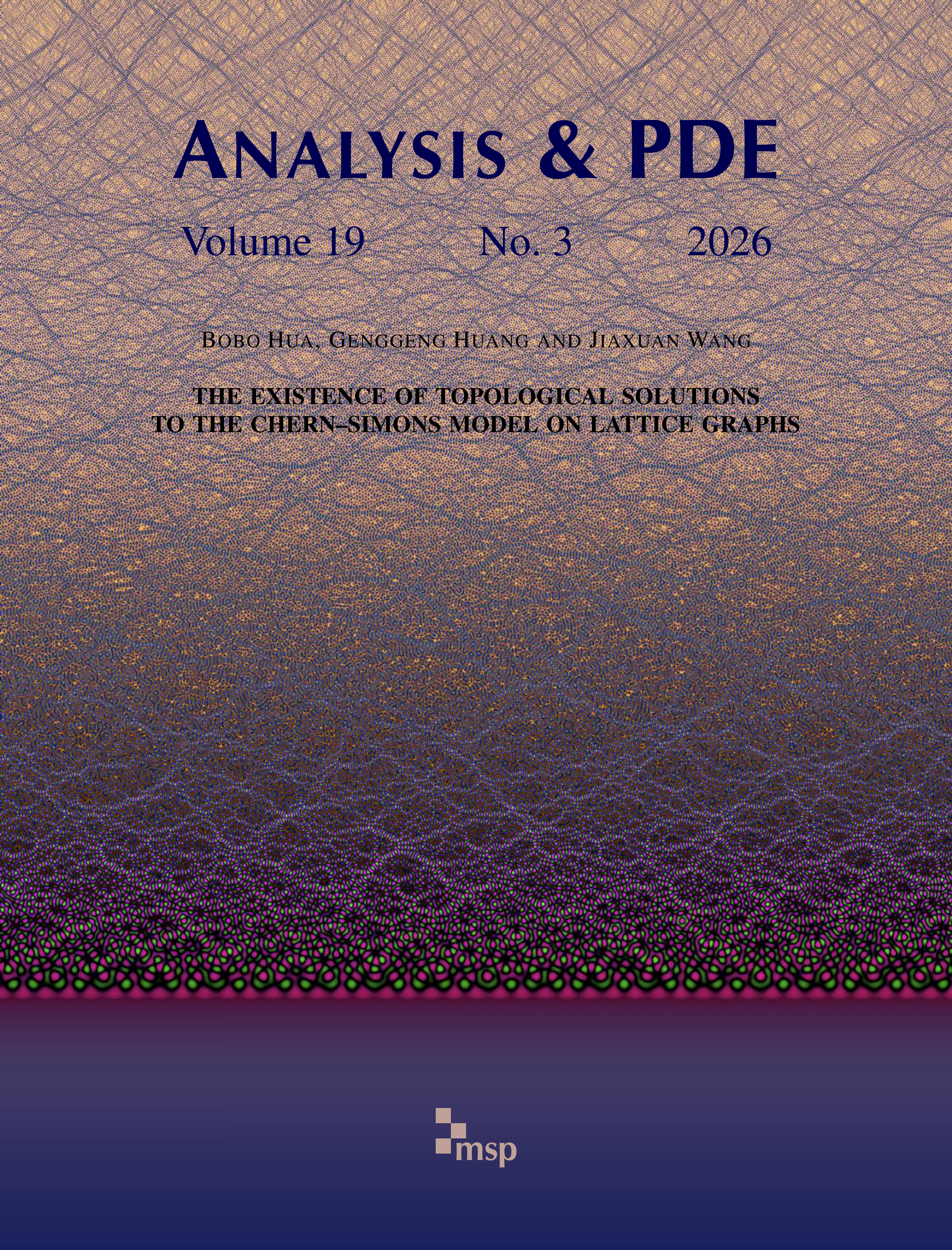

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THE EXISTENCE OF TOPOLOGICAL SOLUTIONS TO THE CHERN–SIMONS MODEL ON LATTICE GRAPHS

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We prove the existence of topological solutions to the self-dual Chern–Simons model and the abelian Higgs system on the lattice graphs $\mathbb{Z}^n$ for $n \geqslant 2$. This extends results of Huang, Lin and Yau (2020) from finite graphs to lattice graphs.

1. Introduction

Various vortex problems have been extensively studied in recent decades, which play important roles in quantum physics, solid state physics and so on. The existence of topological and nontopological solutions in these models has been rigorously proven in mathematics. In $\mathbb{R}^2$, we consider the self-dual Chern–Simons vortex equation

$$\Delta u = \lambda e^u(e^u - 1) + 4\pi \sum_{j=1}^{M} n_j \delta_{p_j} \tag{1}$$

and the abelian Higgs equation

$$\Delta u = \lambda(e^u - 1) + 4\pi \sum_{j=1}^{M} n_j \delta_{p_j} \tag{2}$$

with positive integers $n_1, \dots, n_M$ and distinct vortices $p_1, \dots, p_M \in \mathbb{R}^2$. Here $\lambda > 0$, and δ_{p_j} is the Dirac mass at p_j. A solution of (1) or (2) is called topological if $u(x) \to 0$ as $|x| \to +\infty$, and called nontopological if $u(x) \to -\infty$ as $|x| \to +\infty$.

For the abelian Higgs system (2), Jaffe and Taubes [1980] proved the existence and uniqueness of general finite energy multivortex solutions to the Bogomol’nyi equations, and there have been many studies on this model since then, such as [Jacobs and Rebbi 1979; Jaffe and Taubes 1980; Wang and Yang 1992]. The self-dual Chern–Simons system (1) is the minimal self-dual model containing the Chern–Simons term. The Chern–Simons vortices were discovered in [Jackiw and Weinberg 1990; Hong et al. 1990], which attracted people to investigate the existence problem. The existence of topological solutions in $\mathbb{R}^2$ was established in [Wang 1991; Spruck and Yang 1995] by the variational method and iteration argument, and the existence of self-dual doubly periodic vortex solutions was proved in [Caffarelli and Yang 1995]. Later, nontopological solutions were studied in the literature, e.g., [Chen et al. 1994; Chae and Imanuvilov 2000; Chan et al. 2002; Choe et al. 2011], and see [Dunne 1995; Han 2014; Struwe and Tarantello 1998; Chae and Kim 1997] for other related results.

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Recently, people have paid attention to the elliptic equations on graphs. Grigor'yan, Lin and Yang first studied nonlinear elliptic equations on graphs; see, e.g., [Grigor'yan et al. 2016b; 2017]. In a seminal paper [Huang et al. 2020], Huang, Lin and Yau proved the existence result for solutions to (1) on finite graphs. Furthermore, on a finite graph, the existence of solutions to the generalized self-dual Chern–Simons equation was proved in [Lü and Zhong 2021; Hou and Sun 2022], and the existence of solutions to the Chern–Simons Higgs model has been recently proved in [Li et al. 2024] using topological degree methods. See [Grigor'yan et al. 2016a; Huang et al. 2021; Chao and Hou 2023] for other related results. For infinite graphs, existence results for the Kazdan–Warner equation were proved in [Ge and Jiang 2018; Keller and Schwarz 2018] on graphs with positive spectrum and canonically compactifiable graphs, while lattice graphs $\mathbb{Z}^n$, i.e., discrete analogs of Euclidean spaces $\mathbb{R}^n$, are excluded. In this paper, our main contribution is to extend the results in [Huang et al. 2020] from finite graphs to lattice graphs. We study the Chern–Simons equation (1) on $\mathbb{Z}^n$ for $n \geqslant 2$, and prove the existence of topological solutions. Furthermore, using topological solutions of (1), we prove the existence of topological solutions to the abelian Higgs equation (2) on $\mathbb{Z}^n$ for $n \geqslant 2$.

