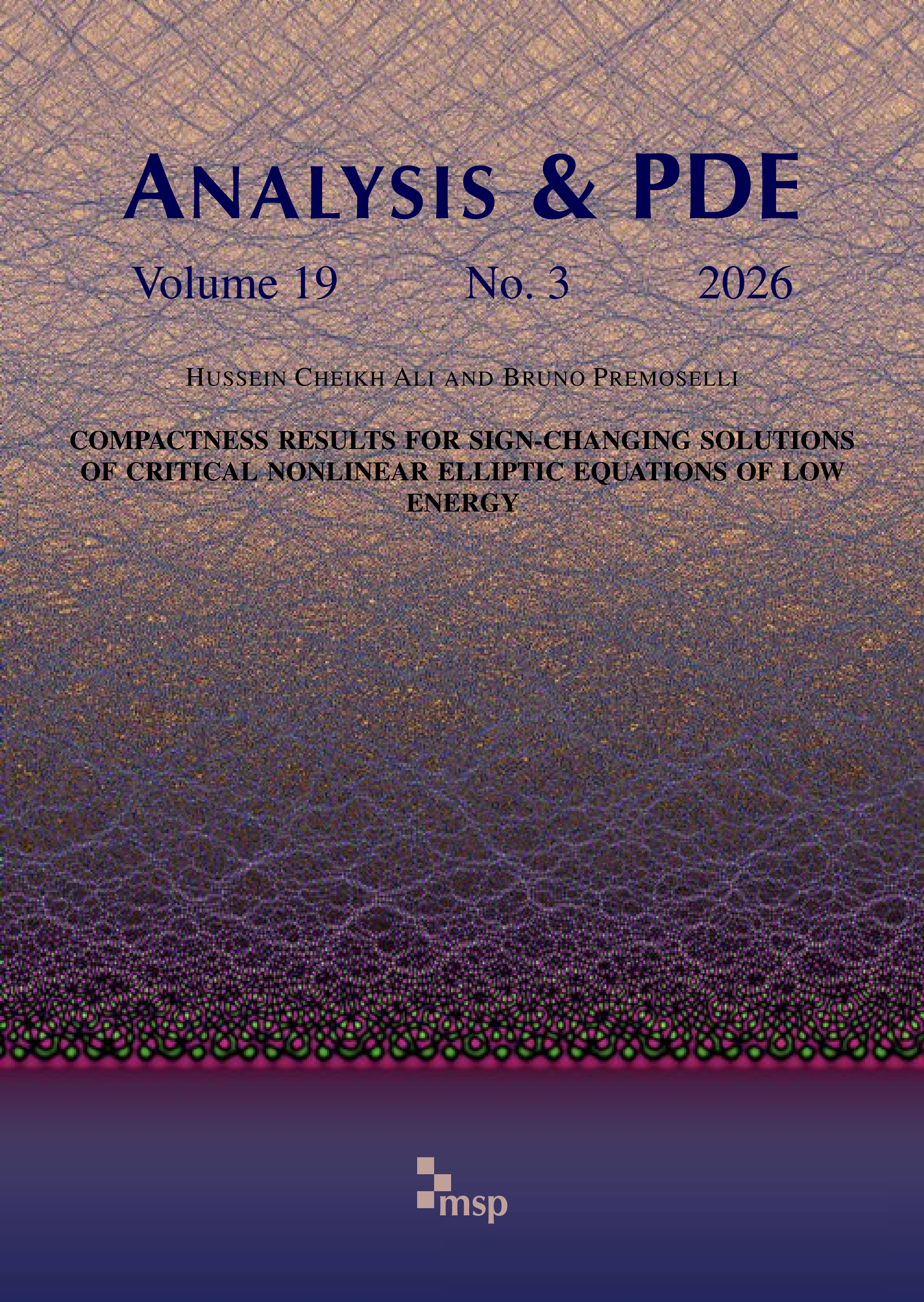

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COMPACTNESS RESULTS FOR SIGN-CHANGING SOLUTIONS OF CRITICAL NONLINEAR ELLIPTIC EQUATIONS OF LOW ENERGY

HUSSEIN CHEIKH ALI AND BRUNO PREMOSELLI

Let Ω be a bounded, smooth connected open domain in $\mathbb{R}^n$ with $n \ge 3$. We investigate compactness properties for the set of sign-changing solutions $v \in H_0^1(\Omega)$ of

$$\begin{cases} -\Delta v + hv = |v|^{2^*-2}v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $h \in C^1(\overline{\Omega})$ and $2^* := 2n/(n-2)$. Our main result establishes that the set of *sign-changing* solutions of the above system at the lowest sign-changing energy level is unconditionally compact in $C^2(\overline{\Omega})$ when $3 \le n \le 5$, and is compact in $C^2(\overline{\Omega})$ when $n \ge 7$ provided h never vanishes in $\overline{\Omega}$. In dimensions $n \ge 7$ our results apply when $h > 0$ in $\overline{\Omega}$ and thus complement the compactness result of Devillanova and Solimini (2002). Our proof is based on a new, global pointwise description of blowing-up sequences of solutions of the above system that holds up to the boundary. We also prove more general compactness results under perturbations of h.

1. Introduction

1.1. ***Statement of the results.*** Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded connected open set in $\mathbb{R}^n$, $n \ge 3$, $h \in C^1(\overline{\Omega})$ and $2^* := 2n/(n-2)$. We investigate solutions $v \in H_0^1(\Omega)$ of

$$\begin{cases} -\Delta v + hv = |v|^{2^*-2}v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1-1}$$

Here and in the sequel, we let $\|\cdot\|_p$ be the usual norm of $L^p(\Omega)$ for $1 \le p \le \infty$, and $H_0^1(\Omega)$ be the completion of $C_c^\infty(\Omega)$ with respect to the norm

$$\|v\|_{H_0^1}^2 := \int_\Omega |\nabla v|^2\,dx.$$

For simplicity we will assume throughout this paper that $-\Delta + h$ is coercive, that is, that there exists $C > 0$ such that

$$\int_\Omega (|\nabla v|^2 + hv^2)\,dx \ge C \int_\Omega |\nabla v|^2\,dx \quad \text{for all } v \in H_0^1(\Omega).$$

Under this assumption, the existence of positive solutions of (1-1) is very well understood. We let

$$I_h(\Omega) := \inf_{v \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega (|\nabla v|^2 + hv^2)\,dx}{\left(\int_\Omega |v|^{2^*}\,dx\right)^{2/2^*}}. \tag{1-2}$$

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Brézis and Nirenberg [1983] proved that, when $n \geq 4$, positive ground states attaining (1-2) exist if and only if $h < 0$ somewhere in Ω. When $n = 3$, Druet [2002] proved that positive ground states attaining (1-2) exist if only if $m_h > 0$ somewhere in Ω, where m_h is the so-called mass function of the operator $-\Delta + h$. This function is defined as follows: let G_h be the Green's function for $-\Delta + h$ with Dirichlet boundary conditions in Ω. Then, when $n = 3$, we have

$$G_h(x, y) = \frac{1}{4\pi |x-y|} + g_h(x, y) \quad \text{for all } y \in \Omega \backslash \{x\}$$

for some $g_h \in C^{0,1}(\bar{\Omega} \times \bar{\Omega})$, and we define $m_h(x) = g_h(x, x)$. Under these assumptions, [Brézis and Nirenberg 1983; Druet 2002] also prove that we have $I_h(\Omega) < K_n^{-2}$, where

$$K_n^{-2} := \inf_{v \in C_c^\infty(\mathbb{R}^n)\backslash\{0\}} \frac{\int_{\mathbb{R}^n} |\nabla v|^2 \, dx}{\left(\int_{\mathbb{R}^n} |v|^{2^*} \, dx\right)^{2/2^*}} \tag{1-3}$$

is the optimal constant in Sobolev's inequality in $\mathbb{R}^n$. An explicit expression of K_n can be found in [Aubin 1976; Talenti 1976]. It is simple to see that if $v \in H_0^1(\Omega)$ attains $I_h(\Omega)$ and is normalised to satisfy (1-1) then

$$\int_\Omega |v|^{2^*} \, dx = I_h(\Omega)^{n/2} < K_n^{-n}. \tag{1-4}$$

The existence of sign-changing solutions for problem (1-1) has also attracted a lot of attention. Existence results for a general function $h \in C^1(\bar{\Omega})$ are in [Bartsch and Weth 2003]. When $h \equiv -\lambda$, for $\lambda \in (0, \lambda_1)$, equation (1-1) is the so-called Brézis–Nirenberg problem

$$\begin{cases} -\Delta v - \lambda v = |v|^{2^*-2} v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1-5}$$

for which existence results have been obtained in [Cerami et al. 1984; Capozzi et al. 1985; Fortunato and Jannelli 1987; Solimini 1995; Devillanova and Solimini 2002; Clapp and Weth 2004; Schechter and Zou 2010]. The existence of a sign-changing solution of least-energy (among all sign-changing solutions) for (1-5) when $\lambda \in (0, \lambda_1)$ — the range in which $-\Delta - \lambda$ is coercive — was proven in [Cerami et al. 1986] when $n \geq 6$ (see also [Chen and Zou 2015] for a new proof) while it was proven in [Roselli and Willem 2009; Tavares et al. 2022] when $n = 4, 5$. The existence of least-energy sign-changing solutions for (1-5) is not yet known when $n = 3$.

In this paper we focus on compactness properties for solutions of (1-1). We let $(h_\alpha)_{\alpha \in \mathbb{N}}$ be a sequence of C^1 functions that converge to h in $C^1(\bar{\Omega})$, and we let $(v_\alpha)_{\alpha \in \mathbb{N}}$ be a sequence of solutions in $H_0^1(\Omega)$ of

$$\begin{cases} -\Delta v_\alpha + h_\alpha v_\alpha = |v_\alpha|^{2^*-2} v_\alpha & \text{in } \Omega, \\ v_\alpha = 0 & \text{on } \partial\Omega \end{cases} \tag{1-6}$$

satisfying $\limsup_{\alpha \to +\infty} \|v_\alpha\|_{H_0^1} < +\infty$. We will say that $(v_\alpha)_\alpha$ is *sign-changing* if $(v_\alpha)_+ = \max(v_\alpha, 0)$ and $(v_\alpha)_- = -\min(v_\alpha, 0)$ are both nonzero for any α. We investigate under which assumptions on h the sequence $(v_\alpha)_{\alpha \in \mathbb{N}}$ converges in a strong topology. Our main result answers this question when $(v_\alpha)_{\alpha \in \mathbb{N}}$ has minimal energy:

Theorem 1.1. *Let Ω be a smooth bounded connected domain of $\mathbb{R}^n$, $n \ge 3$, and $(h_\alpha)_{\alpha\in\mathbb{N}}$ be a sequence that converges in $C^1(\overline{\Omega})$ towards h. Assume that $-\Delta + h$ is coercive and that $I_h(\Omega) < K_n^{-2}$. Let $(v_\alpha)_{\alpha\in\mathbb{N}} \in H_0^1(\Omega)$ be a sequence of solutions of (1-6) such that*

$$\limsup_{\alpha\to+\infty} \int_\Omega |v_\alpha|^{2^*}\,dx \le K_n^{-n} + I_h(\Omega)^{n/2}, \tag{1-7}$$

and assume that either

- *$n \in \{3, 4, 5\}$ and, for all $\alpha \ge 0$, v_α is sign-changing, or*
- *$n \ge 7$ and $h \neq 0$ at every point in $\overline{\Omega}$.*

Then, up to a subsequence, $(v_\alpha)_{\alpha\in\mathbb{N}}$ strongly converges in $C^2(\overline{\Omega})$ to a nonzero solution of (1-1).

Recall that $I_h(\Omega)$ is defined in (1-2). In the particular case where $h_\alpha \equiv h$, Theorem 1.1 implies the following compactness result for solutions of (1-1):

Corollary 1.2. *Let Ω be a smooth bounded connected domain of $\mathbb{R}^n$, $n \ge 3$, and let $h \in C^1(\overline{\Omega})$ be such that $-\Delta + h$ is coercive and $I_h(\Omega) < K_n^{-2}$.*

- *Assume that $n \in \{3, 4, 5\}$. There exists $\varepsilon = \varepsilon(n, \Omega) > 0$ such that the set of* ***sign-changing*** *solutions v of (1-1) satisfying*

  $$\int_\Omega |v|^{2^*}\,dx \le K_n^{-n} + I_h(\Omega)^{n/2} + \varepsilon$$

  is precompact in the $C^2(\overline{\Omega})$-topology.
- *Assume that $n \ge 7$ and $h \neq 0$ in $\overline{\Omega}$. There exists $\varepsilon = \varepsilon(n, h, \Omega) > 0$ such that the set of solutions v of (1-1) satisfying*

  $$\int_\Omega |v|^{2^*}\,dx \le K_n^{-n} + I_h(\Omega)^{n/2} + \varepsilon$$

  is precompact in the $C^2(\overline{\Omega})$-topology.

The energy bound (1-7) is very natural when investigating sign-changing solutions of (1-1). Solutions of (1-6) satisfying (1-7) exist: the least-energy sign-changing solutions of (1-5) constructed in [Cerami et al. 1986; Tavares et al. 2022], for instance, satisfy

$$\int_\Omega |v|^{2^*}\,dx < K_n^{-n} + I_{-\lambda}(\Omega)^{n/2}.$$

A simple application of the celebrated compactness result of Struwe [1984] (see also [Cerami et al. 1986, Lemma 3.1]) shows that if a sequence $(v_\alpha)_{\alpha\in\mathbb{N}}$ of solutions of (1-6) *changes sign* and satisfies $\lim_{\alpha\to+\infty}\|v_\alpha\|_\infty = +\infty$ (we will say in this case that $(v_\alpha)_{\alpha\in\mathbb{N}}$ *blows up*), then

$$\int_\Omega |v_\alpha|^{2^*}\,dx \ge K_n^{-n} + I_h(\Omega)^{n/2} + o(1)$$

as $\alpha \to +\infty$. The threshold $K_n^{-n} + I_h(\Omega)^{n/2}$ is therefore the direct counterpart, for sign-changing solutions, of the minimal energy threshold K_n^{-n} that ensures the existence of positive ground state solutions in (1-4). In this respect, Theorem 1.1 and Corollary 1.2 have to be understood as the first compactness result for (1-6), at the lowest energy-level for sign-changing blow-up, when $I_h(\Omega)$ is attained.

Theorem 1.1 shows that, when $3 \le n \le 5$, *sign-changing* solutions are unconditionally compact in $C^2(\overline{\Omega})$ under assumption (1-7). By contrast, without further assumptions on h, the set of *positive* solutions satisfying (1-7) is not compact in general when $3 \le n \le 5$. For equation (1-5), for instance, families of positive solutions whose energy converges to K_n^{-n} and which are not compact in $C^2(\overline{\Omega})$ have been constructed in [Musso and Pistoia 2002; Rey 1990] when $n \ge 4$ and $\lambda \to 0+$, and in [del Pino et al. 2004] when $n = 3$ and $\lambda \to \lambda_*$ from above, where λ_* satisfies $\max_\Omega m_{\lambda_*} = 0$. When $3 \le n \le 5$, Theorem 1.1 is therefore unexpected since sign-changing solutions of equations like (1-6) are known to exhibit a much richer and more erratic behaviour than positive ones. When $n \ge 7$, Theorem 1.1 applies to positive and sign-changing sequences of solutions $(v_\alpha)_{\alpha \in \mathbb{N}}$ and Corollary 1.2 generalises the well-known compactness theorem for energy-bounded solutions of (1-5) proven in [Devillanova and Solimini 2002]. It is still an open question to know whether Theorem 1.1 holds for any energy-bounded sequence $(v_\alpha)_{\alpha \in \mathbb{N}}$ without the assumption (1-7) when $n \ge 7$ and $h \ne 0$ in $\overline{\Omega}$.

Dimension 6 is excluded from Theorem 1.1. In this case we prove:

Proposition 1.3. *Let Ω be a smooth bounded domain of $\mathbb{R}^6$ and $(h_\alpha)_{\alpha \in \mathbb{N}}$ be a sequence that converges in $C^1(\overline{\Omega})$ towards h. Assume that $-\Delta + h$ is coercive and that $I_h(\Omega) < K_6^{-2}$. Let $(v_\alpha)_{\alpha \in \mathbb{N}} \in H_0^1(\Omega)$ be any sequence of solutions of (1-6) satisfying (1-7), and assume that $\|v_\alpha\|_\infty \to +\infty$ as $\alpha \to +\infty$. Then there exists $v_\infty \in H_0^1(\Omega)$, $v_\infty > 0$ in Ω, attaining $I_h(\Omega)$ such that v_α converges weakly but not strongly to $\pm v_\infty$ in $H_0^1(\Omega)$ and there exists $x_\infty \in \Omega$ such that*

$$h(x_\infty) = \pm 2 v_\infty(x_\infty).$$

Compactness of *sign-changing* solutions of (1-6) satisfying (1-7) does not hold when $n = 6$: in [Pistoia and Vaira 2022], for instance, the authors constructed a noncompact family $(v_\lambda)_\lambda$ of sign-changing solutions of (1-5) which blows up as λ converges to some $\lambda_0 > 0$ that satisfies $\lambda_0 = 2\|v_0\|_\infty$, where v_0 attains $I_{-\lambda_0}(\Omega)$ (the existence of such (λ_0, v_0) is also proven in that work). This six-dimensional phenomenon has been known for a while for positive solutions; see [Druet 2004], where it was first highlighted.

1.2. ***Strategy of proof and outline of the paper.*** For *positive* solutions there is a vast literature addressing the issue of compactness of equations like (1-6) through blow-up analysis. On open sets of $\mathbb{R}^n$ with Dirichlet boundary conditions we mention for instance [Druet 2002; Druet and Laurain 2010; König and Laurain 2022; 2024] for (1-1), [Druet et al. 2012] for Lin–Ni-type problems with Neumann boundary conditions and [Ghoussoub et al. 2023] for singular Hardy–Sobolev-type problems. On closed manifolds we mention [Druet 2003] for compactness of energy-bounded solutions and the series of works related to the compactness of the Yamabe equation: [Li and Zhu 1999; Druet 2003; Marques 2005; Khuri et al. 2009]; see also [Hebey 2014]. On manifolds with boundary we refer to [Mesmar and Robert 2024]. For *sign-changing* solutions of critical elliptic equations on open sets of $\mathbb{R}^n$ the only compactness result available is [Devillanova and Solimini 2002] when $n \ge 7$; this result was generalised on closed manifolds in [Vétois 2007]. In lower dimensions, compactness results on closed manifolds have been obtained more recently: we refer for instance to [Premoselli and Vétois 2019; 2022a; 2022b; 2024; Premoselli and Robert 2025]. Concerning problem (1-5) in particular, there is a vast literature on the construction and the behaviour of blowing-up solutions: we mention for instance [Ben Ayed et al. 2006a; 2006b; Druet 2002; Druet and Laurain 2010; König and Laurain 2022; 2024; Iacopetti and Pacella 2015; Iacopetti and Vaira 2018; Musso and Pistoia 2002; Musso et al. 2024; Premoselli 2022; Vaira 2015].

Our approach in this paper is strongly inspired by these references. We proceed by contradiction: under the assumptions (and with the notations) of Theorem 1.1, and by [Struwe 1984], if $(v_\alpha)_{\alpha\in\mathbb{N}}$ does not strongly converge in $H_0^1(\Omega)$ we have, up to a subsequence,

$$v_\alpha = B_\alpha \pm v_\infty + o(1) \quad \text{in } H_0^1(\Omega) \tag{1-8}$$

as $\alpha \to +\infty$, where $v_\infty \geq 0$ solves (1-1) and where B_α is a positive bubbling profile that concentrates at some point $x_\alpha \in \Omega$ and is modelled on a positive solution of $-\Delta B = B^{2^*-1}$ in $\mathbb{R}^n$; see (2-5) for more details. We perform an asymptotic analysis of v_α near x_α at different scales and obtain necessary conditions on h for blow-up to occur. The contradiction follows from these conditions: to prove Theorem 1.1 when $3 \leq n \leq 5$, for instance, we prove that if (1-8) holds we simultaneously have $v_\infty \equiv 0$ and $v_\infty > 0$ in Ω. In order to investigate the behaviour of v_α near x_α we prove in this paper new pointwise estimates on v_α, up to the boundary, that improve (1-8) in strong spaces. We precisely prove that

$$\left\| \frac{v_\alpha - \Pi B_\alpha \mp v_\infty}{B_\alpha + v_\infty} \right\|_\infty \to 0 \tag{1-9}$$

as $\alpha \to +\infty$, where ΠB_α is the projection of B_α in $H_0^1(\Omega)$ defined by (2-14); see Theorem 2.1 for a precise statement. Estimate (1-9) provides an accurate control on v_α up to $\partial\Omega$ and is particularly useful close to $\partial\Omega$, where, at first order, ΠB_α deviates from B_α and v_∞ vanishes. To the best of our knowledge this is the first time that a similar estimate is proven. We heavily rely on estimate (1-9) to rule out the possibility that the concentration point x_α converges to a point in $\partial\Omega$: this is both the main difficulty that we face in the proof of Theorem 1.1 and the main novelty of our analysis, and is deeply related to the sign-changing nature of the solutions we consider; see Remarks 3.6 and 3.7 for a detailed explanation of this fact.

