_2,K_7} and Λ_{D_2,K_8} with

$$\begin{aligned}\widehat{K}_7(\xi,\eta,\tau)&:=\sum_{j=1}^{J}\int_{\alpha 2^{m_{j-1}}}^{\alpha 2^{m_j}}t^2(\xi+\eta)^2g(t\xi)g(t\eta)\,\frac{dt}{t}\,\hat{g}_{(\beta 2^{m_{j-1}})}(\tau)\hat{g}_{(\beta 2^{m_{j-1}})}(-\tau-\xi-\eta),\\\widehat{K}_8(\xi,\eta,\tau)&:=\sum_{j=1}^{J}\hat{g}_{(\alpha 2^{m_j})}(\xi)\hat{g}_{(\alpha 2^{m_j})}(\eta)\int_{\beta 2^{m_{j-1}}}^{\beta 2^{m_j}}t^2(\xi+\eta)^2g(t\tau)g(t(-\tau-\xi-\eta))\,\frac{dt}{t}.\end{aligned}$$

Note that to arrive at K_8, in place of symmetry arguments, we may also use in place of (8-2) the identity

$$(\xi+\eta)^2+2\tau(\tau+\xi+\eta)=\tau^2+(\tau+\xi+\eta)^2.$$

The forms Λ_{D_2,K_7} and symmetrically Λ_{D_2,K_8} are estimated analogously to Λ_{D_2,K_3} using Proposition 2.9.

Having thus reverted the above steps and having arrived at the analogue of Λ_{D_2,K_2}, we have reduced the bound of Λ_{D_2,K_5+K_6} to a bound on Λ_{D_2,K_9} with

$$\begin{aligned}
&\widehat{K}_9(\xi,\eta,\tau)\\
&\quad=\sum_{j=1}^{J}\big[\hat{g}_{(\alpha 2^{m_{j-1}})}(\xi)\hat{g}_{(\alpha 2^{m_{j-1}})}(\eta)-\hat{g}_{(\alpha 2^{m_j})}(\xi)\hat{g}_{(\alpha 2^{m_j})}(\eta)\big]\hat{g}_{(\beta 2^{m_{j-1}})}(\tau)\hat{g}_{(\beta 2^{m_{j-1}})}(-\tau-\xi-\eta)\\
&\qquad\quad+\sum_{j=1}^{J}\hat{g}_{(\alpha 2^{m_j})}(\xi)\hat{g}_{(\alpha 2^{m_j})}(\eta)\big[\hat{g}_{(\beta 2^{m_{j-1}})}(\tau)\hat{g}_{(\beta 2^{m_{j-1}})}(-\tau-\xi-\eta)-\hat{g}_{(\beta 2^{m_j})}(\tau)\hat{g}_{(\beta 2^{m_j})}(-\tau-\xi-\eta)\big]\\
&\quad=\hat{g}_{(\alpha 2^{m_0})}(\xi)\hat{g}_{(\alpha 2^{m_0})}(\eta)\hat{g}_{(\beta 2^{m_0})}(\tau)\hat{g}_{(\beta 2^{m_0})}(-\tau-\xi-\eta)\\
&\qquad\qquad-\hat{g}_{(\alpha 2^{m_J})}(\xi)\hat{g}_{(\alpha 2^{m_J})}(\eta)\hat{g}_{(\beta 2^{m_J})}(\tau)\hat{g}_{(\beta 2^{m_J})}(-\tau-\xi-\eta),
\end{aligned}$$

where in the last identity we have telescoped the sum. We then obtain

$$|\Lambda_{D_2,K_9}((f_s)_{s\in S})|\lesssim 1$$

by a standard Brascamp–Lieb inequality analogously to the bound (8-1). This completes the bound for Λ_{D_2,K_1}.

It remains to estimate $\Lambda_{D_2,K-K_1}$. We have

$$(\widehat{K}-\widehat{K}_1)(\xi,\eta,\tau)=\alpha^{-N}\sum_{j=1}^{J}(\hat{g}_{(\alpha 2^{m_j})}(\xi)\hat{g}_{(\alpha 2^{m_j})}(\eta)\psi_j(\tau,-\tau-\xi-\eta)$$

with

$$\psi_j=\phi_j\otimes\phi_j-(\chi_{j-1}\otimes\chi_{j-1}-\chi_j\otimes\chi_j).$$

Define

$$\vartheta_{1,j}=\phi_j-\chi_{j-1},\quad \vartheta_{2,j}=\chi_{j-1}-(\chi_j)_{(2^{-4})},\quad \varrho_j=\psi_j-\vartheta_{1,j}\otimes\vartheta_{2,j}-\vartheta_{2,j}\otimes\vartheta_{1,j},$$

$$\widehat{K}_{10}(\xi,\eta,\tau)=\sum_{j=1}^{J}\hat{g}_{(\alpha 2^{m_j})}(\xi)\hat{g}_{(\alpha 2^{m_j})}(\eta)\hat{\vartheta}_{1,j}(\tau)\hat{\vartheta}_{2,j}(-\tau-\xi-\eta),$$

$$\widehat{K}_{11}(\xi,\eta,\tau)=\sum_{j=1}^{J}\hat{g}_{(\alpha 2^{m_j})}(\xi)\hat{g}_{(\alpha 2^{m_j})}(\eta)\hat{\vartheta}_{2,j}(\tau)\hat{\vartheta}_{1,j}(-\tau-\xi-\eta),$$

$$\widehat{K}_{12}(\xi,\eta,\tau)=\alpha^{-N}\sum_{j=1}^{J}\hat{g}_{(\alpha 2^{m_j})}(\xi)\hat{g}_{(\alpha 2^{m_j})}(\eta)\hat{\varrho}_j(\tau,-\tau-\xi-\eta).$$

By the triangle inequality, it remains to estimate $\Lambda_{D_2,K_{10}}$, $\Lambda_{D_2,K_{11}}$, $\Lambda_{D_2,K_{12}}$, separately.

We begin with $\Lambda_{D_2,K_{10}}$. Recall that $(\chi_j)_{(2^{2-k_j})}$ is a left window. If $(\tau, -\tau-\xi-\eta)$ is in the support of $\hat{\vartheta}_{1,j}\otimes\hat{\vartheta}_{2,j}$, then

$$|\tau|\le 2^{-k_j+2},\quad 2^{-k_j+5}\le|\tau+\xi+\eta|\le 2^{-k_{j-1}+2},\quad 2^{-k_j+4}<|\xi+\eta|<2^{-k_{j-1}+3}.$$

Defining

$$\vartheta_{3,j}:=(\phi_j)_{(2^{-2})}$$

we have

$$\widehat{K}_{10}(\xi,\eta,\tau)=\sum_{j=1}^{J}\hat{g}_{(\alpha 2^{m_j})}(\xi)\hat{g}_{(\alpha 2^{m_j})}(\eta)\hat{\vartheta}_{1,j}(\tau)\hat{\vartheta}_{2,j}(-\tau-\xi-\eta)\hat{\vartheta}_{3,j}(\xi+\eta)^2$$

because the additional factor involving $\hat{\vartheta}_{3,j}$ is constant 1 on the support of the original summand in the definition of K_{10}. The bound

$$|\Lambda_{D_2,K_{10}}((f_s)_{s\in S})|\lesssim 1$$

then follows from Proposition 2.8 applied with

$$\rho_j:=\hat{g}_{(\alpha 2^{m_j})}\otimes\hat{g}_{(\alpha 2^{m_j})}\otimes\hat{\vartheta}_{1,j}\otimes\hat{\vartheta}_{2,j}.$$

The form $\Lambda_{D_2,K_{11}}$ is estimated analogously to the form $\Lambda_{D_2,K_{10}}$. It remains to estimate $\Lambda_{D_2,K_{12}}$. This form is a more standard singular Brascamp–Lieb form with a kernel associated with a Hörmander–Mikhlin multiplier and we will apply Theorem 1.1 in [Durcik et al. 2022], which was the reason to set $N=2^{18}$.

That theorem will give

$$|\Lambda_{D_2,K_{12}}((f_s)_{s\in S})|\lesssim 1$$

provided

$$|\partial^\gamma\widehat{K}_{12}(\xi,\eta,\tau)|\lesssim|(\xi,\eta,\tau)|^{-|\gamma|} \tag{8-3}$$

for all multiindices γ of order $0\le|\gamma|\le N$. The assumption of that theorem that $\Pi_s\Pi^T$ is regular for the present datum D_2 is satisfied. It thus remains to show (8-3).

By definition of ψ_j and $\vartheta_{1,j}$, we obtain

$$\psi_j=\chi_{j-1}\otimes\vartheta_{1,j}+\vartheta_{1,j}\otimes\chi_{j-1}+\vartheta_{1,j}\otimes\vartheta_{1,j}+\chi_j\otimes\chi_j.$$

Using further the definition of ϱ_j and $\vartheta_{2,j}$, we obtain

$$\varrho_j=(\chi_j)_{(2^{-4})}\otimes\vartheta_{1,j}+\vartheta_{1,j}\otimes(\chi_j)_{(2^{-4})}+\vartheta_{1,j}\otimes\vartheta_{1,j}+\chi_j\otimes\chi_j. \tag{8-4}$$

Note that $\hat{\vartheta}_{1,j}$ vanishes outside

$$[-2^{-k_j+2},2^{-k_j+2}].$$

Hence $\hat{\varrho}_j$ is supported on the ball of radius 2^{10-k_j} around the origin. In addition, $\hat{\vartheta}_{1,j}$ coincides with -1 on $[-2^{-k_j+1},2^{-k_j+1}]$. Using that $(\chi_j)_{(2^{2-k_j})}$ is a left window, we then see that the Fourier transform of the first two terms on the right-hand side of (8-4) is equal to -1 on $[-2^{-k_j+1},2^{-k_j+1}]^2$ while the Fourier transform of the last two terms coincides with 1 on the same set. Therefore, $\hat{\varrho}_j$ vanishes inside the ball of radius 2^{-k_j} around the origin. The support properties of $\hat{\varrho}_j$ together with the estimates $|\hat{\varrho}_j| \lesssim 1$ and $g \lesssim 1$ yield that $|\widehat{K}_{12}| \lesssim 1$.

Assume next that β is a multiindex with $1 \le |\beta| \le N$. Then $\hat{\varrho}_j$ satisfies symbol estimates adapted to the ball of radius 2^{11-k_j} around the origin, namely

$$|\partial^\beta \hat{\varrho}_j(\tau, \sigma)| \lesssim 2^{k_j|\beta|} 1_{|(\tau,\sigma)| \le 2^{11-k_j}}.$$

Now assume first $|\xi - \eta| \le |(\xi + \eta, \tau)|$. Then, using that all derivatives of g up to order N are $\lesssim 1$, and using that $|m_j - k_j| \le 1$ and $\alpha \ge 1$,

$$|\partial^\beta \widehat{K}_{12}(\xi, \eta, \tau)| \lesssim \alpha^{-N} \sum_{j=1}^{J} (\alpha 2^{k_j})^{|\beta|} 1_{|(\tau, \tau+\xi+\eta)| \le 2^{11-k_j}}.$$

Using further that $\alpha \ge 1$ and $|\beta| \le N$ we estimate the last display by

$$\lesssim |(\tau, \xi + \eta)|^{-|\beta|} \lesssim |(\tau, \xi, \eta)|^{-|\beta|},$$

where in the last inequality we have used $|\xi - \eta| \le |(\tau, \xi + \eta)|$. Now assume to the contrary that $|\xi - \eta| \ge |(\xi + \eta, \tau)|$. Then we use that $|\partial^\beta g(\xi)| \lesssim e^{-|\xi|}$ for all $|\beta| \le N$. Then

$$\begin{aligned} |\partial^\beta \widehat{K}_{12}(\xi, \eta, \tau)| \lesssim \alpha^{-N} \sum_{j=1}^{J} (\alpha 2^{k_j})^{|\beta|} e^{-\alpha 2^{k_j} |\xi-\eta|} &\lesssim |\xi - \eta|^{-|\beta|} \sum_{j=1}^{J} (2^{k_j} |\xi - \eta|)^{|\beta|} e^{-2^{k_j}|\xi-\eta|} \\ &\lesssim |\xi - \eta|^{-|\beta|} \sum_{n \in \mathbb{Z}} 2^{n|\beta|} e^{-2^n} \lesssim |(\xi, \eta, \tau)|^{-|\beta|}. \end{aligned}$$

9. Proof of Propositions 2.8 and 2.9

The proofs of these propositions have some similarities, so we put them into one section and do the second proof analogously to the first.

9.1. ***Proof of Proposition 2.8.*** For $1 \le i \le 2$ let $(a_{i,j})_{j=1}^J$ be increasing sequences of positive real numbers, we choose $a_{i,0} > 0$ so that $(a_{i,j})_{j=0}^J$ is still increasing. For $1 \le j \le J$ let $\rho_j : \mathbb{R}^4 \to \mathbb{R}$ be a continuous function satisfying (2-14), and pick a further such function $\rho_0 : \mathbb{R}^4 \to \mathbb{R}$. Let $(c_j)_{j=0}^J$ be a well separated increasing sequence of positive numbers. Let χ be a left window, and let ϕ_j be a function on $\mathbb{R}$ which for $1 \le j \le J$ satisfy $\widehat{\phi}_j \ge 0$ and

$$(\widehat{\phi}_j)^2 = (\widehat{\chi}_{(c_{j-1})})^2 - (\widehat{\chi}_{(c_j)})^2.$$

Let K be defined by (2-15) and let a tuple $(f_s)_{s\in S}$ be given as in (2-12), (2-13).

The integrand of the integral expressing $\Lambda_{D_2,K}((f_s)_{s\in S})$ factors into functions depending on x_0 and functions depending on x_1. We write the integrals in x_0 and x_1 innermost and separate these. With $p_0 := p$

and $p_1 := q$ we obtain

$$\Lambda_{D_2,K}((f_s)_{s\in S}) = \sum_{j=1}^{J} \int_{\mathbb{R}^7} \Big[\prod_{i=0,1} \int_{\mathbb{R}} \Big[\prod_{q\in\mathcal{C}} f_{(i,q)}(\Pi_{(i,q)}x) \Big] \phi_j(x_i+p_i)\, dx_i \Big] \\ \times \rho_j\big(x_2^0+p_0+p_1+r,\, x_2^1+p_0+p_1+r,\, x_3^0+r,\, x_3^1+r\big)\, dx_2^0\, dx_2^1\, dx_3^0\, dx_3^1\, dp_0\, dp_1\, dr. \tag{9-1}$$

Applying the Cauchy–Schwarz inequality in the seven exterior variables bounds the last display by the geometric mean of two forms, parametrized by $i = 0, 1$, which with the change of variables $p_{1-i} \to p_{1-i} - p_i - r$ we write as

$$\sum_{j=1}^{J} \int_{\mathbb{R}^7} \Big[\int_{\mathbb{R}} \Big[\prod_{q\in\mathcal{C}} f_{(i,q)}(\Pi_{(i,q)}x) \Big] \phi_j(x_i+p_i)\, dx_i \Big]^2 \\ \times |\rho_j|\big(x_2^0+p_{1-i},\, x_2^1+p_{1-i},\, x_3^0+r,\, x_3^1+r\big)\, dx_2^0\, dx_2^1\, dx_3^0\, dx_3^1\, dp_0\, dp_1\, dr. \tag{9-2}$$

Fix i and write f for $f_{(i,j)}$, which thanks to (2-12) does not depend on j.

Using the decay (2-14) for ρ_j, we dominate

$$\int_{\mathbb{R}^2} |\rho_j|\big(x_2^0+p_{1-i},\, x_2^1+p_{1-i},\, x_3^0+r,\, x_3^1+r\big)\, dp_{1-i}\, dr \\ \lesssim \int_1^\infty \int_1^\infty (g*g)_{(\alpha a_{1,j})}(x_2^0-x_2^1)(g*g)_{(\beta a_{2,j})}(x_3^0-x_3^1)\, \frac{d\alpha}{\alpha^2}\, \frac{d\beta}{\beta^2}. \tag{9-3}$$

It suffices to consider fixed α and β, and prove uniform bounds in α and β for (9-2) with (9-3) replaced by

$$(g*g)_{(\alpha a_{1,j})}(x_2^0-x_2^1)(g*g)_{(\beta a_{2,j})}(x_3^0-x_3^1).$$

Modifying the sequences $a_{i,j}$ if necessary, we may assume $\alpha = \beta = 1$.

Expanding the square in (9-2) and integrating in p_i, our task becomes to show

$$\sum_{j=1}^{J} \int_{\mathbb{R}^6} \Big[\prod_{s\in S} f(\Pi_s x) \Big] (\phi_j*\phi_j)(x_1^0-x_1^1)(g*g)_{a_{1,j}}(x_2^0-x_2^1)(g*g)_{a_{2,j}}(x_3^0-x_3^1)\, dx \lesssim 1, \tag{9-4}$$

where S and $(\Pi_s)_{s\in S}$ are as in the datum D_{-I}, which is the datum defined in (2-20) in the case $A = -I$.

Define the kernels

$$K_1 := \sum_{j=1}^{J} \big((\chi*\chi)_{(c_{j-1})} - (\chi*\chi)_{(c_j)}\big) \otimes (g*g)_{(a_{1,j})} \otimes (g*g)_{(a_{2,j})},$$

$$K_2 := \sum_{j=1}^{J} (\chi*\chi)_{(c_{j-1})} \otimes \big((g*g)_{(a_{1,j-1})} - (g*g)_{(a_{1,j})}\big) \otimes (g*g)_{(a_{2,j})},$$

$$K_3 := \sum_{j=1}^{J} (\chi*\chi)_{(c_{j-1})} \otimes (g*g)_{(a_{1,j-1})} \otimes \big((g*g)_{(a_{2,j-1})} - (g*g)_{(a_{2,j})}\big),$$

and for $0 \le j \le J$ also

$$\sigma_j = (\chi*\chi)_{(c_j)} \otimes (g*g)_{(a_{1,j})} \otimes (g*g)_{(a_{2,j})}.$$

We have the telescoping identity

$$K_1 + K_2 + K_3 = \sigma_0 - \sigma_J. \tag{9-5}$$

The form (9-4) to be estimated becomes $\Lambda_{D_{-I},K_1}((f)_{s\in S})$. For each $0 \le j \le J$ one has by a standard Brascamp–Lieb inequality

$$|\Lambda_{D_{-I},\sigma_j}((f)_{s\in S})| \lesssim 1.$$

It then suffices to estimate the forms associated with K_2 and K_3 instead. By symmetry, we will only elaborate on $\Lambda_{D_{-I},K_2}((f)_{s\in S})$.

Next we would like to dominate $|(\chi * \chi)_{(c_{j-1})}|$ in these two forms by superposition of Gaussians in such a way that the cancellation is preserved. To do that, we will use the identity

$$(g*g)_{(a_{1,j-1})} - (g*g)_{(a_{1,j})} = -\frac{1}{\pi}\int_{a_{1,j-1}}^{a_{1,j}} (h*h)_{(t)}\,\frac{dt}{t}, \tag{9-6}$$

which follows by taking the Fourier transform of the identity

$$g(a\xi)^2 - g(b\xi)^2 = -\int_a^b \partial_t g(t\xi)^2\,dt = \frac{1}{\pi}\int_a^b (2\pi t\xi g(t\xi))^2\,\frac{dt}{t} = -\frac{1}{\pi}\int_a^b (\hat{h}(t\xi))^2\,\frac{dt}{t}$$

for any $a, b > 0$. Using further that h is odd and thus

$$-h*h(x-y) = \int_{\mathbb{R}} h(x+p)h(y+p)\,dp,$$

we obtain

$$\Lambda_{D_{-I},K_2}((f)_{s\in S}) = \frac{1}{\pi}\sum_{j=1}^{J}\int_{a_{1,j-1}}^{a_{1,j}}\int_{\mathbb{R}^5}\biggl[\prod_{i=0,1}\int_{\mathbb{R}}\biggl[\prod_{s(1)=i} f(\Pi_s x)\biggr]h_{(t)}(x_2^i+p)\,dx_2^i\biggr]$$
$$\times (\chi*\chi)_{(c_{j-1})}(x_1^0 - x_1^1)(g*g)_{(a_{2,j})}(x_3^0 - x_3^1)\,dx_1^0\,dx_1^1\,dx_3^0\,dx_3^1\,dp\,\frac{dt}{t}. \tag{9-7}$$

The product over $i = 0, 1$ has two identical factors and thus is nonnegative. We may therefore estimate the last display by dominating

$$|(\chi*\chi)_{(c_{j-1})}| \lesssim \int_1^\infty (g*g)_{(\beta c_{j-1})}\beta^{-5}\,d\beta.$$

It suffices to prove bounds of (9-7) with $(\chi*\chi)_{(c_{j-1})}$ replaced by $(g*g)_{(\beta c_{j-1})}$ uniformly in β. Fix β. By changing c_j if necessary, we may assume $\beta = 1$. Define again kernels

$$K_4 := \sum_{j=1}^{J}\bigl((g*g)_{(c_{j-1})} - (g*g)_{(c_j)}\bigr)\otimes(g*g)_{(a_{1,j})}\otimes(g*g)_{(a_{2,j})},$$

$$K_5 := \sum_{j=1}^{J}(g*g)_{(c_{j-1})}\otimes\bigl((g*g)_{(a_{1,j-1})} - (g*g)_{(a_{1,j})}\bigr)\otimes(g*g)_{(a_{2,j})},$$

$$K_6 := \sum_{j=1}^{J}(g*g)_{(c_{j-1})}\otimes(g*g)_{(a_{1,j-1})}\otimes\bigl((g*g)_{(a_{2,j-1})} - (g*g)_{(a_{2,j})}\bigr).$$

Similarly as near (9-5),

$$\Lambda_{D_{-I},K_4}((f)_{s\in S}) + \Lambda_{D_{-I},K_5}((f)_{s\in S}) + \Lambda_{D_{-I},K_6}((f)_{s\in S}) \tag{9-8}$$

telescopes into a form that is $\lesssim 1$ by a standard Brascamp–Lieb inequality. We have seen above that $\Lambda_{D_{-I},K_5}((f)_{s\in S})$ is positive. By symmetric arguments, the other summands in (9-8) are also positive. Hence each summand is $\lesssim 1$. This completes the proof of Proposition 2.8.

9.2. ***Proof of Proposition 2.9.*** Let a positive integer J be given as well as increasing sequences of positive real numbers $(a_j)_{j=0}^J$, $(b_j)_{j=1}^J$. Pick $b_0 > 0$ so that $(b_j)_{j=0}^J$ is an increasing sequence. For $1 \le j \le J$ let ϕ_j be given as in (2-17). Let K be defined by (2-18). Let a tuple $(f_s)_{s\in S}$ be given as in (2-12), (2-13).