We consider the infinite integer lattice $\mathbb{Z}^n$ for $n \geqslant 2$; see Section 2 for details. We define the distance on $\mathbb{Z}^n$ by

$$d(x,y) = \sum_{i=1}^{n} |x_i - y_i|, \quad x, y \in \mathbb{Z}^n,$$

and write $d(x) = d(x,0)$. For any function $u : \mathbb{Z}^n \to \mathbb{R}$, the l^p-norm of u is defined as

$$\|u\|_{l^p(\mathbb{Z}^n)} = \begin{cases} \left(\sum_{x\in\mathbb{Z}^n} |u(x)|^p\right)^{1/p} & \text{for } 1 \leqslant p < \infty, \\ \sup_{x\in\mathbb{Z}^n} |u(x)| & \text{for } p = \infty. \end{cases}$$

The Laplacian is defined as

$$\Delta u(x) = \sum_{d(x,y)=1} (u(y) - u(x)).$$

In the following we mainly consider topological solutions to the self-dual Chern–Simons vortex equation on $\mathbb{Z}^n$

$$\begin{cases} \Delta u = \lambda e^u(e^u - 1) + 4\pi \sum_{j=1}^{M} n_j \delta_{p_j} & \text{on } \mathbb{Z}^n, \\ \lim_{d(x)\to+\infty} u(x) = 0, \end{cases}$$

and construct a topological solution to the above equation, which is maximal among all possible solutions. Our main result is as follows.

Theorem 1.1. *Equation* (1) *has a topological solution* $u \in l^p(\mathbb{Z}^n)$ *on* $\mathbb{Z}^n$ *for* $1 \leqslant p \leqslant \infty$ *and* $n \geqslant 2$, *which is maximal among all possible solutions. Furthermore*, *we have the decay estimate*

$$u = O(e^{-m(1-\epsilon)d(x)}),$$

where $m = \ln(1 + \lambda/(2n))$, $0 < \epsilon < 1$.

In this paper, we provide two proofs of Theorem 1.1. In Proof A, we adopt the exhaustion method and the discrete isoperimetric inequality. Proof A is novel, which relies on the discrete nature of graphs in an essential way. First, by a contradiction argument, we prove the existence of solutions on a finite subset Ω with Dirichlet boundary condition in Lemma 3.2. In order to prove the existence result on $\mathbb{Z}^n$, we apply the exhaustion method. This approach was first introduced by Lin and Yang [2022] to the analysis on graphs. Considering a sequence of finite subsets

$$\Omega_0 \subset \Omega_1 \subset \cdots \subset \Omega_k \subset \cdots, \quad \bigcup_{i=1}^{\infty} \Omega_i = \mathbb{Z}^n,$$

and corresponding monotone solution sequence $\{u_{\Omega_i}\}$ obtained by Lemma 3.2, we denote by $\tilde{u}^i(x)$ the null extension to $\mathbb{Z}^n$ of u_{Ω_i}; see Section 2.1. By passing to the limit, to avoid the triviality of the limit, one needs to prove a uniform bound for all $\tilde{u}^i$. Suppose that it is not true. Then there exists $\lim_{i\to+\infty} \tilde{u}^i(x_i) = -\infty$ for a vertex sequence $\{x_i\}$. Set

$$\begin{aligned} A_1^i &= \{x \in \Omega_i : -C \leqslant \tilde{u}^i(x) \leqslant 0\}, \\ A_2^i &= \{x \in \Omega_i : \tilde{u}^i(x) < -(2n+1)C - \lambda\}, \\ A_3^i &= \{x \in \Omega_i : -(2n+1)C - \lambda \leqslant \tilde{u}^i(x) < -C\}, \end{aligned}$$

where $C = 4\pi \sum_{j=1}^{M} n_j$. Since $|\Delta \tilde{u}^i(x)|$ is bounded on Ω_i, we may prove that

$$A_3^i \neq \varnothing \quad \text{and} \quad \lim_{i\to+\infty} |A_2^i| = +\infty.$$

These imply that $\lim_{i\to+\infty} |A_3^i| = +\infty$ by the isoperimetric inequality on $\mathbb{Z}^n$. However, summing over Ω_i in the equation, we know that $\sum_{x\in\Omega_i} e^{\tilde{u}^i}(1 - e^{\tilde{u}^i})$ is uniformly bounded, which yields a contradiction. Hence we prove the l^∞-convergence $\tilde{u}^i \to u$ on $\mathbb{Z}^n$, and this limit is a topological solution. Applying the maximum principle and Lemma 3.3, we finally get the decay estimate and the maximality of the constructed solution.

In Proof B, we follow the methods in [Spruck and Yang 1995]. We prove a key lemma, Lemma 3.4, which provides the uniform l^2-norm estimate of the solution on a finite subset Ω with Dirichlet boundary condition. To prove this lemma, let $F(u)$ be the natural functional associated to the equation (1) on Ω. By Green's identities on graphs, we prove that $F(u_k)$ decreases with respect to k and has a upper bound which only depends on n, λ and $\sum_{j=1}^{M} n_j$. Applying the discrete Gagliardo–Nirenberg–Sobolev inequality proved in [Porretta 2020], we have

$$\|u_k\|_{l^2(\Omega)} \leqslant C_3(F(u_k) + 1) \leqslant C_4,$$

where C_3, C_4 only depend on n, λ and $\sum_{j=1}^{M} n_j$. Thanks to this lemma, we may pass to the limit and get the solution $u_\Omega \in l^2(\Omega)$ on Ω. Since its l^2-norm is uniformly bounded, we construct the solution u of equation (1) on $\mathbb{Z}^n$ by the exhaustion method. As in Proof A, we prove the decay estimate and the maximality of the solution.

With the help of Theorem 1.1, we prove the existence of topological solutions to the abelian Higgs model.

Theorem 1.2. *Equation* (2) *has a unique topological solution* $u' \in l^p(\mathbb{Z}^n)$ *on* $\mathbb{Z}^n$ *for* $1 \leqslant p \leqslant \infty$ *and* $n \geqslant 2$, *satisfying* $u \leqslant u' \leqslant 0$, *where* u *is constructed in Theorem 1.1. Furthermore*, *there holds the decay estimate*

$$u' = O(e^{-m(1-\epsilon)d(x)}),$$

where $m = \ln(1 + \lambda/(2n))$, $0 < \epsilon < 1$.

To prove Theorem 1.2, we apply the subsupersolution method. By choosing $\omega_1 = 0$ as a supersolution and $\omega_2 = u$ as a subsolution, where u is constructed in Theorem 1.1, we obtain a solution to (2) by the monotone iteration argument.

The paper is organized as follows: In next section, we introduce the setting of graphs. In Section 3, we give two proofs of Theorem 1.1. In Section 4, we prove Theorem 1.2.

2. Preliminaries

2.1. ***The setting of $\mathbb{Z}^n$.*** Consider the infinite integer lattice graph $\mathbb{Z}^n$, $n \geqslant 2$, consisting of the set of vertices

$$V = \mathbb{Z}^n = \left\{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_i \in \mathbb{Z},\ \forall 1 \leqslant i \leqslant n\right\}$$

and the set of edges

$$E = \left\{\{x, y\} : x, y \in \mathbb{Z}^n, \sum_{i=1}^{n} |x_i - y_i| = 1\right\},$$

and we write $x \sim y$ if $\{x, y\} \in E$. We denote by $C(\mathbb{Z}^n) = \{u : \mathbb{Z}^n \to \mathbb{R}\}$ the set of functions on $\mathbb{Z}^n$, by $\operatorname{supp}(u) = \{x \in \mathbb{Z}^n : u(x) \neq 0\}$ the support of u, and by $C_0(\mathbb{Z}^n)$ the set of functions with finite support. For a finite subset $\Omega \subset \mathbb{Z}^n$, we define the boundary of Ω as

$$\delta\Omega := \left\{y \in \mathbb{Z}^n \setminus \Omega : \exists x \in \Omega \text{ such that } y \sim x\right\},$$

and write $\overline{\Omega} = \Omega \cup \delta\Omega$. For $u \in C(\Omega)$, the null extension to $\mathbb{Z}^n$ of u is defined as

$$\tilde{u}(x) = \begin{cases} u(x) & \text{on } \Omega, \\ 0 & \text{on } \Omega^c. \end{cases}$$

We define the difference operator as

$$\nabla_{xy} u = u(y) - u(x),\ u \in C(\mathbb{Z}^n), \quad x, y \in \mathbb{Z}^n.$$

For $f, g \in C(\overline{\Omega})$, we introduce a bilinear form,

$$D_\Omega(f, g) := \tfrac{1}{2} \sum_{\substack{x, y \in \Omega \\ x \sim y}} \nabla_{xy} f \nabla_{xy} g + \sum_{\substack{x \in \Omega,\ y \in \delta\Omega \\ x \sim y}} \nabla_{xy} f \nabla_{xy} g,$$

and we write $D_\Omega(f) = D_\Omega(f, f)$ for the Dirichlet energy of f on Ω. For $f \in C(\overline{\Omega})$, the directional derivative operator $\partial f / \partial \vec{n}$ at $x \in \delta\Omega$ is defined as

$$\frac{\partial f}{\partial \vec{n}}(x) := \sum_{\substack{y \in \Omega \\ x \sim y}} (f(x) - f(y)).$$

The following are Green's identities on graphs; see, e.g., [Grigor'yan 2018].

