ANALYSIS & PDE

Volume 19 No. 3 2026

MICHELE COTI ZELATI AND MATIAS G. DELGADINO

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ANALYSIS AND PDE
Vol. 19 (2026), No. 3, pp. 505–538
DOI: 10.2140/apde.2026.19.505

ENTROPY MAXIMIZATION IN THE TWO-DIMENSIONAL EULER EQUATIONS

MICHELE COTI ZELATI AND MATIAS G. DELGADINO

We consider a variational problem related to entropy maximization in the two-dimensional Euler equations, in order to investigate the long-time dynamics of solutions with bounded vorticity. Using variations on the classical min-max principle and borrowing ideas from optimal transportation and quantitative rearrangement inequalities, we prove results on the structure of entropy maximizers arising in the investigation of the long-time behavior of vortex patches. We further show that the same techniques apply in the study of stability of the canonical Gibbs measure associated to a system of point vortices.

1. Long-time dynamics in two-dimensional perfect fluids

The Euler equations describing the motion of an inviscid and incompressible fluid in a two-dimensional regular simply connected domain $M \subset \mathbb{R}^2$ read

$$\begin{cases} \partial_t \omega + \boldsymbol{u} \cdot \nabla \omega = 0, \\ \omega|_{t=0} = \omega^{\text{in}}, \end{cases} \tag{E}$$

where $\omega(t, x) : \mathbb{R} \times M \to \mathbb{R}$ is the vorticity and $\boldsymbol{u}(t, x) : \mathbb{R} \times M \to \mathbb{R}^2$ is the divergence-free velocity field, related to ω through the *Biot-Savart* law

$$\boldsymbol{u} = \nabla^\perp \psi = (-\partial_{x_2}\psi, \partial_{x_1}\psi), \qquad \begin{cases} \Delta\psi = \omega & \text{in } M, \\ \psi = 0 & \text{on } \partial M. \end{cases} \tag{1-1}$$

In short, $\boldsymbol{u} = \nabla^\perp \Delta^{-1}\omega$. The transport nature of equations (E) translates into the representation of solutions via the method of characteristics,

$$\omega(t, x) = \omega^{\text{in}} \circ \Phi_t^{-1}(x),$$

MSC2020: primary 35Q31; secondary 37K58, 49Q22.
Keywords: Euler equations, entropy maximization, statistical hydrodynamics, rearrangement inequalities, optimal transport.

where

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t(x) = \boldsymbol{u}(t, \Phi_t(x)), \quad \Phi_0(x) = x,$$

is the Lagrangian flow. Thanks to Yudovich theory [1963], the Euler equations can be seen as a well-posed, weak-$*$ continuous dynamical system on the compact metric space given by the unit ball in L^∞

$$X := \{\omega \in L^\infty(M) : \|\omega\|_{L^\infty} \le 1\},$$

endowed with the weak-$*$ topology (see [Nguyen 2022] for a recent proof of this fact). It is then natural to ask what is the generic long-time picture of solutions to (E). A central conjecture due to V. Šverák [2012] posits that generic initial data give rise to solutions whose orbits are not precompact in L^2. While this conjecture, in its generality, remains currently out of reach, the recent *inviscid damping* results [Bedrossian and Masmoudi 2015; Masmoudi and Zhao 2024; Ionescu and Jia 2020; 2022; 2023] validate it in certain perturbative regimes.

A related point of view revolves around the idea that the velocity causes a cascade towards high frequencies that averages (i.e., *mixes*) the vorticity in infinite time. The Euler equations (E) preserve physically relevant quantities, hence said frequency cascade is constrained to be consistent with many conservation laws. These are the *kinetic energy*,

$$\mathsf{E}(\omega) = \tfrac{1}{2}\int_M |\nabla^\perp \Delta^{-1}\omega(x)|^2 \,\mathrm{d}x = \tfrac{1}{2}\int_M |\boldsymbol{u}(x)|^2 \,\mathrm{d}x,$$

the *circulation*,

$$\mathsf{K}(\omega) = \int_{\partial M} \boldsymbol{u} \cdot \mathrm{d}\ell = \int_M \omega(x)\,\mathrm{d}x,$$

and the *Casimirs*,

$$\mathsf{S}_f(\omega) = \frac{1}{|M|}\int_M f(\omega(x))\,\mathrm{d}x, \quad \text{for any continuous } f : \mathbb{R} \to \mathbb{R}.$$

Along any sequence of times tending to infinity, weak compactness implies the existence of subsequential limit points ω_∞ for the dynamics. While E and K are continuous in the weak-$*$ topology, and hence

$$(\mathsf{E}, \mathsf{K})(\omega^{\mathrm{in}}) = (\mathsf{E}, \mathsf{K})(\omega_\infty), \tag{1-2}$$

the Casimirs may lose information at infinite time. In particular, if $\omega(t_j) \overset{*}{\rightharpoonup} \omega_\infty$, along a sequence of time $t_j \to \infty$, then only upper-semicontinuity can be deduced, namely

$$\mathsf{S}_f(\omega^{\mathrm{in}}) = \limsup_{j\to\infty} \mathsf{S}_f(\omega(t_j)) \le \mathsf{S}_f(\omega_\infty), \quad \text{for any continuous concave } f : \mathbb{R} \to \mathbb{R}.$$

A strict inequality above is associated to mixing and is often observed in the long-time limit of the two-dimensional Euler equations.

To give the above observations a robust mathematical framework, one can account for the Euler evolution and its long-time limits by considering the weakly-$*$ closed set

$$\mathcal{O}_{\mathrm{in}} = \overline{\left\{\omega^{\mathrm{in}} \circ \Phi \mid \Phi : M \to M \text{ is an area preserving diffeomorphism}\right\}}^{*}, \tag{1-3}$$

which can be seen as the *orbit* of the natural action of the volume-preserving diffeomorphism group on the vorticity field ω^{in}. Although this set may strictly contain the Ω-limit set of ω^{in}, we can get close to the dynamics of $2d$ Euler by intersecting $\mathcal{O}_{\mathrm{in}}$ with the various conservation laws (1-2). This approach provides the basis of Shnirelman's maximal mixing theory [1993], explored and revisited recently in [Dolce and Drivas 2022].

1.1. ***A statistical mechanics perspective.*** For specific choices of f (e.g., $f(\omega) = -\omega \log \omega$), S_f can be seen as a measure of *entropy*, which is a measure of the number of possible configurations at the microscopic level that leads to the observable macrostate. The *second law of thermodynamics* states that the *entropy* of an isolated system will never decrease, but will instead tend to increase over time until it reaches a maximum value at equilibrium.

L. Onsager argued in his seminal [1949] work that under certain ergodicity assumptions, Euler flows originating from point vortices should relax to vorticities that maximize the Boltzmann entropy

$$\mathsf{S}(\omega) := -\frac{1}{|M|} \int_M \omega(x) \log \omega(x) \, \mathrm{d}x, \tag{1-4}$$

subject to all conservation laws, in analogy with equilibrium statistical mechanics. The field has seen tremendous growth since then, with the development of *statistical hydrodynamics* theories corresponding to variational problems of the type

$$\text{maximize } \mathsf{S}_f(\omega) \text{ subject to } \omega \in \mathcal{O}_{\mathrm{in}} \text{ and } \mathsf{E}(\omega) = \mathsf{E}(\omega^{\mathrm{in}}), \tag{1-5}$$

for suitable choices of f, see the reviews [Robert 1995; Bouchet and Venaille 2012]. A rigorous picture for point vortices has been established in the seminal articles [Caglioti et al. 1992; 1995; Eyink and Spohn 1993; Kiessling 1993]. In this article we center in the choice of the Boltzmann entropy (1-4), but other choices of f are also physically relevant.

In the canonical formalism of statistical mechanics, the maximal entropy functional is computed via a variational problem over a microcanonical ensemble as

$$\mathcal{S}(\mathsf{e}) = \max\{\mathsf{S}_f(\omega) : \omega \in \mathcal{O}_{\mathrm{in}}, \ \mathsf{E}(\omega) = \mathsf{e}\}, \tag{1-6}$$

and is expected to be concave with respect to the energy level e. However, even in the simple scenario of a single vortex patch in a disk, this seems to be an open question [Šverák 2012].

One of the aims of this article is to give a general strategy to show the concavity of $\mathcal{S}$, via a variation of the classical min-max principle. The energy constraint in (1-6) can indeed be rewritten as

$$\mathcal{S}(\mathsf{e}) = \max_{\omega \in \mathcal{O}_{\mathrm{in}}} \Big\{ \mathsf{S}_f(\omega) + \inf_{\beta \in \mathbb{R}} \beta(\mathsf{e} - \mathsf{E}(\omega)) \Big\}, \tag{1-7}$$

since

$$\inf_{\beta \in \mathbb{R}} \beta(\mathsf{e} - \mathsf{E}(\omega)) = \begin{cases} 0 & \text{if } \mathsf{E}(\omega) = \mathsf{e}, \\ -\infty & \text{otherwise.} \end{cases}$$

By formally commuting the max and the inf in (1-7), also known as the min-max principle, we are able to rewrite

$$\mathcal{S}(\mathsf{e}) = \inf_{\beta\in\mathbb{R}}\Big\{\beta\mathsf{e} + \max_{\omega\in\mathcal{O}_{\mathrm{in}}}(\mathsf{S}_f(\omega) - \beta\mathsf{E}(\omega))\Big\} = \inf_{\beta\in\mathbb{R}}\Big\{\beta\mathsf{e} + \max_{\omega\in\mathcal{O}_{\mathrm{in}}}\mathsf{F}_\beta(\omega)\Big\}, \tag{1-8}$$

in terms of the associated free energy

$$\mathsf{F}_\beta(\omega) := \mathsf{S}_f(\omega) - \beta\mathsf{E}(\omega). \tag{1-9}$$

If the min-max principle applies, it follows immediately that $\mathcal{S}$ is concave as it is now written as the infimum over affine functions of e. The rigorous justification of the min-max theorem is included in the Appendix, whose main requirement is the uniqueness of maximizers of F_β in $\mathcal{O}_{\mathrm{in}}$ for every $\beta\in\mathbb{R}$. This discussion leads to the following conditional result.

Theorem 1 (uniqueness implies concavity). *Assume that $\omega^{\mathrm{in}} = \mathbb{1}_A$, for some $A\subset M$, and suppose that $f\in C([0,\infty))\cap C^1((0,\infty))$ is strictly concave with $f'(z)\to-\infty$ as $z\to 0^+$. If for any achievable energy level the constrained entropy maximization problem* (1-5) *has a unique maximizer*, *then the maximal entropy functional $\mathcal{S}$ in* (1-6) *is strictly concave.*

Remark. The uniqueness of the entropy maximization problem (1-5) can be relaxed to allow uniqueness up to transformations that preserve the entropy and the energy, see Remark A.2. For instance, in the case of radially symmetric domains, we can for instance allow for uniqueness up to rotations.

Showing the uniqueness property required in the above Theorem 1 in general settings is an interesting and challenging open question. In statistical physics, nonuniqueness of the equilibrium state is directly related to phase transitions in the associated system [Georgii 2011; Delgadino et al. 2021; 2023]. For interacting particle [Carrillo et al. 2020; Chayes and Panferov 2010] and spin glass systems, several toy models exhibiting discontinuous phase transitions exist. In the case of $2d$ Euler, it could be expected that for nontrivial geometries, a discontinuous phase transitions takes place (specifically, the relevant type are first-order or discontinuous phase transitions).

While in the variational problem it is just a Lagrange multiplier, β plays the role of the inverse temperature in the language of statistical mechanics. More specifically, if there exists a unique $\beta=\beta(\mathsf{e})$ that attains the infimum in (1-8), we obtain the classical statistical mechanics identity (Clausius law)

$$\beta = \frac{\mathrm{d}\mathcal{S}}{\mathrm{de}}.$$

That is to say, the derivative of the maximal entropy with respect to the energy is the inverse temperature of the system. In particular, $\beta(\mathsf{e})$ is strictly decreasing in e, and hence we can write the energy associated to a given inverse temperature level β (see Figure 1).

1.2. ***Main results for vortex patches.*** One of the purposes of this article is to show the uniqueness of maximizers in (1-5) in the case when ω^{in} is a vortex patch on the disk, by exploiting the theory of radial rearrangements. We therefore consider $M=\mathbb{D}$, the unit disk centered at the origin, and ω^{in} the indicator

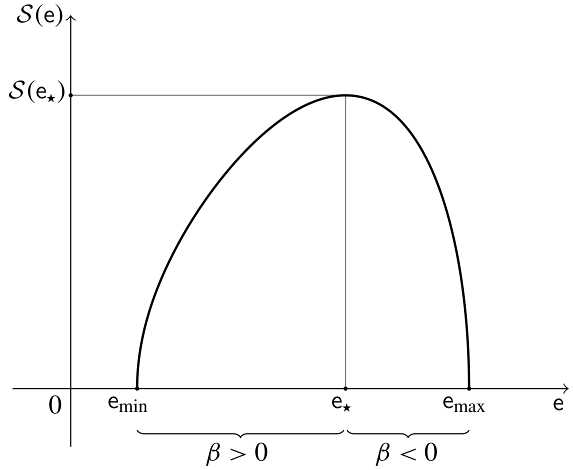

Figure 1. The function $\mathsf{e} \mapsto \mathcal{S}(\mathsf{e})$. The three cases $\beta \to -\infty$, $\beta = 0$ and $\beta \to \infty$ are associated with the energy levels $\mathsf{e}_{\max}$, $\mathsf{e}_\star$ and $\mathsf{e}_{\min}$, respectively.

function of a set $A \subset \mathbb{D}$, i.e.,

$$\omega^{\text{in}}(x) = \mathbb{1}_A(x), \qquad \frac{|A|}{|\mathbb{D}|} = \mathsf{m} \in (0, 1). \tag{1-10}$$

In this context, the corresponding set in (1-3) takes the particularly amenable form [Dolce and Drivas 2022]

$$\mathcal{O} := \left\{ \omega \in L^\infty : 0 \le \omega \le 1, \ \frac{1}{|\mathbb{D}|} \int_{\mathbb{D}} \omega(x)\,\mathrm{d}x = \mathsf{m} \right\}. \tag{1-11}$$