The structure of the paper is as follows. In Section 2 we prove Theorem 2.1 and establish (1-9). In Section 3 we apply it to obtain necessary conditions for the blow-up of $(v_\alpha)_{\alpha\in\mathbb{N}}$ by means of suitable Pohozaev identities at different scales. We separately treat the interior blow-up case (Proposition 3.1) and the boundary blow-up case (Propositions 3.2, 3.4 and 3.5), and we deduce our main result, Theorem 1.1, from this analysis. Finally, the Appendix contains the proof of a few technical results that are used throughout Section 3.

2. The C^0-theory for blow-up

In this section we let $h_\infty \in C^0(\overline{\Omega})$ and consider a family of functions $(h_\alpha)_{\alpha\in\mathbb{N}} \in C^1(\overline{\Omega})$ such that

$$\lim_{\alpha\to+\infty} h_\alpha = h_\infty \quad \text{in } C^0(\overline{\Omega}). \tag{2-1}$$

We assume that $-\Delta + h_\infty$ is coercive in $H_0^1(\Omega)$ and that $I_{h_\infty}(\Omega) < K_n^{-2}$, where $I_{h_\infty}(\Omega)$ is as in (1-2), so that positive ground states of (1-1) with $h = h_\infty$ exist. We consider a sequence of functions $(v_\alpha)_{\alpha\in\mathbb{N}}$ in

$H_0^1(\Omega)$ such that, for all $\alpha \in \mathbb{N}$, v_α is a solution to

$$\begin{cases} -\Delta v_\alpha + h_\alpha v_\alpha = |v_\alpha|^{2^*-2} v_\alpha & \text{in } \Omega, \\ v_\alpha = 0 & \text{in } \partial\Omega. \end{cases} \tag{2-2}$$

We assume that

$$\limsup_{\alpha\to+\infty} \int_\Omega |v_\alpha|^{2^*}\, dx \le K_n^{-n} + I_{h_\infty}(\Omega)^{n/2}. \tag{2-3}$$

We also assume that $(v_\alpha)_{\alpha\in\mathbb{N}}$ blows up, that is

$$\lim_{\alpha\to+\infty} \|v_\alpha\|_\infty = +\infty. \tag{2-4}$$

By (2-3) and (2-4), and following [Struwe 1984] (see also [Struwe 2008]), we get that, up to a subsequence,

$$v_\alpha = B_\alpha \pm v_\infty + \varphi_\alpha \quad \text{in } H_0^1(\Omega), \tag{2-5}$$

where $\|\varphi_\alpha\|_{H_0^1} \to 0$ as $\alpha \to +\infty$. In (2-5), v_∞ is a solution of (1-1) with $h = h_\infty$ and we have let

$$B_\alpha(x) := \mu_\alpha^{-(n-2)/2} B_0(\mu_\alpha^{-1}(x - x_\alpha)) \quad \text{for } x \in \Omega, \tag{2-6}$$

where $(x_\alpha)_{\alpha\in\mathbb{N}}$ and $(\mu_\alpha)_{\alpha\in\mathbb{N}}$ are sequences of points in Ω and the positive real numbers, respectively, and where we have let

$$B_0(x) = \left(1 + \frac{|x|^2}{n(n-2)}\right)^{1-\frac{n}{2}} \quad \text{for any } x \in \mathbb{R}^n. \tag{2-7}$$

It is well known that B_0 satisfies $-\Delta B_0 = B_0^{2^*-1}$ in $\mathbb{R}^n$ and achieves K_n^{-2} in (1-3). As a consequence of (2-5), we have

$$\lim_{\alpha\to+\infty} v_\alpha = \pm v_\infty \quad \text{weakly in } H_0^1(\Omega) \tag{2-8}$$

and

$$\lim_{\alpha\to+\infty} \int_\Omega |v_\alpha|^{2^*}\, dx = K_n^{-n} + \int_\Omega |v_\infty|^{2^*}\, dx.$$

A consequence of (2-3) and of the assumption $I_{h_\infty}(\Omega) < K_n^{-2}$ is that either $v_\infty \equiv 0$ or v_∞ is a least-energy positive solution of

$$\begin{cases} -\Delta v_\infty + h_\infty v_\infty = v_\infty^{2^*-1} & \text{in } \Omega, \\ v_\infty > 0 & \text{in } \Omega, \\ v_\infty = 0 & \text{on } \partial\Omega. \end{cases} \tag{2-9}$$

If v_α is assumed to change sign for all $\alpha \ge 1$, that is if $(v_\alpha)_+$ and $(v_\alpha)_-$ are nonzero, the arguments in [Cerami et al. 1986, Lemma 3.1] show that $v_\infty > 0$ and hence that

$$\lim_{\alpha\to+\infty} \int_\Omega |v_\alpha|^{2^*}\, dx = K_n^{-n} + I_{h_\infty}(\Omega)^{n/2}.$$

This observation will be important in the proof of Theorem 1.1 but will not be used in this section. Without loss of generality we can assume that $(x_\alpha)_{\alpha\in\mathbb{N}}$ and $(\mu_\alpha)_{\alpha\in\mathbb{N}}$ are chosen to satisfy

$$|v_\alpha(x_\alpha)| = \|v_\alpha(x)\|_\infty \quad \text{and} \quad \mu_\alpha := |v_\alpha(x_\alpha)|^{-2/(n-2)}, \tag{2-10}$$

so that $x_\alpha \in \Omega$. Note that (2-4) implies that $\mu_\alpha \to 0$ as $\alpha \to +\infty$. We will denote by $x_\infty \in \overline{\Omega}$ the limit of the x_α as $\alpha \to +\infty$. In the case where $v_\infty > 0$, Hopf's lemma shows that there exists $C_0 > 0$ such that

$$C_0^{-1} d(x, \partial\Omega) \le v_\infty(x) \le C_0 d(x, \partial\Omega) \quad \text{for all } x \in \Omega, \tag{2-11}$$

where $d(x, \partial\Omega) := \inf\{|x-y| : y \in \partial\Omega\}$ is the distance of x to boundary. In (2-5) we used the notation $v_\alpha = B_\alpha \pm v_\infty + \varphi_\alpha$, which classically means either $v_\alpha = B_\alpha + v_\infty + \varphi_\alpha$ or $v_\alpha = B_\alpha - v_\infty + \varphi_\alpha$. It will often be more convenient to subtract $B_\alpha \pm v_\infty$ from v_α (for instance in the statement of Theorem 2.1), which we will thus write as

$$v_\alpha - B_\alpha \mp v_\infty = \varphi_\alpha$$

so that the sign convention is satisfied.

The purpose of this section is to turn (2-5) into a decomposition in strong spaces, and to obtain sharp pointwise estimates on v_α. In order to state our main result we need to introduce more notation. For α large, thanks to (2-1), $-\Delta + h_\alpha$ is coercive in $H_0^1(\Omega)$. We can thus let G_α be the Green's function of $-\Delta + h_\alpha$ in Ω with Dirichlet boundary conditions. By standard properties of the Green's function (see [Robert 2010]), there exists $C > 0$ such that for all $\alpha \ge 1$ we have

$$G_\alpha(y, x) \le \frac{C}{|y-x|^{n-2}} \min\left\{1, \frac{d(y, \partial\Omega) d(x, \partial\Omega)}{|y-x|^2}\right\} \quad \text{for all } x, y \in \Omega, \quad x \ne y, \tag{2-12}$$

and

$$|\nabla G_\alpha(y, x)| \le C|y-x|^{1-n} \quad \text{for all } x, y \in \Omega, \quad x \ne y. \tag{2-13}$$

For $\alpha \ge 1$, we let ΠB_α be the unique solution in $H_0^1(\Omega)$ of

$$\begin{cases} (-\Delta + h_\alpha)\Pi B_\alpha = B_\alpha^{2^*-1} & \text{in } \Omega, \\ \Pi B_\alpha = 0 & \text{on } \partial\Omega. \end{cases} \tag{2-14}$$

Since B_α satisfies $-\Delta B_\alpha = B_\alpha^{2^*-1}$ in $\mathbb{R}^n$ by (2-6) and (2-7), we easily see with (2-14) that $B_\alpha - \Pi B_\alpha \to 0$ in $H_0^1(\Omega)$ as $\alpha \to +\infty$. Thus (2-5) can be rewritten as

$$v_\alpha = \Pi B_\alpha \pm v_\infty + o(1) \quad \text{in } H_0^1(\Omega) \text{ as } \alpha \to +\infty. \tag{2-15}$$

A representation formula for ΠB_α together with (2-12) shows that there exists $C > 0$ such that for all $x \in \Omega$ and all $\alpha \ge 1$ we have

$$0 < \Pi B_\alpha(x) \le C B_\alpha(x), \tag{2-16}$$

where positivity follows from the coercivity of $-\Delta + h_\alpha$. We can now state the main result of this section:

Theorem 2.1. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \ge 3$, and $(h_\alpha)_{\alpha\in\mathbb{N}}$ be a sequence of functions that converges in $C^0(\overline{\Omega})$ to h_∞. We assume that $-\Delta + h_\infty$ is coercive in $H_0^1(\Omega)$ and that $I_{h_\infty}(\Omega) < K_n^{-2}$. Let $(v_\alpha)_{\alpha\in\mathbb{N}} \in H_0^1(\Omega)$ be a sequence of solutions of (2-2) that satisfies (2-3), (2-4) and (2-5). There exists a sequence $(\varepsilon_\alpha)_{\alpha\in\mathbb{N}}$ of positive real numbers converging to 0 such that, up to a subsequence, we have, for any $x \in \Omega$ and $\alpha \ge 1$,*

$$|v_\alpha(x) - \Pi B_\alpha(x) \mp v_\infty(x)| \le \varepsilon_\alpha (B_\alpha(x) + v_\infty(x)). \tag{2-17}$$

Pointwise descriptions of blowing-up solutions as in Theorem 2.1 were first obtained for *positive* solutions of critical Schrödinger-type equations on manifolds without boundary; see for instance [Druet and Hebey 2009; Druet et al. 2004] (see also [Hebey 2014]). For *positive* solutions of equations like (2-2) in bounded open subsets of $\mathbb{R}^n$ they were obtained in [König and Laurain 2022; 2024]. Similar estimates have been obtained for positive solutions of Hardy–Sobolev equations in [Cheikh Ali 2022; Ghoussoub et al. 2023]. These sharp pointwise estimates have proven crucial in order to obtain compactness and stability results for critical stationary elliptic equations [Druet 2003; Druet and Laurain 2010]. When it comes to *sign-changing* blowing-up solutions, a general pointwise description as in Theorem 2.1, on manifolds without boundary, has been obtained in [Premoselli 2024; Premoselli and Robert 2025], and subsequent compactness results have been proven in [Premoselli and Robert 2025; Premoselli and Vétois 2022a; 2022b]. Theorem 2.1 is, to our knowledge, the first instance where sharp pointwise estimates for blowing-up solutions of equations like (2-2) are obtained up to the boundary of Ω. Note indeed that in Theorem 2.1 we do not assume that the concentration point $x_\infty = \lim_{\alpha\to+\infty} x_\alpha$ is an interior point in Ω. It may happen that $x_\infty \in \partial\Omega$: the real novelty of Theorem 2.1 is that (2-17) holds regardless of the speed of convergence of x_α to $\partial\Omega$, uniformly in $x \in \overline{\Omega}$. This creates additional technical difficulties that we overcome in the course of the proof.

We prove Theorem 2.1 by taking inspiration from the arguments in [Druet and Hebey 2009]; see also [Hebey 2014]. Throughout this section we let Ω be a smooth bounded domain in $\mathbb{R}^n$, $n \geq 3$, $(h_\alpha)_{\alpha\in\mathbb{N}} \in C^0(\overline{\Omega})$ and $(v_\alpha)_{\alpha\in\mathbb{N}} \in H_0^1(\Omega)$ be such that (2-1), (2-2), (2-4), and (2-5) hold, and we let $(x_\alpha)_{\alpha\in\mathbb{N}} \in \Omega$ and $(\mu_\alpha)_{\alpha\in\mathbb{N}}$ be as defined as in (2-10). We start with the following simple proposition:

Proposition 2.2. *We have*

$$\lim_{\alpha\to+\infty} \frac{d(x_\alpha, \partial\Omega)}{\mu_\alpha} = +\infty. \tag{2-18}$$

We define the rescaled function

$$\tilde{v}_\alpha(x) := \mu_\alpha^{(n-2)/2} v_\alpha(x_\alpha + \mu_\alpha x) \quad \textit{for all } x \in \Omega_\alpha, \tag{2-19}$$

where $\Omega_\alpha := \{x \in \mathbb{R}^n : x_\alpha + \mu_\alpha x \in \Omega\}$. *Then*

$$\lim_{\alpha\to+\infty} \tilde{v}_\alpha(x) = B_0(x) \quad \textit{in } C^2_{\mathrm{loc}}(\mathbb{R}^n), \tag{2-20}$$

where B_0 *is defined in* (2-7).

Proof. First, (2-18) follows from Struwe's original result [Struwe 1984]; see also [Mazumdar 2017, Theorem 1.2]. We now prove (2-20). For $x \in \Omega_\alpha := \{x \in \mathbb{R}^n : x_\alpha + \mu_\alpha x \in \Omega\}$, it is clear by (2-2) and (2-19) that

$$\begin{cases} -\Delta\tilde{v}_\alpha + \tilde{h}_\alpha \mu_\alpha^2 \tilde{v}_\alpha = |\tilde{v}_\alpha|^{2^*-2}\tilde{v}_\alpha & \text{in } \Omega_\alpha, \\ \tilde{v}_\alpha = 0 & \text{on } \partial\Omega_\alpha, \end{cases}$$

where $\tilde{h}_\alpha(x) = h_\alpha(x_\alpha + \mu_\alpha x)$ and $\tilde{v}_\alpha$ is defined in (2-19). We remark that $|\tilde{v}_\alpha| \leq |\tilde{v}_\alpha(0)| = 1$. It follows from (2-1) and from standard elliptic theory that, after passing to a subsequence, $\tilde{v}_\alpha \to \tilde{v}$ in $C^2_{\mathrm{loc}}(\mathbb{R}^n)$, where $\tilde{v} \in C^2(\mathbb{R}^n)$ is such that

$$-\Delta\tilde{v} = |\tilde{v}|^{2^*-2}\tilde{v} \quad \text{in } \mathbb{R}^n$$

and $|\tilde{v}| \le 1$. Let $K \Subset \mathbb{R}^n$ be a nonempty compact subset of $\mathbb{R}^n$. By (2-5) we have $\tilde{v}_\alpha \to B_0$ in $L^{2^*}(K)$ as $\alpha \to +\infty$, so that $\tilde{v} = B_0$ in K, which proves (2-20). ☐

Using (2-18) and standard elliptic theory, together with (2-14) and (2-16), we also obtain that

$$\mu_\alpha^{(n-2)/2} \Pi B_\alpha(x_\alpha + \mu_\alpha x) \to B_0(x) \quad \text{in } C^2_{\text{loc}}(\mathbb{R}^n) \tag{2-21}$$

as $\alpha \to +\infty$. The following result establishes a first pointwise control on v_α.

Proposition 2.3. *For $x \in \Omega$ we let $D_\alpha(x) := |x - x_\alpha| + \mu_\alpha$. Then*

$$D_\alpha(x)^{(n-2)/2} |v_\alpha - \Pi B_\alpha \mp v_\infty| \to 0 \quad \textit{in } C^0(\overline{\Omega}) \textit{ as } \alpha \to +\infty, \tag{2-22}$$

where v_∞ and ΠB_α are as defined in (2-8), (2-9) *and* (2-14).

To prove Proposition 2.3 we proceed by contradiction: we assume that there exist $\epsilon_0 > 0$ and $(y_\alpha)_{\alpha\in\mathbb{N}} \in \overline{\Omega}$ such that

$$D_\alpha(y_\alpha)^{(n-2)/2} |v_\alpha(y_\alpha) \mp v_\infty(y_\alpha) - \Pi B_\alpha(y_\alpha)| = \max_{x\in\Omega} (D_\alpha(x)^{(n-2)/2} |v_\alpha(x) \mp v_\infty(x) - \Pi B_\alpha(x)|) \ge \epsilon_0, \tag{2-23}$$

and we let $(\nu_\alpha)_{\alpha\in\mathbb{N}} \in (0, +\infty)$ be such that

$$|v_\alpha(y_\alpha)| = \nu_\alpha^{(2-n)/2} \quad \text{for all } \alpha \ge 1. \tag{2-24}$$

Since v_α, ΠB_α and v_∞ vanish in $\partial\Omega$, a first simple observation is that $y_\alpha \in \Omega$.

Step 1. We claim that

$$D_\alpha(y_\alpha)^{(n-2)/2} B_\alpha(y_\alpha) \to 0 \quad \text{as } \alpha \to +\infty.$$

As a consequence, with (2-16) we have

$$D_\alpha(y_\alpha)^{(n-2)/2} \Pi B_\alpha(y_\alpha) \to 0 \quad \text{as } \alpha \to +\infty. \tag{2-25}$$

Proof. Indeed, suppose on the contrary that there exists $\rho_0 > 0$ such that

$$D_\alpha(y_\alpha)^{(n-2)/2} B_\alpha(y_\alpha) \ge \rho_0$$

for all α large enough. Hence we have that

$$1 + \frac{|x_\alpha - y_\alpha|}{\mu_\alpha} = \frac{D_\alpha(y_\alpha)}{\mu_\alpha} \ge \rho_0^{2/(n-2)} \left(1 + \frac{|y_\alpha - x_\alpha|^2}{\mu_\alpha^2}\right).$$

Up to passing to a subsequence we then assume that there exists $R > 0$ such that $\lim_{\alpha\to+\infty} \mu_\alpha^{-1} |y_\alpha - x_\alpha| = R$. This means that

$$D_\alpha(y_\alpha) = O(\mu_\alpha). \tag{2-26}$$

It follows from (2-21) and (2-20) that

$$\lim_{\alpha\to+\infty} \mu_\alpha^{(n-2)/2} |v_\alpha(y_\alpha) - \Pi B_\alpha(y_\alpha)| = 0.$$

With (2-26) we thus get that

$$\lim_{\alpha\to+\infty} D_\alpha(y_\alpha)^{(n-2)/2}|v_\alpha(y_\alpha)\mp v_\infty(y_\alpha)-\Pi B_\alpha(y_\alpha)|=0,$$

which contradicts (2-23). ☐

Step 2. We claim that

$$\nu_\alpha\to 0 \quad \text{as } \alpha\to+\infty, \tag{2-27}$$

where ν_α is defined in (2-24).