Lemma 2.1. *Let $f, g \in C(\mathbb{Z}^n)$ and Ω be a finite subset of $\mathbb{Z}^n$.*

(a) *If $f \in C_0(\mathbb{Z}^n)$, we have*

$$\frac{1}{2}\sum_{\substack{x,y\in\mathbb{Z}^n\\ x\sim y}} \nabla_{xy} f \nabla_{xy} g = -\sum_{x\in\mathbb{Z}^n} f(x)\Delta g(x).$$

(b)
$$D_\Omega(f, g) = -\sum_{x\in\Omega} f(x)\Delta g(x) + \sum_{x\in\delta\Omega} f(x)\frac{\partial g}{\partial\vec{n}}(x).$$

2.2. ***Maximum principle and discrete functional inequalities.*** In this subsection we introduce a maximum principle, the isoperimetric inequality, and the discrete Gagliardo–Nirenberg–Sobolev inequality on $\mathbb{Z}^n$, which play key roles in the proofs of the main results. The following maximum principle is well-known.

Lemma 2.2. *Let Ω be a finite subset of $\mathbb{Z}^n$. For any positive $f \in C(\bar{\Omega})$, suppose that a function $v \in C(\bar{\Omega})$ satisfies*

$$\begin{cases} (\Delta - f)v \geqslant 0 & \text{on } \Omega, \\ v \leqslant 0 & \text{on } \delta\Omega. \end{cases}$$

We have $v \leqslant 0$ on $\bar{\Omega}$.

Proof. We prove the result by contradiction. Suppose that there exists $x \in \Omega$ such that $v(x) = \sup_{y\in\bar{\Omega}} v(y) = c > 0$. By the equation,

$$\Delta v(x) \geqslant f(x)v(x) > 0.$$

This implies that there exists $x_0 \sim x,\ x_0 \in \bar{\Omega}$, such that $v(x_0) > v(x) = c$, which yields a contradiction. □

By the above lemma, we have the following corollary.

Corollary 2.3. *For any positive $f \in C(\mathbb{Z}^n)$, suppose that a function $v \in l^2(\mathbb{Z}^n)$ satisfies*

$$\begin{cases} (\Delta - f)v \geqslant 0 & \text{on } \mathbb{Z}^n, \\ \lim_{d(x)\to+\infty} v(x) \leqslant 0. \end{cases}$$

Then $v \leqslant 0$ on $\mathbb{Z}^n$.

The isoperimetric inequality is well-known on $\mathbb{Z}^n$, see, e.g., [Barlow 2017], which is needed for our Proof A. For $K \subset \mathbb{Z}^n$, we denote by $|K|$ the cardinality of the set K.

Lemma 2.4. *There exists a constant C_n, only depending on the dimension n, such that for any finite $\Omega \subset \mathbb{Z}^n$,*

$$|\delta\Omega| \geqslant C_n|\Omega|^{\frac{n-1}{n}}.$$

For $p \geqslant 1$, we define the $D^{1,p}$-norm as

$$\|u\|_{D^{1,p}(\mathbb{Z}^n)} := \left(\sum_{x\in\mathbb{Z}^n}\sum_{y\sim x} |u(y) - u(x)|^p\right)^{\frac{1}{p}}.$$

In Proof B, we need the discrete Gagliardo–Nirenberg–Sobolev inequality on $\mathbb{Z}^n$. Since $\mathbb{Z}^n$ is a discrete regular mesh, the proof of Theorem 4.1 in [Porretta 2020] yields the following discrete Gagliardo–Nirenberg–Sobolev inequality.

Lemma 2.5 [Porretta 2020]. *Let $n \geqslant 2$, $p > 1$, $\gamma \geqslant p$ and $p' = p/(p-1)$. Then for any $u \in l^p(\mathbb{Z}^n)$, we have*

$$\|u\|^{\gamma}_{l^{\gamma n/(n-1)}(\mathbb{Z}^n)} \leqslant C(p,n,\gamma)\|u\|_{D^{1,p}(\mathbb{Z}^n)}\|u\|^{\gamma-1}_{l^{(\gamma-1)p'}(\mathbb{Z}^n)}.$$

Remark 2.6. Although Theorem 4.1 in [Porretta 2020] requires $p > n$ and $\gamma > p$, the above inequality in fact holds for any $p > 1$ and $\gamma \geqslant p$ by the same argument in [Porretta 2020]. For $n \geqslant 2$, choose

$$p = \gamma = 2, \quad p' = 2.$$

With a well-known fact that for any $q \geqslant p$,

$$\|u\|_{l^q(\mathbb{Z}^n)} \leqslant \|u\|_{l^p(\mathbb{Z}^n)},$$

see, e.g., [Huang et al. 2015], we get for $u \in l^2(\mathbb{Z}^n)$,

$$\|u\|_{l^4(\mathbb{Z}^n)} \leqslant \|u\|_{l^{2n/(n-1)}(\mathbb{Z}^n)} \leqslant C'_n\|u\|^{\frac{1}{2}}_{D^{1,2}(\mathbb{Z}^n)}\|u\|^{\frac{1}{2}}_{l^2(\mathbb{Z}^n)}.$$

3. Existence theorems for the Chern–Simons equation

In this section we consider the existence of topological solutions to the Chern–Simons equation, and we give two proofs of Theorem 1.1. To prove this theorem, we first consider an iterative sequence on a finite subset of $\mathbb{Z}^n$.

Let Ω_0 be a finite subset of $\mathbb{Z}^n$, satisfying $\Omega_0 \supset \{p_j\}_{j=1}^M$, and Ω be an arbitrary connected finite subset such that $\Omega_0 \subset \Omega \subset \mathbb{Z}^n$. We write

$$g = 4\pi\sum_{j=1}^{M} n_j\delta_{p_j}, \quad C = 4\pi\sum_{j=1}^{M} n_j,$$

and it is obvious that $g \in l^p(\mathbb{Z}^n)$ for any $p \geqslant 1$. Choose a constant $K > 2\lambda > 0$. Let $u_0 = 0$ and consider the iterative equations

$$\begin{cases}(\Delta - K)u_k = \lambda e^{u_{k-1}}(e^{u_{k-1}} - 1) + g - Ku_{k-1} & \text{on } \Omega,\\ u_k = 0 & \text{on } \delta\Omega.\end{cases} \tag{3}$$

Lemma 3.1. *Let the sequence $\{u_k\}$ be given in* (3). *Then for each k, u_k is uniquely defined and*

$$0 = u_0 \geqslant u_1 \geqslant u_2 \geqslant \cdots.$$

Proof. First we have

$$\begin{cases}(\Delta - K)u_1 = g & \text{on } \Omega,\\ u_1 = 0 & \text{on } \delta\Omega.\end{cases} \tag{4}$$

One easily sees the existence and uniqueness of the solution u_1 on Ω. Using Lemma 2.2, we obtain that $u_1 \leqslant 0$.

Suppose that $0 = u_0 \geqslant u_1 \geqslant u_2 \geqslant \cdots \geqslant u_i$. Since

$$\lambda e^{u_i}(e^{u_i} - 1) + g - Ku_i \in l^2(\Omega),$$

we have the existence and uniqueness of the solution u_{i+1}. From the equations (3) we get

$$\begin{aligned}(\Delta - K)(u_{i+1} - u_i) &= \lambda(e^{2u_i} - e^{2u_{i-1}}) - \lambda(e^{u_i} - e^{u_{i-1}}) - K(u_i - u_{i-1})\\ &\geqslant 2\lambda e^{2\omega}(u_i - u_{i-1}) - K(u_i - u_{i-1}) \geqslant K(e^{2\omega} - 1)(u_i - u_{i-1}) \geqslant 0,\end{aligned}$$

where ω is a function satisfying $u_i \leqslant \omega \leqslant u_{i-1}$. This implies that $u_{i+1} \leqslant u_i$ by Lemma 2.2 and proves this lemma. □

Proof A of Theorem 1.1. By Lemma 3.1, we prove the convergence of the monotone sequence $\{u_k\}$.