Since $\mathcal{O}$ is weak-$*$ compact, the weak continuity of E implies the existence of a maximum and a minimum, denoted $\mathsf{e}_{\max}$ (resp. $\mathsf{e}_{\min}$), achieved at vorticity $\omega_{\max}$ (resp. $\omega_{\min}$). Also, since $0 \notin \mathcal{O}$ and $\mathsf{E}(\omega) = 0$ if and only if $\omega = 0$, it necessarily holds that $\mathsf{e}_{\min} > 0$. In fact, $\mathsf{e}_{\max}$ and $\mathsf{e}_{\min}$ and corresponding vorticities can be computed explicitly, see Lemma 3.2. Given $\mathsf{e} \in [\mathsf{e}_{\min}, \mathsf{e}_{\max}]$, we are interested in the maximization problem

$$\text{maximize } \mathsf{S}(\omega) \text{ subject to } \omega \in \mathcal{O} \text{ and } \mathsf{E}(\omega) = \mathsf{e}, \tag{1-12}$$

where S denotes the Boltzmann entropy

$$\mathsf{S}(\omega) := -\frac{1}{|\mathbb{D}|} \int_{\mathbb{D}} \omega(x) \log \omega(x)\,\mathrm{d}x. \tag{1-13}$$

Remark (on the choice of entropy). Our analysis does not heavily rely on the specific form (1-13), although for the patch problem, several other entropies can be considered, consistent with classical theories of statistical hydrodynamics. For instance, in the Robert–Sommeria–Miller theory [Robert 1990; 1991; Miller 1990; Robert and Sommeria 1991], the choice of S should be dictated by the form of the initial datum: assume that $\mathbb{D}$ is partitioned into a disjoint union of sets $\{A_\ell\}_{\ell=1}^N$, and $\omega^{\text{in}} = \sum_\ell a_\ell \mathbb{1}_{A_\ell}$ with

$a_\ell \in [0, 1]$. Then one can define the entropy (generated by ω^{in}) as

$$\mathsf{S}_{\mathsf{rsm}}(\omega) := \sup\left\{ -\frac{1}{|\mathbb{D}|}\int_{\mathbb{D}} \sum_\ell \rho_\ell(x)\log\rho_\ell(x)\,\mathrm{d}x \ :\ \omega = \sum_\ell a_\ell\rho_\ell,\ 0 \le \rho_\ell \le 1,\ \sum_\ell \rho_\ell = 1 \right\}. \quad (1\text{-}14)$$

The case of a vortex patch (1-10) leads to explicit computations: since $A_1 = A$, $A_2 = \mathbb{D}\setminus A$, $a_1 = 1$ and $a_2 = 0$, the only possible choice in (1-14) is when $\rho_1 = \omega$ and $\rho_2 = 1-\omega$, leading to

$$\mathsf{S}_{\mathsf{rsm}}(\omega) := -\frac{1}{|\mathbb{D}|}\int_{\mathbb{D}} \big[\omega(x)\log\omega(x) + (1-\omega(x))\log(1-\omega(x))\big]\,\mathrm{d}x,$$

Another possibility, introduced by Turkington [1999], is to consider a different maximization problem compared to (1-14), that, in the case of a vortex patch, allows vorticity to mix on the whole range of small scales $a \in [a_2, a_1] = [0, 1]$. This leads to the entropy

$$\mathsf{S}_{\mathsf{t}}(\omega) := \frac{1}{|\mathbb{D}|}\int_{\mathbb{D}} f_{\mathsf{t}}(\omega(x))\,\mathrm{d}x,$$

with

$$f_{\mathsf{t}}(\omega) = \sup\left\{ -\int_0^1 \rho(y)\log\rho(y)\,\mathrm{d}y \ :\ \rho \ge 0,\ \int_0^1 \rho(y)\,\mathrm{d}y = 1,\ \int_0^1 y\rho(y)\,\mathrm{d}y = \omega \right\}.$$

This amounts to performing an entropy maximization over all probability densities $\rho\,\mathrm{d}y$ in (0, 1) which give ω as their mean value.

The first main result of this article is the following characterization of the maximal entropy functional, in complete analogy with classical statistical mechanics.

Theorem 2. *The function* $\mathcal{S} : [\mathsf{e}_{\min}, \mathsf{e}_{\max}] \to [0, \infty)$ *defined as*

$$\mathcal{S}(\mathsf{e}) = \max\{\mathsf{S}(\omega) \ :\ \omega \in \mathcal{O},\ \mathsf{E}(\omega) = \mathsf{e}\}$$

is strictly concave. Moreover,

(a) $\mathcal{S}$ *has a unique*, *strictly positive global maximum at* $\mathsf{e}_\star = \pi\,\mathsf{m}^2/16$;

(b) $\mathcal{S}(\mathsf{e}_{\min}) = \mathcal{S}(\mathsf{e}_{\max}) = 0$;

(c) $\mathcal{S}$ *is increasing on* $[\mathsf{e}_{\min}, \mathsf{e}_\star]$ *and decreasing on* $[\mathsf{e}_\star, \mathsf{e}_{\max}]$.

Remark. Theorem 2(c) is a consequence of concavity and Theorem 2(a)–(b).

In light of Theorem 1, the rigorous justification of the min-max principle in the Appendix needed for Theorem 2 requires the uniqueness of maximizers of F_β in $\mathcal{O}$ for every $\beta \in \mathbb{R}$. This is the second main result of this article.

Theorem 3. *For any* $\beta \in \mathbb{R}$, *there exists a unique solution* $\omega_\beta \in \mathcal{O}$ *of the maximization problem*

$$\textit{maximize } \mathsf{F}_\beta(\omega) \textit{ subject to } \omega \in \mathcal{O}.$$

Moreover,

- *if* $\beta \ge 0$, *then* ω_β *is radially increasing*,
- *if* $\beta \le 0$, *then* ω_β *is radially decreasing.*

In particular, ω_β *is constant when* $\beta = 0$.

Our uniqueness proof follows closely the developments in uniqueness of steady states for the standard Keller–Segel model [Calvez and Carrillo 2012], and their homogeneous variants [Calvez et al. 2021; Carrillo et al. 2015]. In broad terms, the strategy is to find suitable interpolation curves between two competitor states, such that the free energy over the curve is convex or monotone. The seminal paper of McCann [1997] implements this idea with the interpolation curves given by the geodesics of the optimal transportation distance. We also mention the novel interpolation curve in [Delgadino et al. 2022] for radially decreasing states, which unfortunately is not directly applicable to our setting. Inspired by [Calvez and Carrillo 2012; Calvez et al. 2021; Carrillo et al. 2015], we first use rearrangement theory [Talenti 1976; Kesavan 2006] to reduce the problem to one-dimensional radially decreasing profiles. Then we consider the optimal transportation interpolation between a maximizer and a competitor. Finally, employing Jensen's inequality and the Euler–Lagrange equation for the maximizer, we can show the strict monotonicity of the free energy along the interpolation curve. One of the main differences with [Calvez et al. 2021; Carrillo et al. 2015] is that we need to deal with the L^∞ constraint, imposed by (1-11).

1.3. ***On the conservation of angular momentum.*** The constrained maximization problem (1-12) takes into account the conservation laws (1-2), but neglects the additional symmetries of the disk that give rise to conservation of *angular momentum*

$$\mathsf{A}(\omega) = -\int_{\mathbb{D}} \tfrac{1}{2}(1-|x|^2)\omega(x)\,\mathrm{d}x = \int_{\mathbb{D}} x^{\perp}\cdot \boldsymbol{u}(x)\,\mathrm{d}x, \tag{1-15}$$

which is weak-$*$ continuous, like the energy E.

The inclusion of this extra constraint changes the picture dramatically. Indeed, entropy maximizers are no longer necessarily radially symmetric, as the following heuristics illustrate. Ignoring for the moment the L^∞ constraint that the set $\mathcal{O}$ in (1-11) imposes, the limiting case of maximal angular momentum $\mathsf{a}_{\max} = 0$ is achieved only for states ω which are supported on the boundary $\partial\mathbb{D}$. In which case, if ω_{rad} is radial then $\omega_{\mathrm{rad}} = c\delta_{\partial\mathbb{D}}$ and $\mathsf{E}(\omega_{\mathrm{rad}}) = 0$. On the other hand, the nonradial state $\omega_{x_0} = \delta_{x_0}$ with $x_0 \in \partial\mathbb{D}$ also has zero angular momentum, and formally has unbounded energy $\mathsf{E}(\omega_{x_0}) = +\infty$. This extreme situation hints that constraining the angular momentum implies there are energy levels that are not achievable by radial vorticities. A rigorous statement describing this situation is contained in the following result.

Theorem 4. *For* $\mathsf{m} \in (0, 1)$ *and* $L \geq 1$, *consider the set*

$$\mathcal{O}_L := \left\{ \omega \in L^\infty \,:\, 0 \leq \omega \leq L,\ \frac{1}{|\mathbb{D}|}\int_{\mathbb{D}} \omega(x)\,\mathrm{d}x = \mathsf{m} \right\}.$$

For the radial optimization problem, there exists $C > 1$ *independent of* L, a *and* m *such that*

$$\sup\left\{ \mathsf{E}(\omega) \,:\, \omega \in \mathcal{O}_L,\ \mathsf{A}(\omega) = \mathsf{a},\ \omega \text{ is radial} \right\} \leq C\left(\mathsf{m}|\mathsf{a}| + |\mathsf{a}|^2 \log\left(\frac{L}{|\mathsf{a}|}\right)\right). \tag{1-16}$$

For the nonradial case, if $L \geq 4\pi^2\mathsf{m}^3/|\mathsf{a}|^2$, *we have the lower bound*

$$\sup\{ \mathsf{E}(\omega) \,:\, \omega \in \mathcal{O}_L,\ \mathsf{A}(\omega) = \mathsf{a}\} \geq \frac{\pi\,\mathsf{m}^2}{4} \log\left(\frac{L|\mathsf{a}|^2}{64\pi^2\mathsf{m}^3}\right). \tag{1-17}$$

In particular, there exist $\mathsf{a}_* \in \left(-\frac{1}{2}, 0\right)$ *and* $Q > 2\pi$ *depending on* m, *such that if* $\mathsf{a} \in (\mathsf{a}_*, 0)$ *and* $L = Q^2 m^3/|\mathsf{a}|^2$, *then*

$$\sup\left\{\mathsf{E}(\omega) : \omega \in \mathcal{O}_L,\ \mathsf{A}(\omega) = \mathsf{a},\ \omega \textit{ is radial}\right\} < \sup\left\{\mathsf{E}(\omega)\ :\ \omega \in \mathcal{O}_L,\ \mathsf{A}(\omega) = \mathsf{a}\right\}.$$

The radial bound (1-16) follows by utilizing the formula

$$\mathsf{E}(\omega) = \frac{1}{4\pi}\int_0^1 \frac{1}{r}\left|\int_{B_r} \omega(x)\,\mathrm{d}x\right|^2 \mathrm{d}r,$$

which is valid only for radial functions. The bound follows by estimating the amount of vorticity near the origin, using the L^∞ bound and the angular momentum. The lower bound (1-17) follows by calculating the energy of a vortex patch of the form

$$\omega_{x_0,L} = L\mathbb{1}_{B_{\sqrt{\mathsf{m}/L}}(x_0)}.$$

The complete proof of Theorem 4 is postponed to Section 5.

We note that the proof of Theorem 4 is similar in spirit to [Dolce and Drivas 2022, Theorem 2]. The main difference is that Dolce and Drivas consider the case of a periodic channel (hence not simply connected) instead of the disk. In their case, the conserved quantity of interest is the linear momentum instead of the angular momentum.

1.4. ***On the stability of Onsager solutions.*** The Euler–Lagrange equations associated to the variational problem (1-12) resemble those appearing in the context of mean-field limits of point-vortices studied in [Caglioti et al. 1992; 1995; Eyink and Spohn 1993; Kiessling 1993]. Specifically, in the setting of the unit disk and for $\beta \in (-8\pi, \infty)$, there exists a unique radial solution to the mean field equation

$$\omega_\beta = \frac{\mathrm{e}^{\beta\psi_\beta}}{Z_\beta} \qquad \text{and} \qquad \begin{cases} \Delta\psi_\beta = \omega_\beta & \text{in } \mathbb{D}, \\ \psi_\beta = 0 & \text{on } \partial\mathbb{D}, \end{cases} \tag{1-18}$$

where

$$Z_\beta = \int_{\mathbb{D}} \mathrm{e}^{\beta\psi_\beta(x)}\,\mathrm{d}x,$$

which is given in radial variables by

$$\omega_\beta(r) = \frac{1 - A(\beta)}{\pi}\frac{1}{(1 - A(\beta)r^2)^2}, \quad \text{with } A(\beta) = \frac{\beta}{8\pi + \beta}. \tag{1-19}$$

We call such steady Euler flows *Onsager solutions*, as they appeared first in [Onsager 1949]. A result of [Caglioti et al. 1995], rephrased with the terminology of this article, states that such solutions arise as maximizers of the same free energy F_β in (1-9) over the set

$$\mathcal{P} = \left\{\omega \in L^1\ :\ \omega \geq 0,\ \int_{\mathbb{D}} \omega(x)\,\mathrm{d}x = 1,\ \int_{\mathbb{D}} \omega(x)\log\omega(x)\,\mathrm{d}x < \infty\right\}.$$

Moreover, we notice by (1-19) that as $\beta \to -8\pi$ we have $\omega_\beta \rightharpoonup \delta_0$, see Theorem 6.1 below for a precise statement. This restriction of $\beta > -8\pi$ did not apply to the previous results as we considered the vortex patch problem, which has the conservation of the L^∞ norm which prevents blow-up.

Quantitative stability of these solutions in L^1 has been addressed in [Lemou 2016], using the stability of the Hardy–Littlewood inequality applied to angular momentum. The techniques developed in the proof of Theorem 3 allow us to prove the following qualitative L^2-stability result.

Theorem 5. *For any $\beta \in (-8\pi, \infty)$, the solution ω_β to* (1-18) *is L^2 stable with respect to L^∞ perturbations. That is, for any $\varepsilon > 0$ and any positive $\omega^{\text{in}} \in L^\infty$, there exists $\delta = \delta(\varepsilon, \|\omega^{\text{in}}\|_{L^\infty}) > 0$ such that if $\|\omega^{\text{in}} - \omega_\beta\|_{L^2} < \delta$, the corresponding Euler solution $\omega = \omega(t)$ is such that $\|\omega(t) - \omega_\beta\|_{L^2} < \varepsilon$ for any $t > 0$.*

The case $\beta \geq 0$ follows from the classical method of Arnold [1966; Arnold and Khesin 1998], as the right-hand side of (1-18) is an increasing function (see also [Gallay and Šverák 2024] for a recent revisitation of the method). However, the case $\beta < 0$ is nontrivial, and it will be our main focus. Indeed, Arnold criteria for stability of steady Euler solutions satisfying the semilinear elliptic equation $\Delta\psi = F(\psi)$ require that

$$-\lambda_1 < F' < 0 \quad \text{or} \quad 0 < F' < \infty,$$

where $\lambda_1 > 0$ is the first eigenvalue of the Dirichlet Laplacian. Such a condition is clearly violated by (1-18).

