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DIMENSION-FREE L^p ESTIMATES FOR HIGHER-ORDER MAXIMAL RIESZ TRANSFORMS IN TERMS OF THE RIESZ TRANSFORMS

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We prove a dimension-free $L^p(\mathbb{R}^d)$ estimate, $1 < p < \infty$, for the vector of higher-order maximal Riesz transforms in terms of the corresponding Riesz transforms. This implies a dimension-free $L^p(\mathbb{R}^d)$ estimate for the vector of maximal Riesz transforms in terms of the input function. We also give explicit estimates for the dependencies of the constants on p when the order is fixed. Analogous dimension-free estimates are also obtained for single higher-order Riesz transforms with an improved estimate of the constants.

1. Introduction

Fix a positive integer k and denote by $\mathcal{H}_k = \mathcal{H}_k^d$ the space of spherical harmonics of degree k on the Euclidean sphere S^{d-1}. Throughout the paper we identify $P \in \mathcal{H}_k$ with the corresponding solid spherical harmonic. Via this identification $P \in \mathcal{H}_k$ is a harmonic polynomial on $\mathbb{R}^d$ which is homogeneous of degree k, i.e., satisfies $P(x) = |x|^k P(x/|x|),\ x \in \mathbb{R}^d$.

For $P \in \mathcal{H}_k$ the Riesz transform $R = R_P$ is defined by the kernel

$$K_P(x) = K(x) = \gamma_k \frac{P(x)}{|x|^{d+k}} \quad \text{with } \gamma_k = \frac{\Gamma\left(\frac{1}{2}(k+d)\right)}{\pi^{d/2}\Gamma\left(\frac{1}{2}k\right)}; \tag{1-1}$$

more precisely,

$$R_P f(x) = \lim_{t\to 0^+} R_P^t f(x), \quad \text{where } R_P^t f(x) = \gamma_k \int_{|y|>t} \frac{P(y)}{|y|^{d+k}} f(x-y)\, dy. \tag{1-2}$$

The operator R_P^t is called the truncated Riesz transform. In the particular case of $k = 1$ and $P_j(x) = x_j$ the operators $R_{P_j},\ j = 1, \dots, d$, coincide with the classical first-order Riesz transforms. It is well known, see [Stein 1970, p. 73], that the Fourier multiplier associated with the Riesz transform R_P equals

$$m_P(\xi) = (-i)^k P(\xi/|\xi|), \quad \xi \in \mathbb{R}^d. \tag{1-3}$$

By the above formula m_P is bounded and Plancherel's theorem implies the $L^2(\mathbb{R}^d)$ boundedness of R_P. The $L^p(\mathbb{R}^d)$ boundedness of the single Riesz transforms R_P for $1 < p < \infty$ follows from the Calderón–Zygmund method of rotations [1956].

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The present paper is a merger of our previous two preprints in which we treated separately the cases of even and odd orders k.
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The systematic study of the dimension-free L^p bounds for the Riesz transforms began in the seminal paper of E. M. Stein [1983]. There he proved a dimension-free ℓ^2 vector-valued estimate for the vector of the first-order Riesz transforms

$$\left\| \left(\sum_{j=1}^{d} |R_j f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)} \leqslant C_p \|f\|_{L^p(\mathbb{R}^d)}, \quad 1 < p < \infty. \tag{1-4}$$

In the inequality above, the R_j, $j = 1, \dots, d$, denote the first-order Riesz transforms defined via (1-2) with $P_j(x) = x_j$ and the constant C_p is independent of the dimension d.

Stein's result has been extended to many other settings. The analogue of the dimension-free inequality (1-4) has been also proved for higher-order Riesz transforms; see [Duoandikoetxea and Rubio de Francia 1985, théorème 2]. The optimal constant C_p in (1-4) remains unknown when $d \geqslant 2$; however the best results to date given in [Bañuelos and Wang 1995] (see also [Dragičević and Volberg 2006]) established the correct order of the dependence on p. We note that the explicit values of $L^p(\mathbb{R}^d)$ norms of the single first-order Riesz transforms R_j, $j = 1, \dots, d$, were obtained by Iwaniec and Martin [1996] based on the method of rotations.

In this paper we study the relation between R_P and the maximal Riesz transform defined by

$$R_P^* f(x) = \sup_{t>0} |R_P^t f(x)|.$$

Clearly, we have the pointwise inequality $R_P f(x) \leqslant R_P^* f(x)$. In a series of papers [Mateu and Verdera 2006, Theorem 1] (first-order Riesz transforms), [Mateu et al. 2010, Section 4] (odd-order higher Riesz transforms), and [Mateu et al. 2011, Section 2] (even-order higher Riesz transforms), J. Mateu, J. Orobitg, C. Pérez, and J. Verdera proved that also a reverse inequality holds in the $L^p(\mathbb{R}^d)$ norm. Namely, together the results of [Mateu and Verdera 2006; Mateu et al. 2010; 2011] imply that for each $1 < p < \infty$ there exists a constant $C(p, k, d)$ such that

$$\|R_P^* f\|_{L^p(\mathbb{R}^d)} \leqslant C(p, k, d) \|R_P f\|_{L^p(\mathbb{R}^d)} \tag{1-5}$$

for all $f \in L^p(\mathbb{R}^d)$. As a matter of fact, the estimate (1-5) has been proved in [Mateu and Verdera 2006; Mateu et al. 2010; 2011] for more general singular integral operators with even kernels [Mateu et al. 2011] or with odd kernels [Mateu et al. 2010]. However, even for the higher-order Riesz transforms, the values of $C(p, k, d)$ that follow from these papers grow exponentially with the dimension. In view of [Janakiraman 2004], the question about an improved rate arises naturally.

The first step towards a dimension-free estimate of the constant $C(p, k, d)$ in (1-5) has been made by the first and the second authors, who proved that when $p = 2$ in (1-5) one may take an explicit dimension-free constant $C(2, 1, d) \leqslant 2 \cdot 10^8$; see [Kucharski and Wróbel 2023, Theorem 1.1]. The arguments applied in [Kucharski and Wróbel 2023] relied on Fourier transform estimates together with square function techniques developed by Bourgain [1986], and Bourgain, Mirek, Stein, and Wróbel [Bourgain et al. 2018; 2021], for studying dimension-free estimates for maximal functions associated with symmetric convex bodies.

Recently Liu, Melentijević, and Zhu [Liu et al. 2024] extended the results of [Kucharski and Wróbel 2023] and proved that $C(p, 1, d) \leqslant (2 + 1/\sqrt{2})^{2/p}$ for $p \geq 2$. An important ingredient of their argument is the positivity of the transition kernels M_1^t (see (1-7)), which is not at all clear in [Kucharski and Wróbel 2023].

In this paper we prove that the dimension-free estimate of the form (1-5) and its vector-valued generalization hold for Riesz transforms of arbitrary order k and for all $1 < p < \infty$. The main result of our paper is the following square function estimate of the vector of maximal Riesz transforms in terms of the Riesz transforms.

Theorem 1.1. *Take $p \in (1, \infty)$ and let $k \leqslant d$ be a positive integer. Let $\mathcal{P}_k$ be a subset of $\mathcal{H}_k$. Then there is a constant $A(p, k)$ independent of the dimension d and such that*

$$\bigg\| \bigg(\sum_{P \in \mathcal{P}_k} |R_P^* f|^2 \bigg)^{1/2} \bigg\|_{L^p(\mathbb{R}^d)} \leqslant A(p, k) \bigg\| \bigg(\sum_{P \in \mathcal{P}_k} |R_P f|^2 \bigg)^{1/2} \bigg\|_{L^p(\mathbb{R}^d)},$$

where $f \in L^p(\mathbb{R}^d)$. Moreover, for fixed k we have

$$A(p, k) = O(p^{5/2+k/2}) \quad \text{as } p \to \infty \qquad \text{and} \qquad A(p, k) = O((p-1)^{-5/2-k/2}) \quad \text{as } p \to 1.$$

In particular, if $\mathcal{P}_k$ contains one element P, then Theorem 1.1 immediately gives

$$\|R_P^* f\|_{L^p(\mathbb{R}^d)} \leqslant A(p, k) \|R_P f\|_{L^p(\mathbb{R}^d)}.$$

In this case however, we can slightly improve the constant $A(p, k)$. This is due to the fact that in the proof of Theorem 1.2 below we do not need to apply Khintchine's inequalities twice, which is an important ingredient in the proof of Theorem 1.1.

Theorem 1.2. *Take $p \in (1, \infty)$ and let $k \leqslant d$ be a positive integer. Let P be a spherical harmonic of degree k. Then there is a constant $B(p, k)$ independent of the dimension d and such that*

$$\|R_P^* f\|_{L^p(\mathbb{R}^d)} \leqslant B(p, k) \|R_P f\|_{L^p(\mathbb{R}^d)},$$

where $f \in L^p(\mathbb{R}^d)$. Moreover, for fixed k we have

$$B(p, k) = O(p^{2+k/2}) \quad \text{as } p \to \infty \qquad \text{and} \qquad B(p, k) = O((p-1)^{-2-k/2}) \quad \text{as } p \to 1.$$

Our last main result follows from a combination of Theorem 1.1 with a result of Duoandikoetxea and Rubio de Francia [1985, théorème 2]. Denote by $a(d, k)$ the dimension of $\mathcal{H}_k$ and let $\{Y_j\}_{j=1,\dots,a(d,k)}$ be an orthogonal basis of $\mathcal{H}_k$ normalized by the condition

$$\frac{1}{\sigma(S^{d-1})} \int_{S^{d-1}} |Y_j(\theta)|^2 \, d\sigma(\theta) = \frac{1}{a(d, k)};$$

here $d\sigma$ denotes the (unnormalized) spherical measure.

Corollary 1.3. *Take $p \in (1, \infty)$ and let $k \leqslant d$ be a positive integer. Then there is a constant $G(p, k)$ independent of the dimension d and such that*

$$\bigg\| \bigg(\sum_{j=1}^{a(d,k)} |R_{Y_j}^* f|^2 \bigg)^{1/2} \bigg\|_{L^p(\mathbb{R}^d)} \leqslant G(p, k) \|f\|_{L^p(\mathbb{R}^d)},$$

where $f \in L^p(\mathbb{R}^d)$. Moreover, for fixed and odd k we have

$$G(p,k) = O(p^{7/2+k/2}) \quad \text{as } p \to \infty \qquad \text{and} \qquad G(p,k) = O((p-1)^{-7/2-k}) \quad \text{as } p \to 1,$$

and for even k we have

$$G(p,k) = O(p^{9/2+k/2}) \quad \text{as } p \to \infty \qquad \text{and} \qquad G(p,k) = O((p-1)^{-9/2-k}) \quad \text{as } p \to 1.$$

We finish this section with two remarks.

Remark 1. Corollary 1.3 seems interesting by itself. In the particular case of $k = 1$ it is a direct maximal function counterpart of Stein's inequality (1-4), namely

$$\left\| \left(\sum_{j=1}^{d} |R_j^* f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)} \leqslant C_p \|f\|_{L^p(\mathbb{R}^d)}, \quad 1 < p < \infty. \tag{1-6}$$

It is unclear to us if one can prove Corollary 1.3 or even inequality (1-6) without using Theorem 1.1 as an intermediate step.

Remark 2. We do not know what the sharp rates of the constants $A(p,k)$, $B(p,k)$ and $G(p,k)$ are in terms of p. However, the results of [Mateu et al. 2010; 2011; Mateu and Verdera 2006] suggest that

$$B(p,k) = O(\max(1, (p-1)^{-2})) \quad \text{and} \quad B(p,k) = O(\max(1, (p-1)^{-1}))$$

might be the optimal rates for fixed odd and even k, respectively. In fact, such estimates do follow from these papers at the price of involving upfront constants which depend on the dimension. When $k = 1$, then from [Liu et al. 2024] we indeed have $B(p,1) = O(1)$ as $p \to \infty$. The same holds for $B(p,2)$ because in this case the transition kernel M_2^t (see (1-8)) coincides with the centered Hardy–Littlewood averaging operator over the balls.

We are also unaware what the optimal constants $H(p,k,d)$ in the inequalities for truncated maximal Riesz transforms

$$\|R_P^* f\|_{L^p(\mathbb{R}^d)} \leqslant H(p,k,d) \|f\|_{L^p(\mathbb{R}^d)}$$

are even in the case $k = 1$. In view of the above comment about $B(p,1)$ and known optimal bounds for the Riesz transforms from [Iwaniec and Martin 1996], it might be possible to achieve $H(p,1,d) = O(\max(p, (p-1)^{-3}))$. However, even if this is true, it would not be optimal in dimension $d = 1$ for $p \to 1$. Then the maximal Riesz transform is the maximal Hilbert transform and the sharp order of its $L^p(\mathbb{R})$ norm is $\max(p, (p-1)^{-1})$.

1.1. ***Structure of the paper and our methods.*** The proofs of Theorems 1.1 and 1.2. require four main ingredients.

First, we need a factorization of the truncated Riesz transform $R_P^t = M_k^t(R_P)$. Here, M_k^t, $t > 0$, is a family of radial Fourier multiplier operators. In the case $k = 1$ this factorization on the multiplier level has been one of the key steps in establishing the main results of [Kucharski and Wróbel 2023]. In particular, the operator M_1^t considered here coincides with M^t defined in [Kucharski and Wróbel

2023, equation (3.5)]. For general values of k the factorization on a kernel level is implicit in [Mateu and Verdera 2006, Section 2] ($k=1$), [Mateu et al. 2011, Section 2] (k even), and [Mateu et al. 2010, Section 4] (k odd). In this paper we utilize the factorization on an operator level via an explicit formula for the operator M_k^t in terms of the Riesz transforms R_P and the truncated Riesz transforms R_P^t. Note that for the first-order Riesz transforms ($k=1$) the formulas $R_j^t = M_1^t(R_j)$, $j=1,\dots,d$, together with the identity $I = -\sum_{j=1}^d (R_j)^2$ imply that

$$M_1^t = -\sum_{j=1}^d M_1^t (R_j)^2 = -\sum_{j=1}^d R_j^t R_j. \tag{1-7}$$

It seems that formula (1-7) and its variants for $k>1$ have not been considered before. Yet, they are invaluable when one is interested in dimension-free $L^p(\mathbb{R}^d)$ bounds for M_k^t. Details of the factorization procedure are given in Section 2.

The second ingredient we need is an averaging procedure. It turns out that a useful analogue of (1-7) is not directly available for Riesz transforms of orders higher than 1. The reason behind it is the fact that not all compositions of first-order Riesz transforms are higher-order Riesz transforms according to our definition. For instance, in the case $k=3$ the multiplier symbol of $(R_1)^3 = R_1R_1R_1$ on $L^2(\mathbb{R}^2)$ equals $-i\xi_1^3/|\xi|^3$ and $P(\xi) = -i\xi_1^3$ is not a spherical harmonic. However, the formula

$$I = -\sum_{j_1=1}^d \sum_{j_2=1}^d \sum_{j_3=1}^d (R_{j_1})^2 (R_{j_2})^2 (R_{j_3})^2$$

includes squares of all compositions of Riesz transforms including $(R_1)^6 = ((R_1)^3)^2$. Therefore the above formula does not directly lead to an expression of M_k^t in terms of R_P^t and R_P. To overcome this problem we average over the special orthogonal group $\mathrm{SO}(d)$. Then we obtain

$$M_k^t f(x) = C(d,k) \int_{\mathrm{SO}(d)} \sum_{j\in I} (R_j^t R_j)_U f(x)\, d\mu(U); \tag{1-8}$$

see Proposition 3.1. Here T_U is the conjugation of an operator T by $U \in \mathrm{SO}(d)$, see (3-1), $d\mu$ denotes the normalized Haar measure on $\mathrm{SO}(d)$, while $C(d,k)$ is a constant. The symbol I denotes the set of multi-indices $j = (j_1,\dots,j_k)$ with increasing components while R_j^t and R_j are the truncated Riesz transforms and the Riesz transforms (1-2) corresponding to the monomials $P_j(x) = x_{j_1}\cdots x_{j_d}$. Note that since $j \in I$ the polynomials P_j are spherical harmonics and thus the operators R_j are indeed higher-order Riesz transforms. In view of (1-8), if we demonstrate that $C(d,k)$ is bounded by a universal constant, we are left with estimating the maximal function corresponding to $\sum_{j\in I} R_j^t R_j$. The reduction via the averaging procedure is described in detail in Section 3. It is noteworthy that in order for the averaging approach to work it is essential that for each order k the multiplier symbols of M_k^t are radial functions.

The third main ingredient of our argument is an extension to $\mathbb{C}^d$ followed by the complex method of rotations of Iwaniec and Martin [1996]. We use the complex method of rotations to estimate the maximal function $\widetilde{R}^*$ corresponding to

$$\widetilde{R}^t := \sum_{j\in I} \widetilde{R}_j^t \widetilde{R}_j. \tag{1-9}$$

Here $\widetilde{R}_j^t$ and $\widetilde{R}_j$ denote extensions to $\mathbb{C}^d$ of the truncated Riesz transform R_j^t and the Riesz transform R_j. The definition of $\widetilde{R}_j$ can be given on the multiplier level according to the scheme from [Iwaniec and Martin 1996]. We note, however, that the truncated extended operator $\widetilde{R}_j^t$ needs to be defined differently — on a kernel level. In the context of dimension-free estimates for Riesz transforms the real method of rotations has been employed by Duoandikoetxea and Rubio de Francia [1985]. However, as it can be applied only to operators with odd kernels, for the general case we need the complex version. The method of rotations itself is preceded by a number of other ingredients. In particular we need L^p vector-valued estimates for the maximal directional truncated k-th power of the complex Hilbert transform, see Proposition 4.3, and for the vector of higher-order Riesz transforms, see Proposition 4.4. En route to obtain these results we also need Khintchine's inequalities and specific computations. All of it reflects the size of the constants $A(p,k)$ in Theorem 1.1 and $B(p,k)$ in Theorem 1.2. The extension procedure and the application of the complex method of rotations are described in detail in Section 4.

The last ingredient is a restriction procedure. This allows us to deduce the estimates for R^* on $\mathbb{R}^d$ from the estimates for $\widetilde{R}^*$ on $\mathbb{C}^d$. The restriction of the complex Riesz transforms $\widetilde{R}_j$ in (1-9) can be done on the multiplier level as in [Iwaniec and Martin 1996, Chapter 4]. However, in order to restrict $\widetilde{R}_j^t$ and the maximal function $\widetilde{R}^*$ we need to work on the kernel level. A problem that we encounter here is that the resulting restricted operator of $\widetilde{R}^*$ is not the same as the desired maximal operator R^*. Therefore we need to investigate their difference and estimate it appropriately. The restriction procedure is described in Section 5.

At the first reading it might be helpful to skip the explicit values of constants in terms of k and p and only focus on these constants being independent of the dimension d. An interested reader may trace the exact dependencies of the constants in terms of k and p in the paper.

1.2. *Notation.* We finish the introduction with a description of the notation and conventions used in the rest of the paper.

(1) The letters d and k stand for the dimension and for the order of the Riesz transforms, respectively. In particular we always have $k \leqslant d$, even if this is not stated explicitly.

(2) The symbol $\mathbb{N}$ represents the set of positive integers. Throughout the paper we assume that $k \in \mathbb{N}$. We write $\mathbb{Q}_+$ for the set of positive rational numbers.

(3) By $[d]$ we denote the set $\{1, \dots, d\}$ of positive integers up to d.

(4) For an exponent $p \in [1,\infty]$ we let q be its conjugate exponent satisfying

$$1 = \frac{1}{p} + \frac{1}{q}.$$

When $p \in (1,\infty)$ we set

$$p^* := \max(p, (p-1)^{-1}).$$

(5) We abbreviate $L^p(\mathbb{R}^d)$ to L^p and $\|\cdot\|_{L^p}$ to $\|\cdot\|_p$. For a sublinear operator T on L^p we denote by $\|T\|_{p\to p}$ its norm. We let $\mathcal{S}(\mathbb{R}^d)=\mathcal{S}$ be the space of Schwartz functions on $\mathbb{R}^d$. Slightly abusing the notation we say that a sublinear operator T is bounded on L^p if it is bounded on $\mathcal{S}$ in the L^p norm.

(6) For $k\in\mathbb{N}$ we let $\mathcal{D}(k)$ be the linear span of $\{R_P(f) : P\in\mathcal{H}_k,\ f\in\mathcal{S}\}$. Since R_P is bounded on L^p for $1<p<\infty$, the space $\mathcal{D}(k)$ is then a subspace of each of the L^p spaces.

(7) For a Banach space E the symbol $L^p(\mathbb{R}^d; E)$ stands for the space of weakly measurable functions $f:\mathbb{R}^d\to E$ with the norm $\|f\|_{L^p(\mathbb{R}^d;E)}=\bigl(\int_{\mathbb{R}^d}\|f(x)\|_E^p\,dx\bigr)^{1/p}$. Similarly, for a finite set F, by $\ell^p(F;E)$ we denote the Banach space of E-valued sequences $\{f_s\}_{s\in F}$ with the norm $\|f\|_{\ell^p(F;E)}=\bigl(\sum_{s\in F}\|f_s\|_E^p\bigr)^{1/p}$.

(8) The symbol C_Δ stands for a constant that possibly depends on $\Delta>0$. We write C without a subscript when the constant is universal in the sense that it may depend only on k but not on the dimension d nor on any other quantity.

(9) For two quantities X and Y we write $X\lesssim_\Delta Y$ if $X\leqslant C_\Delta Y$ for some constant $C_\Delta>0$ that depends only on Δ. We abbreviate $X\lesssim Y$ when C is a universal constant. We also write $X\sim Y$ if both $X\lesssim Y$ and $Y\lesssim X$ hold simultaneously. By $X\lesssim^\Delta Y$ we mean that $X\leqslant C^\Delta Y$ with a universal constant C. Note that in this case $X^{1/\Delta}\lesssim Y^{1/\Delta}$.

(10) The symbol S^{d-1} stands for the $(d-1)$-dimensional unit sphere in $\mathbb{R}^d$ and by ω we denote the uniform measure on S^{d-1} normalized by the condition $\omega(S^{d-1})=1$. We also write

$$S_{d-1}=\frac{2\pi^{d/2}}{\Gamma\bigl(\tfrac12 d\bigr)}\tag{1-10}$$

to denote the unnormalized surface area of S^{d-1}. We write ζ for the uniform measure on S^{2d-1} normalized by the condition $\zeta(S^{2d-1})=1$.

(11) We let

$$\gamma_k=\gamma_{k,d}:=\frac{\Gamma\bigl(\tfrac12(k+d)\bigr)}{\pi^{d/2}\Gamma\bigl(\tfrac12 k\bigr)}\quad\text{and}\quad\widetilde{\gamma}_k=\gamma_{k,2d}=\frac{\Gamma\bigl(d+\tfrac12 k\bigr)}{\pi^{d}\Gamma\bigl(\tfrac12 k\bigr)}.\tag{1-11}$$

(12) The Fourier transform is defined for $f\in L^1$ and $\xi\in\mathbb{R}^d$ by the formula

$$\mathcal{F}f(\xi)=\hat{f}(\xi)=\int_{\mathbb{R}^d}f(x)e^{-2\pi i x\cdot\xi}\,dx.$$

(13) The Gamma function is defined for $s>0$ by the formula

$$\Gamma(s)=\int_0^\infty t^{s-1}e^{-t}\,dt.$$

We shall use Stirling's approximation for $\Gamma(s)$:

$$\Gamma(s)\sim\sqrt{2\pi}\,s^{s-1/2}e^{-s},\quad s\to\infty.\tag{1-12}$$

A useful consequence of (1-12) is the formula

$$\Gamma(s+\alpha)\sim s^\alpha\Gamma(s),\quad s\to\infty,\tag{1-13}$$

which is valid for each fixed $\alpha\geq 0$.

(14) We will also need the formula

$$2\int_0^\infty \frac{r^{d-1}}{(1+r^2)^{d+\alpha}}\,dr = B\big(\tfrac12 d,\,\tfrac12 d+\alpha\big) = \frac{\Gamma\big(\frac12 d\big)\Gamma\big(\frac12 d+\alpha\big)}{\Gamma(d+\alpha)}, \tag{1-14}$$

valid for $\alpha \geq 0$. This follows from the change of variables $r^2 \to r$ followed by formulas for Euler's beta function $B(a,b)$ from [Olver et al. 2010, 5.12.1, 5.12.3].

2. Factorization

The goal of this section is to show that a factorization formula for R_P^t in terms of R_P is feasible. Proposition 2.1 below is implicit in [Mateu et al. 2010, Section 4; 2011, pp. 1435–1436].

Proposition 2.1. *Let $k \in \mathbb{N}$. Then there exists a family of operators M_k^t, $t > 0$, which are bounded on L^p, $1 < p < \infty$, and such that for all $P \in \mathcal{H}_k$ we have*

$$R_P^t f = M_k^t(R_P f), \tag{2-1}$$

where $f \in L^p$. Each M_k^t is a convolution operator with a radial convolution kernel b_k^t.

Proof. We consider separately the cases of k odd or even starting with k odd.

Let

$$c_d = \frac{\Gamma\big(\frac12(d-1)\big)}{2\pi^{d/2}\Gamma\big(\frac12\big)}, \qquad N = \tfrac12(k-1),$$

and denote by B the open Euclidean ball of radius 1 in $\mathbb{R}^d$. In [Mateu et al. 2010, pp. 3674–3675] it is justified that the function

$$b(x) = b_{k,d}(x) := \sum_{j=1}^d R_j[y_j \cdot h(y)](x), \tag{2-2}$$

where

$$h(y) = c_d(1-d)\frac{1}{|y|^{d+1}}\mathbb{1}_{B^c}(y) + (\beta_1 + \beta_2|y|^2 + \cdots + \beta_N|y|^{2N-2})\mathbb{1}_B(y),$$

satisfies the formula

$$R_P(b)(x) = K_P(x)\mathbb{1}_{B^c}. \tag{2-3}$$

Here $\beta_1, \ldots, \beta_N$ are constants which depend only on k and d and whose exact value is irrelevant for our considerations, and K_P and R_P have been defined in (1-1) and (1-2), respectively. The important point is that (2-3) remains true for any $P \in \mathcal{H}_k$.

Denote by H the radial profile of the Fourier transform of h, i.e., $H(|\xi|) = \hat{h}(\xi)$ for $\xi \in \mathbb{R}^d$. By taking the Fourier transform of (2-2) it is straightforward to see that b is a radial function. This follows since the multiplier symbol of R_j is $-i\xi_j/|\xi|$ and

$$\widehat{(y_j h(y))}(\xi) = \frac{\xi_j}{-2\pi i|\xi|}\,H'(|\xi|),$$

so that

$$\mathcal{F}b(\xi) = \sum_{j=1}^{d} \frac{\xi_j^2}{2\pi|\xi|^2} \cdot H'(|\xi|) = \frac{1}{2\pi} H'(|\xi|)$$

is indeed radial and so is b.

Let $b^t(x) = b_k^t(x) := t^{-d} b(x/t)$ be the L^1 dilation of b; clearly b^t is still radial. The dilation invariance of R_P together with (2-3) leads us to the expression

$$K_P(x)\mathbb{1}_{B^c}(x/t) = R_P(b^t)(x). \tag{2-4}$$

Let M_k^t be the convolution operator

$$M_k^t f(x) = b^t * f(x).$$

It follows from [Mateu et al. 2010, Section 4] that M_k^t is bounded on L^p spaces whenever $1 < p < \infty$. Moreover, in view of (2-4) we see that

$$R_P^t f = R_P(b^t) * f = b^t * R_P(f) = M_k^t(R_P f).$$

It remains to consider k even. Let $N = k/2$. From (10) and (12) in [Mateu et al. 2011, pp. 1435–1436] it follows that the function

$$b(x) = b_{k,d}(x) := \big(\alpha_0 + \alpha_1|x|^2 + \cdots + \alpha_{N-1}|x|^{2(N-1)}\big)\mathbb{1}_B(x)$$

satisfies the formula

$$R_P(b)(x) = K_P(x)\mathbb{1}_{B^c}(x). \tag{2-5}$$

Here $\alpha_1, \ldots, \alpha_{N-1}$ are constants which depend only on k and d and whose exact value is irrelevant for our considerations. As in the case of odd k, the important point is that (2-5) remains true for any $P \in \mathcal{H}_k$.

Using (2-5) we proceed as in the proof in the case when k. Let $b^t(x) = b_k^t(x) := t^{-d} b(x/t)$ be the L^1 dilation of b. Since b is clearly radial the same is true of b^t. Let M_k^t be the convolution operator

$$M_k^t f(x) = b^t * f(x).$$

It follows from [Mateu et al. 2011, Section 2] that M_k^t is bounded on L^p spaces whenever $1 < p < \infty$. Moreover, in view of (2-5) we see that

$$R_P^t f = R_P(b^t) * f = b^t * R_P(f) = M_k^t(R_P f). \qquad \square$$

As a corollary of Proposition 2.1 we see that in order to justify Theorems 1.1 and 1.2 it suffices to control vector- and scalar-valued maximal functions corresponding to the operators M_k^t. Note that for $f \in \mathcal{S}$, $P \in \mathcal{H}_k$, and $x \in \mathbb{R}^d$, the mapping $t \mapsto R_P^t f(x)$ is continuous on $(0, \infty)$ and thus, by (2-1), so is $t \mapsto M_k^t(R_P f)(x)$. Consequently, for $f \in \mathcal{D}(k)$ (see (6) in the notation section) we have

$$\sup_{t>0} |M_k^t f(x)| = \sup_{t \in \mathbb{Q}_+} |M_k^t f(x)|.$$

In particular $\sup_{t>0}|M_k^t f(x)|$ is measurable for such f, although possibly infinite for some x. Define

$$M_k^* f(x) = \sup_{t\in\mathbb{Q}_+} |M_k^t f(x)|. \tag{2-6}$$

Proposition 2.1 reduces our task to proving the following two theorems.

Theorem 2.2. *Fix $k\in\mathbb{N}$. For each $p\in(1,\infty)$ there is a constant $A(p,k)$ independent of the dimension d and such that for any $S\in\mathbb{N}$ we have*

$$\Big\|\Big(\sum_{s=1}^{S}|M_k^* f_s|^2\Big)^{1/2}\Big\|_p \leqslant A(p,k)\Big\|\Big(\sum_{s=1}^{S}|f_s|^2\Big)^{1/2}\Big\|_p,$$

where $f_1,\dots,f_S\in L^p$. Furthermore $A(p,k)$ satisfies $A(p,k)\lesssim_k (p^*)^{5/2+k/2}$.

Theorem 2.3. *Fix $k\in\mathbb{N}$. For each $p\in(1,\infty)$ there is a constant $B(p,k)$ independent of the dimension d and such that*

$$\|M_k^* f\|_p \leqslant B(p,k)\|f\|_p$$

whenever $f\in L^p$. Moreover $B(p,k)$ satisfies $B(p,k)\lesssim_k (p^*)^{2+k/2}$.

3. Averaging

In this section we describe the averaging procedure. The averaging procedure will allow us to pass from M_k^* to another maximal operator that is better suited for applications in Sections 4 and 5. Before moving on, we establish some notation. We define the set I of multi-indices with increasing components as

$$I=\big\{j\in\{1,\dots,d\}^k : j_i<j_l \text{ for } i<l\big\}.$$

For a multi-index $j=(j_1,\dots,j_k)\in I$ we write

$$P_j(x)=x_j := x_{j_1}\cdots x_{j_k}$$

and denote by R_j the Riesz transform R_{P_j} associated with the monomial P_j. The truncated transform R_j^t and the maximal transform R_j^* are defined analogously. We also abbreviate $K_j(x)=K_{P_j}(x)$ and $K_j^t(x)=K_{P_j}^t(x)$.

The averaging procedure will provide an expression for M_k^t in terms of the Riesz transforms R_j and R_j^t postulated in (1-8). For $f\in L^p$, $1<p<\infty$, let

$$R^t f := \sum_{j\in I} R_j^t R_j f \quad\text{and}\quad R^* f := \sup_{t\in\mathbb{Q}_+}|R^t f|.$$

Note that both R^t and R^* are well defined on all L^p spaces. Indeed, R_j^t and R_j are bounded on L^p and the supremum in the definition of R^* runs over a countable set thus defining a measurable function.

Let $\mathrm{SO}(d)$ be the special orthogonal group in dimension d. Since it is compact, it has a bi-invariant Haar measure μ such that $\mu(\mathrm{SO}(d))=1$. For $U\in\mathrm{SO}(d)$ and a sublinear operator T on L^2 we denote by

T_U the conjugation by U, i.e., the operator acting via

$$T_U f(x) = T(f(U^{-1}\cdot))(Ux). \tag{3-1}$$

Proposition 3.1. *Fix $k \in \mathbb{N}$. Then there is a constant $C(d,k) \in \mathbb{R}$ such that*

$$M_k^t f(x) = C(d,k) \int_{\mathrm{SO}(d)} [(R^t)_U f](x)\, d\mu(U) \tag{3-2}$$

for all $t > 0$ and $f \in L^p$. Moreover, $|C(d,k)|$ has an estimate from above by a constant that depends only on k but not on the dimension d, so that

$$\left(\sum_{s=1}^{S} |M_k^* f_s(x)|^2\right)^{1/2} \lesssim \int_{\mathrm{SO}(d)} \left(\sum_{s=1}^{S} |[(R^*)_U f_s](x)|^2\right)^{1/2} d\mu(U) \tag{3-3}$$

for $S \in \mathbb{N}$ and $f_1, \dots, f_S \in L^p$.

Proof. Let A be the operator

$$A = \sum_{j \in I} (R_j)^2, \tag{3-4}$$

which by (1-3) means that its multiplier symbol equals

$$a(\xi) = (-i)^{2k} \sum_{j\in I} \frac{\xi_j^2}{|\xi|^{2k}} = (-1)^k \sum_{j \in I} \frac{\xi_j^2}{|\xi|^{2k}}.$$

Let $\tilde{A}$ be the operator with the multiplier symbol

$$\tilde{a}(\xi) := \int_{\mathrm{SO}(d)} a(U\xi)\, d\mu(U) = (-1)^k \sum_{j\in I} \int_{\mathrm{SO}(d)} \frac{((U\xi)_j)^2}{|\xi|^{2k}}\, d\mu(U). \tag{3-5}$$

Then $\tilde{a}$ is radial, and since it is homogeneous of order 0, it is constant.

Let now m^t be the multiplier symbol of M_k^t. Then, from Proposition 2.1 we see that $m^t = \hat{b}^t$ is radial, so that

$$m^t(\xi) = \tilde{a}^{-1}\tilde{a} m^t(\xi) = \tilde{a}^{-1} \int_{\mathrm{SO}(d)} m^t(\xi) a(U\xi)\, d\mu(U) = \tilde{a}^{-1} \int_{\mathrm{SO}(d)} m^t(U\xi) a(U\xi)\, d\mu(U).$$

Using properties of the Fourier transform the above equality implies that

$$M_k^t f(x) = \tilde{a}^{-1} \int_{\mathrm{SO}(d)} [(M_k^t A)_U](f)(x)\, d\mu(U).$$

Recalling (3-4) we apply (2-1) from Proposition 2.1 and obtain

$$M_k^t A = \sum_{j\in I} M_k^t R_j R_j = \sum_{j \in I} R_j^t R_j = R^t;$$

here an application of (2-1) is allowed since each R_j corresponds to the monomial x_j which is in $\mathcal{H}_k$ when $j \in I$. In summary, we justified that

$$M_k^t f(x) = \tilde{a}^{-1} \int_{\mathrm{SO}(d)} [(R^t)_U](f)(x)\, d\mu(U), \qquad f \in \mathcal{D}(k), \tag{3-6}$$

which is (3-2) with $C(d,k) = \tilde{a}^{-1}$.

Now we need to prove that

$$|C(d,k)| = |\tilde{a}|^{-1} \sim 1 \tag{3-7}$$

uniformly in the dimension d. Note that each of the integrals on the right-hand side of (3-5) has the same value independent of $j \in I$, so that

$$\tilde{a}(\xi) = (-1)^k |I| \int_{\mathrm{SO}(d)} \frac{((U\xi)_{(1,\dots,k)})^2}{|\xi|^{2k}}\, d\mu(U);$$

here $|I|$ stands for the number of elements in I. Since $\tilde{a}$ is constant, we can integrate it over S^{d-1} with probabilistic measure and change the order of integration to get

$$\tilde{a} = (-1)^k |I| \int_{S^{d-1}} \int_{\mathrm{SO}(d)} (U\omega)^2_{(1,\dots,k)}\, d\mu(U)\, d\omega = (-1)^k |I| \int_{\mathrm{SO}(d)} \int_{S^{d-1}} (U\omega)^2_{(1,\dots,k)}\, d\omega\, d\mu(U).$$

Now notice that the inner integral does not depend on U, which means that

$$\tilde{a} = (-1)^k |I| \int_{S^{d-1}} \omega_1^2 \cdots \omega_k^2\, d\omega. \tag{3-8}$$

Since k is fixed, by an elementary argument we get $|I| = \binom{d}{k} \sim d^k$. Thus it remains to show that

$$\int_{S^{d-1}} \omega_1^2 \cdots \omega_k^2\, d\omega \sim d^{-k}. \tag{3-9}$$

Formula (3-9) is given in [Sýkora 2005, (10)]. It can be also easily computed by the method from [Hörmander 2003, Chapter 3.4]; for the sake of completeness we provide a brief argument. Consider the integral $J = \int_{\mathbb{R}^d} x_1^2 \cdots x_k^2 e^{-|x|^2}\, dx$. Since J is a product of the one-dimensional integrals we calculate $J = \Gamma\big(\frac{3}{2}\big)^k \Gamma\big(\frac{1}{2}\big)^{d-k}$, while using polar coordinates gives $J = S_{d-1} \int_{S^{d-1}} \omega_1^2 \cdots \omega_k^2\, d\omega \int_0^\infty r^{2k+d-1} e^{-r^2}\, dr$, where S_{d-1} is defined by (1-10). Altogether we have justified that

$$\int_{S^{d-1}} \omega_1^2 \cdots \omega_k^2\, d\omega \sim \frac{\Gamma\big(\frac{1}{2}\big)^{d-k}}{S_{d-1} \Gamma\big(k + \frac{1}{2}d\big)}.$$

Since k is fixed and d is arbitrarily large, using (1-10), Stirling's formula for the Γ function (1-12) and the known identity $\Gamma\big(\frac{1}{2}\big) = \sqrt{\pi}$ we obtain

$$\int_{S^{d-1}} \omega_1^2 \cdots \omega_k^2\, d\omega \sim \frac{\sqrt{k + \frac{1}{2}d}\,(d/(2e))^{d/2}}{\sqrt{\frac{1}{2}d}\,((k+d/2)/e)^{k+d/2}} \sim \frac{e^{-d/2}}{e^{-k-d/2}} \left(\frac{k+d/2}{d/2}\right)^{-d/2} (k+d/2)^{-k} \sim d^{-k}.$$

This gives (3-9) and concludes the proof of (3-7).

It remains to justify (3-3). This follows from (2-6), (3-6), and (3-7), together with the norm inequality

$$\left\| \int_{\mathrm{SO}(d)} F_{s,t}(U)\, d\mu(U) \right\|_X \leqslant \int_{\mathrm{SO}(d)} \|F_{s,t}(U)\|_X \, d\mu(U)$$

on the Banach space $X = \ell^2(\{1, \dots, S\}; \ell^\infty(\mathbb{Q}_+))$, with $F_{s,t}(U) = (R^t)_U(f_s)(x)$ and x being fixed. □

Since conjugation by $U \in \mathrm{SO}(d)$ is an isometry on all L^p spaces, in view of $\mu(\mathrm{SO}(d)) = 1$ and Minkowski's integral inequality, equation (3-3) of Proposition 3.1 allows us to deduce Theorems 2.2 and 2.3 from the two theorems below.

Theorem 3.2. *Fix $k \in \mathbb{N}$. For each $p \in (1, \infty)$ there is a constant $A(p, k)$ independent of the dimension d and such that for any $S \in \mathbb{N}$ we have*

$$\left\| \left(\sum_{s=1}^{S} |R^* f_s|^2 \right)^{1/2} \right\|_p \lesssim A(p, k) \left\| \left(\sum_{s=1}^{S} |f_s|^2 \right)^{1/2} \right\|_p,$$

where $f_1, \dots, f_S \in L^p$. Moreover, $A(p, k)$ satisfies $A(p, k) \lesssim_k (p^*)^{5/2+k/2}$.

Theorem 3.3. *Fix $k \in \mathbb{N}$. For each $p \in (1, \infty)$ there is a constant $B(p, k)$ independent of the dimension d and such that*

$$\|R^* f\|_p \lesssim B(p, k) \|f\|_p$$

whenever $f \in L^p$. Moreover, $B(p, k)$ satisfies $B(p, k) \lesssim_k (p^*)^{2+k/2}$.

4. Extension to $\mathbb{C}^d$ and the complex method of rotations

Here we extend the operators R^t acting on $L^p(\mathbb{R}^d)$ to the operators $\widetilde{R}^t$ acting on $L^p(\mathbb{C}^d)$. Then we apply the complex method of rotations of Iwaniec and Martin [1996] to $\widetilde{R}^t$.

Let $P \in \mathcal{H}_k$. For $z = (x_1 + iy_1, \dots, x_d + iy_d)$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^d$ we denote

$$\widetilde{K}_P(z) = \widetilde{\gamma}_k \frac{P(z)}{|z|^{2d+k}} \quad \text{with } \widetilde{\gamma}_k = \frac{\Gamma(d + k/2)}{\pi^d \Gamma(k/2)}, \tag{4-1}$$

and define, for $f \in \mathcal{S}(\mathbb{C}^d)$,

$$\widetilde{R}_P f(z) = \lim_{t \to 0} \widetilde{R}^t_P f(z), \quad \text{where } \widetilde{R}^t_P f(z) = \widetilde{\gamma}_k \int_{w \in \mathbb{C}^d : |w| > t} \frac{P(w)}{|w|^{2d+k}} f(z - w)\, dw. \tag{4-2}$$

Iwaniec and Martin [1996] considered the extension on the multiplier level whereas here we need to write it on the kernel level. This makes no difference for the operator $\widetilde{R}_P$. However, the multiplier symbol corresponding to the truncated transform $\widetilde{R}^t_P$ does not have a simple formula, thus writing the extension on a kernel level seems the only reasonable option here.

Formulas (4-1) and (4-2) lead us to define the extension of R^t by

$$\widetilde{R}^t = \widetilde{R}^t_k := \sum_{j \in I} \widetilde{R}^t_j \widetilde{R}_j. \tag{4-3}$$

Using the complex method of rotations [Iwaniec and Martin 1996, Section 6] we will prove $L^p(\mathbb{C}^d)$ estimates for

$$\widetilde{R}^* f(z) = \sup_{t \in \mathbb{Q}_+} |\widetilde{R}^t f(z)|.$$

Theorem 4.1. *Fix $k \in \mathbb{N}$. For each $p \in (1, \infty)$ there is a constant $A(p, k)$ independent of the dimension d and such that for any $S \in \mathbb{N}$ we have*

$$\left\| \left(\sum_{s=1}^{S} |\widetilde{R}^* f_s|^2 \right)^{1/2} \right\|_{L^p(\mathbb{C}^d)} \leqslant A(p, k) \left\| \left(\sum_{s=1}^{S} |f_s|^2 \right)^{1/2} \right\|_{L^p(\mathbb{C}^d)}$$

whenever $f_1, \ldots, f_S \in L^p(\mathbb{C}^d)$. Moreover, $A(p, k)$ satisfies $A(p, k) \lesssim_k (p^*)^{5/2+k/2}$.

Theorem 4.2. *Fix $k \in \mathbb{N}$. For each $p \in (1, \infty)$ there is a constant $B(p, k)$ independent of the dimension d and such that*

$$\|\widetilde{R}^* f\|_{L^p(\mathbb{C}^d)} \leqslant B(p, k) \|f\|_{L^p(\mathbb{C}^d)}$$

whenever $f \in L^p(\mathbb{C}^d)$. Moreover, $B(p, k)$ satisfies $B(p, k) \lesssim_k (p^*)^{2+k/2}$.

The reminder of this section will be devoted to the proofs of Theorems 4.1 and 4.2. From these results we shall obtain Theorems 2.2 and 2.3 provided we develop a restriction procedure from $\mathbb{C}^d$ to $\mathbb{R}^d$. As we already remarked this is not straightforward, since the restriction of the complex truncated Riesz transform is not the real truncated Riesz transform. Details of the restriction and estimates for the resulting operators are given in Section 5.

We now focus on the proofs of Theorems 4.1 and 4.2. Let $P \in \mathcal{H}_k$. Note that

$$2\pi \int_{\mathbb{C}^d} F(w)\, dw = \int_{S^{2d-1}} \int_{\mathbb{C}} F(\lambda\theta) |\lambda|^{2d-2}\, d\lambda\, d\theta,$$

where $F \in \mathcal{S}(\mathbb{C}^d)$ and $d\theta$ stands for the spherical measure on S^{2d-1} normalized by the condition $\theta(S^{2d-1}) = S_{2d-1}$. Take $f \in \mathcal{S}(\mathbb{C}^d)$. Applying the above identity with

$$F(w) = \widetilde{\gamma}_k \frac{P(w)}{|w|^{2d+k}} \mathbb{1}_{|w| \geq t} f(z - w)$$

gives

$$\begin{aligned} \widetilde{R}^t_P f(z) &= \widetilde{\gamma}_k \int_{\mathbb{C}^d} \frac{P(w)}{|w|^{2d+k}} \mathbb{1}_{|w| \geq t} f(z - w)\, dw \\ &= \frac{\widetilde{\gamma}_k}{2\pi} \int_{S^{2d-1}} \int_{\mathbb{C}} \frac{P(\lambda\theta)}{|\lambda|^{2d+k}} \mathbb{1}_{|\lambda| \geq t} f(z - \lambda\theta) |\lambda|^{2d-2}\, d\lambda\, d\theta \\ &= \frac{\widetilde{\gamma}_k}{2\pi} \int_{S^{2d-1}} P(\theta) \int_{\mathbb{C}} \left(\frac{\lambda}{|\lambda|} \right)^k \frac{f(z - \lambda\theta)}{|\lambda|^2} \mathbb{1}_{|\lambda| \geq t}\, d\lambda\, d\theta, \end{aligned}$$

where in the last equality above we used the k-homogeneity of P. This means that we got

$$\widetilde{R}^t_P f(z) = \frac{\widetilde{\gamma}_k}{2\pi} \int_{S^{2d-1}} P(\theta) H^t_{\theta,k} f(z)\, d\theta, \tag{4-4}$$

where

$$H^t_{\theta,k} f(z) = H^t_\theta f(z) := \int_{\mathbb{C}} \left(\frac{\lambda}{|\lambda|}\right)^k \frac{f(z-\lambda\theta)}{|\lambda|^2} \mathbb{1}_{|\lambda|\ge t}(\lambda)\, d\lambda$$

is the truncated directional k-th power of the complex Hilbert transform.

Identity (4-4) can be written in terms of the probabilistic spherical measure $d\zeta$ on S^{2d-1} in the following way:

$$\widetilde{R}^t_P f(z) = \frac{\Gamma\left(d+\frac{1}{2}k\right)}{\pi\Gamma(d)\Gamma\left(\frac{1}{2}k\right)} \int_{S^{2d-1}} P(\zeta) H^t_\zeta f(z)\, d\zeta. \tag{4-5}$$

The limiting case of (4-5) is then

$$\widetilde{R}_P f(z) = \frac{\Gamma\left(d+\frac{1}{2}k\right)}{\pi\Gamma(d)\Gamma\left(\frac{1}{2}k\right)} \int_{S^{2d-1}} P(\zeta) H_\zeta f(z)\, d\zeta, \tag{4-6}$$

where

$$H_\zeta f = H_{\zeta,k} f = \text{p.v.} \int_{\mathbb{C}} \left(\frac{\lambda}{|\lambda|}\right)^k \frac{f(z-\lambda\zeta)}{|\lambda|^2}\, d\lambda$$

is the directional k-th power of the complex Hilbert transform.

Identities (4-5) and (4-6) were initially established for $f \in \mathcal{S}(\mathbb{C}^d)$. However, a density argument based on the $L^p(\mathbb{C}^d)$ boundedness of H^t_ζ and H_ζ allows us to write these identities for all $f \in L^p(\mathbb{C}^d)$. For further reference we note that when k is fixed, then by (1-13) we have

$$\frac{\Gamma\left(d+\frac{1}{2}k\right)}{\pi\Gamma(d)\Gamma\left(\frac{1}{2}k\right)} \sim d^{k/2}. \tag{4-7}$$

In the proofs of Theorems 3.2 and 3.3 we shall need boundedness properties of the maximal operator

$$H^*_\zeta f(z) = H^*_{\zeta,k} f(z) := \sup_{t\in\mathbb{Q}_+} |H^t_\zeta f(z)|$$

associated to H^t_ζ.

Proposition 4.3. *For each* $1 < p < \infty$ *we have*

$$\left\| \left(\sum_{s=1}^{S} |H^*_\zeta f_s|^2 \right)^{1/2} \right\|_{L^p(\mathbb{C}^d)} \lesssim p^* \left\| \left(\sum_{s=1}^{S} |f_s|^2 \right)^{1/2} \right\|_{L^p(\mathbb{C}^d)}$$

uniformly in $\zeta \in S^{2d-1}$ *and the dimension* d.

The proof of Proposition 4.3 is standard and therefore we omit it here. For the convenience of the reader we include the proof in the Appendix.

We will also need vector-valued estimates for $\{\widetilde{R}_j(f_s)\},\ j \in I,\ s = 1, \ldots, d$.

Proposition 4.4. *Fix $k \in \mathbb{N}$. Then for each $1 < p < \infty$ we have*

$$\bigg\| \Big(\sum_{s=1}^{S} \sum_{j \in I} |\widetilde{R}_j f_s|^2 \Big)^{1/2} \bigg\|_{L^p(\mathbb{C}^d)} \lesssim_k p^* p^{1/2} q^{(k+1)/2} \bigg\| \Big(\sum_{s=1}^{S} |f_s|^2 \Big)^{1/2} \bigg\|_{L^p(\mathbb{C}^d)}, \tag{4-8}$$

$$\bigg\| \Big(\sum_{j \in I} |\widetilde{R}_j f|^2 \Big)^{1/2} \bigg\|_{L^p(\mathbb{C}^d)} \lesssim_k p^* q^{k/2} \|f\|_{L^p(\mathbb{C}^d)} \tag{4-9}$$

uniformly in the dimension d.

Proposition 4.4 can be proved by an iterative application of its $k = 1$ case together with Khintchine's inequalities. However, such an approach produces worse constants than those in (4-8), (4-9). An important ingredient in the proof are properties of the functions

$$\zeta_j = (x_{j_1} + i y_{j_1}) \cdots (x_{j_k} + i y_{j_k}).$$

Note that $\zeta_j,\ j \in I$, are orthogonal with respect to the inner product on S^{2d-1}. Moreover,

$$\int_{S^{2d-1}} |\zeta_j|^2 \, d\zeta \lesssim d^{-k}. \tag{4-10}$$

Indeed, all the integrals on the left-hand side of (4-10) are equal for $j \in I$ and thus

$$\int_{S^{2d-1}} |\zeta_j|^2 \, d\zeta = \frac{1}{|I|} \int_{S^{2d-1}} \sum_{j \in I} |\zeta_j|^2 \, d\zeta \le \frac{1}{|I|} \int_{S^{2d-1}} \sum_{j \in [d]^k} |\zeta_j|^2 \, d\zeta = \frac{1}{|I|} \int_{S^{2d-1}} |\zeta|^{2k} \, d\zeta \lesssim d^{-k}$$

since $|I| \approx d^k$.

We justify (4-8) and (4-9) separately, starting with the latter.

Proof of (4-9). Take numbers $\lambda_j(f, z) = \lambda_j(z),\ j \in I$, such that

$$\Big(\sum_{j \in I} |\widetilde{R}_j f(z)|^2 \Big)^{1/2} = \sum_{j \in I} \lambda_j(z) \widetilde{R}_j f(z), \qquad \sum_{j \in I} \lambda_j^2(z) = 1.$$

Using (4-6) and (4-7) followed by Hölder's inequality we obtain

$$\begin{aligned} \bigg\| \Big(\sum_{j \in I} |\widetilde{R}_j f|^2 \Big)^{1/2} \bigg\|_p^p &= \int_{\mathbb{C}^d} \Big| \sum_{j \in I} \lambda_j(z) \widetilde{R}_j f(z) \Big|^p \, dz \\ &\lesssim^p d^{kp/2} \int_{\mathbb{C}^d} \bigg| \int_{S^{2d-1}} \sum_{j \in I} \lambda_j(z) \zeta_j H_\zeta f(z) \, d\zeta \bigg|^p \, dz \\ &\leqslant d^{kp/2} \int_{\mathbb{C}^d} \bigg(\int_{S^{2d-1}} \Big| \sum_{j \in I} \lambda_j(z) \zeta_j \Big|^q \, d\zeta \bigg)^{p/q} \int_{S^{2d-1}} |H_\zeta f(z)|^p \, d\zeta \, dz. \end{aligned} \tag{4-11}$$

Now we deal with the first inner integral in (4-11). Since $\zeta_j \in \mathcal{H}_k^{2d}$ for $j \in I$, for fixed z the function $\zeta \mapsto \sum_{j \in I} \zeta_j \lambda_j(z)$ also belongs to $\mathcal{H}_k^{2d}$. Using [Duoandikoetxea 1987, Theorem 1], orthogonality of the functions $\zeta_j,\ j\in I$, inequality (4-10), and the formula $\sum_{j\in I}\lambda_j(z)^2=1$, we get

$$\begin{aligned}\biggl(\int_{S^{2d-1}}\Bigl|\sum_{j\in I}\lambda_j(z)\zeta_j\Bigr|^q\,d\zeta\biggr)^{1/q}&\lesssim q^{k/2}\biggl(\int_{S^{2d-1}}\Bigl|\sum_{j\in I}\lambda_j(z)\zeta_j\Bigr|^2\,d\zeta\biggr)^{1/2}\\&=q^{k/2}\biggl(\int_{S^{2d-1}}\sum_{j\in I}\lambda_j(z)^2|\zeta_j|^2\,d\zeta\biggr)^{1/2}\\&\lesssim q^{k/2}\biggl(d^{-k}\sum_{j\in I}\lambda_j(z)^2\biggr)^{1/2}\le q^{k/2}d^{-k/2}.\end{aligned}\tag{4-12}$$

Applying (4-12) and coming back to (4-11) we obtain

$$\biggl\|\biggl(\sum_{j\in I}|\widetilde{R}_j f|^2\biggr)^{1/2}\biggr\|_p\lesssim q^{k/2}\biggl(\int_{S^{2d-1}}\|H_\zeta f\|^p_{L^p(\mathbb{C}^d)}\,d\zeta\biggr)^{1/p}.$$

Now Proposition 4.3 completes the proof of (4-9). □

We are now ready to prove (4-8). This is similar to the proof of (4-9) with an addition of Khintchine's inequalities. For $s=1,2,\dots$ we let $\{r_s\}$ be the Rademacher functions; see [Grafakos 2014, Appendix C]. These form an orthonormal set on $L^2([0,1])$. Moreover we have Khintchine's inequalities [Grafakos 2014, Appendix C.2],

$$\biggl\|\sum_{j=1}^{\infty}a_jr_j\biggr\|_{L^p([0,1])}\lesssim p^{1/2}\biggl(\sum_{j=1}^{\infty}|a_j|^2\biggr)^{1/2}\tag{4-13}$$

and

$$\biggl(\sum_{j=1}^{\infty}|a_j|^2\biggr)^{1/2}\lesssim\biggl\|\sum_{j=1}^{\infty}a_jr_j\biggr\|_{L^p([0,1])},\tag{4-14}$$

for any complex sequence $(a_s)_{s=1}^{\infty}$ and $1\leqslant p<\infty$. The explicit bounds on constants in (4-13) and (4-14) follow from explicit values of the optimal constants established by Haagerup [1981] together with Stirling's formula (1-12).

Proof of (4-8). Take numbers $\lambda_{j,s}(z,\{f_s\})=\lambda_{j,s}(z),\ j\in I,\ s=1,\dots,S$, such that

$$\biggl(\sum_{j\in I}\sum_{s=1}^{S}|\widetilde{R}_j f_s(z)|^2\biggr)^{1/2}=\sum_{s=1}^{S}\sum_{j\in I}\lambda_{j,s}(z)\widetilde{R}_j f_s(z),\quad\sum_{s=1}^{S}\sum_{j\in I}\lambda_{j,s}^2(z)=1.\tag{4-15}$$

Using (4-15), (4-6), and (4-7) we obtain

$$\begin{aligned}\biggl\|\biggl(\sum_{j\in I}\sum_{s=1}^{S}|\widetilde{R}_j f_s|^2\biggr)^{1/2}\biggr\|_p^p&=\int_{\mathbb{C}^d}\Bigl|\sum_{s=1}^{S}\sum_{j\in I}\lambda_{j,s}(z)\widetilde{R}_j f_s(z)\Bigr|^p\,dz\\&\lesssim^p d^{kp/2}\int_{\mathbb{C}^d}\Bigl|\int_{S^{2d-1}}\sum_{s=1}^{S}\sum_{j\in I}\lambda_{j,s}(z)\zeta_jH_\zeta f_s(z)\,d\zeta\Bigr|^p\,dz.\end{aligned}\tag{4-16}$$

Orthogonality of the Rademacher functions $\{r_s\}$ and Hölder's inequality imply

$$\begin{aligned}
&d^{kp/2}\int_{\mathbb{C}^d}\left|\int_{S^{2d-1}}\sum_{s=1}^{S}\sum_{j\in I}\lambda_{j,s}(z)\zeta_j H_\zeta f_s(z)\,d\zeta\right|^p dz\\
&= d^{kp/2}\int_{\mathbb{C}^d}\left|\int_{S^{2d-1}}\int_0^1\Bigl(\sum_{s=1}^{S}\sum_{j\in I} r_s(\xi)\lambda_{j,s}(z)\zeta_j\Bigr)\Bigl(\sum_{s=1}^{S} r_s(\xi)H_\zeta f_s(z)\Bigr)\,d\xi\,d\zeta\right|^p dz\\
&\le d^{kp/2}\int_{\mathbb{C}^d}\biggl(\int_{S^{2d-1}}\int_0^1\left|\sum_{s=1}^{S}\sum_{j\in I} r_s(\xi)\lambda_{j,s}(z)\zeta_j\right|^q d\xi\,d\zeta\biggr)^{p/q}\\
&\qquad\qquad\times\int_{S^{2d-1}}\int_0^1\left|\sum_{s=1}^{S} r_s(\xi)H_\zeta f_s(z)\right|^p d\xi\,d\zeta\,dz. \qquad (4\text{-}17)
\end{aligned}$$

Let

$$Q_{S,q}(z) := \biggl(\int_{S^{2d-1}}\int_0^1\left|\sum_{s=1}^{S}\sum_{j\in I} r_s(\xi)\lambda_{j,s}(z)\zeta_j\right|^q d\xi\,d\zeta\biggr)^{1/q}.$$

Then, coming back to (4-16) and applying Khintchine's inequality (4-13) to the second factor in the last inequality in (4-17), we reach

$$\left\|\Bigl(\sum_{j\in I}\sum_{s=1}^{S}|\widetilde{R}_j f_s|^2\Bigr)^{1/2}\right\|_p^p \lesssim^p p^{p/2}d^{kp/2}\|Q_{S,q}\|^p_{L^\infty(\mathbb{C}^d)}\int_{S^{2d-1}}\int_{\mathbb{C}^d}\Bigl(\sum_{s=1}^{S}|H_\zeta f_s(z)|^2\Bigr)^{p/2} dz\,d\zeta.$$

Thus, Proposition 4.3 implies

$$\left\|\Bigl(\sum_{j\in I}\sum_{s=1}^{S}|\widetilde{R}_j f_s|^2\Bigr)^{1/2}\right\|_p \lesssim p^* p^{1/2}d^{k/2}\|Q_{S,q}\|_{L^\infty(\mathbb{C}^d)}\left\|\Bigl(\sum_{s=1}^{S}|f_s|^2\Bigr)^{1/2}\right\|_{L^p(\mathbb{C}^d)}.$$

Therefore, the proof of (4-8) will be completed if we justify that

$$\|Q_{S,q}\|_{L^\infty(\mathbb{C}^d)} \lesssim q^{(k+1)/2}d^{-k/2}. \qquad (4\text{-}18)$$

The proof of (4-18) splits into two cases.

If $q \geqslant 2$, we proceed similarly as in (4-12). Namely we apply Khintchine's inequality (4-13), Minkowski's inequality and [Duoandikoetxea 1987, Theorem 1], obtaining

$$\begin{aligned}
(Q_{S,q}(z))^q &\lesssim^q q^{q/2}\int_{S^{2d-1}}\Bigl(\sum_{s=1}^{S}\Bigl|\sum_{j\in I}\lambda_{j,s}(z)\zeta_j\Bigr|^2\Bigr)^{q/2} d\zeta \leqslant q^{q/2}\Bigl(\sum_{s=1}^{S}\Bigl(\int_{S^{2d-1}}\Bigl|\sum_{j\in I}\lambda_{j,s}(z)\zeta_j\Bigr|^q d\zeta\Bigr)^{2/q}\Bigr)^{q/2}\\
&\lesssim^q q^{q/2}q^{kq/2}\Bigl(\sum_{s=1}^{S}\int_{S^{2d-1}}\Bigl|\sum_{j\in I}\lambda_{j,s}(z)\zeta_j\Bigr|^2 d\zeta\Bigr)^{q/2},
\end{aligned}$$

uniformly in $z\in\mathbb{C}^d$. Here an application of [Duoandikoetxea 1987, Theorem 1] is justified since $\zeta_j\in\mathcal{H}_k^{2d}$ for $j\in I$ and thus also the sum $\sum_{j\in I}\lambda_{j,s}(z)\zeta_j$ belongs to $\mathcal{H}_k^{2d}$ for each fixed $z\in\mathbb{C}^d$. Now, using the orthogonality of $\zeta_j,\ j\in I$, inequality (4-10) and the formula $\sum_{s=1}^{S}\sum_{j\in I}\lambda_{j,s}^2(z)=1$ we see that

$$(Q_{S,q}(z))^q \lesssim^q q^{q/2}q^{kq/2}\biggl(\sum_{s=1}^{S}\int_{S^{2d-1}}\sum_{j\in I}\lambda_{j,s}(z)^2|\zeta_j|^2\,d\zeta\biggr)^{q/2} = q^{q/2}q^{kq/2}\biggl(d^{-k}\sum_{s=1}^{S}\sum_{j\in I}\lambda_{j,s}(z)^2\biggr)^{q/2} \lesssim q^{q/2}q^{kq/2}d^{-kq/2}.$$

Therefore, (4-18) is justified in the case $q\geqslant 2$.

If on the other hand $1<q<2$, an application of Hölder's inequality together with (4-18) in the case $q=2$ shows that

$$Q_{S,q}(z)\le Q_{S,2}(z)\lesssim d^{-k/2}.$$

This completes the proof of (4-18) and thus also the proof of (4-8) from Proposition 4.4. □

We are now ready to prove Theorems 4.1 and 4.2. In both the proofs we shall need the formula

$$\widetilde{R}^t f(z)=\frac{\Gamma\bigl(d+\frac{1}{2}k\bigr)}{\pi\,\Gamma(d)\Gamma\bigl(\frac{1}{2}k\bigr)}\int_{S^{2d-1}}H_\zeta^t\biggl[\sum_{j\in I}\zeta_j\widetilde{R}_j f\biggr](z)\,d\zeta, \tag{4-19}$$

which follows from (4-3) and (4-5). We start with the proof of Theorem 4.2.

Proof of Theorem 4.2. Using (4-19) and (4-7) we see that

$$|\widetilde{R}^* f(z)|\lesssim d^{k/2}\int_{S^{2d-1}}H_\zeta^*\biggl[\sum_{j\in I}\zeta_j\widetilde{R}_j f\biggr](z)\,d\zeta,\qquad z\in\mathbb{C}^d.$$

Hence, Minkowski's integral inequality followed by Proposition 4.3 shows that

$$\|\widetilde{R}^* f\|_{L^p(\mathbb{C}^d)}\lesssim p^* d^{k/2}\int_{S^{2d-1}}\biggl\|\sum_{j\in I}\zeta_j\widetilde{R}_j f\biggr\|_{L^p(\mathbb{C}^d)}d\zeta.$$

Using Hölder's inequality and Fubini's theorem we obtain

$$\|\widetilde{R}^* f\|_{L^p(\mathbb{C}^d)}\lesssim p^* d^{k/2}\biggl(\int_{\mathbb{C}^d}\int_{S^{2d-1}}\biggl|\sum_{j\in I}\zeta_j\widetilde{R}_j f(z)\biggr|^p d\zeta\,dz\biggr)^{1/p}. \tag{4-20}$$

Since for fixed z the function $\zeta\mapsto\sum_{j\in I}\zeta_j\widetilde{R}_j f(z)$ belongs to $\mathcal{H}_k^{2d}$, applying [Duoandikoetxea 1987, Theorem 1] we obtain

$$\biggl(\int_{S^{2d-1}}\biggl|\sum_{j\in I}\zeta_j\widetilde{R}_j f(z)\biggr|^p d\zeta\biggr)^{1/p}\lesssim p^{k/2}\biggl(\int_{S^{2d-1}}\biggl|\sum_{j\in I}\zeta_j\widetilde{R}_j f(z)\biggr|^2 d\zeta\biggr)^{1/2}.$$

Using orthogonality and (4-10) we thus see that

$$\biggl(\int_{S^{2d-1}}\biggl|\sum_{j\in I}\zeta_j\widetilde{R}_j f(z)\biggr|^p d\zeta\biggr)^{1/p}\lesssim d^{-k/2}\,p^{k/2}\biggl(\sum_{j\in I}|\widetilde{R}_j f(z)|^2\biggr)^{1/2}, \tag{4-21}$$

which, together with (4-20), leads to

$$\|\widetilde{R}^* f\|_{L^p(\mathbb{C}^d)} \lesssim p^* p^{k/2} \bigg\| \Big(\sum_{j\in I} |\widetilde{R}_j f|^2 \Big)^{1/2} \bigg\|_{L^p(\mathbb{C}^d)}.$$

Thus, (4-9) from Proposition 4.4 completes the proof of Theorem 4.2. □

We finish this section with the proof of Theorem 4.1.

Proof of Theorem 4.1. Using (4-19), (4-7), and Minkowski's integral inequality on the space

$$\ell^2(\{1,\dots,S\}; L^\infty(\mathbb{Q}_+))$$

we see that

$$\Big(\sum_{s=1}^S |\widetilde{R}^* f_s(z)|^2\Big)^{1/2} \lesssim d^{k/2} \int_{S^{2d-1}} \bigg(\sum_{s=1}^S \Big(H_\zeta^* \Big[\sum_{j\in I} \zeta_j \widetilde{R}_j f_s \Big](z) \Big)^2 \bigg)^{1/2} d\zeta, \quad z\in\mathbb{C}^d.$$

Thus, another application of Minkowski's integral inequality followed by Proposition 4.3 gives

$$\bigg\| \Big(\sum_{s=1}^S |\widetilde{R}^* f_s|^2\Big)^{1/2} \bigg\|_{L^p(\mathbb{C}^d)} \lesssim p^* d^{k/2} \int_{S^{2d-1}} \bigg\| \Big(\sum_{s=1}^S \Big| \sum_{j\in I} \zeta_j \widetilde{R}_j f_s \Big|^2 \Big)^{1/2} \bigg\|_{L^p(\mathbb{C}^d)} d\zeta.$$

Using Khintchine's inequality (4-14) followed by Hölder's inequality on S^{2d-1} we see that

$$\begin{aligned}\bigg\| \Big(\sum_{s=1}^S |\widetilde{R}^* f_s|^2\Big)^{1/2} \bigg\|_{L^p(\mathbb{C}^d)} &\lesssim p^* d^{k/2} \int_{S^{2d-1}} \bigg(\int_{\mathbb{C}^d}\int_0^1 \Big| \sum_{s=1}^S r_s(\xi) \sum_{j\in I} \zeta_j \widetilde{R}_j f_s(z) \Big|^p d\xi\, dz \bigg)^{1/p} d\zeta \\ &\lesssim p^* d^{k/2} \bigg(\int_{\mathbb{C}^d} \int_0^1 \int_{S^{2d-1}} \Big| \sum_{j\in I} \zeta_j \widetilde{R}_j \Big[\sum_{s=1}^S r_s(\xi) f_s(z) \Big] \Big|^p d\zeta\, d\xi\, dz \bigg)^{1/p}.\end{aligned}$$

Finally, (4-21) followed by (4-9) from Proposition 4.4 and Khintchine's inequality (4-13) gives

$$\begin{aligned}\bigg\| \Big(\sum_{s=1}^S |\widetilde{R}^* f_s|^2\Big)^{1/2} \bigg\|_{L^p(\mathbb{C}^d)} &\lesssim p^* p^{k/2} \bigg(\int_{\mathbb{C}^d}\int_0^1 \Big(\sum_{j\in I} \Big| \widetilde{R}_j \Big[\sum_{s=1}^S r_s(\xi) f_s(z) \Big] \Big|^2 \Big)^{p/2} d\xi\, dz \bigg)^{1/p} \\ &\lesssim (p^*)^{2+k/2} \bigg(\int_{\mathbb{C}^d}\int_0^1 \Big| \sum_{s=1}^S r_s(\xi) f_s(z) \Big|^p d\xi\, dz \bigg)^{1/p} \\ &\lesssim (p^*)^{5/2+k/2} \bigg(\int_{\mathbb{C}^d} \Big(\sum_{s=1}^S |f_s|^2 \Big)^{p/2} dz \bigg)^{1/p}.\end{aligned}$$

□

5. Restriction to the initial Riesz transforms

The purpose of this section is twofold. Firstly, we restrict the maximal operator $\widetilde{R}^*$ acting on $L^p(\mathbb{C}^d)$ to a maximal operator $\mathcal{R}^*$ acting on $L^p(\mathbb{R}^d)$. This is done in a way which preserves estimates for the norms.

However, the restricted maximal operator $\mathcal{R}^*$ is not the same as R^*. Therefore, we need to estimate their difference, which is done in the second part of Section 5.

5.1. ***Bounding the restriction $\mathcal{R}^*$ of $\widetilde{R}^*$.*** In Theorems 4.1 and 4.2, we proved dimension-free estimates for the operator $\widetilde{R}^*$ acting on $L^p(\mathbb{C}^d)$. An approach similar to [Iwaniec and Martin 1996, Chapter 4] leads to dimension-free estimates for the restriction of this operator to $L^p(\mathbb{R}^d)$ which we now describe.

To elaborate, for $x \in \mathbb{R}^d$ and $t > 0$ we define the restricted kernel $\mathcal{K}_j^t(x)$ by

$$\mathcal{K}_j^t(x) = \begin{cases} \widetilde{\gamma}_k S_{d-1} \dfrac{x_j}{|x|^{d+k}} \displaystyle\int_{\sqrt{t^2/|x|^2-1}}^{\infty} \frac{r^{d-1}}{(1+r^2)^{d+k/2}}\, dr & \text{for } |x| < t, \\ K_j^t(x) & \text{for } |x| \geq t. \end{cases} \tag{5-1}$$

Recall that K_j^t is the truncation of the kernel K_j given by (1-1) when $P_j(x) = x_{j_1} \cdots x_{j_k},\ j \in I$. A short computation based on (1-10), (1-11), and (1-14) gives, for $x \neq 0$,

$$\lim_{t\to 0^+} \widetilde{\gamma}_k S_{d-1} \frac{x_j}{|x|^{d+k}} \int_{\sqrt{t^2/|x|^2-1}}^{\infty} \frac{r^{d-1}}{(1+r^2)^{d+k/2}}\, dr = \frac{\Gamma\big(d+\frac{1}{2}k\big)}{\pi^{d/2}\Gamma\big(\frac{1}{2}k\big)\Gamma\big(\frac{1}{2}d\big)} \int_0^{\infty} \frac{2r^{d-1}}{(1+r^2)^{d+k/2}}\, dr \cdot \frac{x_j}{|x|^{d+k}}$$
$$= \gamma_k \frac{x_j}{|x|^{d+k}} = K_j(x). \tag{5-2}$$

For $f \in L^p(\mathbb{R}^d)$ we let $\mathcal{R}_j^t f = f * \mathcal{K}_j^t$ and define

$$\mathcal{R}^t f = \sum_{j\in I} \mathcal{R}_j^t R_j f \quad \text{and} \quad \mathcal{R}^* f = \sup_{t\in\mathbb{Q}_+} |\mathcal{R}^t f|,$$

where R_j is as in (1-2) with $P(x) = P_j(x) = x_{j_1} \cdots x_{j_k}$.

A transference argument leads to the two results below. The proofs of Theorems 5.1 and 5.2 are based on ideas from [Iwaniec and Martin 1996, Section 4], but extra difficulties arise. These complications stem from the fact that we need to restrict compositions of singular integral operators instead of just one singular integral operator. Furthermore, useful formulas for the multiplier symbols of $\widetilde{R}_j^t$ or $\mathcal{R}_j^t$ are not available.

Theorem 5.1. *Fix $k \in \mathbb{N}$. For each $p \in (1, \infty)$ there is a constant $A(p, k)$ independent of the dimension d and such that for any $S \in \mathbb{N}$ we have*

$$\bigg\| \Big(\sum_{s=1}^{S} |\mathcal{R}^* f_s|^2 \Big)^{1/2} \bigg\|_{L^p(\mathbb{R}^d)} \leqslant A(p,k) \bigg\| \Big(\sum_{s=1}^{S} |f_s|^2 \Big)^{1/2} \bigg\|_{L^p(\mathbb{R}^d)}$$

whenever $f_1, \ldots, f_S \in L^p(\mathbb{R}^d)$. Moreover, $A(p, k)$ satisfies $A(p, k) \lesssim_k (p^*)^{5/2+k/2}$.

Theorem 5.2. *Fix $k \in \mathbb{N}$. For each $p \in (1, \infty)$ there is a constant $B(p, k)$ independent of the dimension d and such that*

$$\|\mathcal{R}^* f\|_{L^p(\mathbb{R}^d)} \leqslant B(p,k) \|f\|_{L^p(\mathbb{R}^d)}$$

whenever $f \in L^p(\mathbb{R}^d)$. Moreover, $B(p, k)$ satisfies $B(p, k) \lesssim_k (p^*)^{2+k/2}$.

The restriction procedure from Theorems 4.1 and 4.2 to Theorems 5.1 and 5.2 will result in the kernels $\widetilde{K}_j$ and $\widetilde{K}_j^t$ defined in (4-1) being integrated over their imaginary component iy in $\mathbb{R}^d$. This is the origin of the kernel $\mathcal{K}_j^t$ as the next lemma justifies.

Lemma 5.3. *For each $t > 0$ and $x \in \mathbb{R}^d$,*

$$\int_{\mathbb{R}^d} \widetilde{K}_j^t(x+iy)\,dy = \mathcal{K}_j^t(x). \tag{5-3}$$

Proof. To justify (5-3) consider two cases: $|x| \geq t$ and $|x| < t$. In the first case, integrating in polar coordinates in $\mathbb{R}^d$ and noting that $\int_{S^{d-1}} P_j(x+ir\omega)\,d\omega = P_j(x)$,

$$\begin{aligned}\int_{\mathbb{R}^d} \widetilde{K}_j^t(x+iy)\,dy &= \int_{y\in\mathbb{R}^d : |x+iy|\geq t} \widetilde{\gamma}_k \frac{P_j(x+iy)}{|x+iy|^{2d+k}}\,dy = \int_{\mathbb{R}^d} \widetilde{\gamma}_k \frac{P_j(x+iy)}{|x+iy|^{2d+k}}\,dy \\ &= \widetilde{\gamma}_k S_{d-1} P_j(x) \int_0^\infty \frac{r^{d-1}}{(|x|^2+r^2)^{d+k/2}}\,dr = \widetilde{\gamma}_k S_{d-1} \frac{P_j(x)}{|x|^{d+k}} \int_0^\infty \frac{r^{d-1}}{(1+r^2)^{d+k/2}}\,dr \\ &= K_j(x) = K_j^t(x) = \mathcal{K}_j^t(x).\end{aligned}$$

In the fourth equality above we used the change of variable $r \to r|x|$ and then we used (5-2). Similarly, in the second case $|x| < t$ we obtain

$$\int_{y\in\mathbb{R}^d : |x+iy|\geq t} \widetilde{\gamma}_k \frac{P_j(x+iy)}{|x+iy|^{2d+k}}\,dy = \widetilde{\gamma}_k S_{d-1} P_j(x) \int_{\sqrt{t^2-|x|^2}}^\infty \frac{r^{d-1}}{(|x|^2+r^2)^{d+k/2}}\,dr = \mathcal{K}_j^t(x),$$

where in the second equality we used the change of variable $r \to r|x|$. Thus (5-3) is justified. □

We present only the proof of Theorem 5.1. The proof of Theorem 5.2 is similar. We merely need a simpler duality argument instead of (5-4) below and an application of Theorem 4.2 instead of Theorem 4.1.

Proof of Theorem 5.1. By Lebesgue's monotone convergence theorem we may restrict the supremum in the definition of $\mathcal{R}^*$ to a finite set of positive numbers $\{t_1, \dots, t_N\}$, as long as our final estimate is independent of N. Further, a density argument shows that it suffices to consider $f_1, \dots, f_S \in \mathcal{S}(\mathbb{R}^d)$.

For $F : \mathbb{C}^d \to \mathbb{C}$ and $u > 0$ we let $(\delta_u F)(x+iy) = F(x+iuy)$ and define

$$\widetilde{R}^{t,u}(F)(x+iy) := (\delta_{u^{-1}} \circ \widetilde{R}^t \circ \delta_u)(F)(x+iy) = \widetilde{R}^t(\delta_u F)(x+iu^{-1}y).$$

Using Theorem 4.1 it is straightforward to see that

$$\left\| \left(\sum_{s=1}^S \sup_{n\in\{1,\dots,N\}} |\widetilde{R}^{t_n,u} F_s|^2 \right)^{1/2} \right\|_{L^p(\mathbb{C}^d)} \leqslant A(p,k) \left\| \left(\sum_{s=1}^S |F_s|^2 \right)^{1/2} \right\|_{L^p(\mathbb{C}^d)}.$$

Note that by duality between the spaces $L^p(\mathbb{C}^d; E_\infty)$ and $L^q(\mathbb{C}^d; E_1)$, where

$$E_\infty = \ell^2\big(\{1,\dots,S\}; \ell^\infty(\{t_1,\dots,t_N\})\big) \quad \text{and} \quad E_1 = \ell^2\big(\{1,\dots,S\}; \ell^1(\{t_1,\dots,t_N\})\big),$$

the above inequality can be rewritten in the equivalent form

$$\left| \sum_{s=1}^S \sum_{n=1}^N \langle \widetilde{R}^{t_n,u} F_s, G_{n,s} \rangle_{L^2(\mathbb{C}^d)} \right| \leqslant A(p,k) \left\| \left(\sum_{s=1}^S |F_s|^2 \right)^{1/2} \right\|_{L^p(\mathbb{C}^d)} \left\| \left(\sum_{s=1}^S \left(\sum_{n=1}^N |G_{n,s}| \right)^2 \right)^{1/2} \right\|_{L^q(\mathbb{C}^d)}, \tag{5-4}$$

where $G_{n,s} \in L^q(\mathbb{C}^d, E_1)$.

Let $\eta \in \mathcal{S}(\mathbb{R}^d)$ be a fixed function such that $\|\eta\|_{L^p(\mathbb{R}^d)} = 1$ and take $f \in \mathcal{S}(\mathbb{R}^d)$. Defining

$$F(x+iy) := (f \otimes \eta)(x, y) = f(x) \cdot \eta(y), \quad x, y \in \mathbb{R}^d,$$

we claim that

$$\lim_{u \to 0^+} \langle \widetilde{R}^{t,u} F, G\rangle_{L^2(\mathbb{C}^d)} = \langle \mathcal{R}^t(f) \otimes \eta, G\rangle_{L^2(\mathbb{C}^d)} \tag{5-5}$$

for any function $G \in \mathcal{S}(\mathbb{C}^d)$ and all $t > 0$.

Assume for a moment that the claim holds. Fix $\varepsilon \in (0, 1)$ and let $\psi \in \mathcal{S}(\mathbb{R}^d)$ be a function of $L^q(\mathbb{R}^d)$ norm 1 and such that $|\langle \eta, \psi\rangle_{L^2(\mathbb{R}^d)}| \geqslant (1-\varepsilon)$. Take $f_s \in \mathcal{S}(\mathbb{R}^d)$ and $g_{n,s} \in \mathcal{S}(\mathbb{R}^d)$ for $n = 1, \dots, N$, $s = 1, \dots, S$. Then, substituting $F_s = f_s \otimes \eta$ and $G_{n,s} = g_{n,s} \otimes \psi$ in (5-4) we have

$$\left|\sum_{s=1}^{S}\sum_{n=1}^{N} \langle \widetilde{R}^{t_n,u}(f_s \otimes \eta), g_{n,s} \otimes \psi\rangle_{L^2(\mathbb{C}^d)}\right| \\ \leqslant A(p,k) \left\| \left(\sum_{s=1}^{S} |f_s \otimes \eta|^2\right)^{1/2} \right\|_{L^p(\mathbb{C}^d)} \left\| \left(\sum_{s=1}^{S}\left(\sum_{n=1}^{N} |g_{n,s} \otimes \psi|\right)^2\right)^{1/2} \right\|_{L^q(\mathbb{C}^d)}.$$

At this point the claim (5-5) implies

$$\left|\sum_{s=1}^{S}\sum_{n=1}^{N} \langle \mathcal{R}^{t_n} f_s, g_{n,s}\rangle_{L^2(\mathbb{R}^d)}\right| |\langle \eta, \psi\rangle_{L^2(\mathbb{R}^d)}| \\ \leqslant A(p,k) \left\| \left(\sum_{s=1}^{S} |f_s|^2\right)^{1/2} \right\|_{L^p(\mathbb{R}^d)} \left\| \left(\sum_{s=1}^{S}\left(\sum_{n=1}^{N} |g_{n,s}|\right)^2\right)^{1/2} \right\|_{L^q(\mathbb{R}^d)}.$$

Now, using duality between the spaces $L^p(\mathbb{R}^d; E_\infty)$ and $L^q(\mathbb{R}^d; E_1)$ together with the density of Schwartz functions in $L^q(\mathbb{R}^d)$ we conclude that

$$(1-\varepsilon) \left\| \left(\sum_{s=1}^{S} \sup_{n \in \{1,\dots,N\}} |\mathcal{R}^{t_n} f_s|^2\right)^{1/2} \right\|_{L^p(\mathbb{R}^d)} \leqslant A(p,k) \left\| \left(\sum_{s=1}^{S} |f_s|^2\right)^{1/2} \right\|_{L^p(\mathbb{R}^d)}.$$

Since $\varepsilon \in (0, 1)$ was arbitrary this completes the proof of Theorem 5.1.

It remains to verify the claim (5-5). Since $\widetilde{R}^t = \sum_{j \in I} \widetilde{R}_j^t \widetilde{R}_j$ it is easy to see that

$$\widetilde{R}^{t,u} F = \sum_{j \in I} \widetilde{R}_j^{t,u} \widetilde{R}_j^u F,$$

where, for $F = f \otimes \eta$, we let

$$\widetilde{R}_j^{t,u}(F)(x+iy) = \widetilde{R}_j^t(\delta_u F)(x + iu^{-1}y), \quad \widetilde{R}_j^u(F)(x+iy) = \widetilde{R}_j(\delta_u F)(x+iu^{-1}y).$$

Thus, it is enough to justify that

$$\lim_{u \to 0^+} \langle \widetilde{R}_j^{t,u} \widetilde{R}_j^u F, G\rangle_{L^2(\mathbb{C}^d)} = \langle (\mathcal{R}_j^t R_j f) \otimes \eta, G\rangle_{L^2(\mathbb{C}^d)} \tag{5-6}$$

for $j \in I$, $t > 0$, and $G \in \mathcal{S}(\mathbb{C}^d)$.

Fix $j \in I$ and $t > 0$ and denote by m^t and m the multiplier symbols on $\mathbb{C}^d$ corresponding to the operators $\widetilde{R}_j^t$ and $\widetilde{R}_j$, respectively. Then $\delta_u(m^t)$ and $\delta_u(m)$ are the multiplier symbols corresponding to the operators $\widetilde{R}_j^{t,u}$ and $\widetilde{R}_j^{u}$, respectively. Thus, identifying $\mathbb{C}^d$ with $\mathbb{R}^{2d}$, taking the Fourier transform on $\mathbb{R}^{2d}$, and using Plancherel's theorem, we see that

$$\langle \widetilde{R}_j^{t,u}\widetilde{R}_j^{u} F, G\rangle_{L^2(\mathbb{C}^d)} = \big\langle \delta_u(m)\delta_u(m^t)\mathcal{F}[F], \mathcal{F}[G]\big\rangle_{L^2(\mathbb{C}^d)}. \tag{5-7}$$

By formula (1-3) (applied on $\mathbb{R}^{2d}$) and definitions (4-1), (4-2) for $P_j(z) := z_j = z_{j_1}\cdots z_{j_k}$, we have

$$\delta_u(m)(\xi,\tau) = (-i)^k \frac{P_j(\xi+iu\tau)}{|\xi+iu\tau|^k}$$

for $\xi, \tau \in \mathbb{R}^d$. Hence, for $\xi \neq 0$ and $\tau \in \mathbb{R}^d$ it holds that

$$\lim_{u\to 0^+} m(\xi, u\tau) = m(\xi, 0) = (-i)^k \frac{P_j(\xi)}{|\xi|^k}.$$

Another application of (1-3) (this time on $\mathbb{R}^d$) shows that the function $m_0(\xi) := m(\xi, 0)$ is the multiplier symbol of the operator R_j acting on $L^2(\mathbb{R}^d)$.

Since the operators $\widetilde{R}_j^t$ and $\widetilde{R}_j$ are both bounded on $L^2(\mathbb{C}^d)$ the functions $\delta_u(m)$ and $\delta_u(m^t)$ are in $L^\infty(\mathbb{C}^d)$, uniformly in $u > 0$. Thus, coming back to (5-7) and using Lebesgue's dominated convergence theorem we see that

$$\lim_{u\to 0^+}\langle \widetilde{R}_j^{t,u}\widetilde{R}_j^{u} F, G\rangle_{L^2(\mathbb{C}^d)} = \lim_{u\to 0^+}\big\langle \delta_u(m^t)\mathcal{F}[F], \bar{m}_0\mathcal{F}[G]\big\rangle_{L^2(\mathbb{C}^d)},$$

provided the limit on the right-hand side exists. By definition of m_0 applying again Plancherel's theorem we obtain

$$\lim_{u\to 0^+}\langle \widetilde{R}_j^{t,u}\widetilde{R}_j^{u} F, G\rangle_{L^2(\mathbb{C}^d)} = \lim_{u\to 0^+}\langle \widetilde{R}_j^{t,u} F, (R_j\otimes I)^* G\rangle_{L^2(\mathbb{C}^d)}, \tag{5-8}$$

provided the limit on the right-hand side exists. In the above formula $R_j \otimes I$ denotes the operator R_j acting only on the $\mathbb{R}^d$ coordinates of a function defined on $\mathbb{C}^d$ and the adjoint is taken with respect to the inner product on $L^2(\mathbb{C}^d)$. Now, if we justify that

$$\lim_{u\to 0^+}\big\langle \widetilde{R}_j^{t,u} F, (R_j\otimes I)^* G\big\rangle_{L^2(\mathbb{C}^d)} = \big\langle \mathcal{R}_j^t(f)\otimes\eta, (R_j\otimes I)^* G\big\rangle_{L^2(\mathbb{C}^d)} \tag{5-9}$$

and use the formula

$$\big\langle \mathcal{R}_j^t(f)\otimes\eta, (R_j\otimes I)^* G\big\rangle_{L^2(\mathbb{C}^d)} = \langle (\mathcal{R}_j^t R_j f)\otimes\eta, G\rangle_{L^2(\mathbb{C}^d)}$$

together with (5-8), then we will complete the proof of the claim (5-6).

Since the operators $\widetilde{R}_j^{t,u}$ are uniformly bounded on $L^2(\mathbb{C}^d)$ with respect to $u > 0$ to prove (5-9) it suffices to show that

$$\lim_{u\to 0^+}\langle \widetilde{R}_j^{t,u} F, \widetilde{G}\rangle_{L^2(\mathbb{C}^d)} = \langle \mathcal{R}_j^t(f)\otimes\eta, \widetilde{G}\rangle_{L^2(\mathbb{C}^d)}, \tag{5-10}$$

where $\widetilde{G} \in \mathcal{S}(\mathbb{C}^d)$. For $z = x+iy,\ x, y \in \mathbb{R}^d$, we have

$$\begin{aligned}\widetilde{R}_j^{t,u}(F)(z) &= \widetilde{R}_j^{t,u}(f\otimes\eta)(z) = u^{-d}\delta_{u^{-1}}(\widetilde{K}_j^t) * (f\otimes\eta)(z)\\ &= \int_{\mathbb{R}^d} f(x-x')\int_{y'\in\mathbb{R}^d\,:\,|x'+iu^{-1}y'|\geq t} \widetilde{\gamma}_k u^{-d}\frac{P_j(x'+iu^{-1}y')}{|x'+iu^{-1}y'|^{2d+k}}\eta(y-y')\,dy'\,dx'\\ &= \int_{\mathbb{R}^d}\int_{y'\in\mathbb{R}^d\,:\,|x'+iy'|\geq t} f(x-x')\widetilde{\gamma}_k\frac{P_j(x'+iy')}{|x'+iy'|^{2d+k}}\eta(y-uy')\,dy'\,dx'.\end{aligned} \tag{5-11}$$

Moreover, a computation shows that for fixed $x \in \mathbb{R}^d$ and $t > 0$ it holds that

$$f(x-x')\widetilde{\gamma}_k \frac{P_j(x'+iy')}{|x'+iy'|^{2d+k}} \mathbb{1}_{|x'+iy'|\geq t} \in L^1(\mathbb{C}^d) \tag{5-12}$$

uniformly in $x \in \mathbb{R}^d$. Hence, taking the limit as $u \to 0^+$ in (5-11) and using Lebesgue's dominated convergence theorem followed by Lemma 5.3 we obtain

$$\begin{aligned}\lim_{u\to 0^+} \widetilde{R}_j^{t,u}(F)(z) &= \eta(y) \int_{\mathbb{R}^d} f(x-x') \int_{y'\in\mathbb{R}^d : |x'+iy'|\geq t} \widetilde{\gamma}_k \frac{P_j(x'+iy')}{|x'+iy'|^{2d+k}}\, dy'\, dx' \\ &= \eta(y) \int_{\mathbb{R}^d} f(x-x')\mathcal{K}_j^t(x')\, dx' = \eta(y)\mathcal{R}_j^t f(x) = (\mathcal{R}_j^t(f)\otimes \eta)(x,y)\end{aligned} \tag{5-13}$$

for $x, y \in \mathbb{R}^d$. Moreover, another application of (5-12) shows that $\widetilde{R}_j^{t,u}(F) \in L^\infty(\mathbb{C}^d)$, uniformly in $u > 0$. Now, since $\widetilde{G} \in \mathcal{S}(\mathbb{C}^d)$ using again Lebesgue's dominated convergence theorem followed by (5-13) we reach

$$\lim_{u\to 0^+} \langle \widetilde{R}_j^{t,u} F, \widetilde{G}\rangle_{L^2(\mathbb{C}^d)} = \Big\langle \lim_{u\to 0^+} \widetilde{R}_j^{t,u} F, \widetilde{G}\Big\rangle_{L^2(\mathbb{C}^d)} = \langle \mathcal{R}_j^t(f)\otimes\eta, \widetilde{G}\rangle_{L^2(\mathbb{C}^d)},$$

This justifies (5-10), and hence also the claim (5-6). The proof of Theorem 5.1 is thus complete. □

5.2. ***Bounding the difference between R^t and $\mathcal{R}^t$.*** Define the difference kernels on $\mathbb{R}^d$ by

$$E_j^t(x) := K_j^t(x) - \mathcal{K}_j^t(x). \tag{5-14}$$

Recall that by definitions (1-1) of K_j^t and (5-1) of $\mathcal{K}_j^t$ we have $E_j^t(x) = -\mathcal{K}_j^t(x)$ if $|x| < t$ and $E_j^t(x) = 0$ if $|x| \geq t$. We let D_j be the operator on $L^p(\mathbb{R})$ given by $D_j^t f = f * E_j^t$ and define

$$D^t f = \sum_{j\in I} D_j^t R_j f, \qquad D^* f = \sup_{t\in\mathbb{Q}_+} |D^t f|.$$

Clearly,

$$R^t = \mathcal{R}^t + D^t,$$

so using Theorems 5.1 and 5.2 we reduce Theorems 3.2 and 3.3 to the following two statements.

Theorem 5.4. *Fix $k \in \mathbb{N}$. For each $p \in (1,\infty)$ there is a constant $A(p,k)$ independent of the dimension d and such that for any $S \in \mathbb{N}$ we have*

$$\bigg\| \Big(\sum_{s=1}^{S} |D^* f_s|^2 \Big)^{1/2} \bigg\|_{L^p(\mathbb{R}^d)} \leqslant A(p,k) \bigg\| \Big(\sum_{s=1}^{S} |f_s|^2 \Big)^{1/2} \bigg\|_{L^p(\mathbb{R}^d)}$$

whenever $f_1, \dots, f_S \in L^p(\mathbb{R}^d)$. Moreover, $A(p,k)$ satisfies $A(p,k) \lesssim_k (p^*)^{5/2+k/2}$.

