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SINGULARITIES OF THE CHERN–RICCI FLOW

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We study the nature of finite time singularities for the Chern–Ricci flow, partially answering a question posed by Tosatti and Weinkove. We show that a solution of degenerate parabolic complex Monge–Ampère equations, starting from arbitrarily positive (1,1)-currents, is smooth outside some analytic subset, generalizing works by Di Nezza and Lu. Moreover, we extend Guedj and Lu's recent approach to establish uniform a priori estimates for degenerate complex Monge–Ampère equations on compact Hermitian manifolds. We apply these results to study the Chern–Ricci flow on log terminal varieties starting from a current with mild singularities.

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1. Introduction

Finding canonical metrics on complex varieties has been a central problem in complex geometry over the last few decades. Since Yau's solution to Calabi's conjecture, significant progress has been made in this direction. Cao [1985] introduced a parabolic approach to provide an alternative proof of the existence of Kähler–Einstein metrics on manifolds with numerically trivial or ample canonical line bundle via the Kähler–Ricci flow. This flow is only Hamilton's Ricci flow evolving Kähler metrics. Motivated by the classification of complex varieties, Song and Tian [2012; 2017] have proposed an *analytic minimal model program* to classify algebraic varieties with mild singularities using the Kähler–Ricci flow. This approach necessitates a theory of weak solutions for degenerate parabolic complex Monge–Ampère equations starting from rough initial data. Since then, various results have been achieved in this direction. Song and Tian initiated the study of the Kähler–Ricci flow starting from an initial current with continuous potentials. While Guedj and Zeriahi [2017b] (also [Tô 2017]) showed that the Kähler–Ricci flow could be continued from an initial current with zero Lelong numbers. To the author's knowledge, the best results so far have been obtained by Di Nezza and Lu [2017], who successfully ran the Kähler–Ricci flow from an initial

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current with positive Lelong numbers. There have been several related works in such singular settings from a pluripotential theoretical point of view. For further details, we refer to the recent works [Guedj et al. 2020; Dang 2022].

Beyond the Kähler setting, there more recently has been interest in the study of geometric flows in the context of non-Kähler manifolds. Unlike the Kähler case, Hamilton's Ricci flow does not, in general, preserve the special Hermitian condition. It is thus natural to look for another geometric flow of Hermitian metrics, which somehow specializes in the Ricci flow in the Kähler context. Several parabolic flows on complex manifolds that preserve the Hermitian property have been proposed by Streets and Tian [2010; 2011] and Liu and Yang [Liu and Yang 2012]. Additionally, the anomaly flow of $(n-1, n-1)$-forms has been extensively studied by Phong, Picard, and Zhang [Phong et al. 2018a; 2018b].

This paper is devoted to the Chern–Ricci flow, which is an evolution equation of Hermitian metrics on a complex manifold by their Chern–Ricci form, first introduced by Gill [2011] in the setting of manifolds with vanishing first Bott–Chern class. Let (X, ω_0) be a compact n-dimensional Hermitian manifold. The *Chern–Ricci flow* $\omega = \omega(t)$ starting at ω_0 is an evolution equation of Hermitian metrics,

$$\frac{\partial \omega}{\partial t} = -\mathrm{Ric}(\omega), \quad \omega|_{t=0} = \omega_0, \tag{1-1}$$

where $\mathrm{Ric}(\omega)$ is the *Chern–Ricci form* of ω associated to the Hermitian metric $g = (g_{i\bar{j}})$, which in local coordinates is given by

$$\mathrm{Ric}(\omega) = -dd^c \log \det(g).$$

Here $d = \partial + \bar{\partial}$ and $d^c = i(\bar{\partial} - \partial)/2$ are both real operators, so that $dd^c = i\partial\bar{\partial}$. In the Kähler setting, $\mathrm{Ric}(\omega) = i R_{j\bar{k}}\, dz_j \wedge d\bar{z}_k$, where $R_{j\bar{k}}$ is the usual Ricci curvature of ω. Thus, if ω_0 is Kähler, i.e., $d\omega_0 = 0$, (1-1) coincides with the Kähler–Ricci flow. For complex manifolds with $c_1^{\mathrm{BC}}(X) = 0$, Gill [2011] proved the longtime existence of the flow and smooth convergence of the flow to the unique Chern–Ricci flat metric in the $\partial\bar{\partial}$-class of the initial metric. For general complex manifolds, Tosatti and Weinkove [2015, Theorem 1.3] characterized the maximal existence time $T_{\max}$ of the flow as

$$T_{\max} := \sup\{t > 0 : \exists \psi \in \mathcal{C}^\infty(X) \text{ with } \omega_0 - t\,\mathrm{Ric}(\omega_0) + dd^c\psi > 0\}.$$

Finite time singularities. Suppose that the flow (1-1) exists on a maximal interval $[0, T_{\max})$ with $T_{\max} < \infty$, so the flow develops a finite time singularity. We say that the Chern–Ricci flow does not develop a singularity at a point $x \in X$ if there exist an open neighborhood $U \ni x$ and a smooth metric $\omega_{T_{\max}}$ on U such that $\omega(t)$ converges to $\omega_{T_{\max}}$ in $\mathcal{C}^\infty_{\mathrm{loc}}(U)$ as $t \to T_{\max}^-$.

The following question was asked by Feldman, Ilmanen, and Knopf [Feldman et al. 2003, Question 2, page 204] for the Kähler–Ricci flow and by Tosatti and Weinkove [2022, Question 6.1] for the Chern–Ricci flow.

Question 1.1. Do singularities of the Chern–Ricci flow form a union of all analytic subvarieties of X for which the volume shrinks to zero as $t \to T_{\max}$?

In the Kähler setting, this question was affirmatively answered by Collins and Tosatti [2015]. When X is a compact complex surface and ω_0 is Gauduchon, i.e., $dd^c\omega_0 = 0$, the Chern–Ricci flow preserves the Gauduchon (pluriclosed) condition, in particular, the limiting form $\alpha_{T_{\max}} = \omega_0 - T_{\max}\,\mathrm{Ric}(\omega_0)$ is Gauduchon. The answer is thus affirmative in this case, due to Gill and Smith [2015] (see also [Tosatti and Weinkove 2013]), where they proved that singularities of the Chern–Ricci flow form a finite union of disjoint (-1)-curves.

We partially answer Question 1.1 under two additional assumptions. First, we assume that the limiting form $\alpha_{T_{\max}}$ is *uniformly noncollapsing*:

$$\int_X (\alpha_{T_{\max}} + dd^c\psi)^n \geq c_0 > 0 \quad \text{for all } \psi \in \mathcal{C}^\infty(X),\ \alpha_{T_{\max}} + dd^c\psi > 0. \tag{1-2}$$

We mention that when $\dim X = 2$ and ω_0 is a Gauduchon metric on X, the latter condition is equivalent to $\int_X \alpha^2_{T_{\max}} > 0$ (by Stokes' theorem). A simple example (see [Tosatti and Weinkove 2013, Remark 3.1]) where this condition appears is the following. Let Y be a compact Hermitian manifold and $\pi : X \to Y$ be the blowup of a point with exceptional divisor E. Let ω_X and ω_Y be Gauduchon metrics on X and Y respectively, and fix $T_{\max} > 0$. It is known that there is a metric h on the line bundle $\mathcal{O}(E)$ with curvature R_h such that for $C > 0$ large enough, $\omega' = C\pi^*\omega_Y - T_{\max} R_h + dd^c f$ is a Hermitian metric for some $f \in \mathcal{C}^\infty(X)$. By the adjunction formula, we can choose

$$\omega_0 := (C+1)\pi^*\omega_Y + T_{\max}\,\mathrm{Ric}(\omega_X) + dd^c f$$

which is a Gauduchon metric. Hence $\alpha_{T_{\max}} = \pi^*\widetilde{\omega}_Y + dd^c\tilde{f}$ for some Gauduchon metric $\widetilde{\omega}_Y$ and $\tilde{f} \in \mathcal{C}^\infty(X)$; see [Tosatti and Weinkove 2013, Lemma 3.2] or [Buchdahl 2000]. For any $\psi \in \mathcal{C}^\infty(X)$,

$$\int_X (\alpha_{T_{\max}} + dd^c\psi)^2 = \int_X \alpha^2_{T_{\max}} = \int_Y \widetilde{\omega}^2_Y > 0.$$

The second assumption is that X has *the bounded mass property*, that is, there exists a Hermitian metric ω_X such that $v_+(\omega_X) < +\infty$ (see Definition 2.4). This condition is automatically satisfied for compact complex surfaces (see [Guedj and Lu 2022]). For further examples of non-Kähler manifolds in higher dimensions, we refer the reader to [Angella et al. 2023]. Our main theorem is the following.

Theorem A. *Let (X, ω_0) be an n-dimensional compact Hermitian manifold with bounded mass property, i.e., $v_+(\omega_0) < +\infty$. Assume that the Chern–Ricci flow* (1-1) *starting at ω_0 exists on the maximal interval $[0, T_{\max})$ with $T_{\max} < \infty$ and that the limiting form $\alpha_{T_{\max}}$ is uniformly noncollapsing*:

$$\int_X (\alpha_{T_{\max}} + dd^c\psi)^n \geq c_0 > 0 \quad \textit{for all } \psi \in \mathcal{C}^\infty(X) \textit{ such that } \alpha_{T_{\max}} + dd^c\psi > 0. \tag{1-3}$$

Then as $t \to T^-$ the metrics $\omega(t)$ converge to $\omega_{T_{\max}}$ in $\mathcal{C}^\infty_{\mathrm{loc}}(\Omega)$ for some Zariski open set $\Omega \subset X$.

The strategy of the proof is as follows. Using the uniformly noncollapsing condition of $\alpha_{T_{\max}}$, we show that there exists a quasiplurisubharmonic function ρ with analytic singularities such that $\alpha_{T_{\max}} + dd^c\rho$ dominates a Hermitian metric. This form is called *big* (see Definition 2.6). Then Ω is the set in which ρ is smooth. In particular, it is Zariski open. Next, we establish several uniform local estimates for ω near the maximal time $T_{\max}$, adapting techniques from [Collins and Tosatti 2015; Gill 2011]. The convergence result follows directly from these estimates.

Degenerate parabolic complex Monge–Ampère equations. In the previous paragraph, we studied the behavior of the Chern–Ricci flow at finite singularity time. It is natural to ask whether the flow can pass through this singularity. To do this, we need to define weak solutions of the Chern–Ricci flows starting from degenerate initial currents on a compact complex variety with mild singularities. This leads us to consider several geometric settings arising in the minimal model program, particularly the case of complex varieties with Kawamata log terminal (klt) singularities. From an analytic point of view, this situation naturally involves densities that may blow up but still belong to L^p spaces for some exponent $p > 1$ whose size depends on the algebraic nature of the singularities.

On a compact Hermitian n-manifold (X, ω_X), we consider the following degenerate parabolic complex Monge–Ampère equation,

$$\frac{\partial \varphi_t}{\partial t} = \log\left[\frac{(\theta_t + dd^c\varphi_t)^n}{\mu}\right] \tag{1-4}$$

for $t \in (0, T_{\max})$, where $T_{\max} < \infty$ and

- $\theta_t = \theta + t\chi$ is an affine family of smooth semipositive forms, where χ is a smooth (1,1)-form and θ is a smooth, *big* (1,1)-form, that is there is a quasiplurisubharmonic function ρ with analytic singularities such that
$$\theta + dd^c\rho \geq \delta\omega_X \quad \text{for some } \delta > 0;$$
- μ is a positive measure on X of the form
$$\mu = e^{\psi^+ - \psi^-}\, dV_X$$
with $\psi^{\pm}$ quasiplurisubharmonic functions, being smooth on a given Zariski open subset $U \subset \{\rho > -\infty\}$ and $e^{-\psi^-} \in L^p$ for some $p > 1$ and dV_X a smooth volume form;
- $\varphi : [0, T_{\max}] \times X \to \mathbb{R}$ is the unknown function, with $\varphi_t := \varphi(t, \cdot)$.

We first define the weak solution of the Chern–Ricci flow:

Definition 1.2. A family of functions $\varphi_t : X \to \mathbb{R}$ for $t \in (0, T_{\max})$ is said to be a weak solution of equation (1-4) starting with φ_0 if the following hold:

(1) For each t, φ_t is θ_t-plurisubharmonic on X.

(2) $\varphi_t \to \varphi_0$ in $L^1(X)$ as $t \to 0^+$.

(3) For each $\varepsilon > 0$ there exists a Zariski open set $\Omega_\varepsilon \subset X$ such that the function $(t, x) \mapsto \varphi(t, x) \in \mathcal{C}^\infty([\varepsilon, T_{\max} - \varepsilon] \times \Omega_\varepsilon)$. Furthermore, equation (1-4) satisfies in the classical sense on $[\varepsilon, T_{\max}) \times \Omega_\varepsilon$.

The following theorem establishes the existence of the complex Monge–Ampère flow starting with an initial function φ_0 with small Lelong numbers.

Theorem B. *Let (X, ω_0) be an n-dimensional compact Hermitian manifold and θ a semipositive and big* $(1, 1)$*-form. Let φ_0 be an θ-plurisubharmonic function satisfying $p^*/(2c(\varphi_0)) < T_{\max}$, where p^* is the conjugate exponent of p. Then, there exists a weak solution φ of the flow* (1-4) *starting with φ_0 for* $t \in (0, T_{\max})$.

Here, $c(\varphi_0)$ denotes the integrability index of φ_0, which is the supremum of positive constants $c > 0$ such that $e^{-2c\varphi_0}$ is locally integrable. Thanks to Skoda's integrability theorem, $c(\varphi_0) = +\infty$ if and only if φ_0 has zero Lelong numbers at all points.

Let us briefly outline the strategy for the proof of Theorem B. We first approximate φ_0 by a decreasing sequence of smooth $(\theta + 2^{-j}\omega_X)$-plurisubharmonic functions $\varphi_{0,j}$ thanks to Demailly's regularization theorem. Similarly, $\psi^{\pm}$ are approximated by smooth quasiplurisubharmonic functions. We consider the corresponding solution $\varphi_{t,j}$ to equation (1-4), with $\theta_{t,j} = \theta_t + 2^{-j}\omega_X$. Our goal is to establish several a priori estimates that allow us to take the limit as $j \to +\infty$. Precisely, we aim to show that for any $\varepsilon > 0$, there is a Zariski open set $\Omega_\varepsilon \subset X$ such that for each fixed $0 < T < T_{\max}$ and any compact subset $K \subset \Omega_\varepsilon$,

- $\|\varphi_{t,j}\|_{\mathcal{C}^0([\varepsilon,T]\times K)} \leq C_{\varepsilon,T,K}$;
- $\partial_t\varphi_{t,j}$ is uniformly bounded on $[\varepsilon, T] \times K$;
- $\Delta_{\omega_X}\varphi_{t,j}$ is uniformly bounded on $[\varepsilon, T] \times K$.