To obtain this Lyapunov stability result we make use of a quantitative Jensen's inequality and an adaptation of Talenti's original argument [1976]. In particular, we borrow ideas from [Amato et al. 2024], which combine the arguments of the quantitative versions of Polya–Szegő [Cianchi et al. 2008] and Hardy–Littlewood [Cianchi and Ferone 2008] inequalities, specialized to the solutions of the Poisson equation to obtain a quantitative version of Talenti's inequality.

We mention that the uniqueness of Onsager-type solutions in the sphere $\mathbb{S}^2$ was recently addressed in [Gui and Moradifam 2018] by studying Onofri's inequality [1982], which settled a conjecture in conformal geometry [Chang and Yang 1987]. The stability of Theorem 5 is related to the dual formulation of Onofri's inequality, which was exploited recently in [Carlen 2025] to obtain stability of the log-HLS inequality.

2. Uniqueness implies concavity

The purpose of this section is to prove Theorem 1, for vortex patches of the form (1-10) in a general two-dimensional simply connected domain $M \subset \mathbb{R}^2$. In which case, we will make use of the characterization (1-11) for the orbit of the patch, with the disk $\mathbb{D}$ replaced by M. As mentioned in Section 1.2, the concavity of $\mathcal{S}$ is a consequence of the min-max principle stated in Proposition A.1. The only nontrivial requirement is stated in Proposition A.1(e), which requires the uniqueness of maximizers $\omega_\beta \in \mathcal{O}$ of the functional

$$\mathsf{L}(\omega, \beta) = \beta\mathsf{e} + \mathsf{S}_f(\omega) - \beta\mathsf{E}(\omega)$$

for a given fixed $\beta \in \mathbb{R}$.

In the generality of Section 1.1, the energy functional E still achieves a maximum and minimum values $\mathsf{e}_{\max} \geq \mathsf{e}_{\min} \geq 0$ on $\mathcal{O}$. Theorem 1 is then a consequence of Proposition A.1 and the following result.

Proposition 2.1. *Assume that the free energy F_β has a unique maximizer $\omega_\beta \in \mathcal{O}$ for each $\beta \in \mathbb{R}$. Then the function $\mathsf{e} \mapsto \mathcal{S}(\mathsf{e})$ is strictly concave on* $[\mathsf{e}_{\min}, \mathsf{e}_{\max}]$.

We start by deriving an Euler–Lagrange equation, which we will use in the proof of Proposition 2.1.

Lemma 2.2. *Assume* $f \in C([0,\infty)) \cap C^1((0,\infty))$ *is concave and* $\lim_{z\to 0^+} f'(z) = -\infty$. *Any maximizer* $\bar\omega$ *of* F_β *over* $\mathcal{O}$ *satisfies* $\inf \bar\omega > 0$, *and there exists* $\bar\lambda = \bar\lambda(\bar\omega)$ *such that*

$$\frac{1}{|M|} f'(\bar\omega) + \beta\bar\psi = \bar\lambda \quad \textit{a.e. on } \{\bar\omega < 1\}, \tag{2-1}$$

where

$$\begin{cases} \Delta\bar\psi = \bar\omega & \textit{in } M, \\ \bar\psi = 0 & \textit{on } \partial M. \end{cases}$$

Proof. To prove (2-1), we consider a positive smooth function φ, such that

$$\frac{1}{|M|}\int_M (1-\bar\omega)\varphi = 1. \tag{2-2}$$

We take the perturbation

$$\bar\omega_\varepsilon = \frac{\mathsf{m}}{\mathsf{m}+\varepsilon}(\bar\omega + \varepsilon(1-\bar\omega)\varphi),$$

with $\varepsilon > 0$ small enough to satisfy $\bar\omega_\varepsilon \in \mathcal{O}$. Taking a variation of the entropy, and using concavity, we know

$$\liminf_{\varepsilon\to 0^+} \frac{\mathsf{S}_f(\bar\omega_\varepsilon) - \mathsf{S}_f(\bar\omega)}{\varepsilon} \geq -\frac{1}{|M|}\int_M f'(\bar\omega)(1-\bar\omega)\varphi - \frac{1}{\mathsf{m}}\mathsf{S}_f(\bar\omega).$$

Similarly, taking a variation of the Energy we get

$$\lim_{\varepsilon\to 0^+} \frac{\mathsf{E}(\bar\omega_\varepsilon) - \mathsf{E}(\bar\omega)}{\varepsilon} = -\int_M \bar\psi(1-\bar\omega)\varphi - \frac{2}{\mathsf{m}}\mathsf{E}(\bar\omega).$$

Using the maximality property of $\bar\omega$, we know that

$$\lim_{\varepsilon\to 0^+} \frac{\mathsf{F}_\beta(\bar\omega_\varepsilon) - \mathsf{F}_\beta(\bar\omega)}{\varepsilon} \leq 0,$$

which immediately implies that

$$\int_M \left(\frac{f'(\bar\omega)}{|M|} + \beta\bar\psi\right)(1-\bar\omega)\varphi \geq \bar\lambda(\bar\omega) := \frac{1}{\mathsf{m}}(\mathsf{S}(\bar\omega) - 2\beta\mathsf{E}(\bar\omega))$$

for any positive and smooth test function φ which satisfies (2-2). Therefore,

$$\frac{f'(\bar\omega)}{|M|} + \beta\bar\psi \geq \bar\lambda \quad \text{a.e. on } \{\bar\omega < 1\}.$$

Using that $f'(0) = -\infty$ and that $\bar\psi$ is uniformly bounded, we have that $\bar\omega$ is uniformly bounded below in M. Hence, we have that the perturbation

$$\bar\omega_\varepsilon = \frac{\mathsf{m}}{\mathsf{m}-\varepsilon}(\bar\omega - \varepsilon\varphi) \in \mathcal{O}$$

for any positive smooth φ satisfying

$$\frac{1}{|M|}\int_M \varphi = 1$$

and $\varepsilon > 0$ small enough. Following the same arguments as above, we obtain the reverse inequality

$$\frac{f'(\bar\omega)}{|M|} + \beta\bar\psi \leq \bar\lambda \quad \text{a.e. on } M,$$

and (2-1) follows. ◻

Proof of Proposition 2.1. Applying the min-max principle in Proposition A.1, the maximal entropy function can be written as an infimum over affine functions in e, namely

$$\mathcal{S}(\mathsf{e}) = \inf_{\beta\in\mathbb{R}} \Big\{ \beta\mathsf{e} + \max_{\omega\in\mathcal{O}_{\text{in}}} \mathsf{F}_\beta(\omega) \Big\}, \tag{2-3}$$

and hence it is concave in e. We now verify its strict concavity. For any $\beta \in \mathbb{R}$, let

$$g(\beta) := \max_{\omega\in\mathcal{O}_{\text{in}}} \mathsf{F}_\beta(\omega).$$

Being the supremum of affine functions of β, g is a proper lower semicontinuous convex function. Therefore, for any $\beta \in \mathbb{R}$, the subdifferential $\partial g(\beta)$ is nonempty and monotone. We define the set

$$\beta(\mathsf{e}) := \{\beta \in \mathbb{R} : \mathcal{S}(\mathsf{e}) = \beta\mathsf{e} + g(\beta)\}.$$

We claim that $\beta(\mathsf{e})$ is nonempty in $(\mathsf{e}_{\min}, \mathsf{e}_{\max})$, single-valued and monotone as a function of e.

• $\beta(\mathsf{e})$ **is nonempty.** For $\mathsf{e} \in (\mathsf{e}_{\min}, \mathsf{e}_{\max})$, we first show that

$$\lim_{\beta\to\pm\infty} (\beta\mathsf{e} + g(\beta)) = +\infty. \tag{2-4}$$

The expression for the energy optimizers $\omega_{\max}$ and $\omega_{\min}$ can be computed explicitly (see (3-6) and (3-7)). In particular, $\omega_{\max}, \omega_{\min} \in \{0, 1\}$, and hence $\mathsf{S}_f(\omega_{\min})$, $\mathsf{S}_f(\omega_{\max})$ are finite.

For $\beta > 0$, we have

$$-\beta\mathsf{e}_{\min} + \mathsf{S}_f(\omega_{\min}) = \mathsf{F}_\beta(\omega_{\min}) \le \max_{\omega\in\mathcal{O}_{\text{in}}} \mathsf{F}_\beta(\omega) = g(\beta),$$

which implies the bound

$$\beta(\mathsf{e} - \mathsf{e}_{\min}) + \mathsf{S}_f(\omega_{\min}) \le \beta\mathsf{e} + g(\beta),$$

and shows that $\lim_{\beta\to\infty} (\beta\mathsf{e} + g(\beta)) = +\infty$. Similarly, for $\beta < 0$, we have the bound

$$-\beta\mathsf{e}_{\max} + \mathsf{S}_f(\omega_{\max}) = \mathsf{F}_\beta(\omega_{\max}) \le g(\beta),$$

which implies the bound

$$-\beta(\mathsf{e}_{\max} - \mathsf{e}) + \mathsf{S}_f(\omega_{\max}) \le \beta\mathsf{e} + g(\beta),$$

and the claim (2-4) follows. If $\{\beta_n\} \subset \mathbb{R}$ is a minimizing sequence such that

$$\mathcal{S}(\mathsf{e}) = \lim_{n\to\infty} (\beta_n\mathsf{e} + g(\beta_n)),$$

then by (2-4) we must have that $\{\beta_n\}$ is bounded and therefore has a limit point $\bar{\beta}$. By the lower semicontinuity of f we find from (2-3) that $\bar{\beta}\mathsf{e} + g(\bar{\beta}) = \mathcal{S}(\mathsf{e})$, proving the claim.

• **Characterization.** If $\beta \in \beta(\mathsf{e})$, then

$$-\mathsf{e} \in \partial g(\beta), \tag{2-5}$$

because β is a minimizer. Next we show that the subdifferential of f is given by

$$\partial g(\beta) = \{-\mathsf{E}(\omega_\beta)\}, \tag{2-6}$$

where ω_β is the unique maximizer of F_β over $\mathcal{O}_{\rm in}$. Since g is the pointwise supremum of affine functions $\mathsf{G}_\omega(\beta) := \mathsf{S}_f(\omega) - \beta\mathsf{E}(\omega)$ over $\omega \in \mathcal{O}_{\rm in}$, the subdifferential $\partial g(\beta)$ is in general [Zălinescu 2002, Theorem 2.4.18] given by

$$\partial g(\beta) = \mathrm{co}\biggl(\bigcup_{\omega\in\mathcal{O}_{\rm in}} \{\partial \mathsf{G}_\omega(\beta) \,:\, \mathsf{G}_\omega(\beta) = g(\beta)\}\biggr),$$

where $\mathrm{co}(B)$ denotes the closed convex hull of the set B. Due to our uniqueness assumption and the fact that G_ω is differentiable, in our case the above identity reduces to

$$\partial g(\beta) = \left\{\frac{\mathrm{d}}{\mathrm{d}\beta} G_\omega(\beta)\Big|_{\omega=\omega_\beta}\right\} = \{-\mathsf{E}(\omega_\beta)\},$$

which is (2-6).

• **Strict monotonicity.** Thanks to (2-5)–(2-6), the function $\beta(\mathsf{e})$ is given implicitly by the equation

$$\mathsf{E}(\omega_{\beta(\mathsf{e})}) = \mathsf{e}.$$

Hence $\beta(\mathsf{e})$ is well-defined and monotone if we can prove that the mapping $\beta \to \mathsf{E}(\omega_\beta)$ is strictly monotone.

Given $\beta_1 < \beta_2$, we want to show that $\mathsf{E}(\omega_{\beta_1}) > \mathsf{E}(\omega_{\beta_2})$. By the Euler–Lagrange condition (2-1), we know that $\omega_{\beta_1} \neq \omega_{\beta_2}$. Hence, by uniqueness of maximizers, we have

$$\mathsf{F}_{\beta_1}(\omega_{\beta_1}) > \mathsf{F}_{\beta_1}(\omega_{\beta_2}), \quad \mathsf{F}_{\beta_2}(\omega_{\beta_2}) > \mathsf{F}_{\beta_2}(\omega_{\beta_1}),$$

which implies

$$(\beta_2-\beta_1)(\mathsf{E}(\omega_{\beta_2}) - \mathsf{E}(\omega_{\beta_1})) = (\mathsf{F}_{\beta_1}(\omega_{\beta_2}) - \mathsf{F}_{\beta_1}(\omega_{\beta_1})) + (\mathsf{F}_{\beta_2}(\omega_{\beta_1}) - \mathsf{F}_{\beta_2}(\omega_{\beta_2})) < 0.$$

This shows the desired strict monotonicity. □

Remark. The assumption that $\omega^{\rm in} = \mathbb{1}_A$ for some $A \subset M$ is only used to show that for any $\beta_1 \neq \beta_2$ we have that $\omega_{\beta_1} \neq \omega_{\beta_2}$, which stems from the Euler–Lagrange condition (2-1).

3. Free energy and entropy maximizers

In this section we analyze various aspects of entropy maximization related to the free energy F_β. We specialize from now on in setting of Section 1.2, hence considering the vortex patch problem in the unit disk $M = \mathbb{D}$, with entropy given by (1-13). Since our analysis is based on rearrangements inequality, we first take some time to review the basic concepts of radial rearrangements. The assumptions on f are those of Theorem 1.

