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A SHARP TRACE ADAMS INEQUALITY IN $\mathbb{R}^4$ AND EXISTENCE OF THE EXTREMALS

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Let $\Omega \subseteq \mathbb{R}^4$ be a bounded domain with smooth boundary $\partial\Omega$. In this paper, we establish the following sharp form of the trace Adams inequality in $W^{2,2}(\Omega)$ with zero mean value and zero Neumann boundary condition:

$$S(\alpha) = \sup_{\substack{u\in W^{2,2}(\Omega)\setminus\{0\},\ \|\Delta u\|_2\le 1 \\ \int_\Omega u\,dx=0,\ \frac{\partial u}{\partial \nu}|_{\partial\Omega}=0}} \int_{\partial\Omega} e^{\alpha u^2}\,d\sigma < \infty$$

holds if and only if $\alpha \le 12\pi^2$.

Moreover, we prove a classification theorem for the solutions of a class of nonlinear boundary value problem of biharmonic equations on the half-space $\mathbb{R}^4_+$. With this classification result, we can show that $S(12\pi^2)$ is attained by using the blow-up analysis and capacitary estimate. As an application, we prove a sharp trace Adams–Onofri-type inequality in general four-dimensional bounded domains with smooth boundary.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$ and $W^{m,p}(\Omega)$ denote the usual Sobolev space: the completion of $C^\infty(\bar{\Omega})$ under the norm

$$\|\cdot\| = \left(\sum_{|\alpha|\le m} \int_\Omega |D^\alpha u|^p\,dx\right)^{\frac{1}{p}}.$$

If $1 < p < n/m$, the classical Sobolev embedding asserts that $W^{m,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ for $p^* = np/(n-mp)$, and $W^{m,p}(\Omega) \hookrightarrow L^{p^\star}(\partial\Omega)$ for $p^\star = (n-1)p/(n-mp)$. However, when $p = n/m$, it is known that both $W^{m,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ and $W^{m,p}(\Omega) \hookrightarrow L^\infty(\partial\Omega)$ fail.

It is known that the analogue of optimal Sobolev embedding for $W_0^{m,n/m}(\Omega)$ (the Sobolev space consisting of functions vanishing on the boundary $\partial\Omega$) is given by the famous Trudinger–Moser inequality ($m = 1$) [Moser 1970/71; Trudinger 1967] and Adams inequality ($m > 1$) [Adams 1988], which can be

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stated in the form

$$\sup_{\substack{u\in W_0^{m,\frac{n}{m}}(\Omega)\setminus\{0\}\\ \|\Delta^{m/2}u\|_{\frac{n}{m}}\le 1}} \int_\Omega \exp(\alpha|u(x)|^{\frac{n}{n-m}})\,dx \begin{cases} \le c|\Omega| & \text{if } \alpha\le\alpha(n,m),\\ =+\infty & \text{if } \alpha>\alpha(n,m),\end{cases} \tag{1-1}$$

where

$$\alpha(n,m)=\begin{cases} \dfrac{n}{\omega_{n-1}}\Big[\dfrac{\pi^{n/2}2^m\Gamma\big(\frac{1}{2}(m+1)\big)}{\Gamma\big(\frac{1}{2}(n-m+1)\big)}\Big]^{\frac{n}{n-m}} & \text{when } m \text{ is odd},\\ \dfrac{n}{\omega_{n-1}}\Big[\dfrac{\pi^{n/2}2^m\Gamma\big(\frac{m}{2}\big)}{\Gamma\big(\frac{1}{2}(n-m)\big)}\Big]^{\frac{n}{n-m}} & \text{when } m \text{ is even}.\end{cases}$$

Here ω_{n-1} denotes the $(n-1)$-dimensional surface measure of the unit ball in $\mathbb{R}^n$. So far, the Trudinger–Moser–Adams inequalities (1-1) have been generalized in many other directions such as the Trudinger–Moser inequalities on the unbounded domains, compact or complete and noncompact Riemannian manifolds, CR spheres, hyperbolic spaces, Heisenberg groups, Hardy–Adams-type inequalities on hyperbolic spaces, etc., to just name a few from a long list of extensive works we refer the interested readers to [Adachi and Tanaka 2000; Cohn and Lu 2004; Chen et al. 2020; 2021; 2022b; 2023a; 2023b; Chen 1990; Li 2005; Cohn and Lu 2001; do Ó 1997; 2024; Fontana 1993; Lam and Lu 2012; 2013; Lam et al. 2014; Li and Lu 2021; Li et al. 2018a; 2018b; 2021; Li and Ruf 2008; Li and Zhu 2022; Liang et al. 2020; Lu and Tang 2013; Lu and Yang 2017; Lu et al. 2024; Ma et al. 2021; Mancini et al. 2013; Nguyen 2018; 2024 Ruf and Sani 2013; Wang 2025; Xue et al. 2025; Yang 2007; Zhang and Zhu 2024; Zhang et al. 2025].

An interesting problem related to the Trudinger–Moser–Adams inequalities lies in investigating the existence of extremal functions. Carleson and Chang [1986] first established the existence of extremals for Trudinger–Moser inequalities on the unit ball through a symmetrization rearrangement inequality combined with the ODE technique. After that, the existence of extremals was proved for any bounded domains in $\mathbb{R}^n$ (see [Flucher 1992; Lin 1996; Adimurthi and Druet 2004]). One can also see [Li 2001; 2005; Li and Ruf 2008; Zhu 2014] for existence of extremals for the Trudinger–Moser inequalities on unbounded domains and compact Riemannian manifolds, and see [Lu and Yang 2009a; DelaTorre and Mancini 2021; Chen et al. 2020] for the existence of extremals for Adams inequalities in bounded and unbounded domains. We note that the Trudinger–Moser–Adams inequalities on the Sobolev spaces $W^{m,\frac{n}{m}}(\Omega)$ without the Dirichlet boundary condition have also been established; the interested readers can refer to [Chang and Yang 1988; Leckband 2005; Cianchi 2005; Lu and Yang 2009b; Tarsi 2012].

In this paper, we are interested in the borderline case of Sobolev trace inequality in $W^{m,\frac{n}{m}}(\Omega)$. As mentioned above, from the Sobolev embedding we know that $W^{m,\frac{n}{m}}(\Omega)\hookrightarrow L^q(\partial\Omega)$ for any $q\in[1,\infty)$, but not for $q=\infty$. (See, e.g., Maz'ja's book [1985]). Adams [1973] showed that $W^{m,\frac{n}{m}}(\Omega)$ can be embedded into the Orlicz space $L_\phi(\partial\Omega)$, with $\phi(t)=\exp(|t|^{n/(n-m)}-1)$ (see also [Maz'ja 1985]). The first optimal trace inequality of Moser type on $\partial\Omega$ was obtained in [Chang and Marshall 1985] in a two-dimensional disk D for functions with zero boundary mean value. Namely,

$$\sup_{\substack{u\in W^{1,2}(D)\setminus\{0\}\\ \int_D|\nabla u|^2\,dx=1,\ \int_{\partial D}u\,d\sigma=0}} \int_{\partial D} e^{\alpha u^2}\,d\sigma<+\infty \quad \text{if and only if } \alpha\le\pi. \tag{1-2}$$

Using the technique of blow-up analysis, Li and Liu [2005] extended the result of [Chang and Marshall 1985] to general bounded domains and obtained the existence of corresponding extremals. Yang [2006] obtained another sharp form of trace Trudinger–Moser inequality for functions with zero mean value in $\Omega \subset \mathbb{R}^2$. Furthermore, Cianchi [2008] formulated a unified framework for high-dimensional Trudinger–Moser-type inequalities on a boundary $\partial\Omega$ or smooth submanifold of arbitrary dimension in $\overline{\Omega}$. In particular, this includes the trace Trudinger–Moser inequality with the zero mean value in $W^{1,n}(\Omega)$ for $n \geq 3$.

The main purpose of this paper is to study the second-order trace Adams inequality on a smooth bounded domain Ω with zero mean value. Set

$$\mathcal{H} = \left\{ u \in W^{2,2}(\Omega) : \|\Delta u\|_2 \leq 1,\ \int_\Omega u\,dx = 0,\ \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0 \right\}.$$

Our main results read as follows.

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}^4$ be a bounded smooth domain with smooth boundary $\partial\Omega$. Then if $\alpha \leq 12\pi^2$, we have*

$$S(\alpha) := \sup_{u \in \mathcal{H}\setminus\{0\}} \int_{\partial\Omega} e^{\alpha u^2}\,d\sigma < \infty. \tag{1-3}$$

The constant $12\pi^2$ is sharp in the sense that if $\alpha > 12\pi^2$, then the supremum $S(\alpha)$ is infinity. Moreover, the supremum is attained if $\alpha \leq 12\pi^2$.

As an immediate consequence of Theorem 1.1, we have the following result when Ω is a four-dimensional ball $\mathbb{B}^4$ and $\mathbb{S}^3$ is its boundary.

Corollary 1.2. *If $\alpha \leq 12\pi^2$, we have*

$$S(\alpha, \mathbb{B}^4) := \sup_{u \in \mathcal{H}\setminus\{0\}} \int_{\mathbb{S}^3} e^{\alpha u^2}\,d\sigma < \infty. \tag{1-4}$$

The constant $12\pi^2$ is sharp in the sense that if $\alpha > 12\pi^2$, then the supremum $S(\alpha, \mathbb{B}^4)$ is infinity. Moreover, the supremum $S(\alpha, \mathbb{B}^4)$ is attained if $\alpha \leq 12\pi^2$.

As an application of Theorem 1.1, we can derive the following trace Adams–Onofri-type inequality on any four-dimensional bounded domains.

Theorem 1.3. *Assume that $\Omega \subseteq \mathbb{R}^4$ is a bounded smooth domain with smooth boundary $\partial\Omega$. For any $u \in W^{2,2}(\Omega)$ with $\dfrac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$, there exists a constant C such that*

$$\frac{1}{48\pi^2}\int_\Omega |\Delta u|^2\,dx + \frac{1}{|\Omega|}\int_\Omega u\,dx - \log\left(\int_{\partial\Omega} e^u\,d\sigma\right) \geq C.$$

Remark 1.4. The Moser–Onofri and Adams–Onofri inequalities on the sphere can be obtained by using the endpoint differentiation argument (see Beckner's work [1993]). However, this method cannot be used to establish the sharp trace Adams–Onofri inequality on general four-dimensional bounded domains due to its absence of conformal invariance. Our Theorem 1.3 offers a weak version of such a Moser–Onofri-type inequality on general bounded domains in $\mathbb{R}^4$ from the sharp Adams trace inequality obtained in Theorem 1.1.

Remark 1.5. The first-order Sobolev trace inequality was due to Escobar [1988] and Beckner [1993]. The second-order and higher-order Sobolev trace inequalities were established by Ache and Chang [2017], and subsequently by Ngô, Nguyen, and Phan [Ngô et al. 2020], and Q. Yang [2019], where they established the sharp trace Sobolev inequality of higher-order on the real unit ball $\mathbb{B}^{n+1} \subset \mathbb{R}^{n+1}$. Case [2020] establishes a family of sharp Sobolev trace inequalities involving the $W^{k,2}(\mathbb{R}^n_+, y^\alpha)$-norm, which leads to the well-known embedding

$$W^{k,2}(\mathbb{R}^{n+1}_+) \hookrightarrow \bigoplus_{j=0}^{k-1} W^{k-j-\frac{1}{2},2}(\mathbb{R}^n).$$

More recently, Flynn, the second author, and Q. Yang [Flynn et al. 2023] introduced conformally covariant boundary operators for Poincaré–Einstein manifolds satisfying a mild spectral assumption. Using these boundary operators the authors set up related higher-order trace Sobolev inequalities on these manifolds. They later [Flynn et al. 2025] introduced an appropriate family of conformally covariant boundary operators associated to the Siegel domain $\mathcal{U}^{n+1}$ with the Heisenberg group $\mathbb{H}^n$ as its boundary and the complex ball $\mathbb{B}^{n+1}_{\mathbb{C}}$ with the complex sphere $\mathbb{S}^{2n+1}$ as its boundary and prove all higher-order CR Sobolev trace inequalities for the Siegel domain $\mathcal{U}^{n+1}$ and the complex ball $\mathbb{B}^{n+1}_{\mathbb{C}}$. This generalizes the Sobolev trace inequality in the CR setting by Frank, González, Monticelli, and Tan [Frank et al. 2015] in the case $\gamma \in (0, 1)$ to the general case for all $\gamma \in (0, n+1) \setminus \mathbb{N}$.

Remark 1.6. Besides the trace Trudinger–Moser inequalities on bounded domains as discussed earlier, we also refer to the recent article [Chen et al. 2022a] by the authors and Yang, which studies trace Trudinger–Moser and Adams inequalities under various constraints on the upper half-spaces $\mathbb{R}^{2m}_+$ by the Fourier rearrangement and the polyharmonic extension.

The general strategy we use in this paper is exploiting the blow-up analysis. We first prove the subcritical trace Adams inequalities and the existence of extremals by using the sharp subcritical Adams inequalities involved with zero mean value. Then, we take a sequence $\alpha_k \to 12\pi^2$ and find a maximizing sequence $\{u_k\}_k \subset W^{2,2}(\Omega)$ for $S(12\pi^2)$. If u_k is bounded in $L^\infty(\Omega)$, i.e., $c_k := \max_{x\in\Omega} |u_k(x)| < \infty$, we can easily show that u_k converges to a function u in $W^{2,2}(\Omega)$ by elliptic estimates. If $c_k \to +\infty$, i.e., the blow-up arises, we apply the blow-up analysis method to analyze the asymptotic behavior of u_k near and far away from the blow-up point $p \in \partial\Omega$, and derive an upper bound for the trace Adams functional,

$$S(12\pi^2) \le |\partial\Omega| + 2\pi^2 e^{12\pi^2 A_p - \frac{3}{4}}, \tag{1-5}$$

where A_p is the value at p of the trace of the regular part of the Green function for the operator $\Delta^2 + 1/|\Omega|$. Finally, we construct a function sequence in $\mathcal{H}$ to show that the upper bound can actually be surpassed, which implies that the concentration phenomenon will not happen.

Neither the blow-up strategy in the study of second-order Adams inequality with Dirichlet boundary condition (see [Lu and Yang 2009a]) nor the blow-up method for the first-order trace Trudinger–Moser inequality and existence of their extremals (see [Li and Liu 2005; Yang 2006]) can be easily generalized to second-order trace Adams inequality case. In the following, we will introduce the main difference between the proof of the second-order trace Adams inequality and those for the second-order Adams inequality with Dirichlet boundary condition and the trace Trudinger–Moser blow-up analysis (see [Lu and Yang 2009a; Yang 2006]). We will also outline the elements of novelty when we carry out the blow-up procedure for our second-order trace Adams inequality.

First of all, the main difference between the blow-up analysis for the Adams inequality and that for the Adams trace inequality blowing-up is the location of the blow-up points. For the former, the blow-up points must be at some interior points, while for the latter the blow-up points must lie on the boundary of Ω, which leads to the situation where the related Euler–Lagrange equations of the maximizer sequence are some biharmonic equations with the Neumann boundary condition and the analysis of asymptotic behavior near and far away from the blow-up points is totally different.

Second, unlike the first-order case, one cannot show that the maximum point x_k of u_k lies on the boundary $\partial\Omega$ due to the absence of maximum principle of the biharmonic operator Δ^2. We stress that without this fact one cannot analyze the asymptotic behavior of u_k near the blow up points $p \in \partial\Omega$ if the concentration phenomenon occurs. To overcome this difficulty, we will make use of the assumption that

$$\frac{\partial u_k}{\partial \nu} = 0 \quad \text{on } \partial\Omega$$

to show that there exists some point $\tilde{x}_k \in \partial\Omega$ such that

$$|u_k(\tilde{x}_k) - u_k(x_k)| = o_k(1).$$

This important observation allows us to choose the maximum point x_k on the boundary $\partial\Omega$.