We then apply the parabolic Evans–Krylov–Trudinger theory and Schauder estimates to obtain uniform higher-order local estimates for all derivatives of $\varphi_{t,j}$ (see [Gill 2011] for a recent account in the Chern–Ricci flow context). This allows us to pass to the limit and conclude that

$$\varphi_{t,j} \to \varphi_t \in \mathcal{C}^\infty([\varepsilon, T] \times \Omega_\varepsilon)$$

as $j \to +\infty$. Furthermore, we automatically have the weak convergence $\varphi_t \to \varphi_0$ as $t \to 0^+$. Stronger convergence results are discussed in Section 4.4 when φ_0 has less singularity.

We emphasize here that the mild assumption $p^*/(2c(\varphi_0)) < T_{\max}$ guarantees that the approximating flow is well-defined (i.e., not identically $-\infty$) and is crucial for the smoothing properties of the flow. As noted by Di Nezza and Lu [2017] for the Kähler setting, without this assumption, the Kähler–Ricci flow may still run, but it is likely to lose its regularizing effect due to the presence of positive Lelong numbers. In such cases, they highlighted that the main challenge lies in establishing the a priori $\mathcal{C}^0$-estimate. Their approach relies on Kołodziej's method, which uses generalized Monge–Ampère capacities. In contrast, our approach follows the recent developments of Guedj and Lu [2023; 2025], which have the advantage of being applicable to degenerate (1,1)-forms in the non-Kähler context.

We finally apply the previous analysis to treat the case of mildly singular varieties. This allows us to define a good notion of the weak Chern–Ricci flow on complex compact varieties with log terminal singularities. We will discuss it in Section 6 and prove the following.

Theorem C. *Let Y be a compact complex variety with log terminal singularities. Assume that θ_0 is a Hermitian metric such that*

$$T_{\max} := \sup\{t > 0 : \exists\psi \in \mathcal{C}^\infty(Y) \text{ such that } \theta_0 - t\,\mathrm{Ric}(\theta_0) + dd^c\psi > 0\} > 0.$$

Assume that $S_0 = \theta_0 + dd^c\varphi_0$ is a positive $(1,1)$-current with sufficiently small slopes. Then, there exists a family $(\omega_t)_{t\in[0,T_{\max})}$ of positive (1,1)-currents on Y starting with S_0 such that

(1) $\omega_t = \theta_0 - t\,\mathrm{Ric}(\theta_0) + dd^c\varphi_t$ *are positive* (1,1)*-currents*;

(2) *$\omega_t \to S_0$ weakly as $t \to 0^+$;*

(3) *for each $\varepsilon > 0$ there exists a Zariski open set Ω_ε such that on $[\varepsilon, T_{\max}) \times \Omega_\varepsilon$, ω is smooth and*

$$\frac{\partial \omega}{\partial t} = -\mathrm{Ric}(\omega).$$

This generalizes previous results of Song and Tian [2017], Guedj and Zeriahi [2017a], Tô [2017], Di Nezza and Lu [2017], Guedj, Lu, and Zeriahi [Guedj et al. 2020] and the author [Dang 2022] to the non-Kähler case, and of Tô [2018], Nie [2017] and the author [Dang 2024] to more degenerate initial data.

Organization of the paper. We establish a priori estimates in Section 3, which will be used to prove Theorem B in Section 4. Theorem A will be proved in Section 5, studying the behavior of the Chern–Ricci flow at noncollapsing finite time singularities. In Section 6, we apply these tools to prove the existence of the weak Chern–Ricci flow with initial degenerate data on compact complex varieties with log terminal singularities, proving Theorem C.

2. Preliminaries

2.1. ***Recap on pluripotential theory.*** Let X be a compact complex manifold of dimension n, equipped with a Hermitian metric ω_X. We fix θ a smooth semipositive real $(1, 1)$-form on X.

2.1.1. *Quasiplurisubharmonic functions and Lelong numbers.* A function $u \in L^1(X)$ is quasiplurisubharmonic (quasi-psh for short) if it is locally given as the sum of a smooth function and a plurisubharmonic (psh for short) function.

Definition 2.1. A quasi-psh function $\varphi : X \to [-\infty, +\infty)$ is called *θ-plurisubharmonic* (*θ-psh* for short) if it satisfies $\theta_\varphi := \theta + dd^c\varphi \geq 0$ in the weak sense of currents. We let $\mathrm{PSH}(X, \theta)$ denote the set of all θ-psh functions that are not identically $-\infty$.

The set $\mathrm{PSH}(X, \theta)$ is endowed with the $L^1(X)$-topology. By Hartogs' lemma, the map $\varphi \mapsto \sup_X \varphi$ is continuous with respect to this topology. Since the set of closed positive currents in a fixed dd^c-class is compact (in the weak topology), it follows that the set of $\varphi \in \mathrm{PSH}(X, \theta)$, with $\sup_X \varphi = 0$ is compact. We refer the reader to [Demailly 2012; Guedj and Zeriahi 2017a] for basic properties of θ-psh functions.

Quasi-psh functions are, in general, singular, and a convenient way to measure their singularities is the Lelong numbers.

Definition 2.2. Let $x_0 \in X$. Fixing a holomorphic chart $x_0 \in V_{x_0} \subset X$, the *Lelong number* $\nu(\varphi, x_0)$ of a quasi-psh function φ at $x_0 \in X$ is defined as

$$\nu(\varphi, x_0) := \sup\bigl\{\gamma \geq 0 : \varphi(z) \leq \gamma \log \|z - x_0\| + O(1), \text{ on } V_{x_0}\bigr\}.$$

We remark here that this definition does not depend on the choice of local holomorphic charts. In particular, if $\varphi = \log |f|$ in a neighborhood V_{x_0} of x_0, for some holomorphic function f, then $\nu(\varphi, x_0)$ is equal to the vanishing order $\mathrm{ord}_{x_0}(f) := \sup\{k \in \mathbb{N} : D^\gamma f(x_0) = 0, \ \forall |\gamma| < k\}$.

In some contexts, it is more convenient to work with the integrability index rather than the Lelong numbers. The *integrability index* of a quasi-psh function φ at a point $x \in X$ is defined by

$$c(\varphi, x) := \sup\{c > 0 : e^{-2c\varphi} \in L^1(V_x)\},$$

where V_x is some neighborhood around x. This definition does not depend on the choice of the open neighborhood V_x. We denote by $c(\varphi)$ the infimum of $c(\varphi, x)$ for all $x \in X$. Since X is compact, it follows that $c(\varphi) > 0$.

Skoda's integrability theorem (see [Guedj and Zeriahi 2017a, Chapter 2]) yields the following "optimal" relation between the Lelong number of a quasi-psh function φ at a point $x_0 \in X$ and the local integrability index of φ at x_0:

$$\frac{1}{\nu(\varphi, x_0)} \le c(\varphi, x_0) \le \frac{n}{\nu(\varphi, x_0)}. \tag{2-1}$$

In particular, $c(\varphi) = +\infty$ if and only if $\nu(\varphi, x) = 0$ for all $x \in X$.

2.1.2. *Monge–Ampère measures.* The complex Monge–Ampère measure $(\theta + dd^c u)^n$ is well-defined for any θ-psh function u which is bounded, as follows from the Bedford–Taylor theory: if $\beta = dd^c \rho$ is a Kähler form such that $\beta > \theta$ in a local open chart $U \subset X$, then u is β-psh and the positive currents $(\beta + dd^c u)^j$ are well-defined for $1 \le j \le n$. Thus, the *complex Monge–Ampère measure*,

$$(\theta + dd^c u)^n := \sum_{j=0}^{n} \binom{n}{j} (\beta + dd^c u)^j \wedge (\theta - \beta)^{n-j},$$

is a positive measure on X. Indeed, by Demailly's regularization theorem, we can approximate u by a decreasing sequence of smooth $(\theta + \varepsilon_j \omega_X)$-psh functions u_j. Consequently, $(\theta + dd^c u)^n$ is the limit of positive measures $(\theta + \varepsilon_j \omega_X + dd^c u_j)^n$, ensuring that $(\theta + dd^c u)^n$ is positive.

This definition does not depend on the choice of β by the same arguments. We refer to [Dinew and Kołodziej 2012] for an adaptation of [Bedford and Taylor 1976; 1982] to the Hermitian context. We recall the following maximum principle.

Lemma 2.3. *Let φ, ψ be bounded θ-psh functions such that $\varphi \le \psi$. Then*

$$\mathbf{1}_{\{\varphi=\psi\}} (\theta + dd^c \varphi)^n \le \mathbf{1}_{\{\varphi=\psi\}} (\theta + dd^c \psi)^n.$$

Proof. This is a direct consequence of Bedford–Taylor's maximum principle; see [Guedj and Zeriahi 2017a, Theorem 3.23]. We refer the reader to [Guedj and Lu 2022, Lemma 1.2] for a brief proof. □

2.1.3. *Positivity assumptions.* For our purposes, we need to assume a slightly stronger positivity property of the form θ in the sense of [Guedj and Lu 2023].

Definition 2.4. We consider

$$v_-(\theta) := \inf\left\{ \int_X (\theta + dd^c \varphi)^n : \varphi \in \mathrm{PSH}(X, \theta) \cap L^\infty(X) \right\}$$

and

$$v_+(\theta) := \sup\left\{ \int_X (\theta + dd^c \varphi)^n : \varphi \in \mathrm{PSH}(X, \theta) \cap L^\infty(X) \right\}.$$

We emphasize that when θ is Hermitian, the supremum and infimum in the definition of these quantities can be taken over $\mathrm{PSH}(X,\theta)\cap\mathcal{C}^\infty(X)$ due to Demailly's regularization theorem and Bedford–Taylor's convergence results.

Definition 2.5. A function ρ is said to have *analytic singularities* if there exists a constant $c>0$ such that locally on X,

$$\rho = c\log\sum_{j=1}^N |f_j|^2 + O(1),$$

where the f_j are holomorphic functions.

Definition 2.6. We say θ is *big* if there exists a θ-psh function with analytic singularities such that $\theta+dd^c\rho\geq\delta\omega_X$ for some $\delta>0$. We let Ω denote the open Zariski set in which ρ is locally bounded.

Such a form appears in some contexts of complex differential geometry. For example, if Y is a compact complex space endowed with a Hermitian metric ω_Y and $\pi:X\to Y$ is a log resolution of singularities, then the form $\theta:=\pi^*\omega_Y$ is big; see, e.g., [Fino and Tomassini 2009, Proposition 3.2]. Moreover, we can find a θ-psh function ρ with analytic singularities such that $\theta+dd^c\rho\geq\delta\omega_X$, and

$$\Omega=\{\rho>-\infty\}=X\setminus\mathrm{Exc}(\pi)=\pi^{-1}(Y_{\mathrm{reg}})\simeq Y_{\mathrm{reg}}.$$

2.1.4. *Envelopes.* Recall that a Borel set $E\subset X$ is (*locally*) *pluripolar* if for each $x\in X$, there exists an open neighborhood U of x and a psh function u on U such that $E\cap U\subset\{u=-\infty\}$. As follows from [Vu 2019, Theorem 1.1] or [Guedj and Lu 2022, Lemma 2.6], the set E is globally pluripolar; i.e., there exists $u\in\mathrm{PSH}(X,\omega_X)$ such that $E\subset\{u=-\infty\}$. Since θ is big, the function $u':=\delta u+\rho$ is θ-psh and its $-\infty$-locus contains E.

Definition 2.7. Given a measurable function $h:X\to\mathbb{R}$, we define the *θ-psh envelope* of h by

$$P_\theta(h):=(\sup\{u\in\mathrm{PSH}(X,\theta):u\leq h\text{ on }X\})^*,$$

where the star means that we take the upper semicontinuous regularization.

We note that this definition is equivalent to the one given in [Guedj and Lu 2022, Definition 2.2]; see [Guedj and Lu 2022, Corollary 2.7].

We have the following result, established in [Guedj and Lu 2022, Theorem 2.3].

Theorem 2.8. *If h is bounded from below, quasi-lower-semicontinuous, and $P_\theta(h)<+\infty$, then*

(1) *$P_\theta(h)$ is a bounded θ-psh function;*

(2) *$P_\theta(h)\leq h$ in $X\setminus P$, for some pluripolar set P;*

(3) *$(\theta+dd^cP_\theta(h))^n$ is concentrated on the contact set $\{P_\theta(h)=h\}$.*

The following $\mathcal{C}^0$-estimate is crucial in the sequel.

Lemma 2.9. *Let θ be a smooth real semipositive and big (1,1)-form. Assume $\varphi\in\mathrm{PSH}(X,\theta)\cap L^\infty(X)$ satisfies*

$$(\theta+dd^c\varphi)^n\leq e^{A\varphi-g}f\,dV_X,$$

where $A > 0$ and f, g are measurable functions such that $e^{A\psi - g} f \in L^q(X)$ with $q > 1$, for some $\psi \in \mathrm{PSH}(X, \delta\theta)$, with $\delta \in (0, 1)$. Then we have the estimate

$$\varphi \geq \psi - C,$$

where C is a positive constant only depending on n, A, δ, θ, q and an upper bound for $\int_X e^{q(A\psi - g)} f^q \, dV_X$.

Proof. We apply the approach recently developed by Guedj and Lu [2023; 2025]. Subtracting a large constant, we can assume that $\varphi \leq 0$. Set $u := P_{(1-\delta)\theta}(\varphi - \psi)$. Fix $M > 0$ so large that $E := \{\psi > -M\}$ is not empty and hence it is nonpluripolar. We claim that the global extremal function $V^*_{E,(1-\delta)\theta}$ of E is not identically $+\infty$, where

$$V_{E,(1-\delta)\theta}(x) := \sup\bigl\{\varphi(x) : \varphi \in \mathrm{PSH}(X, (1-\delta)\theta),\ \varphi \leq 0 \text{ on } E\bigr\}.$$

The proof follows almost verbatim from [Guedj and Zeriahi 2017a, Theorem 9.17]. We suppose by contradiction that $\sup_X V_{E,(1-\delta)\theta} = +\infty$. By a lemma of Choquet (see [Guedj and Zeriahi 2017a, Lemma 4.31]), there exists an increasing sequence $u_j \in \mathrm{PSH}(X, (1-\delta)\theta)$ such that $u_j = 0$ on E, $\sup_X u_j \geq 2^j$, and

$$V_{E,(1-\delta)\theta} = (\lim \nearrow u_j)^*.$$

Set $v_j := u_j - \sup_X u_j$. These functions belong to the compact set of $(1-\delta)\theta$-psh functions normalized by $\sup_X w = 0$. Hence, there exists a uniform constant $C > 0$ such that $\int_X v_j \, dV \geq -C$; see [Dinew and Kołodziej 2012, Proposition 2.1]. Since $(1-\delta)\theta \geq 0$, the function $v := \sum_{j \geq 1} 2^{-j} v_j \in \mathrm{PSH}(X, (1-\delta)\theta)$ is a decreasing limit of functions in $\mathrm{PSH}(X, (1-\delta)\theta)$, with $\int_X v \, dV \geq -C$. Since $v = -\infty$ on E, it follows that E is $\mathrm{PSH}(X, (1-\delta)\theta)$-pluripolar. This gives a contradiction.