Lemma 3.2. *Let $\{u_k\}$ be the sequence defined by* (3). *Then there exists $u_\Omega \in C(\overline{\Omega})$ such that*

$$u_k \to u_\Omega \text{ on } \overline{\Omega},$$

which satisfies

$$\begin{cases}\Delta u_\Omega = \lambda e^{u_\Omega}(e^{u_\Omega} - 1) + g & \text{on } \Omega,\\ u_\Omega = 0 & \text{on } \delta\Omega.\end{cases} \tag{5}$$

Proof. Since Ω is finite and the sequence is monotone, the pointwise limit u_Ω of u_k exists. It suffices to show that u_Ω is bounded. We first consider the set

$$B(\Omega) = \{x \in \Omega : \exists y \in \delta\Omega \text{ such that } y \sim x\}.$$

Summing over Ω in (3), by Lemma 2.1 we obtain

$$\sum_{x\in\delta\Omega} \frac{\partial u_k}{\partial \vec{n}}(x) + \lambda \sum_{x\in\Omega} e^{u_{k-1}}(1 - e^{u_{k-1}}) = \sum_{x\in\Omega} g(x) + K\sum_{x\in\Omega}(u_k(x) - u_{k-1}(x)) \leqslant 4\pi \sum_{j=1}^{M} n_j = C.$$

This yields that

$$\sum_{x\in B(\Omega)} |u_k(x)| \leqslant C.$$

In particular, for any $x \in B(\Omega)$, the sequence $\{u_k(x)\}$ is uniformly bounded.

If $x_1 \sim x_0$, $x_1 \in \Omega$ and $x_0 \in B(\Omega)$, we claim that $\{u_k(x_1)\}$ is uniformly bounded. Equation (3) at x_0 shows that

$$\begin{aligned}|\Delta u_k(x_0)| &\leqslant K|u_k(x_0) - u_{k-1}(x_0)| + \lambda|e^{u_{k-1}(x_0)}(e^{u_{k-1}(x_0)} - 1)| + |g(x_0)|\\ &\leqslant K|u_k(x_0)| + \tfrac{1}{4}\lambda + C \leqslant (K+1)C + \tfrac{1}{4}\lambda.\end{aligned}$$

Note that

$$\Delta u_k(x_0) = \sum_{y\sim x_0}(u_k(y) - u_k(x_0)) \leqslant u_k(x_1) - 2nu_k(x_0) \leqslant u_k(x_1) + 2nC$$

and $u_k(x_1) < 0$. We obtain that $\{u_k(x_1)\}$ is uniformly bounded.

Since Ω is connected, we repeat the above process, and get that $\{u_k(x)\}$ is uniformly bounded on Ω, which completes the proof of this lemma. □

Let Ω_i, $1 \leqslant i < \infty$, be finite and connected subsets satisfying

$$\Omega_0 \subset \Omega_1 \subset \cdots \subset \Omega_k \subset \cdots, \quad \bigcup_{i=1}^{\infty} \Omega_i = \mathbb{Z}^n.$$

We write $u^i = u_{\Omega_i}$ for simplicity. To prove Theorem 1.1, we need the following lemma.

Lemma 3.3. *Let Ω be a finite subset of $\mathbb{Z}^n$ and $\{u_k\}$ be the sequence defined by* (3). *For any function $V \in C(\bar{\Omega})$ satisfying*

$$\begin{cases} \Delta V \geqslant \lambda e^V(e^V - 1) + g & \text{on } \Omega, \\ V(x) \leqslant 0 & \text{on } \delta\Omega, \end{cases}$$

we have

$$0 = u_0 \geqslant u_1 \geqslant \cdots \geqslant u_k \geqslant \cdots \geqslant u_\Omega \geqslant V.$$

Proof. First, one has

$$\Delta V \geqslant \lambda e^V(e^V - 1) + g \geqslant \lambda e^V(e^V - 1).$$

We claim that $\sup_{x\in\Omega} V(x) \leqslant 0$. If not, choose $V(x_0) = \sup_{x\in\Omega} V(x) > 0$ for some $x_0 \in \Omega$. Then

$$0 \geqslant \Delta V(x_0) \geqslant \lambda e^{V(x_0)}(e^{V(x_0)} - 1) > 0,$$

which yields a contradiction and proves the claim.

Suppose that $V \leqslant u_k$. Then

$$(\Delta - K)(u_{k+1} - V) \leqslant \lambda(e^{2u_k} - e^{2V}) - \lambda(e^{u_k} - e^V) - K(u_k - V) \leqslant K(e^{2\omega} - 1)(u_k - V) \leqslant 0,$$

where the function ω satisfies $V \leqslant \omega \leqslant u_k \leqslant 0$. This implies that $V \leqslant u_{k+1}$ by Lemma 2.2 and proves this lemma by the induction. □

Finally, we use these lemmas to prove Theorem 1.1.

Proof of Theorem 1.1. For any integers $1 \leqslant j \leqslant k$, we have $\Omega_j \subset \Omega_k$. On $\bar{\Omega}_j$, since $u^k \leqslant 0$, one easily sees that u^k satisfies the conditions in Lemma 3.3, and we obtain

$$u^k \leqslant u^j \quad \text{on } \bar{\Omega}_j.$$

Let $\tilde{u}^i$ be the null extension to $\mathbb{Z}^n$ of u^i on Ω_i. Note that

$$0 \geqslant \tilde{u}^1 \geqslant \tilde{u}^2 \geqslant \cdots \geqslant \tilde{u}^k \geqslant \cdots$$

on $\mathbb{Z}^n$.

In order to prove the convergence of the sequence $\{\tilde{u}^i\}$ to a nontrivial limit, we need to give a uniform lower bound of $\{\tilde{u}^i\}$. We argue by contradiction. Suppose that

$$\lim_{i\to+\infty} \|\tilde{u}^i\|_{l^\infty} = +\infty.$$

We write

$$\begin{aligned} A_1^i &= \{x \in \Omega_i : -C \leqslant \tilde{u}^i(x) \leqslant 0\}, \\ A_2^i &= \{x \in \Omega_i : \tilde{u}^i(x) < -(2n+1)C - \lambda\}, \\ A_3^i &= \{x \in \Omega_i : -(2n+1)C - \lambda \leqslant \tilde{u}^i(x) < -C\}. \end{aligned}$$

By the conditions we assume, we get $A_2^i \neq \varnothing$ when $i \geqslant i_0$ for some i_0. In the following, we only consider $i \geqslant i_0$. Following the proof of Lemma 3.2, we have

$$\sum_{x\in B(\Omega_i)} |\tilde{u}^i(x)| \leqslant C,$$

which yields $B(\Omega_i) \subset A_1^i$ so that $A_1^i \neq \varnothing$. To obtain the uniform l^∞-norm, we show the contradiction in three steps.

(i) We claim that

$$A_3^i \neq \varnothing.$$

Suppose that $A_3^i = \varnothing$, which is equivalent to $A_1^i \cup A_2^i = \Omega_i$, and there exist two vertices $x, y \in \Omega_i$, satisfying $x \sim y,\ x \in A_1^i,\ y \in A_2^i$. Note that

$$\Delta \tilde{u}^i(x) = \sum_{z \sim x} (\tilde{u}^i(z) - \tilde{u}^i(x)) \leqslant \tilde{u}^i(y) - 2n\tilde{u}^i(x) < -(2n+1)C - \lambda + 2nC = -C - \lambda$$

and

$$|\Delta \tilde{u}^i(x)| \leqslant |g(x)| + \lambda |e^{\tilde{u}^i}(1 - e^{\tilde{u}^i})| < C + \lambda.$$

This yields a contradiction. Thus we have $A_3^i \neq \varnothing$.