We write

$$\begin{aligned} t^2(\xi+\eta)^2 g(t\xi)g(t\eta) &= t^2(\xi+\eta)^2 g(2^{-3/2}t(\xi+\eta))^2 g(2^{-1}t(\xi-\eta))g(2^{-1/2}t\xi)g(2^{-1/2}t\eta) \\ &= -\hat{h}(2^{-3/2}t(\xi+\eta))^2\hat{\rho}(2^{-3/2}t(\xi,\eta)) \end{aligned}$$

with

$$\hat{\rho}(2^{-3/2}(\xi,\eta)) := \frac{2}{\pi^2} g(2^{-1}(\xi-\eta))g(2^{-1/2}\xi)g(2^{-1/2}\eta).$$

Hence passing to the spatial side as near (2-16), replacing the arbitrary sequence a_j by $2^{-3/2}a_j$ to avoid the cumbersome factors $2^{-3/2}$,

$$K(u,v,z) = \sum_{j=1}^J \int_{\mathbb{R}^3}\int_{a_{j-1}}^{a_j} h_{(t)}(p)h_{(t)}(q)\rho_{(t)}(u+p+q+r, v+p+q+r)\,\frac{dt}{t}\phi_j(z+r,r)\,dp\,dq\,dr.$$

We thus have analogously to (9-1),

$$\begin{aligned} \Lambda_{D_2,K}((f_s)_{s\in S}) = \sum_{j=1}^J \int_{\mathbb{R}^7}\int_{a_{j-1}}^{a_j} \Big[\prod_{i=0,1}\int_{\mathbb{R}}\Big[\prod_{q\in\mathcal{C}} f_{(i,q)}(\Pi_{(i,q)}x)\Big]h_{(t)}(x_i+p_i)\,dx_i\Big] \\ \times \rho_{(t)}(x_2^0+p_0+p_1+r, x_2^1+p_0+p_1+r)\phi_j(x_3^0+r, x_3^1+r)\,\frac{dt}{t}\,dx_2^0\,dx_2^1\,dx_3^0\,dx_3^1\,dp_0\,dp_1\,dr. \end{aligned}$$

Applying the Cauchy–Schwarz inequality as in (9-2), we need to estimate for $i = 0, 1$,

$$\begin{aligned} \sum_{j=1}^J \int_{\mathbb{R}^7}\int_{a_{j-1}}^{a_j} \Big[\int_{\mathbb{R}}\Big[\prod_{q\in\mathcal{C}} f_{(i,q)}(\Pi_{(i,q)}x)\Big]h_{(t)}(x_i+p_i)\,dx_i\Big]^2 \\ \times |\rho_{(t)}|(x_2^0+p_{1-i}, x_2^1+p_{1-i})|\phi_j|(x_3^0+r, x_3^1+r)\,\frac{dt}{t}\,dx_2^0\,dx_2^1\,dx_3^0\,dx_3^1\,dp_0\,dp_1\,dr. \end{aligned}$$

Thanks to the square, the above integrand is positive and we dominate

$$|\rho_{(t)}| \lesssim g_{(4t)}\otimes g_{(4t)} \quad\text{and}\quad |\phi_j| \lesssim \int_1^\infty g_{(\beta b_j)}\otimes g_{(\beta b_j)}\beta^{-3}\,d\beta.$$

It suffices to prove bounds with $g_{(\beta b_j)}\otimes g_{(\beta b_j)}$ in place of $|\phi_j|$ uniformly in β. Fix β; we may assume $\beta = 1$ by modifying the otherwise arbitrary sequence b_j.

Performing the analogous steps as leading to (9-4) we end up having to estimate $\Lambda_{D_{-I},K_1}((f)_{s\in S})$, where now

$$K_1 := -\sum_{j=1}^J \int_{a_{j-1}}^{a_j} (h*h)_{(t)}\otimes(g*g)_{(4t)}\,\frac{dt}{t}\otimes(g*g)_{(b_j)}.$$

Define

$$K_2 := -\sum_{j=1}^{J} \int_{a_{j-1}}^{a_j} (g*g)_{(t)} \otimes (h*h)_{(4t)} \, \frac{dt}{t} \otimes (g*g)_{(b_j)},$$

$$K_3 := -\sum_{j=1}^{J} (g*g)_{(a_{j-1})} \otimes (g*g)_{(4a_{j-1})} \otimes \int_{b_{j-1}}^{b_j} (h*h)_{(t)} \, \frac{dt}{t},$$

and for $0 \le j \le J$ also

$$\sigma_j = (g*g)_{(a_j)} \otimes (g*g)_{(4a_j)} \otimes (g*g)_{(b_j)}.$$

Then we have the telescoping identity

$$K_1 + K_2 + K_3 = \pi(\sigma_0 - \sigma_J). \tag{9-9}$$

Indeed, this follows with (9-6), which gives

$$K_1 + K_2 = \pi \sum_{j=1}^{J} \int_{a_{j-1}}^{a_j} -t\partial_t\big((g*g)_{(t)} \otimes (g*g)_{(4t)}\big) \frac{dt}{t} \otimes (g*g)_{(b_j)}$$

$$= \pi \sum_{j=1}^{J} \big((g*g)_{(a_{j-1})} \otimes (g*g)_{(4a_{j-1})} - (g*g)_{(a_j)} \otimes (g*g)_{(4a_j)}\big) \otimes (g*g)_{(b_j)},$$

$$K_3 = \pi \sum_{j=1}^{J} (g*g)_{(a_{j-1})} \otimes (g*g)_{(4a_{j-1})} \otimes \big((g*g)_{(b_{j-1})} - (g*g)_{(b_j)}\big).$$

By the identity (9-9),

$$\Lambda_{D_{-I},K_1}((f)_{s\in S}) + \Lambda_{D_{-I},K_2}((f)_{s\in S}) + \Lambda_{D_{-I},K_3}((f)_{s\in S}) \lesssim 1.$$

All quantities on the left-hand side are nonnegative. For $\Lambda_{D_{-I},K_1}((f)_{s\in S})$, this can be seen as it resulted after an application of the Cauchy–Schwarz inequality, while for $\Lambda_{D_{-I},K_2}((f)_{s\in S})$ and $\Lambda_{D_{-I},K_3}((f)_{s\in S})$ it follows by symmetry. This gives the desired upper bound

$$\Lambda_{D_{-I},K_1}((f)_{s\in S}) \lesssim 1.$$

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COMPACTNESS RESULTS FOR SIGN-CHANGING SOLUTIONS OF CRITICAL NONLINEAR ELLIPTIC EQUATIONS OF LOW ENERGY

HUSSEIN CHEIKH ALI AND BRUNO PREMOSELLI

Let Ω be a bounded, smooth connected open domain in $\mathbb{R}^n$ with $n \geq 3$. We investigate compactness properties for the set of sign-changing solutions $v \in H_0^1(\Omega)$ of

$$\begin{cases} -\Delta v + hv = |v|^{2^*-2}v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $h \in C^1(\overline{\Omega})$ and $2^* := 2n/(n-2)$. Our main result establishes that the set of *sign-changing* solutions of the above system at the lowest sign-changing energy level is unconditionally compact in $C^2(\overline{\Omega})$ when $3 \leq n \leq 5$, and is compact in $C^2(\overline{\Omega})$ when $n \geq 7$ provided h never vanishes in $\overline{\Omega}$. In dimensions $n \geq 7$ our results apply when $h > 0$ in $\overline{\Omega}$ and thus complement the compactness result of Devillanova and Solimini (2002). Our proof is based on a new, global pointwise description of blowing-up sequences of solutions of the above system that holds up to the boundary. We also prove more general compactness results under perturbations of h.

1. Introduction

1.1. ***Statement of the results.*** Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded connected open set in $\mathbb{R}^n$, $n \geq 3$, $h \in C^1(\overline{\Omega})$ and $2^* := 2n/(n-2)$. We investigate solutions $v \in H_0^1(\Omega)$ of

$$\begin{cases} -\Delta v + hv = |v|^{2^*-2}v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1-1}$$

Here and in the sequel, we let $\|\cdot\|_p$ be the usual norm of $L^p(\Omega)$ for $1 \leq p \leq \infty$, and $H_0^1(\Omega)$ be the completion of $C_c^\infty(\Omega)$ with respect to the norm

$$\|v\|_{H_0^1}^2 := \int_\Omega |\nabla v|^2 \, dx.$$

For simplicity we will assume throughout this paper that $-\Delta + h$ is coercive, that is, that there exists $C > 0$ such that

$$\int_\Omega (|\nabla v|^2 + hv^2)\, dx \geq C \int_\Omega |\nabla v|^2 \, dx \quad \text{for all } v \in H_0^1(\Omega).$$

Under this assumption, the existence of positive solutions of (1-1) is very well understood. We let

$$I_h(\Omega) := \inf_{v \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega (|\nabla v|^2 + hv^2)\, dx}{\left(\int_\Omega |v|^{2^*} dx\right)^{2/2^*}}. \tag{1-2}$$

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Brézis and Nirenberg [1983] proved that, when $n \geq 4$, positive ground states attaining (1-2) exist if and only if $h < 0$ somewhere in Ω. When $n = 3$, Druet [2002] proved that positive ground states attaining (1-2) exist if only if $m_h > 0$ somewhere in Ω, where m_h is the so-called mass function of the operator $-\Delta + h$. This function is defined as follows: let G_h be the Green's function for $-\Delta + h$ with Dirichlet boundary conditions in Ω. Then, when $n = 3$, we have

$$G_h(x, y) = \frac{1}{4\pi |x - y|} + g_h(x, y) \quad \text{for all } y \in \Omega \backslash \{x\}$$

for some $g_h \in C^{0,1}(\bar{\Omega} \times \bar{\Omega})$, and we define $m_h(x) = g_h(x, x)$. Under these assumptions, [Brézis and Nirenberg 1983; Druet 2002] also prove that we have $I_h(\Omega) < K_n^{-2}$, where

$$K_n^{-2} := \inf_{v \in C_c^\infty(\mathbb{R}^n)\backslash\{0\}} \frac{\int_{\mathbb{R}^n} |\nabla v|^2 \, dx}{\left(\int_{\mathbb{R}^n} |v|^{2^*} \, dx\right)^{2/2^*}} \tag{1-3}$$

is the optimal constant in Sobolev's inequality in $\mathbb{R}^n$. An explicit expression of K_n can be found in [Aubin 1976; Talenti 1976]. It is simple to see that if $v \in H_0^1(\Omega)$ attains $I_h(\Omega)$ and is normalised to satisfy (1-1) then

$$\int_\Omega |v|^{2^*} \, dx = I_h(\Omega)^{n/2} < K_n^{-n}. \tag{1-4}$$

The existence of sign-changing solutions for problem (1-1) has also attracted a lot of attention. Existence results for a general function $h \in C^1(\bar{\Omega})$ are in [Bartsch and Weth 2003]. When $h \equiv -\lambda$, for $\lambda \in (0, \lambda_1)$, equation (1-1) is the so-called Brézis–Nirenberg problem

$$\begin{cases} -\Delta v - \lambda v = |v|^{2^*-2} v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1-5}$$

for which existence results have been obtained in [Cerami et al. 1984; Capozzi et al. 1985; Fortunato and Jannelli 1987; Solimini 1995; Devillanova and Solimini 2002; Clapp and Weth 2004; Schechter and Zou 2010]. The existence of a sign-changing solution of least-energy (among all sign-changing solutions) for (1-5) when $\lambda \in (0, \lambda_1)$ — the range in which $-\Delta - \lambda$ is coercive — was proven in [Cerami et al. 1986] when $n \geq 6$ (see also [Chen and Zou 2015] for a new proof) while it was proven in [Roselli and Willem 2009; Tavares et al. 2022] when $n = 4, 5$. The existence of least-energy sign-changing solutions for (1-5) is not yet known when $n = 3$.

In this paper we focus on compactness properties for solutions of (1-1). We let $(h_\alpha)_{\alpha \in \mathbb{N}}$ be a sequence of C^1 functions that converge to h in $C^1(\bar{\Omega})$, and we let $(v_\alpha)_{\alpha \in \mathbb{N}}$ be a sequence of solutions in $H_0^1(\Omega)$ of

$$\begin{cases} -\Delta v_\alpha + h_\alpha v_\alpha = |v_\alpha|^{2^*-2} v_\alpha & \text{in } \Omega, \\ v_\alpha = 0 & \text{on } \partial\Omega \end{cases} \tag{1-6}$$

satisfying $\limsup_{\alpha \to +\infty} \|v_\alpha\|_{H_0^1} < +\infty$. We will say that $(v_\alpha)_\alpha$ is *sign-changing* if $(v_\alpha)_+ = \max(v_\alpha, 0)$ and $(v_\alpha)_- = -\min(v_\alpha, 0)$ are both nonzero for any α. We investigate under which assumptions on h the sequence $(v_\alpha)_{\alpha \in \mathbb{N}}$ converges in a strong topology. Our main result answers this question when $(v_\alpha)_{\alpha \in \mathbb{N}}$ has minimal energy:

Theorem 1.1. *Let Ω be a smooth bounded connected domain of $\mathbb{R}^n$, $n \geq 3$, and $(h_\alpha)_{\alpha\in\mathbb{N}}$ be a sequence that converges in $C^1(\overline{\Omega})$ towards h. Assume that $-\Delta + h$ is coercive and that $I_h(\Omega) < K_n^{-2}$. Let $(v_\alpha)_{\alpha\in\mathbb{N}} \in H_0^1(\Omega)$ be a sequence of solutions of* (1-6) *such that*

$$\limsup_{\alpha\to+\infty} \int_\Omega |v_\alpha|^{2^*}\,dx \leq K_n^{-n} + I_h(\Omega)^{n/2}, \tag{1-7}$$

and assume that either

- *$n \in \{3, 4, 5\}$ and, for all $\alpha \geq 0$, v_α is sign-changing, or*
- *$n \geq 7$ and $h \neq 0$ at every point in $\overline{\Omega}$.*

Then, up to a subsequence, $(v_\alpha)_{\alpha\in\mathbb{N}}$ strongly converges in $C^2(\overline{\Omega})$ to a nonzero solution of (1-1).

Recall that $I_h(\Omega)$ is defined in (1-2). In the particular case where $h_\alpha \equiv h$, Theorem 1.1 implies the following compactness result for solutions of (1-1):

Corollary 1.2. *Let Ω be a smooth bounded connected domain of $\mathbb{R}^n$, $n \geq 3$, and let $h \in C^1(\overline{\Omega})$ be such that $-\Delta + h$ is coercive and $I_h(\Omega) < K_n^{-2}$.*

- *Assume that $n \in \{3, 4, 5\}$. There exists $\varepsilon = \varepsilon(n, \Omega) > 0$ such that the set of* ***sign-changing*** *solutions v of* (1-1) *satisfying*
$$\int_\Omega |v|^{2^*}\,dx \leq K_n^{-n} + I_h(\Omega)^{n/2} + \varepsilon$$
is precompact in the $C^2(\overline{\Omega})$-topology.
- *Assume that $n \geq 7$ and $h \neq 0$ in $\overline{\Omega}$. There exists $\varepsilon = \varepsilon(n, h, \Omega) > 0$ such that the set of solutions v of* (1-1) *satisfying*
$$\int_\Omega |v|^{2^*}\,dx \leq K_n^{-n} + I_h(\Omega)^{n/2} + \varepsilon$$
is precompact in the $C^2(\overline{\Omega})$-topology.

The energy bound (1-7) is very natural when investigating sign-changing solutions of (1-1). Solutions of (1-6) satisfying (1-7) exist: the least-energy sign-changing solutions of (1-5) constructed in [Cerami et al. 1986; Tavares et al. 2022], for instance, satisfy

$$\int_\Omega |v|^{2^*}\,dx < K_n^{-n} + I_{-\lambda}(\Omega)^{n/2}.$$

A simple application of the celebrated compactness result of Struwe [1984] (see also [Cerami et al. 1986, Lemma 3.1]) shows that if a sequence $(v_\alpha)_{\alpha\in\mathbb{N}}$ of solutions of (1-6) *changes sign* and satisfies $\lim_{\alpha\to+\infty} \|v_\alpha\|_\infty = +\infty$ (we will say in this case that $(v_\alpha)_{\alpha\in\mathbb{N}}$ *blows up*), then

$$\int_\Omega |v_\alpha|^{2^*}\,dx \geq K_n^{-n} + I_h(\Omega)^{n/2} + o(1)$$

as $\alpha \to +\infty$. The threshold $K_n^{-n} + I_h(\Omega)^{n/2}$ is therefore the direct counterpart, for sign-changing solutions, of the minimal energy threshold K_n^{-n} that ensures the existence of positive ground state solutions in (1-4). In this respect, Theorem 1.1 and Corollary 1.2 have to be understood as the first compactness result for (1-6), at the lowest energy-level for sign-changing blow-up, when $I_h(\Omega)$ is attained.

Theorem 1.1 shows that, when $3 \le n \le 5$, *sign-changing* solutions are unconditionally compact in $C^2(\overline{\Omega})$ under assumption (1-7). By contrast, without further assumptions on h, the set of *positive* solutions satisfying (1-7) is not compact in general when $3 \le n \le 5$. For equation (1-5), for instance, families of positive solutions whose energy converges to K_n^{-n} and which are not compact in $C^2(\overline{\Omega})$ have been constructed in [Musso and Pistoia 2002; Rey 1990] when $n \ge 4$ and $\lambda \to 0+$, and in [del Pino et al. 2004] when $n = 3$ and $\lambda \to \lambda_*$ from above, where λ_* satisfies $\max_\Omega m_{\lambda_*} = 0$. When $3 \le n \le 5$, Theorem 1.1 is therefore unexpected since sign-changing solutions of equations like (1-6) are known to exhibit a much richer and more erratic behaviour than positive ones. When $n \ge 7$, Theorem 1.1 applies to positive and sign-changing sequences of solutions $(v_\alpha)_{\alpha\in\mathbb{N}}$ and Corollary 1.2 generalises the well-known compactness theorem for energy-bounded solutions of (1-5) proven in [Devillanova and Solimini 2002]. It is still an open question to know whether Theorem 1.1 holds for any energy-bounded sequence $(v_\alpha)_{\alpha\in\mathbb{N}}$ without the assumption (1-7) when $n \ge 7$ and $h \ne 0$ in $\overline{\Omega}$.

Dimension 6 is excluded from Theorem 1.1. In this case we prove:

Proposition 1.3. *Let Ω be a smooth bounded domain of $\mathbb{R}^6$ and $(h_\alpha)_{\alpha\in\mathbb{N}}$ be a sequence that converges in $C^1(\overline{\Omega})$ towards h. Assume that $-\Delta + h$ is coercive and that $I_h(\Omega) < K_6^{-2}$. Let $(v_\alpha)_{\alpha\in\mathbb{N}} \in H_0^1(\Omega)$ be any sequence of solutions of* (1-6) *satisfying* (1-7), *and assume that $\|v_\alpha\|_\infty \to +\infty$ as $\alpha \to +\infty$. Then there exists $v_\infty \in H_0^1(\Omega)$, $v_\infty > 0$ in Ω, attaining $I_h(\Omega)$ such that v_α converges weakly but not strongly to $\pm v_\infty$ in $H_0^1(\Omega)$ and there exists $x_\infty \in \Omega$ such that*

$$h(x_\infty) = \pm 2 v_\infty(x_\infty).$$

Compactness of *sign-changing* solutions of (1-6) satisfying (1-7) does not hold when $n = 6$: in [Pistoia and Vaira 2022], for instance, the authors constructed a noncompact family $(v_\lambda)_\lambda$ of sign-changing solutions of (1-5) which blows up as λ converges to some $\lambda_0 > 0$ that satisfies $\lambda_0 = 2\|v_0\|_\infty$, where v_0 attains $I_{-\lambda_0}(\Omega)$ (the existence of such (λ_0, v_0) is also proven in that work). This six-dimensional phenomenon has been known for a while for positive solutions; see [Druet 2004], where it was first highlighted.

1.2. ***Strategy of proof and outline of the paper.*** For *positive* solutions there is a vast literature addressing the issue of compactness of equations like (1-6) through blow-up analysis. On open sets of $\mathbb{R}^n$ with Dirichlet boundary conditions we mention for instance [Druet 2002; Druet and Laurain 2010; König and Laurain 2022; 2024] for (1-1), [Druet et al. 2012] for Lin–Ni-type problems with Neumann boundary conditions and [Ghoussoub et al. 2023] for singular Hardy–Sobolev-type problems. On closed manifolds we mention [Druet 2003] for compactness of energy-bounded solutions and the series of works related to the compactness of the Yamabe equation: [Li and Zhu 1999; Druet 2003; Marques 2005; Khuri et al. 2009]; see also [Hebey 2014]. On manifolds with boundary we refer to [Mesmar and Robert 2024]. For *sign-changing* solutions of critical elliptic equations on open sets of $\mathbb{R}^n$ the only compactness result available is [Devillanova and Solimini 2002] when $n \ge 7$; this result was generalised on closed manifolds in [Vétois 2007]. In lower dimensions, compactness results on closed manifolds have been obtained more recently: we refer for instance to [Premoselli and Vétois 2019; 2022a; 2022b; 2024; Premoselli and Robert 2025]. Concerning problem (1-5) in particular, there is a vast literature on the construction and the behaviour of blowing-up solutions: we mention for instance [Ben Ayed et al. 2006a; 2006b; Druet 2002; Druet and Laurain 2010; König and Laurain 2022; 2024; Iacopetti and Pacella 2015; Iacopetti and Vaira 2018; Musso and Pistoia 2002; Musso et al. 2024; Premoselli 2022; Vaira 2015].

Our approach in this paper is strongly inspired by these references. We proceed by contradiction: under the assumptions (and with the notations) of Theorem 1.1, and by [Struwe 1984], if $(v_\alpha)_{\alpha\in\mathbb{N}}$ does not strongly converge in $H_0^1(\Omega)$ we have, up to a subsequence,

$$v_\alpha = B_\alpha \pm v_\infty + o(1) \quad \text{in } H_0^1(\Omega) \tag{1-8}$$

as $\alpha \to +\infty$, where $v_\infty \geq 0$ solves (1-1) and where B_α is a positive bubbling profile that concentrates at some point $x_\alpha \in \Omega$ and is modelled on a positive solution of $-\Delta B = B^{2^*-1}$ in $\mathbb{R}^n$; see (2-5) for more details. We perform an asymptotic analysis of v_α near x_α at different scales and obtain necessary conditions on h for blow-up to occur. The contradiction follows from these conditions: to prove Theorem 1.1 when $3 \leq n \leq 5$, for instance, we prove that if (1-8) holds we simultaneously have $v_\infty \equiv 0$ and $v_\infty > 0$ in Ω. In order to investigate the behaviour of v_α near x_α we prove in this paper new pointwise estimates on v_α, up to the boundary, that improve (1-8) in strong spaces. We precisely prove that

$$\left\| \frac{v_\alpha - \Pi B_\alpha \mp v_\infty}{B_\alpha + v_\infty} \right\|_\infty \to 0 \tag{1-9}$$

as $\alpha \to +\infty$, where ΠB_α is the projection of B_α in $H_0^1(\Omega)$ defined by (2-14); see Theorem 2.1 for a precise statement. Estimate (1-9) provides an accurate control on v_α up to $\partial\Omega$ and is particularly useful close to $\partial\Omega$, where, at first order, ΠB_α deviates from B_α and v_∞ vanishes. To the best of our knowledge this is the first time that a similar estimate is proven. We heavily rely on estimate (1-9) to rule out the possibility that the concentration point x_α converges to a point in $\partial\Omega$: this is both the main difficulty that we face in the proof of Theorem 1.1 and the main novelty of our analysis, and is deeply related to the sign-changing nature of the solutions we consider; see Remarks 3.6 and 3.7 for a detailed explanation of this fact.

The structure of the paper is as follows. In Section 2 we prove Theorem 2.1 and establish (1-9). In Section 3 we apply it to obtain necessary conditions for the blow-up of $(v_\alpha)_{\alpha\in\mathbb{N}}$ by means of suitable Pohozaev identities at different scales. We separately treat the interior blow-up case (Proposition 3.1) and the boundary blow-up case (Propositions 3.2, 3.4 and 3.5), and we deduce our main result, Theorem 1.1, from this analysis. Finally, the Appendix contains the proof of a few technical results that are used throughout Section 3.