Third, when we try to analyze the asymptotic behavior of u_k near the blow up point p, a crucial step is to classify the solutions to the Liouville equation

$$\begin{cases} \Delta^2\psi = 0, & x \in \mathbb{R}^4_+, \\ \dfrac{\partial \Delta\psi}{\partial t} = \exp(24\pi^2\psi), & x \in \partial\mathbb{R}^4_+, \\ \psi(0) = \sup \psi = 0, & \\ \dfrac{\partial\psi}{\partial t} = 0, & x \in \partial\mathbb{R}^4_+, \end{cases}$$

where $\mathbb{R}^4_+$ is the half-space $\{x = (x', t) : x' \in \mathbb{R}^3, t > 0\}$. Instead of assuming $\psi \in W^{2,2}(\mathbb{R}^4_+)$ or other global integrality condition for ψ (see [Ache and Chang 2017; Ndiaye and Sun 2024]), we can prove the classification theorem under the finite growth condition $\int_{B_R^+} |\Delta\psi|\, dx \le CR^2$. This finite growth condition can be verified by the technique of harmonic analysis. Indeed, in order to achieve this goal, we need to prove an important local estimate for Δu_k:

$$\int_{B_\rho(x_k)\cap\Omega} |u_k \Delta u_k|\, dx \le C\rho^2, \tag{1-6}$$

when ρ is small. For this, we first rewrite $u_k\Delta u_k$ in terms of the Riesz potential, then by using the Hardy–Littlewood–Sobolev inequality on the compact manifold with the boundary and the boundedness in $L\log^{\frac{1}{2}} L(\Omega)$of $\Delta^2 u_k$, we can show that $u_k \Delta u_k$ is bounded in the Lorentz space $L^{2,\infty}(\Omega)$, which implies (1-6). Applying this local estimate and a careful computation, we can show that the solution ψ must take the form

$$\psi = -\frac{1}{8\pi^2}\log\Big(\big(1+\big(\tfrac{\pi}{2}\big)^{\frac{2}{3}}t\big)^2+\big(\tfrac{\pi}{2}\big)^{\frac{4}{3}}|x'|^2\Big)+\frac{1}{2^{\frac{8}{3}}\pi^{\frac{4}{3}}}\frac{t}{\big(1+\big(\frac{\pi}{2}\big)^{\frac{2}{3}}t\big)^2+\big(\frac{\pi}{2}\big)^{\frac{4}{3}}|x'|^2}.$$

Fourth, when we try to obtain the upper bound for the trace Adams inequality if the concentration phenomenon occur, an important step consists in finding the sharp lower bounds of the integral of $|\Delta u_k|^2$ on some annular regions when we carry out the capacity estimates. In earlier work [Li and Liu 2005; Yang 2006], this could be achieved by comparing the energy of u_k with the quantity

$$\min_{u\in\{u\,:\,u(R_1)=a,\,u(R_2)=b\}}\int_{\{R_1\le|x|\le R_2\}\cap\mathbb{R}^2_+}|\nabla u|^2\,dx, \tag{1-7}$$

whose extremal function is some harmonic function which can be explicitly obtained by solving some equation on the half-space. However, in the second-order case, finding the explicit expression of the corresponding extremal appears to be very hard. In this work, we will compute the upper bound by directly comparing the Dirichlet energy of u_k with some biharmonic function in the half annular region. In our situation the boundary of the upper half annular region involves some part of $\partial\Omega$ where u_k is not vanishing. This will add a lot of trouble in the comparison of the corresponding calculations since the asymptotic behavior of u_k cannot be obtained in this half annular region. In order to avoid the complicated computations on $\partial\Omega$, we will modify the biharmonic function to cancel the integral on the boundary $\partial\Omega$ (see Section 3.2).

Finally, since $\mathcal{H}$ requires that the test functions not only satisfy $\|\Delta u\|^2 \le 1$, but also satisfy $\partial u/\partial\nu = 0$ on $\partial\Omega$, this makes the construction of test functions more complicated when we try to show that the concentration upper bound can be surpassed.

This paper is organized as follows. Section 2 is devoted to proving the sharp subcritical trace Adams inequality, and showing the existence of extremals. In Section 3, we show that the maximizing sequence must concentrate around the blow-up point when the blow up arises. Moreover, we analyze the asymptotic behavior of the maximizing sequence near and far away from the blow-up point, and derive an upper bound for the trace Adams functional. In Section 4, we prove the existence of extremals by constructing a proper test function sequence, and finish the proof of Theorem 1.1.

2. The best constant for the trace Adams inequality

In this section, we prove that the best constant in Theorem 1.1 is $12\pi^2$. First, we recall the following subcritical Adams inequalities for functions with zero mean value proved by Hang [2022]. Throughout this section, we let $\Omega\subseteq\mathbb{R}^4$ be a bounded smooth domain with smooth boundary $\partial\Omega$. We also recall that

$$\mathcal{H}=\Big\{u\in W^{2,2}(\Omega):\|\Delta u\|_2\le 1,\ \int_\Omega u\,dx=0,\ \frac{\partial u}{\partial\nu}\Big|_{\partial\Omega}=0\Big\}.$$

Lemma 2.1 [Hang 2022, Theorem 3.2]. *For any $\varepsilon > 0$, we have*

$$\sup_{u\in\mathcal{H}\setminus\{0\}} \int_\Omega \exp((16\pi^2-\varepsilon)|u|^2)\,dx < \infty. \tag{2-1}$$

Next, we further prove the following.

Lemma 2.2. *Set $\alpha_2 = \sup\{\alpha : \sup_{u\in\mathcal{H}} \int_\Omega e^{\alpha u^2} < +\infty\}$. Then $\alpha_2 = 16\pi^2$.*

Proof. By Lemma 2.1, we know $\alpha_2 \ge 16\pi^2$. In order to prove the lemma, we only need to show $\alpha \le 16\pi^2$. Taking any $p \in \partial\Omega$, for any $0 < \rho < \delta$, we use the notation $B_\rho = B_\rho(p)$ and set

$$u_k(x) := \begin{cases} \sqrt{\frac{1}{16\pi^2}\log(1/R_k)} - \dfrac{|x|^2}{\rho^2\sqrt{4\pi^2 R_k \log(1/R_k)}} + \dfrac{1}{\sqrt{4\pi^2\log(1/R_k)}} & \text{if } x \in B_{\rho\sqrt[4]{R_k}} \cap \Omega, \\ 1/\sqrt{\pi^2\log(1/R_k)}\,\log(\rho/|x|) & \text{if } x \in (B_\rho \setminus B_{\rho\sqrt[4]{R_k}}) \cap \Omega, \\ \eta_k(|x|) & \text{if } x \in (B_\delta \setminus B_\rho) \cap \Omega, \end{cases}$$

where $\{R_k\}_{k\ge1} \subset \mathbb{R}^+$, $R_k \searrow 0$, and η_k satisfies

$$\frac{\partial\eta_k}{\partial\nu}\Big|_{\partial B_\rho} = -\frac{1}{\rho\sqrt{\pi^2\log(1/R_k)}}, \quad \frac{\partial\eta_k}{\partial\nu}\Big|_{\partial B_\delta} = 0, \quad \eta_k|_{\partial B_\rho} = \eta_k|_{\partial B_\delta} = 0,$$

and η_k, $\Delta\eta_k$ are all $O(1/\sqrt{\log 1/R_k})$. Since u_k is radial, we can choose some function $\varphi_k(x)$ such that

$$\frac{\partial\varphi_k}{\partial\nu} = \frac{\partial u_k}{\partial\nu} = o_\delta(1) \quad \text{for } x \in \partial\Omega,$$

$\varphi_k(x) = o_\delta(1)$ and $\|\Delta\varphi_k\|_2^2 = o_\delta(1)$ as $\delta \to 0$.

For some fixed $r > \delta$, set

$$U_k(x) := \begin{cases} u_k - \varphi_k & \text{if } x \in B_\delta \cap \Omega, \\ t_k\phi_k & \text{if } x \in (B_r \setminus B_\delta) \cap \Omega, \end{cases}$$

where ϕ_k is a smooth function such that $\operatorname{supp}(\phi_k) \subset B_r \setminus B_\delta$, $\frac{\partial\phi_k}{\partial\nu}\big|_{\partial\Omega} = 0$, and t_k is selected such that $\int_\Omega U_k(x)\,dx = 0$. Easy computation directly gives

$$\|U_k\|_2^2 = O_{k,r}(1), \quad \|\Delta U_k\|_2^2 = 1 + O_{k,r}(1).$$

Normalizing U_k by $\widetilde{U}_k = U_k/\|\Delta U_k\|$, we have $\widetilde{U}_k \in \mathcal{H}$. Then it follows that for any fixed $\alpha > 16\pi^2$, there exists some $\varepsilon_0 > 0$, such that

$$\int_\Omega e^{\alpha\widetilde{U}_k^2} \ge \int_{\Omega\cap B_{\rho\sqrt[4]{R_k}}} e^{\alpha\widetilde{U}_k^2} \ge c\rho^4 e^{\varepsilon_0\log(1/R_k)} \to \infty,$$

as $k \to \infty$, and the proof is finished. ☐

Based on Lemma 2.2, we can show that the best constant of the inequality (1-3) is $12\pi^2$.

Lemma 2.3. *Set $I_\alpha(u) = \int_{\partial\Omega} e^{\alpha u^2}\,d\sigma$. Then we have*

$$\sup_{u\in\mathcal{H}\setminus\{0\}} I_\alpha(u) < +\infty \quad \text{for } \alpha < 12\pi^2 \qquad \text{and} \qquad \sup_{u\in\mathcal{H}\setminus\{0\}} I_\alpha(u) = +\infty \quad \text{for } \alpha > 12\pi^2.$$

Proof. Take a smooth vector field $\vec{\nu}(x)$ whose restriction on $\partial\Omega$ is the outward unit normal vector field. Using the divergence theorem and Sobolev embedding theorem, we derive that for any $\varepsilon > 0$,

$$\begin{aligned}\int_{\partial\Omega} e^{(12\pi^2-\varepsilon)u^2}\,d\sigma &= \int_\Omega \operatorname{div}(\vec{\nu}(x)e^{(12\pi^2-\varepsilon)u^2})\,dx\\ &= \int_\Omega \big(\operatorname{div}(\vec{\nu}(x)) + 2(12\pi^2-\varepsilon)u\langle\vec{\nu}(x),\nabla u\rangle\big)e^{(12\pi^2-\varepsilon)u^2}\,dx\\ &\le c\biggl(1+\int_\Omega |\nabla u|\,|u|e^{(12\pi^2-\varepsilon)u^2}\biggr)\,dx\\ &\le c\big(1+\|\nabla u\|_{L^4(\Omega)}\|u\|_{L^p(\Omega)}\|e^{(12\pi^2-\varepsilon)u^2}\|_{L^{(16\pi^2-\varepsilon)/(12\pi^2-\varepsilon)}(\Omega)}\big)\\ &\le c+c(\|\Delta u\|^2_{L^2(\Omega)}+\|u\|^2_{L^2(\Omega)})^{\frac12}\|u\|_{L^p(\Omega)}\|e^{(12\pi^2-\varepsilon)u^2}\|_{L^{(16\pi^2-\varepsilon)/(12\pi^2-\varepsilon)}(\Omega)},\end{aligned} \tag{2-2}$$

where $1/p+\frac14+(12\pi^2-\varepsilon)/(16\pi^2-\varepsilon)=1$. This together with Lemma 2.2 yields

$$\sup_{u\in\mathcal{H}\setminus\{0\}} I_{12\pi^2-\varepsilon}(u) < +\infty$$

for any $\varepsilon > 0$. Using the test function $\widetilde{U}_k$ constructed in Lemma 2.2 again, one can easily check that for any $\alpha > 12\pi^2$, $I_\alpha(\widetilde{U}_k) \to +\infty$ as $k \to \infty$. □

Let α_k be an increasing sequence converging to $12\pi^2$. Then by the weak compactness of the Banach space $L^{12\pi^2/\alpha_k}$, there exists an extremal function $u_k \in \mathcal{H}\setminus\{0\}$ such that

$$\int_{\partial\Omega} \exp(\alpha_k|u_k|^2)\,d\sigma = \sup_{u\in\mathcal{H}\setminus\{0\}} \int_{\partial\Omega}\exp(\alpha_k|u|^2)\,d\sigma.$$

Furthermore, we can show that the extremal function $u_k \in \mathcal{H}$ is smooth. For this, we first recall the following elliptic regularity result.

Lemma 2.4 [Troianiello 1987, Theorem 3.17]. *Suppose that $f \in L^p(\Omega)$ and $h \in W^{1,p}(\Omega)$ for some $p \ge 2$. Let $u \in W^{1,2}(\Omega)$ be a solution of*

$$\begin{cases}\Delta u = f & \text{in } \Omega,\\ \dfrac{\partial u}{\partial\nu} = h & \text{on } \partial\Omega.\end{cases}$$

Then $u \in W^{2,p}(\Omega)$.

Lemma 2.5. *For any $\alpha_k < 12\pi^2$, the functional $I_{\alpha_k}(u)$ defined in $\mathcal{H}$ admits a smooth maximizer.*

Proof. Obviously, there exists $u_k \in \mathcal{H}$ such that

$$I_{\alpha_k}(u_k) = \sup_{u\in\mathcal{H}\setminus\{0\}} I_{\alpha_k}(u).$$

Hence u_k satisfies the Euler–Lagrange equation

$$\begin{cases}\Delta^2 u_k = \gamma_k & \text{for all } x\in\Omega,\\ \dfrac{\partial\Delta u_k}{\partial\nu} = \dfrac{u_k\exp(\alpha_k u_k^2)}{\lambda_k} & \text{for all } x\in\partial\Omega,\\ \int_\Omega |\Delta u_k|^2\,dx = 1,\ \int_\Omega u_k\,dx = 0,\ \dfrac{\partial u_k}{\partial\nu} = 0 & \text{for all } x\in\partial\Omega,\end{cases} \tag{2-3}$$

where

$$\lambda_k = -\int_{\partial\Omega} u_k^2 \exp(\alpha_k u_k^2)\, d\sigma, \quad \gamma_k = \int_{\partial\Omega} \frac{u_k \exp(\alpha_k u_k^2)}{\lambda_k |\Omega|}\, d\sigma. \tag{2-4}$$

By the Orlicz embedding (see Lemma 3.4 in [Hang 2022]), we obtain $\exp(u_k^2) \in L^p(\Omega)$ for any $p > 1$. Therefore, $u_k \exp(\alpha_k u_k^2)/\lambda_k \in W^{1,q}(\Omega)$ for any $1 < q < 2$. We claim that $u_k \in L^\infty(\Omega)$. Indeed, we can rewrite (2-3) as the systems

$$\begin{cases} \Delta u_k = v_k & \text{in } \Omega, \\ \dfrac{\partial u_k}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2-5}$$

and

$$\begin{cases} \Delta v_k = \gamma_k & \text{in } \Omega, \\ \dfrac{\partial v_k}{\partial \nu} = h_k & \text{on } \partial\Omega, \end{cases} \tag{2-6}$$

where $h_k = u_k \exp(\alpha_k u_k^2)/\lambda_k$. Applying Lemma 2.4 for (2-6), we know $v_k \in W^{2,q}(\Omega)$. By the Sobolev embedding theorem, we get $v_k \in L^{4q/(4-2q)}(\Omega)$. Using Lemma 2.4 again for (2-5), we derive that $u_k \in W^{2,4q/(4-2q)}(\Omega)$. Since $q > 1$, we can immediately obtain the claim by the Sobolev embedding theorem.

From the boundedness of u_k, we know that $h_k \in W^{1,2}(\Omega)$. Thus we have $h_k \in W^{2,2}(\Omega)$ from Lemma 2.4, and hence $h_k \in W^{1,p}(\Omega)$ for any $p > 2$. By Lemma 2.4 again we have $u_k \in W^{2,p}(\Omega)$ for any $p > 2$, which implies that $u_k \in C^{1,\alpha}(\Omega)$ for some $\alpha \in (0, 1)$ by the Sobolev compact embedding. Since on the boundary $\partial\Omega$,

$$\frac{\partial v_k}{\partial \nu} = h_k = \frac{u_k \exp(\alpha_k u_k^2)}{\lambda_k} \in C^{1,\alpha}(\partial\Omega)$$

and v_k satisfies (2-6), the elliptic regularity gives that $v_k \in C^{2,\alpha}(\Omega)$. Since $\Delta u_k = v_k \in C^{2,\alpha}(\Omega)$ and by (2-5), we can furthermore derive that $u_k \in C^{4,\alpha}(\Omega)$ using the elliptic regularity again. Repeating the above procedure, we obtain $u_k \in C^\infty(\Omega)$. □

Now, we give the following important observation.

Lemma 2.6. $-\liminf_{k\to\infty} \lambda_k > 0$, *and* $|\gamma_k| < c$ *for some* $c > 0$.

Proof. By the elementary inequality $te^t > e^t - 1$ for all $t > 0$, we have

$$|\partial\Omega| < \sup_{u \in \mathcal{H}\setminus\{0\}} \int_{\partial\Omega} e^{12\pi^2 u^2} = \lim_{k\to\infty} \int_{\partial\Omega} e^{\alpha_k u_k^2} \le |\partial\Omega| - \liminf_{k\to\infty} 12\pi^2 \lambda_k. \tag{2-7}$$

This implies $-\liminf_{k\to\infty} \lambda_k > 0$. By (2-4), (2-7) and Hölder's inequality, we derive

$$|\gamma_k| \le \frac{-1}{\lambda_k |\Omega|} \left(\int_{\partial\Omega} u_k^2 \exp(\alpha_k u_k^2)\, d\sigma \right)^{\frac{1}{2}} \left(\int_{\partial\Omega} \exp(\alpha_k u_k^2)\, d\sigma \right)^{\frac{1}{2}} \le c.$$ □

Set $c_k = |u_k(x_k)| = \max_{x\in\Omega} |u_k(x)|$. If $\{c_k\}$ is bounded, then by the elliptic estimates with respect to (2-3), there exists $u \in \mathcal{H} \cap C^\infty(\Omega)$ such that $u_k \to u$ in $C^\infty(\Omega)$ as $k \to \infty$, and Theorem 1.1 follows immediately. In the sequel, we assume $c_k \to +\infty$ as $k \to \infty$. Passing to a subsequence, we may assume that $u_k(x_k) \ge 0$ for all k, for otherwise we consider $-u_k$ instead of u_k.