3.1. ***Rearrangements and radial symmetry of maximizers.*** A standard technique for studying maximizers is utilizing rearrangements of mass. Given a set $B \subset \mathbb{D}$, its symmetric rearrangement $B^\sharp$ is the open centered ball whose volume agrees with B, namely

$$B^\sharp = \{x \in \mathbb{R}^2 \,:\, \pi|x|^2 < |B|\}.$$

Given a function $\omega \in \mathcal{O}$, its *symmetric decreasing rearrangement* is defined by

$$\omega^\sharp(x) = \int_0^1 \mathbb{1}_{[\omega>t]^\sharp}(x)\,\mathrm{d}t.$$

Notice that $\omega^\sharp$ is radial. The *symmetric increasing rearrangement* of ω is

$$\omega_\sharp(x) = \omega^\sharp(\sqrt{1-|x|^2}\,).$$

It follows directly from a theorem of Talenti [1976, Theorem 1(v)] that

$$\mathsf{E}(\omega) = \tfrac{1}{2}\|\nabla\Delta^{-1}\omega\|_{L^2}^2 \le \tfrac{1}{2}\|\nabla\Delta^{-1}\omega^\sharp\|_{L^2}^2 = \mathsf{E}(\omega^\sharp) \tag{3-1}$$

for any $\omega \in \mathcal{O}$, with equality if and only if $\omega = \omega^\sharp$; see [Carlen and Loss 1992; Kesavan 2006]. We are also interested in comparing the kinetic energy among *radial* vorticities. If ω is radial, then the corresponding stream function can be explicitly derived from (1-1) as

$$\psi(r) = -\int_r^1 \frac{1}{s}\int_0^s \omega(\bar{s})\bar{s}\,\mathrm{d}\bar{s}\,\mathrm{d}s. \tag{3-2}$$

For any radial function $g \in \mathcal{O}$, define

$$M_g(r) = 2\pi\int_0^r g(s)s\,\mathrm{d}s = \int_{B_r} g(x)\,\mathrm{d}x, \quad r \in (0,1], \tag{3-3}$$

where B_r is the ball of radius r centered at the origin. Thanks to (3-2), for any radial $\omega \in \mathcal{O}$, it is not hard to see that

$$\mathsf{E}(\omega) = \pi\int_0^1 |\partial_r\psi(r)|^2 r\,\mathrm{d}r = \frac{1}{4\pi}\int_0^1 \frac{1}{r}|M_\omega(r)|^2\,\mathrm{d}r. \tag{3-4}$$

Moreover, we have the following comparison principle in the radial case.

Lemma 3.1. *Let $\omega \in \mathcal{O}$ be a radial function. Then*

$$\mathsf{E}(\omega_\sharp) \le \mathsf{E}(\omega) \le \mathsf{E}(\omega^\sharp).$$

Proof. Thanks to the Hardy–Littlewood inequality[1] and the fact that $(\mathbb{1}_{B_r})^\sharp = \mathbb{1}_{B_r}$, we have

$$\int_{\mathbb{D}} \mathbb{1}_{B_r}(x)\omega_\sharp(x)\,\mathrm{d}x \le \int_{\mathbb{D}} \mathbb{1}_{B_r}(x)\omega(x)\,\mathrm{d}x \le \int_{\mathbb{D}} \mathbb{1}_{B_r}(x)\omega^\sharp(x)\,\mathrm{d}x,$$

implying that $M_{\omega_\sharp} \le M_\omega \le M_{\omega^\sharp}$. The claim follows from (3-4). □

The representation (3-4) of E is also useful to compute explicitly the energy for specific vorticities. For instance,

$$\omega_\star \equiv \mathsf{m} \quad \Longrightarrow \quad \mathsf{e}_\star := \mathsf{E}(\omega_\star) = \frac{\pi\mathsf{m}^2}{16}. \tag{3-5}$$

[1] For any two measurable functions $f, g : \mathbb{D} \to [0,\infty)$, it holds that $\int_{\mathbb{D}} f^\sharp g_\sharp \le \int_{\mathbb{D}} fg \le \int_{\mathbb{D}} f^\sharp g^\sharp$.

Now, for any radial $\omega \in \mathcal{O}$, we have that $M_\omega(r) \le \pi \min\{r^2, \mathsf{m}\}$. Defining

$$\omega_{\max}(r) = \begin{cases} 1, & r \in (0, \sqrt{\mathsf{m}}), \\ 0, & r \in (\sqrt{\mathsf{m}}, 1), \end{cases} \tag{3-6}$$

we then have that $M_\omega \le M_{\omega_{\max}}$. Similarly,

$$M_\omega(r) = M_\omega(1) - 2\pi \int_r^1 \omega(s) s \, \mathrm{d}s \ge \pi(\mathsf{m} - 1 + r^2),$$

so that $M_\omega \ge M_{\omega_{\min}}$, where

$$\omega_{\min}(r) = \begin{cases} 0, & r \in (0, \sqrt{1-\mathsf{m}}), \\ 1, & r \in (\sqrt{1-\mathsf{m}}, 1). \end{cases} \tag{3-7}$$

As a consequence, by a direct computation of $\mathsf{E}(\omega_{\min})$ and $\mathsf{E}(\omega_{\max})$ we have the following result, which we state without proof.

Lemma 3.2. *For any radial $\omega \in \mathcal{O}$ we have*

$$\mathsf{e}_{\min} \le \mathsf{E}(\omega) \le \mathsf{e}_{\max},$$

where

$$\mathsf{e}_{\min} := \mathsf{E}(\omega_{\min}) = \frac{\pi \mathsf{m}^2}{16} - \frac{\pi}{8}(1-\mathsf{m})\big(\mathsf{m} + (1-\mathsf{m}) \log(1-\mathsf{m})\big)$$

and

$$\mathsf{e}_{\max} := \mathsf{E}(\omega_{\max}) = \frac{\pi \mathsf{m}^2}{16} + \frac{\pi \mathsf{m}^2}{8} |\log \mathsf{m}|.$$

In fact, the above functions satisfy the stronger property below.

Lemma 3.3. *The functions $\omega_{\min}$ and $\omega_{\max}$ are the unique functions that achieve their energy levels, that is*

$$\begin{aligned} \mathsf{E}(\omega) = \mathsf{e}_{\min} \quad &\Longrightarrow \quad \omega = \omega_{\min}, \\ \mathsf{E}(\omega) = \mathsf{e}_{\max} \quad &\Longrightarrow \quad \omega = \omega_{\max}. \end{aligned} \tag{3-8}$$

Proof. Since $\omega \mapsto \mathsf{E}(\omega)$ is a convex function, it has a unique global minimizer in $\mathcal{O}$. Moreover, such minimizer is radially symmetric, since E is invariant under rotation. Thus, Lemma 3.2 implies that $\omega_{\min}$ is the unique minimizer. Turning to (3-8), we know by (3-1) that any energy maximizer is necessarily radially decreasing, and by Lemma 3.2 that $\omega_{\max}$ is one of them. If $\bar\omega$ is another radially decreasing maximizer, it is not hard to see that $\bar\omega = \omega_{\max}$ if and only if $M_{\bar\omega} = M_{\omega_{\max}}$, i.e., if and only if $\mathsf{E}(\bar\omega) = \mathsf{E}(\omega_{\max})$. □

3.2. ***Relaxed maximization problems.*** The strategy to prove Theorem 2 is to study (relaxed versions of) the maximization problem (1-12) and apply a min-max principle. In turn, we will see how this is reduced to study uniqueness of maximizers for the free energy F_β in (1-9), as $\beta \in \mathbb{R}$ varies. We begin with the following observations.

Lemma 3.4. *For every $\mathsf{e} \in [\mathsf{e}_{\min}, \mathsf{e}_{\max}]$, the constrained maximization problem* (1-12) *admits at least one solution.*

Proof. Notice that S_f is strictly concave and upper semicontinuous. Let ω_n be a maximizing sequence. Since $\mathcal{O}$ is weak-$*$ compact, there exists $\bar\omega \in \mathcal{O}$ such that $\omega_n \overset{*}{\rightharpoonup} \bar\omega$ up to subsequences, and since $\mathsf{E}(\omega_n) \to \mathsf{E}(\bar\omega)$ by continuity, $\mathsf{E}(\bar\omega) = \mathsf{e}$. Moreover,

$$\sup\bigl\{\mathsf{S}_f(\omega) \,:\, \omega \in \mathcal{O},\ \mathsf{E}(\omega) = \mathsf{e}\bigr\} = \limsup_{n\to\infty} \mathsf{S}_f(\omega_n) \le \mathsf{S}_f(\bar\omega),$$

as S_f is upper semicontinuous. Hence $\bar\omega$ is a maximizer. □

It is also important to identify the global entropy maximizer, regardless of the energy constraint.

Lemma 3.5. *The unconstrained maximization problem*

$$\textit{maximize } \mathsf{S}_f(\omega) \textit{ subject to } \omega \in \mathcal{O}$$

has a unique maximizer given by the constant solution $\omega_\star \equiv \mathsf{m}$.

Proof. The proof follows from the previous Lemma 3.4, with the uniqueness stemming from the strict concavity of the entropy functional. Assume now that $\bar\omega$ is the unique maximizer, and let $\Phi : \mathbb{D} \to \mathbb{D}$ be a volume-preserving diffeomorphism. Then $\bar\omega \circ \Phi \in \mathcal{O}$, and $\mathsf{S}_f(\bar\omega \circ \Phi) = \mathsf{S}_f(\bar\omega)$. Thus, by uniqueness, $\bar\omega \circ \Phi = \bar\omega$, and since Φ is arbitrary, this implies $\bar\omega$ is constant, hence equal to its average. □

Thanks to the computation in (3-5), Theorem 2(a) is proved. The role of $\mathsf{e}_\star = \mathsf{E}(\omega_\star)$ in (3-5) is quite interesting. Indeed, for a fixed energy level $\mathsf{e} \le \mathsf{e}_\star$, the maximization problem (1-12) can be relaxed into a convex one.

Lemma 3.6. *If* $\mathsf{e}_{\min} \le \mathsf{e} \le \mathsf{e}_\star$, *then the constrained maximization problem* (1-12) *is equivalent to the relaxed problem*

$$\textit{maximize } \mathsf{S}_f(\omega) \textit{ subject to } \omega \in \mathcal{O},\ \mathsf{E}(\omega) \le \mathsf{e}. \tag{3-9}$$

In particular, *problem* (3-9), *and hence* (1-12), *have a unique solution.*

If $\mathsf{e}_\star \le \mathsf{e} \le \mathsf{e}_{\max}$, *then the constrained maximization problem* (1-12) *is equivalent to the relaxed problem*

$$\textit{maximize } \mathsf{S}_f(\omega) \textit{ subject to } \omega \in \mathcal{O},\ \mathsf{E}(\omega) \ge \mathsf{e}. \tag{3-10}$$

Proof. We first look at the case $\mathsf{e} \le \mathsf{e}_\star$ and argue by contradiction. Fix any maximizer $\bar\omega \in \mathcal{O}$ with $\mathsf{E}(\bar\omega) < \mathsf{e}$. For any $t \in [0, 1]$, the convex combination $\omega_t = (1-t)\bar\omega + t\omega_\star$ is in $\mathcal{O}$. Moreover, using the continuity of E, we have that for t small enough $\mathsf{E}(\omega_t) < \mathsf{e}$. Since $\bar\omega$ is a maximizer,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{S}_f(\omega_t)\Big|_{t=0} \le 0.$$

However, a direct computation shows

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{S}_f(\omega_t)\Big|_{t=0} = \frac{1}{|\mathbb{D}|}\int_{\mathbb{D}} (\omega_\star - \bar\omega) f'(\bar\omega) = \frac{1}{|\mathbb{D}|}\int_{\mathbb{D}} (\omega_\star - \bar\omega)(f'(\bar\omega) - f'(\omega_\star)) > 0,$$

hence reaching a contradiction. Since the condition $\mathsf{E}(\omega) \le \mathsf{e}$ in (3-9) is convex and the entropy functional is strictly concave, uniqueness of solution follows immediately. The case $\mathsf{e} \ge \mathsf{e}_\star$ is completely analogous, however (3-10) is not a convex problem and therefore uniqueness of solutions is not as immediate as the previous case. □

4. Uniqueness of maximizers at negative temperature

In light of Proposition 2.1, we have reduced the problem to the investigation of uniqueness of maximizers for the free energy F_β. This is the main result of this section.

Theorem 4.1. *For any $\beta \in \mathbb{R}$, there exists a unique maximizer ω_β of F_β over $\mathcal{O}$.*

For $\beta \geq 0$, F_β is strictly concave and therefore admits a unique maximizer, which is necessarily radial due to the invariance of the functionals under rotations. Hence, this section is devoted to the study of the case of negative inverse temperature $\beta < 0$.

4.1. *The Euler–Lagrange equations.* Despite the loss of concavity of F_β for $\beta < 0$, Talenti's inequality (3-1) implies that if ω is not radially symmetric and decreasing then

$$\mathsf{F}_\beta(\omega) = \mathsf{S}(\omega) - \beta\mathsf{E}(\omega) = \mathsf{S}(\omega^\sharp) - \beta\mathsf{E}(\omega) < \mathsf{S}(\omega^\sharp) - \beta\mathsf{E}(\omega^\sharp) = \mathsf{F}_\beta(\omega^\sharp). \tag{4-1}$$

Thus, any maximizer is radially symmetric and decreasing. First, we derive the Euler–Lagrange equations, which we refine from Lemma 2.2.

Lemma 4.2. *Let $\beta < 0$. Any maximizer $\bar\omega$ of F_β over $\mathcal{O}$ is radially decreasing and satisfies the following: there exist $r_{\bar\omega} \in [0, 1)$ and $\bar\lambda = \bar\lambda(\bar\omega) \in [0, \infty)$ such that*

$$\begin{aligned} \bar\omega(r) &= 1, \quad \text{on } [0, r_{\bar\omega}), \\ -\log\bar\omega(r) - \frac{\beta}{2}\int_r^1 \frac{1}{s} M_{\bar\omega}(s)\,\mathrm{d}s &= \bar\lambda, \quad \text{on } (r_{\bar\omega}, 1). \end{aligned} \tag{4-2}$$

As a consequence, $\bar\omega(r) \geq \mathrm{e}^{-\bar\lambda} > 0$ on $[0, 1]$.

Proof. By (4-1) we know that any maximizer must be radially symmetric and decreasing. Hence, the first property that there exists a $r_{\bar\omega} \in [0, 1)$ such that $\{\bar\omega = 1\} = B_{r_{\bar\omega}}$ follows immediately.

To prove (4-2), we simply rely on Lemma 2.2. In the case of Boltzmann entropy, $f'(z) = -1 - \log(z)$. Moreover, we can make use of the expression (3-2) for radial stream functions with (3-3) and conclude the proof. □

We use the nondegeneracy that stems from (4-2), to show that out of the set of possible maximizers $\bar\omega$, there exists at least one maximizer with the smallest possible radius $r_{\bar\omega}$.