(ii) We claim that

$$\lim_{i \to +\infty} |A_2^i| = +\infty.$$

By

$$\lim_{i \to +\infty} \|\tilde{u}^i\|_{l^\infty} = +\infty,$$

we choose a sequence $\{x_i\}$, where $x_i \in \Omega_i$, satisfying

$$\lim_{i \to +\infty} \tilde{u}^i(x_i) = -\infty.$$

Consider the function

$$w_i(x) = \tilde{u}^i(x - x_1 + x_i)$$

and

$$\Omega_i' = \{x \in \mathbb{Z}^n : x - x_1 + x_i \in \Omega_i\}.$$

We have

$$\lim_{i \to +\infty} w_i(x_1) = -\infty.$$

Let i be sufficiently large. From the proof in (i) we also have the estimate that for any $x \in \Omega_i'$,

$$|\Delta w_i(x)| < C + \lambda.$$

Consider an arbitrary vertex $y_1 \in \Omega_i',\ y_1 \sim x_1$, and by the above facts,

$$\Delta w_i(y_1) = \sum_{z \sim y_1} (w_i(z) - w_i(y_1)) \leqslant -2n w_i(y_1) + w_i(x_1).$$

This implies that

$$w_i(y_1) < \frac{1}{2n} w_i(x_1) + \frac{C+\lambda}{2n}.$$

Since Ω_i' is connected, $x_1 \in \Omega_i'$ and

$$\lim_{i \to +\infty} |\Omega_i'| = +\infty,$$

there exist $y_1 \sim x_1$ and a subsequence, still denoted by $\{\Omega_i'\}$, satisfying $y_1 \in \Omega_i'$. One obtains that

$$\liminf_{i\to+\infty} w_i(y_1) = -\infty.$$

Repeating the above process,

$$\limsup_{i\to+\infty} \bigl|\{y \in \Omega_i' : \liminf_{i\to+\infty} w_i(y) = -\infty\}\bigr| = +\infty.$$

The monotonically decreasing sequence $\{\tilde{u}^i\}$ guarantees that

$$A_2^i \subset A_2^{i+1}.$$

By letting $i \to +\infty$, one easily sees that

$$\lim_{i\to+\infty} |A_2^i| = \limsup_{i\to+\infty} |A_2^i| = +\infty.$$

(iii) From (i) and (ii) we want to prove that

$$\limsup_{i\to+\infty} |A_3^i| = +\infty.$$

We argue by contradiction.

Suppose that

$$\limsup_{i\to+\infty} |A_3^i| = N < +\infty.$$

We focus on the set $\Omega_i \setminus A_3^i = A_1^i \cup A_2^i$, which can be divided into a union of several disjoint connected subsets, i.e.,

$$\Omega_i \setminus A_3^i = \bigcup_{j=1}^{l} O_j,$$

and we have

$$\delta O_j \subset \delta\Omega_i \cup A_3^i.$$

From the proof of (i), we have

$$O_j \subset A_1^i \quad \text{or} \quad O_j \subset A_2^i.$$

For some $1 \leqslant l_1 \leqslant l-1$, without loss of generality, we may assume that

$$A_1^i = \bigcup_{j=1}^{l_1} O_j \quad \text{and} \quad A_2^i = \bigcup_{j=l_1+1}^{l} O_j.$$

Since $B(\Omega_i) \subset A_1^i$, we get

$$\delta\Omega_i \subset \bigcup_{j=1}^{l_1} \delta O_j.$$

Thus for $l_1 + 1 \leqslant j \leqslant l$,

$$\delta O_j \subset A_3^i.$$

For any $x \in \Omega_i$, since $O_1, \dots, O_l$ are disjoint connected sets and $|\delta\{x\}| = 2n$, there are no more than $2n$ sets from the family $\{O_j\}_{j=1}^{l}$ satisfying $x \in \delta O_j$.

By the isoperimetric inequality in Lemma 2.4, we have the estimate

$$|A_2^i| = \sum_{j=l_1+1}^{l} |O_j| \leqslant \left(\frac{1}{C_n}\right)^{\frac{n}{n-1}} \sum_{j=l_1+1}^{l} |\delta O_j|^{\frac{n}{n-1}} \leqslant \left(\frac{1}{C_n}\right)^{\frac{n}{n-1}} \left(\sum_{j=l_1+1}^{l} |\delta O_j|\right)^{\frac{n}{n-1}}$$
$$\leqslant \left(\frac{2n}{C_n}\right)^{\frac{n}{n-1}} |A_3^i|^{\frac{n}{n-1}} \leqslant \left(\frac{2nN}{C_n}\right)^{\frac{n}{n-1}},$$

which contradicts the claim proved in (ii) by letting $i \to +\infty$.

With this fact, we choose a small constant ϵ satisfying

$$0 < \epsilon < \inf_{x \in [-(n+1)C-\lambda, -C]} e^x(1-e^x).$$

From (5), we get

$$\sum_{x \in \delta\Omega_i} \frac{\partial \tilde{u}^i}{\partial \vec{n}}(x) + \lambda \sum_{x \in \Omega_i} e^{\tilde{u}^i}(1-e^{\tilde{u}^i}) = \sum_{x \in \Omega_i} g(x) = 4\pi \sum_{j=1}^{M} n_j = C,$$

which implies that

$$\sum_{x \in \Omega_i} e^{\tilde{u}^i}(1-e^{\tilde{u}^i}) \leqslant \frac{C}{\lambda} \quad \text{and} \quad \sum_{x \in A_3^i} \epsilon \leqslant \frac{C}{\lambda}.$$

This yields a contradiction to (iii). Thus $\{\tilde{u}^i\}$ has a uniform bound in $l^\infty(\mathbb{Z}^n)$, and we have the pointwise convergence

$$\lim_{i \to +\infty} \tilde{u}^i(x) = u(x) \quad \text{for all } x \in \mathbb{Z}^n,$$

where $u \in l^\infty(\mathbb{Z}^n)$ and u satisfies the self-dual Chern–Simons vortex equation (1). From the above inequality and the fact that u has a lower bound, we pass to the limit, and get that for any $i \geqslant 1$,

$$e^{\inf_{x \in \mathbb{Z}^n} u(x)} \sum_{x \in \Omega_i} (1-e^u) \leqslant \sum_{x \in \Omega_i} e^u(1-e^u) \leqslant \frac{C}{\lambda},$$

which yields that u is a topological solution.

That the solution is maximal follows from Lemma 3.3. On any finite subset Ω, by Lemma 3.3, we obtain that the solution u_Ω is maximal. On $\mathbb{Z}^n$, we suppose that there exists another topological solution f of the self-dual Chern–Simons vortex equation. From the proof of Lemma 3.3, we observe that $f \leqslant 0$ on $\mathbb{Z}^n$. Applying Lemma 3.3 on Ω_i, we have

$$f \leqslant u^i.$$

For a fixed integer $k \geqslant 1$ and for $i \geqslant k$ we have

$$f(x) \leqslant \underline{\lim}_{i \to \infty} u^i(x) = u(x) \quad \text{on } \Omega_k.$$

For any $x \in \mathbb{Z}^n$, there exists a sufficiently large integer k satisfying $x \in \Omega_k$ such that $f(x) \leqslant u(x)$. Thus we obtain $f \leqslant u$ on $\mathbb{Z}^n$, and the solution u is maximal among all possible solutions.

The last part is to prove the decay estimate

$$u = O(e^{-m(1-\epsilon)d(x)}),$$

where $m = \ln(1+\lambda/(2n))$. Note that the solution u satisfies

$$\Delta u = \lambda e^u(e^u - 1) \quad \text{on } \overline{\Omega}_0^c.$$

Since

$$\lim_{d(x)\to+\infty} u(x) = 0,$$

for any $0 < \epsilon < 1$, we can choose $R \geqslant 1$ sufficiently large that

$$\lambda e^{2u} \geqslant 2n\Big[\Big(1+\frac{\lambda}{2n}\Big)^{1-\epsilon} - 1\Big], \quad d(x) \geqslant R.$$

Then for $d(x) \geqslant R$,

$$\Delta u = \lambda e^u(e^u - 1) = \lambda e^{u+\omega} u \leqslant \lambda e^{2u} u \leqslant c_3 u,$$

where the function ω satisfies $u \leqslant \omega \leqslant 0$ and $c_3 = 2n[(1+\lambda/(2n))^{1-\epsilon} - 1]$.