3. Blow-up analysis

In this section, we consider the blow-up case, that is $u_k(x_k) \to \infty$ as $k \to \infty$. Applying the Adams inequality [1988], we know that passing to a subsequence, $x_k \to p$ for some $p \in \partial\Omega$. Now, we show that the weak limit of u_k in $W^{2,2}(\Omega)$ is zero. Furthermore, u_k must concentrate around the blow-up point p.

Lemma 3.1. *If $c_k \to +\infty$, then $u_k \rightharpoonup 0$ in $W^{2,2}(\Omega)$ and $u_k \to 0$ in $L^p(\Omega)$ for any $1 \le p < \infty$. Moreover,*

(i) $|\Delta u_k|^2\,dx \to \delta_p$ *in the sense of measures*;

(ii) $e^{\alpha_k u_k^2}$ *is bounded in $L^p(\Omega \setminus B_\delta(p))$, for any $p \ge 1$, $\delta > 0$*;

(iii) $u_k \to 0$ *in $C^{3,\gamma}(\Omega \setminus B_\delta(p))$, for any $\gamma \in (0,1)$, $\delta > 0$.*

Proof. Since u_k is bounded in $W^{2,2}(\Omega)$, we assume that $u_k \rightharpoonup u_0$ in $W^{2,2}(\Omega)$ with some $u_0 \in W^{2,2}(\Omega)$. The compactness of the embedding of $W^{2,2}(\Omega)$ into $L^p(\Omega)$ implies $u_k \to u_0$ in $L^p(\Omega)$ for any $p \ge 1$. If $u_0 \ne 0$, then by the concentration compactness principle (see Proposition 3.2 of [Hang 2022]), $e^{16\pi^2 u_k^2}$ is bounded in $L^p(\Omega)$ for some $p > 1$. Similarly to (2-2), we can find some $\varepsilon_0 > 0$ such that $e^{12\pi^2 u_k^2}$ is bounded in $L^{1+\varepsilon_0}(\partial\Omega)$, hence $\partial\Delta u_k/\partial\nu$ is bounded in $L^{1+\varepsilon_0}(\partial\Omega)$. Using the same argument in Lemma 2.5, we get that u_k is bounded in $L^\infty(\Omega)$. This contradicts $c_k \to +\infty$. Hence, we have $u_0 = 0$.

Now, we show that u_k must concentrate around the blow-up point p. Let

$$A = \left\{ q \in \Omega : \lim_{r\to 0} \liminf_{k\to\infty} \int_{B_r(q)} |\Delta u_k|^2\,dx > 0 \right\}.$$

We claim that A contains only one point. Suppose that the claim does not hold. Then, for any $q \in \Omega$, we have

$$\lim_{r\to 0} \liminf_{k\to\infty} \int_{B_r(q)} |\Delta u_k|^2\,dx < 1.$$

Then there exist positive numbers r and δ such that

$$\int_{B_r(q)} |\Delta u_k|^2\,dx \le \delta(q) < 1.$$

Using the same argument as that in (2-2) again, we see that there exists a constant $\alpha(q) > 12\pi^2$ such that

$$\int_{\partial\Omega \cap B_r(q)} e^{\alpha(q) u_k^2}\,d\sigma \le C_q$$

for some constant C_q depending on q. Hence there exists an $\alpha > 12\pi^2$ such that

$$\int_{\partial\Omega} e^{\alpha u_k^2}\,d\sigma \le C,$$

by using the covering argument. Therefore, it follows from the Vitali convergence lemma that

$$\lim_{k\to+\infty} \int_{\partial\Omega} e^{\alpha_k u_k^2}\,d\sigma = |\partial\Omega|,$$

which is impossible by the choice of u_k. Next we show that $A = \{p\}$ and

$$\lim_{r\to 0}\liminf_{k\to\infty}\int_{B_r(p)} |\Delta u_k|^2 = 1.$$

Suppose not: repeating the argument above, we can obtain that u_k is bounded in $L^\infty(B_\delta(p))$ for some $\delta > 0$, which contradicts with $c_k \to +\infty$, and the statement (i) is proved.

The statement (ii) follows from (i) and Lemma 2.1, and the statement (iii) can be proved by the standard regularity argument; we omit the details. □

To understand the asymptotic behavior of u_k near the blow-up point p, we define

$$r_k^3 = -\frac{\lambda_k}{c_k^2}\exp(-\alpha_k c_k^2).$$

Indeed, r_k decays very fast as $k \to \infty$:

Lemma 3.2. *For any $\gamma < 12\pi^2$, it holds that $e^{\gamma c_k^2} r_k^3 \to 0$ as $k \to \infty$.*

Proof. For any $\gamma < 12\pi^2$, we have

$$\begin{aligned} c_k^2 r_k^3 e^{\gamma c_k^2} &= e^{(\gamma-\alpha_k)c_k^2}\int_{\partial\Omega} u_k^2 e^{\alpha_k u_k^2}\,d\sigma \le \int_{\partial\Omega} u_k^2 e^{\alpha_k u_k^2} e^{(\gamma-\alpha_k)u_k^2}\,d\sigma \\ &= \int_{\partial\Omega} u_k^2 e^{\gamma u_k^2}\,d\sigma \le \left(\int_{\partial\Omega} u_k^s\,d\sigma\right)^{2/s}\left(\int_{\partial\Omega} e^{\gamma s/(s-2)u_k^2}\,d\sigma\right)^{(s-2)/s} \\ &\le c, \end{aligned}$$

provided s is large enough, where we have used the subcritical trace Adams inequality. □

The following important observation allows us to choose suitably approximate $\{x_k\}$ by points on the boundary $\partial\Omega$.

Lemma 3.3. *There exists some point $\tilde{x}_k \in \partial\Omega$ such that*

$$|u_k(\tilde{x}_k) - u_k(x_k)| = o_k(1),$$

as $k \to \infty$.

Proof. If $x_k \notin \partial\Omega$, since $d(x_k, \partial\Omega)$ is sufficiently small, then there exists a unique $y_k \in \Omega$ such that $d_k := d(x_k, \partial\Omega) = |y_k - x_k|$ and $x_k = y_k + d_k\nu_k$, where ν_k is the inner normal vector of boundary $\partial\Omega$ at the point y_k. Hence

$$|u_k(x_k) - u_k(y_k)| \le \int_0^1 \left|\frac{d}{dt}(u_k(y_k + t d_k \nu_k)\right| dt = \int_0^1 \left|\frac{\partial u_k}{\partial \nu_k}(y_k + t d_k, \nu_k) d_k\right| dt \to 0$$

when $k \to +\infty$ by the mean value theorem and the fact that $\frac{\partial u_k}{\partial_{\nu_k}}\big|_{y_k} = 0$, and the proof is finished. □

In view of Lemma 3.3, we can take $x_k = \tilde{x}_k \in \partial\Omega$ and then

$$u_k(x_k) = c_k + o_k(1), \tag{3-1}$$

as $k \to \infty$. Define two sequences of functions on $\partial\Omega$, namely,

$$\begin{cases} \phi_k(x) = u_k(x_k + r_k x)/c_k, & x \in \Omega_k = \{x : x_k + r_k x \in \Omega\}, \\ \psi_k(x) = c_k(u_k(x_k + r_k x) - c_k), & x \in \Omega_k. \end{cases}$$

Up to translation and rotation, we can easily obtain $\Omega_k \to \mathbb{R}^4_+$ as $k \to +\infty$.

Lemma 3.4. $$\phi_k(x) \to 1 \quad \textit{in } C^3_{\mathrm{loc}}(\overline{\mathbb{R}^4_+}).$$

Proof. By (2-3), for k large enough we have

$$\begin{cases} \Delta^2 \phi_k = \dfrac{r_k^4}{c_k} \gamma_k & \text{for all } x \in B_R(0) \cap \Omega_k, \\ \dfrac{\partial}{\partial \nu} \Delta \phi_k = \dfrac{r_k^3 u_k \exp(\alpha_k u_k^2)}{c_k \lambda_k} & \text{for all } x \in B_R(0) \cap \partial\Omega_k, \end{cases} \tag{3-2}$$

for any $R > 0$. By the definition of r_k, we have

$$\left| \frac{r_k^4}{c_k} \gamma_k \right| = \frac{\lambda_k}{c_k^2} \exp(-\alpha_k c_k^2) \frac{r_k}{c_k} \int_{\partial\Omega} \frac{u_k \exp(\alpha_k u_k^2)}{\lambda_k |\Omega|} \, d\sigma \le \frac{|\partial\Omega|}{|\Omega|} \frac{r_k}{c_k^2} \to 0$$

and

$$\left| \frac{r_k^3 u_k \exp(\alpha_k u_k^2)}{c_k \lambda_k} \right| \le \frac{1}{c_k^2} \to 0,$$

as $k \to \infty$. Since ϕ_k is bounded in $L^1_{\mathrm{loc}}(\overline{B_R(0) \cap \Omega_k})$ and $\phi_k(x_k) = 1 + o_k(1)$, by the standard elliptic regularity argument, we have $\phi_k \to 1$ in $C^3_{\mathrm{loc}}(\overline{B_{R/2}(0) \cap \Omega_k})$. □

In order to obtain the limit behavior of ψ_k, we need to check the following growth condition:

Lemma 3.5. $$\int_{B_R(0) \cap \Omega_k} |\Delta \psi_k| \, dx \le C R^2.$$

Proof. Direct computation gives that

$$\int_{B_R(0) \cap \Omega_k} |\Delta \psi_k| \, dx = c_k r_k^{-2} \int_{B_{Rr_k}(x_k) \cap \Omega} |\Delta u_k| \, dx.$$

Since $u_k(r_k x + x_k)/c_k \to 1$ in $C^3(B_R \cap \Omega_k)$ for any $R > 0$, in order to prove this lemma we only need to show that

$$(R r_k)^{-2} \int_{B_{Rr_k}(x_k) \cap \Omega} |u_k \Delta u_k| \, dx \lesssim 1.$$

Applying Hölder's inequality in Lorentz space (see [O'Neil 1963]), we get

$$\begin{aligned} (R r_k)^{-2} \int_{B_{Rr_k}(x_k) \cap \Omega} |u_k \Delta u_k| \, dx &\le (R r_k)^{-2} \| \chi_{B_{Rr_k}(x_k) \cap \Omega_k} \|_{L^{2,1}(B_{Rr_k}(x_k) \cap \Omega)} \| u_k \Delta u_k \|_{L^{2,\infty}(B_{Rr_k}(x_k) \cap \Omega)} \\ &\lesssim \| u_k \Delta u_k \|_{L^{2,\infty}(B_{Rr_k}(x_k))}. \end{aligned} \tag{3-3}$$

Now, we start to prove that $\|u_k\Delta u_k\|_{L^{2,\infty}(\Omega)} \lesssim 1$. Let G denote the Green function of the Laplacian operator with Neumann boundary condition:

$$\begin{cases} -\Delta G_x(y) = \delta_x(y) - \dfrac{1}{|\Omega|}, & x,\, y \in \overline{\Omega}, \\ \dfrac{\partial G}{\partial \nu}\Big|_{\partial\Omega} = 0, & \\ \int_\Omega G_x(y)\,dy = 0, & x \in \overline{\Omega}. \end{cases}$$

Obviously $G_x(y)$ satisfies $G_x(y) \lesssim |x-y|^{-2}$ for any $x,\, y \in \Omega$. By integration by parts together with $\int_\Omega u_k(x)\,dx = 0$ and $\partial u_k/\partial\nu|_{\partial\Omega} = 0$ and using the fact that $\int_\Omega |\Delta u_k|^2\,dx = 1$ (see the Euler–Lagrange equation (2-3)), we derive that

$$|u_k(x)| \lesssim \int_\Omega |\Delta u_k|\,|x-y|^{-2}\,dy$$

and

$$|\Delta u_k(x)| \lesssim \int_{\partial\Omega} |x-y|^{-2} f_k(y)\,d\sigma_y + \int_\Omega |x-y|^{-2}\gamma_k\,dy + \frac{1}{|\Omega|}\int_\Omega |\Delta u_k|\,dy \lesssim 1 + \int_{\partial\Omega} |x-y|^{-2} f_k(y)\,d\sigma_y,$$

where $f_k = u_k \exp(\alpha_k u_k^2)/\lambda_k$. Then it follows that

$$|u_k(x)|\,|\Delta u_k(x)| \lesssim \left(\int_\Omega |(\Delta u_k)(y)|\,|x-y|^{-2}\,dy\right)\left(1 + \int_{\partial\Omega} |x-z|^{-2} f_k(z)\,d\sigma_z\right). \tag{3-4}$$

Now, we claim that

$$\left\|\left(\int_\Omega |\Delta u_k(y)|\,|x-y|^{-2}\,dy\right)\left(1 + \int_{\partial\Omega} |x-z|^{-2} f_k(z)\,d\sigma_z\right)\right\|_{L^{2,\infty}} \lesssim 1.$$

Recall the Hardy–Littlewood–Sobolev inequality in $\mathbb{R}^n$: for any $f \in L^p(\mathbb{R}^n)$,

$$\left\|\int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\theta}}\,dy\right\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)},$$

where $p > 1,\ 0 < \theta < n$ and $\frac{1}{q} = \frac{1}{p} - \frac{\theta}{n}$. Hence it follows that

$$\begin{aligned} \left\|\left(\int_\Omega |\Delta u_k(y)|\,|x-y|^{-2}\,dy\right)\right\|_{L^{2,\infty}(\Omega)} &\lesssim \left\|\left(\int_\Omega |\Delta u_k(y)|\,|x-y|^{-2}\,dy\right)\right\|_{L^6(\mathbb{R}^4)} \\ &\lesssim \|\Delta u\|_{L^{\frac{3}{2}}(\Omega)} \lesssim \|\Delta u\|_{L^2(\Omega)} \lesssim 1. \end{aligned} \tag{3-5}$$

Hence it suffices to prove that

$$\left\|\left(\int_\Omega |\Delta u_k(y)|\,|x-y|^{-2}\,dy\right)\left(\int_{\partial\Omega} |x-z|^{-2} f_k(z)\,d\sigma_z\right)\right\|_{L^{2,\infty}} \lesssim 1.$$

For any $\varepsilon > 0$ sufficiently small, using the estimate (see [Maalaoui et al. 2016])

$$|x-y|^{-2}|x-z|^{-2} \le |x-y|^{-2-\varepsilon}|x-z|^{-2+\varepsilon} + |z-y|^{-2}|x-z|^{-2},$$

we obtain

$$\begin{aligned}
&\left\| \left(\int_\Omega |\Delta u_k(y)|\,|x-y|^{-2}\,dy \right) \left(\int_{\partial\Omega} |x-z|^{-2} f_k(z)\,d\sigma_z \right) \right\|_{L^{2,\infty}} \\
&\quad \le \left\| \left(\int_\Omega |\Delta u_k(y)|\,|x-y|^{-2-\varepsilon}\,dy \right) \left(\int_{\partial\Omega} |x-z|^{-2+\varepsilon} f_k(z)\,d\sigma_z \right) \right\|_{L^{2,\infty}} \\
&\qquad\qquad + \left\| \int_{\partial\Omega} \left(\int_\Omega |\Delta u_k(y)|\,|z-y|^{-2}\,dy \right) f_k(z) |x-z|^{-2}\,d\sigma_z \right\|_{L^{2,\infty}} \\
&:= \mathrm{I}_1 + \mathrm{I}_2. \qquad (3\text{-}6)
\end{aligned}$$

Applying the generalized Hölder's inequality involving the Lorentz norm, we derive that

$$\mathrm{I}_1 \le \left\| \int_\Omega |\Delta u_k(y)|\,|x-y|^{-2-\varepsilon}\,dy \right\|_{L^{4/\varepsilon}(\Omega)} \left\| \int_{\partial\Omega} |x-z|^{-2+\varepsilon} f_k(z)\,d\sigma_z \right\|_{L^{4/(2-\varepsilon),\infty}(\Omega)} := \mathrm{I}_{11} \times \mathrm{I}_{12}.$$

For I_{11}, the boundedness of fractional integral operator directly gives $\mathrm{I}_{11} \lesssim \|\Delta u_k\|_{L^2(\Omega)}$. For I_{12}, we claim that it can be dominated by $\|f_k\|_{L^1(\partial\Omega)}$. Define the auxiliary integral operators

$$T^1_{\varepsilon,r}(x) = \int_{\{\partial\Omega \cap |x-y|<r\}} \frac{f_k(y)}{|x-y|^{2-\varepsilon}}\,d\sigma_y, \quad T^2_{\varepsilon,r}(x) = \int_{\{\partial\Omega \cap |x-y|\ge r\}} \frac{f_k(y)}{|x-y|^{2-\varepsilon}}\,d\sigma_y.$$

Obviously,

$$\int_\Omega |T^1_{\varepsilon,r}(x)|\,dx \le \left(\sup_{y\in\partial\Omega} \int_{\{|x-y|<r\}} \frac{1}{|x-y|^{2-\varepsilon}}\,dx \right) \|f_k\|_{L^1(\partial\Omega)} \lesssim r^{2+\varepsilon} \|f_k\|_{L^1(\partial\Omega)}$$

and

$$\|T^2_{\varepsilon,r}\|_{L^\infty(\Omega)} \le \frac{1}{r^{2-\varepsilon}} \|f_k\|_{L^1(\partial\Omega)}.$$

For any $\lambda > 0$, we can write

$$|\{x : T^1_{\varepsilon,r}(x) + T^2_{\varepsilon,r}(x) > 2\lambda\}| \le |\{x : T^1_{\varepsilon,r}(x) > \lambda\}| + |\{x : T^2_{\varepsilon,r}(x) > \lambda\}|.$$

Choosing r such that $1/r^{2-\varepsilon} \|f_k\|_{L^1(\partial\Omega)} = \lambda$, then $|\{x : T^2_{\varepsilon,r}(x) > \lambda\}| = 0$. Hence, we deduce that

$$|\{x : T^1_{\varepsilon,r}(x) + T^2_{\varepsilon,r}(x) > 2\lambda\}| \lesssim \frac{r^{2+\varepsilon}}{\lambda} \|f_k\|_{L^1(\partial\Omega)} = \frac{1}{\lambda^{4/(2-\varepsilon)}} \|f_k\|^{4/(2-\varepsilon)}_{L^1(\partial\Omega)},$$

which gives that $\mathrm{I}_{12} \lesssim \|f_k\|_{L^1(\partial\Omega)}$, and the claim is proved.