Lemma 4.3. *Let $\beta < 0$. There exists a maximizer ω_β of F_β over $\mathcal{O}$ such that*

(a) $\{\omega_\beta = 1\} = [0, r_\beta)$;

(b) *For $\lambda_\beta = \bar\lambda(\omega_\beta)$ in (4-2), it holds that*

$$-\log\omega_\beta(r) - \frac{\beta}{2}\int_r^1 \frac{1}{s} M_{\omega_\beta}(s)\,\mathrm{d}s = \lambda_\beta \quad \text{on } (r_\beta, 1); \tag{4-3}$$

(c) $r_\beta \leq r_{\bar\omega}$ *for any maximizer $\bar\omega$.*

Proof. By the upper-semicontinuity of the entropy and the continuity of the energy, it follows that the set of maximizers

$$\mathcal{M} = \left\{ \bar{\omega} \in \mathcal{O} \,:\, \mathsf{F}_\beta(\bar{\omega}) = \max_{\omega \in \mathcal{O}} \mathsf{F}_\beta(\omega) \right\}$$

is weak-$*$ compact. As (c) suggests, we want r_β to be defined by

$$r_\beta = \inf_{\bar{\omega} \in \mathcal{M}} r_{\bar{\omega}}.$$

We consider $\{\bar{\omega}_n\}_{n\in\mathbb{N}} \subset \mathcal{M}$ a minimizing sequence such that

$$\lim_{n\to\infty} r_{\bar{\omega}_n} = r_\beta.$$

Using compactness, we know that, up to subsequence, there exist an accumulation point $\bar{\omega}_\beta \in \mathcal{M}$ such that $\bar{\omega}_n \overset{*}{\rightharpoonup} \omega_\beta$. The fact that ω_β satisfies the Euler–Lagrange condition (4-3) follows from the maximality, so that to complete the proof we just need to show (a), namely

$$\{\omega_\beta = 1\} = [0, r_\beta).$$

We notice that $\{\bar{\omega}_n\}_{n\in\mathbb{N}}$ is a sequence of radially decreasing and bounded functions, hence they are of bounded variation, which implies the convergences is strong in $L^p(\mathbb{D})$ for any $p \in [1, \infty)$. Using that $r_\beta \le r_{\bar{\omega}_n}$, we know that $\bar{\omega}_n = 1$ on $r \in [0, r_\beta)$, which implies the inclusion $[0, r_\beta) \subset \{\omega_\beta = 1\}$, for instance by passing to a further subsequence that converges pointwise. We will show the reverse inclusion by using the Euler–Lagrange equation for $\bar{\omega}_n$. Indeed, differentiating in r the Euler–Lagrange condition (4-2) for $\bar{\omega}_n$, we get the equation

$$-\frac{\partial_r \bar{\omega}_n(r)}{\bar{\omega}_n(r)} + \frac{\beta}{2}\frac{1}{r} M_{\omega_\beta}(r) = 0, \quad \text{for } r > r_n.$$

Using the strong convergence of $\bar{\omega}_n \to \omega_\beta$, and the distributional convergence of the derivatives $\partial_r \bar{\omega}_n \rightharpoonup \partial_r \omega_\beta$, we obtain that

$$\partial_r \omega_\beta(r) = \frac{\beta}{2} \omega_\beta(r) \frac{1}{r} M_{\omega_\beta}(r) < 0 \quad \text{for } r > r_\beta,$$

which implies that

$$\omega_\beta < 1 \quad \text{for } r > r_\beta. \qquad \square$$

Written for the corresponding radial stream function ψ_β, the Euler–Lagrange equation (4-3) reads

$$\Delta \psi_\beta = \mathrm{e}^{-\lambda_\beta} \mathrm{e}^{\pi \beta \psi_\beta}. \tag{4-4}$$

4.2. ***Uniqueness of maximizers.*** We will show that ω_β is in fact unique maximizer of the free energy F_β. We start by considering $\bar{\omega}$ a general radial competitor. Below we write an expression for $\mathsf{F}_\beta(\bar{\omega})$ in terms of ω_β and the Brenier [1991] map (or optimal transport map) between them. As both functions are radial (and decreasing), we know that the optimal mapping is also radial and increasing. Just like the one-dimensional case it can be represented implicitly by the cumulative distribution functions. To avoid some pathological regularity situations, from now on we assume that both the source and target measure are bounded above and below, which is satisfied by maximizers; see Equation (4-3). Namely, there exists a unique strictly increasing map $T : [0, 1] \to [0, 1]$ such that $T(0) = 0$, $T(1) = 1$ and

$$\int_0^r \omega_\beta(s) s \,\mathrm{d}s = \int_0^{T(r)} \bar{\omega}(s) s \,\mathrm{d}s \quad \text{for any } r \in [0, 1]. \tag{4-5}$$

In particular, from (3-3) we have

$$M_{\omega_\beta}(r) = 2\pi \int_0^r \omega_\beta(s) s \,\mathrm{d}s = 2\pi \int_0^{T(r)} \bar{\omega}(s) s \,\mathrm{d}s = M_{\bar{\omega}}(T(r)). \tag{4-6}$$

Using the Monge–Ampère equation associated to the change of variable we obtain the relationship

$$\frac{r}{T(r)T'(r)} \omega_\beta(r) = \bar{\omega}(T(r)) \quad \text{for any } r \in [0, 1]. \tag{4-7}$$

To simplify the notation we define the function

$$\phi(r) := \frac{T(r)T'(r)}{r}. \tag{4-8}$$

The next results is a comparison between the energy and entropy of ω_β and $\bar{\omega}$.

Lemma 4.4. *Let ω_β and $\bar{\omega}$ be two radial functions in $\mathcal{O}$ that are bounded below away from zero. Then*

$$\mathsf{S}(\bar{\omega}) - 2\int_0^1 \omega_\beta(r) \log \phi(r) r \,\mathrm{d}r = \mathsf{S}(\omega_\beta), \tag{4-9}$$

$$\mathsf{E}(\bar{\omega}) + \int_0^1 \left(\int_r^1 \frac{1}{s} \omega_\beta(s) M_{\omega_\beta}(s) \,\mathrm{d}s \right) \log \phi(r) r \,\mathrm{d}r \le \mathsf{E}(\omega_\beta), \tag{4-10}$$

where $\phi(r)$ is defined in (4-8).

Remark. We note that in the previous result we do not use the optimality in of $\bar{\omega}$ or ω_β in any strong way, it only requires that both vorticities are radial.

Proof of Lemma 4.4. Using the change of variable $r = T(s)$ given by the Brenier map (4-5), we can rewrite the entropy as

$$\mathsf{S}(\bar{\omega}) = -2\int_0^1 \bar{\omega}(r) \log(\bar{\omega}(r)) r \,\mathrm{d}r = -2\int_0^1 \bar{\omega}(T(s)) \log\big(\bar{\omega}(T(s))\big) T(s)T'(s) \,\mathrm{d}s.$$

Using (4-7), we obtain

$$\mathsf{S}(\bar{\omega}) = -2\int_0^1 \omega_\beta(s) \log\left(\frac{\omega_\beta(s)}{\phi(s)}\right) s \,\mathrm{d}s,$$

which coincides with the desired (4-9).

For the energy, we first integrate by parts to obtain the different representation

$$\mathsf{E}(\bar{\omega}) = \frac{1}{4\pi} \int_0^1 \frac{|M_{\bar{\omega}}(r)|^2}{r} \,\mathrm{d}r = -\int_0^1 M_{\bar{\omega}}(r) \bar{\omega}(r) \log(r) r \,\mathrm{d}r.$$

Next, we perform the change of variables $r = T(s)$ and use (4-6) and (4-7) to obtain

$$\mathsf{E}(\bar{\omega}) = -\int_0^1 M_{\bar{\omega}}(T(s)) \bar{\omega}(T(s)) \log(T(s)) T(s)T'(s) \,\mathrm{d}s = -\int_0^1 M_{\omega_\beta}(s) \omega_\beta(s) \log(T(s)) s \,\mathrm{d}s. \tag{4-11}$$

We use that $\phi(r) = [T^2]'/2r$ to rewrite

$$\log(T(s)) = \tfrac{1}{2}\log(T^2(s)) = \tfrac{1}{2}\log\left(2\int_0^s \phi(a)a\,\mathrm{d}a\right).$$

Then, we normalize the integral to be able to apply Jensen's inequality and we obtain

$$\log(T(s)) = \tfrac{1}{2}\log(s^2) + \tfrac{1}{2}\log\left(\frac{2}{s^2}\int_0^s \phi(a)a\,\mathrm{d}a\right) \geq \log(s) + \frac{1}{s^2}\int_0^s \log(\phi(a))a\,\mathrm{d}a.$$

Using this inequality on (4-11) we have

$$\mathsf{E}(\bar{\omega}) \leq \mathsf{E}(\omega_\beta) - \int_0^1 \left(\frac{1}{s^2}\int_0^s \log(\phi(a))a\,\mathrm{d}a\right) M_{\omega_\beta}(s)\omega_\beta(s)s\,\mathrm{d}s.$$

The result (4-10) follows by applying Fubini's theorem. □

Finally, we show that ω_β is the unique maximizer of F_β.

Lemma 4.5. *Let $\beta < 0$. Let r_β and ω_β be given by Lemma 4.3. Assume that $\bar{\omega} \in \mathcal{O}$ is radial, bounded below away from* 0, *and satisfies* $(0, r_\beta) \subset \{\bar{\omega} = 1\}$. *Then we have the inequality*

$$\mathsf{F}_\beta(\bar{\omega}) - 2\omega_\beta(1)\int_0^1 \log\phi(r)r\,\mathrm{d}r \leq \mathsf{F}_\beta(\omega_\beta), \tag{4-12}$$

where $\phi(r)$ is defined in (4-8). *Moreover, if $\bar{\omega} \neq \omega_\beta$, then the energy difference is positive, namely*

$$-2\omega_\beta(1)\int_0^1 \log\phi(r)r\,\mathrm{d}r > 0. \tag{4-13}$$

Proof. Applying Lemma 4.4, we have

$$\mathsf{F}_\beta(\bar{\omega}) + \int_0^1 \left(-2\omega_\beta(r) - \beta\int_r^1 \frac{1}{s}\omega_\beta(s)M_{\omega_\beta}(s)\,\mathrm{d}s\right)\log(\phi(r))r\,\mathrm{d}r \leq \mathsf{F}_\beta(\omega_\beta). \tag{4-14}$$

Notice that by the hypothesis $(0, r_\beta) \subset \{\bar{\omega} = 1\}$, Brenier's map is trivial on $[0, r_\beta)$. That is to say

$$\phi(r) = 1 \quad \text{on } [0, r_\beta).$$

For $r \in (r_\beta, 1)$, we can use the Euler–Lagrange equation (4-3) to simplify the remainder. More specifically, taking a derivative of (4-3) we obtain

$$\partial_r\omega_\beta(r) = \frac{\beta}{2}\frac{\omega_\beta(r)M_{\omega_\beta}(r)}{r}.$$

Integrating back on $(r, 1)$, we deduce that

$$\omega_\beta(1) = \omega_\beta(r) + \frac{\beta}{2}\int_r^1 \frac{1}{s}\omega_\beta(s)M_{\omega_\beta}(s)\,\mathrm{d}s \quad \text{for any } r \in (r_\beta, 1). \tag{4-15}$$

Replacing back (4-15) into (4-14), we obtain the desired (4-12).

To show (4-13), we apply Jensen's inequality

$$2\int_0^1 \log\phi(r)r\,\mathrm{d}r \leq \log\left(2\int_0^1 \phi(r)r\,\mathrm{d}r\right) = \log\left(2\int_0^1 T(r)T'(r)\,\mathrm{d}r\right) = \log(T^2(1) - T^2(0)) = 0. \tag{4-16}$$

The equality in Jensen's inequality can only occur if $\phi(r) = C$ is constant, which implies that Brenier's map $T(r) = r$ is the identity. The conclusion that the defect is positive if $\bar{\omega} \neq \omega_\beta$ follows directly from the previous argument, and the fact that $\omega_\beta(1) > 0$ by Lemma 4.2. □

We conclude this section with the proof of Theorem 4.1, which is a consequence of the results above.

Proof of Theorem 4.1. We will show that ω_β given in Lemma 4.3 is the unique maximizer of F_β. Assume $\bar{\omega}$ is also maximizer of F_β, then by Lemmas 4.2 and 4.3 it satisfies the hypothesis of Lemma 4.5. Applying Lemma 4.5 and $\mathsf{F}_\beta(\bar{\omega}) = \mathsf{F}_\beta(\omega_\beta)$, we obtain $\omega_\beta = \bar{\omega}$. □

5. Nonradial energy maximizers at fixed angular momentum

This section is dedicated to the proof of Theorem 4. Section 5.1 contains the upper bound on the kinetic energy E for radial functions on $\mathcal{O}_L$, and Section 5.2 contains the lower bound by computing the energy of an explicit vortex patch. The conclusion follows then by choosing L in terms of the angular momentum. As in the statement of Theorem 4, we let $\mathsf{m} \in (0, 1)$ and $L \geq 1$ and set

$$\mathcal{O}_L := \left\{ \omega \in L^\infty \,:\, 0 \leq \omega \leq L,\ \frac{1}{|\mathbb{D}|} \int_{\mathbb{D}} \omega(x)\,\mathrm{d}x = \mathsf{m} \right\}.$$

The proof is carried out in the next sections.