Proof. Indeed, it follows from (2-23) and (2-25) that

$$\epsilon_0\le D_\alpha(y_\alpha)^{(n-2)/2}(|v_\alpha(y_\alpha)|+\|v_\infty\|_\infty)+o(1) \tag{2-28}$$

as $\alpha\to+\infty$. If $D_\alpha(y_\alpha)\to 0$ as $\alpha\to+\infty$, then (2-27) follows from (2-28). Suppose on the contrary that, up to a subsequence, $D_\alpha(y_\alpha)\to c_0$ as $\alpha\to+\infty$ for some $c_0>0$. It follows from (2-23) and (2-25) that

$$|v_\alpha(x)\mp v_\infty(x)|+o(1)\le 2^n|v_\alpha(y_\alpha)\mp v_\infty(y_\alpha)|+o(1) \tag{2-29}$$

for $x\in B_{c_0/2}(y_\alpha)\cap\overline{\Omega}$ and all α sufficiently large. If $v_\alpha(y_\alpha)\to+\infty$ as $\alpha\to+\infty$, it is clear, by the definition of ν_α, that we obtain (2-27). If $v_\alpha(y_\alpha)=O(1)$, standard elliptic theory together with (2-8) and (2-29) proves that $v_\alpha\mp v_\infty\to 0$ in $C^2_{\text{loc}}(B_{c_0/4}(y_\alpha))$ as $\alpha\to+\infty$. This contradicts (2-23) using (2-25). We thus get that (2-27) holds. ☐

For any $x\in\Omega_\alpha:=\{x\in\mathbb{R}^n : y_\alpha+\nu_\alpha x\in\Omega\}$, we set

$$w_\alpha(x)=\nu_\alpha^{(n-2)/2}v_\alpha(y_\alpha+\nu_\alpha x).$$

By (2-2), w_α satisfies

$$\begin{cases}-\Delta w_\alpha+h_\alpha(y_\alpha+\nu_\alpha x)\nu_\alpha^2 w_\alpha=|w_\alpha|^{2^*-2}w_\alpha & \text{in } \Omega_\alpha,\\ w_\alpha=0 & \text{on } \partial\Omega_\alpha.\end{cases} \tag{2-30}$$

Thanks to (2-24), we have that $|w_\alpha(0)|=1$. We define a set S as

$$S=\begin{cases}\left\{\lim_{\alpha\to+\infty}\frac{y_\alpha-x_\alpha}{\nu_\alpha}\right\} & \text{if } |y_\alpha-x_\alpha|=O(\nu_\alpha) \text{ and } \mu_\alpha=o(\nu_\alpha),\\ \varnothing & \text{otherwise,}\end{cases}$$

where it is intended that the limit exists up to passing to a subsequence. Let us fix $K\Subset\mathbb{R}^n\backslash S$ a compact set.

Step 3. As $\alpha\to+\infty$ we have

$$\nu_\alpha^{(n-2)/2}B_\alpha(y_\alpha-\nu_\alpha x)\to 0 \quad \text{for all } x\in K. \tag{2-31}$$

Proof. Let $x\in K$. If $\nu_\alpha=o(\mu_\alpha)$ then (2-31) is true since $B_\alpha(x)\le\mu_\alpha^{-(n-2)/2}$ for any $x\in\overline{\Omega}$. We now assume that $\mu_\alpha=o(\nu_\alpha)$: since $x\in K$, we get that $\nu_\alpha=O(|y_\alpha-x_\alpha-\nu_\alpha x|)$. Thus once again (2-31) holds by definition of B_α. We may thus assume that there exists $C>0$ such that

$$C^{-1}\nu_\alpha\le\mu_\alpha\le C\nu_\alpha \quad \text{for all } \alpha. \tag{2-32}$$

Assume first that $|y_\alpha - x_\alpha - \nu_\alpha x| = O(\mu_\alpha)$. Thus, since $x \in K$ and by (2-32), we get $|y_\alpha - x_\alpha| = O(\mu_\alpha)$. Arguing as in the proof of Step 1 we get a contradiction. Thus, for all $x \in K$, we have

$$\lim_{\alpha\to+\infty} \frac{|y_\alpha - x_\alpha - \nu_\alpha x|}{\mu_\alpha} = +\infty.$$

Together with (2-32) this implies (2-31). □

Step 4. We claim that

$$w_\alpha(x) = O(1) \quad \text{for all } x \in K \cap \Omega_\alpha. \tag{2-33}$$

Proof. Indeed, using (2-23) and (2-25) together with (2-31) yields

$$\left(\frac{D_\alpha(y_\alpha+\nu_\alpha x)}{D_\alpha(y_\alpha)}\right)^{\frac{n-2}{2}} \left|w_\alpha(x) \mp \nu_\alpha^{(n-2)/2} v_\infty(y_\alpha+\nu_\alpha x) - \nu_\alpha^{(n-2)/2} \Pi B_\alpha(y_\alpha+\nu_\alpha x)\right| \le 1 + o(1) \tag{2-34}$$

for all $x \in K \cap \Omega_\alpha$. It then follows from (2-16), (2-27), (2-31) and (2-34) that

$$\left(\frac{D_\alpha(y_\alpha+\nu_\alpha x)}{D_\alpha(y_\alpha)}\right)^{\frac{n-2}{2}} (|w_\alpha(x)| + o(1)) \le 1 + o(1) \quad \text{for all } x \in K \cap \Omega_\alpha. \tag{2-35}$$

We claim that there exists $\eta_K > 0$ such that

$$\lim_{\alpha\to+\infty} D_\alpha(y_\alpha+\nu_\alpha x) D_\alpha(y_\alpha)^{-1} \ge \eta_K$$

for all $x \in K \cap \Omega_\alpha$. Together with (2-35) this will prove that w_α is bounded in $K \cap \Omega_\alpha$. Suppose on the contrary that for a sequence $(z_\alpha)_{\alpha\in\mathbb{N}}$ in $K \cap \Omega_\alpha$ we have

$$|y_\alpha - x_\alpha + \nu_\alpha z_\alpha| + \mu_\alpha = o(|y_\alpha - x_\alpha|) + o(\mu_\alpha).$$

Then $|y_\alpha - x_\alpha| = O(\nu_\alpha)$, $\mu_\alpha = o(\nu_\alpha)$ and

$$\lim_{\alpha\to+\infty} \left|\frac{y_\alpha - x_\alpha}{\nu_\alpha} - z_\alpha\right| = 0,$$

which is a contradiction since $\liminf_{\alpha\to+\infty} d(z_\alpha, S) > 0$. □

We now conclude the proof of Proposition 2.3.

Proof of Proposition 2.3. We first claim that $0 \in \Omega_\alpha \backslash S$. If $S = \varnothing$ this is obvious. Assume thus that $S \ne \varnothing$, which implies that $|y_\alpha - x_\alpha| = O(\nu_\alpha)$ and $\mu_\alpha = o(\nu_\alpha)$ as $\alpha \to +\infty$. Then, since $\nu_\alpha \to 0$ as $\alpha \to +\infty$ and by (2-28), we obtain

$$\epsilon_0^{2/(n-2)} + o(1) \le \nu_\alpha^{-1} D_\alpha(y_\alpha).$$

Hence, we have $\lim_{\alpha\to+\infty} \nu_\alpha^{-1}(y_\alpha - x_\alpha) \ne 0$, and thus $0 \notin S$. By (2-33), for any compact subset $K \subset \mathbb{R}^n \backslash S$ that contains 0, there exists $C_K > 0$ such that

$$|w_\alpha(x)| \le C_K \quad \text{in } K.$$

In particular, by standard elliptic theory, (2-30) and (2-1), we get

$$w_\alpha \to w_0 \in C^1_{\text{loc}}(\mathbb{R}^n \backslash S), \tag{2-36}$$

where w_0 satisfies $-\Delta w_0 = |w_0|^{2^*-2} w_0$ in $\mathbb{R}^n \backslash S$ and $|w_0(0)| = 1$. Independently, it follows from (2-5) and (2-31) that $w_\alpha \to 0$ in $L^{2^*}(K)$ as $\alpha \to +\infty$. Hence, by (2-36), we find that

$$\int_K |w_0|^{2^*} \, dx = 0.$$

Thus $w_0 \equiv 0$ in K, which contradicts $|w_0(0)| = 1$. This ends the proof of Proposition 2.3. $\square$

For $\rho > 0$ small enough, we define

$$\eta_\alpha(\rho) := \sup_{\Omega \backslash B_\rho(x_\alpha)} |v_\alpha(x)|, \tag{2-37}$$

where x_α is given by (2-10). Thanks to (2-22), we obtain

$$\lim_{\alpha \to +\infty} \sup \eta_\alpha(\rho) \le \|v_\infty\|_\infty. \tag{2-38}$$

The next results establishes a first pointwise control on v_α.

Proposition 2.4. *For any $\nu \in \left(0, \frac{1}{2}\right)$ there exists $R_\nu > 0$, $\rho_\nu > 0$, and $C_\nu > 0$ such that for all $\alpha \in \mathbb{N}$*

$$|v_\alpha(x)| \le C_\nu \left(\frac{\mu_\alpha^{(n-2)/2 - \nu(n-2)}}{|x - x_\alpha|^{(n-2)(1-\nu)}} + \frac{\eta_\alpha(\rho_\nu)}{|x - x_\alpha|^{(n-2)\nu}} \right) \tag{2-39}$$

for all $x \in \Omega \backslash B_{R_\nu \mu_\alpha}(x_\alpha)$.

Proof. We divide our proof into two cases, depending on the position of x_∞ with respect to the boundary of Ω.

Case 1: $x_\infty \in \partial\Omega$. Let $U \subset \mathbb{R}^n$ be a smooth bounded open set such that $\bar{\Omega} \Subset U$. For all $\alpha \ge 1$, we extend h_α and h_∞ as functions on U in such a way that

$$h_\alpha \to h_\infty \quad \text{in } C^0(\bar{U}) \tag{2-40}$$

and $-\Delta + h_\infty$ is still coercive in $H_0^1(U)$. Let $\widetilde{G} : \bar{U} \times \bar{U} \backslash \{(x, x) : x \in \bar{U}\} \to \mathbb{R}$ be the Green's function of the operator $-\Delta + h_\infty$ with Dirichlet boundary conditions in U. It exists by coercivity of $-\Delta + h_\infty$ and satisfies, for all $x \in U$,

$$-\Delta \widetilde{G}(x, \cdot) + h_\infty \widetilde{G}(x, \cdot) = \delta_x \quad \text{in } U \backslash \{x\}. \tag{2-41}$$

We now define $\widetilde{G}_\alpha(x) := \widetilde{G}(x_\alpha, x)$ for all $x \in \bar{U} \backslash \{x_\alpha\}$ and $\alpha \in \mathbb{N}$. It follows from [Robert 2010] that there exists $C_1 > 0$ such that

$$0 < \widetilde{G}_\alpha(x) \le C_1 |x - x_\alpha|^{2-n} \quad \text{for all } x \in \bar{U} \backslash \{x_\alpha\} \tag{2-42}$$

and that there exist $\rho > 0$ and $C_2 > 0$ such that

$$\widetilde{G}_\alpha(x) \ge C_2 |x - x_\alpha|^{2-n} \quad \text{and} \quad \frac{|\nabla \widetilde{G}_\alpha(x)|}{|\widetilde{G}_\alpha(x)|} \ge C_2 |x - x_\alpha|^{-1} \tag{2-43}$$

for all $x \in B_\rho(x_\alpha) \backslash \{x_\alpha\} \Subset U$. We define

$$L_\alpha := -\Delta + h_\alpha - |v_\alpha|^{2^*-2}, \tag{2-44}$$

and for a fixed $\nu \in (0, 1)$ we let, for $\alpha \in \mathbb{N}$ and $x \in \overline{U}\backslash\{x_\alpha\}$,

$$\psi_{\nu,\alpha}(x) := \mu_\alpha^{(n-2)/2-\nu(n-2)}\widetilde{G}_\alpha(x)^{1-\nu} + \eta_\alpha(\rho)\widetilde{G}_\alpha(x)^\nu. \tag{2-45}$$

Straightforward computations using (2-40) and (2-41) show that

$$\frac{L_\alpha\psi_{\nu,\alpha}}{\psi_{\nu,\alpha}} \ge -2\|h_\infty\|_\infty + o(1) + \nu(1-\nu)\left|\frac{\nabla\widetilde{G}_\alpha}{\widetilde{G}_\alpha}\right|^2 - |v_\alpha|^{2^*-2}.$$

By using (2-43) we get

$$\frac{L_\alpha\psi_{\nu,\alpha}}{\psi_{\nu,\alpha}} \ge -2\|h_\infty\|_\infty + o(1) + \nu(1-\nu)\frac{C_2^2}{|x-x_\alpha|^2} - |v_\alpha|^{2^*-2} \tag{2-46}$$

for all $x \in B_\rho(x_\alpha)\backslash\{x_\alpha\} \Subset U$, where C_2 is the constant appearing in (2-43). Proposition 2.3 now shows that there exists $R_0 > 0$ such that for any $R > R_0$ and $x \in \Omega\backslash B_{R\mu_\alpha}(x_\alpha)$ we have

$$|x-x_\alpha|^2|v_\alpha(x) \mp v_\infty(x)|^{2^*-2} \le \frac{\nu(1-\nu)C_2^2}{2^{2^*+1}} \tag{2-47}$$

for α sufficiently large. Hence, by (2-47), we get

$$|x-x_\alpha|^2|v_\alpha(x)|^{2^*-2} \le \tfrac{1}{4}\nu(1-\nu)C_2^2 + 2^{2^*-1}\rho^2\|v_\infty\|_\infty^{2^*-2} \tag{2-48}$$

for all $x \in (B_\rho(x_\alpha)\backslash B_{R\mu_\alpha}(x_\alpha)) \cap \Omega$. Choose $\rho_0 > 0$ small enough that for any $\rho \in (0, \rho_0)$ we have

$$2^{2^*-1}\rho^2\|v_\infty\|_\infty^{2^*-2} + 2\rho^2\|h_\infty\|_\infty \le \tfrac{1}{4}\nu(1-\nu)C_2^2. \tag{2-49}$$

Combining (2-48) and (2-49) in (2-46) we finally obtain that, for all $x \in (B_\rho(x_\alpha)\backslash B_{R\mu_\alpha}(x_\alpha)) \cap \Omega$,

$$L_\alpha\psi_{\nu,\alpha} \ge \frac{1}{|x-x_\alpha|^2}\big(o(\rho^2) + \tfrac{1}{2}\nu(1-\nu)C_2^2\big)\psi_{\nu,\alpha} > 0. \tag{2-50}$$

Independently, it follows from (2-20), (2-37) and (2-43) that there exists $C = C(R, \rho, \nu) > 0$ such that

$$|v_\alpha(x)| \le C\psi_{\nu,\alpha}(x) \quad \text{for all } x \in \partial((B_\rho(x_\alpha)\backslash B_{R\mu_\alpha}(x_\alpha)) \cap \Omega). \tag{2-51}$$

By (2-2), v_α satisfies $L_\alpha v_\alpha = 0$. Using (2-50) and (2-51) we thus have

$$\begin{cases} L_\alpha(C\psi_{\nu,\alpha}) \ge 0 = L_\alpha v_\alpha & \text{in } (B_\rho(x_\alpha)\backslash B_{R\mu_\alpha}(x_\alpha)) \cap \Omega, \\ C\psi_{\nu,\alpha} \ge v_\alpha & \text{on } \partial((B_\rho(x_\alpha)\backslash B_{R\mu_\alpha}(x_\alpha)) \cap \Omega), \\ L_\alpha(C\psi_{\nu,\alpha}) \ge 0 = -L_\alpha v_\alpha & \text{in } (B_\rho(x_\alpha)\backslash B_{R\mu_\alpha}(x_\alpha)) \cap \Omega, \\ C\psi_{\nu,\alpha} \ge -v_\alpha & \text{on } \partial((B_\rho(x_\alpha)\backslash B_{R\mu_\alpha}(x_\alpha)) \cap \Omega). \end{cases} \tag{2-52}$$

The operator L_α satisfies the comparison principle on $(B_\rho(x_\alpha)\backslash B_{R\mu_\alpha}(x_\alpha)) \cap \Omega$ since $\psi_{\nu,\alpha} > 0$ and $L_\alpha\psi_{\nu,\alpha} > 0$ (see, e.g., [Berestycki et al. 1994]), and therefore

$$|v_\alpha(x)| \le C\psi_{\nu,\alpha}(x) \quad \text{for all } x \in (B_\rho(x_\alpha)\backslash B_{R\mu_\alpha}(x_\alpha)) \cap \Omega.$$

Using again (2-42) implies (2-39) in this case.

Case 2: $x_\infty \in \Omega$. Let G be the Green's function in Ω of the operator $-\Delta + h_\infty$ with Dirichlet boundary conditions. For $x \in \Omega\backslash\{x_\alpha\}$ define $\widetilde{G}_\alpha := G(x_\alpha, \cdot)$, which satisfies

$$-\Delta\widetilde{G}_\alpha + h_\infty\widetilde{G}_\alpha = 0 \quad \text{in } \Omega\backslash\{x_\alpha\}.$$

Since $x_\infty \in \Omega$, it follows from [Robert 2010] that there exists $C_3 > 0$ such that

$$0 < \widetilde{G}_\alpha(x) \le C_3|x - x_\alpha|^{2-n} \quad \text{for all } x \in \overline{\Omega}\backslash\{x_\alpha\}$$

and there exist $C_4 > 0$ and $\rho > 0$ such that

$$\widetilde{G}_\alpha(x) \ge C_4|x - x_\alpha|^{2-n} \quad \text{and} \quad \frac{|\nabla\widetilde{G}_\alpha(x)|}{|\widetilde{G}_\alpha(x)|} \ge C_4|x - x_\alpha|^{-1}$$

for all $x \in B_\rho(x_\alpha)\backslash\{x_\alpha\} \Subset \Omega$. Define, for a fixed $\nu \in (0, 1)$, for $\alpha \in \mathbb{N}$ and $x \in \overline{\Omega}\backslash\{x_\alpha\}$,

$$\psi_{\nu,\alpha}(x) := \mu_\alpha^{(n-2)/2-\nu(n-2)}\widetilde{G}_\alpha(x)^{1-\nu} + \eta_\alpha(\rho)\widetilde{G}_\alpha(x)^{\nu},$$

and let again $L_\alpha = -\Delta + h_\alpha - |v_\alpha|^{2^*-2}$. Mimicking the arguments in Case 1 we here again have $\psi_{\nu,\alpha} > 0$ and $L_\alpha\psi_{\nu,\alpha} > 0$ in $B_\rho(x_\alpha)\backslash B_{R\mu_\alpha}(x_\alpha)$, and the proof of (2-39) follows in a similar way. ☐

The next results establishes a pointwise control from above on v_α.

Proposition 2.5. *There exists $C > 0$ such that*

$$|v_\alpha(x)| \le C(\mu_\alpha^{(n-2)/2}D_\alpha(x)^{2-n} + \|v_\infty\|_\infty) \tag{2-53}$$

for all $x \in \Omega$.