2. The C^0-theory for blow-up

In this section we let $h_\infty \in C^0(\overline{\Omega})$ and consider a family of functions $(h_\alpha)_{\alpha\in\mathbb{N}} \in C^1(\overline{\Omega})$ such that

$$\lim_{\alpha\to+\infty} h_\alpha = h_\infty \quad \text{in } C^0(\overline{\Omega}). \tag{2-1}$$

We assume that $-\Delta + h_\infty$ is coercive in $H_0^1(\Omega)$ and that $I_{h_\infty}(\Omega) < K_n^{-2}$, where $I_{h_\infty}(\Omega)$ is as in (1-2), so that positive ground states of (1-1) with $h = h_\infty$ exist. We consider a sequence of functions $(v_\alpha)_{\alpha\in\mathbb{N}}$ in

$H_0^1(\Omega)$ such that, for all $\alpha \in \mathbb{N}$, v_α is a solution to

$$\begin{cases} -\Delta v_\alpha + h_\alpha v_\alpha = |v_\alpha|^{2^*-2} v_\alpha & \text{in } \Omega, \\ v_\alpha = 0 & \text{in } \partial\Omega. \end{cases} \tag{2-2}$$

We assume that

$$\limsup_{\alpha \to +\infty} \int_\Omega |v_\alpha|^{2^*} \, dx \le K_n^{-n} + I_{h_\infty}(\Omega)^{n/2}. \tag{2-3}$$

We also assume that $(v_\alpha)_{\alpha \in \mathbb{N}}$ blows up, that is

$$\lim_{\alpha \to +\infty} \|v_\alpha\|_\infty = +\infty. \tag{2-4}$$

By (2-3) and (2-4), and following [Struwe 1984] (see also [Struwe 2008]), we get that, up to a subsequence,

$$v_\alpha = B_\alpha \pm v_\infty + \varphi_\alpha \quad \text{in } H_0^1(\Omega), \tag{2-5}$$

where $\|\varphi_\alpha\|_{H_0^1} \to 0$ as $\alpha \to +\infty$. In (2-5), v_∞ is a solution of (1-1) with $h = h_\infty$ and we have let

$$B_\alpha(x) := \mu_\alpha^{-(n-2)/2} B_0(\mu_\alpha^{-1}(x - x_\alpha)) \quad \text{for } x \in \Omega, \tag{2-6}$$

where $(x_\alpha)_{\alpha \in \mathbb{N}}$ and $(\mu_\alpha)_{\alpha \in \mathbb{N}}$ are sequences of points in Ω and the positive real numbers, respectively, and where we have let

$$B_0(x) = \left(1 + \frac{|x|^2}{n(n-2)}\right)^{1-\frac{n}{2}} \quad \text{for any } x \in \mathbb{R}^n. \tag{2-7}$$

It is well known that B_0 satisfies $-\Delta B_0 = B_0^{2^*-1}$ in $\mathbb{R}^n$ and achieves K_n^{-2} in (1-3). As a consequence of (2-5), we have

$$\lim_{\alpha \to +\infty} v_\alpha = \pm v_\infty \quad \text{weakly in } H_0^1(\Omega) \tag{2-8}$$

and

$$\lim_{\alpha \to +\infty} \int_\Omega |v_\alpha|^{2^*} \, dx = K_n^{-n} + \int_\Omega |v_\infty|^{2^*} \, dx.$$

A consequence of (2-3) and of the assumption $I_{h_\infty}(\Omega) < K_n^{-2}$ is that either $v_\infty \equiv 0$ or v_∞ is a least-energy positive solution of

$$\begin{cases} -\Delta v_\infty + h_\infty v_\infty = v_\infty^{2^*-1} & \text{in } \Omega, \\ v_\infty > 0 & \text{in } \Omega, \\ v_\infty = 0 & \text{on } \partial\Omega. \end{cases} \tag{2-9}$$

If v_α is assumed to change sign for all $\alpha \ge 1$, that is if $(v_\alpha)_+$ and $(v_\alpha)_-$ are nonzero, the arguments in [Cerami et al. 1986, Lemma 3.1] show that $v_\infty > 0$ and hence that

$$\lim_{\alpha \to +\infty} \int_\Omega |v_\alpha|^{2^*} \, dx = K_n^{-n} + I_{h_\infty}(\Omega)^{n/2}.$$

This observation will be important in the proof of Theorem 1.1 but will not be used in this section. Without loss of generality we can assume that $(x_\alpha)_{\alpha \in \mathbb{N}}$ and $(\mu_\alpha)_{\alpha \in \mathbb{N}}$ are chosen to satisfy

$$|v_\alpha(x_\alpha)| = \|v_\alpha(x)\|_\infty \quad \text{and} \quad \mu_\alpha := |v_\alpha(x_\alpha)|^{-2/(n-2)}, \tag{2-10}$$

so that $x_\alpha \in \Omega$. Note that (2-4) implies that $\mu_\alpha \to 0$ as $\alpha \to +\infty$. We will denote by $x_\infty \in \overline{\Omega}$ the limit of the x_α as $\alpha \to +\infty$. In the case where $v_\infty > 0$, Hopf's lemma shows that there exists $C_0 > 0$ such that

$$C_0^{-1} d(x, \partial\Omega) \le v_\infty(x) \le C_0 d(x, \partial\Omega) \quad \text{for all } x \in \Omega, \tag{2-11}$$

where $d(x, \partial\Omega) := \inf\{|x-y| : y \in \partial\Omega\}$ is the distance of x to boundary. In (2-5) we used the notation $v_\alpha = B_\alpha \pm v_\infty + \varphi_\alpha$, which classically means either $v_\alpha = B_\alpha + v_\infty + \varphi_\alpha$ or $v_\alpha = B_\alpha - v_\infty + \varphi_\alpha$. It will often be more convenient to subtract $B_\alpha \pm v_\infty$ from v_α (for instance in the statement of Theorem 2.1), which we will thus write as

$$v_\alpha - B_\alpha \mp v_\infty = \varphi_\alpha$$

so that the sign convention is satisfied.

The purpose of this section is to turn (2-5) into a decomposition in strong spaces, and to obtain sharp pointwise estimates on v_α. In order to state our main result we need to introduce more notation. For α large, thanks to (2-1), $-\Delta + h_\alpha$ is coercive in $H_0^1(\Omega)$. We can thus let G_α be the Green's function of $-\Delta + h_\alpha$ in Ω with Dirichlet boundary conditions. By standard properties of the Green's function (see [Robert 2010]), there exists $C > 0$ such that for all $\alpha \ge 1$ we have

$$G_\alpha(y, x) \le \frac{C}{|y-x|^{n-2}} \min\left\{1, \frac{d(y, \partial\Omega) d(x, \partial\Omega)}{|y-x|^2}\right\} \quad \text{for all } x, y \in \Omega, \quad x \ne y, \tag{2-12}$$

and

$$|\nabla G_\alpha(y, x)| \le C|y-x|^{1-n} \quad \text{for all } x, y \in \Omega, \quad x \ne y. \tag{2-13}$$

For $\alpha \ge 1$, we let ΠB_α be the unique solution in $H_0^1(\Omega)$ of

$$\begin{cases} (-\Delta + h_\alpha)\Pi B_\alpha = B_\alpha^{2^*-1} & \text{in } \Omega, \\ \Pi B_\alpha = 0 & \text{on } \partial\Omega. \end{cases} \tag{2-14}$$

Since B_α satisfies $-\Delta B_\alpha = B_\alpha^{2^*-1}$ in $\mathbb{R}^n$ by (2-6) and (2-7), we easily see with (2-14) that $B_\alpha - \Pi B_\alpha \to 0$ in $H_0^1(\Omega)$ as $\alpha \to +\infty$. Thus (2-5) can be rewritten as

$$v_\alpha = \Pi B_\alpha \pm v_\infty + o(1) \quad \text{in } H_0^1(\Omega) \text{ as } \alpha \to +\infty. \tag{2-15}$$

A representation formula for ΠB_α together with (2-12) shows that there exists $C > 0$ such that for all $x \in \Omega$ and all $\alpha \ge 1$ we have

$$0 < \Pi B_\alpha(x) \le C B_\alpha(x), \tag{2-16}$$

where positivity follows from the coercivity of $-\Delta + h_\alpha$. We can now state the main result of this section:

Theorem 2.1. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \ge 3$, and $(h_\alpha)_{\alpha\in\mathbb{N}}$ be a sequence of functions that converges in $C^0(\overline{\Omega})$ to h_∞. We assume that $-\Delta + h_\infty$ is coercive in $H_0^1(\Omega)$ and that $I_{h_\infty}(\Omega) < K_n^{-2}$. Let $(v_\alpha)_{\alpha\in\mathbb{N}} \in H_0^1(\Omega)$ be a sequence of solutions of* (2-2) *that satisfies* (2-3), (2-4) *and* (2-5). *There exists a sequence $(\varepsilon_\alpha)_{\alpha\in\mathbb{N}}$ of positive real numbers converging to* 0 *such that*, *up to a subsequence*, *we have*, *for any $x \in \Omega$ and $\alpha \ge 1$*,

$$|v_\alpha(x) - \Pi B_\alpha(x) \mp v_\infty(x)| \le \varepsilon_\alpha (B_\alpha(x) + v_\infty(x)). \tag{2-17}$$

Pointwise descriptions of blowing-up solutions as in Theorem 2.1 were first obtained for *positive* solutions of critical Schrödinger-type equations on manifolds without boundary; see for instance [Druet and Hebey 2009; Druet et al. 2004] (see also [Hebey 2014]). For *positive* solutions of equations like (2-2) in bounded open subsets of $\mathbb{R}^n$ they were obtained in [König and Laurain 2022; 2024]. Similar estimates have been obtained for positive solutions of Hardy–Sobolev equations in [Cheikh Ali 2022; Ghoussoub et al. 2023]. These sharp pointwise estimates have proven crucial in order to obtain compactness and stability results for critical stationary elliptic equations [Druet 2003; Druet and Laurain 2010]. When it comes to *sign-changing* blowing-up solutions, a general pointwise description as in Theorem 2.1, on manifolds without boundary, has been obtained in [Premoselli 2024; Premoselli and Robert 2025], and subsequent compactness results have been proven in [Premoselli and Robert 2025; Premoselli and Vétois 2022a; 2022b]. Theorem 2.1 is, to our knowledge, the first instance where sharp pointwise estimates for blowing-up solutions of equations like (2-2) are obtained up to the boundary of Ω. Note indeed that in Theorem 2.1 we do not assume that the concentration point $x_\infty = \lim_{\alpha\to+\infty} x_\alpha$ is an interior point in Ω. It may happen that $x_\infty \in \partial\Omega$: the real novelty of Theorem 2.1 is that (2-17) holds regardless of the speed of convergence of x_α to $\partial\Omega$, uniformly in $x \in \overline{\Omega}$. This creates additional technical difficulties that we overcome in the course of the proof.

We prove Theorem 2.1 by taking inspiration from the arguments in [Druet and Hebey 2009]; see also [Hebey 2014]. Throughout this section we let Ω be a smooth bounded domain in $\mathbb{R}^n$, $n \geq 3$, $(h_\alpha)_{\alpha\in\mathbb{N}} \in C^0(\overline{\Omega})$ and $(v_\alpha)_{\alpha\in\mathbb{N}} \in H_0^1(\Omega)$ be such that (2-1), (2-2), (2-4), and (2-5) hold, and we let $(x_\alpha)_{\alpha\in\mathbb{N}} \in \Omega$ and $(\mu_\alpha)_{\alpha\in\mathbb{N}}$ be as defined as in (2-10). We start with the following simple proposition:

Proposition 2.2. *We have*

$$\lim_{\alpha\to+\infty} \frac{d(x_\alpha, \partial\Omega)}{\mu_\alpha} = +\infty. \tag{2-18}$$

We define the rescaled function

$$\tilde{v}_\alpha(x) := \mu_\alpha^{(n-2)/2} v_\alpha(x_\alpha + \mu_\alpha x) \quad \textit{for all } x \in \Omega_\alpha, \tag{2-19}$$

where $\Omega_\alpha := \{x \in \mathbb{R}^n : x_\alpha + \mu_\alpha x \in \Omega\}$. *Then*

$$\lim_{\alpha\to+\infty} \tilde{v}_\alpha(x) = B_0(x) \quad \textit{in } C^2_{\mathrm{loc}}(\mathbb{R}^n), \tag{2-20}$$

where B_0 *is defined in* (2-7).

Proof. First, (2-18) follows from Struwe's original result [Struwe 1984]; see also [Mazumdar 2017, Theorem 1.2]. We now prove (2-20). For $x \in \Omega_\alpha := \{x \in \mathbb{R}^n : x_\alpha + \mu_\alpha x \in \Omega\}$, it is clear by (2-2) and (2-19) that

$$\begin{cases} -\Delta\tilde{v}_\alpha + \tilde{h}_\alpha \mu_\alpha^2 \tilde{v}_\alpha = |\tilde{v}_\alpha|^{2^*-2}\tilde{v}_\alpha & \text{in } \Omega_\alpha, \\ \tilde{v}_\alpha = 0 & \text{on } \partial\Omega_\alpha, \end{cases}$$

where $\tilde{h}_\alpha(x) = h_\alpha(x_\alpha + \mu_\alpha x)$ and $\tilde{v}_\alpha$ is defined in (2-19). We remark that $|\tilde{v}_\alpha| \leq |\tilde{v}_\alpha(0)| = 1$. It follows from (2-1) and from standard elliptic theory that, after passing to a subsequence, $\tilde{v}_\alpha \to \tilde{v}$ in $C^2_{\mathrm{loc}}(\mathbb{R}^n)$, where $\tilde{v} \in C^2(\mathbb{R}^n)$ is such that

$$-\Delta\tilde{v} = |\tilde{v}|^{2^*-2}\tilde{v} \quad \text{in } \mathbb{R}^n$$

and $|\tilde{v}| \le 1$. Let $K \Subset \mathbb{R}^n$ be a nonempty compact subset of $\mathbb{R}^n$. By (2-5) we have $\tilde{v}_\alpha \to B_0$ in $L^{2^*}(K)$ as $\alpha \to +\infty$, so that $\tilde{v} = B_0$ in K, which proves (2-20). □

Using (2-18) and standard elliptic theory, together with (2-14) and (2-16), we also obtain that

$$\mu_\alpha^{(n-2)/2} \Pi B_\alpha(x_\alpha + \mu_\alpha x) \to B_0(x) \quad \text{in } C^2_{\text{loc}}(\mathbb{R}^n) \tag{2-21}$$

as $\alpha \to +\infty$. The following result establishes a first pointwise control on v_α.

Proposition 2.3. *For $x \in \Omega$ we let $D_\alpha(x) := |x - x_\alpha| + \mu_\alpha$. Then*

$$D_\alpha(x)^{(n-2)/2} |v_\alpha - \Pi B_\alpha \mp v_\infty| \to 0 \quad \text{in } C^0(\overline{\Omega}) \text{ as } \alpha \to +\infty, \tag{2-22}$$

where v_∞ and ΠB_α are as defined in (2-8), (2-9) *and* (2-14).

To prove Proposition 2.3 we proceed by contradiction: we assume that there exist $\epsilon_0 > 0$ and $(y_\alpha)_{\alpha\in\mathbb{N}} \in \overline{\Omega}$ such that

$$D_\alpha(y_\alpha)^{(n-2)/2} |v_\alpha(y_\alpha) \mp v_\infty(y_\alpha) - \Pi B_\alpha(y_\alpha)| = \max_{x\in\Omega} (D_\alpha(x)^{(n-2)/2} |v_\alpha(x) \mp v_\infty(x) - \Pi B_\alpha(x)|) \ge \epsilon_0, \tag{2-23}$$

and we let $(\nu_\alpha)_{\alpha\in\mathbb{N}} \in (0, +\infty)$ be such that

$$|v_\alpha(y_\alpha)| = \nu_\alpha^{(2-n)/2} \quad \text{for all } \alpha \ge 1. \tag{2-24}$$

Since v_α, ΠB_α and v_∞ vanish in $\partial\Omega$, a first simple observation is that $y_\alpha \in \Omega$.

Step 1. We claim that

$$D_\alpha(y_\alpha)^{(n-2)/2} B_\alpha(y_\alpha) \to 0 \quad \text{as } \alpha \to +\infty.$$

As a consequence, with (2-16) we have

$$D_\alpha(y_\alpha)^{(n-2)/2} \Pi B_\alpha(y_\alpha) \to 0 \quad \text{as } \alpha \to +\infty. \tag{2-25}$$

Proof. Indeed, suppose on the contrary that there exists $\rho_0 > 0$ such that

$$D_\alpha(y_\alpha)^{(n-2)/2} B_\alpha(y_\alpha) \ge \rho_0$$

for all α large enough. Hence we have that

$$1 + \frac{|x_\alpha - y_\alpha|}{\mu_\alpha} = \frac{D_\alpha(y_\alpha)}{\mu_\alpha} \ge \rho_0^{2/(n-2)} \left(1 + \frac{|y_\alpha - x_\alpha|^2}{\mu_\alpha^2}\right).$$

Up to passing to a subsequence we then assume that there exists $R > 0$ such that $\lim_{\alpha\to+\infty} \mu_\alpha^{-1} |y_\alpha - x_\alpha| = R$. This means that

$$D_\alpha(y_\alpha) = O(\mu_\alpha). \tag{2-26}$$

It follows from (2-21) and (2-20) that

$$\lim_{\alpha\to+\infty} \mu_\alpha^{(n-2)/2} |v_\alpha(y_\alpha) - \Pi B_\alpha(y_\alpha)| = 0.$$

With (2-26) we thus get that

$$\lim_{\alpha\to+\infty} D_\alpha(y_\alpha)^{(n-2)/2}|v_\alpha(y_\alpha)\mp v_\infty(y_\alpha)-\Pi B_\alpha(y_\alpha)|=0,$$

which contradicts (2-23). □

Step 2. We claim that

$$\nu_\alpha\to 0 \quad \text{as } \alpha\to+\infty, \tag{2-27}$$

where ν_α is defined in (2-24).

Proof. Indeed, it follows from (2-23) and (2-25) that

$$\epsilon_0\le D_\alpha(y_\alpha)^{(n-2)/2}(|v_\alpha(y_\alpha)|+\|v_\infty\|_\infty)+o(1) \tag{2-28}$$

as $\alpha\to+\infty$. If $D_\alpha(y_\alpha)\to 0$ as $\alpha\to+\infty$, then (2-27) follows from (2-28). Suppose on the contrary that, up to a subsequence, $D_\alpha(y_\alpha)\to c_0$ as $\alpha\to+\infty$ for some $c_0>0$. It follows from (2-23) and (2-25) that

$$|v_\alpha(x)\mp v_\infty(x)|+o(1)\le 2^n|v_\alpha(y_\alpha)\mp v_\infty(y_\alpha)|+o(1) \tag{2-29}$$

for $x\in B_{c_0/2}(y_\alpha)\cap\overline{\Omega}$ and all α sufficiently large. If $v_\alpha(y_\alpha)\to+\infty$ as $\alpha\to+\infty$, it is clear, by the definition of ν_α, that we obtain (2-27). If $v_\alpha(y_\alpha)=O(1)$, standard elliptic theory together with (2-8) and (2-29) proves that $v_\alpha\mp v_\infty\to 0$ in $C^2_{\text{loc}}(B_{c_0/4}(y_\alpha))$ as $\alpha\to+\infty$. This contradicts (2-23) using (2-25). We thus get that (2-27) holds. □

For any $x\in\Omega_\alpha:=\{x\in\mathbb{R}^n : y_\alpha+\nu_\alpha x\in\Omega\}$, we set

$$w_\alpha(x)=\nu_\alpha^{(n-2)/2}v_\alpha(y_\alpha+\nu_\alpha x).$$

By (2-2), w_α satisfies

$$\begin{cases}-\Delta w_\alpha+h_\alpha(y_\alpha+\nu_\alpha x)\nu_\alpha^2 w_\alpha=|w_\alpha|^{2^*-2}w_\alpha & \text{in } \Omega_\alpha,\\ w_\alpha=0 & \text{on } \partial\Omega_\alpha.\end{cases} \tag{2-30}$$

Thanks to (2-24), we have that $|w_\alpha(0)|=1$. We define a set S as

$$S=\begin{cases}\left\{\lim_{\alpha\to+\infty}\frac{y_\alpha-x_\alpha}{\nu_\alpha}\right\} & \text{if } |y_\alpha-x_\alpha|=O(\nu_\alpha) \text{ and } \mu_\alpha=o(\nu_\alpha),\\ \varnothing & \text{otherwise,}\end{cases}$$

where it is intended that the limit exists up to passing to a subsequence. Let us fix $K\Subset\mathbb{R}^n\backslash S$ a compact set.

Step 3. As $\alpha\to+\infty$ we have

$$\nu_\alpha^{(n-2)/2}B_\alpha(y_\alpha-\nu_\alpha x)\to 0 \quad \text{for all } x\in K. \tag{2-31}$$

Proof. Let $x\in K$. If $\nu_\alpha=o(\mu_\alpha)$ then (2-31) is true since $B_\alpha(x)\le\mu_\alpha^{-(n-2)/2}$ for any $x\in\overline{\Omega}$. We now assume that $\mu_\alpha=o(\nu_\alpha)$: since $x\in K$, we get that $\nu_\alpha=O(|y_\alpha-x_\alpha-\nu_\alpha x|)$. Thus once again (2-31) holds by definition of B_α. We may thus assume that there exists $C>0$ such that

$$C^{-1}\nu_\alpha\le\mu_\alpha\le C\nu_\alpha \quad \text{for all } \alpha. \tag{2-32}$$

Assume first that $|y_\alpha - x_\alpha - \nu_\alpha x| = O(\mu_\alpha)$. Thus, since $x \in K$ and by (2-32), we get $|y_\alpha - x_\alpha| = O(\mu_\alpha)$. Arguing as in the proof of Step 1 we get a contradiction. Thus, for all $x \in K$, we have

$$\lim_{\alpha\to+\infty} \frac{|y_\alpha - x_\alpha - \nu_\alpha x|}{\mu_\alpha} = +\infty.$$

Together with (2-32) this implies (2-31). □

Step 4. We claim that

$$w_\alpha(x) = O(1) \quad \text{for all } x \in K \cap \Omega_\alpha. \tag{2-33}$$

Proof. Indeed, using (2-23) and (2-25) together with (2-31) yields

$$\left(\frac{D_\alpha(y_\alpha+\nu_\alpha x)}{D_\alpha(y_\alpha)}\right)^{\frac{n-2}{2}} \left|w_\alpha(x) \mp \nu_\alpha^{(n-2)/2} v_\infty(y_\alpha+\nu_\alpha x) - \nu_\alpha^{(n-2)/2}\Pi B_\alpha(y_\alpha+\nu_\alpha x)\right| \le 1+o(1) \tag{2-34}$$

for all $x \in K \cap \Omega_\alpha$. It then follows from (2-16), (2-27), (2-31) and (2-34) that

$$\left(\frac{D_\alpha(y_\alpha+\nu_\alpha x)}{D_\alpha(y_\alpha)}\right)^{\frac{n-2}{2}} (|w_\alpha(x)| + o(1)) \le 1+o(1) \quad \text{for all } x \in K \cap \Omega_\alpha. \tag{2-35}$$

We claim that there exists $\eta_K > 0$ such that

$$\lim_{\alpha\to+\infty} D_\alpha(y_\alpha+\nu_\alpha x) D_\alpha(y_\alpha)^{-1} \ge \eta_K$$

for all $x \in K \cap \Omega_\alpha$. Together with (2-35) this will prove that w_α is bounded in $K \cap \Omega_\alpha$. Suppose on the contrary that for a sequence $(z_\alpha)_{\alpha\in\mathbb{N}}$ in $K \cap \Omega_\alpha$ we have

$$|y_\alpha - x_\alpha + \nu_\alpha z_\alpha| + \mu_\alpha = o(|y_\alpha - x_\alpha|) + o(\mu_\alpha).$$

Then $|y_\alpha - x_\alpha| = O(\nu_\alpha)$, $\mu_\alpha = o(\nu_\alpha)$ and

$$\lim_{\alpha\to+\infty} \left|\frac{y_\alpha - x_\alpha}{\nu_\alpha} - z_\alpha\right| = 0,$$

which is a contradiction since $\liminf_{\alpha\to+\infty} d(z_\alpha, S) > 0$. □

We now conclude the proof of Proposition 2.3.