Consider the function $h(x) = -e^{-m(1-\epsilon)d(x)}$. Let e_i be the vector whose i-th component is 1 and the others are 0. For $x \in \overline{\Omega}_0^c$ and $d(x) \geqslant R$, suppose that $d(x) = t \geqslant R \geqslant 1$, and we have

$$\Delta h(x) = \sum_{y\sim x}\big(h(y) - h(x)\big) = \sum_{i=1}^{n}\big(h(x+e_i) + h(x-e_i) - 2h(x)\big).$$

If $x_i \neq 0$, then

$$h(x+e_i) + h(x-e_i) - 2h(x) = -e^{-m(1-\epsilon)(t-1)} - e^{-m(1-\epsilon)(t+1)} + 2e^{-m(1-\epsilon)t}.$$

If $x_i = 0$, we have

$$\begin{aligned} h(x+e_i) + h(x-e_i) - 2h(x) &= -2e^{-m(1-\epsilon)(t+1)} + 2e^{-m(1-\epsilon)t} \\ &\geqslant -e^{-m(1-\epsilon)(t-1)} - e^{-m(1-\epsilon)(t+1)} + 2e^{-m(1-\epsilon)t}. \end{aligned}$$

Therefore, we have the inequality

$$\begin{aligned} \Delta h(x) &\geqslant n\big[-e^{-m(1-\epsilon)(t-1)} - e^{-m(1-\epsilon)(t+1)} + 2e^{-m(1-\epsilon)t}\big] = n[e^{-m(1-\epsilon)} + e^{m(1-\epsilon)} - 2]h(x) \\ &= n\bigg[\Big(1+\frac{c_3}{2n}\Big) + \frac{1}{1+c_3/(2n)} - 2\bigg]h(x) \geqslant n\bigg[2\Big(1+\frac{c_3}{2n}\Big) - 2\bigg]h(x) = c_3 h(x). \end{aligned}$$

Fix a subset

$$\Omega_0' = \{x \in \mathbb{Z}^n : d(x) \geqslant R_1 \geqslant R\},$$

which satisfies $\Omega_0' \cap \overline{\Omega}_0 = \varnothing$. By choosing a large constant $C(\epsilon)$, we obtain

$$(\Delta - c_3)(C(\epsilon)h - u) \geqslant 0 \quad \text{on } \Omega_0',$$

$$\lim_{|x|\to+\infty}(C(\epsilon)h - u)(x) = 0 \quad \text{and} \quad C(\epsilon)h(x) - u(x) \leqslant 0 \qquad \text{if } d(x) = R_1.$$

These imply that

$$0 \geqslant u(x) \geqslant -C(\epsilon)e^{-m(1-\epsilon)d(x)} \quad \text{on } \Omega_0',$$

completing Theorem 1.1. As a consequence, we obtain $u \in l^p(\mathbb{Z}^n),\ 1 \leqslant p \leqslant \infty$. □

Proof B of Theorem 1.1. Proof B follows the methods in [Spruck and Yang 1995]. We mainly prove the following key lemma.

Lemma 3.4. *Let $n \geqslant 2$, $\lambda > 0$, and Ω_0 be a finite subset of $\mathbb{Z}^n$ containing the distinct points $\{p_j\}_{j=1}^M$. For any finite subset $\Omega \supset \Omega_0$, the boundary value problem*

$$\begin{cases} \Delta u = \lambda e^u(e^u - 1) + 4\pi \sum_{j=1}^M n_j \delta_{p_j} & \text{on } \Omega, \\ u(x) = 0 & \text{on } \delta\Omega, \end{cases}$$

has a solution $u_\Omega : \bar{\Omega} \to \mathbb{R}$. This solution is maximal among all possible solutions and satisfies that $\|u_\Omega\|_{l^2(\Omega)} \leqslant C_0$, where C_0 only depends on n, λ and C.

By Green's identities in Lemma 2.1, we consider the following functional on Ω:

$$F(u) = \tfrac{1}{2} D_\Omega(u) + \sum_{x\in\Omega} \bigl[\tfrac{1}{2}\lambda(e^{u(x)} - 1)^2 + g(x)u(x)\bigr].$$

We prove the following lemma which states that $F(u_k)$ deceases with respect to k.

Lemma 3.5. *Let $\{u_k\}$ be the sequence defined by* (3). *Then*

$$c_0 \geqslant F(u_1) \geqslant F(u_2) \geqslant \cdots \geqslant F(u_k) \geqslant \cdots,$$

where the constant c_0 only depends on n, C, λ.

Proof. Multiplying (3) by $u_k - u_{k-1}$ and summing over Ω, we obtain

$$\begin{aligned} &\sum_{x\in\Omega} (\Delta - K)u_k(x)[u_k(x) - u_{k-1}(x)] \\ &\qquad = \sum_{x\in\Omega} \bigl[\lambda e^{u_{k-1}}(e^{u_{k-1}} - 1)(u_k - u_{k-1}) - K u_{k-1}(u_k - u_{k-1}) + g(u_k - u_{k-1})\bigr](x). \end{aligned} \tag{6}$$

By Green's identities in Lemma 2.1,

$$\sum_{x\in\Omega} \Delta u_k(x)(u_k(x) - u_{k-1}(x)) = -D_\Omega(u_k - u_{k-1}, u_k) = -D_\Omega(u_k) + D_\Omega(u_{k-1}, u_k).$$

Combining it with equation (6), we get

$$\begin{aligned} &D_\Omega(u_k) - D_\Omega(u_{k-1}, u_k) + \sum_{x\in\Omega} K(u_k(x) - u_{k-1}(x))^2 \\ &\qquad = -\sum_{x\in\Omega} [\lambda e^{u_{k-1}}(e^{u_{k-1}} - 1)(u_k - u_{k-1}) + g(u_k - u_{k-1})](x). \end{aligned}$$

Consider the function

$$\varphi(x) = \frac{\lambda}{2}(e^x - 1)^2 - \frac{K}{2}x^2,$$

which is concave for any $x \leqslant 0$. Hence

$$\frac{\varphi(u_{k-1}) - \varphi(u_k)}{u_{k-1} - u_k} \geqslant \varphi'(u_{k-1}) = \lambda e^{u_{k-1}}(e^{u_{k-1}} - 1) - K u_{k-1}.$$

That is,

$$\frac{\lambda}{2}(e^{u_k} - 1)^2 \leqslant \frac{\lambda}{2}(e^{u_{k-1}} - 1)^2 + \frac{K}{2}(u_k - u_{k-1})^2 + \lambda e^{u_{k-1}}(e^{u_{k-1}} - 1)(u_k - u_{k-1}).$$

By the fact

$$\begin{aligned}|D_\Omega(u_{k-1},u_k)| &\leqslant \tfrac{1}{2}\sum_{\substack{x,y\in\Omega\\ x\sim y}}|\nabla_{xy}u_{k-1}\nabla_{xy}u_k| + \sum_{\substack{x\in\Omega,y\in\delta\Omega\\ x\sim y}}|\nabla_{xy}u_{k-1}\nabla_{xy}u_k|\\ &\leqslant \tfrac{1}{4}\sum_{\substack{x,y\in\Omega\\ x\sim y}}(|\nabla_{xy}u_{k-1}|^2+|\nabla_{xy}u_k|^2) + \tfrac{1}{2}\sum_{\substack{x\in\Omega,y\in\delta\Omega\\ x\sim y}}(|\nabla_{xy}u_{k-1}|^2+|\nabla_{xy}u_k|^2)\\ &= \tfrac{1}{2}D_\Omega(u_{k-1})+\tfrac{1}{2}D_\Omega(u_k),\end{aligned}$$

we obtain that

$$F(u_k)\leqslant F(u_k)+\frac{K}{2}\|u_{k-1}-u_k\|^2_{l^2(\Omega)}\leqslant F(u_{k-1}).$$