Since $u \leq \varphi - \psi \leq M$ on E, hence $u - M$ is a candidate defining $V_{E,(1-\delta)\theta}$. Therefore, $\sup_X u \leq M + \sup_X V^*_{E,(1-\delta)\theta}$ is uniformly bounded from above.

Since $\varphi - \psi$ is bounded from below and quasicontinuous, it follows from Theorem 2.8 that

$$((1-\delta)\theta + dd^c u)^n$$

is supported on the contact set $D := \{u + \psi = \varphi\}$. We observe that $u + \psi$ and φ are both θ-psh functions satisfying $u + \psi \leq \varphi$, it follows from Lemma 2.3 that

$$\mathbf{1}_D(\theta + dd^c(u + \psi))^n \leq \mathbf{1}_D(\theta + dd^c\varphi)^n.$$

From these, we have

$$\begin{aligned} ((1-\delta)\theta + dd^c u)^n = \mathbf{1}_D((1-\delta)\theta + dd^c u)^n \leq \mathbf{1}_D(\theta + dd^c(u+\psi))^n &\leq \mathbf{1}_D(\theta + dd^c\varphi)^n \\ &\leq \mathbf{1}_D e^{A\varphi - g} f \, dV_X \\ &= \mathbf{1}_D e^{Au} e^{A\psi - g} f \, dV_X. \end{aligned}$$

By assumption, $F := e^{A\psi - g} f \in L^q(X)$, with $q > 1$. Since $(1-\delta)\theta$ is semipositive and big, it follows from [Guedj and Lu 2023, Lemma 2.1] that there exists a uniform constant $m > 0$ only depending on dV_X,

n, q, θ, δ, and $\|e^{A\psi-g}f\|_{L^q}$, such that we can find $v \in \mathrm{PSH}(X, (1-\delta)\theta) \cap L^\infty(X)$ satisfying $-1 \le v \le 0$ and

$$((1-\delta)\theta + dd^c v)^n \ge mF\,dV_X.$$

Hence

$$e^{-A(v+A^{-1}\ln m)}((1-\delta)\theta + dd^c v)^n \ge F\,dV_X \ge e^{-Au}((1-\delta)\theta + dd^c u)^n.$$

The domination principle (see [Guedj and Lu 2023, Proposition 1.14]) yields $u \ge v + A^{-1}\ln m$. This completes the proof. □

2.2. ***Equisingular approximation.*** Fix φ a θ-psh function on X. We aim at approximating φ by a decreasing sequence of quasi-psh functions which are less singular than φ and such that their singularities are somehow comparable to those of φ. This leads us to apply Demailly's equisingular approximation theorem. For each $c > 0$, we define the *Lelong superlevel sets*

$$E_c(\varphi) := \{x \in X : \nu(\varphi, x) \ge c\}.$$

We also use the notation $E_c(T)$ for a closed positive $(1,1)$-current T. A well-known result of Siu [1974] asserts that the Lelong superlevel sets $E_c(\varphi)$ are analytic subsets of X. We refer the reader to [Demailly 1992, Remark 3.2] for an alternative proof.

The following result of Demailly on the equisingular approximation of a quasi-psh function by quasi-psh functions with analytic singularities is crucial.

Theorem 2.10 (Demailly's equisingular approximation). *Let φ be a θ-psh function on X. There exists a decreasing sequence of quasi-psh functions $(\varphi_m)_{m\in\mathbb{N}}$ such that*

(1) *(φ_m) converges pointwise and in $L^1(X)$ to φ as $m \to +\infty$,*

(2) *φ_m has the same singularities as $1/(2m)$ times a logarithm of a sum of squares of holomorphic functions,*

(3) *$dd^c\varphi_m \ge -\theta - \varepsilon_m\omega_X$, where $\varepsilon_m > 0$ decreases to 0 as $m \to +\infty$,*

(4) *$\int_X e^{2m(\varphi_m-\varphi)}\,dV < +\infty$,*

(5) *φ_m is smooth outside the analytic subset $E_{1/m}(\varphi)$.*

Proof. We briefly outline the idea for the reader's convenience, as it is likely already known to experts. We follow the proof of [Demailly 1992] by applying with the current $T = dd^c\varphi$ and the smooth real (1,1)-form $\gamma = -\theta$. We also borrow notation from there.

For $\delta > 0$ small, let us cover X by $N = N(\delta)$ geodesic balls $B_{2r}(a_j)$ with respect to ω_X such that $X = \bigcup_j B_r(a_j)$ and in terms of coordinates $z^j = (z^j_1, \dots, z^j_n)$,

$$\sum_{\ell=1}^n \lambda^j_\ell i\,dz^j_\ell \wedge d\bar z^j_\ell \le \gamma|_{B_{2r}(a_j)} \le \sum_{\ell=1}^n (\lambda^j_\ell + \delta) i\,dz^j_\ell \wedge d\bar z^j_\ell,$$

where we have diagonalized $\gamma(a_j)$ at the center a_j. Here, N and r are taken to be uniform. Set $\varphi^j := \varphi|_{B_{2r}(a_j)} - \sum_{\ell=1}^n \lambda_\ell^j |z_\ell^j|^2$. On each $B_{2r}(a_j)$, we define

$$\varphi_{j,\delta,m} := \frac{1}{2m} \log \sum_{k\in\mathbb{N}} |f_{j,m,k}|^2,$$

where $(f_{j,m,k})_{k\in\mathbb{N}}$ is an orthogonal basis of the Hilbert space $\mathcal{H}_{B_{2r}(a_j)}(m\varphi^j)$ of holomorphic functions on $B_{2r}(a_j)$ with finite L^2-norm $\|u\|^2 = \int_{B_{2r}(a_j)} |u|^2 e^{-2m\varphi^j}\, dV(z^j)$. Note that since $dd^c\varphi \geq \gamma$ it follows that $\varphi - \sum_{\ell=1}^n \lambda_\ell^j |z_\ell^j|^2$ is psh on $B_{2r}(a_j)$. The Bergman kernel process applied on each ball $B_{2r}(a_j)$ has provided approximations $\varphi_{j,\delta,m}$ of $\varphi^j = \varphi|_{B_{2r}(a_j)} - \sum_{\ell=1}^n \lambda_\ell^j |z_\ell^j|^2$; it thus remains to glue these functions into a function $\varphi_{\delta,m}$ globally defined on X. For this, we set

$$\varphi_{\delta,m}(x) = \frac{1}{2m} \log\biggl(\sum_j \theta_j(x)^2 \exp\biggl(2m\biggl(\varphi_{j,\delta,m} + \sum_\ell (\lambda_\ell^j - \delta)|z_\ell^j|^2\biggr)\biggr)\biggr),$$

where $(\theta_j)_{1\leq j\leq N}$ is the partition of unity subordinate to the $B_r(a_j)$. Now we take $\delta = \delta_m \searrow 0$ slowly and $\varphi_m = \varphi_{\delta_m,m}$ the same computations as in [Demailly 1992, page 16] ensure that

$$dd^c\varphi_m \geq \gamma - \varepsilon(\delta_m)\omega_X$$

for $m \geq m_0$ sufficiently large and $\varepsilon_m = \varepsilon(\delta_m) \searrow 0$ as $m \to +\infty$. By construction, the properties (1), (2), (3), and (5) are satisfied.

Property (4) is crucial for later use; its proof should be provided. The argument originates from [Demailly et al. 2001, Theorem 2.3, Step 2], using local uniform convergence and the strong Noetherian property. By the properties of the functions φ_m, it suffices to show that on each ball $B_j = B_r(a_j)$,

$$\int_{B_j} e^{2m\varphi_m - 2m\varphi}\, dV = \int_{B_j} \biggl(\sum_{k\in\mathbb{N}} |f_{j,m,k}|^2\biggr) e^{-2m\varphi}\, dV(z^j) < +\infty.$$

We let $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_k \subset \cdots \subset \mathcal{O}(B_{2r}(a_j) \times B_{2r}(a_j))$ denote the sequence of ideal coherent sheaves generated by the holomorphic functions $(f_{j,m,\ell}(z)\overline{f_{j,m,\ell}(\overline{w})})_{\ell\leq k}$ on $B_{2r}(a_j) \times B_{2r}(a_j)$. By the strong Noetherian property (see, e.g., [Demailly 2012, C. II, 3.22]), the sequence $(\mathcal{F}_k)$ is stationary on a compact subset $B_j \times B_j \Subset B_{2r}(a_j) \times B_{2r}(a_j)$ at an index k_0 large enough. Using the Cauchy–Schwarz inequality we have that the sum of the series $U(z, w) = \sum_{k\in\mathbb{N}} f_{j,m,k}(z)\overline{f_{j,m,k}(\overline{w})}$ is bounded from above by

$$\biggl(\sum_{k\in\mathbb{N}} |f_{j,m,k}(z)|^2 \sum_{k\in\mathbb{N}} |f_{j,m,k}(\overline{w})|^2\biggr)^{\frac{1}{2}}$$

hence uniformly convergent on every compact subset of $B_{2r}(a_j) \times B_{2r}(a_j)$. Since the space of sections of a coherent ideal sheaf is closed under the topology of uniform convergence on compact subsets, the Noetherian property guarantees $U(z, w) \in \mathcal{F}_{k_0}(B_j \times B_j)$. Restricting to the conjugate diagonal $w = \bar{z}$, we obtain

$$\sum_{k\in\mathbb{N}} |f_{j,m,k}(z)|^2 \leq C_0\biggl(\sum_{k\leq k_0} |f_{j,m,k}(z)|^2\biggr)$$

on B_j. Since all terms $f_{j,m,k}$ have the L^2-norm equal to 1 with respect to the weight $e^{-2m\varphi}$, this completes the proof. □

Using this, one obtains the following lemma, which is slightly more general than the one in [Di Nezza and Lu 2017].

Lemma 2.11. *Let θ be a big* (1,1)*-form. Assume $\varphi \in \mathrm{PSH}(X, \theta)$. Then for each $\varepsilon > 0$ there exist $c(\varepsilon) > 0$ and $\psi_\varepsilon \in \mathrm{PSH}(X, \theta) \cap \mathcal{C}^\infty\big(X \setminus (\{\rho = -\infty\} \cup E_{c(\varepsilon)}(\varphi))\big)$ such that*

$$\int_X e^{\frac{2}{\varepsilon}(\psi_\varepsilon - \varphi)}\, dV_X < +\infty. \tag{2-2}$$

Proof. The proof is quite close to that of [Di Nezza and Lu 2017, Lemma 2.7]. Recall that the bigness of θ implies that there exists ρ an θ-psh function with analytic singularities and $\sup_X \rho = 0$ such that

$$\theta + dd^c\rho \geq 3\delta_0\omega_X \quad \text{for a fixed constant } \delta_0 > 0.$$

Let $c(\varphi)$ be the integrability index of φ. We can assume that $c(\varphi) < +\infty$; otherwise we are done. By Theorem 2.10, we can find (φ_m) a Demailly's equisingular approximant of φ. We have that φ_m is smooth in the complement of the analytic subset $E_{1/m}(\varphi)$ and

$$\theta + dd^c\varphi_m \geq -\varepsilon_m\delta_0\omega_X$$

for $\varepsilon_m > 0$ decreasing to zero as m goes to $+\infty$. We notice that the errors $\varepsilon_m > 0$ appear in the gluing process; see Theorem 2.10. We choose $m = m(\varepsilon)$ to be the smallest positive integer such that

$$m > \frac{2}{\varepsilon(1+\varepsilon_m)}, \quad \frac{2\varepsilon_m}{\varepsilon(1+\varepsilon_m)} < c(\varphi).$$

We now set

$$\psi_\varepsilon := \frac{\varphi_m}{1+\varepsilon_m} + \frac{\varepsilon_m}{1+\varepsilon_m}\rho. \tag{2-3}$$

Thus, we have

$$\theta + dd^c\psi_\varepsilon \geq \frac{\varepsilon_m}{1+\varepsilon_m} 2\delta_0\omega_X := 2\kappa\omega_X.$$

Holder's inequality ensures that (2-2) holds, noticing that $\rho \leq 0$. We easily see that ψ_ε is smooth in the complement of $\{\rho = -\infty\} \cup E_{c(\varepsilon)}(\varphi)$ with $c(\varepsilon) = m(\varepsilon)^{-1}$. □

3. A priori estimates

3.1. ***Notation.*** We use some notation from [Di Nezza and Lu 2017, Section 3.1]. Until further notice, X denotes a compact complex manifold of dimension n, endowed with a reference Hermitian form ω_X. Following the strategy outlined in the introductory section, we assume, in this part, that θ and $\theta_t = \theta + t\chi$, $t \in (0, T_{\max})$, are Hermitian metrics, and φ_0 is a smooth strictly θ-psh function. We denote by $\mu := f\, dV_X$ a positive measure with density $\|f\|_{L^p} \leq C$ uniformly, for some $p > 1$. For higher-order estimates, we assume moreover that

$$f = e^{\psi^+ - \psi^-},$$

where $\psi^\pm$ are smooth quasi-psh functions. Recall that ρ is a θ-psh function with analytic singularities such that $\theta + dd^c\rho$ dominates a Hermitian form. We may assume that $\sup_X \rho = 0$. We remark that as follows from [Guedj and Lu 2023], a priori bounds below remain valid when θ is semipositive and big, $f \in L^p(X, \omega_X)$, and φ_0 is merely bounded and θ-psh.

We consider φ_t a smooth solution of the parabolic complex Monge–Ampère equation

$$\frac{\partial \varphi_t}{\partial t} = \log\left[\frac{(\theta_t + dd^c\varphi_t)^n}{\mu}\right], \quad \varphi|_{t=0} = \varphi_0 \tag{3-1}$$

on $[0, T_{\max})$; see, e.g., [Tosatti and Weinkove 2015]. We should keep in mind that φ_t plays the role of its approximants $\varphi_{t,j}$ in establishing a priori estimates. For brevity, we will suppress the index j.

We fix T and S such that

$$\frac{p^*}{2c(\varphi_0)} < T < S < T_{\max},$$

where p^* is the conjugate exponent of p, i.e., $\frac{1}{p} + \frac{1}{p^*} = 1$. Since we are interested in the behavior of the flow (3-1) near zero, we can assume that

$$\theta_S \geq (1-a)\theta \quad \text{for } a \in \left[0, \tfrac{1}{2}\right).$$

It is truly natural in several geometric contexts; for example, θ_t are the pullback of Hermitian forms. Thus, for each $t \in [0, S]$, we have

$$\theta_t = \frac{t\theta_S}{S} + \frac{S-t}{S}\theta \geq \left(1 - \frac{at}{S}\right)\theta.$$

During the proof, we use the notation $\omega_t := \theta_t + dd^c\varphi_t$ for the smooth path of Hermitian forms and denote $\Delta_t = \operatorname{tr}_{\omega_t} dd^c$ the corresponding time-dependent Laplacian operator on functions.