Proof of Proposition 2.3. We first claim that $0 \in \Omega_\alpha \backslash S$. If $S = \varnothing$ this is obvious. Assume thus that $S \ne \varnothing$, which implies that $|y_\alpha - x_\alpha| = O(\nu_\alpha)$ and $\mu_\alpha = o(\nu_\alpha)$ as $\alpha \to +\infty$. Then, since $\nu_\alpha \to 0$ as $\alpha \to +\infty$ and by (2-28), we obtain

$$\epsilon_0^{2/(n-2)} + o(1) \le \nu_\alpha^{-1} D_\alpha(y_\alpha).$$

Hence, we have $\lim_{\alpha\to+\infty} \nu_\alpha^{-1}(y_\alpha - x_\alpha) \ne 0$, and thus $0 \notin S$. By (2-33), for any compact subset $K \subset \mathbb{R}^n \backslash S$ that contains 0, there exists $C_K > 0$ such that

$$|w_\alpha(x)| \le C_K \quad \text{in } K.$$

In particular, by standard elliptic theory, (2-30) and (2-1), we get

$$w_\alpha \to w_0 \in C^1_{\text{loc}}(\mathbb{R}^n \backslash S), \tag{2-36}$$

where w_0 satisfies $-\Delta w_0 = |w_0|^{2^*-2} w_0$ in $\mathbb{R}^n \backslash S$ and $|w_0(0)| = 1$. Independently, it follows from (2-5) and (2-31) that $w_\alpha \to 0$ in $L^{2^*}(K)$ as $\alpha \to +\infty$. Hence, by (2-36), we find that

$$\int_K |w_0|^{2^*} \, dx = 0.$$

Thus $w_0 \equiv 0$ in K, which contradicts $|w_0(0)| = 1$. This ends the proof of Proposition 2.3. □

For $\rho > 0$ small enough, we define

$$\eta_\alpha(\rho) := \sup_{\Omega \backslash B_\rho(x_\alpha)} |v_\alpha(x)|, \tag{2-37}$$

where x_α is given by (2-10). Thanks to (2-22), we obtain

$$\lim_{\alpha \to +\infty} \sup \eta_\alpha(\rho) \le \|v_\infty\|_\infty. \tag{2-38}$$

The next results establishes a first pointwise control on v_α.

Proposition 2.4. *For any $\nu \in \left(0, \frac{1}{2}\right)$ there exists $R_\nu > 0$, $\rho_\nu > 0$, and $C_\nu > 0$ such that for all $\alpha \in \mathbb{N}$*

$$|v_\alpha(x)| \le C_\nu \left(\frac{\mu_\alpha^{(n-2)/2 - \nu(n-2)}}{|x - x_\alpha|^{(n-2)(1-\nu)}} + \frac{\eta_\alpha(\rho_\nu)}{|x - x_\alpha|^{(n-2)\nu}} \right) \tag{2-39}$$

for all $x \in \Omega \backslash B_{R_\nu \mu_\alpha}(x_\alpha)$.

Proof. We divide our proof into two cases, depending on the position of x_∞ with respect to the boundary of Ω.

Case 1: $x_\infty \in \partial\Omega$. Let $U \subset \mathbb{R}^n$ be a smooth bounded open set such that $\overline{\Omega} \Subset U$. For all $\alpha \ge 1$, we extend h_α and h_∞ as functions on U in such a way that

$$h_\alpha \to h_\infty \quad \text{in } C^0(\overline{U}) \tag{2-40}$$

and $-\Delta + h_\infty$ is still coercive in $H_0^1(U)$. Let $\widetilde{G} : \overline{U} \times \overline{U} \backslash \{(x, x) : x \in \overline{U}\} \to \mathbb{R}$ be the Green's function of the operator $-\Delta + h_\infty$ with Dirichlet boundary conditions in U. It exists by coercivity of $-\Delta + h_\infty$ and satisfies, for all $x \in U$,

$$-\Delta \widetilde{G}(x, \cdot) + h_\infty \widetilde{G}(x, \cdot) = \delta_x \quad \text{in } U \backslash \{x\}. \tag{2-41}$$

We now define $\widetilde{G}_\alpha(x) := \widetilde{G}(x_\alpha, x)$ for all $x \in \overline{U} \backslash \{x_\alpha\}$ and $\alpha \in \mathbb{N}$. It follows from [Robert 2010] that there exists $C_1 > 0$ such that

$$0 < \widetilde{G}_\alpha(x) \le C_1 |x - x_\alpha|^{2-n} \quad \text{for all } x \in \overline{U} \backslash \{x_\alpha\} \tag{2-42}$$

and that there exist $\rho > 0$ and $C_2 > 0$ such that

$$\widetilde{G}_\alpha(x) \ge C_2 |x - x_\alpha|^{2-n} \quad \text{and} \quad \frac{|\nabla \widetilde{G}_\alpha(x)|}{|\widetilde{G}_\alpha(x)|} \ge C_2 |x - x_\alpha|^{-1} \tag{2-43}$$

for all $x \in B_\rho(x_\alpha) \backslash \{x_\alpha\} \Subset U$. We define

$$L_\alpha := -\Delta + h_\alpha - |v_\alpha|^{2^*-2}, \tag{2-44}$$

and for a fixed $\nu \in (0,1)$ we let, for $\alpha \in \mathbb{N}$ and $x \in \overline{U}\backslash\{x_\alpha\}$,

$$\psi_{\nu,\alpha}(x) := \mu_\alpha^{(n-2)/2-\nu(n-2)}\widetilde{G}_\alpha(x)^{1-\nu} + \eta_\alpha(\rho)\widetilde{G}_\alpha(x)^\nu. \tag{2-45}$$

Straightforward computations using (2-40) and (2-41) show that

$$\frac{L_\alpha\psi_{\nu,\alpha}}{\psi_{\nu,\alpha}} \geq -2\|h_\infty\|_\infty + o(1) + \nu(1-\nu)\left|\frac{\nabla\widetilde{G}_\alpha}{\widetilde{G}_\alpha}\right|^2 - |v_\alpha|^{2^*-2}.$$

By using (2-43) we get

$$\frac{L_\alpha\psi_{\nu,\alpha}}{\psi_{\nu,\alpha}} \geq -2\|h_\infty\|_\infty + o(1) + \nu(1-\nu)\frac{C_2^2}{|x-x_\alpha|^2} - |v_\alpha|^{2^*-2} \tag{2-46}$$

for all $x \in B_\rho(x_\alpha)\backslash\{x_\alpha\} \Subset U$, where C_2 is the constant appearing in (2-43). Proposition 2.3 now shows that there exists $R_0 > 0$ such that for any $R > R_0$ and $x \in \Omega\backslash B_{R\mu_\alpha}(x_\alpha)$ we have

$$|x-x_\alpha|^2|v_\alpha(x) \mp v_\infty(x)|^{2^*-2} \leq \frac{\nu(1-\nu)C_2^2}{2^{2^*+1}} \tag{2-47}$$

for α sufficiently large. Hence, by (2-47), we get

$$|x-x_\alpha|^2|v_\alpha(x)|^{2^*-2} \leq \tfrac{1}{4}\nu(1-\nu)C_2^2 + 2^{2^*-1}\rho^2\|v_\infty\|_\infty^{2^*-2} \tag{2-48}$$

for all $x \in (B_\rho(x_\alpha)\backslash B_{R\mu_\alpha}(x_\alpha)) \cap \Omega$. Choose $\rho_0 > 0$ small enough that for any $\rho \in (0, \rho_0)$ we have

$$2^{2^*-1}\rho^2\|v_\infty\|_\infty^{2^*-2} + 2\rho^2\|h_\infty\|_\infty \leq \tfrac{1}{4}\nu(1-\nu)C_2^2. \tag{2-49}$$

Combining (2-48) and (2-49) in (2-46) we finally obtain that, for all $x \in (B_\rho(x_\alpha)\backslash B_{R\mu_\alpha}(x_\alpha)) \cap \Omega$,

$$L_\alpha\psi_{\nu,\alpha} \geq \frac{1}{|x-x_\alpha|^2}\big(o(\rho^2) + \tfrac{1}{2}\nu(1-\nu)C_2^2\big)\psi_{\nu,\alpha} > 0. \tag{2-50}$$

Independently, it follows from (2-20), (2-37) and (2-43) that there exists $C = C(R,\rho,\nu) > 0$ such that

$$|v_\alpha(x)| \leq C\psi_{\nu,\alpha}(x) \quad \text{for all } x \in \partial((B_\rho(x_\alpha)\backslash B_{R\mu_\alpha}(x_\alpha)) \cap \Omega). \tag{2-51}$$

By (2-2), v_α satisfies $L_\alpha v_\alpha = 0$. Using (2-50) and (2-51) we thus have

$$\begin{cases} L_\alpha(C\psi_{\nu,\alpha}) \geq 0 = L_\alpha v_\alpha & \text{in } (B_\rho(x_\alpha)\backslash B_{R\mu_\alpha}(x_\alpha)) \cap \Omega, \\ C\psi_{\nu,\alpha} \geq v_\alpha & \text{on } \partial((B_\rho(x_\alpha)\backslash B_{R\mu_\alpha}(x_\alpha)) \cap \Omega), \\ L_\alpha(C\psi_{\nu,\alpha}) \geq 0 = -L_\alpha v_\alpha & \text{in } (B_\rho(x_\alpha)\backslash B_{R\mu_\alpha}(x_\alpha)) \cap \Omega, \\ C\psi_{\nu,\alpha} \geq -v_\alpha & \text{on } \partial((B_\rho(x_\alpha)\backslash B_{R\mu_\alpha}(x_\alpha)) \cap \Omega). \end{cases} \tag{2-52}$$

The operator L_α satisfies the comparison principle on $(B_\rho(x_\alpha)\backslash B_{R\mu_\alpha}(x_\alpha)) \cap \Omega$ since $\psi_{\nu,\alpha} > 0$ and $L_\alpha\psi_{\nu,\alpha} > 0$ (see, e.g., [Berestycki et al. 1994]), and therefore

$$|v_\alpha(x)| \leq C\psi_{\nu,\alpha}(x) \quad \text{for all } x \in (B_\rho(x_\alpha)\backslash B_{R\mu_\alpha}(x_\alpha)) \cap \Omega.$$

Using again (2-42) implies (2-39) in this case.

Case 2: $x_\infty \in \Omega$. Let G be the Green's function in Ω of the operator $-\Delta + h_\infty$ with Dirichlet boundary conditions. For $x \in \Omega \backslash \{x_\alpha\}$ define $\widetilde{G}_\alpha := G(x_\alpha, \cdot)$, which satisfies

$$-\Delta \widetilde{G}_\alpha + h_\infty \widetilde{G}_\alpha = 0 \quad \text{in } \Omega \backslash \{x_\alpha\}.$$

Since $x_\infty \in \Omega$, it follows from [Robert 2010] that there exists $C_3 > 0$ such that

$$0 < \widetilde{G}_\alpha(x) \le C_3 |x - x_\alpha|^{2-n} \quad \text{for all } x \in \overline{\Omega} \backslash \{x_\alpha\}$$

and there exist $C_4 > 0$ and $\rho > 0$ such that

$$\widetilde{G}_\alpha(x) \ge C_4 |x - x_\alpha|^{2-n} \quad \text{and} \quad \frac{|\nabla \widetilde{G}_\alpha(x)|}{|\widetilde{G}_\alpha(x)|} \ge C_4 |x - x_\alpha|^{-1}$$

for all $x \in B_\rho(x_\alpha) \backslash \{x_\alpha\} \Subset \Omega$. Define, for a fixed $\nu \in (0, 1)$, for $\alpha \in \mathbb{N}$ and $x \in \overline{\Omega} \backslash \{x_\alpha\}$,

$$\psi_{\nu,\alpha}(x) := \mu_\alpha^{(n-2)/2 - \nu(n-2)} \widetilde{G}_\alpha(x)^{1-\nu} + \eta_\alpha(\rho) \widetilde{G}_\alpha(x)^\nu,$$

and let again $L_\alpha = -\Delta + h_\alpha - |v_\alpha|^{2^*-2}$. Mimicking the arguments in Case 1 we here again have $\psi_{\nu,\alpha} > 0$ and $L_\alpha \psi_{\nu,\alpha} > 0$ in $B_\rho(x_\alpha) \backslash B_{R\mu_\alpha}(x_\alpha)$, and the proof of (2-39) follows in a similar way. □

The next results establishes a pointwise control from above on v_α.

Proposition 2.5. *There exists $C > 0$ such that*

$$|v_\alpha(x)| \le C(\mu_\alpha^{(n-2)/2} D_\alpha(x)^{2-n} + \|v_\infty\|_\infty) \tag{2-53}$$

for all $x \in \Omega$.

Proof. Recall that $D_\alpha(x) = \mu_\alpha + |x - x_\alpha|$ for $x \in \Omega$. We first prove that there exists $\rho > 0$ and $C > 0$ such that

$$|v_\alpha(x)| \le C(\mu_\alpha^{(n-2)/2} D_\alpha(x)^{2-n} + \eta_\alpha(\rho)), \tag{2-54}$$

where $\eta_\alpha(\rho)$ is defined in (2-37). We fix $0 < \nu < 1/(n+2)$, and we let $R_\nu > 0$ and $\rho_\nu > 0$ be given by Proposition 2.4. We let $\rho = \rho_\nu$. Proving (2-54) amounts to proving that, for any sequence $y_\alpha \in \Omega$, we have

$$\frac{|v_\alpha(y_\alpha)|}{\mu_\alpha^{(n-2)/2} D_\alpha(y_\alpha)^{2-n} + \eta_\alpha(\rho)} = O(1) \quad \text{as } \alpha \to +\infty. \tag{2-55}$$

We let in this proof $r_\alpha := |y_\alpha - x_\alpha|$. First, if $r_\alpha \ge \rho$, it is clear that (2-55) is satisfied by definition of $\eta_\alpha(\rho)$. If now $r_\alpha = O(\mu_\alpha)$ we also have $D_\alpha(y_\alpha) = O(\mu_\alpha)$, and (2-21) and (2-22) yield

$$D_\alpha(y_\alpha)^{n-2} \mu_\alpha^{-(n-2)/2} |v_\alpha(y_\alpha)| = O(1),$$

which proves (2-55). We thus assume from now on that

$$r_\alpha \le \rho \quad \text{and} \quad \lim_{\alpha \to +\infty} \frac{r_\alpha}{\mu_\alpha} = +\infty. \tag{2-56}$$

Green's representation formula and (2-12) yield the existence of $C > 0$ such that

$$|v_\alpha(y_\alpha)| \le C \int_\Omega |y_\alpha - x|^{2-n} |v_\alpha(x)|^{2^*-1} \, dx \tag{2-57}$$

for all $\alpha \geq 1$. We write

$$\int_{\Omega} |y_\alpha - x|^{2-n} |v_\alpha(x)|^{2^*-1}\, dx \\ \leq \int_{\Omega\cap\{|x-x_\alpha|\leq R_\nu\mu_\alpha\}} |y_\alpha - x|^{2-n} |v_\alpha(x)|^{2^*-1}\, dx + \int_{\Omega\cap\{|x-x_\alpha|\geq R_\nu\mu_\alpha\}} |y_\alpha - x|^{2-n} |v_\alpha(x)|^{2^*-1}\, dx. \quad (2\text{-}58)$$

Fix $C_0 > R_\nu$. For α sufficiently large we have using (2-56) that

$$r_\alpha \geq C_0\mu_\alpha \geq \frac{C_0}{R_\nu}|x - x_\alpha| \quad \text{for all } x \in \Omega \cap \{|x - x_\alpha| \leq R_\nu\mu_\alpha\},$$

so that $|y_\alpha - x| \geq (1 - R_\nu C_0^{-1})r_\alpha$ for all such x. Therefore, using Hölder's inequality and (2-3) yields

$$\int_{\Omega\cap\{|x-x_\alpha|\leq R_\nu\mu_\alpha\}} |y_\alpha - x|^{2-n} |v_\alpha(x)|^{2^*-1}\, dx = O\left(\frac{\mu_\alpha^{(n-2)/2}}{|y_\alpha - x_\alpha|^{n-2}}\right). \quad (2\text{-}59)$$

Now, we deal with the second term of (2-58). From (2-39), we get

$$\int_{\Omega\cap\{|x-x_\alpha|\geq R_\nu\mu_\alpha\}} |y_\alpha - x|^{2-n} |v_\alpha(x)|^{2^*-1}\, dx = O\left(\mu_\alpha^{(n+2)(1-2\nu)/2} \int_{\Omega\cap\{|x-x_\alpha|\geq R_\nu\mu_\alpha\}} \frac{|y_\alpha - x|^{2-n}}{|x - x_\alpha|^{(n+2)(1-\nu)}}\, dx\right) \\ + O\left(\eta_\alpha(\rho_\nu)^{2^*-1} \int_{\Omega\cap\{|x-x_\alpha|\geq R_\nu\mu_\alpha\}} \frac{|y_\alpha - x|^{2-n}}{|x - x_\alpha|^{(n+2)\nu}}\, dx\right).$$

Since $2 - (n+2)\nu > 0$, using Giraud's lemma (see [Hebey 2014, Lemma 7.5]) yields

$$\int_{\Omega} |y_\alpha - x|^{2-n} |x - x_\alpha|^{-(n+2)\nu}\, dx = O(1). \quad (2\text{-}60)$$

Independently, letting $\tilde{y}_\alpha = (y_\alpha - x_\alpha)/\mu_\alpha$ we have

$$\int_{\Omega\cap\{|x-x_\alpha|\geq R_\nu\mu_\alpha\}} \frac{1}{|y_\alpha - x|^{n-2}} \frac{1}{|x - x_\alpha|^{(n+2)(1-\nu)}}\, dx \leq \mu_\alpha^{2-(n+2)(1-\nu)} \int_{\mathbb{R}^n\setminus B(0,R_\nu)} \frac{1}{|\tilde{y}_\alpha - x|^{n-2}} \frac{1}{|x|^{(n+2)(1-\nu)}}\, dx \\ = O\left(\frac{\mu_\alpha^{2-(n+2)(1-\nu)}}{(1+|\tilde{y}_\alpha|)^{n-2}}\right) = O\left(\frac{\mu_\alpha^{n-(n+2)(1-\nu)}}{|x_\alpha - y_\alpha|^{n-2}}\right), \quad (2\text{-}61)$$

where the second line again follows from Giraud's lemma in $\mathbb{R}^n$ since $(n+2)(1-\nu) > n$. Combining (2-60) and (2-61) finally shows that

$$\int_{\Omega\cap\{|x-x_\alpha|\geq R_\nu\mu_\alpha\}} |y_\alpha - x|^{2-n} |v_\alpha(x)|^{2^*-1}\, dx = O\left(\frac{\mu_\alpha^{(n-2)/2}}{|x_\alpha - y_\alpha|^{n-2}}\right) + O(\eta_\alpha(\rho)),$$

which together with (2-59) concludes the proof of (2-54).

We now conclude the proof of (2-53). First, if $v_\infty > 0$, (2-53) simply follows from (2-38) and (2-54). We may thus assume that $v_\infty \equiv 0$. We now prove that for α large enough

$$\eta_\alpha(\rho) = O(\mu_\alpha^{(n-2)/2}). \quad (2\text{-}62)$$

Together with (2-54) this will conclude the proof of (2-53) in this case. We prove (2-62) by contradiction: we assume that

$$\frac{\eta_\alpha(\rho)}{\mu_\alpha^{(n-2)/2}} \to +\infty \tag{2-63}$$

as $\alpha \to +\infty$, and we let $V_\alpha = v_\alpha/\eta_\alpha(\rho)$. For any α we let $z_\alpha \in \Omega\backslash B_\rho(x_\alpha)$ be such that $|v_\alpha(z_\alpha)| = \eta_\alpha(\rho)$. By the definition of $D_\alpha(x)$ and by (2-54) we see that for any $\delta > 0$ fixed we have $|V_\alpha(z_\alpha)| = 1$ and

$$|V_\alpha(x)| \le C + o(1) \quad \text{for } x \in \Omega\backslash B_\delta(x_\alpha). \tag{2-64}$$

Now, the function V_α satisfies

$$-\Delta V_\alpha + h_\alpha V_\alpha = \eta_\alpha(\rho)^{2^*-2}|V_\alpha|^{2^*-2}V_\alpha$$

in Ω. Since $\eta_\alpha(\rho) \to 0$ by (2-38), (2-64) and standard elliptic theory show that $V_\alpha \to V_\infty$ in $C^2_{\text{loc}}(\overline{\Omega}\backslash\{x_\infty\}$ as $\alpha \to +\infty$, where V_∞ satisfies $|V_\infty(x)| \le C$ for any $x \ne x_\infty$ and

$$-\Delta V_\infty + h_\infty V_\infty = 0 \quad \text{in } \Omega\backslash\{x_\infty\}.$$

In particular, the singularity of V_∞ at x_∞ is removable and V_∞ satisfies weakly $-\Delta V_\infty + h_\infty V_\infty = 0$ in Ω. Since $-\Delta + h_\infty$ is coercive by assumption, this shows that $V_\infty \equiv 0$. Independently, if we let $z_\infty = \lim_{\alpha\to+\infty} z_\alpha$, the C^2_{loc} convergence shows that $|V_\infty(z_\infty)| = 1$; hence $V_\infty \not\equiv 0$. This is a contradiction, which concludes the proof of (2-62). □

The next result will be frequently used in the proof of Theorem 2.1.

Proposition 2.6. *Let $U \subset \Omega$ be an open set. There exists a constant $C(U)$ such that* $\lim_{|U|\to 0} C(U) = 0$ *and such that, for all $y \in \Omega$ and for all $\alpha \ge 1$,*

$$\int_U G_\alpha(y, x)\, dx \le C(U) d(y, \partial\Omega). \tag{2-65}$$

Proof. We let $C(U) = \sup_{y\in\Omega} \int_U |x-y|^{1-n}\, dx$. Since Ω is bounded and $y \mapsto |y|^{1-n} \in L^1_{\text{loc}}(\mathbb{R}^n)$ we have $C(U) \to 0$ as $|U| \to 0$ by absolute continuity of the integral. Using (2-12) yields

$$\int_U G_\alpha(y, x)\, dx = O(I_1(y) + I_2(y)), \tag{2-66}$$

where we have let, for $i = 1, 2$,

$$I_i(y) := \int_{U_i} \frac{1}{|y-x|^{n-2}} \min\left\{1, \frac{d(y,\partial\Omega)d(x,\partial\Omega)}{|y-x|^2}\right\} dx,$$

and

$$U_1 := U \cap \left\{|y-x| < \tfrac{1}{2}d(y,\partial\Omega)\right\} \quad \text{and} \quad U_2 := U \cap \left\{|y-x| > \tfrac{1}{2}d(y,\partial\Omega)\right\}.$$

When $x \in U_1$ we have $|y-x| < \frac{1}{2}d(y,\partial\Omega)$, so that

$$I_1(y) \le \int_{U_1} \frac{1}{|y-x|^{n-2}}\, dx \le \tfrac{1}{2}d(y,\partial\Omega) \int_U \frac{1}{|y-x|^{n-1}}\, dx \le \tfrac{1}{2}C(U)d(y,\partial\Omega).$$

When $x \in U_2$ we get that $d(x, \partial\Omega) \le 3|y-x|$. We then get

$$I_2(y) \le d(y, \partial\Omega) \int_{U_2} \frac{d(x, \partial\Omega)}{|y-x|^n} \le 3d(y, \partial\Omega) \int_U \frac{1}{|y-x|^{n-1}}\, dx \le 3C(U)d(y, \partial\Omega).$$

Combining these estimates proves Proposition 2.6. □

The next result improves the upper estimate in Proposition 2.5.