Proof. Recall that $D_\alpha(x) = \mu_\alpha + |x - x_\alpha|$ for $x \in \Omega$. We first prove that there exists $\rho > 0$ and $C > 0$ such that

$$|v_\alpha(x)| \le C(\mu_\alpha^{(n-2)/2}D_\alpha(x)^{2-n} + \eta_\alpha(\rho)), \tag{2-54}$$

where $\eta_\alpha(\rho)$ is defined in (2-37). We fix $0 < \nu < 1/(n+2)$, and we let $R_\nu > 0$ and $\rho_\nu > 0$ be given by Proposition 2.4. We let $\rho = \rho_\nu$. Proving (2-54) amounts to proving that, for any sequence $y_\alpha \in \Omega$, we have

$$\frac{|v_\alpha(y_\alpha)|}{\mu_\alpha^{(n-2)/2}D_\alpha(y_\alpha)^{2-n} + \eta_\alpha(\rho)} = O(1) \quad \text{as } \alpha \to +\infty. \tag{2-55}$$

We let in this proof $r_\alpha := |y_\alpha - x_\alpha|$. First, if $r_\alpha \ge \rho$, it is clear that (2-55) is satisfied by definition of $\eta_\alpha(\rho)$. If now $r_\alpha = O(\mu_\alpha)$ we also have $D_\alpha(y_\alpha) = O(\mu_\alpha)$, and (2-21) and (2-22) yield

$$D_\alpha(y_\alpha)^{n-2}\mu_\alpha^{-(n-2)/2}|v_\alpha(y_\alpha)| = O(1),$$

which proves (2-55). We thus assume from now on that

$$r_\alpha \le \rho \quad \text{and} \quad \lim_{\alpha\to+\infty}\frac{r_\alpha}{\mu_\alpha} = +\infty. \tag{2-56}$$

Green's representation formula and (2-12) yield the existence of $C > 0$ such that

$$|v_\alpha(y_\alpha)| \le C\int_\Omega |y_\alpha - x|^{2-n}|v_\alpha(x)|^{2^*-1}\,dx \tag{2-57}$$

for all $\alpha \ge 1$. We write

$$\begin{aligned}\int_\Omega |y_\alpha - x|^{2-n}|v_\alpha(x)|^{2^*-1}\,dx \\ \le \int_{\Omega\cap\{|x-x_\alpha|\le R_\nu\mu_\alpha\}} |y_\alpha - x|^{2-n}|v_\alpha(x)|^{2^*-1}\,dx + \int_{\Omega\cap\{|x-x_\alpha|\ge R_\nu\mu_\alpha\}} |y_\alpha - x|^{2-n}|v_\alpha(x)|^{2^*-1}\,dx.\end{aligned} \tag{2-58}$$

Fix $C_0 > R_\nu$. For α sufficiently large we have using (2-56) that

$$r_\alpha \ge C_0\mu_\alpha \ge \frac{C_0}{R_\nu}|x - x_\alpha| \quad \text{for all } x \in \Omega\cap\{|x - x_\alpha| \le R_\nu\mu_\alpha\},$$

so that $|y_\alpha - x| \ge (1 - R_\nu C_0^{-1})r_\alpha$ for all such x. Therefore, using Hölder's inequality and (2-3) yields

$$\int_{\Omega\cap\{|x-x_\alpha|\le R_\nu\mu_\alpha\}} |y_\alpha - x|^{2-n}|v_\alpha(x)|^{2^*-1}\,dx = O\left(\frac{\mu_\alpha^{(n-2)/2}}{|y_\alpha - x_\alpha|^{n-2}}\right). \tag{2-59}$$

Now, we deal with the second term of (2-58). From (2-39), we get

$$\begin{aligned}\int_{\Omega\cap\{|x-x_\alpha|\ge R_\nu\mu_\alpha\}} |y_\alpha - x|^{2-n}|v_\alpha(x)|^{2^*-1}\,dx = O\left(\mu_\alpha^{(n+2)(1-2\nu)/2}\int_{\Omega\cap\{|x-x_\alpha|\ge R_\nu\mu_\alpha\}} \frac{|y_\alpha - x|^{2-n}}{|x - x_\alpha|^{(n+2)(1-\nu)}}\,dx\right) \\ + O\left(\eta_\alpha(\rho_\nu)^{2^*-1}\int_{\Omega\cap\{|x-x_\alpha|\ge R_\nu\mu_\alpha\}} \frac{|y_\alpha - x|^{2-n}}{|x - x_\alpha|^{(n+2)\nu}}\,dx\right).\end{aligned}$$

Since $2 - (n+2)\nu > 0$, using Giraud's lemma (see [Hebey 2014, Lemma 7.5]) yields

$$\int_\Omega |y_\alpha - x|^{2-n}|x - x_\alpha|^{-(n+2)\nu}\,dx = O(1). \tag{2-60}$$

Independently, letting $\tilde{y}_\alpha = (y_\alpha - x_\alpha)/\mu_\alpha$ we have

$$\begin{aligned}\int_{\Omega\cap\{|x-x_\alpha|\ge R_\nu\mu_\alpha\}} \frac{1}{|y_\alpha - x|^{n-2}}\frac{1}{|x - x_\alpha|^{(n+2)(1-\nu)}}\,dx &\le \mu_\alpha^{2-(n+2)(1-\nu)}\int_{\mathbb{R}^n\setminus B(0,R_\nu)} \frac{1}{|\tilde{y}_\alpha - x|^{n-2}}\frac{1}{|x|^{(n+2)(1-\nu)}}\,dx \\ &= O\left(\frac{\mu_\alpha^{2-(n+2)(1-\nu)}}{(1+|\tilde{y}_\alpha|)^{n-2}}\right) = O\left(\frac{\mu_\alpha^{n-(n+2)(1-\nu)}}{|x_\alpha - y_\alpha|^{n-2}}\right),\end{aligned} \tag{2-61}$$

where the second line again follows from Giraud's lemma in $\mathbb{R}^n$ since $(n+2)(1-\nu) > n$. Combining (2-60) and (2-61) finally shows that

$$\int_{\Omega\cap\{|x-x_\alpha|\ge R_\nu\mu_\alpha\}} |y_\alpha - x|^{2-n}|v_\alpha(x)|^{2^*-1}\,dx = O\left(\frac{\mu_\alpha^{(n-2)/2}}{|x_\alpha - y_\alpha|^{n-2}}\right) + O(\eta_\alpha(\rho)),$$

which together with (2-59) concludes the proof of (2-54).

We now conclude the proof of (2-53). First, if $v_\infty > 0$, (2-53) simply follows from (2-38) and (2-54). We may thus assume that $v_\infty \equiv 0$. We now prove that for α large enough

$$\eta_\alpha(\rho) = O(\mu_\alpha^{(n-2)/2}). \tag{2-62}$$

Together with (2-54) this will conclude the proof of (2-53) in this case. We prove (2-62) by contradiction: we assume that

$$\frac{\eta_\alpha(\rho)}{\mu_\alpha^{(n-2)/2}} \to +\infty \tag{2-63}$$

as $\alpha \to +\infty$, and we let $V_\alpha = v_\alpha/\eta_\alpha(\rho)$. For any α we let $z_\alpha \in \Omega \backslash B_\rho(x_\alpha)$ be such that $|v_\alpha(z_\alpha)| = \eta_\alpha(\rho)$. By the definition of $D_\alpha(x)$ and by (2-54) we see that for any $\delta > 0$ fixed we have $|V_\alpha(z_\alpha)| = 1$ and

$$|V_\alpha(x)| \le C + o(1) \quad \text{for } x \in \Omega \backslash B_\delta(x_\alpha). \tag{2-64}$$

Now, the function V_α satisfies

$$-\Delta V_\alpha + h_\alpha V_\alpha = \eta_\alpha(\rho)^{2^*-2} |V_\alpha|^{2^*-2} V_\alpha$$

in Ω. Since $\eta_\alpha(\rho) \to 0$ by (2-38), (2-64) and standard elliptic theory show that $V_\alpha \to V_\infty$ in $C^2_{\text{loc}}(\overline{\Omega} \backslash \{x_\infty\}$ as $\alpha \to +\infty$, where V_∞ satisfies $|V_\infty(x)| \le C$ for any $x \ne x_\infty$ and

$$-\Delta V_\infty + h_\infty V_\infty = 0 \quad \text{in } \Omega \backslash \{x_\infty\}.$$

In particular, the singularity of V_∞ at x_∞ is removable and V_∞ satisfies weakly $-\Delta V_\infty + h_\infty V_\infty = 0$ in Ω. Since $-\Delta + h_\infty$ is coercive by assumption, this shows that $V_\infty \equiv 0$. Independently, if we let $z_\infty = \lim_{\alpha \to +\infty} z_\alpha$, the C^2_{loc} convergence shows that $|V_\infty(z_\infty)| = 1$; hence $V_\infty \not\equiv 0$. This is a contradiction, which concludes the proof of (2-62). □

The next result will be frequently used in the proof of Theorem 2.1.

Proposition 2.6. *Let $U \subset \Omega$ be an open set. There exists a constant $C(U)$ such that* $\lim_{|U| \to 0} C(U) = 0$ *and such that, for all $y \in \Omega$ and for all $\alpha \ge 1$,*

$$\int_U G_\alpha(y, x)\, dx \le C(U) d(y, \partial\Omega). \tag{2-65}$$

Proof. We let $C(U) = \sup_{y \in \Omega} \int_U |x-y|^{1-n}\, dx$. Since Ω is bounded and $y \mapsto |y|^{1-n} \in L^1_{\text{loc}}(\mathbb{R}^n)$ we have $C(U) \to 0$ as $|U| \to 0$ by absolute continuity of the integral. Using (2-12) yields

$$\int_U G_\alpha(y, x)\, dx = O(I_1(y) + I_2(y)), \tag{2-66}$$

where we have let, for $i = 1, 2$,

$$I_i(y) := \int_{U_i} \frac{1}{|y-x|^{n-2}} \min\left\{1, \frac{d(y, \partial\Omega) d(x, \partial\Omega)}{|y-x|^2}\right\} dx,$$

and

$$U_1 := U \cap \left\{|y-x| < \tfrac{1}{2} d(y, \partial\Omega)\right\} \quad \text{and} \quad U_2 := U \cap \left\{|y-x| > \tfrac{1}{2} d(y, \partial\Omega)\right\}.$$

When $x \in U_1$ we have $|y-x| < \frac{1}{2} d(y, \partial\Omega)$, so that

$$I_1(y) \le \int_{U_1} \frac{1}{|y-x|^{n-2}}\, dx \le \tfrac{1}{2} d(y, \partial\Omega) \int_U \frac{1}{|y-x|^{n-1}}\, dx \le \tfrac{1}{2} C(U) d(y, \partial\Omega).$$

When $x \in U_2$ we get that $d(x, \partial\Omega) \le 3|y-x|$. We then get

$$I_2(y) \le d(y, \partial\Omega) \int_{U_2} \frac{d(x, \partial\Omega)}{|y-x|^n} \le 3d(y, \partial\Omega) \int_U \frac{1}{|y-x|^{n-1}}\, dx \le 3C(U)d(y, \partial\Omega).$$

Combining these estimates proves Proposition 2.6. ☐

The next result improves the upper estimate in Proposition 2.5.

Proposition 2.7. *There exists $C > 0$ such that*

$$|v_\alpha(x)| \le C(B_\alpha(x) + v_\infty(x)) \quad \textit{for all } \alpha \textit{ and all } x \in \Omega. \tag{2-67}$$

Proof. First, if $v_\infty \equiv 0$, (2-67) simply follows from (2-53). We may thus assume in the following that $v_\infty > 0$ in Ω. Proving (2-67) in Theorem 2.1 is equivalent to proving that, for any sequence $(y_\alpha)_{\alpha\in\mathbb{N}} \in \Omega$, we have

$$\frac{|v_\alpha(y_\alpha)|}{B_\alpha(y_\alpha) + v_\infty(y_\alpha)} = O(1) \quad \text{as } \alpha \to +\infty. \tag{2-68}$$

Assume first that $|y_\alpha - x_\alpha| = O(\mu_\alpha)$. It follows from (2-21) and Proposition 2.3 that

$$|v_\alpha(y_\alpha)| = O(v_\infty(y_\alpha) + B_\alpha(y_\alpha)) + o(D_\alpha(y_\alpha)^{-(n-2)/2}) = O(v_\infty(y_\alpha) + B_\alpha(y_\alpha)),$$

which proves (2-67) in this case. We thus assume from now on that

$$\lim_{\alpha\to+\infty} \frac{|y_\alpha - x_\alpha|}{\mu_\alpha} = +\infty. \tag{2-69}$$

Using Proposition 2.3 and standard elliptic theory, we have that

$$v_\alpha \to \mp v_\infty \quad \text{in } C^2_{\mathrm{loc}}(\overline{\Omega}\backslash\{x_\infty\}) \text{ as } \alpha \to +\infty. \tag{2-70}$$

Therefore, there exists $\rho_\alpha > 0$, $\rho_\alpha \to 0$ as $\alpha \to +\infty$, such that, up to a subsequence,

$$\|v_\alpha \pm v_\infty\|_{C^2(\{|x-x_\alpha|>\rho_\alpha\}\cap\Omega)} = o(1). \tag{2-71}$$

Using again Green's representation formula and (2-12) we have

$$|v_\alpha(y_\alpha)| = O\left(\int_{\{|x-x_\alpha|\le\rho_\alpha\}\cap\Omega} G_\alpha(y_\alpha, x)|v_\alpha(x)|^{2^*-1}\, dx + \int_{\{|x-x_\alpha|>\rho_\alpha\}\cap\Omega} G_\alpha(y_\alpha, x)|v_\alpha(x)|^{2^*-1}\, dx\right). \tag{2-72}$$

Thanks to (2-11), (2-65) and (2-71), we get

$$\int_{\{|x-x_\alpha|>\rho_\alpha\}\cap\Omega} G_\alpha(y_\alpha, x)|v_\alpha(x)|^{2^*-1}\, dx = O(v_\infty(y_\alpha)). \tag{2-73}$$

We fix $R > 0$, and we now write

$$\int_{\Omega\cap\{|x-x_\alpha|\le\rho_\alpha\}} G_\alpha(y_\alpha, x)|v_\alpha(x)|^{2^*-1}\, dx$$
$$= O\left(\int_{\Omega\cap\{|x-x_\alpha|\le R\mu_\alpha\}} |y_\alpha-x|^{2-n}|v_\alpha(x)|^{2^*-1}\, dx + \int_{\Omega\cap\{R\mu_\alpha\le|x-x_\alpha|\le\rho_\alpha\}} G_\alpha(y_\alpha, x)|v_\alpha(x)|^{2^*-1}\, dx\right). \tag{2-74}$$

As in the proof of (2-59), thanks to (2-3) and to Hölder's inequality, we obtain

$$\int_{\Omega\cap\{|x-x_\alpha|\le R\mu_\alpha\}} |y_\alpha - x|^{2-n}|v_\alpha(x)|^{2^*-1}\,dx = O\left(\frac{\mu_\alpha^{(n-2)/2}}{|y_\alpha - x_\alpha|^{n-2}}\right). \tag{2-75}$$

By (2-53), there exists $C > 0$ such that

$$|v_\alpha(x)|^{2^*-1} \le C\big(\mu_\alpha^{(n+2)/2} D_\alpha(x)^{-2-n} + \|v_\infty\|_\infty^{2^*-1}\big),$$

where $D_\alpha(x) := \mu_\alpha + |x - x_\alpha|$ for all $x \in \Omega$. Therefore, using again (2-11), we have

$$\begin{aligned}
&\int_{\Omega\cap\{R\mu_\alpha\le|x-x_\alpha|\le\rho_\alpha\}} G_\alpha(y_\alpha, x)|v_\alpha(x)|^{2^*-1}\,dx \\
&\quad = O\left(\mu_\alpha^{(n+2)/2}\int_{\Omega\cap\{|x-x_\alpha|\ge R\mu_\alpha\}} |y_\alpha - x|^{2-n}|x-x_\alpha|^{-2-n}\,dx\right) + O\left(\int_{\Omega\cap\{R\mu_\alpha\le|x-x_\alpha|\le\rho_\alpha\}} G_\alpha(y_\alpha, x)\,dx\right) \\
&\quad = O\left(\frac{\mu_\alpha^{(n-2)/2}}{|x_\alpha - y_\alpha|^{n-2}}\right) + O(v_\infty(y_\alpha)).
\end{aligned} \tag{2-76}$$

Combining (2-75) and (2-76) in (2-74) finally shows that

$$\int_{\Omega\cap\{|x-x_\alpha|\le\rho_\alpha\}} G_\alpha(y_\alpha, x)|v_\alpha(x)|^{2^*-1}\,dx = O(\mu_\alpha^{(n-2)/2}|x_\alpha - y_\alpha|^{2-n}) + O(v_\infty(y_\alpha))$$

as $\alpha \to +\infty$. Together with (2-72) and (2-73) this proves (2-68) and concludes the proof of (2-67). □

We are now in position to conclude the proof of Theorem 2.1.

Proof of Theorem 2.1. Proving Theorem 2.1 is equivalent to proving that, for any sequence $(y_\alpha)_{\alpha\in\mathbb{N}} \in \Omega$, we have

$$v_\alpha(y_\alpha) = \Pi B_\alpha(v_\alpha) \pm v_\infty(y_\alpha) + o(B_\alpha(y_\alpha)) + o(v_\infty(y_\alpha)) \tag{2-77}$$

as $\alpha \to +\infty$. Throughout this proof it will be intended that all the terms involving v_∞ disappear if $v_\infty \equiv 0$. If $|x_\alpha - y_\alpha| = O(\mu_\alpha)$ or if $|x_\alpha - y_\alpha| \not\to 0$, (2-77) follows from Proposition 2.3. We may thus assume in the following that

$$|x_\alpha - y_\alpha| \to 0 \quad\text{and}\quad \frac{|x_\alpha - y_\alpha|}{\mu_\alpha} \to +\infty \tag{2-78}$$

as $\alpha \to +\infty$. We write three representation formulae for v_α, ΠB_α and v_∞, using (2-2), (2-9) and (2-14), respectively, and we subtract them to get

$$\begin{aligned}
&v_\alpha(y_\alpha) - \Pi B_\alpha(y_\alpha) \mp v_\infty(y_\alpha) \\
&\quad = \int_\Omega G_\alpha(y_\alpha,\cdot)(|v_\alpha|^{2^*-2}v_\alpha - B_\alpha^{2^*-1} \mp v_\infty^{2^*-1})\,dx \pm \int_\Omega (G_\alpha(y_\alpha,\cdot) - G_\infty(y_\alpha,\cdot))v_\infty^{2^*-1}\,dx,
\end{aligned} \tag{2-79}$$

where we have denoted by G_∞ the Green's function for $-\Delta + h_\infty$.

Case 1: $v_\infty \equiv 0$. In this case the second integral in (2-79) vanishes and we only have to estimate the first one. Let $R > 1$ be fixed. Using (2-12) and (2-53) and letting $\check{y}_\alpha = (y_\alpha - x_\alpha)/\mu_\alpha$, a simple change of variables and direct computations give

$$\left|\int_{\Omega\backslash B_{R\mu_\alpha}(x_\alpha)} G_\alpha(y_\alpha,\cdot)(|v_\alpha|^{2^*-2}v_\alpha - B_\alpha^{2^*-1})\,dx\right| \le C\mu_\alpha^{-(n-2)/2}\int_{\mathbb{R}^n\backslash B_R(0)} \frac{1}{|\check{y}_\alpha - x|^{n-2}} B_0^{2^*-1}\,dx$$
$$= O(\varepsilon_R B_\alpha(y_\alpha)) \tag{2-80}$$

as $\alpha \to +\infty$, where ε_R denotes a positive number satisfying $\lim_{R\to+\infty}\varepsilon_R = 0$. Independently, (2-21) and (2-20) show that

$$\left\|\frac{v_\alpha - B_\alpha}{B_\alpha}\right\|_{L^\infty(B_{R\mu_\alpha}(x_\alpha))} \to 0$$

as $\alpha \to +\infty$. As a consequence, using (2-12),

$$\left|\int_{B_{R\mu_\alpha}(x_\alpha)} G_\alpha(y_\alpha,\cdot)(|v_\alpha|^{2^*-2}v_\alpha - B_\alpha^{2^*-1})\,dx\right| = o\left(\int_{B_{R\mu_\alpha}(x_\alpha)} |y_\alpha - y|^{2-n} B_\alpha^{2^*-1}\,dx\right)$$
$$= o(B_\alpha(y_\alpha)). \tag{2-81}$$

Up to passing to a subsequence, combining (2-80) and (2-81) proves (2-77) in the $v_\infty \equiv 0$ case.