Theorem 5.5. *Fix $k \in \mathbb{N}$. For each $p \in (1,\infty)$ there is a constant $B(p,k)$ independent of the dimension d and such that*

$$\|D^* f\|_{L^p(\mathbb{R}^d)} \leqslant B(p,k) \|f\|_{L^p(\mathbb{R}^d)}$$

whenever $f \in L^p(\mathbb{R}^d)$. Moreover, $B(p,k)$ satisfies $B(p,k) \lesssim_k (p^*)^{2+k/2}$.

The proofs of the above two theorems will follow the scheme of the proofs of Theorems 4.1 and 4.2. The main difference lies in the application of the real method of rotations. The reason for taking complex extensions (4-3) of the operators R^t is due to the fact that the real method of rotations is only applicable to singular integrals with odd kernels. Using this method for k odd one may express R^t as an integral of directional truncated Hilbert transforms. In the case of the operator D^t the cancellations are not important. We can use the real method of rotations to estimate D^* by an integral of one-dimensional directional Hardy–Littlewood maximal functions.

For $t > 0$ we let I^t be the function on $(0, \infty)$ given by

$$I^t(r) = \mathbb{1}_{(0,t)}(r) \int_{\sqrt{t^2/r^2-1}}^{\infty} \frac{s^{d-1}}{(1+s^2)^{d+k/2}}\, ds, \qquad r > 0. \tag{5-15}$$

Using the definitions (5-1) and (5-14) and integrating in polar coordinates in $\mathbb{R}^d$ we obtain

$$\begin{aligned} -D_j^t f(x) &= \int_{\mathbb{R}^d} \widetilde{\gamma}_k S_{d-1} \frac{y_j}{|y|^{d+k}} I^t(|y|) f(x-y)\, dy \\ &= \widetilde{\gamma}_k S_{d-1}^2 \int_0^t \int_{S^{d-1}} \frac{\omega_j}{r} I^t(r) f(x - r\omega)\, d\omega\, dr \\ &= \gamma_k S_{d-1} \int_{S^{d-1}} \omega_j \mathcal{H}_\omega^t f(x)\, d\omega = \frac{2\Gamma\big(\frac{1}{2}(k+d)\big)}{\Gamma\big(\frac{1}{2}k\big)\Gamma\big(\frac{1}{2}d\big)} \int_{S^{d-1}} \omega_j \mathcal{H}_\omega^t f(x)\, d\omega, \end{aligned} \tag{5-16}$$

where

$$\mathcal{H}_\omega^t f(x) = \frac{\widetilde{\gamma}_k}{\gamma_k} S_{d-1} \int_0^t I^t(r) \frac{f(x-r\omega)}{r}\, dr. \tag{5-17}$$

Let now $\mathcal{H}_\omega^* f(x) = \sup_{t \in \mathbb{Q}_+} |\mathcal{H}_\omega^t f(x)|$. The next proposition serves as a replacement for Proposition 4.3.

Proposition 5.6. *For each $1 < p < \infty$ we have*

$$\bigg\| \Big(\sum_{s=1}^{S} |\mathcal{H}_\omega^* f_s|^2 \Big)^{1/2} \bigg\|_{L^p(\mathbb{R}^d)} \lesssim p^* \bigg\| \Big(\sum_{s=1}^{S} |f_s|^2 \Big)^{1/2} \bigg\|_{L^p(\mathbb{R}^d)} \tag{5-18}$$

uniformly in $\omega \in S^{d-1}$ and the dimension d.

Proof. For $\omega \in S^{d-1}$ and $t > 0$ we let

$$\mathcal{M}_\omega^t f(x) = \frac{1}{t} \int_{-t}^{t} |f(x - r\omega)|\, dr \quad \text{and} \quad \mathcal{M}_\omega^* f(x) = \sup_{t>0} |\mathcal{M}_\omega^t f(x)|$$

be the directional Hardy–Littlewood averaging operator and the directional Hardy–Littlewood maximal function. Using Fubini's theorem and one-dimensional estimates for the Hardy–Littlewood maximal function, see, e.g., [Grafakos 2014, Theorem 5.6.6], we obtain

$$\bigg\| \Big(\sum_{s=1}^{S} |\mathcal{M}_\omega^* f_s|^2 \Big)^{1/2} \bigg\|_{L^p(\mathbb{R}^d)} \lesssim p^* \bigg\| \Big(\sum_{s=1}^{S} |f_s|^2 \Big)^{1/2} \bigg\|_{L^p(\mathbb{R}^d)},$$

uniformly in $\omega \in S^{d-1}$. Thus, to prove (5-18) it suffices to show the pointwise estimate

$$\mathcal{H}_\omega^t f(x) \lesssim \mathcal{M}_\omega^t f(x)$$

uniformly in $x \in \mathbb{R}^d$, $\omega \in S_{d-1}$, with in-explicit constants independent of the dimension.

This bound will follow if we justify that

$$\frac{\widetilde{\gamma}_k}{\gamma_k} S_{d-1} \frac{I^t(r)}{r} \lesssim \frac{1}{t}, \tag{5-19}$$

with the implicit constant being uniform in $t > 0$, $0 \le r \leqslant t$, and the dimension d. Note that for $s \geqslant (t^2/r^2 - 1)^{1/2}$ we have $\frac{1}{r} \leqslant \sqrt{s^2+1}/t$. Hence, recalling (5-15) and using (1-14), we obtain

$$\begin{aligned} \frac{\widetilde{\gamma}_k}{\gamma_k} S_{d-1} \frac{I^t(r)}{r} &\leqslant \frac{\widetilde{\gamma}_k}{\gamma_k} S_{d-1} \frac{1}{t} \int_{\sqrt{t^2/r^2-1}}^{\infty} \frac{s^{d-1}}{(1+s^2)^{d+(k-1)/2}}\, ds \\ &\leqslant S_{d-1} \frac{\widetilde{\gamma}_k}{\gamma_k} \frac{1}{t} \int_0^{\infty} \frac{s^{d-1}}{(1+s^2)^{d+(k-1)/2}}\, ds = S_{d-1} \frac{\widetilde{\gamma}_k}{\gamma_k} \frac{\Gamma\big(\frac12(d+k-1)\big)\Gamma\big(\frac12 d\big)}{2\Gamma\big(d+\frac12(k-1)\big)} \cdot \frac{1}{t}. \end{aligned}$$

Applying (1-10) and (1-11) we reach

$$\begin{aligned} S_{d-1} \frac{\widetilde{\gamma}_k}{\gamma_k} \frac{I^t(r)}{r} &\le \frac{2\pi^{d/2}}{\Gamma\big(\frac12 d\big)} \frac{\Gamma\big(d+\frac12 k\big)}{\pi^{d/2}\Gamma\big(\frac12(d+k)\big)} \frac{\Gamma\big(\frac12(d+k-1)\big)\Gamma\big(\frac12 d\big)}{2\Gamma\big(d+\frac12(k-1)\big)} \cdot \frac{1}{t} \\ &= \frac{\Gamma\big(d+\frac12 k\big)}{\Gamma\big(d+\frac12(k-1)\big)} \cdot \frac{\Gamma\big(\frac12(d+k-1)\big)}{\Gamma\big(\frac12(d+k)\big)} \cdot \frac{1}{t}. \end{aligned}$$

Since k is fixed, using (1-13) we conclude that

$$S_{d-1} \frac{\widetilde{\gamma}_k}{\gamma_k} \frac{I^t(r)}{r} \lesssim \frac{\big(d+\frac12(k-1)\big)^{1/2}}{\big(\frac12 d+\frac12(k-1)\big)^{1/2}} \cdot \frac{1}{t} \lesssim \frac{1}{t}.$$

Thus, we have completed the proof of (5-19) and hence also the proof of Proposition 5.6. □

We will also need vector-valued estimates for $\{R_j(f_s)\}$, $j \in I$, $s = 1, \dots, d$. The following proposition can be deduced from Proposition 4.4 if we proceed along the lines of [Iwaniec and Martin 1996, Section 4].

Proposition 5.7. *For each $1 < p < \infty$ we have*

$$\Big\| \Big(\sum_{s=1}^{S} \sum_{j \in I} |R_j f_s|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^d)} \lesssim p^* p^{1/2} q^{(k+1)/2} \Big\| \Big(\sum_{s=1}^{S} |f_s|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^d)}, \tag{5-20}$$

$$\Big\| \Big(\sum_{j \in I} |R_j f|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^d)} \lesssim p^* q^{k/2} \|f\|_{L^p(\mathbb{R}^d)}, \tag{5-21}$$

uniformly in the dimension d.

Proof. In contrast to the proofs of Theorems 5.1 and 5.2, here we apply the methods from [Iwaniec and Martin 1996, Section 4] in a direct way. Therefore we shall be brief. Let $n = k = d$ and identify $\mathbb{C}^d$ with $\mathbb{R}^{2d}$.

For the proof of (5-20) we take $E = \ell^2(\{1, \dots, S\})$ and $F = \ell^2(\{1, \dots, S\} \times I)$. The operator $\boldsymbol{T}$ is defined by

$$\boldsymbol{T}(\{f_s\}_{s=1,\dots,S}) = \{\widetilde{R}_j(f_s)\}_{(s,j)\in\{1,\dots,S\}\times I}.$$

Using (1-3) for $P(z)=z_{j_1}\cdots z_{j_k}$ one can check that the restricted operator $\boldsymbol{T}_0$ is then

$$\boldsymbol{T}_0(\{f_s\}_{s=1,\dots,S})=\{R_j(f_s)\}_{(s,j)\in\{1,\dots,S\}\times I}.$$

Hence, [Iwaniec and Martin 1996, equation (45)] together with (4-8) lead to (5-20).

The proof of (5-21) is similar. We take $E=\mathbb{C}$ and $F=\ell^2(I)$. The operators $\boldsymbol{T}$ and $\boldsymbol{T}_0$ are defined as above. The desired inequality follows from [Iwaniec and Martin 1996, equation (45)] together with (4-9). □

We are finally ready to justify Theorems 5.4 and 5.5. At this point the proofs mimic the corresponding proofs of Theorems 4.1 and 4.2. Therefore we shall be brief and only point out the differences.

Proof of Theorem 5.4. We proceed analogously to the proof of Theorem 4.1 on page 646. In particular, we replace $\mathbb{C}^d$ with $\mathbb{R}^d$, $\widetilde{R}_j^{t_n}$ with $D_j^{t_n}$ and $\widetilde{R}_j$ with R_j. The most important difference is that (5-16) replaces (4-5). This leads to the replacement of (4-19) by

$$D^t f(x)=-\frac{2\Gamma\big(\frac{1}{2}(k+d)\big)}{\Gamma\big(\frac{1}{2}k\big)\Gamma\big(\frac{1}{2}d\big)}\int_{S^{d-1}}\mathcal{H}_\omega^t\Big[\sum_{j\in I}\omega_j R_j f\Big](x)\,d\omega. \tag{5-22}$$

In the proof we also use (5-20) in place of (4-8) and Proposition 5.6 instead of Proposition 4.3. □

Proof of Theorem 5.5. We proceed analogously to the proof of Theorem 4.2 on page 645, making the replacements as in the proof of Theorem 5.4. In particular we use (5-22), (5-21), and Proposition 5.6. □

Appendix

Proof of Proposition 4.3. A (complex) rotational invariance argument reduces the inequality to its one-dimensional case:

$$\bigg\|\Big(\sum_{s=1}^S|H^*f_s|^2\Big)^{1/2}\bigg\|_{L^p(\mathbb{C})}\lesssim p^*\bigg\|\Big(\sum_{s=1}^S|f_s|^2\Big)^{1/2}\bigg\|_{L^p(\mathbb{C})}.$$

Here

$$H^*f(z):=\sup_{t\in\mathbb{Q}_+}|H_k^tf(z)|,\quad \text{with } H_k^tf(z)=\int_{\mathbb{C}}\Big(\frac{\lambda}{|\lambda|}\Big)^k\frac{f(z-\lambda)}{|\lambda|^2}\mathbb{1}_{|\lambda|\geq t}(\lambda)\,d\lambda,$$

is the k-th power of the complex Hilbert transform on $\mathbb{C}$.

We split the operator H^* into two parts. To this end let $\varphi:\mathbb{C}\to\mathbb{R}$ be a smooth radial function satisfying $\varphi(z)=1$ for $|z|<2$, and $\varphi(z)=0$ for $|z|>4$. Define $\varphi_t(z)=\varphi(z/t)$ and let

$$\chi_t(z)=\Big(\frac{z}{|z|}\Big)^k\frac{1}{|z|^2}\mathbb{1}_{|z|\geqslant t}$$

be the kernel of H_k^t. Then

$$H^*f(z)\leqslant\sup_{t>0}|(\varphi_t\chi_t*f)(z)|+\sup_{t>0}\big|((1-\varphi_t)\chi_t*f)(z)\big|=:H_\varphi^*f(z)+H_{1-\varphi}^*f(z)\lesssim\mathcal{M}f(z)+H_{1-\varphi}^*f(z),$$

where $\mathcal{M}$ denotes the Hardy–Littlewood maximal operator on $\mathbb{R}^2$. Since [Grafakos 2014, Theorem 5.6.6] gives us vector-valued estimates for $\mathcal{M}$, we get

$$\Big\|\Big(\sum_{s=1}^{S}|H_{\varphi}^{*}f_s|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{C})}\lesssim p^*\Big\|\Big(\sum_{s=1}^{S}|f_s|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{C})}.$$

The remaining ingredient is to prove

$$\Big\|\Big(\sum_{s=1}^{S}|H_{1-\varphi}^{*}f_s|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{C})}\lesssim p^*\Big\|\Big(\sum_{s=1}^{S}|f_s|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{C})}. \tag{A-1}$$

We will apply [Grafakos 2014, Theorem 5.6.1] with

$$\mathcal{B}_1=\ell^2(\{1,\dots,S\})\quad\text{and}\quad\mathcal{B}_2=\ell^2(\{1,\dots,S\};L^\infty(\mathbb{Q}_+))$$

and

$$\vec{K}(z)(u)=\big((1-\varphi_t)\chi_t(z)\cdot u_1,\dots,(1-\varphi_t)\chi_t(z)\cdot u_S\big)\in\mathcal{B}_2 \tag{A-2}$$

for any sequence $u=(u_s)_{s=1}^S\in\mathcal{B}_1$. Then, taking $e_s=(0,\dots,1,\dots,0)$, with 1 on the s-th coordinate, we see that the operator $\vec{T}$ defined in [Grafakos 2014, 5.6.4] satisfies

$$\vec{T}\Big(\sum_{s=1}^{S}f_se_s\Big)(z)=\big(H_{1-\varphi}^{t}f_1(z),\dots,H_{1-\varphi}^{t}f_S(z)\big) \tag{A-3}$$

and

$$\Big\|\vec{T}\Big(\sum_{s=1}^{S}f_se_s\Big)(z)\Big\|_{\mathcal{B}_2}=\Big(\sum_{s=1}^{S}|H_{1-\varphi}^{*}f_s(z)|^2\Big)^{1/2}$$

for any sequence $(f_s)_{s=1}^S$ of smooth functions that vanish at infinity. In order to use [Grafakos 2014, Theorem 5.6.1] we need to verify conditions (5.6.1), (5.6.2) and (5.6.3) from [Grafakos 2014] and check that $\vec{T}$ is bounded from $L^2(\mathbb{C},\mathcal{B}_1)$ to $L^2(\mathbb{C},\mathcal{B}_2)$.

Condition (5.6.1) is a straightforward consequence of (A-2). It is also not hard to verify that $\int_{\varepsilon\le|z|\le1}\vec{K}(z)\,dz=0$, so that condition (5.6.3) is satisfied with $\vec{K}_0=0$.

We shall now justify (5.6.2). Let $\widetilde{\varphi}_t:=1-\varphi_t$ and $g_t=\widetilde{\varphi}_t\chi_t$ so that

$$g_t(z)=\widetilde{\varphi}_t(z)\frac{z^k}{|z|^{k+2}}.$$

Since

$$\|\vec{K}(z-w)-\vec{K}(z)\|_{\mathcal{B}_1\to\mathcal{B}_2}=\sup_{t>0}|g_t(z-w)-g_t(z)|,$$

we have

$$\begin{aligned}\|\vec{K}(z-w)-\vec{K}(z)\|_{\mathcal{B}_1\to\mathcal{B}_2}&\\&=\sup_{t>0}\Big|\widetilde{\varphi}_t(z-w)\frac{(z-w)^k}{|z-w|^{k+2}}-\widetilde{\varphi}_t(z)\frac{z^k}{|z|^{k+2}}\Big|\\&\leqslant\sup_{t>0}\Big|(\widetilde{\varphi}_t(z-w)-\widetilde{\varphi}_t(z))\frac{(z-w)^k}{|z-w|^{k+2}}\Big|+\sup_{t>0}\Big|\widetilde{\varphi}_t(z)\Big(\frac{(z-w)^k}{|z-w|^{k+2}}-\frac{z^k}{|z|^{k+2}}\Big)\Big|.\end{aligned} \tag{A-4}$$

Hence, the proof of (5.6.2) boils down to estimating the two terms in (A-4) under the assumption $|z| \geq 2|w|$. We begin with the first term. Since $|z| \geq 2|w|$ we have $|z| \approx |z-w|$. Hence, in order for the expression inside the absolute value to be nonzero, t has to be comparable to $|z|$ and $|z-w|$. In that case, using the smoothness of φ we obtain

$$\left|(\widetilde{\varphi}_t(z-w)-\widetilde{\varphi}_t(z))\frac{(z-w)^k}{|z-w|^{k+2}}\right| \lesssim \frac{|w|}{2t}\frac{1}{|z-w|^2} \approx \frac{|w|}{|z|\,|z-w|^2} \approx \frac{|w|}{|z|^3}.$$

In the second term of (A-4) we omit $\widetilde{\varphi}_t$ and get

$$\begin{aligned}\left|\frac{(z-w)^k}{|z-w|^{k+2}}-\frac{z^k}{|z|^{k+2}}\right| &\leqslant \left|\frac{(z-w)^k}{|z-w|^{k+2}}-\frac{(z-w)^k}{|z|^{k+2}}\right|+\left|\frac{(z-w)^k}{|z|^{k+2}}-\frac{z^k}{|z|^{k+2}}\right| \\ &= |z-w|^k\frac{\big||z|^{k+2}-|z-w|^{k+2}\big|}{|z-w|^{k+2}|z|^{k+2}}+\frac{1}{|z|^{k+2}}\big|(z-w)^k-z^k\big| \approx \frac{|w|}{|z|^3}.\end{aligned}$$

This means that we have proved that

$$\|\vec{K}(z-w)-\vec{K}(z)\|_{\mathcal{B}_1\to\mathcal{B}_2} \lesssim \frac{|w|}{|z|^3}$$

for $|z| \geqslant 2|w|$. Integrating this yields

$$\int_{|z|\geqslant 2|w|}\|\vec{K}(z-w)-\vec{K}(z)\|_{\mathcal{B}_1\to\mathcal{B}_2}\,dz \lesssim |w|\int_{|z|\geqslant 2|w|}\frac{1}{|z|^3}\,dz \approx 1,$$

so that condition (5.6.2) is satisfied.

It remains to justify the boundedness of $\vec{T}$ from $L^2(\mathbb{C},\mathcal{B}_1)$ to $L^2(\mathbb{C},\mathcal{B}_2)$. We have the pointwise bound

$$H^*_{1-\varphi}f(z) \lesssim \mathcal{M}f(z)+H^*f(z).$$

Therefore the desired L^2 boundedness of $\vec{T}$ is a consequence of (A-3) and the $L^2(\mathbb{C})$ boundedness of H^*. This allows us to use [Grafakos 2014, Theorem 5.6.1] and completes the proof of (A-1), and hence also the proof of Proposition 4.3. □

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A SHARP STABILITY CRITERION FOR EULER EQUATIONS VIA SPARSENESS

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We introduce sparse versions of function spaces that are relevant to characterize the solutions of Euler equations without concentration. The standard Sobolev space H^{-1} is given a sparse structure that allows measuring the degree of compactness of embeddings into H^{-1} and provides new quantitative general criteria for H^{-1}-stability. Indices of sparseness are defined, and function spaces whose indices have prescribed decay are constructed, resulting in an improvement of the classical H^{-1}-stability results: sparse stability. The analysis relies on the introduction of sparse Riesz–Morrey–Tadmor spaces, that are characterized via maximal operators and new sparse domination theorems, together with extrapolation techniques. Our methods also yield improvements on recent results on the conservation of energy of physically realizable solutions of 2D-Euler.

1. Preamble

The classical Euler equations for incompressible fluid flow are given by

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p, \\ \operatorname{div} u = 0, \\ \text{initial and boundary conditions,} \end{cases} \tag{1-1}$$

where $u = (u_1, \dots, u_n)$ is the *velocity* field and p is the (scalar) *pressure*. Although the Euler equations have been studied for more than two and half centuries, many important open problems remain unanswered.

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In particular, while it is easy to see that smooth solutions of (1-1) conserve kinetic energy, the existence of weak solutions that conserve energy or the uniqueness of weak solutions are more subtle issues.

1.1. ***H^{-1}-stability for approximate solutions of Euler equations.*** Research on conservation of energy has been considerably influenced by the work of DiPerna and Majda [1987a; 1987b; 1988]. These authors introduced the key concept of *approximate solutions* $\{u^\varepsilon\}_{\varepsilon>0}$ (see Definition 16) that weakly converge to u. If $u^\varepsilon \to u$ strongly in L^2, then u is a weak solution to (1-1). Otherwise the energy concentrates on sets. Despite this, u may be still an Euler solution due to the presence of subtle cancellations. This is the so-called *concentration-cancellation phenomenon*.

In their foundational papers, Lopes Filho, Nussenzveig Lopes, and Tadmor [Lopes Filho et al. 2000] and Tadmor [2001] develop H^{-1}-stability[1] (see Definition 19) into a very powerful unifying framework to study lack of concentrations in approximate solutions. To be more precise, these authors obtained the following result.

Theorem 1 [Lopes Filho et al. 2000]. *Suppose that $\{u^\varepsilon\}_{\varepsilon>0}$ is an H^{-1}-stable approximate family of Euler solutions. Then $\{u^\varepsilon\}_{\varepsilon>0}$ converges strongly (possibly passing to a subfamily) to a weak solution of the Euler equation u in $L^\infty([0,T]; L^2_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n))$.*

The implementation of H^{-1}-stability depends on having at one's disposal sharp criteria to characterize the compact sets of H^{-1} and, in particular, leverage this knowledge to decide which function spaces, among those relevant in the description of physical phenomena connected with the Euler equations, embed compactly into H^{-1}. In this direction, the H^{-1}-criteria, as it applies to rearrangement invariant spaces, was extensively developed in [Lopes Filho et al. 2000], recovering and extending earlier results from [DiPerna and Majda 1987a; Lions 1996].

As shown in [DiPerna and Majda 1987a], solutions to 2D Euler equations when the initial vorticity is supported in a curve play a central role in fluid dynamics. These solutions are called *vortex sheets* and their regularity can be naturally measured in terms of the Morrey spaces $M^{p,\alpha}$ (see (2-2) below).

Remark. We use standard notation: If X is a function space on $\mathbb{R}^n$, we let X_c be the subspace of compactly supported functions; and we let X_{loc} be the set of functions f such that $f\mathbf{1}_{Q_0} \in X$, for all cubes Q_0. We write

$$X_c \hookrightarrow H^{-1}_{\mathrm{loc}}(\mathbb{R}^n) \quad \text{or} \quad X_c \xhookrightarrow{\text{compactly}} H^{-1}_{\mathrm{loc}}(\mathbb{R}^n)$$

if for all Q_0

$$X(Q_0) \hookrightarrow H^{-1}(\mathbb{R}^n) \quad \text{or} \quad X(Q_0) \xhookrightarrow{\text{compactly}} H^{-1}(\mathbb{R}^n),$$

respectively, where $X(Q_0) = \{f \in X : \operatorname{supp} f \subset Q_0\}$ with $\|f\|_{X(Q_0)} = \|f\mathbf{1}_{Q_0}\|_X$.

The H^{-1}-stability theory for Morrey spaces is also treated in [Lopes Filho et al. 2000], and relies on a compactness result[2] due independently to DeVore and Tao (a proof was given in [Lopes Filho et al. 2000, Theorem 4.2]),

$$M^{p,\alpha}_c(\mathbb{R}^n) \xhookrightarrow{\text{compactly}} H^{-1}_{\mathrm{loc}}(\mathbb{R}^n), \tag{1-2}$$

[1] More precisely, H^{-1}_{loc}-stability.

[2] The same statement holds, mutatis mutandis, for the Morrey space of measures [Lopes Filho et al. 2000, Theorem 4.3].

provided that one of the following conditions is satisfied:

(a) $$p > \frac{n}{2},$$

(b) $$p = \frac{n}{2} \quad \text{and} \quad \alpha > 1.$$

Once in possession of these statements, Theorem 1 can be applied to establish that for families of approximate solutions, with uniformly bounded vorticities in $M^{p,\alpha}(\mathbb{R}^n)$, one can extract convergent subsequences to a solution of the Euler equation (1-1), without concentration. In the special case $n = 2$, $p = 1$ and $\alpha > 1$, this result[3] was first obtained by DiPerna and Majda [1987a, Theorem 3.1] using tools from elliptic theory. On the other hand, the case $n = 3$ and $p = \frac{3}{2}$ is connected with the work of Giga and Miyakawa [1989] on well-posedness of 3D Navier–Stokes equations with initial singular data such as vortex filaments.

At present time, the picture is not completely understood for all the values of the parameters involved in (1-2). Specifically in 2D, it is known that for $p = 1$ and $\alpha = \frac{1}{2}$, (1-2) does not hold (see [DiPerna and Majda 1987a; Majda 1993]). To the best of our knowledge, it remains an open problem to decide whether (1-2) with $p = 1$ still holds for $\alpha \in \left(\frac{1}{2}, 1\right]$ (the so-called "gap problem"), leaving open the existence of solutions without concentrations in $M^{1,\alpha}$. Similar types of gaps also appear when dealing with higher dimensions.

In an effort to understand the nature of these gaps, and their impact on the convergence of approximate solutions of the Euler equations, Tadmor [2001] introduced the finer scale of RMT spaces, $R_{p,q}\log^{\alpha}$, that sharpen (1-2); see Definition 20.

1.2. ***Tadmor's refinement of H^{-1}-stability.*** It is shown in [Tadmor 2001] that RMT spaces "interpolate the compactness gap" in the sense that

$$R_{p,2}\log^{\alpha}(\mathbb{R}^n)_c \xrightarrow{\text{compactly}} H^{-1}_{\text{loc}}(\mathbb{R}^n) \tag{1-3}$$

provided that one of the following conditions is satisfied:

(a) $$p > \frac{2n}{n+2},$$

(b) $$p = \frac{2n}{n+2} \quad \text{and (crucially)} \quad \alpha > \tfrac{1}{2}.$$

The $R_{p,2}\log^{\alpha}$ scale is sharp, with respect to the H^{-1}- stability, in the sense that for approximate solutions, with vorticities uniformly bounded in $R_{2n/(n+2),2}\log^{\alpha}$, $\alpha > \frac{1}{2}$, we can extract solutions without concentration, while for $\alpha \in \left(0, \frac{1}{2}\right]$ there is a weak limit solution (i.e., a concentration-cancellation effect). On the other hand, the original gap problem for Morrey spaces $M^{1,\alpha}$, is apparently not resolved in this fashion, since[4] (see [Tadmor 2001, page 519 and the discussion after (3.5)])

$$R_{1,2}\log^{\alpha}(\mathbb{R}^2) \subset M^{1,\alpha}(\mathbb{R}^2).$$

[3]The original statement from [DiPerna and Majda 1987a] involves a certain additional assumption on weak decay at infinity of vorticities.

[4]In other words, Tadmor's scale requires a stronger regularity condition than Morrey regularity on the set of vorticities to achieve compactness.

1.3. ***Paving the way to sparseness.*** The presence of logarithms in (1-3) (and (1-2)) is very natural and is connected with some implicit *extrapolation* constructions that are needed since $R_{1,2}(\mathbb{R}^2)$ (or more generally, $R_{2n/(n+2),2}(\mathbb{R}^n)$) is not suitable for H^{-1}-stability. In fact, we have (see (4-9))

$$R_{\frac{2n}{n+2},2}(\mathbb{R}^n) \not\subseteq H^{-1}(\mathbb{R}^n). \tag{1-4}$$

To overcome this obstacle, in this paper we propose a different methodology based on *sparseness*.

1.4. ***A new framework for H^{-1}-stability: Sparse stability.*** The main goal of this paper is to reformulate H^{-1}-stability applying the theory of sparse spaces, that we recently introduced in [Domínguez and Milman 2021]. In a nutshell, we show that the "defect" of RMT spaces exhibited by (1-4) can be overcome if the geometry of testing cubes in the definition of these spaces is changed. More precisely, let $SR_{2n/(n+2),2}$ the space that is obtained by replacing pairwise disjoint cubes in $R_{2n/(n+2),2}$ by sparse[5] families of cubes (see Definition 4 below). Then, somewhat informally, the following surprising formula holds (see Theorem 6 for the precise statement):

$$SR_{\frac{2n}{n+2},2}(\mathbb{R}^n) = H^{-1}(\mathbb{R}^n). \tag{1-5}$$

Armed with formula (1-5) we provide a sparse structure to H^{-1}, which we exploit to develop new methods to characterize compact sets in H^{-1}. In particular, we introduce *indices of sparseness*, associated to function spaces, that measure the degree of compactness into H^{-1}. Conversely, given a *decay* Ψ, i.e., a positive decreasing function on $[0,\infty)$ satisfying

$$\lim_{t\to\infty} \Psi(t) = 0, \tag{1-6}$$

we construct *sparse spaces* S_Ψ, whose sparse indices have the prescribed decay Ψ. This leads to the introduction of the concept of Ψ-*sparse stability* for approximate solutions; see Definition 11. As a consequence, we create a refined scale that exhausts the classical H^{-1}-stability in the following sense (compare with Theorem 1).

Theorem 2. *Let $\{u^\varepsilon\}_{\varepsilon>0}$ be a family of approximate solutions. The following are equivalent*:

(i) $\{u^\varepsilon\}_{\varepsilon>0}$ *is H^{-1}-stable*,

(ii) $\{u^\varepsilon\}_{\varepsilon>0}$ *is sparse stable.*

As a consequence, *if* (ii) *holds then* (*possibly passing to a subfamily*) $u^\varepsilon \to u$ *strongly in* L^2, *where* u *is a solution of* (1-1).

We show that sparse stability not only provides a simplified approach to all previously known existence results from [DiPerna and Majda 1987a], [Lopes Filho et al. 2000] and [Tadmor 2001] but, more importantly, it leads to the sharpening of the classical results.

We next detour to present in detail the construction of sparse spaces (including their connection with negative Sobolev spaces; see (1-5)), sparse indices and sparse stability.

[5] Loosely speaking, sparse cubes are not necessarily disjoint but possible overlappings can be controlled in a sharp fashion; see Definition 3.

1.5. ***Negative Sobolev spaces via sparse RMT spaces.*** The norms of many familiar spaces in analysis are defined in terms of coverings by disjoint cubes or "packings" (e.g., spaces like BMO, John–Nirenberg spaces, Morrey spaces, Campanato spaces, Brudnyi spaces, Lipschitz spaces, Garsia–Rodemich spaces, …). In [Domínguez and Milman 2021], we initiated the analysis of "sparse versions" of classical spaces, obtained modifying the requirements on the coverings: we replaced the usual packings of cubes by the slightly bigger class of "sparse coverings". We briefly recall the definition of sparse family of cubes.

Let Q_0 be a (not necessarily dyadic) cube in $\mathbb{R}^n$ of sidelength $\ell > 0$ and corner $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, i.e.,

$$Q_0 = [x_1, x_1 + \ell] \times \cdots \times [x_n, x_n + \ell].$$

A (dyadic) child of Q_0 is any of the 2^n cubes obtained by partitioning Q_0 by n median hyperplanes (i.e., the hyperplanes parallel to the faces of Q_0 and dividing each edge into 2 equal parts). Iterating this process, from Q_0 to its children, then to the children of the children, …, we construct the lattice $\mathcal{D}(Q_0)$ of dyadic subcubes in Q_0.

Definition 3 (sparse cubes). A (countable) family $(Q_i)_{i\in I} \subset \mathcal{D}(Q_0)$ is called *η-sparse*,[6] $\eta \in (0, 1)$, if for every Q_i there exists a measurable subset E_{Q_i} such that

(i) the sets E_{Q_i} are pairwise disjoint,

(ii) $\eta |Q_i| \le |E_{Q_i}|$.

We let $S(Q_0)$ be the collection of all sparse families of dyadic cubes in Q_0. Analogously, one can introduce $S(\mathbb{R}^n)$, the set formed by all sparse families of dyadic cubes in $\mathbb{R}^n$.

In particular, the sparse spaces $SR_{p,q} \log^\alpha(Q_0)$ are constructed modifying standard RMT spaces (see (2-1)) by replacing families of packings, "$(Q_i)_{i\in I} \in \Pi(Q_0)$", by sparse families, "$(Q_i)_{i\in I} \in S(Q_0)$".

Definition 4 (sparse RMT spaces). Let $1 \le p, q \le \infty$ and $\alpha \in \mathbb{R}$. The *sparse RMT space $SR_{p,q} \log^\alpha(Q_0)$* is formed by all those $f \in L^1(Q_0)$ such that

$$\|f\|_{SR_{p,q}\log^\alpha(Q_0)} = \sup_{(Q_i)_{i\in I}\in S(Q_0)} \left\{ \sum_{i\in I} \left[\frac{(1 - (\log |Q_i|)_-)^\alpha}{|Q_i|^{1/p'}} \int_{Q_i} |f| \right]^q \right\}^{1/q} < \infty. \tag{1-7}$$

In particular, we let $SR_{p,q}(Q_0) = SR_{p,q} \log^0(Q_0)$. The corresponding spaces on $\mathbb{R}^n$ are introduced analogously.

Remark 5. The definition of $SR_{p,q} \log^\alpha$ is well adapted to work with signed measures $\omega \in BM^+$. Indeed, we simply replace $\int_{Q_i} |f|$ in (1-7) by $\omega(Q_i)$.

Since we trivially have $\Pi(Q_0) \subset S(Q_0)$, it follows that

$$SR_{p,q} \log^\alpha(Q_0) \subset R_{p,q} \log^\alpha(Q_0).$$

[6] In this paper the parameter η will not play a role, so in what follows we shall let $\eta = \frac{1}{2}$.

However, in general, the sparse spaces are different[7] from their parent spaces. In our context, the differences manifest themselves through the behavior of the maximal operators M_{λ,Q_0} (see (3-1)). The theory developed in Section 3 will play a crucial role in our analysis.

Suppose that $1 \le p < q < \infty$; then a special case of Proposition 25 below shows that there exists $\{Q_i\}_{i\in I} \in S(Q_0)$ and a constant c depending only on p and q such that

$$M_{n(\frac{1}{p}-\frac{1}{q}),Q_0} f(x) \le c \sum_{i\in I} \left(\frac{1}{|Q_i|^{1/p'+1/q}} \int_{Q_i} |f(y)|\, dy \right) \mathbf{1}_{Q_i}(x). \tag{1-8}$$

In the literature the process of constructing such coverings is referred as *sparse domination*; see [Lerner and Nazarov 2019; Hytönen 2021]. From (1-8) and more or less standard arguments, we obtain the following remarkable result connecting $SR_{p,q}(\mathbb{R}^n)$ and classical Sobolev spaces $H_q^{-\lambda}(\mathbb{R}^n)$, $\lambda \in (0,n)$ (see (4-1)).

Theorem 6. *Let* $1 \le p < q < \infty$. *If* $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $f \ge 0$ *a.e., then*

$$\|f\|_{SR_{p,q}(\mathbb{R}^n)} \approx \|f\|_{H_q^{-n(\frac{1}{p}-\frac{1}{q})}(\mathbb{R}^n)}.$$

In general

$$SR_{p,q}(\mathbb{R}^n) \hookrightarrow H_q^{-n(\frac{1}{p}-\frac{1}{q})}(\mathbb{R}^n).$$

In particular, the canonical choice of parameters $p = \dfrac{2n}{n+2}$, $n \ge 2$, *and* $q = 2$ *gives*[8]

$$\|f\|_{SR_{\frac{2n}{n+2},2}(\mathbb{R}^n)} \approx \|f\|_{H^{-1}(\mathbb{R}^n)} \quad \text{if } f \ge 0. \tag{1-9}$$

Consequently

$$SR_{\frac{2n}{n+2},2}(\mathbb{R}^n) \hookrightarrow H^{-1}(\mathbb{R}^n). \tag{1-10}$$

1.6. ***Sparse indices.*** Since we deal with local problems, most of the analysis will be carried out on cubes, but similar constructions are also possible in $\mathbb{R}^n$. Let $Q_0 \subset \mathbb{R}^n$, $n \ge 2$, be a fixed cube and let $\mathcal{Q} \in S(Q_0)$. For $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, let $\mathbb{D}_{\le k;Q_0} := \{Q : Q \in \mathcal{D}(Q_0) \text{ with sidelength } \ell(Q) \le 2^{-k}\ell(Q_0)\}$, $\mathbb{D}_{\le k;Q_0}(\mathcal{Q}) := \mathbb{D}_{\le k;Q_0} \cap \mathcal{Q}$. When there is no danger of confusion, we use the simplified notation $\mathbb{D}_{\le k}$ and $\mathbb{D}_{\le k}(\mathcal{Q})$.

Definition 7 (sparse indices). (i) The *sparse indices of* $f \in L^1(Q_0)$ are defined by[9]

$$s_N(f) = \sup_{\mathcal{Q}\in S(Q_0)} \left[\sum_{Q\in\mathbb{D}_{\le N-1}(\mathcal{Q})} \left(|Q|^{\frac{1}{n}-\frac{1}{2}} \int_Q |f| \right)^2 \right]^{\frac{1}{2}}, \quad N \in \mathbb{N}. \tag{1-11}$$

[7] Note that $SR_{p,\infty}\log^\alpha(Q_0) = R_{p,\infty}\log^\alpha(Q_0) = M^{p,\alpha}(Q_0)$; see Remark 21.

[8] The role of sparseness is crucial here. In particular, for the classical space $R_{2n/(n+2),2}$, this approach fails dramatically (see (1-4)).

[9] Sparse indices may depend on the given cube Q_0. However, since this dependance will not play a role in our arguments, it will be safely omitted in the corresponding notation.

X	upper estimate for $s_N(X)$
$L^p, \quad p > \dfrac{2n}{n+2}$	$2^{-N(\frac{2+n}{2n}-\frac{1}{\min\{2,p\}})n}$
$M^{p,\alpha}, \quad p > \dfrac{n}{2}, \quad \alpha \in \mathbb{R}$	$2^{-N(\frac{2}{n}-\frac{1}{p})\frac{n}{2}}N^{-\frac{\alpha}{2}}$
$M^{\frac{n}{2},\alpha}, \quad \alpha > 1$	$N^{\frac{1-\alpha}{2}}$
$R_{p,2}\log^{\alpha}, \quad p > \dfrac{2n}{n+2}, \quad \alpha \in \mathbb{R}$	$2^{-N(\frac{2+n}{2n}-\frac{1}{p})n}N^{-\alpha}$
$R_{\frac{2n}{n+2},2}\log^{\alpha}, \quad \alpha > \frac{1}{2}$	$N^{\frac{1}{2}-\alpha}$

Table 1. Sparse indices for some classical function spaces.

(ii) Let X be a function space, $X \subset L^1_{\mathrm{loc}}(\mathbb{R}^n)$. The *sparse indices of* $X(Q_0)$ are defined by

$$s_N(X) = \sup_{\|f\|_{X(Q_0)} \le 1} s_N(f). \tag{1-12}$$

Remark 8. The definitions above can be extended in a natural way to the setting of measures with distinguished sign.

Compactness of embeddings into H^{-1} can be characterized in terms of sparse indices. Specifically, we have the following result.

Theorem 9. *Let X be a function space*[10] $X \subset L^1_{\mathrm{loc},+}(\mathbb{R}^n)$, $n \ge 2$, (*or more generally*, $X \subset BM_c^+$). *Then*:[11]

(i) $s_1(X) < \infty \quad \Leftrightarrow \quad X_c \hookrightarrow H^{-1}_{\mathrm{loc}}(\mathbb{R}^n)$.

(ii) $\lim_{N\to\infty} s_N(X) = 0 \quad \Leftrightarrow \quad X_c \xhookrightarrow{\text{compactly}} H^{-1}_{\mathrm{loc}}(\mathbb{R}^n)$.

The proof of this result is given in Section 4.3.

Sparse indices provide a very satisfactory criteria for H^{-1}-stability (see Theorem 1).

Corollary 10. *Let X be a function space $X \subset L^1_{\mathrm{loc},+}(\mathbb{R}^n)$, $n \ge 2$, (or more generally, $X \subset BM_c^+$) such that*

$$\lim_{N\to\infty} s_N(X) = 0. \tag{1-13}$$

Suppose that $\{u^\varepsilon\}_{\varepsilon>0}$ is an approximate family of Euler solutions with related set of vorticities $\{\omega^\varepsilon\}_{\varepsilon>0}$ uniformly bounded in $C((0,T);X)$. Then (passing to a subfamily if necessary) $u^\varepsilon \to u$ strongly in $L^\infty([0,T];L^2_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n))$, where u is a solution to (1-1).

The new indices pose a challenge: can we compute them? In Section 5 we provide the explicit calculation of sparse indices for classical spaces like Lebesgue, Morrey, and RMT spaces. The results are presented in Table 1. These computations, combined with Theorem 9, give a unified proof of (1-2) and (1-3).

[10] As usual, $L^1_{\mathrm{loc},+}(\mathbb{R}^n) = \{f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : f \ge 0 \text{ a.e.}\}$. The additional assumption $f \ge 0$ is not restrictive since $s_N(f) = s_N(|f|)$.

[11] Note that $s_1(X) = \sup_{N\in\mathbb{N}} s_N(X)$.

Furthermore, understanding the rates of decay of the sparse indices will allow us to measure the degree of H^{-1}-compactness, and pave the way to extend the known results as we now explain.

1.7. ***Sparse spaces.*** So far, given a function space X, we analyzed the decay of its sparse indices $s_N(X)$ (see (1-12)) in order to guarantee H^{-1}-stability; see Corollary 10. However, note that the definition of sparse indices $s_N(f)$ (see (1-11)) is independent of any particular space X. This simple observation leads to the following question: can we use "reverse engineering" to create new function spaces whose sparse indices have prescribed decay? A natural construction associated with this idea can be described as follows.

Definition 11 (sparse spaces). Let Ψ be a decay (see (1-6)). The *sparse space* $S_\Psi(Q_0)$ is formed by all $f \in L^1(Q_0)$ such that

$$\|f\|_{S_\Psi(Q_0)} = \sup_{N \in \mathbb{N}} \frac{s_N(f)}{\Psi(N)} < \infty. \tag{1-14}$$

The counterparts on $\mathbb{R}^n$ as well as for positive measures can be introduced analogously.

Note, parenthetically, the superficial similarity with the constructions of Yudovich spaces and extrapolation spaces in [Domínguez and Milman 2024]. As we shall soon see this connection goes deeper and, moreover, some concrete calculations can be effected which lead to the introduction of new Euler relevant function spaces.

From (1-14), we clearly have

$$s_N(S_\Psi(Q_0)) \le \Psi(N),$$

therefore, by Theorem 9,

$$S_\Psi(\mathbb{R}^n)_c \xhookrightarrow{\text{compactly}} H^{-1}_{\text{loc}}(\mathbb{R}^n). \tag{1-15}$$

In particular, using sparse embeddings we can formulate a new H^{-1}-criteria.

Theorem 12 (H^{-1}-stability via sparse embeddings). *Suppose that*

$$X_c \hookrightarrow S_\Psi(\mathbb{R}^n)_c \tag{1-16}$$

holds for some decay Ψ, *then* X *is* H^{-1}*-stable*, *in the sense that*

$$X_c \xhookrightarrow{\text{compactly}} H^{-1}_{\text{loc}}(\mathbb{R}^n).$$

In particular, *suppose that* $\{u^\varepsilon\}_{\varepsilon>0}$ *is a family of approximate solutions of the Euler equations*, *such that the related set of vorticities* $\{\omega^\varepsilon\}_{\varepsilon>0}$ *is uniformly bounded in* X. *Then there exists a subfamily of* $\{u^\varepsilon\}_{\varepsilon>0}$ *which converges strongly to a weak Euler solution in* $L^\infty([0,T]; L^2_{\text{loc}}(\mathbb{R}^n))$.

The proof of this result is an immediate consequence of (1-15) and Theorem 1.

Next we go a step further and show that assumption (1-16) in Theorem 12 is in fact necessary to establish H^{-1}-stability. To do this, we need to introduce the natural generalization[12] (say, function space-free) of (1-16) to approximate solutions: *sparse stability*. As already anticipated in Theorem 2 (see

[12] Recall that H^{-1}-stability does not involve any function space X, but only approximate solutions; see Definition 19.

Section 6 for its proof), this new concept provides us with a remarkable characterization of H^{-1}-stability in terms of sparseness.

As usual, let $\mathbb{A}^n$ be the set of all antisymmetric matrices of order n with real entries.

Definition 13 (sparse stability). We say that a family $\{u^\varepsilon\}_{\varepsilon>0}$ of approximate solutions of the Euler equation is *sparse stable* if there exists a decay Ψ such that the corresponding set of vorticities $\{\omega^\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in[13] $C(0,T;S_\Psi(\mathbb{R}^n;\mathbb{A}^n))$. In particular, $\{u^\varepsilon\}_{\varepsilon>0}$ satisfies the *admissible sparse stability* property if Ψ is an admissible[14] decay.

Applying sparse stability, it is thus possible, at least theoretically, to improve all the known H^{-1}-stability results of [DiPerna and Majda 1987a; Lopes Filho et al. 2000; Tadmor 2001] (see Sections 1.1 and 1.2). However, to make the implied extensions meaningful, we need to exhibit concrete instantiations. In fact, we obtain significant improvements on the classical results and we show that our constructions unexpectedly connect with the theory of Yudovich spaces [1995], Vishik spaces [1999] and more specifically with the extrapolation spaces of [Jawerth and Milman 1991; Domínguez and Milman 2024].

1.8. ***New extrapolation spaces guaranteeing strong convergence to Euler solutions.*** As already mentioned in Section 1.3, extrapolation constructions seem to be implicit in (1-2)–(1-3). In Sections 7 and 8 we confirm this belief and show how the extrapolation theory of Jawerth and Milman [1991] (more precisely, the updated account given recently in [Domínguez and Milman 2024]) can be successfully applied to construct concrete examples of function spaces with prescribed sparse decay. In particular, these new spaces strictly contain the limiting spaces involved in (1-2) and (1-3), but are still H^{-1}-stable. As a consequence, we are able to extend the existence results for vortex sheets of [DiPerna and Majda 1987a; Tadmor 2001] to larger classes of vorticities.

1.8.1. *Sharpening Morrey regularity of DiPerna–Majda.* We introduce the distributional space $V_\Psi(\mathbb{R}^n)$ given by (see Definition 34)

$$\sum_{j=N}^{\infty} 2^{-2j}\|\Delta_j f\|_{L^\infty(\mathbb{R}^n)} \lesssim \Psi(N)^2, \quad N\in\mathbb{N}_0.$$

These spaces may be considered as “dual” counterparts of classical Vishik spaces proposed in [Vishik 1999] in connection with uniqueness issues for Euler flows; see Remark 35 for further explanations. Applying the set of techniques explained in previous sections, we show sparse stability[15] and nonconcentration phenomenon in V_Ψ; see Theorem 37. A crucial point in our arguments is that V_Ψ can be characterized as an extrapolation space of classical Besov spaces (see Theorem 39). In particular, for the special decay $\Psi(t)=t^{(1-\alpha)/2}$, $\alpha>1$, we have (see Theorem 38)

$$M^{\frac{n}{2},\alpha}(\mathbb{R}^n) \hookrightarrow V_\Psi(\mathbb{R}^n).$$

[13] In what follows, we will use the simplified notation $S_\Psi(\mathbb{R}^n)$ rather than $S_\Psi(\mathbb{R}^n;\mathbb{A}^n)$.

[14] see Definition 36(i).

[15] In fact, sparse numbers of V_Ψ behave like Ψ.

Furthermore, this result is sharp, i.e., we give a constructive method to produce functions in $V_\Psi(\mathbb{R}^n)$ but not in $M^{n/2,\alpha}(\mathbb{R}^n)$. As a by-product, we get a nontrivial improvement of (1-2).

1.8.2. *Sharpening Tadmor regularity.* The results stated in Section 1.8.1 for Morrey spaces admit counterparts for RMT spaces. In this setting, the role of V_Ψ is played by the new space $T_\Psi(\mathbb{R}^n)$ (see Definition 44), which admits the following nice characterization in terms of Fourier integrals:

$$\int_{|\xi|>2^N} (1+|\xi|^2)^{-1} |\widehat{f}(\xi)|^2 \, d\xi \lesssim \Psi(N)^2, \quad N \in \mathbb{N}_0.$$

Then we establish sparse stability and nonconcentration phenomenon in T_Ψ; see Theorem 47. In particular, for the special decay $\Psi(t) = t^{1/2-\alpha}$, $\alpha > \frac{1}{2}$, we have (see Theorem 48)

$$R_{\frac{2n}{n+2},2} \log^\alpha(\mathbb{R}^n) \hookrightarrow T_\Psi(\mathbb{R}^n).$$

Again, this result is sharp. As a consequence, we improve Tadmor's embedding (1-3).

1.9. ***Energy conservation for physically realizable solutions via sparse stability.*** In Section 10 we show that our methods are sufficiently robust to provide criteria for the preservation of energy by *physically realizable solutions*[16] of 2D Euler equations on the two-dimensional torus $\mathbb{T}^2$. Indeed, extending $L^p(\mathbb{T}^2)$-results,[17] $p > 1$, obtained by Cheskidov, Lopes Filho, Nussenzveig Lopes, and Shvydkoy [Cheskidov et al. 2016] (see also [Ciampa et al. 2021] for the case on the whole plane $\mathbb{R}^2$), we show that our framework can be used to provide conditions for physically realizable solutions to conserve energy.

Theorem 14. *Let u be a physically realizable weak solution of the 2D Euler equations with a physical realization $\{u^\varepsilon\}_{\varepsilon>0}$ satisfying admissible sparse stability. Then u is conservative, i.e.,* $\|u(t)\|_{L^2(\mathbb{T}^2)} = \|u_0\|_{L^2(\mathbb{T}^2)}$.

Very recently, Lanthaler, Mishra, and Parés-Pulido [Lanthaler et al. 2021] proposed an interesting approach to energy conservation based on the so-called structure functions (i.e., the L^2-modulus of smoothness) of $\{u^\varepsilon\}_{\varepsilon>0}$. On the other hand, Theorem 14 relies on the sparse indices of $\{\omega^\varepsilon\}_{\varepsilon>0}$. Switching from u^ε to ω^ε has important advantages from the point of view of applications, as it is illustrated by the following.

Corollary 15. *Let X be a function space $X \subset L^1(\mathbb{T}^2)$ (or more generally, $X \subset BM^+$) with sparse indices $s_N(X)$ satisfying* (1-13) *and the admissibility condition* (*see Definition 36*). *Let $\{u^\varepsilon\}_{\varepsilon>0}$ be a physical realization of the Euler solution u. If $\{\omega^\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in X, then u is conservative.*

This result follows immediately from Theorem 14 from the fact that $X \hookrightarrow S_\Psi(\mathbb{T}^2)$, where $\Psi(N) = s_N(X)$. In particular, Corollary 15 can be applied to all the classical function spaces exhibited in Table 1, as well as the new spaces $X = V_\Psi$ and $X = T_\Psi$.

[16] Roughly speaking, physically realizable solutions are weak solutions of Euler equations that can be obtained as weak limits of vanishing viscosity; see Definition 61.

[17] For general L^p solutions with $p \geq \frac{3}{2}$, conservation of energy can be derived from the well-known Besov-type criterion of Cheskidov, Constantin, Friedlander and Shvydkoy [Cheskidov et al. 2008]; see also [Cheskidov et al. 2016, Theorem 1] for an alternative and streamlined proof.

1.10. ***Brief interlude: some references.*** Existence and uniqueness of weak solutions for the 2D Euler equations are well established. In particular, we mention the concentration-cancellation result by Delort [1991] (resp. Vecchi and Wu [1993]) proving existence of weak solutions for initial vorticities in $BM_c^+ \cap H^{-1}$ (resp. in $L_c^1 \cap H^{-1}$); existence and uniqueness results of weak solutions for initial bounded vorticities (resp. vorticities in Yudovich spaces) were established by Yudovich [1963] (resp. [Yudovich 1995]), and the corresponding results for vorticities in BMO (and related spaces) obtained by Vishik [1999]. For vorticities in L^p the uniqueness problem remains open, although substantial progress has been achieved recently. Relaxing assumptions in the sense of forced 2D Euler equations, Vishik [2018a; 2018b] (see also [Albritton et al. 2024; Castro et al. 2025]) established nonuniqueness for vorticities in L^p, $p < \infty$. More recently, using a newly developed refined version of the convex integration technique, the nonuniqueness of weak solutions under vorticity in $L^{1+\varepsilon}$, with ε sufficiently small, was shown by Bruè, Colombo and Kumar [Bruè et al. 2024].

We close this introduction stating our belief that, given the central role of negative Sobolev spaces in PDEs, our methods could find applications elsewhere.

Notation. Given two normed spaces X and Y, the symbol $X \hookrightarrow Y$ means that the identity operator from X into Y is continuous. Given two positive quantities A and B, we write $A \lesssim B$ if there is a constant $C > 0$ such that $A \le CB$. We also use $A \approx B$ if $A \lesssim B$ and $A \gtrsim B$. For $a \in \mathbb{R}$, $a_- = \min\{a, 0\}$, p' denotes the dual exponent of p given by $1/p + 1/p' = 1$ and $|E|$ is the (Lebesgue) measure of a measurable set E.

2. Background

2.1. ***Approximate solutions and H^{-1}-stability.*** For convenience of the reader, we recall the well-known concepts of approximate solutions of Euler equations and their H^{-1}-stability, as introduced in [DiPerna and Majda 1987a] and [Lopes Filho et al. 2000], respectively.

Definition 16. A family of velocity vector fields $\{u^\varepsilon(\cdot\,,t)\}_{\varepsilon>0}$, $t \in [0,T]$, defines an *approximate solution* of (1-1) if for some $L > 1$, it is uniformly bounded[18] in

$$L^\infty([0,T]; L_c^2(\mathbb{R}^n;\mathbb{R}^n)) \cap \mathrm{Lip}((0,T); H_{\mathrm{loc}}^{-L}(\mathbb{R}^n;\mathbb{R}^n)),$$

with $\operatorname{div} u^\varepsilon = 0$ (in the distributional sense), and is weakly consistent with (1-1), in the sense that[19]

$$\int_0^T\!\!\int_{\mathbb{R}^n} \varphi_t \cdot u^\varepsilon + (D\varphi\, u^\varepsilon)\cdot u^\varepsilon \,dx\,dt + \int_{\mathbb{R}^n} \varphi(x,0)\cdot u^\varepsilon(x,0)\,dx \to 0 \quad \text{as } \varepsilon \to 0$$

for every test field $\varphi \in C_c^\infty([0,T) \times \mathbb{R}^n;\mathbb{R}^n)$ with $\operatorname{div}\varphi = 0$. Here, $D\varphi$ is the Jacobian matrix of φ.

Remark 17. If the family is constant, $u^\varepsilon \equiv u$ for all $\varepsilon > 0$, then u is in fact a classical *weak solution* to (1-1).

[18] The uniform bound in $\mathrm{Lip}((0,T); H_{\mathrm{loc}}^{-L}(\mathbb{R}^n;\mathbb{R}^n))$ is a technical assumption in order to guarantee that initial vector fields $u^\varepsilon(\cdot\,,0)$ are well-defined. In practice, this follows easily from the uniform energy bound $L^\infty([0,T]; L_c^2(\mathbb{R}^n;\mathbb{R}^n))$; see [DiPerna and Majda 1987a].

[19] The weak formulation related to domains with boundary is analogous to that of $\mathbb{R}^n$, taking into account the additional boundary condition $u^\varepsilon \cdot n = 0$ (in the trace sense).

Remark 18. There are standard methodologies to construct approximation solution families, e.g., through mollification of initial data, Navier–Stokes approximate solutions (also known as vanishing viscosity method), vortex blob approximations, discrete methods, ...

Definition 19 (H^{-1}-stability). We say that a family $\{u^\varepsilon\}_{\varepsilon>0}$ of approximate solutions of the Euler equation is *H^{-1}-stable* if the corresponding set of vorticities $\{\omega^\varepsilon = \operatorname{curl} u^\varepsilon\}_{\varepsilon>0}$ (i.e., $\omega^\varepsilon_{i,j} = (u^\varepsilon_i)_{x_j} - (u^\varepsilon_j)_{x_i}$ for $i, j = 1, \dots, n$) is precompact in $C((0,T); H^{-1}_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{A}^n))$.

2.2. ***Riesz–Morrey–Tadmor spaces.*** Let $\Pi(Q_0)$ be the set of families of packings[20] $(Q_i)_{i\in I}$, with $Q_i \in \mathcal{D}(Q_0)$.

Definition 20 [Tadmor 2001]. The *Riesz–Morrey–Tadmor spaces*[21] (RMT spaces, in short) $R_{p,q}\log^\alpha(Q_0)$, $1 \le p, q \le \infty$, $\alpha \in \mathbb{R}$, are defined through the condition

$$\|f\|_{R_{p,q}\log^\alpha(Q_0)} = \sup_{(Q_i)_{i\in I}\in\Pi(Q_0)} \left\{ \sum_{i\in I} \left[\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'}} \int_{Q_i} |f| \right]^q \right\}^{\frac{1}{q}} < \infty. \tag{2-1}$$

The corresponding spaces on $\mathbb{R}^n$ are defined analogously.

Remark 21. In particular, *Morrey spaces* are part of this scale. Let $1 \le p \le \infty$, $\alpha \in \mathbb{R}$. The Morrey space $M^{p,\alpha}(Q_0)$ is defined by

$$\|f\|_{M^{p,\alpha}(Q_0)} = \sup_{Q\in\mathcal{D}(Q_0)} \frac{(1-(\log|Q|)_-)^\alpha}{|Q|^{1/p'}} \int_Q |f| < \infty. \tag{2-2}$$

Consequently, $M^{p,\alpha}(Q_0) = R_{p,\infty}\log^\alpha(Q_0)$.

Remark 22. A similar comment to Remark 5 also applies to $R_{p,q}\log^\alpha$ and $M^{p,\alpha}$.

3. Characterization of sparse RMT spaces via maximal operators

Let $0 \le \lambda < n$ and $\alpha \in \mathbb{R}$. For $f \in L^1(Q_0)$, consider the maximal operator

$$M_{\lambda,\alpha,Q_0} f(x) = \sup_{\substack{Q\in\mathcal{D}(Q_0)\\ x\in Q}} |Q|^{\frac{\lambda}{n}-1} (1-(\log|Q|)_-)^\alpha \int_Q |f(y)|\,dy, \quad x \in Q_0. \tag{3-1}$$

In the absence of logarithmic weight (i.e., $\alpha = 0$), we simply write M_{λ,Q_0}. In addition, if $\lambda = 0$ then one recovers the classical (dyadic) maximal operator M_{Q_0}.

In this section we show that, under some natural conditions, the sparse $SR_{p,q}\log^\alpha$ spaces (see Definition 4) admit simple characterizations in terms of maximal operators (3-1). This is in sharp contrast with the parent spaces $R_{p,q}\log^\alpha$.

[20] Families of pairwise disjoint cubes.

[21] Our notation differs from [Tadmor 2001] where the space $R_{p,q}\log^\alpha(Q_0)$ is instead denoted by $V^{pq}(\log V)^\alpha(Q_0)$ (or simply by $V^{pq,\alpha}(Q_0)$). The reason behind this change of notation comes from the Riesz theorem (see (4-7)).

Theorem 23. *Suppose that p, q, α satisfy*

$$1 \leq p \leq q < \infty \quad \textit{and} \quad \alpha \in \mathbb{R} \quad (\alpha \leq 0 \textit{ if } p = q). \tag{3-2}$$

Then

$$SR_{p,q} \log^{\alpha}(Q_0) = M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_0} L^q(Q_0).$$

More precisely,

$$\|f\|_{SR_{p,q} \log^{\alpha}(Q_0)} \approx \|M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_0} f\|_{L^q(Q_0)}, \tag{3-3}$$

where the hidden constants of equivalence are independent of f and Q_0.

Remark 24. When $p = q$, the restriction $\alpha \leq 0$ is necessary to avoid trivial cases. To be more precise, if $\alpha > 0$ and $p = q$, then

$$\|M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_0} f\|_{L^q(Q_0)} < \infty \implies f = 0 \quad \text{a.e. on } Q_0.$$

Indeed, since $M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_0} f(x) = M_{0,\alpha,Q_0} f(x) < \infty$ a.e. $x \in Q_0$, we have

$$\frac{1}{|Q|} \int_Q |f| \leq (1 - (\log |Q|)_-)^{-\alpha} M_{0,\alpha,Q_0} f(x) \tag{3-4}$$

for every $Q \in \mathcal{D}(Q_0)$, with $x \in Q$, and $|Q|$ sufficiently small. Taking limits on both sides of (3-4) as $|Q| \to 0$, and applying the Lebesgue differentiation theorem, we conclude that $f(x) = 0$ for every Lebesgue point x.

Sparse domination principles underlie the characterizations of sparse spaces via maximal functions.

Proposition 25. *Suppose that p, q and α satisfy* (3-2), *and let $f \in L^1(Q_0)$. Then there exists a family $(Q_i)_{i \in I} \in S(Q_0)$ (depending on f and the parameters p, q and α) such that, for almost every $x \in Q_0$,*

$$M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_0} f(x) \leq 2 \max\left\{1, e^{\frac{1}{p}-\frac{1}{q}-\alpha}\left(\frac{pq\alpha}{q-p}\right)^{\alpha}\right\} \sum_{i \in I} \left(\frac{(1 - (\log |Q_i|)_-)^{\alpha}}{|Q_i|^{1/p'+1/q}} \int_{Q_i} |f(y)|\, dy \right) \mathbf{1}_{Q_i}(x) \tag{3-5}$$

Remark 26. As usual, if $p = q$ the constant $\max\{1, e^{1/p-1/q-\alpha}(pq\alpha/(q-p))^{\alpha}\}$ in (3-5) should be interpreted to be equal to 1.

Proof of Proposition 25. The desired decomposition will be obtained by a standard process of exhaustion, whereby for each cube of the starting decomposition we shall apply the process again and again. We set up the selection process by letting $\mathcal{Q}_f = \mathcal{Q}_{f,p,q,\alpha}$ be the collection of $Q \in \mathcal{D}(Q_0)$ such that the following condition is satisfied:

$$\frac{(1 - (\log |Q|)_-)^{\alpha}}{|Q|^{1/p'+1/q}} \int_Q |f(y)|\, dy \geq 2 \max\left\{1, e^{\lambda-\alpha}\left(\frac{\alpha}{\lambda}\right)^{\alpha}\right\} \frac{(1 - (\log |Q_0|)_-)^{\alpha}}{|Q_0|^{1/p'+1/q}} \int_{Q_0} |f(y)|\, dy, \tag{3-6}$$

where $\lambda := \frac{1}{p} - \frac{1}{q}$. If the collection $\mathcal{Q}_f$ is empty then we let $E_{Q_0} = \{Q_0\}$, and we readily verify that (3-5) holds. Otherwise we continue the process selecting $(Q_i)_{i \in \mathbb{N}}$, the family of maximal dyadic cubes in $\mathcal{Q}_f$. By construction, the selected family $(Q_i)_{i\in\mathbb{N}}$ is pairwise disjoint and, therefore, for almost every $x \in Q_0$,

$$\begin{aligned} M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_0} f(x) &= M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_0} f(x) \mathbf{1}_{Q_0 \setminus \bigcup_{i=1}^\infty Q_i}(x) + \sum_{i=1}^\infty M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_0} f(x) \mathbf{1}_{Q_i}(x) \\ &=: \text{(A)} + \text{(B)}. \end{aligned}$$

Next we estimate each of the terms (A) and (B) separately.

Estimate (A). We claim that, for $x \notin \bigcup_{i=1}^\infty Q_i$,

$$M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_0} f(x) \leq 2 \max\Big\{1, e^{\lambda-\alpha}\Big(\frac{\alpha}{\lambda}\Big)^\alpha\Big\} \frac{(1-(\log|Q_0|)_-)^\alpha}{|Q_0|^{1/p'+1/q}} \int_{Q_0} |f(y)|\, dy. \tag{3-7}$$

Indeed, suppose, to the contrary, that for some $x \notin \bigcup_{i=1}^\infty Q_i$, (3-7) does not hold. Then, by the definition of $M_{n(1/p-1/q),\alpha,Q_0}$ (see (3-1)), there exists a dyadic cube $Q \subset Q_0$, such that $x \in Q$, and $Q \in \mathcal{Q}_f$. Consequently, there exists a maximal cube Q_i such that $Q \subset Q_i$, but this leads to a contradiction since $x \notin Q_i$. Therefore, for $x \notin \bigcup_{i=1}^\infty Q_i$, we have

$$\text{(A)} \leq 2 \max\Big\{1, e^{\lambda-\alpha}\Big(\frac{\alpha}{\lambda}\Big)^\alpha\Big\} \frac{(1-(\log|Q_0|)_-)^\alpha}{|Q_0|^{1/p'+1/q}} \int_{Q_0} |f(y)|\, dy.$$

Moreover, from $(Q_i)_{i\in\mathbb{N}} \subset \mathcal{Q}_f$ (see (3-6)) we see that

$$\varphi(|Q_i|)|Q_i|^{-1} \int_{Q_i} |f(y)|\, dy \geq 2 \max\Big\{1, e^{\lambda-\alpha}\Big(\frac{\alpha}{\lambda}\Big)^\alpha\Big\} \varphi(|Q_0|)|Q_0|^{-1} \int_{Q_0} |f(y)|\, dy, \tag{3-8}$$

where

$$\varphi(t) := t^{\frac{1}{p}-\frac{1}{q}}(1-(\log t)_-)^\alpha, \quad t > 0.$$

We distinguish two possible cases.

(I) Suppose first that $\alpha \leq \lambda$. Routine computations show, under this assumption, that φ is a nondecreasing function. It follows from (3-8) that

$$\begin{aligned} \sum_{i=1}^\infty |Q_i| &\leq \frac{1}{2\max\{1, e^{\lambda-\alpha}(\alpha/\lambda)^\alpha\}} \frac{|Q_0|}{\varphi(|Q_0|)\int_{Q_0}|f(y)|\,dy} \sum_{i=1}^\infty \varphi(|Q_i|) \int_{Q_i} |f(y)|\,dy \\ &\leq \frac{1}{2\max\{1, e^{\lambda-\alpha}(\alpha/\lambda)^\alpha\}} \frac{|Q_0|}{\int_{Q_0}|f(y)|\,dy} \sum_{i=1}^\infty \int_{Q_i} |f(y)|\,dy \leq \frac{|Q_0|}{2}. \end{aligned}$$

Note that, in the first step of above computations, we assume that f is not identically zero (almost everywhere) on Q_0; otherwise the desired result (3-5) holds trivially. Therefore, if we assign to the cube Q_0 the set

$$E_{Q_0} := Q_0 \Big\backslash \bigcup_{i=1}^\infty Q_i, \tag{3-9}$$

then

$$\frac{|Q_0|}{2} \leq |E_{Q_0}|, \tag{3-10}$$

i.e., the sparseness condition given in Definition 3(ii) holds.

(II) Suppose now that $\alpha > \lambda$. Let

$$\psi(t) := \begin{cases} t^{\lambda}(1-\log t)^{\alpha} & \text{if } t \in (0, e^{1-\alpha/\lambda}), \\ e^{\lambda-\alpha}(\alpha/\lambda)^{\alpha} & \text{if } t \in [e^{1-\alpha/\lambda}, \infty). \end{cases}$$

It is plain that ψ is a nondecreasing function such that, moreover, $\varphi(t) \le \psi(t)$ for $t > 0$. By (3-8), we have

$$\begin{aligned} \sum_{i=1}^{\infty} |Q_i| &\le \frac{1}{2\max\{1, e^{\lambda-\alpha}(\alpha/\lambda)^{\alpha}\}} \frac{|Q_0|}{\varphi(|Q_0|)\int_{Q_0}|f(y)|\,dy} \sum_{i=1}^{\infty} \varphi(|Q_i|) \int_{Q_i} |f(y)|\,dy \\ &\le \frac{1}{2\max\{1, e^{\lambda-\alpha}(\alpha/\lambda)^{\alpha}\}} \frac{|Q_0|}{\varphi(|Q_0|)\int_{Q_0}|f(y)|\,dy} \sum_{i=1}^{\infty} \psi(|Q_i|) \int_{Q_i} |f(y)|\,dy \\ &\le \frac{1}{2\max\{1, e^{\lambda-\alpha}(\alpha/\lambda)^{\alpha}\}} \frac{|Q_0|\psi(|Q_0|)}{\varphi(|Q_0|)\int_{Q_0}|f(y)|\,dy} \sum_{i=1}^{\infty} \int_{Q_i} |f(y)|\,dy \\ &\le \frac{|Q_0|}{2\max\{1, e^{\lambda-\alpha}(\alpha/\lambda)^{\alpha}\}} \frac{\psi(|Q_0|)}{\varphi(|Q_0|)}. \end{aligned}$$

Furthermore, using the estimate

$$\frac{\psi(|Q_0|)}{\varphi(|Q_0|)} \le \max\Big\{1, e^{\lambda-\alpha}\Big(\frac{\alpha}{\lambda}\Big)^{\alpha}\Big\},$$

we obtain

$$\sum_{i=1}^{\infty} |Q_i| \le \frac{|Q_0|}{2}.$$

Hence the set E_{Q_0} defined by (3-9) satisfies the required sparseness condition (3-10).

Estimate (B). We will show that the procedure used to estimate (A) can be iterated to estimate each term of the sum (B). Fix $i \in \mathbb{N}$. Observe that for $x \in Q_i$, the maximality of the Q_i's and the nesting property of dyadic cubes, yield

$$M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_0} f(x) = M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_i} f(x). \tag{3-11}$$

Indeed, we only need to prove that the right-hand side is $\ge$ the left-hand side. Consider Q a generic dyadic cube such that $x \in Q \subset Q_0$. In particular, $Q \cap Q_i \neq \varnothing$. Now there are two possible situations. Firstly, if $Q \subseteq Q_i$, the cube Q enters in the competition for computing both, the left- and right-hand sides of (3-11), which is consistent with what we wish to prove. Assume now that $Q_i \subset Q$. In this case, since Q_i is a maximal element of $\mathcal{Q}_f$, we must have that $Q \notin \mathcal{Q}_f$. Therefore (see (3-6))

$$\frac{(1-(\log|Q|)_-)^{\alpha}}{|Q|^{1/p'+1/q}} \int_Q |f(y)|\,dy < 2\max\Big\{1, e^{\lambda-\alpha}\Big(\frac{\alpha}{\lambda}\Big)^{\alpha}\Big\} \frac{(1-(\log|Q_0|)_-)^{\alpha}}{|Q_0|^{1/p'+1/q}} \int_{Q_0} |f(y)|\,dy. \tag{3-12}$$

On the other hand, since $Q_i \in \mathcal{Q}_f$, it follows that

$$\begin{aligned} 2\max\Big\{1, e^{\lambda-\alpha}\Big(\frac{\alpha}{\lambda}\Big)^{\alpha}\Big\} \frac{(1-(\log|Q_0|)_-)^{\alpha}}{|Q_0|^{1/p'+1/q}} \int_{Q_0} |f(y)|\,dy &\le \frac{(1-(\log|Q_i|)_-)^{\alpha}}{|Q_i|^{1/p'+1/q}} \int_{Q_i} |f(y)|\,dy \\ &\le M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_i} f(x). \end{aligned} \tag{3-13}$$

Putting together (3-12) and (3-13),

$$\frac{(1-(\log|Q|)_-)^\alpha}{|Q|^{1/p'+1/q}}\int_Q |f(y)|\,dy < M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_i} f(x),$$

and taking now the supremum over all possible dyadic cubes $Q \subset Q_0$ with $x \in Q$, we arrive at the desired upper estimate $\leq$ in (3-11).

By (3-11), we can write (B) as

$$(\mathrm{B}) = \sum_{i=1}^{\infty} M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_i} f(x)\mathbf{1}_{Q_i}(x). \tag{3-14}$$

The proof can be now completed applying the procedure used to estimate (A) to each of the terms that appear on the right-hand side of (3-14). □

Proof of Theorem 23. Let $f \in SR_{p,q}\log^\alpha(Q_0)$. In light of Proposition 25, there exists $(Q_i)_{i\in I} \in S(Q_0)$ (depending, in particular, on f) such that the estimate (3-5) holds. Then, taking L^q-norms on both sides of this estimate, we find

$$\|M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_0} f\|_{L^q(Q_0)} \lesssim \left\| \sum_{i\in I}\left(\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'+1/q}}\int_{Q_i}|f(y)|\,dy\right)\mathbf{1}_{Q_i}\right\|_{L^q(Q_0)}.$$

To estimate the right-hand side, we shall use duality, the properties of sparseness and the Hardy–Littlewood maximal theorem (recall that $q<\infty$). This requires a number of elementary manipulations, but to facilitate the reading we present all the steps,

$$\begin{aligned}
&\|M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_0} f\|_{L^q(Q_0)} \\
&\quad\lesssim \sup_{\|g\|_{L^{q'}(Q_0)}\leq 1}\int_{Q_0}\left(\sum_{i\in I}\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'+1/q}}\int_{Q_i}|f(y)|\,dy\,\mathbf{1}_{Q_i}(x)\right)|g(x)|\,dx \\
&\quad= \sup_{\|g\|_{L^{q'}(Q_0)}\leq 1}\sum_{i\in I}\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'+1/q}}\int_{Q_i}|f(y)|\,dy\int_{Q_i}|g(x)|\,dx \\
&\quad\lesssim \sup_{\|g\|_{L^{q'}(Q_0)}\leq 1}\sum_{i\in I}\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'+1/q}}\frac{|E_{Q_i}|}{|Q_i|}\int_{Q_i}|f(y)|\,dy\int_{Q_i}|g(x)|\,dx \\
&\quad= \sup_{\|g\|_{L^{q'}(Q_0)}\leq 1}\sum_{i\in I}\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'+1/q}}\int_{Q_i}|f(y)|\,dy\int_{E_{Q_i}}\left(\frac{1}{|Q_i|}\int_{Q_i}|g(u)|\,du\right)dx \\
&\quad\leq \sup_{\|g\|_{L^{q'}(Q_0)}\leq 1}\sum_{i\in I}\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'+1/q}}\int_{Q_i}|f(y)|\,dy\int_{E_{Q_i}}M_{Q_0}g(x)\,dx \\
&\quad= \sup_{\|g\|_{L^{q'}(Q_0)}\leq 1}\int_{Q_0}\left(\sum_{i\in I}\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'+1/q}}\int_{Q_i}|f(y)|\,dy\,\mathbf{1}_{E_{Q_i}}(x)\right)M_{Q_0}g(x)\,dx \\
&\quad\leq \sup_{\|g\|_{L^{q'}(Q_0)}\leq 1}\|M_{Q_0}g\|_{L^{q'}(Q_0)}\left\|\sum_{i\in I}\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'+1/q}}\int_{Q_i}|f(y)|\,dy\,\mathbf{1}_{E_{Q_i}}\right\|_{L^q(Q_0)}
\end{aligned}$$

$$\begin{aligned}
&\lesssim q\Big\|\sum_{i\in I}\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'+1/q}}\int_{Q_i}|f(y)|\,dy\,\mathbf{1}_{E_{Q_i}}\Big\|_{L^q(Q_0)}\\
&= q\bigg(\sum_{i\in I}\bigg(\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'+1/q}}\int_{Q_i}|f(y)|\,dy\bigg)^q|E_{Q_i}|\bigg)^{1/q}\\
&\le q\bigg(\sum_{i\in I}\bigg(\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'}}\int_{Q_i}|f(y)|\,dy\bigg)^q\bigg)^{1/q}\\
&\le q\|f\|_{SR_{p,q}\log^\alpha(Q_0)}.
\end{aligned}$$

Conversely, for any $(Q_i)_{i\in I}\in S(Q_0)$, we have (recalling the sparseness condition in Definition 3(ii))

$$\begin{aligned}
\sum_{i\in I}\bigg(\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'}}\int_{Q_i}|f(y)|\,dy\bigg)^q &\le 2\sum_{i\in I}\bigg(\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'+1/q}}\int_{Q_i}|f(y)|\,dy\bigg)^q|E_{Q_i}|\\
&= 2\int_{Q_0}\sum_{i\in I}\bigg(\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'+1/q}}\int_{Q_i}|f(y)|\,dy\bigg)^q\mathbf{1}_{E_{Q_i}}(x)\,dx\\
&\le 2\|M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_0}f\|^q_{L^q(Q_0)}.
\end{aligned}$$

Taking the supremum over all $(Q_i)_{i\in I}\in S(Q_0)$, we arrive at

$$\|f\|^q_{SR_{p,q}\log^\alpha(Q_0)}\le 2\|M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_0}f\|^q_{L^q(Q_0)}. \qquad \square$$

Remark 27. Note that, since the Hardy–Littlewood maximal function is not bounded on L^1, the above proof does not work if $q=\infty$. However, Theorem 23 holds trivially if $q=\infty$ (for any value of $\alpha\in\mathbb{R}$) since

$$\|f\|_{SR_{p,\infty}\log^\alpha(Q_0)}=\|f\|_{M^{p,\alpha}(Q_0)}=\|M_{\frac{n}{p},\alpha,Q_0}f\|_{L^\infty(Q_0)};$$

see (2-2) and (3-1).

3.1. ***Spaces defined on the whole space.*** The analogue of Theorem 23 for $SR_{p,q}\log^\alpha(\mathbb{R}^n)$ can be now formulated in terms of the (dyadic) maximal function

$$M_{\lambda,\alpha}f(x)=\sup_{\substack{Q\in\mathcal{D}(\mathbb{R}^n)\\ x\in Q}}|Q|^{\frac{\lambda}{n}-1}(1-(\log|Q|)_-)^\alpha\int_Q|f(y)|\,dy,\quad x\in\mathbb{R}^n, \tag{3-15}$$

where $\lambda\in[0,n)$ and[22] $\alpha\in\mathbb{R}$. For $q\in[1,\infty)$, we define

$$M_{\lambda,\alpha}L^q(\mathbb{R}^n)=\{f\in L^1_{\mathrm{loc}}(\mathbb{R}^n):\|M_{\lambda,\alpha}f\|_{L^q(\mathbb{R}^n)}<\infty\}.$$

Theorem 28. *Suppose that* p,q,α *satisfy* (3-2). *Then*

$$SR_{p,q}\log^\alpha(\mathbb{R}^n)=M_{n(\frac{1}{p}-\frac{1}{q}),\alpha}L^q(\mathbb{R}^n).$$

[22] In the absence of the log-parameter (i.e., $\alpha=0$), we simply write M_λ instead of $M_{\lambda,0}$. If, in addition, $\lambda=0$ then we get back the classical Hardy–Littlewood maximal function M.

Proof. To facilitate the reading we have divided the proof into four steps, which we now outline. The general goal is to extend the local estimate (3-3) to a global one. For this purpose in Step 1 we construct a suitable nested sequence of cubes Q_k such that $\bigcup_k Q_k = \mathbb{R}^n$ and invoke (3-3) for each Q_k. The quantities involved in (3-3) are local maximal operators and local sparse RMT functionals related to each cube Q_k. In Steps 2 and 3 we develop the asymptotic analysis that will enable us to take limits[23] when $k \to \infty$ in Step 4, and in this manner effect the required transference from local to global estimates for sparse RMT functionals.

Step 1. Consider the sequence of (not dyadic) cubes

$$Q_k := [-2^k, 2^k]^n, \quad k \in \mathbb{N}.$$

According to Theorem 23, with equivalence constants independent of f and k,

$$\|f \mathbf{1}_{Q_k}\|_{SR_{p,q} \log^\alpha(Q_k)} \approx \big\| M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_k}(f \mathbf{1}_{Q_k}) \big\|_{L^q(Q_k)}. \tag{3-16}$$

Step 2. We claim that, for every $k \in \mathbb{N}$ and $x \in Q_k$,

$$M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_k}(f \mathbf{1}_{Q_k})(x) \approx M_{n(\frac{1}{p}-\frac{1}{q}),\alpha}(f \mathbf{1}_{Q_k})(x), \tag{3-17}$$

(see (3-1) and (3-15)). Indeed, the estimate $\lesssim$ follows from the simple observation that

$$\mathcal{D}(Q_k) \backslash \{Q_k\} \subset \mathcal{D}(\mathbb{R}^n),$$

and the fact that the first (dyadic) generation of Q_k, say $\{Q^1_{k,l} : l = 1, \dots, 2^n\}$, gives a pairwise disjoint decomposition of Q_k. In particular,

$$Q_k = \bigcup_{l=1}^{2^n} Q^1_{k,l}$$

and $|Q^1_{k,l}| = 2^{-n}|Q_k|$. Hence, given any $x \in Q_k$,

$$\begin{aligned} |Q_k|^{\frac{1}{p}-\frac{1}{q}-1} \int_{Q_k} |f(y)|\, dy &= 2^{n(\frac{1}{p}-\frac{1}{q}-1)} \sum_{l=1}^{2^n} |Q^1_{k,l}|^{\frac{1}{p}-\frac{1}{q}-1} \int_{Q^1_{k,l}} |f(y)|\, dy \\ &\leq 2^{n(\frac{1}{p}-\frac{1}{q})} M_{n(\frac{1}{p}-\frac{1}{q}),\alpha}(f \mathbf{1}_{Q_k})(x). \end{aligned}$$

Accordingly,

$$\begin{aligned} &M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_k}(f \mathbf{1}_{Q_k})(x) \\ &\quad\leq \sup_{\substack{Q \in \mathcal{D}(Q_k) \backslash \{Q_k\} \\ x \in Q}} |Q|^{\frac{1}{p}-\frac{1}{q}-1}(1-(\log|Q|)_-)^\alpha \int_{Q \cap Q_k} |f(y)|\, dy + |Q_k|^{\frac{1}{p}-\frac{1}{q}-1} \int_{Q_k} |f(y)|\, dy \\ &\quad\lesssim \sup_{\substack{Q \in \mathcal{D}(\mathbb{R}^n) \\ x \in Q}} |Q|^{\frac{1}{p}-\frac{1}{q}-1}(1-(\log|Q|)_-)^\alpha \int_Q |f(y)| \mathbf{1}_{Q_k}(y)\, dy + M_{n(\frac{1}{p}-\frac{1}{q}),\alpha}(f \mathbf{1}_{Q_k})(x) \\ &\quad\approx M_{n(\frac{1}{p}-\frac{1}{q}),\alpha}(f \mathbf{1}_{Q_k})(x). \end{aligned}$$

[23] In particular, to justify the passage to the limit requires estimates that are independent of Q_k.

Next we focus on the estimate $\gtrsim$ in (3-17). Consider $x \in Q_k$, and $Q \in \mathcal{D}(\mathbb{R}^n)$ with $Q \ni x$ and moreover $Q \not\subset Q_k$ (indeed, if $Q \subset Q_k$ then $Q \in \mathcal{D}(Q_k)$). We cannot assert that $Q_k \subset Q$ (recall that Q_k is not dyadic), but what is certainly true is that there exists $Q_k^1 \in \mathcal{D}(Q_k)$, in the first dyadic generation (i.e., $2\,\ell(Q_k^1) = \ell(Q_k)$), such that $Q_k^1 \subset Q$ (because $\mathcal{D}(Q_k)\setminus\{Q_k\} \subset \mathcal{D}(\mathbb{R}^n)$ and $x \in Q_k$). Note that, in particular, $|Q| \geq |Q_k^1| = \ell(Q_k^1)^n = 2^{-n}\ell(Q_k)^n = 2^{-n}|Q_k| = 2^{kn} > 1$. Since $\frac{1}{p} - \frac{1}{q} - 1 < 0$, we have

$$\begin{aligned}|Q|^{\frac{1}{p}-\frac{1}{q}-1}(1-(\log|Q|)_-)^{\alpha} = |Q|^{\frac{1}{p}-\frac{1}{q}-1} \leq |Q_k^1|^{\frac{1}{p}-\frac{1}{q}-1} &= 2^{n(1+\frac{1}{q}-\frac{1}{p})}|Q_k|^{\frac{1}{p}-\frac{1}{q}-1} \\ &= 2^{n(1+\frac{1}{q}-\frac{1}{p})}|Q_k|^{\frac{1}{p}-\frac{1}{q}-1}(1-(\log|Q_k|)_-)^{\alpha},\end{aligned}$$

which yields

$$\begin{aligned}|Q|^{\frac{1}{p}-\frac{1}{q}-1}(1-(\log|Q|)_-)^{\alpha} \int_{Q\cap Q_k} |f(y)|\,dy &\\ \leq 2^{n(1+\frac{1}{q}-\frac{1}{p})}|Q_k|^{\frac{1}{p}-\frac{1}{q}-1}&(1-(\log|Q_k|)_-)^{\alpha} \int_{Q_k} |f(y)|\,dy \\ \leq 2^{n(1+\frac{1}{q}-\frac{1}{p})} &M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_k}(f\mathbf{1}_{Q_k})(x).\end{aligned}$$

Consequently,

$$M_{n(\frac{1}{p}-\frac{1}{q}),\alpha}(f\mathbf{1}_{Q_k})(x) \leq 2^{n(1+\frac{1}{q}-\frac{1}{p})} M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_k}(f\mathbf{1}_{Q_k})(x).$$

This completes the proof of (3-17).

On the other hand, since

$$Q_k \subset Q_{k+1} \quad \text{and} \quad \bigcup_{k\in\mathbb{N}} Q_k = \mathbb{R}^n, \tag{3-18}$$

we have

$$\lim_{k\to\infty} M_{n(\frac{1}{p}-\frac{1}{q}),\alpha}(f\mathbf{1}_{Q_k})(x)\mathbf{1}_{Q_k}(x) = M_{n(\frac{1}{p}-\frac{1}{q}),\alpha} f(x), \quad x \in \mathbb{R}^n. \tag{3-19}$$

Indeed, given any fixed $x \in \mathbb{R}^n$, we have

$$M_{n(\frac{1}{p}-\frac{1}{q}),\alpha}(f\mathbf{1}_{Q_k})(x)\mathbf{1}_{Q_k}(x) \leq M_{n(\frac{1}{p}-\frac{1}{q}),\alpha} f(x), \quad \text{for all } k \in \mathbb{N},$$

and (see (3-18))

$$M_{n(\frac{1}{p}-\frac{1}{q}),\alpha}(f\mathbf{1}_{Q_k})(x)\mathbf{1}_{Q_k}(x) \leq M_{n(\frac{1}{p}-\frac{1}{q}),\alpha}(f\mathbf{1}_{Q_{k+1}})(x)\mathbf{1}_{Q_{k+1}}(x). \tag{3-20}$$

By the monotone convergence theorem for sequences of real numbers, we derive

$$\begin{aligned}\lim_{k\to\infty} M_{n(\frac{1}{p}-\frac{1}{q}),\alpha}(f\mathbf{1}_{Q_k})(x)\mathbf{1}_{Q_k}(x) &= \sup_{k\in\mathbb{N}} M_{n(\frac{1}{p}-\frac{1}{q}),\alpha}(f\mathbf{1}_{Q_k})(x) \\ &= \sup_{\substack{Q\in\mathcal{D}(\mathbb{R}^n)\\ x\in Q}} |Q|^{\frac{1}{p}-\frac{1}{q}-1}(1-(\log|Q|)_-)^{\alpha} \sup_{k\in\mathbb{N}} \int_{Q\cap Q_k} |f(y)|\,dy \\ &= M_{n(\frac{1}{p}-\frac{1}{q}),\alpha} f(x),\end{aligned}$$

where we have used (3-18) in the last step.

It follows from (3-17) that

$$\left\| M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_k}(f\mathbf{1}_{Q_k})\right\|_{L^q(Q_k)} \approx \left\| M_{n(\frac{1}{p}-\frac{1}{q}),\alpha}(f\mathbf{1}_{Q_k})\mathbf{1}_{Q_k}\right\|_{L^q(\mathbb{R}^n)},$$

uniformly with respect to k. Consequently, applying the monotone convergence theorem (see (3-19) and (3-20)):

$$\lim_{k\to\infty}\left\| M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_k}(f\mathbf{1}_{Q_k})\right\|_{L^q(Q_k)} \approx \| M_{n(\frac{1}{p}-\frac{1}{q}),\alpha} f\|_{L^q(\mathbb{R}^n)}. \tag{3-21}$$

Step 3. Next we deal with the left-hand side of (3-16). We claim that

$$\| f\mathbf{1}_{Q_k}\|_{SR_{p,q}\log^\alpha(Q_k)} \approx \| f\mathbf{1}_{Q_k}\|_{SR_{p,q}\log^\alpha(\mathbb{R}^n)} \tag{3-22}$$

uniformly with respect to k and f.

The estimate $\lesssim$ can be obtained as follows. Given any $(Q_i)_{i\in I}\in S(Q_k)$, there are two possible scenarios. (I) $Q_i\neq Q_k$ for every $i\in I$. Then $(Q_i)_{i\in I}\in S(\mathbb{R}^n)$, since $\mathcal{D}(Q_k)\backslash\{Q_k\}\subset\mathcal{D}(\mathbb{R}^n)$. Clearly, this implies $\| f\mathbf{1}_{Q_k}\|_{SR_{p,q}\log^\alpha(Q_k)} \le \| f\mathbf{1}_{Q_k}\|_{SR_{p,q}\log^\alpha(\mathbb{R}^n)}$. (II) Suppose now that there is $i_0\in I$ such that $Q_{i_0}=Q_k$. In particular, $(Q_i)_{i\in I\backslash\{i_0\}}\in S(\mathbb{R}^n)$ and $Q_i\subset Q_k$ for $i\in I\backslash\{i_0\}$. Now, the first dyadic decomposition of Q_k (i.e., $\{Q^1_{k,l}: l=1,\dots,2^n\}$) is formed by pairwise disjoint cubes in $\mathcal{D}(\mathbb{R}^n)$ (so, in particular, $\{Q^1_{k,l}: l=1,\dots,2^n\}\in S(\mathbb{R}^n)$) with $|Q^1_{k,l}|=2^{-n}|Q_k|$. Hence, in this case, we can split the sum related to the $SR_{p,q}\log^\alpha(Q_k)$-norm as

$$\begin{aligned}
\sum_{i\in I}&\left(\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'}}\int_{Q_i}|f(y)|\,dy\right)^q \\
&= \sum_{\substack{i\in I\\ i\neq i_0}}\left(\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'}}\int_{Q_i}|f(y)|\,dy\right)^q + \left(\frac{1}{|Q_k|^{1/p'}}\int_{Q_k}|f(y)|\,dy\right)^q \\
&\le \| f\mathbf{1}_{Q_k}\|^q_{SR_{p,q}\log^\alpha(\mathbb{R}^n)} + \frac{1}{2^{nq/p'}}\left(\sum_{l=1}^{2^n}\frac{1}{|Q^1_{k,l}|^{1/p'}}\int_{Q^1_{k,l}}|f(y)|\,dy\right)^q \\
&\lesssim \| f\mathbf{1}_{Q_k}\|^q_{SR_{p,q}\log^\alpha(\mathbb{R}^n)} + \sum_{l=1}^{2^n}\left(\frac{1}{|Q^1_{k,l}|^{1/p'}}\int_{Q^1_{k,l}}|f(y)|\,dy\right)^q \\
&\lesssim \| f\mathbf{1}_{Q_k}\|^q_{SR_{p,q}\log^\alpha(\mathbb{R}^n)}.
\end{aligned}$$

Therefore, taking the supremum over all possible $(Q_i)_{i\in I}\in S(Q_k)$, we achieve

$$\| f\mathbf{1}_{Q_k}\|_{SR_{p,q}\log^\alpha(Q_k)} \lesssim \| f\mathbf{1}_{Q_k}\|_{SR_{p,q}\log^\alpha(\mathbb{R}^n)},$$

i.e., the estimate $\lesssim$ in (3-22) is shown.