Gathering the estimates of I_{11} and I_{12}, we derive that $\mathrm{I}_1 \lesssim \|\Delta u_k\|_{L^2(\Omega)} \|f_k\|_{L^1(\partial\Omega)}$. For I_2, obviously

$$\mathrm{I}_2 \lesssim \left\| \int_\Omega |\Delta u_k(y)|\,|z-y|^{-2} f_k(z)\,dy \right\|_{L^1(\partial\Omega)} \lesssim \|\Delta u_k\|_{L^2(\Omega)} \left\| \int_{\partial\Omega} |z-y|^{-2} f_k(z)\,d\sigma_z \right\|_{L^2(\Omega)}.$$

According to Corollary 6.16 in [Bennett and Sharpley 1988], we derive that

$$\left\| \int_{\partial\Omega} |z-y|^{-2} f_k(z)\,d\sigma_z \right\|_{L^2(\Omega)} \lesssim \int_{\partial\Omega} f_k(z) \log^{\frac{1}{2}}(1 + f_k(z))\,d\sigma_z.$$

Since $f_k = u_k \exp(\alpha_k u_k^2)/\lambda_k$, it is easy to check that $\int_{\partial\Omega} f_k(z)\log^{\frac{1}{2}}(1+f_k(z))\,d\sigma_z \lesssim 1$.Combining the estimates of I_1 and I_2, we find that

$$\left\| \left(\int_{\Omega} |\Delta u_k(y)||x-y|^{-2}\,dy \right) \left(\int_{\partial\Omega} |x-z|^{-2} f_k(z)\,d\sigma_z \right) \right\|_{L^{2,\infty}} \lesssim 1,$$

which accomplishes the proof of Lemma 3.5. □

Lemma 3.6. *We have* $\psi_k(x) \to \psi(x',t)$ *in* $C^3_{\mathrm{loc}}(\overline{B_R^+(0)})$ $(x' \in \partial\mathbb{R}^4_+,\ t \in \mathbb{R}^+)$, *where* $\psi(x',t)$ *satisfies the equations*

$$\begin{cases} \Delta^2 \psi = 0, & x \in \mathbb{R}^4_+, \\ \dfrac{\partial \Delta \psi}{\partial t} = \exp(24\pi^2 \psi), & x \in \partial\mathbb{R}^4_+, \\ \psi(0) = \sup \psi = 0, & \\ \dfrac{\partial \psi}{\partial t} = 0, & x \in \partial\mathbb{R}^4_+. \end{cases}$$

Furthermore, ψ *must take the form*

$$\psi = -\frac{1}{8\pi^2}\log\Bigl(\bigl(1+\bigl(\tfrac{\pi}{2}\bigr)^{\frac{2}{3}} t\bigr)^2 + \bigl(\tfrac{\pi}{2}\bigr)^{\frac{4}{3}}|x'|^2\Bigr) + \frac{1}{2^{\frac{8}{3}}\pi^{\frac{4}{3}}} \frac{t}{\bigl(1+\bigl(\tfrac{\pi}{2}\bigr)^{\frac{2}{3}} t\bigr)^2 + \bigl(\tfrac{\pi}{2}\bigr)^{\frac{4}{3}}|x'|^2}.$$

Proof. By (2-3), we can easily obtain

$$\begin{cases} \Delta^2 \psi_k = c_k r_k^4 \gamma_k & \text{for all } x \in B_R(0) \cap \Omega_k, \\ \dfrac{\partial \psi_k}{\partial t} = 0 & \text{for all } x \in B_R(0) \cap \partial\Omega_k, \\ \dfrac{\partial \Delta \psi_k}{\partial t} = \dfrac{u_k \exp(\alpha_k \psi_k (1+u_k/c_k))}{c_k} & \text{for all } x \in B_R(0) \cap \partial\Omega_k, \end{cases} \tag{3-7}$$

for any $R > 0$. Let $-\Delta\psi_k = v_k$; then ψ_k and v_k respectively satisfy the equations

$$\begin{cases} -\Delta \psi_k = v_k & \text{for all } x \in B_R(0) \cap \Omega_k, \\ \dfrac{\partial \psi_k}{\partial t} = 0 & \text{for all } x \in B_R(0) \cap \partial\Omega_k, \end{cases} \tag{3-8}$$

and

$$\begin{cases} -\Delta v_k = c_k r_k^4 \gamma_k & \text{for all } x \in B_R(0) \cap \Omega_k, \\ \dfrac{\partial v_k}{\partial t} = \dfrac{u_k \exp(\alpha_k \psi_k (1+u_k/c_k))}{c_k} & \text{for all } x \in B_R(0) \cap \partial\Omega_k. \end{cases} \tag{3-9}$$

Noticing

$$\frac{\partial v_k}{\partial t} = \frac{u_k \exp(\alpha_k \psi_k (1+u_k/c_k))}{c_k} \in L^\infty(B_R(0) \cap \partial\Omega_k),$$

applying Lemma 3.5 and the standard elliptic regularity, we deduce that

$$\|v_k\|_{C^{1,\alpha}(\overline{B_{R/2}(0)\cap\Omega_k})} \lesssim 1.$$

Then there exists some $v \in C^{1,\alpha}(\overline{B_{R/2}(0)\cap\Omega_k})$ such that $v_k \to v$ in $C^{1,\beta}(\overline{B_{R/2}(0)\cap\Omega_k})$ for any $\beta < \alpha$. Let $\widetilde{\psi_k(x)}$ be the even extension of ψ_k with respect to the boundary $\partial B_R^+(0) \cap \partial\mathbb{R}^4_+$; then we have $-\Delta\widetilde{\psi_k} \in C^{1,\alpha}(B_R(0)\cap\Omega_k)$, $\widetilde{\psi_k(x)} \le \psi_k(0) = 0$. Using the Harnack inequality and elliptic regularity estimates, we get $\|\widetilde{\psi_k}\|_{C^{3,\alpha}(B_R(0)\cap\Omega_k)} \lesssim C$. Hence there exists $\psi \in C^{3,\beta}(\overline{B_R(0)\cap\Omega_k})$ such that $\psi_k \to \psi$ in $C^{3,\beta}(\overline{B_R(0)\cap\Omega_k})$ for any $\beta < \alpha$, where ψ satisfies the equation

$$\begin{cases} \Delta^2\psi = 0 & \text{in } \mathbb{R}^4_+, \\ \dfrac{\partial\Delta\psi}{\partial t} = \exp(24\pi^2\psi) & \text{on } \partial\mathbb{R}^4_+, \\ \psi(0) = \sup\psi = 0, & \\ \dfrac{\partial\psi}{\partial t} = 0, & \text{on } \partial\mathbb{R}^4_+. \end{cases}$$

From (3-1), it is not difficult to see that

$$\int_{B_R\cap\partial\mathbb{R}^4_+} \exp(24\pi^2\psi) \le -\!\!\!\!\!\!\int_{B_{Rr_k}\cap\partial\Omega} \frac{u_k^2\exp(\alpha_k u_k^2)}{\lambda_k} \le 1. \tag{3-10}$$

Next, we will prove that ψ must take the form

$$\psi = -\frac{1}{8\pi^2}\log\Bigl(\bigl(1+\bigl(\tfrac{\pi}{2}\bigr)^{\frac{2}{3}}t\bigr)^2 + \bigl(\tfrac{\pi}{2}\bigr)^{\frac{4}{3}}|x'|^2\Bigr) + \frac{1}{2^{\frac{8}{3}}\pi^{\frac{4}{3}}}\frac{t}{\bigl(1+\bigl(\frac{\pi}{2}\bigr)^{\frac{2}{3}}t\bigr)^2 + \bigl(\frac{\pi}{2}\bigr)^{\frac{4}{3}}|x'|^2}.$$

Indeed, let $\phi(x) = \int_{\partial\mathbb{R}^+_4} P(x,y')\psi(y',0)\,dy'$, where $x = (x',t),\ y = (y',t)$ and

$$P(x,y') = \frac{4}{\pi^2}\frac{t^3}{|x-y'|^6}$$

is the Poisson kernel for the bi-Laplace operator on the upper half-space. It is not difficult to check that ϕ satisfies the equations

$$\begin{cases} (-\Delta)^2\phi = 0, & x \in \mathbb{R}^4_+, \\ \phi = \psi(x), & x \in \partial\mathbb{R}^4_+, \\ \dfrac{\partial\phi}{\partial t} = 0, & x \in \partial\mathbb{R}^4_+, \end{cases} \tag{3-11}$$

and $\int_{B^+_R(0)} |\Delta\phi|\,dx \le CR^2$.

Let $w = \psi - \phi$. Then w satisfies

$$\begin{cases} (-\Delta)^2 w = 0, & x \in \mathbb{R}^4_+, \\ w = 0, & x \in \partial\mathbb{R}^4_+, \\ \dfrac{\partial w}{\partial t} = 0, & x \in \partial\mathbb{R}^4_+. \end{cases} \tag{3-12}$$

Noticing $\int_{B^+_R(0)} |\Delta w|\,dx \le \int_{B^+_R(0)} |\Delta\psi|\,dx + \int_{B^+_R(0)} |\Delta\phi|\,dx \le CR^2$, one can deduce that w must be equal to zero. Hence $\psi(x) = \int_{\partial\mathbb{R}^+_4} P(x,y')\psi(y',0)\,d\xi$. Set $\psi_0(x') = \psi(x',0)$. Then we know that $\frac{1}{2}\partial\Delta\psi/\partial t|_{\partial\mathbb{R}^4_+} = (-\Delta)^{\frac{3}{2}}\psi_0$ and $\psi_0(x')$ satisfies the following equation in the distributional sense:

$$\begin{cases} (-\Delta)^{\frac{3}{2}}\psi_0 = \frac{1}{2}e^{24\pi^2\psi_0}, & x' \in \partial\mathbb{R}^3, \\ \int_{\mathbb{R}^3} e^{24\pi^2\psi_0(x')}\,dx' \le \frac{1}{2}. \end{cases} \tag{3-13}$$

Let $\eta_0(x') = 8\pi^2\psi_0(x') + \frac{1}{3}\log(2\pi^2)$. Then η_0 satisfies

$$\begin{cases} (-\Delta)^{\frac{3}{2}}\eta_0 = 2e^{3\eta_0}, & x' \in \mathbb{R}^3, \\ \int_{\mathbb{R}^3} e^{3\eta_0}\,dx' \le \pi^2. \end{cases} \tag{3-14}$$

From the result of Hyder [2019], we known that $\eta_0(x')$ can be decomposed as $\eta_0 = v + p$, where p is a polynomial of degree at most 2 and $v(x') = -\alpha \log|x'| + o(\log|x'|)$ as $|x'| \to +\infty$. Furthermore, $\eta_0(x') = \log(2\lambda/(1+\lambda^2|x'-x_0'|^2))$ if and only if p is a constant. Noticing that $\psi(x', t)$ is a Poisson extension of ψ_0 on $\mathbb{R}^4_+$ and $\int_{B_R^+} |\Delta\psi|\,dx \le CR^2$, we deduce that p must be equal to constant. This proves

$$\eta_0(x') = \log \frac{2\lambda}{1+\lambda^2|x'-x_0'|^2}.$$

Since $\psi(x) \le \psi(0) = \sup_{x\in\mathbb{R}^4_+} \psi(x) = 0$, it follows that ψ has the form

$$\psi(x', t) = -\frac{1}{8\pi^2} \log\Big(\big(1+\big(\tfrac{\pi}{2}\big)^{\frac{2}{3}} t\big)^2 + \big(\tfrac{\pi}{2}\big)^{\frac{4}{3}} |x'|^2\Big) + \frac{1}{2^{\frac{8}{3}}\pi^{\frac{4}{3}}} \frac{t}{\big(1+\big(\frac{\pi}{2}\big)^{\frac{2}{3}} t\big)^2 + \big(\frac{\pi}{2}\big)^{\frac{4}{3}} |x'|^2},$$

where the second term ensures $\frac{\partial\psi}{\partial t}\big|_{\partial\mathbb{R}^4_+} = 0$. By an easy computation, one can see that

$$\int_{\partial\mathbb{R}^4_+} \exp(24\pi^2\psi(x'))\,dx' = 1. \qquad \square$$

3.1. ***Polyharmonic truncation functions.*** We first introduce some notation. If $x_0 \in \partial\Omega$, for small $\delta > 0$, let $M_{\delta,x_0} = B_\delta(x_0) \cap \bar{\Omega}$. We can choose a Fermi coordinate (see [Manasse and Misner 1963]) for M_{δ,x_0} by the map $\theta : M_\delta \to \overline{B^3_\delta(0)} \times [0, \delta]$, where $\theta(x_0) = 0$. We will identify M_{δ,x_0} with $\overline{B^3_\delta(0)} \times [0, \delta]$ through the map θ. Under the Fermi coordinate, we can write the metric on the $M_{\delta,p}$ as

$$g = g_{ij}\,dx_i \otimes dx_j + dt \otimes dt \quad (i, j \in \{1, 2, 3\}), \tag{3-15}$$

where $(1-\varepsilon)\delta_{i,j} \le g_{ij} \le (1+\varepsilon)\delta_{i,j}$ for small $\varepsilon > 0$.

We choose a Fermi coordinate system (U_k, θ_k) near the point x_k such that $\theta_k(x_k) = 0$, and $\theta_k(U_k \cap \Omega) \subseteq \mathbb{R}^4_+ = \{x = (x', t) \in \mathbb{R}^4 : t > 0\}$, and $\theta_k(U_k \cap \partial\Omega) \subseteq \partial\mathbb{R}^4_+$. In the following, we make an even extension for $u_k \circ \theta_k^{-1}$ in the direction of t under the Fermi coordinate system (U_k, θ_k):

$$\begin{cases} \tilde{u}_k(x) = u_k \circ \theta_k^{-1}(x', t) & \text{if } t \ge 0, \\ \tilde{u}_k(x) = u_k \circ \theta_k^{-1}(x', -t) & \text{if } t < 0. \end{cases}$$

Then $\tilde{u}_k(x) \in W^{2,2}(B_r(0))$ with $\|\Delta\tilde{u}_k\|_{L^2(B_r(0))} = 2\|\Delta\tilde{u}_k\|_{L^2(B_r^+(0))}$ for small $r > 0$.

Now, we need some biharmonic truncation functions $\tilde{u}_k^M$ which was studied in [DelaTorre and Mancini 2021]. Roughly speaking, the value of the truncations functions $\tilde{u}_k^M$ are close to c_k/M in a small neighborhood of 0, and coincide with $\tilde{u}_k$ outside the same neighborhood.