5.1. *Upper bounds on the kinetic energy for radial functions.*

We claim that there exists a constant $C \geq 1$, independent of L, a, and m, such that

$$\sup\left\{ \mathsf{E}(\omega) \,:\, \omega \in \mathcal{O}_L,\ \mathsf{A}(\omega) = \mathsf{a},\ \omega \text{ is radial} \right\} \leq C(\mathsf{m}|\mathsf{a}| + |\mathsf{a}|^2 \log(L/|\mathsf{a}|)). \tag{5-1}$$

Now, if $\omega \in \mathcal{O}_L$ is a radial function, from (1-15) we deduce that

$$\mathsf{A}(\omega) = -\frac{1}{2}\left[2\pi \int_0^1 \omega(r)(1 - r^2) r\,\mathrm{d}r \right] = -\frac{1}{2}\left[\pi\mathsf{m} - \int_0^1 \partial_r M_\omega(r) r^2\,\mathrm{d}r \right] = -\int_0^1 M_\omega(r) r\,\mathrm{d}r.$$

Thus, for every $r \in [0, 1]$, we have the pointwise identity

$$\tfrac{1}{2} M_\omega(r)(1 - r^2) = \tfrac{1}{2} \int_0^r \partial_s [M_\omega(s)(1 - s^2)]\,\mathrm{d}s = \pi \int_0^r \omega(s)(1 - s^2) s\,\mathrm{d}s - \int_0^r M_\omega(s) s\,\mathrm{d}s.$$

In particular,

$$\tfrac{1}{2} M_\omega(r)(1 - r^2) \leq \pi \int_0^r \omega(s)(1 - s^2) s\,\mathrm{d}s \leq -\mathsf{A}(\omega),$$

so that any radial function $\omega \in \mathcal{O}_L$ with $\mathsf{A}(\omega) = \mathsf{a}$ satisfies

$$M_\omega(r) \leq \min\left\{ \pi L r^2,\ \frac{2|\mathsf{a}|}{1 - r^2},\ \pi\mathsf{m} \right\}.$$

This implies

$$M_\omega(r) \leq \begin{cases} \pi L r^2, & r^2 \leq 1 - \sqrt{1 - \dfrac{8|\mathsf{a}|}{\pi L}}, \\ \dfrac{2|\mathsf{a}|}{1 - r^2}, & 1 - \sqrt{1 - \dfrac{8|\mathsf{a}|}{\pi L}} < r^2 \leq 1 - \dfrac{2|\mathsf{a}|}{\pi\mathsf{m}}, \\ \pi\mathsf{m}, & 1 - \dfrac{2|\mathsf{a}|}{\pi\mathsf{m}} < r^2 \leq 1. \end{cases}$$

Thanks to (3-4), we then have $\mathsf{E}(\omega) \le \mathsf{E}_1 + \mathsf{E}_2 + \mathsf{E}_3$, where

$$\mathsf{E}_1 = \frac{1}{4\pi}\int_0^{r^2=1-\sqrt{1-8|\mathsf{a}|/(\pi L)}} \frac{1}{r}|\pi L r^2|^2\,\mathrm{d}r = \frac{\pi L^2}{16}\left(1-\sqrt{1-\frac{8|\mathsf{a}|}{\pi L}}\right)^2 \lesssim |\mathsf{a}|^2,$$

$$\begin{aligned}\mathsf{E}_2 &= \frac{1}{4\pi}\int_{r^2=1-\sqrt{1-8|\mathsf{a}|/(\pi L)}}^{r^2=1-2|\mathsf{a}|/(\pi\mathsf{m})} \frac{1}{r}\left|\frac{2|\mathsf{a}|}{1-r^2}\right|^2\mathrm{d}r \\ &= \frac{|\mathsf{a}|^2}{2\pi}\left[\frac{\pi\mathsf{m}}{2|\mathsf{a}|}+\log\left(\frac{\pi\mathsf{m}}{2|\mathsf{a}|}-1\right)-\frac{1}{\sqrt{1-8|\mathsf{a}|/(\pi L)}}-\log\left(\frac{1-\sqrt{1-8|\mathsf{a}|/(\pi L)}}{\sqrt{1-8|\mathsf{a}|/(\pi L)}}\right)\right] \\ &\lesssim \mathsf{m}|\mathsf{a}|+|\mathsf{a}|^2\log(L/|\mathsf{a}|),\end{aligned}$$

$$\mathsf{E}_3 = \frac{1}{4\pi}\int_{r^2=1-2|\mathsf{a}|/(\pi\mathsf{m})}^{1} \frac{1}{r}|\mathsf{m}|^2\,\mathrm{d}r = -\frac{\pi\mathsf{m}^2}{8}\log\left(1-\frac{2|\mathsf{a}|}{\pi\mathsf{m}}\right) \lesssim \mathsf{m}|\mathsf{a}|.$$

Thus (5-1) follows by collecting the above three bounds.

5.2. ***Kinetic energy of a vortex patch near the boundary.*** Next, we compute the energy of a vortex approximation near the boundary. We consider the vortex patch of height $L > 0$ around $x_0 \in \mathbb{D}$, given by

$$\omega_{x_0,L} = L\mathbb{1}_{B_{\sqrt{\mathsf{m}/L}}(x_0)},$$

where we impose that L satisfies

$$L \ge \frac{4\mathsf{m}}{(1-|x_0|)^2}, \tag{5-2}$$

so that $\omega_{x_0,L} \in \mathcal{O}_L$. To estimate the kinetic energy of $\omega_{x_0,L}$, we use the explicit Green's function of the Laplace operator on the unit disk, so that

$$\psi_{x_0,L}(x) = \frac{1}{2\pi}\int_{\mathbb{D}} \log\frac{|x-y|}{|y|\,|x-y_*|}\omega_{x_0,L}(y)\,\mathrm{d}y = \frac{L}{2\pi}\int_{B_{\sqrt{\mathsf{m}/L}}(x_0)} \log\frac{|x-y|}{|y|\,|x-y_*|}\,\mathrm{d}y,$$

where $y_* = y/|y|^2$. Thus

$$\mathsf{E}(\omega_{x_0,L}) = -\tfrac{1}{2}\int_{\mathbb{D}} \psi_{x_0,L}(x)\omega_{x_0,M}(x)\,\mathrm{d}x = \frac{L^2}{2\pi}\int_{B_{\sqrt{\mathsf{m}/L}}(x_0)}\int_{B_{\sqrt{\mathsf{m}/L}}(x_0)} \log\frac{|y|\,|x-y_*|}{|x-y|}\,\mathrm{d}y\,\mathrm{d}x. \tag{5-3}$$

For $x, y \in B_{\sqrt{\mathsf{m}/L}}(x_0)$, we have the bound

$$|x-y| \le 2\sqrt{\frac{\mathsf{m}}{L}}. \tag{5-4}$$

Using that $|y| > \frac{1}{2}$ and $y_* \notin \mathbb{D}$, from (5-2) we deduce the bound

$$|y|\,|x-y_*| \ge \frac{1}{2}\left[1-|x_0|-\sqrt{\frac{\mathsf{m}}{L}}\right] = \frac{1}{2}\left[1-|x_0|-\frac{1-|x_0|}{2}\right] \ge \tfrac{1}{4}(1-|x_0|). \tag{5-5}$$

Plugging (5-4)–(5-5) into (5-3) we obtain the bound

$$\mathsf{E}(\omega_{x_0,L}) \ge \frac{\pi\mathsf{m}^2}{2}\log\left(\sqrt{\frac{L}{\mathsf{m}}}\frac{(1-|x_0|)}{8}\right). \tag{5-6}$$

Computing explicitly the angular momentum, we find

$$\mathsf{A}(\omega_{x_0,L}) = -\tfrac{1}{2}\int_{\mathbb{D}} (1-|x|^2)\omega_{x_0,L}(x)\,\mathrm{d}x = -\frac{\pi\,\mathsf{m}}{2}\Big(1-|x_0|^2-\frac{\mathsf{m}}{2L}\Big) \geq -\pi\,\mathsf{m}(1-|x_0|).$$

Hence, from (5-6) we have the bound

$$\mathsf{E}(\omega_{x_0,L}) \geq \frac{\pi\,\mathsf{m}^2}{2}\log\bigg(\frac{\sqrt{L}\,|\mathsf{A}(\omega_{x_0,L})|}{8\pi\sqrt{m^3}}\bigg), \tag{5-7}$$

as long as

$$L \geq \frac{4\pi^2\mathsf{m}^3}{|\mathsf{A}(\omega_{x_0,L})|^2}$$

to satisfy (5-2).

5.3. ***Proof of Theorem 4.*** The bounds (1-16) and (1-17) are included in the sections above. Given an angular momentum a, we consider the height

$$L = Q^2\pi^2\frac{\mathsf{m}^3}{|\mathsf{a}|^2}, \tag{5-8}$$

with $Q > 2$ to be chosen below. By (5-7) and (5-8), we have the bound

$$\sup\{\mathsf{E}(\omega) \,:\, \omega\in\mathcal{O}_L,\ \mathsf{A}(\omega)=\mathsf{a}\} \geq \frac{\pi\,\mathsf{m}^2}{2}\log\big(\tfrac{1}{8}Q\big)$$

which is independent of a. We pick

$$Q = 8\mathrm{e}^{\frac{2}{\pi\,\mathsf{m}^2}},$$

which implies

$$\sup\{\mathsf{E}(\omega) \,:\, \omega\in\mathcal{O}_L,\ \mathsf{A}(\omega)=\mathsf{a}\} \geq 1.$$

For radial functions, we use (5-1) to get the bound

$$\sup\{\mathsf{E}(\omega) \,:\, \omega\in\mathcal{O}_L,\ \mathsf{A}(\omega)=\mathsf{a},\ \omega \text{ is radial}\} \leq C\bigg(\mathsf{m}|\mathsf{a}| + |\mathsf{a}|^2\log\bigg(Q^2\pi^2\frac{\mathsf{m}^3}{|\mathsf{a}|^3}\bigg)\bigg).$$

So to finish the proof we need to pick $\mathsf{a}_* < 0$ close enough to zero depending only on m such that for any $\mathsf{a}\in(\mathsf{a}_*,0)$, we have

$$C\bigg(\mathsf{m}|\mathsf{a}| + |\mathsf{a}|^2\log\bigg(Q^2\pi^2\frac{\mathsf{m}^3}{|\mathsf{a}|^3}\bigg)\bigg) < 1,$$

which implies the desired inequality.

6. Stability of Onsager solutions with negative inverse temperature

Equations of Liouville-type such as (4-4) arise in the classical setting of mean-field limits of the canonical Gibbs measure associated to a system of point vortices. We state below an important result from [Caglioti et al. 1995] for Onsager solutions, namely solutions to the mean-field equation (1-18).

Theorem 6.1 [Caglioti et al. 1995, Section 5]. *Let $\beta \in (-8\pi, \infty)$. Onsager solutions*

$$\omega_\beta(r) = \frac{1 - A(\beta)}{\pi} \frac{1}{(1 - A(\beta) r^2)^2} \quad \textit{with} \quad A(\beta) = \frac{\beta}{8\pi + \beta}$$

are the unique maximizer of

$$\mathsf{F}_\beta(\omega) = \mathsf{S}(\omega) - \beta \mathsf{E}(\omega) = -\int_{\mathbb{D}} \omega \log \omega \, \mathrm{d}x - \frac{\beta}{2} \int_{\mathbb{D}} |\nabla \Delta^{-1} \omega|^2 \, \mathrm{d}x$$

over the set

$$\mathcal{P} = \left\{ \omega \in L^1 \,:\, \omega \geq 0, \, \int_{\mathbb{D}} \omega(x) \, \mathrm{d}x = 1, \, \int_{\mathbb{D}} \omega(x) \log \omega(x) \, \mathrm{d}x < \infty \right\}.$$

Moreover, we have the convergence

$$\lim_{\beta \to -8\pi^-} \omega_\beta \to \delta_0,$$

weakly in the sense of measures.

The purpose of this section is to prove Theorem 5. As mentioned already, the ideas related to (quantitative) rearrangement inequalities and elliptic equations from [Talenti 1976; Amato et al. 2024; Cianchi et al. 2008; Cianchi and Ferone 2008] are here revisited in the case of the disk $\mathbb{D}$ and vorticities ω satisfying an L^∞ bound. The key result for us is the following stability result with respect to the H^{-1} norm and its radially decreasing rearrangement.

Lemma 6.2. *Consider a positive vorticity distribution $\omega \in L^\infty$ such that $\int_{\mathbb{D}} \omega \, \mathrm{d}x = \mathsf{m} > 0$. Then, for every $\varepsilon > 0$, there exists $\delta > 0$ such that if*

$$\mathsf{E}(\omega^\sharp) - \mathsf{E}(\omega) < \delta,$$

then there exists $x_* \in \mathbb{R}^2$ such that $|x_*| \leq \varepsilon$ and

$$\|\omega - \omega^\sharp(\cdot - x_*)\|_{L^1(\mathbb{R}^2)} < \varepsilon,$$

where ω and $\omega^\sharp$ are extended by zero outside the disk.

The proof is deferred until after the proof of Arnold's stability.

6.1. *Proof of Theorem 5.*

We now proceed with the proof of the main result in this section.

Proof of Theorem 5. Throughout the proof we take $\|\omega^{\mathrm{in}} - \omega_\beta\|_{L^2} < \delta$ progressively smaller, and we keep changing ε accordingly and without renaming it. We will also omit the dependence on t of the solution $\omega = \omega(t)$, as the proof is carried for any arbitrary $t \geq 0$. We proceed in several steps. To simplify the notation, we first assume that

$$\int_{\mathbb{D}} \omega_{\mathrm{in}}(x) \, \mathrm{d}x = \mathsf{m} = 1 \tag{6-1}$$

so that we have unit mass, and in the last step we generalize to the case $\mathsf{m} \neq 1$.

Step 0. We show that for any $\varepsilon > 0$ small enough, we can pick $\|\omega_\beta - \omega^{\text{in}}\|_{L^2} < \delta$ small enough that the corresponding Euler solution $\omega = \omega(t)$ is such that

$$0 \le \mathsf{F}_\beta[\omega_\beta] - \mathsf{F}_\beta(\omega(t)) < \varepsilon \quad \text{for all } t \in [0, \infty). \tag{6-2}$$

Proof of Step 0. We notice that F_β is continuous with respect to the L^2 norm. For what concerns the kinetic energy part, by the triangle and Poincaré inequalities there exists $C_0 = C_0(\|\omega'\|_{L^2}, \|\omega\|_{L^2}) > 0$ such that

$$|\mathsf{E}(\omega) - \mathsf{E}(\omega')| = \tfrac{1}{2}\Big|\|\omega\|_{H^{-1}} - \|\omega'\|_{H^{-1}}\Big|(\|\omega\|_{H^{-1}} + \|\omega'\|_{H^{-1}}) \le C_0\|\omega - \omega'\|_{L^2} \quad \text{for all } \omega, \omega' \in L^\infty \cap \mathcal{P}.$$

For the Boltzmann entropy part, we notice that

$$|\omega \log \omega| \lesssim 1 + |\omega|^2. \tag{6-3}$$

Thus if $\omega_n \to \omega$ in L^2, then up to subsequences $\omega_n \log \omega_n \to \omega \log \omega$ almost everywhere and (6-3) implies uniform integrability. Thus, the Vitali convergence theorem implies that $\mathsf{S}(\omega_n) \to \mathsf{S}(\omega)$.

Hence, given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, \beta)$ such that if $\|\omega_\beta - \omega^{\text{in}}\|_{L^2} < \delta$, then

$$|\mathsf{F}_\beta(\omega_\beta) - \mathsf{F}_\beta(\omega^{\text{in}})| < \varepsilon.$$

Using assumption (6-1), that ω_β is the unique maximizer over $\mathcal{P}$ of F_β, and that the mass and free energy F_β are conserved along the evolution of the Euler equations, we obtain the desired inequality (6-2).