To deal with the converse estimate, for any $\mathcal{Q}=(Q_i)_{i\in I}\in S(\mathbb{R}^n)$, we consider the index set

$$I_k := \{i\in I: Q_i\subset Q_k\}.$$

Therefore we can split

$$\begin{aligned}\sum_{i\in I}\biggl(\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'}}\int_{Q_i}|f(y)|\mathbf{1}_{Q_k}(y)\,dy\biggr)^q \\ =\sum_{i\in I_k}\biggl(\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'}}\int_{Q_i}|f(y)|\,dy\biggr)^q+\sum_{i\in I\setminus I_k}\biggl(\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'}}\int_{Q_i\cap Q_k}|f(y)|\,dy\biggr)^q \\ =: R_1+R_2.\end{aligned} \tag{3-23}$$

Note that $(Q_i)_{i\in I_k}\in S(Q_k)$ (since $(Q_i)_{i\in I_k}\in S(\mathbb{R}^n)\cap\mathcal{D}(Q_k)$). Accordingly

$$R_1\le\|f\mathbf{1}_{Q_k}\|^q_{SR_{p,q}\log^\alpha(Q_k)}. \tag{3-24}$$

Concerning R_2, we argue as follows. Let $i\in I\setminus I_k$, i.e., $Q_i\not\subset Q_k$. Assume further that $Q_i\cap Q_k\neq\varnothing$. Note that Q_k is not a dyadic cube in $\mathbb{R}^n$, but its first dyadic generation $\{Q^1_{k,l}:l=1,\dots,2^n\}$ is formed by dyadic cubes in $\mathbb{R}^n$. Since Q_k can be expressed as the disjoint union of the cubes $Q^1_{k,l}$, we can assert that there exists a unique $l(i)$ such that $Q_i\cap Q^1_{k,l(i)}\neq\varnothing$. By the structure of dyadic cubes in $\mathbb{R}^n$, we have either $Q_i\subset Q^1_{k,l(i)}$ or $Q^1_{k,l(i)}\subset Q_i$. The former is not possible; otherwise, $Q_i\subset Q_k$ but $i\notin I_k$. Hence $Q^1_{k,l(i)}\subset Q_i$, therefore

$$\int_{Q_i\cap Q_k}|f(y)|\,dy=\int_{Q^1_{k,l(i)}}|f(y)|\,dy. \tag{3-25}$$

For $l\in\{1,\dots,2^n\}$, we define

$$\mathcal{Q}_{k,l}:=\{Q_i:i\in I\setminus I_k\text{ and }Q^1_{k,l}\subset Q_i\}.$$

The above argument leads to

$$(Q_i)_{i\in I\setminus I_k}=\bigcup_{l=1}^{2^n}\mathcal{Q}_{k,l}. \tag{3-26}$$

Moreover, since the Q_i's are dyadic cubes in $\mathbb{R}^n$, the elements of $\mathcal{Q}_{k,l}=\{Q_1,Q_2,\dots\}$ can be ordered in such a way that $Q^1_{k,l}\subset Q_1\subset Q_2\subset\cdots$ We cannot exclude the possibility that some of the cubes in $\mathcal{Q}_{k,l}$ coincide, but the number of these cubes is uniformly bounded by the sparse constant η in Definition 3. Therefore, by (3-25) and (3-26),

$$\begin{aligned}R_2&\le\sum_{i\in I\setminus I_k}\biggl(\frac{1}{|Q_i|^{1/p'}}\int_{Q^1_{k,l(i)}}|f(y)|\,dy\biggr)^q=\sum_{l=1}^{2^n}\biggl(\int_{Q^1_{k,l}}|f(y)|\,dy\biggr)^q\sum_{Q_i\in\mathcal{Q}_{k,l}}\frac{1}{|Q_i|^{q/p'}}\\&\lesssim\sum_{l=1}^{2^n}\biggl(\int_{Q^1_{k,l}}|f(y)|\,dy\biggr)^q\sum_{j=k}^{\infty}2^{-jnq/p'}\approx\sum_{l=1}^{2^n}\biggl(\int_{Q^1_{k,l}}|f(y)|\,dy\biggr)^q2^{-knq/p'}\\&=\sum_{l=1}^{2^n}\biggl(\frac{1}{|Q^1_{k,l}|^{1/p'}}\int_{Q^1_{k,l}}|f(y)|\,dy\biggr)^q\le\|f\mathbf{1}_{Q_k}\|^q_{SR_{p,q}\log^\alpha(Q_k)}.\end{aligned} \tag{3-27}$$

Combining (3-23), (3-24) and (3-27),

$$\sum_{i\in I}\biggl(\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'}}\int_{Q_i}|f(y)|\mathbf{1}_{Q_k}(y)\,dy\biggr)^q \lesssim \|f\mathbf{1}_{Q_k}\|^q_{SR_{p,q}\log^\alpha(Q_k)}$$

for all $(Q_i)_{i\in I}\in S(\mathbb{R}^n)$. In particular,

$$\|f\mathbf{1}_{Q_k}\|_{SR_{p,q}\log^\alpha(\mathbb{R}^n)} \lesssim \|f\mathbf{1}_{Q_k}\|_{SR_{p,q}\log^\alpha(Q_k)},$$

completing the proof of (3-22).

By the lattice property of sparse RMT spaces, we have (recall (3-18))

$$\|f\mathbf{1}_{Q_k}\|_{SR_{p,q}\log^\alpha(\mathbb{R}^n)} \le \|f\mathbf{1}_{Q_{k+1}}\|_{SR_{p,q}\log^\alpha(\mathbb{R}^n)}$$

and

$$\|f\mathbf{1}_{Q_k}\|_{SR_{p,q}\log^\alpha(\mathbb{R}^n)} \le \|f\|_{SR_{p,q}\log^\alpha(\mathbb{R}^n)}, \quad k\in\mathbb{N}.$$

Hence, applying the monotone convergence theorem and the dominated convergence theorem, we obtain

$$\begin{aligned}\lim_{k\to\infty}\|f\mathbf{1}_{Q_k}\|_{SR_{p,q}\log^\alpha(\mathbb{R}^n)} &= \sup_{k\in\mathbb{N}}\|f\mathbf{1}_{Q_k}\|_{SR_{p,q}\log^\alpha(\mathbb{R}^n)}\\ &= \sup_{(Q_i)_{i\in I}\in S(\mathbb{R}^n)}\ \sup_{k\in\mathbb{N}}\biggl\{\sum_{i\in I}\biggl[\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'}}\int_{Q_i\cap Q_k}|f|\biggr]^q\biggr\}^{\frac{1}{q}}\\ &= \sup_{(Q_i)_{i\in I}\in S(\mathbb{R}^n)}\ \lim_{k\to\infty}\biggl\{\sum_{i\in I}\biggl[\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'}}\int_{Q_i\cap Q_k}|f|\biggr]^q\biggr\}^{\frac{1}{q}}\\ &= \sup_{(Q_i)_{i\in I}\in S(\mathbb{R}^n)}\biggl\{\sum_{i\in I}\biggl[\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'}}\int_{Q_i}|f|\biggr]^q\biggr\}^{\frac{1}{q}}.\end{aligned}$$

In other words, we have shown that

$$\lim_{k\to\infty}\|f\mathbf{1}_{Q_k}\|_{SR_{p,q}\log^\alpha(\mathbb{R}^n)} = \|f\|_{SR_{p,q}\log^\alpha(\mathbb{R}^n)}. \tag{3-28}$$

Step 4. Finally, taking limits on both sides of (3-16) as $k\to\infty$, and invoking (3-21), (3-22) and (3-28), we achieve the desired estimate

$$\|f\|_{SR_{p,q}\log^\alpha(\mathbb{R}^n)} \approx \|M_{n(\frac{1}{p}-\frac{1}{q}),\alpha}f\|_{L^q(\mathbb{R}^n)}.$$

□

4. A sparse approach to H^{-1}-stability

As already mentioned in Section 1.5, one of the main features of the theory of sparse function spaces lies in the fact that, unlike their classical parent spaces, they often admit complete explicit characterizations. Indeed, Theorem 6 provides us with the following surprising (informal) characterization: *$SR_{p,q}$ spaces can be identified with negative Sobolev spaces*. Before we give the proof of this result, we introduce some basic notation.

Consider the *Riesz potential operators* I_λ, $\lambda \in (0, n)$, formally defined, for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, by

$$I_\lambda f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\lambda}}\, dy, \quad x \in \mathbb{R}^n.$$

For $1 < q < \infty$, we let

$$H_q^{-\lambda}(\mathbb{R}^n) := \{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{H_q^{-\lambda}} = \|I_\lambda f\|_{L^q(\mathbb{R}^n)} < \infty \}, \tag{4-1}$$

the *Riesz potential space*, and its associated lattice

$$\mathcal{H}_q^{-\lambda}(\mathbb{R}^n) := \{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{\mathcal{H}_q^{-\lambda}} = \|I_\lambda(|f|)\|_{L^q(\mathbb{R}^n)} < \infty \}. \tag{4-2}$$

It is plain that

$$\mathcal{H}_q^{-\lambda}(\mathbb{R}^n) \subset H_q^{-\lambda}(\mathbb{R}^n).$$

Furthermore, as it is customary, we shall suppress the subindex $q = 2$ and simply write

$$\mathcal{H}^{-\lambda}(\mathbb{R}^n) := \mathcal{H}_2^{-\lambda}(\mathbb{R}^n) \quad \text{and} \quad H^{-\lambda}(\mathbb{R}^n) := H_2^{-\lambda}(\mathbb{R}^n). \tag{4-3}$$

4.1. ***Proof of Theorem 6.*** In order to be able to use a result of Muckenhoupt and Wheeden [1974] we introduce the fractional maximal[24] operator, defined for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$,

$$\mathcal{M}_\lambda f(x) := \sup_{x \in Q} |Q|^{\frac{\lambda}{n}-1} \int_Q |f(y)|\, dy, \quad x \in \mathbb{R}^n,$$

where the supremum runs over all (not necessarily dyadic) cubes Q in $\mathbb{R}^n$, with $x \in Q$. It is plain (see (3-15)) that $M_{\lambda,0} f(x) \le \mathcal{M}_\lambda f(x)$, and, although this pointwise inequality cannot be reversed, it is well-known that by the $\frac{1}{3}$-translation trick (see [Christ 1988]) we have the equivalence

$$\|M_{\lambda,0} f\|_{L^q(\mathbb{R}^n)} \approx \|\mathcal{M}_\lambda f\|_{L^q(\mathbb{R}^n)}. \tag{4-4}$$

Putting together Theorem 28, with $\alpha = 0$ and $\lambda = n\left(\frac{1}{p} - \frac{1}{q}\right)$, and (4-4), we get

$$\|f\|_{SR_{p,q}(\mathbb{R}^n)} \approx \|\mathcal{M}_{n(\frac{1}{p}-\frac{1}{q})} f\|_{L^q(\mathbb{R}^n)}. \tag{4-5}$$

For the maximal operator $\mathcal{M}_\lambda$ we have the trivial estimate

$$\mathcal{M}_\lambda f(x) \le c_n I_\lambda(|f|)(x), \quad x \in \mathbb{R}^n,$$

where c_n depends only on n. In fact, via the Muckenhoupt–Wheeden theorem [1974, Theorem 1], we achieve, for $0 < q < \infty$,

$$\|\mathcal{M}_\lambda f\|_{L^q(\mathbb{R}^n)} \approx \|I_\lambda(|f|)\|_{L^q(\mathbb{R}^n)}. \tag{4-6}$$

Combining (4-2), (4-5) and (4-6) we arrive at

$$\|f\|_{SR_{p,q}(\mathbb{R}^n)} \approx \|f\|_{\mathcal{H}_q^{-n(\frac{1}{p}-\frac{1}{q})}(\mathbb{R}^n)},$$

as we wished to show. □

[24] Compare with the dyadic local version M_{λ,Q_0} defined in (3-1).

Our next result refers to the limiting case $p = q$ in Theorem 6 and it can be viewed as the sparse counterpart of the Riesz's theorem (see [Domínguez and Milman 2021, p. 1062]),

$$R_{p,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n). \tag{4-7}$$

Theorem 29. *Let* $1 < p < \infty$. *Then*

$$SR_{p,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n).$$

Proof. This is an immediate consequence of Theorem 28 (with $p = q$ and $\alpha = 0$) and the classical Hardy–Littlewood maximal theorem:

$$\|Mf\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}, \quad p > 1. \qquad \square$$

4.2. ***On the difference between*** $R_{1,2}(\mathbb{R}^2)$ ***and*** $SR_{1,2}(\mathbb{R}^2)$**.** In view of Theorem 29 and (4-7),

$$SR_{p,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n) = R_{p,p}(\mathbb{R}^n), \quad 1 < p < \infty.$$

One may be tempted to think that $SR_{p,q}(\mathbb{R}^n) = R_{p,q}(\mathbb{R}^n)$ for general values of p and q. However, this is far from being true. Next we concentrate on the most relevant case for the purposes of this paper, i.e., we will show that the embedding

$$SR_{1,2}(\mathbb{R}^2) \subset R_{1,2}(\mathbb{R}^2)$$

is strict, in the sense that,

$$SR_{1,2}(\mathbb{R}^2) \neq R_{1,2}(\mathbb{R}^2). \tag{4-8}$$

As a by-product (see (1-9)),

$$R_{1,2}(\mathbb{R}^2) \not\subseteq H^{-1}(\mathbb{R}^2). \tag{4-9}$$

We shall use an elementary but indirect method. It is well known (see, e.g., [Lions 1996, p. 141]) that the largest rearrangement invariant space embedded in $H^{-1}_{\text{loc}}(\mathbb{R}^2)$ is the *Lorentz space* defined for a given cube Q_0[25] by

$$L^{(1,2)}(Q_0) := \left\{ f : \|f\|_{L^{(1,2)}(Q_0)} = \left[\int_0^1 (t f^{**}_{Q_0}(t))^2 \frac{dt}{t} \right]^{\frac{1}{2}} < \infty \right\}.$$

Here, we use standard notation: $f^*_{Q_0}$ is the nonincreasing rearrangement of a measurable function f restricted to Q_0, more precisely, $f^*_{Q_0}$ is the generalized inverse of the distribution function

$$\lambda_f(\alpha) = \big|\{x \in Q_0 : |f(x)| \geq \alpha\}\big|, \quad \alpha > 0,$$

and $f^{**}_{Q_0}$ is the maximal function given by $f^{**}_{Q_0}(t) = t^{-1} \int_0^t f^*_{Q_0}(s)\, ds$. When there is no danger of confusion, we shall simply drop Q_0 and use the notation f^* and f^{**} rather than $f^*_{Q_0}$ and $f^{**}_{Q_0}$, respectively.

[25] Without loss of generality, we may assume that $|Q_0| = 1$.

Using that $t \mapsto tf^{**}(t)$ is an increasing function, we have, for every $u \in (0,1)$,

$$\|f\|_{L^{(1,2)}(Q_0)} \geq \left[\int_u^1 (tf^{**}(t))^2 \frac{dt}{t}\right]^{\frac{1}{2}} \geq (-\log u)^{\frac{1}{2}} u f^{**}(u).$$

It follows that

$$L^{(1,2)}(Q_0) \subset L^{(1,\infty)}(\log L)^{\frac{1}{2}}(Q_0),$$

where $L^{(1,\infty)}(\log L)^{\frac{1}{2}}(Q_0)$ is the *Lorentz–Zygmund space* defined by[26]

$$\|f\|_{L^{(1,\infty)}(\log L)^{\frac{1}{2}}(Q_0)} := \sup_{0<t<1} t(1-\log t)^{\frac{1}{2}} f^{**}(t).$$

Then

$$L^{(1,\infty)}(\log L)^{\frac{1}{2}}(Q_0) \not\subseteq H^{-1}_{\text{loc}}(\mathbb{R}^2),$$

which in turn yields (see Theorem 6)

$$L^{(1,\infty)}(\log L)^{\frac{1}{2}}(Q_0) \not\subseteq SR_{1,2,\text{loc}}(\mathbb{R}^2). \tag{4-10}$$

On the other hand, by the Hardy–Littlewood inequality for rearrangements (see, e.g., [Bennett and Sharpley 1988, Lemma 2.1, p. 44]) and (4-10),

$$\begin{aligned}
\|f\|^2_{R_{1,2}(Q_0)} &= \sup_{(Q_i)_{i\in I}\in\Pi(Q_0)} \sum_{i\in I}\left(\int_{Q_i}|f|\right)^2 \\
&\leq \sup_{(Q_i)_{i\in I}\in\Pi(Q_0)} \sum_{i\in I}\left(\int_0^{|Q_i|} f^*\right)\left(\int_{Q_i}|f|\right) \\
&= \sup_{(Q_i)_{i\in I}\in\Pi(Q_0)} \sum_{i\in I} |Q_i| f^{**}(|Q_i|) \int_{Q_i}|f| \\
&\leq \|f\|_{L^{(1,\infty)}(\log L)^{\frac{1}{2}}(Q_0)} \sup_{(Q_i)\in\Pi(Q_0)} \sum_{i\in I}(1-\log|Q_i|)^{-\frac{1}{2}} \int_{Q_i}|f| \\
&\leq \|f\|_{L^{(1,\infty)}(\log L)^{\frac{1}{2}}(Q_0)} \sup_{(Q_i)_{i\in I}\in\Pi(Q_0)} \sum_{i\in I}\int_{Q_i}|f| \\
&\leq \|f\|_{L^{(1,\infty)}(\log L)^{\frac{1}{2}}(Q_0)} \|f\|_{L^1(Q_0)} \\
&\leq \|f\|^2_{L^{(1,\infty)}(\log L)^{\frac{1}{2}}(Q_0)}.
\end{aligned}$$

It follows that

$$L^{(1,\infty)}(\log L)^{\frac{1}{2}}(Q_0) \subset R_{1,2,\text{loc}}(\mathbb{R}^2). \tag{4-11}$$

Consequently, (4-8) now follows from (4-10) and (4-11).

[26] A basic reference to Lorentz–Zygmund spaces is [Bennett and Rudnick 1980].

4.3. ***Proof of Theorem 9.*** (i) Let Q_0 be a cube. Recalling that

$$\|f\|_{SR_{\frac{2n}{n+2},2}(Q_0)} = \sup_{(Q_i)_{i\in I}\in S(Q_0)} \left\{\sum_{i\in I}\left(|Q_i|^{\frac{1}{n}-\frac{1}{2}}\int_{Q_i}|f|\right)^2\right\}^{\frac{1}{2}}, \tag{4-12}$$

we observe $s_1(f) = \|f\|_{SR_{\frac{2n}{n+2},2}(Q_0)}$ (see (1-11)). Then

$$s_1(X) = \sup_{\|f\|_{X(Q_0)}\le 1} \|f\|_{SR_{\frac{2n}{n+2},2}(Q_0)} < \infty \quad\Longleftrightarrow\quad X(Q_0) \hookrightarrow SR_{\frac{2n}{n+2},2}(Q_0).$$

The desired assertions follow immediately from (1-9) and (1-10).

(ii) We need to introduce some notation: Given $\mathcal{Q} = (Q_i)_{i\in I} \in S(Q_0)$, note that $\mathcal{Q}$ can be split as $\mathcal{Q} = \bigcup_{k=0}^{\infty} \mathbb{D}_{k;Q_0}(\mathcal{Q})$, where

$$\mathbb{D}_{k;Q_0} := \{Q \in \mathcal{D}(Q_0) : \ell(Q) = 2^{-k}\ell(Q_0)\}, \quad \mathbb{D}_{k;Q_0}(\mathcal{Q}) := \mathbb{D}_{k;Q_0} \cap \mathcal{Q}.$$

When there is no danger of confusion, we use the simplified notation $\mathbb{D}_k$ and $\mathbb{D}_k(\mathcal{Q})$. By construction, $\mathbb{D}_k(\mathcal{Q})$ is formed by pairwise disjoint cubes (i.e., $\mathbb{D}_k(\mathcal{Q}) \subset \Pi(Q_0)$), and for $Q_i \in \mathbb{D}_k(\mathcal{Q})$, we have $|Q_i| = 2^{-kn}|Q_0|$.

Assume that $\lim_{N\to\infty} s_N(X) = 0$, i.e., given any $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$, such that for all $N > N_0$,

$$\sup_{\|f\|_{X(Q_0)}\le 1} s_N(f) \le \varepsilon. \tag{4-13}$$

Let $\mathcal{Q} = (Q_i)_{i\in I} \in S(Q_0)$ and let $f \in X(Q_0)$ be such that $\|f\|_{X(Q_0)} \le 1$. Then

$$\left\{\sum_{i\in I}\left(|Q_i|^{\frac{1}{n}-\frac{1}{2}}\int_{Q_i}|f|\right)^2\right\}^{\frac{1}{2}} = \left\{\sum_{k=0}^{\infty}\sum_{i\in I:Q_i\in\mathbb{D}_k(\mathcal{Q})}\left(|Q_i|^{\frac{1}{n}-\frac{1}{2}}\int_{Q_i}|f|\right)^2\right\}^{\frac{1}{2}} \le \mathrm{I} + \mathrm{II}, \tag{4-14}$$

where

$$\mathrm{I} := \left\{\sum_{k=0}^{N_0}\sum_{i\in I:Q_i\in\mathbb{D}_k(\mathcal{Q})}\left(|Q_i|^{\frac{1}{n}-\frac{1}{2}}\int_{Q_i}|f|\right)^2\right\}^{\frac{1}{2}}$$

and

$$\mathrm{II} := \left\{\sum_{k=N_0+1}^{\infty}\sum_{i\in I:Q_i\in\mathbb{D}_k(\mathcal{Q})}\left(|Q_i|^{\frac{1}{n}-\frac{1}{2}}\int_{Q_i}|f|\right)^2\right\}^{\frac{1}{2}}.$$

It follows from (1-11) and (4-13) that

$$\mathrm{II} = \left\{\sum_{i\in I:Q_i\in\mathbb{D}_{\le N_0+1}(\mathcal{Q})}\left(|Q_i|^{\frac{1}{n}-\frac{1}{2}}\int_{Q_i}|f|\right)^2\right\}^{\frac{1}{2}} \le s_{N_0+2}(f) \le \varepsilon. \tag{4-15}$$

On the other hand, we obviously have

$$\mathrm{I} \le \left\{\sum_{k=0}^{N_0}\sum_{Q\in\mathbb{D}_k}\left(|Q|^{\frac{1}{n}-\frac{1}{2}}\int_{Q}|f|\right)^2\right\}^{\frac{1}{2}}. \tag{4-16}$$

Combining (4-14), (4-15) and (4-16), we find that, for all $(Q_i)_{i\in I}\in S(Q_0)$ and for all f in the unit ball of $X(Q_0)$,

$$\left\{\sum_{i\in I}\left(|Q_i|^{\frac{1}{n}-\frac{1}{2}}\int_{Q_i}|f|\right)^2\right\}^{\frac{1}{2}}\le\left\{\sum_{k=0}^{N_0}\sum_{Q\in\mathbb{D}_k}\left(|Q|^{\frac{1}{n}-\frac{1}{2}}\int_{Q}|f|\right)^2\right\}^{\frac{1}{2}}+\varepsilon.$$

Therefore, by (4-12),

$$\|f\|_{SR_{\frac{2n}{n+2},2}(Q_0)}\le\left\{\sum_{k=0}^{N_0}\sum_{Q\in\mathbb{D}_k}\left(|Q|^{\frac{1}{n}-\frac{1}{2}}\int_{Q}|f|\right)^2\right\}^{\frac{1}{2}}+\varepsilon. \tag{4-17}$$

Let $\mathbb{D}_{\ge N_0;Q_0}=\mathbb{D}_{\ge N_0}=\bigcup_{k=0}^{N_0}\mathbb{D}_k$; then the cardinality of $\mathbb{D}_{\ge N_0}$ is

$$L:=\frac{2^{n(N_0+1)}-1}{2^n-1}.$$

Consider the linear operator

$$T:f\in X(Q_0)\mapsto\left(|Q|^{\frac{1}{n}-\frac{1}{2}}\int_Q f\right)_{Q\in\mathbb{D}_{\ge N_0}}\in\ell_2^L.$$

It is easy to see that T is well-defined: if $f\in X(Q_0)$ (and hence $f\ge 0$) then

$$\begin{aligned}\|Tf\|_{\ell_2^L}&=\left\{\sum_{k=0}^{N_0}\sum_{Q\in\mathbb{D}_k}\left(|Q|^{\frac{1}{n}-\frac{1}{2}}\int_{Q}|f|\right)^2\right\}^{\frac{1}{2}}\\&=|Q_0|^{\frac{1}{n}-\frac{1}{2}}\left\{\sum_{k=0}^{N_0}2^{-kn(\frac{1}{n}-\frac{1}{2})2}\sum_{Q\in\mathbb{D}_k}\left(\int_Q|f|\right)^2\right\}^{\frac{1}{2}}\\&\le|Q_0|^{\frac{1}{n}-\frac{1}{2}}\left\{\sum_{k=0}^{N_0}2^{-kn(\frac{1}{n}-\frac{1}{2})2}\left(\sum_{Q\in\mathbb{D}_k}\int_Q|f|\right)^2\right\}^{\frac{1}{2}}\\&=|Q_0|^{\frac{1}{n}-\frac{1}{2}}\left\{\sum_{k=0}^{N_0}2^{-kn(\frac{1}{n}-\frac{1}{2})2}\right\}^{\frac{1}{2}}\|f\|_{L^1(Q_0)}\\&\lesssim 2^{N_0(\frac{n}{2}-1)}\|f\|_{X(Q_0)}.\end{aligned}$$

Furthermore, T is compact, since it is a finite rank operator. We can equivalently rewrite (4-17) in terms of T as follows: for every $f\in X(Q_0)$, $\|f\|_{X(Q_0)}\le 1$,

$$\|f\|_{SR_{\frac{2n}{n+2},2}(Q_0)}\le\|Tf\|_{\ell_2^L}+\varepsilon. \tag{4-18}$$

Let $\{f_l\}_{l\in\mathbb{N}}$ be a bounded sequence in $X(Q_0)$ (without loss we may assume that, for all l, $\|f_l\|_X\le\frac{1}{2}$). The compactness of $T:X(Q_0)\to\ell_2^L$ guarantees (modulo passing to a subsequence) that $\{Tf_l\}_{l\in\mathbb{N}}$ is convergent in ℓ_2^L. Accordingly, there exists l_0 such that

$$\|Tf_l - Tf_{l'}\|_{\ell_2^L} \le \varepsilon, \quad \text{if } l, l' \ge l_0.$$

Therefore, by (4-18),

$$\|f_l - f_{l'}\|_{SR_{\frac{2n}{n+2},2}(Q_0)} \le 2\varepsilon.$$

Consequently, from (1-10) we see that $\{f_l\}_{l\in\mathbb{N}}$ is a Cauchy sequence in H^{-1}.

Next we show the converse statement, i.e., if[27] $X_c \xrightarrow{\text{compactly}} \mathcal{H}^{-1}_{\text{loc}}(\mathbb{R}^n)$ then

$$\lim_{N\to\infty} s_N(X) = 0. \tag{4-19}$$

By assumption $U_{X(Q_0)}$, the closure of the unit ball of $X(Q_0)$, is a compact set in $\mathcal{H}^{-1}(\mathbb{R}^n)$. In particular, for any $\delta > 0$ there exist $f_1, \dots, f_L \in U_{X(Q_0)}$ such that

$$U_{X(Q_0)} \subset \bigcup_{l=1}^{L} B\Big(f_l, \frac{\delta}{2}\Big),$$

where $B(f_l, \delta/2)$ denotes the ball in $\mathcal{H}^{-1}$ centered at f_l and radius $\delta/2$. Hence, for any $f \in X(Q_0)$, $\|f\|_{X(Q_0)} \le 1$, there exists $l \in \{1, \dots, L\}$ such that

$$\|f - f_l\|_{\mathcal{H}^{-1}(\mathbb{R}^n)} < \frac{\delta}{2}.$$

As a consequence (see (1-9))

$$\begin{aligned} s_N(f) \le s_N(f - f_l) + s_N(f_l) \le s_1(f - f_l) + s_N(f_l) &\lesssim \|f - f_l\|_{\mathcal{H}^{-1}(\mathbb{R}^n)} + s_N(f_l) \\ &< \frac{\delta}{2} + \sup_{l\in\{1,\dots,L\}} s_N(f_l). \end{aligned} \tag{4-20}$$

Assume momentarily that

$$\lim_{N\to\infty} s_N(\omega) = 0 \quad \text{for every } \omega \in \mathcal{H}^{-1}(\mathbb{R}^n). \tag{4-21}$$

In particular, we have, for N sufficiently large depending only on δ,

$$\sup_{l\in\{1,\dots,L\}} s_N(f_l) \le \frac{\delta}{2}.$$

Inserting this estimate into (4-20), we conclude that (4-19) holds.

To complete the proof, it remains to show (4-21): Fix $\chi \in C^\infty(\mathbb{R}^n)$ with[28] $\operatorname{supp} \chi \subset B(0,1)$ and $\chi \ge 0$. For every dyadic cube $Q_{jm} \in \mathbb{D}_j$, we let[29]

$$\chi_{jm}(f) := (\chi_{jm}, f) = \int_{\mathbb{R}^n} \chi_{jm}(x) f(x)\, dx, \quad m \in \mathbb{Z}^n, \tag{4-22}$$

[27]Recall that $X \subset L^1_{\text{loc},+}(\mathbb{R}^n)$; see (4-3).

[28]One may think that $\chi(x) = e^{-1/(1-|x|^2)}\mathbf{1}_{B(0,1)}(x)$.

[29]$\chi_{jm}(f)$ should be adequately interpreted in the distributional sense.

where $\chi_{jm}(x) := 2^{jn}\chi(2^j x - m)$. Without loss of generality, we may assume that $\operatorname{supp}\chi_{jm} \subset dQ_{jm}$ for a fixed constant $d > 1$ and

$$\inf_{x\in Q_{jm}} \chi_{jm}(x) \gtrsim 2^{jn}. \tag{4-23}$$

Then, we have

$$\begin{aligned} s_N(\omega) &= \sup_{\mathcal{Q}\in S(\mathbb{R}^n)} \biggl[\sum_{k=N-1}^{\infty} \sum_{i\in I: Q_i\in\mathbb{D}_k(\mathcal{Q})} \Bigl(|Q_i|^{\frac{1}{n}-\frac{1}{2}} \int_{Q_i} d\omega\Bigr)^2\biggr]^{\frac{1}{2}} \\ &\le \biggl[\sum_{k=N-1}^{\infty} \sum_{Q\in\mathbb{D}_k} \Bigl(|Q|^{\frac{1}{n}-\frac{1}{2}} \int_{Q} d\omega\Bigr)^2\biggr]^{\frac{1}{2}} \\ &= \biggl[\sum_{k=N-1}^{\infty} \sum_{Q\in\mathbb{D}_k} \Bigl(2^{-kn(\frac{1}{n}+\frac{1}{2})} \int_{Q} 2^{kn}\, d\omega\Bigr)^2\biggr]^{\frac{1}{2}} \\ &\le \biggl[\sum_{k=N-1}^{\infty} 2^{k(-1-\frac{n}{2})2} \sum_{Q\in\mathbb{D}_k} \Bigl(\int_{Q} \chi_Q(x)\, d\omega\Bigr)^2\biggr]^{\frac{1}{2}}. \end{aligned} \tag{4-24}$$

Note that the last step is true because both $\chi_Q \ge 0$ and $\omega \ge 0$. Furthermore, using well-known estimates of function spaces in terms of local means (see, e.g., [Triebel 2008, Theorem 1.15]), we get

$$\biggl[\sum_{k=0}^{\infty} 2^{k(-1-\frac{n}{2})2} \sum_{Q\in\mathbb{D}_k} \Bigl(\int_{Q} \chi_Q(x)\, d\omega\Bigr)^2\biggr]^{\frac{1}{2}} \lesssim \|\omega\|_{H^{-1}(\mathbb{R}^n)}.$$

In particular, this implies

$$\lim_{N\to\infty} \biggl[\sum_{k=N-1}^{\infty} 2^{k(-1-\frac{n}{2})2} \sum_{Q\in\mathbb{D}_k} \Bigl(\int_{Q} \chi_Q(x)\, d\omega\Bigr)^2\biggr]^{\frac{1}{2}} = 0$$

provided that $\omega \in H^{-1}(\mathbb{R}^n) \cap BM_c^+$ and (see (4-24))

$$\lim_{N\to\infty} s_N(\omega) = 0.$$

This shows the desired result (4-21). □

5. Computability of sparse indices

In this section we estimate the sparse indices for familiar scales of spaces.

Proposition 30 (sparse indices for L^p). *Let $n \ge 2$ and $p > \frac{2n}{n+2}$. Then, for every $N \in \mathbb{N}$,*

$$s_N(L^p) \lesssim 2^{-Nn(\frac{2+n}{2n} - \frac{1}{\min\{2,p\}})}. \tag{5-1}$$

Proof. We can estimate $s_N(f)$ (see (1-11)) as follows: Let $\mathcal{Q}=(Q_i)_{i\in I}\in S(Q_0)$. By Hölder's inequality we have

$$\sum_{Q\in\mathbb{D}_{\le N-1}(\mathcal{Q})}\Big(|Q|^{\frac{1}{n}-\frac{1}{2}}\int_Q|f|\Big)^2=\sum_{k=N-1}^{\infty}\sum_{i\in I:Q_i\in\mathbb{D}_k(\mathcal{Q})}\Big(|Q_i|^{\frac{1}{n}-\frac{1}{2}}\int_{Q_i}|f|\Big)^2$$
$$\le\sum_{k=N-1}^{\infty}\sum_{i\in I:Q_i\in\mathbb{D}_k(\mathcal{Q})}|Q_i|^{\frac{2}{n}-\frac{2}{p}+1}\Big(\int_{Q_i}|f|^p\Big)^{\frac{2}{p}}$$
$$=|Q_0|^{(\frac{2+n}{2n}-\frac{1}{p})2}\sum_{k=N-1}^{\infty}2^{-kn(\frac{2+n}{2n}-\frac{1}{p})2}\sum_{i\in I:Q_i\in\mathbb{D}_k}\Big(\int_{Q_i}|f|^p\Big)^{\frac{2}{p}}. \tag{5-2}$$

We distinguish two possible cases. First, assume that $p\le 2$. Then

$$\sum_{k=N-1}^{\infty}2^{-kn(\frac{2+n}{2n}-\frac{1}{p})2}\sum_{i\in I:Q_i\in\mathbb{D}_k}\Big(\int_{Q_i}|f|^p\Big)^{\frac{2}{p}}\le\sum_{k=N-1}^{\infty}2^{-kn(\frac{2+n}{2n}-\frac{1}{p})2}\Big(\sum_{i\in I:Q_i\in\mathbb{D}_k}\int_{Q_i}|f|^p\Big)^{\frac{2}{p}}$$
$$\le\|f\|^2_{L^p(Q_0)}\sum_{k=N-1}^{\infty}2^{-kn(\frac{2+n}{2n}-\frac{1}{p})2}$$
$$\le c_n\Big(\frac{2+n}{2n}-\frac{1}{p}\Big)^{-1}2^{-Nn(\frac{2+n}{2n}-\frac{1}{p})2}\|f\|^2_{L^p(Q_0)}. \tag{5-3}$$

On the other hand, if $p>2$ then, by Hölder's inequality,

$$\sum_{k=N-1}^{\infty}2^{-kn(\frac{2+n}{2n}-\frac{1}{p})2}\sum_{i\in I:Q_i\in\mathbb{D}_k}\Big(\int_{Q_i}|f|^p\Big)^{\frac{2}{p}}\le\sum_{k=N-1}^{\infty}2^{-kn(\frac{2+n}{2n}-\frac{1}{2})2}\Big(\sum_{i\in I:Q_i\in\mathbb{D}_k}\int_{Q_i}|f|^p\Big)^{\frac{2}{p}}$$
$$\le\|f\|^2_{L^p(Q_0)}\sum_{k=N-1}^{\infty}2^{-kn(\frac{2+n}{2n}-\frac{1}{2})2}$$
$$\le c_n2^{-Nn(\frac{2+n}{2n}-\frac{1}{2})2}\|f\|^2_{L^p(Q_0)}. \tag{5-4}$$

Combining (5-2)–(5-4) (and noting that all estimates are uniform with respect to $\mathcal{Q}$), we obtain

$$s_N(f)\lesssim 2^{-Nn(\frac{2+n}{2n}-\frac{1}{\min\{2,p\}})}\|f\|_{L^p(Q_0)}.$$

Taking the supremum over all $f\in L^p(Q_0)$, $\|f\|_{L^p(Q_0)}\le 1$, we achieve the desired estimate (5-1). □

Proposition 31 (sparse indices for $M^{p,\alpha}$). *Let $n\ge 2$. If N is sufficiently large*[30] *then*

$$s_N(M^{p,\alpha})\lesssim\begin{cases}2^{-N(\frac{2}{n}-\frac{1}{p})\frac{n}{2}}N^{-\frac{\alpha}{2}} & \text{if } p>\frac{n}{2},\ \alpha\in\mathbb{R},\\ N^{\frac{-\alpha+1}{2}} & \text{if } p=\frac{n}{2},\ \alpha>1.\end{cases}$$

[30]To be more precise, $2^{(N-1)n}>|Q_0|$. This assumption is not restrictive since we are only interested in the asymptotic behavior of indices. For the sake of completeness, we mention that $s_N(M^{p,\alpha})$ with $2^{(N-1)n}\le|Q_0|$ can be also computed using the same ideas, but now the log-parameter α does not play any role.

Proof. By definition (see (2-2)),

$$\int_Q |f| \le |Q|^{\frac{1}{p'}}(1-(\log|Q|)_-)^{-\alpha}\,\|f\|_{M^{p,\alpha}(Q_0)} \quad \text{for all } Q\in\mathcal{D}(Q_0).$$

Therefore, for any $\mathcal{Q}=(Q_i)_{i\in I}\in S(Q_0)$,

$$\begin{aligned}
\sum_{Q\in\mathbb{D}_{\le N-1}(\mathcal{Q})}\Big(|Q|^{\frac{1}{n}-\frac{1}{2}}\int_Q|f|\Big)^2
&= \sum_{k=N-1}^{\infty}\ \sum_{i\in I:Q_i\in\mathbb{D}_k(\mathcal{Q})}\Big(|Q_i|^{\frac{1}{n}-\frac{1}{2}}\int_{Q_i}|f|\Big)^2\\
&\le \|f\|_{M^{p,\alpha}(Q_0)}\sum_{k=N-1}^{\infty}\ \sum_{i\in I:Q_i\in\mathbb{D}_k(\mathcal{Q})}|Q_i|^{\frac{2}{n}-\frac{1}{p}}(1-(\log|Q_i|)_-)^{-\alpha}\int_{Q_i}|f|\\
&\approx \|f\|_{M^{p,\alpha}(Q_0)}\sum_{k=N-1}^{\infty}2^{-kn(\frac{2}{n}-\frac{1}{p})}k^{-\alpha}\sum_{i\in I:Q_i\in\mathbb{D}_k(\mathcal{Q})}\int_{Q_i}|f|\\
&\le \|f\|_{M^{p,\alpha}(Q_0)}\|f\|_{L^1(Q_0)}\sum_{k=N-1}^{\infty}2^{-kn(\frac{2}{n}-\frac{1}{p})}k^{-\alpha}\\
&\le \|f\|^2_{M^{p,\alpha}(Q_0)}\sum_{k=N-1}^{\infty}2^{-kn(\frac{2}{n}-\frac{1}{p})}k^{-\alpha}. \qquad (5\text{-}5)
\end{aligned}$$

Furthermore

$$\sum_{k=N-1}^{\infty}2^{-kn(\frac{2}{n}-\frac{1}{p})}k^{-\alpha}\approx\begin{cases}2^{-Nn(\frac{2}{n}-\frac{1}{p})}N^{-\alpha} & \text{if } p>\frac{n}{2},\ \alpha\in\mathbb{R},\\ N^{-\alpha+1} & \text{if } p=\frac{n}{2},\ \alpha>1.\end{cases} \qquad (5\text{-}6)$$

The desired result follows then from (5-5) and (5-6). ☐

Proposition 32 (sparse indices for RMT spaces). *Let $n\ge 2$. If N is sufficiently large then*

$$s_N(R_{p,2}\log^\alpha)\lesssim\begin{cases}2^{-Nn(\frac{n+2}{2n}-\frac{1}{p})}N^{-\alpha} & \text{if } p>\frac{2n}{n+2},\ \alpha\in\mathbb{R},\\ N^{-\alpha+\frac{1}{2}} & \text{if } p=\frac{2n}{n+2},\ \alpha>\frac{1}{2}.\end{cases}$$

Proof. Let $f\in R_{p,2}\log^\alpha(Q_0)$ and $\mathcal{Q}=(Q_i)_{i\in I}\in S(Q_0)$. Since $\mathbb{D}_{k;Q_0}(\mathcal{Q})\subset\mathbb{D}_{k;Q_0}\subset\Pi(Q_0)$, we get

$$\begin{aligned}
\sum_{Q\in\mathbb{D}_{\le N-1}(\mathcal{Q})}\Big(|Q|^{\frac{1}{n}-\frac{1}{2}}\int_Q|f|\Big)^2 &= \sum_{k=N-1}^{\infty}\ \sum_{i\in I:Q_i\in\mathbb{D}_k(\mathcal{Q})}\Big(|Q_i|^{\frac{1}{n}-\frac{1}{2}}\int_{Q_i}|f|\Big)^2\\
&\lesssim \sum_{k=N-1}^{\infty}2^{-kn2(\frac{n+2}{2n}-\frac{1}{p})}(1+k)^{-2\alpha}\sum_{Q\in\mathbb{D}_k}\Big(\frac{(1-(\log|Q|)_-)^\alpha}{|Q|^{1/p'}}\int_Q|f|\Big)^2\\
&\le \|f\|^2_{R_{p,2}\log^\alpha(Q_0)}\sum_{k=N-1}^{\infty}2^{-kn2(\frac{n+2}{2n}-\frac{1}{p})}(1+k)^{-2\alpha}.
\end{aligned}$$

Since the previous estimates are uniform with respect to $\mathcal{Q}$, we derive

$$s_N(f)^2 \lesssim \mathcal{I}_N \, \|f\|^2_{R_{p,2}\log^\alpha(Q_0)}, \tag{5-7}$$

where

$$\mathcal{I}_N := \sum_{k=N-1}^{\infty} 2^{-kn2(\frac{n+2}{2n}-\frac{1}{p})}(1+k)^{-2\alpha}.$$

Observe that

$$\mathcal{I}_N \approx \begin{cases} 2^{-Nn2(\frac{n+2}{2n}-\frac{1}{p})}N^{-2\alpha} & \text{if } p > \frac{2n}{n+2},\ \alpha \in \mathbb{R}, \\ N^{-2\alpha+1} & \text{if } p = \frac{2n}{n+2},\ \alpha > \frac{1}{2}. \end{cases} \tag{5-8}$$

Plugging (5-8) into (5-7) and taking the supremum over all $f \in R_{p,2}\log^\alpha$, $\|f\|_{R_{p,2}\log^\alpha} \le 1$, we arrive at the desired estimate for $s_N(R_{p,2}\log^\alpha)$. □

Remark 33. The proof of Proposition 32 gives a slightly stronger result using the refined class $CR_{p,q}\log^\alpha$. These spaces are defined using congruent cubes, by[31]

$$\|f\|_{CR_{p,q}\log^\alpha(Q_0)} := \sup_{k\in\mathbb{N}_0} \sup_{(Q_i)_{i\in I}\in\mathbb{D}_{k;Q_0}} \left\{\sum_{i\in I}\left[\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{\frac{1}{p'}}}\int_{Q_i}|f|\right]^q\right\}^{\frac{1}{q}} < \infty.$$

Clearly $\|f\|_{CR_{p,q}\log^\alpha(Q_0)} \le \|f\|_{R_{p,q}\log^\alpha(Q_0)}$ and hence

$$s_N(R_{p,q}\log^\alpha(Q_0)) \le s_N(CR_{p,q}\log^\alpha(Q_0)).$$

Then Proposition 32 with $CR_{p,2}\log^\alpha$ also holds.

6. Proof of Theorem 2

(ii) ⇒ (i). Assume that $\{u^\varepsilon\}_{\varepsilon>0}$ is sparse stable, i.e., $\{\omega^\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $S_\Psi(\mathbb{R}^n)$ for some decay Ψ. It follows from (1-15) that $\{\omega^\varepsilon\}_{\varepsilon>0}$ is a precompact set in $H^{-1}_{\text{loc}}(\mathbb{R}^n)$, i.e., $\{u^\varepsilon\}_{\varepsilon>0}$ is H^{-1}-stable.

(i) ⇒ (ii). Define

$$\Psi(N) := \sup_{\varepsilon>0} s_N(\omega^\varepsilon). \tag{6-1}$$

It is clear that Ψ is decreasing. Furthermore $s_N(\omega^\varepsilon) \le \Psi(N)$, which yields that for all $\varepsilon > 0$,

$$\|\omega^\varepsilon\|_{S_\Psi(\mathbb{R}^n)} = \sup_{N\in\mathbb{N}} \frac{s_N(\omega^\varepsilon)}{\Psi(N)} \le 1,$$

i.e., $\{\omega^\varepsilon\}_{\varepsilon>0}$ is bounded in $S_\Psi(\mathbb{R}^n)$.

[31]Note that $CR_{p,q}\log^\alpha(Q_0)$ can be equivalently introduced as the set of all $f \in L^1(Q_0)$ such that, for every $k \ge 0$,

$$\left\{\sum_{Q_i\in\mathbb{D}_{k;Q_0}}\left(\int_{Q_i}|f|\right)^q\right\}^{1/q} \lesssim \begin{cases} 2^{kn/p'}(1+k)^{-\alpha} & \text{if } 2^k \ge \ell(Q_0), \\ 2^{kn/p'} & \text{if } 2^k \le \ell(Q_0). \end{cases}$$

It remains to show that $\Psi(N)\to 0$ as $N\to\infty$. The proof follows closely the one of (4-19). On account of (i), the set $W=\overline{\{\omega^\varepsilon\}_{\varepsilon>0}}\subset BM_c^+$ is compact (in $\mathcal{H}^{-1}(\mathbb{R}^n)$). In particular, for any $\delta>0$ there exist $\omega_1,\dots,\omega_L\in W$ such that

$$W\subset\bigcup_{l=1}^{L}B\Big(\omega_l,\frac{\delta}{2}\Big).$$

As a by-product, for any $\varepsilon>0$ we can find $l\in\{1,\dots,L\}$ such that

$$\|\omega^\varepsilon-\omega_l\|_{\mathcal{H}^{-1}(\mathbb{R}^n)}<\frac{\delta}{2}. \tag{6-2}$$

Therefore

$$s_N(\omega^\varepsilon)\le s_N(\omega^\varepsilon-\omega_l)+s_N(\omega_l)\lesssim\|\omega^\varepsilon-\omega_l\|_{\mathcal{H}^{-1}(\mathbb{R}^n)}+s_N(\omega_l),$$

where the hidden equivalence constant is independent of ε. As a consequence (see (6-1) and (6-2))

$$\Psi(N)\lesssim\frac{\delta}{2}+\sup_{l\in\{1,\dots,L\}}s_N(\omega_l).$$

Then, by (4-21), $\lim_{N\to\infty}\Psi(N)=0$. □

7. Sharpening Morrey regularity of DiPerna–Majda via V_Ψ

As already mentioned in Section 1.1, a famous 2D result due to DiPerna and Majda [1987a] asserts strong convergence of approximate solutions with initial vortex sheet satisfying Morrey regularity $M^{1,\alpha}(\mathbb{R}^2)$ with $\alpha>1$. A proof can be obtained from the compactness assertion (see (1-2))

$$M_c^{\frac{n}{2},\alpha}(\mathbb{R}^n)\xhookrightarrow{\text{compactly}}H^{-1}_{\text{loc}}(\mathbb{R}^n),\quad \alpha>1,\ n\ge 2. \tag{7-1}$$

The goal of this section is to show that these results can be further improved using the sparse techniques developed in previous sections combined with extrapolation techniques.

7.1. V_Ψ-spaces. Given a decay function Ψ, in this section we construct a new Besov-type space V_Ψ, whose sparse indices are controlled by Ψ.

Definition 34. Let $V_\Psi(\mathbb{R}^n)$ be the space formed by all tempered distributions $f\in\mathcal{S}'(\mathbb{R}^n)$ such that[32]

$$\|f\|_{V_\Psi(\mathbb{R}^n)}:=\sup_{N\in\mathbb{N}_0}\frac{1}{\Psi(N)^2}\sum_{j=N}^{\infty}2^{-2j}\|\Delta_j f\|_{L^\infty(\mathbb{R}^n)}<\infty.$$

Let $V_\Psi^+(\mathbb{R}^n)=V_\Psi(\mathbb{R}^n)\cap BM_c^+$.

Remark 35. The construction of the space V_Ψ is in some sense “dual” to the one used to define the classical Vishik space[33] B_Γ, where $\Gamma:[0,\infty)\to(0,\infty)$ is an increasing function with $\lim_{t\to\infty}\Gamma(t)=\infty$

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[32] As usual, $\{\Delta_j\}_{j\in\mathbb{N}_0}$ refers to standard (inhomogeneous) Littlewood–Paley operators on $\mathbb{R}^n$.
[33] See [Domínguez and Milman 2024].

(a "growth function") and the norm of B_Γ is given by

$$\|f\|_{B_\Gamma(\mathbb{R}^n)} := \sup_{N\in\mathbb{N}_0} \frac{1}{\Gamma(N)} \sum_{j=0}^{N} \|\Delta_j f\|_{L^\infty(\mathbb{R}^n)}. \tag{7-2}$$

Note that we could have elements $f \in B_\Gamma(\mathbb{R}^n)$ such that $\sum_{j=0}^{\infty} \|\Delta_j f\|_{L^\infty(\mathbb{R}^n)} = \infty$ (i.e., f does not belong to the Besov space[34] $B^0_{\infty,1}(\mathbb{R}^n)$), as long as the growth of the corresponding partial sums is controlled by Γ. On the other hand, $V_\Psi(\mathbb{R}^n)$ is formed by elements $f \in B^{-2}_{\infty,1}(\mathbb{R}^n)$, the classical Besov space of negative order, equipped with

$$\|f\|_{B^{-2}_{\infty,1}(\mathbb{R}^n)} = \sum_{j=0}^{\infty} 2^{-2j} \|\Delta_j f\|_{L^\infty(\mathbb{R}^n)}, \tag{7-3}$$

such that the remainder of the corresponding series in (7-3) has a prescribed decay given by $\Psi(N)^2$. The connection of these spaces becomes apparent through the use of stream functions. Let $\omega \in V_\Psi(\mathbb{R}^n)$, and let ψ be a stream function, i.e., $\Delta\psi = \omega$. Using Fourier multipliers one can show that

$$\omega \in V_\Psi(\mathbb{R}^n) \quad \Longleftrightarrow \quad \sup_{N\in\mathbb{N}_0} \frac{1}{\Psi(N)^2} \sum_{j=N}^{\infty} \|\Delta_j \psi\|_{L^\infty(\mathbb{R}^n)} < \infty.$$

In other words, the space $V_\Psi(\mathbb{R}^n)$ is formed by vorticities ω with corresponding stream functions ψ satisfying the "dual" of the Vishik condition (7-2).

7.2. *V_Ψ-regularity of Euler flows.* In this section, we restrict ourselves to the following sufficiently rich class of decays.

Definition 36 (admissible/doubling decays). Let Ψ be a decay. We say that Ψ is

(i) *admissible*[35] provided that

$$\sum_{r=0}^{N} (2^r \Psi(r))^2 \lesssim (2^N \Psi(N))^2, \quad N \in \mathbb{N}_0;$$

(ii) *doubling* provided that $\Psi(ct) \gtrsim \Psi(t)$ for some $c > 1$.

We are now ready to state the main result of this section.

Theorem 37. *Let Ψ be an admissible doubling decay. Then*:

(i) $V^+_\Psi(\mathbb{R}^n)_c \hookrightarrow S_\Psi(\mathbb{R}^n)_c$. *As a consequence* (*see* (1-15)), $V^+_\Psi(\mathbb{R}^n)_c \xrightarrow{\text{compactly}} H^{-1}_{\text{loc}}(\mathbb{R}^n)$.

[34] Recall that the Besov spaces $B^s_{p,q}(\mathbb{R}^n)$, $s \in \mathbb{R}$, $p, q \in [1,\infty]$, are endowed with the norm

$$\|f\|_{B^s_{p,q}(\mathbb{R}^n)} = \Big(\sum_{j=0}^{\infty} 2^{jsq} \|\Delta_j f\|^q_{L^p(\mathbb{R}^n)}\Big)^{1/q}.$$

See, e.g., [Stein 1970; Bennett and Sharpley 1988; Triebel 2008].

[35] This is a very weak assumption on the monotonicity properties of Ψ. Basic examples of admissible decays are $\Psi(t) = t^{-\lambda}$, $\Psi(t) = (\log t)^{-\lambda}$, where $\lambda > 0$, (or more generally, concatenations of logarithms) and their products. Exponential decays $\Psi(t) = 2^{-Ct}$, $C \geq 1$, are excluded. However, this is not restrictive since $V_{2^{-Ct}}(\mathbb{R}^n) \hookrightarrow V_{t^{-\lambda}}(\mathbb{R}^n)$.

(ii) *Let $\{u^\varepsilon\}_{\varepsilon>0}$ be a family of approximate solutions to Euler equations* (1-1) *such that the related family of vorticities $\{\omega^\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $L^\infty([0,T]; V_\Psi^+(\mathbb{R}^n)_c)$. Then $\{u^\varepsilon\}_{\varepsilon>0}$ has a strong limit u in $L^\infty([0,T]; L^2_{\mathrm{loc}}(\mathbb{R}^n))$, where u is a solution with no concentrations.*

Specializing the previous result with $\Psi(t) = t^{(1-\alpha)/2}$, we are able to improve[36] (7-1) in the following sense.

Theorem 38. *Assume that $\alpha > 1$. Then*

$$M^{\frac{n}{2},\alpha}(\mathbb{R}^n) \hookrightarrow V_\Psi(\mathbb{R}^n).$$

Furthermore, this embedding is strict in the sense that $M^{\frac{n}{2},\alpha}(\mathbb{R}^n) \neq V_\Psi(\mathbb{R}^n)$.

The proofs of these results are based on extrapolation methods developed in the following section.

7.3. *Extrapolation characterization of V_Ψ.* Let (A_0, A_1) be an interpolation pair[37] of Banach spaces. Recall that the *K-functional* relative to (A_0, A_1) is defined by

$$K(t, f; A_0, A_1) = \|f\|_{A_0+tA_1} = \inf_{f=f_0+f_1} (\|f_0\|_{A_0} + t\|f_1\|_{A_1})$$

for $t > 0$ and $f \in A_0 + A_1$.

Theorem 39. *Suppose that Ψ is an admissible doubling decay. Then*

$$\|f\|_{V_\Psi(\mathbb{R}^n)} \approx \sup_{t\in(0,1)} \frac{K(t, f; B^{-2}_{\infty,1}(\mathbb{R}^n), B^0_{\infty,1}(\mathbb{R}^n))}{\Psi(-\log t)^2}. \tag{7-4}$$

Remark 40. This result shows that the V_Ψ spaces can be described as extrapolation spaces[38] relative to the classical Besov pair $(B^{-2}_{\infty,1}, B^0_{\infty,1})$, a fact that will be very useful later, since it enables the transfer of fundamental properties of the classical Besov spaces to V_Ψ.

Remark 41. The assumption that Ψ is doubling is necessary in order to ensure that the right-hand side of (7-4) is nontrivial. For instance, for the admissible decay $\Psi(t) = 2^{-Ct}$, $C \in \left(\frac{1}{2}, 1\right)$,

$$\sup_{t\in(0,1)} \frac{K(t, f; B^{-2}_{\infty,1}(\mathbb{R}^n), B^0_{\infty,1}(\mathbb{R}^n))}{t^{2C}} < \infty \iff f = 0.$$

Proof of Theorem 39. We use the retraction method of interpolation theory. Recall that $\ell^s_1(L^\infty(\mathbb{R}^n))$, $s \in \mathbb{R}$, is the vector-valued sequence space equipped with the norm

$$\|\{f_j\}_{j\in\mathbb{N}_0}\|_{\ell^s_1(L^\infty(\mathbb{R}^n))} = \sum_{j=0}^\infty 2^{js}\|f_j\|_{L^\infty(\mathbb{R}^n)}.$$

[36]We only focus on the most interesting case $p = n/2$, but similar improvements can also be obtained in the noncritical regime $p > n/2$.

[37]Loosely speaking, $A_0 + A_1$ makes sense.

[38]with respect to the so-called Σ-method of extrapolation (see [Jawerth and Milman 1991]).

It is well known (see [Domínguez and Milman 2024, Appendix A1]) that $B^s_{\infty,1}(\mathbb{R}^n)$ is a retract of $\ell^s_1(L^\infty(\mathbb{R}^n))$ via

$$f \mapsto \{\Delta_j f\}_{j \in \mathbb{N}_0}.$$

In particular this yields

$$K(t, f; B^{-2}_{\infty,1}(\mathbb{R}^n), B^s_{\infty,1}(\mathbb{R}^n)) \approx K\big(t, \{\Delta_j f\}_{j\in\mathbb{N}_0}; \ell_1^{-2}(L^\infty(\mathbb{R}^n)), \ell_1^s(L^\infty(\mathbb{R}^n))\big).$$

Furthermore, a well-known estimate for K-functionals asserts

$$K\big(t, \{f_j\}_{j\in\mathbb{N}_0}; \ell_1^{-2}(L^\infty(\mathbb{R}^n)), \ell_1^s(L^\infty(\mathbb{R}^n))\big) \approx \sum_{j=0}^\infty 2^{js} \min\{2^{(-2-s)j}, t\} \|f_j\|_{L^\infty(\mathbb{R}^n)}.$$

Hence (letting $s = 0$)

$$\begin{aligned} K\big(2^{-2N}, f; B^{-2}_{\infty,1}(\mathbb{R}^n), B^0_{\infty,1}(\mathbb{R}^n)\big) &\approx \sum_{j=0}^\infty \min\{2^{-2j}, 2^{-2N}\} \|\Delta_j f\|_{L^\infty(\mathbb{R}^d)} \\ &= 2^{-2N} \sum_{j=0}^N \|\Delta_j f\|_{L^\infty(\mathbb{R}^n)} + \sum_{j=N+1}^\infty 2^{-2j} \|\Delta_j f\|_{L^\infty(\mathbb{R}^n)}. \end{aligned}$$

This implies that

$$\sup_{N\in\mathbb{N}_0} \frac{K\big(2^{-2N}, f; B^{-2}_{\infty,1}(\mathbb{R}^n), B^s_{\infty,1}(\mathbb{R}^n)\big)}{\Psi(N)^2} \approx \mathcal{A} + \|f\|_{V_\Psi(\mathbb{R}^n)}, \tag{7-5}$$

where

$$\mathcal{A} := \sup_{N\in\mathbb{N}_0} \frac{2^{-2N} \sum_{j=0}^N \|\Delta_j f\|_{L^\infty(\mathbb{R}^n)}}{\Psi(N)^2}.$$

Furthermore, we claim that

$$\mathcal{A} \lesssim \|f\|_{V_\Psi(\mathbb{R}^n)}. \tag{7-6}$$

Indeed,

$$\begin{aligned} \sum_{j=0}^N \|\Delta_j f\|_{L^\infty(\mathbb{R}^n)} &= \sum_{j=0}^N 2^{2j} \Psi(j)^2 \frac{2^{-2j}}{\Psi(j)^2} \|\Delta_j f\|_{L^\infty(\mathbb{R}^n)} \\ &\le \Big(\sup_{j\in\mathbb{N}_0} \frac{2^{-2j}}{\Psi(j)^2} \|\Delta_j f\|_{L^\infty(\mathbb{R}^n)}\Big) \sum_{j=0}^N 2^{2j} \Psi(j)^2 \\ &\lesssim 2^{2N} \Psi(N)^2 \|f\|_{V_\Psi(\mathbb{R}^n)} \quad \text{(by the admissibility of } \Psi\text{)}, \end{aligned}$$

which yields the desired estimate (7-6).

Combining (7-5) and (7-6), we see that

$$\sup_{N\in\mathbb{N}_0} \frac{K\big(2^{-2N}, f; B^{-2}_{\infty,1}(\mathbb{R}^n), B^s_{\infty,1}(\mathbb{R}^n)\big)}{\Psi(N)^2} \approx \|f\|_{V_\Psi(\mathbb{R}^n)}.$$

Note that, by basic monotonicity properties of the expressions involved (recall that Ψ is doubling),

$$\sup_{t\in(0,1)} \frac{K(t,f;B^{-2}_{\infty,1}(\mathbb{R}^n),B^{0}_{\infty,1}(\mathbb{R}^n))}{\Psi(-\log t)^2} \approx \sup_{N\in\mathbb{N}_0} K\big(2^{-2N},f;B^{-2}_{\infty,1}(\mathbb{R}^n),B^{0}_{\infty,1}(\mathbb{R}^n)\big) \sup_{t\in(2^{-2(N+1)},2^{-2N})} \frac{1}{\Psi(-\log t)^2} \approx \sup_{N\in\mathbb{N}_0} \frac{K\big(2^{-2N},f;B^{-2}_{\infty,1}(\mathbb{R}^n),B^{0}_{\infty,1}(\mathbb{R}^n)\big)}{\Psi(N)^2}. \qquad \square$$

Remark 42. The proof above shows that $B^0_{\infty,1}(\mathbb{R}^n)$ plays an auxiliary role in Theorem 39, in the sense that the same result is obtained if we replace it by $B^s_{\infty,1}(\mathbb{R}^n)$, for any $s>-2$. More precisely, let $s>-2$, and replace the admissibility condition given in Definition 36(i) by

$$\sum_{r=0}^{N} 2^{(2+s)r}\Psi(r)^2 \lesssim 2^{(2+s)N}\Psi(N)^2, \quad N\in\mathbb{N}_0.$$

Then using the same methodology we can readily show that

$$\|f\|_{V_\Psi(\mathbb{R}^n)} \approx \sup_{t\in(0,1)} \frac{K(t,f;B^{-2}_{\infty,1}(\mathbb{R}^n),B^{s}_{\infty,1}(\mathbb{R}^n))}{\Psi(-\log t)^2}.$$

7.4. ***Proof of Theorem 37.*** The proof relies strongly on the extrapolation description of V_Ψ. In particular, Theorem 39 will be applied to decompose functions in V_Ψ in terms of wavelets (see Proposition 43 below).

Let $\{\Upsilon^G_{Nl} : N\in\mathbb{N}_0,\ G\in G^N,\ l\in\mathbb{Z}^n\}$ be an orthonormal wavelet basis in $L^2(\mathbb{R}^n)$.

Remark. We briefly recall that orthonormal wavelet bases may be constructed in a standard way from two compactly supported (Daubechies) wavelets $\psi_F\in C^1(\mathbb{R})$ (*father wavelet*) and $\psi_M\in C^1(\mathbb{R})$ (*mother wavelet*) satisfying certain moment conditions. More precisely

$$\Upsilon^G_{Nl}(x) = 2^{Nn/2}\prod_{r=1}^{n}\psi_{G_r}(2^N x_r - l_r).$$

Here $G^0=\{F,M\}^n$ and $G^N=\{F,M\}^{n*}$, $N\in\mathbb{N}$, where $*$ indicates that at least one of the components of $G\in G^N$ must be an M. The role played by the tensor index $G\in G^N$ is auxiliary (note that $\operatorname{card} G^N\approx 1$). To simplify the exposition, the index G may be safely removed from our computations.

Proposition 43. *Suppose that Ψ is an admissible doubling decay. Then, $f\in V_\Psi(\mathbb{R}^n)$ if and only if*

$$f = \sum_{\substack{N\in\mathbb{N}_0,\,G\in G^N\\ l\in\mathbb{Z}^n}} \lambda^G_{Nl}2^{-Nn/2}\Upsilon^G_{Nl}, \quad \{\lambda^G_{Nl}\}\in v_\Psi \tag{7-7}$$

(*unconditional convergence in the sense of* $\mathcal{S}'(\mathbb{R}^n)$), *where*

$$\|\{\lambda^G_{Nl}\}\|_{v_\Psi} := \sup_{N\in\mathbb{N}_0}\frac{1}{\Psi(N)^2}\sum_{k=N}^{\infty}2^{-2k}\sup_{G\in G^k,\,l\in\mathbb{Z}^n}|\lambda^G_{kl}| < \infty.$$

The representation of f is unique, and the coefficients λ^G_{Nl} are determined by

$$\lambda^G_{Nl} = 2^{Nn/2}(f, \Upsilon^G_{Nl}), \tag{7-8}$$

and the operator

$$I : f \mapsto \{\lambda^G_{Nl}\} \tag{7-9}$$

defines an isomorphism of $V_\Psi(\mathbb{R}^n)$ onto v_Ψ. Furthermore

$$\|f\|_{V_\Psi(\mathbb{R}^n)} \approx \|\{\lambda_{Nl}\}\|_{v_\Psi}. \tag{7-10}$$

Proof. By the classical wavelet theory for Besov spaces [Triebel 2008, Theorem 1.20], the operator I given by (7-9) acts as an isomorphism

$$I : B^{-2}_{\infty,1}(\mathbb{R}^n) \to \ell^{-2}_1(\ell_\infty) \quad \text{and} \quad I : B^0_{\infty,1}(\mathbb{R}^n) \to \ell_1(\ell_\infty). \tag{7-11}$$

Here, as usual, $\ell^s_1(\ell_\infty)$, $s \in \mathbb{R}$, is the mixed sequence space formed by all those $\{\lambda_{Nl}\}$ such that

$$\|\{\lambda_{Nl}\}\|_{\ell^s_1(\ell_\infty)} = \sum_{N=0}^{\infty} 2^{Ns} \sup_{l\in\mathbb{Z}^n} |\lambda_{Nl}| < \infty.$$

We also let $\ell_1(\ell_\infty) = \ell^0_1(\ell_\infty)$.

It follows from (7-11) that

$$K\big(t, f; B^{-2}_{\infty,1}(\mathbb{R}^n), B^0_{\infty,1}(\mathbb{R}^n)\big) \approx K\big(t, \{\lambda_{Nl}\}; \ell^{-2}_1(\ell_\infty), \ell_1(\ell_\infty)\big).$$

Consequently, combining with Theorem 39 yields

$$\|f\|_{V_\Psi(\mathbb{R}^n)} \approx \sup_{t\in(0,1)} \frac{K\big(t, \{\lambda_{Nl}\}; \ell^{-2}_1(\ell_\infty), \ell_1(\ell_\infty)\big)}{\Psi(-\log t)^2}. \tag{7-12}$$

At this point the method of proof developed in Theorem 39 can be applied line by line (ℓ_∞ now playing the role previously played by $L^\infty(\mathbb{R}^n)$) to show that

$$\sup_{t\in(0,1)} \frac{K\big(t, \{\lambda_{Nl}\}; \ell^{-2}_1(\ell_\infty), \ell_1(\ell_\infty)\big)}{\Psi(-\log t)^2} \approx \|\{\lambda_{Nl}\}\|_{v_\Psi}. \tag{7-13}$$

Combining (7-12) and (7-13), we obtain (7-10).

For $f \in V_\Psi(\mathbb{R}^n)$, the convergence of the wavelet expansion (7-7), as well as the uniqueness of wavelet coefficients given by (7-8), is guaranteed by the fact that $V_\Psi(\mathbb{R}^n) \hookrightarrow B^{-2}_{\infty,1}(\mathbb{R}^n)$ (see Remark 35) since the corresponding assertions are valid for classical Besov spaces. □

Proof of Theorem 37. (i) Without loss of generality, we may assume that $Q_0 = (0,1)^n$. Let $L \in \mathbb{N}_0$ and $\mathcal{Q} = (Q_i)_{i\in I} \in S(Q_0)$. If $f \geq 0$ is compactly supported in Q_0, then

$$\bigg[\sum_{Q\in\mathbb{D}_{\leq L-1}(\mathcal{Q})} \Big(|Q|^{\frac{1}{n}-\frac{1}{2}} \int_Q |f|\Big)^2\bigg]^{\frac{1}{2}} = \bigg[\sum_{k=L-1}^{\infty} \sum_{i\in I: Q_i\in\mathbb{D}_k(\mathcal{Q})} \Big(|Q_i|^{\frac{1}{n}-\frac{1}{2}} \int_{Q_i} f\Big)^2\bigg]^{\frac{1}{2}}$$

$$\leq \|f\|_{L^1(Q_0)}^{\frac{1}{2}} \Bigg[\sum_{k=L-1}^{\infty} 2^{-kn(\frac{1}{n}-\frac{1}{2})2} \sup_{Q\in\mathbb{D}_k} \int_Q f \Bigg]^{\frac{1}{2}}$$

$$\leq \|f\|_{L^1(Q_0)}^{\frac{1}{2}} \Psi(L) \sup_{N\in\mathbb{N}_0} \Bigg[\frac{1}{\Psi(N)^2} \sum_{k=N-1}^{\infty} 2^{k(-2+n)} \sup_{Q\in\mathbb{D}_k} \int_Q f \Bigg]^{\frac{1}{2}}$$

$$\leq \Psi(L) \sup_{N\in\mathbb{N}_0} \frac{1}{\Psi(N)^2} \sum_{k=N-1}^{\infty} 2^{k(-2+n)} \sup_{Q\in\mathbb{D}_k} \int_Q f.$$

As a by-product (see (1-14))

$$\|f\|_{S_\Psi(Q_0)} = \sup_{L\in\mathbb{N}} \frac{s_L(f)}{\Psi(L)} \leq \sup_{N\in\mathbb{N}_0} \frac{1}{\Psi(N)^2} \sum_{k=N-1}^{\infty} 2^{k(-2+n)} \sup_{Q\in\mathbb{D}_k} \int_Q f.$$

Hence the desired embedding $(V_\Psi^+(\mathbb{R}^n))_c \hookrightarrow (S_\Psi(\mathbb{R}^n))_c$ follows if we show that

$$\sup_{N\in\mathbb{N}_0} \frac{1}{\Psi(N)^2} \sum_{k=N-1}^{\infty} 2^{k(-2+n)} \sup_{Q\in\mathbb{D}_k} \int_Q f \lesssim \|f\|_{V_\Psi(\mathbb{R}^n)}. \tag{7-14}$$

Let $f \in V_\Psi(\mathbb{R}^n)$. According to Proposition 43, f can be expressed as

$$f = \sum_{r=0}^{\infty} \sum_{l\in\mathbb{Z}^n} \lambda_{rl} 2^{-rn/2} \Upsilon_{rl}, \tag{7-15}$$

where λ_{rl} is given by (7-8). The wavelets Υ_{rl} can be chosen such that

$$\operatorname{supp} \Upsilon_{rl} \subset c Q_{rl} = c(2^{-r} l + 2^{-r} Q_0), \tag{7-16}$$

$$|\Upsilon_{rl}(x)| \lesssim 2^{rn/2}, \quad r \in \mathbb{N}_0,\ l \in \mathbb{Z}^n, \tag{7-17}$$

and there exists $A \in \mathbb{N},\ A > 2$, satisfying

$$\int_{\mathbb{R}^n} x^\beta \Upsilon_{rl}(x)\, dx = 0, \quad |\beta| < A,\ r \in \mathbb{N},\ l \in \mathbb{Z}^n. \tag{7-18}$$

We are going to compute $\chi_{jm}(f)$ given by (4-22). For every $j \in \mathbb{N}_0$, we can split f as (see (7-15))

$$f = f_j + f^j := \sum_{r=0}^{j} \sum_{l\in\mathbb{Z}^n} \lambda_{rl} 2^{-rn/2} \Upsilon_{rl} + \sum_{r=j+1}^{\infty} \sum_{l\in\mathbb{Z}^n} \lambda_{rl} 2^{-rn/2} \Upsilon_{rl}. \tag{7-19}$$

Then, for $m \in \mathbb{Z}^n$,

$$\chi_{jm}(f) = \chi_{jm}(f_j) + \chi_{jm}(f^j). \tag{7-20}$$

Next, we estimate each of these terms separately.

We first estimate $\chi_{jm}(f_j)$. For $r \leq j$, we let (recall that $\operatorname{supp} \chi_{jm} \subset dQ_{jm}$)

$$\ell_r^j(m) := \{l \in \mathbb{Z}^n : dQ_{jm} \cap cQ_{rl} \neq \varnothing\}. \tag{7-21}$$

Obviously (see (7-16))

$$\chi_{jm}(\Upsilon_{rl})=0 \quad \text{for } l\notin \ell_r^j(m). \tag{7-22}$$

On the other hand, if $l\in \ell_r^j(m)$ then (noting that $|\chi_{jm}(x)|\lesssim 2^{jn}$)

$$|\chi_{jm}(\Upsilon_{rl})|\le \int_{dQ_{jm}}|\chi_{jm}(x)||\Upsilon_{rl}(x)|\,dx\lesssim 2^{jn}\int_{dQ_{jm}}|\Upsilon_{rl}(x)|\,dx\lesssim 2^{rn/2}, \tag{7-23}$$

where we have used the property (7-17) in the last step. In light of (7-22) and (7-23), we derive

$$\begin{aligned}|\chi_{jm}(f_j)|\le \sum_{r=0}^{j}\sum_{l\in\mathbb{Z}^n}|\lambda_{rl}|2^{-rn/2}|\chi_{jm}(\Upsilon_{rl})| &= \sum_{r=0}^{j}\sum_{l\in\ell_r^j(m)}|\lambda_{rl}|2^{-rn/2}|\chi_{jm}(\Upsilon_{rl})|\\ &\lesssim \sum_{r=0}^{j}\sum_{l\in\ell_r^j(m)}|\lambda_{rl}|. \end{aligned}\tag{7-24}$$

Note that since $\operatorname{card}\ell_r^j(m)\approx 1$, if $0\le r\le j$, from (7-24) it follows that

$$|\chi_{jm}(f_j)|\lesssim \sum_{r=0}^{j}\sup_{l\in\mathbb{Z}^n}|\lambda_{rl}|. \tag{7-25}$$

Next we deal with $\chi_{jm}(f^j)$. Let $r>j$ and $l\in\ell_r^j(m)$ (see (7-21)). Using the Taylor expansion of χ_{jm} around $2^{-r}l$ and using the cancellation conditions (7-18), one can show that

$$\begin{aligned}|\chi_{jm}(\Upsilon_{rl})| &= \left|\int_{\mathbb{R}^n}\chi_{jm}(x)\Upsilon_{rl}(x)\,dx\right|\\ &\le \sum_{|\gamma|=A}\sup_{x\in\mathbb{R}^n}|D^\gamma\chi_{jm}(x)|\int_{\mathbb{R}^n}|\Upsilon_{rl}(x)||x-2^{-r}l|^A\,dx\\ &\lesssim 2^{j(n+A)}\int_{\mathbb{R}^n}2^{rn/2}|\psi(2^rx-l)|2^{-rA}|2^rx-l|^A\,dx\\ &= 2^{(j-r)(n+A)}2^{rn/2}\int_{\mathbb{R}^n}|\psi(x)||x|^A\,dx\\ &\approx 2^{(j-r)(n+A)}2^{rn/2}.\end{aligned}$$

Using this estimate we obtain

$$|\chi_{jm}(f^j)|\le\sum_{r=j}^{\infty}\sum_{l\in\ell_r^j(m)}|\lambda_{rl}|2^{-rn/2}|\chi_{jm}(\Upsilon_{rl})|\lesssim\sum_{r=j}^{\infty}2^{(j-r)(n+A)}\sum_{l\in\ell_r^j(m)}|\lambda_{rl}|. \tag{7-26}$$

Note that $\operatorname{card}\ell_r^j(m)\approx 2^{n(r-j)}$ if $r\ge j$. Hence, by (7-26),

$$|\chi_{jm}(f^j)|\lesssim\sum_{r=j}^{\infty}2^{(j-r)(n+A)}\sup_{l\in\mathbb{Z}^n}|\lambda_{rl}|2^{n(r-j)}=2^{jA}\sum_{r=j}^{\infty}2^{-rA}\sup_{l\in\mathbb{Z}^n}|\lambda_{rl}|. \tag{7-27}$$

Putting together (7-20), (7-25) and (7-27),

$$|\chi_{jm}(f)| \le |\chi_{jm}(f_j)| + |\chi_{jm}(f^j)| \lesssim \sum_{r=0}^{j} \sup_{l\in\mathbb{Z}^n} |\lambda_{rl}| + 2^{jA} \sum_{r=j}^{\infty} 2^{-rA} \sup_{l\in\mathbb{Z}^n} |\lambda_{rl}|$$

uniformly with respect to $m \in \mathbb{Z}^n$. Consequently,

$$\sup_{m\in\mathbb{Z}^n} |\chi_{jm}(f)| \lesssim \sum_{r=0}^{j} \sup_{l\in\mathbb{Z}^n} |\lambda_{rl}| + 2^{jA} \sum_{r=j}^{\infty} 2^{-rA} \sup_{l\in\mathbb{Z}^n} |\lambda_{rl}|. \tag{7-28}$$

Let $N \in \mathbb{N}_0$. From (7-28), changing the order of summation and using $A > 2$, we get

$$\begin{aligned}\sum_{j=N}^{\infty} 2^{-2j} \sup_{m\in\mathbb{Z}^n} |\chi_{jm}(f)| &\lesssim \sum_{j=N}^{\infty} 2^{-2j} \sum_{r=0}^{j} \sup_{l\in\mathbb{Z}^n} |\lambda_{rl}| + \sum_{j=N}^{\infty} 2^{(A-2)j} \sum_{r=j}^{\infty} 2^{-rA} \sup_{l\in\mathbb{Z}^n} |\lambda_{rl}| \\ &\lesssim 2^{-2N} \sum_{r=0}^{N} \sup_{l\in\mathbb{Z}^n} |\lambda_{rl}| + \sum_{r=N}^{\infty} 2^{-2r} \sup_{l\in\mathbb{Z}^n} |\lambda_{rl}|.\end{aligned}$$

It follows that

$$\begin{aligned}\sup_{N\in\mathbb{N}_0} \frac{1}{\Psi(N)^2} \sum_{j=N}^{\infty} 2^{-2j} \sup_{m\in\mathbb{Z}^n} |\chi_{jm}(f)| &\lesssim \sup_{N\in\mathbb{N}_0} \frac{2^{-2N}}{\Psi(N)^2} \sum_{r=0}^{N} \sup_{l\in\mathbb{Z}^n} |\lambda_{rl}| \\ &\qquad + \sup_{N\in\mathbb{N}_0} \frac{1}{\Psi(N)^2} \sum_{r=N}^{\infty} 2^{-2r} \sup_{l\in\mathbb{Z}^n} |\lambda_{rl}| \\ &=: \mathcal{I} + \mathcal{II}.\end{aligned} \tag{7-29}$$

Next we show $\mathcal{I} \lesssim \mathcal{II}$. Indeed, by condition (i) from Definition 36,

$$\begin{aligned}\sum_{r=0}^{N} \sup_{l\in\mathbb{Z}^n} |\lambda_{rl}| = \sum_{r=0}^{N} 2^{2r}\Psi(r)^2 \frac{2^{-2r}}{\Psi(r)^2} \sup_{l\in\mathbb{Z}^n} |\lambda_{rl}| &\le \left(\sup_{M\in\mathbb{N}_0} \left(\frac{2^{-2M}}{\Psi(M)^2} \sup_{l\in\mathbb{Z}^n} |\lambda_{Ml}| \right)\right) \sum_{r=0}^{N} 2^{2r}\Psi(r)^2 \\ &\lesssim \left(\sup_{M\in\mathbb{N}_0} \left(\frac{2^{-2M}}{\Psi(M)^2} \sup_{l\in\mathbb{Z}^n} |\lambda_{Ml}| \right)\right) 2^{2N}\Psi(N)^2.\end{aligned}$$

Consequently,

$$\frac{2^{-2N}}{\Psi(N)^2} \sum_{r=0}^{N} \sup_{l\in\mathbb{Z}^n} |\lambda_{rl}| \lesssim \sup_{M\in\mathbb{N}_0} \left(\frac{1}{\Psi(M)^2} \sum_{r=M}^{\infty} 2^{-2r} \sup_{l\in\mathbb{Z}^n} |\lambda_{rl}| \right) = \mathcal{II}.$$

Now, taking supremum over all $N \in \mathbb{N}_0$, we arrive at

$$\mathcal{I} \lesssim \mathcal{II}.$$

Consequently (7-29) reads as

$$\sup_{N\in\mathbb{N}_0} \frac{1}{\Psi(N)^2} \sum_{j=N}^{\infty} 2^{-2j} \sup_{m\in\mathbb{Z}^n} |\chi_{jm}(f)| \lesssim \mathcal{II}. \tag{7-30}$$

Since $f \geq 0$ and (4-23) holds, we have (using the notation $\chi_Q = \chi_{jm}$ if $Q = Q_{jm} \in \mathbb{D}_j$)

$$\sup_{N\in\mathbb{N}_0} \frac{1}{\Psi(N)^2} \sum_{k=N-1}^{\infty} 2^{k(-2+n)} \sup_{Q\in\mathbb{D}_k} \int_Q f \lesssim \sup_{N\in\mathbb{N}_0} \frac{1}{\Psi(N)^2} \sum_{k=N-1}^{\infty} 2^{-2k} \sup_{Q\in\mathbb{D}_k} \chi_Q(f).$$

Consequently by (7-29)–(7-30),

$$\sup_{N\in\mathbb{N}_0} \frac{1}{\Psi(N)^2} \sum_{k=N-1}^{\infty} 2^{k(-2+n)} \sup_{Q\in\mathbb{D}_k} \int_Q f \lesssim \sup_{N\in\mathbb{N}_0} \frac{1}{\Psi(N)^2} \sum_{r=N}^{\infty} 2^{-2r} \sup_{l\in\mathbb{Z}^n} |\lambda_{rl}|,$$

which combined with Proposition 43 yields

$$\sup_{N\in\mathbb{N}_0} \frac{1}{\Psi(N)^2} \sum_{k=N-1}^{\infty} 2^{k(-2+n)} \sup_{Q\in\mathbb{D}_k} \int_Q f \lesssim \|f\|_{V_\Psi(\mathbb{R}^n)},$$

concluding the proof of (7-14).

(ii) Combine (i) with Theorem 12. ☐

7.5. *Proof of Theorem 38.* Let $\Psi(t) = t^{(1-\alpha)/2}$, $\alpha > 1$. It follows from (7-8), (7-16) and (7-17) that

$$\begin{aligned}
\sum_{k=N}^{\infty} 2^{-2k} \sup_{l\in\mathbb{Z}^n} |\lambda_{kl}| &\lesssim \sum_{k=N}^{\infty} 2^{-2k} \sup_{l\in\mathbb{Z}^n} 2^{kn/2} \int_{cQ_{kl}} |f||\Upsilon_{kl}| \\
&\lesssim \sum_{k=N}^{\infty} 2^{-2k} \sup_{l\in\mathbb{Z}^n} 2^{kn} \int_{cQ_{kl}} |f| \\
&\lesssim \|f\|_{M^{n/2,\alpha}(\mathbb{R}^n)} \sum_{k=N}^{\infty} 2^{-2k} \sup_{l\in\mathbb{Z}^n} 2^{kn} |Q_{kl}|^{1-\frac{2}{n}} (1-(\log|Q_{kl}|)_-)^{-\alpha} \\
&\approx \|f\|_{M^{n/2,\alpha}(\mathbb{R}^n)} \sum_{k=N}^{\infty} (1+k)^{-\alpha} \\
&\approx N^{-\alpha+1} \|f\|_{M^{n/2,\alpha}(\mathbb{R}^n)} = \Psi(N)^2 \|f\|_{M^{n/2,\alpha}(\mathbb{R}^n)}.
\end{aligned}$$

Taking the supremum over all $N \in \mathbb{N}_0$ and invoking Proposition 43, we get

$$M^{\frac{n}{2},\alpha}(\mathbb{R}^n) \hookrightarrow V_\Psi(\mathbb{R}^n) \tag{7-31}$$

as desired.

To show that (7-31) is strict, we apply again Proposition 43, and use the fact that $\alpha > 1$, to derive

$$\|2^{(2-\frac{n}{2})K} \Upsilon_{K(0,\dots,0)}\|_{V_\Psi(\mathbb{R}^n)} \approx \sup_{0\leq N\leq K} \frac{1}{N^{-\alpha+1}} = \frac{1}{K^{-\alpha+1}}. \tag{7-32}$$

On the other hand, we have

$$\|\Upsilon_{K(0,\dots,0)}\|_{M^{n/2,\alpha}(\mathbb{R}^n)} = \sup_Q \frac{(1-(\log|Q|)_-)^\alpha}{|Q|^{(n-2)/n}} \int_Q |\Upsilon_{K,(0,\dots,0)}|$$

$$\begin{aligned}
&\geq \frac{(1-(\log|Q_{K(0,\dots,0)}|)_-)^\alpha}{|Q_{K(0,\dots,0)}|^{(n-2)/n}} \int_{Q_{K(0,\dots,0)}} |\Upsilon_{K,(0,\dots,0)}| \\
&\approx \frac{K^\alpha}{2^{-K(n-2)}} \int_{Q_{K(0,\dots,0)}} |\Upsilon_{K,(0,\dots,0)}| \\
&\gtrsim \frac{K^\alpha}{2^{-K(n-2)}} 2^{-Kn} 2^{Kn/2} = 2^{K(\frac{n}{2}-2)} K^\alpha,
\end{aligned}$$

where we have used the wavelet properties in the penultimate step. Consequently,

$$\|2^{(2-\frac{n}{2})K} \Upsilon_{K(0,\dots,0)}\|_{M^{n/2,\alpha}(\mathbb{R}^n)} \gtrsim K^\alpha. \tag{7-33}$$

We now argue by contradiction. Suppose that, to the contrary,

$$V_\Psi(\mathbb{R}^n) \hookrightarrow M^{\frac{n}{2},\alpha}(\mathbb{R}^n).$$

In particular, we have, uniformly with respect to K,

$$\|2^{(2-\frac{n}{2})K} \Upsilon_{K(0,\dots,0)}\|_{M^{n/2,\alpha}(\mathbb{R}^n)} \lesssim \|2^{(2-\frac{n}{2})K} \Upsilon_{K(0,\dots,0)}\|_{V_\Psi(\mathbb{R}^n)}.$$

Combining the last inequality with (7-32) and (7-33) yields

$$K^\alpha \lesssim K^{\alpha-1},$$

and letting $K \to \infty$ we arrive at a contradiction. □

8. Sharpening Tadmor's regularity via T_Ψ

As already mentioned in Section 1.2, Tadmor [2001] proposed an approach, based on $R_{p,2}\log^\alpha$-spaces, guaranteeing existence of Euler solutions with no concentration. In particular, in the distinguished 2D case, the author was able to improve the Morrey regularity of vortex sheets obtained in [DiPerna and Majda 1987a] from $\alpha = 1$ to the borderline regularity $\alpha = \frac{1}{2}$. This is an application of the H^{-1}-stability method since (see (1-3))

$$R_{2n/(n+2),2}\log^\alpha(\mathbb{R}^n)_c \xrightarrow{\text{compactly}} H^{-1}_{\text{loc}}(\mathbb{R}^n), \quad \alpha > \tfrac{1}{2}. \tag{8-1}$$

The goal of this section is to show that the results from [Tadmor 2001] admit improvements in terms of new scales of extrapolation spaces (T_Ψ-spaces; see Definition 44 below) and the method of sparse stability developed in previous sections.

8.1. ***T_Ψ-spaces.*** To motivate the constructions that follow it is instructive to compare the scalings of the spaces involved in the critical embeddings (1-2). For a function space $X(\mathbb{R}^n)$, let the scaling parameter of X be the number i_X such for all $\lambda > 0$, and all $\|f\|_X = 1$, we have $\lambda^{-i_X}\|f(\lambda\,\cdot)\|_X = 1$. For the Morrey spaces $M^p(\mathbb{R}^n)$, for $\lambda > 0$, and $\|f\|_{M^p(\mathbb{R}^n)} = 1$, we have $\|f(\lambda\,\cdot)\|_{M^p(\mathbb{R}^n)} = \lambda^{-n/p}$ so that $i_{M^p(\mathbb{R}^n)} = -n/p$. Likewise, for $H^{-1}(\mathbb{R}^n)$, $i_{H^{-1}(\mathbb{R}^n)} = -1-n/2$. Comparing scaling parameters in the critical case $p = n/2$, we see that $i_{M^{n/2}(\mathbb{R}^n)} = i_{H^{-1}(\mathbb{R}^n)}$ only when $n = 2$, in which case the common value of the parameter is -2. This suggests to seek for an alternative to (1-2) where the involved spaces have the same scaling parameter as H^{-1} (i.e., $-1-n/2$). With this in mind, we propose a new space T_Ψ, a variant of V_Ψ introduced in Definition 34, which is obtained by measuring the dyadic frequencies $\Delta_j f$ in the L^2-norm rather than the L^∞-norm.

Definition 44. Let $T_\Psi(\mathbb{R}^n)$ be the space formed by all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|^2_{T_\Psi(\mathbb{R}^n)} := \sup_{N\in\mathbb{N}_0} \frac{1}{\Psi(N)^2} \sum_{j=N}^{\infty} 2^{-2j} \|\Delta_j f\|^2_{L^2(\mathbb{R}^n)} < \infty.$$

Let $T^+_\Psi(\mathbb{R}^n) = T_\Psi(\mathbb{R}^n) \cap BM^+_c$.

Remark 45. Similarly as in Remark 35, the space T_Ψ can be seen as a dual counterpart of classical Vishik spaces involving the Besov space $B^{-1}_{2,2}$. Note that $B^{-1}_{2,2}$ can be identified with the inhomogeneous version of H^{-1}.

Remark 46. The space T_Ψ admits a somewhat simpler characterization in terms of Fourier transforms. Namely

$$\|f\|^2_{T_\Psi(\mathbb{R}^n)} \approx \sup_{N\in\mathbb{N}_0} \frac{1}{\Psi(N)^2} \int_{|\xi|>2^{N-1}} (1+|\xi|^2)^{-1} |\hat{f}(\xi)|^2 \, d\xi.$$

Indeed, this is a consequence of Plancherel's and Fubini's theorem, together with basic properties of[39] $\{\varphi_j\}_{j\in\mathbb{N}_0}$,

$$\begin{aligned}\|f\|^2_{T_\Psi(\mathbb{R}^n)} &\approx \sup_{N\in\mathbb{N}_0} \frac{1}{\Psi(N)^2} \int_{\mathbb{R}^n} \sum_{j=N}^{\infty} [2^{-j}\varphi_j(\xi)]^2 |\hat{f}(\xi)|^2 \, d\xi \\ &\approx \sup_{N\in\mathbb{N}_0} \frac{1}{\Psi(N)^2} \int_{\mathbb{R}^n} (1+|\xi|^2)^{-1} \mathbf{1}_{(2^{N-1},\infty)}(|\xi|) |\hat{f}(\xi)|^2 \, d\xi.\end{aligned}$$

8.2. *T_Ψ-regularity of Euler flows.* We state now the main results of this section.

Theorem 47. *Let Ψ be an admissible doubling decay. Then*:

(i) $T^+_\Psi(\mathbb{R}^n)_c \hookrightarrow S_\Psi(\mathbb{R}^n)_c$. *As a consequence* (*see* (1-15)), $T^+_\Psi(\mathbb{R}^n)_c \xrightarrow{\text{compactly}} H^{-1}_{\text{loc}}(\mathbb{R}^n)$.

(ii) *Let $\{u^\varepsilon\}_{\varepsilon>0}$ be a family of approximate solutions to Euler equations* (1-1) *such that the related family of vorticities $\{\omega^\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $L^\infty([0,T]; T^+_\Psi(\mathbb{R}^n)_c)$. Then $\{u^\varepsilon\}_{\varepsilon>0}$ has a strong limit u in $L^\infty([0,T]; L^2_{\text{loc}}(\mathbb{R}^n))$, where u is a solution with no concentrations.*

Specializing the previous result with $\Psi_\alpha(t) = t^{-\alpha+1/2}$, we are able to improve (8-1) in the following sense.

Theorem 48. *Assume that $\alpha > \frac{1}{2}$. Then*

$$R_{2n/(n+2),2} \log^\alpha(\mathbb{R}^n) \hookrightarrow T_{\Psi_\alpha}(\mathbb{R}^n).$$

Furthermore, this embedding is strict in the sense that $R_{2n/(n+2),2} \log^\alpha(\mathbb{R}^n) \neq T_{\Psi_\alpha}(\mathbb{R}^n)$.

[39] Here φ_j denotes the Fourier multiplier associated with Δ_j, i.e., $\widehat{\Delta_j f}(\xi) = \varphi_j(\xi)\hat{f}(\xi)$.

8.3. ***Extrapolation characterization of T_Ψ.*** The next result represents the T_Ψ spaces as extrapolation spaces for the pair (H^{-1}, L^2). Since the proof follows closely the one for Theorem 39, we shall leave the details to the interested reader.

Theorem 49. *Suppose that Ψ is an admissible doubling decay. Then*[40]

$$\|f\|_{T_\Psi(\mathbb{R}^n)} \approx \sup_{t\in(0,1)} \frac{K(t, f; H^{-1}(\mathbb{R}^n), L^2(\mathbb{R}^n))}{\Psi(-\log t)}. \tag{8-2}$$

Remark 50. A variant of Remark 42 also holds for the $T_\Psi(\mathbb{R}^n)$ spaces. Indeed, suppose that the admissibility condition stated Definition 36(i) is replaced by

$$\sum_{r=0}^{N} (2^{(1+s)r}\Psi(r))^2 \lesssim (2^{(1+s)N}\Psi(N))^2.$$

Let $H^s(\mathbb{R}^n)$ be the standard (fractional) Sobolev space endowed with the norm

$$\|f\|_{H^s(\mathbb{R}^n)} = \|(I-\Delta)^{s/2} f\|_{L^2(\mathbb{R}^n)},$$

then formula (8-2) holds if we replace the pair $(H^{-1}(\mathbb{R}^n), L^2(\mathbb{R}^n))$ by $(H^{-1}(\mathbb{R}^n), H^s(\mathbb{R}^n))$, where $s > -1$.

8.4. ***Proof of Theorem 47.*** For the proof we will use the following analogue of Proposition 43, that can be obtained mutatis mutandis and we therefore leave its proof to the interested reader.

Proposition 51. *Let $\{\Upsilon^G_{Nl} : N \in \mathbb{N}_0,\ G \in G^N,\ l \in \mathbb{Z}^n\}$ be a wavelet system satisfying* (7-16)–(7-18) *with*[41] $A > 1$. *Assume that Ψ is an admissible doubling decay. Then, $f \in T_\Psi(\mathbb{R}^n)$ if and only if*

$$f = \sum_{\substack{N\in\mathbb{N}_0,\, G\in G^N\\ l\in\mathbb{Z}^n}} \lambda^G_{Nl} 2^{-Nn/2} \Upsilon^G_{Nl}, \quad \{\lambda^G_{Nl}\} \in t_\Psi \tag{8-3}$$

(*unconditional convergence in the sense of $\mathcal{S}'(\mathbb{R}^n)$*), *where*

$$\|\{\lambda^G_{Nl}\}\|^2_{t_\Psi} := \sup_{N\in\mathbb{N}_0} \frac{1}{\Psi(N)^2} \sum_{k=N}^{\infty} 2^{k(-2-n)} \sum_{G\in G^k,\, l\in\mathbb{Z}^n} |\lambda^G_{kl}|^2 < \infty. \tag{8-4}$$

This representation is unique in the sense that the coefficients λ^G_{Nl} are determined by (7-8) *and the operator I given by* (7-9) *defines an isomorphism from $T_\Psi(\mathbb{R}^n)$ onto t_Ψ. Furthermore*

$$\|f\|_{T_\Psi(\mathbb{R}^n)} \approx \|\{\lambda^{N,G}_l\}\|_{t_\Psi}. \tag{8-5}$$

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[40] Since T_Ψ is not homogeneous, the space $H^{-1}(\mathbb{R}^n)$ in (8-2) should be adequately interpreted from the context as the inhomogeneous counterpart of (4-3), equipped with the norm $\|f\|_{H^{-1}(\mathbb{R}^n)} = \|(I-\Delta)^{-1/2} f\|_{L^2(\mathbb{R}^n)}$. Recall the well-known fact that $\|(-\Delta)^{-1/2} f\|_{L^2(\mathbb{R}^n)} \approx \|I_1 f\|_{L^2(\mathbb{R}^n)}$.