Case 2: $v_\infty > 0$. We first estimate the first integral in (2-79) by decomposing it in three domains: $B_{R\mu_\alpha}(x_\alpha)$, $(\Omega\cap B_{1/R}(x_\alpha))\backslash B_{R\mu_\alpha}(x_\alpha)$ and $\Omega\backslash B_{1/R}(x_\alpha)$. We first have

$$\int_{B_{R\mu_\alpha}(x_\alpha)} G_\alpha(y_\alpha,\cdot)(|v_\alpha|^{2^*-2}v_\alpha - B_\alpha^{2^*-1} \mp v_\infty^{2^*-1})\,dx$$
$$= \int_{B_{R\mu_\alpha}(x_\alpha)} G_\alpha(y_\alpha,\cdot)(|v_\alpha|^{2^*-2}v_\alpha - B_\alpha^{2^*-1})\,dx + O\left(\int_{B_{R\mu_\alpha}(x_\alpha)} G_\alpha(y_\alpha,\cdot)\,dx\right)$$
$$= o(B_\alpha(y_\alpha)) + o(v_\infty(y_\alpha)), \tag{2-82}$$

where the last line follows from (2-81) and from (2-11) and (2-65) with $U = B_{R\mu_\alpha}(x_\alpha)$. Using (2-71) we now have

$$\int_{\Omega\backslash B_{1/R}(x_\alpha)} G_\alpha(y_\alpha,\cdot)(|v_\alpha|^{2^*-2}v_\alpha - B_\alpha^{2^*-1} \mp v_\infty^{2^*-1})\,dx$$
$$= \int_{\Omega\backslash B_{1/R}(x_\alpha)} G_\alpha(y_\alpha,\cdot)(|v_\alpha|^{2^*-2}v_\alpha \mp v_\infty^{2^*-1})\,dx + O(\mu_\alpha^{(n+2)/2})$$
$$= o\left(\int_\Omega G_\alpha(y_\alpha, y)\,dy\right) + o(B_\alpha(y_\alpha)) = o(B_\alpha(y_\alpha)) + o(v_\infty(y_\alpha)), \tag{2-83}$$

where the last equality again follows from (2-11) and (2-65). Finally, using (2-12) and (2-53) we have

$$\left|\int_{(\Omega\cap B_{1/R}(x_\alpha))\backslash B_{R\mu_\alpha}(x_\alpha)} G_\alpha(y_\alpha,\cdot)(|v_\alpha|^{2^*-2}v_\alpha - B_\alpha^{2^*-1} \mp v_\infty^{2^*-1})\,dx\right|$$
$$= O\left(\int_{\Omega\backslash B_{R\mu_\alpha}(x_\alpha)} |y_\alpha - y|^{2-n} B_\alpha^{2^*-1}\,dx\right) + O\left(\int_{\Omega\cap B_{1/R}(x_\alpha)} G_\alpha(y_\alpha, y)\,dy\right)$$
$$= O(\varepsilon_R B_\alpha(y_\alpha)) + O(\varepsilon_R v_\infty(y_\alpha)), \tag{2-84}$$

where the last line follows from (2-80) and (2-65) with $U = \Omega \cap B_{1/R}(x_\alpha)$. Combining (2-82), (2-83) and (2-84) proves that

$$\int_\Omega G_\alpha(y_\alpha, \cdot)(|v_\alpha|^{2^*-2} v_\alpha - B_\alpha^{2^*-1} \mp v_\infty^{2^*-1})\, dx \\ = o(B_\alpha(y_\alpha)) + o(v_\infty(y_\alpha)) + O(\varepsilon_R B_\alpha(y_\alpha)) + O(\varepsilon_R v_\infty(y_\alpha)) \qquad (2\text{-}85)$$

as $\alpha \to +\infty$, where $\lim_{R\to+\infty} \varepsilon_R = 0$. We now estimate the second integral in (2-79). For $y \in \Omega$ and for all α, we let

$$F_{1,\alpha}(y) = \int_\Omega G_\alpha(y, \cdot) v_\infty^{2^*-1}\, dx \quad \text{and} \quad F_2(y) = \int_\Omega G_\infty(y, \cdot) v_\infty^{2^*-1}\, dx.$$

By definition of G_α and G_∞, these functions satisfy $(-\Delta + h_\alpha)F_{1,\alpha} = v_\infty^{2^*-1}$ and $(-\Delta + h_\infty)F_2 = v_\infty^{2^*-1}$, respectively, so that by (2-1) and standard elliptic theory $(F_{1,\alpha})_{\alpha\in\mathbb{N}}$ is uniformly bounded in $L^\infty(\Omega)$. We also have

$$(-\Delta + h_\infty)(F_{1,\alpha} - F_2) = (h_\infty - h_\alpha)F_{1,\alpha}.$$

A representation formula for $F_{1,\alpha} - F_2$ applied at y_α then shows

$$\int_\Omega (G_\alpha(y_\alpha, \cdot) - G_\infty(y_\alpha, \cdot)) v_\infty^{2^*-1}\, dx = F_{1,\alpha}(y_\alpha) - F_2(y_\alpha) = \int_\Omega G_\infty(y_\alpha, \cdot)(h_\infty - h_\alpha)F_{1,\alpha}\, dx.$$

Using (2-1), (2-11) and (2-65) we thus obtain

$$\left| \int_\Omega (G_\alpha(y_\alpha, \cdot) - G_\infty(y_\alpha, \cdot)) v_\infty^{2^*-1}\, dx \right| = o\left(\int_\Omega G_\infty(y_\alpha, x)\, dx \right) = o(v_\infty(y_\alpha)). \qquad (2\text{-}86)$$

Plugging (2-85) and (2-86) into (2-79) finally proves that

$$|v_\alpha(y_\alpha) - \Pi B_\alpha(y_\alpha) \mp v_\infty(y_\alpha)| = o(B_\alpha(y_\alpha)) + o(v_\infty(y_\alpha)) + O(\varepsilon_R B_\alpha(y_\alpha)) + O(\varepsilon_R v_\infty(y_\alpha))$$

as $\alpha \to +\infty$, where $\lim_{R\to+\infty} \varepsilon_R = 0$. Passing to a subsequence proves (2-77) and concludes the proof of Theorem 2.1. ☐

3. Necessary conditions for blow-up and proof of Theorem 1.1

Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$. Throughout this section we let $(h_\alpha)_{\alpha\in\mathbb{N}}$ be a sequence of functions that converges in $C^1(\overline{\Omega})$ to h_∞, where $-\Delta + h_\infty$ is coercive in $H_0^1(\Omega)$ and where $I_{h_\infty}(\Omega) < K_n^{-2}$, and we let $(v_\alpha)_{\alpha\in\mathbb{N}} \in H_0^1(\Omega)$ be a sequence of solutions of (2-2) that satisfies (2-3), (2-4) and (2-5). Equation (2-15) is thus also satisfied, and we have

$$v_\alpha = \Pi B_\alpha \pm v_\infty + o(1) \quad \text{in } H_0^1(\Omega) \text{ as } \alpha \to +\infty,$$

where ΠB_α is given by (2-14) and where $(x_\alpha)_{\alpha\in\mathbb{N}}$ and $(\mu_\alpha)_{\alpha\in\mathbb{N}}$ are sequences of points in Ω and $(0, +\infty)$ satisfying (2-10) and with $\lim_{\alpha\to+\infty} \mu_\alpha = 0$. We let again $x_\infty = \lim_{\alpha\to+\infty} x_\alpha$, and we identify in this section necessary blow-up conditions that constrain the localisation of x_∞. We recall for this the celebrated

Pohozaev identity, that for our sequence $(v_\alpha)_{\alpha\in\mathbb{N}}$ is as follows: for any family U_α of smooth domains such that $x_\alpha \in U_\alpha \subset \Omega$ for $\alpha \in \mathbb{N}$ we have

$$\int_{U_\alpha} \big(h_\alpha(x) + \tfrac{1}{2}\langle \nabla h_\alpha(x), x - x_\alpha\rangle\big) v_\alpha^2 \, dx$$
$$= \int_{\partial U_\alpha} \langle x - x_\alpha, \nu\rangle \left(\frac{|\nabla v_\alpha|^2}{2} + h_\alpha \frac{v_\alpha^2}{2} - \frac{|v_\alpha|^{2^*}}{2^*} \right) d\sigma(x) - \int_{\partial U_\alpha} \big(\langle x - x_\alpha, \nabla v_\alpha\rangle + \tfrac{1}{2}(n-2) v_\alpha\big) \partial_\nu v_\alpha \, d\sigma(x), \tag{3-1}$$

where ν is the outer unit normal to the boundary of U_α and $\langle\,\cdot\,,\,\cdot\,\rangle$ is the Euclidean scalar product; see for instance [Hebey 2014, Lemma 6.5]. We distinguish two cases according to whether x_∞ is a boundary blow-up point or not.

3.1. *Interior blow-up case*: $x_\infty \in \Omega$.

If x_∞ is an interior point we prove the following result:

Proposition 3.1. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \ge 3$. Let $(h_\alpha)_{\alpha\in\mathbb{N}}$ be a sequence of functions that converges in $C^1(\overline{\Omega})$ to h_∞, where $-\Delta + h_\infty$ is coercive in $H_0^1(\Omega)$ and where $I_{h_\infty}(\Omega) < K_n^{-2}$, and we let $(v_\alpha)_{\alpha\in\mathbb{N}} \in H_0^1(\Omega)$ be a sequence of solutions of (2-2) that satisfies (2-3), (2-4) and (2-5). Let $x_\infty = \lim_{\alpha\to+\infty} x_\alpha$ and assume that $x_\infty \in \Omega$. Then*

- *if $n = 3$, we have $v_\infty \equiv 0$ and $m_{h_\infty}(x_\infty) = 0$,*
- *if $n = 4, 5$, we have $v_\infty \equiv 0$ and $h_\infty(x_\infty) = 0$,*
- *if $n = 6$, we have $h_\infty(x_\infty) = \pm 2 v_\infty(x_\infty)$,*
- *if $n \ge 7$, we have $h_\infty(x_\infty) = 0$.*

Proof. First, since $x_\infty \in \Omega$, we have $B_{\delta\sqrt{\mu_\alpha}}(x_\alpha) \subset \Omega$ for all α large enough. The Pohozaev identity (3-1) yields

$$\int_{B_{\delta\sqrt{\mu_\alpha}}(x_\alpha)} \big(h_\alpha(x) + \tfrac{1}{2}\langle \nabla h_\alpha(x), x - x_\alpha\rangle\big) v_\alpha^2 \, dx = \int_{\partial B_{\delta\sqrt{\mu_\alpha}}(x_\alpha)} F_\alpha(x) \, d\sigma(x), \tag{3-2}$$

where we have let

$$F_\alpha(x) := \langle x - x_\alpha, \nu\rangle \left(\frac{|\nabla v_\alpha|^2}{2} + h_\alpha \frac{v_\alpha^2}{2} - \frac{|v_\alpha|^{2^*}}{2^*} \right) - \big(\langle x - x_\alpha, \nabla v_\alpha\rangle + \tfrac{1}{2}(n-2) v_\alpha\big) \partial_\nu v_\alpha. \tag{3-3}$$

For any $x \in (\Omega - x_\alpha)/\sqrt{\mu_\alpha}$ we let

$$\hat{v}_\alpha(x) = v_\alpha(x_\alpha + \sqrt{\mu_\alpha} x).$$

Using (2-2) it is easily seen that $\hat{v}_\alpha$ satisfies

$$\begin{cases} -\Delta \hat{v}_\alpha + \mu_\alpha \hat{h}_\alpha \hat{v}_\alpha = \mu_\alpha |\hat{v}_\alpha|^{2^*-2} \hat{v}_\alpha & \text{in } (\Omega - x_\alpha)/\sqrt{\mu_\alpha}, \\ \hat{v}_\alpha = 0 & \text{on } \partial((\Omega - x_\alpha)/\sqrt{\mu_\alpha}), \end{cases}$$

where we have let $\hat{h}_\alpha(x) = h(x_\alpha + \sqrt{\mu_\alpha} x)$. By (2-67) and standard elliptic theory there thus exists $\hat{v}_\infty \in C^2(\mathbb{R}^n \backslash \{0\})$ such that $\hat{v}_\alpha \to \hat{v}_\infty$ in $C^2_{\mathrm{loc}}(\mathbb{R}^n \backslash \{0\})$, and Theorem 2.1 shows that for any $x \in \mathbb{R}^n \backslash \{0\}$ we have

$$\hat{v}_\infty(x) = (n(n-2))^{(n-2)/2} |x|^{2-n} \pm v_\infty(x_\infty).$$

The change of variables $x = x_\alpha + \sqrt{\mu_\alpha}\, y$ and straightforward computations then show that

$$\begin{aligned}
&\mu_\alpha^{-(n-2)/2} \int_{\partial B_{\delta\sqrt{\mu_\alpha}}(x_\alpha)} F_\alpha(x)\, d\sigma(x) \\
&\quad = \int_{\partial B_\delta(0)} \langle x, \nu\rangle \left(\frac{|\nabla \hat{v}_\alpha|^2}{2} + \mu_\alpha \hat{h}_\alpha \frac{\hat{v}_\alpha^2}{2} - \mu_\alpha \frac{|\hat{v}_\alpha|^{2^*}}{2^*} \right) d\sigma(x) - \int_{\partial B_\delta(0)} \bigl(\langle x, \nabla \hat{v}_\alpha \rangle + \tfrac{1}{2}(n-2)\hat{v}_\alpha \bigr) \partial_\nu \hat{v}_\alpha \, d\sigma(x) \\
&\quad = \pm \tfrac{1}{2} \omega_{n-1} n^{(n-2)/2} (n-2)^{(n+2)/2} v_\infty(x_\infty) + \varepsilon_\delta + o(1)
\end{aligned} \tag{3-4}$$

as $\alpha \to +\infty$, where ε_δ denotes a quantity such that $\lim_{\delta \to 0} \varepsilon_\delta = 0$ and where ω_{n-1} is the area of the round sphere $\mathbb{S}^{n-1}$. We now claim that

$$\int_{B_{\delta\sqrt{\mu_\alpha}}(x_\alpha)} \bigl(h_\alpha(x) + \tfrac{1}{2} \langle \nabla h_\alpha(x), x - x_\alpha \rangle \bigr) v_\alpha^2 \, dx = \begin{cases} O(\mu_\alpha^{3/2}) & \text{if } n = 3, \\ O(\mu_\alpha^2 \ln(1/\mu_\alpha)) & \text{if } n = 4, \\ \mu_\alpha^2 \bigl(h_\infty(x_\infty) \int_{\mathbb{R}^n} B_0(x)^2 \, dx + o(1) \bigr) & \text{if } n \geq 5, \end{cases} \tag{3-5}$$

where B_0 is defined in (2-7). We prove (3-5). First, using (2-16) and Theorem 2.1, straightforward computations show that

$$\int_{B_{\delta\sqrt{\mu_\alpha}}(x_\alpha)} \tfrac{1}{2} \langle \nabla h_\alpha(x), x - x_\alpha \rangle v_\alpha^2 \, dx = \begin{cases} O(\mu_\alpha^2) & \text{if } n = 3, 4, \\ O(\mu_\alpha^3 |\ln \mu_\alpha|) & \text{if } n \geq 5, \end{cases} \tag{3-6}$$

and that

$$\int_{B_{\delta\sqrt{\mu_\alpha}}(x_\alpha)} h_\alpha(x) v_\alpha^2 \, dx = \begin{cases} O(\mu_\alpha^{3/2}) & \text{if } n = 3, \\ O(\mu_\alpha^2 \ln(1/\mu_\alpha)) & \text{if } n = 4. \end{cases} \tag{3-7}$$

If $n \geq 5$, using Theorem 2.1, we have

$$\int_{B_{\delta\sqrt{\mu_\alpha}}(x_\alpha)} h_\alpha(x) v_\alpha^2 \, dx = \int_{B_{\delta\sqrt{\mu_\alpha}}(x_\alpha)} h_\alpha(x) (\Pi B_\alpha)^2 \, dx + o(\mu_\alpha^2).$$

Dominated convergence together with (2-21) now shows that

$$\int_{B_{\delta\sqrt{\mu_\alpha}}(x_\alpha)} h_\alpha(x) (\Pi B_\alpha)^2 \, dx = h_\infty(x_\infty) \int_{\mathbb{R}^n} \mu_\alpha^2 B_0(x)^2 \, dx + o(\mu_\alpha^2).$$

Combining the latter with (3-6) and (3-7) proves (3-5). Combining (3-2), (3-4) and (3-5) now shows that

$$\begin{aligned}
&\pm \tfrac{1}{2} \omega_{n-1} n^{(n-2)/2} (n-2)^{(n+2)/2} v_\infty(x_\infty) \mu_\alpha^{(n-2)/2} + \varepsilon_\delta \mu_\alpha^{(n-2)/2} + o(\mu_\alpha^{(n-2)/2}) \\
&\qquad = \begin{cases} O(\mu_\alpha^{3/2}) & \text{if } n = 3, \\ O(\mu_\alpha^2 \ln(1/\mu_\alpha)) & \text{if } n = 4, \\ \mu_\alpha^2 \bigl(h_\infty(x_\infty) \int_{\mathbb{R}^n} B_0^2 \, dx + o(1) \bigr) & \text{if } n \geq 5. \end{cases}
\end{aligned} \tag{3-8}$$

Assume first that $n \in \{3, 4, 5\}$. Equation (3-8) then gives

$$v_\infty(x_\infty) + \varepsilon_\delta + o(1) = \begin{cases} O(\mu_\alpha) & \text{if } n = 3, \\ O(\mu_\alpha \ln(1/\mu_\alpha)) & \text{if } n = 4, \\ O(\sqrt{\mu_\alpha}) & \text{if } n = 5 \end{cases}$$

as $\alpha \to +\infty$. Letting first $\alpha \to +\infty$ then $\delta \to 0$ shows that $v_\infty(x_\infty) = 0$. Since $v_\infty \geq 0$ by (2-3) and the assumption $I_{h_\infty}(\Omega) < K_n^{-2}$, the strong maximum principle then shows that $v_\infty \equiv 0$.

Assume now that $n = 6$. Integrating $-\Delta B_0 = B_0^2$ shows that

$$\int_{\mathbb{R}^6} B_0^2\,dx = 6^2 4^3 \omega_5.$$

Therefore, it follows from (3-8) that

$$\pm\tfrac{1}{2}\omega_5 6^2 4^4 v_\infty(x_\infty)\mu_\alpha^2 + \varepsilon_\delta \mu_\alpha^2 + o(\mu_\alpha^2) = 6^2 4^3 \omega_5 h_\infty(x_\infty)\mu_\alpha^2 + o(\mu_\alpha^2).$$

Letting $\alpha \to +\infty$ and then $\delta \to 0$ shows that

$$h_\infty(x_\infty) = \pm 2 v_\infty(x_\infty).$$

Assume finally that $n \geq 7$. Then $\mu_\alpha^{(n-2)/2} = o(\mu_\alpha^2)$ as $\alpha \to +\infty$, and (3-8) then gives, after letting $\alpha \to +\infty$,

$$h_\infty(x_\infty) = 0.$$

These considerations prove Proposition 3.1 in the case $n \geq 6$.

To conclude the proof of Proposition 3.1 we now consider the case where $3 \leq n \leq 5$ and $v_\infty \equiv 0$. We let $\delta > 0$ be small enough that $B_\delta(x_\alpha) \subset \Omega$ for all α, and we write a Pohozaev identity in $B_\delta(x_\alpha)$,

$$\int_{B_\delta(x_\alpha)} \bigl(h_\alpha(x) + \tfrac{1}{2}\langle \nabla h_\alpha(x), x - x_\alpha\rangle\bigr) v_\alpha^2\,dx = \int_{B_\delta(x_\alpha)} F_\alpha(x)\,d\sigma(x), \tag{3-9}$$

where F_α is again as in (3-3). For $x \in \Omega$ we let in this case

$$\hat{v}_\alpha(x) = \mu_\alpha^{-(n-2)/2} v_\alpha(x).$$

Using (2-2) it is easily seen that $\hat{v}_\alpha$ satisfies

$$\begin{cases} -\Delta \hat{v}_\alpha + h_\alpha \hat{v}_\alpha = \mu_\alpha^2 |\hat{v}_\alpha|^{2^*-2}\hat{v}_\alpha & \text{in } \Omega,\\ \hat{v}_\alpha = 0 & \text{on } \partial\Omega,\end{cases}$$

and (2-16) and (2-67) show that

$$|\hat{v}_\alpha(x)| \leq \frac{C}{|x - x_\alpha|^{n-2}} \quad \text{for all } x \in \Omega\backslash\{x_\alpha\},$$

where C is a positive constant independent of α. Standard elliptic theory with (2-20) then shows that $\hat{v}_\alpha \to \hat{v}_\infty$ in $C^2_{\mathrm{loc}}(\overline{\Omega}\backslash\{x_\infty\})$, where

$$\hat{v}_\infty(x) = (n-2)\omega_{n-1}(n(n-2))^{(n-2)/2} G_\infty(x_\infty, x)$$

and where G_∞ is the Green's function for $-\Delta + h_\infty$ with Dirichlet boundary conditions in Ω, which is the only solution to

$$\begin{cases} -\Delta_y G_{h_\infty}(x, y) + h G_{h_\infty}(x, y) = \delta_x & \text{in } \Omega,\\ G_{h_\infty}(x, y) = 0 & \text{for } y \in \partial\Omega, \quad x \in \Omega.\end{cases}$$

When $n = 3$ it is well known that

$$G_\infty(x_\infty, y) = \frac{1}{4\pi|x - y|} + m_{h_\infty}(x_\infty) + O(|x_\infty - y|) \quad \text{for all } y \in \Omega\backslash\{x_\infty\}.$$

Straightforward computations with the latter then show that

$$\mu_\alpha^{2-n}\int_{B_\delta(x_\alpha)} F_\alpha(x)\,d\sigma(x) = \begin{cases} 24\pi^2 m_{h_\infty}(x_\infty)+\varepsilon_\delta+o(1), & n=3,\\ O(1), & n=4,5, \end{cases} \tag{3-10}$$

where $\lim_{\delta\to 0}\varepsilon_\delta = 0$. Independently, straightforward computations using Theorem 2.1 (see, e.g., [Premoselli and Robert 2025, Section 5]) show that

$$\int_{B_\delta(x_\alpha)} \big(h_\alpha(x)+\tfrac{1}{2}\langle\nabla h_\alpha(x), x-x_\alpha\rangle\big)v_\alpha^2\,dx = \begin{cases} O(\delta\mu_\alpha) & \text{if } n=3,\\ 64\omega_3 h_\infty(x_\infty)\mu_\alpha^2\ln(1/\mu_\alpha)+O(\mu_\alpha^2) & \text{if } n=4,\\ \mu_\alpha^2\big(h_\infty(x_\infty)\int_{\mathbb{R}^n} B_0(x)^2\,dx+o(1)\big) & \text{if } n\ge 5 \end{cases} \tag{3-11}$$

as $\alpha\to+\infty$. If $n\in\{4,5\}$, combining (3-10) and (3-11) in (3-9) shows that

$$h_\infty(x_\infty)+o(1) = \begin{cases} O(\ln(1/\mu_\alpha)^{-1}), & n=4,\\ O(\mu_\alpha), & n=5, \end{cases}$$

as $\alpha\to+\infty$, which shows that $h_\infty(x_\infty)=0$. If $n=3$, combining (3-10) and (3-11) in (3-9) shows that

$$m_{h_\infty}(x_\infty)+o(1)+\varepsilon_\delta = O(\delta)$$

as $\alpha\to+\infty$. Letting first $\alpha\to+\infty$ then $\delta\to 0$ proves that $m_{h_\infty}(x_\infty)=0$, which concludes the proof of Proposition 3.1. ☐