Proposition 2.7. *There exists $C > 0$ such that*

$$|v_\alpha(x)| \le C(B_\alpha(x) + v_\infty(x)) \quad \text{for all } \alpha \text{ and all } x \in \Omega. \tag{2-67}$$

Proof. First, if $v_\infty \equiv 0$, (2-67) simply follows from (2-53). We may thus assume in the following that $v_\infty > 0$ in Ω. Proving (2-67) in Theorem 2.1 is equivalent to proving that, for any sequence $(y_\alpha)_{\alpha\in\mathbb{N}} \in \Omega$, we have

$$\frac{|v_\alpha(y_\alpha)|}{B_\alpha(y_\alpha) + v_\infty(y_\alpha)} = O(1) \quad \text{as } \alpha \to +\infty. \tag{2-68}$$

Assume first that $|y_\alpha - x_\alpha| = O(\mu_\alpha)$. It follows from (2-21) and Proposition 2.3 that

$$|v_\alpha(y_\alpha)| = O(v_\infty(y_\alpha) + B_\alpha(y_\alpha)) + o(D_\alpha(y_\alpha)^{-(n-2)/2}) = O(v_\infty(y_\alpha) + B_\alpha(y_\alpha)),$$

which proves (2-67) in this case. We thus assume from now on that

$$\lim_{\alpha\to+\infty} \frac{|y_\alpha - x_\alpha|}{\mu_\alpha} = +\infty. \tag{2-69}$$

Using Proposition 2.3 and standard elliptic theory, we have that

$$v_\alpha \to \mp v_\infty \quad \text{in } C^2_{\text{loc}}(\overline{\Omega}\backslash\{x_\infty\}) \text{ as } \alpha \to +\infty. \tag{2-70}$$

Therefore, there exists $\rho_\alpha > 0$, $\rho_\alpha \to 0$ as $\alpha \to +\infty$, such that, up to a subsequence,

$$\|v_\alpha \pm v_\infty\|_{C^2(\{|x-x_\alpha|>\rho_\alpha\}\cap\Omega)} = o(1). \tag{2-71}$$

Using again Green's representation formula and (2-12) we have

$$|v_\alpha(y_\alpha)| = O\left(\int_{\{|x-x_\alpha|\le\rho_\alpha\}\cap\Omega} G_\alpha(y_\alpha, x)|v_\alpha(x)|^{2^*-1}\, dx + \int_{\{|x-x_\alpha|>\rho_\alpha\}\cap\Omega} G_\alpha(y_\alpha, x)|v_\alpha(x)|^{2^*-1}\, dx\right). \tag{2-72}$$

Thanks to (2-11), (2-65) and (2-71), we get

$$\int_{\{|x-x_\alpha|>\rho_\alpha\}\cap\Omega} G_\alpha(y_\alpha, x)|v_\alpha(x)|^{2^*-1}\, dx = O(v_\infty(y_\alpha)). \tag{2-73}$$

We fix $R > 0$, and we now write

$$\int_{\Omega\cap\{|x-x_\alpha|\le\rho_\alpha\}} G_\alpha(y_\alpha, x)|v_\alpha(x)|^{2^*-1}\, dx = O\left(\int_{\Omega\cap\{|x-x_\alpha|\le R\mu_\alpha\}} |y_\alpha - x|^{2-n}|v_\alpha(x)|^{2^*-1}\, dx + \int_{\Omega\cap\{R\mu_\alpha\le|x-x_\alpha|\le\rho_\alpha\}} G_\alpha(y_\alpha, x)|v_\alpha(x)|^{2^*-1}\, dx\right). \tag{2-74}$$

As in the proof of (2-59), thanks to (2-3) and to Hölder's inequality, we obtain

$$\int_{\Omega\cap\{|x-x_\alpha|\le R\mu_\alpha\}} |y_\alpha - x|^{2-n}|v_\alpha(x)|^{2^*-1}\,dx = O\biggl(\frac{\mu_\alpha^{(n-2)/2}}{|y_\alpha - x_\alpha|^{n-2}}\biggr). \tag{2-75}$$

By (2-53), there exists $C > 0$ such that

$$|v_\alpha(x)|^{2^*-1} \le C(\mu_\alpha^{(n+2)/2} D_\alpha(x)^{-2-n} + \|v_\infty\|_\infty^{2^*-1}),$$

where $D_\alpha(x) := \mu_\alpha + |x - x_\alpha|$ for all $x \in \Omega$. Therefore, using again (2-11), we have

$$\begin{aligned}\int_{\Omega\cap\{R\mu_\alpha\le|x-x_\alpha|\le\rho_\alpha\}} & G_\alpha(y_\alpha, x)|v_\alpha(x)|^{2^*-1}\,dx \\ &= O\biggl(\mu_\alpha^{(n+2)/2}\int_{\Omega\cap\{|x-x_\alpha|\ge R\mu_\alpha\}} |y_\alpha - x|^{2-n}|x - x_\alpha|^{-2-n}\,dx\biggr) + O\biggl(\int_{\Omega\cap\{R\mu_\alpha\le|x-x_\alpha|\le\rho_\alpha\}} G_\alpha(y_\alpha, x)\,dx\biggr) \\ &= O\biggl(\frac{\mu_\alpha^{(n-2)/2}}{|x_\alpha - y_\alpha|^{n-2}}\biggr) + O(v_\infty(y_\alpha)). \end{aligned}\tag{2-76}$$

Combining (2-75) and (2-76) in (2-74) finally shows that

$$\int_{\Omega\cap\{|x-x_\alpha|\le\rho_\alpha\}} G_\alpha(y_\alpha, x)|v_\alpha(x)|^{2^*-1}\,dx = O(\mu_\alpha^{(n-2)/2}|x_\alpha - y_\alpha|^{2-n}) + O(v_\infty(y_\alpha))$$

as $\alpha \to +\infty$. Together with (2-72) and (2-73) this proves (2-68) and concludes the proof of (2-67). □

We are now in position to conclude the proof of Theorem 2.1.

Proof of Theorem 2.1. Proving Theorem 2.1 is equivalent to proving that, for any sequence $(y_\alpha)_{\alpha\in\mathbb{N}} \in \Omega$, we have

$$v_\alpha(y_\alpha) = \Pi B_\alpha(v_\alpha) \pm v_\infty(y_\alpha) + o(B_\alpha(y_\alpha)) + o(v_\infty(y_\alpha)) \tag{2-77}$$

as $\alpha \to +\infty$. Throughout this proof it will be intended that all the terms involving v_∞ disappear if $v_\infty \equiv 0$. If $|x_\alpha - y_\alpha| = O(\mu_\alpha)$ or if $|x_\alpha - y_\alpha| \not\to 0$, (2-77) follows from Proposition 2.3. We may thus assume in the following that

$$|x_\alpha - y_\alpha| \to 0 \quad\text{and}\quad \frac{|x_\alpha - y_\alpha|}{\mu_\alpha} \to +\infty \tag{2-78}$$

as $\alpha \to +\infty$. We write three representation formulae for v_α, ΠB_α and v_∞, using (2-2), (2-9) and (2-14), respectively, and we subtract them to get

$$\begin{aligned} v_\alpha(y_\alpha) &- \Pi B_\alpha(y_\alpha) \mp v_\infty(y_\alpha) \\ &= \int_\Omega G_\alpha(y_\alpha, \cdot)(|v_\alpha|^{2^*-2}v_\alpha - B_\alpha^{2^*-1} \mp v_\infty^{2^*-1})\,dx \pm \int_\Omega (G_\alpha(y_\alpha, \cdot) - G_\infty(y_\alpha, \cdot))v_\infty^{2^*-1}\,dx, \end{aligned}\tag{2-79}$$

where we have denoted by G_∞ the Green's function for $-\Delta + h_\infty$.

Case 1: $v_\infty \equiv 0$. In this case the second integral in (2-79) vanishes and we only have to estimate the first one. Let $R > 1$ be fixed. Using (2-12) and (2-53) and letting $\check{y}_\alpha = (y_\alpha - x_\alpha)/\mu_\alpha$, a simple change of variables and direct computations give

$$\left|\int_{\Omega\backslash B_{R\mu_\alpha}(x_\alpha)} G_\alpha(y_\alpha,\cdot)(|v_\alpha|^{2^*-2}v_\alpha - B_\alpha^{2^*-1})\,dx\right| \le C\mu_\alpha^{-(n-2)/2}\int_{\mathbb{R}^n\backslash B_R(0)} \frac{1}{|\check{y}_\alpha - x|^{n-2}} B_0^{2^*-1}\,dx$$
$$= O(\varepsilon_R B_\alpha(y_\alpha)) \tag{2-80}$$

as $\alpha \to +\infty$, where ε_R denotes a positive number satisfying $\lim_{R\to+\infty}\varepsilon_R = 0$. Independently, (2-21) and (2-20) show that

$$\left\|\frac{v_\alpha - B_\alpha}{B_\alpha}\right\|_{L^\infty(B_{R\mu_\alpha}(x_\alpha))} \to 0$$

as $\alpha \to +\infty$. As a consequence, using (2-12),

$$\left|\int_{B_{R\mu_\alpha}(x_\alpha)} G_\alpha(y_\alpha,\cdot)(|v_\alpha|^{2^*-2}v_\alpha - B_\alpha^{2^*-1})\,dx\right| = o\left(\int_{B_{R\mu_\alpha}(x_\alpha)} |y_\alpha - y|^{2-n} B_\alpha^{2^*-1}\,dx\right)$$
$$= o(B_\alpha(y_\alpha)). \tag{2-81}$$

Up to passing to a subsequence, combining (2-80) and (2-81) proves (2-77) in the $v_\infty \equiv 0$ case.

Case 2: $v_\infty > 0$. We first estimate the first integral in (2-79) by decomposing it in three domains: $B_{R\mu_\alpha}(x_\alpha)$, $(\Omega\cap B_{1/R}(x_\alpha))\backslash B_{R\mu_\alpha}(x_\alpha)$ and $\Omega\backslash B_{1/R}(x_\alpha)$. We first have

$$\int_{B_{R\mu_\alpha}(x_\alpha)} G_\alpha(y_\alpha,\cdot)(|v_\alpha|^{2^*-2}v_\alpha - B_\alpha^{2^*-1} \mp v_\infty^{2^*-1})\,dx$$
$$= \int_{B_{R\mu_\alpha}(x_\alpha)} G_\alpha(y_\alpha,\cdot)(|v_\alpha|^{2^*-2}v_\alpha - B_\alpha^{2^*-1})\,dx + O\left(\int_{B_{R\mu_\alpha}(x_\alpha)} G_\alpha(y_\alpha,\cdot)\,dx\right)$$
$$= o(B_\alpha(y_\alpha)) + o(v_\infty(y_\alpha)), \tag{2-82}$$

where the last line follows from (2-81) and from (2-11) and (2-65) with $U = B_{R\mu_\alpha}(x_\alpha)$. Using (2-71) we now have

$$\int_{\Omega\backslash B_{1/R}(x_\alpha)} G_\alpha(y_\alpha,\cdot)(|v_\alpha|^{2^*-2}v_\alpha - B_\alpha^{2^*-1} \mp v_\infty^{2^*-1})\,dx$$
$$= \int_{\Omega\backslash B_{1/R}(x_\alpha)} G_\alpha(y_\alpha,\cdot)(|v_\alpha|^{2^*-2}v_\alpha \mp v_\infty^{2^*-1})\,dx + O(\mu_\alpha^{(n+2)/2})$$
$$= o\left(\int_\Omega G_\alpha(y_\alpha, y)\,dy\right) + o(B_\alpha(y_\alpha)) = o(B_\alpha(y_\alpha)) + o(v_\infty(y_\alpha)), \tag{2-83}$$

where the last equality again follows from (2-11) and (2-65). Finally, using (2-12) and (2-53) we have

$$\left|\int_{(\Omega\cap B_{1/R}(x_\alpha))\backslash B_{R\mu_\alpha}(x_\alpha)} G_\alpha(y_\alpha,\cdot)(|v_\alpha|^{2^*-2}v_\alpha - B_\alpha^{2^*-1} \mp v_\infty^{2^*-1})\,dx\right|$$
$$= O\left(\int_{\Omega\backslash B_{R\mu_\alpha}(x_\alpha)} |y_\alpha - y|^{2-n} B_\alpha^{2^*-1}\,dx\right) + O\left(\int_{\Omega\cap B_{1/R}(x_\alpha)} G_\alpha(y_\alpha, y)\,dy\right)$$
$$= O(\varepsilon_R B_\alpha(y_\alpha)) + O(\varepsilon_R v_\infty(y_\alpha)), \tag{2-84}$$

where the last line follows from (2-80) and (2-65) with $U = \Omega \cap B_{1/R}(x_\alpha)$. Combining (2-82), (2-83) and (2-84) proves that

$$\int_\Omega G_\alpha(y_\alpha, \cdot)(|v_\alpha|^{2^*-2} v_\alpha - B_\alpha^{2^*-1} \mp v_\infty^{2^*-1})\, dx = o(B_\alpha(y_\alpha)) + o(v_\infty(y_\alpha)) + O(\varepsilon_R B_\alpha(y_\alpha)) + O(\varepsilon_R v_\infty(y_\alpha)) \quad (2\text{-}85)$$

as $\alpha \to +\infty$, where $\lim_{R\to+\infty} \varepsilon_R = 0$. We now estimate the second integral in (2-79). For $y \in \Omega$ and for all α, we let

$$F_{1,\alpha}(y) = \int_\Omega G_\alpha(y, \cdot) v_\infty^{2^*-1}\, dx \quad \text{and} \quad F_2(y) = \int_\Omega G_\infty(y, \cdot) v_\infty^{2^*-1}\, dx.$$

By definition of G_α and G_∞, these functions satisfy $(-\Delta + h_\alpha)F_{1,\alpha} = v_\infty^{2^*-1}$ and $(-\Delta + h_\infty)F_2 = v_\infty^{2^*-1}$, respectively, so that by (2-1) and standard elliptic theory $(F_{1,\alpha})_{\alpha\in\mathbb{N}}$ is uniformly bounded in $L^\infty(\Omega)$. We also have

$$(-\Delta + h_\infty)(F_{1,\alpha} - F_2) = (h_\infty - h_\alpha)F_{1,\alpha}.$$

A representation formula for $F_{1,\alpha} - F_2$ applied at y_α then shows

$$\int_\Omega (G_\alpha(y_\alpha, \cdot) - G_\infty(y_\alpha, \cdot)) v_\infty^{2^*-1}\, dx = F_{1,\alpha}(y_\alpha) - F_2(y_\alpha) = \int_\Omega G_\infty(y_\alpha, \cdot)(h_\infty - h_\alpha) F_{1,\alpha}\, dx.$$

Using (2-1), (2-11) and (2-65) we thus obtain

$$\left| \int_\Omega (G_\alpha(y_\alpha, \cdot) - G_\infty(y_\alpha, \cdot)) v_\infty^{2^*-1}\, dx \right| = o\left(\int_\Omega G_\infty(y_\alpha, x)\, dx \right) = o(v_\infty(y_\alpha)). \quad (2\text{-}86)$$

Plugging (2-85) and (2-86) into (2-79) finally proves that

$$|v_\alpha(y_\alpha) - \Pi B_\alpha(y_\alpha) \mp v_\infty(y_\alpha)| = o(B_\alpha(y_\alpha)) + o(v_\infty(y_\alpha)) + O(\varepsilon_R B_\alpha(y_\alpha)) + O(\varepsilon_R v_\infty(y_\alpha))$$

as $\alpha \to +\infty$, where $\lim_{R\to+\infty} \varepsilon_R = 0$. Passing to a subsequence proves (2-77) and concludes the proof of Theorem 2.1. □

3. Necessary conditions for blow-up and proof of Theorem 1.1

Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$. Throughout this section we let $(h_\alpha)_{\alpha\in\mathbb{N}}$ be a sequence of functions that converges in $C^1(\overline{\Omega})$ to h_∞, where $-\Delta + h_\infty$ is coercive in $H_0^1(\Omega)$ and where $I_{h_\infty}(\Omega) < K_n^{-2}$, and we let $(v_\alpha)_{\alpha\in\mathbb{N}} \in H_0^1(\Omega)$ be a sequence of solutions of (2-2) that satisfies (2-3), (2-4) and (2-5). Equation (2-15) is thus also satisfied, and we have

$$v_\alpha = \Pi B_\alpha \pm v_\infty + o(1) \quad \text{in } H_0^1(\Omega) \text{ as } \alpha \to +\infty,$$

where ΠB_α is given by (2-14) and where $(x_\alpha)_{\alpha\in\mathbb{N}}$ and $(\mu_\alpha)_{\alpha\in\mathbb{N}}$ are sequences of points in Ω and $(0, +\infty)$ satisfying (2-10) and with $\lim_{\alpha\to+\infty} \mu_\alpha = 0$. We let again $x_\infty = \lim_{\alpha\to+\infty} x_\alpha$, and we identify in this section necessary blow-up conditions that constrain the localisation of x_∞. We recall for this the celebrated

Pohozaev identity, that for our sequence $(v_\alpha)_{\alpha\in\mathbb{N}}$ is as follows: for any family U_α of smooth domains such that $x_\alpha \in U_\alpha \subset \Omega$ for $\alpha \in \mathbb{N}$ we have

$$\int_{U_\alpha} \big(h_\alpha(x) + \tfrac{1}{2}\langle \nabla h_\alpha(x), x - x_\alpha\rangle\big) v_\alpha^2 \, dx$$
$$= \int_{\partial U_\alpha} \langle x - x_\alpha, \nu\rangle \left(\frac{|\nabla v_\alpha|^2}{2} + h_\alpha \frac{v_\alpha^2}{2} - \frac{|v_\alpha|^{2^*}}{2^*} \right) d\sigma(x) - \int_{\partial U_\alpha} \big(\langle x - x_\alpha, \nabla v_\alpha\rangle + \tfrac{1}{2}(n-2) v_\alpha\big) \partial_\nu v_\alpha \, d\sigma(x), \tag{3-1}$$

where ν is the outer unit normal to the boundary of U_α and $\langle\,\cdot\,,\cdot\,\rangle$ is the Euclidean scalar product; see for instance [Hebey 2014, Lemma 6.5]. We distinguish two cases according to whether x_∞ is a boundary blow-up point or not.

3.1. ***Interior blow-up case*: $x_\infty \in \Omega$.** If x_∞ is an interior point we prove the following result:

Proposition 3.1. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$. Let $(h_\alpha)_{\alpha\in\mathbb{N}}$ be a sequence of functions that converges in $C^1(\overline{\Omega})$ to h_∞, where $-\Delta + h_\infty$ is coercive in $H_0^1(\Omega)$ and where $I_{h_\infty}(\Omega) < K_n^{-2}$, and we let $(v_\alpha)_{\alpha\in\mathbb{N}} \in H_0^1(\Omega)$ be a sequence of solutions of* (2-2) *that satisfies* (2-3), (2-4) *and* (2-5)*. Let $x_\infty = \lim_{\alpha\to+\infty} x_\alpha$ and assume that $x_\infty \in \Omega$. Then*

- *if $n = 3$, we have $v_\infty \equiv 0$ and $m_{h_\infty}(x_\infty) = 0$,*
- *if $n = 4, 5$, we have $v_\infty \equiv 0$ and $h_\infty(x_\infty) = 0$,*
- *if $n = 6$, we have $h_\infty(x_\infty) = \pm 2 v_\infty(x_\infty)$,*
- *if $n \geq 7$, we have $h_\infty(x_\infty) = 0$.*

Proof. First, since $x_\infty \in \Omega$, we have $B_{\delta\sqrt{\mu_\alpha}}(x_\alpha) \subset \Omega$ for all α large enough. The Pohozaev identity (3-1) yields

$$\int_{B_{\delta\sqrt{\mu_\alpha}}(x_\alpha)} \big(h_\alpha(x) + \tfrac{1}{2}\langle \nabla h_\alpha(x), x - x_\alpha\rangle\big) v_\alpha^2 \, dx = \int_{\partial B_{\delta\sqrt{\mu_\alpha}}(x_\alpha)} F_\alpha(x) \, d\sigma(x), \tag{3-2}$$

where we have let

$$F_\alpha(x) := \langle x - x_\alpha, \nu\rangle \left(\frac{|\nabla v_\alpha|^2}{2} + h_\alpha \frac{v_\alpha^2}{2} - \frac{|v_\alpha|^{2^*}}{2^*} \right) - \big(\langle x - x_\alpha, \nabla v_\alpha\rangle + \tfrac{1}{2}(n-2) v_\alpha\big) \partial_\nu v_\alpha. \tag{3-3}$$

For any $x \in (\Omega - x_\alpha)/\sqrt{\mu_\alpha}$ we let

$$\hat{v}_\alpha(x) = v_\alpha(x_\alpha + \sqrt{\mu_\alpha}\, x).$$

Using (2-2) it is easily seen that $\hat{v}_\alpha$ satisfies

$$\begin{cases} -\Delta \hat{v}_\alpha + \mu_\alpha \hat{h}_\alpha \hat{v}_\alpha = \mu_\alpha |\hat{v}_\alpha|^{2^*-2} \hat{v}_\alpha & \text{in } (\Omega - x_\alpha)/\sqrt{\mu_\alpha}, \\ \hat{v}_\alpha = 0 & \text{on } \partial((\Omega - x_\alpha)/\sqrt{\mu_\alpha}), \end{cases}$$

where we have let $\hat{h}_\alpha(x) = h(x_\alpha + \sqrt{\mu_\alpha}\, x)$. By (2-67) and standard elliptic theory there thus exists $\hat{v}_\infty \in C^2(\mathbb{R}^n \backslash \{0\})$ such that $\hat{v}_\alpha \to \hat{v}_\infty$ in $C^2_{\mathrm{loc}}(\mathbb{R}^n \backslash \{0\})$, and Theorem 2.1 shows that for any $x \in \mathbb{R}^n \backslash \{0\}$ we have

$$\hat{v}_\infty(x) = (n(n-2))^{(n-2)/2} |x|^{2-n} \pm v_\infty(x_\infty).$$

The change of variables $x = x_\alpha + \sqrt{\mu_\alpha}\, y$ and straightforward computations then show that

$$\begin{aligned}
&\mu_\alpha^{-(n-2)/2} \int_{\partial B_{\delta\sqrt{\mu_\alpha}}(x_\alpha)} F_\alpha(x)\, d\sigma(x) \\
&\quad = \int_{\partial B_\delta(0)} \langle x, \nu\rangle \left(\frac{|\nabla \hat{v}_\alpha|^2}{2} + \mu_\alpha \hat{h}_\alpha \frac{\hat{v}_\alpha^2}{2} - \mu_\alpha \frac{|\hat{v}_\alpha|^{2^*}}{2^*} \right) d\sigma(x) - \int_{\partial B_\delta(0)} \bigl(\langle x, \nabla \hat{v}_\alpha \rangle + \tfrac{1}{2}(n-2)\hat{v}_\alpha \bigr) \partial_\nu \hat{v}_\alpha\, d\sigma(x) \\
&\quad = \pm \tfrac{1}{2} \omega_{n-1} n^{(n-2)/2} (n-2)^{(n+2)/2} v_\infty(x_\infty) + \varepsilon_\delta + o(1)
\end{aligned} \tag{3-4}$$

as $\alpha \to +\infty$, where ε_δ denotes a quantity such that $\lim_{\delta \to 0} \varepsilon_\delta = 0$ and where ω_{n-1} is the area of the round sphere $\mathbb{S}^{n-1}$. We now claim that

$$\int_{B_{\delta\sqrt{\mu_\alpha}}(x_\alpha)} \bigl(h_\alpha(x) + \tfrac{1}{2}\langle \nabla h_\alpha(x), x - x_\alpha\rangle \bigr) v_\alpha^2\, dx = \begin{cases} O(\mu_\alpha^{3/2}) & \text{if } n = 3, \\ O(\mu_\alpha^2 \ln(1/\mu_\alpha)) & \text{if } n = 4, \\ \mu_\alpha^2 \bigl(h_\infty(x_\infty) \int_{\mathbb{R}^n} B_0(x)^2\, dx + o(1) \bigr) & \text{if } n \ge 5, \end{cases} \tag{3-5}$$

where B_0 is defined in (2-7). We prove (3-5). First, using (2-16) and Theorem 2.1, straightforward computations show that

$$\int_{B_{\delta\sqrt{\mu_\alpha}}(x_\alpha)} \tfrac{1}{2}\langle \nabla h_\alpha(x), x - x_\alpha\rangle v_\alpha^2\, dx = \begin{cases} O(\mu_\alpha^2) & \text{if } n = 3, 4, \\ O(\mu_\alpha^3 |\ln \mu_\alpha|) & \text{if } n \ge 5, \end{cases} \tag{3-6}$$

and that

$$\int_{B_{\delta\sqrt{\mu_\alpha}}(x_\alpha)} h_\alpha(x) v_\alpha^2\, dx = \begin{cases} O(\mu_\alpha^{3/2}) & \text{if } n = 3, \\ O(\mu_\alpha^2 \ln(1/\mu_\alpha)) & \text{if } n = 4. \end{cases} \tag{3-7}$$