[41] The explanation behind $A > 1$ comes from Theorem 49 and well-known wavelet assumptions on H^{-1}.

Proof of Theorem 47. (i) Let $L \in \mathbb{N}_0$. Given $f \geq 0$ compactly supported on Q_0 (without loss of generality, we may assume $Q_0 = (0,1)^n$), we observe that

$$s_L(f) = \sup_{\mathcal{Q} \in S(Q_0)} \left[\sum_{k=L}^{\infty} 2^{k(-2+n)} \sum_{i \in I : Q_i \in \mathbb{D}_k(\mathcal{Q})} \left(\int_{Q_i} f \right)^2 \right]^{1/2}.$$

Using Proposition 51 we will show that

$$\sup_{L \in \mathbb{N}_0} \frac{1}{\Psi(L)} \left[\sum_{k=L}^{\infty} 2^{k(-2+n)} \sum_{i \in I : Q_i \in \mathbb{D}_k(\mathcal{Q})} \left(\int_{Q_i} f \right)^2 \right]^{1/2} \lesssim \|f\|_{T_\Psi(\mathbb{R}^n)}, \tag{8-6}$$

with a constant independent of the sparse family $\mathcal{Q}$. Then the desired (local) embedding

$$\|f\|_{S_\Psi(Q_0)} \lesssim \|f\|_{T_\Psi(\mathbb{R}^n)}$$

follows readily.

Let χ be a smooth cut-off function introduced in the proof of Theorem 9, and define the corresponding coefficients $\chi_{jm}(f)$ via (4-22). According to (7-19), (7-20), (7-24) and (7-26), these coefficients can be estimated as

$$|\chi_{jm}(f)| \lesssim \sum_{r=0}^{j} \sum_{l \in \ell_r^j(m)} |\lambda_{rl}| + \sum_{r=j}^{\infty} 2^{(j-r)(n+A)} \sum_{l \in \ell_r^j(m)} |\lambda_{rl}|, \tag{8-7}$$

where $\ell_r^j(m)$, which was introduced in (7-21), satisfies

$$\operatorname{card} \ell_r^j(m) \approx \begin{cases} 2^{n(r-j)} & \text{if } r \geq j, \\ 1 & \text{if } r \leq j. \end{cases} \tag{8-8}$$

Using Hölder's inequality and (8-8),

$$\sum_{l \in \ell_r^j(m)} |\lambda_{rl}| \lesssim \left(\sum_{l \in \ell_r^j(m)} |\lambda_{rl}|^2 \right)^{1/2} \times \begin{cases} 2^{n(r-j)/2} & \text{if } r \geq j, \\ 1 & \text{if } r \leq j, \end{cases}$$

and inserting this into (8-7), we achieve

$$|\chi_{jm}(f)| \lesssim \sum_{r=0}^{j} \left(\sum_{l \in \ell_r^j(m)} |\lambda_{rl}|^2 \right)^{1/2} + 2^{j(A+\frac{n}{2})} \sum_{r=j}^{\infty} 2^{-r(A+\frac{n}{2})} \left(\sum_{l \in \ell_r^j(m)} |\lambda_{rl}|^2 \right)^{1/2}. \tag{8-9}$$

Let $\varepsilon \in (0, \min\{1, A-1\})$ (recall that $A > 1$ is fixed). By Hölder's inequality, the two terms given in the right-hand side of (8-9) can be bounded by

$$\sum_{r=0}^{j} \left(\sum_{l \in \ell_r^j(m)} |\lambda_{rl}|^2 \right)^{1/2} \lesssim 2^{j\varepsilon} \left(\sum_{r=0}^{j} 2^{-r\varepsilon 2} \sum_{l \in \ell_r^j(m)} |\lambda_{rl}|^2 \right)^{1/2}$$

and

$$\sum_{r=j}^{\infty} 2^{-r(A+\frac{n}{2})} \left(\sum_{l \in \ell_r^j(m)} |\lambda_{rl}|^2 \right)^{1/2} \lesssim 2^{-j\varepsilon} \left(\sum_{r=j}^{\infty} 2^{-r(A+\frac{n}{2}-\varepsilon)2} \sum_{l \in \ell_r^j(m)} |\lambda_{rl}|^2 \right)^{1/2}.$$

Hence

$$|\chi_{jm}(f)|^2 \lesssim 2^{j\varepsilon 2} \sum_{r=0}^{j} 2^{-r\varepsilon 2} \sum_{l\in \ell_r^j(m)} |\lambda_{rl}|^2 + 2^{j(A+\frac{n}{2}-\varepsilon)2} \sum_{r=j}^{\infty} 2^{-r(A+\frac{n}{2}-\varepsilon)2} \sum_{l\in \ell_r^j(m)} |\lambda_{rl}|^2$$

and summing up over all $m \in \mathbb{Z}^n$, we have

$$\sum_{m\in\mathbb{Z}^n} |\chi_{jm}(f)|^2 \lesssim 2^{j\varepsilon 2} \sum_{r=0}^{j} 2^{-r\varepsilon 2} \sum_{m\in\mathbb{Z}^n} \sum_{l\in \ell_r^j(m)} |\lambda_{rl}|^2 + 2^{j(A+\frac{n}{2}-\varepsilon)2} \sum_{r=j}^{\infty} 2^{-r(A+\frac{n}{2}-\varepsilon)2} \sum_{m\in\mathbb{Z}^n} \sum_{l\in \ell_r^j(m)} |\lambda_{rl}|^2. \tag{8-10}$$

Furthermore, we remark that

$$\operatorname{card}\{m \in \mathbb{Z}^n : l \in \ell_r^j(m)\} \approx \begin{cases} 2^{n(j-r)} & \text{if } r \le j, \\ 1 & \text{if } r \ge j. \end{cases}$$

Using this information and changing the order of summation in (8-10), we arrive at

$$\sum_{m\in\mathbb{Z}^n} |\chi_{jm}(f)|^2 \lesssim 2^{j\varepsilon 2} \sum_{r=0}^{j} 2^{-r\varepsilon 2} 2^{n(j-r)} \sum_{l\in\mathbb{Z}^n} |\lambda_{rl}|^2 + 2^{j(A+\frac{n}{2}-\varepsilon)2} \sum_{r=j}^{\infty} 2^{-r(A+\frac{n}{2}-\varepsilon)2} \sum_{l\in\mathbb{Z}^n} |\lambda_{rl}|^2,$$

where the equivalence constant is independent of j. Summing up the last estimate over all $j \ge L$ and using Fubini's theorem (recall that $\varepsilon < \min\{1, A-1\}$), we have

$$\begin{aligned}
\sum_{j=L}^{\infty} 2^{j(-2-n)} \sum_{m\in\mathbb{Z}^n} |\chi_{jm}(f)|^2
&\lesssim \sum_{j=L}^{\infty} 2^{j(-1+\varepsilon)2} \sum_{r=0}^{j} 2^{-r(\varepsilon 2+n)} \sum_{l\in\mathbb{Z}^n} |\lambda_{rl}|^2 + \sum_{j=L}^{\infty} 2^{j2(-1+A-\varepsilon)} \sum_{r=j}^{\infty} 2^{-r(A+\frac{n}{2}-\varepsilon)2} \sum_{l\in\mathbb{Z}^n} |\lambda_{rl}|^2 \\
&\lesssim 2^{L(-1+\varepsilon)2} \sum_{r=0}^{L} 2^{-r(\varepsilon 2+n)} \sum_{l\in\mathbb{Z}^n} |\lambda_{rl}|^2 + \sum_{r=L}^{\infty} 2^{-r(\varepsilon 2+n)} \sum_{l\in\mathbb{Z}^n} |\lambda_{rl}|^2 \sum_{j=r}^{\infty} 2^{j(-1+\varepsilon)2} \\
&\qquad + \sum_{r=L}^{\infty} 2^{-r(A+\frac{n}{2}-\varepsilon)2} \sum_{l\in\mathbb{Z}^n} |\lambda_{rl}|^2 \sum_{j=L}^{r} 2^{j2(-1+A-\varepsilon)} \\
&\lesssim 2^{L(-1+\varepsilon)2} \sum_{r=0}^{L} 2^{-r(\varepsilon 2+n)} \sum_{l\in\mathbb{Z}^n} |\lambda_{rl}|^2 + \sum_{r=L}^{\infty} 2^{-r(2+n)} \sum_{l\in\mathbb{Z}^n} |\lambda_{rl}|^2.
\end{aligned}$$

Hence

$$\sup_{L\in\mathbb{N}_0} \frac{1}{\Psi(L)} \left[\sum_{j=L}^{\infty} 2^{j(-2-n)} \sum_{m\in\mathbb{Z}^n} |\chi_{jm}(f)|^2 \right]^{1/2} \lesssim \mathcal{I} + \mathcal{II}, \tag{8-11}$$

where

$$\mathcal{I} := \sup_{L\in\mathbb{N}_0} \frac{2^{L(-1+\varepsilon)}}{\Psi(L)} \left[\sum_{r=0}^{L} 2^{-r(\varepsilon 2+n)} \sum_{l\in\mathbb{Z}^n} |\lambda_{rl}|^2 \right]^{1/2}$$

and

$$\mathcal{II} := \sup_{L\in\mathbb{N}_0} \frac{1}{\Psi(L)}\Bigg[\sum_{r=L}^{\infty} 2^{-r(2+n)} \sum_{l\in\mathbb{Z}^n} |\lambda_{rl}|^2\Bigg]^{1/2}.$$

We have

$$\begin{aligned}\Bigg[\sum_{r=0}^{L} 2^{-r(\varepsilon 2+n)} \sum_{l\in\mathbb{Z}^n} |\lambda_{rl}|^2\Bigg]^{1/2} &\le \Bigg[\sum_{r=0}^{L} 2^{r(1-\varepsilon)2}\Psi(r)^2\Bigg]^{1/2} \sup_{M\in\mathbb{N}_0} \frac{2^{-M(1+\frac{n}{2})}}{\Psi(M)}\Bigg[\sum_{l\in\mathbb{Z}^n} |\lambda_{Ml}|^2\Bigg]^{1/2} \\ &\le \Bigg[\sum_{r=0}^{L} 2^{r(1-\varepsilon)2}\Psi(r)^2\Bigg]^{1/2} \mathcal{II}. \end{aligned} \tag{8-12}$$

Furthermore, the following estimate holds:

$$\Bigg[\sum_{r=0}^{L} 2^{r(1-\varepsilon)2}\Psi(r)^2\Bigg]^{1/2} \lesssim 2^{L(1-\varepsilon)}\Psi(L). \tag{8-13}$$

Indeed, by monotonicity properties, a simple change of variables and the doubling property of Ψ (see (ii) in Definition 36),

$$\begin{aligned}\sum_{r=0}^{L} 2^{r(1-\varepsilon)2}\Psi(r)^2 \approx \int_0^L 2^{t(1-\varepsilon)2}\Psi(t)^2\,dt &\approx \int_0^{L(1-\varepsilon)} 2^{t2}\Psi\Big(\frac{t}{1-\varepsilon}\Big)^2\,dt \\ &\approx \int_0^{L(1-\varepsilon)} 2^{t2}\Psi(t)^2\,dt \approx \sum_{r=0}^{\lfloor L(1-\varepsilon)\rfloor} 2^{r2}\Psi(r)^2 \\ &\lesssim 2^{L(1-\varepsilon)2}\Psi(L)^2,\end{aligned}$$

where[42] the last step follows from (i) in Definition 36. This proves (8-13). Applying now (8-13) in (8-12), we find

$$\Bigg[\sum_{r=0}^{L} 2^{-r(\varepsilon 2+n)} \sum_{l\in\mathbb{Z}^n} |\lambda_{rl}|^2\Bigg]^{1/2} \lesssim 2^{L(1-\varepsilon)}\Psi(L)\,\mathcal{II},$$

i.e., we have shown that $\mathcal{I} \lesssim \mathcal{II}$. As a consequence (see (8-11))

$$\sup_{L\in\mathbb{N}_0} \frac{1}{\Psi(L)}\Bigg[\sum_{j=L}^{\infty} 2^{j(-2-n)} \sum_{m\in\mathbb{Z}^n} |\chi_{jm}(f)|^2\Bigg]^{1/2} \lesssim \mathcal{II},$$

or equivalently (see (8-4))

$$\sup_{L\in\mathbb{N}_0} \frac{1}{\Psi(L)}\Bigg[\sum_{j=L}^{\infty} 2^{j(-2-n)} \sum_{m\in\mathbb{Z}^n} |\chi_{jm}(f)|^2\Bigg]^{1/2} \lesssim \|\{\lambda_{rl}\}\|_{t_\Psi}.$$

[42] As usual, $\lfloor x\rfloor$ denotes the integer part of $x \in \mathbb{R}$.

Consequently, invoking Proposition 51,

$$\sup_{L\in\mathbb{N}_0} \frac{1}{\Psi(L)}\Bigg[\sum_{j=L}^{\infty} 2^{j(-2-n)} \sum_{m\in\mathbb{Z}^n} |\chi_{jm}(f)|^2\Bigg]^{1/2} \lesssim \|f\|_{T_\Psi(\mathbb{R}^n)}. \tag{8-14}$$

On the other hand, the assumption $f \geq 0$ and (4-23) enable us to derive

$$\begin{aligned}\sup_{L\in\mathbb{N}_0} \frac{1}{\Psi(L)}\Bigg[\sum_{k=L}^{\infty} 2^{k(-2+n)} \sum_{i\in I:Q_i\in\mathbb{D}_k(\mathcal{Q})} \bigg(\int_{Q_i} f\bigg)^2\Bigg]^{1/2} &= \sup_{L\in\mathbb{N}_0} \frac{1}{\Psi(L)}\Bigg[\sum_{k=L}^{\infty} 2^{k(-2-n)} \sum_{i\in I:Q_i\in\mathbb{D}_k(\mathcal{Q})} \bigg(\int_{Q_i} 2^{kn} f\bigg)^2\Bigg]^{1/2} \\ &\lesssim \sup_{L\in\mathbb{N}_0} \frac{1}{\Psi(L)}\Bigg[\sum_{k=L}^{\infty} 2^{k(-2-n)} \sum_{i\in I:Q_i\in\mathbb{D}_k(\mathcal{Q})} \chi_{Q_i}(f)^2\Bigg]^{1/2} \\ &\lesssim \|f\|_{T_\Psi(\mathbb{R}^n)},\end{aligned}$$

where in the last step we used (8-14). This concludes the proof of (8-6) and hence (i).

(ii) Invoking Theorem 12, statement (ii) is a consequence of (i). □

8.5. *Proof of Theorem 48.* To avoid unnecessary technicalities, we assume, without loss of generality, that the constant c in (7-16) is equal to 1. From (7-8) and (7-17), we find

$$\begin{aligned}\sum_{k=N}^{\infty} 2^{k(-2-n)} \sum_{G\in G^k,\, l\in\mathbb{Z}^n} |\lambda_{kl}^G|^2 &\lesssim \sum_{k=N}^{\infty} 2^{k(-2+n)} \sum_{l\in\mathbb{Z}^n} \bigg(\int_{Q_{kl}} |f|\bigg)^2 \\ &\approx \sum_{k=N}^{\infty} 2^{k(-2+n)} 2^{-k(-2+n)} k^{-2\alpha} \sum_{l\in\mathbb{Z}^n} \bigg(\frac{\big|\log|Q_{kl}|\big|^\alpha}{|Q_{kl}|^{1/(2n/(n+2))'}} \int_{Q_{kl}} |f|\bigg)^2 \\ &\leq \|f\|^2_{R_{2n/(n+2),2}\log^\alpha(\mathbb{R}^n)} \sum_{k=N}^{\infty} k^{-2\alpha} \\ &\approx N^{-2\alpha+1} \|f\|^2_{R_{2n/(n+2),2}\log^\alpha(\mathbb{R}^n)}.\end{aligned}$$

Consequently,

$$\sup_{N\in\mathbb{N}_0} \frac{1}{\Psi_\alpha(N)^2} \sum_{k=N}^{\infty} 2^{k(-2-n)} \sum_{G\in G^k,\, l\in\mathbb{Z}^n} |\lambda_{kl}^G|^2 \lesssim \|f\|^2_{R_{2n/(n+2),2}\log^\alpha(\mathbb{R}^n)},$$

where $\Psi_\alpha(t) = t^{-\alpha+1/2}$. Then by (8-4) and (8-5) it follows that

$$R_{\frac{2n}{n+2},2}\log^\alpha(\mathbb{R}^n) \hookrightarrow T_{\Psi_\alpha}(\mathbb{R}^n), \quad \Psi_\alpha(t) = t^{-\alpha+\frac{1}{2}}. \tag{8-15}$$

To show that the embedding (8-15) is strict we argue by contradiction. Suppose to the contrary that

$$R_{\frac{2n}{n+2},2}\log^\alpha(\mathbb{R}^n) = T_{\Psi_\alpha}(\mathbb{R}^n).$$

In particular (for a fixed G)

$$\|2^K \Upsilon^G_{K(0,\dots,0)}\|_{T_{\Psi_\alpha}(\mathbb{R}^n)} \approx \|2^K \Upsilon^G_{K(0,\dots,0)}\|_{R_{2n/(n+2),2}\log^\alpha(\mathbb{R}^n)} \quad \text{for every } K \in \mathbb{N}. \tag{8-16}$$

Using Proposition 51, we compute

$$\|2^K \Upsilon^G_{K(0,\dots,0)}\|_{T_{\Psi_\alpha}(\mathbb{R}^n)} \approx \sup_{N \le K} \frac{1}{N^{-\alpha+1/2}} = \frac{1}{K^{-\alpha+1/2}},$$

which combined with (8-16) results in

$$\begin{aligned}\frac{1}{K^{-\alpha+1/2}} &\approx \|2^K \Upsilon^G_{K(0,\dots,0)}\|_{R_{2n/(n+2),2}\log^\alpha(\mathbb{R}^n)} \\ &\gtrsim \frac{\left|\log|Q_{K(0,\dots,0)}|\right|^\alpha}{|Q_{K(0,\dots,0)}|^{(n-2)/(2n)}} \int_{Q_{K(0,\dots,0)}} 2^K |\Upsilon^G_{K(0,\dots,0)}| \\ &\approx \frac{K^\alpha}{2^{-Kn/2}} \int_{Q_{K(0,\dots,0)}} |\Upsilon^G_{K(0,\dots,0)}| \\ &\gtrsim \frac{K^\alpha}{2^{-Kn/2}} 2^{Kn/2} |Q_{K(0,\dots,0)}| = K^\alpha.\end{aligned}$$

Taking the limit as $K \to \infty$ we arrive at a contradiction. □

Next we seek a strategy to establish a priori T_Ψ-bounds as required in Theorem 47(ii). In particular, we shall focus on the critical case $\alpha = \frac{1}{2}$ in the prototypical choice $\Psi_\alpha(t) = t^{-\alpha+1/2}$ (see Theorem 48). This case presents several intrinsic difficulties. In particular, according to Definition 44,

$$T_1(\mathbb{R}^n) = H^{-1}(\mathbb{R}^n) \quad \text{if } \Psi_{1/2}(t) = 1, \tag{8-17}$$

so we shall redefine T_{Ψ_α} to avoid trivial statements. In order to do this, we first observe that, for $\alpha > \frac{1}{2}$, the following characterization of T_{Ψ_α} holds (see Proposition 51):

$$\|f\|^2_{T_{\Psi_\alpha}(\mathbb{R}^n)} \approx \sup_{N \in \mathbb{N}_0} \frac{1}{N^{-2\alpha+1}} \sum_{k=N}^\infty 2^{k(-2-n)} \sum_{G \in G^k,\, l \in \mathbb{Z}^n} |\lambda^G_{kl}|^2. \tag{8-18}$$

Putting (at least formally) $\alpha = \frac{1}{2}$ in the previous characterization, we would arrive at the wavelet counterpart of (8-17), i.e.,

$$\|f\|^2_{T_1(\mathbb{R}^n)} \approx \sum_{k=0}^\infty 2^{k(-2-n)} \sum_{G \in G^k,\, l \in \mathbb{Z}^n} |\lambda^G_{kl}|^2 \approx \|f\|^2_{H^{-1}(\mathbb{R}^n)},$$

where the last equivalence corresponds to the classical wavelet characterization of $H^{-1}(\mathbb{R}^n)$. By basic monotonicity properties, (8-18) can be rewritten as

$$\|f\|^2_{T_{\Psi_\alpha}(\mathbb{R}^n)} \approx \sup_{j \in \mathbb{N}_0} \frac{1}{2^{j(-2\alpha+1)}} \sum_{k=2^j-1}^{2^{j+1}-2} 2^{k(-2-n)} \sum_{G \in G^k,\, l \in \mathbb{Z}^n} |\lambda^G_{kl}|^2 \tag{8-19}$$

if $\alpha > \frac{1}{2}$. When compared with the standard characterization (8-18) of $T_{\Psi_\alpha}(\mathbb{R}^n)$, the alternative characterization given by (8-19) has the important advantage of providing us with nontrivial spaces (i.e., different from H^{-1}) in the critical case $\alpha = \frac{1}{2}$, and leads to the following:

Definition 52. The space $T_1(\mathbb{R}^n)$ is formed by all those $f \in \mathcal{S}'(\mathbb{R}^n)$ with wavelet decomposition (8-3) such that

$$\|f\|^2_{T_1(\mathbb{R}^n)} = \sup_{j\in\mathbb{N}_0} \sum_{k=2^j-1}^{2^{j+1}-2} 2^{k(-2-n)} \sum_{G\in G^k,\, l\in\mathbb{Z}^n} |\lambda^G_{kl}|^2 < \infty,$$

where $\{\lambda^G_{kl}\}$ is the sequence of wavelet coefficients of f given by (7-8).

Proposition 53. *Let $\{u^\varepsilon\}_{\varepsilon>0}$ be a family of approximate solutions of the 2D Euler equations with corresponding initial vorticities $\{\omega^\varepsilon_0\}_{\varepsilon>0}$ of positive sign. Then, for every $\varepsilon > 0$ and $t > 0$,*

$$\|\omega^\varepsilon(t)\|_{T_1(\mathbb{R}^2)_c} \lesssim 1. \tag{8-20}$$

Proof. Consider the pseudoenergy

$$H(\omega) = -\frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log|x-y|\omega(x)\omega(y)\,dx\,dy.$$

For every fixed $k \in \mathbb{N}_0$, we can express $H(\omega^\varepsilon(t))$ as

$$\begin{aligned} H(\omega^\varepsilon(t)) = -\frac{1}{2\pi}\sum_{l\in\mathbb{Z}^2}\int_{Q_{kl}}\int_{Q_{kl}} &\log|x-y|\omega^\varepsilon(x,t)\omega^\varepsilon(y,t)\,dx\,dy \\ &-\frac{1}{2\pi}\sum_{l\neq m}\int_{Q_{kl}}\int_{Q_{km}} \log|x-y|\omega^\varepsilon(x,t)\omega^\varepsilon(y,t)\,dx\,dy \\ =: H_{si,k}(\omega^\varepsilon(t)) + H_{ie,k}(\omega^\varepsilon(t)).& \end{aligned} \tag{8-21}$$

Here, $H_{si,k}$ and $H_{ie,k}$ refer to the self-induced part and the interaction energy at the dyadic level k, respectively.

Since $(\log|x-y|)_+ \le 2(|x|^2+|y|^2)$ (apply the parallelogram rule!), we can estimate $H_{ie,k}$ as

$$\begin{aligned} -H_{ie,k}(\omega^\varepsilon(t)) &\le \frac{1}{\pi}\sum_{l\neq m}\int_{Q_{kl}}\int_{Q_{km}} (|x|^2+|y|^2)\omega^\varepsilon(x,t)\omega^\varepsilon(y,t)\,dx\,dy \\ &\le \frac{2}{\pi} I_0(\omega^\varepsilon(t)) I_2(\omega^\varepsilon(t)), \end{aligned} \tag{8-22}$$

where

$$I_0(\omega) := \int_{\mathbb{R}^2}\omega(x)\,dx, \quad I_2(\omega) := \int_{\mathbb{R}^2}|x|^2\omega(x)\,dx.$$

Let $j \in \mathbb{N}_0$. By the positivity assumption of ω^ε and using (7-8), (7-16) and (7-17), we have

$$\sum_{k=2^j-1}^{2^{j+1}-2} 2^{-4k}\sum_{l\in\mathbb{Z}^2}|\lambda_{kl}(\omega^\varepsilon(t))|^2 \le \sum_{k=2^j-1}^{2^{j+1}-2}\sum_{l\in\mathbb{Z}^2}\left(\int_{Q_{kl}}|\omega^\varepsilon(x,t)|\,dx\right)^2$$

$$= \sum_{k=2^j-1}^{2^{j+1}-2} \sum_{l\in\mathbb{Z}^2} \int_{Q_{kl}} \int_{Q_{kl}} \omega^\varepsilon(x,t)\omega^\varepsilon(y,t)\,dx\,dy$$
$$\lesssim \sum_{k=2^j-1}^{2^{j+1}-2} \frac{1}{k} \sum_{l\in\mathbb{Z}^2} \int_{Q_{kl}} \int_{Q_{kl}} \big|\log|x-y|\big|\omega^\varepsilon(x,t)\omega^\varepsilon(y,t)\,dx\,dy$$
$$\approx \sum_{k=2^j-1}^{2^{j+1}-2} \frac{1}{k} H_{si,k}(\omega^\varepsilon(t)).$$

Then, by (8-21) and (8-22),

$$\sum_{k=2^j-1}^{2^{j+1}-2} 2^{-4k} \sum_{l\in\mathbb{Z}^2} |\lambda_{kl}(\omega^\varepsilon(t))|^2 \lesssim \sum_{k=2^j-1}^{2^{j+1}-2} \frac{1}{k}\big[H(\omega^\varepsilon(t)) - H_{ie,k}(\omega^\varepsilon(t))\big]$$
$$\le \Big[H(\omega^\varepsilon(t)) + \frac{2}{\pi} I_0(\omega^\varepsilon(t)) I_2(\omega^\varepsilon(t))\Big] \sum_{k=2^j-1}^{2^{j+1}-2} \frac{1}{k}$$
$$\lesssim H(\omega^\varepsilon(t)) + I_0(\omega^\varepsilon(t)) I_2(\omega^\varepsilon(t)).$$

Taking now the supremum over all j, we arrive at

$$\|\omega^\varepsilon(t)\|^2_{T_{1/2}(\mathbb{R}^2)} \lesssim H(\omega^\varepsilon(t)) + I_0(\omega^\varepsilon(t)) I_2(\omega^\varepsilon(t)) \lesssim 1,$$

where the last step follows from well-known conservation laws and ε-independence bounds for the quantities H, I_0 and I_2; see [Majda 1993, Section 3]. □

Remark 54. An alternative proof of Proposition 53 can be obtained from RMT spaces. To be more precise, we claim that

$$R_{1,2} \log^{\frac{1}{2}}(\mathbb{R}^2) \hookrightarrow T_1(\mathbb{R}^2). \tag{8-23}$$

Assuming momentarily the validity of this embedding, (8-20) follows from the a priori estimates given in [Tadmor 2001, Lemma 4.1] that assert (under the same assumptions of Proposition 53)

$$\|\omega^\varepsilon(t)\|_{R_{1,2}\log^{1/2}(\mathbb{R}^2)_c} \lesssim 1.$$

Next we show (8-23): for every $j \in \mathbb{N}_0$, we have

$$\sum_{k=2^j-1}^{2^{j+1}-2} 2^{-4k} \sum_{l\in\mathbb{Z}^2} |\lambda_{kl}|^2 \le \sum_{k=2^j-1}^{2^{j+1}-2} \sum_{l\in\mathbb{Z}^2} \Big(\int_{Q_{kl}} |f|\Big)^2$$
$$\lesssim \|f\|^2_{R_{1,2}\log^{1/2}(\mathbb{R}^2)} \sum_{k=2^j-1}^{2^{j+1}-2} k^{-1} \approx \|f\|^2_{R_{1,2}\log^{1/2}(\mathbb{R}^2)}.$$

This gives the desired result (8-23).

Remark 55. A priori estimates in the spirit of Proposition 53 but now for the V_Ψ-spaces introduced in Section 7 also hold. In particular, similar to in Definition 52, the distributional space $V_1(\mathbb{R}^2)$ is endowed with[43]

$$\|f\|_{V_1(\mathbb{R}^n)} = \sup_{j\in\mathbb{N}_0} \sum_{k=2^j-1}^{2^{j+1}-2} 2^{-2k}(1+k)^{-1/2} \sup_{G\in G^k,\, l\in\mathbb{Z}^n} |\lambda_{kl}^G|,$$

where $\{\lambda_{kl}^G\}$ is the sequence of wavelet coefficients of f given by (7-8); see Proposition 43. If $\{u^\varepsilon\}_{\varepsilon>0}$ is a family of approximate solutions of the 2D Euler equations with corresponding initial vorticities $\{\omega_0^\varepsilon\}_{\varepsilon>0}$ of positive sign then, for every $\varepsilon > 0$ and $t > 0$,

$$\|\omega^\varepsilon(t)\|_{V_1(\mathbb{R}^2)_c} \lesssim 1.$$

The proof of this result is similar but easier than the one given for Proposition 53.

9. Comparison between V_Ψ and T_Ψ

In view of the results obtained in Sections 7 and 8, it is natural to compare[44] the spaces $V_\Psi(\mathbb{R}^2)$ and $T_\Psi(\mathbb{R}^2)$ for a fixed decay Ψ. In this section we show that neither space contains the other. More precisely, we construct explicit examples of functions showing that $T_\Psi(\mathbb{R}^2)\backslash V_\Psi(\mathbb{R}^2) \neq \varnothing$ and $V_\Psi(\mathbb{R}^2)\backslash T_\Psi(\mathbb{R}^2) \neq \varnothing$.

Example 56 ($T_\Psi(\mathbb{R}^2)\backslash V_\Psi(\mathbb{R}^2) \neq \varnothing$). Given a scalar sequence $\{c_N\}_{N\in\mathbb{N}_0}$, let f be given by (7-7), where

$$\lambda_{Nl} = \begin{cases} 2^{2N} c_N, & N \in \mathbb{N}_0,\ l = (0,\dots,0), \\ 0, & \text{otherwise.} \end{cases}$$

We compute the norms of f in V_Ψ and T_Ψ using Propositions 43 and 51, respectively,

$$\|f\|_{V_\Psi(\mathbb{R}^2)} \approx \sup_{N\in\mathbb{N}_0} \frac{1}{\Psi(N)^2} \sum_{k=N}^{\infty} |c_k|$$

and

$$\|f\|_{T_\Psi(\mathbb{R}^2)} \approx \sup_{N\in\mathbb{N}_0} \frac{1}{\Psi(N)} \biggl(\sum_{k=N}^{\infty} |c_k|^2\biggr)^{1/2}.$$

Therefore, if we select $\{c_N\}_{N\in\mathbb{N}_0} \in \ell_2$ such that

$$\sum_{k=N}^{\infty} c_k^2 \approx \Psi(N)^2, \tag{9-1}$$

then $\|f\|_{T_\Psi(\mathbb{R}^2)} \approx 1$. The existence of such sequences (even with $\approx$ replaced by $=$ in (9-1)) is guaranteed by classical results in approximation theory (see, e.g., [Timan 1963, Section 2.5]). On the other hand, since

$$\sum_{k=N}^{\infty} |c_k| \geq \Bigl(\inf_{l\geq N} \frac{1}{|c_l|}\Bigr) \sum_{k=N}^{\infty} |c_k|^2 \approx \frac{\Psi(N)^2}{\sup_{l\geq N} |c_l|},$$

[43] Note that the additional log-smoothness of order $\frac{1}{2}$ (i.e., $(1+k)^{-1/2}$) is now introduced in the definition of $V_1(\mathbb{R}^n)$; see also Proposition 43. This modification is rather natural in view of the classical a priori estimates for $M^{1,1/2}(\mathbb{R}^2)$ obtained in [Majda 1993, Proposition, p. 928].

[44] According to the discussion at the beginning of Section 8.1, we may restrict our attention to the 2D setting.

we have

$$\frac{1}{\Psi(N)^2}\sum_{k=N}^{\infty}|c_k| \gtrsim \frac{1}{\sup_{l\geq N}|c_l|}, \tag{9-2}$$

but since $\{c_N\}_{N\in\mathbb{N}_0}\in\ell_2$, we have $\lim_{N\to\infty}|c_N|=0$ and therefore the left-hand side of (9-2) tends to ∞ as $N\to\infty$, showing that $f\notin V_\Psi(\mathbb{R}^2)$.

Example 57 ($V_\Psi(\mathbb{R}^2)\backslash T_\Psi(\mathbb{R}^2)\neq\varnothing$). Given a scalar sequence $\{c_l\}_{l\in\mathbb{Z}^n}$, define f using (7-7) with

$$\lambda_{Nl}=\begin{cases} c_l, & N=0,\ l\in\mathbb{Z}^n,\\ 0, & \text{otherwise.}\end{cases}$$

Computing norms using Propositions 43 and 51, we find

$$\|f\|_{V_\Psi(\mathbb{R}^2)}\approx\sup_{l\in\mathbb{Z}^n}|c_l| \quad\text{and}\quad \|f\|_{T_\Psi(\mathbb{R}^2)}\approx\Big(\sum_{l\in\mathbb{Z}^n}|c_l|^2\Big)^{1/2}.$$

Thus, if we select $\{c_l\}_{l\in\mathbb{Z}^n}\in\ell_\infty\backslash\ell_2$ we obtain an example of $f\in V_\Psi(\mathbb{R}^2)$ but $f\notin T_\Psi(\mathbb{R}^2)$.

Remark 58. The above computations show a stronger assertion: given any decays Ψ and Φ, one can always construct $f\in V_\Psi(\mathbb{R}^2)$ such that $f\notin T_\Phi(\mathbb{R}^2)$.

The previous remark shows that for different decays $\Psi\neq\Phi$, $V_\Psi(\mathbb{R}^2)$ cannot be contained in $T_\Phi(\mathbb{R}^2)$. The next result shows that under some additional condition the reverse inclusion is possible.

Proposition 59. *Suppose that*

$$\sum_{j=N}^{\infty}\Phi(j)\lesssim\Psi(N)^2,\quad N\in\mathbb{N}_0. \tag{9-3}$$

Then

$$T_\Phi(\mathbb{R}^2)\hookrightarrow V_\Psi(\mathbb{R}^2).$$

Proof. We use Nikolskii's inequality for entire functions of exponential type (see, e.g., [Timan 1963, Section 4.9.53]) to estimate

$$\begin{aligned}\sum_{j=N}^{\infty}2^{-2j}\|\Delta_j f\|_{L^\infty(\mathbb{R}^2)} &\lesssim \sum_{j=N}^{\infty}2^{-j}\|\Delta_j f\|_{L^2(\mathbb{R}^2)}\\ &\leq\Big(\sup_{M\in\mathbb{N}_0}\frac{2^{-M}}{\Phi(M)}\|\Delta_M f\|_{L^2(\mathbb{R}^2)}\Big)\sum_{j=N}^{\infty}\Phi(j)\\ &\lesssim\Psi(N)^2\sup_{M\in\mathbb{N}_0}\frac{2^{-M}}{\Phi(M)}\|\Delta_M f\|_{L^2(\mathbb{R}^2)}\\ &\leq\Psi(N)^2\sup_{M\in\mathbb{N}_0}\frac{1}{\Phi(M)}\Big(\sum_{j=M}^{\infty}2^{-2j}\|\Delta_j f\|^2_{L^2(\mathbb{R}^2)}\Big)^{1/2}\\ &=\Psi(N)^2\,\|f\|_{T_\Phi(\mathbb{R}^2)}.\end{aligned}$$

Therefore $T_\Phi(\mathbb{R}^2)\hookrightarrow V_\Psi(\mathbb{R}^2)$. $\square$

Remark 60. The assumption (9-3) is quite restrictive. In particular, it forces the series $\sum_{j=0}^{\infty}\Phi(j)$ to be convergent. This automatically excludes many interesting examples of spaces T_Φ such as the corresponding one to the decay $\Phi(j)=j^{-\alpha}$ with $\alpha\in(0,1]$, which are connected with R-spaces (see Theorem 48). In conclusion, the analysis of T_Φ cannot be, in general, reduced to study of the simpler spaces V_Ψ.

10. A sparse approach to energy conservation

Throughout this section, we work with the following special class of approximate solutions on $\mathbb{T}^2\equiv[0,2\pi]^2$ introduced in [Cheskidov et al. 2016, Definition 3].

Definition 61. Let $u\in C(0,T;L^2(\mathbb{T}^2))$ with $u_0\in L^2(\mathbb{T}^2)$. We say that a weak solution u of Euler equations is *physically realizable* with initial velocity u_0 provided that there exists a family $\{u^\varepsilon\}_{\varepsilon>0}$ of solutions of Navier–Stokes equations with viscosity ε, such that $u^\varepsilon\rightharpoonup u$ weakly* in $L^\infty(0,T;L^2(\mathbb{T}^2))$ and $u_0^\varepsilon\to u_0$ strongly in $L^2(\mathbb{T}^2)$. In this case, $\{u^\varepsilon\}_{\varepsilon>0}$ is called a *physical realization* of u.

Next we provide the proof of Theorem 14. In this regard, the following interpolation inequality involving sparse function spaces plays a crucial role. This result is of independent interest.

Lemma 62. *Let Q_0 be a cube in $\mathbb{R}^2$ or $Q_0=\mathbb{T}^2$, and let Ψ be an admissible decay. Assume that $f\in S_\Psi(Q_0)\cap\dot H^1(Q_0)$. Then, with absolute constants, we have*

$$\|f-f_{Q_0}\|_{L^2(Q_0)}\lesssim\frac{\Psi(-\log r)}{r}\|f\|_{S_\Psi(Q_0)}+r\|\nabla f\|_{L^2(Q_0)}\quad\textit{for all } r\in(0,1).\tag{10-1}$$

Remark 63 (see equation (11) in [Cheskidov et al. 2016]). Let us show how (10-1) can be applied to produce a classical Gagliardo–Nirenberg inequality. Let $f\in L^p$, $p\in(1,2)$. Then by Proposition 30, with decay $\Psi(t)=2^{-2t(1-1/p)}$, $L^p(Q_0)\hookrightarrow S_\Psi(Q_0)$. Applying (10-1) for this special decay, we obtain

$$\|f-f_{Q_0}\|_{L^2(Q_0)}\lesssim r^{1-\frac{2}{p}}\|f\|_{L^p(Q_0)}+r\|\nabla f\|_{L^2(Q_0)}$$

for all $r\in(0,1)$. Optimizing the right-hand side by equating both terms, i.e., selecting

$$r=\left(\frac{\|\nabla f\|_{L^2(Q_0)}}{\|f\|_{L^p(Q_0)}}\right)^{-\frac{p}{2}},$$

we find

$$\|f-f_{Q_0}\|_{L^2(Q_0)}\lesssim\|\nabla f\|_{L^2(Q_0)}^{1-\frac{p}{2}}\|f\|_{L^p(Q_0)}^{\frac{p}{2}}.$$

Proof of Lemma 62. We will use the sparse characterization of L^2 (see Theorem 29 and [Domínguez and Milman 2021]):

$$\|f-f_{Q_0}\|_{L^2(Q_0)}\approx\sup_{\mathcal{Q}=(Q_i)_{i\in I}\in S(Q_0)}\left\{\sum_{i\in I}\left(\frac{1}{|Q_i|}\int_{Q_i}|f-f_{Q_i}|\right)^2|Q_i|\right\}^{1/2}\tag{10-2}$$

and

$$\|f\|_{L^2(Q_0)}\approx\sup_{\mathcal{Q}\in S(Q_0)}\left\{\sum_{i\in I}\left(\frac{1}{|Q_i|}\int_{Q_i}|f|\right)^2|Q_i|\right\}^{1/2}.\tag{10-3}$$

To estimate the left-hand side of (10-1) we use (10-2). Let $f \in S_\Psi(Q_0)$ and $\mathcal{Q} \in S(Q_0)$. Then, for $M \in \mathbb{N}_0$ we have

$$\sum_{i\in I}\biggl(\frac{1}{|Q_i|}\int_{Q_i}|f-f_{Q_i}|\biggr)^2|Q_i| = J_1(M)+J_2(M), \tag{10-4}$$

where

$$J_1(M) := \sum_{k=0}^{M}\sum_{Q_i\in\mathbb{D}_k(\mathcal{Q})}\biggl(\frac{1}{|Q_i|}\int_{Q_i}|f-f_{Q_i}|\biggr)^2|Q_i|,$$
$$J_2(M) := \sum_{k=M+1}^{\infty}\sum_{Q_i\in\mathbb{D}_k(\mathcal{Q})}\biggl(\frac{1}{|Q_i|}\int_{Q_i}|f-f_{Q_i}|\biggr)^2|Q_i|.$$

We estimate $J_1(M)$ and $J_2(M)$. Since $f \in S_\Psi(Q_0)$, we find

$$\begin{aligned} J_1(M) &\lesssim \sum_{k=0}^{M}\sum_{Q_i\in\mathbb{D}_k(\mathcal{Q})}\biggl(\frac{1}{|Q_i|}\int_{Q_i}|f|\biggr)^2|Q_i| \approx \sum_{k=0}^{M}2^{2k}\sum_{Q_i\in\mathbb{D}_k(\mathcal{Q})}\biggl(\int_{Q_i}|f|\biggr)^2 \\ &\le \sum_{k=0}^{M}2^{2k}s_{k+1}(f)^2 \\ &\lesssim \|f\|^2_{S_\Psi(Q_0)}\sum_{k=1}^{M}2^{2k}\Psi(k)^2 \\ &\lesssim \|f\|^2_{S_\Psi(Q_0)}2^{2M}\Psi(M)^2, \end{aligned} \tag{10-5}$$

where we have used Definition 36(i) in the last estimate.

To estimate $J_2(M)$ we will make use of the classical Poincaré inequality,

$$\int_Q |f-f_Q| \lesssim \ell(Q)\int_Q|\nabla f|. \tag{10-6}$$

Then, by (10-6) and (10-3) applied to $|\nabla f|$,

$$\begin{aligned} J_2(M) &\lesssim \sum_{k=M+1}^{\infty}\sum_{Q_i\in\mathbb{D}_k(\mathcal{Q})}\biggl(\frac{\ell(Q_i)}{|Q_i|}\int_{Q_i}|\nabla f|\biggr)^2|Q_i| \\ &\approx \sum_{k=M+1}^{\infty}2^{-k2}\sum_{Q_i\in\mathbb{D}_k(\mathcal{Q})}\biggl(\frac{1}{|Q_i|}\int_{Q_i}|\nabla f|\biggr)^2|Q_i| \\ &\lesssim \|\nabla f\|^2_{L^2(Q_0)}\sum_{k=M+1}^{\infty}2^{-k2} \quad \text{(by Hölder's inequality)} \\ &\approx \|\nabla f\|^2_{L^2(Q_0)}2^{-M2}. \end{aligned} \tag{10-7}$$

Inserting the estimates (10-5) and (10-7) into (10-4), we achieve

$$\left\{\sum_{i\in I}\left(\frac{1}{|Q_i|}\int_{Q_i}|f-f_{Q_i}|\right)^2|Q_i|\right\}^{1/2}\lesssim \|f\|_{S_\Psi(Q_0)}2^M\Psi(M)+\|\nabla f\|_{L^2(Q_0)}2^{-M}.$$

Since this bound is independent of the sparse family $\mathcal{Q}$, we arrive at (see (10-2))

$$\|f-f_{Q_0}\|_{L^2(Q_0)}\lesssim \|f\|_{S_\Psi(Q_0)}2^M\Psi(M)+\|\nabla f\|_{L^2(Q_0)}\,2^{-M}.$$

Since Ψ is decreasing, the previous estimate can be expressed as

$$\|f-f_{Q_0}\|_{L^2(Q_0)}\lesssim \frac{\Psi(-\log r)}{r}\|f\|_{S_\Psi(Q_0)}+r\|\nabla f\|_{L^2(Q_0)}$$

for all $r\in(0,1)$. □

We are now ready to present the proof of Theorem 14. The strategy of proof is inspired by [Lanthaler et al. 2021]; we have replaced the role played there by structure functions with our decays of sparse indices. We provide full details for the sake of completeness.

Proof of Theorem 14. Let $\{u^\varepsilon\}_{\varepsilon>0}$ be a physical realization of u and let $\{\omega^\varepsilon\}_{\varepsilon>0}$ be the related vorticities. By assumption, there exists an admissible decay Ψ such that

$$M:=\sup_{\varepsilon>0}\|\omega^\varepsilon\|_{C(0,T;S_\Psi(\mathbb{T}^2))}<\infty. \tag{10-8}$$

Furthermore, ω^ε satisfies the transport equation,

$$\omega^\varepsilon_t+u^\varepsilon\cdot\nabla\omega^\varepsilon=\varepsilon\Delta\omega^\varepsilon,$$

and $\operatorname{div}u^\varepsilon=0$. Multiplying both sides of the previous equation by ω^ε and integrating on $\mathbb{T}^2$ yields

$$\frac{d}{dt}\,\|\omega^\varepsilon\|^2_{L^2(\mathbb{T}^2)}=-2\varepsilon\|\nabla\omega^\varepsilon\|^2_{L^2(\mathbb{T}^2)}.$$

Consequently, for any $\delta\in(0,T)$ and $t\in(\delta,T)$,

$$\|\omega^\varepsilon(t)\|^2_{L^2(\mathbb{T}^2)}=\|\omega^\varepsilon(\delta)\|^2_{L^2(\mathbb{T}^2)}-2\varepsilon\int_\delta^t\|\nabla\omega^\varepsilon(s)\|^2_{L^2(\mathbb{T}^2)}\,ds. \tag{10-9}$$

According to Lemma 62 (with $Q_0=\mathbb{T}^2$ and[45] $f=\omega^\varepsilon=\omega^\varepsilon(\,\cdot\,,t),\ t\in(0,T)$) and (10-8), there exists a universal constant $C>0$ such that

$$\|\omega^\varepsilon\|^2_{L^2(\mathbb{T}^2)}\le C\frac{\Psi(-\log r)^2}{r^2}M^2+Cr^2\|\nabla\omega^\varepsilon\|^2_{L^2(\mathbb{T}^2)}\quad\text{for all } r\in(0,1).$$

Integrating,

$$\int_\delta^t\|\omega^\varepsilon(s)\|^2_{L^2(\mathbb{T}^2)}\,ds\le CTM^2\frac{\Psi(-\log r)^2}{r^2}+Cr^2\int_\delta^t\|\nabla\omega^\varepsilon(s)\|^2_{L^2(\mathbb{T}^2)}\,ds \tag{10-10}$$

[45]Note that ω^ε has mean zero, i.e., $\omega^\varepsilon_{\mathbb{T}^2}=0$.

for $r \in (0,1)$. In fact, letting $\Psi(t)=\Psi(0)$ for $t<0$, (10-10) with $r \geq 1$ follows immediately from Poincaré's inequality.[46] Next we optimize the right-hand side of (10-10), setting

$$r_0 := \log \frac{\bigl(\int_\delta^t \|\nabla\omega^\varepsilon(s)\|^2_{L^2(\mathbb{T}^2)}\,ds\bigr)^{1/4}}{\Psi(0)^{1/2}} \quad \text{and} \quad r = \frac{\Psi(r_0)^{1/2}}{\bigl(\int_\delta^t \|\nabla\omega^\varepsilon(s)\|^2_{L^2(\mathbb{T}^2)}\,ds\bigr)^{1/4}}.$$

Note that $-\log r \geq r_0$ (since Ψ is decreasing) and thus $\Psi(-\log r) \leq \Psi(r_0)$. Accordingly, it follows from (10-10) that

$$\biggl(\int_\delta^t \|\omega^\varepsilon(s)\|^2_{L^2(\mathbb{T}^2)}\,ds\biggr)^2 \leq C^2(TM^2+1)^2 \Psi\biggl(\log \frac{\bigl(\int_\delta^t \|\nabla\omega^\varepsilon(s)\|^2_{L^2(\mathbb{T}^2)}\,ds\bigr)^{1/4}}{\Psi(0)^{1/2}}\biggr)^2 \int_\delta^t \|\nabla\omega^\varepsilon(s)\|^2_{L^2(\mathbb{T}^2)}\,ds.$$

Setting $x_\varepsilon = x_\varepsilon(t) = \varepsilon \int_\delta^t \|\omega^\varepsilon(s)\|^2_{L^2(\mathbb{T}^2)}\,ds$ and $y_\varepsilon = y_\varepsilon(t) = \int_\delta^t \|\nabla\omega^\varepsilon(s)\|^2_{L^2(\mathbb{T}^2)}\,ds$, the previous estimate can be rewritten as

$$\Bigl(\frac{x_\varepsilon}{\varepsilon}\Bigr)^2 \leq f(y_\varepsilon), \tag{10-11}$$

where

$$f(y) = C^2(TM^2+1)^2 y \Psi\biggl(\log \frac{y^{1/4}}{\Psi(0)^{1/2}}\biggr)^2.$$

The function f satisfies

$$\sup_{y>0} \frac{f(y)}{y} = C^2(TM^2+1)^2 \sup_{y>0} \Psi\biggl(\log \frac{y^{1/4}}{\Psi(0)^{1/2}}\biggr)^2 = C^2(TM^2+1)^2\Psi(0)^2 < \infty,$$

and (recall that $\lim_{y\to\infty}\Psi(y)=0$)

$$\limsup_{y\to\infty} \frac{f(y)}{y} = C^2(TM^2+1)^2 \limsup_{y\to\infty} \Psi\biggl(\log \frac{y^{1/4}}{\Psi(0)^{1/2}}\biggr)^2 = 0.$$

In addition $f(0)=0$ (note that $\lim_{y\to-\infty}\Psi(y)=\Psi(0)$). Hence [Lanthaler et al. 2021, Lemma C.1] guarantees the existence of a strictly increasing function F with $F(y) \geq f(y)$ such that the corresponding inverse function of F can be expressed as $F^{-1}(x) = \sigma(\sqrt{x})x$, where σ is a continuous increasing function with $\sigma(\sqrt{x}) \geq \sigma_0 > 0$ and $\lim_{x\to\infty}\sigma(x)=\infty$. From (10-11), we have

$$\Bigl(\frac{x_\varepsilon}{\varepsilon}\Bigr)^2 \leq F(y_\varepsilon),$$

and thus

$$\sigma\Bigl(\frac{x_\varepsilon}{\varepsilon}\Bigr)\Bigl(\frac{x_\varepsilon}{\varepsilon}\Bigr)^2 = F^{-1}\Bigl(\Bigl(\frac{x_\varepsilon}{\varepsilon}\Bigr)^2\Bigr) \leq y_\varepsilon$$

---

[46] By the Poincaré inequality $\|\omega^\varepsilon\|_{L^2(\mathbb{T}^2)} \lesssim \|\nabla\omega^\varepsilon\|_{L^2(\mathbb{T}^2)}$, we have, for $r \geq 1$,

$$\int_\delta^t \|\omega^\varepsilon(s)\|^2_{L^2(\mathbb{T}^2)}\,ds \lesssim \int_t^\delta \|\nabla\omega^\varepsilon(s)\|^2_{L^2(\mathbb{T}^2)}\,ds \leq r^2 \int_t^\delta \|\nabla\omega^\varepsilon(s)\|^2_{L^2(\mathbb{T}^2)}\,ds.$$

or equivalently

$$-\varepsilon^2 \int_\delta^t \|\nabla\omega^\varepsilon(s)\|^2_{L^2(\mathbb{T}^2)}\, ds \le -\sigma\Big(\frac{x_\varepsilon}{\varepsilon}\Big)x_\varepsilon^2. \tag{10-12}$$

Note that (10-9) can be rewritten as

$$\frac{d}{dt}\, x_\varepsilon = \varepsilon\|\omega^\varepsilon(\delta)\|^2_{L^2(\mathbb{T}^2)} - 2\varepsilon^2 \int_\delta^t \|\nabla\omega^\varepsilon(s)\|^2_{L^2(\mathbb{T}^2)}\, ds$$

and then, by (10-12) and well-known a priori L^2-estimates[47] for Navier–Stokes solutions (see [Lanthaler et al. 2021, Lemma A.2]),

$$\frac{d}{dt}\, x_\varepsilon \le \varepsilon\|\omega^\varepsilon(\delta)\|^2_{L^2(\mathbb{T}^2)} - 2\sigma\Big(\frac{x_\varepsilon}{\varepsilon}\Big)x_\varepsilon^2 \le \frac{\|u_0\|^2_{L^2(\mathbb{T}^2)}}{\delta} - 2\sigma\Big(\frac{x_\varepsilon}{\varepsilon}\Big)x_\varepsilon^2. \tag{10-13}$$

Next we show that

$$\lim_{\varepsilon\to 0} \varepsilon \int_\delta^t \|\omega^\varepsilon(s)\|^2_{L^2(\mathbb{T}^2)}\, ds = 0 \tag{10-14}$$

uniformly with respect to $t \in [\delta, T]$. For $\eta > 0$ arbitrary, consider the set

$$A_{\eta,t} := \{\varepsilon > 0 : x_\varepsilon(t) \ge \eta\}.$$

Assume first that $0 \in \bar{A}_{\eta,t}$. In particular, there exists $\{\varepsilon_l\}_{l\in\mathbb{N}} \subset A_{\eta,t}$ with $\lim_{l\to\infty} \varepsilon_l = 0$. Since σ is increasing, $\sigma(x_{\varepsilon_l}(t)/\varepsilon_l) \ge \sigma(\eta/\varepsilon_l)$ and (see (10-13))

$$\frac{d}{dt}\, x_{\varepsilon_l} \le \frac{\|u_0\|^2_{L^2(\mathbb{T}^2)}}{\delta} - 2\sigma\Big(\frac{\eta}{\varepsilon_l}\Big)\eta^2.$$

Observe that $\lim_{l\to\infty} \sigma(\eta/\varepsilon_l) = \infty$, which yields that $x_{\varepsilon_l}(t)$ is decreasing with respect to t whenever $l \ge l_0 = l_0(\eta, \sigma, \|u_0\|_{L^2(\mathbb{T}^2)}, \delta)$. Since $x_{\varepsilon_l}(\delta) = 0$ and $x_{\varepsilon_l}(t) \ge 0$, we conclude that $x_{\varepsilon_l}(t) = 0$ for all $t > \delta$ and $l \ge l_0$. Therefore there exists $\varepsilon_0 = \varepsilon_0(\eta, \sigma, \|u_0\|_{L^2(\mathbb{T}^2)}, \delta) > 0$ such that

$$x_\varepsilon(t) \le \eta \quad \text{if } \varepsilon \le \varepsilon_0. \tag{10-15}$$

On the other hand, if $0 \notin \bar{A}_{\varepsilon,t}$ then (10-15) holds trivially. Either way, we have shown that (10-14) is fulfilled.

Recall the energy formula for 2D Navier–Stokes solutions

$$\frac{d}{dt}\, \|u^\varepsilon\|^2_{L^2(\mathbb{T}^2)} = -2\varepsilon\|\omega^\varepsilon\|^2_{L^2(\mathbb{T}^2)}.$$

Integrating over $[\delta, t]$:

$$\|u^\varepsilon(t)\|^2_{L^2(\mathbb{T}^2)} - \|u^\varepsilon(\delta)\|^2_{L^2(\mathbb{T}^2)} = -2\varepsilon \int_\delta^t \|\omega^\varepsilon(s)\|^2_{L^2(\mathbb{T}^2)}\, ds.$$

In light of (10-14), we get (uniformly in t)

$$\lim_{\varepsilon\to 0} \|u^\varepsilon(t)\|^2_{L^2(\mathbb{T}^2)} - \|u^\varepsilon(\delta)\|^2_{L^2(\mathbb{T}^2)} = 0. \tag{10-16}$$

[47] Recall that $u_0 \in L^2(\mathbb{T}^2)$; see Definition 61.

Since $\{u^\varepsilon\}_{\varepsilon>0}$ is sparse stable, by virtue of Theorem 2, one can find a sequence of ε's with $\varepsilon \to 0$ such that

$$\lim_{\varepsilon\to 0} \|u^\varepsilon(t)\|^2_{L^2(\mathbb{T}^2)} = \|u(t)\|^2_{L^2(\mathbb{T}^2)}$$

for all $t \in (0, T)$. This combined with (10-16) lead to

$$\|u(t)\|_{L^2(\mathbb{T}^2)} = \|u(\delta)\|_{L^2(\mathbb{T}^2)} \quad \text{for any } t \in (\delta, T). \tag{10-17}$$

On the other hand, since $\|u(t)\|_{L^2(\mathbb{T}^2)}$ is right-continuous at $t = 0$, given any $\eta > 0$ one can find $\delta > 0$ (depending on η) such that

$$0 \le \|u(t)\|_{L^2(\mathbb{T}^2)} - \|u_0\|_{L^2(\mathbb{T}^2)} \le \eta \quad \text{for all } t \in (0, \delta].$$

Since (10-17) also holds, we have

$$0 \le \|u(t)\|_{L^2(\mathbb{T}^2)} - \|u_0\|_{L^2(\mathbb{T}^2)} \le \eta \quad \text{for all } t \in (0, T].$$

Clearly this shows that u is conservative, i.e., $\|u(t)\|_{L^2(\mathbb{T}^2)} = \|u_0\|_{L^2(\mathbb{T}^2)}$, $t \in [0, T]$. □

Added in proof

After this paper was submitted, the sparse spaces technology was successfully applied by Domínguez and D. Spector [2024] to resolve in the negative the DiPerna–Majda gap problem (see Section 1.1): $M^{1,1}$ is the borderline regularity space regarding existence of approximate solution sequences with concentrations for the 2D Euler equations. This closes the gap between lack of concentration (and hence existence of weak solutions with energy conservation) in $M^{1,\alpha}$ with $\alpha > 1$ [DiPerna and Majda 1987a] and the concentration-cancellation phenomenon in $M^{1,1/2}$ [Delort 1991; Majda 1993]. As a consequence, the sufficient conditions for the H^{-1}-compactness assertion (1-2) turn out to be also necessary. Hence the sparse methods introduced in this paper provide an optimal strategy to characterize lack of concentration/concentration-cancellation.

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ILL-POSEDNESS FOR DISPERSIVE EQUATIONS: DEGENERATE DISPERSION AND THE TAKEUCHII–MIZOHATA CONDITION

IN-JEE JEONG AND SUNG-JIN OH

We provide a unified viewpoint on two ill-posedness mechanisms for dispersive equations in one spatial dimension, namely degenerate dispersion and (the failure of) the Takeuchi–Mizohata condition. Our approach is based on a robust energy- and duality-based method introduced in an earlier work of the authors in the setting of Hall-magnetohydrodynamics. Concretely, the main results in this paper concern strong ill-posedness of the Cauchy problem (e.g., nonexistence and unboundedness of the solution map) in high-regularity Sobolev spaces for various quasilinear degenerate Schrödinger- and KdV-type equations, including the Hunter–Smothers equation, $K(m,n)$ models of Rosenau–Hyman, and the inviscid surface growth model. The mechanism behind these results may be understood in terms of the combination of two effects: degenerate dispersion — which is a property of the principal term in the presence of degenerating coefficients — and the evolution of the amplitude governed by the Takeuchi–Mizohata condition — which concerns the subprincipal term. We also demonstrate how the same techniques yield a more quantitative version of the classical L^2-ill-posedness result by Mizohata for linear variable-coefficient Schrödinger equations with failed Takeuchi–Mizohata condition.

1. Introduction

1.1. ***Quasilinear degenerate dispersive equations.*** We study the issue of ill-posedness of the Cauchy problem for various quasilinear dispersive equations in one spatial dimension in the presence of *degenerate dispersion*. We consider both Schrödinger- and KdV-type equations. Examples of Schrödinger-type equations we treat include, for instance, the equation of Hunter and Smothers [2019]

$$i\partial_t\phi + \partial_x(|\phi|^2\partial_x\phi) = 0, \tag{1-1}$$

which was derived from the equation of Majda, Rosales and Schonbek [Majda et al. 1988] describing the resonant reflection of sound waves off a sawtooth entropy wave, as well as its Hamiltonian variant

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considered by Germain, Harrop-Griffith and Marzuola [Germain et al. 2020; Harrop-Griffiths and Marzuola 2022]

$$i\partial_t\phi + \bar{\phi}\partial_x(\phi\partial_x\phi) - \mu_0|\phi|^2\phi = 0, \quad \mu_0 = -1, 0, 1, \tag{1-2}$$

where $\phi : \mathbb{R}_+ \times \mathbb{T} \to \mathbb{C}$. A degenerate Schrödinger-type equation similar in form to (1-1) and (1-2) (but with different lower-order terms) also arose earlier in the work of Rosenau and Schochet [2005] in the study of compact breathers (see also Remark 1.3 below). In the KdV case, our results cover the $K(m, n)$ equation of Rosenau and Hyman [1993] with $n = 2$, i.e.,

$$\partial_t u + \left(\frac{1}{m}u^m\right)_x + \left(\frac{1}{2}u^2\right)_{xxx} = 0, \quad m \text{ a nonnegative integer}, \tag{1-3}$$

which has been studied extensively in connection with the remarkable nonlinear phenomenon of the existence of *compactons* (solitons with compact spatial support) [Rosenau 1994; 2005; 2006; Rosenau and Hyman 1993; Zilburg and Rosenau 2017; 2018] (see [Rosenau and Zilburg 2018] for a recent review), as well as the inviscid surface growth model (see [Blömker and Romito 2009; 2012; Choi and Yang 2021; Ożański and Robinson 2019] for the full surface growth model, with the dissipation $-\nu h_{xxxx}$ on the right-hand side)

$$\partial_t h + ((h_x)^2)_{xx} = 0, \tag{1-4}$$

where $u, h : \mathbb{R}_+ \times \mathbb{T} \to \mathbb{R}$. Indeed, degenerate KdV-type equations similar in form to (1-3) appear in various subjects including sedimentation models [Zumbrun 1999; Betancourt et al. 2011], the shoreline problem in water waves [Lannes and Métivier 2018] and magma dynamics [Simpson et al. 2007; 2008], to name a few. A more extensive list of references on degenerate KdV equations can be found in [Germain et al. 2019; 2020].

In each of these equations, observe that the highest-order term is nonlinear — more specifically, quadratic or cubic — in the solution. Vanishing of the solution, therefore, leads to some kind of "degeneracy" of the highest-order term, which in turn gives rise to delicate issues in the (local) well-posedness of the associated Cauchy problem.

Indeed, for initial data that are uniformly bounded away from 0 (a property henceforth referred to as *nondegeneracy*), one expects local well-posedness in high-regularity L^2-based Sobolev spaces $H^s(\mathbb{T})$. For example, in the case of (1-1), for a sufficiently regular solution ϕ, one has the conservation of the L^2 norm:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{T}} |\phi|^2 \,\mathrm{d}x\right) = 0.$$

Obtaining higher-regularity a priori estimates is a much more nontrivial task. One can observe the following bound for $n \geq 1$ (in operator notation) at each t:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{T}} |(\partial_x|\phi|^2\partial_x)^n\phi|^2 \,\mathrm{d}x\right) \lesssim_n \|\phi\|_{H^{2n}}^{4n+2}.$$

Furthermore, as long as the solution stays nondegenerate at t, in the sense that $\inf_x |\phi(x, t)|^2 > c$ for some $c > 0$, a standard argument involving the ellipticity of $(\partial_x|\phi(x, t)|^2\partial_x)^n$ allows us to bound $\|\phi(\cdot, t)\|_{H^{2n}}^2$ by the integral on the left-hand side up to errors of the form $O(\|\phi(\cdot, t)\|_{H^{2n}}^{4n+2})$. Putting these together, one can establish a short-time H^{2n} a priori estimate for the solution ϕ with nondegenerate initial data. However, in the case of *degenerate* initial data (i.e., those without a uniform bound away from 0), the above scheme for a short-time H^{2n} a priori estimate with $n \geq 1$ clearly breaks down.

In this paper, we show that this failure of proof of higher-derivative a priori estimates is, in fact, a manifestation of genuine ill-posedness in standard function spaces. Despite the formal conservation of the L^2 norm, we demonstrate that all of the equations above are rather strongly ill-posed — in the sense of nonexistence of solutions and unboundedness of the data-to-solution map in suitable set-ups — in a neighborhood of degenerate initial data (e.g., zero data) in high-regularity spaces ($C^{k-1,1}$, Sobolev or Hölder spaces).

1.2. *Main results for quasilinear degenerate dispersive equations.*

1.2.1. *Results for Schrödinger-type equations.* To treat the Hunter–Smothers equation (1-1) and the Hamiltonian equation (1-2) simultaneously, we shall consider the general equation

$$i\partial_t\phi + |\phi|^2\partial_{xx}\phi + \alpha_1\phi|\partial_x\phi|^2 + \beta_1\bar{\phi}(\partial_x\phi)^2 + \mu_1|\phi|^2\phi = 0, \tag{1-5}$$

where $\phi : \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{C}$, $\alpha_1, \beta_1 \in \mathbb{R}$ and $\mu_1 \in \mathbb{C}$. Indeed, the case $\alpha_1 = \beta_1 = 1$ and $\mu_1 = 0$ corresponds to (1-1), while the case $\alpha_1 = 0$ and $\beta_1 = 1$ corresponds to (1-2).

For the statement of the main results, we need to introduce the following exponents. Given $\alpha_1, \beta_1 \in \mathbb{R}$, we introduce the exponent

$$\sigma_c = -\left(\frac{\alpha_1}{2} + \beta_1 - 1\right) \tag{1-6}$$

and let s_c be the smallest integer greater than 1 and $\sigma_c - \frac{1}{2}$, i.e.,

$$s_c = \max\bigl\{2, \lfloor \sigma_c - \tfrac{1}{2} \rfloor + 1\bigr\}. \tag{1-7}$$

Note that $\sigma_c = -\frac{1}{2}$ and $s_c = 2$ for (1-1), while $\sigma_c = 0$ and $s_c = 2$ for (1-2). For the significance of σ_c and $\lfloor \sigma_c - \frac{1}{2} \rfloor + 1$, see Remark 1.4 and Section 1.3. We note already that the lower bound $s_c \geq 2$ is a technical byproduct of our proof, which we have not attempted to optimize.

Our first result is unboundedness of the solution map (i.e., norm inflation) in C^{s_c} near any solution with a linear degeneracy.

Theorem 1.1 (unboundedness of the solution map near a linearly degenerate solution). *Assume that there exists a solution $f \in L^\infty([0, \delta]; C^{s_c+1,1}(\mathbb{T}))$ to* (1-5) *with some $\delta > 0$ such that $f(t = 0) = f_0$ is linearly degenerate; that is, there exists $x_0 \in \mathbb{T}$ with $f(x_0) = 0$ and $f_0'(x_0) \neq 0$.*

Then, for any $\epsilon > 0$, $s_0 \geq s_c$, and $0 < \delta' \leq \delta$, we can find $\tilde{\phi}_0 \in C^\infty(\mathbb{T})$ such that $\|\tilde{\phi}_0\|_{C^{s_0}} \leq \epsilon$ and one of the following holds:

- *there exists **no** $L^\infty([0, \delta']; C^{s_c}(\mathbb{T}))$ solution to* (1-5) *with initial data $f_0 + \tilde{\phi}_0$; or*
- ***any** $L^\infty([0, \delta']; C^{s_c}(\mathbb{T}))$ solution ϕ with $\phi(t = 0) = f_0 + \tilde{\phi}_0$ satisfies*

$$\sup_{0<t<\delta'} \|\phi(t, \cdot) - f(t, \cdot)\|_{C^{s_c}(\mathbb{T})} > c_0(\delta')^{-1/2},$$

with some $c_0 > 0$ depending only on f.

While the solutions considered in this theorem and below are assumed to be only L^∞ in time, it is immediate from the equation and the high spatial regularity (i.e., $s_c \geq 2$ in the present case, and $s_c \geq 3$ in the KdV-type case below) that they are in fact (at least) continuous as a function of (t, x). Hence, there is no ambiguity in the notion of the initial data (i.e., the restriction to $\{t = 0\}$) for such solutions.

We remark that the norm inflation assertion immediately implies the inflation of any norm that controls C^{s_c}, such as H^σ with $\sigma > s_c + \frac{1}{2}$. In fact, our proof readily extends to norm inflation in H^σ for any $\sigma > \sigma_c$ in the second alternative, which is expected to be sharp according to Remark 1.4 and Section 1.3 below; see Remark 2.10 for further details. We also remark that the statement of Theorem 1.1 should extend over to the case of solutions with orders of degeneracy other than 1. For simplicity, however, we restrict ourselves to the linearly degenerate case, which is "critical" in some sense; see Section 1.3.

Our second result is the nonexistence of a regular local-in-time solution in arbitrarily high-regularity C^{s_0} spaces.

Theorem 1.2 (nonexistence of regular local-in-time solution). *For any $\epsilon > 0$ and $s_0 \geq s_c + 2$, there exists an initial data $\phi_0 \in C^\infty(\mathbb{T})$ satisfying $\|\phi_0\|_{C^{s_0}} < \epsilon$ for which there is no corresponding solution to* (1-5) *belonging to $L^\infty([0, \delta]; C^{s_c+2}(\mathbb{T}))$ with any $\delta > 0$.*

As an immediate corollary of the above, we have that (1-1) and (1-2) are ill-posed in the strongest sense of Hadamard in function spaces which contain C^∞ and control the C^4 norm (where $4 = s_c + 2$): there exists C^∞ initial data without a local solution in $L^\infty_t W^{s,p}$ with $s - \frac{1}{p} > 4$ and $L^\infty_t C^{k,\alpha}$ with $k + \alpha \geq 4$.

Remark 1.3. We give a few simple remarks regarding the above.

- In all of the above, the physical domain could be taken to be $\mathbb{R}$ instead of $\mathbb{T}$.
- As one may expect, the lower-order term $\mu_1|\phi|^2\phi$ in (1-5) does not play any essential role.
- In [Rosenau and Schochet 2005], the following equation (with $\mu = 1$) was studied:

$$i\partial_t w + \tfrac{3}{8}(\partial_x(|\partial_x w|^2 \partial_x w) + \mu|w|^2 w) = 0. \tag{1-8}$$

If the coefficient μ in front of the lower-order term is zero, then observe that $\phi := \partial_x w$ obeys exactly an equation of the form (1-5), to which our theorems apply. In view of the preceding remark, we expect our method to be readily extendible to (1-8) for $\mu = 1$ as well.

Remark 1.4 (exponents σ_c, s_c, Takeuchi–Mizohata condition and degenerate dispersion). Observe that the ill-posedness results, Theorems 1.1 and 1.2, hold for all possible coefficients α_1, β_1 in front of subprincipal terms, although these possibly *alter* the exponents σ_c and s_c. Heuristically, σ_c is the expected critical L^2-Sobolev regularity exponent above which the linearization of (1-5) around a regular linearly degenerate solution is ill-posed. In fact, the negativity of σ_c already signals L^2-ill-posedness of the linearized equation by the classical Takeuchi–Mizohata condition [Mizohata 1985, Chapter VII]! Even if σ_c is positive, it turns out to be L^∞-ill-posed after taking k many derivatives with $k > \sigma_c - \frac{1}{2}$. This consideration motivates the exponent s_c and our ill-posedness results. We shall elaborate on this remark in Section 1.3.

1.2.2. *Results for KdV-type equations.* To unify our treatment of KdV-type equations, we consider the general equation

$$\partial_t u + u u_{xxx} + \alpha_1 u_x u_{xx} + \frac{\mu_1}{m}(u^m)_x = 0, \tag{1-9}$$

where $u : \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{R}$, $\alpha_1 \in \mathbb{R}$, $\mu_1 \in \mathbb{R}$ and m is an integer greater than or equal to 2. Note that there is no need to separately consider the case $m = 1$, as then this term can be easily removed by the change of variables $(t, x) = (t', x' + \mu_1 t')$.

Note that $\alpha_1 = 3$ and $\mu_1 = 1$ corresponds to the $K(m, 2)$ equation (1-3). The inviscid surface growth model (1-4) reduces to the case $\alpha_1 = 3$ and $\mu_1 = 0$ after making the change of variables $u = \sqrt{2} h_x$.

In the present case, the role of linear degeneracy in the Schrödinger case is played by cubic degeneracy, see Section 1.3. As before, we introduce the constant

$$\sigma_c = -\left(\alpha_1 - \tfrac{3}{2}\right)$$

and the regularity exponent

$$s_c = \max\left\{5, \left\lfloor \sigma_c - \tfrac{1}{2} \right\rfloor + 1\right\}.$$

Here, σ_c is the critical L^2-Sobolev regularity exponent above which the linearization of (1-9) around a regular cubically degenerate solution is ill-posed (see Section 1.3). The linearized equation is L^∞-ill-posed after taking k many derivatives with $k > \sigma_c - \frac{1}{2}$; this motivates the exponent s_c. The lower bound $s_c \geq 5$ is again a nonoptimal technical byproduct of our proof; see Proposition 3.2 for where it is used.

Theorem 1.5 (unboundedness of the solution map near a cubically degenerate solution). *Assume that there exists a solution $f \in L^\infty([0, \delta]; C^{s-1,1}(I))$ of* (1-9) *with some $\delta > 0$ and $I = [a, b]$, such that the initial data f_0 is positive on $I \setminus \partial I$, vanishes cubically on ∂I and $f_0 \in C^{s_0-1,1}$, where $s_c \leq s \leq s_0$. Then, for any $\epsilon > 0$, $s \leq m_0 \leq s_0$, and $0 < \delta' \leq \delta$, we can find $\phi_0 \in C^\infty(\mathbb{T}; \mathbb{R})$ such that* $\operatorname{supp} \phi_0 \subseteq I \setminus \partial I$, *$\|\phi_0\|_{C^{m_0}} \leq \epsilon$, and one of the following holds*:

- *there exists no solution to* (1-9) *with initial data $f_0 + \phi_0$ that belongs to $L^\infty([0, \delta']; C^{s-1,1}(I))$; or*
- *any solution u with $u(0) = f_0 + \phi_0$ and belonging to $L^\infty([0, \delta']; C^{s-1,1}(I))$ satisfies, for every $s_c \leq s' \leq 2\left\lfloor \frac{1}{2}s \right\rfloor$,*

$$\sup_{0<t<\delta'} \|u(t, \cdot) - f(t, \cdot)\|_{C^{s'}(I)} > (\delta')^{-1/2}.$$

That $s' \leq 2\left\lfloor \frac{1}{2}s \right\rfloor$ is not essential and is expected to be replaceable by $s' \leq s$; however, it is assumed here to simplify the proof (see the proof of Theorem 1.5 below).

As in the Schrödinger case, the statement of Theorem 1.5 should extend over to the case of solutions with orders of degeneracy other than 3, provided that s_c is modified suitably. We however focus on the cubic degeneracy case for simplicity, which is "critical"; see Section 1.3.

The nonexistence result for (1-9) is as follows.

Theorem 1.6 (nonexistence of regular local-in-time solutions). *For any $\epsilon > 0$ and $s_c \leq s \leq s_0$, where s is an even integer, there exists an initial data $u_0 \in C^\infty(\mathbb{T})$ satisfying $\|u_0\|_{C^{s_0}} < \epsilon$ for which there is no corresponding solution to* (1-9) *belonging to $L^\infty([0, \delta]; C^s(\mathbb{T}))$ for any $\delta > 0$.*

That s is an even integer is not essential but is assumed here to simplify the proof (see the proof of Theorem 1.6 below).

Remark 1.7. We now give a few simple remarks regarding the above.

- As in the Schrödinger case, the physical domain could be taken to be $\mathbb{R}$ instead of $\mathbb{T}$. Also, the lower-order term $(u^m)_x$, for any $m \geq 2$, does not play any essential role in the proof of ill-posedness of (1-9).
- Our proof easily extends to norm inflation in H^σ for any $\sigma > \sigma_c$ in the second alternative in Theorem 1.5. Moreover, in contrast to the Schrödinger case, we may also easily extend Theorem 1.6 to the nonexistence of solutions in H^σ for any $\sigma > \max\{\sigma_c, 5 + \frac{1}{2}\}$ (see Remark 1.15 for why the situations are different). These numerologies are expected to be sharp, as we shall discuss in Section 1.3 below. We refer the reader to Remark 2.10 for more details on this modification (which is for norm inflation in the Schrödinger case, but the overall idea is the same).
- We expect our results to generalize to $K(m, n)$ with $n > 2$, as well as $\mathcal{C}(m, a, b)$ equations [Rosenau 2006] with $n := a + b > 2$, by considering degeneracies of order $\frac{3}{n-1}$ (which are critical).

Remark 1.8 (comparison with the work of Ambrose, Simpson, Wright and Yang [Ambrose et al. 2012]). In the pioneering paper [Ambrose et al. 2012], the ill-posedness of $u_t = uu_{xxx}$ in the (fairly low-regularity) Sobolev space H^2 has been proved based on the construction of a compactly supported H^2 (but not smooth) self-similar solution A. However, the existence of such a solution is specific to the equation $u_t = uu_{xxx}$, and the proof does not extend to the more general class of equations (1-9), nor to higher-regularity Sobolev spaces, as in our results. Our approach is distinct from that of [Ambrose et al. 2012]: it does not involve self-similar solutions, but is rather based on appropriate smooth wave packet-type approximate solutions traveling towards the degeneracy; see Section 1.3. While our results (Theorems 1.5 and 1.6) do not cover Sobolev regularities as low as H^2 due to technical reasons, our heuristics suggest that our ill-posedness mechanism should extend to H^σ with $\sigma > \sigma_c = \frac{3}{2}$.

Nevertheless, we point out that a key heuristic consideration of our approach, namely, the combined effect of degenerate dispersion and subprincipal terms, can already be found in [Ambrose et al. 2012], albeit with a different viewpoint.

Remark 1.9 (comparison with the works of Germain, Harrop-Griffith and Marzuola [Germain et al. 2019] and Harrop-Griffith and Marzuola [2022]). For solutions to (1-1) and (1-2) with degenerate initial data (i.e., initial data with a zero), our proof identifies and exploits, in a nonlinear fashion, a mechanism by which energies in low frequencies are transferred to high frequencies at arbitrarily fast rates, where the frequencies are defined with respect to the original variable x. We emphasize, however, that it does *not* rule out the possibility of well-posedness in regularity classes adapted to the degeneracies of the initial data, by working with a renormalized variable and/or suitable weights. Indeed, such positive results have been proved in the interesting works of Germain, Harrop-Griffith and Marzuola [Germain et al. 2019] for a KdV-type quasilinear dispersive equation, and Harrop-Griffith and Marzuola [2022] for (1-2), where Lagrangian-type coordinates adapted to the solution were used to formulate the function spaces.

1.3. ***Key mechanism*: *degenerate dispersion and the Takeuchi–Mizohata condition.*** The nonlinear ill-posedness results in this paper are firmly based on a detailed and quantitative understanding of ill-posedness for the linearized equation around a background solution f whose initial data contains a degeneracy. For simplicity, in this subsection we shall assume that the linearization takes the form

$$\begin{cases} \partial_t u - i\partial_x(a\partial_x u) - ib\partial_x u = \text{(lower-order)} & \text{in the Schrödinger case,} \\ \partial_t u + \frac{1}{2}(\partial_x^3(au) + a\partial_x^3 u) + b\partial_x^2 u = \text{(lower-order)} & \text{in the KdV case,} \end{cases} \tag{1-10}$$

where $a = a(x)$ is real-valued in both cases and $b = b(x)$ is also real-valued in the KdV case.[1]

Remark 1.10 (on time independence of the coefficients in (1-10)). While we assumed that $a(x)$ and $b(x)$ are time independent, the actual linearization of (1-5) and (1-9) on a dynamic background solution $f(t, x)$ would, of course, have time-dependent coefficients. Nevertheless, the timescale of the ill-posedness mechanism is arbitrarily short, and hence we may effectively approximate these coefficients by the initial values for the purpose of our discussion.

It is conceptually useful to distinguish two intertwined mechanisms for ill-posedness, *degenerate dispersion* and *Takeuchi–Mizohata instability*, which can be seen from the principal and subprincipal terms, respectively. Both phenomena must be taken into account to obtain a comprehensive picture of the ill-posedness of (1-5) and (1-9) in the presence of a degeneracy in the initial data (and, more concretely, to explain the relevance of the exponents σ_c and s_c).

(1) *Principal term*: *dynamics of bicharacteristics*. The ill-posedness of (1-10) from degenerate dispersion can be most easily described at the level of the *bicharacteristic ODE system* associated with the principal symbol p of the spatial part of (1-10), which is given by

$$\begin{cases} \dot{X} = \partial_\xi p(X, \Xi), \\ \dot{\Xi} = -\partial_x p(X, \Xi), \end{cases} \tag{1-11}$$

where $p(x, \xi)$ equals $-a(x)\xi^2$ in the Schrödinger case and $-a(x)\xi^3$ in the KdV case. By geometric optics, the trajectory $(X(t), \Xi(t))$ describes (at least on sufficiently short timescales) wave packets concentrated near $X(t)$ in the physical space and $\Xi(t)$ in the frequency space; see (1-15) and (1-16) below for further discussion. If (1-11) admits the growth of $|\Xi|$ by a definite factor (e.g., 2) in arbitrarily short timescales, we would have a strong indication of ill-posedness of (1-10) in high-regularity Sobolev spaces. In turn, such a growth may come from some degeneracy of p in X — this phenomenon is what we shall refer to as *degenerate dispersion*.

To be concrete, let us assume that the dynamics is given in $(x, \xi) \in \mathbb{R} \times \mathbb{R}$ and the coefficient a in p is of the degenerate form $a(x) \approx Ax^n$ $(n > 0)$ for $|x|$ small, so that

$$p(x, \xi) \approx -Ax^n\xi^m \quad \text{for } |x| \text{ small and } |\xi| \text{ large.} \tag{1-12}$$

[1] For Schrödinger-type problems, we regard first-order terms of the form $\tilde{b}(x)\partial_x\bar{u}$ as (lower-order), as it can be removed by a suitable change of the dependent variable; see the introduction of ψ in Section 2.3.2 below.

Note that m equals 2 in the Schrödinger case and 3 in the KdV case. The associated bicharacteristic ODE system is

$$\begin{cases} \dot{X} \approx -AmX^n \Xi^{m-1}, \\ \dot{\Xi} \approx AnX^{n-1}\Xi^m. \end{cases} \tag{1-13}$$

In view of the fact that the group velocity $\dot{X}$ vanishes at the point $x = 0$ (since $n > 0$), we shall say that p is *degenerate* at $x = 0$.

We shall now describe the ill-posedness mechanism of degenerate dispersion in this concrete case. (This analysis can be found in the introduction of [Germain et al. 2019] as well.) With a change of the time variable, we may take $A = 1$. Assume, for the sake of this heuristic discussion, that the $\approx$ above are exact equalities. Consider the solution to (1-13) with initial conditions (X_0, Ξ_0), where $0 < X_0 \ll 1$ and $\Xi_0 \gg 1$. Then, appealing to the fact that $X^n\Xi^m$ is conserved in time, we have

$$\Xi(t) = \Xi_0(1 + (n-m)X_0^{n-1}\Xi_0^{m-1}t)^{m/(m-n)}$$

for $m \neq n$. When $m = n$, which will be referred to as the *critical* case, we have instead

$$\Xi(t) = \Xi_0 \exp(mX_0^{m-1}\Xi_0^{m-1}t). \tag{1-14}$$

In all cases, the frequency magnitude doubles (i.e., $|\Xi(\tau_2)| = 2|\Xi_0|$) at time $\tau_2 \simeq |\Xi_0|^{1-m}|X_0|^{1-n}$. If the order of p is greater than 1 (i.e., $m > 1$), the doubling time τ_2 may be taken to be arbitrarily small by choosing $|\Xi_0|$ large, as we desired. Such an arbitrarily fast growth of Ξ suggests that high derivatives of the solution following this bicharacteristic flow would also grow arbitrarily fast — this is what we shall refer to as *ill-posedness via degenerate dispersion*.

Finally, let us connect the above model case to the equations considered in this work. Recall that the principal coefficient a in the linearized operator is determined by the background solution f, where $a = |f|^2$ for (1-5) and $a = f$ for (1-9). Since the relevant frequency doubling timescale τ_2 is arbitrarily small, it is reasonable to make the approximation $f \approx f_0$. Assuming that f_0 is degenerate at $x = 0$, in the sense that $|f_0|$ vanishes to some finite order at 0, we arrive at the ansatz $a(x) \approx Ax^n$ for some $A \neq 0$ and $n > 0$.

Remark 1.11 (critical degeneracy). In this work, for simplicity, we shall consider only background solutions with critical degeneracy $n = m$. The heuristics suggest, however, that a similar arbitrary fast growth of $|\Xi|$ is expected for any order of degeneracy $n > 0$. The techniques in this paper should be generalizable to these cases.

(2) *Subprincipal term*: *evolution of wave packet amplitude and the Takeuchi–Mizohata condition*. While subprincipal terms do not enter in the dynamics of bicharacteristics, they need to be considered in order to fully understand the well- and ill-posedness issues for (1-10). In fact, the subprincipal term may already cause ill-posedness in L^2 even when the principal term is *nondegenerate*! This phenomenon is captured by the classical *Takeuchi–Mizohata condition* (after the works [Takeuchi 1980; Mizohata 1981] in the Schrödinger case); see (1-19) below.

To understand this phenomenon, it is instructive to delve a little deeper into the construction of wave packet (approximate) solutions for (1-10). Consider the ansatz[2] $u = \boldsymbol{a}(t,x)e^{i\boldsymbol{\Phi}(t,x)}$ with the following properties: (i) $\boldsymbol{\Phi}(t,x)$ is real-valued, $\partial_x \boldsymbol{\Phi}(0,x) = \Xi_0$ on the support of $\boldsymbol{a}(0,x)$, and (ii) $\boldsymbol{a}(t,x)$ is complex-valued, and $\boldsymbol{a}(0,x)$ is a smooth bump function adapted to a small ball centered at X_0. With the expectation that the $\partial_x \boldsymbol{\Phi}(t,x)$ stays large compared to the characteristic frequencies of $\boldsymbol{a}$, a and b, we may write

$$e^{-i\boldsymbol{\Phi}}(\partial_t - i\partial_x a(x)\partial_x - ib\partial_x)(\boldsymbol{a}e^{i\boldsymbol{\Phi}}) \\ = i(\partial_t\boldsymbol{\Phi} + a(\partial_x\boldsymbol{\Phi})^2)\boldsymbol{a} + \partial_t\boldsymbol{a} + 2a\partial_x\boldsymbol{\Phi}\partial_x\boldsymbol{a} + \left(\partial_x a + b + a\frac{\partial_x^2\boldsymbol{\Phi}}{\partial_x\boldsymbol{\Phi}}\right)\partial_x\boldsymbol{\Phi}\boldsymbol{a} + \cdots$$

in the Schrödinger case (where we omitted terms that do not involve $\partial_x\boldsymbol{\Phi}$) and

$$e^{-i\boldsymbol{\Phi}}\big(\partial_t + \tfrac{1}{2}(\partial_x^3 a + a\partial_x^3) + b\partial_x^2\big)(\boldsymbol{a}e^{i\boldsymbol{\Phi}}) \\ = i(\partial_t\boldsymbol{\Phi} - a(\partial_x\boldsymbol{\Phi})^3)\boldsymbol{a} + \partial_t\boldsymbol{a} - 3a(\partial_x\boldsymbol{\Phi})^2\partial_x\boldsymbol{a} - \left(b + \tfrac{3}{2}\partial_x a - 3a\frac{\partial_x^2\boldsymbol{\Phi}}{\partial_x\boldsymbol{\Phi}}\right)(\partial_x\boldsymbol{\Phi})^2\boldsymbol{a} + \cdots$$

in the KdV case (where we omitted terms of order 0 and 1 in $\partial_x\boldsymbol{\Phi}$). To eliminate the main terms on the right-hand sides, we are led to impose the following classical *Hamilton–Jacobi* and *transport equations* for $\boldsymbol{\Phi}$ and $\boldsymbol{a}$:

$$\begin{cases}\partial_t\boldsymbol{\Phi} + a(\partial_x\boldsymbol{\Phi})^2 = 0, \\ \partial_t\boldsymbol{a} + 2a\partial_x\boldsymbol{\Phi}\partial_x\boldsymbol{a} + \partial_x(a\partial_x\boldsymbol{\Phi})\boldsymbol{a} = -b\partial_x\boldsymbol{\Phi}\boldsymbol{a}\end{cases} \quad \text{in the Schrödinger case,} \tag{1-15}$$

$$\begin{cases}\partial_t\boldsymbol{\Phi} - a(\partial_x\boldsymbol{\Phi})^3 = 0, \\ \partial_t\boldsymbol{a} - 3a(\partial_x\boldsymbol{\Phi})^2\partial_x\boldsymbol{a} - \tfrac{3}{2}\partial_x(a(\partial_x\boldsymbol{\Phi})^2)\boldsymbol{a} = b(\partial_x\boldsymbol{\Phi})^2\boldsymbol{a}\end{cases} \quad \text{in the KdV case.} \tag{1-16}$$

Observe that $(X(t), \Xi(t))$ solving (1-11) are precisely the bicharacteristics for the above equations in the method of characteristics [Evans 2010, Chapter 3], which explains the relevance of (1-11). Moreover, the transport equations show how b influences the evolution of the amplitude $\boldsymbol{a}$. In fact, we may easily check that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{a}\|_{L^2}^2 = \begin{cases}-\langle \operatorname{Re} b\,\partial_x\boldsymbol{\Phi}\boldsymbol{a}, \boldsymbol{a}\rangle & \text{in the Schrödinger case,} \\ \langle b(\partial_x\boldsymbol{\Phi})^2\boldsymbol{a}, \boldsymbol{a}\rangle & \text{in the KdV case,}\end{cases} \tag{1-17}$$

which clearly demonstrates how b influences the evolution of the L^2 norm (here $\langle\cdot,\cdot\rangle$ is the standard L^2-inner product).

We are now ready to give a heuristic derivation of the Takeuchi–Mizohata conditions. By the method of characteristics, we expect, at least for a short time, that $\partial_x\boldsymbol{\Phi}(t, X(t)) = \Xi(t)$ and $\boldsymbol{a}$ remains a bump function adapted to a ball centered at $X(t)$. Hence, on $\operatorname{supp}\boldsymbol{a}$, we expect

$$-\langle \operatorname{Re} b\,\partial_x\boldsymbol{\Phi}\boldsymbol{a}, \boldsymbol{a}\rangle \approx -\operatorname{Re} b(X(t))\,\Xi(t)\|\boldsymbol{a}\|_{L^2}^2 = -\frac{\operatorname{Re} b(X(t))}{2a(X(t))}\dot{X}(t)\|\boldsymbol{a}\|_{L^2}^2$$

[2]In the KdV case, we can take the real or imaginary part of $\boldsymbol{a}e^{i\boldsymbol{\Phi}}$ at the end to obtain a real-valued wave packet.

in the Schrödinger case and

$$\langle b(\partial_x \boldsymbol{\Phi})^2 \boldsymbol{a}, \boldsymbol{a}\rangle \approx b(X(t))\Xi(t)^2\|\boldsymbol{a}\|_{L^2}^2 = -\frac{b(X(t))}{3a(X(t))}\dot{X}(t)\|\boldsymbol{a}\|_{L^2}^2$$

in the KdV case, where we used (1-11) for the last equalities. Using (1-17) and $\|u\|_{L^2} = \|\boldsymbol{a}\|_{L^2}$, we arrive at the expectations

$$\|u(t)\|_{L^2} \simeq \begin{cases} \exp\bigl(-\int_{X_0}^{X(t)} (\operatorname{Re} b)/(2a)\,\mathrm{d}x\bigr)\|u(t=0)\|_{L^2} & \text{in the Schrödinger case,} \\ \exp\bigl(\int_{X(t)}^{X_0} b/(3a)\,\mathrm{d}x\bigr)\|u(t=0)\|_{L^2} & \text{in the KdV case.} \end{cases} \tag{1-18}$$

The *Takeuchi–Mizohata conditions* (see [Takeuchi 1980; Mizohata 1981] in the Schrödinger case; see [Akhunov 2014; Ambrose and Wright 2013] and Remark 1.13 for the KdV case) are simply sufficient conditions for the forward-in-time boundedness of $\|\boldsymbol{a}\|_{L^2}$ read off from (1-18):

$$\sup_{x_0<x_1}\left|\int_{x_0}^{x_1} \frac{\operatorname{Re} b}{2a}\,\mathrm{d}x\right| < +\infty \quad \text{in the Schrödinger case,} \tag{1-19}$$

$$\sup_{x_0<x_1}\int_{x_0}^{x_1} \frac{b}{3a}\,\mathrm{d}x < +\infty \quad \text{in the KdV case.} \tag{1-20}$$

Conversely, the failure of the Takeuchi–Mizohata conditions (1-19) and (1-20) signals arbitrarily fast growth (i.e., norm inflation) of the L^2 norm of u, since $X(t)$ may travel arbitrarily far from X_0 in any fixed duration of time if Ξ_0 is large. In this paper, we shall refer to this norm inflation (or ill-posedness) mechanism as the *Takeuchi–Mizohata instability*. Below, we shall consider the interaction of degenerate dispersion and the Takeuchi–Mizohata instability, which provides us with a detailed heuristic understanding of the ill-posedness properties of the linearization of (1-5) and (1-9) in the presence of a (critical) degeneracy in the initial data.