3.2. *Boundary blow-up case*: $x_\infty\in\partial\Omega$. We assume in this subsection that $x_\infty\in\partial\Omega$. For $\alpha\ge 1$, we let

$$d_\alpha = d(x_\alpha,\partial\Omega)\to 0 \tag{3-12}$$

as $\alpha\to+\infty$, since $x_\infty\in\partial\Omega$. We know from (2-18) that $d_\alpha\gg\mu_\alpha$ as $\alpha\to+\infty$. For $\alpha\ge 1$ we also let

$$r_\alpha = \frac{\sqrt{\mu_\alpha}}{d_\alpha^{1/(n-2)}}, \tag{3-13}$$

and we analyse the bubbling behaviour of v_α at the scale r_α. The idea to consider the scale r_α comes from the following heuristic. Recall that when $v_\infty>0$, Hopf's lemma shows that

$$v_\infty(x_\infty - t\nu(x_\infty)) = (-\partial_\nu v_\infty(x_\infty))t+o(t)$$

as $t\to 0$. At distance d_α from $\partial\Omega$, v_∞ thus behaves at first order as $(-\partial_\nu v_\infty(x_\infty))d_\alpha$. The scale r_α thus defines the distance from x_α at which B_α and v_∞ become of the same size. We analyse the boundary blow-up of v_α according to the value of d_α/r_α. We first prove the following result, which states that boundary blow-up points cannot get too close to $\partial\Omega$:

Proposition 3.2. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n\ge 3$. Let $(h_\alpha)_{\alpha\in\mathbb{N}}$ be a sequence of functions that converges in $C^1(\overline{\Omega})$ to h_∞, where $-\Delta+h_\infty$ is coercive in $H_0^1(\Omega)$ and where $I_{h_\infty}(\Omega)<K_n^{-2}$, and we let $(v_\alpha)_{\alpha\in\mathbb{N}}\in H_0^1(\Omega)$ be a sequence of solutions of (2-2) that satisfies (2-3), (2-4) and (2-5). Let $x_\infty=\lim_{\alpha\to+\infty}x_\alpha$, and assume that $x_\infty\in\partial\Omega$. If $n\ge 6$, assume in addition that $h_\infty\ne 0$ in $\overline{\Omega}$. Then, up to a subsequence,*

$$\frac{d_\alpha}{r_\alpha}\to+\infty$$

as $\alpha\to+\infty$.

Proof. We proceed by contradiction, and we assume that, up to a subsequence,

$$\lim_{\alpha\to+\infty}\frac{d_\alpha}{r_\alpha}=\rho\in[0,+\infty). \tag{3-14}$$

In this case we define, for all $x\in(\Omega-x_\alpha)/d_\alpha$,

$$\bar v_\alpha(x):=\frac{d_\alpha^{n-2}}{\mu_\alpha^{(n-2)/2}}v_\alpha(x_\alpha+d_\alpha x). \tag{3-15}$$

Equation (2-2) and the definition of $\bar v_\alpha$ show that $\bar v_\alpha$ satisfies

$$\begin{cases}-\Delta\bar v_\alpha-d_\alpha^2\bar h_\alpha\bar v_\alpha=(\mu_\alpha/d_\alpha)^2|\bar v_\alpha|^{2^*-2}\bar v_\alpha & \text{in } (\Omega-x_\alpha)/d_\alpha,\\ \bar v_\alpha=0 & \text{on } \partial((\Omega-x_\alpha)/d_\alpha),\end{cases} \tag{3-16}$$

where $\bar v_\alpha$ is as in (3-15) and $\bar h_\alpha(x):=h(x_\alpha+d_\alpha x)$. By (3-13) and (3-14) we have

$$d_\alpha=O(\mu_\alpha^{(n-2)(n-1)/2})\quad\text{or, equivalently,}\quad\frac{d_\alpha^{n-2}}{\mu_\alpha^{(n-2)/2}}\cdot d_\alpha=O(1). \tag{3-17}$$

By Hopf's lemma we have

$$v_\infty(x_\alpha+d_\alpha x)=v_\infty(x_\alpha)+O(d_\alpha)=O(d_\alpha) \tag{3-18}$$

as $\alpha\to+\infty$, and the latter remains obviously true if $v_\infty\equiv0$. The latter with (2-16) and Theorem 2.1 show that

$$|\bar v_\alpha(x)|\le C(1+|x|^{2-n})\quad\text{for all } x\in\frac{\Omega-x_\alpha}{d_\alpha}\backslash\{0\} \tag{3-19}$$

for some positive constant C. Since Ω is smooth and since $d_\alpha\to0$ as $\alpha\to+\infty$ by assumption, standard elliptic theory shows that, up to a rotation, $\bar v_\alpha\to\bar v_\infty\in C^2(\overline{\Omega}_0\backslash\{0\})$, where we have let

$$\Omega_0:=\,]-\infty,1[\,\times\mathbb{R}^{n-1}\quad\text{as } \alpha\to+\infty \tag{3-20}$$

and where $\bar v_\infty$ satisfies

$$-\Delta\bar v_\infty=0\quad\text{in } \Omega_0\backslash\{0\},\qquad \bar v_\infty=0\quad\text{on } \partial\Omega_0, \tag{3-21}$$

and

$$|\bar v_\infty(x)|\le C(1+|x|^{2-n})\quad\text{for all } x\in\Omega_0. \tag{3-22}$$

Lemma 3.3. *We have*

$$\bar v_\infty(x)=\frac{(n(n-2))^{(n-2)/2}}{|x|^{n-2}}+\mathcal{H}(x)\quad\textit{for all } x\in\Omega_0\backslash\{0\}, \tag{3-23}$$

where $\mathcal{H}$ satisfies

$$-\Delta\mathcal{H}=0\quad\textit{in } \Omega_0,\qquad \mathcal{H}=-(n(n-2))^{-(n-2)/2}|\cdot|^{2-n}\quad\textit{on } \partial\Omega_0, \tag{3-24}$$

and $\mathcal{H}(0)<0$.

Proof of Lemma 3.3. Let $0 < \delta < 1$ be fixed, and let $x \in \partial B_\delta(0) \backslash \{0\}$. For $\alpha \geq 1$, Lemma A.1 shows that

$$\frac{d_\alpha^{n-2}}{\mu_\alpha^{(n-2)/2}} \Pi B_\alpha(x_\alpha + d_\alpha x) = \frac{(n(n-2))^{(n-2)/2}}{|x|^{n-2}} + o(1) + \frac{\varepsilon(|x|)}{|x|^{n-2}} \tag{3-25}$$

as $\alpha \to +\infty$, where $\varepsilon(|x|)$ denotes a function that satisfies $\lim_{|x| \to 0} \varepsilon(|x|) = 0$. We now consider $\bar{v}_\infty$ satisfying (3-21). By (3-22) and Bôcher's theorem [Axler et al. 1992; Bôcher 1903] there exist $\Lambda \neq 0$ and a harmonic function $\mathcal{H}$ in Ω_0 such that

$$\bar{v}_\infty(x) = \Lambda |x|^{2-n} + \mathcal{H}(x) \quad \text{for } x \in \Omega_0. \tag{3-26}$$

Theorem 2.1 together with (3-17) shows that

$$\left| \bar{v}_\alpha(x) - \frac{d_\alpha^{n-2}}{\mu_\alpha^{(n-2)/2}} \Pi B_\alpha(x_\alpha + d_\alpha x) \right| \leq C + o(1)$$

for $x \in B_\delta(0) \backslash \{0\}$, for some fixed $C > 0$ as $\alpha \to +\infty$. Multiplying the latter by $|x|^{n-2}$ and passing to the limit as $\alpha \to +\infty$ then shows, using (3-25), that

$$\left| |x|^{n-2} \bar{v}_\infty(x) - (1 + \varepsilon(|x|))(n(n-2))^{(n-2)/2} \right| \leq C |x|^{n-2}.$$

Letting $x \to 0$ then shows that $\Lambda = (n(n-2))^{(n-2)/2}$ and proves (3-23). That $\mathcal{H}$ satisfies (3-24) is a simple consequence of the Dirichlet boundary conditions.

To conclude the proof of Lemma 3.3 we thus need to show that $\mathcal{H}(0) < 0$. For $x \in \Omega_0$ as in (3-20) we define

$$\widetilde{\mathcal{H}}(x) = 2 \frac{n^{(n-4)/2}(n-2)^{(n-2)/2}}{\omega_{n-1}} (x_1 - 1) \int_{\partial \Omega_0} |y|^{2-n} |x-y|^{-n} \, d\sigma(y). \tag{3-27}$$

If $y \in \Omega_0$, we let $y^* := (2 - y_1, y') \in \mathbb{R}^n$ be its reflection with respect to the hyperplane $\{y_1 = 1\}$. For $x, y \in \Omega_0$, $x \neq y$, we let

$$G_0(x, y) = \frac{1}{(n-2)\omega_{n-1}} \left(|x-y|^{2-n} - |x-y^*|^{2-n} \right)$$

be the Green's function of $-\Delta$ in Ω_0 with Dirichlet boundary conditions. Straightforward computations show that

$$\partial_\nu G_0(x, y) = \frac{2(x_1 - 1)}{n w_{n-1}} \frac{1}{|x-y|^n} \quad \text{for } x \in \Omega_0 \text{ and } y \in \partial \Omega_0,$$

so that $\widetilde{\mathcal{H}}$ can be rewritten as

$$\widetilde{\mathcal{H}}(x) = \int_{\partial \Omega_0} \frac{(n(n-2))^{(n-2)/2}}{|y|^{n-2}} \partial_\nu G_0(x, y) \, d\sigma(y).$$

In particular, $\widetilde{\mathcal{H}}$ satisfies

$$-\Delta \widetilde{\mathcal{H}} = 0 \quad \text{in } \Omega_0, \qquad \widetilde{\mathcal{H}} = -(n(n-2))^{-(n-2)/2} |\cdot|^{2-n} \quad \text{on } \partial \Omega_0,$$

and we have

$$\widetilde{\mathcal{H}}(0) = -2 \frac{(n(n-2))^{(n-2)/2}}{n w_{n-1}} \int_{\mathbb{R}^{n-1}} (1 + |y'|^2)^{1-n} \, dy' < 0. \tag{3-28}$$

We now claim that

$$\mathcal{H} = \widetilde{\mathcal{H}} \quad \text{in } \Omega_0. \tag{3-29}$$

To prove (3-29) we first prove that $\widetilde{\mathcal{H}} \in L^\infty(\Omega_0)$. We write any $y \in \partial\Omega_0$ as $y = (1, y')$ with $y' \in \mathbb{R}^n$. We similarly write $x \in \Omega_0$ as $x = (x_1, x')$ with $x_1 < 1$. If $x \in \Omega_0$, with (3-27) and a simple change of variables we thus have, for some positive constant $C = C(n)$,

$$|\widetilde{\mathcal{H}}(x)| \le C(1-x_1)\int_{\partial\Omega_0} \frac{1}{((x_1-1)^2+|y'|^2)^{n/2}}\,dy' \le C\int_{\partial\Omega_0}\frac{1}{(1+|y'|^2)^{n/2}}\,dy' < +\infty,$$

where the last line again follows from a change of variables. Thus $\widetilde{\mathcal{H}}$ is bounded in $\Omega_0 \backslash B_{\varepsilon_0}(1)$. We can now conclude the proof of Lemma 3.3. Since $\mathcal{H}$ is harmonic in Ω_0 it is bounded in $B_{1/2}(0)$. Equations (3-22) and (3-23) also show that $\mathcal{H}$ is bounded in Ω_0. Independently, we just proved that $\widetilde{\mathcal{H}} \in L^\infty(\Omega_0)$. The function $\mathcal{H} - \widetilde{\mathcal{H}}$ is thus harmonic in Ω_0, bounded in Ω_0 and vanishes on $\partial\Omega_0$. Since $\partial\Omega_0$ is a hyperplane a simple reflection argument allows to apply Liouville's theorem, which shows that $\mathcal{H} \equiv \widetilde{\mathcal{H}}$. This proves (3-29) and by (3-28) conclude the proof of Lemma 3.3. □

We are now in position to prove Proposition 3.2. Let $\delta > 0$ be fixed. We write Pohozaev's identity (3-1) in $U_\alpha = B_{\delta d_\alpha}(x_\alpha)$: this gives

$$\int_{B_{\delta d_\alpha}(x_\alpha)} \bigl(h_\alpha(x) + \tfrac{1}{2}\langle \nabla h_\alpha(x), x - x_\alpha\rangle\bigr) v_\alpha^2\,dx = \int_{\partial B_{\delta d_\alpha}(x_\alpha)} F_\alpha(x)\,d\sigma(x), \tag{3-30}$$

where F_α is defined in (3-3). Changing variables we get that

$$\begin{aligned}
&\left(\frac{\mu_\alpha}{d_\alpha}\right)^{2-n} \int_{\partial B_{\delta d_\alpha}(x_\alpha)} F_\alpha(x)\,d\sigma(x)\\
&= \int_{\partial B_\delta(0)} \langle x,\nu\rangle \left(\frac{|\nabla \bar v_\alpha|^2}{2} + \bar h_\alpha d_\alpha^2 \frac{\bar v_\alpha^2}{2} - d_\alpha^2 \frac{|\bar v_\alpha|^{2^*}}{2^*}\right) d\sigma(x) - \int_{\partial B_\delta(0)} \bigl(\langle x, \nabla \bar v_\alpha\rangle + \tfrac{1}{2}(n-2)\bar v_\alpha\bigr)\partial_\nu \bar v_\alpha\,d\sigma(x),
\end{aligned} \tag{3-31}$$

where $\bar v_\alpha$ is defined in (3-15). Direct calculations using (3-17) and (3-19) yield, since $h_\alpha \in L^\infty(\Omega)$,

$$\begin{aligned}
d_\alpha^2 \int_{\partial B_\delta(0)} \langle x,\nu\rangle \bar h_\alpha \bar v_\alpha^2\,d\sigma(x) &= O(d_\alpha^2 \delta^{4-n} + \mu_\alpha^{(n-2)/(n-1)}\delta^n) = o(1),\\
d_\alpha^2 \int_{\partial B_\delta(0)} \langle x,\nu\rangle |v_\alpha|^{2^*}\,d\sigma(x) &= O(\delta^{-n} d_\alpha^2 + \mu_\alpha^{(n-2)/(n-1)}\delta^n) = o(1)
\end{aligned} \tag{3-32}$$

as $\alpha \to +\infty$. Plugging (3-32) into (3-31) gives, since $\bar v_\alpha \to \bar v_\infty \in C^2(\overline{\Omega}_0 \backslash \{0\})$,

$$\begin{aligned}
\lim_{\alpha\to+\infty} &\left(\frac{\mu_\alpha}{d_\alpha}\right)^{2-n} \int_{\partial B_{\delta d_\alpha}(x_\alpha)} F_\alpha(x)\,d\sigma(x)\\
&= \int_{\partial B_\delta(0)} |x|\bigl(\tfrac{1}{2}|\nabla \bar v_\infty|^2 - (\partial_\nu \bar v_\infty)^2\bigr)\,d\sigma(x) - \tfrac{1}{2}(n-2)\int_{\partial B_\delta(0)} \bar v_\infty \partial_\nu \bar v_\infty\,d\sigma(x)\\
&= \tfrac{1}{2}\omega_{n-1} n^{(n-2)/2}(n-2)^{(n+2)/2}\mathcal{H}(0) + \varepsilon(\delta),
\end{aligned} \tag{3-33}$$

where $\varepsilon(\delta) \to 0$ as $\delta \to 0$ and where the last equality follows from Lemma 3.3. Independently, direct computations using (2-1), (2-20) and (2-67) show that

$$\int_{B_{\delta d_\alpha}(x_\alpha)} \big(h_\alpha(x) + \tfrac{1}{2}\langle \nabla h_\alpha(x), x - x_\alpha\rangle\big) v_\alpha^2 \, dx = \begin{cases} O(\delta^3 d_\alpha^5 + \delta \mu_\alpha d_\alpha) & \text{if } n = 3, \\ O(\delta^4 d_\alpha^6 + \mu_\alpha^2 \ln(d_\alpha/\mu_\alpha)) & \text{if } n = 4, \\ \mu_\alpha^2 h_\infty(x_\infty) \int_{\mathbb{R}^n} B_0(x)^2 \, dx + o(\mu_\alpha^2) + O(\delta^n d_\alpha^{n+2}) & \text{if } n \ge 5. \end{cases} \tag{3-34}$$

Combining (3-33) and (3-34) into (3-30) we finally obtain

$$\tfrac{1}{2}\omega_{n-1} n^{(n-2)/2}(n-2)^{(n+2)/2}\mathcal{H}(0) + \epsilon(\delta) = \left(\frac{d_\alpha}{\mu_\alpha}\right)^{n-2} \begin{cases} O(\delta^3 d_\alpha^5 + \delta \mu_\alpha d_\alpha) & \text{if } n = 3, \\ O(\delta^4 d_\alpha^6 + \mu_\alpha^2 \ln(d_\alpha/\mu_\alpha)) & \text{if } n = 4, \\ \mu_\alpha^2 h_\infty(x_\infty) \int_{\mathbb{R}^n} B_0(x)^2 \, dx + o(\mu_\alpha^2) + O(\delta^n d_\alpha^{n+2}) & \text{if } n \ge 5. \end{cases} \tag{3-35}$$

Using (3-17), and since $d_\alpha \to 0$, we easily obtain, when $n \in \{3, 4, 5\}$, that (3-35) shows

$$\mathcal{H}(0) + \epsilon(\delta) = o(1)$$

as $\alpha \to +\infty$, which contradicts Lemma 3.3. If now $n \ge 6$, (3-17) shows that $d_\alpha^{n+2} = o(\mu_\alpha^2)$. Since $\mathcal{H}(0) < 0$ by Lemma 3.3, we can choose δ fixed but small enough that $\mathcal{H}(0) + \varepsilon(\delta) < 0$. By (3-35) we then have

$$h_\infty(x_\infty) \int_{\mathbb{R}^n} B_0(x)^2 \, dx + o(1) \le 0.$$

Letting $\alpha \to +\infty$ implies $h_\infty(x_\infty) \le 0$. In the case where $h_\infty > 0$ in $\overline{\Omega}$ this is a contradiction and concludes the proof of Proposition 3.2.