Step 1. We can choose $\|\omega^{\text{in}} - \omega_\beta\|_{L^2} < \delta$, such that

$$\|\omega(t, \cdot) - \omega^\sharp(t, \cdot - x_*)\|_{L^2(\mathbb{R}^2)} < \varepsilon \quad \text{for all } t \in [0, \infty), \tag{6-4}$$

where $x_* = x_*(t) \in \mathbb{R}$ satisfies $|x_*| < \varepsilon$, and both functions are extended to $\mathbb{R}^2$ by zero outside the disk.

Proof of Step 1. Using (6-2), we have

$$0 \le \underbrace{\mathsf{F}_\beta(\omega_\beta) - \mathsf{F}_\beta(\omega^\sharp(t))}_{\ge 0} + \underbrace{\mathsf{F}_\beta(\omega^\sharp(t)) - \mathsf{F}_\beta(\omega(t))}_{\ge 0} < \varepsilon \quad \text{for all } t \in [0, \infty), \tag{6-5}$$

where the positivity of each term follows from the fact that ω_β is the optimizer of F_β over $\mathcal{P}$, and that $\mathsf{F}_\beta(\omega^\sharp(t)) \ge \mathsf{F}_\beta(\omega(t))$ for $\beta < 0$, in view of (4-1). Hence, using that $\mathsf{S}(\omega^\sharp) = \mathsf{S}(\omega)$ we can conclude that

$$0 \le \mathsf{E}(\omega^\sharp(t)) - \mathsf{E}(\omega(t)) < \varepsilon.$$

Up to notation, the conclusion of (6-4) follows from the quantitative Talenti's inequality in Lemma 6.2. To get the stability in L^2 we just need to interpolate the above bound with the bounds in L^∞.

Step 2. We consider the Brenier map $T : [0, 1] \to [0, 1]$ such that $T(0) = 0$, $T(1) = 1$ and

$$\int_0^r \omega_\beta(s)s \, \mathrm{d}s = \int_0^{T(r)} \omega^\sharp(s)s \, \mathrm{d}s \quad \text{for any } r \in [0, 1]. \tag{6-6}$$

For every $\varepsilon > 0$, we can pick $\|\omega_\beta - \omega^{\text{in}}\|_{L^2} < \delta$ small enough such that

$$\int_0^1 |\omega_\beta(r) - \omega^\sharp(T(r))|^2 \, r \, \mathrm{d}r < \varepsilon \quad \text{and} \quad \int_0^1 H\Big(\frac{TT'}{r}\Big) \, r \, \mathrm{d}r < \varepsilon, \tag{6-7}$$

where $H(u) = -\log u + u - 1$.

Proof of Step 2. By (6-5) we have that

$$0 \leq \mathsf{F}_\beta(\omega_\beta) - \mathsf{F}_\beta(\omega^\sharp(t)) < \varepsilon.$$

Applying Lemma 4.5, we have that

$$0 \leq -2\omega_\beta(1)\int_0^1 \log\Big(\frac{TT'}{r}\Big) r \,\mathrm{d}r \leq \mathsf{F}_\beta(\omega_\beta) - \mathsf{F}_\beta(\omega^\sharp) < \varepsilon.$$

The first inequality follows from an application of Jensen's inequality as in (4-16).

Next, we apply a quantitative version of Jensen's inequality. Given $G(\cdot) = -\log(\cdot)$, considering the random variable $X = TT'/r$ and the probability measure $2r\,\mathrm{d}r$ in $[0, 1]$, we have

$$\mathbb{E}X = 2\int_0^1 TT' \,\mathrm{d}r = T^2(1) - T^2(0) = 1.$$

Hence, for the function $G(\cdot) = -\log(\cdot)$, we have

$$\mathbb{E}G(X) = \mathbb{E}\big[G(X) - G(\mathbb{E}X) - G'(\mathbb{E}X)(X - \mathbb{E}X)\big].$$

We define the convex function

$$H(x) := -\log x + x - 1,$$

which satisfies the inequality

$$\tfrac{1}{4}\min\{|x-1|^2, |x-1|\} \leq H(x) \quad \text{for all } x > 0. \tag{6-8}$$

Using the observations above, we have

$$2\int_0^1 H\Big(\frac{TT'}{r}\Big) r\,\mathrm{d}r \leq -2\int_0^1 \log\Big(\frac{TT'}{r}\Big) r\,\mathrm{d}r < \varepsilon.$$

Differentiating (6-6), we have

$$\frac{T(r)T'(r)}{r} = \frac{\omega_\beta(r)}{\omega^\sharp(T(r))}.$$

Applying (6-8) and using that $\omega^\sharp, \omega_\beta \in L^\infty$, we can pick $\tilde{a} = \tilde{a}(\|\omega^\sharp\|_{L^\infty}, \|\omega_\beta\|_{L^\infty}) > 0$ small enough so that

$$\tilde{a}|\omega_\beta(r) - \omega^\sharp(T(r))|^2 \leq \tfrac{1}{4}\min\bigg\{\bigg|\frac{\omega_\beta(r)}{\omega^\sharp(T(r))} - 1\bigg|^2, \bigg|\frac{\omega_\beta(r)}{\omega^\sharp(T(r))} - 1\bigg|\bigg\} \leq H\bigg(\frac{\omega_\beta(r)}{\omega^\sharp(T(r))}\bigg).$$

Therefore, up to relabelling ε we have

$$\int_0^1 |\omega_\beta(r) - \omega^\sharp(T(r))|^2 r\,\mathrm{d}r \leq \varepsilon.$$

Step 3. We exhibit a control on how far the Brenier map is from the identity. That is to say, we can choose $\|\omega_\beta - \omega^{\mathrm{in}}\|_{L^2} < \delta$ small enough so that

$$|T(s) - s| < \varepsilon \quad \text{for all } s \in [0, 1]. \tag{6-9}$$

Proof of Step 3. By (6-7), for δ small enough we have the inequality

$$\int_0^s H\Big(\frac{TT'}{r}\Big) r \,\mathrm{d}r \le \int_0^1 H\Big(\frac{TT'}{r}\Big) r \,\mathrm{d}r < \varepsilon.$$

Applying Jensen's inequality, we have

$$\frac{s^2}{2} H\bigg(\frac{T^2(s)}{s^2}\bigg) = \frac{s^2}{2} H\bigg(\frac{2}{s^2}\int_0^s \frac{TT'}{r} r \,\mathrm{d}r\bigg) \le \int_0^s H\Big(\frac{TT'}{r}\Big) r \,\mathrm{d}r < \varepsilon.$$

Using (6-8), we have

$$\tfrac{1}{4} s^2 \min\bigg\{\bigg|\frac{T^2(s)}{s^2} - 1\bigg|^2, \bigg|\frac{T^2(s)}{s^2} - 1\bigg|\bigg\} \le \frac{s^2}{2} H\bigg(\frac{T^2(s)}{s^2}\bigg).$$

Using the mass constraint and the L^∞ bound on ω we can take $r > 0$ small enough depending on $\|\omega^{\text{in}}\|_{L^\infty}$ such that

$$\omega^\sharp(x) \ge \frac{1 - \|\omega^{\text{in}}\|_{L^\infty} |x|^2}{1 - |x|^2} \ge \tfrac{1}{2} \quad \text{for all } |x| < r.$$

Hence, using (6-6), we have that for r small enough

$$C^{-1} T(r) \le \int_0^{T(r)} \omega^\sharp(s) s \,\mathrm{d}s = \int_0^r \omega_\beta(s) s \,\mathrm{d}s \le Cr,$$

which implies that uniformly

$$\sup_{r\in[0,1]} \bigg|\frac{T^2(r)}{r^2} - 1\bigg| < C(\|\omega^\sharp\|_{L^\infty}).$$

Hence, up to relabeling ε, we have

$$|T(s) - s|^2 \bigg|\frac{T(s)}{s} + 1\bigg|^2 = s^2 \bigg|\frac{T^2(s)}{s^2} - 1\bigg|^2 < \varepsilon,$$

and the conclusion follows.

Step 4. For every $\varepsilon > 0$, we can pick $\|\omega_\beta - \omega^{\text{in}}\|_{L^2} < \delta$ small enough so that

$$\int_0^1 |\omega_\beta(r) - \omega^\sharp(r)| r \,\mathrm{d}r < \varepsilon. \tag{6-10}$$

Proof of Step 4. Changing variables and applying the triangle inequality, we notice that

$$\begin{aligned}\int_0^1 |\omega_\beta(r) - \omega^\sharp(r)| r \,\mathrm{d}r &= \int_0^1 |\omega_\beta(T(r)) - \omega^\sharp(T(r))|\, T(r) T'(r) \,\mathrm{d}r \\ &\le \underbrace{\int_0^1 |\omega_\beta(T(r)) - \omega_\beta(r)| T(r) T'(r) \,\mathrm{d}r}_{\mathrm{I}} + \underbrace{\int_0^1 |\omega_\beta(r) - \omega^\sharp(T(r))| T(r) T'(r) \,\mathrm{d}r}_{\mathrm{II}}.\end{aligned}$$

Applying (6-9) and the smoothness of ω_β, we get the bound

$$\mathrm{I} \le \|\partial_r \omega_\beta\|_{L^\infty} \|T - r\|_{L^\infty} \int_0^1 T(r) T'(r) \,\mathrm{d}r < \|\partial_r \omega_\beta\|_{L^\infty} \varepsilon.$$

We manipulate the second term to get

$$\mathrm{II} \le \int_0^1 |\omega_\beta(r) - \omega^\sharp(T(r))| r \,\mathrm{d}r + \int_0^1 |\omega_\beta(r) - \omega^\sharp(T(r))|(TT' - r)\,\mathrm{d}r \le \sqrt{\frac{\varepsilon}{2}} + \varepsilon,$$

where we have used Cauchy–Schwarz and (6-7). Up to renaming ε, (6-10) now follows.

Step 5. We now conclude the proof of the theorem, under the unit mass assumption (6-1).

Proof of Step 5. We combine (6-4) and (6-10) to obtain that

$$\|\omega_\beta(\,\cdot\, - x_*) - \omega(t)\|_{L^1(\mathbb{R}^2)} < 2\varepsilon, \quad \text{for all } t \ge 0,$$

for some $x_* \in \mathbb{R}^2$ such that $|x_*| \le \varepsilon$, where we have extended the functions by zero outside the disk $\mathbb{D}$. Using the continuity of the L^1 norm over translations for ω_β, we can conclude that, up to renaming ε,

$$\|\omega_\beta - \omega(t)\|_{L^1(\mathbb{R}^2)} < \varepsilon.$$

To get the stability in L^2 we just need to interpolate the above bound with the bounds in L^∞.

Step 6. We conclude the proof of the theorem, without the unit mass assumption (6-1).

Proof of Step 6. We start by showing that the maximizer of the problem

$$\max_{\omega \in \mathcal{P}_\mathsf{m}} \mathsf{F}_\beta(\omega) = \mathsf{S}(\omega) - \beta \mathsf{E}(\omega) = -\int_{\mathbb{D}} \omega \log \omega \,\mathrm{d}x - \frac{\beta}{2}\int_{\mathbb{D}} |\nabla \Delta^{-1}\omega|^2 \,\mathrm{d}x, \tag{6-11}$$

where

$$\mathcal{P}_\mathsf{m} = \left\{ \omega \in L^1 : \omega \ge 0, \int_{\mathbb{D}} \omega(x)\,\mathrm{d}x = \mathsf{m}, \int_{\mathbb{D}} \omega(x) \log \omega(x)\,\mathrm{d}x < \infty \right\},$$

is continuous with respect to the mass parameter m in any L^p with $p \in [1, \infty)$, as the long the parameter $\mathsf{m}\beta > -8\pi$. By rescaling, we find

$$\max_{\omega \in \mathcal{P}_\mathsf{m}} -\int_{\mathbb{D}} \omega \log \omega \,\mathrm{d}x - \frac{\beta}{2}\int_{\mathbb{D}} |\nabla \Delta^{-1}\omega|^2 \,\mathrm{d}x = \max_{\omega \in \mathcal{P}_1} -\mathsf{m}\int_{\mathbb{D}} \omega \log \omega \,\mathrm{d}x - \frac{\beta \mathsf{m}^2}{2}\int_{\mathbb{D}} |\nabla \Delta^{-1}\omega|^2 \,\mathrm{d}x - \mathsf{m}\log \mathsf{m}.$$

Hence, for general m, we get that the maximizer of (6-11) is given by

$$\omega_* = \mathsf{m}\omega_{\mathsf{m}\beta},$$

which by (1-19) is continuous in any topology as long as it does not blow up $\mathsf{m}\beta > -8\pi$.

Using the continuity of the mass with respect to the L^2 norm, we can pick δ small enough so that $\|\omega_\beta - \omega_{\mathrm{in}}\|_{L^2} < \delta$ implies

$$\|\mathsf{m}\omega_{\mathsf{m}\beta} - \omega_\beta\|_{L^2} < \varepsilon \quad \text{and} \quad \|\mathsf{m}\omega_{\mathsf{m}\beta} - \omega_{\mathrm{in}}\|_{L^2} < \varepsilon, \tag{6-12}$$

where

$$\mathsf{m} = \int_{\mathbb{D}} \omega_{\mathrm{in}}(x)\,\mathrm{d}x.$$

To conclude the proof, we need to repeat *Steps 0–5* replacing the role of ω_β by $\mathsf{m}\omega_{\mathsf{m}\beta}$. We conclude that we can pick δ small enough that $\|\omega_\beta - \omega_{\mathrm{in}}\|_{L^2} < \delta$ implies $\|\mathsf{m}\omega_{\mathsf{m}\beta} - \omega(t)\|_{L^2} < \varepsilon$ for all $t > 0$, and the result follows, up to relabeling ε, by (6-12). □

6.2. ***Stability of rearrangements.*** We now proceed to prove Lemma 6.2, which constitutes the crucial step in the proof of Theorem 5.

Proof of Lemma 6.2. We consider (up to signs!) the associated stream functions to ω and its rearrangement $\omega^{\#}$

$$\begin{cases} -\Delta\phi = \omega & \text{in } \mathbb{D}, \\ \phi = 0 & \text{on } \partial\mathbb{D}, \end{cases} \qquad \begin{cases} -\Delta\bar{\phi} = \omega^{\sharp} & \text{in } \mathbb{D}, \\ \bar{\phi} = 0 & \text{on } \partial\mathbb{D}. \end{cases} \tag{6-13}$$

A celebrated theorem of Talenti [1976, Theorem 1] states that

$$\phi^{\sharp}(x) \le \bar{\phi}(x) \quad \text{for all } x \in \mathbb{D},$$

and

$$\mathsf{E}(\omega) = \tfrac{1}{2}\|\nabla\phi\|_{L^2}^2 \le \tfrac{1}{2}\|\nabla\bar{\phi}\|_{L^2}^2 = \mathsf{E}(\omega^{\sharp}).$$

To prove the lemma, we again proceed in steps.