Remark 1.12 (rigorous results on Takeuchi–Mizohata-type conditions). The necessity of (1-19) for the L^2-well-posedness of (1-10) in the Schrödinger case has been known since the early works [Takeuchi 1980; Mizohata 1981]; see also [Akhunov 2014] for the KdV case. On the other hand, whether such a condition alone is sufficient for L^2 boundedness in general is less clear, especially in higher dimensions. Nevertheless, some strengthened form of the Takeuchi–Mizohata condition underlies many works on the well-posedness of the Cauchy problem for linear and even nonlinear Schrödinger- and KdV-type equations; see, e.g., [Akhunov 2014; Akhunov et al. 2019; Ambrose and Wright 2013; Harrop-Griffiths 2015a; 2015b: Kenig et al. 1998; 2004; Marzuola et al. 2012; 2014; 2021; Mizohata 1985].

Remark 1.13 (role of sgn b in the KdV case). Observe that the absolute value is needed in the Schrödinger case (1-19) since $X(t)$ may travel in both directions, while it is not necessary in the KdV case (1-20) since $X(t)$ is *always* decreasing if a is positive (resp. increasing if a is negative) according to (1-11). In particular, in the KdV case, (1-20) is always satisfied if $b < 0$, and even when b has some positive parts, it is possible that the Takeuchi–Mizohata condition is still satisfied (e.g., when b oscillates). This phenomenon has been explored by Ambrose and Wright [2013], who prove well-posedness of some variable coefficient linear KdV-type equations in the periodic setting in the presence of the positive part of b (referred to as "antidiffusion" in that paper).

Remark 1.14. While our main focus is the interaction of Takeuchi–Mizohata instability with degenerate dispersion, the method developed in this paper also provides a new and effective way to rigorously establish the necessity of (1-19) and (1-20) for the L^2-well-posedness of (1-10). We refer the reader to Section 1.5 and the Appendix for sample results in the Schrödinger case for $a = 1$ (but in arbitrarily dimensions).

(3) *Combined effect of degenerate dispersion and Takeuchi–Mizohata instability.* We are now ready to discuss the combined effect of the principal and subprincipal terms in (1-10) obtained by linearizing around a background solution f with a degeneracy. Keeping Remarks 1.10 and 1.11 in mind, we consider the linearization of (1-5) and (1-9) around $f(t,x) = x$ and x^3 (for $|x|$ small), respectively. Then we arrive at (1-10) with

$$\begin{aligned} a(x) &= x^2, \quad b(x) = 3\big(\tfrac{1}{2}\alpha_1 + \beta_1 - 1\big)x && \text{in the Schrödinger case,} \\ a(x) &= x^3, \quad b(x) = 3\big(\alpha_1 - \tfrac{3}{2}\big)x^2 && \text{in the KdV case.} \end{aligned}$$

Recall from the above that we are considering bicharacteristics $(X(t), \Xi(t))$ with $X_0 > 0$ and $X(t)$ traveling to the degeneracy 0 in both cases. Wave packets corresponding to such bicharacteristics shall be called *degenerating wave packets*.

The relevant Takeuchi–Mizohata condition (see (1-21) below with $\sigma = 0$) for $\|u\|_{L^2}$ may or may not hold, meaning that degenerating wave packets may or may not remain bounded in L^2. Nevertheless, it *always fails for high enough derivatives*, which is consistent with the heuristic $\Xi(t) \to \infty$! Indeed, observe that commutation of (1-10) with ∂_x^σ leads to a similar equation for $\partial_x^\sigma u$ but with the following coefficients:

$$\begin{aligned} a(x) &= x^2, \quad b(x) = 3\big(\tfrac{1}{2}\alpha_1 + \beta_1 - 1 + \sigma\big)x && \text{in the Schrödinger case,} \\ a(x) &= x^3, \quad b(x) = 3\big(\alpha_1 - \tfrac{3}{2} + \sigma\big)x^2 && \text{in the KdV case.} \end{aligned}$$

In view of $0 < X(t) < X_0$, the Takeuchi–Mizohata condition for boundedness of $\|u\|_{H^\sigma}$ is

$$\begin{aligned} \sup_{0<x_0<x_1\ll 1} \int_{x_0}^{x_1} \big(\tfrac{1}{2}\alpha_1 + \beta_1 - 1 + \sigma\big)\frac{\mathrm{d}x}{x} &< +\infty && \text{in the Schrödinger case,} \\ \sup_{0<x_0<x_1\ll 1} \int_{x_0}^{x_1} \big(\alpha_1 - \tfrac{3}{2} + \sigma\big)\frac{\mathrm{d}x}{x} &< +\infty && \text{in the KdV case,} \end{aligned} \tag{1-21}$$

which fails exactly when $\sigma > \sigma_c$ in both cases. Moreover, the preceding heuristic analysis suggests that the H^σ norm of the degenerating wave packet grows if $\sigma > \sigma_c$, stays constant if $\sigma = \sigma_c$, and decays if $\sigma < \sigma_c$. This consideration explains the relevance of the exponent σ_c.

Working directly with the transport equations for $\boldsymbol{a}$ in (1-15)–(1-16) in place of (1-17), we may also see that, for $\tilde{s}_c = \sigma_c + \frac{1}{2}$, the $W^{s,\infty}$ norm of the wave packet grows if $s > \tilde{s}_c$, stays constant if $s = \tilde{s}_c$, and decays if $s < \tilde{s}_c$. This consideration motivates the integer exponent s_c in our results.

1.4. ***Discussion of the proof.*** Our discussion so far has been rather formal; deriving actual nonlinear ill-posedness in standard function spaces requires more ideas. Our main technical contribution in this work is developing a robust scheme for establishing quantitative ill-posedness, which is not only able to deduce strong ill-posedness in quasilinear cases but also yields much stronger statements for linear equations. The scheme largely consists of three parts: (1) construction of degenerating wave packets for the linearized equation, (2) duality testing argument and (3) incorporation of the nonlinearity.

(1) *Degenerating wave packets.* We first describe the ideas for construction of a degenerating wave packet. Compared to the heuristic discussion above, the actual construction of such an approximate solution $\tilde{u}$ to (1-10) has to (i) allow for time-dependent coefficients $a = a(t, x)$ and $b = b(t, x)$ (as the background solution may depend on time, see Part (3) below), and (ii) solve the linearized equation up to an equation error $\epsilon_{\tilde{u}}$ of size $\mathcal{O}(\Xi_0^{m-1-\delta})$ (in a suitable norm) for some $\delta > 0$ (here, m equals 2 for Schrödinger and 3 for KdV). Property (ii) is necessary to justify the approximation on a longer timescale than Ξ_0^{1-m}, which is the instability timescale; see (1-14).

Our idea is to make appropriate changes of the independent and dependent variables from (x, u) to (y, v) to reduce the problem to the constant coefficient case, for which the construction is standard. For time-dependent coefficients $a = a(t, x)$ and $b = b(t, x)$, the transformation $(x, u) \mapsto (y, \check{v})$ is of the form

$$\mathrm{d}x = (a(t, x))^{1/m}\,\mathrm{d}y, \quad u = (w\check{w})^{-1}\check{v},$$

where

$$w^{-1}\,\partial_x w = \frac{\operatorname{Re} b}{ma}, \quad \check{w}^{-1}\,\partial_x \check{w} = \frac{\partial_x a}{2ma}.$$

Roughly speaking, the Takeuchi–Mizohata instability is renormalized by the conjugation of the dependent variable by the weight w, in the sense that $v(t, x) := wu(t, x)$ solves (1-10) with $b = 0$ (with possibly different lower-order terms). Similarly, degenerate dispersion is renormalized by the change of variables $x \mapsto y$ accompanied with the conjugation of the dependent variable by the weight $\check{w}$, in the sense that $\check{v}(t, y)$ solves the constant coefficient problem $(\partial_t + i(i\partial_y)^m)\check{v} =$ (lower-order terms). Now, starting from a standard wave packet for the constant coefficient problem traveling towards the degeneracy and returning to original variables, we obtain a degenerating wave packet $\tilde{u}$ with the desired properties.

In order to make the above heuristic discussion precise, there are several more factors to consider. For instance, we need to make sure that the contribution of $\partial_t a$, $\partial_t b$ are indeed acceptable, which ultimately relies on the estimates we have on the time derivative of the background solution $f(t, x)$ in applications; see Part (3) below. In the Schrödinger case, we need the following two additional ideas: (a) an extra change of dependent variables to treat terms of the form $\tilde{b}\partial_x\bar{u}$, and (b) an extra phase rotation $e^{i\lambda S}$ for the wave packet $\check{v}$ to treat terms of the form $-(\operatorname{Im} b + \partial_t y)\partial_y\check{v}$, both of which are potentially problematic for achieving Property (ii). For more details, see the proofs of Propositions 2.7 and 3.3 for details.

Remark 1.15 (numerologies in the Schrödinger vs. KdV cases). In order to justify the properties of $\tilde{u}$ needed for the proof of the H^s or C^s norm growth, (b) above forces the technical restriction that $f \in C^{s+1,1}$ with $s \geq 2$ in the Schrödinger case, while $f \in C^{s-1,1}$ with $s \geq 4$ is sufficient in the KdV case; compare the degeneration bounds in Propositions 2.7 and 3.3. This point explains the different numerologies in Theorems 1.1 and 1.2 in the Schrödinger case versus Theorems 1.5 and 1.6 in the KdV case.

(2) *Modified energy estimate and duality testing argument*. In order to upgrade the norm growth for a degenerating wave packet $\tilde{u}$ to an actual solution u to (1-10), we adapt the *energy estimate and duality method* introduced in our previous work [Jeong and Oh 2022] on Hall-magnetohydrodynamics (Hall-MHD). Here we shall briefly explain the argument, in the simplest setup.

Given a degenerating wave packet $\tilde{u}$ for (1-10), denote by u any[3] solution to (1-10) with $u(t=0)=\tilde{u}_0$, where $\tilde{u}_0 := \tilde{u}(t=0)$. In view of the aforementioned fact that $v = wu$ solves (1-10) with $b = 0$, the following *modified energy estimates* should hold (at least when u is sufficiently regular):

$$\|wu\|_{L_t^\infty([0,t_0];L^2)} \lesssim \|wu_0\|_{L^2}, \quad \|w\tilde{u}\|_{L_t^\infty([0,t_0];L^2)} \lesssim \|w\tilde{u}_0\|_{L^2}, \tag{1-22}$$

where $0 < t_0 < 1$. By the same token, the following *generalized* (*bilinear*) *energy estimate* should also hold:

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\langle wu, w\tilde{u}\rangle\right| \lesssim \|w\boldsymbol{\epsilon}_u\|_{L^2}\|w\tilde{u}\|_{L^2} + \|wu\|_{L^2}\|w\boldsymbol{\epsilon}_{\tilde{u}}\|_{L^2}.$$

(We remark that $\partial_t w$ also arises, but in applications we shall have $|\partial_t w| \lesssim w$.) Here, $\boldsymbol{\epsilon}_u$ and $\boldsymbol{\epsilon}_{\tilde{u}}$ are the errors associated with u and $\tilde{u}$ viewed as approximate solutions to (1-10). Then, as long as the error terms are bounded, we may deduce that $\langle wu, w\tilde{u}\rangle \simeq \langle wu_0, w\tilde{u}_0\rangle = \|w\tilde{u}_0\|_{L^2}^2$, which allows us to obtain behavior of u in various norms by simply estimating the degenerating wave packet $\tilde{u}$ and using duality.

In actual applications, the errors often contain derivatives and hence $\|w\boldsymbol{\epsilon}_{\tilde{u}}\|_{L^2}$ (resp. $\|w\boldsymbol{\epsilon}_u\|_{L^2}$) may diverge as $|\Xi_0| \to \infty$. Nevertheless, in view of the fact the instability time-scale is $\simeq |\Xi_0|^{1-m}$, for the above argument to work it suffices to have $\int_0^{t_0}\|w\boldsymbol{\epsilon}_{\tilde{u}}\|_{L^2} \lesssim 1$ $\big($resp. $\int_0^{t_0}\|w\boldsymbol{\epsilon}_u\|_{L^2} \lesssim 1\big)$ for $t_0 > |\Xi_0|^{1-m+\delta}$. For $\tilde{u}$, this is precisely Property (ii) in the preceding discussion. For an actual solution u to (1-10), this follows from the fact that $\boldsymbol{\epsilon}_u$ does not contain principal nor subprincipal terms (except $\tilde{b}\partial_x\bar{u}$ in the Schrödinger case, which may be eliminated using integration by parts).

(3) *Incorporation of the nonlinearity*. The ideas discussed so far explain how to prove the ill-posedness of (1-10) that arises from linearizing (1-5) and (1-9) around a regular solution $f(t,x)$ whose initial data has a critical degeneracy at $x = 0$. As in [Jeong and Oh 2022], the nonlinear norm inflation results (Theorems 1.1 and 1.5) are derived by assuming the existence of a nonlinear perturbation u around f (i.e., $f + u$ solves the nonlinear equation) without the instability behavior, then applying the above argument. Moreover, the nonlinear nonexistence results (Theorems 1.2 and 1.6) are proved by superposition of infinitely many configurations exhibiting norm inflation (with unbounded rates of growth), with disjoint supports in physical space. We refer to [Jeong and Oh 2022, Section 1.6] for a more detailed summary of the ideas involved, and to Sections 2.5, 2.6, 3.5 and 3.6 for details. A key new feature of the present paper compared to [Jeong and Oh 2022], however, is that the background solution f need not be stationary solutions, and are given as a part of the contradiction hypothesis in the proof of the nonexistence theorems.

1.5. ***Revisiting L^2-ill-posedness à la Takeuchi–Mizohata.*** In view of the extensive appearance of the Takeuchi–Mizohata instability in this paper, it is perhaps not surprising that our techniques also apply to the original setting considered by Takeuchi and Mizohata of L^2-ill-posedness of linear nondegenerate

[3]Note that it is a priori possible that uniqueness of the Cauchy problem for (1-10) fails. Nevertheless, the method is still applicable and establishes the norm growth of every solution u with the same initial data satisfying (1-22).

Schrödinger-type equations. In the Appendix, we provide a few results concerning the Takeuchi–Mizohata condition obtained through our approach. In particular, we recover the following result of Mizohata [1985, §VII.2]:

Proposition 1.16. *Consider the linear first-order perturbation of the Schrödinger equation on* $\mathbb{R}^d$

$$i\partial_t u + \Delta u + b^j(x)\partial_j u = 0, \tag{1-23}$$

where $b \in C^{1,1}(\mathbb{R}^d)$. *Suppose that the Takeuchi–Mizohata condition for* (1-23) *fails*, *i.e.*,

$$\sup\left\{\int_0^T \operatorname{Re} b^j(x-2s\omega)\omega_j \,\mathrm{d}s : x \in \mathbb{R}^d,\ \omega \in \mathbb{S}^{d-1},\ T > 0\right\} = +\infty. \tag{1-24}$$

Then, *for any* $\delta > 0$, *every solution map* $L^2 \to L^\infty_t([0,\delta]; L^2)$ *for* (1-23), *if it exists*, *is unbounded.*

Note that Proposition 1.16 clearly implies the result proved in [Mizohata 1985, §VII.2], namely, the impossibility of having a solution map for the inhomogeneous equation

$$i\partial_t u + \Delta u + b^j(x)\partial_j u = f \tag{1-25}$$

satisfying

$$\|u\|_{L^\infty([0,\delta];L^2)} \le C_0(\|u_0\|_{L^2} + \|f\|_{L^1([0,\delta];L^2)})$$

for some $C_0 < +\infty$. In fact, via Duhamel's principle, this result is equivalent to Proposition 1.16. Nonetheless, our techniques generalize easily to other situations when such an equivalence is not obvious, e.g., when b depends on time.

More interestingly, we also provide some new unconditional quantitative lower bounds for (1-23), which are valid way past the trivial $O\left(\frac{1}{\lambda}\right)$ timescale (where λ is the initial characteristic frequency), up to a time when the L^2 norm may grow at a *quantitative* rate depending on λ; see Propositions A.1 and A.3. These results should be contrasted with the proofs of Proposition 1.16 and [Mizohata 1985, §VII.2], which rely on *qualitative* contradiction arguments up to $O\left(\frac{1}{\lambda}\right)$ timescales.

Organization of the paper. The Schrödinger- and KdV-type equations are treated respectively in Sections 2 and 3. In the Appendix, we prove Proposition 1.16 and related results for the linear nondegenerate Schrödinger-type equation (1-23).

2. Schrödinger-type equations

This section is organized as follows. To motivate our approach, we analyze in Section 2.1 a model problem (2-2) derived from (1-1). In Section 2.2, we study the properties of linearly degenerate solutions — typically denoted by f — and in Section 2.3, we construct degenerating wave packets for the linearized equation around f. In Section 2.4, we establish a modified and generalized (bilinear) energy estimates for the perturbation (solving the nonlinear difference equation) around f. Finally, in Sections 2.5 and 2.6, we prove Theorems 1.1 and 1.2, respectively.

2.1. ***Degenerating wave packets for model linear equation.*** To motivate what is to follow, consider the case (1-1) (i.e., (1-5) with $\alpha_1 = \beta_1 = 1$ and $\mu_1 = 0$), which we recall here for convenience:

$$i\partial_t \phi + \partial_x(|\phi|^2 \partial_x \phi) = 0. \tag{1-1}$$

We note that when the domain is taken to be $\mathbb{R}$, $f(t,x) = xe^{2it}$ — which degenerates (i.e., vanishes) linearly at 0 — is *formally* a solution to (1-1).[4] Indeed, even in the absence of any well-posedness results for the Cauchy problem, it is not difficult to show that any hypothetical smooth solution to (1-1) with initial data $f_0(x)$ that equals x in some neighborhood of the origin should approximate xe^{2it} uniformly for small $|x|$ and $|t|$. To illustrate our ill-posedness mechanism for initial data close to $f(0,x)$, we consider the linearization of (1-1) around the background solution $f(t,x) = xe^{2it}$. To wit, by plugging in the ansatz

$$\phi(t,x) = xe^{2it} + \tilde{\phi}(t,x) \tag{2-1}$$

into (1-1) and dropping quadratic or higher terms in $\tilde{\phi}$, we obtain the linearization of (1-1) around the explicit solution xe^{2it}:

$$i\partial_t\tilde{\phi} + \partial_x(x^2\,\partial_x\tilde{\phi}) + 2\partial_x(x\,\mathrm{Re}(\tilde{\phi})) + (e^{4it}-1)\partial_x(x\bar{\tilde{\phi}}) = 0.$$

Freezing the coefficients of the linearized equation at $t = 0$ and dropping zeroth-order terms in $\tilde{\phi}$, which can be readily incorporated into our ill-posedness proof if desired (see Remark 2.3), we arrive at the *model linear equation*:

$$i\partial_t\tilde{\phi} + \mathcal{L}[\tilde{\phi}] = 0, \quad \mathcal{L}[\,\cdot\,] = \partial_x(x^2\,\partial_x(\,\cdot\,)) + 2x\,\partial_x\,\mathrm{Re}(\,\cdot\,). \tag{2-2}$$

The goal of this section is to sketch a proof of the fact that this model linear equation is *ill-posed*; see Proposition 2.1.

An important observation regarding the operator $\mathcal{L}$ is that for any sufficiently regular v, we have the estimate

$$|\langle |x|^{1/2} i\mathcal{L}[v], |x|^{1/2}v\rangle| \le C\||x|^{1/2}v\|_{L^2}^2. \tag{2-3}$$

To see this, we first expand $\mathcal{L}$ and perform an integration by parts to get

$$\begin{aligned}
&\langle |x|^{1/2} i\mathcal{L}[v], |x|^{1/2}v\rangle \\
&\quad = \int \mathrm{Re}[(i\partial_x(x^2\,\partial_x v) + ix\,\partial_x v + x\,\partial_x\bar{v})(\mathrm{sgn}\,x)x\bar{v}]\,\mathrm{d}x \\
&\quad = \int \mathrm{Re}[-i(\mathrm{sgn}\,x)x^3\,\partial_x v\,\partial_x\bar{v} - i(\mathrm{sgn}\,x)(x^2\,\partial_x v)(\partial_x x)\bar{v} + i(\mathrm{sgn}\,x)x^2\,\partial_x v\bar{v} + i(\mathrm{sgn}\,x)x^2\,\partial_x\bar{v}\bar{v}]\,\mathrm{d}x,
\end{aligned}$$

where the contribution of $\partial_x\,\mathrm{sgn}\,x$ is zero thanks to the vanishing integrand at $x = 0$. Inside $\mathrm{Re}[\,\cdot\,s]$, the first term vanishes since it is purely imaginary, and the second and third terms exactly cancel (which

[4] In this section, we take the domain to be $\mathbb{R}$ rather than $\mathbb{T}$.

dictates the power $\frac{1}{2}$ in (2-3)). For the fourth term, we write $\partial_x \bar{v}\bar{v} = \frac{1}{2}\partial_x(\bar{v}^2)$ and perform another integration by parts to obtain

$$\langle |x|^{1/2} i\mathcal{L}[v], |x|^{1/2} v\rangle = \int \mathrm{Re}\big[-i\tfrac{1}{2}(\operatorname{sgn} x)(\partial_x x^2)\bar{v}^2\big]\,\mathrm{d}x,$$

whose absolute value is clearly estimated by $\||x|^{1/2}v\|_{L^2}^2$, as desired.

Equation (2-3) suggests that the correct way to measure regularity for solutions of (2-2) is to use $|x|^{1/2}$-weighted spaces.[5] To this end, we set $\|v\|_{L^2_w} = \||x|^{1/2}v\|_{L^2}$. We are now ready to state the main result of this section.

Proposition 2.1. *Equation* (2-2) *is ill-posed in* L^2. *More specifically, for any profile* $g_0 \in C^\infty\big(\frac{1}{2}, 1\big)$, ***any*** $L^\infty_t L^2_w$ *solution* $\tilde{\phi}_{(\lambda)}$ *to* (2-2) *with initial data*

$$\tilde{\phi}_{(\lambda),0}(x) = g_0(x)\exp(i\lambda \ln|x|), \quad \lambda < 0,$$

satisfies the growth

$$\|\tilde{\phi}_{(\lambda)}\|_{L^2}(t) \ge c_0 \exp(|\lambda| t) \quad \textit{for any } 0 < t < T,$$

with constants $c_0, T > 0$ *depending only on* g_0.

Remark 2.2. By an $L^\infty_t L^2_w$ solution to (2-2), we mean a weak solution $\tilde{\phi}$ which satisfies the bound

$$\|\tilde{\phi}\|_{L^2_w}(t) \le \exp(Ct)\|\tilde{\phi}_0\|_{L^2_w},$$

where $C > 0$ is the constant from (2-3) and attains the initial data in the weak sense. Existence of an $L^\infty_t L^2_w$ solution given an L^2_w initial data follows from a standard argument involving the Aubin–Lions lemma (see [Jeong and Oh 2022, Appendix A] for instance). Note that $\|\tilde{\phi}_{(\lambda),0}\|_{L^2}, \|\tilde{\phi}_{(\lambda),0}\|_{L^2_w} \lesssim 1$ uniformly in λ. While we cannot rule out the possibility of nonuniqueness, the above result applies to *all* $L^\infty_t L^2_w$ solutions.

In following the proof, the reader may find Parts (1) (degenerating wave packets) and (2) (modified energy estimate and duality testing argument) in Section 1.4 expanded in detail. See also the remarks following the proof, which discuss additional ideas that go into the proof of Theorems 1.1 and 1.2.

Proof. We demonstrate how to construct approximate solutions to (2-2), from which Proposition 2.1 naturally follows. To begin with, we make a change of variable $y = \ln x$ for $x \ge 0$. Then using $x\,\partial_x = \partial_y$, (2-2) transforms into

$$i\partial_t\tilde{\phi} + \partial_{yy}\tilde{\phi} + \partial_y\tilde{\phi} + 2\partial_y \mathrm{Re}(\tilde{\phi}) = 0.$$

Defining $\varphi = e^y\tilde{\phi}$,

$$i\partial_t\varphi + \partial_{yy}\varphi + \partial_y\bar{\varphi} - 2\varphi - \bar{\varphi} = 0.$$

We then introduce

$$\psi = \varphi + \mathcal{A}\bar{\varphi}, \tag{2-4}$$

[5] As we shall see below, the exponents 2 and $\frac{1}{2}$ in xe^{2it} and $|x|^{1/2}$, respectively, should be replaced by appropriate constants for (1-5) in general.

where $\mathcal{A} = \frac{1}{2}\partial_y^{-1}$ is (formally) an operator of order -1. Then

$$\begin{aligned} i\partial_t\psi &= -\partial_{yy}\varphi + \mathcal{A}\,\partial_{yy}\bar{\varphi} - \partial_y\bar{\varphi} + \mathcal{A}\,\partial_y\varphi - (-2\varphi - \bar{\varphi} + 2\mathcal{A}\bar{\varphi} + \mathcal{A}\varphi) \\ &= -\partial_{yy}\psi + \mathcal{A}\,\partial_{yy}\bar{\varphi} + \partial_{yy}\mathcal{A}\bar{\varphi} - \partial_y\bar{\varphi} + \mathcal{A}\,\partial_y\varphi - (-2\varphi - \bar{\varphi} + 2\mathcal{A}\bar{\varphi} + \mathcal{A}\varphi). \end{aligned}$$

Then we see that

$$i\partial_t\psi + \partial_{yy}\psi = \mathcal{A}\,\partial_y\varphi - (-2\varphi - \bar{\varphi} + 2\mathcal{A}\bar{\varphi} + \mathcal{A}\varphi).$$

Since the right-hand side is of order zero, it suggests that a degenerating wave packet φ may be constructed by taking ψ to be an approximate wave packet solution to the one-dimensional Schrödinger equation, and then going back to φ. More precisely, take

$$\psi_{(\lambda)}^{\text{app}}(t, y) = \exp(i\lambda y - i\lambda^2 t)a_0(y - 2\lambda t), \tag{2-5}$$

where we fix a_0 to be C^∞-smooth and supported in $\{-2 < y < -1\}$. We need to take $\lambda < 0$, so that the support of $\psi^{\text{app}}(t, \cdot)$ is confined to $\{y < -1\}$ for all $t \geq 0$. To invert (2-4), we wish to take $\varphi \approx \psi - \mathcal{A}\bar{\psi}$. Since $\mathcal{A} = \frac{1}{2}\partial_y^{-1}$ acts like $-\frac{1}{2i\lambda}$ on $\bar{\psi}$, we are motivated to take

$$\varphi^{\text{app}}(t, y) = \psi_{(\lambda)}^{\text{app}}(t, y) + \frac{1}{2i\lambda}\overline{\psi_{(\lambda)}^{\text{app}}}(t, y) \tag{2-6}$$

and then set

$$\tilde{\phi}^{\text{app}} = e^{-y}\varphi^{\text{app}} = e^{-y}\biggl(e^{i\lambda(y-\lambda t)}a_0(y - 2\lambda t) + \frac{1}{2i\lambda}e^{-i\lambda(y-\lambda t)}\overline{a_0}(y - 2\lambda t)\biggr).$$

Returning to the x-coordinates and defining the error by $\boldsymbol{\epsilon}_{\tilde{\phi}} = [i\partial_t + \mathcal{L}]\tilde{\phi}^{\text{app}}$, we have

$$\|\boldsymbol{\epsilon}_{\tilde{\phi}}(t)\|_{L_w^2} \lesssim \|a_0\|_{H_x^2}, \quad t \geq 0, \tag{2-7}$$

uniformly in λ. In this sense, $\tilde{\phi}^{\text{app}}$ is an approximate solution of (2-2). Moreover, $\tilde{\phi}^{\text{app}}$ itself satisfies the bound $\|\tilde{\phi}^{\text{app}}(t)\|_{L_w^2} \lesssim \|a_0\|_{L_x^2}$. The last key property is degeneration: with a weight higher than $|x|^{1/2}$, $\tilde{\phi}^{\text{app}}(t)$ decays in the $O(|\lambda|^{-1})$-timescale: for example, with the weight $|x|$, we have

$$\||x|\tilde{\phi}^{\text{app}}(t, x)\|_{L_x^2} \lesssim e^{-|\lambda|t}\|a_0\|_{L_x^2}. \tag{2-8}$$

Interpolating (2-8) with the L_w^2-estimate shows that $\|\tilde{\phi}^{\text{app}}(t, \cdot)\|_{L^2} \gtrsim e^{|\lambda|t}$. Now, let $\tilde{\phi}$ be an $L_t^\infty L_w^2$ solution of (2-2). Then, with a direct computation, we have the *generalized energy estimate* for the weighted L^2-estimate (see Section 2.4.2)

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle |x|^{1/2}\tilde{\phi}, |x|^{1/2}\tilde{\phi}^{\text{app}}\rangle = \langle |x|^{1/2}\tilde{\phi}, |x|^{1/2}\boldsymbol{\epsilon}_{\tilde{\phi}}\rangle,$$

which gives, together with (2-7),

$$\operatorname{Re}\langle |x|^{1/2}\tilde{\phi}, |x|^{1/2}\tilde{\phi}^{\text{app}}\rangle(t) \geq \operatorname{Re}\langle |x|^{1/2}\tilde{\phi}, |x|^{1/2}\tilde{\phi}^{\text{app}}\rangle(t = 0) - Ct\|\tilde{\phi}\|_{L_t^\infty L_w^2}\|a_0\|_{H_x^2}.$$

At the initial time, by choosing a_0 in a way depending only on g_0, we can guarantee that

$$\operatorname{Re}\langle |x|^{1/2}\tilde{\phi}, |x|^{1/2}\tilde{\phi}^{\text{app}}\rangle(t = 0) \geq \tfrac{1}{2}\|\tilde{\phi}_0\|_{L_w^2}\|\tilde{\phi}_0^{\text{app}}\|_{L_w^2}.$$

Then, for $0 < t < C\|a_0\|_{H^2_x}/(4\|\tilde{\phi}_0\|_{L^2_w})$, we obtain with (2-8) that

$$\frac{1}{C}e^{-|\lambda|t}\|a_0\|_{L^2}\|\tilde{\phi}(t)\|_{L^2} \geq \mathrm{Re}\langle\tilde{\phi}, |x|\tilde{\phi}^{\mathrm{app}}\rangle(t) = \mathrm{Re}\langle|x|^{1/2}\tilde{\phi}, |x|^{1/2}\tilde{\phi}^{\mathrm{app}}\rangle(t) \geq \tfrac{1}{4}\|\tilde{\phi}_0\|_{L^2_w}\|\tilde{\phi}_0^{\mathrm{app}}\|_{L^2_w},$$

which gives the claimed exponential growth of $\|\tilde{\phi}(t)\|_{L^2}$. □

Remark 2.3. (1) *Ill-posedness of the linearization of* (1-1). A small modification of the above proof gives an analogous ill-posedness result for the linearization of (1-1) around xe^{2it}, which takes the form

$$i\partial_t\phi + \mathcal{L}\phi + (e^{4it} - 1)x\partial_x\bar{\tilde{\phi}} = (\text{zeroth-order in } \tilde{\phi}).$$

Note that the additional zeroth-order terms in $\tilde{\phi}$ do not affect argument in any way; the main modification is due to the presence of the additional term $(e^{4it} - 1)x\partial_x\bar{\tilde{\phi}}$. Specifically, to cancel the contribution of $\partial_y\bar{\varphi}$ (where y and $\varphi = e^y\tilde{\phi}$ are as before), the operator in (2-4) needs to be modified to $\mathcal{A} = \frac{1}{2}e^{4it}\partial_y^{-1}$, which in turn motivates the modified ansatz

$$\varphi^{\mathrm{app}}(t, y) = \psi^{\mathrm{app}}_{(\lambda)}(t, y) + \frac{e^{4it}}{2i\lambda}\overline{\psi^{\mathrm{app}}_{(\lambda)}}(t, y),$$

with $\psi^{\mathrm{app}}_{(\lambda)}(t, y)$ as before. The remainder of the proof proceeds similarly as before; we leave the details to the interested reader.

(2) *Ill-posedness in H^m for $m > 0$.* In fact, another small modification of the above proof shows that (2-2) is ill-posed in H^m for $m > 0$. More precisely, we have the growth

$$\|\partial_x^m\tilde{\phi}_{(\lambda)}\|_{L^2}(t) \geq c_0\exp((1 + 2m)|\lambda|t) \quad \text{for any } m \geq 0 \text{ and } 0 < t < T,$$

with $c_0, T > 0$ depending only on g_0 and m.

We now sketch the needed modification; see Section 2.5 for the complete proof. We would like to modify the last part of the proof of Proposition 2.1 using "differentiation by parts": under the assumption that $\partial_x^{-m}(|x|\tilde{\phi}^{\mathrm{app}}) \in L^2$,

$$\|\partial_x^m\tilde{\phi}\|_{L^2}\|\partial_x^{-m}(|x|\tilde{\phi}^{\mathrm{app}})\|_{L^2} \geq \mathrm{Re}\langle\partial_x^m\tilde{\phi}, \partial_x^{-m}(|x|\tilde{\phi}^{\mathrm{app}})\rangle(t) = \mathrm{Re}\langle|x|^{1/2}\tilde{\phi}, |x|^{1/2}\tilde{\phi}^{\mathrm{app}}\rangle(t).$$

Now the point is that $\partial_x^{-1} = x\partial_y^{-1} = e^y\partial_y^{-1}$ and $y \simeq -2|\lambda|t$ on the support of $\tilde{\phi}^{\mathrm{app}}(t)$, which gives a faster rate of degeneration $\|\partial_x^{-m}(|x|\tilde{\phi}^{\mathrm{app}})\|_{L^2} \lesssim \exp(-(2m + 1)|\lambda|t)$. This gives the claimed lower bound for $\|\partial_x^m\tilde{\phi}\|_{L^2}$. In general, there could be some low frequency part of $|x|\tilde{\phi}^{\mathrm{app}}$ which does not degenerate, and for this reason we introduce a decomposition of $|x|\tilde{\phi}^{\mathrm{app}}$ into high and low frequency parts in the actual proof in Section 2.5.

Remark 2.4 (additional ideas in the proof of Theorems 1.1 and 1.2). For a general equation of the form (1-5), we do not have access to a stationary solution with a linear degeneracy in general (furthermore, we shall also require that f_0 be compactly supported, which rules out xe^{2it}, too). Hence, we shall carry out the above analysis (degenerating wave packet construction, modified energy estimate and duality), where the background solution f is merely a regular (most likely) *time-dependent* solution to (1-5), which has compactly supported initial data f_0 with a linear degeneracy, in place of xe^{2it}.

Theorem 1.1 is proved by considering a perturbation $f+\tilde{\phi}$ of such an f, and arguing that if $f+\tilde{\phi}$ exists as a regular solution (i.e., if we are in the second case in Theorem 1.1), then the above growth mechanism for $\tilde{\phi}$ can be justified. To prove Theorem 1.2, we consider initial data $\tilde{\phi}_0$ consisting of a superposition of an infinitude of configurations as above (i.e., $\sum_k (f_{k,0}+\tilde{\phi}_{k,0})$, where $f_{k,0}$ has a linear degeneracy and $\tilde{\phi}_{k,0}$ is a degenerating wave packet adapted to $f_{k,0}$) with unbounded rates (i.e., the initial frequencies of the degenerating wave packets are unbounded), disjoint supports (i.e., $\{\operatorname{supp} f_{k,0} \cup \operatorname{supp} \phi_{k,0}\}_k$ is pairwise disjoint), yet with an ϵ-small C^{m_0} norm. Then we perform a contradiction argument: if a regular solution ϕ to such initial data exists, then we may justify the growth mechanism (as in Proposition 2.1), which is absurd. For details, see Sections 2.5 and 2.6 below.

2.2. ***Properties of a regular linearly degenerate solution.*** We shall assume that there exists a smooth solution to (1-5) which is linearly degenerate and analyze its properties. To be precise, we will let $f : [0, \delta] \times [-x_1, x_1] \to \mathbb{C}$ be a $L^\infty([0, \delta]; C^{3,1}([x_0 - x_1, x_0+x_1]))$ solution to (1-5) with some $x_1, \delta > 0$ satisfying

$$f_0 \in C^{3,1}([x_0 - x_1, x_0 + x_1]), \quad f_0(x_0) = 0, \quad |f_0'(x_0)| > 0$$

at $t = 0$ for some $x_0 \in \mathbb{T}$.

Owing to the symmetries of (1-5) (translation and phase rotation), as well as its behavior under the transformation $\phi \mapsto c\phi$, we may assume without loss of generality that $x_0 = 0$ and $f_0'(0) = 1$. Then, from the equation it is easy to see that, on the time interval $[0, \delta]$,

$$f(t, 0) = 0$$

and

$$i \frac{\mathrm{d}}{\mathrm{d}t} f'(t, 0) = -(\alpha_1 + \beta_1)|f'(t, 0)|^2 f'(t, 0),$$

which implies in particular that

$$|f'(t, 0)| = 1 \quad \text{and} \quad |f(t, x)| = x + O(|x|^2) \ \text{ uniformly in } t.$$

More generally, we have the following lemma.

Lemma 2.5. *Let $s \geq 2$ be an integer, and let $f \in C_t([0, \delta]; C^{s-1,1}(\mathbb{T}))$ be a solution to* (1-5). *Then*:

(1) *The zero set of $f(t, x)$ is preserved in time, i.e., $a \in \mathbb{T}$ is a zero of $f(0, x)$ if and only if it is a zero for $f(t, x)$ for all $t \in [0, \delta]$.*

(2) *Let $a \in \mathbb{T}$ be a zero of $f(0, x)$. Then $\{\partial_x^k f(t, a)\}_{k=0}^{s-1}$ is determined by the initial data at $x = a$, i.e., $\{\partial_x^k f(0, a)\}_{k=0}^{s-1}$.*

Here, the important point is that, thanks to the regularity assumption, $f(0, x)$ vanishes at least linearly at each zero $x = a$, which is *critical* for (1-5) in the senses discussed in Section 1.3.

Proof. By the regularity assumption (in particular, that $s \geq 2$), it follows from (1-5) that $|\partial_t f(t, x)| \leq C|f(t, x)|$; hence the first statement follows. To prove the second statement, consider a zero a of $f(0, x)$. Without any loss of generality, we may assume that $a = 0$. By the assumption and Taylor expansion, we have

$$f(t,x)=\sum_{k=1}^{s-1}\frac{1}{k!}\partial_x^k f(t,0)x^k+O(|x|^s),$$

$$f_x(t,x)=\sum_{k=0}^{s-2}\frac{1}{k!}\partial_x^{k+1} f(t,0)x^k+O(|x|^{s-1}),$$

$$f_{xx}(t,x)=\sum_{k=0}^{s-3}\frac{1}{k!}\partial_x^{k+2} f(t,0)x^k+O(|x|^{s-2}),$$

where the implicit constants depend only on $\|f\|_{L_t^\infty C_x^{s-1,1}}$ (and in particular are independent of (t,x)). Plugging this into (1-5) and matching the coefficients of x^k for $k=1,\dots,s-1$, we formally obtain a determined system of first-order ODEs for $\{\partial_x^k f(t,0)\}_{k=1}^{s-1}$; here, the fact that $f(t,x)$ vanishes at least linearly is crucially used to ensure that no $\partial_x^k f(t,0)$ with $k>s-1$ arises. Indeed, these ODEs may be justified in the sense of distributions by testing (1-5) against a test function of the form $\eta(t)(-1)^k\partial_x^k\chi_\epsilon(x)$, where $\eta\in C_c^\infty(0,\delta)$, $\chi\in C_c^\infty\big(-\frac12,\frac12\big)$ with $\int\chi=1$, and $\chi_\epsilon(x)=\epsilon^{-1}\chi(\epsilon^{-1}x)$. By the uniqueness of this ODE system, the desired statement follows. □

From now on, given f_0 which are linearly degenerate at $x=0$ and $f_x(0,0)$ positive real, we are going to take $0<x_1<1$ smaller if necessary, so that

$$\Big(\sup_{x\in[-x_1,x_1]}|f_{xx}(0,x)|\Big)x_1<\tfrac12 f_x(0,0). \tag{2-9}$$

In particular, we have

$$\tfrac12 f_x(0,0)<|f_x(0,x)|<2f_x(0,0)\quad\text{for all } x\in[-x_1,x_1].$$

Proposition 2.6. *Let $f\in L_t^\infty([0,\delta];C^{3,1}([-x_1,x_1])$ be a solution to* (1-5), *and set $M=\|f\|_{L^\infty([0,\delta];C^{3,1})}$. Then, we have the pointwise bounds*

$$|f(t,x)|\le|f_0(x)|\exp(CM^2t) \tag{2-10}$$

and

$$|\partial_t(|f(t,x)|^2)|\le CM\exp(CM^2\delta)(1+(f_x(0,0))^{-1}M)^3(|f_0(x)|^3+tM^3|f_0(x)|^2) \tag{2-11}$$

for all $t\in[0,\delta]$ and $|x|\le x_1$.

Proof. We first note directly from (1-5) that $\big|\partial_t|f(t,x)|\big|\le C\|f\|^2_{L_t^\infty C_x^{1,1}}|f(t,x)|$ holds, which gives (2-10). Now note that $f\in L^\infty([0,\delta];C^{3,1}([-x_1,x_1]))$ implies, via (1-5), that

$$|\partial_t f(t,x)|\le C|f(t,x)|\|f\|^2_{L_t^\infty C_x^{1,1}},\quad |\partial_{tt}f(t,x)|\le C|f(t,x)|^2\|f\|^3_{L_t^\infty C_x^{3,1}}. \tag{2-12}$$

Then, the Taylor expansion in time of $f(t,x)$ gives

$$|f(t,x)|^2=|f_0(x)|^2+2\,\mathrm{Re}\bigg(\overline{f_0(x)}\int_0^t(\partial_t f)(t',x)\,dt'\bigg)+\bigg|\int_0^t(\partial_t f)(t',x)\,dt'\bigg|^2.$$

Taking the time derivative,

$$\partial_t(|f(t,x)|^2) = 2\,\mathrm{Re}(\overline{f_0(x)}(\partial_t f)(t,x)) + 2\,\mathrm{Re}\left(\overline{(\partial_t f)(t,x)} \int_0^t (\partial_t f)(t',x)\,\mathrm{d}t'\right). \tag{2-13}$$

Using (2-10), the last term in (2-13) is bounded by

$$C\|f\|^4_{L^\infty_t C^{3,1}_x} |f(t,x)| \int_0^t |f(t',x)|\,\mathrm{d}t' \le CM^4 t \exp(CM^2\delta)|f_0(x)|^2.$$

For the other term in the right-hand side of (2-13), we further rewrite it as

$$2\,\mathrm{Re}(\overline{f_0(x)}(\partial_t f)(t,x)) = 2\,\mathrm{Re}(\overline{f_0(x)}(\partial_t f)(0,x)) + 2\,\mathrm{Re}\left(\overline{f_0(x)} \int_0^t (\partial_{tt} f)(t',x)\,\mathrm{d}t'\right)$$

and note that the last term is bounded using (2-12) by $C\exp(CM^2\delta)M^3 t|f_0(x)|^3$. On the other hand, the first term on the right-hand side equals

$$\begin{aligned}\mathrm{Im}\big(\overline{f_0(x)}\big(|f_0(x)|^2\partial_{xx} f_0(x) + \alpha_1 f_0(x)|\partial_x f_0(x)|^2 + \beta_1 \overline{f_0(x)}(\partial_x f_0(x))^2 + \mu_1|f_0(x)|^2 f_0(x)\big)\big)& \\ = \beta_1\,\mathrm{Im}\big(\overline{f_0(x)}^2(\partial_x f_0(x))^2\big) + O(\|f_0\|_{C^{1,1}})|f_0(x)|^3&,\end{aligned}$$

and we see that the leading term in the Taylor expansion of $\overline{f_0(x)}^2(\partial_x f_0(x))^2$ is purely real, with remainder bounded by

$$C(f_x(0,0)^3|x|^3\|f_0\|_{\dot C^{1,1}} + f_x(0,0)^2|x|^4\|f_0\|^2_{\dot C^{1,1}} + f_x(0,0)^3|x|^5\|f_0\|^2_{\dot C^{1,1}}) \le C(1+f_x(0,0)^{-1}M)^3|f_0(x)|^3,$$

where we have used $|x| < x_1$ and the smallness of x_1 from (2-9). Collecting the bounds, we obtain the proposition. □

2.3. ***Degenerating wave packets for the linearized equation.*** In this subsection, our goal is to construct approximate solutions, called degenerating wave packets, for the linearization of (1-5) around a (possibly hypothetical) regular linearly degenerate solution, which possess the desired degeneration property that is responsible for the ill-posedness of (1-5); see Proposition 2.7 below.

2.3.1. *Properties of degenerating wave packets.* Given a smooth solution f to (1-5), let us write $\phi = f + \tilde\phi$, where ϕ is another smooth solution to (1-5). The equation for $\tilde\phi$ is given by

$$i\partial_t\tilde\phi + |f|^2\,\partial_{xx}\tilde\phi + \alpha_1 f(\overline{\partial_x f}\,\partial_x\tilde\phi + \partial_x f\,\partial_x\bar{\tilde\phi}) + 2\beta_1\bar f\,\partial_x f\,\partial_x\tilde\phi + V_f\tilde\phi + W_f\bar{\tilde\phi} = Q_f[\tilde\phi], \tag{2-14}$$

with

$$\begin{aligned} V_f &= \bar f\,\partial_{xx} f + \alpha_1|\partial_x f|^2 + 2\mu_1|f|^2,\\ W_f &= f\,\partial_{xx} f + \beta_1(\partial_x f)^2 + \mu_1 f^2,\\ Q_f[\tilde\phi] &= -(\bar f\tilde\phi + f\bar{\tilde\phi})\partial_{xx}\tilde\phi - \alpha_1\tilde\phi(\partial_x\bar f\,\partial_x\tilde\phi + \partial_x f\,\partial_x\bar{\tilde\phi}) - \alpha_1 f|\partial_x\tilde\phi|^2 - 2\beta_1\bar{\tilde\phi}\,\partial_x f\,\partial_x\tilde\phi - \beta_1\bar f(\partial_x\tilde\phi)^2\\ &\quad - |\tilde\phi|^2\,\partial_{xx}\tilde\phi - \alpha_1\tilde\phi|\partial_x\tilde\phi|^2 - \beta_1\bar{\tilde\phi}(\partial_x\tilde\phi)^2 - \mu_0|\tilde\phi|^2\tilde\phi - 2\mu_1 f|\tilde\phi|^2 - \mu_1\bar f(\tilde\phi)^2. \end{aligned} \tag{2-15}$$

Note that $Q_f[\tilde{\phi}]$ is at least quadratic in $\tilde{\phi}$ and its derivatives. Dropping the right-hand side, we obtain the linearized equation around f:

$$i\partial_t\tilde{\phi}+|f|^2\,\partial_{xx}\tilde{\phi}+\alpha_1 f(\overline{\partial_x f}\,\partial_x\tilde{\phi}+\partial_x f\,\partial_x\bar{\tilde{\phi}})+2\beta_1\bar{f}\,\partial_x f\,\partial_x\tilde{\phi}+V_f\tilde{\phi}+W_f\bar{\tilde{\phi}}=0. \tag{2-16}$$

We now state the key proposition of this section, which shows properties of degenerating wave packets for (2-16). Given a positive number L, we introduce the notation

$$\|g\|_{W^{s,p}_{(L)}}=\sum_{j=0}^{s}\|(L\partial_x)^j g\|_{L^p}$$

and write $H^s_{(L)}$ when $p=2$.

Proposition 2.7. *Let $f\in L^\infty([0,\delta];C^{s_0-1,1}([0,x_1]))$ be a solution to* (1-5) *with $s_0\ge 4$ satisfying*

$$f(0,0)=0,\qquad f'(0,0)=A \tag{2-17}$$

for some $A>0$. By taking $x_1<1$ small if necessary, assume that (2-9) *holds. Then, to any $\lambda\le -1$ and a C^∞-smooth complex-valued profile g_0 supported in $\left(\tfrac{1}{2}x_1,x_1\right)$, we may associate a function $\tilde{\phi}^{\mathrm{app}}_{(\lambda)}=\tilde{\phi}^{\mathrm{app}}_{(\lambda)}[g_0,f]$ defined in $[0,\delta]\times\mathbb{R}$ satisfying the following properties*:

- *Linearity*: *the map $g_0\mapsto\tilde{\phi}^{\mathrm{app}}_{(\lambda)}$ is* (*real*) *linear*;
- *Support property*: $\operatorname{supp}(\tilde{\phi}^{\mathrm{app}}_{(\lambda)}[g_0])\subset(0,e^{-|\lambda|A^2t}x_1)$;
- *Initial data*: *for any $1\le p\le\infty$,*

$$\frac{1}{C}\|g_0\|_{L^2}\le A^{\sigma_c-1}\||f|^{-\sigma_c}\tilde{\phi}^{\mathrm{app}}(0,x)\|_{L^2}\le C\|g_0\|_{L^2},$$
$$A^{\sigma_c-1}\||f|^{\sigma_c}\tilde{\phi}^{\mathrm{app}}(0,x)\|_{L^p}\le Cx_1^{1/p-1/2}\|g_0\|_{L^p};$$

- *Regularity*: *for $0\le n\le s_0-2$, we have*

$$\||f|^{-\sigma_c}(|f|\,\partial_x)^n\tilde{\phi}^{\mathrm{app}}_{(\lambda)}(t,x)\|_{L^2}\le C_{f,\delta}A^{-\sigma_c+n+1}|\lambda|^n\|g_0\|_{H^n_{(x_1)}},\quad t\le\min\{A^{-2}|\lambda|^{-1/2},\delta\}; \tag{2-18}$$

- *Degeneration*: *for any $1\le p\le 2$, $0\le s\le s_0-2$, and $\gamma'\ge -s-\frac{1}{p}+\frac{1}{2}$, we have*

$$|f|^{-\sigma_c+\gamma'}\tilde{\phi}^{\mathrm{app}}_{(\lambda)}=\partial_x^s\left(\frac{|f|^{\gamma'+s-1/2}}{i^s\lambda^s(1+|f|\,\partial_x S)^s}\psi^{\mathrm{app}}_{(\lambda)}\right)+|f|^{-\sigma_c+\gamma'}\tilde{\phi}^{\mathrm{small}}_{(\lambda)} \tag{2-19}$$

for some $\psi^{\mathrm{app}}_{(\lambda)}$, $\tilde{\phi}^{\mathrm{small}}_{(\lambda)}$, and S, where $\psi^{\mathrm{app}}_{(\lambda)}$ is independent of p, s and γ', and

$$\left\|\frac{|f|^{\gamma'+s-1/2}}{\lambda^s(1+|f|\,\partial_x S)^s}\psi^{\mathrm{app}}_{(\lambda)}(t,x)\right\|_{L^p}\le C^{1+\gamma'}_{f,\delta}A^{\gamma'+s+1/2}|\lambda|^{-s}\times\exp\!\left(-2|\lambda|\left(\gamma'+s+\tfrac{1}{p}-\tfrac{1}{2}\right)A^2t\right)\|g_0\|_{L^2}, \tag{2-20}$$

$$\||f|^{-\sigma_c}\tilde{\phi}^{\mathrm{small}}_{(\lambda)}(t,x)\|_{L^2}\le C_{f,\delta}A^{-\sigma_c+1}|\lambda|^{-1}\|g_0\|_{H^s_{(x_1)}} \tag{2-21}$$

for $t\le\min\{A^{-2}|\lambda|^{-1/2},\delta\}$, after taking $\delta>0$ smaller in a way that $\delta\|f\|^2_{L^\infty([0,\delta];C^{1,1})}$ is small in terms of $A^{-1}\|f\|_{L^\infty([0,\delta];C^{3,1})}$;

- *<u>Error estimate</u>: defining the error* $\boldsymbol{\epsilon}[\tilde{\phi}^{\mathrm{app}}_{(\lambda)}]$ *by the left-hand side of* (2-16) *with* $\tilde{\phi} = \tilde{\phi}^{\mathrm{app}}_{(\lambda)}$, *we have the estimate*

$$\||f|^{-\sigma_c}\boldsymbol{\epsilon}[\tilde{\phi}^{\mathrm{app}}_{(\lambda)}](t)\|_{L^2} \le C_{f,\delta} A^{-\sigma_c+3}\|g_0\|_{H^2_{(x_1)}}, \quad t \le \min\{A^{-2}|\lambda|^{-1/2}, \delta\}. \tag{2-22}$$

In the above estimates, the constant $C_{f,\delta}$ *satisfies*

$$C_{f,\delta} \le C_0(1 + A^{-1}\|f\|_{L^\infty_t C^{s_0-1,1}})^{N_0} \exp(C_0\|f\|^2_{L^\infty_t C^{s_0-1,1}}\delta) \tag{2-23}$$

for some C_0, $N_0 > 0$ *depending on* α_1, β_1, μ_1 *and* s_0 *but not on* f *and* x_1.

We fix $A = 1$ and prove Proposition 2.7 in the remainder of this subsection. In the general case, given f we can define $\tilde{f}(t, x) := A^{-1} f(A^{-2}t, x)$ which is another $L^\infty_t C^{s-1,1}$ solution to (1-5) satisfying $\tilde{f}_x(0, 0) = 1$. Then, we simply define

$$\tilde{\phi}^{\mathrm{app}}_{(\lambda)}[g_0, f](t, x) := \tilde{\phi}^{\mathrm{app}}_{(\lambda)}[Ag_0, \tilde{f}](A^2t, x)$$

and verify the claimed properties of $\tilde{\phi}^{\mathrm{app}}_{(\lambda)}[g_0, f]$ using those for $\tilde{\phi}^{\mathrm{app}}_{(\lambda)}[Ag_0, \tilde{f}]$. In the proof, it will be seen that $|f|\,\partial_x S$ remains invariant under this rescaling.

2.3.2. *Renormalization and wave packet construction.* With $x_1 > 0$ given in Proposition 2.7, we define the variable y for $t \in [0, \delta]$ and $x \in (0, x_1]$ by

$$y(t, x) = -\int_x^{x_1} \frac{1}{|f(t, x')|}\,\mathrm{d}x' \le 0.$$

For each $t \ge 0$, the inverse of $x \mapsto y(t, x)$ is denoted by $x = x(t, y)$. From $|f(t, x)| = x + O(|x|^2)$, we have

$$y(t, x) - \ln\frac{x}{x_1} = B(t, x), \quad x(t, y) = x_1 e^{y-B}, \quad |B(t, x)| \le Cx_1\|f\|_{L^\infty_t C^{1,1}}. \tag{2-24}$$

Using $|f|\,\partial_x = \partial_y$, we rewrite (2-16) in (t, y)-coordinates:

$$i\partial_t\tilde{\phi} + ih\,\partial_y\tilde{\phi} + \partial_y^2\tilde{\phi} + \frac{\alpha_1 f\,\overline{\partial_y f} + 2\beta_1\bar{f}\,\partial_y f - |f|\,\partial_y|f|}{|f|^2}\partial_y\tilde{\phi} + \alpha_1\frac{f\,\partial_y f}{|f|^2}\partial_y\bar{\tilde{\phi}} + V_f\tilde{\phi} + W_f\bar{\tilde{\phi}} = 0. \tag{2-25}$$

Here, we have introduced $h(t, y) = \partial_t y$ so that $\partial_t\tilde{\phi}(t, x) = \partial_t\tilde{\phi}(t, y) + h(t, y)\partial_y\tilde{\phi}(t, y)$. Now defining

$$G(t, y) = \left(-\sigma_c + \tfrac{1}{2}\right)\ln|f|(t, y), \quad (\partial_y G)(t, y) = \left(-\sigma_c + \tfrac{1}{2}\right)\frac{\mathrm{Re}(\bar{f}\,\partial_y f)}{|f|^2}(t, y)$$

and introducing the conjugation $\varphi = e^G\tilde{\phi}$, we obtain $\big($recall from (1-6) that $\sigma_c = -\left(\tfrac{1}{2}\alpha_1 + \beta_1 - 1\right)\big)$

$$i\partial_t\varphi + \partial_{yy}\varphi + \alpha_1\frac{f\,\partial_y f}{|f|^2}\overline{\partial_y\varphi} + \left((-\alpha_1 + 2\beta_1)\frac{\mathrm{Im}(\bar{f}\,\partial_y f)}{|f|^2} + h\right)i\partial_y\varphi = \mathcal{B}_0[\varphi] \tag{2-26}$$

with

$$\begin{aligned}\mathcal{B}_0[\varphi] = {} & i(\partial_t G)\varphi + (\partial_{yy}G + (\partial_y G)^2)\varphi \\ & + \alpha_1\frac{f\,\partial_y f}{|f|^2}\partial_y G\bar{\varphi} + \left((-\alpha_1 + 2\beta_1)\frac{\mathrm{Im}(\bar{f}\,\partial_y f)}{|f|^2} + h\right)i(\partial_y G)\varphi - V_f\varphi - W_f\bar{\varphi}.\end{aligned}$$

Note that the terms in $\mathcal{B}_0$ do not contain derivatives of φ. To handle the term containing $\overline{\partial_y \varphi}$ in the left-hand side of (2-26), we make yet another change of variables: introducing formally

$$\psi = \varphi + \frac{\alpha_1}{2} \frac{f\,\partial_y f}{|f|^2} \partial_y^{-1} \bar{\varphi},$$

we have that (2-26) turns into

$$i\partial_t \psi + \partial_{yy}\psi + \left((-\alpha_1 + 2\beta_1) \frac{\operatorname{Im}(\bar{f}\,\partial_y f)}{|f|^2} + h \right) i\partial_y \psi = \cdots, \tag{2-27}$$

where the terms on the right-hand side do not contain any derivatives of ψ. Indeed, introducing the shorthand

$$\mathcal{A}\bar{\varphi} = \frac{\alpha_1}{2} \frac{f\,\partial_y f}{|f|^2} \partial_y^{-1} \bar{\varphi}$$

and omitting any zeroth-order terms in φ, we have the formal computation

$$\begin{aligned} i\partial_t \psi &= -\partial_{yy}\varphi + i[\partial_t, \mathcal{A}]\bar{\varphi} + \mathcal{A}\,\partial_{yy}\bar{\varphi} - \alpha_1 \frac{f\,\partial_y f}{|f|^2} \overline{\partial_y \varphi} - \left((-\alpha_1 + 2\beta_1) \frac{\operatorname{Im}(\bar{f}\,\partial_y f)}{|f|^2} + h \right) i\partial_y \varphi + \cdots \\ &= -\partial_{yy}\psi + \partial_{yy}\mathcal{A}\bar{\varphi} + \mathcal{A}\,\partial_{yy}\bar{\varphi} - \alpha_1 \frac{f\,\partial_y f}{|f|^2} \overline{\partial_y \varphi} - \left((-\alpha_1 + 2\beta_1) \frac{\operatorname{Im}(\bar{f}\,\partial_y f)}{|f|^2} + h \right) i\partial_y \psi + \cdots \\ &= -\partial_{yy}\psi - \left((-\alpha_1 + 2\beta_1) \frac{\operatorname{Im}(\bar{f}\,\partial_y f)}{|f|^2} + h \right) i\partial_y \psi + \cdots. \end{aligned}$$

Motivated by this computation, we construct a wave packet approximate solution for (2-26) by starting with a wave packet for the preceding equation for ψ, then coming back to φ. More precisely, given $g_0(x)$ as in Proposition 2.7, we take

$$a_0(y) = x_1^{1/2} g_0(x(0, y)),$$

which is supported in $y \in \left(-\frac{1}{2}\ln 2, 0\right)$ by (2-24). For each $\lambda < 0$, we define

$$\psi^{\mathrm{app}}_{(\lambda)}(t, y) := e^{i\lambda(y - \lambda t)} a_{(\lambda)}(t, y), \tag{2-28}$$

where $a_{(\lambda)}(t, y)$ is the unique solution to

$$\partial_t a_{(\lambda)} + 2\lambda\,\partial_y a_{(\lambda)} = \frac{\lambda}{i} \left((-\alpha_1 + 2\beta_1) \frac{\operatorname{Im}(\bar{f}\,\partial_y f)}{|f|^2} + h \right) a_{(\lambda)} \tag{2-29}$$

with initial data $a_{(\lambda)}(0, y) = a_0(y)$. The function $\psi^{\mathrm{app}}_{(\lambda)}$ defined via (2-28) and (2-29) turns out to be a suitable approximate solution to (2-27) (more precisely, (2-38) holds). Next, given ψ^{app}, set

$$\varphi^{\mathrm{app}}_{(\lambda)} = \psi^{\mathrm{app}}_{(\lambda)} + \frac{\alpha_1}{2i\lambda} \frac{f\,\partial_y f}{|f|^2} \overline{\psi^{\mathrm{app}}_{(\lambda)}}, \tag{2-30}$$

which will be shown to be a suitable approximate solution to (2-26) (for more details, see the end of the proof of Proposition 2.7). Finally, the degenerating wave packet is defined by

$$\tilde{\phi}^{\mathrm{app}}_{(\lambda)}[g_0, f] = e^{-G} \varphi^{\mathrm{app}}_{(\lambda)} = |f|^{\sigma_c - 1/2} \varphi^{\mathrm{app}}_{(\lambda)}. \tag{2-31}$$

2.3.3. *Proof of Proposition 2.7.* Now that we have defined the wave packet solution, let us proceed to confirm the properties stated in Proposition 2.7.

Linearity and support property. From the definition, linearity is clear. Furthermore, note from (2-31), (2-30) and (2-28) that the support of $\tilde{\phi}^{\mathrm{app}}_{(\lambda)}(t,\cdot)$ coincides with that of $a_{(\lambda)}(t,\cdot)$. (From now on, we shall refrain from writing out the subscript λ.) On the other hand, note the following formulae for a:

$$a(t,y) = e^{i\lambda S(t,y)} a_0(y-2\lambda t), \tag{2-32}$$

$$S(t,y) = \int_0^t \Big((-\alpha_1 + 2\beta_1) \frac{\operatorname{Im}(\bar{f}\,\partial_y f)}{|f|^2} + h \Big)(t', y-2\lambda(t-t'))\,\mathrm{d}t'. \tag{2-33}$$

Since $\lambda < 0$, the support of $a(t,\cdot)$ is contained in the interval $\big(-\frac{1}{2}\ln 2 + 2\lambda t, 2\lambda t\big) \subseteq (-\infty, 0)$ for $t \geq 0$, which verifies the support property of $\tilde{\phi}^{\mathrm{app}}$ via (2-24).

Regularity estimates. To begin with, we obtain estimates on $h := \partial_t y$. Recalling (2-13), we have

$$h = -\int_x^{x_1} \partial_t \Big(\frac{1}{|f(t,x')|} \Big)\,\mathrm{d}x' = \int_x^{x_1} \frac{\partial_t(|f|^2)}{|f|^3}\,\mathrm{d}x', \qquad \partial_y h = |f| \partial_x h = -\frac{\partial_t(|f|^2)}{|f|^2}.$$

Applying (2-10) and (2-11), we obtain the pointwise estimates

$$|h| \leq C_{f,\delta}\Big(1 + t\ln\frac{1}{x}\Big)x_1, \qquad |\partial_y h| \leq C_{f,\delta}(x+t). \tag{2-34}$$

We now estimate a. Observing that the right-hand side in (2-29) is purely imaginary,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|a|^2(t,y) = -2\lambda\operatorname{Re}(\partial_y a\bar{a})(t,y), \quad \text{which gives } \frac{\mathrm{d}}{\mathrm{d}t}\|a\|^2_{L^2(\mathrm{d}y)} = 0.$$

In what follows, we use the notation $L^2(\mathrm{d}y)$ to denote the L^2 norm taken with respect to the y variable, to avoid confusion with the corresponding norm in the original x variable. Similarly, we use the notation $H^1(\mathrm{d}y)$ and so on. Now, taking a y-derivative and then integrating in y, we see that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_y a\|^2_{L^2(\mathrm{d}y)} \leq C_{f,\delta}|\lambda|(e^{-|\lambda|t} + t)\|a\|_{L^2(\mathrm{d}y)}\|\partial_y a\|_{L^2(\mathrm{d}y)},$$

where we have used

$$\Big| \partial_y \Big((-\alpha_1 + 2\beta_1)\frac{\operatorname{Im}(\bar{f}\,\partial_y f)}{|f|^2} + h \Big) \Big| \leq C_{f,\delta}(x(t,y) + t) \leq C_{f,\delta}(\exp(-|\lambda|t) + t) \tag{2-35}$$

on the support of $a(t,\cdot)$. This estimate follows from (2-34) and

$$\Big| \partial_y \frac{\operatorname{Im}(\bar{f}\,\partial_y f)}{|f|^2} \Big| \leq \Big| \partial_x \partial_y \frac{\operatorname{Im}(\bar{f}\,\partial_y f)}{|f|^2} \Big| x.$$

Therefore, by integrating in time, we obtain

$$\|\partial_y a(t)\|_{L^2(\mathrm{d}y)} \leq C_{f,\delta}\|a_0\|_{H^1(\mathrm{d}y)}$$

uniformly in λ, for (t, λ) satisfying $t \le |\lambda|^{-1/2}$. A similar argument applies to the estimate of $\partial_y^k a$, as long as $k \le s_0 - 2$; one can proceed by an induction in k, using the bound

$$\left|\partial_y^k\left((-\alpha_1 + 2\beta_1)\frac{\operatorname{Im}(\bar{f}\,\partial_y f)}{|f|^2} + h\right)\right| \le C_{f,\delta}(x(t, y) + t) \le C_{f,\delta}(\exp(-|\lambda| t) + t)$$

on the support of $a(t, \cdot)$. The estimate for $|\partial_y^k h|$ readily follows from the explicit decomposition $\partial_y h = h_1 + t h_2$, where h_1 and h_2 are $L_t^\infty C^{s_0-2,1}$-smooth functions defined by

$$h_1(t, x) = 2|f|^{-2} \operatorname{Re}(\overline{f_0(x)}(\partial_t f)(t, x)), \quad h_2(t, x) = 2|f|^{-2} \operatorname{Re}\left(\overline{(\partial_t f)(t, x)}\frac{1}{t}\int_0^t (\partial_t f)(t', x)\,\mathrm{d}t'\right).$$

Hence we conclude

$$\|a(t)\|_{H^k(\mathrm{d}y)} \le C_{f,\delta}\|a_0\|_{H^k(\mathrm{d}y)}, \quad 0 \le t \le \min\{|\lambda|^{-1/2}, \delta\}. \tag{2-36}$$

In what follows, we shall restrict the variable t to $[0, \min\{|\lambda|^{-1/2}, \delta\}]$.

Initial data and regularity estimates. At the initial time, from $a_0(y) = x_1^{1/2} g_0(x(0, y))$ we have that

$$\int |a_0(y)|^2\,\mathrm{d}y = \int x_1 |f_0(x)|^{-1} |g_0(x)|^2\,\mathrm{d}x,$$

and we note that the right-hand side is equivalent up to constants with $\|g_0\|_{L_x^2}^2$. This gives the claimed initial data estimate in the case $p = 2$, and the case of general p can be proved similarly. Next, with $\partial_y = |f_0(x)|\partial_x$ at the initial time, we note the bound

$$|\partial_y^k a_0(x)| \le C_k x_1^{1/2}\left(\sum_{j=1}^k \|f_0\|_{C^{k-2,1}}^{k-j} |f_0(x)|^j |\partial_x^j g_0(x)|\right),$$

which gives

$$\|a_0\|_{H^k(\mathrm{d}y)} \le C_k(1 + \|f_0\|_{C^{k-2,1}})^{k-1}\|g_0\|_{H^k_{(x_1)}}, \quad k \le s_0 - 1. \tag{2-37}$$

Let us now check the regularity estimate (2-18) in the case $n = 0$: using $|f|^{-\sigma_c+1/2} = e^G$,

$$\begin{aligned}\||f|^{-\sigma_c}\tilde{\phi}^{\mathrm{app}}_{(\lambda)}(t, x)\|_{L^2}^2 &= \int_0^{x_1} |\tilde{\phi}^{\mathrm{app}}_{(\lambda)}(t, x)|^2 |f(t, x)|^{-2\sigma_c}\,\mathrm{d}x = \int_{-\infty}^0 |\tilde{\phi}^{\mathrm{app}}_{(\lambda)}(t, y)|^2 |f(t, y)|^{-2\sigma_c+1}\,\mathrm{d}y \\ &= \int_{-\infty}^0 |\varphi^{\mathrm{app}}_{(\lambda)}(t, y)|^2\,\mathrm{d}y \le C(1 + |\lambda|^{-1})\int_{-\infty}^0 |\psi^{\mathrm{app}}_{(\lambda)}(t, y)|^2\,\mathrm{d}y \\ &\le C\|a(t)\|_{L^2(\mathrm{d}y)}^2 \le C\|a_0\|_{L^2(\mathrm{d}y)} \le C\|g_0\|_{L^2}.\end{aligned}$$

The cases $1 \le n \le s_0 - 2$ can be handled similarly, using (2-36) and (2-37).

Degeneration estimate. Next, we check the degeneration property (2-19). To simplify the notation, we introduce the notation

$$H = q\,O_k(a_0) \iff \sup_{t \in [0, \min\{\lambda^{-1/2}, \delta\}]}\left\||f|^{1/2}\frac{H}{q}\right\|_{L^2(\mathrm{d}y)} \le C_{f,\delta}\|a_0\|_{H^k(\mathrm{d}y)}.$$

Note that $\| |f|^{1/2}(\,\cdot\,)\|_{L^2(\mathrm{d}y)} = \|\cdot\|_{L^2(\mathrm{d}x)}$ for each t. The terms that are abbreviated as $\frac{1}{\lambda} O_k(a_0)$ (for $k \le s$) will constitute $|f|^{-\sigma_c} \tilde{\phi}^{\mathrm{small}}_{(\lambda)}$; the desired estimate (2-21) would be an immediate consequence of the L^2 norm estimate embedded in the $O_k(\,\cdot\,)$ notation. Recalling the definitions of $\tilde{\phi}^{\mathrm{app}}_{(\lambda)}$, $\varphi^{\mathrm{app}}_{(\lambda)}$, and $\psi^{\mathrm{app}}_{(\lambda)}$, and arguing as in the proof of the regularity estimate, we have

$$|f|^{-\sigma_c+\gamma'} \tilde{\phi}^{\mathrm{app}}_{(\lambda)} = |f(t,y)|^{\gamma'-1/2} \psi^{\mathrm{app}}_{(\lambda)} + \frac{|f|^{\gamma'}}{\lambda} O_0(a_0).$$

For the first term, we have

$$\begin{aligned}
\| |f(t,x)|^{\gamma'-1/2} \psi^{\mathrm{app}}_{(\lambda)}(t,x)\|_{L^p}^p &\le C \int_{-\infty}^0 |\psi^{\mathrm{app}}_{(\lambda)}(t,y)|^p |f(t,y)|^{p\gamma'-p/2+1}\,\mathrm{d}y \\
&\le C\biggl(\int_{-\infty}^0 |\psi^{\mathrm{app}}_{(\lambda)}(t,y)|^2\,\mathrm{d}y\biggr)^{p/2} \biggl(\int_{\operatorname{supp}\psi^{\mathrm{app}}_{(\lambda)}(t,\cdot)} |f(t,y)|^{\frac{p}{1-p/2}\gamma'+1}\,\mathrm{d}y\biggr)^{1-p/2} \\
&\le C\|a_0\|_{L^2(\mathrm{d}y)}^p \biggl(\int_{\operatorname{supp}\psi^{\mathrm{app}}_{(\lambda)}(t,\cdot)} |f(t,y)|^{\frac{p}{1-p/2}\gamma'+1}\,\mathrm{d}y\biggr)^{p(1/p-1/2)},
\end{aligned}$$

so it remains to estimate the last factor. Note that, since $|f|^{-1}\partial_y|f| = \partial_x|f| = 1 + O(x)$ and $x \le x_1 e^{y/2}$ for $y \in (-\infty, 0)$, we have

$$|f(t,y)| \le Ce^y \quad \text{for } y \in (-\infty, 0).$$

Using the support property $\operatorname{supp}\psi^{\mathrm{app}}_{(\lambda)}(t,\cdot) \subseteq (-\infty, -2|\lambda|t)$, we see that

$$\biggl(\int_{\operatorname{supp}\psi^{\mathrm{app}}_{(\lambda)}(t,\cdot)} |f(t,y)|^{\frac{p}{1-p/2}\gamma'+1}\,\mathrm{d}y\biggr)^{1/p-1/2} \lesssim \exp\bigl(-2|\lambda|\bigl(\gamma' + \tfrac{1}{p} - \tfrac{1}{2}\bigr)t\bigr).$$

Hence the desired estimate (2-19) in the case $s = 0$ now follows.

To treat the cases $s > 0$, we begin by recalling that $\psi^{\mathrm{app}}_{(\lambda)} = \psi^{\mathrm{app}} = \exp\bigl(i\lambda(y - \lambda t + S(t,y))\bigr) a_0(y - 2\lambda t)$. Note the identity

$$\exp(i\lambda(y - \lambda t + S)) = \frac{|f|}{i\lambda(1+\partial_y S)} \biggl(\frac{1}{|f|}\partial_y\biggr) \exp(i\lambda(y - \lambda t + S)).$$

For the expression $\partial_y S$ in the denominator, recalling (2-33) and (2-35), we have

$$|\partial_y S| \le C_{f,\delta} t x,$$

and in particular we note that $1 + \partial_y S \ge \frac{1}{2}$ when t is sufficiently small, which can be arranged by taking $\delta > 0$ smaller. Commuting $\frac{1}{|f|}\partial_y$ (which equals ∂_x in the (t,x)-coordinates) outside, we have

$$|f(t,y)|^{\gamma'-1/2} \psi^{\mathrm{app}}_{(\lambda)} = \frac{1}{|f|}\partial_y \biggl(\frac{|f(t,y)|^{\gamma'+1-1/2}}{i\lambda(1+\partial_y S)} \psi^{\mathrm{app}}_{(\lambda)}\biggr) + \frac{|f|^{\gamma'}}{\lambda} O_1(a_0).$$

By arguing as in the case of $s = 0$, the expression inside the parentheses can be shown to obey the degeneration bound (2-20). The cases $s > 1$ are handled similarly.

Error estimate. To begin with, at the level of $\psi^{\mathrm{app}}_{(\lambda)}$, the point of choosing $a(t, y)$ as the solution of (2-29) is to have

$$i\partial_t\psi^{\mathrm{app}}_{(\lambda)} + \partial_{yy}\psi^{\mathrm{app}}_{(\lambda)} + \biggl((-\alpha_1+2\beta_1)\frac{\mathrm{Im}(\bar f\,\partial_y f)}{|f|^2}+h\biggr)i\partial_y\psi^{\mathrm{app}}_{(\lambda)} = O_2(a_0), \tag{2-38}$$

which can be checked with a direct computation using (2-27). We now see that $\varphi^{\mathrm{app}}_{(\lambda)}$ is an approximate solution to (2-26), which is motivated by the following heuristics: recalling (2-28), we have

$$\varphi^{\mathrm{app}}_{(\lambda)} \simeq \psi^{\mathrm{app}}_{(\lambda)} - \frac{\alpha_1}{2}\frac{f\,\partial_y f}{|f|^2}\partial_y^{-1}\overline{\psi^{\mathrm{app}}_{(\lambda)}} \simeq \psi^{\mathrm{app}}_{(\lambda)} + \frac{\alpha_1}{2i\lambda}\frac{f\,\partial_y f}{|f|^2}\overline{\psi^{\mathrm{app}}_{(\lambda)}}.$$

To this end, (2-38) gives

$$-i\overline{\partial_t\psi^{\mathrm{app}}_{(\lambda)}} + \overline{\partial_{yy}\psi^{\mathrm{app}}_{(\lambda)}} - \biggl((-\alpha_1+2\beta_1)\frac{\mathrm{Im}(\bar f\,\partial_y f)}{|f|^2}+h\biggr)i\overline{\partial_y\psi^{\mathrm{app}}_{(\lambda)}} = O_2(a_0)$$

and from this it is not difficult to see that

$$\frac{1}{2i\lambda}[i\partial_t+\partial_{yy}]\overline{\psi^{\mathrm{app}}_{(\lambda)}} = \frac{1}{i\lambda}\partial_{yy}\overline{\psi^{\mathrm{app}}_{(\lambda)}} + O_2(a_0) = -\partial_y\overline{\psi^{\mathrm{app}}_{(\lambda)}} + O_2(a_0), \tag{2-39}$$

so that

$$(i\partial_t+\partial_{yy})\biggl(\frac{\alpha_1}{2i\lambda}\frac{f\,\partial_y f}{|f|^2}\overline{\psi^{\mathrm{app}}_{(\lambda)}}\biggr) = -\alpha_1\frac{f\,\partial_y f}{|f|^2}\partial_y\overline{\psi^{\mathrm{app}}_{(\lambda)}} + O_2(a_0).$$

Using (2-38), (2-39), and

$$\biggl((-\alpha_1+2\beta_1)\frac{\mathrm{Im}(\bar f\,\partial_y f)}{|f|^2}+h\biggr)i\partial_y\biggl(\frac{\alpha_1}{2i\lambda}\frac{f\,\partial_y f}{|f|^2}\overline{\psi^{\mathrm{app}}_{(\lambda)}}\biggr) = O_2(a_0), \qquad \frac{\alpha_1^2 f\,\partial_y f}{2i\lambda|f|^2}\partial_y\biggl(\frac{\overline{f\,\partial_y f}}{|f|^2}\psi^{\mathrm{app}}_{(\lambda)}\biggr) = O_2(a_0),$$

we simplify

$$\begin{aligned}
&\biggl[i\partial_t+\partial_{yy}+\biggl((-\alpha_1+2\beta_1)\frac{\mathrm{Im}(\bar f\,\partial_y f)}{|f|^2}+h\biggr)i\partial_y\biggr]\varphi^{\mathrm{app}}_{(\lambda)} + \alpha_1\frac{f\,\partial_y f}{|f|^2}\overline{\partial_y\varphi^{\mathrm{app}}_{(\lambda)}}\\
&\quad= \biggl[i\partial_t+\partial_{yy}+\biggl((-\alpha_1+2\beta_1)\frac{\mathrm{Im}(\bar f\,\partial_y f)}{|f|^2}+h\biggr)i\partial_y\biggr]\biggl(\psi^{\mathrm{app}}_{(\lambda)} + \frac{\alpha_1}{2i\lambda}\frac{f\,\partial_y f}{|f|^2}\overline{\psi^{\mathrm{app}}_{(\lambda)}}\biggr)\\
&\qquad\qquad + \alpha_1\frac{f\,\partial_y f}{|f|^2}\overline{\partial_y\psi^{\mathrm{app}}_{(\lambda)}} - \frac{\alpha_1^2 f\,\partial_y f}{2i\lambda|f|^2}\partial_y\biggl(\frac{\overline{f\,\partial_y f}}{|f|^2}\psi^{\mathrm{app}}_{(\lambda)}\biggr)\\
&\quad= -\alpha_1\frac{f\,\partial_y f}{|f|^2}\partial_y\overline{\psi^{\mathrm{app}}_{(\lambda)}} + \alpha_1\frac{f\,\partial_y f}{|f|^2}\partial_y\overline{\psi^{\mathrm{app}}_{(\lambda)}} + O_2(a_0) = O_2(a_0).
\end{aligned}$$

Moreover, it is easy to see that $\mathcal{B}_0[\varphi^{\mathrm{app}}_{(\lambda)}] = O_2(a_0)$, and finally the error estimate (2-22) follows from

$$\biggl[i\partial_t+\partial_{yy}+\biggl((-\alpha_1+2\beta_1)\frac{\mathrm{Im}(\bar f\,\partial_y f)}{|f|^2}+h\biggr)i\partial_y\biggr]\varphi^{\mathrm{app}}_{(\lambda)} + \alpha_1\frac{f\,\partial_y f}{|f|^2}\overline{\partial_y\varphi^{\mathrm{app}}_{(\lambda)}} - \mathcal{B}_0[\varphi^{\mathrm{app}}_{(\lambda)}] = O_2(a_0)$$

and (2-37). This completes the proof of Proposition 2.7. □

2.4. ***Modified and generalized energy estimates.*** In Sections 2.4.1 and 2.4.2, we establish the modified and generalized (or bilinear) energy estimates that we shall need in the proofs of Theorems 1.1 and 1.2, respectively.

2.4.1. *Modified energy estimate.* Assume that f and $\phi = f + \tilde{\phi}$ are solutions to (1-1) on some time interval. Then, recall that $\tilde{\phi}$ solves

$$i\partial_t\tilde{\phi} + |f|^2\,\partial_{xx}\tilde{\phi} + \alpha_1 f(\overline{\partial_x f}\,\partial_x\tilde{\phi} + \partial_x f\,\partial_x\bar{\tilde{\phi}}) + 2\beta_1\bar{f}\,\partial_x f\,\partial_x\tilde{\phi} + V_f\tilde{\phi} + W_f\bar{\tilde{\phi}} = Q_f[\tilde{\phi}], \tag{2-40}$$

where V_f, W_f and $Q_f[\,\cdot\,]$ are defined in (2-15). To deal with solutions of (2-40), it turns out that the following time-dependent Hermitian product and norm are very natural (which will be referred to as the modified energy): given some f, we define

$$\langle v, u\rangle_{L^2_f}(t) := \int |f(t,\cdot)|^{-2\sigma_c} v(t,\cdot)\overline{u(t,\cdot)}\,\mathrm{d}x, \quad \|v\|^2_{L^2_f}(t) := \int |f(t,\cdot)|^{-2\sigma_c}|v(t,\cdot)|^2\,\mathrm{d}x.$$

Regarding this modified energy, we have the following estimate.

Proposition 2.8. *Let $f \in L^\infty([0,\delta]; C^{s_c-1,1}))$ be a solution to* (1-5) *and $\tilde{\phi} \in L^\infty([0,\delta']; C^{s_c-1,1}))$ be a solution to* (2-40) *for some $0 < \delta' \le \delta$. When $\sigma_c \ge \frac{1}{2}$, assume furthermore that at every zero a of f_0, we have $\partial_x f_0(a) \neq 0$ and $\tilde{\phi}_0(x)$ vanishes up to order $\lfloor \sigma_c - \frac{1}{2}\rfloor$ at a. Then, on $t \in [0,\delta']$, we have*

$$\|\tilde{\phi}\|_{L^2_f}(t) \le \|\tilde{\phi}\|_{L^2_f}(0)\exp(C(\|f\|^2_{L^\infty_t C^{1,1}} + \|\tilde{\phi}\|^2_{L^\infty_t C^{1,1}})t), \tag{2-41}$$

where σ_c is as in (1-6) *and $C > 0$ is an absolute constant.*

Proof. We first present a formal computation without worrying about the finiteness of the modified energy and the validity of integration by parts, and discuss its justification below. We begin with

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{\phi}\|^2_{L^2_f}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int |\tilde{\phi}|^2|f|^{-2\sigma_c}\,\mathrm{d}x = \int |\tilde{\phi}|^2\,\partial_t(|f|^{-2\sigma_c})\,\mathrm{d}x + \int \partial_t(|\tilde{\phi}|^2)|f|^{-2\sigma_c}\,\mathrm{d}x.$$

The term involving $\partial_t(|f|^{-2\sigma_c})$ in the right-hand side can be bounded using the pointwise inequality

$$\left|\frac{\partial_t|f|}{|f|}\right| \lesssim \|f\|^2_{L^\infty_t C^{1,1}}.$$

To handle the second term, we write

$$\begin{aligned}\partial_t|\tilde{\phi}|^2 = \mathrm{Re}(i|f|^2\,\partial_{xx}\tilde{\phi}\bar{\tilde{\phi}}) + \alpha_1\,\mathrm{Re}(if(\partial_x\bar{f}\,\partial_x\tilde{\phi} + \partial_x f\,\partial_x\bar{\tilde{\phi}})\bar{\tilde{\phi}}) + 2\beta_1\,\mathrm{Re}(i\bar{f}\,\partial_x f\,\partial_x\tilde{\phi}\bar{\tilde{\phi}})\\ + \mathrm{Re}(iV_f\tilde{\phi}\bar{\tilde{\phi}}) + \mathrm{Re}(iW_f(\bar{\tilde{\phi}})^2) - \mathrm{Re}(iQ_f[\tilde{\phi}]\bar{\tilde{\phi}}).\end{aligned} \tag{2-42}$$

We multiply both sides by $|f|^{-2\sigma_c}$ and integrate in x. From the first term on the right-hand side, we obtain, after an integration by parts,

$$\begin{aligned}\int \mathrm{Re}(i|f|^2\,\partial_{xx}\tilde{\phi}\bar{\tilde{\phi}})|f|^{-2\sigma_c}\,\mathrm{d}x &= -\int |f|^{2-2\sigma_c}\,\mathrm{Re}(i\partial_x\tilde{\phi}\,\partial_x\bar{\tilde{\phi}})\,\mathrm{d}x - (2-2\sigma_c)\int |f|^{1-2\sigma_c}\,\partial_x|f|\,\mathrm{Re}(i\partial_x\tilde{\phi}\bar{\tilde{\phi}})\\ &= -(2-2\sigma_c)\int |f|^{-2\sigma_c}\,\mathrm{Re}(\bar{f}\,\partial_x f)\,\mathrm{Re}(i\partial_x\tilde{\phi}\bar{\tilde{\phi}}).\end{aligned} \tag{2-43}$$

From the second and third terms on the right-hand side of (2-42), we have

$$\begin{aligned}
&\int [\alpha_1 \operatorname{Re}(if(\partial_x \bar{f}\, \partial_x \tilde{\phi} + \partial_x f\, \partial_x \bar{\tilde{\phi}})\bar{\tilde{\phi}}) + 2\beta_1 \operatorname{Re}(i\bar{f}\, \partial_x f\, \partial_x \tilde{\phi}\bar{\tilde{\phi}})]|f|^{-2\sigma_c}\, \mathrm{d}x \\
&= (\alpha_1 + 2\beta_1) \int |f|^{-2\sigma_c} \operatorname{Re}(\bar{f}\, \partial_x f) \operatorname{Re}(i\partial_x \tilde{\phi}\bar{\tilde{\phi}})\, \mathrm{d}x + \frac{\alpha_1 - 2\beta_1}{2} \int |f|^{-2\sigma_c} \operatorname{Im}(\bar{f}\, \partial_x f)\partial_x |\tilde{\phi}|^2\, \mathrm{d}x \\
&\qquad + \frac{\alpha_1}{2} \int |f|^{-2\sigma_c} \operatorname{Re}(if\, \partial_x f\, \partial_x (\bar{\tilde{\phi}})^2)\, \mathrm{d}x \\
&= (\alpha_1 + 2\beta_1) \int |f|^{-2\sigma_c} \operatorname{Re}(\bar{f}\, \partial_x f) \operatorname{Re}(i\partial_x \tilde{\phi}\bar{\tilde{\phi}})\, \mathrm{d}x - \frac{\alpha_1 - 2\beta_1}{2} \int \big(\partial_x (|f|^{-2\sigma_c} \operatorname{Im}(\bar{f}\, \partial_x f))\big)|\tilde{\phi}|^2\, \mathrm{d}x \\
&\qquad - \frac{\alpha_1}{4} \int (i\partial_x(|f|^{-2\sigma_c} f\, \partial_x f))(\bar{\tilde{\phi}})^2\, \mathrm{d}x + \frac{\alpha_1}{4} \int (i\partial_x(|f|^{-2\sigma_c} \overline{f\, \partial_x f}))(\tilde{\phi})^2\, \mathrm{d}x.
\end{aligned}$$

By our choice of σ_c in (1-6), the first term on the right-hand side cancels exactly with (2-43). The remaining terms are estimated from the above by $C\|f\|^2_{L^\infty_t C^{1,1}} \|\tilde{\phi}\|^2_{L^2_f}$. Next, it is easy to see that

$$\left| \int \operatorname{Re}(iV_f \tilde{\phi}\bar{\tilde{\phi}})|f|^{-2\sigma_c}\, \mathrm{d}x \right| + \left| \int \operatorname{Re}(iW_f \bar{\tilde{\phi}}\bar{\tilde{\phi}})|f|^{-2\sigma_c}\, \mathrm{d}x \right| \lesssim \|f\|^2_{L^\infty_t C^{1,1}} \|\tilde{\phi}\|^2_{L^2_f}.$$

It remains to estimate $\int \operatorname{Re}(iQ_f[\tilde{\phi}]\bar{\tilde{\phi}})|f|^{-2\sigma_c}\, \mathrm{d}x$. The contribution of any term with at least one factor of $\tilde{\phi}$ (without any derivatives) may be easily estimated by $(\|f\|^2_{C^{1,1}} + \|\tilde{\phi}\|^2_{C^{1,1}}) \int |\tilde{\phi}|^2 |f|^{-2\sigma_c}\, \mathrm{d}x$. Recalling the expression for Q_f from (2-15), we may estimate

$$\left| \int \operatorname{Re}(iQ_f[\tilde{\phi}]\bar{\tilde{\phi}})|f|^{-2\sigma_c}\, \mathrm{d}x \right| \lesssim (\|f\|_{C^{1,1}} \|\tilde{\phi}\|_{C^{1,1}} + \|\tilde{\phi}\|^2_{C^{1,1}}) \|\tilde{\phi}\|^2_{L^2_f} + \left(\int |\partial_x \tilde{\phi}|^4 |f|^{2-2\sigma_c} \right)^{1/2} \|\tilde{\phi}\|_{L^2_f}.$$

Integrating by parts and using Hölder's inequality, we have

$$\begin{aligned}
\int (\partial_x \tilde{\phi})^4 |f|^{2-2\sigma_c} &= \int \tilde{\phi}(\partial_x \tilde{\phi})^2 (-3\partial_{xx}\tilde{\phi}|f| - (2-2\sigma_c)\partial_x \tilde{\phi}\partial_x |f|)|f|\,|f|^{-2\sigma_c}\, \mathrm{d}x \\
&\le C\|\tilde{\phi}\|_{C^{1,1}} \|f\|_{C^{0,1}} \|\tilde{\phi}\|_{L^2_f} \left(\int (\partial_x \tilde{\phi})^4 |f|^{2-2\sigma_c}\, \mathrm{d}x \right)^{1/2}.
\end{aligned}$$

Hence

$$\left| \int \operatorname{Re}(iQ_f[\tilde{\phi}]\bar{\tilde{\phi}})|f|^{-2\sigma_c}\, \mathrm{d}x \right| \lesssim (\|f\|_{C^{1,1}} \|\tilde{\phi}\|_{C^{1,1}} + \|\tilde{\phi}\|^2_{C^{1,1}}) \|\tilde{\phi}\|^2_{L^2_f}.$$

Collecting all the terms, we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\tilde{\phi}\|^2_{L^2_f} \lesssim (\|f\|^2_{L^\infty_t C^{1,1}} + \|\tilde{\phi}\|^2_{L^\infty_t C^{1,1}}) \|\tilde{\phi}\|^2_{L^2_f}.$$

Integrating in time gives the desired conclusion.

We now sketch the observations needed to make the above computation rigorous. Note that, in order for (2-41) to be nontrivial, the right-hand side must be finite, i.e., $\|\tilde{\phi}\|_{L^2_f}(t=0) = \||f_0|^{-\sigma_c}\tilde{\phi}_0\|_{L^2} < +\infty$. When $\sigma_c \ge \frac{1}{2}$, this implies the vanishing of $\tilde{\phi}_0$ at each zero a of f (which is isolated by the assumption in this case) up to order $\lfloor \sigma_c - \frac{1}{2} \rfloor$. Applying Lemma 2.5 to the $L^\infty_t([0,\delta]; C^{s_c-1,1}))$ solutions f and $f + \tilde{\phi}$, it follows that the zero set of $f(t,x)$, as well as the nonvanishing of $f'(t,a)$ and the vanishing of $\tilde{\phi}(t,x)$

up to order $\lfloor \sigma_c - \frac{1}{2} \rfloor$ at each zero a of f, is preserved in $t \in [0, \delta]$. As a consequence, $\|\tilde{\phi}\|_{L^2_f} < +\infty$ for every $t \in [0, \delta]$ as well. Using the vanishing properties of f and $\tilde{\phi}$ (the latter is needed only when $\sigma_c \geq \frac{1}{2}$), the above computation can then be justified. □

2.4.2. *Generalized (bilinear) energy estimate.* We proceed to prove the generalized energy estimate.