We may thus assume $h_\infty < 0$ in $\overline{\Omega}$ and $n \ge 6$. With (3-35) we obtain

$$d_\alpha = (C_0 + o(1))\mu_\alpha^{(n-4)/(n-2)} \tag{3-36}$$

for some constant $C_0 > 0$ that depend on n and h_∞. Integrating (2-2) against ∇v_α in U_α yields the Pohozaev identity

$$\int_{\partial U_\alpha} \left(\frac{1}{2}|\nabla v_\alpha|^2 \nu - \partial_\nu v_\alpha \nabla v_\alpha - \frac{1}{2^*} v_\alpha^{2^*} \nu\right) d\sigma(x) = -\frac{1}{2}\int_{U_\alpha} h_\alpha \nabla(v_\alpha^2)\, dx, \tag{3-37}$$

where ν is the outer unit normal to U_α. Straightforward computations using Theorem 2.1, (2-16) and (3-18) show that

$$\int_{\partial U_\alpha} \frac{1}{2^*} v_\alpha^{2^*} \nu \, d\sigma = O(\mu_\alpha^n d_\alpha^{-n-1}) + O(d_\alpha^{n+1}),$$

while integrating by parts and using Theorem 2.1 and (2-16) shows that

$$\int_{U_\alpha} h_\alpha \nabla(v_\alpha^2)\, dx = \int_{\partial U_\alpha} h_\alpha v_\alpha^2 \nu \, d\sigma(x) - \int_{U_\alpha} v_\alpha^2 \nabla h_\alpha \, dx = O(\mu_\alpha^{n-2} d_\alpha^{3-n}) + O(d_\alpha^{n+1}) + O(\mu_\alpha^2).$$

Independently, (3-22) and (3-23) show that

$$\begin{aligned}\int_{\partial U_\alpha}\left(\tfrac{1}{2}|\nabla v_\alpha|^2\nu-\partial_\nu v_\alpha\nabla v_\alpha\right)d\sigma(x)&=\frac{\mu_\alpha^{n-2}}{d_\alpha^{n-1}}\left(\int_{\partial B_\delta(0)}\left(\tfrac{1}{2}|\nabla\bar{v}_\infty|^2\nu-\partial_\nu\bar{v}_\infty\nabla\bar{v}_\infty\right)d\sigma(x)+o(1)\right)\\&=\frac{\mu_\alpha^{n-2}}{d_\alpha^{n-1}}\left(n^{(n-2)/2}(n-2)^{(n+2)/2}\omega_{n-1}\nabla\mathcal{H}(0)+\varepsilon(\delta)+o(1)\right)\end{aligned}$$

as $\alpha\to+\infty$. Plugging these estimates into (3-37) finally gives

$$\nabla\mathcal{H}(0)+\varepsilon(\delta)=O\left(\left(\frac{\mu_\alpha}{d_\alpha}\right)^2+\frac{d_\alpha^{2n}}{\mu_\alpha^{n-2}}+d_\alpha^2+\frac{d_\alpha^{n-1}}{\mu_\alpha^{n-4}}\right)=o(1),$$

where in the last line we used (3-36). Passing to the limit as $\alpha\to+\infty$ and as $\delta\to0$ shows that $\nabla\mathcal{H}(0)=0$. But going back to (3-27), and since $\mathcal{H}=\widetilde{\mathcal{H}}$, we have $\partial_1\mathcal{H}(0)<0$ by Lemma A.2, which is a contradiction. This concludes the proof of Proposition 3.2. ☐

We now investigate more precisely what happens at the scale r_α. This is the content of the following result:

Proposition 3.4. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n\geq3$. Let $(h_\alpha)_{\alpha\in\mathbb{N}}$ be a sequence of functions that converges in $C^1(\overline{\Omega})$ to h_∞, where $-\Delta+h_\infty$ is coercive in $H_0^1(\Omega)$ and where $I_{h_\infty}(\Omega)<K_n^{-2}$, and we let $(v_\alpha)_{\alpha\in\mathbb{N}}\in H_0^1(\Omega)$ be a sequence of solutions of (2-2) that satisfies (2-3), (2-4) and (2-5). Let $x_\infty=\lim_{\alpha\to+\infty}x_\alpha$, and assume that $x_\infty\in\partial\Omega$. Assume that*

$$\frac{d_\alpha}{r_\alpha}\to+\infty$$

as $\alpha\to+\infty$. Then

- *if $n\in\{3,4,5\}$, we have $v_\infty\equiv0$,*
- *if $n\geq6$, we have $h_\infty(x_\infty)=0$.*

Proof. We assume that

$$\lim_{\alpha\to+\infty}\frac{d_\alpha}{r_\alpha}=+\infty. \tag{3-38}$$

Using (3-13) we define, for $x\in(\Omega-x_\alpha)/r_\alpha$,

$$\bar{v}_\alpha(x)=\frac{r_\alpha^{n-2}}{\mu_\alpha^{(n-2)/2}}v_\alpha(x_\alpha+r_\alpha x)=d_\alpha^{-1}v_\alpha(x_\alpha+r_\alpha x). \tag{3-39}$$

Since v_α satisfies (2-2), $\bar{v}_\alpha$ solves

$$\begin{cases}-\Delta\bar{v}_\alpha+r_\alpha^2\bar{h}_\alpha\bar{v}_\alpha=r_\alpha^2 d_\alpha^{4/(n-2)}|\bar{v}_\alpha|^{2^*-2}\bar{v}_\alpha & \text{in } (\Omega-x_\alpha)/r_\alpha,\\ \bar{v}_\alpha=0 & \text{on } \partial((\Omega-x_\alpha)/r_\alpha),\end{cases}$$

where we have let $\bar{h}_\alpha(x)=h(x_\alpha+r_\alpha x)$. By Hopf's lemma and by (3-38) we have

$$v_\infty(x_\alpha+r_\alpha x)=v_\infty(x_\alpha)+O(r_\alpha)=-\partial_\nu v_\infty(x_\infty)d_\alpha+o(d_\alpha) \tag{3-40}$$

as $\alpha \to +\infty$, and (3-40) obviously remains true if $v_\infty \equiv 0$. Using (2-16), Theorem 2.1, (3-13) and (3-40) we thus have

$$|\bar{v}_\alpha(x)| \le C(|x|^{2-n}+1) \quad \text{for all } x \in \frac{\Omega - x_\alpha}{r_\alpha}\backslash\{0\}.$$

Standard elliptic theory then shows that $\bar{v}_\alpha$ converges to some $\bar{v}_\infty$ in $C^2_{\mathrm{loc}}(\mathbb{R}^n\backslash\{0\})$. Let $x \in \mathbb{R}^n\backslash\{0\}$ be fixed. First, as a consequence of Lemma A.1,

$$\frac{r_\alpha^{n-2}}{\mu_\alpha^{(n-2)/2}}\Pi B_\alpha(x_\alpha + r_\alpha x) \to (n(n-2))^{(n-2)/2}|x|^{2-n} \quad \text{in } C^2_{\mathrm{loc}}(\mathbb{R}^n\backslash\{0\})$$

as $\alpha \to +\infty$. The latter with (3-40) and Theorem 2.1 then shows that

$$\bar{v}_\infty = (n(n-2))^{(n-2)/2}|x|^{2-n} \pm \partial_\nu v_\infty(x_\infty). \tag{3-41}$$

For α large enough we let $U_\alpha = B_{r_\alpha}(x_\alpha) \subset \Omega$, and we apply the Pohozaev identity (3-1). We get

$$\int_{B_{r_\alpha}(x_\alpha)} \big(h_\alpha(x) + \tfrac{1}{2}\langle \nabla h_\alpha(x), x - x_\alpha\rangle\big) v_\alpha^2 \, dx = \int_{\partial B_{r_\alpha}(x_\alpha)} F_\alpha(x)\, d\sigma(x), \tag{3-42}$$

where F_α is defined in (3-3). By changing x into $x_\alpha + d_\alpha x$, we can write

$$\begin{aligned} &d_\alpha^{-2} r_\alpha^{2-n} \int_{\partial B_{r_\alpha}(x_\alpha)} F_\alpha(x)\, d\sigma(x) \\ &= \int_{\partial B_1(0)} \langle x, \nu\rangle \bigg(\frac{|\nabla \bar{v}_\alpha|^2}{2} + \bar{h}_\alpha r_\alpha^2 \frac{\bar{v}_\alpha^2}{2} - r_\alpha^2 \frac{|\bar{v}_\alpha|^{2^*}}{2^*}\bigg) d\sigma(x) - \int_{\partial B_1(0)} \big(\langle x, \nabla \bar{v}_\alpha\rangle + \tfrac{1}{2}(n-2)\bar{v}_\alpha\big)\partial_\nu \bar{v}_\alpha \, d\sigma(x), \end{aligned}$$

where $\bar{v}_\alpha$ is as in (3-39). Direct calculations with (2-67) and (3-40) give

$$r_\alpha^2 \int_{\partial B_1(0)} \langle x, \nu\rangle \bar{h}_\alpha \bar{v}_\alpha^2 \, d\sigma(x) = O(r_\alpha^2) \quad \text{and} \quad r_\alpha^2 \int_{\partial B_1(0)} \langle x, \nu\rangle |\bar{v}_\alpha|^{2^*} \, d\sigma(x) = O(r_\alpha^2).$$

Together with (3-41), the latter then shows that

$$\lim_{\alpha\to+\infty} d_\alpha^{-2} r_\alpha^{2-n} \int_{\partial B_{r_\alpha}(x_\alpha)} F_\alpha(x)\, d\sigma(x) = \pm\tfrac{1}{2}\omega_{n-1} n^{(n-2)/2}(n-2)^{(n+2)/2}\partial_\nu v_\infty(x_\infty). \tag{3-43}$$

Since $\lim_{\alpha\to+\infty} r_\alpha \mu_\alpha^{-1} = +\infty$, direct computations using (2-1), (2-20), (2-67), (3-13) and (3-40) show that

$$\int_{B_{r_\alpha}(x_\alpha)} \big(h_\alpha(x) + \tfrac{1}{2}\langle \nabla h_\alpha(x), x - x_\alpha\rangle\big) v_\alpha^2 \, dx = \begin{cases} O(\mu_\alpha^{3/2}/d_\alpha) & \text{if } n = 3, \\ O(\mu_\alpha^2 \ln(r_\alpha/\mu_\alpha) + \mu_\alpha^2) & \text{if } n = 4, \\ \mu_\alpha^2\big(h_\infty(x_\infty) \int_{\mathbb{R}^n} B_0(x)^2\, dx + o(1)\big) & \text{if } n \ge 5. \end{cases} \tag{3-44}$$

Returning now to (3-42) with (3-43) and (3-44), and since $d_\alpha^2 r_\alpha^{n-2} = d_\alpha \mu_\alpha^{(n-2)/2}$ by (3-13), we have that

$$\begin{aligned} &\pm\tfrac{1}{2}\omega_{n-1}(n-2)^{(n+2)/2} n^{(n-2)/2} \partial_\nu v_\infty(x_\infty) d_\alpha \mu_\alpha^{(n-2)/2} + o(d_\alpha \mu_\alpha^{(n-2)/2}) \\ &\qquad = \begin{cases} O(\mu_\alpha^{3/2}/d_\alpha) & \text{if } n = 3, \\ O(\mu_\alpha^2 \ln(r_\alpha/\mu_\alpha)) & \text{if } n = 4, \\ \mu_\alpha^2\big(h_\infty(x_\infty) \int_{\mathbb{R}^n} B_0(x)^2\, dx + o(1)\big) & \text{if } n \ge 5. \end{cases} \end{aligned} \tag{3-45}$$

Independently, since $r_\alpha = o(d_\alpha)$ by (3-38), and by (3-13), we get

$$\sqrt{\mu_\alpha} = o(d_\alpha^{(n-1)/(n-2)}) \quad \text{as } \alpha \to +\infty. \tag{3-46}$$

Assume first that $n = 3$. Then (3-45) shows that

$$\partial_\nu v_\infty(x_\infty) + o(1) = O\left(\frac{\mu_\alpha}{d_\alpha^2}\right) = o(1)$$

by (3-46). If $n = 4$, (3-45) shows that

$$\partial_\nu v_\infty(x_\infty) + o(1) = O\left(\frac{\mu_\alpha}{d_\alpha} \ln\left(\frac{r_\alpha}{\mu_\alpha}\right)\right) = O\left(\mu_\alpha^{2/3} \ln\left(\frac{r_\alpha}{\mu_\alpha}\right)\right) = o(1)$$

by (3-46). If $n = 5$, (3-45) shows that

$$\partial_\nu v_\infty(x_\infty) + o(1) = O\left(\frac{\mu_\alpha^{1/2}}{d_\alpha}\right) = o(1)$$

again by (3-46). We thus obtain, when $n \in \{3, 4, 5\}$, that

$$\partial_\nu v_\infty(x_\infty) = 0,$$

which shows that $v_\infty \equiv 0$ by Hopf's lemma. Assume now that $n \geq 6$. Then (3-45) shows that

$$h_\infty(x_\infty) \int_{\mathbb{R}^n} B_0(x)^2 \, dx = O(d_\alpha \mu_\alpha^{(n-6)/2}) + o(1) = o(1)$$

since $d_\alpha \to 0$ as $\alpha \to +\infty$. This concludes the proof of Proposition 3.4. ☐

The next result finally shows that, in small dimensions, the concentration point cannot occur on $\partial\Omega$.

Proposition 3.5. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$. Let $(h_\alpha)_{\alpha\in\mathbb{N}}$ be a sequence of functions that converges in $C^1(\bar{\Omega})$ to h_∞, where $-\Delta + h_\infty$ is coercive in $H_0^1(\Omega)$ and where $I_{h_\infty}(\Omega) < K_n^{-2}$, and we let $(v_\alpha)_{\alpha\in\mathbb{N}} \in H_0^1(\Omega)$ be a sequence of solutions of (2-2) that satisfies (2-3), (2-4) and (2-5). Let $x_\infty = \lim_{\alpha\to+\infty} x_\alpha$. Assume that $n \in \{3, 4\}$ or that $n = 5$ and $h_\infty \neq 0$ in $\bar{\Omega}$. Then $x_\infty \in \Omega$.*

Proof. We proceed by contradiction and assume that $x_\infty \in \partial\Omega$. Under the assumptions of Proposition 3.5, Propositions 3.2 and 3.4 also apply. They show in particular that

$$\frac{d_\alpha}{r_\alpha} \to +\infty \tag{3-47}$$

as $\alpha \to +\infty$ and that $v_\infty \equiv 0$. For $x \in (\Omega - x_\alpha)/d_\alpha$ we define again

$$\bar{v}_\alpha(x) := \frac{d_\alpha^{n-2}}{\mu_\alpha^{(n-2)/2}} v_\alpha(x_\alpha + d_\alpha x). \tag{3-48}$$

Equation (2-2) then shows that $\bar{v}_\alpha$ satisfies

$$\begin{cases} -\Delta \bar{v}_\alpha - d_\alpha^2 \bar{h}_\alpha \bar{v}_\alpha = (\mu_\alpha/d_\alpha)^2 |\bar{v}_\alpha|^{2^*-2} \bar{v}_\alpha & \text{in } (\Omega - x_\alpha)/d_\alpha, \\ \bar{v}_\alpha = 0 & \text{on } \partial((\Omega - x_\alpha)/d_\alpha), \end{cases}$$

where $\bar{h}_\alpha(x) := h(x_\alpha + d_\alpha x)$. Since $v_\infty \equiv 0$, (2-16) and Theorem 2.1 show that

$$|\bar{v}_\alpha(x)| \le C|x|^{2-n} \quad \text{for all } x \in \frac{\Omega - x_\alpha}{d_\alpha} \backslash \{0\} \tag{3-49}$$

for some positive constant C. Since Ω is smooth and since $d_\alpha \to 0$ as $\alpha \to +\infty$ by assumption, standard elliptic theory shows that, up to a rotation, $\bar{v}_\alpha \to \bar{v}_\infty \in C^2(\bar{\Omega}_0 \backslash \{0\})$ as $\alpha \to +\infty$, where $\Omega_0 :=]-\infty, 1[\times \mathbb{R}^{n-1}$ and where $\bar{v}_\infty$ satisfies

$$-\Delta \bar{v}_\infty = 0 \quad \text{in } \Omega_0 \backslash \{0\}, \qquad \bar{v}_\infty = 0 \quad \text{on } \partial\Omega_0$$

and

$$|\bar{v}_\infty(x)| \le C|x|^{2-n} \quad \text{for all } x \in \Omega_0.$$

The arguments in the proof of Lemma 3.3 again show that

$$\bar{v}_\infty(x) = \frac{(n(n-2))^{(n-2)/2}}{|x|^{n-2}} + \mathcal{H}(x) \quad \text{for all } x \in \Omega_0 \backslash \{0\}, \tag{3-50}$$

where $\mathcal{H}$ satisfies

$$-\Delta \mathcal{H} = 0 \quad \text{in } \Omega_0, \qquad \mathcal{H} = -(n(n-2))^{-(n-2)/2} |\cdot|^{2-n} \quad \text{on } \partial\Omega_0$$

and is given for any $x \in \Omega$ by

$$\mathcal{H}(x) = 2\frac{n^{(n-4)/2}(n-2)^{(n-2)/2}}{\omega_{n-1}}(x_1 - 1) \int_{\partial\Omega_0} |y|^{2-n} |x-y|^{-n} \, d\sigma(y) \tag{3-51}$$

and also satisfies

$$\mathcal{H}(0) < 0. \tag{3-52}$$

In the following we let $0 < \delta < 1$ and $U_\alpha = B_{\delta d_\alpha}(x_\alpha)$. We write Pohozaev's identity (3-1) in U_α: this gives

$$\int_{B_{\delta d_\alpha}(x_\alpha)} \big(h_\alpha(x) + \tfrac{1}{2}\langle \nabla h_\alpha(x), x - x_\alpha \rangle\big) v_\alpha^2 \, dx = \int_{\partial B_{\delta d_\alpha}(x_\alpha)} F_\alpha(x) \, d\sigma(x),$$

where F_α is defined in (3-3). Mimicking the computations that led to (3-31), (3-32) and (3-33) we obtain

$$\int_{\partial B_{\delta d_\alpha}(x_\alpha)} F_\alpha(x) \, d\sigma(x) = \left(\frac{\mu_\alpha}{\delta d_\alpha}\right)^{n-2} \big(\tfrac{1}{2}\omega_{n-1} n^{(n-2)/2} (n-2)^{(n+2)/2} \mathcal{H}(0) + \varepsilon(\delta) + o(1)\big) \tag{3-53}$$

as $\alpha \to +\infty$, where $\varepsilon(\delta) \to 0$. Independently, direct computations using (2-1), (2-20) and (2-67) show

$$\int_{B_{\delta d_\alpha}(x_\alpha)} \big(h_\alpha(x) + \tfrac{1}{2}\langle \nabla h_\alpha(x), x - x_\alpha \rangle\big) v_\alpha^2 \, dx = \begin{cases} O(\mu_\alpha r_\alpha) & \text{if } n = 3, \\ 64\omega_3 h_\infty(x_\infty) \mu_\alpha^2 \ln(d_\alpha/\mu_\alpha) + O(\mu_\alpha^2) & \text{if } n = 4, \\ \mu_\alpha^2 \big(h_\infty(x_\infty) \int_{\mathbb{R}^n} B_0(x)^2 \, dx + o(1)\big) & \text{if } n \ge 5. \end{cases} \tag{3-54}$$

If $n = 3$, combining (3-53) and (3-54) gives

$$\mathcal{H}(0) = O(\sqrt{\mu_\alpha});$$

hence $\mathcal{H}(0)=0$, which contradicts (3-52). This proves Proposition 3.5 when $n=3$. If $n=4,5$, using (3-52), we obtain $h_\infty(x_\infty)\le 0$. If $h_\infty>0$ in $\overline{\Omega}$ this is a contradiction and concludes the proof of Proposition 3.5.

We assume from now on that $h_\infty<0$ in $\overline{\Omega}$ and $n=4,5$. In this case the proof is similar to the proof of Proposition 3.2 when $n\ge 6$. Using again (3-52) the previous Pohozaev's identity then shows the existence of a constant $C_0>0$ depending on $n,\ h_\infty$ and δ such that

$$d_\alpha^2\ln(d_\alpha/\mu_\alpha)=C_0+o(1)\quad\text{if } n=4\qquad\text{and}\qquad d_\alpha=(C_0+o(1))\mu_\alpha^{1/3}\quad\text{if } n=5. \tag{3-55}$$

We recall the gradient Pohozaev identity (3-37),

$$\int_{\partial U_\alpha}\Big(\frac{1}{2}|\nabla v_\alpha|^2\nu-\partial_\nu v_\alpha\nabla v_\alpha-\frac{1}{2^*}v_\alpha^{2^*}\nu\Big)\,d\sigma(x)=-\frac{1}{2}\int_{U_\alpha}h_\alpha\nabla(v_\alpha^2)\,dx,$$

where ν is the outer unit normal to U_α. Straightforward computations using Theorem 2.1 and (2-16) show

$$\int_{\partial U_\alpha}\frac{1}{2^*}v_\alpha^{2^*}\nu\,d\sigma(x)=O(\mu_\alpha^n d_\alpha^{-n-1}),$$

while integrating by parts and using Theorem 2.1 and (2-16) shows

$$\int_{U_\alpha}h_\alpha\nabla(v_\alpha^2)\,dx=\int_{\partial U_\alpha}h_\alpha v_\alpha^2\nu\,d\sigma(x)-\int_{U_\alpha}v_\alpha^2\nabla h_\alpha\,dx=O(\mu_\alpha^{n-2}d_\alpha^{3-n})+\begin{cases}O(\mu_\alpha^2\ln(d_\alpha/\mu_\alpha)) & \text{if } n=4,\\ O(\mu_\alpha^2) & \text{if } n=5.\end{cases}$$

Independently, (3-49) and (3-50) show that

$$\begin{aligned}\int_{\partial U_\alpha}\big(\tfrac{1}{2}|\nabla v_\alpha|^2\nu-\partial_\nu v_\alpha\nabla v_\alpha\big)\,d\sigma(x)&=\frac{\mu_\alpha^{n-2}}{d_\alpha^{n-1}}\Big(\int_{\partial B_\delta(0)}\big(\tfrac{1}{2}|\nabla\bar v_\infty|^2\nu-\partial_\nu\bar v_\infty\nabla\bar v_\infty\big)\,d\sigma(x)+o(1)\Big)\\&=\frac{\mu_\alpha^{n-2}}{d_\alpha^{n-1}}\big(n^{(n-2)/2}(n-2)^{(n+2)/2}\omega_{n-1}\nabla\mathcal{H}(0)+\varepsilon(\delta)+o(1)\big)\end{aligned}$$

as $\alpha\to+\infty$. Plugging these estimates into (3-37) finally gives

$$\begin{aligned}\nabla\mathcal{H}(0)+\varepsilon(\delta)&=O\bigg(\Big(\frac{\mu_\alpha}{d_\alpha}\Big)^2\bigg)+O(d_\alpha^2)+\begin{cases}O(d_\alpha^3\ln(d_\alpha/\mu_\alpha)) & \text{if } n=4,\\ O(d_\alpha^4/\mu_\alpha) & \text{if } n=5,\end{cases}\\&=o(1),\end{aligned}$$

where in the last line we used (3-55). Passing to the limit as $\alpha\to+\infty$ and as $\delta\to 0$ shows that $\nabla\mathcal{H}(0)=0$. But going back to (3-51) we again have $\partial_1\mathcal{H}(0)<0$ by Lemma A.2, which is a contradiction. This concludes the proof of Proposition 3.5 when $n=4,5$ and $h_\infty<0$.