Thus we only need to prove $F(u_1)\leqslant c_0$. Note that

$$\begin{aligned}D_\Omega(u_1) &= \tfrac{1}{2}\sum_{\substack{x,y\in\Omega\\ x\sim y}}|\nabla_{xy}u_1|^2 + \sum_{\substack{x\in\Omega,y\in\delta\Omega\\ x\sim y}}|\nabla_{xy}u_1|^2\\ &\leqslant \sum_{\substack{x,y\in\Omega\\ x\sim y}}(u_1(x)^2+u_1(y)^2) + 2\sum_{\substack{x\in\Omega,y\in\delta\Omega\\ x\sim y}}(u_1(x)^2+u_1(y)^2)\\ &\leqslant 4n\|u_1\|^2_{l^2(\Omega)}\end{aligned}$$

and $|e^{u_1}-1|=1-e^{u_1}\leqslant -u_1$. Then we have the estimate

$$\begin{aligned}F(u_1) &\leqslant \tfrac{1}{2}\cdot 4n\|u_1\|^2_{l^2(\Omega)}+\frac{\lambda}{2}\sum_{x\in\Omega}u_1(x)^2+\tfrac{1}{2}\sum_{x\in\Omega}[g(x)^2+u_1(x)^2]\\ &= c_1+c_2\|u_1\|^2_{l^2(\Omega)},\end{aligned}$$

where c_1, c_2 are constants that only depend on n, λ and C. Multiplying (4) by u_1 and summing over Ω, we have

$$D_\Omega(u_1)+K\sum_{x\in\Omega}u_1(x)^2=-\sum_{x\in\Omega}g(x)u_1(x).$$

This yields

$$K\sum_{x\in\Omega}u_1(x)^2\leqslant\frac{1}{2K}\sum_{x\in\Omega}g(x)^2+\frac{K}{2}\sum_{x\in\Omega}u_1(x)^2.$$

Hence,

$$\sum_{x\in\Omega}u_1(x)^2\leqslant\frac{\|g\|^2_{l^2(\mathbb{Z}^n)}}{K^2},$$

which completes the proof. ☐

Our aim is a uniform control of the l^2-norm of $\{u_k\}$. By Lemma 3.5, we can use the functional $F(u_k)$ to control the l^2-norm of u_k. In fact, we prove the following lemma, which states that the functional F is coercive.

Lemma 3.6. *Let $v \in l^2(\bar{\Omega})$ and $v(x) = 0$ for all $x \in \delta\Omega$. Then*

$$\|v\|_{l^2(\Omega)} \leqslant C_2(F(v)+1),$$

where C_2 only depends on n, C, λ. In particular, let $\{u_k\}$ be the sequence defined by (3). *We have for any $k \geqslant 1$,*

$$\|u_k\|_{l^2(\Omega)} \leqslant C_2(F(u_k)+1) \leqslant C_0,$$

where C_0 only depends on n, C, λ.

Proof. For any function $v \in l^2(\bar{\Omega})$ with $v(x) = 0$ for all $x \in \delta\Omega$. Let $\tilde{v}$ be the null extension to $\mathbb{Z}^n$ of v on Ω. Hence $\tilde{v} \in l^2(\mathbb{Z}^n)$. By Lemma 2.5, we have

$$\|\tilde{v}\|^4_{l^4(\mathbb{Z}^n)} \leqslant C'_n \|\tilde{v}\|^2_{D^{1,2}(\mathbb{Z}^n)} \|\tilde{v}\|^2_{l^2(\mathbb{Z}^n)}.$$

Note that

$$\|\tilde{v}\|^4_{l^4(\mathbb{Z}^n)} = \sum_{x\in\Omega} v(x)^4, \quad \|\tilde{v}\|^2_{l^2(\mathbb{Z}^n)} = \sum_{x\in\Omega} v(x)^2, \quad \text{and} \quad \|\tilde{v}\|_{D^{1,2}(\mathbb{Z}^n)} \leqslant (2D_\Omega(v))^{\frac{1}{2}}.$$

This yields that

$$\sum_{x\in\Omega} v(x)^4 \leqslant C_3 D_\Omega(v) \sum_{x\in\Omega} v(x)^2, \tag{7}$$

where $C_3 = 2C'_n$. Since $e^v - 1 \geqslant v$ and $1 - e^{-v} \geqslant v/(1+v)$ for $v \geqslant 0$,

$$|e^v - 1|^2 \geqslant \left(\frac{|v|}{1+|v|}\right)^2.$$

By (7) we obtain

$$\begin{aligned}
F(v) &= \tfrac{1}{2}D_\Omega(v) + \sum_{x\in\Omega}\left[\tfrac{1}{2}\lambda(e^{v(x)}-1)^2 + g(x)v(x)\right] \\
&\geqslant \tfrac{1}{2}D_\Omega(v) + \tfrac{1}{2}\lambda \sum_{x\in\Omega}\left(\frac{|v(x)|}{1+|v(x)|}\right)^2 - \|g\|_{l^{4/3}(\mathbb{Z}^n)}\|v\|_{l^4(\Omega)} \\
&\geqslant \tfrac{1}{2}D_\Omega(v) + \tfrac{1}{2}\lambda \sum_{x\in\Omega}\left(\frac{|v(x)|}{1+|v(x)|}\right)^2 - C_4(D_\Omega(v))^{\frac{1}{4}}\left(\sum_{x\in\Omega} v(x)^2\right)^{\frac{1}{4}} \\
&\geqslant \tfrac{1}{2}D_\Omega(v) + \tfrac{1}{2}\lambda \sum_{x\in\Omega}\left(\frac{|v(x)|}{1+|v(x)|}\right)^2 - \epsilon\|v\|_{l^2(\Omega)} - \frac{C_4}{\epsilon}(D_\Omega(v))^{\frac{1}{2}} \\
&\geqslant \tfrac{1}{2}D_\Omega(v) + \tfrac{1}{2}\lambda \sum_{x\in\Omega}\left(\frac{|v(x)|}{1+|v(x)|}\right)^2 - \epsilon\|v\|_{l^2(\Omega)} - \tfrac{1}{4}D_\Omega(v) - C_4 \\
&= \tfrac{1}{4}D_\Omega(v) + \tfrac{1}{2}\lambda \sum_{x\in\Omega}\left(\frac{|v(x)|}{1+|v(x)|}\right)^2 - \epsilon\|v\|_{l^2(\Omega)} - C_4,
\end{aligned} \tag{8}$$

where $\epsilon > 0$ is a sufficiently small constant which will be chosen below, and C_4 is a uniform constant only depending on ϵ, C, C'_n which may change its value from line to line.

By the inequality (7), we have the estimate

$$\begin{aligned}\Big(\sum_{x\in\Omega} v(x)^2\Big)^2 &= \Big[\sum_{x\in\Omega}\frac{|v(x)|}{1+|v(x)|}(1+|v(x)|)|v(x)|\Big]^2\\ &\leqslant \sum_{x\in\Omega}\Big(\frac{|v(x)|}{1+|v(x)|}\Big)^2\sum_{x\in\Omega}(1+|v(x)|)^2v(x)^2\\ &\leqslant 2\sum_{x\in\Omega}\Big(\frac{|v(x)|}{1+|v(x)|}\Big)^2\sum_{x\in\Omega}(v(x)^2+v(x)^4)\\ &\leqslant 2\sum_{x\in\Omega}\Big(\frac{|v(x)|}{1+|v(x)|}\Big)^2\sum_{x\in\Omega}v(x)^2+2C_3\sum_{x\in\Omega}\Big(\frac{|v(x)|}{1+|v(x)|}\Big)^2D_\Omega(v)\sum_{x\in\Omega}v(x)^2\\ &\leqslant \tfrac{1}{2}\Big(\sum_{x\in\Omega}v(x)^2\Big)^2+C_5\Big[\Big(\sum_{x\in\Omega}\Big(\frac{|v(x)|}{1+|v(x)|}\Big)^2\Big)^2+\Big(\sum_{x\in\Omega}\Big(\frac{|v(x)|}{1+|v(x)|}\Big)^2\Big)^2D_\Omega(v)^2\Big]\\ &\leqslant \tfrac{1}{2}\Big(\sum_{x\in\Omega}v(x)^2\Big)^2+C_5\Big[1+\Big(\sum_{x\in\Omega}\Big(\frac{|v(x)|}{1+|v(x)|}\Big)^2\Big)^4+D_\Omega(v)^4\Big],\end{aligned}$$

where C_5, C_6 are uniform constants only depending on C'_n. This yields that

$$\|v\|_{l^2(\Omega)} \leqslant C_6\Big[1+\sum_{x\in\Omega}\Big(\frac{|v(x)|}{1+|v(x)|}\Big)^2+D_\Omega(v)\Big]. \tag{9}$$