We fix $\varepsilon_0 > 0$ small and let $\psi_0 := \psi_{\varepsilon_0}$ as established in Lemma 2.11. By construction, ψ_0 is smooth outside an analytic subset $\{\rho = -\infty\} \cup E_{c(\varepsilon)}(\varphi_0)$ and satisfies

$$\theta + dd^c\psi_0 \geq 2\kappa\omega_X. \tag{3-2}$$

We let E_1, E_2 denote the quantities

$$E_1 := \int_X e^{2(\psi_0 - \varphi_0)/\varepsilon_0}\, dV_X < +\infty, \quad E_2 := \int_X e^{-p^*\psi_0/T}\, dV_X < +\infty.$$

Observe that E_1 is finite by Lemma 2.11, while E_2 is finite since $p^*/(2c(\varphi_0)) < T$ and that ψ_0 is less singular than φ_0. We should emphasize that φ_0 in this a priori estimate section plays a role in its approximating sequence $\varphi_{0,j}$ (which are smooth strictly θ-psh functions decreasing to φ_0). The corresponding sequence E_1^j is uniformly bounded from above in j. Hence we can pass the limit.

In what follows, we use C to denote a positive constant whose value may change from line to line but is uniformly controlled.

3.2. ***Uniform estimate.*** We first look for an upper a priori bound for φ_t. We recall that

$$\tfrac{1}{2}\theta \leq \theta_t \leq A\omega_X \quad \text{for all } t \in [0, T]$$

for $A > 0$ sufficiently large. It follows from [Tosatti and Weinkove 2010] that there exist a constant c and a smooth $A\omega_X$-psh function Φ normalized by $\inf_X \Phi = 0$ such that

$$(A\omega_X + dd^c\Phi)^n = e^c f \, dV_X.$$

Proposition 3.1. *For any* $(t, x) \in [0, T] \times X$, *there exists a uniform constant* $C > 0$ *such that*

$$\varphi_t(x) \le C.$$

Proof. For any $(t, x) \in [0, T] \times X$, we set $v(t, x) = \Phi(x) + ct + \sup_X \varphi_0$. Then, we can check that

$$\frac{\partial v}{\partial t} = \log\left[\frac{(A\omega_X + dd^c v_t)^n}{\mu}\right], \quad \text{while } \frac{\partial \varphi}{\partial t} \le \log\left[\frac{(A\omega_X + dd^c \varphi_t)^n}{\mu}\right],$$

and $v_0 \ge \varphi_0$. Hence, by the classical maximum principle, we have $v(t, x) \ge \varphi(t, x)$ for $(t, x) \in [0, T] \times X$. Consequently, this provides an upper bound for $\varphi(t, x)$:

$$\sup_X |\Phi| + \max(c, 0)T + \sup_X \varphi_0. \qquad \square$$

We fix two positive constants α, β such that

$$\frac{p^*}{2c(\varphi_0)} < \frac{1}{\alpha} < \frac{1}{\alpha - \beta} < T_{\max} \quad \text{and} \quad \theta + (\alpha - \beta)\chi \ge 0.$$

We observe that the density $e^{-\alpha\varphi_0} f$ belongs to L^q for $q > 1$. Indeed, we choose $\delta > 0$ so small ($\alpha(p^* + \delta) < 2c(\varphi_0)$) that

$$\frac{1}{q} = \frac{1}{p} + \frac{1}{p^* + \delta} \quad \text{with } q > 1.$$

Applying Hölder's inequality and Skoda's theorem, we have

$$\int_X e^{-\alpha q \varphi_0} f^q \, dV \le \|f\|_{L^p}^q \left(\int_X e^{-\alpha(p^*+\delta)\varphi_0} \, dV\right)^{q/p^*+\delta} < +\infty.$$

Thus, by [Tosatti and Weinkove 2010], there exists a smooth θ-psh function u such that

$$\beta^n(\theta + dd^c u)^n = e^{\beta u - \alpha\varphi_0} f \, dV.$$

Proposition 3.2. *For* $t \in (0, \alpha^{-1})$,

$$(1 - \alpha t)\varphi_0 + \beta t u + n(t \log t - t) \le \varphi_t.$$

In particular, there exists a uniform constant $C > 0$ *such that*

$$\varphi_0 - C(t - t \log t) \le \varphi_t \quad \textit{for all } t \in (1, \alpha^{-1}).$$

Proof. The proof is identical to that of [Guedj and Zeriahi 2017b, Lemma 2.9]. $\square$

Before establishing a lower bound for the solution φ_t, we first prove an upper bound for its time derivative, $\dot{\varphi}_t := \dfrac{\partial \varphi}{\partial t}$.

Proposition 3.3. *For all* $(t, x) \in (0, T] \times X$,

$$\dot{\varphi}_t(x) \le \frac{\varphi_t(x) - \varphi_0(x)}{t} + n. \tag{3-3}$$

Proof. The proof is identical to that of [Guedj and Zeriahi 2017b, Proposition 3.1] (also in [Guedj et al. 2020]). □

We follow the approach in [Di Nezza and Lu 2017] to derive the following uniform estimate for the complex parabolic Monge–Ampère equation.

Theorem 3.4. *Fix* $\varepsilon > p^*\varepsilon_0$. *For* $t \in [\varepsilon, T]$, *we obtain the estimate*

$$\varphi_t \ge \left(1 - \frac{bt}{T}\right)\psi_0 - C,$$

where $b \in \left(a, \frac{1}{2}\right)$ *and* $C > 0$ *is a uniform constant.*

Proof. Fixing $t \in [\varepsilon, T]$, it follows from Proposition 3.3 that

$$(\theta_t + dd^c\varphi_t)^n = e^{\dot{\varphi}_t} \le e^{n+(\varphi_t-\varphi_0)/t} f\, dV.$$

We set

$$\psi_t := \left(1 - \frac{bt}{T}\right)\psi_0$$

for $b \in \left(a, \frac{1}{2}\right)$ close to a. We recall that

$$\theta_t \ge \left(1 - \frac{at}{S}\right)\theta.$$

It then follows that ψ_t is $\delta\theta_t$-psh with $\delta \in (0, 1)$ only depending on ε_0, a, b, T, S. More precisely,

$$\delta = \frac{TS - bS\varepsilon_0}{TS - aT\varepsilon_0}.$$

Using similar arguments as in the proof of [Di Nezza and Lu 2017, Theorem 3.2], we can bound the quantity

$$\int_X e^{q(\psi_t-\varphi_0)/t} f^q\, dV < +\infty \tag{3-4}$$

for some $q > 1$, in terms of $\|f\|_{L^p}$, E_1 and E_2. To establish this, we fix $\gamma > 0$ small enough and choose $q > 1$ such that

$$\frac{1}{q} = \frac{1}{p} + \frac{1}{2p^* + \gamma} + \frac{1}{2p^* + \gamma}.$$

By Hölder's inequality, we obtain

$$\int_X e^{q(\psi_t-\varphi_0)/t} f^q\, dV \le \|f\|^q_{L^p} \left(\int_X e^{(2p^*+\gamma)(\psi_0-\varphi_0)/t}\, dV\right)^{q/(2p^*+\gamma)} \left(\int_X e^{-(2p^*+\gamma)b\psi_0/T}\, dV\right)^{q/(2p^*+\gamma)}.$$

The second term on the right-hand side is finite due to the construction of ψ_0 in Lemma 2.11. Also, since ψ_0 is less singular than φ_0, the third term is finite.

From (3-4), we apply Lemma 2.9 with $A = 1/t$ and $g = \varphi_0/t - n$ to obtain the desired estimate. It is important to note that our $\mathcal{C}^0$-estimate depends only on n, θ, q, the fixed parameters ε_0, ε, T, S, and an upper bound for E_1 and E_2. □

Remark 3.5. When φ_0 is bounded or, more generally, has zero Lelong numbers, it was shown in [Tô 2018] (generalizing the result of [Guedj and Zeriahi 2017b] in the Kähler context) that the estimate (3-3) ensures a lower bound for φ_t using the Kolodziej–Nguyen theorem [2015]. Unfortunately, this method cannot be applied in more general cases, such as when φ_0 is more singular, for example, when it has a positive Lelong number. To analyze the singularities of the initial potential φ_0 in such cases, Guedj and Lu's approach [2023] could help.

3.3. *Laplacian estimate.* We recall the following classical inequality.

Lemma 3.6. *Let α, β be two positive* (1,1)*-forms. Then*

$$n\left(\frac{\alpha^n}{\beta^n}\right)^{\frac{1}{n}} \leq \operatorname{tr}_\beta(\alpha) \leq n\left(\frac{\alpha^n}{\beta^n}\right)(\operatorname{tr}_\alpha(\beta))^{n-1}.$$

We define

$$\Psi_t := \left(1 - \frac{bt}{S}\right)\psi_0,$$

where ψ_0 is defined in Lemma 2.11 with $\varepsilon_0 > 0$ fixed.

To establish the $\mathcal{C}^2$-estimate, it is necessary to derive a lower bound for $\dot{\varphi}_t = \dfrac{\partial\varphi}{\partial t}$.

Proposition 3.7. *Fix $\varepsilon > p^*\varepsilon_0$. For $(t, x) \in (\varepsilon, T] \times X$,*

$$\dot{\varphi}_t(x) \geq n\log(t-\varepsilon) + A(\Psi_t - \varphi_t) - C,$$

where $A, C > 0$ are positive constants only depending on ε, T, $\|f\|_{L^p}$, and an upper bound for E_1 and E_2.

Proof. The proof is almost identical to that of [Di Nezza and Lu 2017, Proposition 3.5]. The only difference is that we use Theorem 3.4 instead of the corresponding result in [Di Nezza and Lu 2017, Theorem 3.2]. We include the proof for the reader's convenience.

Since $\mu = f\,dV$ is a smooth volume form, the main result of Tosatti and Weinkove [2010] ensures that there exists a constant c_1 and $\phi_1 \in \mathrm{PSH}(X,\theta) \cap \mathcal{C}^\infty(X)$ such that

$$(\theta + dd^c\phi_1)^n = e^{c_1}\mu, \quad \sup_X \phi_1 = 0.$$

From [Guedj and Lu 2023, Theorems 2.2, 3.4], it follows that $|c_1| + \|\phi_1\|_{L^\infty} \leq C$, where $C > 0$ depends only on the semipositivity and bigness of θ, n, dV_X, p and $\|f\|_p$.

We define

$$G(t,x) := \dot{\varphi}_t(x) + A(\varphi_t - \Psi_t) - \phi_1 - n\log(t-\varepsilon)$$

for a constant $A > 0$ to be determined hereafter. Observe that G achieves its minimum on $[\varepsilon, T] \times X$ at some point $(t_0, x_0) \in (\varepsilon, T] \times (X \backslash \{\psi_0 = -\infty\})$. In the following, all computations will be performed at this point. We compute

$$\Big(\frac{\partial}{\partial t}-\Delta_t\Big)G = A\dot{\varphi}_t - \frac{n}{t-\varepsilon} + A\frac{b\psi_0}{S} - nA + A\,\mathrm{tr}_{\omega_t}(\theta_t + dd^c\Psi_t) + \mathrm{tr}_{\omega_t}(\chi + dd^c\phi_1).$$

We observe that

$$\theta_t + dd^c\Psi_t = \frac{t(b-a)}{S}\theta + \Big(1-\frac{bt}{S}\Big)(\theta + dd^c\psi_0) \geq \frac{\varepsilon(b-a)}{S}\theta + \tfrac{1}{2}2\kappa\omega_X.$$

We now choose $A > 0$ so big that

$$A(\theta_t + dd^c\Psi_t) + \chi \geq \theta.$$

Therefore

$$\Big(\frac{\partial}{\partial t}-\Delta_t\Big)G \geq A\dot{\varphi}_t - \frac{n}{t-\varepsilon} + A\frac{b\psi_0}{S} - nA + \mathrm{tr}_{\omega_t}(\theta + dd^c\phi_1). \tag{3-5}$$

On the other hand, Lemma 3.6 ensures that

$$\mathrm{tr}_{\omega_t}(\theta + dd^c\phi_1) \geq n\bigg(\frac{(\theta + dd^c\phi_1)^n}{\omega_t^n}\bigg)^{\frac{1}{n}} = ne^{-\dot{\varphi}_t + c_1/(n).}$$

Using the elementary inequality $\gamma y - \log y \geq -C_\gamma$ for any small constant $\gamma > 0$ and $y > 0$, we observe that

$$A\dot{\varphi}_t + ne^{(-\dot{\varphi}_t + c_1)/n} \geq e^{-\dot{\varphi}_t/n - C_1} - C_2.$$

Substituting this into (3-5), it follows from the minimum principle that at (t_0, x_0),

$$\dot{\varphi}_t \geq -n\log\bigg(C_2 + \frac{n}{t-\varepsilon} - \frac{Ab\psi_0}{S} + nA\bigg) - nC_1,$$

and hence

$$G(t_0, x_0) \geq -C_3 - n\log\bigg(C_2(t_0-\varepsilon) + n - \frac{Ab(t_0-\varepsilon)\psi_0}{S}\bigg) - \frac{Abt_0(S-T)}{ST}\psi_0,$$

where we have used Theorem 3.4. Thus, we obtain a uniform lower bound for $G(t_0, x_0)$, and the desired lower bound follows. □

We are now in a position to establish the $\mathcal{C}^2$-estimate. We follow the computations of [Tosatti and Weinkove 2015, Lemma 4.1] (see also [Tô 2018, Lemma 6.4]), where they use the technical trick introduced by Phong and Sturm [2010]. Recall that the measure μ is of the form

$$\mu = e^{\psi^+ - \psi^-}\, dV_X,$$

where $\psi^\pm$ are smooth $K\omega_X$-psh functions on X for uniform constant $K > 0$. For simplicity, we assume $K = 1$ and normalize $\sup_X \psi^\pm = 0$.

Theorem 3.8. *Fix $\varepsilon > p^*\varepsilon_0$. For $(t, x) \in [\varepsilon, T] \times X$ we have the bound*

$$(t-\varepsilon)\log \mathrm{tr}_{\omega_X}(\omega_t) \leq -B\psi_0 - C\psi^- + C,$$

where B, C are positive constants depending only on ε, T, $\|e^{-\psi^-}\|_{L^p}$, and an upper bound for E_1 and E_2.

Proof. We follow the computations of [Gill 2011; Tô 2018] (which are due to the trick of Phong and Sturm [2010]) with modification to deal with unbounded functions. The constant C denotes various uniform constants, which may differ throughout the argument.