If $n \ge 5$, using Theorem 2.1, we have

$$\int_{B_{\delta\sqrt{\mu_\alpha}}(x_\alpha)} h_\alpha(x) v_\alpha^2\, dx = \int_{B_{\delta\sqrt{\mu_\alpha}}(x_\alpha)} h_\alpha(x) (\Pi B_\alpha)^2\, dx + o(\mu_\alpha^2).$$

Dominated convergence together with (2-21) now shows that

$$\int_{B_{\delta\sqrt{\mu_\alpha}}(x_\alpha)} h_\alpha(x) (\Pi B_\alpha)^2\, dx = h_\infty(x_\infty) \int_{\mathbb{R}^n} \mu_\alpha^2 B_0(x)^2\, dx + o(\mu_\alpha^2).$$

Combining the latter with (3-6) and (3-7) proves (3-5). Combining (3-2), (3-4) and (3-5) now shows that

$$\begin{aligned}
&\pm \tfrac{1}{2} \omega_{n-1} n^{(n-2)/2} (n-2)^{(n+2)/2} v_\infty(x_\infty) \mu_\alpha^{(n-2)/2} + \varepsilon_\delta \mu_\alpha^{(n-2)/2} + o(\mu_\alpha^{(n-2)/2}) \\
&\qquad = \begin{cases} O(\mu_\alpha^{3/2}) & \text{if } n = 3, \\ O(\mu_\alpha^2 \ln(1/\mu_\alpha)) & \text{if } n = 4, \\ \mu_\alpha^2 \bigl(h_\infty(x_\infty) \int_{\mathbb{R}^n} B_0^2\, dx + o(1) \bigr) & \text{if } n \ge 5. \end{cases}
\end{aligned} \tag{3-8}$$

Assume first that $n \in \{3, 4, 5\}$. Equation (3-8) then gives

$$v_\infty(x_\infty) + \varepsilon_\delta + o(1) = \begin{cases} O(\mu_\alpha) & \text{if } n = 3, \\ O(\mu_\alpha \ln(1/\mu_\alpha)) & \text{if } n = 4, \\ O(\sqrt{\mu_\alpha}) & \text{if } n = 5 \end{cases}$$

as $\alpha \to +\infty$. Letting first $\alpha \to +\infty$ then $\delta \to 0$ shows that $v_\infty(x_\infty) = 0$. Since $v_\infty \ge 0$ by (2-3) and the assumption $I_{h_\infty}(\Omega) < K_n^{-2}$, the strong maximum principle then shows that $v_\infty \equiv 0$.

Assume now that $n = 6$. Integrating $-\Delta B_0 = B_0^2$ shows that

$$\int_{\mathbb{R}^6} B_0^2\,dx = 6^2 4^3 \omega_5.$$

Therefore, it follows from (3-8) that

$$\pm\tfrac{1}{2}\omega_5 6^2 4^4 v_\infty(x_\infty)\mu_\alpha^2 + \varepsilon_\delta \mu_\alpha^2 + o(\mu_\alpha^2) = 6^2 4^3 \omega_5 h_\infty(x_\infty)\mu_\alpha^2 + o(\mu_\alpha^2).$$

Letting $\alpha \to +\infty$ and then $\delta \to 0$ shows that

$$h_\infty(x_\infty) = \pm 2 v_\infty(x_\infty).$$

Assume finally that $n \geq 7$. Then $\mu_\alpha^{(n-2)/2} = o(\mu_\alpha^2)$ as $\alpha \to +\infty$, and (3-8) then gives, after letting $\alpha \to +\infty$,

$$h_\infty(x_\infty) = 0.$$

These considerations prove Proposition 3.1 in the case $n \geq 6$.

To conclude the proof of Proposition 3.1 we now consider the case where $3 \leq n \leq 5$ and $v_\infty \equiv 0$. We let $\delta > 0$ be small enough that $B_\delta(x_\alpha) \subset \Omega$ for all α, and we write a Pohozaev identity in $B_\delta(x_\alpha)$,

$$\int_{B_\delta(x_\alpha)} \big(h_\alpha(x) + \tfrac{1}{2}\langle \nabla h_\alpha(x), x - x_\alpha\rangle\big) v_\alpha^2\,dx = \int_{\partial B_\delta(x_\alpha)} F_\alpha(x)\,d\sigma(x), \tag{3-9}$$

where F_α is again as in (3-3). For $x \in \Omega$ we let in this case

$$\hat{v}_\alpha(x) = \mu_\alpha^{-(n-2)/2} v_\alpha(x).$$

Using (2-2) it is easily seen that $\hat{v}_\alpha$ satisfies

$$\begin{cases} -\Delta \hat{v}_\alpha + h_\alpha \hat{v}_\alpha = \mu_\alpha^2 |\hat{v}_\alpha|^{2^*-2}\hat{v}_\alpha & \text{in } \Omega,\\ \hat{v}_\alpha = 0 & \text{on } \partial\Omega,\end{cases}$$

and (2-16) and (2-67) show that

$$|\hat{v}_\alpha(x)| \leq \frac{C}{|x - x_\alpha|^{n-2}} \quad \text{for all } x \in \Omega\backslash\{x_\alpha\},$$

where C is a positive constant independent of α. Standard elliptic theory with (2-20) then shows that $\hat{v}_\alpha \to \hat{v}_\infty$ in $C^2_{\mathrm{loc}}(\overline{\Omega}\backslash\{x_\infty\})$, where

$$\hat{v}_\infty(x) = (n-2)\omega_{n-1}(n(n-2))^{(n-2)/2} G_\infty(x_\infty, x)$$

and where G_∞ is the Green's function for $-\Delta + h_\infty$ with Dirichlet boundary conditions in Ω, which is the only solution to

$$\begin{cases} -\Delta_y G_{h_\infty}(x,y) + h G_{h_\infty}(x,y) = \delta_x & \text{in } \Omega,\\ G_{h_\infty}(x,y) = 0 & \text{for } y \in \partial\Omega,\quad x \in \Omega.\end{cases}$$

When $n = 3$ it is well known that

$$G_\infty(x_\infty, y) = \frac{1}{4\pi|x-y|} + m_{h_\infty}(x_\infty) + O(|x_\infty - y|) \quad \text{for all } y \in \Omega\backslash\{x_\infty\}.$$

Straightforward computations with the latter then show that

$$\mu_\alpha^{2-n}\int_{B_\delta(x_\alpha)} F_\alpha(x)\,d\sigma(x)=\begin{cases}24\pi^2 m_{h_\infty}(x_\infty)+\varepsilon_\delta+o(1), & n=3,\\ O(1), & n=4,5,\end{cases}\tag{3-10}$$

where $\lim_{\delta\to 0}\varepsilon_\delta=0$. Independently, straightforward computations using Theorem 2.1 (see, e.g., [Premoselli and Robert 2025, Section 5]) show that

$$\int_{B_\delta(x_\alpha)}\big(h_\alpha(x)+\tfrac{1}{2}\langle\nabla h_\alpha(x),x-x_\alpha\rangle\big)v_\alpha^2\,dx=\begin{cases}O(\delta\mu_\alpha) & \text{if } n=3,\\ 64\omega_3 h_\infty(x_\infty)\mu_\alpha^2\ln(1/\mu_\alpha)+O(\mu_\alpha^2) & \text{if } n=4,\\ \mu_\alpha^2\big(h_\infty(x_\infty)\int_{\mathbb{R}^n}B_0(x)^2\,dx+o(1)\big) & \text{if } n\geq 5\end{cases}\tag{3-11}$$

as $\alpha\to+\infty$. If $n\in\{4,5\}$, combining (3-10) and (3-11) in (3-9) shows that

$$h_\infty(x_\infty)+o(1)=\begin{cases}O(\ln(1/\mu_\alpha)^{-1}), & n=4,\\ O(\mu_\alpha), & n=5,\end{cases}$$

as $\alpha\to+\infty$, which shows that $h_\infty(x_\infty)=0$. If $n=3$, combining (3-10) and (3-11) in (3-9) shows that

$$m_{h_\infty}(x_\infty)+o(1)+\varepsilon_\delta=O(\delta)$$

as $\alpha\to+\infty$. Letting first $\alpha\to+\infty$ then $\delta\to 0$ proves that $m_{h_\infty}(x_\infty)=0$, which concludes the proof of Proposition 3.1. □

3.2. *Boundary blow-up case*: $x_\infty\in\partial\Omega$. We assume in this subsection that $x_\infty\in\partial\Omega$. For $\alpha\geq 1$, we let

$$d_\alpha=d(x_\alpha,\partial\Omega)\to 0\tag{3-12}$$

as $\alpha\to+\infty$, since $x_\infty\in\partial\Omega$. We know from (2-18) that $d_\alpha\gg\mu_\alpha$ as $\alpha\to+\infty$. For $\alpha\geq 1$ we also let

$$r_\alpha=\frac{\sqrt{\mu_\alpha}}{d_\alpha^{1/(n-2)}},\tag{3-13}$$

and we analyse the bubbling behaviour of v_α at the scale r_α. The idea to consider the scale r_α comes from the following heuristic. Recall that when $v_\infty>0$, Hopf's lemma shows that

$$v_\infty(x_\infty-t\nu(x_\infty))=(-\partial_\nu v_\infty(x_\infty))t+o(t)$$

as $t\to 0$. At distance d_α from $\partial\Omega$, v_∞ thus behaves at first order as $(-\partial_\nu v_\infty(x_\infty))d_\alpha$. The scale r_α thus defines the distance from x_α at which B_α and v_∞ become of the same size. We analyse the boundary blow-up of v_α according to the value of d_α/r_α. We first prove the following result, which states that boundary blow-up points cannot get too close to $\partial\Omega$:

Proposition 3.2. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n\geq 3$. Let $(h_\alpha)_{\alpha\in\mathbb{N}}$ be a sequence of functions that converges in $C^1(\overline{\Omega})$ to h_∞, where $-\Delta+h_\infty$ is coercive in $H_0^1(\Omega)$ and where $I_{h_\infty}(\Omega)<K_n^{-2}$, and we let $(v_\alpha)_{\alpha\in\mathbb{N}}\in H_0^1(\Omega)$ be a sequence of solutions of* (2-2) *that satisfies* (2-3), (2-4) *and* (2-5). *Let $x_\infty=\lim_{\alpha\to+\infty}x_\alpha$, and assume that $x_\infty\in\partial\Omega$. If $n\geq 6$, assume in addition that $h_\infty\neq 0$ in $\overline{\Omega}$. Then, up to a subsequence,*

$$\frac{d_\alpha}{r_\alpha}\to+\infty$$

as $\alpha\to+\infty$.

Proof. We proceed by contradiction, and we assume that, up to a subsequence,

$$\lim_{\alpha\to+\infty}\frac{d_\alpha}{r_\alpha}=\rho\in[0,+\infty). \tag{3-14}$$

In this case we define, for all $x\in(\Omega-x_\alpha)/d_\alpha$,

$$\bar v_\alpha(x):=\frac{d_\alpha^{n-2}}{\mu_\alpha^{(n-2)/2}}v_\alpha(x_\alpha+d_\alpha x). \tag{3-15}$$

Equation (2-2) and the definition of $\bar v_\alpha$ show that $\bar v_\alpha$ satisfies

$$\begin{cases}-\Delta\bar v_\alpha-d_\alpha^2\bar h_\alpha\bar v_\alpha=(\mu_\alpha/d_\alpha)^2|\bar v_\alpha|^{2^*-2}\bar v_\alpha & \text{in } (\Omega-x_\alpha)/d_\alpha,\\ \bar v_\alpha=0 & \text{on } \partial((\Omega-x_\alpha)/d_\alpha),\end{cases} \tag{3-16}$$

where $\bar v_\alpha$ is as in (3-15) and $\bar h_\alpha(x):=h(x_\alpha+d_\alpha x)$. By (3-13) and (3-14) we have

$$d_\alpha=O(\mu_\alpha^{(n-2)(n-1)/2})\quad\text{or, equivalently,}\quad\frac{d_\alpha^{n-2}}{\mu_\alpha^{(n-2)/2}}\cdot d_\alpha=O(1). \tag{3-17}$$

By Hopf's lemma we have

$$v_\infty(x_\alpha+d_\alpha x)=v_\infty(x_\alpha)+O(d_\alpha)=O(d_\alpha) \tag{3-18}$$

as $\alpha\to+\infty$, and the latter remains obviously true if $v_\infty\equiv 0$. The latter with (2-16) and Theorem 2.1 show that

$$|\bar v_\alpha(x)|\le C(1+|x|^{2-n})\quad\text{for all } x\in\frac{\Omega-x_\alpha}{d_\alpha}\backslash\{0\} \tag{3-19}$$

for some positive constant C. Since Ω is smooth and since $d_\alpha\to 0$ as $\alpha\to+\infty$ by assumption, standard elliptic theory shows that, up to a rotation, $\bar v_\alpha\to\bar v_\infty\in C^2(\bar\Omega_0\backslash\{0\})$, where we have let

$$\Omega_0:=\,]-\infty,1[\,\times\mathbb{R}^{n-1}\quad\text{as }\alpha\to+\infty \tag{3-20}$$

and where $\bar v_\infty$ satisfies

$$-\Delta\bar v_\infty=0\quad\text{in }\Omega_0\backslash\{0\},\qquad\bar v_\infty=0\quad\text{on }\partial\Omega_0, \tag{3-21}$$

and

$$|\bar v_\infty(x)|\le C(1+|x|^{2-n})\quad\text{for all } x\in\Omega_0. \tag{3-22}$$

Lemma 3.3. *We have*

$$\bar v_\infty(x)=\frac{(n(n-2))^{(n-2)/2}}{|x|^{n-2}}+\mathcal{H}(x)\quad\textit{for all } x\in\Omega_0\backslash\{0\}, \tag{3-23}$$

where $\mathcal{H}$ satisfies

$$-\Delta\mathcal{H}=0\quad\textit{in }\Omega_0,\qquad\mathcal{H}=-(n(n-2))^{-(n-2)/2}|\cdot|^{2-n}\quad\textit{on }\partial\Omega_0, \tag{3-24}$$

and $\mathcal{H}(0)<0$.

Proof of Lemma 3.3. Let $0 < \delta < 1$ be fixed, and let $x \in \partial B_\delta(0)\backslash\{0\}$. For $\alpha \geq 1$, Lemma A.1 shows that

$$\frac{d_\alpha^{n-2}}{\mu_\alpha^{(n-2)/2}} \Pi B_\alpha(x_\alpha + d_\alpha x) = \frac{(n(n-2))^{(n-2)/2}}{|x|^{n-2}} + o(1) + \frac{\varepsilon(|x|)}{|x|^{n-2}} \tag{3-25}$$

as $\alpha \to +\infty$, where $\varepsilon(|x|)$ denotes a function that satisfies $\lim_{|x|\to 0} \varepsilon(|x|) = 0$. We now consider $\bar{v}_\infty$ satisfying (3-21). By (3-22) and Bôcher's theorem [Axler et al. 1992; Bôcher 1903] there exist $\Lambda \neq 0$ and a harmonic function $\mathcal{H}$ in Ω_0 such that

$$\bar{v}_\infty(x) = \Lambda |x|^{2-n} + \mathcal{H}(x) \quad \text{for } x \in \Omega_0. \tag{3-26}$$

Theorem 2.1 together with (3-17) shows that

$$\left| \bar{v}_\alpha(x) - \frac{d_\alpha^{n-2}}{\mu_\alpha^{(n-2)/2}} \Pi B_\alpha(x_\alpha + d_\alpha x) \right| \leq C + o(1)$$

for $x \in B_\delta(0)\backslash\{0\}$, for some fixed $C > 0$ as $\alpha \to +\infty$. Multiplying the latter by $|x|^{n-2}$ and passing to the limit as $\alpha \to +\infty$ then shows, using (3-25), that

$$\left| |x|^{n-2} \bar{v}_\infty(x) - (1 + \varepsilon(|x|))(n(n-2))^{(n-2)/2} \right| \leq C|x|^{n-2}.$$

Letting $x \to 0$ then shows that $\Lambda = (n(n-2))^{(n-2)/2}$ and proves (3-23). That $\mathcal{H}$ satisfies (3-24) is a simple consequence of the Dirichlet boundary conditions.

To conclude the proof of Lemma 3.3 we thus need to show that $\mathcal{H}(0) < 0$. For $x \in \Omega_0$ as in (3-20) we define

$$\widetilde{\mathcal{H}}(x) = 2\frac{n^{(n-4)/2}(n-2)^{(n-2)/2}}{\omega_{n-1}} (x_1 - 1) \int_{\partial\Omega_0} |y|^{2-n} |x-y|^{-n} \, d\sigma(y). \tag{3-27}$$

If $y \in \Omega_0$, we let $y^* := (2 - y_1, y') \in \mathbb{R}^n$ be its reflection with respect to the hyperplane $\{y_1 = 1\}$. For $x, y \in \Omega_0$, $x \neq y$, we let

$$G_0(x, y) = \frac{1}{(n-2)\omega_{n-1}} (|x-y|^{2-n} - |x-y^*|^{2-n})$$

be the Green's function of $-\Delta$ in Ω_0 with Dirichlet boundary conditions. Straightforward computations show that

$$\partial_\nu G_0(x, y) = \frac{2(x_1 - 1)}{n w_{n-1}} \frac{1}{|x-y|^n} \quad \text{for } x \in \Omega_0 \text{ and } y \in \partial\Omega_0,$$

so that $\widetilde{\mathcal{H}}$ can be rewritten as

$$\widetilde{\mathcal{H}}(x) = \int_{\partial\Omega_0} \frac{(n(n-2))^{(n-2)/2}}{|y|^{n-2}} \partial_\nu G_0(x, y) \, d\sigma(y).$$

In particular, $\widetilde{\mathcal{H}}$ satisfies

$$-\Delta \widetilde{\mathcal{H}} = 0 \quad \text{in } \Omega_0, \qquad \widetilde{\mathcal{H}} = -(n(n-2))^{-(n-2)/2} |\cdot|^{2-n} \quad \text{on } \partial\Omega_0,$$

and we have

$$\widetilde{\mathcal{H}}(0) = -2\frac{(n(n-2))^{(n-2)/2}}{n w_{n-1}} \int_{\mathbb{R}^{n-1}} (1 + |y'|^2)^{1-n} \, dy' < 0. \tag{3-28}$$

We now claim that

$$\mathcal{H} = \widetilde{\mathcal{H}} \quad \text{in } \Omega_0. \tag{3-29}$$

To prove (3-29) we first prove that $\widetilde{\mathcal{H}} \in L^\infty(\Omega_0)$. We write any $y \in \partial\Omega_0$ as $y = (1, y')$ with $y' \in \mathbb{R}^n$. We similarly write $x \in \Omega_0$ as $x = (x_1, x')$ with $x_1 < 1$. If $x \in \Omega_0$, with (3-27) and a simple change of variables we thus have, for some positive constant $C = C(n)$,

$$|\widetilde{\mathcal{H}}(x)| \le C(1 - x_1) \int_{\partial\Omega_0} \frac{1}{((x_1 - 1)^2 + |y'|^2)^{n/2}}\, dy' \le C \int_{\partial\Omega_0} \frac{1}{(1 + |y'|^2)^{n/2}}\, dy' < +\infty,$$

where the last line again follows from a change of variables. Thus $\widetilde{\mathcal{H}}$ is bounded in $\Omega_0 \backslash B_{\varepsilon_0}(1)$. We can now conclude the proof of Lemma 3.3. Since $\mathcal{H}$ is harmonic in Ω_0 it is bounded in $B_{1/2}(0)$. Equations (3-22) and (3-23) also show that $\mathcal{H}$ is bounded in Ω_0. Independently, we just proved that $\widetilde{\mathcal{H}} \in L^\infty(\Omega_0)$. The function $\mathcal{H} - \widetilde{\mathcal{H}}$ is thus harmonic in Ω_0, bounded in Ω_0 and vanishes on $\partial\Omega_0$. Since $\partial\Omega_0$ is a hyperplane a simple reflection argument allows to apply Liouville's theorem, which shows that $\mathcal{H} \equiv \widetilde{\mathcal{H}}$. This proves (3-29) and by (3-28) conclude the proof of Lemma 3.3. □