Proposition 2.9. *Let $\tilde{\phi}$ be a solution of*

$$[i\partial_t + \mathcal{L}_f]\tilde{\phi} = Q_f[\tilde{\phi}],$$

where $[i\partial_t + \mathcal{L}_f]\tilde{\phi}$ denotes the left-hand side of (2-14), *and let $\tilde{\phi}^{\text{app}} = \tilde{\phi}^{\text{app}}[g_0, f]$ be the degenerating wave packet constructed in Proposition 2.7. Then, we have the following estimate on $t \in [0, \min\{|\lambda|^{-1/2}, \delta\}]$:*

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Re}\langle \tilde{\phi}, \tilde{\phi}^{\text{app}} \rangle_{L^2_f} \right| \leq (C(\|f\|^2_{L^\infty_t C^{1,1}_x} + \|\tilde{\phi}\|^2_{L^\infty_t C^{1,1}_x})\|\tilde{\phi}^{\text{app}}\|_{L^2_f} + C_{f,\delta} A^{-\sigma_c+3} \|g_0\|_{H^2_{(x_1)}})\|\tilde{\phi}\|_{L^2_f}. \tag{2-44}$$

Proof. In the proof, the time variable t will be restricted to the interval $[0, \min\{|\lambda|^{-1/2}, \delta\}]$. Before we proceed, let us recall that $\mathcal{L}_f$ is given by

$$\begin{aligned} \mathcal{L}_f[\tilde{\phi}] &:= |f|^2 \partial_{xx}\tilde{\phi} + \alpha_1 f(\overline{\partial_x f}\partial_x\tilde{\phi} + \partial_x f \partial_x \bar{\tilde{\phi}}) + 2\beta_1 \bar{f}\partial_x f \partial_x \tilde{\phi} + V_f \tilde{\phi} + W_f \bar{\tilde{\phi}} \\ &= |f|^2 \partial_{xx}\tilde{\phi} + (\alpha_1 + 2\beta_1)\operatorname{Re}(\bar{f}\partial_x f)\partial_x\tilde{\phi} + (-\alpha_1 + 2\beta_1) i \operatorname{Im}(\bar{f}\partial_x f)\partial_x\tilde{\phi} + \alpha_1 f \partial_x f \partial_x \bar{\tilde{\phi}} + V_f\tilde{\phi} + W_f \bar{\tilde{\phi}} \end{aligned}$$

and that $\tilde{\phi}^{\text{app}}$ satisfies $[i\partial_t + \mathcal{L}_f]\tilde{\phi}^{\text{app}} = \boldsymbol{\epsilon}_{\tilde{\phi}}$. We compute[6]

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Re}\langle \tilde{\phi}, \tilde{\phi}^{\text{app}} \rangle_{L^2_f} = \operatorname{Re}\bigg(\int & -2\sigma_c |f|^{-2\sigma_c - 1} \partial_t |f| \tilde{\phi} \overline{\tilde{\phi}^{\text{app}}} \\ & + \int i|f|^{-2\sigma_c}(\mathcal{L}_f[\tilde{\phi}] - Q_f[\tilde{\phi}])\overline{\tilde{\phi}^{\text{app}}} - \int i|f|^{-2\sigma_c}\tilde{\phi}(\overline{\mathcal{L}_f[\tilde{\phi}^{\text{app}}] - \boldsymbol{\epsilon}_{\tilde{\phi}}}) \bigg). \end{aligned}$$

Using the estimates for $|\partial_t |f||$, $Q_f[\tilde{\phi}]$, and $\boldsymbol{\epsilon}_{\tilde{\phi}}$, we can bound

$$\left| \operatorname{Re} \int |f|^{-2\sigma_c - 1}\partial_t |f| \tilde{\phi}\overline{\tilde{\phi}^{\text{app}}} \right| \lesssim \|f\|^2_{L^\infty_t C^{1,1}} \|\tilde{\phi}\|_{L^2_f} \|\tilde{\phi}^{\text{app}}\|_{L^2_f},$$

$$\left| \operatorname{Re} \int i|f|^{-2\sigma_c} Q_f[\tilde{\phi}]\overline{\tilde{\phi}^{\text{app}}} \right| \lesssim (\|f\|^2_{L^\infty_t C^{1,1}} + \|\tilde{\phi}\|^2_{L^\infty_t C^{1,1}})\|\tilde{\phi}\|_{L^2_f}\|\tilde{\phi}^{\text{app}}\|_{L^2_f}$$

and

$$\left| \operatorname{Re} \int i|f|^{-2\sigma_c}\tilde{\phi}\overline{\boldsymbol{\epsilon}_{\tilde{\phi}}} \right| \leq C_{f,\delta}\|\tilde{\phi}^{\text{app}}\|_{L^2_f} A^{-\sigma_c+3}\|g_0\|_{H^2_{(x_1)}}.$$

We now consider the remaining expression

$$\int |f|^{-2\sigma_c} \operatorname{Re}(i\mathcal{L}_f[\tilde{\phi}]\overline{\tilde{\phi}^{\text{app}}})\,\mathrm{d}x - \int |f|^{-2\sigma_c}\operatorname{Re}(i\tilde{\phi}\overline{\mathcal{L}_f[\tilde{\phi}^{\text{app}}]})\,\mathrm{d}x.$$

[6]Here, since $\tilde{\phi}^{\text{app}}$ is smooth and compactly supported away from the zeroes of f at each t, there are no issues whatsoever in justifying the computation that follows.

For the contribution of the principal term $|f|^2\,\partial_{xx}$, we obtain

$$\begin{aligned}
&\int |f|^{-2\sigma_c}\operatorname{Re}(i|f|^2\,\partial_{xx}\tilde{\phi}\overline{\tilde{\phi}^{\mathrm{app}}})\,\mathrm{d}x-\int |f|^{-2\sigma_c}\operatorname{Re}(i\tilde{\phi}\overline{|f|^2\,\partial_{xx}\tilde{\phi}^{\mathrm{app}}})\,\mathrm{d}x\\
&\quad=-\int |f|^{2-2\sigma_c}\operatorname{Re}(i\,\partial_x\phi\,\overline{\partial_x\tilde{\phi}^{\mathrm{app}}})\,\mathrm{d}x-\int(2-2\sigma_c)|f|^{1-2\sigma_c}\,\partial_x|f|\operatorname{Re}(i\,\partial_x\phi\overline{\tilde{\phi}^{\mathrm{app}}})\,\mathrm{d}x\\
&\qquad\qquad+\int |f|^{2-2\sigma_c}\operatorname{Re}(i\,\partial_x\phi\,\overline{\partial_x\tilde{\phi}^{\mathrm{app}}})\,\mathrm{d}x+\int(2-2\sigma_c)|f|^{1-2\sigma_c}\,\partial_x|f|\operatorname{Re}(i\phi\,\overline{\partial_x\tilde{\phi}^{\mathrm{app}}})\,\mathrm{d}x\\
&\quad=-\int(2-2\sigma_c)|f|^{1-2\sigma_c}\,\partial_x|f|(\operatorname{Re}(i\,\partial_x\phi\overline{\tilde{\phi}^{\mathrm{app}}})-\operatorname{Re}(i\phi\,\overline{\partial_x\tilde{\phi}^{\mathrm{app}}}))\,\mathrm{d}x=:\mathrm{I}.
\end{aligned}$$

Since $|f|\,\partial_x|f|=\operatorname{Re}(\bar f\,\partial_x f)$, this term cancels with some of the first-order terms, i.e.,

$$\begin{aligned}
&\int(\alpha_1+2\beta_1)|f|^{-2\sigma_c}\operatorname{Re}(\bar f\,\partial_x f)\operatorname{Re}(i\,\partial_x\tilde{\phi}\overline{\tilde{\phi}^{\mathrm{app}}})\,\mathrm{d}x\\
&\qquad\qquad-\int(\alpha_1+2\beta_1)|f|^{-2\sigma_c}\operatorname{Re}(\bar f\,\partial_x f)\operatorname{Re}(i\tilde{\phi}\,\overline{\partial_x\tilde{\phi}^{\mathrm{app}}})\,\mathrm{d}x=-\mathrm{I}.
\end{aligned}$$

For the remaining first-order terms, we have, after integrating by parts,

$$\begin{aligned}
&-\int(-\alpha_1+2\beta_1)|f|^{-2\sigma_c}\operatorname{Im}(\bar f\,\partial_x f)\operatorname{Re}(\partial_x\tilde{\phi}\overline{\tilde{\phi}^{\mathrm{app}}})\,\mathrm{d}x-\int(-\alpha_1+2\beta_1)|f|^{-2\sigma_c}\operatorname{Im}(\bar f\,\partial_x f)\operatorname{Re}(\tilde{\phi}\,\overline{\partial_x\tilde{\phi}^{\mathrm{app}}})\,\mathrm{d}x\\
&\qquad\qquad=(-\alpha_1+2\beta_1)\int(\partial_x(|f|^{-2\sigma_c}\operatorname{Im}(\bar f\,\partial_x f)))\operatorname{Re}(\tilde{\phi}\overline{\tilde{\phi}^{\mathrm{app}}})\,\mathrm{d}x,
\end{aligned}$$

$$\begin{aligned}
&\int\alpha_1|f|^{-2\sigma_c}\operatorname{Re}(if\,\partial_x f\,\partial_x\bar{\tilde{\phi}}\overline{\tilde{\phi}^{\mathrm{app}}})\,\mathrm{d}x-\int\alpha_1|f|^{-2\sigma_c}\operatorname{Re}(i\phi\overline{f\,\partial_x f\,\partial_x\bar{\tilde{\phi}}^{\mathrm{app}}})\,\mathrm{d}x\\
&=\frac{\alpha_1}{2}\int|f|^{-2\sigma_c}(if\,\partial_x f\,\partial_x\bar{\tilde{\phi}}\overline{\tilde{\phi}^{\mathrm{app}}}-i\overline{f\,\partial_x f}\,\partial_x\tilde{\phi}\tilde{\phi}^{\mathrm{app}})\,\mathrm{d}x-\frac{\alpha_1}{2}\int|f|^{-2\sigma_c}(i\overline{f\,\partial_x f}\,\phi\,\partial_x\tilde{\phi}^{\mathrm{app}}-if\,\partial_x f\,\bar{\phi}\,\partial_x\overline{\tilde{\phi}^{\mathrm{app}}})\,\mathrm{d}x\\
&=-\frac{\alpha_1}{2}\int(\partial_x(i|f|^{-2\sigma_c}f\,\partial_x f))\bar{\tilde{\phi}}\overline{\tilde{\phi}^{\mathrm{app}}}\,\mathrm{d}x+\frac{\alpha_1}{2}\int(\partial_x(i|f|^{-2\sigma_c}\overline{f\,\partial_x f}))\tilde{\phi}\tilde{\phi}^{\mathrm{app}}\,\mathrm{d}x.
\end{aligned}$$

Both expressions may be bounded from above by $C\|f\|_{C^{1,1}}^2\|\tilde{\phi}\|_{L^2_f}\|\tilde{\phi}^{\mathrm{app}}\|_{L^2_f}$. Finally, for the zeroth-order terms, we easily have

$$\begin{aligned}
&\left|\int|f|^{-2\sigma_c}\operatorname{Re}(i(V_f\tilde{\phi}+W_f\bar{\tilde{\phi}})\overline{\tilde{\phi}^{\mathrm{app}}})\,\mathrm{d}x-\int|f|^{-2\sigma_c}\operatorname{Re}(i\tilde{\phi}\overline{(V_f\tilde{\phi}^{\mathrm{app}}+W_f\overline{\tilde{\phi}^{\mathrm{app}}})})\,\mathrm{d}x\right|\\
&\qquad\qquad\lesssim\|f\|_{C^{1,1}}^2\|\tilde{\phi}\|_{L^2_f}\|\tilde{\phi}^{\mathrm{app}}\|_{L^2_f}.
\end{aligned}$$

This gives (2-44), which concludes the proposition. □

2.5. ***Proof of Theorem 1.1.*** We are now in a position to conclude the proof of Theorem 1.1 for equation (1-5). To begin with, let f satisfy the assumptions of the theorem with $f_0=f(t=0)$. We may assume that $f_0(0)=0$ and $f_0'(0)=:A>0$ by translation and phase rotation if necessary. We also fix x_1 as in Proposition 2.7.

Now let $\epsilon > 0$, $s_0 \geq s_c$ and $0 < \delta' \leq \delta$ be given. We take some $\lambda \leq -1$ and g_0 satisfying the assumptions of Proposition 2.7, and define $\tilde{\phi}_0$ by

$$\tilde{\phi}_0 = \epsilon c(s_0)|\lambda|^{-s_0}\tilde{\phi}^{\mathrm{app}}_{(\lambda)}(t=0)[g_0; f].$$

Here, $\tilde{\phi}^{\mathrm{app}}_{(\lambda)} = \tilde{\phi}^{\mathrm{app}}_{(\lambda)}[g_0; f]$ is the degenerating wave packet constructed in Proposition 2.7 using g_0. It is not difficult to check that $\tilde{\phi}^{\mathrm{app}}_{(\lambda)}(t=0) \in C^\infty_c$ and $\|\tilde{\phi}^{\mathrm{app}}_{(\lambda)}(t=0)\|_{C^{s_0}} \lesssim_{s_0} |\lambda|^{s_0}$. Hence, by taking a sufficiently small $c(s_0) > 0$, we can ensure that $\|\tilde{\phi}_0\|_{C^{s_0}} \leq \epsilon$ uniformly for all $\lambda \leq -1$, as required by the statement of the theorem. We observe that

$$\mathrm{Re}\langle \tilde{\phi}_0, \tilde{\phi}^{\mathrm{app}}_{(\lambda)}(t=0)\rangle_{L^2_f} \geq c_0\|\tilde{\phi}_0\|_{L^2_f}\|\tilde{\phi}^{\mathrm{app}}(t=0)\|_{L^2_f} \tag{2-45}$$

for some $c_0 > 0$ independent of λ. To proceed, let us assume that the first option in the theorem does not hold; namely, there exists a solution ϕ to (1-5) satisfying $\|\phi - f\|_{L^\infty([0,\delta'];C^{s_c})} < +\infty$ and $\phi(t=0) = f_0 + \tilde{\phi}_0$. On $[0, \delta']$, we write $\tilde{\phi} = \phi - f$ and set

$$M_2 = \sup_{t\in[0,\delta']} (\|f(t)\|_{C^{1,1}} + \|\tilde{\phi}(t)\|_{C^{1,1}}).$$

We shall now establish the claimed norm inflation statement for $\tilde{\phi}$ by taking $|\lambda|$ sufficiently large but in a way depending only on f and δ'.

On the time interval $[0, \delta']$, using Proposition 2.8 and (2-18) we obtain that

$$\|\tilde{\phi}(t)\|_{L^2_f} \leq \exp(CM_2^2 t)\|\tilde{\phi}_0\|_{L^2_f}, \quad \|\tilde{\phi}^{\mathrm{app}}(t)\|_{L^2_f} \leq C_{f,\delta}A^{-\sigma_c+1}\|g_0\|_{L^2}.$$

In particular, we note that $\|\tilde{\phi}_0\|_{L^2_f} < +\infty$ since $\tilde{\phi}_0$ is supported away from the zeroes of f, and as discussed in Section 2.4.1, $\tilde{\phi}(t)$ vanishes sufficiently fast at the zeroes of f (ultimately due to Lemma 2.5) so that $\|\tilde{\phi}(t)\|_{L^2_f}$ is well-defined and obeys the above bound. Applying (2-44), integrating in time on the interval $[0, \min\{\delta', cM_2^{-2}, A^{-2}|\lambda|^{-1/2}\}]$ for a sufficiently small $c > 0$ and using (2-45), we have

$$\mathrm{Re}\langle \tilde{\phi}(t), \tilde{\phi}^{\mathrm{app}}(t)\rangle_{L^2_f} \geq \tfrac{1}{2}c_0\|\tilde{\phi}_0\|_{L^2_f}\|g_0\|_{L^2} \quad \text{for } |t| \leq \min\{\delta', cM_2^{-2}, A^{-2}|\lambda|^{-1/2}\}. \tag{2-46}$$

Next, applying (2-19)–(2-21) with $\gamma' = -\sigma_c$ and $s = s_c$, we have

$$\begin{aligned}
&\mathrm{Re}\langle \tilde{\phi}(t), \tilde{\phi}^{\mathrm{app}}(t)\rangle_{L^2_f} - C_{f,\delta}A^{-\sigma_c+1}|\lambda|^{-1}\|\tilde{\phi}_0\|_{L^2_f}\|g_0\|_{H^{|s_c|}_{(x_1)}} \\
&\qquad \leq \mathrm{Re}\left\langle \tilde{\phi}(t), \partial_x^{s_c}\left(\frac{|f|^{-\sigma_c+s_c-1/2}}{i^{s_c}\lambda^{s_c}(1+|f|\partial_x S)^{s_c}}\psi^{\mathrm{app}}(t)\right)\right\rangle \\
&\qquad \leq \|\partial_x^{s_c}\tilde{\phi}(t)\|_{L^\infty}|\lambda|^{-s_c}\|(1+|f|\partial_x S)^{-s_c}|f|^{s_c-\sigma_c-1/2}\psi^{\mathrm{app}}(t)\|_{L^1} \\
&\qquad \leq C_{f,\delta}^{1-\sigma_c}A^{s_c-\sigma_c+1/2}|\lambda|^{-s_c}\exp\bigl(-2|\lambda|\bigl(s_c-\sigma_c+\tfrac{1}{2}\bigr)A^2t\bigr)\|\partial_x^{s_c}\tilde{\phi}(t)\|_{L^\infty}\|g_0\|_{L^2}.
\end{aligned}$$

Taking $|\lambda|$ sufficiently large, we may ensure that

$$C_{f,\delta}A^{-\sigma_c+1}|\lambda|^{-1}\|g_0\|_{H^{|s_c|}_{(x_1)}} \leq \tfrac{1}{4}c_0\|g_0\|_{L^2} \quad \text{and} \quad A^{-2}|\lambda|^{-1/2} < \delta', \tag{2-47}$$

which gives, after combining the previous two inequalities with (2-46),

$$\tfrac{1}{4}c_0C_{f,\delta}^{-\sigma_c+1}A^{-(s_c-\sigma_c+1/2)}|\lambda|^{s_c}\exp\bigl(2|\lambda|\bigl(s_c-\sigma_c+\tfrac{1}{2}\bigr)A^2t\bigr)\|\tilde{\phi}_0\|_{L^2_f} \leq \|\partial_x^{s_c}\tilde{\phi}(t)\|_{L^\infty}$$

for $|t| \le \min\{cM_2^{-2}, A^{-2}|\lambda|^{-1/2}\}$. For each $|\lambda|$ satisfying (2-47) there are two cases; either (i) $cM_2^{-2} < A^{-2}|\lambda|^{-1/2}$ or (ii) $cM_2^{-2} \ge A^{-2}|\lambda|^{-1/2}$. In the case (i), we obtain that $M_2 \gtrsim_A |\lambda|^{1/4} \gtrsim_A (\delta')^{-1/2}$ using (2-47). Here, we could have assumed that $|\lambda|$ is sufficiently large from the beginning so that $\sup_{t\in[0,\delta']}\|f(t)\|_{C^{1,1}} \ll_A |\lambda|^{1/4}$. Then, $M_2 \simeq \sup_{t\in[0,\delta']}\|\tilde{\phi}(t)\|_{C^{1,1}}$ and the desired norm inflation follows simply from our assumption in (1-7) that $s_c \ge 2$. In the case (ii), we simply take $t = A^{-2}|\lambda|^{-1/2}$ in (2-46), which gives the claimed norm inflation (actually, in this case we obtain a much stronger growth in terms of $1/\delta'$) using that $s_c > \sigma_c - \frac{1}{2}$. This finishes the proof of Theorem 1.1. □

Remark 2.10. At the end of the above proof, observe that we could have followed the same argument but have used (2-19)–(2-21) with $\gamma' = -\sigma_c$, $s = \sigma$ and $p = 2$ to derive

$$\tfrac{1}{4}c_0 C_{f,\delta}^{-\sigma_c+1} A^{-(\sigma-\sigma_c)}|\lambda|^{s_c}\exp(2|\lambda|(\sigma-\sigma_c)A^2 t)\|\tilde{\phi}_0\|_{L^2_f} \le \|\tilde{\phi}(t)\|_{H^\sigma}$$

for $t \le \min\{cM_2^{-2}, A^{-2}|\lambda|^{-1/2}\}$. This can be used to prove the inflation of the H^σ norm for any $\sigma > \sigma_c$ in the second alternative of Theorem 1.1.

2.6. *Proof of Theorem 1.2.* Let us divide the proof of Theorem 1.2 into several steps.

Choice of background solution. Towards a contradiction, we shall assume that there exist $\epsilon > 0$ and $s_0 \ge s_c + 2$ such that, for any $\phi_0 \in C^\infty(\mathbb{T})$ satisfying $\|\phi_0\|_{C^{s_0}} < \epsilon$, there exist $\delta = \delta(\phi_0) > 0$ and a solution $\phi \in L^\infty([0,\delta]; C^{s_c+1,1})$ to (1-5) with initial data $\phi(t=0) = \phi_0$.

Under this assumption, let us fix a function $\mathring{f}_0 \in C^\infty(\mathbb{T})$ which is supported in $\left(-\frac{1}{2}, \frac{1}{2}\right)$ and $\mathring{f}_0(x) = x$ in $\left[-\frac{1}{4}, \frac{1}{4}\right]$. We then set

$$f_0 := \sum_{k=k_0}^{\infty} f_{k,0} := \sum_{k=k_0}^{\infty} A_k 2^{-k}\mathring{f}_0(2^k(x-x_k)), \quad A_k = 2^{-k^2}, \ \ x_k = 2^{-k/2},$$

where $k_0 = k_0(s_0, \epsilon, \mathring{f}_0) \ge 1$ is taken sufficiently large to achieve $\|f_0\|_{C^{s_0}} < \frac{1}{2}\epsilon$. It is not difficult to see that $f_0 \in C^\infty(\mathbb{T})$. Furthermore, since the supports of $f_{k,0}$ are disjoint from each other, for each $k \ge k_0$, we may choose $\chi_k \in C^\infty(\mathbb{T})$ to be a cutoff function satisfying $\chi_k = 1$ on $\mathrm{supp}(f_{k,0})$ and $\chi_k = 0$ on $\mathrm{supp}(f_{k',0})$ for any $k' \ne k$. From the contradiction hypothesis, we have a solution $f(t) \in L^\infty([0,\delta]; C^{s_c+1,1})$ to (1-5) with initial data f_0 for some $\delta > 0$. The estimate $|\partial_t f| \lesssim |f|$ from (2-12) shows that $\mathrm{supp}(f(t)) = \mathrm{supp}(f_0)$ on $[0,\delta]$, and since χ_k equals either 0 or 1 on $\mathrm{supp}(f(t)) = \mathrm{supp}(f_0)$, we have that

$$\chi_k = \chi_k^3, \quad \partial_x \chi_k = 0$$

on $\mathrm{supp}(f(t))$ for any $k \ge k_0$. Using these observations, it follows, for each $k \ge k_0$, that $f_k := \chi_k f$ is again a solution to (1-5) with initial data $f_k(t=0) = f_{k,0}$. Furthermore, the $L^\infty([0,\delta]; C^{s_c+1,1})$ norm of f_k is bounded uniformly in k.

In the following, as in the above, we use the notation

$$\langle a, b\rangle_f(t) := \int_{\mathbb{T}} |f(t,x)|^{-2\sigma_c} a(t,x)\overline{b(t,x)}\,\mathrm{d}x, \quad \|a\|^2_{L^2_f} := \langle a, a\rangle_f.$$

Let us also use the shorthand $\|a_0\|_{L^2_{f_0}} = \|a\|_{L^2_f}(t=0)$.

Choice of wave packet solutions. We now fix some nonzero function $g_0 \in C^\infty$ supported in $\left(\frac{1}{8}, \frac{1}{4}\right)$ and take $g_k(x) := 2^{k/2} g_0(2^k(x - x_k))$. For some sequence $\{\lambda_k\}_{k \geq k_0}$ to be determined (for now, we take $-\lambda_k \geq A_k^{10}$), we consider the sequence of wave packet solutions

$$\tilde{\phi}_k^{\text{app}} := \tilde{\phi}_{(\lambda_k)}^{\text{app}}[g_k;\, f_k],$$

where $\tilde{\phi}_{(\lambda_k)}^{\text{app}}[g_k;\, f_k]$ is the wave packet solution from Proposition 2.7 with data g_k, λ_k, adapted to the linearly degenerate solution f_k, with $A = A_k$ and $x_1 = 2^{-k-2}$. We define the corresponding error by

$$[i\partial_t + \mathcal{L}_{f_k}]\tilde{\phi}_k^{\text{app}} = \boldsymbol{\epsilon}_k, \tag{2-48}$$

where the operator $[i\partial_t + \mathcal{L}_{f_k}]$ is obtained from (2-14) by replacing f with f_k. Applying Proposition 2.7, we obtain the following bounds: with $\delta_k := \min\{\delta, A_k^{-2}|\lambda_k|^{-1/2}\} = A_k^{-2}|\lambda_k|^{-1/2}$ (by our choice of $-\lambda_k$ in the above),

- $\|\tilde{\phi}_k^{\text{app}}\|_{L^\infty([0,\delta_k];L^2_f)} \lesssim C_{f_k,\delta_k}\|\tilde{\phi}_k^{\text{app}}(t=0)\|_{L^2_{f_0}} \lesssim C_{f_k,\delta_k} A_k^{-\sigma_c+1}\|g_0\|_{L^2}$;
- $\|\boldsymbol{\epsilon}_k\|_{L^\infty([0,\delta_k],L^2_f)} \lesssim C_{f_k,\delta_k} A_k^{-\sigma_c+3}\|g_k\|_{H^2_{(2^{-k-2})}} \lesssim C_{f_k,\delta_k} A_k^{-\sigma_c+3}\|g_0\|_{H^2}$;

and

$$|f|^{-2\sigma_c}\tilde{\phi}_k^{\text{app}} = \partial_x^{s_c}\left(\frac{|f|^{-\sigma_c+s_c-1/2}}{(i\lambda_k)^{s_c}(1+|f|\,\partial_x S)^{s_c}}\psi_k^{\text{app}}\right) + |f|^{-2\sigma_c}\tilde{\phi}_k^{\text{small}}$$

with

$$\left\|\frac{|f|^{-\sigma_c+s_c-1/2}}{(i\lambda_k)^{s_c}(1+|f|\,\partial_x S)^{s_c}}\psi_k^{\text{app}}\right\|_{L^1} \lesssim C_{f_k,\delta_k}^{1-\sigma_c} A_k^{-\sigma_c+s_c+1/2}|\lambda_k|^{-s_c}\exp\bigl(-2|\lambda_k|\bigl(-\sigma_c+s_c+\tfrac{1}{2}\bigr)A_k t\bigr)\|g_0\|_{L^2}$$

for $0 \leq t \leq \delta_k$ and

$$\begin{aligned}\|\tilde{\phi}_k^{\text{small}}\|_{L^\infty([0,\delta_k];L^2_f)} &\lesssim C_{f_k,\delta_k} A_k^{-\sigma_c+1}|\lambda_k|^{-1}\|g_k\|_{H^{s_c}_{(2^{-k-2})}} \\ &\lesssim C_{f_k,\delta_k} A_k^{-\sigma_c+1}|\lambda_k|^{-1}\|g_0\|_{H^{s_c}}.\end{aligned}$$

From (2-23) we see that

$$C_{f_k,\delta_k} \lesssim (1 + A_k^{-1}M)^{N_0}\exp(C_0 M^2\delta_k), \quad M = \sup_{t\in[0,\delta]}\|f(t,\cdot)\|_{C^{s_c+1,1}},$$

where the implicit constant and N_0 depends on $g_0, \alpha_1, \beta_1, \mu_1, s_c$, but not on k and λ_k. Then, simply using $\delta_k \leq \delta$ and recalling $A_k = 2^{-k^2}$, we see that $C_{f_k,\delta_k} \lesssim 2^{N_0 k^2}$ holds, where the implicit constant depends further on M and δ but not on k and λ_k. In turn, this gives an upper bound on the constants in the estimates above; for instance

$$C_{f_k,\delta_k}^{1-\sigma_c} A_k^{-\sigma_c+s_c+1/2} \lesssim 2^{N_1 k^2}$$

with some $N_1 > 0$ depending additionally on σ_c and s_c. *In the following, we shall write $\lesssim$ as long as the implicit constant does not depend on k and λ_k.*

Choice of initial data. We now take

$$\tilde{\phi}_0(x) = \sum_{k=k_0}^{\infty} \tilde{\phi}_{k,0}(x) := \sum_{k=k_0}^{\infty} \exp(-|\lambda_k|^{1/4})\tilde{\phi}_k^{\mathrm{app}}(t=0,x), \tag{2-49}$$

which belongs to $C^\infty(\mathbb{T})$. By taking k_0 even larger if necessary, we can guarantee that $\|\tilde{\phi}_0\|_{C^m} < \frac{1}{2}\epsilon$. Then we set $\phi_0 = f_0 + \tilde{\phi}_0$, which satisfies $\|\phi_0\|_{C^m} < \epsilon$. Again from the contradiction hypothesis, we have a $L_t^\infty C^{s_c+1,1}$ solution $\phi(t)$ to (1-5) with initial data ϕ_0 on some time interval $[0, \delta']$. We may assume that $0 < \delta' \le \delta$ and define

$$\tilde{\phi}(t) := \phi(t) - f(t), \quad \tilde{\phi}_k(t) := \chi_k \tilde{\phi}(t)$$

for all $k \ge k_0$. We have that $\sum_{k=k_0}^{\infty} \tilde{\phi}_k = \tilde{\phi}$; this follows from $\partial_t|f+\tilde{\phi}| \lesssim |f+\tilde{\phi}|$ and the uniform pointwise estimate $|f+\tilde{\phi}|(t,x) \lesssim |f_0+\tilde{\phi}_0|(x) \lesssim |f_0|(x)$. Then we see that $\tilde{\phi}_k$ solves

$$[i\partial_t + \mathcal{L}_{f_k}]\tilde{\phi}_k = Q_{f_k}[\tilde{\phi}_k],$$

(which is (2-40) with f and $\tilde{\phi}$ replaced with f_k and $\tilde{\phi}_k$, respectively). We note that the $L^\infty([0,\delta']; C^{s_0+1,1})$ norm is uniformly bounded for $\{f_k\}_{k\ge k_0}$ and $\{\tilde{\phi}_k\}_{k\ge k_0}$. Therefore, from Proposition 2.8, we obtain the estimate

$$\|\tilde{\phi}_k\|_{L^\infty([0,\delta'];L^2_f)} \lesssim \|\tilde{\phi}_{k,0}\|_{L^2_{f_0}} \tag{2-50}$$

uniformly in $k \ge k_0$. Now, combining this with the generalized energy estimate (2-44) for $\tilde{\phi}_k$ and $\tilde{\phi}_k^{\mathrm{app}}$, we obtain that

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Re}\langle \tilde{\phi}_k, \tilde{\phi}_k^{\mathrm{app}} \rangle_f \right| \lesssim 2^{N_1 k^2} \|\tilde{\phi}_{k,0}\|_{L^2_{f_0}} \|g_0\|_{H^2} \tag{2-51}$$

for $t \in [0, \delta_k]$. We shall now take $|\lambda_k|$ larger so that $\delta_k = A_k^{-2}|\lambda_k|^{-1/2}$ satisfies $2^{N_1 k^2}\delta_k$ is very small with respect to the implicit constants in (2-50) and (2-51). Then, since at $t=0$ we have

$$\operatorname{Re}\langle \tilde{\phi}_k, \tilde{\phi}_k^{\mathrm{app}} \rangle_f (t=0) \ge \tfrac{1}{4} \|\tilde{\phi}_{k,0}\|_{L^2_{f_0}} \|\tilde{\phi}_k^{\mathrm{app}}(t=0)\|_{L^2_{f_0}},$$

by integrating (2-51) in time from $t=0$ to δ_k, we obtain

$$\operatorname{Re}\langle \tilde{\phi}_k, \tilde{\phi}_k^{\mathrm{app}} \rangle_f (\delta_k) \ge \tfrac{1}{8} \|\tilde{\phi}_{k,0}\|_{L^2_{f_0}} \|\tilde{\phi}_{k,0}^{\mathrm{app}}\|_{L^2_{f_0}}. \tag{2-52}$$

At $t = \delta_k$ we write

$$\operatorname{Re}\langle \tilde{\phi}_k, \tilde{\phi}_k^{\mathrm{app}} \rangle_f (\delta_k) = (-1)^{s_c} \operatorname{Re}\left\langle \partial_x^{s_c}\tilde{\phi}_k, \frac{|f|^{-\sigma_c+s_c-1/2}}{(i\lambda_k)^{s_c}(1+|f|\partial_x S)^{s_c}} \psi_k^{\mathrm{app}} \right\rangle + \operatorname{Re}\langle \tilde{\phi}_k, \tilde{\phi}_k^{\mathrm{small}} \rangle_f,$$

and then combining the estimates of the right-hand side with (2-52), we get

$$\|\tilde{\phi}_{k,0}\|_{L^2_{f_0}} \|\tilde{\phi}_{k,0}^{\mathrm{app}}\|_{L^2_{f_0}} \lesssim 2^{N_1 k^2} \big(\|\partial_x^{s_c}\tilde{\phi}_k\|_{L^\infty} |\lambda_k|^{-s_c} \exp\big(-2|\lambda_k|^{1/2}\big(-\sigma_c + s_c + \tfrac{1}{2}\big)\big) \|g_0\|_{L^2} + |\lambda_k|^{-1} \|g_0\|_{H^{s_c}} \big).$$

By taking $|\lambda_k|$ even larger if necessary, we can guarantee that $|\lambda_k|^{-1} 2^{N_1 k^2} \|g_0\|_{H^{s_c}} \ll \|\tilde{\phi}_{k,0}^{\mathrm{app}}\|_{L^2_{f_0}}$ holds ($\ll$ is defined in terms of the implicit constant in the previous inequality), so that we deduce

$$\|\tilde{\phi}_{k,0}\|_{L^2_{f_0}} \|\tilde{\phi}_{k,0}^{\mathrm{app}}\|_{L^2_{f_0}} \lesssim 2^{N_1 k^2} \|\partial_x^{s_c}\tilde{\phi}_k\|_{L^\infty} |\lambda_k|^{-s_c} \exp\big(-2|\lambda_k|^{1/2}\big(-\sigma_c + s_c + \tfrac{1}{2}\big)\big) \|g_0\|_{L^2},$$

and then recalling the form of $\tilde{\phi}_{k,0}$ from (2-49),

$$\|\partial_x^{s_c}\tilde{\phi}_k(t=\delta_k)\|_{L^\infty} \gtrsim 2^{-N_1 k^2}|\lambda_k|^{s_c}\exp\bigl(2|\lambda_k|^{1/2}\bigl(-\sigma_c+s_c+\tfrac{1}{2}\bigr)-|\lambda_k|^{1/4}\bigr).$$

Note that the left-hand side is bounded by

$$\|f\|_{L^\infty([0,\delta'];C^{s_c+1,1})}+\|\phi\|_{L^\infty([0,\delta'];C^{s_c+1,1})}$$

for all k sufficiently large. This is a contradiction since the right-hand side diverges as $k\to\infty$. The proof is now complete. □

3. KdV-type equations

This section is organized as follows. After setting up some pieces of notation in Section 3.1, we study the properties of regular cubically degenerate solutions — typically denoted by f — in Section 3.2. Then in Section 3.3, we carry out the key construction of degenerating wave packets for the linearized equation around f, and in Section 3.4, we establish a modified energy estimate for the perturbation (solving the nonlinear difference equation) around f. Finally, in Sections 3.5 and 3.6, we prove Theorems 1.5 and 1.6, respectively.

3.1. ***Preliminaries.*** We introduce the following quantity defined for a $C^{2,1}$ function f on an interval I:

$$\|f\|_{Y(I)}=\|f^{-2/3}\,\partial_x f\|^3_{L^\infty(I)}+\|f^{-1/3}\,\partial_{xx}f\|^{3/2}_{L^\infty(I)}+\|f\|_{L^\infty(I)}+\|\partial_{xxx}f\|_{L^\infty(I)}.$$

We shall write $f\in Y(I)$ if $\|f\|_{Y(I)}$ is finite. This quantity is appropriate to handle solutions with degeneracies of order at least 3. For convenience we set

$$\|f\|_{\widetilde{C}^{k,\alpha}(I)}=\|f\|_{C^{k,\alpha}(I)}+\|f\|_{Y(I)}.$$

For f depending on time, we say $f\in L^\infty([0,\delta];\widetilde{C}^{k,\alpha}(I))$ (resp. $f\in L^\infty([0,\delta];Y(I))$ if

$$\|f\|_{L^\infty([0,\delta];\widetilde{C}^{k,\alpha}(I))}:=\sup_{t\in[0,\delta]}\|f(t)\|_{\widetilde{C}^{k,\alpha}(I)}<+\infty\quad\bigl(\text{resp. }\|f\|_{L^\infty([0,\delta];Y(I))}:=\sup_{t\in[0,\delta]}\|f(t)\|_{Y(I)}<+\infty\bigr).$$

It is easy to see using the Taylor expansion that any $C^{3,\alpha}$ function which vanishes cubically at its zeroes must belong to Y. However, *propagation* of Y-boundedness for (1-9) in general requires higher regularity, e.g., $C^{4,1}$ (see Proposition 3.2).

For later use, we introduce the notation

$$\langle a,b\rangle_f(t):=\int_I f(t,x)^{-2\sigma_c/3}a(t,x)b(t,x)\,\mathrm{d}x,\qquad \|a\|^2_{L^2_f}(t):=\langle a,a\rangle_f(t). \tag{3-1}$$

For the motivation behind the power $f^{-2\sigma_c/3}$, see Section 3.4.

3.2. ***Properties of a regular cubically degenerate solution.*** We first discuss a few basic properties of a regular cubically degenerate solution f to (1-9), which shall serve as the background for our ill-posedness mechanism.

Under the assumption $f\in L^\infty_t\widetilde{C}^{3,\alpha}(I)$ with any $\alpha>0$, we can propagate the information that f vanishes cubically on an endpoint of I and compute the coefficient.

Lemma 3.1. *Let $f \in L^\infty([0,\delta]; \widetilde{C}^{3,\alpha}(I))$ be a solution of* (1-9) *with initial data f_0 that is positive on $I \setminus \partial I$ and vanishes to order at least* 3 *on each point in ∂I, where $0 < \alpha \le 1$. Then the following statements hold*:

(1) *The zeroes and the sign of $f(t,x)$ are preserved in time, i.e., $f(t,x)$ vanishes on ∂I and $f(t,x) > 0$ for $x \in I \setminus \partial I$ for all $t \in [0,\delta]$.*

(2) *Let $I = [a,b]$. Then, the set of t-dependent functions $\{\partial_x^k f(t,a)\}_{k=0}^3$ for $t \in [0,\delta]$ is determined by the initial data at $x = a$, i.e., $\{\partial_x^k f(0,a)\}_{k=0}^3$. In particular,*

$$f(t,x) = (\beta(t)(x-a))^3 + O(\|f\|_{L_t^\infty([0,\delta];C^{3,\alpha}(I))}|x-a|^{3+\alpha}), \quad x \to a^+, \tag{3-2}$$

where $\beta(t)$ is the solution of

$$\dot\beta(t) = -(2+6\alpha_1)\beta^4(t), \quad 6\beta^3(0) = f_{0,xxx}(a), \tag{3-3}$$

and the implicit constant in $O(\cdot)$ is universal. The same statement applies to $b \in \partial I$.

Proof. Since we are assuming that $f(t,\cdot) \in C^3$, from (1-9), we have

$$|\partial_t f| \le C(|f_{xxx}||f| + |f_x||f_{xx}| + |f_x||f|^{m-1}).$$

From the assumption that $f(t,\cdot) \in Y$, we have the pointwise estimate

$$|\partial_t f| \le C(\|f\|_Y + \|f\|_{C^3}(1 + \|f\|_{L^\infty}^{m-2}))|f|.$$

This shows that

$$f_0(x)\exp(-Ct\|f\|_{L_t^\infty \tilde{C}^{3,\alpha}}(1+\|f\|_{L^\infty}^{m-2})) \le f(t,x) \le f_0(x)\exp(Ct\|f\|_{L_t^\infty \tilde{C}^{3,\alpha}}(1+\|f\|_{L^\infty}^{m-2})) \tag{3-4}$$

for any $x \in I$, which proves the first statement. The second statement follows from simply evaluating equation (1-9) at $x = a, b$ and carrying out a minor modification of the proof of Lemma 2.5. Here, the fact that f vanishes at least cubically at $x = a$ ensures that no $\partial_x^k f(t,a)$ with $k > 3$ occurs in the ODEs for the Taylor coefficients. We omit the details. □

Before we proceed further, let us note that the assumptions of Theorem 1.5 on the solution f are automatically satisfied for any sufficiently smooth solutions of (1-9).

Proposition 3.2. *Consider an interval $I = [a,b] \subseteq \mathbb{T}$. Let $f_0 \in C^{4,1}(\mathbb{T})$ satisfy $f_0 > 0$ on $I \setminus \{a\}$ (resp. $I \setminus \{b\}$) and vanishes at least cubically at a (resp. b), so that $f_0 \in Y(I)$. Then there exists $\delta > 0$ depending on $\|f_0\|_{Y(I)}$ such that, if f is a solution to* (1-9) *with initial data f_0 satisfying $f \in L^\infty([0,\delta]; C^{4,1}(\mathbb{T}))$, then f satisfies $f|_I \in L^\infty([0,\delta]; Y(I))$ with the bound $\|f\|_{L^\infty([0,\delta];Y(I))} \le CC_0\exp(CM\delta)$.*

Furthermore, for this value of δ, let u be another solution to (1-9) *belonging to $L^\infty([0,\delta]; C^{4,1}(\mathbb{T}))$ with initial data u_0 satisfying $u_0 \in Y(I)$ and*

$$|u_0(x)| \le C_1 f_0(x) \tag{3-5}$$

for some $C_1 > 0$ uniformly for $x \in I$. Then, for some $0 < \delta' \le \delta$ depending only on $\|u_0\|_Y$ and $\|f_0\|_Y$,

$$|u(t,x)| \le C_1(1 + CC_0 t)\exp(CMt) f(t,x), \quad t \in [0,\delta'], \tag{3-6}$$

uniformly for $x \in I$, where $C_0 = C_0(\|f_0\|_Y, \|u_0\|_Y)$ and $M = M(\|f\|_{L_t^\infty C^{4,1}}, \|u\|_{L_t^\infty C^{4,1}})$.

Proof. Without loss of generality, we consider the case $f_0 > 0$ on $I \setminus \{a\}$ with f_0 vanishing at least cubically at a. We compute that

$$\begin{aligned} \partial_t f &= -\mu_1 f^{m-1} f_x - \alpha_1 f_x f_{xx} - f f_{xxx}, \\ \partial_t f_x &= -\mu_1 (f^{m-1} f_x)_x - \alpha_1 (f_{xx})^2 - (\alpha_1 + 1) f_x f_{xxx} - f f_{xxxx}, \\ \partial_t f_{xx} &= -\mu_1 (f^{m-1} f_x)_{xx} - (3\alpha_1 + 1) f_{xx} f_{xxx} - (\alpha_1 + 2) f_x f_{xxxx} - f f_{xxxxx}. \end{aligned}$$

Upon $f \in L_t^\infty C^{4,1}$, we have the pointwise estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}(|f|^2 + |f_x|^3 + |f_{xx}|^6) \le CM(|f|^2 + |f_x|^3 + |f_{xx}|^6),$$

where we introduce the shorthand

$$M = 1 + \|f\|^4_{L_t^\infty C^{4,1}} + \|f\|^{6m-2}_{L_t^\infty C^{2,1}}$$

for simplicity. By Gronwall's inequality, we have the pointwise estimate

$$(|f|^2 + |f_x|^3 + |f_{xx}|^6)(t, x) \le \exp(CMt)(|f_0|^2 + |f_{0,x}|^3 + |f_{0,xx}|^6)(x). \tag{3-7}$$

From the assumptions on the initial data, we have that

$$|f_{0,x}(x)|^3 + |f_{0,xx}(x)|^6 \le C_0^2 (f_0(x))^2 \tag{3-8}$$

holds pointwise on I, with some $C_0 > 0$ depending only on $\|f_0\|_Y$. Returning to (3-7) and applying Young's inequality, we deduce the pointwise bound

$$|f_x(t, x) f_{xx}(t, x)| \le CC_0 \exp(CMt) f_0(x)$$

for all $x \in I$. In turn, using this bound in the equation for $\partial_t f$, we obtain for all $x \in I$ that

$$|\partial_t f(t, x)| \le CM|f(t, x)| + CC_0 \exp(CMt) f_0(x). \tag{3-9}$$

Dividing by f_0 and applying Gronwall's inequality to $f/f_0 - 1$, we obtain

$$\left| \frac{f}{f_0} - 1 \right| \le CC_0 t \exp(CMt)$$

which, after some simplification, implies

$$f(t, x) \le (1 + CC_0 t) \exp(CMt) f_0(x) \tag{3-10}$$

as well as

$$f(t, x) \ge (1 - CC_0 t) \exp(-CMt) f_0(x). \tag{3-11}$$

This guarantees that $f \in L_t^\infty([0, \delta]; Y)$, provided that δ is sufficiently small depending on $C_0 = C_0(\|f_0\|_Y)$.

For the second statement, we note that the assumption (3-5) implies

$$|u_0(x)|^2 + |u_{0,x}(x)|^3 + |u_{0,xx}(x)|^6 \le C_1^2(1 + C\|u_0\|_Y)|f_0(x)|^2$$

for some absolute constant $C > 0$. With this bound, we may apply the above argument to u instead of f and obtain the bound

$$|u(t,x)| \le (1 + CC_0 t)\exp(CMt)|u_0(x)| \le C_1(1 + C_0 t)\exp(CMt) f_0(x), \quad t \in [0, \delta'],$$

for some $0 < \delta' \le \delta$ depending only on $C_0 = C_0(\|u_0\|_Y)$. Here, $M = 1 + \|u\|^4_{L^\infty_t C^{4,1}} + \|u\|^{6m-2}_{L^\infty_t C^{2,1}}$. Using this bound together with (3-11), we obtain the desired estimate (3-6), by taking δ' smaller in a way depending only on $\|u_0\|_Y$ and $\|f_0\|_Y$ if necessary. This finishes the proof. □

3.3. ***Degenerating wave packets for the linearized equation.*** In this subsection, our goal is to construct degenerating wave packets for the linearization of (1-9) around a (possibly hypothetical) regular cubic degenerate solution; see Proposition 3.3.

3.3.1. *Linearized equation and degenerating wave packets.* In the following, we fix some function f that satisfies all the assumptions from Theorem 1.5 and further assume for simplicity that the interval is given by $I = [0, b]$ for some $b > 0$. We fix some $0 < x_1 < b$ such that

$$\tfrac{1}{2} f_{0,xxx}(x) < f_{0,xxx}(0) < 2 f_{0,xxx}(x) \quad \text{for all } x \in [0, x_1]. \tag{3-12}$$

We now write $u = f + \phi$, where u is a solution to (1-9). Then, we have that ϕ must solve

$$\partial_t \phi + \mathcal{L}_f \phi = Q[\phi], \tag{3-13}$$

with

$$\mathcal{L}_f \phi = f\phi_{xxx} + \alpha_1 f_x \phi_{xx} + (\alpha_1 f_{xx} + \mu_1 f^{m-1})\phi_x + (f_{xxx} + (m-1)\mu_1 f^{m-2} f_x)\phi, \tag{3-14}$$

$$Q[\phi] = -\phi\phi_{xxx} - \alpha_1 \phi_x \phi_{xx} - \frac{\mu_1}{m}((f+\phi)^m - f^m - m f^{m-1}\phi)_x. \tag{3-15}$$

We are concerned with constructing wave packets to the linearized equation

$$\partial_t \phi + \mathcal{L}_f \phi = 0. \tag{3-16}$$

Recall the notation $\|g\|_{W^{s,p}_{(L)}} = \sum_{j=0}^{s} \|(L\partial_x)^j g\|_{L^p(\mathrm{d}x)}$ and $H^s_{(L)} = W^{s,2}_{(L)}$ from the previous section. Our aim is to prove the following result.

Proposition 3.3. *Let $f \in L^\infty_t([0,\delta]; \widetilde{C}^{s-1,1}(I))$ be a solution to* (1-9) *with initial data f_0 satisfying $f_0 > 0$ on $I \setminus \{0\}$ and vanishing cubically at* 0 *and $f_0 \in \widetilde{C}^{s_0-1,1}(I)$, where $4 \le s \le s_0$. Let $A = \frac{1}{6} f_{0,xxx}(0)$ and fix $0 < x_1 < 1$ so that* (3-12) *holds. Then, given $\lambda \in \mathbb{N}$ and $g_0 \in C^\infty_c$ supported in $\left(\frac{1}{2}x_1, x_1\right)$, we may associate a function $\phi^{\mathrm{app}}_{(\lambda)}[g_0, f]$ defined in $[0,\delta] \times I$ satisfying the following properties*:

- <u>*Linearity*</u>: *the map $g_0 \mapsto \phi^{\mathrm{app}}_{(\lambda)}[g_0, f]$ is linear*;
- <u>*Support property*</u>: $\mathrm{supp}(\phi^{\mathrm{app}}_{(\lambda)}[g_0, f](t,\cdot)) \subset (0, x_1) \cap \left(0, C_{\tilde{f}} x_1 \exp(-3\beta(t) A^{2/3} \lambda^2 t)\right)$;
- <u>*Initial data estimates*</u>: *for $0 \le n \le s_0$ and $1 \le p \le \infty$, we have*

$$\frac{1}{C_{\tilde{f}_0}}(\|g_0\|_{L^2} - \lambda^{-1}\|g_0\|_{H^1_{(x_1)}}) \le \|f_0^{-\sigma_c/3}(x)\phi^{\mathrm{app}}_{\lambda}(0,x)\|_{L^2} \le C_{\tilde{f}_0}\|g_0\|_{L^2}, \tag{3-17}$$

$$\|f_0^{-\sigma_c/3}(x)(A^{-1/3} f_0^{1/3}(x)\partial_x)^n \phi^{\mathrm{app}}_{(\lambda)}(0,x)\|_{L^p} \le C_{\tilde{f}_0} x_1^{1/p-1/2} |\lambda|^n \|g_0\|_{W^{n,p}_{(x_1)}}; \tag{3-18}$$

- <u>Regularity</u>: *for* $t \in [0, \delta]$ *and* $0 \le n \le s$,

$$\| f^{-\sigma_c/3} (A^{-1/3} f^{1/3}(t,x)\partial_x)^n \phi^{\mathrm{app}}_{(\lambda)}(t,x)\|_{L^2} \le C_{\tilde f} |\lambda|^n \|g_0\|_{H^n_{(x_1)}}; \tag{3-19}$$

- <u>Degeneration</u>: *for any* $1 \le p \le 2$, *a nonnegative even integer* $s' \le s$, *and* $\gamma' \ge -s' - \frac{1}{p} + \frac{1}{2}$, *we have*

$$f^{(-\sigma_c+\gamma')/3}\phi^{\mathrm{app}}_{(\lambda)} = \partial_x^{s'}\left(\frac{f^{(-\sigma_c+\gamma'+s')/3}}{(-1)^{s'/2}A^{s'/3}\lambda^{s'}}\phi^{\mathrm{app}}_{(\lambda)}\right) + f^{(-\sigma_c+\gamma')/3}\phi^{\mathrm{small}}_{(\lambda)}, \tag{3-20}$$

where, for $0 \le j \le 1$ *and* $t \in [0, \delta]$, *we have*

$$\begin{aligned} \|\partial_x^j (A^{-s'/3}\lambda^{-s'} f^{(-\sigma_c+\gamma'+s')/3}\phi^{\mathrm{app}}_{(\lambda)})(t,x)\|_{L^p} &\le C_{\tilde f}^{1+\gamma'} x_1^{(\gamma'+(s'-j)+1/p-1/2)} A^{\gamma'/3}\lambda^{-(s'-j)} \\ &\quad \times \exp\bigl(-3\beta(t)A^{2/3}\lambda^2\bigl(\gamma'+(s'-j)+\tfrac{1}{p}-\tfrac{1}{2}\bigr)t\bigr)\|g_0\|_{H^1_{(x_1)}}, \end{aligned} \tag{3-21}$$

$$\| f^{-\sigma_c/3}\tilde\phi^{\mathrm{small}}_{(\lambda)}(t,x)\|_{L^2} \le C_{\tilde f}\lambda^{-1}\|g_0\|_{H^{s'}_{(x_1)}}; \tag{3-22}$$

- <u>Error estimate</u>: *letting*

$$\boldsymbol{\epsilon}[\phi^{\mathrm{app}}_{(\lambda)}] = (\partial_t + \mathcal{L}_f)\phi^{\mathrm{app}}_{(\lambda)},$$

for $t \in [0, \delta]$, *we have*

$$\| f^{-\sigma_c/3}\boldsymbol{\epsilon}[\phi^{\mathrm{app}}_{(\lambda)}](t,x)\|_{L^2} \le C_{\tilde f}(1 + \|f\|^{m-2}_{L^\infty_t C^{0,1}})(1+|\lambda|^2 t)|\lambda|\|g_0\|_{H^3_{(x_1)}}. \tag{3-23}$$

In the above properties, each constant referred to as $C_{\tilde f}$ *(resp.* $C_{\tilde f_0}$*) obeys the estimate*

$$C_{\tilde f} \le C_s \exp(N_s A^{-1}\|f\|_{L^\infty_t \widetilde{C}^{s-1,1}}) \quad (\textit{resp. } C_{\tilde f_0} \le C_s \exp(N_s A^{-1}\|f_0\|_{\widetilde{C}^{s_0-1,1}}))$$

for some $C_s > 0$ *and* $N_s \in \mathbb{N}$ *independent of* f *and* x_1 *(but possibly dependent on* s*).*

When f or g_0 are clear from the context, we shall often simply omit them in $\phi^{\mathrm{app}}_{(\lambda)}[g_0, f]$.

3.3.2. *Renormalization and conjugation.* For the construction, we introduce the normalization

$$\tilde f(t,x) := \frac{f(t,x)}{A}.$$

By this normalization, we have $\tilde f(0,x) = x^3 + o_f(x^3)$. Using $\tilde f$, we define for $t \in [0,\delta]$ and $x \in (0, x_1]$

$$y(t,x) = -\int_x^{x_1} \frac{1}{\tilde f(t,x')^{1/3}}\,\mathrm{d}x' \le 0.$$

Note that $\partial_x y > 0$. Then, we compute from $\partial_y = \tilde f^{1/3}\,\partial_x$ that

$$\tilde f^{2/3}\,\partial_{xx} = \partial_{yy} - \tfrac{1}{3} f^{-1} f_y\,\partial_y, \qquad \tilde f\,\partial_{xxx} = \partial_{yyy} - f^{-1} f_y\,\partial_{yy} + \bigl(-\tfrac{1}{3} f^{-1} f_{yy} + \tfrac{5}{9} f^{-2}(f_y)^2\bigr)\partial_y.$$

Furthermore, in the time derivative of ϕ,

$$\partial_t\phi(t,x) = \partial_t\phi(t,y) + (\partial_t y)\partial_y\phi(t,y),$$

and we set $q := \partial_t y$ for simplicity. Note that in the (t, x)-coordinates, we have

$$A^{-1}q = \frac{1}{3}\int_x^{x_1} \frac{-\tilde{f}\tilde{f}_{xxx} - \alpha_1 \tilde{f}_x \tilde{f}_{xx} - \frac{\mu_1}{m}(f^{m-2}\tilde{f}^2)_x}{\tilde{f}(t,x')^{4/3}}\,\mathrm{d}x'. \tag{3-24}$$

Then, in the (t, y)-coordinates, (3-16) transforms into

$$\begin{aligned} &A^{-1}\partial_t\phi + \phi_{yyy} + (\alpha_1 - 1)\tilde{f}^{-1}\tilde{f}_y\phi_{yy} \\ &\quad = -A^{-1}q\phi_y + \left(\left(\tfrac{1}{3} - \alpha_1\right)\tilde{f}^{-1}\tilde{f}_{yy} + \left(-\tfrac{5}{9} + \tfrac{2}{3}\alpha_1\right)\tilde{f}^{-2}(\tilde{f}_y)^2 + \mu_1 f^{m-2}\tilde{f}^{2/3}\right)\phi_y \\ &\qquad\qquad - ((\tilde{f}^{-1/3}\,\partial_y)^3\tilde{f} + (m-1)\mu_1 f^{m-3} f_y \tilde{f}^{2/3})\phi. \end{aligned} \tag{3-25}$$

We shall regard the expressions on the right-hand side of (3-25) as error terms. To remove the last term on the left-hand side, we introduce the conjugated variable

$$\varphi = e^{-G}\phi, \tag{3-26}$$

where G shall be determined below. We compute

$$\begin{aligned} \partial_y\phi &= e^G(\partial_y\varphi + \partial_y G\varphi), \\ \partial_y^2\phi &= e^G(\partial_y^2\varphi + 2\partial_y G\,\partial_y\varphi + (\partial_y^2 G + (\partial_y G)^2)\varphi), \\ \partial_y^3\phi &= e^G(\partial_y^3\varphi + 3\partial_y G\,\partial_y^2\varphi + (3\partial_y^2 G + 3(\partial_y G)^2)\partial_y\varphi + (\partial_y^3 G + 3\partial_y G\,\partial_y^2 G + (\partial_y G)^3)\varphi). \end{aligned}$$

Hence, the left-hand side of (3-25), after factoring out e^G, becomes

$$\begin{aligned} &A^{-1}\partial_t\varphi + \varphi_{yyy} + (3G_y + (\alpha_1 - 1)f^{-1}f_y)\varphi_{yy} + (3G_{yy} + 3G_y^2 + 2(\alpha_1 - 1)G_y f^{-1}f_y)\varphi_y \\ &\qquad + (A^{-1}G_t + G_{yyy} + 3G_yG_{yy} + G_y^3 + (\alpha_1 - 1)(G_{yy} + G_y^2)f^{-1}f_y)\varphi. \end{aligned}$$

The right-hand side, after factoring out e^G, becomes

$$\begin{aligned} &-A^{-1}q\varphi_y + \left(\left(\tfrac{1}{3} - \alpha_1\right)f^{-1}f_{yy} + \left(-\tfrac{5}{9} + \tfrac{2}{3}\alpha_1\right)f^{-2}(f_y)^2 + \mu_1 A^{-1}f^{m-1}\tilde{f}^{-1/3}\right)\varphi_y \\ &\qquad - \Big(q - \left(\left(\tfrac{1}{3} - \alpha_1\right)f^{-1}f_{yy} + \left(-\tfrac{5}{9} + \tfrac{2}{3}\alpha_1\right)f^{-2}(f_y)^2 + \mu_1 A^{-1}f^{m-1}\tilde{f}^{-1/3}\right)\Big)G_y\varphi \\ &\qquad\qquad - ((f^{-1/3}\partial_y)^3 f + (m-1)\mu_1 A^{-1}f^{m-2}f_y\tilde{f}^{-1/3})\varphi. \end{aligned}$$

To remove the second-order term, we are motivated to choose

$$G_y = -\frac{\alpha_1 - 1}{3}f^{-1}f_y = \frac{\sigma_c - \frac{1}{2}}{3}f^{-1}f_y.$$

Noting that $f^{-1}f_y = (\ln f)_y$, we see that $G(t, y) = \left(\sigma_c - \frac{1}{2}\right)\frac{1}{3}\ln f(t, y) + C$ for some choice of C. We choose $C = \frac{1}{6}\ln A$ so that

$$e^{G(t,y)} = f^{\sigma_c/3}\tilde{f}^{-1/6}, \tag{3-27}$$

in view of the weight $f^{-2\sigma_c/3}$ in the modified energy estimate we shall prove later (see also the definition of L_f^2 in Section 3.1).

In conclusion, (3-25), after factoring out e^G, may be rewritten as

$$\begin{aligned} &A^{-1}\partial_t\varphi+\varphi_{yyy}\\ &=(-A^{-1}q+C_{1,1}f^{-1}f_{yy}+C_{1,2}f^{-2}(f_y)^2+\mu_1A^{-1}f^{m-1}\tilde{f}^{-1/3})\varphi_y\\ &\quad+(-A^{-1}G_t+C_{0,1}f^{-1}f_{yyy}+C_{0,2}f^{-2}f_yf_{yy}+C_{0,3}f^{-3}(f_y)^3+C_{0,4}\mu_1A^{-1}f^{m-2}f_y\tilde{f}^{-1/3})\varphi, \end{aligned} \tag{3-28}$$

where $C_{j,k}\in\mathbb{R}$ are constants that depend on α_1, μ_1 and m.

3.3.3. *Specification of the wave packet and the proof of Proposition 3.3.* Given $g_0(x)$, we set $h_0(y)=x_1^{1/2}g_0(x(0,y))$, and for each $\lambda\in\mathbb{N}$, we first take $\varphi^{\mathrm{app}}_{(\lambda)}[g_0,f]$ to be the standard wave packet for the Airy equation with time rescaled by A and with frequency λ, i.e.,

$$\varphi^{\mathrm{app}}_{(\lambda)}[g_0,f](t,y)=\mathrm{Re}(e^{i\lambda(y+A\lambda^2t)}h_0(y+3A\lambda^2t))=\cos(\lambda(y+A\lambda^2t))h_0(y+3A\lambda^2t). \tag{3-29}$$

Then, we define the degenerating wave packet $\phi^{\mathrm{app}}_{(\lambda)}[g_0,f]$ by $e^G\varphi^{\mathrm{app}}_{(\lambda)}[g_0,f]$. Explicitly, we have

$$\phi^{\mathrm{app}}_{(\lambda)}[g_0,f]=f(t,y)^{\sigma_c/3}\tilde{f}(t,y)^{-1/6}\cos(\lambda(y+A\lambda^2t))h_0(y+3A\lambda^2t). \tag{3-30}$$

Proof of Proposition 3.3. Now that we have specified the construction of $\phi^{\mathrm{app}}_{(\lambda)}[g_0,f]$, we verify its properties claimed in Proposition 3.3. In what follows, the dependence of constants on f, A and x_1 has been made explicit. Moreover, we shall use the notation $C_{\tilde{f}}$ introduced in Proposition 3.3.

Linearity and support property. To begin with, the linearity property is clear. To prove the support property, we first note from (3-2) and the positivity of $\tilde{f}$ that $y<0$ implies $x<x_1$, and vice versa. Note also that

$$y(t,x)=\frac{1}{\tilde{\beta}(t)}\Big(\ln\frac{x}{x_1}+B(t,x)\Big),$$

where

$$\tilde{\beta}(t):=\frac{\beta(t)}{A^{1/3}},\quad |B(t,x)|\le Cx_1\|\tilde{f}\|_{L^\infty_tC^{3,1}}.$$

Here, $\beta(t)^3=\frac{1}{6}f_{xxx}(t,0)$ as in Lemma 3.1; observe that $\tilde{\beta}(0)=1$ by definition. The above formula for y gives

$$x(t,y)=x_1e^{\tilde{\beta}(t)y-B}, \tag{3-31}$$

from which the rest of the support property follows.

From (3-31), it follows that

$$\begin{aligned} \tilde{f}(t,y)&=x_1^3\tilde{\beta}^3(t)e^{3\tilde{\beta}(t)y-3B}(1+O(x_1e^{\tilde{\beta}(t)y-B}\|\tilde{f}\|_{L^\infty_tC^{3,1}}))\\ &\le\exp(C\|f\|_{L^\infty_t\tilde{C}^{3,1}})x_1^3\tilde{\beta}^3(t)e^{3\tilde{\beta}(t)y}. \end{aligned} \tag{3-32}$$

Using the control of $\|f\|_{L^\infty_tY}$, we furthermore have

$$|f_y|\le C\|\tilde{f}\|^{1/3}_{L^\infty_tY}|f|,\quad |f_{yy}|\le C\|\tilde{f}\|^{2/3}_{L^\infty_tY}|f|,\quad |f_{yyy}|\le C\|\tilde{f}\|_{L^\infty_tY}|f|. \tag{3-33}$$

For higher derivatives, it is straightforward to verify by induction that

$$|\partial_y^k f| \le C_k \|\tilde{f}\|_{L_t^\infty \widetilde{C}^{k-1,1}}^{k/3} |f| \quad \text{for } k \ge 4. \tag{3-34}$$

We furthermore claim that, for any integer $0 \le k \le s+1$,

$$\|h_0\|_{H^k(\mathrm{d}y)} \le C_k (1 + \|\tilde{f}\|_{L_t^\infty \widetilde{C}^{s-1,1}}^{(k-1)/3}) \|g_0\|_{H^k_{(x_1)}}. \tag{3-35}$$

Indeed, arguing via induction in a similar fashion as above, we may verify that, for any $0 \le k \le s+1$,

$$\sum_{j'=0}^{k} |\partial_y^{j'} h_0| \le x_1^{1/2} \biggl(|g_0| + C_k \sum_{j'=1}^{k} \sum_{j=1}^{j'} \|\tilde{f}_0\|_{L_t^\infty \widetilde{C}^{s-1,1}}^{(j'-j)/3} \tilde{f}_0^{j/3} |\partial_x^j g_0| \biggr).$$

Note furthermore that, by (3-12), $C^{-1} x_1^3 \le \tilde{f}(x) \le C x_1^3$ on $\operatorname{supp} g_0$. Taking the $L^2(\mathrm{d}y)$ norm of both sides and changing variables, we are led to (3-35).

3.3.4. *Initial data and regularity estimates.* Let us now verify the initial data and regularity estimates. We begin by noting that

$$\begin{aligned} \int f^{-2\sigma_c/3} \phi^{\mathrm{app}}(t,x)^2 \,\mathrm{d}x &= \int (f^{-\sigma_c/3} \tilde{f}^{1/6} \phi^{\mathrm{app}}(t,y))^2 \,\mathrm{d}y \\ &= \int \varphi^{\mathrm{app}}(t,y)^2 \,\mathrm{d}y = \|h_0\|_{L^2(\mathrm{d}y)}^2, \end{aligned}$$

from which the regularity estimate in the case $n = 0$ follows. Moreover, from this identity it is clear that

$$\|f^{-\sigma_c/3} \phi^{\mathrm{app}}(0,x)\|_{L^2} \le C_{\tilde{f}_0} \|g_0\|_{L^2}.$$

To obtain the claimed lower bound, first note that

$$\|\cos(\lambda y) h_0\|_{L^2(\mathrm{d}y)}^2 = \int \frac{1}{\lambda} \partial_y (\sin(2\lambda y)) h_0^2(y) \,\mathrm{d}y + \frac{1}{2} \|h_0\|_{L^2(\mathrm{d}y)}^2,$$

and then one can integrate by parts in the first term on the right-hand side, with $\|h_0\|_{L^2(\mathrm{d}y)} \gtrsim_{\tilde{f}_0} \|g_0\|_{L^2}$.

Next, when $n = 1$, we note that

$$\begin{aligned} \partial_y \phi^{\mathrm{app}}(t,x) = \mathrm{Re}(&i\lambda f(t,y)^{\sigma_c/3} \tilde{f}^{1/6} e^{i\lambda(y + A\lambda^2 t)} h_0(y + 3A\lambda^2 t)) \\ &+ \frac{\sigma_c - \frac{1}{2}}{3} \frac{\partial_y f}{f} f(t,y)^{\sigma_c/3} \tilde{f}^{1/6} \,\mathrm{Re}(e^{i\lambda(y + A\lambda^2 t)} h_0(y + 3A\lambda^2 t)) \\ &+ f(t,y)^{\sigma_c/3} \tilde{f}^{1/6} \,\mathrm{Re}(e^{i\lambda(y + A\lambda^2 t)} \partial_y h_0(y + 3A\lambda^2 t)). \end{aligned}$$

From this expression, the regularity estimate follows by the earlier computation; the power $|\lambda|$ arises from the first term, the second term is estimated using (3-33), and the need for the $H^1_{(x_1)}$ norm of g_0 is due to the third term. The case of higher n can be handled similarly; we omit the details. Lastly, the initial data estimate can be proved simply by taking $t = 0$; see the computations below for $s' = 0$.

Degeneration property. When $s'=0$, we simply set $\phi^{\mathrm{small}}_{(\lambda)}=0$. Arguing as in the proof of the regularity property, we have

$$\begin{aligned}
&\|f(t,x)^{(-\sigma_c+\gamma')/3}\phi^{\mathrm{app}}_{(\lambda)}(t,x)\|^p_{L^p(\mathrm{d}x)}\\
&\quad=\int|\varphi^{\mathrm{app}}_{(\lambda)}(t,y)|^p f(t,y)^{p\gamma'/3}\tilde f(t,y)^{\frac{-p/2+1}{3}}\,\mathrm{d}y\\
&\quad\le A^{p\gamma'/3}\Big(\int|\varphi^{\mathrm{app}}_{(\lambda)}(t,y)|^2\,\mathrm{d}y\Big)^{p/2}\Big(\int_{\operatorname{supp}\varphi^{\mathrm{app}}_{(\lambda)}(t,\cdot)}\tilde f(t,y)^{\frac{\frac{1}{1/p-1/2}\gamma'+1}{3}}\,\mathrm{d}y\Big)^{p(1/p-1/2)}\\
&\quad\le\|h_0\|^p_{L^2}A^{p\gamma'/3}(C_{\tilde f}x_1\tilde\beta)^{p(\gamma'+1/p-1/2)}\Big(\int_{-\infty}^{-3A\lambda^2t}\exp\Big(\Big(\frac{1}{1/p-1/2}\gamma'+1\Big)\tilde\beta(t)y\Big)\,\mathrm{d}y\Big)^{p(1/p-1/2)}\\
&\quad\le\|h_0\|^p_{L^2}(C_{\tilde f}x_1)^{p(\gamma'+1/p-1/2)}(A^{1/3}\tilde\beta)^{p\gamma'}\exp\big(-3p\tilde\beta(t)A\lambda^2\big(\gamma'+\tfrac{1}{p}-\tfrac{1}{2}\big)t\big),
\end{aligned}$$

where we have simply used (3-32) to bound $\tilde f(t,y)$. This proves (3-21) in the case $s=0$.

To handle the case $s>0$, it is convenient to introduce the following notation (as in the Schrödinger case): given some function $r=r(t,y)$,

$$H=r\,O_k(h_0)\quad\Longleftrightarrow\quad\sup_{t\in[0,\delta]}\Big\|\tilde f^{1/6}\frac{H}{r}\Big\|_{L^2(\mathrm{d}y)}\le C_{\tilde f}\|h_0\|_{H^k(\mathrm{d}y)}.$$

In this case, note that $\|\tilde f^{1/6}(\cdot)\|_{L^2(\mathrm{d}y)}=\|\cdot\|_{L^2(\mathrm{d}x)}$ for each t. We shall also freely use (3-35) to relate the right-hand side with $\|g_0\|_{H^k_{(L)}}$. In what follows, the expression abbreviated as $\frac{1}{\lambda}O_k(h_0)$ constitutes $f^{-\sigma_c/3}\phi^{\mathrm{small}}_{(\lambda)}$; the desired estimate (3-22) would be an immediate consequence of the L^2 boundedness property embedded in the $O_k(\cdot)$ notation.

We treat the case $s=2$. We begin with the identity

$$\cos(\lambda(y+A\lambda^2t))=-\frac{f^{2/3}}{A^{2/3}\lambda^2}(\tilde f^{-1/3}\partial_y)^2\cos(\lambda(y+A\lambda^2t))-\frac{1}{3\lambda}f^{-1}\partial_yf\,\sin(\lambda(y+A\lambda^2t)).$$

Plugging this identity into the expression (3-30) for $\phi^{\mathrm{app}}_{(\lambda)}[g_0,f]$ and commuting $(\tilde f^{-1/3}\partial_y)^2$ (which equals ∂_x^2 in the (t,x)-coordinates) outside, we have

$$f(t,y)^{(-\sigma_c+\gamma')/3}\phi^{\mathrm{app}}_{(\lambda)}=(\tilde f^{-1/3}\partial_y)^2\Big(\frac{f(t,y)^{(-\sigma_c+\gamma'+2)/3}}{(-1)A^{2/3}\lambda^2}\phi^{\mathrm{app}}_{(\lambda)}\Big)+\frac{f(t,y)^{\gamma'/3}}{\lambda}O_2(h_0).$$

Arguing as in the case $s=0$, the expression inside the parentheses can be shown to obey the degeneration bound (3-21). The cases $s>2$ are handled similarly.

Error bound. We begin by noticing that, by our construction, we have

$$\|f^{-\sigma_c/3}\boldsymbol{\epsilon}[\phi^{\mathrm{app}}_{(\lambda)}]\|_{L^2(\mathrm{d}x)}\le\|(\partial_t+\partial_{yyy})\varphi^{\mathrm{app}}_{(\lambda)}\|_{L^2(\mathrm{d}y)}+\|(\text{RHS of (3-28)})\|_{L^2(\mathrm{d}y)}.$$

The first term is the error for the standard wave packet for the Airy equation with frequency λ; it is easily bounded by $C|\lambda|\|h_0\|_{H^3_y}$, which is acceptable. Now, it only remains to estimate the $L^2(\mathrm{d}y)$ norm of each term on the right-hand side of (3-28). The worst contribution turns out to be $-q\varphi_y$, which we turn to first.

By (3-32), (3-33), and the definition of q, it follows that

$$|A^{-1}q(t,y)| \le C_{\tilde{f}}(1+\|f\|^{m-2}_{L^\infty_t C^{0,1}}) \int_y^0 \mathrm{d}y' \le C_{\tilde{f}}(1+\|f\|^{m-2}_{L^\infty_t C^{0,1}})|y|. \tag{3-36}$$

By the support property of $\varphi^{\mathrm{app}}_{(\lambda)}$, we have

$$\|A^{-1}q(\varphi^{\mathrm{app}}_{(\lambda)})_y\|_{L^2} \le C_{\tilde{f}}(1+\|f\|^{m-2}_{L^\infty_t C^{0,1}})(1+A\lambda^2 t)|\lambda|\|h_0\|_{H^1(\mathrm{d}y)},$$

which is acceptable. The remaining terms on the right-hand side of (3-28) involving φ_y are bounded by $C_{\tilde{f}}(1+\|f\|^{m-2}_{L^\infty_t C^{0,1}})|\lambda|\|h_0\|_{H^1(\mathrm{d}y)}$, which are strictly better. Next, since $|A^{-1}\partial_t f| \le C_{\tilde{f}}(1+\|f\|^{m-2}_{L^\infty_t C^{0,1}})|f|$ (as in the estimate for q), we have

$$|A^{-1}\partial_t G| \le C_{\tilde{f}}(1+\|f\|^{m-2}_{L^\infty_t C^{0,1}}).$$

Using this bound, as well as (3-32) and (3-33), the terms on the right-hand side of (3-28) involving φ are bounded by $C_{\tilde{f}}(1+\|f\|^{m-2}_{L^\infty_t C^{0,1}})\|\varphi\|_{L^2(\mathrm{d}y)}$, which is good. This completes the proof of (3-23). □

3.4. ***Modified energy estimate for the perturbation.*** Recall the equation satisfied by ϕ:

$$\begin{aligned}\partial_t\phi + f\phi_{xxx} + \alpha_1 f_x\phi_{xx} + (\alpha_1 f_{xx} + \mu_1 f^{m-1})\phi_x + (f_{xxx} + (m-1)\mu_1 f^{m-2} f_x)\phi \\ = -\phi\phi_{xxx} - \alpha_1\phi_x\phi_{xx} - \mu_1((f+\phi)^m - f^m - mf^{m-1}\phi)_x.\end{aligned} \tag{3-37}$$

Regarding a solution ϕ of the above and recalling the notation $\|\cdot\|_{L^2_f}$ from (3-1), we have the modified energy estimate

$$\|\phi\|^2_{L^2_f}(t) = \int_I \phi^2(t,x) f(t,x)^{-2\sigma_c/3}\,\mathrm{d}x, \quad \|\phi_0\|^2_{L^2_{f_0}} = \int_I \phi^2(0,x) f(0,x)^{-2\sigma_c/3}\,\mathrm{d}x$$

assuming that f is defined on I.

Proposition 3.4. *Assume that f is a solution to* (1-9) *satisfying $f \in L^\infty([0,\delta];\widetilde{C}^{3,\alpha}(I))$ with initial data f_0 that is positive on $I \setminus \partial I$ and vanishes to order at least* 3 *on each point in ∂I. Moreover, assume that $\phi \in L^\infty([0,\delta]; C^{3,\alpha}(I))$ is a solution to* (3-37) *satisfying*

$$f+\phi \in L^\infty([0,\delta];\widetilde{C}^{3,\alpha}(I)), \quad f^{-1}(f+\phi) \in L^\infty([0,\delta]; L^\infty(I)).$$

Then we have the estimate

$$\|\phi\|_{L^2_f}(t) \le \exp(C_{f,f+\phi}t)\|\phi_0\|_{L^2_{f_0}}$$

for $t \in [0,\delta]$, where $\phi_0(x) = \phi(0,x)$ and

$$C_{f,f+\phi} \le C \sup_{t\in[0,\delta]} \big(\|f\|_Y + (1+\|f^{-1}(f+\phi)\|^{1/2}_{L^\infty})\|f+\phi\|_Y + (\|f\|_{C^{0,1}} + \|f+\phi\|_{C^{0,1}})^{m-1}\big), \tag{3-38}$$

with $C>0$ an absolute constant.

Proof. In what follows, we shall simply present a formal computation without worrying about the validity of the expressions and manipulations. Also, all integrals are taken over I. As in Section 2.4.1, the assumption $|\phi_0(x)| \le C|f_0(x)|$ and the finiteness of the right-hand side $\|\phi_0\|_{L^2_{f_0}} < +\infty$ would imply, via

Lemma 3.1, the vanishing property of $\phi(t,\cdot)$ on ∂I that is necessary to justify the computation; we shall leave the routine details to the reader.

To prove the proposition, we compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\phi\|^2_{L^2_f} = \frac{\mathrm{d}}{\mathrm{d}t}\int \phi^2 f(t,x)^{-2\sigma_c/3}\,\mathrm{d}x = \int 2\phi\,\partial_t\phi f(t,x)^{-2\sigma_c/3}\,\mathrm{d}x + \int \phi^2\,\partial_t(f(t,x)^{-2\sigma_c/3})\,\mathrm{d}x,$$

and the last term can be bounded as in the proof of (3-4); we have

$$\left|\int \phi^2\partial_t(f(t,x)^{-2\sigma_c/3})\,\mathrm{d}x\right| \le C\|f\|_Y\|\phi\|^2_{L^2_f}.$$

We decompose the other term in the right-hand side as follows, up to a factor of 2:

$$\begin{aligned}
\mathrm{I} &= -\int \phi(f\phi_{xxx} + \phi f_{xxx} + \alpha_1 f_x\phi_{xx} + \alpha_1 f_{xx}\phi_x) f^{-2\sigma_c/3}\,\mathrm{d}x,\\
\mathrm{II} &= -\mu_1\int \phi((m-1)\phi f^{m-2} f_x + f^{m-1}\phi_x) f^{-2\sigma_c/3}\,\mathrm{d}x,\\
\mathrm{III} &= -\int \phi Q[\phi] f^{-2\sigma_c/3}\,\mathrm{d}x.
\end{aligned}$$

To estimate I, we observe the following chain of inequalities:

$$\begin{aligned}
\left|\int \phi^2 f_{xxx} f^{-2\sigma_c/3}\,\mathrm{d}x\right| &\le C\|f\|_Y\|\phi\|^2_{L^2_f},\\
\left|\int \alpha_1\phi\phi_x f_{xx} f^{-2\sigma_c/3}\,\mathrm{d}x\right| &= \left|\frac{\alpha_1}{2}\int \phi^2\partial_x(f_{xx} f^{-2\sigma_c/3})\,\mathrm{d}x\right|,\\
\int \alpha_1\phi\phi_{xx} f_x f^{-2\sigma_c/3}\,\mathrm{d}x &= -\alpha_1\int (\phi_x)^2 f_x f^{-2\sigma_c/3}\,\mathrm{d}x + \frac{\alpha_1}{2}\int \phi^2\partial_{xx}(f_x f^{-2\sigma_c/3})\,\mathrm{d}x,
\end{aligned}$$

and lastly

$$\begin{aligned}
\int \phi\phi_{xxx} f f^{-2\sigma_c/3}\,\mathrm{d}x &= -\int \phi_{xx}\phi_x f f^{-2\sigma_c/3} - \phi_{xx}\phi\partial_x(f f^{-2\sigma_c/3})\,\mathrm{d}x\\
&= \frac{3}{2}\int (\phi_x)^2\partial_x(ff^{-2\sigma_c/3})\,\mathrm{d}x - \frac{1}{2}\int \phi^2\partial_{xxx}(f f^{-2\sigma_c/3})\,\mathrm{d}x.
\end{aligned}$$

From

$$\partial_x(f f^{-2\sigma_c/3}) = \left(1-\tfrac{2}{3}\sigma_c\right) f^{-2\sigma_c/3} f_x = \tfrac{2}{3}\alpha_1 f^{-2\sigma_c/3} f_x,$$

we obtain a cancellation of terms involving $(\phi_x)^2$ and then we observe

$$|\partial_x(f_{xx} f^{-2\sigma_c/3})| + |\partial_{xx}(f_x f^{-2\sigma_c/3})| + |\partial_{xxx}(f f^{-2\sigma_c/3})| \le C\|f\|_Y f^{-2\sigma_c/3}$$

to conclude the estimate

$$|\mathrm{I}| \le C\|f\|_Y\|\phi\|^2_{L^2_f}.$$

Next, to treat II we simply integrate by parts:

$$|\mathrm{II}| = \left| \mu_1 \int \phi^2 \Big((m-1) f^{m-2} f_x - \frac{-\frac{2}{3}\sigma_c + m - 1}{2} f^{m-2} f_x \Big) f^{-2\sigma_c/3} \,\mathrm{d}x \right| \le C \|f\|_{C^{0,1}}^{m-1} \|\phi\|_{L^2_f}^2 .$$

Finally, we turn to III. Observe that we may use $\|\phi\|_{C^{3,\alpha}}$ since it is controlled by $\|f\|_{C^{3,\alpha}} + \|f+\phi\|_{C^{3,\alpha}}$; similarly for $\|\phi\|_{C^{0,1}}$. Recall the expression for $Q[\phi]$ given in (3-15). We first estimate

$$\left| \int \phi(-\phi\phi_{xxx}) f^{-2\sigma_c/3} \,\mathrm{d}x \right| \le C \|\phi\|_{C^{2,1}} \|\phi\|_{L^2_f}^2$$

and

$$\begin{aligned}
&\left| \int \phi(-\alpha_1 \phi_x \phi_{xx}) f^{-2\sigma_c/3} \,\mathrm{d}x \right| \\
&\le \left| \int \frac{\alpha_1}{2} \phi^2 \phi_{xxx} f^{-2\sigma_c/3} \,\mathrm{d}x \right| + \left| \int \frac{\alpha_1 \sigma_c}{3} \phi^2 (\phi+f)_{xx} f^{(-2\sigma_c-3)/3} f_x \,\mathrm{d}x \right| + \left| \int \frac{\alpha_1 \sigma_c}{3} \phi^2 f_{xx} f^{(-2\sigma_c-3)/3} f_x \,\mathrm{d}x \right| \\
&\le \left| \int \frac{\alpha_1}{2} \phi^2 \phi_{xxx} f^{-2\sigma_c/3} \,\mathrm{d}x \right| + \left| \int \frac{\alpha_1 \sigma_c}{3} \phi^2 (\phi+f)_{xx} f^{(-2\sigma_c-3)/3} f_x \,\mathrm{d}x \right| + \left| \int \frac{\alpha_1 \sigma_c}{3} \phi^2 f_{xx} f^{(-2\sigma_c-3)/3} f_x \,\mathrm{d}x \right| \\
&\le C(\|\phi\|_{C^{2,1}} + \|f^{-1}(f+\phi)\|_{L^\infty}^{1/3} \|f+\phi\|_Y^{2/3} \|f\|_Y^{1/3} + \|f\|_Y) \|\phi\|_{L^2_f}^2 .
\end{aligned}$$

The remaining terms in $Q[\phi]$ are easier to treat, and collecting the estimates, we conclude

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} \|\phi\|_{L^2_f}^2 \right| \le C_{f,f+\phi} \|\phi\|_{L^2_f}^2$$

for some $C_{f,f+\phi}$ satisfying (3-38). The proposition follows by integrating in time. □

3.5. ***Generalized energy estimate and the proof of Theorem 1.5.*** Let f satisfy the assumptions of Theorem 1.5. Without loss of generality, we may assume that $a = 0$ and $0 < \epsilon \le 1$. For simplicity, we shall focus on the case $\beta(0) = \big(\frac{1}{6} f_{0,xxx}(0)\big)^{1/3} = 1$, the general case being analogous. Fix $0 < x_1 < b$ so that (3-12) holds. Fix $g_0 \in C_c^\infty$ supported in $\big(\frac{1}{2}x_1, x_1\big)$ with normalization $\|g_0\|_{L^2} = 1$. In what follows, *we shall suppress the dependence of constants on f and g_0, in addition to α_1, μ_1 and m as before.* Also, *we write $C(M)$ for a positive strictly increasing function of $M \in (0, \infty)$ such that $C(M) \to \infty$ as $M \to \infty$, which may vary from line to line.*

Let $\phi^{\mathrm{app}}_{(\lambda)} = \phi^{\mathrm{app}}_{(\lambda)}[g_0, f]$ according to Proposition 3.3. We shall take

$$\phi_0 = c_0 \epsilon \lambda^{-m_0} x_1^{1/2} \phi^{\mathrm{app}}_{(\lambda)}(0),$$

where c_0 is chosen so that $\|\phi_0\|_{C^{m_0}} \le \epsilon$ using (3-19) and λ is to be determined below. Furthermore, by the normalization $\|g_0\|_{L^2} = 1$, we have

$$\frac{1}{C_0} |\lambda|^{-m_0} \le \|\phi_0\|_{L^2_{f_0}} \le C_0 |\lambda|^{-m_0}, \quad \langle \phi_0, \phi^{\mathrm{app}}_{(\lambda)}(0) \rangle_{f_0} \ge \frac{1}{C_0} \|\phi_0\|_{L^2_{f_0}} \tag{3-39}$$

for some constant $C_0 > 0$ (which, in fact, depends on $\|f\|_{\widetilde{C}^{s_0-1,1}}$). At this point, it is easy to ensure that (3-5) is satisfied with $C_1 = 2$, where $u_0 = f_0 + \phi_0$.

Fix also $0 < \delta' \le \delta$. To prove the theorem, we assume that the first alternative does not hold, i.e., there exists a solution $f+\phi \in L^\infty([0,\delta'];C^{s-1,1}(I))$ to (1-9). By Proposition 3.2 (and since $s \ge 5$), there exists $0 < t_0 \le \delta'$ depending only on $\|f(0)\|_Y$ and $\|(f+\phi)(0)\|_Y$ such that f, $f+\phi \in L^\infty([0,t_0];Y(I))$ and $f^{-1}(f+\phi) \in L^\infty([0,t_0];L^\infty(I))$. Moreover, by the same proposition, we have

$$\sup_{0<t<t_0} (\|f(t)\|_{\widetilde{C}^{4,1}(I)} + \|(f+\phi)(t)\|_{\widetilde{C}^{4,1}(I)} + \|f^{-1}(f+\phi)(t)\|_{L^\infty(I)}) \le C(M_5), \tag{3-40}$$

where $M_5 := \|\phi\|_{L^\infty([0,t_0];C^{4,1}(I))}$. (Here, we remind the reader of our convention of omitting the dependence on f in this proof.) By Proposition 3.4 and the preceding bound, we have

$$\|\phi(t)\|_{L^2_f} \le \exp(C_1(M_5)t)\|\phi_0\|_{L^2_{f_0}} \tag{3-41}$$

for some positive strictly increasing function $C_1(\,\cdot\,)$ that diverges at infinity. We emphasize that this function is *independent* of λ, although $M_5 = \|\phi\|_{L^\infty([0,t_0];C^{4,1}(I))}$ might be dependent on λ.

Now using that ϕ is a solution to (3-37) and $(\partial_t + \mathcal{L}_f)\phi^{\mathrm{app}}_{(\lambda)} = \boldsymbol{\epsilon}[\phi^{\mathrm{app}}_{(\lambda)}]$, we compute that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}&\langle \phi, \phi^{\mathrm{app}}_{(\lambda)}\rangle_f \\ &= -\langle \phi, \mathcal{L}_f[\phi^{\mathrm{app}}_{(\lambda)}]\rangle_f + \langle \phi, \boldsymbol{\epsilon}[\phi^{\mathrm{app}}_{(\lambda)}]\rangle_f - \langle \mathcal{L}_f[\phi], \phi^{\mathrm{app}}_{(\lambda)}\rangle_f + \langle Q_f[\phi], \phi^{\mathrm{app}}_{(\lambda)}\rangle_f - \tfrac{2}{3}\sigma_c\langle f^{-1}\partial_t f\phi, \phi^{\mathrm{app}}_{(\lambda)}\rangle_f.\end{aligned}$$

We first uncover some cancellations between the two terms involving the linearized operator $\mathcal{L}_f$, which resemble those in the proof of Proposition 3.4. We write

$$\begin{aligned}-\langle \phi, &\mathcal{L}_f[\phi^{\mathrm{app}}_{(\lambda)}]\rangle_f - \langle \mathcal{L}_f[\phi], \phi^{\mathrm{app}}_{(\lambda)}\rangle_f \\ &= -\int \phi(f\phi^{\mathrm{app}}_{(\lambda)xxx} + \alpha_1 f_x\phi^{\mathrm{app}}_{(\lambda)xx} + \alpha_1 f_{xx}\phi^{\mathrm{app}}_{(\lambda)x} + f_{xxx}\phi^{\mathrm{app}}_{(\lambda)})f^{-2\sigma_c/3}\,\mathrm{d}x \\ &\qquad - \mu_1\int \phi(f^{m-1}\phi^{\mathrm{app}}_{(\lambda)x} + (m-1)f^{m-2}f_x\phi^{\mathrm{app}}_{(\lambda)})f^{-2\sigma_c/3}\,\mathrm{d}x \\ &\qquad\qquad - \int (f\phi_{xxx} + \alpha_1 f_x\phi_{xx} + \alpha_1 f_{xx}\phi_x + f_{xxx}\phi)\phi^{\mathrm{app}}_{(\lambda)} f^{-2\sigma_c/3}\,\mathrm{d}x \\ &\qquad\qquad\qquad - \mu_1\int (f^{m-1}\phi_x + (m-1)f^{m-2}f_x\phi)\phi^{\mathrm{app}}_{(\lambda)} f^{-2\sigma_c/3}\,\mathrm{d}x \\ &=: \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}.\end{aligned}$$

We begin with $\mathrm{I}+\mathrm{III}$. The zeroth-order terms (in both ϕ and $\phi^{\mathrm{app}}_{(\lambda)}$) are not dangerous, but we need to perform some integration by parts for the higher-order terms. For the third-order terms, we have

$$\begin{aligned}-\int &(\phi\phi^{\mathrm{app}}_{(\lambda)xxx} + \phi_{xxx}\phi^{\mathrm{app}}_{(\lambda)})f^{1-2\sigma_c/3}\,\mathrm{d}x \\ &= \int (\phi_x\phi^{\mathrm{app}}_{(\lambda)xx} + \phi_{xx}\phi^{\mathrm{app}}_{(\lambda)x})f^{1-2\sigma_c/3}\,\mathrm{d}x + \big(1-\tfrac{2}{3}\sigma_c\big)\int (\phi\phi^{\mathrm{app}}_{(\lambda)xx} + \phi_{xx}\phi^{\mathrm{app}})f^{-1}f_x f^{1-2\sigma_c/3}\,\mathrm{d}x \\ &= -3\big(1-\tfrac{2}{3}\sigma_c\big)\int \phi_x\phi^{\mathrm{app}}_{(\lambda)x} f_x f^{-2\sigma_c/3}\,\mathrm{d}x - \big(1-\tfrac{2}{3}\sigma_c\big)\int (\phi\phi^{\mathrm{app}}_{(\lambda)x} + \phi_x\phi^{\mathrm{app}})(f_x f^{-2\sigma_c/3})_x\,\mathrm{d}x \\ &= -3\big(1-\tfrac{2}{3}\sigma_c\big)\int \phi_x\phi^{\mathrm{app}}_{(\lambda)x} f_x f^{-2\sigma_c/3}\,\mathrm{d}x + \big(1-\tfrac{2}{3}\sigma_c\big)\int \phi\phi^{\mathrm{app}}_{(\lambda)}(f_x f^{-2\sigma_c/3})_{xx}\,\mathrm{d}x;\end{aligned}$$

for the second-order terms, we have

$$
\begin{aligned}
-\int (\alpha_1 \phi \phi^{\mathrm{app}}_{(\lambda)xx} f_x + \alpha_1 \phi_{xx} \phi^{\mathrm{app}}_{(\lambda)} f_x) f^{-2\sigma_c/3} \,\mathrm{d}x
&= 2\alpha_1 \int \phi_x \phi^{\mathrm{app}}_{(\lambda)x} f_x f^{-2\sigma_c/3} \,\mathrm{d}x + \alpha_1 \int (\phi \phi^{\mathrm{app}}_{(\lambda)x} + \phi_x \phi^{\mathrm{app}}_{(\lambda)})(f_x f^{-2\sigma_c/3})_x \\
&= 2\alpha_1 \int \phi_x \phi^{\mathrm{app}}_{(\lambda)x} f_x f^{-2\sigma_c/3} \,\mathrm{d}x - \alpha_1 \int \phi \phi^{\mathrm{app}}_{(\lambda)} (f_x f^{-2\sigma_c/3})_{xx};
\end{aligned}
$$

and for the first-order terms, we have

$$
-\int (\alpha_1 \phi \phi^{\mathrm{app}}_{(\lambda)x} f_{xx} + \alpha_1 \phi_x \phi^{\mathrm{app}}_{(\lambda)} f_{xx}) f^{-2\sigma_c/3} \,\mathrm{d}x = \alpha_1 \int \phi \phi^{\mathrm{app}}_{(\lambda)} (f_{xx} f^{-2\sigma_c/3})_x \,\mathrm{d}x.
$$

In particular, since $3\bigl(1-\tfrac{2}{3}\sigma_c\bigr) = 2\alpha_1$, integrals that involve $\phi_x \phi^{\mathrm{app}}_{(\lambda)x}$ cancel and we are left with

$$
|\mathrm{I} + \mathrm{III}| \le C \|f\|_Y \|\phi\|_{L^2_f} \|\phi^{\mathrm{app}}_{(\lambda)}\|_{L^2_f}.
$$

Next, $\mathrm{II} + \mathrm{IV}$ consist of first- and zeroth-order terms, where the former may be treated as above and the latter are already acceptable. We have

$$
|\mathrm{II} + \mathrm{IV}| \le C \|f\|^{m-1}_{C^{0,1}} \|\phi\|_{L^2_f} \|\phi^{\mathrm{app}}_{(\lambda)}\|_{L^2_f}.
$$

For the remaining terms in $\frac{\mathrm{d}}{\mathrm{d}t}\langle \phi, \phi^{\mathrm{app}}_{(\lambda)}\rangle_f$, we have

$$
\begin{aligned}
\|\epsilon[\phi^{\mathrm{app}}_{(\lambda)}]\|_{L^2_f} &\le C(1+\lambda^2 t)|\lambda|, \\
\|Q[\phi]\|_{L^2_f} &\le C\|\phi\|_{L^2_f}, \\
\|f^{-1}\partial_t f\|_{L^\infty} &\le C.
\end{aligned}
$$

We conclude that

$$
\left| \frac{\mathrm{d}}{\mathrm{d}t} \langle \phi, \phi^{\mathrm{app}}_{(\lambda)}\rangle_f \right| \le C(1 + (1+\lambda^2 t)|\lambda|) \|\phi\|_{L^2_f} \le C(1 + (1+\lambda^2 t)|\lambda|) \exp(C_1(M_5)t) \frac{\|\phi_0\|_{L^2}}{4C_0}. \tag{3-42}
$$

Integrating this estimate in time and using (3-39), we have

$$
\langle \phi, \phi^{\mathrm{app}}_{(\lambda)}\rangle_f(t) \ge \frac{3}{4C_0} \|\phi_0\|_{L^2} \quad \text{for } |t| \le \min\{t_0, C_1(M_5)^{-1}, c|\lambda|^{-3/2}\} \tag{3-43}
$$

for some $c > 0$.