Consider

$$H := (t-\varepsilon)\log \operatorname{tr}_{\omega_X}(\omega_t) - \gamma(u), \quad (t,x)\in[\varepsilon,T]\times X,$$

where $\gamma:\mathbb{R}\to\mathbb{R}$ is a smooth, concave, increasing function such that $\lim_{t\to+\infty}\gamma(t)=+\infty$, and

$$u(t,x) := \varphi_t(x) - \Psi_t(x) - \kappa\psi^- + 1 \ge 1,$$

as follows from Theorem 3.4, and $\psi_0, \psi^- \le 0$. We will show that H is uniformly bounded from above for an appropriate choice of γ.

We let g denote the Riemann metric associated with ω_X and $\tilde{g}$ the one associated with $\omega_t := \theta_t + dd^c\varphi_t$. Since H goes to $-\infty$ on the boundary of $X_0 := \{x\in X : \psi_0(x) > -\infty\}$, H achieves its maximum on $[\varepsilon,T]\times X$ at some point $(t_0,x_0)\in(\varepsilon,T]\times X_0$.

At this maximum point, we use the following local coordinate systems due to Guan and Li [2010, Lemma 2.1, (2.19)]:

$$g_{i\bar{j}} = \delta_{ij}, \quad \frac{\partial g_{i\bar{i}}}{\partial z_j} = 0 \quad \text{and} \quad \tilde{g}_{i\bar{j}} \text{ is diagonal.}$$

Following the computations in [Tô 2018, equation (3.20)], we have

$$\Delta_t \operatorname{tr}_{\omega_X}(\omega_t) \ge \sum_{i,j} \tilde{g}^{i\bar{i}}\tilde{g}^{j\bar{j}}\tilde{g}_{i\bar{j}j}\tilde{g}_{j\bar{i}\bar{j}} - \operatorname{tr}_{\omega_X}\operatorname{Ric}(\omega_t) - C_1 \operatorname{tr}_{\omega_X}(\omega_t)\operatorname{tr}_{\omega_t}(\omega_X). \tag{3-6}$$

From standard arguments, as in [Guedj and Lu 2023, equation (4.5)], we obtain

$$\frac{|\partial \operatorname{tr}_{\omega_X}(\omega_t)|^2_{\tilde{\omega}_t}}{(\operatorname{tr}_{\omega_X}(\omega_t))^2} \le \frac{1}{\operatorname{tr}_{\omega_X}(\omega_t)}\biggl(\sum_{i,j}\tilde{g}^{i\bar{i}}\tilde{g}^{j\bar{j}}\tilde{g}_{i\bar{j}j}\tilde{g}_{j\bar{i}\bar{j}}\biggr) + C\frac{\operatorname{tr}_{\omega_t}(\omega_X)}{(\operatorname{tr}_{\omega_X}(\omega_t))^2} + \frac{2}{(\operatorname{tr}_{\omega_X}(\omega_t))^2}\operatorname{Re}\sum_{i,j,k}\tilde{g}^{i\bar{i}}T_{ij\bar{j}}\tilde{g}_{k\bar{i}\bar{k}}, \tag{3-7}$$

where $T_{ij\bar{j}} := \tilde{g}_{j\bar{j}i} - \tilde{g}_{i\bar{j}j}$ is the torsion term corresponding to θ_t which is controlled: $|T_{ij\bar{j}}|\le C$. Now at the point (t_0,x_0), we have $\partial_{\bar{i}}H = 0$; hence

$$(t-\varepsilon)\sum_k \tilde{g}_{k\bar{k}\bar{i}} = \operatorname{tr}_{\omega_X}(\omega_t)\gamma'(u)u_{\bar{i}}.$$

The Cauchy–Schwarz inequality yields

$$\biggl|\frac{2}{(\operatorname{tr}_{\omega_X}(\omega_t))^2}\operatorname{Re}\sum_{i,j,k}\tilde{g}^{i\bar{i}}T_{ij\bar{j}}\tilde{g}_{k\bar{k}\bar{i}}\biggr| \le C\frac{\gamma'(u)(t_0-\varepsilon)}{-\gamma''(u)}\frac{\operatorname{tr}_{\omega_t}(\omega_X)}{(\operatorname{tr}_{\omega_X}(\omega_t))^2} + \frac{-\gamma''(u)}{t_0-\varepsilon}|\partial u|^2_{\omega_t},$$

and hence

$$\biggl|\frac{2}{(\operatorname{tr}_{\omega_X}(\omega_t))^2}\operatorname{Re}\sum_{i,j,k}\tilde{g}^{i\bar{i}}T_{ij\bar{j}}\tilde{g}_{k\bar{i}\bar{k}}\biggr| \le C\biggl(\frac{\gamma'(u)T}{-\gamma''(u)}+1\biggr)\frac{\operatorname{tr}_{\omega_t}(\omega_X)}{(\operatorname{tr}_{\omega_X}(\omega_t))^2} + \frac{-\gamma''(u)}{t_0-\varepsilon}|\partial u|^2_{\omega_t},$$

using that $|\tilde{g}_{k\bar{k}\bar{i}} - \tilde{g}_{k\bar{i}\bar{k}}| \le C$. From this, the inequality (3-7) becomes

$$\frac{|\partial \operatorname{tr}_{\omega_X}(\omega_t)|^2_{\omega_t}}{(\operatorname{tr}_{\omega_X}(\omega_t))^2} \le \frac{1}{\operatorname{tr}_{\omega_X}(\omega_t)}\Big(\sum_{i,j} \tilde{g}^{i\bar{i}}\tilde{g}^{j\bar{j}}\tilde{g}_{i\bar{j}j}\tilde{g}_{j\bar{i}\bar{j}}\Big) + C\Big(\frac{\gamma'(u)T}{-\gamma''(u)}+2\Big)\frac{\operatorname{tr}_{\omega_t}(\omega_X)}{(\operatorname{tr}_{\omega_X}(\omega_t))^2} + \frac{-\gamma''(u)}{t_0-\varepsilon}|\partial u|^2_{\omega_t}. \tag{3-8}$$

Set $\alpha := \operatorname{tr}_{\omega_X}(\omega_t)$. We compute

$$\begin{aligned}\dot\alpha &= \operatorname{tr}_{\omega_X}(\chi) - \operatorname{tr}_{\omega_X}\operatorname{Ric}(\omega_t) - \operatorname{tr}_{\omega_X} dd^c(\psi^+ - \psi^-) + \operatorname{tr}_{\omega_X}(\operatorname{Ric}(\omega_X))\\ &\le \operatorname{tr}_{\omega_X}(C_1\omega_X + dd^c\psi^-) - \operatorname{tr}_{\omega_X}\operatorname{Ric}(\omega_t),\end{aligned}$$

where we have used the fact that $\operatorname{tr}_{\omega_X}(\chi)$ is bounded from above, together with the trivial inequality $n \le \operatorname{tr}_{\omega_X}(\omega_t)\operatorname{tr}_{\omega_t}(\omega_X)$. Combining this with (3-6) and (3-8), we infer that

$$\begin{aligned}\frac{\dot\alpha}{\alpha} - \Delta_t \log\alpha &= \frac{\dot\alpha}{\alpha} - \frac{\Delta_t\alpha}{\alpha} + \frac{|\partial\alpha|^2_{\omega_t}}{\alpha^2}\\ &\le \frac{\operatorname{tr}_{\omega_t}(C_1\omega_X + dd^c\psi^-)}{\alpha} + C\Big(\frac{\gamma'(u)T}{-\gamma''(u)}+2\Big)\frac{\operatorname{tr}_{\omega_t}(\omega_X)}{\alpha^2} + \frac{-\gamma''(u)}{t_0-\varepsilon}|\partial u|^2_{\omega_t}.\end{aligned} \tag{3-9}$$

From this, at the maximum point (t_0, x_0),

$$\begin{aligned}0 \le \Big(\frac{\partial}{\partial t} - \Delta_t\Big)H &= \log\alpha + (t-\varepsilon)\Big(\frac{\dot\alpha}{\alpha} - \Delta_t\log\alpha\Big) - \gamma'(u)\dot u + \gamma'(u)\Delta_t u + \gamma''(u)|\partial u|^2_{\omega_t}\\ &\le \log\alpha + \frac{C_3\operatorname{tr}_{\omega_t}(\omega_X + dd^c\psi^-)}{\alpha} + C_4\Big(\frac{\gamma'(u)T}{-\gamma''(u)}+2\Big)\frac{\operatorname{tr}_{\omega_t}(\omega_X)}{\alpha^2}\\ &\qquad - \gamma'(u)\dot\varphi_t + \gamma'(u)\dot\Psi_t + \gamma'(u)\Delta_{\omega_t}(\varphi_t - \Psi_t - \kappa\psi^-),\end{aligned} \tag{3-10}$$

with $C_3, C_4 > 0$ under control. Moreover, since $\theta_t \ge (1 - at/S)\theta$,

$$\theta_t + dd^c\Psi_t \ge \Big(1 - \frac{bt}{S}\Big)2\kappa\omega_X.$$

Thus we obtain

$$\Delta_t(\varphi_t - \Psi_t) \le n - \kappa\operatorname{tr}_{\omega_t}(\omega_X). \tag{3-11}$$

Substituting (3-11) into (3-10), we obtain

$$\begin{aligned}0 \le \log\alpha + \frac{C_3\operatorname{tr}_{\omega_t}(\omega_X + dd^c\psi^-)}{\alpha} - \gamma'(u)(n - \kappa\operatorname{tr}_{\omega_t}(\omega_X + dd^c\psi^-)) - \gamma'(u)\dot\varphi_t - \gamma'(u)\frac{b\psi_0}{S}\\ + C_4\Big(\frac{\gamma'(u)T}{-\gamma''(u)}+2\Big)\frac{\operatorname{tr}_{\omega_t}(\omega_X)}{(\operatorname{tr}_{\omega_X}(\omega_t))^2} + C_5.\end{aligned}$$

We now choose the function γ to obtain a simplified formulation. We set

$$\gamma(u) := \frac{C_3+3}{\min(\kappa, 1)}u + \log(u).$$

Since $u \ge 1$ we have

$$\frac{C_3+3}{\min(\kappa,1)} \le \gamma'(u) \le 1 + \frac{C_3+3}{\min(\kappa,1)}, \qquad \frac{\gamma'(u)T}{-\gamma''(u)} + 2 \le C_5u^2.$$

Using $\operatorname{tr}_{\omega_X}(\omega_X + dd^c\psi^-) \le \operatorname{tr}_{\omega_t}(\omega_X + dd^c\psi^-)\operatorname{tr}_{\omega_X}(\omega_t)$ we obtain

$$0 \le \log\alpha - \gamma'(u)\dot{\varphi}_t - \gamma'(u)\frac{b\psi_0}{S} - 3\operatorname{tr}_{\omega_t}(\omega_X) + C_6(u^2+1)\frac{\operatorname{tr}_{\omega_t}(\omega_X)}{\alpha^2}. \tag{3-12}$$

If at the point (t_0, x_0) we have $\alpha^2 \le C_6(u^2+1)$, then

$$H(t_0, x_0) \le T\log\sqrt{C_6(u^2+1)} - \gamma(u) \le C_7,$$

and we are done. Otherwise, we assume that, at (t_0, x_0), $\alpha^2 \ge C_6(u^2+1)$. Applying Lemma 3.6, we obtain

$$\log\alpha = \log\operatorname{tr}_{\omega_X}(\omega_t) \le (n-1)\log\operatorname{tr}_{\omega_t}(\omega_X) + \log n + \dot{\varphi}_t - \psi^-$$

using that $\sup_X \psi^+ = 0$. Plugging this into (3-12), we obtain

$$0 \le C_5 + (n-1)\log\operatorname{tr}_{\omega_t}(\omega_X) - 2\operatorname{tr}_{\omega_t}(\omega_X) - (\gamma'(u)-1)\dot{\varphi}_t - \gamma'(u)\frac{b\psi_0}{S} - \psi^-,$$

or equivalently,

$$\operatorname{tr}_{\omega_t}(\omega_X) \le C_8 - (\gamma'(u)-1)\dot{\varphi}_t - \gamma'(u)\frac{b\psi_0}{S} - \psi^- \tag{3-13}$$

since $(n-1)\log y - 2y \le -y + O(1)$ for $y > 0$. In particular, we have

$$\dot{\varphi}_t \le \frac{C_5}{\gamma'(u)-1} - \frac{\gamma'(u)}{\gamma'(u)-1}\frac{b\psi_0}{S} \le \frac{C_5}{A-1} - \frac{bA\psi_0}{(A-1)S} - \frac{\psi^-}{A-1} \tag{3-14}$$

at (t_0, x_0), since $\operatorname{tr}_{\omega_t}(\omega_X) \ge 0$ and $A \le \gamma'(u) \le A+1$ with $A =: (C_3+3)/\min(\kappa, 1)$. It follows from Lemma 3.6 that

$$\operatorname{tr}_{\omega_t}(\omega_X) \ge n\exp\left(\frac{-\dot{\varphi}_t + \psi^-}{n}\right).$$

Plugging this into (3-13), we obtain

$$\operatorname{tr}_{\omega_t}(\omega_X) \le C_9 - \gamma'(u)\frac{b\psi_0}{S} - \gamma'(u)\psi^- \le C_9 - \frac{(A+1)b\psi_0}{S} - (A+1)\psi^-$$

with $C_9 > 0$ under control; since $e^y - Dy \ge -C$ for $y \in \mathbb{R}$, $D > 0$, we apply with $y = (-\dot{\varphi}_t + \psi^-)/n$. Again Lemma 3.6 yields

$$\log\alpha \le (n-1)\log\left(C_9 - \frac{b(A+1)\psi_0}{S} - (A+1)\psi^-\right) + \log n + \dot{\varphi}_t - \psi^-.$$

Combining this together with (3-14), we have at (t_0, x_0)

$$H \le C_{10} - A\left[\varphi_t - \left(1 - \frac{bt}{S} - \frac{b(t-\varepsilon)}{(A-1)S}\right)\psi_0\right] + \left(A\kappa - 1 - \frac{1}{A-1}\right)\psi^- \\ + (t-\varepsilon)(n-1)\log\left(C_9 - \frac{b(A+1)\psi_0}{S} - (A+1)\psi^-\right).$$

Up to increasing $A > 0$ if necessary, so that

$$\eta := \frac{b\varepsilon}{T} - \frac{b\varepsilon}{S} - \frac{bT}{(A-1)S} > 0,$$

and since $\psi_0 \le 0$, we obtain, at (t_0, x_0),

$$H \le C_{10} - A\left[\varphi_t - \left(1 - \frac{bt}{T}\right)\psi_0\right] + A\eta\psi_0 + A\kappa/2\psi^- \\ + (t-\varepsilon)(n-1)\log\left(C_9 - \frac{b(A+1)\psi_0}{S} - (A+1)\psi^-\right).$$

The second term is uniformly bounded from above by Theorem 3.4. Since $-\gamma y + \log y$ is bounded from above for $y > 0$, we conclude that H achieves a uniform bound at (t_0, x_0). This completes the proof. □