Lemma 3.7 [DelaTorre and Mancini 2021, Lemma 4.20]. *For any $M > 1$ and $k \in \mathbb{N}$, there exists a radius $\tilde{\rho}_k^M > 0$ and a constant $c = c(M)$ such that*

(1) $\tilde{u}_k \ge \frac{c_k}{M}$ *in* $B_{\tilde{\rho}_k^M}(0)$;

(2) $\left|\tilde{u}_k - \frac{c_k}{M}\right| \le \frac{c}{c_k}$ *on* $\partial B_{\tilde{\rho}_k^M}(0)$;

(3) $|\nabla^l \tilde{u}_k| \le \frac{c}{c_k(\tilde{\rho}_k^M)^l}$ *on* $\partial B_{\tilde{\rho}_k^M}(0)$ *for any* $1 \le l \le 3$;

(4) $\tilde{\rho}_k^M \to 0$, *and* $\frac{\tilde{\rho}_k^M}{r_k} \to +\infty$, *as* $k \to \infty$.

Let $\tilde{v}_k^M \in C^4(\overline{B_{\tilde{\rho}_k^M}(0)})$ be the unique solution of

$$\begin{cases} \Delta^2(\tilde{v}_k^M) = 0 & \text{in } B_{\tilde{\rho}_k^M}(0), \\ \partial_\nu^i(\tilde{v}_k^M) = \partial_\nu^i(\tilde{u}_k) & \text{on } \partial B_{\tilde{\rho}_k^M}(0),\ i = 0, 1. \end{cases} \tag{3-16}$$

We consider the function

$$u_k^M = \begin{cases} \tilde{v}_k^M \circ \theta_k & \text{in } \theta_k^{-1}(B_{\tilde{\rho}_k^M}(0)), \\ u_k & \text{in } \Omega \setminus \theta_k^{-1}(B_{\tilde{\rho}_k^M}(0)). \end{cases} \tag{3-17}$$

Lemma 3.8 [DelaTorre and Mancini 2021, Lemma 4.21]. *For any $M > 1$, we have*

$$u_k^M = \frac{c_k}{M} + O(c_k^{-1}),$$

uniformly on $\theta_k^{-1}(\overline{B_{\tilde{\rho}_k^M}(0)})$.

Remark 3.9. Using the explicit form of the Green function of Δ^2 on balls, namely Boggio's formula [1905], and the representation formula of solutions for (3-16), one can see that $\partial u_k^M/\partial\nu = 0$ for any $x \in \theta_k^{-1}(B_{\tilde{\rho}_k^M}(0)) \cap \partial\Omega$.

Lemma 3.10. *For any $M > 1$,*

$$\limsup_{k\to\infty} \int_\Omega |\Delta u_k^M|^2\, dx \le \frac{1}{M}.$$

Proof. Testing (3-2) with $(u_k - u_k^M)$, by Lemmas 3.7, 2.6 and Remark 3.9, for any $R > 0$, we have

$$\begin{aligned}
&\int_{\theta_k^{-1}(B_{\tilde{\rho}_k^M}(0))\cap\Omega} \Delta u_k \Delta(u_k - u_k^M)\, dx \\
&= \int_{\theta_k^{-1}(B_{\tilde{\rho}_k^M}(0))\cap\Omega} \gamma_k(u_k - u_k^M)\, dx - \int_{\partial(\theta_k^{-1}(B_{\tilde{\rho}_k^M}(0))\cap\Omega)} (u_k - u_k^M)\frac{\partial}{\partial\nu}\Delta u_k\, d\sigma \\
&\qquad\qquad + \int_{\partial(\theta_k^{-1}(B_{\tilde{\rho}_k^M}(0))\cap\Omega)} \frac{\partial}{\partial\nu}(u_k - u_k^M)\Delta u_k\, d\sigma \\
&= \int_{\theta_k^{-1}(B_{\tilde{\rho}_k^M}(0))\cap\Omega} \gamma_k(u_k - u_k^M)\, dx - \int_{\theta_k^{-1}(B_{\tilde{\rho}_k^M}(0))\cap\partial\Omega} (u_k - u_k^M)\frac{u_k \exp(\alpha_k u_k^2)}{\lambda_k}\, d\sigma \\
&\ge -\int_{\theta_k^{-1}(B_{\tilde{\rho}_k^M}(0))\cap\partial\Omega} \lambda_k^{-1} u_k \exp\{\alpha_k u_k^2\}(u_k - u_k^M)\, d\sigma + o_k(1) \\
&\ge -\int_{B_{Rr_k(x_k)}\cap\partial\Omega} \lambda_k^{-1} c_k \exp\{\alpha_k u_k^2\}\Big(c_k - \frac{c_k}{M}\Big)\, d\sigma + o_k(1) \\
&= \int_{B_R^+(0)\cap\partial\mathbb{R}_+^4} \Big(1 - \frac{1}{M}\Big) \exp\Big\{\frac{u_k(x_k + r_k x) + c_k}{c_k}\alpha_k \psi_k(x)\Big\}\, d\sigma + o_k(1) \\
&\ge \Big(1 - \frac{1}{M}\Big) \int_{B_R^+(0)\cap\partial\mathbb{R}_+^4} \exp\{24\pi^2 \psi(x)\}\, d\sigma + o_k(1).
\end{aligned}$$

Letting $R \to \infty$, we get

$$\int_{\theta_k^{-1}(B_{\tilde{\rho}_k^M}(0))\cap\Omega} \Delta u_k \Delta(u_k - u_k^M)\, dx \geq 1 - \frac{1}{M} + o_k(1). \tag{3-18}$$

Observing that

$$\begin{aligned}\int_\Omega |\Delta u_k^M|^2\, dx &= \int_{\theta_k^{-1}(B_{\tilde{\rho}_k^M}(0))\cap\Omega} |\Delta v_k^M|^2\, dx + \int_{\Omega\setminus\theta_k^{-1}(B_{\tilde{\rho}_k^M}(0))} |\Delta u_k|^2\, dx \\ &= \int_{\theta_k^{-1}(B_{\tilde{\rho}_k^M}(0))\cap\Omega} |\Delta v_k^M|^2\, dx + 1 - \int_{\theta_k^{-1}(B_{\tilde{\rho}_k^M}(0))\cap\Omega} |\Delta u_k|^2\, dx \\ &= \int_{\theta_k^{-1}(B_{\tilde{\rho}_k^M}(0))\cap\Omega} |\Delta v_k^M|^2\, dx + 1 - \int_{\theta_k^{-1}(B_{\tilde{\rho}_k^M}(0))\cap\Omega} \Delta u_k \Delta(u_k - u_k^M)\, dx \\ &\qquad - \int_{\theta_k^{-1}(B_{\tilde{\rho}_k^M}(0))\cap\Omega} \Delta u_k \Delta u_k^M\, dx,\end{aligned}$$

by (3-18) and (3-17), we have

$$\begin{aligned}\int_\Omega |\Delta u_k^M|^2\, dx &\leq \frac{1}{M} + \frac{1}{2}\int_{\theta_k^{-1}(B_{\tilde{\rho}_k^M}(0))\cap\Omega} |\Delta v_k^M|^2\, dx - \frac{1}{2}\int_{\theta_k^{-1}(B_{\tilde{\rho}_k^M}(0))\cap\Omega} \Delta u_k \Delta v_k^M\, dx + o_k(1) \\ &\leq \frac{1}{M} + \frac{1}{2}\int_{\theta_k^{-1}(B_{\tilde{\rho}_k^M}(0))\cap\Omega} \Delta v_k^M \Delta(v_k^M - u_k)\, dx + o_k(1) \\ &= \frac{1}{M} + o_k(1).\end{aligned}$$

□

Lemma 3.11. *We have*

$$\lim_{k\to\infty}\int_{\partial\Omega} \exp(\alpha_k u_k^2)\, d\sigma = \lim_{L\to\infty}\lim_{k\to\infty}\int_{B_{Lr_k(x_k)}\cap\partial\Omega} \exp(\alpha_k u_k^2)\, d\sigma = \lim_{k\to\infty}\frac{-\lambda_k}{c_k^2} + |\partial\Omega|,$$

and consequently,

$$\frac{-\lambda_k}{c_k} \to \infty \quad \textit{and} \quad \sup_k \frac{-c_k^2}{\lambda_k} < \infty.$$

Proof. By Lemmas 3.7 and 3.1, we have

$$\begin{aligned}\int_{\partial\Omega} \exp(\alpha_k u_k^2)\, d\sigma &= \int_{\partial\Omega\cap\theta_k^{-1}(B_{\tilde{\rho}_k^M}(0))} \exp(\alpha_k u_k^2)\, d\sigma + \int_{\partial\Omega\setminus\theta_k^{-1}(B_{\tilde{\rho}_k^M}(0))} \exp(\alpha_k (u_k^M)^2)\, d\sigma \\ &\leq \frac{-M^2\lambda_k(1+o_k(1))}{c_k^2}\int_{\theta_k^{-1}(B_{\tilde{\rho}_k^M}(0))\cap\partial\Omega} \frac{u_k^2}{-\lambda_k}\exp(\alpha_k u_k^2)\, d\sigma + |\partial\Omega| \\ &\leq -(1+o_k(1))M^2\frac{\lambda_k}{c_k^2} + |\partial\Omega|,\end{aligned}$$

Let $k \to +\infty$ and $M \to 1$; we find that

$$\lim_{k\to+\infty}\int_{\partial\Omega} \exp(\alpha_k u_k^2)\, d\sigma \leq -\lim_{k\to\infty}\frac{\lambda_k}{c_k^2} + |\partial\Omega|.$$

On the other hand, we also have

$$\int_{\partial\Omega} \exp(\alpha_k u_k^2)\, d\sigma = \left(\int_{\partial\Omega\setminus B_{Rr_k}(x_k)} + \int_{B_{Rr_k}(x_k)\cap\partial\Omega}\right) \exp(\alpha_k u_k^2)\, d\sigma$$
$$\geq |\partial\Omega| - |B_{Rr_k}\cap\partial\Omega| - \frac{\lambda_k}{c_k^2}\int_{B_R(0)\cap\partial\mathbb{R}^4_+} \exp(\psi_k + o_k(1))\, d\sigma.$$

Letting $k\to+\infty$ and $R\to+\infty$, we get that

$$\lim_{k\to+\infty}\int_{\partial\Omega} \exp(\alpha_k u_k^2)\, d\sigma \geq -\lim_{k\to\infty}\frac{\lambda_k}{c_k^2} + |\partial\Omega|.$$

Combining the above estimates, we accomplish the proof of Lemma 3.11. □

Lemma 3.12. *For any $\varphi\in C^\infty(\partial\Omega)$, one has*

$$-\lim_{k\to\infty}\int_{\partial\Omega} \varphi(x)\frac{c_k u_k}{\lambda_k}\exp(\alpha_k u_k^2)\, d\sigma = \varphi(p). \tag{3-19}$$

Proof. For any fixed $M>1$, and k large enough, we divide $\partial\Omega$ into three parts,

$$\Omega_1 = \big(\theta_k^{-1}(B_{\tilde\rho_k^M}(0))\setminus B_{Rr_k}(x_k)\big)\cap\partial\Omega, \quad \Omega_2 = \partial\Omega\setminus\theta_k^{-1}(B_{\tilde\rho_k^M}(0)), \quad \Omega_3 = B_{Rr_k}(x_k)\cap\partial\Omega,$$

and split the integral as

$$\int_{\partial\Omega}\varphi(x)\frac{c_k u_k}{-\lambda_k}\exp(\alpha_k u_k^2)\, d\sigma = \left(\int_{\Omega_1} + \int_{\Omega_2} + \int_{\Omega_2}\right)\varphi(x)\frac{c_k u_k}{-\lambda_k}\exp(\alpha_k u_k^2)\, d\sigma$$
$$= \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3. \tag{3-20}$$

For I_1, we have

$$\begin{aligned}|\mathrm{I}_1| &\leq M\sup_{\partial\Omega}|\varphi|\int_{\Omega_1}\frac{u_k^2}{-\lambda_k}\exp(\alpha_k u_k^2)\, d\sigma\\
&\leq M\sup_{\partial\Omega}|\varphi|(1+o_k(1))\left(1 - \int_{B_{Rr_k}\cap\partial\Omega}\frac{u_k^2}{-\lambda_k}\exp(\alpha_k u_k^2)\, d\sigma\right)\\
&\leq M\sup_{\partial\Omega}|\varphi|\left(1 - \int_{B_R^+\cap\partial\mathbb{R}^4_+}\exp(24\pi^2\psi)\, d\sigma + o_k(1)\right)\\
&\to 0 \quad \text{as } k, R\to\infty.\end{aligned} \tag{3-21}$$

Next, by Lemma 3.10, Hölder's inequality, Sobolev embedding theorem and Lemma 3.11, we have

$$\begin{aligned}|\mathrm{I}_2| \leq \sup_{\partial\Omega}|\varphi|\frac{c_k}{-\lambda_k}\int_{\partial\Omega}|u_k|e^{\alpha_k(u_k^M)^2}\, d\sigma &\leq \sup_{\partial\Omega}|\varphi|\frac{c_k}{-\lambda_k}\|u_k\|_{L^{p'}(\partial\Omega)}\|e^{\alpha_k(u_k^M)^2}\|_{L^p(\partial\Omega)}\\
&\leq c\sup_{\partial\Omega}|\varphi|\left|\frac{c_k}{\lambda_k}\right| \to 0 \quad \text{as } k\to\infty,\end{aligned} \tag{3-22}$$

for some $p>1$ and p' with $\frac{1}{p}+\frac{1}{p'}=1$.

Finally, we have

$$\begin{aligned} \mathrm{I}_3 &= \int_{B_{Rr_k}\cap\partial\Omega} \varphi(x)\frac{c_k u_k}{-\lambda_k}\exp(\alpha_k u_k^2)\,d\sigma \\ &= \int_{B_R^+\cap\partial\mathbb{R}^4_+} \varphi(r_k x + x_k)\exp\{(\phi_k+1)\alpha_k\psi_k(x)\}\,dx + o_k(1) \\ &= \varphi(p)\int_{B_R^+\cap\partial\mathbb{R}^4_+} \exp\{24\pi^2\psi(x)\}\,d\sigma + o_k(1) \\ &= \varphi(p) + o_{k,R}(1). \end{aligned} \tag{3-23}$$

Combining (3-21), (3-22) and (3-23), we obtain (3-19) and the proof is finished. □

Lemma 3.13. *For any $1 < q < 2$, $c_k u_k \rightharpoonup G$ weakly in $W^{2,q}(\Omega)$. Furthermore, for any $\Omega' \Subset \overline{\Omega}\setminus p$, we have $c_k u_k \to G$ in $C^\infty(\overline{\Omega}')$, where G satisfies*

$$\begin{cases} \Delta^2 G = \delta_p - \frac{1}{|\Omega|} & \text{in } \Omega, \\ \int_\Omega G = 0,\ \dfrac{\partial G}{\partial\nu} = 0,\ \dfrac{\partial\Delta G}{\partial\nu}|_{\partial\Omega\setminus\{p\}} = 0. \end{cases} \tag{3-24}$$

Moreover, we have

$$G = -\frac{1}{4\pi^2}\ln|x-p| + A_p + \varphi(x), \tag{3-25}$$

where A_p is some constant depending on p, $\varphi(x) \in C^3(\Omega)\cap C^1(\overline{\Omega})$ and $\varphi(p) = 0$.

Proof. From (2-3), we have

$$\begin{cases} \Delta^2(c_k u_k) = c_k\gamma_k & \text{for all } x\in\Omega, \\ \dfrac{\partial}{\partial\nu}\Delta(c_k u_k) = c_k u_k\exp(\alpha_k u_k^2)/\lambda_k & \text{for all } x\in\partial\Omega. \end{cases} \tag{3-26}$$

Integrating both sides on Ω, one has

$$\int_\Omega c_k\gamma_k\,dx = \int_\Omega \Delta^2(c_k u_k)\,dx = \int_{\partial\Omega}\frac{\partial\Delta(c_k u_k)}{\partial\nu}\,d\sigma = \int_{\partial\Omega}\frac{c_k u_k\exp(\alpha_k u_k^2)}{\lambda_k}\,d\sigma, \tag{3-27}$$

which together with Lemma 3.12 gives $c_k\gamma_k \to -\dfrac{1}{|\Omega|}$ as $k\to\infty$. For any $q\in(1,2)$, we have

$$\int_\Omega |\Delta c_k u_k|^q\,dx = \sup\left\{\int_\Omega \Delta(c_k u_k)\Delta\varphi\,dx : \|\varphi\|_{W^{2,q'}} = 1\right\},$$

where $\frac{1}{q}+\frac{1}{q'} = 1$. By the Sobolev embedding theorem, we have $\sup_{x\in\Omega}|\varphi(x)| < \infty$. Using Lemma 3.12, we have

$$\begin{aligned} \int_\Omega \Delta(c_k u_k)\Delta\varphi\,dx &= \int_\Omega \Delta^2(c_k u_k)\varphi\,dx - \int_{\partial\Omega}\frac{\partial\Delta(c_k u_k)}{\partial\nu}\varphi\,d\sigma \\ &= \int_\Omega c_k\gamma_k\varphi(x)\,dx - \int_{\partial\Omega}\frac{c_k u_k\varphi\exp(\alpha_k u_k^2)}{\lambda_k}\,d\sigma \\ &= -\frac{1}{|\Omega|}\int_\Omega\varphi(x)\,dx + \varphi(p) + o_k(1) \\ &\le c\sup_{x\in\Omega}|\varphi(x)| < c, \end{aligned} \tag{3-28}$$

which implies that

$$\int_\Omega |\Delta c_k u_k|^q\,dx < c.$$

Combining this and the condition $\int_\Omega c_k u_k\,dx = 0$, $\int_\Omega u_k\,dx = 0$, we derive that $c_k u_k$ is bounded in $W^{2,q}(\Omega)$ for any $1 \le q < 2$. Thus, there exists some $G \in W^{2,q}(\Omega)$ such that $c_k u_k \rightharpoonup G$ in $W^{2,q}(\Omega)$ as $k \to \infty$. Now, letting $k \to \infty$ in (3-28), we have

$$\int_\Omega \Delta G \Delta\varphi\,dx = -\frac{1}{|\Omega|}\int_\Omega \varphi(x)\,dx + \varphi(p).$$

Combining the assumptions on u_k, (3-24) is proved.