To proceed, let m be the smallest even integer greater than or equal to s', and define $j = m - s'$. Applying Proposition 3.3 with $\gamma' = -\sigma_c$ and $s' = m$, we have

$$
\langle \phi, \phi^{\mathrm{app}}_{(\lambda)}\rangle_f \le \int \phi \partial_x^{s'+j} \left(\frac{f^{(-2\sigma_c + s' + j)/3}}{(-1)^{s_0+j} \lambda^{s_0+j}} \phi^{\mathrm{app}}_{(\lambda)} \right) \mathrm{d}x + \|\phi\|_{L^2_f} \|\phi^{\mathrm{small}}_{(\lambda)}\|_{L^2_f}. \tag{3-44}
$$

By (3-22) and (3-41), the last term may be bounded as follows for some $C_2 > 0$:

$$
\begin{aligned}
\|\phi\|_{L^2_f} \|\phi^{\mathrm{small}}_{(\lambda)}\|_{L^2_f} &\le C_2 \lambda^{-1} \exp(C_1(M_5)t) \frac{1}{4C_0} \|\phi_0\|_{L^2} \\
&\le \frac{1}{4C_0} \|\phi_0\|_{L^2} \qquad \text{if } |t| \le C_1(M_5)^{-1} \text{ and } |\lambda| > C_2.
\end{aligned} \tag{3-45}
$$

We are now ready to conclude the proof of the theorem. For each λ, there are two possible cases: (i) $C_1(M_5)^{-1} \le c|\lambda|^{-3/2}$, or (ii) $C_1(M_5)^{-1} > c|\lambda|^{-3/2}$. In case (i), we have $c^{-1}|\lambda|^{3/2} \le C_1(M_5)$, so $M_5 > (\delta')^{-1/2}$ if $|\lambda|$ is chosen large enough depending on $C_1(\,\cdot\,)$ and δ'. Since $s' \ge s_c \ge 5$, the desired norm inflation follows. Hence, it only remains to consider case (ii). Then, by (3-43), (3-44) and (3-45), we have

$$\frac{1}{2C_0}\|\phi_0\|_{L^2_f} \le \int \phi\,\partial_x^{s'+j}\biggl(\frac{f^{(-2\sigma_c+s'+j)/3}}{(-1)^{s_0+j}\lambda^{s_0+j}}\phi^{\mathrm{app}}_{(\lambda)}\biggr)\,\mathrm{d}x \quad \text{for } |t| \le \min\{t_0, c|\lambda|^{-3/2}\}.$$

Using duality and applying (3-21), we arrive at

$$\begin{aligned}\frac{1}{2C_0}\|\phi_0\|_{L^2_f} &\le \|\partial_x^{s'}\phi\|_{L^\infty}\|\partial_x^{j}(\lambda^{-s_0-j}f^{(-2\sigma_c+s'+j)/3}\phi^{\mathrm{app}}_{(\lambda)})\|_{L^1}\\ &\le C\lambda^{-s'}\exp\bigl(-3\beta(t)\lambda^2\bigl(-\sigma_c+s'+\tfrac{1}{2}\bigr)t\bigr)\|\partial_x^{s'}\phi\|_{L^\infty}.\end{aligned}$$

Rearranging the factors, we finally arrive at

$$\|\partial_x^{s'}\phi(t)\|_{L^\infty} \ge \frac{1}{C}\lambda^{s'-m_0}\exp\bigl(3\beta(t)\lambda^2\bigl(-\sigma_c+s'+\tfrac{1}{2}\bigr)t\bigr) \quad \text{for } 0 \le t \le \min\{t_0, c|\lambda|^{-3/2}\}.$$

Fix $t = c|\lambda|^{-3/2}$; by taking λ sufficiently large, we may clearly ensure that $t \le t_0$. Since $s' \ge s_c > \sigma_c - \frac{1}{2}$ and $\lambda^2 t = c|\lambda|^{1/2}$, by taking λ larger we may also ensure that the right-hand side is at least $(\delta')^{-1/2}$ for each s'. This completes the proof of Theorem 1.5. □

3.6. ***Proof of Theorem 1.6.*** We are now in a position to complete the proof of Theorem 1.6. As we shall see, the argument is parallel to that for Theorem 1.2 in the Schrödinger case.

Let $s_0 \ge s \ge s_c$ and $\epsilon > 0$ be given as in the statement of Theorem 1.6. Suppose, for contradiction, that for every $u_0 \in C^\infty(\mathbb{T})$ satisfying $\|u_0\|_{C^{s_0}} < \epsilon$ there exists $\delta = \delta(u_0) > 0$ and a corresponding solution u to (1-9) belonging to $L^\infty([0,\delta]; C^s(\mathbb{T}))$.

We shall fix a function $\mathring{f}_0 \in C^\infty(\mathbb{T})$ supported in $[-4x_1, 4x_1]$ which satisfies $\mathring{f}_0(x) = x^3$ in $[-x_1, x_1]$, $\mathring{f}_0 > 0$ on $(0, 2x_1)$ and the cubic vanishing property at $2x_1$ for some small $0 < x_1 < \frac{1}{100}$; in what follows, we omit the dependence of constants on $\mathring{f}_0$. Then, we take

$$f_0 := \sum_{k=k_0}^{\infty} f_{k,0} := \sum_{k=k_0}^{\infty} A_k 2^{-3k}\mathring{f}_0(2^k(x-x_k)), \quad x_k = 2^{-k/2}, \;\; A_k = 2^{-k^2},$$

where k_0 shall be fixed below. We see that $f_0 \in C^\infty$ and $\|f_0\|_{C^{s_0}} < \frac{1}{2}\epsilon$ provided that k_0 is sufficiently large depending on s_0 and ϵ. Moreover, we can check that there exists a constant $C_0 = C_0(f_0) > 0$ such that (3-8) holds for f_0. For each $k \in \mathbb{N}$, we take cutoff functions χ_k which are equal to 1 on the support of $f_{k,0}$ and vanish on the support of $f_{k',0}$ for all $k' \ne k$.

Now, let $f \in L^\infty([0,\delta]; C^s(\mathbb{T}))$ be a solution to (1-9) with $f(t=0) = f_0$. Using the equation and $C^{4,1}$-regularity, we have the pointwise estimate $|\partial_t f| \lesssim |f_0| + |f|$, which guarantees that the support of $f(t,\cdot)$ is preserved in time. For simplicity, we set $M_f = \|f\|_{L^\infty([0,\delta];C^{4,1})}$ and replace δ with $\min\{\delta, c\}$ with some small constant $c = c(M_f) > 0$, so that we have uniformly

$$\tfrac{1}{2}f_0(x) \le f(t,x) \le \tfrac{3}{2}f_0(x), \quad t \in [0,\delta], \tag{3-46}$$

whenever $f_0(x) > 0$. The existence of such a constant c follows from (3-10) and (3-11).

Since χ_k is either 0 or 1 on $\operatorname{supp}(f_0) = \operatorname{supp}(f(t,\cdot))$, we have $\partial_x \chi_k \equiv 0$ on the support of f and $\chi_k f = \chi_k^2 f$. From these observations, it follows that $\chi_k f =: f_k$ provides a solution to (1-9) for any $k \geq k_0$ with initial data $f_{k,0}$. Let $I_k = [x_k, x_k + 2^{-k} 2x_1]$, and observe that

$$\|f_{k,0}\|_{Y(I_k)} \leq A_k (\|\mathring{f}_0\|_{Y([0,2x_1])} + 2^{-3k} \|\mathring{f}_0\|_{L^\infty([0,2x_1])}) \to 0 \quad \text{as } k \to \infty. \tag{3-47}$$

Taking k_0 larger and δ smaller if necessary, by Proposition 3.2, we have, for every $k \geq k_0$,

$$\|f_k\|_{L^\infty([0,\delta];Y(I_k))} \leq C(M_f).$$

Let us fix a C^∞-smooth profile g_0 supported in $\left(\frac{1}{2}x_1, x_1\right)$ and normalized in L^2; in what follows, we omit the dependence of constants on g_0. We take $g_k(x) = 2^{k/2} g_0(2^k(x - x_k))$. For a strictly increasing sequence $\{\lambda_k\}_{k \geq k_0}$ ($\lambda_k \gg 1$) to be determined, we consider the wave packets

$$\phi_k^{\mathrm{app}}(t,x) := \phi_{(\lambda_k)}^{\mathrm{app}}[g_k, f_k],$$

where $\phi_{(\lambda_k)}^{\mathrm{app}}[g_k, f_k]$ denotes the wave packet constructed in Proposition 3.3 using the solution f_k with profile g_k and frequency λ_k. We define the corresponding error by

$$[\partial_t + \mathcal{L}_{f_k}]\phi_k^{\mathrm{app}} = \boldsymbol{\epsilon}_{\phi_k}, \tag{3-48}$$

where $\mathcal{L}_{f_k}$ is simply (3-14) with f replaced with f_k. Recall also the definition of the L^2_f norm from (3-1), and observe that $f = f_k$ on the support of ϕ_k^{app}. Applying Proposition 3.3, we obtain the following properties of ϕ_k^{app} for all $t \in [0,\delta]$:

- $\|\phi_k^{\mathrm{app}}(t,x)\|_{L^2_f} \leq C_{\tilde{f}_k} \|\phi_{k,0}^{\mathrm{app}}\|_{L^2_{f_k}} \leq C_{\tilde{f}_k} \|g_0\|_{L^2}$,
- $\|(\tilde{f}^{1/3}\partial_x)^n \phi_k^{\mathrm{app}}(t,x)\|_{L^2_f} \leq C_{\tilde{f}_k} \|\phi_{k,0}^{\mathrm{app}}\|_{L^2_{f_k}} \leq C_{\tilde{f}_k}$,
- $\|\boldsymbol{\epsilon}_{\phi_k}(t,x)\|_{L^2_f} \leq C_{\tilde{f}_k}(1 + A_k^{m-2})\lambda_k(1 + \lambda_k^2 t)$,

and, since s is even, we have

$$f^{-2\sigma_c/3}\phi_k^{\mathrm{app}} = \partial_x^s \left(\frac{f^{(-2\sigma_c+s)/3}}{(-1)^{s/2} A_k^{s/3} \lambda_k^s} \phi_k^{\mathrm{app}} \right) + f^{-2\sigma_c/3} \phi_k^{\mathrm{small}}, \tag{3-49}$$

with

- $\|(A_k^{-s/3}\lambda^{-s} f^{(-2\sigma_c+s)/3}\phi_k^{\mathrm{app}})(t,\cdot)\|_{L^1} \leq C_{\tilde{f}_k} A_k^{-\sigma_c/3} \lambda_k^{-s} \exp\bigl(-3\beta_k(t) A_k^{2/3} \lambda_k^2 \bigl(-\sigma_c + s + \frac{1}{2}\bigr)t\bigr)$,
- $\|f^{-\sigma_c/3}\tilde{\phi}_k^{\mathrm{small}}(t,\cdot)\|_{L^2} \leq C_{\tilde{f}_k} \lambda_k^{-1}$,

where $\beta_k(t)$ is the solution to (3-3) with f replaced by f_k. Observe that $C_{\tilde{f}_k}$ depends on M_f, k and s, but *not* on λ_k. In view of (3-17), note that λ_k should be sufficiently large depending on k and g_0 to ensure that the first inequality in the first item holds. Define

$$\phi_0(x) := \sum_{k=k_0}^{\infty} \phi_{k,0}(x) := \sum_{k=k_0}^{\infty} \exp(-\lambda_k^{1/8}) \phi_{k,0}^{\mathrm{app}}(x).$$

By ensuring some growth of λ_k (e.g., $\lambda_k \geq 2^k$) and by taking k_0 even larger if necessary, we can guarantee that $\|\phi_0\|_{C^s} < \frac{1}{2}\epsilon$, so that $u_0 := f_0 + \phi_0$ is C^∞-smooth and satisfies $\|u_0\|_{C^{s_0}} < \epsilon$. From the contradiction hypothesis, we have a $L_t^\infty C^s$ solution $u(t, x)$ to (1-9) with initial data u_0 on some time interval $[0, \delta']$. By shrinking either δ or δ', we may assume that $0 < \delta' = \delta$. We now set

$$\phi(t) := u(t) - f(t), \quad \phi_k(t) := \chi_k \phi(t)$$

for all $k \geq k_0$. Moreover, taking k_0 larger if necessary, we can easily arrange that $f_0(x)$ and $u_0(x)$ are uniformly comparable; for all x,

$$\tfrac{7}{8}u_0(x) \leq f_0(x) \leq \tfrac{9}{8}u_0(x).$$

From the conservation of the support in time, we have that $\sum_{k=k_0}^{\infty} \phi_k = \phi$. We now introduce

$$M = 1 + \sup_{t \in [0,\delta]} (\|f(t)\|_{C^s} + \|\phi(t)\|_{C^s}) \geq M_f, \tag{3-50}$$

which is finite by the contradiction hypothesis, and further replace δ with $\min\{\delta, c\}$, where $c = c(M) > 0$ is large enough that we have

$$\tfrac{1}{2}u_0(x) \leq u(t, x) \leq \tfrac{3}{2}u_0(x), \quad t \in [0, \delta], \tag{3-51}$$

whenever $u_0(x) > 0$. Note also that, by a computation similar to (3-47), we have $\|u_0\|_{Y(I_k)} \to 0$ as $k \to \infty$. Choosing k_0 sufficiently large and δ small enough, by Proposition 3.2, we have, for every $k \geq k_0$,

$$\|u_k\|_{L^\infty([0,\delta];Y(I_k))} \leq C(M).$$

We see that $\chi_k u$ is a solution to (1-9) and it follows that ϕ_k solves

$$[\partial_t + \mathcal{L}_{f_k}]\phi_k = Q_{f_k}[\phi_k],$$

where Q_{f_k} is simply (3-15) with f replaced with f_k.

From Proposition 3.4, we have the modified energy estimate for ϕ_k,

$$\|\phi_k(t)\|_{L^2_f(I_k)} \leq C(M)\|\phi_{k,0}\|_{L^2_{f_0}(I_k)}.$$

Proceeding as in the proof of (3-42), we obtain

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\langle \phi_k, \phi_k^{\mathrm{app}}\rangle_f\right| \leq C(M, k, s)(1 + \lambda_k(1 + \lambda_k^2 t))\|\phi_{k,0}\|_{L^2_{f_0}(I_k)}, \quad t \in [0, \delta]. \tag{3-52}$$

We shall take $t_k := \lambda_k^{-5/3} \ll 1$ and make sure that k_0 is large enough so that $t_k \leq \delta$. Integrating (3-52) from $t = 0$ to t_k,

$$\langle \phi_k, \phi_k^{\mathrm{app}}\rangle_f(t_k) \geq \big(1 - C(M, k, s)(1 + \lambda_k(1 + \lambda_k^2 t_k))t_k\big)\|\phi_{k,0}\|_{L^2_{f_0}} \geq \tfrac{1}{2}\|\phi_{k,0}\|_{L^2_{f_0}},$$

by taking λ_k larger if necessary. On the other hand, we write

$$\langle \phi_k, \phi_k^{\mathrm{app}}\rangle_f(t_k) = \frac{1}{(-1)^{s/2}A_k^{s/3}\lambda_k^s}\langle \phi_k, \partial_x^s(f^{(-2\sigma_c+s)/3}\phi_k^{\mathrm{app}})\rangle(t_k) + \langle f^{-\sigma_c/3}\phi_k, f^{-\sigma_c/3}\phi_k^{\mathrm{small}}\rangle(t_k).$$

Using the above estimates for ϕ_k and ϕ_k^{small} at $t = t_k$, for λ_k sufficiently large, the last term on the right-hand side is bounded by $\frac{1}{4}\|\phi_{k,0}\|_{L^2_{f_0}}$. We may therefore obtain

$$\begin{aligned}\tfrac{1}{4}\|\phi_{k,0}\|_{L^2_{f_0}} &\le A_k^{-s/3}\lambda_k^{-s}\|\partial_x^s\phi_k(t_k)\|_{L^\infty}\|f^{(-2\sigma_c+s)/3}\phi_k^{\mathrm{app}}(t_k)\|_{L^1}\\ &\le C(M,k,s)A_k^{-\sigma_c/3}\lambda_k^{-s}\exp\bigl(-3\beta_k(t)A_k^{2/3}\bigl(-\sigma_c+s+\tfrac{1}{2}\bigr)\lambda_k^{2-5/3}\bigr)\|\partial_x^s\phi_k(t_k)\|_{L^\infty}.\end{aligned}$$

Recalling that $\|\phi_{k,0}\|_{L^2_{f_0}} \ge c(k)\exp(-\lambda_k^{1/8})$, we arrive at the lower bound

$$\|\phi_k(t_k)\|_{C^s} \ge c(M,k,s)\lambda_k^s\exp\bigl(3\beta_k(t)A_k^{2/3}\bigl(-\sigma_c+s+\tfrac{1}{2}\bigr)\lambda_k^{1/3}-\lambda_k^{-1/8}\bigr).$$

Finally choosing λ_k to be sufficiently large, we may guarantee that

$$M \ge \sup_{t\in[0,\delta']}\|\phi(t)\|_{C^s} \ge \|\phi_k(t_k)\|_{C^s} \to \infty \quad \text{as } k\to\infty.$$

This contradicts the finiteness of M in (3-50), which completes the proof of Theorem 1.6. □

Appendix: Takeuchi–Mizohata ill-posedness via duality

In this appendix, we show how an application of the duality (or generalized energy) argument from [Jeong and Oh 2022] and this paper leads to simple proofs of quantitative ill-posedness results for first-order perturbations of the free Schrödinger equation related to the Takeuchi–Mizohata condition, including Proposition 1.16 (see Section A.2).

A.1. ***One-dimensional case.*** We begin with the one-dimensional first-order perturbation of the free Schrödinger equation,

$$i\partial_t u + \partial_{xx}u + b(x)\partial_x u = 0. \tag{A-1}$$

Fix $x_0 \in \mathbb{R}$. For $T > 0$ and $\mu \ge 1$, we define the weight

$$w(x) = \exp\biggl(\int_0^x \mathrm{Re}\,\frac{b(x')}{2}\,\mathrm{d}x'\biggr)$$

and the growth factor

$$M(T,\mu) = \inf_{(y,y_0):|y|\le\mu^{-1},|y_0|\le\mu^{-1}}\exp\biggl(\int_{x_0+y_0}^{x_0+2T+y}\mathrm{Re}\,\frac{b(x')}{2}\,\mathrm{d}x'\biggr).$$

Fix also $\psi_1 \in C^\infty(\mathbb{R})$ with $\operatorname{supp}\psi_1 \subseteq \{x : |x| < 1\}$ and $\|\psi_1\|_{L^2} = 1$. Given $\mu \ge 1$ (inverse spatial scale), define $\psi_{\mu,x_0} = \mu^{d/2}\psi_1(\mu(x-x_0))$. Given also $\lambda \ge 1$ (frequency, or inverse semiclassical parameter) — which in practice would be much larger than μ — define

$$\tilde{u}(t,x) = w^{-1}(x)e^{i\lambda x - i\lambda^2 t}\exp\biggl(-\int_0^{\lambda t} i\,\mathrm{Im}\,b(x-2s)\,\mathrm{d}s\biggr)\psi_{\mu,x_0}(x-2\lambda t). \tag{A-2}$$

This is a wave packet that approximately solves (A-1); see (A-7)–(A-9) below.

Proposition A.1. *Let $\tilde{u}$ be as in* (A-2), *and let u_0 satisfy*

$$\int \operatorname{Re}(u_0 \overline{\tilde{u}(0)}) w^2 \, \mathrm{d}x = 1, \qquad \operatorname{supp} u_0 \subseteq [x_0 - \mu^{-1}, x_0 + \mu^{-1}].$$

Then there exists at least one corresponding solution u of (A-1) *belonging to $L^\infty_{\mathrm{loc},t}(\mathbb{R}; L^2_w)$. Assume that it furthermore satisfies $u \in L^\infty_t([0, t_f]; L^2)$ with*

$$t_f \leq c\mu^{-1}, \tag{A-3}$$

where c is a constant depending only on $\|b\|_{C^{1,1}}$ and $\|\psi_1\|_{H^2}$. Then, $u(t)$ necessarily satisfies the pointwise lower bound

$$\|u(t)\|_{L^2} \geq \tfrac{1}{2} M(\lambda t, \mu) \|u_0\|_{L^2} \quad \textit{for all } 0 \leq t \leq t_f. \tag{A-4}$$

Proof. As discussed in Section 1.4, we consider the conjugation $v = wu$. Introducing the formally self-adjoint operator

$$\widetilde{\mathcal{L}} = \Delta + i \operatorname{Im} b(x) \partial_x + \frac{i}{2} \operatorname{Im} b_x,$$

we have the conjugation identity

$$(i\partial_t + \widetilde{\mathcal{L}})v = \Big(w^{-1}\,\partial_x^2 w + b w^{-1}\,\partial_x w + \frac{i}{2}\operatorname{Im} b_x\Big)v + w(i\partial_t + \Delta + b(x)\partial_x)u.$$

Under the assumption that $v(t) \in L^2$, we have for $t \geq 0$

$$\|v(t)\|_{L^2} \leq e^{C_0 t} \|v_0\|_{L^2}, \tag{A-5}$$

where C_0 depends only on $\|b\|_{C^{1,1}}$ and $\|\psi_1\|_{H^2}$. Moreover, we also have

$$\|(i\partial_t + \widetilde{\mathcal{L}})v(t)\|_{L^2} \leq C_0 e^{C_0 t} \|v_0\|_{L^2}, \tag{A-6}$$

where C_0 depends only on $\|b\|_{C^{1,1}}$ and $\|\psi_1\|_{H^2}$.

Next, we consider the standard wave packet $\tilde{v}$ for $i\partial_t + \widetilde{\mathcal{L}}$ obtained by solving (1-15) with $a = 1$, $\operatorname{Re} b = 0$, $\mathbf{\Phi}(0, x) = \lambda x$ and $\boldsymbol{a}(0, x) = \psi_{\mu, x_0}(x)$. It is given explicitly by

$$\tilde{v} = e^{i\lambda x - i\lambda^2 t} \exp\Big(-\int_0^{\lambda t} i \operatorname{Im} b(x - 2s)\, \mathrm{d}s\Big) \psi_{\mu, x_0}(x - 2\lambda t).$$

(Note, furthermore, that $\tilde{u} = w^{-1}\tilde{v}$.) By the definition, we clearly have, for all t,

$$\|\tilde{v}(t)\|_{L^2} = 1. \tag{A-7}$$

Moreover, by the support property of $\tilde{v}$, it follows that

$$\|w\tilde{v}(t)\|_{L^2} \leq \sup_{y:|y| \leq \mu^{-1}} w(x - 2\lambda t + y). \tag{A-8}$$

Finally, we consider the error incurred by $\tilde{v}$. A straightforward computation gives the following:

Lemma A.2. *We have*

$$\|(i\partial_t + \widetilde{\mathcal{L}})\tilde{v}\|_{L^2} \leq \widetilde{C}_0 \mu^2, \tag{A-9}$$

where $\widetilde{C}_0$ depends only on $\|b\|_{C^{1,1}}$ and $\|\psi_1\|_{H^2}$.

Given the above lemma, by the self-adjointness of $\widetilde{\mathcal{L}}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle v, \tilde{v}\rangle = -\langle (i\partial_t + \widetilde{\mathcal{L}})v, i\tilde{v}\rangle + \langle iv, i(i\partial_t + \widetilde{\mathcal{L}})\tilde{v}\rangle.$$

By (A-5), (A-6), (A-7) and (A-9), we have

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\langle v, \tilde{v}\rangle\right| \le (C_0 + \widetilde{C}_0\mu^2)e^{C_0 t}\|v_0\|_{L^2}.$$

Provided that (A-3) holds with c sufficiently small compared to C_0 and $\widetilde{C}_0$, we see that

$$\langle v, \tilde{v}\rangle(t) \ge \tfrac{1}{2}\|wu_0\|_{L^2} \quad \text{for all } 0 \le t \le t_f.$$

On the one hand, by duality (i.e., Cauchy–Schwartz) and (A-8),

$$\langle v, \tilde{v}\rangle(t) = \langle u, w\tilde{v}\rangle(t) \le \sup_{y:|y|\le\mu^{-1}} w(x - 2\lambda t + y)\|u(t)\|_{L^2}.$$

On the other hand, by the support property of u_0, we have

$$\|u_0\|_{L^2} \le \sup_{y:|y|\le\mu^{-1}} w^{-1}(x_0 - 2\lambda t + y)\|wu_0\|_{L^2}.$$

Combining the preceding three inequalities, we arrive at (A-4). □

Proof of Lemma A.2. We compute, with $\psi = \psi_{\mu,x_0}$,

$$i\partial_t\tilde{v} = \lambda^2\tilde{v} + \lambda\tilde{v}\,\mathrm{Im}\,b(x-2\lambda t) - 2i\lambda\tilde{v}\frac{\psi_x(x-2\lambda t)}{\psi(x-2\lambda t)},$$

$$\partial_x\tilde{v} = \left(i\lambda - \int_0^{\lambda t} i\,\mathrm{Im}\,b_x(x-2s)\,\mathrm{d}s + \frac{\psi_x(x-2\lambda t)}{\psi(x-2\lambda t)}\right)\tilde{v},$$

and

$$\begin{aligned}\partial_{xx}\tilde{v} = -\left(\lambda - \int_0^{\lambda t}\mathrm{Im}\,b_x(x-2s)\,\mathrm{d}s\right)^2\tilde{v} + 2\left(i\lambda - \int_0^{\lambda t} i\,\mathrm{Im}\,b_x(x-2s)\,\mathrm{d}s\right)\tilde{v}\frac{\psi_x(x-2\lambda t)}{\psi(x-2\lambda t)}\\ - \tilde{v}\int_0^{\lambda t} i\,\mathrm{Im}\,b_{xx}(x-2s)\,\mathrm{d}s + \tilde{v}\frac{\psi_{xx}(x-2\lambda t)}{\psi(x-2\lambda t)}.\end{aligned}$$

Then, after several direct cancellations, we have

$$\begin{aligned}(i\partial_t + \widetilde{\mathcal{L}})\tilde{v} = \lambda\left(\mathrm{Im}\,b(x-2\lambda t) - \mathrm{Im}\,b(x) + 2\int_0^{\lambda t}\mathrm{Im}\,b_x(x-2s)\,\mathrm{d}s\right)\tilde{v}\\ - \mathrm{Im}\,b\left(-\int_0^{\lambda t} i\,\mathrm{Im}\,b_x(x-2s)\,\mathrm{d}s + \frac{\psi_x(x-2\lambda t)}{\psi(x-2\lambda t)}\right)\tilde{v} + \left(\int_0^{\lambda t} i\,\mathrm{Im}\,b_x(x-2s)\,\mathrm{d}s\right)^2\tilde{v}\\ + 2\left(-\int_0^{\lambda t} i\,\mathrm{Im}\,b_x(x-2s)\,\mathrm{d}s\right)\tilde{v}\frac{\psi_x(x-2\lambda t)}{\psi(x-2\lambda t)} + \left(-\int_0^{\lambda t} i\,\mathrm{Im}\,b_{xx}(x-2s)\,\mathrm{d}s\right)\tilde{v}\\ + \frac{\psi_{xx}(x-2\lambda t)}{\psi(x-2\lambda t)}\tilde{v} + \frac{i}{2}\,\mathrm{Im}\,b_x\tilde{v}.\end{aligned}$$

Using

$$\int_0^{\lambda t} \operatorname{Im} b_x(x-2s)\,\mathrm{d}s = -\tfrac{1}{2}\operatorname{Im}(b(x-2\lambda t)-b(x))$$

we get a cancellation of remaining terms of order λ. Moreover, the same identity eliminates all integrals on the domain $[0, \lambda t]$. Using $\mu \geq 1$ to bound all the other terms by $O(\mu^2)$, the proof is complete. □

A.2. *Multidimensional case.* We consider the following equation on $\mathbb{R}^d$:

$$i\partial_t u + \Delta u + b^j(x)\partial_j u = 0. \tag{A-10}$$

Unfortunately, the proof of a pointwise lower bound in Proposition A.1 breaks down due to the lack of a simple physical space conjugation that removes $\operatorname{Re} b^j(x)\partial_j$. Instead, we shall prove two (conceptually) weaker statements using the duality method, including Proposition 1.16.

The first result is an unconditional *integrated* lower bound that is valid for nontrivially long (i.e., $t \gg \lambda^{-1}$) timescales. To state this result, given $x_0 \in \mathbb{R}^d$, $\omega_0 \in \mathbb{S}^{d-1}$, $\mu \geq 1$ and $T > 0$, define

$$M_{x_0,\omega_0}(T,\mu) = \inf_{y:|y|\leq \mu^{-1}} \exp\biggl(-\int_0^T \operatorname{Re} b^j(x_0+y-2s\omega_0)(\omega_0)_j\,\mathrm{d}s\biggr).$$

Fix $\psi_1 \in C^\infty(\mathbb{R}^d)$ with $\operatorname{supp}\psi_1 \subseteq \{x : |x| < 1\}$ and $\|\psi_1\|_{L^2} = 1$. Given $\mu \geq 1$, define

$$\psi_{\mu,x_0} = \mu^{d/2}\psi_1(\mu(x-x_0)).$$

Given also $\lambda \geq 1$, define (see [Mizohata 1985, §VII.2])

$$\tilde{u}(t,x) = e^{i\lambda\omega_0\cdot x - i\lambda^2 t}\exp\biggl(-\int_0^{\lambda t} b^j(x-2s\omega_0)(\omega_0)_j\,\mathrm{d}s\biggr)\psi_{\mu,x_0}(x-2\lambda\omega_0 t).$$

Proposition A.3. *Let $u \in L_t^\infty([0,t_f]; L^2)$ be a solution to equation* (A-10) *with initial data u_0 satisfying $\langle u_0, \tilde{u}(0)\rangle = 1$, where $\tilde{u}$ is determined from μ, ν and ψ_1 as above. Then as long as*

$$\mu \leq c\lambda^{1/3} \quad \textit{and} \quad t_f \leq c\lambda^{-2/3}, \tag{A-11}$$

where c is a constant depending only on $\|b\|_{C^{1,1}}$ and $\|\psi_1\|_{H^2}$, we have that $u(t)$ necessarily satisfies the averaged lower bound

$$\frac{1}{t_f}\int_0^{t_f} \frac{(1+\mu^{-1}\lambda t)^2}{(1+\mu^{-1}\lambda t_f)^2} M_{x_0,\omega_0}(\lambda t,\mu)^{-1}\|u(t)\|_{L^2}\,\mathrm{d}t \geq \tfrac{1}{6}\|u_0\|_{L^2}. \tag{A-12}$$

An example of an initial data u_0 satisfying the above hypothesis is, of course, $u_0 = \tilde{u}(0)$, in which case u is expected to behave like $\tilde{u}$. An argument similar to the proof of (A-14) shows that

$$\|\tilde{u}(t)\|_{L^2} \leq \sup_{y:|y|\leq\mu^{-1}} \exp\biggl(-\int_0^{\lambda t} \operatorname{Re} b^j(x_0+y-2s\omega_0)(\omega_0)_j\,\mathrm{d}s\biggr)\|\tilde{u}(0)\|_{L^2}.$$

Thence, provided we choose μ^{-1} to be sufficiently small depending on b, (A-12) is sharp for $\tilde{u}$ up to a constant.

Proof. We introduce $\mathcal{L}$ and its formal L^2-adjoint $\mathcal{L}^*$ (in operator notation),

$$\mathcal{L} = \Delta + b^j(x)\partial_j, \quad \mathcal{L}^* = \Delta - \partial_j \bar{b}^j(x).$$

The basis of the proof of Proposition A.3 is the generalized energy identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u_1, u_2\rangle = -\langle (i\partial_t + \mathcal{L})u_1, iu_2\rangle - \langle iu_1, (i\partial_t + \mathcal{L}^*)u_2\rangle, \tag{A-13}$$

which is a consequence of the Leibniz rule for ∂_t and $0 = \langle i\mathcal{L}u_1, u_2\rangle + \langle iu_1, i\mathcal{L}^* u_2\rangle$. A simple but important observation is that (A-13) holds even under the weak assumption $u_1 = u \in L^\infty_t([0, t_f]; L^2)$, provided that u_2 is nice enough, e.g., smooth in t, x and compactly supported in space for each fixed time.

The identity (A-13) motivates us to consider *not* a wave packet for $i\partial_t + \mathcal{L}$, but rather its adjoint $i\partial_t + \mathcal{L}^*$. Given $1 \le \mu \le \lambda$, consider

$$\tilde{u}^*(t, x) = e^{i\lambda\omega_0\cdot x - i\lambda^2 t} \exp\biggl(\int_0^{\lambda t} \bar{b}^j(x - 2s\omega_0)(\omega_0)_j \,\mathrm{d}s\biggr)\psi_{\mu,x_0}(x - 2\lambda\omega_0 t).$$

Observe that

$$\exp\biggl(\int_0^{\lambda t} \operatorname{Re} \bar{b}^j(x - 2s\omega_0)(\omega_0)_j \,\mathrm{d}s\biggr) \le M(\lambda t, \mu)^{-1} \quad \text{for } x \in \operatorname{supp} \psi_{\mu,x_0}(\cdot - 2\lambda\omega_0 t),$$

where we have introduced the abbreviation $M(\lambda t, \mu) = M_{x_0,\omega_0}(\lambda t, \mu)$. Hence, using also that $\operatorname{Re} \bar{b}^j = \operatorname{Re} b^j$, it follows that

$$\|\tilde{u}^*(t)\|_{L^2} \le M(\lambda t, \mu)^{-1}. \tag{A-14}$$

The following lemma quantifies the error $\boldsymbol{\epsilon}[\tilde{u}^*] = (i\partial_t + \Delta)\tilde{u}^* - \partial_j(\bar{b}^j(x)\tilde{u}^*)$ incurred by $\tilde{u}^*$.

Lemma A.4. *There exists a constant C_0, which depends only on $\|b\|_{C^{1,1}}$ and $\|\psi_1\|_{H^2}$, such that*

$$\|\boldsymbol{\epsilon}[\tilde{u}^*](t)\|_{L^2} \le C_0(\mu + \lambda t)^2 M(\lambda t, \mu)^{-1}. \tag{A-15}$$

We are now ready to implement the duality method. Assume for the moment that, for some $B > 0$, we have

$$\frac{1}{t_f}\biggl\| \frac{(1 + \mu^{-1}\lambda t)^2}{(1 + \mu^{-1}\lambda t_f)^2} M(\lambda t, \mu)^{-1} u \biggr\|_{L^1_t([0,t_f];L^2)} < B\|u_0\|_{L^2}. \tag{A-16}$$

By (A-13) and (A-15), we then have

$$\biggl|\frac{\mathrm{d}}{\mathrm{d}t}\langle u, \tilde{u}^*\rangle\biggr| \le C_0(\mu + \lambda t)^2 M(\lambda t, \mu)^{-1}\|u(t)\|_{L^2}.$$

Integrating in t and using the contradiction assumption, we arrive at

$$\langle u, \tilde{u}^*\rangle(t) \ge \|v_0\|_{L^2}(1 - BC_0\mu^2(1 + \mu^{-1}\lambda t_f)^2 t_f).$$

Suppose that

$$\mu \le \Bigl(\frac{1}{8BC_0}\Bigr)^{1/3}\lambda^{1/3}, \quad t_f \le \Bigl(\frac{1}{8BC_0}\Bigr)^{1/3}\lambda^{-2/3}. \tag{A-17}$$

Dividing into two cases $t_f \le \mu/\lambda$ and $t_f \ge \mu/\lambda$, it follows that

$$BC_0\mu^2(1+\mu^{-1}\lambda t_f)^2 t_f \le \tfrac{1}{2}.$$

Therefore,

$$\langle u, \tilde{u}^*\rangle(t) \ge \tfrac{1}{2}\|u_0\|_{L^2} \quad \text{for all } 0 \le t \le t_f.$$

Then, applying (A-14), we obtain the lower bound

$$\|u(t)\|_{L^2} \ge \tfrac{1}{2} M(\lambda t, \mu)\|u_0\|_{L^2}. \tag{A-18}$$

We now multiply both sides by $(1+\mu^{-1}\lambda t)^2 M(\lambda t, \mu)^{-1}$ and integrate. Since

$$\int_0^{t_f} (1+\mu^{-1}\lambda t)^2 \,\mathrm{d}t \ge \tfrac{1}{3} t_f (1+\mu^{-1}\lambda t_f)^2,$$

we arrive at

$$\frac{1}{t_f}\left\| \frac{(1+\mu^{-1}\lambda t)^2}{(1+\mu^{-1}\lambda t_f)^2} M(\lambda t, \mu)^{-1} u \right\|_{L_t^1([0,t_f];L^2)} \ge \tfrac{1}{6}\|u_0\|_{L^2}. \tag{A-19}$$

To complete the proof, we assume, for the purpose of contradiction, that (A-16) holds with $B = \frac{1}{6}$. Take $c = (3/(4C_0))^{1/3}$ in (A-11) so that (A-17) is satisfied. Then by the preceding argument, we arrive at (A-19), which is a contradiction that establishes (A-10). □

Proof of Lemma A.4. As in the proof of Lemma A.2, we compute with $\psi = \psi_{\mu,x_0}$ that

$$i\partial_t \tilde{u}^* = \lambda^2 \tilde{u}^* + i\lambda \bar{b}^j(x - 2\lambda t\omega_0)(\omega_0)_j \tilde{u}^* - 2i\lambda \frac{\omega_0 \cdot \nabla\psi}{\psi}\tilde{u}^*.$$

Introducing for simplicity

$$I_k(x) = -\int_0^{\lambda t} \partial_k \bar{b}^j(x - 2s\omega_0)(\omega_0)_j \,\mathrm{d}s, \qquad I(x) = (I_1, \ldots, I_d),$$

we have

$$\partial_k \tilde{u}^* = \left(i\lambda(\omega_0)_k - I_k + \frac{\partial_k\psi}{\psi}\right)\tilde{u}^*,$$

$$\Delta\tilde{u}^* = -\lambda^2|\omega_0|^2\tilde{u}^* - 2i\lambda\omega_0\cdot I\tilde{u}^* + 2i\lambda\frac{\omega_0\cdot\nabla\psi}{\psi}\tilde{u}^* + \mathcal{R}[\tilde{u}^*],$$

where

$$\mathcal{R}[\tilde{u}^*] = \left(|I|^2 - \frac{2I\cdot\nabla\psi}{\psi} + \frac{|\nabla\psi|^2}{\psi^2} + \sum_k \partial_k \frac{\partial_k\psi}{\psi} - \nabla\cdot I\right)\tilde{u}^*.$$

Then, after several direct cancellations, we have

$$\begin{aligned}(i\partial_t + \widetilde{\mathcal{L}})\tilde{u}^* - \partial_j(\bar{b}^j(x)\tilde{u}^*) = (-2i\lambda\omega_0\cdot I + i\lambda\bar{b}^j(x-2\lambda t\omega_0)(\omega_0)_j - i\lambda\bar{b}^j(x)(\omega_0)_j)\tilde{u}^* \\ + \mathcal{R}[\tilde{u}^*] - \bar{b}^j(x)I_j(x)\tilde{u}^* + \frac{\bar{b}^j(x)\partial_j\psi}{\psi}\tilde{u}^* - (\partial_j\bar{b}^j(x))\tilde{u}^*\end{aligned}$$

and, as in the one-dimensional case, we use that

$$\tfrac{1}{2}(\bar{b}^j(x-2\lambda t\omega_0) - \bar{b}^j(x)) = \int_0^{\lambda t} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\bar{b}^j(x-2s\omega_0)\,\mathrm{d}s = \sum_k I_k(x)(\omega_0)_k$$

to get cancellations among the $O(\lambda)$ terms.[7] It is now not difficult to see that the remaining terms are bounded in L^2 by the right-hand side of (A-15). □

As alluded to before, the second result we shall prove using essentially the same argument is Proposition 1.16, i.e., that the failure of the Takeuchi–Mizohata condition (see (1-24)) implies *norm inflation* for (A-10).

Proof of Proposition 1.16. Assume, for contradiction, that there exists $B_0 < +\infty$ such that, for every $u_0 \in L^2$,

$$\|u\|_{L^\infty([0,\delta];L^2)} \leq B_0 \|u_0\|_{L^2}. \tag{A-20}$$

By (1-24), there exists a sequence (x_n, ω_n, T_n) such that

$$M_{x_n,\omega_n}(T_n) := \exp\left(\int_0^{T_n} \operatorname{Re} b^j(x_n - 2s\omega_n)(\omega_n)_j \,\mathrm{d}x\right) \geq e^{2n}.$$

By restarting from the point $x_n + 2T\omega_n$, where $M_{x_n,\omega_n}(T) = 1$ if necessary, we may assume also that $M_{x_n,\omega_n}(T) \geq 1$ for all $0 \leq T \leq T_n$. Since $M_{x_n,\omega_n}(T_n, \mu) \to M_{x_n,\omega_n}(T_n)$ as $\mu \to \infty$, we may choose μ_n so that

$$M_{x_n,\omega_n}(T_n, \mu_n) \geq e^n.$$

We shall apply the argument in the proof of Proposition A.3 with the parameters

$$t_f = \frac{T_n}{\lambda_n}, \quad x_0 = x_n, \quad \omega_0 = \omega_n, \quad \mu = \mu_n, \quad \lambda = \lambda_n,$$

where λ_n shall be determined below. We denote by $\tilde{u}_n^*$ the wave packet for $i\partial_t + \widetilde{\mathcal{L}}^*$ with these parameters, and by u_n the solution to (A-10) with initial data $u_0 = \tilde{u}_n^*(0)$ satisfying (A-20). Taking λ_n to be large enough, we may guarantee that $t_f = T_n/\lambda_n \leq \delta$. Then by the contradiction assumption and the bound $M(T) \geq 1$, it follows that (A-16) is satisfied for $u = u_n$ with $B = C_1 B_0$, where C_1 depends only on $\|\partial b\|_{L^\infty}$. Furthermore, choosing λ_n sufficiently large depending on B_0, C_0, C_1 and T_n, we may ensure that (A-17) holds (here, it is important that the power of λ in the second inequality is greater than -1). Thence, it follows from (A-18) and our choices of parameters that

$$\left\| u_n\left(\frac{T_n}{\lambda_n}\right) \right\|_{L^2} \geq \tfrac{1}{2} e^n \|u_n(0)\|_{L^2}.$$

Taking $n \to \infty$, we arrive at a contradiction. □

Remark A.5. We note that when $d = 1$, Proposition 1.16 is essentially a consequence of Proposition A.1, although pedantically the notion of solution is slightly different due to the presence of a conjugation when $d = 1$. On the other hand, the preceding proof applies to all $d \geq 1$.

[7] Unlike the one-dimensional case, however, we cannot eliminate the integral on the domain $[0, \lambda t]$ in I. Hence, we let the right-hand side of (A-15) depend on λt.

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DISCRETE-TO-CONTINUUM CRYSTALLINE CURVATURE FLOWS

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We consider here a fully discrete variant of the implicit variational scheme for mean curvature flow, see Almgren et al. (1993) and Luckhaus and Sturzenhecker (1995), in a setting where the flow is governed by a crystalline surface tension defined by the limit of pairwise interactions energy on the discrete grid. The algorithm is based on a new discrete distance from the evolving sets, which prevents the occurrence of the spatial drift and pinning phenomena identified in Misiats and Yip (2016) and Braides et al. (2010) in a similar discrete framework. We provide the first rigorous convergence result holding in any dimension, for any initial set and for a large class of purely crystalline anisotropies, in which the spatial discretization mesh can be of the same order or coarser than the time step.

1. Introduction

We analyze a space- and time-discrete approximation of crystalline mean curvature flows of the form

$$V(x,t) = -\phi(\nu_{E(t)}(x))\kappa^{\phi}_{E(t)}(x), \quad x \in \partial E(t), \ t \geq 0, \tag{1-1}$$

for a class of crystalline norms ϕ. We recall that an anisotropy ϕ is said to be crystalline if and only if $\{\phi \leq 1\}$ is a polytope (or, equivalently, ϕ is the support function of a polytope). Moreover, in the current paper we restrict ourselves to the case where $\{\phi \leq 1\}$ is a zonotope with rational generators [McMullen 1971; Braides and Chambolle 2024]. Here $V(x,t)$ stands for the (outer) normal velocity of the boundary $\partial E(t)$ at x, ϕ is a crystalline norm on $\mathbb{R}^N$ representing the surface tension, $\kappa^{\phi}_{E(t)}$ is the crystalline mean curvature of $\partial E(t)$ associated to ϕ, and $\nu_{E(t)}$ is the outer unit normal to $\partial E(t)$. The evolution law (1-1) has been considered to describe some phenomena in materials science and crystal growth; see, e.g., [Gurtin 1993; Taylor 1978]. Our main result is a convergence result of the discrete approximation to the continuous evolution, as the time and space steps go to zero, even in the somewhat surprising case where the space step is greater or equal to the time-step.

From the mathematical point of view, the lack of regularity of the differential operator involved in the definition of the crystalline curvature (see [Bellettini et al. 2001; Bellettini and Paolini 1996]) is the main reason why the well-posedness of the crystalline mean curvature flow in every dimension has been a long-standing open problem. After some partial results (see for instance [Almgren and Taylor 1995; Angenent and Gurtin 1989; Bellettini et al. 2006; Caselles and Chambolle 2006; Giga and Giga 2001; Giga et al. 1998; 2014]), important breakthroughs have been obtained simultaneously in [Giga and Požár 2016; 2018; 2020], where a suitable crystalline theory of viscosity solutions was developed, and with a different approach in [Chambolle et al. 2017; 2019a; 2019b], where a new notion of distributional solutions was proposed.

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Keywords: crystalline flows, discrete-to-continuum approximation, minimizing movements, distributional solutions.

Let us focus on the definition of distributional solutions, referring to the nice review [Giga and Požár 2022] for further information on viscosity solutions to (1-1): we just note that the two notions are equivalent in the setting of [Chambolle et al. 2019a, Remark 6.1]. The exact definition of distributional solutions will be recalled in Definition 2.1, but when ϕ is smooth it can be motivated as follows: it is known (see for instance [Soner 1993] for the isotropic case) that $E(t)$ evolves according to (1-1) if and only if the signed distance function $d(\cdot\,,t) := \mathrm{sd}^{\phi^\circ}_{E(t)}$ to $\partial E(t)$ induced by the polar norm ϕ°,[1] satisfies

$$\partial_t d \geq \operatorname{div}(\nabla\phi(\nabla d)) \quad \text{in } \{d > 0\}, \tag{1-2}$$

$$\partial_t d \leq \operatorname{div}(\nabla\phi(\nabla d)) \quad \text{in } \{d < 0\} \tag{1-3}$$

in the viscosity sense. The idea of the new definition introduced in [Chambolle et al. 2017] is to reinterpret the equations above in the distributional sense. In particular, note that replacing $\nabla\phi(\nabla u)$ by a vector field $z \in L^\infty(\{d > 0\}; \mathbb{R}^N)$ such that $z(x) \in \partial\phi(\nabla d)$ for a.e. x, where $\partial\phi$ denotes the subdifferential of ϕ, means equations (1-2) and (1-3) make sense even when ϕ is crystalline. The corresponding notion of super- and subsolutions admits a comparison principle, which yields uniqueness of the motion up to fattening. Existence is obtained either by a variant of the minimizing movements scheme of [Almgren et al. 1993; Luckhaus and Sturzenhecker 1995] in the spirit of [Chambolle 2004], which consists in building a discrete-in-time evolution obtained by a recursive minimization procedure [Chambolle et al. 2017; 2019a], or by approximation with smooth anisotropies [Chambolle et al. 2019b]. We observe that the convergence of such time-discrete approaches to a motion characterized by (1-2)–(1-3) in the *viscosity sense* was shown in [Ishii 2014], including in the two-dimensional crystalline setting, while convergence in a distributional sense was established in [Caselles and Chambolle 2006] in the convex case only. Briefly, given a time step $h > 0$ and an initial closed set $E_0 =: E^{h,0}$, one defines $E^{h,k+1} = \{u^{h,k+1} \leq 0\}$, where $u^{h,k+1}$ is defined as the minimizer of a so-called "Rudin–Osher–Fatemi" [Rudin et al. 1992] problem:

$$u^{h,k+1} \in \operatorname{argmin}\left\{\int_{\mathbb{R}^N} \phi(Du) + \frac{1}{2h}\int_{\mathbb{R}^N} |u - \mathrm{sd}^{\phi^\circ}_{E^{h,k}}|^2\right\}. \tag{1-4}$$

The idea of the present work is to combine this discretization in time with a simultaneous discretization in space for the particular class of purely crystalline anisotropies ϕ of the form

$$\phi(\nu) = \sum_{i\in\mathscr{E}} \beta(i)|i\cdot\nu|, \tag{1-5}$$

where $\beta(i) > 0$ and $\mathscr{E} \subseteq \mathbb{Z}^N \setminus \{0\}$ is a finite set of generators such that $\operatorname{Span}\mathscr{E} = \mathbb{R}^N$. These kinds of convex polytopes are known in the literature as *rational zonotopes*. The class of rational zonotopes is dense in the class of symmetric convex sets if $N = 2$, while for $N \geq 3$ it is nowhere dense. This fact is due to the strong symmetry properties of zonotopes, as every facet of a zonotope is itself a zonotope [McMullen 1971]. Note however that the Euclidean ball may be approximated by rational zonotopes in every dimension.

We now specify the discrete setting we are interested in, referring the reader to [Braides and Solci 2021] for a more thorough introduction to related topics. We consider an ε-spaced square lattice $\varepsilon\mathbb{Z}^N$

[1] The norm is defined by $\phi^\circ(x) = \sup_{\phi(\nu)\leq 1} \nu\cdot x$ and satisfies $\phi(x) = \sup_{\phi^\circ(x)\leq 1} \nu\cdot x$.

and discrete functions $u : \varepsilon\mathbb{Z}^N \to \mathbb{R}$, and we define $u_i := u(i)$. We observe that we could also consider a general finite-dimensional Bravais lattice, at the expense of more tedious notation. A natural discrete version of total variation-like energies are those appearing in Ising systems, namely energies of the form

$$TV^\varepsilon_\beta(v) := \varepsilon^{N-1} \sum_{i,j\in\varepsilon\mathbb{Z}^N} \beta(i/\varepsilon - j/\varepsilon)|v_i - v_j|, \tag{1-6}$$

where β is as in (1-5), extended to 0 in $\mathbb{Z}^N \setminus \mathcal{E}$. Under the hypotheses above on β, the functionals TV^ε_β are shown to Γ-converge[2] as $\varepsilon \to 0$ to the total variation functional

$$TV_\phi(v) = \int_{\mathbb{R}^N} \phi(Dv),$$

where ϕ is as in (1-5); see, e.g., [Chambolle and Kreutz 2023]. It is thus natural to define a minimizing movements scheme based on TV^ε_β which is the discrete counterpart of the minimizing procedure (1-4) as follows: given $E_0 \subseteq \mathbb{R}^N$, we define $E^0_{\varepsilon,h} = \{i \in \varepsilon\mathbb{Z}^N \mid (i + [0, \varepsilon)^N) \cap E_0 \neq \varnothing\}$, and for every $k \in \mathbb{N}$ we let $u^{k+1}_{\varepsilon,h}$ be such that

$$u^{k+1}_{\varepsilon,h} \in \operatorname{argmin}\left\{ TV^\varepsilon_\beta(v) + \frac{1}{2h} \sum_{i\in\varepsilon\mathbb{Z}^N} |v_i - (\mathrm{sd}^k_{\varepsilon,h})_i|^2 \;\middle|\; v : \varepsilon\mathbb{Z}^N \to \mathbb{R} \right\}, \tag{1-7}$$

where $\mathrm{sd}^k_{\varepsilon,h}$ denotes a suitable signed ϕ°-distance function to $E^k_{\varepsilon,h}$ defined on $\varepsilon\mathbb{Z}^N$. (Actually, the energy in (1-7) is infinite and we would rather consider the Euler–Lagrange equation of the problem.) Then, one sets $E^{k+1}_{\varepsilon,h} := \{u^{k+1}_{\varepsilon,h} \le 0\}$.

The idea is to study the asymptotic behavior of the discrete evolutions $E^k_{\varepsilon,h}$ as both $\varepsilon, h \to 0$. A similar analysis has been performed in [Braides et al. 2010], in the planar case, for $\phi = \|\cdot\|_1$ and $\mathrm{sd}^k_{\varepsilon,h}$ the continuous signed distance function from the discrete sets $E^k_{\varepsilon,h}$ restricted to the lattice $\varepsilon\mathbb{Z}^N$; see also [Misiats and Yip 2016; Braides et al. 2016; Braides and Scilla 2013; Braides and Solci 2016; Malusa and Novaga 2018; Scilla 2020] for further related results. With this choice, if $\varepsilon \gg h$ it is easy to see that the dissipation-like term in (1-7)

$$\frac{1}{2h} \sum_{i\in\varepsilon\mathbb{Z}^N} |v_i - (\mathrm{sd}^{k+1}_{\varepsilon,h})_i|^2$$

forces the functions $u^k_{\varepsilon,h}$ to be constant as k varies, therefore producing *pinning* on the moving interfaces. Moreover, when the two scales ε, h are going to zero at the same speed it is shown in [Braides et al. 2010] that a direct implementation of the standard scheme, with the choice above for the distance, introduces a systematic error of order $\varepsilon = h$ at each step, which accumulates and produces a drift in the limiting evolution. As a result, low curvature shapes remain pinned, while sets with higher curvature evolve with a law which is a nonlinear modification of the crystalline curvature flow (1-1). Thus, the evolution law (1-1) can be approximated with the scheme of [Braides et al. 2010] only if $\varepsilon \ll h$. In [Misiats and Yip 2016], similar results are derived, still in dimension 2, for the isotropic (Euclidean) mean curvature flow.

[2]Note that we do not need to assume that the lattice generated by $\{e_k\}_{k=1,\dots,m}$ is $\mathbb{Z}^N$, which is necessary to ensure the equicoercivity of the discrete functionals.

We show in our main result, Theorem 5.2, that with a new appropriate definition of the distance $\mathrm{sd}^k_{\varepsilon,h}$ we can recover in the limit ε, $h \to 0$ the actual distributional solution to (1-1) for every initial set $E_0 \subseteq \mathbb{R}^N$, for every purely crystalline anisotropy ϕ of the form (1-5) with rational coefficients, in any dimension and irrespective of relative size of the space and time steps. In fact, the assumption of the rational character of β can be removed in the regime $\varepsilon \leq O(h)$. To the best of our knowledge this is the first general rigorous convergence result for a fully discrete scheme without restrictions on the dimension, on the initial sets and in which the spatial mesh is allowed to be of the same order or even coarser than the time step.

Let us further comment on the analysis carried out in [Braides et al. 2010] in the planar case; see also [Braides and Solci 2021] for many more references on the topic. One important change between these older results and ours is that we consider distributional solutions to the crystalline mean curvature flow (1-1) instead of relying on the characterization of the motion via ODEs, which dates back to [Almgren and Taylor 1995; Angenent and Gurtin 1989]. The latter notion of solutions is indeed suited only for planar evolutions, thus the limitation $N = 2$ in the past works. With the ODE definition and for $\phi = \|\cdot\|_1$, the authors of [Braides et al. 2010] precisely prove the following results: if $\varepsilon \ll h$ then the limiting motion is consistent with (1-1), while if $h \ll \varepsilon$ pinning happens for any nonempty initial data. As already mentioned, in the critical case $\varepsilon = h$, the limit planar motion is not driven by (1-1) but instead by a slightly modified nonlinear crystalline mean curvature flow, and pinning may happen for some particular (low curvature) initial data. This striking difference with our result may be (vaguely) justified by the following remark: While in [Braides et al. 2010] the focus is on discrete sets, we rather evolve, in accordance with the definition of distributional solutions, the *signed distance functions* to the boundaries. In this way we can effectively achieve a subpixel precision in our approximation, as $u_{\varepsilon,h}$ and the signed distance function carry more information than the evolving level set $\{u_{\varepsilon,h}(t) \leq 0\}$. Our new definition of the interpolated signed distance is detailed in Section 4.

The consistency result in this paper validates the numerical experiments which we carry out in Section 6 to illustrate our results. These experiments are derived from previous experiments in [Chambolle and Darbon 2009], which however used a different redistancing operation for which no consistency was proven. Numerical schemes based on the variational approach [Almgren et al. 1993; Luckhaus and Sturzenhecker 1995] have been introduced for crystal growth [Almgren 1993]. Since then, there have been many attempts to implement implicit schemes based on this approach for isotropic and anisotropic curvature flows in various settings [Chambolle 2004; Eto et al. 2012; Oberman et al. 2011; Požár 2018; Eto and Giga 2024]. We are however not aware of a formal convergence proof for these schemes in the fully discrete setting that does not rely on the consistency of the spatial discretization with respect to the time-discrete scheme (and hence, assuming $\varepsilon \ll h$, even if in practice these implementations seem very robust).

Many other techniques have been considered to simulate crystalline flows after [Taylor 1991; 1993]; see, e.g., [Girão 1995; Girão and Kohn 1996; Dziuk 1999] for the evolution of planar curves and [Novaga and Paolini 1999; Paolini and Pasquarelli 2000] for higher-dimensional algorithms.

Let us conclude this introduction with two comments. The first one concerns the hypothesis that ϕ is purely crystalline. It seems quite technical as it implies that the associated interaction function β (in the sense of (1-5)) has finite range. While this is not necessary to carry out the existence part for the discrete minimizing movements scheme, it is essential for building a calibration which yields a bound on the speed of Wulff shapes; see Appendix A. In practice, since the closed Wulff shape $\mathcal{W} := \{\phi^\circ \le 1\}$ is a finite Minkowski sum of (rational) segments (which is called a *zonotope*), we can effectively handcraft a calibration along the directions identified by these segments. It is a remarkable difference between this discrete setting and the continuous one, where instead the vector field $x/\phi^\circ(x)$ in $\mathbb{R}^N$ is the right calibration *for any* anisotropy ϕ.

The second one is on possible generalizations of the present analysis to more general evolution laws than (1-1). The more general evolution law which is shown to admit a unique distributional solution is

$$V(x,t) = \psi(\nu_{E(t)}(x))(-\kappa^{\phi}_{E(t)}(x) + f(x,t)), \quad x \in \partial E(t), \ \ t \ge 0, \tag{1-8}$$

where ψ is a norm (usually referred to as the *mobility*) and f is a forcing term; see [Chambolle et al. 2017; 2019a]. We expect most of the present analysis to be valid even if $\psi \ne \phi$, under suitable compatibility assumptions on ψ (see the same two works for details), and it should not be difficult to consider a driving force f as long as it is Lipschitz in space and globally bounded; see [Chambolle et al. 2019a] again.

The paper is organized as follows: In Section 2, we recall the definition of distributional crystalline curvature flows from [Chambolle et al. 2017; 2019a]. Then, we study the discrete "Rudin–Osher–Fatemi" problem and its Euler–Lagrange equation in Section 3. In Section 4, we introduce the discrete minimizing movement scheme, with our particular definition of the signed distance function. We study in detail the properties of these distances, then in Section 4.3 we analyze the particular case of an initial Wulff shape. In the continuous setting, it is well known that under the law (1-1) it decreases in a self-similar way with a speed proportional to the inverse of its radius. We show an estimate bounding the decay of the discrete Wulff shapes; it relies on the delicate construction of a calibration z for the Rudin–Osher–Fatemi problem with datum ϕ°, detailed in Appendix A.

Our main result — which is that, in the limit $\varepsilon, h \to 0$, the motion defined in Section 4 converges to a crystalline flow — is stated, and proved, in Section 5. We implemented the discrete scheme in two dimensions and show some numerical simulations in Section 6. Some technical results are collected in the appendices.

2. Distributional crystalline curvature flows

We recall the distributional formulation for the crystalline mean curvature motion of sets evolving with normal velocity (1-1) introduced in [Chambolle et al. 2017]; see also [Chambolle et al. 2019a]. Here and in what follows ϕ is any norm, ϕ° denotes the polar (or dual) norm of ϕ and, given a closed set $F \subseteq \mathbb{R}^N$, $\mathrm{dist}^{\phi^\circ}(\,\cdot\,, F)$ stands for the ϕ°-distance function from F defined by

$$\mathrm{dist}^{\phi^\circ}(x, F) := \min\{\phi^\circ(x-y) \mid y \in F\}.$$

Analogously, for any E, F closed, we set

$$\mathrm{dist}^{\phi^\circ}(E, F) := \min\{\phi^\circ(x-y) \mid x \in E,\ y \in F\}.$$

We recall that a sequence of closed sets $(E_k)_{k\geq 1}$ in $\mathbb{R}^N$ converges to a closed set E in the *Kuratowski sense* if the following conditions are satisfied:

(1) If $x_k \in E_k$ for each k, any limit point of $\{x_k\}$ belongs to E.

(2) For all $x \in E$ there exists a sequence $\{x_k\}$ such that $x_k \in E_k$ for each k and $x_k \to x$.

We will write in this case

$$E_k \xrightarrow{\mathcal{K}} E.$$

One can easily verify that $E_k \xrightarrow{\mathcal{K}} E$ if and only if (for any norm ψ) $\mathrm{dist}^\psi(\,\cdot\,, E_k) \to \mathrm{dist}^\psi(\,\cdot\,, E)$ locally uniformly in $\mathbb{R}^N$. Hence, by the Ascoli–Arzelà theorem, we have that any sequence of closed sets admits a converging subsequence in the Kuratowski sense (possibly to $\varnothing$, when $\mathrm{dist}^\psi(\,\cdot\,, E_k) \to +\infty$).

Definition 2.1. Let $E_0 \subseteq \mathbb{R}^N$ be a closed set. Let E be a closed set in $\mathbb{R}^N \times [0, +\infty)$, and for each $t \geq 0$ define $E(t) := \{x \in \mathbb{R}^N \mid (x, t) \in E\}$. We say that E is a *superflow* for (1-1) with initial datum E_0 if the following conditions are satisfied:

(a) $E(0) \subseteq E_0$.

(b) $E(s) \xrightarrow{\mathcal{K}} E(t)$ as $s \nearrow t$ for all $t > 0$.

(c) If $E(t) = \varnothing$ for some $t \geq 0$, then $E(s) = \varnothing$ for all $s > t$.

(d) Set $T^* := \inf\{t > 0 \mid E(s) = \varnothing \text{ for } s \geq t\}$ and

$$d(x, t) := \mathrm{dist}^{\phi^\circ}(x, E(t)) \quad \text{for all } (x, t) \in \mathbb{R}^N \times (0, T^*) \setminus E.$$

Then,

$$\partial_t d \geq \operatorname{div} z \tag{2-1}$$

in the distributional sense in $\mathbb{R}^N \times (0, T^*) \setminus E$ for a suitable $z \in L^\infty(\mathbb{R}^N \times (0, T^*))$ such that $z \in \partial\phi(\nabla d)$ a.e., $\operatorname{div} z$ is a Radon measure in $\mathbb{R}^N \times (0, T^*) \setminus E$, and

$$(\operatorname{div} z)^+ \in L^\infty(\{(x, t) \in \mathbb{R}^N \times (0, T^*) \mid d(x, t) \geq \delta\})$$

for every $\delta \in (0, 1)$.

We say that A, an open set in $\mathbb{R}^N \times [0, +\infty)$, is a *subflow* for (1-1) with initial datum E_0 if $\mathbb{R}^N \times [0, +\infty) \setminus A$ is a superflow for (1-1) with initial datum $\mathbb{R}^N \setminus \mathrm{int}(E_0)$.

Finally, we say that E, a closed set in $\mathbb{R}^N \times [0, +\infty)$, is a *weak flow* for (1-1) with initial datum E_0 if it is a superflow and if $\mathrm{int}(E)$ is a subflow,[3] both with initial datum E_0.

In [Chambolle et al. 2017] the next crucial inclusion principle between sub- and superflows is proven.

Theorem 2.2. *Let E be a superflow with initial datum E_0 and F be a subflow with initial datum F_0 in the sense of Definition 2.1. Assume that* $\mathrm{dist}^{\phi^\circ}(E^0, \mathbb{R}^N \setminus F^0) =: \Delta > 0$. *Then,*

$$\mathrm{dist}^{\phi^\circ}(E(t), \mathbb{R}^N \setminus F(t)) \geq \Delta \quad \textit{for all } t \geq 0$$

(*with the convention that* $\mathrm{dist}^{\phi^\circ}(G, \varnothing) = \mathrm{dist}^{\phi^\circ}(\varnothing, G) = +\infty$ *for any* G).

[3] Here we are taking the interior with respect to $\mathbb{R}^N \times [0, +\infty)$.

We also recall the corresponding notion of sub- and supersolutions to the level set flow associated with (1-1). In what follows $\mathrm{UC}(\mathbb{R}^N)$ stands for the space of uniformly continuous functions on $\mathbb{R}^N$.

Definition 2.3 (level set subsolutions and supersolutions). Let $u_0 \in \mathrm{UC}(\mathbb{R}^N)$. A lower-semicontinuous function $u : \mathbb{R}^N \times [0, +\infty) \to \mathbb{R}$ is called a *level set superflow* for (1-1), with initial datum u_0, if $u(\cdot, 0) \geq u_0$ and if for a.e. $\lambda \in \mathbb{R}$ the closed sublevel set $\{u(\cdot, t) \leq \lambda\}$ is a superflow for (1-1) in the sense of Definition 2.1, with initial datum $\{u_0 \leq \lambda\}$.

An upper-semicontinuous function $u : \mathbb{R}^N \times [0, +\infty) \to \mathbb{R}$ is called a *level set subflow* for (1-1), with initial datum u_0, if $-u$ is a level set superflow in the previous sense, with initial datum $-u_0$.

Finally, a continuous function $u : \mathbb{R}^N \times [0, +\infty) \to \mathbb{R}$ is called a *level set flow* for (1-1) if it is both a level set sub- and superflow.

Using Theorem 2.2, it is not difficult to deduce the following parabolic comparison principle between level set sub- and superflows, which yields in particular the uniqueness of level set flows (in the sense of Definition 2.3); see [Chambolle et al. 2019a].

Theorem 2.4. *Let $u_0, v_0 \in \mathrm{UC}(\mathbb{R}^N)$ and let u and v be a level set subflow starting from u_0 and a level set superflow starting from v_0, respectively. If $u_0 \leq v_0$, then $u \leq v$.*

We finally recall that in [Chambolle et al. 2017] (see also [Chambolle et al. 2019a]) the existence of level set flows is established by implementing a level-by-level minimizing movements scheme. This in turn yields existence and uniqueness (up to fattening) for weak flows. This is made precise in the following statement; see [Chambolle et al. 2017, Corollary 4.6; Chambolle et al. 2019a, Theorem 4.8].

Theorem 2.5. *Let $u_0 \in \mathrm{UC}(\mathbb{R}^N)$. Then the following hold*:

(i) *There exists a unique level set flow u in the sense of Definition 2.3 starting from u_0.*

(ii) *For all $\lambda \in \mathbb{R}$ the sets $\{(x, t) \mid u(x, t) \leq \lambda\}$ and $\{(x, t) \mid u(x, t) < \lambda\}$ are the maximal superflow and minimal subflow with initial datum $\{u_0 \leq \lambda\}$, respectively.*

(iii) *For all but countably many $\lambda \in \mathbb{R}$, the fattening phenomenon does not occur*; *that is*,

$$\begin{aligned} \{(x, t) \mid u(x, t) < \lambda\} &= \mathrm{int}(\{(x, t) \mid u(x, t) \leq \lambda\}), \\ \mathrm{cl}(\{(x, t) \mid u(x, t) < \lambda\}) &= \{(x, t) \mid u(x, t) \leq \lambda\}, \end{aligned} \tag{2-2}$$

where interior and closure are relative to space-time.

For all such λ, $\{(x, t) \mid u(x, t) \leq \lambda\}$ is the unique weak flow in the sense of Definition 2.1, starting from $\{u_0 \leq \lambda\}$.

The aim of this paper is to show that the convergence to the continuum level set flow also holds when the Euler implicit time discretization is combined with a suitable spatial discretization procedure.

3. The discrete "Rudin–Osher–Fatemi" problem

In this section, we describe our discrete setting. We then introduce and analyze the discrete variant (1-7) of the Rudin–Osher–Fatemi (ROF) problem (1-4).

3.1. ***Discrete function spaces and operators.*** For $\varepsilon > 0$, we define the function spaces $X_\varepsilon = \mathbb{R}^{\varepsilon\mathbb{Z}^N}$ and $Y_\varepsilon = \mathbb{R}^{\varepsilon\mathbb{Z}^N \times \varepsilon\mathbb{Z}^N}$. Given a function $u \in X_\varepsilon$ and a discrete "vector field" $z \in Y_\varepsilon$, with a slight abuse of notation we will write $u_i = u(i)$ and $z_{ij} = z(i, j),\ i, j \in \varepsilon\mathbb{Z}^N$. The discrete gradient $D_\varepsilon : X_\varepsilon \to Y_\varepsilon$ is defined, for $u \in X_\varepsilon$, as

$$(D_\varepsilon u)_{ij} = \frac{u_i - u_j}{\varepsilon}.$$

We denote its adjoint operator by $D_\varepsilon^* : Y_\varepsilon \to X_\varepsilon$, which is namely the operator that, for $\eta \in X_\varepsilon$ compactly supported and for $z \in Y_\varepsilon$, is defined as

$$\sum_i (D_\varepsilon^* z)_i \eta_i := \sum_{ij} z_{ij} (D_\varepsilon \eta)_{ij} = \sum_{ij} z_{ij} \frac{\eta_i - \eta_j}{\varepsilon},$$

where the indexes, here and throughout the paper, range over $\varepsilon\mathbb{Z}^N$ if not otherwise stated. In particular, taking $\eta = \chi_{\{i\}}$, one finds that

$$(D_\varepsilon^* z)_i = \sum_j \frac{z_{ij} - z_{ji}}{\varepsilon}, \tag{3-1}$$

which can be seen as a discrete divergence operator.

3.2. ***Discrete ROF problem.*** In this subsection we consider the discrete anisotropic ROF problem associated with the discrete total variation functional. Without loss of generality, we consider $\varepsilon = 1$ in this subsection, and define $X := X_1,\ Y := Y_1$ and $D := D_1$. Given a nonnegative $\beta \in X$, which will be called the *interaction function*, satisfying

$$\sum_{i \in \mathbb{Z}^N} \beta(i) =: c_\beta < +\infty, \tag{3-2}$$

we set $\alpha_{ij} = \beta(i - j)$ and, for any $u \in X$, we define

$$TV(u) = \sum_{i,j \in \mathbb{Z}^N} \alpha_{ij} |u_i - u_j| = \sum_{i,j} \alpha_{ij} |(Du)_{i,j}|. \tag{3-3}$$

We also consider the discrete perimeter $\mathcal{P}$ defined for every $E \subseteq \mathbb{Z}^N$ as

$$\mathcal{P}(E) := TV(\chi^E) = \sum_{i,j \in \mathbb{Z}^N} \alpha_{ij} |\chi_i^E - \chi_j^E|.$$

We also consider a suitable localization of the perimeter: namely, for any set $A \subseteq \mathbb{R}^N$, we define

$$\mathcal{P}(E; A) = \sum_{i \in A \cap \mathbb{Z}^N \text{ or } j \in A \cap \mathbb{Z}^N} \alpha_{ij} |\chi_i^E - \chi_j^E|.$$

Note that the quantities above may well be infinite.

Then, given $g \in X$, we consider the following problem: find a pair $(u, z) \in X \times Y$ such that

$$\begin{cases} D^* z + u = g, \\ z_{ij}(u_i - u_j) = \alpha_{ij} |u_i - u_j|,\ |z_{ij}| \le \alpha_{ij} \quad \text{for all } i, j \in \mathbb{Z}^N. \end{cases} \tag{3-4}$$

The equation above is the Euler–Lagrange equation of the discrete ROF functional

$$\mathrm{ROF}_g(v) = TV(v) + \frac{1}{2}\sum_{i\in\mathbb{Z}^N}(v_i - g_i)^2. \tag{3-5}$$

However, (3-4) makes sense also for those g such that $\mathrm{ROF}_g \equiv +\infty$. That (3-4) is the first-order condition for optimality in (3-5) follows from standard convex analysis: the idea is that, since

$$TV(v) = \sup\{\langle z, Dv\rangle \mid |z_{i,j}| \le \alpha_{i,j}\ \forall (i, j)\},$$

the subgradients $\partial TV(v)$ of TV at v are precisely given by the vectors D^*z for those z which realize the supremum in this expression. Then, for g with bounded support (such that there is at least some u with finite energy), (3-4) requires that $0 \in \partial\mathrm{ROF}_g(u)$, which by definition is the condition for the minimality of u.

We will also consider the following geometric minimization problem. Given $g \in X$, find

$$\min_{F\subseteq\mathbb{Z}^N} \mathcal{P}(F) + \sum_{i\in\mathbb{Z}^N}\chi_i^F g_i. \tag{3-6}$$

In order to deal with unbounded sets, possibly with infinite perimeter, we will consider the following notion of global minimality with respect to compactly supported perturbations:

Definition 3.1. A set $E \subseteq \mathbb{Z}^N$ is a global minimizer for the problem (3-6) if for every $R > 0$

$$\mathcal{P}(E; B_R) + \sum_{|i|<R}\chi_i^E g_i \le \mathcal{P}(F; B_R) + \sum_{|i|<R}\chi_i^F g_i \tag{3-7}$$

for every $F \subseteq \mathbb{Z}^N$ such that $F \triangle E \subseteq B_R$. Here $B_R = \{x \in \mathbb{R}^N \mid |x| < R\}$ is the open ball of radius R centered at the origin.

Proposition 3.2. *Let $g, g' \in X$ be such that $g' - g \ge \delta > 0$. Let E and E' be two global minimizers of problem* (3-7), *in the sense of Definition 3.1, corresponding to g and g', respectively. Then, $E' \subseteq E$.*

Proof. Let us define in the following $\chi := \chi^{E_s}$ and $\chi' := \chi^{E'_s}$. For a given $R > 0$ we define the competitor sets $F = (E_s \setminus B_R) \cup ((E'_s \cup E_s) \cap B_R)$ and $F' = (E'_s \setminus B_R) \cup ((E'_s \cap E_s) \cap B_R)$. By minimality of E_s and E'_s in B_R one has

$$\sum_{|i|<R \text{ or } |j|<R} \alpha_{ij}|\chi'_i - \chi'_j| + \sum_{|i|<R} g'_i(\chi'_i - \chi'_i \wedge \chi_i)$$
$$\le \sum_{\substack{|i|<R\\|j|<R}} \alpha_{ij}|\chi'_i \wedge \chi_i - \chi'_j \wedge \chi_j| + \sum_{\substack{|i|<R\\|j|\ge R}} (\alpha_{ij} + \alpha_{ji})|\chi'_i \wedge \chi_i - \chi'_j|, \tag{3-8}$$

$$\sum_{|i|<R \text{ or } |j|<R} \alpha_{ij}|\chi_i - \chi_j| + \sum_{|i|<R} g_i(\chi_i - \chi'_i \vee \chi_i)$$
$$\le \sum_{\substack{|i|<R\\|j|<R}} \alpha_{ij}|\chi'_i \vee \chi_i - \chi'_j \vee \chi_j| + \sum_{\substack{|i|<R\\|j|\ge R}} (\alpha_{ij} + \alpha_{ji})|\chi'_i \vee \chi_i - \chi_j|. \tag{3-9}$$

Using the inequality[4] $|a\wedge b-c\wedge d|+|a\vee b-c\vee d|\le |a-c|+|b-d|$ and summing together (3-8) and (3-9) we obtain

$$\sum_{\substack{|i|<R\\|j|\ge R}}(\alpha_{ij}+\alpha_{ji})(|\chi_i-\chi_j|+|\chi_i'-\chi_j'|)+2\sum_{|i|<R}(g_i'-g_i)(\chi_i'-\chi_i)^+ \le \sum_{\substack{|i|<R\\|j|\ge R}}(\alpha_{ij}+\alpha_{ji})(|\chi_i'\wedge\chi_i-\chi_j'|+|\chi_i'\vee\chi_i-\chi_j|). \tag{3-10}$$

We then remark that $|\chi_i'\wedge\chi_i-\chi_j'|\le|\chi_i'\wedge\chi_i-\chi_i'|+|\chi_i'-\chi_j'|=(\chi_i'-\chi_i)^++|\chi_i'-\chi_j'|$ and analogously $|\chi_i'\vee\chi_i-\chi_j|\le(\chi_i'-\chi_i)^++|\chi_i-\chi_j|$. Therefore, (3-10) implies

$$\sum_{|i|<R}(g_i'-g_i)(\chi_i'-\chi_i)^+\le\sum_{|i|<R}(\chi_i'-\chi_i)^+\sum_{|j|\ge R}(\alpha_{ij}+\alpha_{ji}). \tag{3-11}$$

Fix now $R_\delta>0$ such that

$$\sum_{|k|\ge R_\delta}\beta(k)\le\tfrac14\delta,$$

and define $V_R:=\sum_{|i|<R}(\chi_i'-\chi_i)^+$. Assuming $R>R_\delta$, for every $\ell<R$, we use (3-11) and $g+\delta\le g'$ to get

$$\begin{aligned}\delta V_R&\le\sum_{|i|<\ell}(\chi_i'-\chi_i)^+\sum_{|j|\ge R}(\alpha_{ij}+\alpha_{ji})+2c_\beta\sum_{\ell\le|i|<R}(\chi_i'-\chi_i)^+\\&\le 2\sum_{|i|<\ell}(\chi_i'-\chi_i)^+\sum_{|k|\ge R-\ell}\beta(k)+2c_\beta(V_R-V_\ell).\end{aligned} \tag{3-12}$$

Therefore, choosing $\ell=R-R_\delta$ in (3-12), we obtain

$$\tfrac12\delta V_R\le 2c_\beta(V_R-V_{R-R_\delta}), \tag{3-13}$$

which implies that for every $k,\ell\in\mathbb{N}$

$$V_{kR_\delta}\le\left(1-\frac{\delta}{4c_\beta}\right)^{\ell}V_{(k+\ell)R_\delta}. \tag{3-14}$$

Letting $\ell\to+\infty$, since $V_{(k+\ell)R_\delta}=O(\ell^N)$, we infer that $V_{kR_\delta}=0$ for every $k\in\mathbb{N}$. In particular, this implies that $(\chi'-\chi)^+=0$, i.e., $\chi'\le\chi$. □

We will prove the following theorem.

Theorem 3.3. *Given $g\in X$ there exists a unique function $u^g\in X$ and there exists a discrete vector field $z\in Y$ such that (u^g,z) is a solution of* (3-4)*. Moreover, the following comparison principle holds: if $g\le g'$ then $u^g\le u^{g'}$. Finally, for any $R>0$ and $s\in\mathbb{R}$, the sublevel set $E_s:=\{i\in\mathbb{Z}^N\mid u_i^g\le s\}$ is a global minimizer (in the sense of Definition 3.1) for* (3-6) *with g replaced by $g-s$.*

[4]Indeed, if $a\ge b$ and $c\ge d$, this is an equality, while if $a>b$ and $c<d$, one deduces that $b-d<a-d<a-c$, $b-d<b-c<a-c$ so that there exists $t\in(0,1)$ with $a-d=t(b-d)+(1-t)(a-c)$, $b-c=(1-t)(b-d)+t(a-c)$: the conclusion follows by convexity of $|\cdot|$.

Proof. **Step 1.** (existence). For every $n \in \mathbb{N}$ set $g^n := g\chi^{B_n}$ and note that $g^n \in \ell^2(\mathbb{Z}^N)$. Therefore, by standard methods and by strict convexity, the functional (3-5), with g replaced by g^n, admits a unique minimizer u^n and, as previously observed, the optimality condition is the existence of a discrete field z^n such that (u^n, z^n) solves (3-4) (with g^n in place of g). Note that, for any $k \in \mathbb{Z}^N$, by (3-4),

$$|u^n_k| \le |g^n_k| + |(D^* z)_k| \le |g_k| + c_\beta \quad \text{for every } n \in \mathbb{N}, \tag{3-15}$$

where the last inequality follows from the definition (3-1) and from $|z_{ij}| \le \alpha_{ij}$ and $|g^n| \le |g|$. Now, it is clear that we can extract a subsequence n_k and find (u, z) such that $u^{n_k}_i \to u_i$ and $z^{n_k}_{ij} \to z_{ij}$ as $k \to +\infty$. Clearly we have that $|z_{ij}| \le \alpha_{ij}$ and $z_{ij}(u_i - u_j) = \alpha_{ij}|u_i - u_j|$, and it is immediate to check that (u, z) satisfies (3-4).

Step 2. (minimality of the sublevel sets). Let $R > 0$, $s \in \mathbb{R}$ and let $F \subseteq \mathbb{Z}^N$ such that $E_s \triangle F \Subset B_R$. We first remark that $\alpha_{ij}|\chi^{E_s}_i - \chi^{E_s}_j| = -z_{ij}(\chi^{E_s}_i - \chi^{E_s}_j)$, which follows easily from the definition of E_s and $z_{ij}(u_i - u_j) = \alpha_{ij}|u_i - u_j|$.

We set $I_R := \{(i, j) \in \mathbb{Z}^N \times \mathbb{Z}^N \mid |i| < R \text{ or } |j| < R\}$ and compute

$$\begin{aligned}
\mathcal{P}(F; B_R) - \mathcal{P}(E_s; B_R) &= \sum_{(i,j)\in I_R} \alpha_{ij}|\chi^F_i - \chi^F_j| - \sum_{(i,j)\in I_R} \alpha_{ij}|\chi^{E_s}_i - \chi^{E_s}_j| \\
&\ge -\sum_{(i,j)\in I_R} z_{ij}(\chi^F_i - \chi^F_j) + \sum_{(i,j)\in I_R} z_{ij}(\chi^{E_s}_i - \chi^{E_s}_j) \\
&= \sum_{(i,j)\in I_R} z_{ij}(\chi^{E_s}_i - \chi^F_i - (\chi^{E_s}_j - \chi^F_j)) \\
&= \sum_{ij} z_{ij}(\chi^{E_s}_i - \chi^F_i - (\chi^{E_s}_j - \chi^F_j)),
\end{aligned} \tag{3-16}$$

where in the last equality we used the fact that $\chi^{E_s}_i = \chi^F_i$ if $|i| \ge R$. Noting that the function $\chi^{E_s} - \chi^F$ is compactly supported, we may use it as a test function for (3-4). Therefore, from (3-16) we deduce

$$\begin{aligned}
\mathcal{P}(F; B_R) - \mathcal{P}(E_s; B_R) &\ge \sum_{ij} z_{ij}(\chi^{E_s}_i - \chi^F_i - (\chi^{E_s}_j - \chi^F_j)) \\
&= \sum_i (\chi^{E_s}_i - \chi^F_i)(g_i - u_i) \ge \sum_{i\in E_s\setminus F} (g_i - s) - \sum_{i \in F\setminus E_s} (g_i - s),
\end{aligned}$$

which shows the minimality of E_s.

Step 3. (comparison and uniqueness for (3-4)). Assume $g \le g'$, and let (u, z) and (u', z') be two corresponding solutions for (3-4). Let $s > s'$, and recall that by Step 2 $\{u' \le s'\}$ and $\{u \le s\}$ are global minimizers for (3-6) according to Definition 3.1, with g replaced by $g' - s'$ and $g - s$, respectively. Since $g' - s' - (g - s) \ge s - s' > 0$, from Proposition 3.2 we obtain $\{u' \le s'\} \subseteq \{u \le s\}$. By the arbitrariness of s and s' we conclude that $u \le u'$. □

Remark 3.4. We remark that, given $g \in X$, clearly $u^{-g} = -u^g$.

4. The minimizing movements scheme

In this section we provide a combined spatial and time discretization of the flow (1-1) for a particular class of norms ϕ and show the convergence of the scheme to the continuum flow. In what follows, we consider $\{e_1, \dots, e_m\} \subseteq \mathbb{Z}^N$, a finite number of integer vectors spanning the whole $\mathbb{R}^N$, and set $\mathcal{E} = \{\pm e_k\}_{k=1}^m$. We let $\beta \in X$ be a nonnegative function such that

$$\beta(-i) = \beta(i) \quad \text{and} \quad \beta(i) > 0 \text{ if and only if } i \in \mathcal{E}.$$

One can naturally associate an anisotropy ϕ with the function β by setting

$$\phi(v) = \sum_{i\in\mathcal{E}} \beta(i)|i\cdot v| = \sum_{k=1}^{m} 2\beta(e_k)|v\cdot e_k|. \tag{4-1}$$

Note that, in particular,

$$\#\{k \in \mathbb{Z}^N \mid \beta(k) \neq 0\} < +\infty. \tag{4-2}$$

We recall that the ϕ-perimeter associated with (4-1),

$$P_\phi(E) = \int_{\partial^* E} \phi(\nu_E)\, d\mathcal{H}^{N-1},$$

(defined for every $E \subseteq \mathbb{R}^N$ of finite perimeter) is the Γ-limit (in a suitable sense) as $\varepsilon \to 0$ of the scaled discrete perimeters

$$\mathcal{P}^\varepsilon(E) := \varepsilon^{N-1} \sum_{i,j\in\varepsilon\mathbb{Z}^N} \alpha^\varepsilon_{ij} |\chi^E_i - \chi^E_j| = \varepsilon^N \sum_{i,j\in\varepsilon\mathbb{Z}^N} \alpha^\varepsilon_{i,j} |(D_\varepsilon \chi^E)_{i,j}|$$

defined for all $E \subseteq \varepsilon\mathbb{Z}^N$; see for instance [Braides and Chambolle 2024]. Here we have set

$$\alpha^\varepsilon_{ij} := \beta\left(\frac{i}{\varepsilon} - \frac{j}{\varepsilon}\right). \tag{4-3}$$

Given ϕ a norm on $\mathbb{R}^N$ and a closed set $E \notin \{\varnothing, \mathbb{R}^N\}$, we denote by $\mathrm{sd}^{\phi^\circ}_E$ the signed ϕ°-distance function from E and define it as

$$\mathrm{sd}^{\phi^\circ}_E(x) := \min_{y\in E} \phi^\circ(x-y) - \min_{y\notin E} \phi^\circ(x-y).$$

We also set $\mathrm{sd}^{\phi^\circ}_\varnothing \equiv +\infty$ and $\mathrm{sd}^{\phi^\circ}_{\mathbb{R}^N} \equiv -\infty$. We write

$$C_\phi = \min_{i\in\mathbb{Z}^N\setminus\{0\}} \phi^\circ(i) > 0 \tag{4-4}$$

and define the ϕ-Wulff shape $\mathcal{W}_R(x)$ of radius $R > 0$ and center $x \in \mathbb{R}^N$ as $\mathcal{W}_R(x) = \{y \in \mathbb{R}^N \mid \phi^\circ(x-y) \le R\}$.

4.1. ***A discrete redistancing operator.*** In this subsection we introduce a discrete proxy for the signed distance function to a set and study some of its properties.

Given $u \in X_\varepsilon$ we define the operators $d_\pm^{\varepsilon,\phi^\circ}$, $\mathrm{sd}_\pm^{\varepsilon,\phi^\circ}$, $\mathrm{sd}^{\varepsilon,\phi^\circ} : X_\varepsilon \to X_\varepsilon$ in the following way: letting $E = \{i \in \varepsilon\mathbb{Z}^N \mid u_i \le 0\}$, we first set

$$\begin{aligned}
(d_-^{\varepsilon,\phi^\circ}(u))_i &= \sup_{j\in\{u\ge 0\}} \{u_j - \phi^\circ(i-j)\},\\
(\mathrm{sd}_-^{\varepsilon,\phi^\circ}(u))_i &= \inf_{j\in\{u\le 0\}} \{(d_-^{\varepsilon,\phi^\circ}(u))_j + \phi^\circ(i-j)\},\\
(d_+^{\varepsilon,\phi^\circ}(u))_i &= \inf_{j\in\{u\le 0\}} \{u_j + \phi^\circ(i-j)\},\\
(\mathrm{sd}_+^{\varepsilon,\phi^\circ}(u))_i &= \sup_{j\in\{u\ge 0\}} \{(d_+^{\varepsilon,\phi^\circ}(u))_j - \phi^\circ(i-j)\},\\
(\mathrm{sd}^{\varepsilon,\phi^\circ}(u))_i &= \tfrac{1}{2}(\mathrm{sd}_+^{\varepsilon,\phi^\circ}(u))_i + \tfrac{1}{2}(\mathrm{sd}_-^{\varepsilon,\phi^\circ}(u))_i.
\end{aligned} \tag{4-5}$$

Note that $d_+^{\varepsilon,\phi^\circ}(u) = -d_-^{\varepsilon,\phi^\circ}(-u)$ and $\mathrm{sd}_+^{\varepsilon,\phi^\circ}(u) = -\mathrm{sd}_-^{\varepsilon,\phi^\circ}(-u)$.

We will say that $f \in X_\varepsilon$ is (L, ϕ°)-Lipschitz if for all $i, j \in \varepsilon\mathbb{Z}^N$ we have $|f_i - f_j| \le L\phi^\circ(i-j)$.

Remark 4.1. We assume in what follows that u is $(1, \phi^\circ)$-Lipschitz. Then, concerning $d_-^{\varepsilon,\phi^\circ}$ and $\mathrm{sd}_-^{\varepsilon,\phi^\circ}$, we remark that

$$d_-^{\varepsilon,\phi^\circ}(u) = \min\{f \in X_\varepsilon \mid f \ge u \text{ in } \{u \ge 0\},\ f \text{ is } (1,\phi^\circ)\text{-Lipschitz}\}, \tag{4-6}$$

and analogously

$$\mathrm{sd}_-^{\varepsilon,\phi^\circ}(u) = \max\{f \in X_\varepsilon \mid f \le d_-^{\varepsilon,\phi^\circ}(u) \text{ in } \{u \le 0\},\ f \text{ is } (1,\phi^\circ)\text{-Lipschitz}\}. \tag{4-7}$$

Correspondingly we have

$$\begin{aligned}
d_+^{\varepsilon,\phi^\circ}(u) &= \max\{f \in X_\varepsilon \mid f \le u \text{ in } \{u \le 0\},\ f \text{ is } (1,\phi^\circ)\text{-Lipschitz}\},\\
\mathrm{sd}_+^{\varepsilon,\phi^\circ}(u) &= \min\{f \in X_\varepsilon \mid f \ge d_+^{\varepsilon,\phi^\circ}(u) \text{ in } \{u \ge 0\},\ f \text{ is } (1,\phi^\circ)\text{-Lipschitz}\}.
\end{aligned} \tag{4-8}$$

In particular, the functions $d_\pm^{\varepsilon,\phi^\circ}(u)$, $\mathrm{sd}_\pm^{\varepsilon,\phi^\circ}(u)$, $\mathrm{sd}^{\varepsilon,\phi^\circ}(u)$ are also $(1, \phi^\circ)$-Lipschitz. Let us show (4-6), the other identities being analogous. To this aim, denote by $\hat{d}$ the function defined by the right-hand side of (4-6). Since $d_-^{\varepsilon,\phi^\circ}(u)$ is the pointwise supremum of $(1, \phi^\circ)$-Lipschitz functions, we clearly have that $d_-^{\varepsilon,\phi^\circ}(u)$ is itself $(1, \phi^\circ)$-Lipschitz. Moreover, testing with $j = i$ in the definition of $d_-^{\varepsilon,\phi^\circ}(u)$, we get $d_-^{\varepsilon,\phi^\circ}(u) \ge u$ in $\{u \ge 0\}$. Thus, we infer $\hat{d} \le d_-^{\varepsilon,\phi^\circ}(u)$. For the opposite inequality, let f be any function as in the minimization problem on the right-hand side of (4-6). Then for any $i \in \varepsilon\mathbb{Z}^N$ and $j \in \{u \ge 0\}$ we have

$$f_i \ge f_j - \phi^\circ(i-j) \ge u_j - \phi^\circ(i-j).$$

By maximizing with respect to $j \in \{u \ge 0\}$, we get $f \ge d_-^{\varepsilon,\phi^\circ}(u)$ and in turn, by the arbitrariness of f, $\hat{d} \ge d_-^{\varepsilon,\phi^\circ}(u)$, which concludes the proof of (4-6)

Since the functions $d_\pm^{\varepsilon,\phi^\circ}(u)$, $\mathrm{sd}_\pm^{\varepsilon,\phi^\circ}(u)$, $\mathrm{sd}^{\varepsilon,\phi^\circ}(u)$ are $(1, \phi^\circ)$-Lipschitz, from (4-6) it follows that

$$d_-^{\varepsilon,\phi^\circ}(u) \le u \ \text{ in } \varepsilon\mathbb{Z}^N, \quad d_-^{\varepsilon,\phi^\circ}(u) = u \ \text{ in } \{u \ge 0\}, \tag{4-9}$$

while (4-7) implies that

$$\mathrm{sd}_-^{\varepsilon,\phi^\circ}(u) \ge d_-^{\varepsilon,\phi^\circ}(u) \ \text{ in } \varepsilon\mathbb{Z}^N, \quad \mathrm{sd}_-^{\varepsilon,\phi^\circ}(u) = d_-^{\varepsilon,\phi^\circ}(u) \ \text{ in } \{u \le 0\}. \tag{4-10}$$

Reasoning in the same way, we see that

$$\begin{aligned} d_+^{\varepsilon,\phi^\circ}(u) &\ge u && \text{in } \varepsilon\mathbb{Z}^N, & d_+^{\varepsilon,\phi^\circ}(u) &= u && \text{in } \{u \le 0\},\\ \mathrm{sd}_+^{\varepsilon,\phi^\circ}(u) &\le d_+^{\varepsilon,\phi^\circ}(u) && \text{in } \varepsilon\mathbb{Z}^N, & \mathrm{sd}_+^{\varepsilon,\phi^\circ}(u) &= d_+^{\varepsilon,\phi^\circ}(u) && \text{in } \{u \ge 0\}. \end{aligned} \tag{4-11}$$

In particular we conclude

$$\mathrm{sd}^{\varepsilon,\phi^\circ}(u) \ge u \ \text{ in } \{u \ge 0\}, \quad \mathrm{sd}^{\varepsilon,\phi^\circ}(u) \le u \ \text{ in } \{u \le 0\}. \tag{4-12}$$

Note that (4-12) implies $\{\mathrm{sd}_+^{\varepsilon,\phi^\circ}(u) \ge 0\} \supseteq \{u \ge 0\}$, and (4-11) yields $\{\mathrm{sd}_+^{\varepsilon,\phi^\circ}(u) < 0\} \supseteq \{u < 0\}$; thus $\{\mathrm{sd}_+^{\varepsilon,\phi^\circ}(u) \ge 0\} = \{u \ge 0\}$ (and analogously for $\mathrm{sd}_-^{\varepsilon,\phi^\circ}$). Similarly, one shows that $\{\mathrm{sd}_\pm^{\varepsilon,\phi^\circ}(u) \le 0\} = \{u \le 0\}$. In particular, if the level set 0 of u is "fat", then this is preserved by these discrete "signed distance functions". Further properties of these discrete signed distance functions are presented in Lemma 4.3 below and in Remark 4.9

Moreover, it follows directly from the definition of $d_\pm^{\varepsilon,\phi^\circ}(u)$, $\mathrm{sd}_\pm^{\varepsilon,\phi^\circ}(u)$ that the function $\mathrm{sd}^{\varepsilon,\phi^\circ}(u)$ is invariant under integer translations, meaning that, for any $i, \tau \in \varepsilon\mathbb{Z}^N$,

$$\big(\mathrm{sd}^{\varepsilon,\phi^\circ}(u(\,\cdot\,+\tau))\big)_i = (\mathrm{sd}^{\varepsilon,\phi^\circ}(u))_{i+\tau}. \tag{4-13}$$

We now show that the redistancing operator $\mathrm{sd}^\varepsilon(u)$ is indeed a discrete approximation of the signed distance function to the 0-sublevel set of the function u.