3.4. ***Estimates near the zero time.*** Recall that there exists a θ-psh function ρ with analytic singularities such that $\sup_X \rho = 0$ and

$$\theta + dd^c\rho \ge 3\delta_0\omega_X$$

for some $\delta_0 > 0$. The main result of Tosatti and Weinkove [2010] ensures that there exists a constant c_1 and $\phi_1 \in \mathrm{PSH}(X, \theta) \cap \mathcal{C}^\infty(X)$ such that

$$(\theta + dd^c\phi_1)^n = e^{c_1} d\mu, \quad \sup_X \phi_1 = 0.$$

Proposition 3.9. *Assume that ψ_1, ψ_2 are two smooth ω_X-psh functions satisfying*

$$\dot\varphi_0 \ge C_1\psi_1, \quad \varphi_0 \ge \tfrac{1}{2}(\rho + \delta_0\psi_2)$$

for some constants $C_1 > 0$. Fix $T_1 \in (0, T_{\max})$ such that $\theta_t > \frac{1}{2}\theta$ for all $t \in [0, T_1]$. Then there exists a uniform constant $C_2 > 0$ only depending on C_1, δ_0, T_1 and $\sup_X |\phi_1|$ such that

$$\dot\varphi_t \ge C_2(\rho + \delta_0\psi_2 + 1) + C_1\psi_1 \quad \textit{for all } t \in [0, T_1].$$

Proof. The proof is identical to that of Proposition 3.7. We consider

$$H(t,x) := \dot\varphi_t - C_1\psi_1 + A\big(\varphi_t - \tfrac{1}{2}(\rho + \delta_0\psi_2)\big) - \phi_1$$

for $A > 0$ to be chosen later. We observe that H achieves its minimum at some point $(t_0, x_0) \in [0, T_1] \times X$. If $t_0 = 0$, we are done by assumptions. Otherwise, by the minimum principle, we have at (t_0, x_0),

$$0 \ge \left(\frac{\partial}{\partial t} - \Delta_t\right)H \ge -An + A\dot\varphi_t + (-C_1 + A\delta_0)\,\mathrm{tr}_{\omega_t}(\omega_X) + \mathrm{tr}_{\omega_t}(dd^c\phi_1)$$

using $\theta_t + dd^c\frac{1}{2}(\rho + \delta_0\psi_2) \ge \delta_0\omega_X$. Now, we choose $A = \delta_0(C_1 + 1)$, thus

$$\mathrm{tr}_{\omega_t}(\omega_X + dd^c\phi_1) \ge n\left(\frac{(\theta + dd^c\phi_1)^n}{\omega_t^n}\right)^{1/n} = ne^{(-\dot\varphi_t + c_1)/n}$$

using Lemma 3.6. Together with the inequality $e^y \ge By - C_B$, we obtain a uniform lower bound for $\dot\varphi_t$ at (t_0, x_0). On the other hand, by Proposition 3.2 we see that $\varphi_t \ge \varphi_0 - c(t)$, so

$$\varphi_t \ge \tfrac{1}{2}(\rho + \delta_0\psi_2) - c(t),$$

where $c(t) \to 0$ as $t \to 0$. The lower bound for $H(t_0, x_0)$ thus follows, finishing the proof. □

Proposition 3.10. *Assume that ψ_1, ψ_2 are two smooth ω_X-psh functions satisfying*

$$\Delta_{\omega_X}\varphi_0 \le e^{-C_1\psi_1}, \quad \varphi_0 \ge \tfrac{1}{2}(\rho+\delta_0\psi_2)$$

for some constants $C_1>0$. Fix $T_1\in(0,T_{\max})$ such that $\theta_t > \frac{1}{2}\theta$ for all $t\in[0,T_1]$. Then there exist uniform constants $C_2>0,\ C_3>0$ only depending on C_1, δ_0 and T_1 such that

$$\mathrm{tr}_{\omega_X}(\omega_t) \le C_3 e^{-C_1\psi_1 - C_2(\rho+\delta_0\psi_2+\delta_0\psi^-)} \quad \textit{for all } t\in[0,T_1].$$

Proof. Consider the function

$$H(t,\cdot) = \log \mathrm{tr}_{\omega_X}(\omega_t) + C_1\psi_1 - \gamma(u),$$

where $\gamma:\mathbb{R}\to\mathbb{R}$ is a smooth concave increasing function such that $\lim_{t\to+\infty}\gamma(t)=+\infty$ and

$$u(t,x) := \varphi_t(x) - \tfrac{1}{2}(\rho(x)+\delta_0\psi_2(x)) + \delta_0\psi^-(x) + 1.$$

We suppose that H achieves its maximum at a point $(t_0,x_0)\in[0,T_1]\times X$, with $x_0\in\{\rho>-\infty\}$. If $t_0=0$, then $H(0,\cdot)\le \log n - \gamma(1)$. Otherwise, assume $t_0>0$. We proceed by computing at this point. By the maximum principle and the arguments in Theorem 3.8, we have

$$\begin{aligned} 0 \le \Big(\frac{\partial}{\partial t} - \Delta_t\Big)H \le \frac{C\,\mathrm{tr}_{\omega_t}(\omega_X+dd^c\psi^-)}{\mathrm{tr}_{\omega_X}(\omega_t)} &- \gamma'(u)(n-\delta_0\,\mathrm{tr}_{\omega_t}(\omega_X+dd^c\psi^-)) + C \\ &- C_1\,\mathrm{tr}_{\omega_t}(dd^c\psi_1) - \gamma'(u)\dot{\varphi}_t + C\bigg(\frac{\gamma'(u)}{-\gamma''(u)}+2\bigg)\frac{\mathrm{tr}_{\omega_t}(\omega_X)}{(\mathrm{tr}_{\omega_X}(\omega_t))^2}. \end{aligned} \tag{3-15}$$

Here, we use $\theta_t + dd^c\frac{1}{2}(\rho+\delta_0\psi_2)\ge\delta_0\omega_X$. We set

$$\gamma(u) := \frac{C+C_1+3}{\min(\kappa,1)}u + \ln(u).$$

We then proceed in the same way as in the proof of Theorem 3.8 to obtain the uniform upper bound for $H(t_0,x_0)$. This finishes the proof. ☐

4. Degenerate Monge–Ampère flows

4.1. ***Proof of Theorem B.*** By Demailly's regularization theorem (Theorem 2.10), we can find two sequences $\psi_j^\pm\in\mathcal{C}^\infty(X)$ such that

- $\psi_j^\pm$ decreases pointwise to $\psi^\pm$ on X and the convergence is in $\mathcal{C}^\infty_{\mathrm{loc}}(U)$;
- $dd^c\psi^\pm \ge -\omega_X$.

We note that $|\sup_X\psi_j^\pm|$ is uniformly bounded, and for all j,

$$\|e^{-\psi_j^-}\|_{L^p} \le \|e^{-\psi^-}\|_{L^p}.$$

Thanks to Demailly's regularization theorem again, we can find a smooth sequence $(\varphi_{0,j})$ of strictly $(\theta+2^{-j}\omega_X)$-psh functions decreasing towards φ_0. We set $\theta_{t,j}=\theta_t+2^{-j}\omega_X$ and $\mu_j = e^{\psi_j^+-\psi_j^-}$. It follows from [Tosatti and Weinkove 2015, Theorem 1.2] (see also [Tô 2018]) that there exists a unique function $\varphi_j \in \mathcal{C}^\infty([0,T) \times X)$ such that

$$\begin{cases} \dfrac{\partial \varphi_{t,j}}{\partial t} = \log\left[\dfrac{(\theta_{t,j} + dd^c \varphi_{t,j})^n}{\mu_j}\right], \\ \varphi_j|_{t=0} = \varphi_{0,j}. \end{cases} \tag{4-1}$$

It follows from the maximum principle that the sequence $\varphi_{t,j}$ decreases with respect to j. Moreover, Proposition 3.1 ensures that $\sup_X \varphi_{t,j}$ is uniformly bounded from above. By Proposition 3.2, as $j \to +\infty$, the family $\varphi_{t,j}$ decreases to φ_t, which is a well-defined θ_t-psh function on X. Following the same arguments as in [Tô 2018, Section 4.1], we conclude that $\varphi_t \to \varphi_0$ in $L^1(X)$ as $t \to 0^+$.

Next, we study the partial regularity of φ_t for small t. We fix $\varepsilon_0 > 0$ and $\varepsilon > p^*\varepsilon_0$. Let T and S be as defined in Section 3.1. Let ρ be a θ-psh function with analytic singularities along D such that $\theta + dd^c\rho$ dominates a Hermitian form, where $D := \{\rho = -\infty\}$. By Lemma 2.11, there is a function $\psi_0 \in \mathrm{PSH}(X,\theta) \cap \mathcal{C}^\infty\big(X \setminus (D \cup E_c(\varphi_0))\big)$, where $c = c(\varepsilon_0) > 0$, such that

$$\int_X e^{2(\psi_0 - \varphi_0)/\varepsilon_0}\, dV_X < +\infty.$$

We assume without loss of generality that $\psi_0 \le 0$. Since $\dfrac{p^*}{2c(\varphi_0)} < T$ and ψ_0 is less singular than φ_0, we also have

$$\int_X e^{-p^*\psi_0/T}\, dV_X < +\infty.$$

We note that since φ_0 is a decreasing limit of a smooth sequence $\varphi_{0,j}$, the corresponding constants for $\varphi_{0,j}$ are uniformly bounded (in j), and we can pass to the limit as $j \to +\infty$.

Recall that $\psi^\pm$ are smooth (merely locally bounded) in a Zariski open set $U \subset X \backslash D$. We will show that φ_t is smooth on $U \setminus E_c(\varphi_0)$ for each $t > \varepsilon$. Let K be an arbitrarily compact subset of $U \setminus E_c(\varphi_0)$. It follows from Proposition 3.1, Theorem 3.4, and the remark above that

$$\sup_{[\varepsilon,T]\times K} |\varphi_j| \le C(\varepsilon, T, K).$$

Next, Proposition 3.7 yields

$$\sup_{[\varepsilon,T]\times K} |\dot{\varphi}_j| \le C(\varepsilon, T, K).$$

Moreover, by Theorem 3.8, we also obtain a uniform bound for $\Delta\varphi_t^j$:

$$\sup_{[\varepsilon,T]\times K} |\Delta\varphi_j| \le C(\varepsilon, T, K).$$

Using the complex parabolic Evans–Krylov–Trudinger theory, together with parabolic Schauder estimates (see, e.g., [Boucksom and Guedj 2013, Theorem 4.1.4]), we derive higher-order estimates for φ_j on $[\varepsilon, T] \times K$:

$$\|\varphi_j\|_{\mathcal{C}^k([\varepsilon,T]\times K)} \le C(\varepsilon, T, K, k).$$

This ensures that φ_j is relatively compact in $\mathcal{C}^\infty([\varepsilon, T] \times (U \setminus E_c(\varphi_0))$ since K was taken arbitrarily. By passing to the limit in (4-1), we deduce that φ satisfies (1-4) in the classical sense on $[\varepsilon, T] \times \Omega_\varepsilon$ with $\Omega_\varepsilon = U \setminus E_{c(\varepsilon)}(\varphi_0)$.

4.2. ***Uniqueness.*** We now follow the argument in [Guedj and Zeriahi 2017b] to prove that the solution φ to equation (1-4) constructed in the previous part is the unique maximal solution in the following sense:

Proposition 4.1. *Let ψ_t be a weak solution to equation* (1-4) *with initial data φ_0. Then $\psi_t \le \varphi_t$ for all $t \in (0, T_{\max})$.*

Proof. By construction in the previous paragraph, $\varphi_{t,j}$ are smooth $(\theta_t + 2^{-j}\omega_X)$-psh functions decreasing pointwise to φ_t. It thus suffices to show that $\psi_t \le \varphi_{t,j}$ for all fixed j.

Fix $0 < T < T_{\max}$ and $2^{-j} > \varepsilon > \delta > 0$. We let $U_\varepsilon \subset X$ denote the Zariski open set in which $\psi_{t+\varepsilon}$ is smooth. We can find a ω_X-psh function ϕ with analytic singularities along $X \setminus U_\varepsilon$; see, e.g., [Demailly and Paun 2004]. We apply the maximum principle to the function $H := \psi_{t+\varepsilon} - \varphi_{t+\varepsilon,j} + \delta\phi$. Suppose that H achieves its maximum on $[0, T - \varepsilon] \times X$ at $(t_\varepsilon, x_\varepsilon)$ with $t_\varepsilon > 0$. Note that $x_\varepsilon \in U_\varepsilon$. We thus have

$$0 \le \frac{\partial}{\partial t} H \le \log\left[\frac{(\theta_{t+\varepsilon} + dd^c\varphi_{t+\varepsilon,j} - \delta dd^c\phi)^n}{(\theta_{t+\varepsilon} + 2^{-j}\omega_X + dd^c\varphi_{t+\varepsilon,j})^n}\right] < 0$$

using that $-dd^c\phi \le \omega_X$, which is a contradiction. Letting $\delta \searrow 0$, we obtain

$$\psi_{t+\varepsilon}(x) - \varphi_{t+\varepsilon,j}(x) \le \sup_X(\psi_\varepsilon - \varphi_{\varepsilon,j}).$$

Moreover, since $(\varepsilon, x) \mapsto \varphi_{\varepsilon,j}(x)$ is continuous, it follows from Hartogs' lemma (see [Guedj and Zeriahi 2017a, Proposition 8.4]) that

$$\sup_X(\psi_\varepsilon - \varphi_{\varepsilon,j}) \xrightarrow{\varepsilon\to 0} \sup_X(\varphi_0 - \varphi_{0,j}) \le 0.$$

Letting $\varepsilon \to 0$, the desired inequality follows. □

The uniqueness we have just shown is referred to as "maximally stretched" by P. Topping [2010, Remark 1.9] in the context of Riemann surfaces.

4.3. ***Short time behavior.*** In this subsection, we study the behavior of the solution to the degenerate Monge–Ampère flow in a short time. We show that the flow φ_t starting from a current with positive Lelong numbers also has positive Lelong numbers for a sufficiently short time. This result follows almost verbatim from the Kähler case, as discussed in [Di Nezza and Lu 2017, Section 4.2].

Theorem 4.2. *If φ_0 has positive Lelong numbers, then*

$$E_c(\varphi_0) \subset E_{c(t)}(\varphi_t), \quad c(t) = c - 2nt.$$

In particular, the maximal solution φ_t has positive Lelong numbers for any $t < \dfrac{1}{2nc(\varphi_0)}$.