We choose $\epsilon = \dfrac{\min\{1/8, \lambda/4\}}{C_6}$, and by combining (8) with (9), we obtain

$$\|v\|_{l^2(\Omega)} \leqslant C_2(F(v)+1).$$

By Lemma 3.5, we have

$$\|u_k\|_{l^2(\Omega)} \leqslant C_2(F(u_k)+1) \leqslant C_0,$$

where C_2, C_0 only depend on n, C, λ. ☐

Proof of Lemma 3.4. By Lemmas 3.6 and 3.1, we obtain

$$u_k \to u_\Omega \quad \text{in } l^2(\Omega), \qquad \text{and} \qquad \|u_\Omega\|_{l^2(\Omega)} \leqslant C_0.$$

Since Δ is a local operator, by the pointwise convergence the function $u_\Omega \in l^2(\Omega)$ is the solution to the equation

$$\begin{cases}\Delta u = \lambda e^u(e^u-1)+g & \text{on } \Omega,\\ u(x)=0 & \text{on } \delta\Omega.\end{cases}$$

This finishes the main proof of Lemma 3.4. For the rest, it remains to prove that this solution is maximal, which we argue the same as in Proof A. ☐

Let Ω_i be finite and connected subsets satisfying

$$\Omega_0 \subset \Omega_1 \subset \cdots \subset \Omega_k \subset \cdots, \quad \bigcup_{i=1}^{\infty} \Omega_i = \mathbb{Z}^n,$$

and we write $u^i = u_{\Omega_i}$. Finally we use these lemmas to prove Theorem 1.1.

Proof of Theorem 1.1. As in Proof A, for any integer $1 \leqslant j \leqslant k$, one easily sees that

$$u^k \leqslant u^j \quad \text{on } \overline{\Omega}_j.$$

Let $\tilde{u}^k$ be the null extension to $\mathbb{Z}^n$ of u_k. Then

$$0 \geqslant \tilde{u}^1 \geqslant \tilde{u}^2 \geqslant \cdots \geqslant \tilde{u}^k \geqslant \cdots$$

on $\mathbb{Z}^n$. Noting that $\|\tilde{u}^k\|_{l^2(\mathbb{Z}^n)} \leqslant C_0$ for any $k \geqslant 1$, we have the pointwise convergence

$$\tilde{u}^k(x) \to u(x), \quad \text{for all } x \in \mathbb{Z}^n,$$

and $u \in l^2(\mathbb{Z}^n)$. Hence u satisfies the equations

$$\begin{cases} \Delta u = \lambda e^u(e^u - 1) + 4\pi \sum_{j=1}^{M} \delta_{p_j} & \text{on } \mathbb{Z}^n, \\ \lim_{d(x)\to+\infty} u(x) = 0, \end{cases}$$

which is a topological solution. Analogous to Proof A, one can show that this solution is maximal and satisfies the decay estimate. This implies that $u \in l^p(\mathbb{Z}^n)$ for any $1 \leqslant p \leqslant \infty$, and we finish Proof B. □

4. Existence theorems of the abelian Higgs equation

Note that the topological solution to the Chern–Simons model, obtained in Theorem 1.1, serves as a subsolution of the abelian Higgs equation. In this section we will prove the existence of topological solutions to the abelian Higgs equation (2) on $\mathbb{Z}^n$ for $n \geqslant 2$ using the subsupersolution approach. We prove the existence of topological solutions to (2) by a monotone iteration method.

Definition 4.1. We call a function ω a supersolution or a subsolution of (2) if, on $\mathbb{Z}$,

$$\Delta\omega \leqslant \lambda(e^\omega - 1) + g \quad \text{or} \quad \Delta\omega \geqslant \lambda(e^\omega - 1) + g, \quad \text{respectively}.$$

Lemma 4.2. *The function $\omega_1 = 0$ is a supersolution of* (2), *and the function $\omega_2 = u$ given by Theorem 1.1 is a subsolution of* (2).

Proof. Since $g \geqslant 0$, we have

$$\Delta\omega_1 = 0 \leqslant \lambda(e^{\omega_1} - 1) + g.$$

Noting that $\omega_2 \leqslant 0$, we have

$$\Delta\omega_2 = \lambda e^{\omega_2}(e^{\omega_2} - 1) + g \geqslant \lambda(e^{\omega_2} - 1) + g.$$

□

Similar to Section 3, we define an iterative sequence as follows. For $K > \lambda$, let $u'_0 = 0$ and consider the following equations, $k \geqslant 1$,

$$\begin{cases} (\Delta - K)u'_k = \lambda(e^{u'_{k-1}} - 1) + g - Ku'_{k-1} \quad \text{on } \mathbb{Z}^n, \\ \lim\limits_{d(x)\to+\infty} u'_k(x) = 0. \end{cases} \tag{10}$$

We have the following lemma.

Lemma 4.3. *Let* $\{u'_k\}$ *be the sequence defined by* (10). *Then for each* k*,* u'_k *is uniquely defined and*

$$\omega_1 = 0 = u'_0 \geqslant u'_1 \geqslant u'_2 \geqslant \cdots \geqslant \omega_2.$$

Proof. It is clear that u'_1 is unique and $u'_1 \in l^2(\mathbb{Z}^n)$. Since

$$(\Delta - K)(\omega_2 - u'_1) \geqslant \lambda(e^{\omega_2} - 1) - K\omega_2 \geqslant (\lambda - K)\omega_2 \geqslant 0,$$

with the boundary conditions, we prove that $u'_1 \geqslant \omega_2$ by Corollary 2.3.

Suppose that

$$0 = u'_0 \geqslant u'_1 \geqslant u'_2 \geqslant \cdots \geqslant u'_i \geqslant \omega_2$$

and we have the existence and uniqueness of $u'_{i+1} \in l^2(\mathbb{Z}^n)$. By calculation, we obtain

$$\begin{aligned}(\Delta - K)(u'_{i+1} - u'_i) = \lambda(e^{u'_i} - e^{u'_{i-1}}) - K(u'_i - u'_{i-1}) &\geqslant \lambda e^{\eta_1}(u'_i - u'_{i-1}) - K(u'_i - u'_{i-1}) \\ &\geqslant K(e^{\eta_1} - 1)(u'_i - u'_{i-1}) \geqslant 0\end{aligned}$$

and

$$(\Delta - K)(\omega_2 - u'_{i+1}) \geqslant \lambda(e^{\omega_2} - e^{u'_i}) - K(\omega_2 - u'_i) \geqslant (\lambda e^{\eta_2} - K)(\omega_2 - u'_i) \geqslant 0,$$

where the functions η_1, η_2 satisfy

$$u'_i \leqslant \eta_1 \leqslant u'_{i-1} \leqslant 0 \quad \text{and} \quad \omega_2 \leqslant \eta_2 \leqslant u'_i \leqslant 0.$$

These yield that

$$\omega_2 \leqslant u'_{i+1} \leqslant u'_i. \qquad \square$$

Finally, we give a sketch of the proof of Theorem 1.2.

Proof of Theorem 1.2. By Lemma 4.3, the monotone sequence $\{u'_k\}$ is bounded in $l^2(\mathbb{Z}^n)$. Hence, we get the pointwise convergence

$$u'_k(x) \to u'(x) \quad \text{for all } x \in \mathbb{Z}^n,$$

and we obtain that $u' \in l^2(\mathbb{Z}^n)$ and u' is a topological solution to (2). In addition, if there exists another topological solution f, then

$$\Delta(u' - f) = \lambda(e^{u'} - e^f).$$

Hence there exists a function f', satisfying $\min\{u', f\} \leqslant f' \leqslant \max\{u', f\}$, such that

$$(\Delta - \lambda e^{f'})(u' - f) = 0.$$

By the maximum principle, we obtain that this solution is unique.

Furthermore, since $0 \geqslant u' \geqslant \omega_2 = u$ and

$$u = O(e^{-m(1-\epsilon)d(x)}),$$

we have

$$u' = O(e^{-m(1-\epsilon)d(x)})$$

and $u' \in l^p(\mathbb{Z}^n)$ for any $1 \leqslant p \leqslant \infty$. □

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