Step 1. We show that under our hypothesis, there exists $C(\|\omega\|_{L^\infty}, \mathsf{m}) > 0$ such that

$$\|\bar{\phi} - \phi^{\sharp}\|_{L^\infty} \le C(\mathsf{E}(\omega^{\sharp}) - \mathsf{E}(\omega)). \tag{6-14}$$

Proof of Step 1. For $h \ge 0$, we consider the distribution function

$$u(h) = |\{x \in \mathbb{D} : \phi(x) > h\}|,$$

whose derivative is

$$u'(h) = -\int_{\partial[\phi>h]} \frac{1}{|\nabla\phi|}\, \mathrm{d}\mathcal{H}^1.$$

Considering the perimeter of the level sets and the isoperimetric inequality we obtain

$$2\pi^{\frac{1}{2}} u(h)^{\frac{1}{2}} \le \mathrm{Per}([\phi > h]) = \int_{\partial[\phi>h]} \mathrm{d}\mathcal{H}^1 \le \left(-u'(h)\int_{\partial[\phi>h]} |\nabla\phi|\, \mathrm{d}\mathcal{H}^1\right)^{\frac{1}{2}}.$$

Next, we use the first equation in (6-13) to compute the last integral,

$$\int_{\partial[\phi>h]} |\nabla\phi|\, \mathrm{d}\mathcal{H}^1 = \int_{[\phi>h]} -\Delta\phi\, \mathrm{d}x = \int_{[\phi>h]} \omega\, \mathrm{d}x \le \int_0^{(u(h)/\pi)^{1/2}} \omega^{\sharp}(s)s\, \mathrm{d}s.$$

Putting the last two equations together, we get the inequality

$$4\pi u(h) \le -u'(h)\int_0^{(u(h)/\pi)^{1/2}} \omega^{\sharp}(s)s\, \mathrm{d}s. \tag{6-15}$$

Noting that for the rearranged vorticity $\omega^{\sharp}$ all the inequalities are in fact equalities, we obtain that the distribution $v(h) = |\{x \in \mathbb{D} : \bar{\phi}(x) > h\}|$ function satisfies

$$4\pi v(h) = -v'(h)\int_0^{(v(h)/\pi)^{1/2}} \omega^{\sharp}(s)s\, \mathrm{d}s. \tag{6-16}$$

Using the boundary condition we have that

$$u(0) = v(0) = \pi,$$

and hence we can use the derivative equations (6-15) and (6-16) to conclude that

$$u(h) \le v(h) \quad \text{for all } h \ge 0.$$

Using the inequality above with

$$u(\phi^\sharp(x)) = \pi |x|^2 = v(\bar{\phi}(x)) \quad \text{and} \quad v'(h) < 0, \tag{6-17}$$

we get Talenti's inequality

$$\phi^\sharp(x) \le \bar{\phi}(x). \tag{6-18}$$

Using (6-17), with (6-15) and (6-16), we obtain that

$$\partial_r \phi^\sharp(r) \ge \partial_r \bar{\phi}(r) \quad \text{for all } r \in [0, 1],$$

where we have abused notation and considered $\phi^\sharp$ and $\bar{\phi}$ with respect to the radial variable. This implies

$$\max_{x \in \mathbb{D}} |\bar{\phi}(x) - \phi^\sharp(x)| = \bar{\phi}(0) - \phi^\sharp(0).$$

Since $\omega,\ \omega^\sharp \in L^\infty$, the corresponding stream functions $\phi,\ \bar{\phi}$ are Lipschitz-continuous. As radial rearrangements are contractive in the Lipschitz norm, we have

$$\|\phi^\sharp\|_{W^{1,\infty}} \le \|\phi\|_{W^{1,\infty}} \lesssim \|\omega\|_{L^\infty}.$$

Hence, there exists $r > 0$ small enough depending only on $\|\omega\|_{L^\infty}$ such that

$$|\bar{\phi}(x) - \phi^\sharp(x)| \ge \tfrac{1}{2} \|\bar{\phi} - \phi^\sharp\|_{L^\infty} \quad \text{for all } |x| < r. \tag{6-19}$$

Using the mass constraint and the L^∞ bound on ω, we can take $r > 0$ small enough depending on $\|\omega\|_{L^\infty}$ and m such that

$$\omega^\sharp(x) \ge \frac{\mathsf{m} - \|\omega\|_{L^\infty} |x|^2}{1 - |x|^2} \ge \frac{\mathsf{m}}{2} \quad \text{for all } |x| < r. \tag{6-20}$$

Now we are ready to show (6-14). By the Hardy–Littlewood inequality,

$$\begin{aligned} \mathsf{E}(\omega^\sharp) - \mathsf{E}(\omega) = \tfrac{1}{2} \int_{\mathbb{D}} \bar{\phi}\omega^\sharp - \phi\omega \, \mathrm{d}x = \tfrac{1}{2} \int_{\mathbb{D}} \phi^\sharp \omega^\sharp - \phi\omega + (\bar{\phi} - \phi^\sharp)\omega^\sharp \, \mathrm{d}x &\ge \tfrac{1}{2} \int_{\mathbb{D}} (\bar{\phi} - \phi^\sharp)\omega^\sharp \, \mathrm{d}x \\ &\ge \tfrac{1}{2} \int_{B_r} (\bar{\phi} - \phi^\sharp)\omega^\sharp \, \mathrm{d}x \\ &\ge c(\|\omega\|_{L^\infty}, \mathsf{m}) \|\bar{\phi} - \phi^\sharp\|_{L^\infty}, \end{aligned}$$

where we used (6-18), (6-19) and (6-20).

Step 2. For any $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|\bar{\phi} - \phi^\sharp\|_{L^\infty} < \delta$, there exists $x_* \in \mathbb{R}^2$ such that $|x_*| < \varepsilon$ and

$$\|\omega - \omega^\sharp(\,\cdot\, - x_*)\|_{L^1(\mathbb{R}^2)} < \varepsilon,$$

where the functions ω and $\omega^\sharp$ are extended by zero outside the disk.

Proof of Step 2. We choose x_* to be the optimizer of

$$\|\phi - \phi^\sharp(\cdot - x_*)\|_{L^2(\mathbb{R}^2)} = \inf_{x_0 \in \mathbb{R}^2} \|\phi - \phi^\sharp(\cdot - x_0)\|_{L^2(\mathbb{R}^2)}.$$

Applying [Amato et al. 2024, Section 5, equation (82)], we have that there exists a $C > 0$ such that

$$C^{-1} \min\bigl(|x_*|, \tfrac{1}{2}\bigr) \le |\mathbb{D} \triangle (\mathbb{D} + x_*)| \le C \|\bar{\phi} - \phi^\sharp\|_{L^\infty}^{1/4}.$$

The proof of [Amato et al. 2024, Theorem 1.4] shows exactly that we can pick $\|\bar{\phi} - \phi^\sharp\|_{L^\infty} < \delta$ small enough that

$$\inf_{x_0 \in \mathbb{R}^2} \|\omega - \omega^\sharp(\cdot - x_0)\|_{L^1} \le \|\omega - \omega^\sharp(\cdot - x_*)\|_{L^1} < \varepsilon. \qquad \square$$

Appendix: A min-max principle

We show a variation of the classical min-max principle, which can be found in [Ekeland and Témam 1999, Chapter VI].

Proposition A.1. *Let A and B be closed convex sets of a Banach space $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$, and consider a proper functional $\mathsf{L} : A \times B \to \mathbb{R}$. Assume the following*:

(a) *For every $\beta \in A$, the function $\omega \mapsto \mathsf{L}(\beta, \omega)$ is weakly upper semicontinuous.*

(b) *For every $\omega \in B$, the function $\beta \mapsto \mathsf{L}(\beta, \omega)$ is convex and lower semicontinuous.*

(c) *The functional L is coercive in β. More specifically, there exists a function $g : [0, \infty) \to [0, \infty)$ such that $\lim_{u \to \infty} g(u) = \infty$ and for any $\beta \in A$ there exists $\omega \in B$ such that*

$$\mathsf{L}(\beta, \omega) \ge g(\|\beta\|_1).$$

(d) *The set B is bounded, and hence weakly compact.*

(e) *For every $\beta \in A$, the function $\omega \mapsto \mathsf{L}(\beta, \omega)$ has a unique maximizer ω_β.*

Then

$$\inf_{\beta \in A} \sup_{\omega \in B} \mathsf{L}(\beta, \omega) = \sup_{\omega \in B} \inf_{\beta \in A} \mathsf{L}(\beta, \omega).$$

Remark A.2. The (e) can be weakened to the following: for any $\beta_* \in A$, any two maximizers ω_1^* and ω_2^* satisfy

$$\mathsf{L}(\beta, \omega_1^*) = \mathsf{L}(\beta, \omega_2^*) \quad \text{for any other } \beta \in A.$$

In the case of the Euler equation in a radial domain this means that the min-max principle applies if we know that the maximizers are unique up to rigid rotations, which preserves the energy and the entropy.

Proof of Proposition A.1. First of all, we observe that

$$\mathsf{L}(\beta, \omega) \ge \inf_{\beta \in A} \mathsf{L}(\beta, \omega) \quad \text{for all } \omega \in B,$$

so that

$$\sup_{\omega \in B} \mathsf{L}(\beta, \omega) \ge \sup_{\omega \in B} \inf_{\beta \in A} \mathsf{L}(\beta, \omega),$$

and thus

$$\inf_{\beta\in A}\sup_{\omega\in B}\mathsf{L}(\beta,\omega)\geq\sup_{\omega\in B}\inf_{\beta\in A}\mathsf{L}(\beta,\omega),$$

so we only need to prove the reverse inequality. Define

$$f(\beta):=\mathsf{L}(\beta,\omega_\beta)=\sup_{\omega\in B}\mathsf{L}(\beta,\omega),$$

where $\omega_\beta\in B$ is assumed to be the unique maximizer from (e). The function $\beta\mapsto f(\beta)$ is convex and lower semicontinuous, being the envelope of convex lower semicontinuous functions by (b). Therefore by convexity and coercivity (c) it attains its lower bound at some $\bar\beta\in A$, so that

$$f(\bar\beta)=\min_{\beta\in A}f(\beta)=\min_{\beta\in A}\max_{\omega\in B}\mathsf{L}(\beta,\omega)$$

and

$$f(\bar\beta)\geq\mathsf{L}(\bar\beta,\omega)\quad\text{for all }\omega\in B.$$

Now, by convexity (b), for every $\beta\in A,\ \omega\in B$ and $t\in(0,1)$, we have

$$\mathsf{L}((1-t)\bar\beta+t\beta,\omega)\leq(1-t)\mathsf{L}(\bar\beta,\omega)+t\mathsf{L}(\beta,\omega).$$

In particular, taking $\beta_t=(1-t)\bar\beta+t\beta$ we consider $\omega=\omega_{\beta_t}$ given by (e) and we find

$$f(\bar\beta)\leq f(\beta_t)=\mathsf{L}(\beta_t,\omega_{\beta_t})\leq(1-t)\mathsf{L}(\bar\beta,\omega_{\beta_t})+t\mathsf{L}(\beta,\omega_{\beta_t})\leq(1-t)f(\bar\beta)+t\mathsf{L}(\beta,\omega_{\beta_t}),$$

implying

$$f(\bar\beta)\leq\mathsf{L}(\beta,\omega_{\beta_t})\quad\text{for all }\beta\in A.\tag{A-1}$$

Now, by compactness (d), as $t\to0,\ \omega_{\beta_t}$ converges weakly to some $\bar\omega\in B$, up to subsequences. Next, we claim that $\bar\omega=\omega_{\bar\beta}$. Indeed,

$$\mathsf{L}(\beta_t,\omega_{\beta_t})\geq\mathsf{L}(\beta_t,\omega)\quad\text{for all }\omega\in B,$$

and from convexity (b) we have

$$(1-t)\mathsf{L}(\bar\beta,\omega_{\beta_t})+t\mathsf{L}(\beta,\omega_{\beta_t})\geq\mathsf{L}(\beta_t,\omega)\quad\text{for all }\omega\in B.$$

Since $\mathsf{L}(\beta,\omega_{\beta_t})\leq f(\beta)<\infty$, we can use the semicontinuity (a) and (b) to pass to the limit as $t\to0$ and obtain

$$\mathsf{L}(\bar\beta,\bar\omega)\geq\limsup_{t\to0}(1-t)\mathsf{L}(\bar\beta,\omega_{\beta_t})+t\mathsf{L}(\beta,\omega_{\beta_t})\geq\liminf_{t\to0}\mathsf{L}(\beta_t,\omega)\geq\mathsf{L}(\bar\beta,\omega)\quad\text{for all }\omega\in B,$$

proving the claim that $\bar\omega=\omega_{\bar\beta}$, the unique maximizer by (e). We can now pass to the limit in (A-1), using weak upper semicontinuity (a), to get

$$f(\bar\beta)\leq\mathsf{L}(\beta,\omega_{\bar\beta})\quad\text{for all }\beta\in A.$$

Thus

$$\min_{\beta\in A}\max_{\omega\in B}\mathsf{L}(\beta,\omega)\leq\min_{\beta\in A}\mathsf{L}(\beta,\omega_{\bar\beta})\leq\max_{\omega\in B}\min_{\beta\in A}\mathsf{L}(\beta,\omega),$$

as we needed, and the proof is complete. □

Acknowledgements

The research of Coti Zelati was partially supported by the Royal Society URF\R1\191492 and EPSRC Horizon Europe Guarantee EP/X020886/1. The research of Delgadino was partially supported by NSF-DMS-2205937 and NSF-DMS RTG 1840314. The authors would also like to thank AIMS Senegal for their hospitality in the early stages of this project. We would like to thank J. A. Carrillo, E. Caglioti, T. D. Drivas, T. Hmidi and V. Šverák for illuminating discussions that helped improve this work.

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Received 30 May 2024. Revised 20 Feb 2025. Accepted 14 Apr 2025.

MICHELE COTI ZELATI: m.coti-zelati@imperial.ac.uk
Department of Mathematics, Imperial College London, United Kingdom

MATIAS G. DELGADINO: matias.delgadino@math.utexas.edu
Department of Mathematics, University of Texas at Austin, United States

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004, is published continuously online.

APDE peer review and production are managed by EditFlow® from MSP.

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ANALYSIS & PDE

Volume 19 No. 3 2026