Proof. The proof is identical to that of [Di Nezza and Lu 2017, Theorem 4.5]. We give a sketch of the proof here. Fix $x_0 \in E_c(\varphi_0)$. We can find a cutoff function $\chi \in \mathcal{C}^\infty(X)$ with support near x_0 and $\chi = 1$

on a neighborhood of x_0. Define $\phi := \chi(x) c \log \|x - x_0\|$, which is $B\omega_X$-psh, and $e^{2\phi/c} \in \mathcal{C}^\infty(X)$. Since $x_0 \in E_c(\varphi_0)$ we can choose ϕ so that $\phi \geq \varphi_0$ by adding a positive constant. By Lemma 4.3, we obtain

$$\varphi_t \leq (1 - 2nt/c)\phi + Ct,$$

which implies $\nu(\varphi_t, x_0) \geq c - 2nt$. If $t < 1/(2nc(\varphi_0))$, then by Skoda's integrability theorem, $e^{-2\varphi_0/c}$ is not integrable for $2nt < c < 1/c(\varphi_0)$. Therefore, $E_c(\varphi_0)$ is not empty, neither is $E_{c(t)}(\varphi_t)$ for sufficiently small $t > 0$. □

Lemma 4.3. *Assume that $\phi \in \mathrm{PSH}(X, \omega_X)$ satisfies $e^{\gamma\phi} \in \mathcal{C}^\infty(X)$ for some constant $\gamma > 0$, and $0 \geq \psi^\pm \geq \phi \geq \varphi_0$. Then, there exists a positive constant C depending on an upper bound for $dd^c e^{\gamma\phi}$ such that*

$$\varphi(t) \leq (1 - (n\gamma + 1)t)\phi + Ct \quad \textit{for all } t \in [0, 1/n\gamma].$$

Proof. Assume that $\theta_t \leq \omega_X$ for $t \in [0, 1/(n\gamma + 1)]$. As argued in [Di Nezza and Lu 2017, Lemma 4.4], we can assume that ϕ is smooth and work with the approximants $\varphi_{t,j}$ instead. We choose $C > 0$ depending only on an upper bound for $dd^c e^{\gamma\phi}$, such that $dd^c \phi \leq C e^{-\gamma\phi}\omega_X$. Consider the function

$$\phi_t := (1 - (n\gamma + 1)t)\phi + t \log(2^n C^n).$$

We observe that

$$0 \leq \omega_X + dd^c\phi \leq 2Ce^{-\gamma\phi}\omega_X,$$

and hence

$$(\omega_X + dd^c\phi_t)^n \leq (2C)^n e^{-n\gamma\phi}\omega_X^n \leq e^{\dot{\phi}_t + \psi^+ - \psi^-}\omega_X^n.$$

Therefore, ϕ_t is a supersolution to the parabolic equation

$$(\omega_X + dd^c u_t)^n = e^{\dot{u}_t + \psi^+ - \psi^-}\omega_X^n,$$

while $\varphi_{t,j}$ is a subsolution. By the classical maximum principle, it follows that $\varphi_{t,j} \leq \phi_t$ for any fixed j. This completes the proof. □

4.4. ***Convergence at time zero.*** We study in this part the convergence at zero of the degenerate complex Monge–Ampère flow.

We recall the quasimonotone convergence in the sense of Guedj and Trusiani [2023]: φ_j converges quasimonotonically to φ if $P_\theta(\inf_{\ell \geq j} \varphi_\ell)$ is a sequence of θ-psh functions that increases to φ.

Theorem 4.4. *The flow φ_t converges quasimonotonically to φ_0 as $t \to 0^+$.*

Proof. By Proposition 3.2, we have that for small $t > 0$,

$$\varphi_t \geq \varphi_0 - C(t - t\log t).$$

It follows that

$$P_\theta\Big(\inf_{0<s\leq t} \varphi_s\Big) \geq \varphi_0 - C(t - t\log t),$$

which completes the proof. □

Theorem 4.5. *Assume that φ_0 is continuous in an open set $U \subset X$. Then φ_t converges to φ_0 in $L^\infty_{\mathrm{loc}}(U)$.*

Proof. The proof closely follows the arguments in the Kähler case [Di Nezza and Lu 2017]. Without loss of generality, we assume that $\varphi_t \le 0$. By Proposition 3.2, there exists a uniform constant $C > 0$ and a small time t_0 such that for $0 \le s < t \le t_0$,

$$\varphi_s - C(t-s)\log(t-s) - C(t-s) \le \varphi_t.$$

Set $u_t := \varphi_t + (C + \log 4)t - Ct\log t$. Substituting $s = t/2$, we deduce that $u_t \ge u_{t/2}$, hence the sequence $u_{t_0 2^{-j}}$ decreases to $u_0 = \varphi_0$. The conclusion follows from Dini's theorem. □

We also have the same result as in the Kähler case [Di Nezza and Lu 2017, Theorem 6.3]. We assume that θ is a big form and that $f = e^{\psi^+ - \psi^-} \in L^p$, for some $p > 1$, where $\psi^\pm$ are quasi-psh functions. Assume moreover that $\psi^- \in L^\infty_{\mathrm{loc}}(X \setminus D)$ for some closed set $D \subset X$. It follows from [Guedj and Lu 2023, Theorem 4.1] that there exists a bounded θ-psh function φ_0 such that $\sup_X \varphi_0 = 0$ and

$$(\theta + dd^c\varphi_0)^n = cf\,dV.$$

We recall that there is $\rho \in \mathrm{PSH}(X,\theta)$ with analytic singularities along a closed subset E such that $\theta + dd^c\rho \ge 2\delta\omega_X$ for some $\delta > 0$. Set $U := X \setminus (D \cup E)$.

Theorem 4.6. *Assume φ_0 is as above. Let φ_t be the weak solution of equation* (1-4) *with the initial data φ_0. Then φ_t converges to φ_0 in $\mathcal{C}^\infty_{\mathrm{loc}}(U)$.*

Proof. The proof is quite close to [Di Nezza and Lu 2017, Theorem 6.3]. We sketch the key steps for the reader's convenience. First, we approximate $\psi^\pm$ by their smooth approximants $\psi_j^\pm$, thanks to [Demailly 1992]. We then apply the Tosatti–Weinkove theorem [2010] to obtain smooth $(\theta + 2^{-j}\omega_X)$-psh functions $\varphi_{0,j}$ such that $\sup_X \varphi_{0,j} = 0$ and

$$(\theta + 2^{-j}\omega_X + dd^c\varphi_{0,j})^n = c_j e^{\psi_j^+ - \psi_j^-}\,dV.$$

Note here that the $f_j = e^{\psi_j^+ - \psi_j^-}$ have uniform L^p-norms. The same arguments as in [Guedj and Lu 2023, Theorem 4.2] show that

- $c_j \to c > 0$;
- for any $\varepsilon > 0$, $\varphi_{0,j} \ge \varepsilon(\rho + \delta\psi^-) - C(\varepsilon)$;
- $\Delta_{\omega_X}\varphi_{0,j} \le e^{-C(\varepsilon)(\rho+\delta\psi^-)}$.

Let $\varphi_{t,j}$ be a smooth solution to equation (1-4) with initial data $\varphi_{0,j}$. The sequence $\varphi_{t,j}$ converges to the unique weak solution φ_t. We apply Propositions 3.9 and 3.10, together with bootstrapping arguments, to obtain locally uniform estimates for all derivatives of $\varphi_{t,j}$. This leads to convergence in $\mathcal{C}^\infty_{\mathrm{loc}}(U)$. □

5. Finite time singularities

In this section, we study the finite time singularities of the Chern–Ricci flow and provide a proof of Theorem A.

We consider a family of Hermitian metrics $\omega(t)$ evolving under the Chern–Ricci flow (1-1) with the initial Hermitian metric ω_0. Suppose that the maximal existence time of the flow is finite, i.e., $T_{\max} < \infty$.

The form $\alpha_{T_{\max}} := \omega_0 - T_{\max}\operatorname{Ric}(\omega_0)$ is nef in the sense of [Guedj and Lu 2022]; i.e., for each $\varepsilon > 0$ there exists $\psi_\varepsilon \in \mathcal{C}^\infty(X)$ such that $\alpha_{T_{\max}} + dd^c\psi_\varepsilon \geq -\varepsilon\omega_0$. Indeed, for $\varepsilon > 0$,

$$\alpha_{T_{\max}} + \varepsilon\omega_0 = (1+\varepsilon)\left(\omega_0 - \frac{T_{\max}}{1+\varepsilon}\operatorname{Ric}(\omega_0)\right),$$

and since $T_{\max}/(1+\varepsilon) < T_{\max}$, we have $\omega_0 - T_{\max}/(1+\varepsilon)\operatorname{Ric}(\omega_0) + dd^c\psi > 0$ for some smooth function ψ. We assume that $\alpha_{T_{\max}}$ is *uniformly noncollapsing*, i.e.,

$$\int_X (\alpha_{T_{\max}} + dd^c\psi)^n \geq c_0 > 0 \quad \text{for all } \psi \in \operatorname{PSH}(X, \alpha_{T_{\max}}) \cap \mathcal{C}^\infty(X). \tag{5-1}$$

This condition implies that the volume of $(X, \omega(t))$ does not collapse to zero as $t \to T_{\max}^-$.

Theorem 5.1. *Let α be a nef* (1,1)*-form satisfying the uniformly noncollapsing condition* (5-1). *If X admits a Hermitian metric ω_X such that $v_+(\omega_X) < +\infty$ then α is big.*

Conversely, if α is big and $v_-(\omega_X) > 0$ then α is uniformly noncollapsing.

When α is semipositive or closed the result was proved by Guedj and Lu [2022, Theorems 4.6, 4.9], answering the transcendental Grauert–Riemenschneider conjecture [Demailly and Paun 2004, Conjecture 0.8]. For our purposes, we would like to extend it when α is no longer closed.

Proof. The proof of this theorem follows the same lines as in [Guedj and Lu 2022, Theorem 4.6], which is based on ideas from Chiose [2016], so we omit it here. □

Remark 5.2. When ω_0 is closed, or more generally, is a Guan–Li metric, i.e., $dd^c\omega_0 = dd^c\omega_0^2 = 0$, the condition (5-1) is simply written as $\int_X \alpha_{T_{\max}}^n > 0$. The assumption $v_+(\omega_X) < \infty$ or $v_-(\omega_X) > 0$ is independent of the choice of the Hermitian ω_X, as shown in [Guedj and Lu 2022, Proposition 3.2]. For additional examples of manifolds where such conditions hold, we refer the reader to [Angella et al. 2023].

This result is a slight generalization of [Nguyen 2016, Theorem 4.3], where α is closed semipositive, and X admits a pluriclosed metric, i.e., $dd^c\omega_X = 0$.

As a consequence of Theorem 5.1, we give a slight improvement of the main result in [Tosatti and Weinkove 2012] (see also [Nguyen 2016, Theorem 4.1]) which extends the one of Demailly [1993] to the non-Kähler setting.

Theorem 5.3. *Let X be a compact complex n-manifold equipped with a Hermitian metric ω_X satisfying $v_+(\omega_X) < \infty$. Let α be a nef* (1,1)*-form. Assume that $x_1, \dots, x_N \in X$ are fixed points and $\tau_1, \dots, \tau_N$ are positive constants such that*

$$0 < \sum_{j=1}^N \tau_j^n < \int_X (\alpha + dd^c\psi)^n \quad \textit{for all } \psi \in \operatorname{PSH}(X, \alpha) \cap \mathcal{C}^\infty(X).$$

Then, there exists an α-psh function φ with logarithmic poles at $x_1, \dots, x_N \in X$,

$$\varphi(z - x_j) \leq \tau_j \log\|z - x_j\| + O(1)$$

in local coordinates near x_j, for all $j = 1, \dots, N$.

Proof. By Theorem 5.1, we know that α is big. The rest of the proof follows in the same manner as in [Tosatti 2016, Theorem 1.3]. □

We go back to the Chern–Ricci flow. Observe that one can deduce the Chern–Ricci flow (1-1) to a parabolic complex Monge–Ampère equation

$$\frac{\partial \varphi_t}{\partial t} = \log\left[\frac{(\alpha_t + dd^c\varphi_t)^n}{\omega_0^n}\right], \quad \alpha_t + dd^c\varphi > 0, \quad \varphi(0) = 0,$$

where $\alpha_t := \omega_0 - t\,\mathrm{Ric}(\omega_0)$. We assume that the form $\alpha_{T_{\max}}$ is uniformly noncollapsing. By Theorem 5.1, there exists a function ρ with analytic singularities such that

$$\alpha_{T_{\max}} + dd^c\rho \geq 2\delta_0\omega_0$$

for some $\delta_0 > 0$. We observe that

$$\alpha_t + dd^c\rho = \frac{1}{T_{\max}}\big((T_{\max} - t)(\omega_0 + dd^c\rho) + t(\alpha_{T_{\max}} + dd^c\rho)\big) \geq \delta_0\omega_0 \tag{5-2}$$

for $t \in [T_{\max} - \varepsilon, T_{\max}]$ with sufficiently small $\varepsilon > 0$. Set

$$\Omega := X \setminus \{\rho = -\infty\}.$$

We establish uniform $\mathcal{C}^\infty_{\mathrm{loc}}$ estimates on Ω.

Lemma 5.4. *There is a uniform constant $C_0 > 0$ such that on $[0, T_{\max}) \times X$ we have*

(i) $\varphi \leq C_0$;

(ii) $\dot{\varphi} \leq C_0$;

(iii) $\varphi \geq \rho - C_0$;

(iv) $\dot{\varphi} \geq C_0\rho - C_0$.

Proof. The proofs of (i) and (ii) follow directly from the classical maximum principle; see, e.g., [Tosatti and Weinkove 2015, Lemma 4.1] or [Tian and Zhang 2006].

For (iii), we set $\psi := \varphi - \rho$. Note that the function $\psi + At \geq -C$ holds on $[0, T_{\max} - \varepsilon]$ with ε as above. Fix $T_{\max} - \varepsilon < T' < T_{\max}$, and assume that $\psi + At$ achieves its minimum at $(t_0, x_0) \in [0, T'] \times X$ with $t_0 \in (T_{\max} - \varepsilon, T']$. Note that $x_0 \in \Omega$. We compute at (t_0, x_0),

$$0 \geq \frac{\partial\psi}{\partial t} + A = \log\frac{(\alpha_t + dd^c\rho + dd^c\psi)^n}{\omega_0^n} + A \geq \log\frac{(\delta_0\omega_0)^n}{\omega_0^n} + A \geq -C + A,$$

where we have used the estimate (5-2). If we choose $A > C$, then we get a contradiction. Thus, we obtain the lower bound for ψ, completing the proof.