We are now in position to prove Proposition 3.2. Let $\delta > 0$ be fixed. We write Pohozaev's identity (3-1) in $U_\alpha = B_{\delta d_\alpha}(x_\alpha)$: this gives

$$\int_{B_{\delta d_\alpha}(x_\alpha)} \big(h_\alpha(x) + \tfrac{1}{2}\langle \nabla h_\alpha(x), x - x_\alpha\rangle\big) v_\alpha^2\, dx = \int_{\partial B_{\delta d_\alpha}(x_\alpha)} F_\alpha(x)\, d\sigma(x), \tag{3-30}$$

where F_α is defined in (3-3). Changing variables we get that

$$\begin{aligned}&\left(\frac{\mu_\alpha}{d_\alpha}\right)^{2-n} \int_{\partial B_{\delta d_\alpha}(x_\alpha)} F_\alpha(x)\, d\sigma(x)\\ &= \int_{\partial B_\delta(0)} \langle x, \nu\rangle \left(\frac{|\nabla \bar{v}_\alpha|^2}{2} + \bar{h}_\alpha d_\alpha^2 \frac{\bar{v}_\alpha^2}{2} - d_\alpha^2 \frac{|\bar{v}_\alpha|^{2^*}}{2^*}\right) d\sigma(x) - \int_{\partial B_\delta(0)} \big(\langle x, \nabla \bar{v}_\alpha\rangle + \tfrac{1}{2}(n-2)\bar{v}_\alpha\big)\partial_\nu \bar{v}_\alpha\, d\sigma(x),\end{aligned} \tag{3-31}$$

where $\bar{v}_\alpha$ is defined in (3-15). Direct calculations using (3-17) and (3-19) yield, since $h_\alpha \in L^\infty(\Omega)$,

$$\begin{aligned} d_\alpha^2 \int_{\partial B_\delta(0)} \langle x, \nu\rangle \bar{h}_\alpha \bar{v}_\alpha^2\, d\sigma(x) &= O(d_\alpha^2 \delta^{4-n} + \mu_\alpha^{(n-2)/(n-1)}\delta^n) = o(1),\\ d_\alpha^2 \int_{\partial B_\delta(0)} \langle x, \nu\rangle |v_\alpha|^{2^*}\, d\sigma(x) &= O(\delta^{-n} d_\alpha^2 + \mu_\alpha^{(n-2)/(n-1)}\delta^n) = o(1)\end{aligned} \tag{3-32}$$

as $\alpha \to +\infty$. Plugging (3-32) into (3-31) gives, since $\bar{v}_\alpha \to \bar{v}_\infty \in C^2(\overline{\Omega}_0 \backslash \{0\})$,

$$\begin{aligned}&\lim_{\alpha\to+\infty} \left(\frac{\mu_\alpha}{d_\alpha}\right)^{2-n} \int_{\partial B_{\delta d_\alpha}(x_\alpha)} F_\alpha(x)\, d\sigma(x)\\ &\qquad = \int_{\partial B_\delta(0)} |x| \big(\tfrac{1}{2}|\nabla \bar{v}_\infty|^2 - (\partial_\nu \bar{v}_\infty)^2\big)\, d\sigma(x) - \tfrac{1}{2}(n-2) \int_{\partial B_\delta(0)} \bar{v}_\infty \partial_\nu \bar{v}_\infty\, d\sigma(x)\\ &\qquad = \tfrac{1}{2}\omega_{n-1} n^{(n-2)/2}(n-2)^{(n+2)/2}\mathcal{H}(0) + \varepsilon(\delta),\end{aligned} \tag{3-33}$$

where $\varepsilon(\delta) \to 0$ as $\delta \to 0$ and where the last equality follows from Lemma 3.3. Independently, direct computations using (2-1), (2-20) and (2-67) show that

$$\int_{B_{\delta d_\alpha}(x_\alpha)} \big(h_\alpha(x) + \tfrac{1}{2}\langle \nabla h_\alpha(x), x - x_\alpha\rangle\big) v_\alpha^2\, dx = \begin{cases} O(\delta^3 d_\alpha^5 + \delta \mu_\alpha d_\alpha) & \text{if } n = 3, \\ O(\delta^4 d_\alpha^6 + \mu_\alpha^2 \ln(d_\alpha/\mu_\alpha)) & \text{if } n = 4, \\ \mu_\alpha^2 h_\infty(x_\infty) \int_{\mathbb{R}^n} B_0(x)^2\, dx + o(\mu_\alpha^2) + O(\delta^n d_\alpha^{n+2}) & \text{if } n \geq 5. \end{cases} \tag{3-34}$$

Combining (3-33) and (3-34) into (3-30) we finally obtain

$$\tfrac{1}{2}\omega_{n-1} n^{(n-2)/2}(n-2)^{(n+2)/2} \mathcal{H}(0) + \epsilon(\delta) = \left(\frac{d_\alpha}{\mu_\alpha}\right)^{n-2} \begin{cases} O(\delta^3 d_\alpha^5 + \delta \mu_\alpha d_\alpha) & \text{if } n = 3, \\ O(\delta^4 d_\alpha^6 + \mu_\alpha^2 \ln(d_\alpha/\mu_\alpha)) & \text{if } n = 4, \\ \mu_\alpha^2 h_\infty(x_\infty) \int_{\mathbb{R}^n} B_0(x)^2\, dx + o(\mu_\alpha^2) + O(\delta^n d_\alpha^{n+2}) & \text{if } n \geq 5. \end{cases} \tag{3-35}$$

Using (3-17), and since $d_\alpha \to 0$, we easily obtain, when $n \in \{3, 4, 5\}$, that (3-35) shows

$$\mathcal{H}(0) + \epsilon(\delta) = o(1)$$

as $\alpha \to +\infty$, which contradicts Lemma 3.3. If now $n \geq 6$, (3-17) shows that $d_\alpha^{n+2} = o(\mu_\alpha^2)$. Since $\mathcal{H}(0) < 0$ by Lemma 3.3, we can choose δ fixed but small enough that $\mathcal{H}(0) + \varepsilon(\delta) < 0$. By (3-35) we then have

$$h_\infty(x_\infty) \int_{\mathbb{R}^n} B_0(x)^2\, dx + o(1) \leq 0.$$

Letting $\alpha \to +\infty$ implies $h_\infty(x_\infty) \leq 0$. In the case where $h_\infty > 0$ in $\overline{\Omega}$ this is a contradiction and concludes the proof of Proposition 3.2.

We may thus assume $h_\infty < 0$ in $\overline{\Omega}$ and $n \geq 6$. With (3-35) we obtain

$$d_\alpha = (C_0 + o(1)) \mu_\alpha^{(n-4)/(n-2)} \tag{3-36}$$

for some constant $C_0 > 0$ that depend on n and h_∞. Integrating (2-2) against ∇v_α in U_α yields the Pohozaev identity

$$\int_{\partial U_\alpha} \Big(\frac{1}{2}|\nabla v_\alpha|^2 \nu - \partial_\nu v_\alpha \nabla v_\alpha - \frac{1}{2^*} v_\alpha^{2^*} \nu\Big)\, d\sigma(x) = -\frac{1}{2}\int_{U_\alpha} h_\alpha \nabla(v_\alpha^2)\, dx, \tag{3-37}$$

where ν is the outer unit normal to U_α. Straightforward computations using Theorem 2.1, (2-16) and (3-18) show that

$$\int_{\partial U_\alpha} \frac{1}{2^*} v_\alpha^{2^*} \nu\, d\sigma = O(\mu_\alpha^n d_\alpha^{-n-1}) + O(d_\alpha^{n+1}),$$

while integrating by parts and using Theorem 2.1 and (2-16) shows that

$$\int_{U_\alpha} h_\alpha \nabla(v_\alpha^2)\, dx = \int_{\partial U_\alpha} h_\alpha v_\alpha^2 \nu\, d\sigma(x) - \int_{U_\alpha} v_\alpha^2 \nabla h_\alpha\, dx = O(\mu_\alpha^{n-2} d_\alpha^{3-n}) + O(d_\alpha^{n+1}) + O(\mu_\alpha^2).$$

Independently, (3-22) and (3-23) show that

$$\int_{\partial U_\alpha} \left(\tfrac{1}{2}|\nabla v_\alpha|^2 \nu - \partial_\nu v_\alpha \nabla v_\alpha\right) d\sigma(x) = \frac{\mu_\alpha^{n-2}}{d_\alpha^{n-1}}\left(\int_{\partial B_\delta(0)} \left(\tfrac{1}{2}|\nabla \bar{v}_\infty|^2 \nu - \partial_\nu \bar{v}_\infty \nabla \bar{v}_\infty\right) d\sigma(x) + o(1)\right)$$
$$= \frac{\mu_\alpha^{n-2}}{d_\alpha^{n-1}}\left(n^{(n-2)/2}(n-2)^{(n+2)/2}\omega_{n-1}\nabla\mathcal{H}(0) + \varepsilon(\delta) + o(1)\right)$$

as $\alpha \to +\infty$. Plugging these estimates into (3-37) finally gives

$$\nabla\mathcal{H}(0) + \varepsilon(\delta) = O\left(\left(\frac{\mu_\alpha}{d_\alpha}\right)^2 + \frac{d_\alpha^{2n}}{\mu_\alpha^{n-2}} + d_\alpha^2 + \frac{d_\alpha^{n-1}}{\mu_\alpha^{n-4}}\right) = o(1),$$

where in the last line we used (3-36). Passing to the limit as $\alpha \to +\infty$ and as $\delta \to 0$ shows that $\nabla\mathcal{H}(0) = 0$. But going back to (3-27), and since $\mathcal{H} = \widetilde{\mathcal{H}}$, we have $\partial_1\mathcal{H}(0) < 0$ by Lemma A.2, which is a contradiction. This concludes the proof of Proposition 3.2. □

We now investigate more precisely what happens at the scale r_α. This is the content of the following result:

Proposition 3.4. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$. Let $(h_\alpha)_{\alpha\in\mathbb{N}}$ be a sequence of functions that converges in $C^1(\overline{\Omega})$ to h_∞, where $-\Delta + h_\infty$ is coercive in $H_0^1(\Omega)$ and where $I_{h_\infty}(\Omega) < K_n^{-2}$, and we let $(v_\alpha)_{\alpha\in\mathbb{N}} \in H_0^1(\Omega)$ be a sequence of solutions of* (2-2) *that satisfies* (2-3), (2-4) *and* (2-5). *Let $x_\infty = \lim_{\alpha\to+\infty} x_\alpha$, and assume that $x_\infty \in \partial\Omega$. Assume that*

$$\frac{d_\alpha}{r_\alpha} \to +\infty$$

as $\alpha \to +\infty$. Then

- *if $n \in \{3, 4, 5\}$, we have $v_\infty \equiv 0$,*
- *if $n \geq 6$, we have $h_\infty(x_\infty) = 0$.*

Proof. We assume that

$$\lim_{\alpha\to+\infty} \frac{d_\alpha}{r_\alpha} = +\infty. \tag{3-38}$$

Using (3-13) we define, for $x \in (\Omega - x_\alpha)/r_\alpha$,

$$\bar{v}_\alpha(x) = \frac{r_\alpha^{n-2}}{\mu_\alpha^{(n-2)/2}} v_\alpha(x_\alpha + r_\alpha x) = d_\alpha^{-1} v_\alpha(x_\alpha + r_\alpha x). \tag{3-39}$$

Since v_α satisfies (2-2), $\bar{v}_\alpha$ solves

$$\begin{cases} -\Delta\bar{v}_\alpha + r_\alpha^2 \bar{h}_\alpha \bar{v}_\alpha = r_\alpha^2 d_\alpha^{4/(n-2)} |\bar{v}_\alpha|^{2^*-2}\bar{v}_\alpha & \text{in } (\Omega - x_\alpha)/r_\alpha, \\ \bar{v}_\alpha = 0 & \text{on } \partial((\Omega - x_\alpha)/r_\alpha), \end{cases}$$

where we have let $\bar{h}_\alpha(x) = h(x_\alpha + r_\alpha x)$. By Hopf's lemma and by (3-38) we have

$$v_\infty(x_\alpha + r_\alpha x) = v_\infty(x_\alpha) + O(r_\alpha) = -\partial_\nu v_\infty(x_\infty) d_\alpha + o(d_\alpha) \tag{3-40}$$

as $\alpha \to +\infty$, and (3-40) obviously remains true if $v_\infty \equiv 0$. Using (2-16), Theorem 2.1, (3-13) and (3-40) we thus have

$$|\bar{v}_\alpha(x)| \le C(|x|^{2-n}+1) \quad \text{for all } x \in \frac{\Omega - x_\alpha}{r_\alpha}\backslash\{0\}.$$

Standard elliptic theory then shows that $\bar{v}_\alpha$ converges to some $\bar{v}_\infty$ in $C^2_{\text{loc}}(\mathbb{R}^n\backslash\{0\})$. Let $x \in \mathbb{R}^n\backslash\{0\}$ be fixed. First, as a consequence of Lemma A.1,

$$\frac{r_\alpha^{n-2}}{\mu_\alpha^{(n-2)/2}} \Pi B_\alpha(x_\alpha + r_\alpha x) \to (n(n-2))^{(n-2)/2}|x|^{2-n} \quad \text{in } C^2_{\text{loc}}(\mathbb{R}^n\backslash\{0\})$$

as $\alpha \to +\infty$. The latter with (3-40) and Theorem 2.1 then shows that

$$\bar{v}_\infty = (n(n-2))^{(n-2)/2}|x|^{2-n} \pm \partial_\nu v_\infty(x_\infty). \tag{3-41}$$

For α large enough we let $U_\alpha = B_{r_\alpha}(x_\alpha) \subset \Omega$, and we apply the Pohozaev identity (3-1). We get

$$\int_{B_{r_\alpha}(x_\alpha)} \big(h_\alpha(x) + \tfrac{1}{2}\langle \nabla h_\alpha(x), x - x_\alpha\rangle\big) v_\alpha^2\, dx = \int_{\partial B_{r_\alpha}(x_\alpha)} F_\alpha(x)\, d\sigma(x), \tag{3-42}$$

where F_α is defined in (3-3). By changing x into $x_\alpha + d_\alpha x$, we can write

$$\begin{aligned} d_\alpha^{-2} r_\alpha^{2-n} &\int_{\partial B_{r_\alpha}(x_\alpha)} F_\alpha(x)\, d\sigma(x) \\ &= \int_{\partial B_1(0)} \langle x, \nu\rangle \left(\frac{|\nabla \bar{v}_\alpha|^2}{2} + \bar{h}_\alpha r_\alpha^2 \frac{\bar{v}_\alpha^2}{2} - r_\alpha^2 \frac{|\bar{v}_\alpha|^{2^*}}{2^*}\right) d\sigma(x) - \int_{\partial B_1(0)} \big(\langle x, \nabla \bar{v}_\alpha\rangle + \tfrac{1}{2}(n-2)\bar{v}_\alpha\big)\partial_\nu \bar{v}_\alpha\, d\sigma(x), \end{aligned}$$

where $\bar{v}_\alpha$ is as in (3-39). Direct calculations with (2-67) and (3-40) give

$$r_\alpha^2 \int_{\partial B_1(0)} \langle x, \nu\rangle \bar{h}_\alpha \bar{v}_\alpha^2\, d\sigma(x) = O(r_\alpha^2) \quad \text{and} \quad r_\alpha^2 \int_{\partial B_1(0)} \langle x, \nu\rangle |\bar{v}_\alpha|^{2^*}\, d\sigma(x) = O(r_\alpha^2).$$

Together with (3-41), the latter then shows that

$$\lim_{\alpha\to+\infty} d_\alpha^{-2} r_\alpha^{2-n} \int_{\partial B_{r_\alpha}(x_\alpha)} F_\alpha(x)\, d\sigma(x) = \pm\tfrac{1}{2}\omega_{n-1} n^{(n-2)/2}(n-2)^{(n+2)/2}\partial_\nu v_\infty(x_\infty). \tag{3-43}$$

Since $\lim_{\alpha\to+\infty} r_\alpha \mu_\alpha^{-1} = +\infty$, direct computations using (2-1), (2-20), (2-67), (3-13) and (3-40) show that

$$\int_{B_{r_\alpha}(x_\alpha)} \big(h_\alpha(x) + \tfrac{1}{2}\langle \nabla h_\alpha(x), x - x_\alpha\rangle\big) v_\alpha^2\, dx = \begin{cases} O(\mu_\alpha^{3/2}/d_\alpha) & \text{if } n = 3, \\ O(\mu_\alpha^2 \ln(r_\alpha/\mu_\alpha) + \mu_\alpha^2) & \text{if } n = 4, \\ \mu_\alpha^2\big(h_\infty(x_\infty)\int_{\mathbb{R}^n} B_0(x)^2\, dx + o(1)\big) & \text{if } n \ge 5. \end{cases} \tag{3-44}$$

Returning now to (3-42) with (3-43) and (3-44), and since $d_\alpha^2 r_\alpha^{n-2} = d_\alpha \mu_\alpha^{(n-2)/2}$ by (3-13), we have that

$$\begin{aligned} \pm\tfrac{1}{2}\omega_{n-1}(n-2)^{(n+2)/2} n^{(n-2)/2} \partial_\nu v_\infty(x_\infty) d_\alpha \mu_\alpha^{(n-2)/2} &+ o(d_\alpha \mu_\alpha^{(n-2)/2}) \\ &= \begin{cases} O(\mu_\alpha^{3/2}/d_\alpha) & \text{if } n = 3, \\ O(\mu_\alpha^2 \ln(r_\alpha/\mu_\alpha)) & \text{if } n = 4, \\ \mu_\alpha^2\big(h_\infty(x_\infty)\int_{\mathbb{R}^n} B_0(x)^2\, dx + o(1)\big) & \text{if } n \ge 5. \end{cases} \end{aligned} \tag{3-45}$$

Independently, since $r_\alpha = o(d_\alpha)$ by (3-38), and by (3-13), we get

$$\sqrt{\mu_\alpha} = o(d_\alpha^{(n-1)/(n-2)}) \quad \text{as } \alpha \to +\infty. \tag{3-46}$$

Assume first that $n = 3$. Then (3-45) shows that

$$\partial_\nu v_\infty(x_\infty) + o(1) = O\left(\frac{\mu_\alpha}{d_\alpha^2}\right) = o(1)$$

by (3-46). If $n = 4$, (3-45) shows that

$$\partial_\nu v_\infty(x_\infty) + o(1) = O\left(\frac{\mu_\alpha}{d_\alpha}\ln\left(\frac{r_\alpha}{\mu_\alpha}\right)\right) = O\left(\mu_\alpha^{2/3}\ln\left(\frac{r_\alpha}{\mu_\alpha}\right)\right) = o(1)$$

by (3-46). If $n = 5$, (3-45) shows that

$$\partial_\nu v_\infty(x_\infty) + o(1) = O\left(\frac{\mu_\alpha^{1/2}}{d_\alpha}\right) = o(1)$$

again by (3-46). We thus obtain, when $n \in \{3, 4, 5\}$, that

$$\partial_\nu v_\infty(x_\infty) = 0,$$

which shows that $v_\infty \equiv 0$ by Hopf's lemma. Assume now that $n \geq 6$. Then (3-45) shows that

$$h_\infty(x_\infty)\int_{\mathbb{R}^n} B_0(x)^2\,dx = O(d_\alpha \mu_\alpha^{(n-6)/2}) + o(1) = o(1)$$

since $d_\alpha \to 0$ as $\alpha \to +\infty$. This concludes the proof of Proposition 3.4. □

The next result finally shows that, in small dimensions, the concentration point cannot occur on $\partial\Omega$.

Proposition 3.5. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$. Let $(h_\alpha)_{\alpha\in\mathbb{N}}$ be a sequence of functions that converges in $C^1(\bar{\Omega})$ to h_∞, where $-\Delta + h_\infty$ is coercive in $H_0^1(\Omega)$ and where $I_{h_\infty}(\Omega) < K_n^{-2}$, and we let $(v_\alpha)_{\alpha\in\mathbb{N}} \in H_0^1(\Omega)$ be a sequence of solutions of* (2-2) *that satisfies* (2-3), (2-4) *and* (2-5). *Let $x_\infty = \lim_{\alpha\to+\infty} x_\alpha$. Assume that $n \in \{3, 4\}$ or that $n = 5$ and $h_\infty \neq 0$ in $\bar{\Omega}$. Then $x_\infty \in \Omega$.*

Proof. We proceed by contradiction and assume that $x_\infty \in \partial\Omega$. Under the assumptions of Proposition 3.5, Propositions 3.2 and 3.4 also apply. They show in particular that

$$\frac{d_\alpha}{r_\alpha} \to +\infty \tag{3-47}$$

as $\alpha \to +\infty$ and that $v_\infty \equiv 0$. For $x \in (\Omega - x_\alpha)/d_\alpha$ we define again

$$\bar{v}_\alpha(x) := \frac{d_\alpha^{n-2}}{\mu_\alpha^{(n-2)/2}} v_\alpha(x_\alpha + d_\alpha x). \tag{3-48}$$

Equation (2-2) then shows that $\bar{v}_\alpha$ satisfies

$$\begin{cases} -\Delta\bar{v}_\alpha - d_\alpha^2 \bar{h}_\alpha \bar{v}_\alpha = (\mu_\alpha/d_\alpha)^2 |\bar{v}_\alpha|^{2^*-2}\bar{v}_\alpha & \text{in } (\Omega - x_\alpha)/d_\alpha, \\ \bar{v}_\alpha = 0 & \text{on } \partial((\Omega - x_\alpha)/d_\alpha), \end{cases}$$

where $\bar{h}_\alpha(x) := h(x_\alpha + d_\alpha x)$. Since $v_\infty \equiv 0$, (2-16) and Theorem 2.1 show that

$$|\bar{v}_\alpha(x)| \le C|x|^{2-n} \quad \text{for all } x \in \frac{\Omega - x_\alpha}{d_\alpha}\backslash\{0\} \tag{3-49}$$

for some positive constant C. Since Ω is smooth and since $d_\alpha \to 0$ as $\alpha \to +\infty$ by assumption, standard elliptic theory shows that, up to a rotation, $\bar{v}_\alpha \to \bar{v}_\infty \in C^2(\overline{\Omega}_0\backslash\{0\})$ as $\alpha \to +\infty$, where $\Omega_0 :=]-\infty, 1[\times \mathbb{R}^{n-1}$ and where $\bar{v}_\infty$ satisfies

$$-\Delta \bar{v}_\infty = 0 \quad \text{in } \Omega_0\backslash\{0\}, \qquad \bar{v}_\infty = 0 \quad \text{on } \partial\Omega_0$$

and

$$|\bar{v}_\infty(x)| \le C|x|^{2-n} \quad \text{for all } x \in \Omega_0.$$

The arguments in the proof of Lemma 3.3 again show that

$$\bar{v}_\infty(x) = \frac{(n(n-2))^{(n-2)/2}}{|x|^{n-2}} + \mathcal{H}(x) \quad \text{for all } x \in \Omega_0\backslash\{0\}, \tag{3-50}$$

where $\mathcal{H}$ satisfies

$$-\Delta \mathcal{H} = 0 \quad \text{in } \Omega_0, \qquad \mathcal{H} = -(n(n-2))^{-(n-2)/2}|\cdot|^{2-n} \quad \text{on } \partial\Omega_0$$

and is given for any $x \in \Omega$ by

$$\mathcal{H}(x) = 2\frac{n^{(n-4)/2}(n-2)^{(n-2)/2}}{\omega_{n-1}}(x_1 - 1)\int_{\partial\Omega_0} |y|^{2-n}|x-y|^{-n}\, d\sigma(y) \tag{3-51}$$

and also satisfies

$$\mathcal{H}(0) < 0. \tag{3-52}$$

In the following we let $0 < \delta < 1$ and $U_\alpha = B_{\delta d_\alpha}(x_\alpha)$. We write Pohozaev's identity (3-1) in U_α: this gives

$$\int_{B_{\delta d_\alpha}(x_\alpha)} \big(h_\alpha(x) + \tfrac{1}{2}\langle \nabla h_\alpha(x), x - x_\alpha\rangle\big) v_\alpha^2\, dx = \int_{\partial B_{\delta d_\alpha}(x_\alpha)} F_\alpha(x)\, d\sigma(x),$$

where F_α is defined in (3-3). Mimicking the computations that led to (3-31), (3-32) and (3-33) we obtain

$$\int_{\partial B_{\delta d_\alpha}(x_\alpha)} F_\alpha(x)\, d\sigma(x) = \left(\frac{\mu_\alpha}{\delta d_\alpha}\right)^{n-2} \big(\tfrac{1}{2}\omega_{n-1} n^{(n-2)/2}(n-2)^{(n+2)/2}\mathcal{H}(0) + \varepsilon(\delta) + o(1)\big) \tag{3-53}$$

as $\alpha \to +\infty$, where $\varepsilon(\delta) \to 0$. Independently, direct computations using (2-1), (2-20) and (2-67) show

$$\int_{B_{\delta d_\alpha}(x_\alpha)} \big(h_\alpha(x) + \tfrac{1}{2}\langle \nabla h_\alpha(x), x - x_\alpha\rangle\big) v_\alpha^2\, dx = \begin{cases} O(\mu_\alpha r_\alpha) & \text{if } n = 3, \\ 64\omega_3 h_\infty(x_\infty)\mu_\alpha^2 \ln(d_\alpha/\mu_\alpha) + O(\mu_\alpha^2) & \text{if } n = 4, \\ \mu_\alpha^2\big(h_\infty(x_\infty)\int_{\mathbb{R}^n} B_0(x)^2\, dx + o(1)\big) & \text{if } n \ge 5. \end{cases} \tag{3-54}$$

If $n = 3$, combining (3-53) and (3-54) gives

$$\mathcal{H}(0) = O(\sqrt{\mu_\alpha});$$

hence $\mathcal{H}(0)=0$, which contradicts (3-52). This proves Proposition 3.5 when $n=3$. If $n=4,5$, using (3-52), we obtain $h_\infty(x_\infty)\le 0$. If $h_\infty>0$ in $\overline{\Omega}$ this is a contradiction and concludes the proof of Proposition 3.5.