For any $\Omega' \Subset \overline{\Omega} \setminus p$, we can choose some function $\phi \in C^\infty(\mathbb{R}^4)$ such that $\phi(x) = 1$ for $x \in \Omega'$ and $\phi(x) = 0$ for x belonging to a small neighborhood of p. By Lemma 3.1, we know that $\phi u_k \to 0$ in $L^2(\Omega')$ as $k \to +\infty$. This together with the convergence $\Delta u_k \to 0$ in $L^2(\Omega')$ as $k \to \infty$ implies that $e^{\alpha_k u_k^2}$ is uniformly bounded in $L^s(\overline{\Omega}')$ for any $s > 1$. Standard elliptic regularity gives that $c_k u_k \to G$ in $C^k(\overline{\Omega}')$ for any positive integer k.

Next, we prove (3-25). Fix $r > 0$, without loss of generality, we assume $p = 0$, and choose some cutoff function $\phi \in C_0^\infty(B_{2r}(0))$ such that $\phi = 1$ in $B_r(0)$. Let

$$g(x) = G(x) + \frac{1}{4\pi^2}\phi(x)\ln|x|.$$

Then we have

$$\Delta^2 g(x) = f \quad \text{in } \Omega,$$

where

$$f(x) = \frac{1}{4\pi^2}\big(\Delta^2\phi\cdot\ln|x| + 2\nabla\Delta\phi\cdot\nabla\ln|x| + 2\Delta(\nabla\phi\cdot\nabla\ln|x|) + 2\nabla\phi\cdot\nabla\Delta\ln|x| + \phi\cdot\Delta^2\ln|x|\big) + \delta(x) - \frac{1}{|\Omega|}.$$

Since $1/(4\pi^2)\phi\cdot\Delta^2\ln|x| = \delta(x)$ in $\mathbb{R}^4_+$, a careful computation yields

$$f(x) = \frac{1}{4\pi^2}\big(\Delta^2\phi\cdot\ln|x| + 2\nabla\Delta\phi\cdot\nabla\ln|x| + 2\Delta(\nabla\phi\cdot\nabla\ln|x|) + 2\nabla\phi\cdot\nabla\Delta\ln|x|\big) - \frac{1}{|\Omega|}.$$

Observing $G \in W^{2,s}(\Omega)$ for any $1 < s < 2$, we obtain $f(x) \in L^p(\Omega)$ for any $p > 2$. By the standard regularity theory, we get $g(x) \in C^3_{\mathrm{loc}}(\Omega) \cap C^1(\overline{\Omega})$. Let $A_p = g(0)$ and

$$\varphi(x) = g(x) - g(0) + \frac{1}{4\pi^2}(1-\phi)\ln|x|.$$

Then we have

$$G = -\frac{1}{4\pi^2}\ln|x| + A_p + \varphi(x), \tag{3-29}$$

where A_p is some constant depending on p, $\varphi(x) \in C^3(\Omega) \cap C^1(\overline{\Omega})$ and $\varphi(0) = 0$, and the proof is finished. □

3.2. ***Neck analysis.*** In this subsection, we will use the capacity technique to derive the upper bound of $I_{12\pi^2}(u_k)$ when $c_k \to \infty$. The capacity technique applied to the existence of extremals for Adams inequalities was first used by Lu and Yang in [2009a], and was improved by DelaTorre and Mancini [2021] by comparing the Dirichlet energy of maximizing sequence with the energy of a suitable polyharmonic function.

Based on Lemma 3.11, we only need to give the sharp upper bound of $\lim_{k\to\infty} -\lambda_k/c_k^2$. Let us fix a large $R > 0$ and a small $\delta > 0$ and consider the annular region

$$A_k(R, \delta) := \{x \in \Omega : r_k R \le |x - x_k| \le \delta\}.$$

Our strategy is to compare the Dirichlet energy of u_k on $A_k(R, \delta)$ with the energy of the function

$$\mathcal{W}_k(x) := -\frac{1}{4\pi^2 c_k}(\log|x - x_k| + \rho_k(x)),$$

where $\rho_k(x) \in C^\infty(\overline{\Omega})$ is chosen such that

$$\frac{\partial \mathcal{W}_k(x)}{\partial \nu} = \frac{\partial \Delta \mathcal{W}_k(x)}{\partial \nu} = 0 \quad \text{for } x \in \partial\Omega$$

and $\|\rho_k(x)\|_{C^3} = O(\delta)$.

As a consequence of Lemma 3.6, on $\partial B_{Rr_k}(x_k) \cap \Omega$, we have that

$$u_k(x) = c_k + \frac{\psi((x - x_k)/r_k)}{c_k} + o(c_k^{-1}) = c_k - \frac{1}{4\pi^2 c_k}\log R - \frac{1}{6\pi^2 c_k}\log\frac{\pi}{2} + \frac{O(R^{-1})}{c_k} + o(c_k^{-1}),$$

provided k is large enough. Similarly, a direct computation also gives

$$\Delta^{\frac{j}{2}} u_k = \frac{(\Delta^{j/2}\psi)((x - x_k)/r_k)}{r_k^j c_k} + o(r_k^{-j} c_k^{-1}) = -\frac{K_{2,j/2}}{4\pi^2 r_k^j c_k R^j} e_j(x - x_k) + \frac{O(R^{-j-1})}{r_k^j c_k} + o(r_k^{-j} c_k^{-1})$$

for any $1 \le j \le 3$, where

$$K_{2,j/2} = \begin{cases} 1 & \text{if } j = 1, \\ 2 & \text{if } j = 2, \\ -4 & \text{if } j = 3, \end{cases} \quad \text{and} \quad e_j(x) := \begin{cases} 1 & \text{if } j \text{ is even}, \\ \dfrac{x}{|x|} & \text{if } j \text{ is odd}. \end{cases}$$

Recalling the definition of $\mathcal{W}_k$, we have on $\partial B_{Rr_k}(x_k) \cap \Omega$ that

$$\mathcal{W}_k = \frac{\alpha_k}{12\pi^2} c_k - \frac{1}{12\pi^2 c_k}\log\frac{-\lambda_k}{c_k^2} - \frac{1}{4\pi^2 c_k}\log R + \frac{O(\delta)}{c_k} \tag{3-30}$$

and

$$\Delta^{j/2}\mathcal{W}_k = -\frac{K_{2,j/2}}{4\pi^2 c_k r_k^j R^j} e_j(x - x_k) + \frac{O(\delta)}{c_k} \quad \text{for any } 1 \le j \le 3. \tag{3-31}$$

Hence, we conclude that on $\partial B_{Rr_k}(x_k) \cap \Omega$,

$$u_k(x) - \mathcal{W}_k = \frac{1}{12\pi^2 c_k}\log\frac{-\lambda_k}{c_k^2} - \frac{1}{6\pi^2 c_k}\log\frac{\pi}{2} + \frac{O(R^{-1})}{c_k} + \frac{O(\delta)}{c_k} + o(c_k^{-1}) + \left(1 - \frac{\alpha_k}{12\pi^2}\right)c_k$$

and

$$\Delta^{j/2}(u_k - \mathcal{W}_k) = \frac{O(R^{-j-1})}{r_k^j c_k} + o(r_k^{-j} c_k^{-1}) \quad \text{for any } 1 \le j \le 3.$$

Similarly, in view of Lemma 3.13, we also derive that on $\partial B_\delta(x_k) \cap \Omega$,

$$u_k(x) - \mathcal{W}_k = \frac{A_p}{c_k} + \frac{O(\delta)}{c_k} + o(c_k^{-1})$$

and

$$\Delta^{j/2}(u_k(x) - \mathcal{W}_k) = \frac{O(1)}{c_k} + o(c_k^{-1}) \quad \text{for any } 1 \le j \le 3,$$

where we have also used that $|x - x_k|/|x - p| \to 1$ uniformly on $\partial B_\delta(x_k)$.

Now, we compare $\|\Delta u_k\|_{L^2(A_k(R,\delta))}$ and $\|\Delta \mathcal{W}_k\|_{L^2(A_k(R,\delta))}$. Obviously,

$$\|\Delta u_k\|^2_{L^2(A_k(R,\delta))} - \|\Delta \mathcal{W}_k\|^2_{L^2(A_k(R,\delta))} \ge 2\int_{A_k(R,\delta)} \Delta(u_k - \mathcal{W}_k) \cdot \Delta \mathcal{W}_k \, dx. \tag{3-32}$$

Step 1. Estimates for the right-hand side of (3-32).

Integrating by parts, the integral in the right-hand side equals to

$$2\int_{A_k(R,\delta)} \Delta(u_k - \mathcal{W}_k) \cdot \Delta \mathcal{W}_k \, dx = -2\int_{\partial A_k(R,\delta)\setminus\partial\Omega} \nu \cdot ((u_k - \mathcal{W}_k)\Delta^{\frac{3}{2}} \mathcal{W}_k) \, d\sigma + 2\int_{\partial A_k(R,\delta)\setminus\partial\Omega} \nu \cdot (\Delta^{\frac{1}{2}}(u_k - \mathcal{W}_k)\Delta \mathcal{W}_k) \, d\sigma,$$

where we have used the fact that $\frac{\partial \mathcal{W}_k}{\partial \nu} = \frac{\partial \Delta \mathcal{W}_k}{\partial \nu} = 0$ on $\partial A_k(R,\delta) \cap \partial\Omega$.

On $\partial B_{Rr_k}(x_k) \cap \Omega$, we have

$$\begin{aligned}
(u_k - \mathcal{W}_k)\Delta^{\frac{3}{2}} \mathcal{W}_k \cdot \nu
&= \frac{1}{4\pi^2}\left(\frac{1}{12\pi^2 c_k^2}\log\frac{-\lambda_k}{c_k^2} - \frac{1}{6\pi^2 c_k^2}\log\frac{\pi}{2} + \frac{O(R^{-1})}{c_k^2} + \left(1 - \frac{\alpha_k}{12\pi^2}\right) + \frac{O(\delta)}{c_k^2} + o(c_k^{-2})\right)\frac{K_{2,\frac{3}{2}}}{(r_k R)^3} \\
&= -\frac{1}{\pi^2 (r_k R)^3}\left(\frac{1}{12\pi^2 c_k^2}\log\frac{-\lambda_k}{c_k^2} - \frac{1}{6\pi^2 c_k^2}\log\frac{\pi}{2} + \frac{O(R^{-1})}{c_k^2} + \left(1 - \frac{\alpha_k}{12\pi^2}\right) + \frac{O(\delta)}{c_k^2} + o(c_k^{-2})\right)
\end{aligned}$$

and

$$\Delta^{\frac{1}{2}}(u_k - \mathcal{W}_k)\Delta \mathcal{W}_k \cdot \nu = \left(\frac{O(R^{-1})}{c_k^2} + o(c_k^{-2})\right) O(r_k R)^{-3}.$$

Similarly, on $\partial B_\delta(x_k) \cap \Omega$, we have

$$(u_k - \mathcal{W}_k)\Delta^{\frac{3}{2}} \mathcal{W}_k \cdot \nu = \frac{1}{\pi^2 \delta^3}\left(\frac{A_p}{c_k^2} + \frac{O(\delta)}{c_k^2} + o(c_k^{-2})\right)$$

and

$$\Delta^{\frac{1}{2}}(u_k - \mathcal{W}_k)\Delta \mathcal{W}_k \cdot \nu = \left(\frac{O(1)}{c_k^2} + o(c_k^{-2})\right) O(\delta^{-2}).$$

Then we can obtain

$$\int_{A_k(R,\delta)} \Delta(u_k - \mathcal{W}_k)\cdot\Delta\mathcal{W}_k\,dx = \frac{1}{12\pi^2 c_k^2}\log\frac{-\lambda_k}{c_k^2} - \frac{1}{6\pi^2 c_k^2}\log\frac{\pi}{2} - \frac{A_p}{c_k^2} + \frac{O(R^{-1})}{c_k^2} + \frac{O(\delta)}{c_k^2} + \Big(1 - \frac{\alpha_k}{12\pi^2}\Big) + o(c_k^{-2}).$$

Combining the above estimates, we derive that

$$\|\Delta u_k\|^2_{L^2(A_k(R,\delta))} - \|\Delta\mathcal{W}_k\|^2_{L^2(A_k(R,\delta))} \ge \frac{1}{6\pi^2 c_k^2}\log\frac{-\lambda_k}{c_k^2} - \frac{1}{3\pi^2 c_k^2}\log\frac{\pi}{2} - \frac{2A_p}{c_k^2} + \frac{O(R^{-1})}{c_k^2} + \frac{O(\delta)}{c_k^2} + \Big(2 - \frac{\alpha_k}{6\pi^2}\Big) + o(c_k^{-2}). \tag{3-33}$$

Step 2. Estimates for $\|\Delta u_k\|^2_{L^2(A_k(R,\delta))}$.

We rewrite $\|\Delta u_k\|^2_{L^2(A_k(R,\delta))}$ as

$$\|\Delta u_k\|^2_{L^2(A_k(R,\delta))} = 1 - \int_{\Omega\setminus B_\delta(x_k)} |\Delta u_k|^2\,dx - \int_{\Omega\cap B_{Rr_k}(x_k)} |\Delta u_k|^2\,dx. \tag{3-34}$$

Since

$$\Delta^{\frac12}(\log|x|) = \frac{x}{|x|^2},\quad \Delta(\log|x|) = \frac{2}{|x|^2},\quad \Delta^{1+\frac12}(\log|x|) = -4\frac{x}{|x|^4},$$

we have

$$\begin{aligned}\nu\cdot G(\delta)\Delta^{\frac32}G(\delta) &= -\Big(-\frac{1}{4\pi^2}\ln|\delta| + A_p + o_\delta(1)\Big)\Big(\frac{1}{\pi^2}\cdot\frac{1}{\delta^3} + O(1)\Big)\\ &= -\frac{1}{\pi^2}\frac{1}{\delta^3}\Big(-\frac{1}{4\pi^2}\ln\delta + A_p + o_\delta(1)\Big)\end{aligned} \tag{3-35}$$

and

$$\nu\cdot\Delta^{\frac12}G(\delta)\Delta G(\delta) = -\Big(-\frac{1}{4\pi^2}\frac{1}{\delta} + O(1)\Big)\Big(-\frac{1}{4\pi^2}\frac{2}{\delta^2} + O(1)\Big) = -\frac{1}{8\pi^4}\frac{1}{\delta^3}(1 + o_\delta(1)). \tag{3-36}$$

Since

$$\int_{\Omega\setminus B_\delta(x_k)} |\Delta G|^2\,dx = \int_{\Omega\cap\partial B_\delta(x_k)} \nu(-G\Delta^{\frac32}G + \Delta^{\frac12}G\Delta G)\,d\sigma,$$

we have by Lemma 3.13,

$$\int_{\Omega\setminus B_\delta(x_k)} |\Delta u_k|^2\,dx = \frac{1}{c_k^2}\Big(-\frac{1}{4\pi^2}\log\delta - \frac{1}{8\pi^2} + A_p + o_\delta(1) + o_k(1)\Big). \tag{3-37}$$

By Lemma 3.6, we derive that

$$\begin{aligned}\int_{\Omega\cap B_{Rr_k}(x_k)} |\Delta u_k|^2\,dx &= \frac{1}{c_k^2}\int_{B_R^+} |\Delta\psi|^2\,dx + o\Big(\frac{1}{c_k^2}\Big) = \frac{1}{c_k^2}\Big(\int_{\partial B_R^+} \nu(\Delta^{\frac12}\psi\Delta\psi - \psi\Delta^{\frac32}\psi)\,d\sigma\Big) + o\Big(\frac{1}{c_k^2}\Big)\\ &:= \frac{1}{c_k^2}(\mathrm{I} - \mathrm{II}) + o\Big(\frac{1}{c_k^2}\Big).\end{aligned} \tag{3-38}$$