For (iv), we apply the minimum principle to

$$Q = \dot{\varphi} + A\psi + Bt,$$

where A and B are large constants that will be chosen later. Our goal is to show that $Q \geq -C$ on $X \times [0, T_{\max})$. As above, we observe that $Q \geq -C$ on $[0, T_{\max} - \varepsilon] \times X$. Given any $T_{\max} - \varepsilon < T' < T_{\max}$, suppose that Q achieves its minimum on $[0, T'] \times X$ at some point (t_0, x_0) with $t_0 \in (T_{\max} - \varepsilon, T']$. Note that $x_0 \in \Omega$. At this point, we have

$$0 \geq \Big(\frac{\partial}{\partial t} - \Delta_\omega\Big) Q = -\mathrm{tr}_\omega \operatorname{Ric}(\omega_0) + A\dot{\varphi} - An + A\,\mathrm{tr}_\omega(\alpha_t + dd^c\rho) + B$$
$$\geq \delta_0\, \mathrm{tr}_\omega\, \omega_0 + A \log \frac{\omega^n}{\omega_0^n} + \mathrm{tr}_\omega\, \omega_0 - An + B,$$

where A is chosen so large that

$$(A-1)(\alpha_t + dd^c\rho) - \operatorname{Ric}(\omega_0) \geq \omega_0$$

for $t \in [T_{\max} - \varepsilon, T_{\max}]$. But since $A \log y - \delta_0 y^{1/n}$ is bounded from above for $y > 0$, the arithmetic-geometric inequality yields

$$\delta_0\, \mathrm{tr}_\omega\, \omega_0 + A \log \frac{\omega^n}{\omega_0^n} \geq \delta_0 \bigg(\frac{\omega_0^n}{\omega^n}\bigg)^{1/n} + A \log \frac{\omega^n}{\omega_0^n} \geq -C_1$$

for a uniform constant $C_1 > 0$. If we choose $B = C_1 + An$, we obtain

$$0 \geq \Big(\frac{\partial}{\partial t} - \Delta_\omega\Big) Q \geq \mathrm{tr}_\omega\, \omega_0 > 0$$

which leads to a contradiction. Thus, the desired estimate follows. □

Lemma 5.5. *There is a uniform constant $C > 0$ such that on* $[0, T_{\max}) \times X$ *we have*

$$\mathrm{tr}_{\omega_0}\, \omega(t) \leq C e^{-C\rho}.$$

Proof. Set $\psi = \varphi - \rho + C_0 \geq 0$. We apply the maximum principle to

$$Q = \log \mathrm{tr}_{\omega_0}\, \omega - A\psi + e^{-\psi},$$

where $A > 0$ will be determined later. The idea of using the last term in Q is due to Phong and Sturm [2010] and was used in the context of the Chern–Ricci flow in [Tosatti and Weinkove 2013; 2015; Tô 2018]. Note that $e^{-\psi} \in (0, 1]$.

It suffices to show that Q is uniformly bounded from above. Again, it follows from the definition of Q that $Q \leq C$ on $[0, T_{\max} - \varepsilon] \times X$ for a uniform $C > 0$. Fix $T_{\max} - \varepsilon < T' < T_{\max}$, and suppose that Q achieves its maximum at some point $(t_0, x_0) \in [0, T'] \times X$ with $t \in (T_{\max} - \varepsilon, T']$. In what follows, we compute at this point. From [Tosatti and Weinkove 2015, Proposition 3.1, also (4.2)] we have

$$\Big(\frac{\partial}{\partial t} - \Delta_\omega\Big) \log \mathrm{tr}_{\omega_0}\, \omega \leq \frac{2}{(\mathrm{tr}_{\omega_0}\, \omega)^2} \operatorname{Re}(g^{\bar{q}k} (T_0)_{kp}^p \partial_{\bar{q}}\, \mathrm{tr}_{\omega_0}\, \omega) + C\, \mathrm{tr}_\omega\, \omega_0,$$

where $(T_0)_{kp}^p$ denotes the torsion terms corresponding to ω_0. At the maximum point (x_0, t_0) of Q, we have $\partial_i Q = 0$; hence

$$\frac{1}{\mathrm{tr}_{\omega_0}\, \omega} \partial_i\, \mathrm{tr}_{\omega_0}\, \omega - A\partial_i \psi - e^{-\psi} \partial_i \psi = 0.$$

Thus, the Cauchy–Schwarz inequality yields

$$\begin{aligned}\left|\frac{2}{(\operatorname{tr}_{\omega_0}\omega)^2}\operatorname{Re}(g^{\bar{q}k}(T_0)_{kp}^{p}\partial_{\bar{q}}\operatorname{tr}_{\omega_0}\omega)\right| &\le \left|\frac{2}{(\operatorname{tr}_{\omega_0}\omega)^2}\operatorname{Re}\big((A+e^{-\psi})g^{\bar{q}k}(T_0)_{kp}^{p}\partial_{\bar{q}}\psi\big)\right| \\ &\le e^{-\psi}|\partial\psi|^2_{\omega}+C(A+1)^2e^{\psi}\frac{\operatorname{tr}_{\omega}\omega_0}{(\operatorname{tr}_{\omega_0}\omega)^2}\end{aligned}$$

for uniform $C > 0$ only depending on the torsion term. It thus follows that, at the point (t_0, x_0),

$$\begin{aligned}0 \le \Big(\frac{\partial}{\partial t}-\Delta_{\omega}\Big)Q &\le C(A+1)^2e^{\psi}\frac{\operatorname{tr}_{\omega}\omega_0}{(\operatorname{tr}_{\omega_0}\omega)^2}+C\operatorname{tr}_{\omega}\omega_0-(A+e^{-\psi})\dot{\varphi}+(A+e^{-\psi})\operatorname{tr}_{\omega}(\omega-(\alpha_t+dd^c\rho)) \\ &\le C(A+1)^2e^{\psi}\frac{\operatorname{tr}_{\omega}\omega_0}{(\operatorname{tr}_{\omega_0}\omega)^2}+(C-A\delta_0)\operatorname{tr}_{\omega}\omega_0+(A+1)\log\frac{\omega_0^n}{\omega^n},\end{aligned} \tag{5-3}$$

where we have used $\alpha_t + dd^c\rho \ge \delta_0\omega_0$. If at (x_0, t_0), $(\operatorname{tr}_{\omega_0}\omega)^2 \le e^{\psi}C(A+1)^2$ then at the same point we obtain

$$Q \le C + \tfrac{1}{2}\psi - A\psi + e^{-\psi} \le C + 1.$$

Since $\psi \ge 0$, we are done. Otherwise, we choose $A = \delta_0^{-1}(C+2)$. Hence, from (5-3) one gets

$$\operatorname{tr}_{\omega}\omega_0 \le C\log\frac{\omega_0^n}{\omega^n} + C.$$

By Lemma 3.6, we obtain

$$\operatorname{tr}_{\omega_0}\omega \le n(\operatorname{tr}_{\omega}\omega_0)^{n-1}\frac{\omega^n}{\omega_0^n} \le C\frac{\omega^n}{\omega_0^n}\left(\log\frac{\omega_0^n}{\omega^n}\right)^{n-1} + C \le C'$$

since $\omega^n/\omega_0^n \le C_0$ by Lemma 5.4, and $y \mapsto y|\log y|^{n-1}$ is bounded from above as $y \to 0$. Thanks to Lemma 5.4(iii), Q is bounded from above at its maximum, finishing the proof. □

Proof of Theorem A. Let $K \subset \Omega$ be any compact set. It follows from Lemmas 5.4 and 5.5 that there exists a constant $C_K > 0$ such that on $[0, T_{\max}) \times K$,

$$C_K^{-1}\omega_0 \le \omega(t) \le C_K\omega_0.$$

Applying the local higher-order estimates of Gill [2011, Section 4], we obtain uniform $\mathcal{C}^{\infty}$ estimates for $\omega(t)$ on compact subsets of Ω. Consequently, there exists a constant c_K such that

$$\frac{\partial}{\partial t}\omega = -\operatorname{Ric}(\omega) \le c_K\omega \quad \text{on } [0, T_{\max}) \times K.$$

This implies that $e^{-c_K t}\omega(t)$ decreases in t and is bounded from below. Hence $\omega(t)$ converges to $\omega_{T_{\max}}$ as $t \to T_{\max}$, and since we have uniform estimates on compact subsets of Ω, we see that the convergence is in $\mathcal{C}^{\infty}_{\mathrm{loc}}(\Omega)$. This finishes the proof. □

6. The Chern–Ricci flow on varieties with log terminal singularities

In this section, we extend our previous analysis to the case of compact complex varieties with *mild singularities*. We refer the reader to [Eyssidieux et al. 2009, Section 5] for a brief introduction to the complex analysis on mildly singular varieties.

We assume here that Y is a $\mathbb{Q}$-Gorenstein variety, i.e., Y is a normal complex space such that its canonical divisor K_Y is $\mathbb{Q}$-Cartier. We denote the singular set of Y by Y_{sing} and let $Y_{\text{reg}} := Y \setminus Y_{\text{sing}}$. Given a log resolution of singularities $\pi : X \to Y$ (which may and will always be chosen to be an isomorphism over Y_{reg}), there exists a unique (exceptional) $\mathbb{Q}$-divisor $\sum a_i E_i$ with simple normal crossings (snc for short) such that

$$K_X = \pi^* K_Y + \sum_i a_i E_i.$$

The coefficients $a_i \in \mathbb{Q}$ are called the *discrepancies* of Y along E_i.

Definition 6.1. We say that Y has *log terminal* (*lt* for short) singularities if and only if $a_i > -1$ for all i.

The following definition of *adapted measure* is introduced in [Eyssidieux et al. 2009, Section 6]:

Definition 6.2. Let h be a smooth Hermitian metric on the $\mathbb{Q}$-line bundle $\mathcal{O}_Y(K_Y)$. The corresponding adapted measure $\mu_{Y,h}$ on Y_{reg} is locally defined by choosing a nowhere vanishing section σ of mK_Y over a small open set U and setting

$$\mu_{Y,h} := \frac{(i^{mn^2}\sigma \wedge \bar{\sigma})^{1/m}}{|\sigma|_{h^m}^{2/m}}.$$

The point of the definition is that the measure $\mu_{Y,h}$ does not depend on the choice of σ, so it is globally defined. The arguments above show that Y has log terminal singularities if and only if $\mu_{Y,h}$ has a finite total mass on Y, which can be considered as a Radon measure on the whole of Y. Then $\chi = dd^c \log \mu_{Y,h}$ is a well-defined smooth closed (1, 1)-form on Y.

Given a Hermitian form ω_Y on Y, there exists a unique Hermitian metric $h = h(\omega_Y)$ of K_Y such that

$$\omega_Y^n = \mu_{Y,h}.$$

We have the following definition.

Definition 6.3. The *Ricci curvature form* of ω_Y is $\operatorname{Ric}(\omega_Y) := -dd^c \log h$.

We recall the *slope* of a quasi-psh function ϕ at y in the sense of [Berman et al. 2019]. Choosing local generators (f_j) of the maximal ideal $\mathfrak{m}_y$ of $\mathcal{O}_{Y,y}$, we define

$$s(\phi, y) = \sup\left\{s \geq 0 : \varphi \leq s \log \sum |f_j| + O(1)\right\}.$$

Note that this definition is independent of the choice of (f_j). By [Berman et al. 2019, Theorem A.2] there is $C > 0$ such that for any log resolution $\pi : X \to Y$,

$$\nu(\phi \circ \pi, E) \leq C s(\phi, y),$$

with E a prime divisor lying above y. In particular, the Lelong numbers of $\phi\circ\pi$ are sufficiently small if the $s(\phi, y)$ is also sufficiently small at all points $y\in Y$. We refer to [Pan 2025] for related results.

Applying the analysis in the previous section, we obtain the existence of the Chern–Ricci flow on log terminal singularities. This generalizes the result in [Dang 2024, Theorem E].

Theorem 6.4. *Let Y be a compact complex variety with log terminal singularities. Assume that θ_0 is a Hermitian metric such that*

$$T_{\max} := \sup\{t>0 : \exists\psi\in\mathcal{C}^\infty(Y) \text{ such that } \theta_0 - t\operatorname{Ric}(\theta_0) + dd^c\psi > 0\} > 0.$$

Assume that $S_0 = \theta_0 + dd^c\phi_0$ is a positive (1,1)-current with small slopes. Then, there exists a family $(\omega_t)_{t\in[0,T_{\max})}$ of positive (1,1)-currents on Y starting with S_0 such that

(1) $\omega_t = \theta_0 - t\operatorname{Ric}(\theta_0) + dd^c\varphi_t$ *are positive (1,1)-currents*;

(2) $\omega_t \to S_0$ *weakly as* $t\to 0^+$;

(3) *for each $\varepsilon>0$, there exists a Zariski open set Ω_ε such that on $[\varepsilon, T_{\max})\times\Omega_\varepsilon$, ω is smooth and*

$$\frac{\partial\omega}{\partial t} = -\operatorname{Ric}(\omega).$$

Proof. It is classical that solving the (weak) Chern–Ricci flow is equivalent to solving a complex Monge–Ampère equation flow. Let χ be a closed smooth (1,1)-form that represents $c_1^{\mathrm{BC}}(K_Y)$. Given $T\in(0,T_{\max})$, there is a function $\psi_T\in\mathcal{C}^\infty(Y)$ such that $\theta_0 - t\operatorname{Ric}(\theta_0) + dd^c\psi_T > 0$. We set for $t\in[0,T]$,

$$\hat{\theta}_t := \theta_0 + t\chi, \quad \text{with } \chi = -\operatorname{Ric}(\theta_0) + dd^c\frac{\psi_T}{T},$$

which defines an affine path of Hermitian forms. Since χ is a smooth representative of $c_1^{\mathrm{BC}}(K_Y)$, one can find a smooth metric h on the $\mathbb{Q}$-line bundle $\mathcal{O}_Y(K_Y)$ with curvature form χ. We obtain $\mu_{Y,h}$, the adapted measure corresponding to h. The Chern–Ricci flow is equivalent to the following complex Monge–Ampère flow:

$$(\hat{\theta}_t + dd^c\phi_t)^n = e^{\partial_t\phi}\mu_{Y,h}. \tag{6-1}$$

Let $\pi: X\to Y$ be a log resolution of singularities. We have seen that the measure

$$\mu := \pi^*\mu_{Y,h} = f\,dV \quad \text{where } f = \prod_i |s_i|^{2a_i}$$

has poles (corresponding to $a_i<0$) or zeroes (corresponding to $a_i>0$) along the exceptional divisors $E_i = (s_i = 0)$, and dV is a smooth volume form. Passing to the resolution, the flow (6-1) becomes

$$\frac{\partial\varphi}{\partial t} = \log\left[\frac{(\theta_t + dd^c\varphi_t)^n}{\mu}\right], \tag{6-2}$$

where $\theta_t := \pi^*\hat{\theta}_t$ and $\varphi := \pi^*\phi$. Since $(\hat{\theta}_t)_{t\in[0,T]}$ is a smooth family of Hermitian forms, it follows that the family of semipositive forms $[0,T]\ni t\mapsto\theta_t$ satisfies all our requirements. We also have that $\theta := \pi^*\theta_0$, the latter is smooth, semipositive, and big but no longer Hermitian. We fix a θ-psh function ρ with analytic singularities along a divisor $E = \pi^{-1}(Y_{\mathrm{sing}})$ such that $\theta + dd^c\rho \geq 2\delta\omega_X$ with $\delta>0$. We observe that since the Lelong numbers $\nu(\varphi_0, x)$ are sufficiently small, we have the assumption $p^*/(2c(\varphi_0)) < T_{\max}$ by Skoda's integrability theorem. The result therefore follows from Theorem B. □

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