We assume from now on that $h_\infty<0$ in $\overline{\Omega}$ and $n=4,5$. In this case the proof is similar to the proof of Proposition 3.2 when $n\ge 6$. Using again (3-52) the previous Pohozaev's identity then shows the existence of a constant $C_0>0$ depending on n, h_∞ and δ such that

$$d_\alpha^2\ln(d_\alpha/\mu_\alpha)=C_0+o(1)\quad\text{if } n=4\qquad\text{and}\qquad d_\alpha=(C_0+o(1))\mu_\alpha^{1/3}\quad\text{if } n=5. \tag{3-55}$$

We recall the gradient Pohozaev identity (3-37),

$$\int_{\partial U_\alpha}\Big(\frac{1}{2}|\nabla v_\alpha|^2\nu-\partial_\nu v_\alpha\nabla v_\alpha-\frac{1}{2^*}v_\alpha^{2^*}\nu\Big)\,d\sigma(x)=-\frac{1}{2}\int_{U_\alpha}h_\alpha\nabla(v_\alpha^2)\,dx,$$

where ν is the outer unit normal to U_α. Straightforward computations using Theorem 2.1 and (2-16) show

$$\int_{\partial U_\alpha}\frac{1}{2^*}v_\alpha^{2^*}\nu\,d\sigma(x)=O(\mu_\alpha^n d_\alpha^{-n-1}),$$

while integrating by parts and using Theorem 2.1 and (2-16) shows

$$\int_{U_\alpha}h_\alpha\nabla(v_\alpha^2)\,dx=\int_{\partial U_\alpha}h_\alpha v_\alpha^2\nu\,d\sigma(x)-\int_{U_\alpha}v_\alpha^2\nabla h_\alpha\,dx=O(\mu_\alpha^{n-2}d_\alpha^{3-n})+\begin{cases}O(\mu_\alpha^2\ln(d_\alpha/\mu_\alpha)) & \text{if } n=4,\\ O(\mu_\alpha^2) & \text{if } n=5.\end{cases}$$

Independently, (3-49) and (3-50) show that

$$\begin{aligned}\int_{\partial U_\alpha}\big(\tfrac{1}{2}|\nabla v_\alpha|^2\nu-\partial_\nu v_\alpha\nabla v_\alpha\big)\,d\sigma(x)&=\frac{\mu_\alpha^{n-2}}{d_\alpha^{n-1}}\bigg(\int_{\partial B_\delta(0)}\big(\tfrac{1}{2}|\nabla\bar v_\infty|^2\nu-\partial_\nu\bar v_\infty\nabla\bar v_\infty\big)\,d\sigma(x)+o(1)\bigg)\\&=\frac{\mu_\alpha^{n-2}}{d_\alpha^{n-1}}(n^{(n-2)/2}(n-2)^{(n+2)/2}\omega_{n-1}\nabla\mathcal{H}(0)+\varepsilon(\delta)+o(1))\end{aligned}$$

as $\alpha\to+\infty$. Plugging these estimates into (3-37) finally gives

$$\begin{aligned}\nabla\mathcal{H}(0)+\varepsilon(\delta)&=O\bigg(\Big(\frac{\mu_\alpha}{d_\alpha}\Big)^2\bigg)+O(d_\alpha^2)+\begin{cases}O(d_\alpha^3\ln(d_\alpha/\mu_\alpha)) & \text{if } n=4,\\ O(d_\alpha^4/\mu_\alpha) & \text{if } n=5,\end{cases}\\&=o(1),\end{aligned}$$

where in the last line we used (3-55). Passing to the limit as $\alpha\to+\infty$ and as $\delta\to 0$ shows that $\nabla\mathcal{H}(0)=0$. But going back to (3-51) we again have $\partial_1\mathcal{H}(0)<0$ by Lemma A.2, which is a contradiction. This concludes the proof of Proposition 3.5 when $n=4,5$ and $h_\infty<0$.

To conclude the proof of Proposition 3.5 we finally assume that $n=4$. If $h_\infty(x_\infty)\ne 0$ in $\overline{\Omega}$ the proof of Proposition 3.5 follows from the previous arguments. We may then assume that $h_\infty(x_\infty)=0$. In this case combining (3-53) and (3-54) shows

$$\mathcal{H}(0)=O(d_\alpha^2)=o(1)$$

as $\alpha\to+\infty$, which contradicts (3-52). This concludes the proof of Proposition 3.5. □

Remark 3.6. Assume that $(v_\alpha)_{\alpha\in\mathbb{N}} \in H_0^1(\Omega)$ is any sequence of solutions of (2-2) that satisfies (2-3) and (2-4), so that (2-5), (2-6) and (2-8) also hold. Let $x_\infty = \lim_{\alpha\to\infty} x_\alpha$ be the concentration point of u_α. Propositions 3.2, 3.4 and 3.5 prove that $x_\infty \in \Omega$, i.e., that x_∞ is an interior blow-up point, in the following cases (regardless of the value of v_∞): either when $n \in \{3, 4\}$ or when $n \geq 5$ and under the assumption $h_\infty \neq 0$ in $\overline{\Omega}$. If h_∞ is allowed to vanish somewhere in $\partial\Omega$ the property that $x_\infty \in \Omega$ is unlikely to remain true, and concentration points may arise on the boundary in large dimensions. When $n \geq 7$, for instance, *sign-changing* solutions of (1-5) that blow-up with one concentration point in $\partial\Omega$ as $\lambda \to 0_+$ (which corresponds to $h_\infty \equiv 0$) have been constructed in [Vaira 2015]; see also [Musso et al. 2024] for a more recent construction with an arbitrary number of bubbles.

Remark 3.7. We mentioned in Remark 3.6 that, when $n \geq 7$ and $h_\infty \equiv 0$, *sign-changing* solutions of (1-5) that blow-up with one concentration point in $\partial\Omega$ as $\lambda \to 0_+$ exist; see [Vaira 2015]. By contrast, it is important to point out that, in any dimension $n \geq 4$, *positive* solutions of (1-5) as $\lambda \to 0_+$ may only blow-up with interior concentration points and do not possess concentration points in $\partial\Omega$. This is shown in [König and Laurain 2024, Proposition 2.1] and heavily relies on the positivity of the solutions. The issue of boundary concentration points thus arises when working with *sign-changing* solutions of (1-6).

We are now in position to prove Theorem 1.1.

End of the proof of Theorem 1.1. Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$, and $(h_\alpha)_{\alpha\in\mathbb{N}}$ be a sequence that converges in $C^1(\overline{\Omega})$ towards h_∞. Assume that $-\Delta + h_\infty$ is coercive and that $I_{h_\infty}(\Omega) < K_n^{-2}$. Let $(v_\alpha)_{\alpha\in\mathbb{N}} \in H_0^1(\Omega)$ be a sequence of solutions of (2-2) that satisfies (2-3). Assume first that $(v_\alpha)_{\alpha\in\mathbb{N}}$ is, up to a subsequence, uniformly bounded in $L^\infty(\Omega)$. By standard elliptic theory it then strongly converges, again up to a subsequence, to some v_0 in $C^2(\overline{\Omega})$ as $\alpha \to +\infty$. That $v_0 \neq 0$ simply follows from the coercivity of $-\Delta + h_\infty$ which easily implies, by Sobolev's inequality, that $\liminf_{\alpha\to+\infty} \|v_\alpha\|_{H_0^1} > 0$. This concludes the proof of Theorem 1.1.

We thus proceed by contradiction and assume that, up to a subsequence, (2-4) holds, and hence that (2-5), (2-6) and (2-8) hold for some sequence $(x_\alpha)_{\alpha\in\mathbb{N}}$ of points in Ω and $(\mu_\alpha)_{\alpha\in\mathbb{N}}$ of positive real numbers converging to 0 satisfying (2-10). In particular,

$$v_\alpha = B_\alpha \pm v_\infty + o(1) \quad \text{in } H_0^1(\Omega),$$

where $v_\infty \equiv 0$ or v_∞ is a positive solution of (2-9). We let $x_\infty = \lim_{\alpha\to+\infty} x_\alpha \in \overline{\Omega}$. Under these assumptions, the analysis of Section 3 applies.

We first assume that $n \geq 7$ and that $h_\infty \neq 0$ at every point of $\overline{\Omega}$. Propositions 3.2 and 3.4 first show that $x_\infty \in \Omega$. Proposition 3.1 then applies and shows that $h_\infty(x_\infty) = 0$, which is a contradiction.

We now assume that $3 \leq n \leq 5$ and that $(v_\alpha)_{\alpha\in\mathbb{N}} \in H_0^1(\Omega)$ is *sign-changing* for all $\alpha \geq 0$. We then claim that

$$v_\infty > 0 \quad \text{in } \Omega. \tag{3-56}$$

This is a strong consequence of the assumption that v_α changes sign. We adapt an argument from [Cerami et al. 1986, Lemma 3.1]. Since v_α does not strongly converge to v_∞, $(v_\alpha)_+$ and $(v_\alpha)_-$ may not simultaneously strongly converge to $(v_\infty)_+$ and $(v_\infty)_-$. Assume for simplicity that $(v_\alpha)_+$ weakly but not strongly converges to $(v_\infty)_+$ in $H_0^1(\Omega)$. Integrating (2-2) against $(v_\alpha)_+$ and using the Brézis–Lieb lemma shows that

$$\int_\Omega |\nabla((v_\alpha)_+ - (v_\infty)_+)|^2\,dx + o(1) = \int_\Omega |(v_\alpha)_+ - (v_\infty)_+|^{2^*}\,dx,$$

from which we deduce that $\int_\Omega |(v_\alpha)_+ - (v_\infty)_+|^{2^*}\,dx \geq K_n^{-n} + o(1)$ as $\alpha \to +\infty$ by (1-3). Independently, since $(v_\alpha)_-$ is nonzero, integrating (2-2) against $(v_\alpha)_-$ and using (1-2) yields

$$\int_\Omega |(v_\alpha)_-|^{2^*}\,dx \geq I_{h_\alpha}(\Omega)^{n/2}.$$

Thus, again by Brézis–Lieb's lemma,

$$\begin{aligned}\int_\Omega |v_\alpha|^{2^*}\,dx &= \int_\Omega |(v_\alpha)_+|^{2^*}\,dx + \int_\Omega |(v_\alpha)_-|^{2^*}\,dx \\ &= \int_\Omega |(v_\alpha)_+ - (v_\infty)_+|^{2^*}\,dx + \int_\Omega |(v_\infty)_+|^{2^*}\,dx + \int_\Omega |(v_\alpha)_-|^{2^*}\,dx + o(1) \\ &\geq K_n^{-n} + I_{h_\infty}(\Omega)^{n/2} + o(1)\end{aligned}$$

as $\alpha \to +\infty$. This shows that $v_\infty \not\equiv 0$ and hence that $v_\infty > 0$ in Ω and attains $I_{h_\infty}(\Omega)$. As before, the analysis of Section 3 applies to v_α. First, using (3-56), Propositions 3.2 and 3.4 show that $x_\infty \in \Omega$. We may thus apply Proposition 3.1, which shows that $v_\infty \equiv 0$ and contradicts (3-56). Thus $(v_\alpha)_{\alpha\in\mathbb{N}}$ is again uniformly bounded in $L^\infty(\Omega)$ and Theorem 1.1 is proven. □

We now prove Corollary 1.2.

Proof of Corollary 1.2. We assume that Ω and h are as in the assumptions of Corollary 1.2. We observed in the proof of Theorem 1.1 that any sequence $(v_\alpha)_{\alpha\in\mathbb{N}}$ of solutions of (1-1) which is bounded in $L^\infty(\Omega)$ up to a subsequence is precompact in $C^2(\overline{\Omega})$. With this observation we proceed by contradiction: if no ε as in the statement of Corollary 1.2 exists, we can find a sequence $(v_\alpha)_{\alpha\in\mathbb{N}}$ of solutions of

$$\begin{cases} -\Delta v_\alpha + h v_\alpha = |v_\alpha|^{2^*-2} v_\alpha & \text{in } \Omega, \\ v_\alpha = 0 & \text{in } \partial\Omega, \end{cases}$$

which satisfies $\lim_{\alpha\to+\infty} \|v_\alpha\|_\infty = +\infty$ and $\limsup_{\alpha\to+\infty} \int_\Omega |v_\alpha|^{2^*}\,dx \leq K_n^{-n} + I_h(\Omega)^{n/2}$. When $3 \leq n \leq 5$ we have in addition that $(v_\alpha)_{\alpha\in\mathbb{N}}$ changes sign. We may now apply Theorem 1.1 to the sequence $(v_\alpha)_{\alpha\in\mathbb{N}}$ with $h_\alpha \equiv h$ for all $\alpha \geq 0$, which gives a contradiction. □

We now consider the six-dimensional case and prove Proposition 1.3.

Proof of Proposition 1.3. Assume indeed that $(v_\alpha)_{\alpha\in\mathbb{N}}$ is a sequence of solutions of (2-2) that satisfies (2-3) and (2-4). Hence (2-5), (2-6) and (2-8) hold for some sequence $(x_\alpha)_\alpha$ of points in Ω and $(\mu_\alpha)_\alpha$ of positive real numbers converging to 0 satisfying (2-10). Then

$$v_\alpha = B_\alpha \pm v_\infty + o(1) \quad \text{in } H_0^1(\Omega),$$

where $v_\infty \equiv 0$ or v_∞ is a positive solution of (2-9). We let $x_\infty = \lim_{\alpha\to+\infty} x_\alpha \in \overline{\Omega}$. First, Propositions 3.2 and 3.4 show that $x_\infty \in \Omega$. Proposition 3.1 then applies and shows that $h_\infty(x_\infty) = \pm 2 v_\infty(x_\infty)$. □

Remark 3.8. When $n \in \{3, 4, 5\}$, Theorem 1.1 is likely to be false if (1-7) is not satisfied. On a closed Riemannian manifold and when $3 \le n \le 5$, blowing-up solutions of equations like (1-6) of the form $B_\alpha + v_\infty$, where v_∞ is a *sign-changing* solution of (1-1), may exist; see [Premoselli and Vétois 2022b, Theorem 1.4]. The examples in that result are constructed on a closed manifold with symmetries and B_α concentrates at a point where v_∞ vanishes. These examples are likely to adapt to the case of a symmetric bounded open set when $3 \le n \le 5$ and $h_\infty \ne 0$ in $\overline{\Omega}$. They suggest that, even when $3 \le n \le 5$, sign-changing solutions may exhibit noncompactness at a higher energy level than $K_n^{-n} + I_{h_\infty}(\Omega)^{n/2}$.

Appendix: Technical results

A.1. ***Pointwise estimates on ΠB_α.*** Let ΠB_α be given by (2-14). We prove a technical result that was used several times throughout the paper and that provides an asymptotic expansion of ΠB_α close to $\partial\Omega$.

Lemma A.1. *Let $(x_\alpha)_{\alpha\in\mathbb{N}}$ and $(\mu_\alpha)_{\alpha\in\mathbb{N}}$ be sequences of points in Ω and positive real numbers, respectively, satisfying $d(x_\alpha, \partial\Omega) \gg \mu_\alpha$ as $\alpha \to +\infty$. Let B_α be given by* (2-6) *and ΠB_α be given by* (2-14)*. Let $(y_\alpha)_{\alpha\in\mathbb{N}}$ be a sequence of points in Ω satisfying*

$$d(y_\alpha, \partial\Omega) \to 0, \quad |x_\alpha - y_\alpha| \le \tfrac{1}{2} d(x_\alpha, \partial\Omega) \quad \text{and} \quad \frac{|x_\alpha - y_\alpha|}{\mu_\alpha} \to +\infty \tag{A-1}$$

as $\alpha \to +\infty$. Let $\ell = \lim_{\alpha\to+\infty} |x_\alpha - y_\alpha|/d(x_\alpha, \partial\Omega)$, which exists up to a subsequence. Then, as $\alpha \to +\infty$, we have

$$\Pi B_\alpha(y_\alpha) = \big((n(n-2))^{(n-2)/2} + o(1) + \varepsilon(\ell)\big) \frac{\mu_\alpha^{(n-2)/2}}{|x_\alpha - y_\alpha|^{n-2}},$$

where $\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$ denotes a function satisfying $\varepsilon(0) = 0$ and $\lim_{x\to 0} \varepsilon(x) = 0$.

Proof. We write a representation formula for ΠB_α using (2-14),

$$\Pi B_\alpha(y_\alpha) = \int_\Omega G_\alpha(y_\alpha, \cdot) B_\alpha^{2^*-1}\, dx, \tag{A-2}$$

where as before G_α denotes the Green's function of $-\Delta + h_\alpha$ with Dirichlet boundary conditions in Ω. Using (A-1), (2-12) and arguing as in (2-80) we have

$$\int_{\Omega \setminus B_{|x_\alpha - y_\alpha|/2}(x_\alpha)} G_\alpha(y_\alpha, \cdot) B_\alpha^{2^*-1}\, dx = o(B_\alpha(y_\alpha)) \tag{A-3}$$

as $\alpha \to +\infty$. We let in what follows

$$I_\alpha := |x_\alpha - y_\alpha|^{n-2} \mu_\alpha^{-(n-2)/2} \int_{B_{|x_\alpha - y_\alpha|/2}(x_\alpha)} G_\alpha(y_\alpha, \cdot) B_\alpha^{2^*-1}\, dx.$$

By a change of variable we have

$$I_\alpha = \int_{B_{|x_\alpha - y_\alpha|/(2\mu_\alpha)}(0)} |x_\alpha - y_\alpha|^{n-2} G_\alpha(y_\alpha, x_\alpha + \mu_\alpha z) B_0(z)^{2^*-1}\, dz, \tag{A-4}$$

where B_0 is as in (2-7). Using (2-12) there is $C > 0$ such that, for any $z \in B_{|x_\alpha - y_\alpha|/(2\mu_\alpha)}(0)$,

$$|x_\alpha - y_\alpha|^{n-2} G_\alpha(y_\alpha, x_\alpha + \mu_\alpha z) \le C.$$

Let $z \in \mathbb{R}^n$ be fixed. Since $\mu_\alpha = o(d_\alpha)$ we have by (A-1)

$$D := \lim_{\alpha\to+\infty} \frac{d(y_\alpha, \partial\Omega) d(x_\alpha + \mu_\alpha z, \partial\Omega)}{|y_\alpha - (x_\alpha + \mu_\alpha z)|^2} \ge \frac{1}{\ell^2}(1-\ell)$$

as $\alpha \to +\infty$, where we have let $\ell = \lim_{\alpha\to+\infty} |x_\alpha - y_\alpha|/d(x_\alpha, \partial\Omega)$, and we use the convention that the right-hand side is equal to $+\infty$ if $\ell = 0$. Note that $\ell \le \frac{1}{2}$ by (A-1). Since $\mu_\alpha = o(d_\alpha)$ and $\lim_{\alpha\to+\infty} |y_\alpha - (x_\alpha + \mu_\alpha z)| = 0$ uniformly in $z \in B_R(0)$, Proposition 12 in [Robert 2010] applies and shows that, for any fixed $z \in \mathbb{R}^n$,

$$\begin{aligned}\lim_{\alpha\to+\infty} |x_\alpha - y_\alpha|^{n-2} G_\alpha(y_\alpha, x_\alpha + \mu_\alpha z) &= \frac{1}{(n-2)\omega_{n-1}} \left(1 - \frac{1}{(1+4D)^{(n-2)/2}}\right) \\ &= \frac{1}{(n-2)\omega_{n-1}}(1 + O(\ell)). \end{aligned} \tag{A-5}$$

Plugging (A-5) into (A-4) we get by dominated convergence that

$$I_\alpha = (1 + \varepsilon(\ell) + o(1)) \frac{1}{(n-2)\omega_{n-1}} \int_{\mathbb{R}^n} B_0^{2^*-1}\, dx = (1 + \varepsilon(\ell) + o(1))(n(n-2))^{(n-2)/2}$$

as $\alpha \to +\infty$, where $\varepsilon(\ell)$ denotes a function such that $\varepsilon(0) = 0$ and $\varepsilon(\ell) \to 0$ as $\ell \to 0$. In the latter estimate we used

$$\int_{\mathbb{R}^n} B_0^{2^*-1}\, dx = (n-2)\omega_{n-1}(n(n-2))^{(n-2)/2},$$

which follows from integrating the equation $-\Delta B_0 = B_0^{2^*-1}$. Going back to the definition of I_α proves the lemma. □

A.2. *Sign of $\partial_1 \mathcal{H}(0)$.* We will finally prove the following simple result that was used in the proof of Propositions 3.2 and 3.5.

Lemma A.2. *Let $\widetilde{\mathcal{H}}$ be given by* (3-27). *Then* $\partial_1 \widetilde{\mathcal{H}}(0) < 0$.

Proof. Straightforward computations show that

$$\frac{1}{D_0} \partial_1 \widetilde{\mathcal{H}}(0) = \int_{\partial\Omega_0} |y|^{2-2n}\, d\sigma(y) - n \int_{\partial\Omega_0} |y|^{-2n}\, d\sigma(y),$$

where we have let $D_0 = 2n^{(n-4)/2}(n-2)^{(n-2)/2}/\omega_{n-1}$ and where $\partial\Omega_0 = \{1\} \times \mathbb{R}^{n-1}$. Simple changes of variable then yield

$$\int_{\partial\Omega_0} |y|^{2-2n}\, d\sigma(y) = \tfrac{1}{2}\omega_{n-2} I_{n-1}^{(n-3)/2} \quad \text{and} \quad \int_{\partial\Omega_0} |y|^{-2n}\, d\sigma(y) = \tfrac{1}{2}\omega_{n-2} I_n^{(n-3)/2},$$

where ω_{n-2} is the area of the round sphere $\mathbb{S}^{n-2}$ and where we have let, for $p, q > 0,\ p > q + 1$,

$$I_p^q = \int_0^{+\infty} \frac{r^q}{(1+r)^p}\, dr.$$

Classical induction formulae (see, e.g., [Aubin 1976]) show that $I_n^{(n-3)/2} = \frac{1}{2} I_{n-1}^{(n-3)/2}$. Combining these computations finally shows that

$$\frac{1}{D_0}\partial_1 \widetilde{\mathcal{H}}(0) = \tfrac{1}{2}\omega_{n-2} I_{n-1}^{(n-3)/2}\big(1 - \tfrac{1}{2}n\big) = -\tfrac{1}{2}(n-2)\int_{\partial\Omega_0} |y|^{2-2n}\, d\sigma(y) < 0,$$

which proves the lemma. □

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