Observe that on $\partial B_R^+ \cap \mathbb{R}^4_+$, we also have

$$\psi(x) = -\frac{1}{6\pi^2}\log\frac{\pi}{2} - \frac{1}{4\pi^2}\log R + O\Big(\frac{1}{R}\Big),\quad \nu\Delta^{\frac12}\psi(x) = -\frac{1}{4\pi^2}\frac{1}{R} + O\Big(\frac{1}{R^2}\Big)$$

and

$$\begin{aligned}\nu\Delta^{\frac{3}{2}}\psi &= -\frac{1}{4\pi^2}\left(\frac{-4}{\left(\left(t+\left(\frac{2}{\pi}\right)^{\frac{2}{3}}\right)^2+|x'|^2\right)^2}\right)\frac{\left(x', t+\left(\frac{2}{\pi}\right)^{\frac{2}{3}}\right)\cdot(x',t)}{R}+O\left(\frac{1}{R^4}\right)\\
&= \frac{1}{\pi^2}\left(\frac{1}{\left(\left(t+\left(\frac{2}{\pi}\right)^{\frac{2}{3}}\right)^2+|x'|^2\right)^2}\right)\frac{\left(x', t+\left(\frac{2}{\pi}\right)^{\frac{2}{3}}\right)\cdot(x',t)}{R}+O\left(\frac{1}{R^4}\right)\\
&= \frac{1}{\pi^2}\left(\frac{1}{R^4}+O\left(\frac{1}{R^5}\right)\right)(R+O(1))+O\left(\frac{1}{R^4}\right)\\
&= \frac{1}{\pi^2}\frac{1}{R^3}+O\left(\frac{1}{R^4}\right).\end{aligned}$$

Hence we can write

$$\mathrm{II} = \int_{\partial B_R^+\cap\mathbb{R}^4_+}\nu\psi\,\Delta^{\frac{3}{2}}\psi\,d\sigma+\int_{\partial B_R^+\cap\partial\mathbb{R}^4_+}\nu\psi\,\Delta^{\frac{3}{2}}\psi\,d\sigma := \mathrm{II}_1+\mathrm{II}_2,$$

where

$$\begin{aligned}\mathrm{II}_1 &= \int_{\partial B_R^+\cap\mathbb{R}^4_+}\nu\psi\,\Delta^{\frac{3}{2}}\psi\,d\sigma\\
&= \pi^2R^3\left(-\frac{1}{6\pi^2}\log\frac{\pi}{2}-\frac{1}{4\pi^2}\log R+O\left(\frac{1}{R}\right)\right)\cdot\left(\frac{1}{\pi^2}\frac{1}{R^3}+O\left(\frac{1}{R^4}\right)\right)\\
&= -\frac{1}{4\pi^2}\log R-\frac{1}{6\pi^2}\log\frac{\pi}{2}+O\left(\frac{\log R}{R}\right).\end{aligned}$$

Since $\frac{\partial}{\partial t}\Delta\psi = \exp(24\pi^2\psi)$ for $x=(x',0)\in\partial\mathbb{R}^4_+$, set $\psi_0(x')=\psi(x',0)$. We have

$$\begin{aligned}\mathrm{II}_2 &= \int_{\partial B_R^+\cap\partial\mathbb{R}^4_+}\nu\psi\,\Delta^{\frac{3}{2}}\psi\,d\sigma\\
&= \int_{B_R^3}-\exp(24\pi^2\psi_0(x'))\psi_0(x')\,dx'\\
&= -\int_{\mathbb{R}^3}\frac{\left(\left(\frac{2}{\pi}\right)^{\frac{2}{3}}\right)^3}{\left(\frac{\pi}{2}\right)^2\left(|x'|^2+\left(\left(\frac{2}{\pi}\right)^{\frac{2}{3}}\right)^2\right)^3}\psi_0(x')\,dx'+O\left(\frac{1}{R}\right)\\
&= -\psi\left(0,\left(\frac{2}{\pi}\right)^{\frac{2}{3}}\right)+O\left(\frac{1}{R}\right)\\
&= \frac{1}{4\pi^2}\log 2-\frac{1}{16\pi^2}+O\left(\frac{1}{R}\right),\end{aligned}$$

where B_R^3 denotes the three-dimensional balls with radius R.

So, we have

$$\begin{aligned}\mathrm{II} &= -\frac{1}{4\pi^2}\log R-\frac{1}{6\pi^2}\log\frac{\pi}{2}+\frac{1}{4\pi^2}\log 2-\frac{1}{16\pi^2}+O\left(\frac{\log R}{R}\right)\\
&= -\frac{1}{4\pi^2}\log\frac{R}{2}-\frac{1}{6\pi^2}\log\frac{\pi}{2}-m\frac{1}{16\pi^2}+O\left(\frac{\log R}{R}\right).\end{aligned}\tag{3-39}$$

Now, we estimate I, and rewrite it as

$$\mathrm{I}=\int_{\partial B_R^+}\nu\Delta^{\frac12}\psi\,\Delta\psi\,d\sigma=\int_{\partial B_R^+\cap\mathbb{R}_+^4}\nu\Delta^{\frac12}\psi\,\Delta\psi\,d\sigma+\int_{\partial B_R^+\cap\partial\mathbb{R}_+^4}\nu\Delta^{\frac12}\psi\,\Delta\psi\,d\sigma:=\mathrm{I}_1+\mathrm{I}_2.$$

Since on $\partial B_R^+\cap\mathbb{R}_+^4$, we have

$$\nu\Delta^{\frac12}\psi(x)=-\frac{1}{4\pi^2}\frac{1}{R}+O\Big(\frac{1}{R^2}\Big)$$

and $\Delta\psi=-\frac{1}{2\pi^2}\frac{1}{R^2}$, we therefore get

$$\mathrm{I}_1=\int_{\partial B_R^+\cap\mathbb{R}_+^4}\nu\Delta^{\frac12}\psi\,\Delta\psi\,d\sigma=\pi^2R^3\Big(-\frac{1}{4\pi^2}\frac{1}{R}+O\Big(\frac{1}{R^2}\Big)\Big)\Big(-\frac{1}{4\pi^2}\Big(\frac{2}{R^2}\Big)\Big)=\frac{1}{8\pi^2}+O\Big(\frac{1}{R}\Big).$$

Using the fact that $\dfrac{\partial\psi}{\partial t}=0$ on $\partial\mathbb{R}_+^4$, we obtain

$$\mathrm{I}_2=\int_{\partial B_R^+\cap\partial\mathbb{R}_+^4}\nu\Delta^{\frac12}\psi\,\Delta\psi\,d\sigma=0.$$

Hence we have

$$\mathrm{I}=\frac{1}{8\pi^2}+O\Big(\frac{1}{R}\Big). \tag{3-40}$$

By (3-38), (3-39) and (3-40), we get

$$\begin{aligned}\int_{\Omega\cap B_{Rr_k(x_k)}}|\Delta u_k|^2\,dx&=\frac{1}{c_k^2}\bigg(\frac{1}{8\pi^2}+O\Big(\frac{1}{R}\Big)-\Big(-\frac{1}{4\pi^2}\log\frac{R}{2}-\frac{1}{6\pi^2}\log\frac{\pi}{2}-\frac{1}{16\pi^2}+O\Big(\frac{\log R}{R}\Big)\Big)\bigg)+o\Big(\frac{1}{c_k^2}\Big)\\&=\frac{1}{c_k^2}\Big(\frac{3}{16\pi^2}+\frac{1}{4\pi^2}\log\frac{R}{2}+\frac{1}{6\pi^2}\log\frac{\pi}{2}\Big)+\frac{1}{c_k^2}O\Big(\frac{\log R}{R}\Big).\end{aligned}\tag{3-41}$$

Combining (3-37) and (3-34), we derive that

$$\begin{aligned}\|\Delta u_k\|^2_{A_k(R,\delta)}&=1-\frac{1}{c_k^2}\Big(-\frac{1}{4\pi^2}\log\delta-\frac{1}{8\pi^2}+A_p+o_{\delta,k}(1)\Big)\\&\qquad-\frac{1}{c_k^2}\Big(\frac{3}{16\pi^2}+\frac{1}{4\pi^2}\log\frac{R}{2}+\frac{1}{6\pi^2}\log\frac{\pi}{2}\Big)+\frac{1}{c_k^2}O\Big(\frac{\log R}{R}\Big)\\&=1-\frac{1}{c_k^2}\Big(\frac{1}{16\pi^2}+\frac{1}{4\pi^2}\log\frac{R}{2\delta}+\frac{1}{6\pi^2}\log\frac{\pi}{2}+A_p+O\Big(\frac{\log R}{R}\Big)+o_{\delta,k}(1)\Big).\end{aligned}\tag{3-42}$$

Step 3. Estimates for $\|\Delta\mathcal{W}_k\|^2_{L^2(A_k(R,\delta))}$.

Since

$$\begin{aligned}\int_{A_k(R,\delta)}|\Delta\mathcal{W}_k|^2\,dx&=-\int_{\partial A_k(R,\delta)}\nu(\mathcal{W}_k\Delta^{\frac32}\mathcal{W}_k-\Delta^{\frac12}\mathcal{W}_k\Delta\mathcal{W}_k)\,d\sigma\\&=-\int_{\Omega\cap\partial B_\delta(x_k)}\nu(\mathcal{W}_k\Delta^{\frac32}\mathcal{W}_k-\Delta^{\frac12}\mathcal{W}_k\Delta\mathcal{W}_k)\,d\sigma\\&\qquad+\int_{\Omega\cap\partial B_{Rr_k(x_k))}}\nu(\mathcal{W}_k\Delta^{\frac32}\mathcal{W}_k-\Delta^{\frac12}\mathcal{W}_k\Delta\mathcal{W}_k)\,d\sigma\\&:=-\mathrm{III}_1+\mathrm{III}_2.\end{aligned}\tag{3-43}$$

From (3-30) and (3-31), we have

$$\begin{aligned}\mathrm{III}_2 &= \Bigg(\bigg(\frac{\alpha_k}{12\pi^2}c_k - \frac{1}{12\pi^2 c_k}\log\frac{-\lambda_k}{c_k^2} - \frac{1}{4\pi^2 c_k}\log R\bigg) \\ &\qquad\qquad \cdot\bigg(-\frac{K_{2,\frac{3}{2}}}{4\pi^2 c_k R^3 r_k^3}\bigg) - \frac{K_{2,\frac{1}{2}}}{4\pi^2 c_k R r_k}\frac{K_{2,1}}{4\pi^2 c_k R^2 r_k^2} + O(\delta)\Bigg)\pi^2 R^3 r_k^3 \\ &= 1 - \frac{1}{c_k^2}\bigg(\frac{1}{12\pi^2}\log\frac{-\lambda_k}{c_k^2} + \frac{1}{4\pi^2}\log R + \frac{1}{8\pi^2} + O(\delta)\bigg). \qquad (3\text{-}44)\end{aligned}$$

Similarly, we can also obtain

$$\begin{aligned}\mathrm{III}_1 &= \int_{\Omega\cap\partial B_\delta} \nu(\mathcal{W}_k\Delta^{\frac{3}{2}}\mathcal{W}_k - \Delta^{\frac{1}{2}}\mathcal{W}_k\Delta\mathcal{W}_k)\,d\sigma \\ &= \frac{\nu}{c_k^2}\bigg(\Big(\frac{-1}{4\pi^2}\log\delta + O(\delta)\Big)\Big(\frac{-K_{2,\frac{3}{2}}}{4\pi^2\delta^3}e_3(x-x_k) + O(\delta)\Big) - \frac{-K_{2,\frac{1}{2}}e_1(x-x_k)}{4\pi^2\delta}\frac{-K_{2,\frac{2}{2}}}{4\pi^2\delta^2}\bigg)\pi^2\delta^3 + \frac{O(\delta)}{c_k^2} \\ &= \frac{1}{c_k^2}\Big(\frac{-1}{4\pi^2}\log\delta - \frac{1}{8\pi^2} + O(\delta)\Big).\end{aligned}$$

Combining (3-43) and (3-44), we derive that

$$\begin{aligned}\int_{A_k(R,\delta)} |\Delta\mathcal{W}_k|^2\,dx &= 1 - \frac{1}{c_k^2}\bigg(\frac{-1}{4\pi^2}\log\delta - \frac{1}{8\pi^2} + \frac{1}{12\pi^2}\log\frac{-\lambda_k}{c_k^2} + \frac{1}{4\pi^2}\log R + \frac{1}{8\pi^2} + O(\delta)\bigg) \\ &= 1 - \frac{1}{c_k^2}\bigg(\frac{-1}{4\pi^2}\log\frac{\delta}{R} + \frac{1}{12\pi^2}\log\frac{-\lambda_k}{c_k^2} + O(\delta)\bigg). \qquad (3\text{-}45)\end{aligned}$$

Now, we are in position to give the sharp upper bound for $\lim_{k\to\infty}\frac{-\lambda_k}{c_k^2}$. Indeed, from (3-42), (3-45) and (3-33), we can get

$$\begin{aligned}&\|\Delta u_k\|^2_{A_k(R,\delta)} - \int_{A_k(R,\delta)} |\Delta\mathcal{W}_k|^2\,dx \\ &\quad = 1 - \frac{1}{c_k^2}\Big(\frac{1}{16\pi^2} + \frac{1}{4\pi^2}\log\frac{R}{2\delta} + \frac{1}{6\pi^2}\log\frac{\pi}{2} + A_p\Big) - 1 + \frac{1}{c_k^2}\Big(\frac{-1}{4\pi^2}\log\frac{\delta}{R} + \frac{1}{12\pi^2}\log\frac{-\lambda_k}{c_k^2} + O(\delta)\Big) \\ &\quad = \frac{1}{c_k^2}\Big(\frac{-1}{4\pi^2}\log\frac{\delta}{R} + \frac{1}{12\pi^2}\log\frac{-\lambda_k}{c_k^2} - \frac{1}{16\pi^2} - \frac{1}{4\pi^2}\log\frac{R}{2\delta} - \frac{1}{6\pi^2}\log\frac{\pi}{2} - A_p + O(\delta)\Big) \\ &\quad = \frac{1}{c_k^2}\Big(\frac{-1}{4\pi^2}\log\frac{\delta}{R} + \frac{1}{12\pi^2}\log\frac{-\lambda_k}{c_k^2} - \frac{1}{16\pi^2} - \frac{1}{4\pi^2}\log\frac{R}{2\delta} - \frac{1}{6\pi^2}\log\frac{\pi}{2} - A_p + O(\delta)\Big) \\ &\quad \geq \frac{1}{c_k^2}\Big(\frac{1}{6\pi^2}\log\frac{-\lambda_k}{c_k^2} - 2A_p - \frac{1}{3\pi^2}\log\frac{\pi}{2} + o_{\delta,k}(1) + o(R^{-1})\Big) + 2 - \frac{\alpha_k}{6\pi^2},\end{aligned}$$

which implies

$$\lim_{k\to\infty}\frac{-\lambda_k}{c_k^2} \leq 2\pi^2\exp\big(-\tfrac{3}{4} + 12\pi^2 A_p\big).$$

Therefore, we can conclude with the following.

Proposition 3.14. *If* $c_k \to \infty$, *then*

$$\sup_{u\in W^{2,2}(\Omega),\ \|\Delta u\|_2\leqslant 1} \int_{\partial\Omega} e^{12\pi^2 u^2}\,dx \leqslant |\partial\Omega| + 2\pi^2 e^{12\pi^2 A_p - \frac{3}{4}}.$$

4. A test functions argument and the proof of Theorem 1.1

In this section, we assume $A_p = \max_{p\in\partial\Omega} A_p$ for some $p\in\partial\Omega$. Now we construct a blowing up sequence ϕ_ε with $\int_\Omega |\Delta\phi_\varepsilon|^2 = 1$, and

$$\int_{\partial\Omega} e^{12\pi^2(\phi_\varepsilon-\bar{\phi}_\varepsilon)^2}\,d\sigma > |\partial\Omega| + 2\pi^2 e^{12\pi^2 A_p-\frac{3}{4}}, \quad \text{where } \bar{\phi}_\varepsilon = \frac{1}{|\Omega|}\int_\Omega \phi_\varepsilon\,dx. \tag{4-1}$$

Take a Fermi coordinate system (U,θ) around p such that $\theta(p)=(0,0)$, θ maps $\partial\Omega\cap U$ inside $\partial\mathbb{R}^4_+$, and for any $\varepsilon>0$ and $x\in\partial\Omega$, there exists $\delta>0$ such that

$$(1-\varepsilon)\theta \le g = g_{ij}\,dx_i\otimes dx_j + dt\otimes dt \le (1+\varepsilon)\theta \quad \text{in } M_\delta,$$

where $M_\delta = \{x\in\Omega_\delta : \operatorname{dist}(\pi(x),p)\le\delta\}$.