To conclude the proof of Proposition 3.5 we finally assume that $n=4$. If $h_\infty(x_\infty)\ne 0$ in $\overline{\Omega}$ the proof of Proposition 3.5 follows from the previous arguments. We may then assume that $h_\infty(x_\infty)=0$. In this case combining (3-53) and (3-54) shows

$$\mathcal{H}(0)=O(d_\alpha^2)=o(1)$$

as $\alpha\to+\infty$, which contradicts (3-52). This concludes the proof of Proposition 3.5. ☐

Remark 3.6. Assume that $(v_\alpha)_{\alpha\in\mathbb{N}} \in H_0^1(\Omega)$ is any sequence of solutions of (2-2) that satisfies (2-3) and (2-4), so that (2-5), (2-6) and (2-8) also hold. Let $x_\infty = \lim_{\alpha\to\infty} x_\alpha$ be the concentration point of u_α. Propositions 3.2, 3.4 and 3.5 prove that $x_\infty \in \Omega$, i.e., that x_∞ is an interior blow-up point, in the following cases (regardless of the value of v_∞): either when $n \in \{3, 4\}$ or when $n \geq 5$ and under the assumption $h_\infty \neq 0$ in $\overline{\Omega}$. If h_∞ is allowed to vanish somewhere in $\partial\Omega$ the property that $x_\infty \in \Omega$ is unlikely to remain true, and concentration points may arise on the boundary in large dimensions. When $n \geq 7$, for instance, *sign-changing* solutions of (1-5) that blow-up with one concentration point in $\partial\Omega$ as $\lambda \to 0_+$ (which corresponds to $h_\infty \equiv 0$) have been constructed in [Vaira 2015]; see also [Musso et al. 2024] for a more recent construction with an arbitrary number of bubbles.

Remark 3.7. We mentioned in Remark 3.6 that, when $n \geq 7$ and $h_\infty \equiv 0$, *sign-changing* solutions of (1-5) that blow-up with one concentration point in $\partial\Omega$ as $\lambda \to 0_+$ exist; see [Vaira 2015]. By contrast, it is important to point out that, in any dimension $n \geq 4$, *positive* solutions of (1-5) as $\lambda \to 0_+$ may only blow-up with interior concentration points and do not possess concentration points in $\partial\Omega$. This is shown in [König and Laurain 2024, Proposition 2.1] and heavily relies on the positivity of the solutions. The issue of boundary concentration points thus arises when working with *sign-changing* solutions of (1-6).

We are now in position to prove Theorem 1.1.

End of the proof of Theorem 1.1. Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$, and $(h_\alpha)_{\alpha\in\mathbb{N}}$ be a sequence that converges in $C^1(\overline{\Omega})$ towards h_∞. Assume that $-\Delta + h_\infty$ is coercive and that $I_{h_\infty}(\Omega) < K_n^{-2}$. Let $(v_\alpha)_{\alpha\in\mathbb{N}} \in H_0^1(\Omega)$ be a sequence of solutions of (2-2) that satisfies (2-3). Assume first that $(v_\alpha)_{\alpha\in\mathbb{N}}$ is, up to a subsequence, uniformly bounded in $L^\infty(\Omega)$. By standard elliptic theory it then strongly converges, again up to a subsequence, to some v_0 in $C^2(\overline{\Omega})$ as $\alpha \to +\infty$. That $v_0 \neq 0$ simply follows from the coercivity of $-\Delta + h_\infty$ which easily implies, by Sobolev's inequality, that $\liminf_{\alpha\to+\infty} \|v_\alpha\|_{H_0^1} > 0$. This concludes the proof of Theorem 1.1.

We thus proceed by contradiction and assume that, up to a subsequence, (2-4) holds, and hence that (2-5), (2-6) and (2-8) hold for some sequence $(x_\alpha)_{\alpha\in\mathbb{N}}$ of points in Ω and $(\mu_\alpha)_{\alpha\in\mathbb{N}}$ of positive real numbers converging to 0 satisfying (2-10). In particular,

$$v_\alpha = B_\alpha \pm v_\infty + o(1) \quad \text{in } H_0^1(\Omega),$$

where $v_\infty \equiv 0$ or v_∞ is a positive solution of (2-9). We let $x_\infty = \lim_{\alpha\to+\infty} x_\alpha \in \overline{\Omega}$. Under these assumptions, the analysis of Section 3 applies.

We first assume that $n \geq 7$ and that $h_\infty \neq 0$ at every point of $\overline{\Omega}$. Propositions 3.2 and 3.4 first show that $x_\infty \in \Omega$. Proposition 3.1 then applies and shows that $h_\infty(x_\infty) = 0$, which is a contradiction.

We now assume that $3 \leq n \leq 5$ and that $(v_\alpha)_{\alpha\in\mathbb{N}} \in H_0^1(\Omega)$ is *sign-changing* for all $\alpha \geq 0$. We then claim that

$$v_\infty > 0 \quad \text{in } \Omega. \tag{3-56}$$

This is a strong consequence of the assumption that v_α changes sign. We adapt an argument from [Cerami et al. 1986, Lemma 3.1]. Since v_α does not strongly converge to v_∞, $(v_\alpha)_+$ and $(v_\alpha)_-$ may not simultaneously strongly converge to $(v_\infty)_+$ and $(v_\infty)_-$. Assume for simplicity that $(v_\alpha)_+$ weakly but not strongly converges to $(v_\infty)_+$ in $H_0^1(\Omega)$. Integrating (2-2) against $(v_\alpha)_+$ and using the Brézis–Lieb lemma shows that

$$\int_\Omega |\nabla((v_\alpha)_+ - (v_\infty)_+)|^2\,dx + o(1) = \int_\Omega |(v_\alpha)_+ - (v_\infty)_+|^{2^*}\,dx,$$

from which we deduce that $\int_\Omega |(v_\alpha)_+ - (v_\infty)_+|^{2^*}\,dx \geq K_n^{-n} + o(1)$ as $\alpha \to +\infty$ by (1-3). Independently, since $(v_\alpha)_-$ is nonzero, integrating (2-2) against $(v_\alpha)_-$ and using (1-2) yields

$$\int_\Omega |(v_\alpha)_-|^{2^*}\,dx \geq I_{h_\alpha}(\Omega)^{n/2}.$$

Thus, again by Brézis–Lieb's lemma,

$$\begin{aligned}\int_\Omega |v_\alpha|^{2^*}\,dx &= \int_\Omega |(v_\alpha)_+|^{2^*}\,dx + \int_\Omega |(v_\alpha)_-|^{2^*}\,dx\\ &= \int_\Omega |(v_\alpha)_+ - (v_\infty)_+|^{2^*}\,dx + \int_\Omega |(v_\infty)_+|^{2^*}\,dx + \int_\Omega |(v_\alpha)_-|^{2^*}\,dx + o(1)\\ &\geq K_n^{-n} + I_{h_\infty}(\Omega)^{n/2} + o(1)\end{aligned}$$

as $\alpha \to +\infty$. This shows that $v_\infty \not\equiv 0$ and hence that $v_\infty > 0$ in Ω and attains $I_{h_\infty}(\Omega)$. As before, the analysis of Section 3 applies to v_α. First, using (3-56), Propositions 3.2 and 3.4 show that $x_\infty \in \Omega$. We may thus apply Proposition 3.1, which shows that $v_\infty \equiv 0$ and contradicts (3-56). Thus $(v_\alpha)_{\alpha\in\mathbb{N}}$ is again uniformly bounded in $L^\infty(\Omega)$ and Theorem 1.1 is proven. ☐

We now prove Corollary 1.2.

Proof of Corollary 1.2. We assume that Ω and h are as in the assumptions of Corollary 1.2. We observed in the proof of Theorem 1.1 that any sequence $(v_\alpha)_{\alpha\in\mathbb{N}}$ of solutions of (1-1) which is bounded in $L^\infty(\Omega)$ up to a subsequence is precompact in $C^2(\overline{\Omega})$. With this observation we proceed by contradiction: if no ε as in the statement of Corollary 1.2 exists, we can find a sequence $(v_\alpha)_{\alpha\in\mathbb{N}}$ of solutions of

$$\begin{cases}-\Delta v_\alpha + h v_\alpha = |v_\alpha|^{2^*-2} v_\alpha & \text{in } \Omega,\\ v_\alpha = 0 & \text{in } \partial\Omega,\end{cases}$$

which satisfies $\lim_{\alpha\to+\infty} \|v_\alpha\|_\infty = +\infty$ and $\limsup_{\alpha\to+\infty} \int_\Omega |v_\alpha|^{2^*}\,dx \leq K_n^{-n} + I_h(\Omega)^{n/2}$. When $3 \leq n \leq 5$ we have in addition that $(v_\alpha)_{\alpha\in\mathbb{N}}$ changes sign. We may now apply Theorem 1.1 to the sequence $(v_\alpha)_{\alpha\in\mathbb{N}}$ with $h_\alpha \equiv h$ for all $\alpha \geq 0$, which gives a contradiction. ☐

We now consider the six-dimensional case and prove Proposition 1.3.

Proof of Proposition 1.3. Assume indeed that $(v_\alpha)_{\alpha\in\mathbb{N}}$ is a sequence of solutions of (2-2) that satisfies (2-3) and (2-4). Hence (2-5), (2-6) and (2-8) hold for some sequence $(x_\alpha)_\alpha$ of points in Ω and $(\mu_\alpha)_\alpha$ of positive real numbers converging to 0 satisfying (2-10). Then

$$v_\alpha = B_\alpha \pm v_\infty + o(1) \quad \text{in } H_0^1(\Omega),$$

where $v_\infty \equiv 0$ or v_∞ is a positive solution of (2-9). We let $x_\infty = \lim_{\alpha\to+\infty} x_\alpha \in \overline{\Omega}$. First, Propositions 3.2 and 3.4 show that $x_\infty \in \Omega$. Proposition 3.1 then applies and shows that $h_\infty(x_\infty) = \pm 2 v_\infty(x_\infty)$. ☐

Remark 3.8. When $n \in \{3,4,5\}$, Theorem 1.1 is likely to be false if (1-7) is not satisfied. On a closed Riemannian manifold and when $3 \le n \le 5$, blowing-up solutions of equations like (1-6) of the form $B_\alpha + v_\infty$, where v_∞ is a *sign-changing* solution of (1-1), may exist; see [Premoselli and Vétois 2022b, Theorem 1.4]. The examples in that result are constructed on a closed manifold with symmetries and B_α concentrates at a point where v_∞ vanishes. These examples are likely to adapt to the case of a symmetric bounded open set when $3 \le n \le 5$ and $h_\infty \not\equiv 0$ in $\bar{\Omega}$. They suggest that, even when $3 \le n \le 5$, sign-changing solutions may exhibit noncompactness at a higher energy level than $K_n^{-n} + I_{h_\infty}(\Omega)^{n/2}$.

Appendix: Technical results

A.1. *Pointwise estimates on ΠB_α.* Let ΠB_α be given by (2-14). We prove a technical result that was used several times throughout the paper and that provides an asymptotic expansion of ΠB_α close to $\partial\Omega$.

Lemma A.1. *Let $(x_\alpha)_{\alpha\in\mathbb{N}}$ and $(\mu_\alpha)_{\alpha\in\mathbb{N}}$ be sequences of points in Ω and positive real numbers, respectively, satisfying $d(x_\alpha, \partial\Omega) \gg \mu_\alpha$ as $\alpha \to +\infty$. Let B_α be given by (2-6) and ΠB_α be given by (2-14). Let $(y_\alpha)_{\alpha\in\mathbb{N}}$ be a sequence of points in Ω satisfying*

$$d(y_\alpha, \partial\Omega) \to 0, \quad |x_\alpha - y_\alpha| \le \tfrac{1}{2} d(x_\alpha, \partial\Omega) \quad \textit{and} \quad \frac{|x_\alpha - y_\alpha|}{\mu_\alpha} \to +\infty \tag{A-1}$$

as $\alpha \to +\infty$. Let $\ell = \lim_{\alpha\to+\infty} |x_\alpha - y_\alpha|/d(x_\alpha, \partial\Omega)$, which exists up to a subsequence. Then, as $\alpha \to +\infty$, we have

$$\Pi B_\alpha(y_\alpha) = \big((n(n-2))^{(n-2)/2} + o(1) + \varepsilon(\ell)\big) \frac{\mu_\alpha^{(n-2)/2}}{|x_\alpha - y_\alpha|^{n-2}},$$

where $\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$ denotes a function satisfying $\varepsilon(0) = 0$ and $\lim_{x\to 0} \varepsilon(x) = 0$.

Proof. We write a representation formula for ΠB_α using (2-14),

$$\Pi B_\alpha(y_\alpha) = \int_\Omega G_\alpha(y_\alpha, \cdot) B_\alpha^{2^*-1}\, dx, \tag{A-2}$$

where as before G_α denotes the Green's function of $-\Delta + h_\alpha$ with Dirichlet boundary conditions in Ω. Using (A-1), (2-12) and arguing as in (2-80) we have

$$\int_{\Omega \setminus B_{|x_\alpha - y_\alpha|/2}(x_\alpha)} G_\alpha(y_\alpha, \cdot) B_\alpha^{2^*-1}\, dx = o(B_\alpha(y_\alpha)) \tag{A-3}$$

as $\alpha \to +\infty$. We let in what follows

$$I_\alpha := |x_\alpha - y_\alpha|^{n-2} \mu_\alpha^{-(n-2)/2} \int_{B_{|x_\alpha - y_\alpha|/2}(x_\alpha)} G_\alpha(y_\alpha, \cdot) B_\alpha^{2^*-1}\, dx.$$

By a change of variable we have

$$I_\alpha = \int_{B_{|x_\alpha - y_\alpha|/(2\mu_\alpha)}(0)} |x_\alpha - y_\alpha|^{n-2} G_\alpha(y_\alpha, x_\alpha + \mu_\alpha z) B_0(z)^{2^*-1}\, dz, \tag{A-4}$$

where B_0 is as in (2-7). Using (2-12) there is $C > 0$ such that, for any $z \in B_{|x_\alpha - y_\alpha|/(2\mu_\alpha)}(0)$,

$$|x_\alpha - y_\alpha|^{n-2} G_\alpha(y_\alpha, x_\alpha + \mu_\alpha z) \le C.$$

Let $z \in \mathbb{R}^n$ be fixed. Since $\mu_\alpha = o(d_\alpha)$ we have by (A-1)

$$D := \lim_{\alpha \to +\infty} \frac{d(y_\alpha, \partial\Omega) d(x_\alpha + \mu_\alpha z, \partial\Omega)}{|y_\alpha - (x_\alpha + \mu_\alpha z)|^2} \ge \frac{1}{\ell^2}(1 - \ell)$$

as $\alpha \to +\infty$, where we have let $\ell = \lim_{\alpha \to +\infty} |x_\alpha - y_\alpha| / d(x_\alpha, \partial\Omega)$, and we use the convention that the right-hand side is equal to $+\infty$ if $\ell = 0$. Note that $\ell \le \frac{1}{2}$ by (A-1). Since $\mu_\alpha = o(d_\alpha)$ and $\lim_{\alpha \to +\infty} |y_\alpha - (x_\alpha + \mu_\alpha z)| = 0$ uniformly in $z \in B_R(0)$, Proposition 12 in [Robert 2010] applies and shows that, for any fixed $z \in \mathbb{R}^n$,

$$\begin{aligned} \lim_{\alpha \to +\infty} |x_\alpha - y_\alpha|^{n-2} G_\alpha(y_\alpha, x_\alpha + \mu_\alpha z) &= \frac{1}{(n-2)\omega_{n-1}} \left(1 - \frac{1}{(1+4D)^{(n-2)/2}}\right) \\ &= \frac{1}{(n-2)\omega_{n-1}} (1 + O(\ell)). \end{aligned} \tag{A-5}$$

Plugging (A-5) into (A-4) we get by dominated convergence that

$$I_\alpha = (1 + \varepsilon(\ell) + o(1)) \frac{1}{(n-2)\omega_{n-1}} \int_{\mathbb{R}^n} B_0^{2^*-1}\, dx = (1 + \varepsilon(\ell) + o(1)) (n(n-2))^{(n-2)/2}$$

as $\alpha \to +\infty$, where $\varepsilon(\ell)$ denotes a function such that $\varepsilon(0) = 0$ and $\varepsilon(\ell) \to 0$ as $\ell \to 0$. In the latter estimate we used

$$\int_{\mathbb{R}^n} B_0^{2^*-1}\, dx = (n-2)\omega_{n-1} (n(n-2))^{(n-2)/2},$$

which follows from integrating the equation $-\Delta B_0 = B_0^{2^*-1}$. Going back to the definition of I_α proves the lemma. ☐

A.2. ***Sign of $\partial_1 \mathcal{H}(0)$.*** We will finally prove the following simple result that was used in the proof of Propositions 3.2 and 3.5.

Lemma A.2. *Let $\widetilde{\mathcal{H}}$ be given by (3-27). Then $\partial_1 \widetilde{\mathcal{H}}(0) < 0$.*

Proof. Straightforward computations show that

$$\frac{1}{D_0} \partial_1 \widetilde{\mathcal{H}}(0) = \int_{\partial\Omega_0} |y|^{2-2n}\, d\sigma(y) - n \int_{\partial\Omega_0} |y|^{-2n}\, d\sigma(y),$$

where we have let $D_0 = 2n^{(n-4)/2}(n-2)^{(n-2)/2}/\omega_{n-1}$ and where $\partial\Omega_0 = \{1\} \times \mathbb{R}^{n-1}$. Simple changes of variable then yield

$$\int_{\partial\Omega_0} |y|^{2-2n}\, d\sigma(y) = \tfrac{1}{2}\omega_{n-2} I_{n-1}^{(n-3)/2} \quad \text{and} \quad \int_{\partial\Omega_0} |y|^{-2n}\, d\sigma(y) = \tfrac{1}{2}\omega_{n-2} I_n^{(n-3)/2},$$

where ω_{n-2} is the area of the round sphere $\mathbb{S}^{n-2}$ and where we have let, for $p, q > 0$, $p > q + 1$,

$$I_p^q = \int_0^{+\infty} \frac{r^q}{(1+r)^p}\, dr.$$

Classical induction formulae (see, e.g., [Aubin 1976]) show that $I_n^{(n-3)/2} = \frac{1}{2} I_{n-1}^{(n-3)/2}$. Combining these computations finally shows that

$$\frac{1}{D_0}\partial_1 \widetilde{\mathcal{H}}(0) = \tfrac{1}{2}\omega_{n-2} I_{n-1}^{(n-3)/2}\bigl(1 - \tfrac{1}{2}n\bigr) = -\tfrac{1}{2}(n-2)\int_{\partial\Omega_0} |y|^{2-2n}\, d\sigma(y) < 0,$$

which proves the lemma. □

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