Set

$$\widetilde{\phi}_\varepsilon(x',t) = C + \frac{-1/(8\pi^2)\log\Big(\big(\frac{\pi}{2}\big)^{4/3}|x'|^2/\varepsilon^2 + \Big(\big(\frac{\pi}{2}\big)^{2/3}t/\varepsilon+1\Big)^2\Big) + B + g_\varepsilon(x',t)}{C}$$

for some constants B, C, where

$$g_\varepsilon(x',t) = \frac{1}{2^{\frac{8}{3}}\pi^{\frac{4}{3}}}\frac{t/\varepsilon}{(1+(\pi/2)^{\frac{2}{3}}t/\varepsilon)^2 + (\pi/2)^{\frac{4}{3}}|x'|^2/\varepsilon^2}.$$

Let $B_r^+ = B_r(p)\cap\Omega$ and R be a function of ε such that $R\to+\infty$ and $R\varepsilon\to 0$ as $\varepsilon\to 0$. Set

$$\phi_\varepsilon = \begin{cases} \widetilde{\phi}_\varepsilon\circ\theta(x) & \text{if } x\in B^+_{R\varepsilon}, \\ (G-\eta\beta)/C & \text{if } x\in B^+_{2R\varepsilon}\setminus B^+_{R\varepsilon}, \\ G/C & \text{if } x\in\Omega\setminus B^+_{2R\varepsilon}, \end{cases}$$

where $\beta = G - C\widetilde{\phi}_\varepsilon\circ\theta(x)$, η is some radial function in $C_0^\infty(B_{2R\varepsilon}(p))$ with $\eta\equiv 1$ on $B_{R\varepsilon}(p)$, and $|\nabla\eta| = O(1/R\varepsilon)$, $|\Delta\eta| = O(1/(R\varepsilon)^2)$. One can easily verify that $\partial\phi_\varepsilon(x)/\partial\nu = 0$ for any $x\in\partial\Omega$.

Now, we estimate $\int_\Omega |\Delta\phi_\varepsilon|^2\,dx$; rewrite it as

$$\int_\Omega |\Delta\phi_\varepsilon|^2\,dx = \bigg(\int_{B^+_{R\varepsilon}} + \int_{\Omega\setminus B^+_{R\varepsilon}}\bigg)|\Delta\phi_\varepsilon|^2\,dx := \mathrm{I}_1 + \mathrm{I}_2. \tag{4-2}$$

Since

$$\begin{aligned} \mathrm{I}_2 &= \int_{\Omega\setminus B^+_{R\varepsilon}} |\Delta\phi_\varepsilon|^2 \\ &= \int_{\Omega\setminus B^+_{R\varepsilon}} \frac{|\Delta G|^2}{C^2} + \int_{B^+_{2R\varepsilon}\setminus B^+_{R\varepsilon}} \frac{\big|\Delta\big(\eta(G-C\widetilde{\phi}_\varepsilon\circ\theta(x))\big)\big|^2}{C^2} - \frac{2}{C^2}\int_{B^+_{2R\varepsilon}\setminus B^+_{R\varepsilon}} |\nabla G\nabla(G-C\widetilde{\phi}_\varepsilon\circ\theta(x))|^2 \\ &:= \mathrm{II}_1 + \mathrm{II}_2 + \mathrm{II}_3. \end{aligned} \tag{4-3}$$

Let C satisfy

$$C+\frac{-\frac{1}{8\pi^2}\log\big(\big(\frac{\pi}{2}\big)^{4/3}R^2\big)+B}{C}=\frac{-\frac{1}{4\pi^2}\log R\varepsilon+A_p}{C}, \tag{4-4}$$

by direct computing, one can easily verify that

$$|\mathrm{II}_2|,|\mathrm{II}_3|=\frac{1}{C^2}(O(R\varepsilon)+O(R^{-1})). \tag{4-5}$$

Similar as (3-37) and (3-41), we can obtain

$$\begin{aligned}\mathrm{II}_1=\int_{\Omega\setminus B^+_{R\varepsilon}}\frac{|\Delta G|^2}{C^2}&=\int_{\partial(\Omega\setminus B^+_{R\varepsilon})}\nu(-G\Delta^{\frac{3}{2}}G+\Delta^{\frac{1}{2}}G\Delta G)\,d\sigma\\&=\frac{1}{C^2}\Big(-\frac{1}{4\pi^2}\log R\varepsilon-\frac{1}{8\pi^2}+A_p+O(R\varepsilon)\Big)\end{aligned} \tag{4-6}$$

and

$$\mathrm{I}_1=\int_{B^+_{R\varepsilon}}|\Delta\phi_\varepsilon|^2=\frac{1}{C^2}\bigg(\frac{3}{16\pi^2}+\frac{1}{4\pi^2}\log\frac{R}{2}+\frac{1}{6\pi^2}\log\frac{\pi}{2}+O\Big(\frac{\log R}{R}\Big)\bigg). \tag{4-7}$$

Combining (4-2), (4-3), (4-5), (4-6) and (4-7), we have

$$\begin{aligned}\int_\Omega|\Delta\phi_\varepsilon|^2&=\frac{1}{C^2}\bigg(\frac{3}{16\pi^2}+\frac{1}{4\pi^2}\log\frac{R}{2}+\frac{1}{6\pi^2}\log\frac{\pi}{2}+O\Big(\frac{\log R}{R}\Big)\bigg)\\&\qquad+\frac{1}{C^2}\Big(-\frac{1}{4\pi^2}\log R\varepsilon-\frac{1}{8\pi^2}+A_p+O(R\varepsilon)+O\Big(\frac{1}{R}\Big)\Big)\\&=\frac{1}{C^2}\bigg(\frac{1}{16\pi^2}+\frac{1}{6\pi^2}\log\frac{\pi}{2}+\frac{1}{4\pi^2}\log\frac{1}{2\varepsilon}+A_p+O(R\varepsilon)+O\Big(\frac{\log R}{R}\Big)\bigg).\end{aligned}$$

To ensure that $\int_\Omega|\Delta\phi_\varepsilon|^2=1$, we set

$$C^2=\frac{1}{16\pi^2}+\frac{1}{6\pi^2}\log\frac{\pi}{2}+\frac{1}{4\pi^2}\log\frac{1}{2\varepsilon}+A_p+O(R\varepsilon)+O\Big(\frac{\log R}{R}\Big). \tag{4-8}$$

On the other hand, from (4-4), we have

$$C^2=-\frac{1}{4\pi^2}\log\varepsilon+A_p-B+\frac{1}{6\pi^2}\log\frac{\pi}{2}.$$

Therefore,

$$B=\frac{1}{4\pi^2}\log 2-\frac{1}{16\pi^2}+O(R\varepsilon)+O\Big(\frac{\log R}{R}\Big). \tag{4-9}$$

A straightforward computation gives

$$\begin{aligned}\bar{\phi}_\varepsilon=\frac{1}{|\Omega|}\int_\Omega\phi_\varepsilon&=\frac{1}{C}\big(O((R\varepsilon)^4\log R)+O((R\varepsilon)^4\log\varepsilon)+O((R\varepsilon)^4\log R\varepsilon)\big)\\&=\frac{1}{C}\big(O((R\varepsilon)^4\log R)+O((R\varepsilon)^4\log\varepsilon)\big).\end{aligned}$$

Then

$$\begin{aligned}
&\int_{\partial\Omega} \exp(12\pi^2(\phi_\varepsilon - \overline{\phi}_\varepsilon)^2)\, d\sigma \\
&\qquad \geq \int_{\partial B^+_{R\varepsilon}\cap\partial\mathbb{R}^+_4} \exp\big(12\pi^2(\widetilde{\phi}_\varepsilon - \overline{\phi}_\varepsilon)^2(x',t)\big)\, dx'\, dt \\
&\qquad \geq \int_{\partial B^+_{R\varepsilon}\cap\partial\mathbb{R}^+_4} \exp\bigg(12\pi^2C^2 - 3\log\bigg(\Big(\frac{\pi}{2}\Big)^{4/3}\frac{|x'|^2}{\varepsilon^2}+1\bigg) + 24\pi^2 B - 24\pi^2 C\overline{\phi}_\varepsilon\bigg)\, dx' \\
&\qquad = \exp\big(12\pi^2C^2 + 24\pi^2B + O((R\varepsilon)^4\log R) + O((R\varepsilon)^4\log\varepsilon)\big)\int_{B^3_{R\varepsilon}} \frac{1}{\big((\frac{\pi}{2})^{4/3}|x'|^2/\varepsilon^2+1\big)^3}\, dx'.
\end{aligned}$$

Let $\left(\frac{\pi}{2}\right)^{\frac{2}{3}}\frac{x'}{\varepsilon} = \tilde{x}$. Then

$$\begin{aligned}
\int_{B^3_{R\varepsilon}} \frac{1}{\big((\frac{\pi}{2})^{4/3}|x'|^2/\varepsilon^2+1\big)^3}\, dx' = \Big(\frac{2}{\pi}\Big)^2\varepsilon^3\int_{B^3_{(\pi/2)^{2/3}R}} \frac{1}{(\tilde{x}^2+1)^3}\, d\tilde{x} &= \Big(\frac{2}{\pi}\Big)^2\varepsilon^3\int_0^{(\pi/2)^{2/3}R}\frac{4\pi r^2}{(r^2+1)^3}\, dr \\
&= \Big(\frac{2}{\pi}\Big)^2\varepsilon^3 4\pi\int_0^{(\pi/2)^{2/3}R}\frac{r^2}{(r^2+1)^3}\, dr \\
&= \varepsilon^3\Big(1+O\Big(\frac{1}{R}\Big)\Big),
\end{aligned}$$

where we have used the fact that

$$\int_0^\infty \frac{r^2}{(r^2+1)^3}\, dr = \tfrac{1}{16}\pi.$$

Hence, it follows from (4-8) and (4-9) that

$$\begin{aligned}
&\int_{\partial B^+_{R\varepsilon}\cap\partial\mathbb{R}^+_4} \exp\big(12\pi^2(\phi_\varepsilon - \overline{\phi}_\varepsilon)^2(x',t)\big)\, dx'\, dt \\
&\geq \varepsilon^3\Big(1+O\Big(\frac{1}{R}\Big)\Big)\exp\big(12\pi^2C^2 + 24\pi^2B + O((R\varepsilon)^4\log R) + O((R\varepsilon)^4\log\varepsilon)\big) \\
&= \varepsilon^3\exp\bigg(12\pi^2\Big(\frac{1}{16\pi^2} + \frac{1}{6\pi^2}\log\frac{\pi}{2} + \frac{1}{4\pi^2}\log\frac{1}{2\varepsilon} + A_p\Big) + 24\pi^2\Big(\frac{1}{4\pi^2}\log 2 - \frac{1}{16\pi^2}\Big)\bigg) \\
&\qquad\qquad + O(R\varepsilon) + O\bigg(\frac{\log R}{R} + O((R\varepsilon)^4\log R) + O((R\varepsilon)^4\log\varepsilon)\bigg) \\
&= \exp\big(-\tfrac{3}{4} + 2\log\pi + \log 2 + 12\pi^2A_p\big) + O((R\varepsilon)^4\log R) + O((R\varepsilon)^4\log\varepsilon) + O(R\varepsilon) + O\Big(\frac{\log R}{R}\Big) \\
&= 2\pi^2\exp\big(-\tfrac{3}{4} + 12\pi^2A_p\big) + O((R\varepsilon)^4\log R) + O((R\varepsilon)^4\log\varepsilon) + O(R\varepsilon) + O\Big(\frac{\log R}{R}\Big).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\int_{\Omega\setminus\partial B^+_{R\varepsilon}} \exp(12\pi^2(\phi_\varepsilon - \overline{\phi}_\varepsilon)^2)\, d\sigma &\geq \int_{\partial\Omega\setminus\partial B^+_{R\varepsilon}} (1 + 12\pi^2(\phi_\varepsilon - \overline{\phi}_\varepsilon)^2)\, d\sigma \\
&\geq |\partial\Omega\setminus\partial B^+_{R\varepsilon}| + \frac{12\pi^2}{C^2}\int_{\partial\Omega\setminus\partial B^+_{2R\varepsilon}} (G - C\overline{\phi}_\varepsilon)^2\, d\sigma.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{\partial\Omega} \exp(12\pi^2(\phi_\varepsilon - \bar{\phi}_\varepsilon)^2)\, d\sigma \\
&\geq |\partial\Omega| - O((R\varepsilon)^3) + \frac{12\pi^2}{C^2}\int_{\partial\Omega\setminus\partial B^+_{2R\varepsilon}} \big(G - C(O((R\varepsilon)^4\log R) + O((R\varepsilon)^4 \log\varepsilon))\big)^2\, d\sigma \\
&\qquad + 2\pi^2\exp\big(-\tfrac{3}{4} + 12\pi^2 A_p\big) + O((R\varepsilon)^4\log R) + O((R\varepsilon)^4\log\varepsilon) + O\Big(\frac{\log R}{R}\Big) + O(R\varepsilon) \\
&= |\partial\Omega| + 2\pi^2\exp\big(-\tfrac{3}{4} + 12\pi^2 A_p\big) + \frac{12\pi^2}{C^2}\int_{\partial\Omega} G^2 + O((R\varepsilon)^4\log R) \\
&\qquad + O((R\varepsilon)^4\log\varepsilon) + O(R\varepsilon) + O\Big(\frac{\log R}{R}\Big).
\end{aligned}$$

Let $R = \log^2\varepsilon$. Then we have $R \to \infty$ and $R\varepsilon \to 0$, and

$$(R\varepsilon)^4\log R + (R\varepsilon)^4\log\frac{1}{\varepsilon} + O\Big(\frac{\log R}{R}\Big) + O(R\varepsilon) = o\Big(\frac{1}{C^2}\Big).$$

Hence

$$\int_{\partial\Omega} \exp(12\pi^2(\phi_\varepsilon - \bar{\phi}_\varepsilon)^2)\, d\sigma > |\partial\Omega| + 2\pi^2\exp\big(-\tfrac{3}{4} + 12\pi^2 A_p\big),$$

as ε is small enough.

Proof of Theorem 1.1. In the subcritical case $\alpha < 12\pi^2$, the inequality (1-3) and the sharpness of the constant $12\pi^2$ can be obtained from Lemma 2.3. In the critical case, that is $\alpha = 12\pi^2$, we will address the problem by dividing it into two cases. If $c_k = \max_{x\in\Omega}|u_k(x)|$ is bounded, then the inequality (1-3) is obvious, and by the elliptic estimates with respect to (2-3), there exists $u \in \mathcal{H}\cap C^\infty(\Omega)$ such that $u_k \to u$ in $C^\infty(\Omega)$ as $k \to \infty$, and Theorem 1.1 follows immediately. While if we assume that $c_k \to +\infty$ as $k \to \infty$, one can find a contradiction between Proposition 3.14 and the arguments of the test functions for (4-1) in Section 4, this means that c_k must be bounded, and the proof is finished. □

Proof of Theorem 1.3. For any $u \in W^{2,2}(\Omega)$ with $\frac{\partial u}{\partial\nu} = 0$ on $\partial\Omega$, define $\bar{u} = \frac{1}{|\Omega|}\int_\Omega u\, dx$. Then we can write

$$\begin{aligned}
\int_{\partial\Omega} e^{u-\bar{u}}\, d\sigma &= \int_{\partial\Omega} \exp\Big(\frac{u-\bar{u}}{\|\Delta(u-\bar{u})\|_2}\|\Delta(u-\bar{u})\|_2\Big)\, d\sigma \\
&\leq \int_{\partial\Omega} \exp\Big(12\pi^2\frac{|u-\bar{u}|^2}{\|\Delta(u-\bar{u})\|_2^2}\Big)\exp\Big(\frac{1}{48\pi^2}\|\Delta(u-\bar{u})\|_2^2\Big)\, d\sigma \leq C_0 e^{1/(48\pi^2)\|\Delta u\|_2^2}, \quad (4\text{-}10)
\end{aligned}$$

where we have used the elementary inequality $ab \leq 12\pi^2 a^2 + \frac{1}{48\pi^2}b^2$ and the trace Adams inequality in Theorem 1.1. Then we have

$$\log\Big(\int_{\partial\Omega} e^{u-\bar{u}}\, d\sigma\Big) \leq \log C_0 + \frac{1}{48\pi^2}\|\Delta u\|_2^2.$$

That is

$$\log\Big(\int_{\partial\Omega} e^{u}\, d\sigma\Big) \leq \frac{1}{48\pi^2}\int_\Omega |\Delta u|^2\, dx + \frac{1}{|\Omega|}\int_\Omega u\, dx + \log C_0. \qquad \square$$

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