Given a set $E \subseteq \varepsilon\mathbb{Z}^N$, we will denote with $\widehat{E} \subseteq \mathbb{R}^N$ the closed set defined by

$$\widehat{E} := E + [0, \varepsilon]^N.$$

Lemma 4.2. *Given a $(1, \phi^\circ)$-Lipschitz function $u \in X_\varepsilon$, we have*

$$\sup_{\varepsilon\mathbb{Z}^N \setminus E} |\mathrm{sd}_\pm^{\varepsilon,\phi^\circ}(u) - \mathrm{sd}_{\widehat{E}}^{\phi^\circ}| \le c_\phi \varepsilon \tag{4-14}$$

for a suitable positive constant c_ϕ, where $E = \{i \in \varepsilon\mathbb{Z}^N \mid u_i \le 0\}$. Moreover,

$$\mathrm{sd}_\pm^{\varepsilon,\phi^\circ}(u) \ge \mathrm{sd}_{\widehat{E}}^{\phi^\circ} - c_\phi\varepsilon \quad \textit{in } \varepsilon\mathbb{Z}^N. \tag{4-15}$$

Proof. In this proof we let c_ϕ denote a positive constant which depends on ϕ and that may change from line to line and also within the same line.

We start by introducing a slightly modified definition of the discrete signed distance $\mathrm{sd}^{\varepsilon,\phi^\circ}(u)$. Namely, setting

$$\begin{aligned} \partial_\varepsilon^+ E &:= \{i \in \varepsilon\mathbb{Z}^N \setminus E \mid \exists j \in E \text{ with } \|i-j\|_\infty = \varepsilon\},\\ \partial_\varepsilon^- E &:= \{i \in E \mid \exists j \in \varepsilon\mathbb{Z}^N \setminus E \text{ with } \|i-j\|_\infty = \varepsilon\}, \end{aligned} \tag{4-16}$$

we define

$$\tilde{d}_i = \begin{cases} \inf\{u_j + \phi^\circ(i-j) \mid j \in \partial_\varepsilon^- E\} & \text{for } i \in \varepsilon\mathbb{Z}^N \setminus E,\\ \sup\{u_j - \phi^\circ(i-j) \mid j \in \partial_\varepsilon^+ E\} & \text{for } i \in E. \end{cases} \tag{4-17}$$

We start by showing that

$$\begin{aligned} \mathrm{sd}_{\pm}^{\varepsilon,\phi^\circ}(u) &\geq \tilde{d} \quad \text{in } E, \\ \mathrm{sd}_{\pm}^{\varepsilon,\phi^\circ}(u) &\leq \tilde{d} \quad \text{in } \varepsilon\mathbb{Z}^N \setminus E. \end{aligned} \tag{4-18}$$

Indeed, we note that for every $i \in E$ we have

$$(\mathrm{sd}_{-}^{\varepsilon,\phi^\circ}(u))_i = (d_{-}^{\varepsilon,\phi^\circ}(u))_i = \sup_{j\in\{u\geq 0\}}\{u_j - \phi^\circ(i-j)\} \geq \sup_{j\in\partial_\varepsilon^+ E}\{u_j - \phi^\circ(i-j)\} = \tilde{d}_i.$$

On the other hand, recalling that $d_{-}^{\varepsilon,\phi^\circ}(u) \leq u$ in E, for every $i \in \varepsilon\mathbb{Z}^N \setminus E$ we see

$$(\mathrm{sd}_{-}^{\varepsilon,\phi^\circ}(u))_i = \inf_{j\in\{u\leq 0\}}\{(d_{-}^{\varepsilon,\phi^\circ}(u))_j + \phi^\circ(i-j)\} \leq \inf_{j\in\partial_\varepsilon^- E}\{u_j + \phi^\circ(i-j)\} = \tilde{d}_i.$$

Reasoning analogously we show the same inequalities between $\mathrm{sd}_{+}^{\varepsilon,\phi^\circ}$ and $\tilde{d}$ and thus prove (4-18).

Next, we prove

$$\sup_{\varepsilon\mathbb{Z}^N} |\tilde{d} - \mathrm{sd}_{\widehat{E}}^{\phi^\circ}| \leq c_\phi \varepsilon. \tag{4-19}$$

Recall that by definition (4-16), since $u \leq 0$ in E and $u > 0$ in $\varepsilon\mathbb{Z}^N \setminus E$ and since u is $(1, \phi^\circ)$-Lipschitz, we have

$$|u_j| \leq c_\phi \varepsilon \quad \text{for } j \in \partial_\varepsilon^{\pm} E.$$

Then, for every $i \in \varepsilon\mathbb{Z}^N \setminus E$, we have

$$\tilde{d}_i = \inf_{j\in\partial_\varepsilon^- E}\{u_j + \phi^\circ(i-j)\} \geq \inf_{j\in\partial_\varepsilon^- E} \phi^\circ(i-j) - c_\phi\varepsilon \geq \mathrm{sd}_{\widehat{E}}^{\phi^\circ}(i) - c_\phi\varepsilon. \tag{4-20}$$

On the other hand, by definition of $\mathrm{sd}_{\widehat{E}}^{\phi^\circ}$ there exists $x \in \partial\widehat{E}$ such that $\mathrm{sd}_{\widehat{E}}^{\phi^\circ}(i) = \phi^\circ(i-x)$. Let $k \in \varepsilon\mathbb{Z}^N$ be the closest point to x in $\partial_\varepsilon^- E$. We have

$$\mathrm{sd}_{\widehat{E}}^{\phi^\circ}(i) = \phi^\circ(i-x) \geq \phi^\circ(i-k) - c_\phi\varepsilon \geq \phi^\circ(i-k) + u_k - c_\phi\varepsilon \geq \tilde{d}_i - c_\phi\varepsilon. \tag{4-21}$$

Finally, equations (4-20) and (4-21) imply (4-19) outside E. The other case is analogous.

We now finally prove (4-14) outside E. From (4-18) and (4-19) we have

$$d_{-}^{\varepsilon,\phi^\circ}(u) = \mathrm{sd}_{-}^{\varepsilon,\phi^\circ}(u) \geq \tilde{d} \geq \mathrm{sd}_{\widehat{E}}^{\phi^\circ} - c_\phi\varepsilon \quad \text{in } E.$$

In particular, $\mathrm{sd}_{\widehat{E}}^{\phi^\circ} - c_\phi\varepsilon$ is an admissible competitor in (4-7), thus $\mathrm{sd}_{-}^{\varepsilon,\phi^\circ}(u) \geq \mathrm{sd}_{\widehat{E}}^{\phi^\circ} - c_\phi\varepsilon$ in $\varepsilon\mathbb{Z}^N$. On the other hand, in $\varepsilon\mathbb{Z}^N \setminus E$ we have (4-18); thus we conclude (4-14) for $\mathrm{sd}_{-}^{\varepsilon,\phi^\circ}(u)$. Concerning $\mathrm{sd}_{+}^{\varepsilon,\phi^\circ}(u)$, we note that by Remark 4.1 and the equation above we have

$$u \geq \mathrm{sd}_{-}^{\varepsilon,\phi^\circ}(u) \geq \mathrm{sd}_{\widehat{E}}^{\phi^\circ} - c_\phi\varepsilon \quad \text{in } E.$$

The function $\mathrm{sd}_{\widehat{E}}^{\phi^\circ} - c_\phi\varepsilon$ is therefore admissible in (4-8). Thus by maximality

$$d_{+}^{\varepsilon,\phi^\circ}(u) \geq \mathrm{sd}_{\widehat{E}}^{\phi^\circ} - c_\phi\varepsilon.$$

Since $\mathrm{sd}_{+}^{\varepsilon,\phi^\circ}(u) = d_{+}^{\varepsilon,\phi^\circ}(u)$ in $\varepsilon\mathbb{Z}^N \setminus E$, we conclude (4-14), taking also into account again (4-18) and (4-19). Finally, (4-15) follows by combining (4-14), (4-18) and (4-19). □

We conclude the subsection with some further properties of the operator $\mathrm{sd}^{\varepsilon,\phi^\circ}$.

Lemma 4.3. *Given $u \in X_\varepsilon$ that is $(1, \phi^\circ)$-Lipschitz, we have*

$$\mathrm{sd}^{\varepsilon,\phi^\circ}(-u) = -\mathrm{sd}^{\varepsilon,\phi^\circ}(u). \tag{4-22}$$

Furthermore, if $u_1, u_2 \in X_\varepsilon$ are $(1, \phi^\circ)$-Lipschitz and $u_1 \le u_2$ then

$$\mathrm{sd}^{\varepsilon,\phi^\circ}(u_1) \le \mathrm{sd}^{\varepsilon,\phi^\circ}(u_2). \tag{4-23}$$

Finally, for any $s > 0$ and $u \in X_\varepsilon$ that is $(1, \phi^\circ)$-Lipschitz, we have

$$\mathrm{sd}^{\varepsilon,\phi^\circ}(u - s) \le \mathrm{sd}^{\varepsilon,\phi^\circ}(u) - s. \tag{4-24}$$

Proof. For every $i \in \varepsilon\mathbb{Z}^N$ we have

$$(d_-^{\varepsilon,\phi^\circ}(-u))_i = \max_{j\in\{(-u)\ge 0\}}\{-u_j - \phi^\circ(i-j)\} = -\min_{j\in\{u\le 0\}}\{u_j + \phi^\circ(i-j)\} = -(d_+^{\varepsilon,\phi^\circ}(u))_i.$$

In turn,

$$(\mathrm{sd}_-^{\varepsilon,\phi^\circ}(-u))_i = \min_{j\in\{(-u)\le 0\}}\{(d_-^{\varepsilon,\phi^\circ}(-u))_j + \phi^\circ(i-j)\} = -\max_{j\in\{u\ge 0\}}\{(d_+^{\varepsilon,\phi^\circ}(u))_j - \phi^\circ(i-j)\} = -(\mathrm{sd}_+^{\varepsilon,\phi^\circ}(u))_i.$$

Reasoning in the same way for $d_+^{\varepsilon,\phi^\circ}$ and $\mathrm{sd}_+^{\varepsilon,\phi^\circ}$ we arrive at

$$\mathrm{sd}_\pm^{\varepsilon,\phi^\circ}(-u) = -\mathrm{sd}_\mp^{\varepsilon,\phi^\circ}(u) \tag{4-25}$$

and thus $\mathrm{sd}^{\varepsilon,\phi^\circ}(-u) = -\mathrm{sd}^{\varepsilon,\phi^\circ}(u)$. The monotonicity property (4-23) follows easily from the definitions in (4-5). The proofs of the other results also follow from the definitions in (4-5); we present only the one concerning (4-24). Fix $s > 0$, and let $u \in X_\varepsilon$ be a $(1, \phi^\circ)$-Lipschitz function. By definition of $d_-^{\varepsilon,\phi^\circ}(u)$ we have

$$(d_-^{\varepsilon,\phi^\circ}(u))_i = \sup_{j\in\{u\ge 0\}}\{u_j - \phi^\circ(i-j)\} \ge s + \sup_{j\in\{u\ge s\}}\{(u_j - s) - \phi^\circ(i-j)\} = (d_-^{\varepsilon,\phi^\circ}(u-s))_i + s.$$

Analogously,

$$\begin{aligned}(\mathrm{sd}_-^{\varepsilon,\phi^\circ}(u))_i &= \inf_{j\in\{u\le 0\}}\{(d_-^{\varepsilon,\phi^\circ}(u))_j + \phi^\circ(i-j)\} \\ &\ge s + \inf_{j\in\{u\le s\}}\{(d_-^{\varepsilon,\phi^\circ}(u-s))_j + \phi^\circ(i-j)\} = s + (\mathrm{sd}_+^{\varepsilon,\phi^\circ}(u-s))_i.\end{aligned}$$

Since the proofs for $d_+^{\varepsilon,\phi^\circ}(u)$ and $\mathrm{sd}_+^{\varepsilon,\phi^\circ}(u)$ are analogous, we conclude. □

4.2. ***The discrete scheme.*** We now describe our minimizing movements scheme, discretized in both time and space. A particularity of our scheme is that, in practice, it evolves the distance function to a set rather than the set itself. In particular, at the discrete level, it may depend on the initialization (even if in the limit the flow is geometric and only depends on the initial set).

Recalling (4-3), we rescale (3-4) on the lattice $\varepsilon\mathbb{Z}^N$ in the following way: We recall that $X_\varepsilon = \mathbb{R}^{\varepsilon\mathbb{Z}^N}$ and $Y_\varepsilon = \mathbb{R}^{\varepsilon\mathbb{Z}^N\times\varepsilon\mathbb{Z}^N}$. Given $g \in X_\varepsilon$ and a time step $h > 0$, the problem (3-4) now becomes to find

$(u, z) \in X_\varepsilon \times Y_\varepsilon$ satisfying

$$\begin{cases} hD^*_\varepsilon z + u = g & \text{on } \varepsilon\mathbb{Z}^N, \\ z_{ij}(u_i - u_j) = \alpha^\varepsilon_{ij}|u_i - u_j|, \ |z_{ij}| \le \alpha^\varepsilon_{ij}, \end{cases} \tag{4-26}$$

where $D^*_\varepsilon z$ is defined in (3-1). For ease of notation we assume $\varepsilon = \varepsilon(h)$, with $\varepsilon \to 0$ as $h \to 0$, and we will specify the dependence on h only.

Let $E_0 \subseteq \mathbb{R}^N$ be a closed set. We define $E^{h,0} := \{i \in \varepsilon\mathbb{Z}^N \mid (i + [0, \varepsilon)^N) \cap E_0 \neq \varnothing\}$. We note that

$$\widehat{E}^{h,0} \to E_0, \quad E^{h,0} \to E_0 \tag{4-27}$$

as $h \to 0$ in the Kuratowski sense, where with a slight abuse of notation we write $\widehat{E}^{h,0}$ to denote the set $E^{h,0} + [0, \varepsilon]^N$.

Given a closed set $E_0 \subseteq \mathbb{R}^N$ with $E_0 \notin \{\varnothing, \mathbb{R}^N\}$, we consider $u^{h,0}$, a $(1, \phi^\circ)$-Lipschitz function on $\varepsilon\mathbb{Z}^N$ which is negative inside $E^{h,0}$ and positive outside. For instance, we set

$$u^{h,0} := \tfrac{1}{2}C_\phi\varepsilon(1 - \chi_{E^{h,0}}) - \tfrac{1}{2}C_\phi\varepsilon\chi_{E^{h,0}},$$

where C_ϕ is defined in (4-4), so that $u^{h,0}$ is $(1, \phi^\circ)$-Lipschitz. Let us set $(z^{h,0})_{ij} = 0$ for all $i, j \in \varepsilon\mathbb{Z}^N$. Then, as long as $E^{h,k} \notin \{\varnothing, \mathbb{R}^N\}$, we can iteratively define $u^{h,k+1}$, $z^{h,k+1}$ for $k \in \mathbb{N}$ by solving (4-26) with $g = \mathrm{sd}^{\varepsilon,\phi^\circ}(u^{h,k})$; i.e.,

$$\begin{cases} hD^*_\varepsilon z^{h,k+1} + u^{h,k+1} = \mathrm{sd}^{\varepsilon,\phi^\circ}(u^{h,k}) & \text{on } \varepsilon\mathbb{Z}^N, \\ z^{h,k+1}_{ij}(u^{h,k+1}_i - u^{h,k+1}_j) = \alpha^\varepsilon_{ij}|u^{h,k+1}_i - u^{h,k+1}_j|, \ |z^{h,k+1}_{ij}| \le \alpha^\varepsilon_{ij}. \end{cases} \tag{4-28}$$

We recall that the redistancing operator $\mathrm{sd}^{\varepsilon,\phi^\circ}$ has been introduced in the previous section. We then set

$$E^{h,k+1} = \{i \in \varepsilon\mathbb{Z}^N \mid u^{h,k+1}_i \le 0\}.$$

If either $E^{h,k} = \varnothing$ or $E^{h,k} = \mathbb{R}^N$, we define $E^{h,k+1} = E^{h,k}$. We denote by T^*_h the first discrete time hk such that $E^{h,k} = \varnothing$, if any; otherwise we let $T^*_h = +\infty$. Analogously, we set T'^*_h to be the first discrete time hk such that $E^{h,k} = \mathbb{R}^N$, if any; otherwise we let $T'^*_h = +\infty$.

For ease of notation we will set

$$\begin{gathered} E^h(t) := E^{h,[t/h]} \subseteq \varepsilon\mathbb{Z}^N, \quad d^h(t) := \mathrm{sd}^{\varepsilon,\phi^\circ}(u^{h,[t/h]}) \in X_\varepsilon, \quad u^h(t) := u^{h,[t/h]} \in X_\varepsilon, \\ z^h(t) := z^{h,[t/h]} \in Y_\varepsilon, \quad \hat{d}^h(\,\cdot\,, t) := \mathrm{sd}^{\phi^\circ}_{\widehat{E}^h(t)} \in \mathrm{Lip}(\mathbb{R}^N), \end{gathered} \tag{4-29}$$

where again, with a slight abuse of notation, $\widehat{E}^h(t)$ stands for $\widehat{E^h(t)}$. Note that in the definition of $\hat{d}^h(\,\cdot\,, t)$ we are possibly using the convention $\mathrm{sd}^{\phi^\circ}_\varnothing \equiv +\infty$ and $\mathrm{sd}^{\phi^\circ}_{\mathbb{R}^N} \equiv -\infty$. Note also that $z^h(t)$ is well defined only for $0 \le t < \min\{T^*_h, T'^*_h\}$; however, if needed, we can set $z^h(t) = 0$ for $t \ge \min\{T^*_h, T'^*_h\}$.

Remark 4.4. If u is the solution of (4-26) with (L, ϕ°)-Lipschitz datum g, by standard arguments, based on the comparison principle and translation invariance, one can show that u satisfies the same Lipschitz bound of g. Indeed, given $j \in \varepsilon\mathbb{Z}^N$, the function $u(\,\cdot - j) \pm L\phi^\circ(j)$ solves (4-26) with datum $g(\,\cdot - j) \pm L\phi^\circ(j)$. By comparison one concludes, as $g(\,\cdot - j) - L\phi^\circ(j) \le g(\,\cdot\,) \le g(\,\cdot - j) + L\phi^\circ(j)$.

Lemma 4.5. *Let u^h, E^h and d^h be defined as in* (4-29). *Then, for every $t \geq 0$, $d^h(t)$ is $(1, \phi^\circ)$-Lipschitz and satisfies*

$$\begin{cases} u^h(t) \leq d^h(t) & \text{in } \varepsilon\mathbb{Z}^N \setminus E^h(t), \\ u^h(t) \geq d^h(t) & \text{in } E^h(t). \end{cases} \tag{4-30}$$

Proof. It follows from Remarks 4.1 and 4.4. □

Remark 4.6 (evolution of the complement). Let $E^h(t)$ and $u^h(t)$ be as in (4-29). We note that, if $F_0 \subseteq \mathbb{R}^N$ is a closed set such that $F^{h,0} = \varepsilon\mathbb{Z}^N \setminus E^{h,0}$, then the discrete evolution starting from F_0 coincides with $\{u^h(t) \geq 0\}$ for every $t \geq 0$. Indeed, denoting by v^h the discrete evolution starting from F_0, by definition $v^{h,0} = -u^{h,0}$. Thus recalling (4-22) we have

$$\mathrm{sd}^{\varepsilon,\phi^\circ}(v^{h,0}) = -\mathrm{sd}^{\varepsilon,\phi^\circ}(u^{h,0})$$

and, by uniqueness for (4-26), it follows that $v^h(h) = -u^h(h)$. Then we can iterate to conclude.

Remark 4.7 (comparison principle). Let E_0 and F_0 be closed sets in $\mathbb{R}^N$ such that $E^{h,0} \subseteq F^{h,0}$ (note that this condition is satisfied if $E_0 \subseteq F_0$). Let $E^h(t)$ and $F^h(t)$ be the corresponding discrete evolutions, and let $u^h(t)$ and $v^h(t)$ be the associated functions as in (4-29). Then, for every $t \geq 0$, we have $E^h(t) \subseteq F^h(t)$. This follows easily by iteration from the monotonicity property (4-23) and from the comparison principle for (4-26). One in fact could also consider the "open" discrete evolution given by

$$\mathring{E}^h(t) := \{u^h(t) < 0\} \quad \text{and} \quad \mathring{F}^h(t) := \{v^h(t) < 0\}.$$

Then, by the same argument one also has that $\mathring{E}^h(t) \subseteq \mathring{F}^h(t)$.

Remark 4.8 (avoidance principle). Let $E_0, F_0 \subseteq \mathbb{R}^N$ be closed sets such that $E^{h,0} \cap F^{h,0} = \varnothing$ (which is, for example, implied by $\mathrm{dist}(E_0, F_0) > c_\phi \varepsilon$ for a suitable $c_\phi > 0$). Let E^h, u^h and $\mathring{F}^h(t)$, v^h be the closed and open discrete evolutions starting from E_0 and F_0, respectively (where the open discrete evolution has been defined in Remark 4.7). Then,

$$\mathring{F}^h(t) \subseteq \varepsilon\mathbb{Z}^N \setminus E^h(t).$$

Indeed, $F^{h,0} \subseteq \varepsilon\mathbb{Z}^N \setminus E^{h,0}$ implies that $-u^{h,0} \leq v^{h,0}$, and thus by (4-22) and (4-23)

$$-\mathrm{sd}^{\varepsilon,\phi^\circ}(u^{h,0}) = \mathrm{sd}^{\varepsilon,\phi^\circ}(-u^{h,0}) \leq \mathrm{sd}^{\varepsilon,\phi^\circ}(v^{h,0}).$$

By the comparison principle for (4-26) and iterating one sees that $-u^h(t) \leq v^h(t)$ for all $t \geq 0$, which implies

$$\mathring{F}^h(t) = \{v^h(t) < 0\} \subseteq \{u^h(t) > 0\} = \varepsilon\mathbb{Z}^N \setminus E^h(t).$$

Remark 4.9. We conclude this subsection by observing that we could have made different choices of the distance function without affecting the final convergence result. In definition (4-5) we could have set

$$\begin{aligned} (d^<(u))_i &= \inf_{j \in \{u<0\}} \{u_j + \phi^\circ(i-j)\}, & (d^\leq(u))_i &= \inf_{j \in \{u\leq 0\}} \{u_j + \phi^\circ(i-j)\}, \\ (\mathrm{sd}^<(u))_i &= \sup_{j \in \{u\geq 0\}} \{(d^<(u))_j - \phi^\circ(i-j)\}, & (\mathrm{sd}^\leq(u))_i &= \sup_{j \in \{u>0\}} \{(d^<(u))_j - \phi^\circ(i-j)\}. \end{aligned} \tag{4-31}$$

One can see that $\mathrm{sd}^{\le}(u)$ mimics the signed distance function to the boundary of $\{u \le 0\}$ while $\mathrm{sd}^{<}(u)$ mimics the signed distance function to the boundary of $\{u < 0\}$. Defining the algorithm as in (4-28) but with $\mathrm{sd}^{<}$, $\mathrm{sd}^{\le}$ replacing $\mathrm{sd}^{\varepsilon,\phi^\circ}$, adapting our proof one can conclude the same convergence result. Let us further comment on the relation between $\mathrm{sd}^{\varepsilon,\phi^\circ}$, $\mathrm{sd}^{\le}$, $\mathrm{sd}^{<}$. One can prove that, for any $(1,\phi^\circ)$-Lipschitz function $u \in X_\varepsilon$,

$$\mathrm{sd}^{\le}(u) \le \mathrm{sd}_-^{\varepsilon,\phi^\circ}(u) \le \mathrm{sd}_+^{\varepsilon,\phi^\circ}(u) \le \mathrm{sd}^{<}(u). \tag{4-32}$$

Thus, between the many possible choices we could have performed in (4-5), it turns out that $\mathrm{sd}^{<}$ is the "maximal" one, while $\mathrm{sd}^{\le}$ is the "minimal". Indeed, let us show that $\mathrm{sd}_-^{\varepsilon,\phi^\circ}(u) \le \mathrm{sd}_+^{\varepsilon,\phi^\circ}(u)$. By definition (4-5) and equations (4-9) and (4-11), for every $i \in \{u \ge 0\}$ we have

$$(\mathrm{sd}_-^{\varepsilon,\phi^\circ}(u))_i = \inf_{j\in\{u\le 0\}}\{(d_-^{\varepsilon,\phi^\circ}(u))_j + \phi^\circ(i-j)\} \le \inf_{j\in\{u\le 0\}}\{u_j + \phi^\circ(i-j)\} = (\mathrm{sd}_+^{\varepsilon,\phi^\circ}(u))_i.$$

Reasoning analogously, for every $i \in \{u \le 0\}$ we have

$$(\mathrm{sd}_+^{\varepsilon,\phi^\circ}(u))_i = \sup_{j\in\{u\ge 0\}}\{(d_+^{\varepsilon,\phi^\circ}(u))_j - \phi^\circ(i-j)\} \ge \sup_{j\in\{u\ge 0\}}\{u_j - \phi^\circ(i-j)\} = (\mathrm{sd}_-^{\varepsilon,\phi^\circ}(u))_i.$$

Furthermore, for any two $(1,\phi^\circ)$-Lipschitz functions $u, u' \in X_\varepsilon$, if $u \le u' - s$ for $s > 0$ then

$$\mathrm{sd}^{<}(u) \le \mathrm{sd}^{\le}(u') - s.$$

In particular, this implies that, for any $(1,\phi^\circ)$-Lipschitz function $u \in X_\varepsilon$ and $s' > s$,

$$\mathrm{sd}^{\varepsilon,\phi^\circ}(u-s) \le \mathrm{sd}^{\varepsilon,\phi^\circ}(u-s') + s' - s.$$

Fix $u_0 \in X_\varepsilon$, a $(1,\phi^\circ)$-Lipschitz function. Using the properties above and standard arguments, one can see that for all but countably many $s \in \mathbb{R}$ the discrete evolutions starting from $\{u_0 \le s\}$ and corresponding to the three possible choices of distances in (4-32) coincide.

4.3. ***Discrete evolution of Wulff shapes.*** In this section we provide some control on the evolution speed of discrete Wulff shapes. The first result estimates the solution of (4-26) for the distance to the Wulff shape.

Lemma 4.10. *There exists a constant $C = C(\phi) > 0$ with the following property: if u is the solution of* (4-26) *with $g = \phi^\circ$, then $u \le \phi^h$, where $\phi^h \in X_\varepsilon$ is defined as*

$$\phi_i^h := \begin{cases} \phi^\circ(i) + \frac{Ch}{\phi^\circ(i)} & \text{if } \phi^\circ(i) \ge C(\sqrt{h}\vee\varepsilon), \\ C(\sqrt{h}\vee\varepsilon) + \frac{Ch}{\sqrt{h}\vee\varepsilon} & \text{otherwise}. \end{cases} \tag{4-33}$$

The proof of Lemma 4.10, based on the construction of a calibration, is postponed until Appendix A. We now prove a useful lemma used to estimate the redistancing step in our algorithm for functions of the form (4-33).

Lemma 4.11. *Let $R \ge \delta > 0$, and set*

$$u := (\phi^\circ - R) \vee \left(\tfrac{1}{2}\delta - R\right).$$

Then, for ε small enough depending on δ we have

$$\mathrm{sd}^{\varepsilon,\phi^\circ}(u) \le \phi^\circ - R + \hat{c}\varepsilon \quad \text{in } \varepsilon\mathbb{Z}^N \tag{4-34}$$

for a suitable positive constant $\hat{c}$, depending on ϕ. Furthermore, if we assume (B-1), *we have*

$$\mathrm{sd}^{\varepsilon,\phi^\circ}(u) \le \phi^\circ - R \quad \text{in } \varepsilon\mathbb{Z}^N. \tag{4-35}$$

Proof. By (4-32), it is sufficient to prove the claim for $\mathrm{sd}_+^{\varepsilon,\phi^\circ}$. We start by showing that $d_+^{\varepsilon,\phi^\circ}(u) = u$ and noting that by (4-11) it suffices to prove $d_+^{\varepsilon,\phi^\circ}(u) \le u$ in $\{u \ge 0\} = \{\phi^\circ \ge R\}$. Assuming (B-1), given $i \in \{u \ge 0\}$ we note that $\phi^\circ(i) \ge R$; thus by Lemma B.1 there exists $j \in \mathcal{W}_R \setminus \mathcal{W}_{R-2\varepsilon\ell_1}$ satisfying

$$\phi^\circ(j) + \phi^\circ(i-j) = \phi^\circ(i).$$

Taking $\varepsilon = \varepsilon(\delta)$ we can ensure that $R - 2\varepsilon\ell_1 \ge \frac{1}{2}\delta$, so that $j \in (\mathcal{W}_R \setminus \mathcal{W}_{\delta/2}) \cap \varepsilon\mathbb{Z}^N$. By definition (4-5) and the equation above we conclude that

$$d_+^{\varepsilon,\phi^\circ}(u) \le u_j + \phi^\circ(i-j) = \phi^\circ(j) - R + \phi^\circ(i-j) = \phi^\circ(i) - R,$$

and hence we have shown that $d_+^{\varepsilon,\phi^\circ}(u) = u$. Finally, from definition (4-5) and since $d_+^{\varepsilon,\phi^\circ}(u) = u = \phi^\circ - R$ on $\{u \ge 0\}$, we conclude by the triangular inequality that $\mathrm{sd}_+^{\varepsilon,\phi^\circ}(u) \le \phi^\circ - R$. All in all, we have obtained (4-35).

If instead (B-1) does not hold, using the first part of Lemma B.1 and reasoning as above, one concludes that

$$\mathrm{sd}_+^{\varepsilon,\phi^\circ}(u) \le \phi^\circ - R + \hat{c}\varepsilon$$

for a positive constant $\hat{c}$, and then the conclusion follows. □

Combining the two results above we can provide a bound on the evolution speed of Wulff shapes in the algorithm (4-28).

Proposition 4.12. *Assume either $\varepsilon \le O(h)$ or that* (B-1) *holds. For every $\delta > 0$ there exist positive constants ε_0, h_0, c_0 depending on δ with the following property: if $R \ge \delta$, $\varepsilon \le \varepsilon_0$ and $h \le h_0$, then the discrete evolution of $\mathcal{W}_R$ defined in* (4-28), *denoted $\mathcal{W}^h(t)$, satisfies*

$$\mathcal{W}^h(t) \supseteq \mathcal{W}_{R-c_0(t+\varepsilon)} \cap \varepsilon\mathbb{Z}^N \tag{4-36}$$

as long as $R - c_0(t+\varepsilon) \ge \frac{1}{2}\delta$.

Proof. Let $\mathring{\mathcal{W}}^h(t)$ be the open discrete evolution (see Remark 4.7) starting from the closure of $\mathcal{W}_R$ for some $R > 0$ and let $v^h(t)$ be the associated function as in the third equation in (4-29). Using the definition of $v^{h,0}$, (4-10) and the first definition in (4-5), it is easy to see that

$$(\mathrm{sd}_-^{\varepsilon,\phi^\circ}(v^{h,0}))_0 = (d_-^{\varepsilon,\phi^\circ}(v^{h,0}))_0 \le -R + c_\phi\varepsilon. \tag{4-37}$$

On the other hand, consider $i \in \{v^{h,0} \ge 0\}$ and let $x' \in \partial\mathcal{W}_R$ be such that

$$\phi^\circ(i-x') = \phi^\circ(i) - \phi^\circ(x') = \phi^\circ(i) - R.$$

Since there exists $j' \in \{v^{h,0} \le 0\}$ such that $\phi^\circ(j'-x') \le c_\phi \varepsilon$, then by the triangular inequality

$$\phi^\circ(i-j') \le \phi^\circ(i) - R + c_\phi \varepsilon.$$

Thus, using again definition (4-5), we get

$$(d_+^{\varepsilon,\phi^\circ}(v^{h,0}))_i \le \inf_{j \in \{v^{h,0} \le 0\}} \phi^\circ(i-j) \le \phi^\circ(i) - R + c_\phi \varepsilon,$$

which implies

$$(\mathrm{sd}_+^{\varepsilon,\phi^\circ}(v^{h,0}))_0 \le \sup_{j \in \{v^{h,0} \ge 0\}} (d_+^{h,0}(v^{h,0}))_j - \phi^\circ(j) \le -R + c_\phi \varepsilon. \tag{4-38}$$

Therefore, since $\mathrm{sd}^{\varepsilon,\phi^\circ}(v^{h,0})$ is a $(1,\phi^\circ)$-Lipschitz function, from (4-37) and (4-38) we get that

$$\mathrm{sd}^{\varepsilon,\phi^\circ}(v^{h,0}) \le \phi^\circ - R + c_\phi \varepsilon \quad \text{in } \varepsilon\mathbb{Z}^N.$$

By comparison and Lemma 4.10 we obtain

$$v^h(h) \le \phi^h - R + c_\phi \varepsilon, \tag{4-39}$$

where $\phi^h \in X_\varepsilon$ is defined in (4-33). Considering $R \ge \delta$ and $h = h(\delta)$, $\varepsilon = \varepsilon(\delta)$ small enough, the equation above implies that

$$v^h(h) \le (\phi^\circ - R + c_0 h + c_\phi \varepsilon) \vee \left(\tfrac{1}{2}\delta - R\right), \tag{4-40}$$

where $c_0 = 4C/\delta$, with C the same as in (4-33). Assume first (B-1). From Lemma 4.11, with R replaced by $R - c_0 h - c_\phi \varepsilon$, we get

$$\mathrm{sd}^{\varepsilon,\phi^\circ}(v^h(h)) \le \phi^\circ - R + c_0 h + c_\phi \varepsilon, \tag{4-41}$$

and therefore by comparison and Lemma 4.10 we get

$$v^h(2h) \le \phi^h - R + c_0 h + c_\phi \varepsilon,$$

which, reasoning as above, implies for $\varepsilon(\delta)$ and $h(\delta)$ small

$$v^h(2h) \le (\phi^\circ - R + 2c_0 h + c_\phi \varepsilon) \vee \left(\tfrac{1}{2}\delta - R\right).$$

Hence we can iterate the argument to conclude that

$$v^h(t) \le (\phi^\circ - R + c_0 t + c_\phi \varepsilon) \vee \left(\tfrac{1}{2}\delta - R\right) \tag{4-42}$$

as long as $R - c_0 t - c_\phi \varepsilon \ge \frac{1}{2}\delta$ and ε, h are sufficiently small. In particular, this implies (4-36) (possibly changing the value of c_0).

If instead (B-1) does not hold and $\varepsilon \le O(h)$, we obtain (4-39) and (4-40) in the same way. Then, using the first part of Lemma 4.11 we get

$$\mathrm{sd}^{\varepsilon,\phi^\circ}(v^h(h)) \le \phi^\circ - R + c_0 h + \hat{c}\varepsilon + c_\phi \varepsilon, \tag{4-43}$$

and then iterating we get

$$v^h(kh) \le (\phi^\circ - R + kc_0 h + k\hat{c}\varepsilon + c_\phi \varepsilon) \vee \left(\tfrac{1}{2}\delta - R\right).$$

Hence, recalling that $\varepsilon \le O(h)$, we conclude (4-42) and (4-36) as long as $R - c_0 t - c_\phi \varepsilon \ge \frac{1}{2}\delta$, with ε, h sufficiently small and possibly changing the value of c_0. □

As a corollary of the previous result, we deduce an estimate of the evolution of the distance function $\hat{d}^h$ at a distance from the evolving boundary, which we show next.

Corollary 4.13. *Let $E_0 \subseteq \mathbb{R}^N$ be a closed set, and consider the discrete evolution defined in* (4-29). *Assume either that $\varepsilon \leq O(h)$ or that* (B-1) *holds. Then, for every $\delta > 0$ there exist $c_0 = c_0(\delta) > 0$, $h_0 = h_0(\delta) > 0$ and $\varepsilon_0 = \varepsilon_0(\delta)$ such that the following holds. If $\hat{d}^h(x,t) \geq \delta$, then, for $s \geq t$,*

$$\hat{d}^h(x,s) \geq \hat{d}^h(x,t) - c_0(s - t + \varepsilon + h) \tag{4-44}$$

provided $0 < h \leq h_0$, $0 < \varepsilon < \varepsilon_0$ and as long as $\hat{d}^h(x,t) - c_0(s-t+\varepsilon+h) \geq \frac{1}{2}\delta$. Similarly, if $\hat{d}^h(x,t) \leq -\delta$, then, for $s \geq t$,

$$\hat{d}^h(x,s) \leq \hat{d}^h(x,t) + c_0(s - t + \varepsilon + h) \tag{4-45}$$

provided $0 < h \leq h_0$ and as long as $\hat{d}^h(x,t) + c_0(s - t + \varepsilon + h) \leq -\frac{1}{2}\delta$.

Proof. As usual, in this proof we denote by c_ϕ a positive constant depending on ϕ whose value may change from line to line and also within the same line.

Assume $\hat{d}^h(x,t) \geq \delta$. Without loss of generality we may assume $t \in [0, T_h^*)$ so that $\hat{d}^h(x,t)$ is finite. Denote by $x_\varepsilon \in \varepsilon\mathbb{Z}^N$ an element of the lattice such that $x \in x_\varepsilon + [0,\varepsilon)^N$. Note that there exists a constant $c_\phi > 0$ such that, setting $R := \hat{d}^h(x,t) - c_\phi\varepsilon$, one has $(\mathcal{W}_R(x_\varepsilon))^{h,0} \cap E^h(t) = \varnothing$ and $R > \frac{1}{2}\delta$ (if ε, h are sufficiently small, depending on δ). By the avoidance principle stated in Remark 4.8, we deduce that the open discrete evolution of $\mathcal{W}_R(x_\varepsilon)$, which we denote by $F(\tau)$, lies outside $E^h([t/h]h + \tau)$ for all $\tau \geq 0$. By Proposition 4.12 we deduce

$$F(\tau) \supseteq \mathcal{W}_{R - c_0(\tau+\varepsilon)}(x_\varepsilon) \cap \varepsilon\mathbb{Z}^N \tag{4-46}$$

provided that $R - c_0(\tau + \varepsilon) \geq \frac{1}{2}\delta$. Note that in particular

$$\mathcal{W}_{R - c_0(\tau+h+\varepsilon)}(x_\varepsilon) \cap \varepsilon\mathbb{Z}^N \subseteq \varepsilon\mathbb{Z}^N \setminus E^h(t+\tau)$$

as long as $R - c_0(\tau + h + \varepsilon) \geq \frac{1}{2}\delta$. In turn, we get

$$\hat{d}^h(x_\varepsilon, t+\tau) \geq R - c_0(\tau + h + \varepsilon) \tag{4-47}$$

provided $R - c_0(\tau + h + \varepsilon) \geq \frac{1}{2}\delta$ (for a possibly larger value of c_0). Recalling the definition of R and x_ε and possibly increasing the value of c_0, we infer

$$\hat{d}^h(x, t+\tau) \geq \hat{d}^h(x,t) - c_0(\tau + h + \varepsilon) \tag{4-48}$$

as long as $\hat{d}^h(x,t) - c_0(\tau + h + \varepsilon) \geq \delta$. The case $\hat{d}^h(x,t) \leq -\delta$ is analogous. □

5. Convergence of the scheme

We now are ready to study the convergence of the scheme as $\varepsilon \to 0$, $h \to 0$. Recall that we assumed that $\varepsilon = \varepsilon(h)$ goes to 0 as $h \to 0$. In this section we assume that either $\varepsilon \leq O(h)$ or that (B-1) holds. Let $E^h(\cdot)$ be the discrete evolution defined in (4-29), and recall that $\widehat{E}^h(\cdot) = E^h(\cdot) + [0,\varepsilon]^N$. We introduce the closed space-time tubes

$$\overline{E}^h := \mathrm{cl}(\{(x,t) \in \mathbb{R}^N \times [0,+\infty) : x \in \widehat{E}^h(t)\}), \tag{5-1}$$

where the closure is in space-time. Then, there exist A and E, open and closed (respectively) subsets of $\mathbb{R}^N \times [0, +\infty)$, with $A \subseteq E$, and a subsequence $h_k \to 0$ such that

$$\bar{E}^{h_k} \xrightarrow{\mathcal{K}} E \quad \text{and} \quad \mathbb{R}^N \times [0, +\infty) \setminus \operatorname{int}(\bar{E}^{h_k}) \xrightarrow{\mathcal{K}} \mathbb{R}^N \times [0, +\infty) \setminus A,$$

where taking the interior and Kuratowski convergence are meant in space-time. Let $E(t)$ and $A(t)$ be the t-time slice of E and A, respectively.

Note that if $E(t) = \varnothing$ for some $t \geq 0$, then (4-44) implies $E(s) = \varnothing$ for all $s \geq t$ so that we can define, as in Definition 2.1, the extinction time T^* of E. In the same fashion one can define the extinction time T'^* of $\mathbb{R}^N \times [0, +\infty) \setminus A$ (notice that at least one of T^* and T'^* is $+\infty$). Possibly extracting a further (not relabeled) subsequence and arguing exactly as in [Chambolle et al. 2017, Proof of Proposition 4.4] (and relying on the bounds (4-44) and (4-45)), one can in fact show the following result.

Proposition 5.1. *There exists a countable set $\mathcal{N} \subseteq (0, +\infty)$ such that $\hat{d}^{h_k}(\cdot, t)^+ \to \operatorname{dist}^{\phi^\circ}(\cdot, E(t))$ and $\hat{d}^{h_k}(\cdot, t)^- \to \operatorname{dist}^{\phi^\circ}(\cdot, \mathbb{R}^N \setminus A(t))$ locally uniformly for all $t \in (0, +\infty) \setminus \mathcal{N}$. Moreover, E and $\mathbb{R}^N \times [0, +\infty) \setminus A$ satisfy the continuity properties* (b) *and* (c) *of Definition 2.1. In addition, if $T^* > 0$, then $\{\hat{d}^{h_k}\}$ is locally uniformly bounded in $\mathbb{R}^N \times (0, T^*) \setminus E$, and analogously $\{\hat{d}^{h_k}\}$ is locally uniformly bounded in $\mathbb{R}^N \times (0, T'^*) \cap A$ if $T'^* > 0$. Finally, $E(0) = E_0$ and $A(0) = \operatorname{int}(E_0)$.*

Theorem 5.2. *The set E is a superflow in the sense of Definition 2.1 with initial datum E_0, while A is a subflow with initial datum E_0.*

The proof of this result follows the main lines of the proof of [Chambolle et al. 2017, Theorem 4.5]. One important difference with respect to the local, continuous setting is that the variable z^{h_k} is defined on the *edges* (i, j) between the vertices $i, j \in \varepsilon\mathbb{Z}^N$, and it is therefore unclear how to pass to the limit in this variable to obtain the limiting vector field $z(x, t)$. In order to do so, we associate with the discrete vector field $z_{ij}^h(t) \in Y_\varepsilon$ a vector field $\mathbf{z}^h(\cdot, t)$ in $\mathbb{R}^N$ defined as

$$\mathbf{z}^h(x, t) := \frac{1}{\varepsilon} \sum_{j \in \varepsilon\mathbb{Z}^N} z_{ij}^h(t)(i - j), \tag{5-2}$$

where $i \in \varepsilon\mathbb{Z}^N$ is such that $x \in i + [0, \varepsilon)^N$. Recall that we can take $z_{ij}^h(t)$ and thus $\mathbf{z}^h(\cdot, t)$ identically zero for $t \geq \min\{T_h^*, T_h'^*\}$. First, we show the following:

Lemma 5.3. *The vector field $\mathbf{z}^h$ satisfies*

$$\phi^\circ(\mathbf{z}^h) \leq 1. \tag{5-3}$$

Proof. Take $v \neq 0$ in $\mathbb{R}^N$. Recalling that $\phi(v) = \sum_{\ell \in \mathbb{Z}^N} \beta(\ell)|v \cdot \ell|$, one has, for any $x \in \mathbb{R}^N$ and $i \in \varepsilon\mathbb{Z}^N$ such that $x \in i + [0, \varepsilon)^N$,

$$\mathbf{z}^h(x, t) \cdot v = \frac{1}{\varepsilon} \sum_{j \in \varepsilon\mathbb{Z}^N} z_{ij}^h(t)(i - j) \cdot v = \sum_{\ell \in \mathbb{Z}^N} z_{i,i+\varepsilon\ell}^h(t)\ell \cdot v \leq \phi(v), \tag{5-4}$$

where we used that $|z_{i,i+\varepsilon\ell}^h(t)| \leq \beta(\ell)$. □

Hence, being globally bounded, this vector field is weakly-$*$ compact in $L^\infty(\mathbb{R}^N \times (0, T); \mathbb{R}^N)$ for any $T > 0$. The following lemma establishes a relationship between the divergence of its limits and the limits of the discrete divergences of z^h.

Lemma 5.4. *Assume that $z^{h_k} \overset{*}{\rightharpoonup} z$ in $L^\infty(\mathbb{R}^N \times (0, T); \mathbb{R}^N)$ along a subsequence $h_k \to 0$. Then, for every $\varphi \in C^\infty(\mathbb{R}^N \times (0, T))$ and $\eta \in C_c^\infty(\mathbb{R}^N \times (0, T))$, we have*

$$\lim_{k\to\infty} \left(\varepsilon_k^N \int \sum_{i,j\in\varepsilon_k\mathbb{Z}^N} z_{ij}^{h_k}(t)\eta(i,t) \frac{\varphi(i,t)-\varphi(j,t)}{\varepsilon_k} \,\mathrm{d}t \right) = \iint \eta z \cdot \nabla\varphi \,\mathrm{d}x \,\mathrm{d}t.$$

Proof. Let $\varphi \in C^\infty(\mathbb{R}^N \times (0, T))$ and $\eta \in C_c^\infty(\mathbb{R}^N \times (0, T))$, and write $S(t) = \operatorname{supp}(\eta(t))$ and $Q_k := [0, \varepsilon_k)^N$. We have

$$\varepsilon_k^N \sum_{i,j\in\varepsilon_k\mathbb{Z}^N} z_{ij}^{h_k}(t)\eta(i,t) \frac{\varphi(i,t)-\varphi(j,t)}{\varepsilon_k} = \varepsilon_k^N \sum_{i,j\in\varepsilon_k\mathbb{Z}^N} \frac{z_{ij}^{h_k}(t)}{\varepsilon_k} \eta(i,t)\nabla\varphi(x_{ij}) \cdot (i-j), \tag{5-5}$$

where x_{ij} belongs to the segment between i and j. Furthermore we have

$$\begin{aligned}
&\left| \varepsilon_k^N \sum_{i,j\in\varepsilon_k\mathbb{Z}^N} z_{ij}^{h_k}(t)\eta(i,t) \frac{\varphi(i,t)-\varphi(j,t)}{\varepsilon_k} - \sum_{i,j\in\varepsilon_k\mathbb{Z}^N} \frac{z_{ij}^{h_k}(t)}{\varepsilon_k} \int_{i+Q_k} \eta\nabla\varphi\cdot(i-j)\,\mathrm{d}x \right| \\
&\qquad \le \sum_{i,j\in\varepsilon_k\mathbb{Z}^N} \frac{\alpha_{ij}^{\varepsilon_k}}{\varepsilon_k} |\eta(i,t)| \int_{i+Q_k} |(\nabla\varphi(x_{ij},t) - \nabla\varphi(x,t))\cdot(i-j)| \,\mathrm{d}x + O(\varepsilon_k^N) \\
&\qquad \le 2\|\eta\|_\infty \sum_{i\in S(t)\cap\varepsilon_k\mathbb{Z}^N} \sum_{j\in\varepsilon_k\mathbb{Z}^N} \frac{\alpha_{ij}^{\varepsilon_k}}{\varepsilon_k} \int_{i+Q_k} |(\nabla\varphi(x_{ij},t) - \nabla\varphi(x,t))\cdot(i-j)| \,\mathrm{d}x + O(\varepsilon_k^N) \\
&\qquad \le c\varepsilon_k^N \sum_{i\in S(t)\cap\varepsilon_k\mathbb{Z}^N} \sum_{j\in\varepsilon_k\mathbb{Z}^N} \frac{\alpha_{ij}^{\varepsilon_k}}{\varepsilon_k} |i-j|^2 + O(\varepsilon_k^N) \\
&\qquad = c\varepsilon_k^{N+1} \sum_{\substack{i\in\mathbb{Z}^N \\ \varepsilon_k i\in S(t)}} \sum_{j\in\mathbb{Z}^N} \alpha_{ij}|i-j|^2 + O(\varepsilon_k^N) \\
&\qquad \le c\varepsilon_k^{N+1} \left(\sum_{\ell\in\mathbb{Z}^N} \beta(\ell)|\ell|^2 \right) (\#S(t)\cap\varepsilon_k\mathbb{Z}^N) + O(\varepsilon_k^N) \\
&\qquad \le c\varepsilon_k \sum_{\ell\in\mathbb{Z}^N} \beta(\ell)|\ell|^2 + O(\varepsilon_k^N),
\end{aligned} \tag{5-6}$$

where in the second line we used the Lipschitz property of η and (4-2), while in the fourth line we used the Lipschitz property of $\nabla\varphi$ and $|x_{ij} - x| \le (1+\sqrt{N})|i-j|$ for $i \ne j$ and $x \in i + Q_k$, and finally in the last line we used that $\#(S(t) \cap \varepsilon\mathbb{Z}^N) = O(\varepsilon_k^{-N})$, which holds locally uniformly in time. Moreover, note that the estimate provided above is uniform as t varies in compact subsets of $(0, T)$. Recalling (4-2), we conclude by integrating in time and sending $k \to \infty$. □

At this point, we may proceed with the proof of Theorem 5.2.

Proof of Theorem 5.2. As usual, in this proof we denote by c_ϕ a positive constant depending on ϕ whose value may change from line to line and also within the same line.

We only show that E is a superflow, as the subflow property of A can be proven analogously. Points (a), (b) and (c) of Definition 2.1 follow from Proposition 5.1. We are left with showing (d). Without loss of generality we may assume $T^* > 0$ (which follows from Corollary 4.13 if the initial set is not trivial). Note also that by Proposition 5.1 we have $\liminf_k T^*_{h_k} \geq T^*$.

Step 1. (proof of (2-1)). For $(x,t) \in \mathbb{R}^N \times (0,T^*) \setminus E$ we set $d(x,t) := \mathrm{dist}^{\phi^\circ}(\,\cdot\,, E(t))$. By Lemma 4.2 and Proposition 5.1 we have

$$\sup_{\varepsilon_k\mathbb{Z}^N \cap K} |d^{h_k}(t) - d(\,\cdot\,,t)| \to 0 \text{ as } k\to\infty \text{ for } t \in (0,T^*)\setminus \mathcal{N} \text{ and for any compact } K \subseteq \mathbb{R}^N \setminus E(t). \tag{5-7}$$

Moreover, d^{h_k} and d are locally uniformly bounded in $\mathbb{R}^N \times (0,T^*)\setminus E$. Set $z^{h_k}(\,\cdot\,,t) := 0$ for $t > T^*_{h_k}$ if $T^*_{h_k} < T^*$. Extracting a further subsequence, if needed, and recalling Lemma 5.3, we may assume that z^{h_k} converges weakly-$*$ in $L^\infty(\mathbb{R}^N \times (0,T^*);\mathbb{R}^N)$ to some vector-field z satisfying

$$\phi^\circ(z) \leq 1 \tag{5-8}$$

almost everywhere. Recall that by (4-30) we have $u^{h_k}(t) \leq d^{h_k}(t)$ in $\varepsilon_k\mathbb{Z}^N \setminus E^{h_k}(t)$, i.e., in the region where $d^{h_k}(t)$ is nonnegative. Combining with (4-28) (and recalling (4-29)) we infer that for $t < T^*_{h_k}$

$$-D^*_{\varepsilon_k} z^{h_k}(t+h_k) \leq \frac{d^{h_k}(t+h_k) - d^{h_k}(t)}{h_k} \quad \text{in } \varepsilon_k\mathbb{Z}^N \setminus E^{h_k}(t). \tag{5-9}$$

Consider a nonnegative test function $\varphi \in C^\infty_c((\mathbb{R}^N \times (0,T^*))\setminus E)$. If k is large enough, then the distance of the support of φ from $\bar{E}^{h_k}$ is bounded away from zero. In particular, d^{h_k} is finite and positive on $\operatorname{supp}\varphi$. We deduce from (5-9) that

$$\begin{aligned}
&\varepsilon_k^N \int \sum_{i\in\varepsilon_k\mathbb{Z}^N} \varphi(i,t)\Big(\frac{d_i^{h_k}(t+h_k) - d_i^{h_k}(t)}{h_k} + (D^*_{\varepsilon_k} z^{h_k}(t+h_k))_i\Big)\,\mathrm{d}t \\
&\quad = -\varepsilon_k^N \int \sum_{i\in\varepsilon_k\mathbb{Z}^N} \frac{\varphi(i,t)-\varphi(i,t-h_k)}{h_k} d_i^{h_k}(t)\,\mathrm{d}t + \varepsilon_k^N \int \sum_{i,j\in\varepsilon_k\mathbb{Z}^N} \frac{z_{ij}^{h_k}(t+h_k) - z_{ji}^{h_k}(t+h_k)}{h_k}\varphi(i,t)\,\mathrm{d}t \\
&\quad = -\varepsilon_k^N \int \sum_{i\in\varepsilon_k\mathbb{Z}^N} \frac{\varphi(i,t)-\varphi(i,t-h_k)}{h_k} d_i^{h_k}(t)\,\mathrm{d}t + \varepsilon_k^N \int \sum_{i,j\in\varepsilon_k\mathbb{Z}^N} z_{ij}^{h_k}(t+h_k)\frac{\varphi(i,t)-\varphi(j,t)}{h_k}\,\mathrm{d}t \\
&\quad \geq 0.
\end{aligned} \tag{5-10}$$

It is easy to check that the first integral in (5-10) converges to $-\iint d\,\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t$ as $k\to\infty$ thanks to (5-7) and since d^{h_k} and d are uniformly bounded. Recalling that z^{h_k} converges weakly-$*$ in $L^\infty(\mathbb{R}^N \times (0,T^*))$ to z, we use Lemma 5.4 to conclude that the second integral in (5-10) converges to $\iint z\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t$. We thus conclude (2-1).

Step 2. (convergence of u^{h_k} to d). Firstly, we establish an upper bound for $-D^*_{\varepsilon_k} z_{h_k}$ away from E^{h_k}. We start by noting that definition (4-5) implies

$$\mathrm{sd}^{\varepsilon,\phi^\circ}(u) \le \tfrac{1}{2}((d_-^{\varepsilon,\phi^\circ}(u))_j + u_\ell + \phi^\circ(\,\cdot - j) + \phi^\circ(\,\cdot - \ell)) \quad \text{in } \varepsilon\mathbb{Z}^N \setminus \{u \le 0\} \tag{5-11}$$

for every $(1, \phi^\circ)$-Lipschitz function $u \in X_\varepsilon$ and $j, \ell \in \{u \le 0\}$. Therefore, specifying the inequality above for $u^{h_k}(t)$, by the comparison principle and Lemma 4.10 we conclude

$$u_i^{h_k}(t+h_k) \le \tfrac{1}{2}\big(\phi^{h_k}_{i-j} + \phi^{h_k}_{i-\ell} + \big(d_-^{\varepsilon,\phi^\circ}(u^{h_k}(t))_j\big) + u_\ell^{h_k}(t)\big) \quad \text{for all } i \in \varepsilon_k\mathbb{Z}^N \setminus E^{h_k}(t), \tag{5-12}$$

where $j, \ell \in E^{h_k}(t)$. If $\hat{d}^{h_k}(i,t) \ge R > 0$, recalling the definition of ϕ^h, we get

$$u_i^{h_k}(t+h_k) \le \tfrac{1}{2}\big(\phi^\circ(i-j) + \phi^\circ(i-\ell) + (d_-^{\varepsilon,\phi^\circ}(u^{h_k}(t))_j) + u_\ell^{h_k}(t)\big) + \frac{Ch_k}{R - c_\phi\varepsilon} \tag{5-13}$$

for all $i \in \varepsilon_k\mathbb{Z}^N \setminus E^{h_k}(t)$. Taking the infimum in j, ℓ over $E^{h_k}(t)$ in (5-13) and using again (4-5) and (4-11), we conclude

$$u_i^{h_k}(t+h_k) \le d_i^{h_k}(t) + h_k\frac{C}{R - c_\phi\varepsilon_k} \le d_i^{h_k}(t) + h_k\frac{C}{R} \tag{5-14}$$

provided h_k and ε_k are small enough depending on R, and for a possibly larger value of C. As a consequence of (5-14), we obtain

$$-D^*_{\varepsilon_k} z^{h_k}(t+h_k) \le \frac{C}{R} \quad \text{in } \{\hat{d}^{h_k}(\,\cdot\,,t) \ge R\} \cap \varepsilon_k\mathbb{Z}^N. \tag{5-15}$$

Using again Lemma 5.4 and the convergences of E_{h_k} and d_{h_k}, it follows that

$$\operatorname{div} z \le \frac{C}{R} \quad \text{in } \{(x,t) \in \mathbb{R}^N \times (0,T^*) \mid d(x,t) > R\} \tag{5-16}$$

in the sense of distributions. Hence we have that $\operatorname{div} z$ is a Radon measure in $\mathbb{R}^N \times (0,T^*) \setminus E$ and $(\operatorname{div} z)^+ \in L^\infty(\{(x,t) \in \mathbb{R}^N \times (0,T^*) \mid d(x,t) \ge \delta\})$ for every $\delta > 0$.

On the other hand, note that for every $i \in \varepsilon_k\mathbb{Z}^N$ we have

$$d^{h_k}(t) \ge d_i^{h_k}(t) - \phi^\circ(\,\cdot - i). \tag{5-17}$$

Thus, by Lemma 4.10 and by comparison as before, we get

$$u_i^{h_k}(t+h_k) \ge d_i^{h_k}(t) - \phi_0^{h_k} = d_i^{h_k}(t) - (C+1)\sqrt{h_k}.$$

Combining the above inequality with (5-14), we deduce for all $t \in (0,T^*) \setminus \mathcal{N}$ and any $\delta > 0$ that

$$\sup_{\{\hat{d}_{h_k}(\cdot,t)\ge\delta\}\cap\varepsilon_k\mathbb{Z}^N} |u^{h_k}(t+h_k) - d^{h_k}(t)| \le \sqrt{h_k}(C+2)$$

provided that k is large enough. In particular, recalling also (5-7), we deduce that

$$\sup_{\varepsilon_k\mathbb{Z}^N\cap K} |u^{h_k}(t) - d(\,\cdot\,,t)| \to 0 \text{ as } k\to\infty \text{ for } t \in (0,T^*)\setminus\mathcal{N} \text{ and for any compact } K \subseteq \mathbb{R}^N\setminus E(t), \tag{5-18}$$

also with the sequence $\{u^{h_k}\}$ locally (in space and time) uniformly bounded.

Step 3. (the subdifferential inclusion). It remains to show that

$$z \in \partial\phi(\nabla d) \quad \text{a.e. in } \mathbb{R}^N \times (0, T^*) \setminus E. \tag{5-19}$$

Recall that $\xi \in \partial\phi(\eta)$ if and only if $\xi \in \{v \mid \phi^\circ(v) \le 1,\ v \cdot \eta \ge \phi(\eta)\}$. Since one inequality has been proved in (5-8), we show the other one. Consider a test function $\eta \ge 0,\ \eta \in C_c^\infty((\mathbb{R}^N \times (0, T^*)) \setminus E)$. Let $\sigma > 0$, and set $d_\sigma \in C^\infty(\mathbb{R}^N \times (0, T^*))$, as $d_\sigma = d * \rho_\sigma$, where the ρ_σ are space-time mollifiers. Obviously

$$\begin{aligned}\sum_{i,j\in\varepsilon_k\mathbb{Z}^N} z_{ij}^{h_k}(t)\eta(i,t)(u_i^{h_k}(t) - u_j^{h_k}(t)) &= \sum_{i,j\in\varepsilon_k\mathbb{Z}^N} z_{ij}^{h_k}(t)\eta(i,t)(d_\sigma(i,t) - d_\sigma(j,t)) \\ &\quad + \sum_{i,j\in\varepsilon_k\mathbb{Z}^N} z_{ij}^{h_k}(t)\eta(i,t)(u_i^{h_k}(t) - d_\sigma(i,t) - u_j^{h_k}(t) + d_\sigma(j,t)).\end{aligned} \tag{5-20}$$

In turn, Lemma 5.4 implies that

$$\lim_{k\to\infty} \varepsilon_k^N \int \Bigg(\sum_{i,j\in\varepsilon_k\mathbb{Z}^N} z_{ij}^{h_k}(t)\eta(i,t)\frac{d_\sigma(i,t) - d_\sigma(j,t)}{\varepsilon_k} \Bigg) \mathrm{d}t = \iint z \cdot \nabla d_\sigma \eta \,\mathrm{d}x\,\mathrm{d}t. \tag{5-21}$$

Let us thus show that

$$\lim_{\sigma\to 0}\lim_{k\to\infty} \varepsilon_k^N \int \sum_{i,j\in\varepsilon_k\mathbb{Z}^N} \Bigg(z_{ij}^{h_k}(t)\eta(i,t)\frac{u_i^{h_k}(t) - d_\sigma(i,t) - u_j^{h_k}(t) - d_\sigma(j,t)}{\varepsilon_k} \Bigg) \mathrm{d}t = 0. \tag{5-22}$$

We set for every $t \in (0, T_h^*)$ and $\sigma > 0$

$$\begin{aligned} m_{k,\sigma}(t) &:= \min_{i\in \mathrm{supp}(\eta)\cap\varepsilon_k\mathbb{Z}^N} (u_i^{h_k}(t) - d_\sigma(i,t)), \\ M_{k,\sigma}(t) &:= \max_{i\in \mathrm{supp}(\eta)\cap\varepsilon_k\mathbb{Z}^N} (u_i^{h_k}(t) - d_\sigma(i,t)). \end{aligned}$$

The convergence (5-18) implies that these quantities are uniformly bounded and

$$\begin{aligned} \lim_{\sigma\to 0}\lim_{k\to+\infty} m_{k,\sigma}(t) &= 0, \\ \lim_{\sigma\to 0}\lim_{k\to+\infty} M_{k,\sigma}(t) &= 0 \end{aligned} \tag{5-23}$$

uniformly for all $t \notin \mathcal{N}$. For all times $t \in (0, T^*) \setminus \mathcal{N}$

$$\begin{aligned} &\varepsilon_k^N \sum_{i,j\in\varepsilon_k\mathbb{Z}^N} z_{ij}^{h_k}(t)\eta(i,t)\frac{u_i^{h_k}(t) - d_\sigma(i,t) - u_j^{h_k}(t) + d_\sigma(j,t)}{\varepsilon_k} \\ &\quad = \varepsilon_k^N \sum_{i,j\in\varepsilon_k\mathbb{Z}^N} z_{ij}^{h_k}(t)\eta(i,t)\frac{(u_i^{h_k}(t) - d_\sigma(i,t) - m_{k,\sigma}(t)) - (u_j^{h_k}(t) - d_\sigma(j,t) - m_{k,\sigma}(t))}{\varepsilon_k} \\ &\quad = \varepsilon_k^N \sum_{i\in\varepsilon_k\mathbb{Z}^N} (u_i^{h_k}(t) - d_\sigma(i,t) - m_{k,\sigma}(t)) \sum_{j\in\varepsilon_k\mathbb{Z}^N} \Bigg(\frac{z_{ij}^{h_k}(t) - z_{ji}^{h_k}(t)}{\varepsilon_k}\eta(i,t) + z_{ji}^{h_k}(t)\frac{\eta(i,t) - \eta(j,t)}{\varepsilon_k} \Bigg). \end{aligned} \tag{5-24}$$

For k large enough, since the support of η is at positive distance from E, by the bound (5-15) one has $D^*_{\varepsilon_k} z^{h_k}(t) \ge -c(\delta)$ on the support for h_k small enough. Thus

$$\varepsilon_k^N \sum_{i\in\varepsilon_k\mathbb{Z}^N} (u_i^{h_k}(t) - d_\sigma(i,t) - m_{k,\sigma}(t))\eta(i,t) \sum_{j\in\varepsilon_k\mathbb{Z}^N} \frac{z_{ij}^{h_k}(t) - z_{ji}^{h_k}(t)}{\varepsilon_k} \ge -c(\delta)\varepsilon_k^N \sum_{i\in\varepsilon_k\mathbb{Z}^N} (u_i^{h_k}(t) - d_\sigma(i,t) - m_{k,\sigma}(t))\eta(i,t).$$

Recalling that

$$\#(\mathrm{supp}(\eta)\cap\varepsilon_k\mathbb{Z}^N) = O(h_k^{-N})$$

uniformly in time, by uniform convergence and (5-18) we conclude that

$$\lim_{\sigma\to 0}\liminf_{k\to\infty} \varepsilon_k^N \int \sum_{i\in\varepsilon_k\mathbb{Z}^N} (u_i^{h_k}(t) - d_\sigma(i,t) - m_{\varepsilon,k}(t))\eta(i,t) \sum_{j\in\varepsilon_k\mathbb{Z}^N} \frac{z_{ij}^{h_k}(t) - z_{ji}^{h_k}(t)}{\varepsilon_k}\,\mathrm{d}t \ge 0. \tag{5-25}$$

The other term in (5-24) can be estimated using the Lipschitz constant of η as

$$\left|\int \sum_{i,j\in\varepsilon_k\mathbb{Z}^N} \varepsilon_k^N (u_i^{h_k}(t) - d_\sigma(i,t) - m_{\varepsilon,k}(t)) z_{ji}^{h_k}(t) \frac{\eta(i,t)-\eta(j,t)}{\varepsilon_k}\,\mathrm{d}t\right| \le \|\nabla\eta\|_\infty \varepsilon_k^N \int \sum_{i,j\in\varepsilon_k\mathbb{Z}^N} (u_i^{h_k}(t) - d_\sigma(i,t) - m_{\varepsilon,k}(t))\alpha_{ji}^{h_k} \frac{|i-j|}{\varepsilon_k}\,\mathrm{d}t \to 0,$$

letting first $k\to+\infty$ and then $\sigma\to 0$, thanks to (5-18) and (5-23). Note that reasoning as in (5-22) but using $M_{\varepsilon,k}(t)$ instead of $m_{\varepsilon,k}(t)$, one proves that

$$\begin{aligned} &\lim_{\sigma\to 0}\limsup_{k\to\infty} \varepsilon_k^N \int \biggl(\sum_{i\in\varepsilon_k\mathbb{Z}^N} (u_i^{h_k}(t) - d_\sigma(i,t) - M_{\varepsilon,k}(t))\eta(i,t) \sum_{j\in\varepsilon_k\mathbb{Z}^N} \frac{z_{ij}^{h_k}(t) - z_{ji}^{h_k}(t)}{\varepsilon_k} \biggr)\,\mathrm{d}t \le 0, \\ &\lim_{\varepsilon\to 0}\lim_{k\to\infty} \varepsilon_k^N \int \biggl| \sum_{i,j\in\varepsilon_k\mathbb{Z}^N} (u_i^{h_k}(t) - d_\sigma(i,t) - M_{\varepsilon,k}(t)) z_{ji}^{h_k}(t) \frac{\eta(i,t)-\eta(j,t)}{\varepsilon_k} \biggr|\,\mathrm{d}t = 0. \end{aligned} \tag{5-26}$$

Combining (5-24), (5-25) and (5-26), we conclude (5-22).

Integrating in time (5-20) and combining (5-21) and (5-22), since the $\nabla d_\sigma = \rho_\sigma * \nabla d$ go to ∇d pointwise a.e. and are uniformly bounded in $L^\infty(\mathbb{R}^N\times(0,T^*);\mathbb{R}^N)$, we have

$$\lim_{k\to\infty} \varepsilon_k^N \int \biggl(\sum_{i,j\in\varepsilon_k\mathbb{Z}^N} \eta(i,t) z_{ij}^{h_k}(t) \frac{u_i^{h_k}(t) - u_j^{h_k}(t)}{\varepsilon_k} \biggr)\,\mathrm{d}t = \iint z\cdot\nabla d\,\eta\,\mathrm{d}x\,\mathrm{d}t.$$

The convergence above can be paired with the lower semicontinuity of the Γ-convergence of the discrete total variations (which follows from an adaptation of classical arguments; we suggest the reader consult, e.g., [Chambolle and Kreutz 2023]) and $z^{\varepsilon}_{ij}(u^{\varepsilon}_i - u^{\varepsilon}_j) = \alpha^{\varepsilon}_{ij}|u^{\varepsilon}_i - u^{\varepsilon}_j|$ to obtain

$$\iint \phi(\nabla d)\eta \le \liminf_{k\to\infty} \varepsilon_k^N \int \Bigg(\sum_{i,j\in\varepsilon_k\mathbb{Z}^N} \eta(i,t)\alpha^{h_k}_{ij} \frac{|u^{h_k}_i(t) - u^{h_k}_j(t)|}{\varepsilon_k} \Bigg)\,\mathrm{d}t$$
$$= \liminf_{k\to\infty} \varepsilon_k^N \int \Bigg(\sum_{i,j\in\varepsilon_k\mathbb{Z}^N} \eta(i,t) z^{h_k}_{ij}(t) \frac{u^{h_k}_i(t) - u^{h_k}_j(t)}{\varepsilon_k} \Bigg)\,\mathrm{d}t = \iint z\cdot\nabla d\,\eta,$$

which shows that $\phi(\nabla d) = z\cdot\nabla d$ a.e. on the support of η, from which we deduce (5-19). □

We conclude this section by observing that the discrete scheme converges to the unique weak flow (in the sense of Definition 2.1) starting from E_0 for "generic" initial data E_0, i.e., whenever fattening does not occur. More precisely, we have the following corollary.

Corollary 5.5. *Let $u_0 \in \mathrm{UC}(\mathbb{R}^N)$, and for every $\lambda\in\mathbb{R}$ let $\bar{E}^h_\lambda$ be the closed space-time tube of the h-discrete evolution starting from $\{u_0 \le \lambda\}$, i.e., as in* (5-1) *with $E_0 = \{u_0\le\lambda\}$. Then, there exists a countable set $\mathcal{N}$ such that for all $\lambda\in\mathbb{R}^N\setminus\mathcal{N}$*

$$\bar{E}^h_\lambda \xrightarrow{\mathcal{K}} E_\lambda \quad \text{in } \mathbb{R}^N\times[0,+\infty)$$

as $h\to 0$, where E_λ is the unique weak flow in the sense of Definition 2.1 starting from $\{u_0\le\lambda\}$.

Proof. It follows by combining Theorems 5.2 and 2.5. □

6. Numerical experiments

We show some numerical experiments to illustrate our results in dimension 2 (an implementation in three dimensions is currently being developed). We follow the implementation described in [Chambolle and Darbon 2009] (see also [Chambolle and Darbon 2012]), except that now the distance is properly computed using the inf/sup-convolution formulas (4-5). The (exact) numerical resolution of the discrete ROF functional is computed using the parametric maximum flow algorithm of Hochbaum [2001; 2013], implemented upon the maxflow/mincut implementation of Boykov and Kolmogorov [2004]. This particular algorithm has the advantage to provide an *exact* solution of the ROF problem, up to computer precision. Other implementations of the algorithm yielding approximate minimizers have been considered for instance in [Chambolle 2004; Oberman et al. 2011]: of course they work in practice and allow one to address more (an)isotropies than the current work, yet the joint convergence as $\varepsilon = h \to 0$ is not clear in these contexts. For numerical speedup, the infimum and supremum of the definitions in (4-5) are computed only in a neighborhood of fixed size and not on the whole grid. We expect this to yield, in general, an error of order $C\varepsilon$ with C getting smaller as the width of the strip increases; however, we observe that Corollary B.2 justifies this restriction (showing that $C = 0$) in some cases, notably the case $\phi = \|\cdot\|_{\ell^1}$, $\phi^\circ = \|\cdot\|_{\ell^\infty}$; see Figure 1, left, for which the sup/inf are in fact min/max which are reached very close to the evolving boundary (as one can chose $\ell_1 = 1$ in Lemma B.1). Similarly, the ROF minimization is only performed in a neighborhood of the boundary (one can show that this does not affect the solution in a smaller neighborhood, hence the overall error is the same as when computing the distance in a strip only).

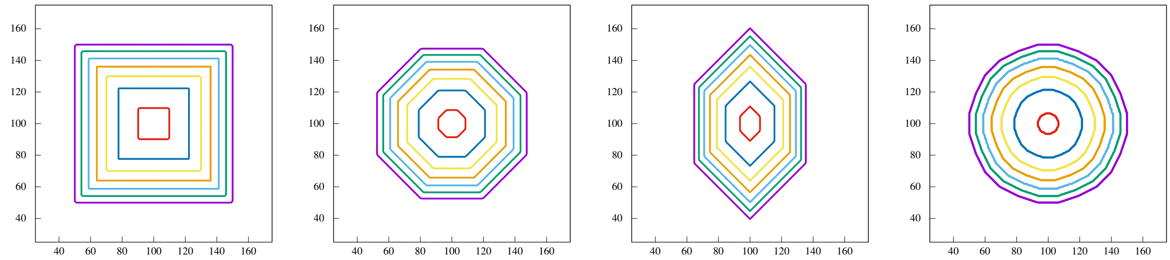

Figure 1. Wulff shapes of initial radius $R_0 = 50$ evolved at times $t = 0, 200, 400, \ldots, 1200$ for four different anisotropies: square, octagon, diamond and "almost isotropic".

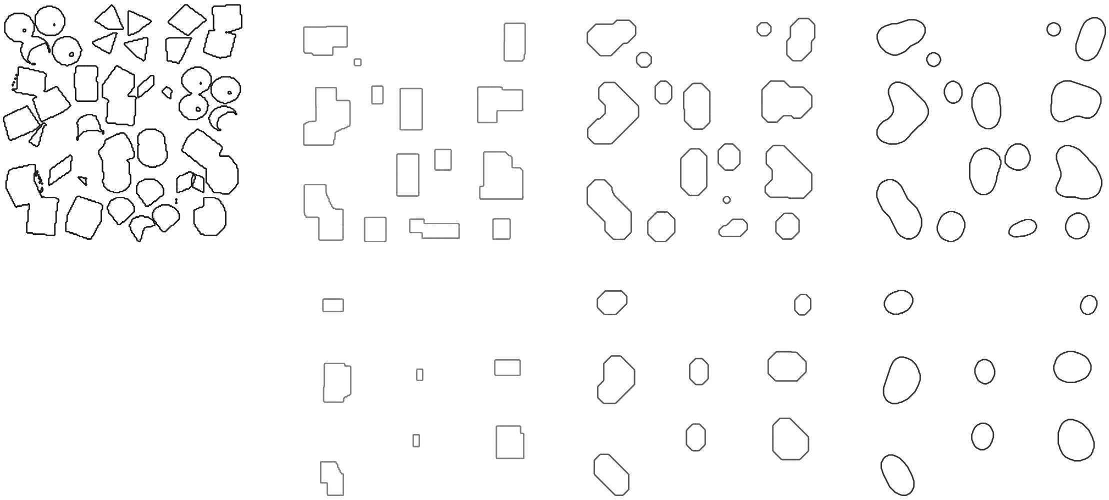

Figure 2. An initial datum and evolutions for square, octagonal and "almost isotropic" anisotropies, at two different times.

The code is available at https://plmlab.math.cnrs.fr/chambolle/discretecrystals/ (implemented in `C/C++` and running on GNU/linux with `gcc`).

Figure 2 shows three examples of flows from the same starting set, composed of random shapes. The anisotropies are square (nearest neighbors interactions), octagonal (next nearest neighbors, weighted so that the corresponding Wulff shape is a regular octagon), and "almost isotropic", which is generated by the interactions in the 12 directions $e \in \{(0, \pm 1), (\pm 1, 0), (\pm 1, \pm 2), (\pm 1, \pm 3)\}$ weighted so that the Wulff shape is a polygon with 24 facets of equal lengths. This is obtained by setting the weights in the discrete total variation as $0.131/\mathrm{length}(e)$ for each direction e, so that the total perimeter of the unit Wulff shape is $24 \times (2 \times 0.131) \approx 2\pi$, in the hope that the corresponding crystalline curvature will be close to the Euclidean one.

Then, we estimate the decay of the radius of an initial Wulff shape $\mathcal{W}_{R_0} = \{\phi \leq R_0\}$ along the evolution, up to extinction. In our experiment, $R_0 = 50$. It is well known that the solution is the Wulff shape of radius $R(t) = \sqrt{R_0^2 - 2(N-1)t}$ (where here $N = 2$). The evolutions are depicted in Figure 1. We use the same anisotropies as in Figure 2, with additionally a "diamond" Wulff shape generated by the directions

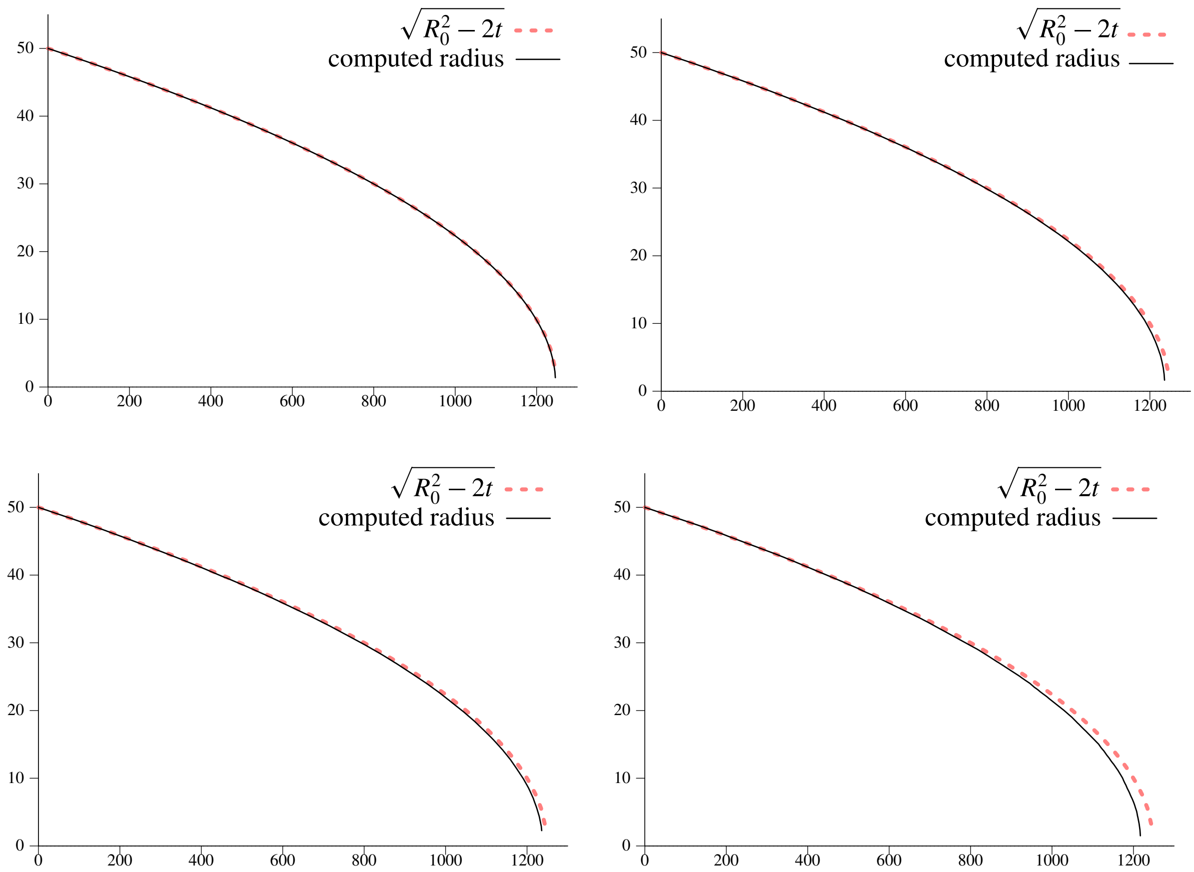

Figure 3. Top: Evolution of the radius for the square (left) and octogonal (right) anisotropies. Bottom: Evolution of the radius for the diamond (left) and "almost isotropic" (right) anisotropies.

(0, ±1), (±1, ±2) and with sides of equal lengths. In all cases, the weights have been calibrated so that the perimeters of the Wulff shapes are $6.28 \approx 2\pi$.

The plots in Figure 3 show that the decay of the radii is remarkably close to the theoretical prediction, even if this is less precise when more directions of interactions are involved, near extinction. This might be due in part to the fact that the computation of the distance through truncated variants of (4-5) becomes less precise.

Finally, we perform the same experiment with varying ε and h. We observe that the results look remarkably close even if, at low resolution, the error becomes huge when the size of the Wulff shape is of the order of the discretization. Figure 4 shows the shapes. Observe that the shape at time $t = 49$ is only computed for $\varepsilon = 0.1$ and $h = 0.1$ (the shape vanishes before for the two other experiments). On the other hand, this computation took more than one hour, while the case $\varepsilon = 1$ took less than a minute and the case $\varepsilon = 0.1$ and $h = 0.5$ a bit less than an hour. Figure 5 shows the decay of the radii, which should be $\sqrt{R_0^2 - 2t}$ for $R_0 = 10$ and $t \in [0, 50]$.

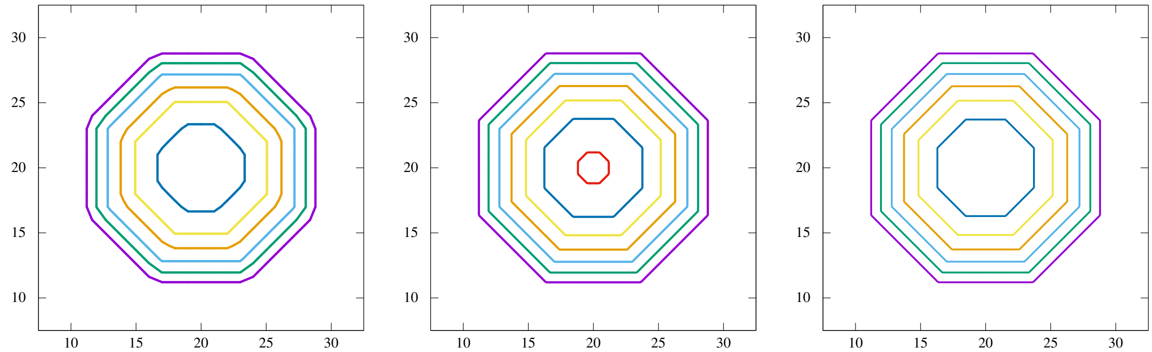

Figure 4. Evolution of an initial octagon with $R_0 = 10$ at times $0, 7, 14, \ldots$ (left: $\varepsilon = 1,\ h = 0.1$; middle: $\varepsilon = 0.1,\ h = 0.1$; right: $\varepsilon = 0.1,\ h = 0.5$).

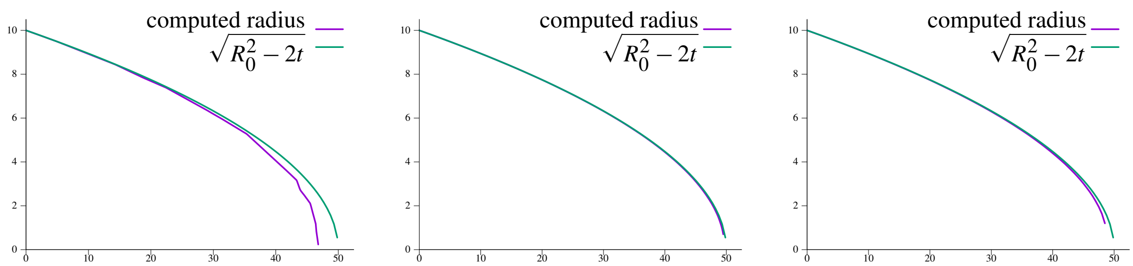

Figure 5. Evolution of the radius for an initial octagon with $R_0 = 10$ until the vanishing time $t = 50$ (left: $\varepsilon = 1,\ h = 0.1$; middle: $\varepsilon = 0.1,\ h = 0.1$; right: $\varepsilon = 0.1,\ h = 0.5$).

Appendix A: Proof of Lemma 4.10

We build here a supersolution to Problem (4-26) when $g = \phi^\circ$. Let us first recall some notation and results concerning zonotopes; see, e.g., [McMullen 1971]. Recall that $\mathcal{E} = \{\pm e_k\}_{k=1}^m \subseteq \mathbb{Z}^N$, where, without loss of generality, the vectors $e_1, \ldots, e_m$ span the whole $\mathbb{R}^N$. Given a nonnegative interaction function $\beta \in X$, we assume that $\beta = 0$ on $\mathbb{Z}^N \setminus \mathcal{E}$ and that $\beta(-i) = \beta(i)$ for every $i \in \mathbb{Z}^N$. The anisotropy ϕ associated to β, as defined in (1-5), is such that its 1-Wulff shape $\mathcal{W}_1 \subseteq \mathbb{R}^N$ is a zonotope, which can be expressed as the Minkowski sum

$$\mathcal{W}_1 = \sum_{e \in \mathcal{E}} \beta(e)[-e, e] = \sum_{k=1}^{m} 2\beta(e_k)[-e_k, e_k],$$

where $[-e, e] \subseteq \mathbb{R}$ denotes the closed segment from $-e$ to e. Alternatively, one can define the zonotope $\mathcal{W}_1$ as the image of a cube under an affine map. Indeed,

$$\mathcal{W}_1 = V(Q^{(m)}), \tag{A-1}$$

where $V = (2\beta(e_1)e_1, \ldots, 2\beta(e_m)e_m) \in \mathbb{R}^{N \times m}$ and $Q^{(m)} = [-1, 1]^m$. Since the set $\mathcal{E}$ is uniquely defined up to sign changes, the matrix V is also uniquely determined up to permutations of columns or sign changes.

Note that by definition of zonotope any element $x \in \mathcal{W}_\ell$ for $\ell > 0$ can be written as

$$x = \ell \sum_{k=1}^{m} 2\beta(e_k)\lambda_k e_k$$

for suitable coefficients $|\lambda_k| \le 1$. We note that (the closure of) a facet F (of nonzero dimension) of the zonotope $\mathcal{W}_\ell$ can be described in the form

$$F = \ell \sum_{j=1}^{r} 2\beta(e_{\sigma(j)})[-e_{\sigma(j)}, e_{\sigma(j)}] + \ell \sum_{j=r+1}^{m} 2\beta(e_{\sigma(j)})\varepsilon_{\sigma(j)} e_{\sigma(j)}, \tag{A-2}$$

where σ is a permutation of $\{1, \dots, m\}$, $1 \le r \le m$ and $|\varepsilon_j| = 1$. Moreover (see [McMullen 1971, p. 206] for details), the vectors $e_{\sigma(1)}, \dots, e_{\sigma(r)}$ uniquely identify

$$\{e \in \mathcal{E} \mid e \parallel F\},$$

and r is uniquely defined as the number of vectors in the family $\mathcal{E}$ which are parallel to the facet F. Analogously, any vertex v of the zonotope $\mathcal{W}_\ell$ is of the form

$$v = \ell \sum_{j=1}^{m} 2\beta(e_{\sigma(j)})\varepsilon_{\sigma(j)} e_{\sigma(j)}, \tag{A-3}$$

where $\varepsilon_j \in \{\pm 1\}$ for every $j = 1, \dots, m$ and σ is a permutation of $\{1, \dots, m\}$. Note however that not every point of this form is a vertex of the zonotope.

Lemma A.1. *There exists $\ell_0 > 0$ such that, for every $\varepsilon > 0$ and every $\ell \ge \ell_0$, if $i \in \varepsilon\mathbb{Z}^N$ belongs to $\partial\mathcal{W}_{\varepsilon\ell}$, then for each $k \in \{1, \dots, m\}$ one of the following holds*:

(i) *Neither $i + \varepsilon e_k$ nor $i - \varepsilon e_k$ belong to $\partial\mathcal{W}_{\varepsilon\ell}$. In this case either $\phi^\circ(i + \varepsilon e_k) > \phi^\circ(i) > \phi^\circ(i - \varepsilon e_k)$ or $\phi^\circ(i - \varepsilon e_k) > \phi^\circ(i) > \phi^\circ(i + \varepsilon e_k)$.*

(ii) *One of $i \pm \varepsilon e_k$ belongs to $\partial\mathcal{W}_{\varepsilon\ell}$. In this case $\phi^\circ(i \pm \varepsilon e_k) \ge \ell$ and*

$$\#((i + \varepsilon\mathbb{Z}e_k) \cap \partial\mathcal{W}_{\varepsilon\ell}) \ge 2[\ell/\ell_0]. \tag{A-4}$$

Proof. By scaling, it suffices to prove the result in the case $\varepsilon = 1$. We take ℓ_0 such that

$$\ell_0 \ge \max_{k=1,\dots,m} \frac{1}{2\beta(e_k)} \tag{A-5}$$

and remark that $\ell_0 \in (0, +\infty)$. Note that the choice (A-5) implies for every $j = 1, \dots, m$ that

$$|[-2\ell\beta(e_j)e_j, 2\ell\beta(e_j)e_j]| = 4\ell\beta(e_j)|e_j| \ge 2\frac{\ell}{\ell_0}|e_j|.$$

We then fix $i \in \partial\mathcal{W}_\ell \cap \mathbb{Z}^N$ and $e_k \in \mathcal{E}$. We have to distinguish two cases.

Case 1. There exists a facet $F \ni i$ of $\mathcal{W}_\ell$ such that $e_k \parallel F$. By (A-2) we then see that

$$i \in 2\ell\beta(e_k)[-e_k, e_k] + j,$$

where $j \in F$. This implies in particular that $\{n \in \mathbb{Z} \mid i + ne_k \in F\}$ is an interval of $\mathbb{Z}$ containing 0. Furthermore, by the assumption (A-5), it contains at least $[2\ell|e_k|/\ell_0]$ points and we conclude (A-4). Since i and one of $i \pm e_k$ belong to $\partial \mathcal{W}_\ell$, we have that $\phi^\circ(i \pm e_k) \geq \ell$ by convexity.

Case 2. For every facet $F \ni i$ of $\mathcal{W}_\ell$, we have $e_k \nparallel F$. Let us fix a facet $F \ni i$ and note that, by (A-2) and up to relabeling the indexes,

$$i \in \ell \sum_{j=1}^{r} 2\beta(e_j)[-e_j, e_j] + \ell \sum_{j=r+1}^{m} 2\beta(e_j)\varepsilon_j e_j,$$

with $k > r$ and $|\varepsilon_j| = 1$ for $j = r+1, \ldots, m$. Recalling (A-1), we see that

$$i - \varepsilon_k e_k = \ell V\Big(y - \frac{\varepsilon_k}{\ell\beta(e_k)}\tilde{e}_k\Big),$$

where $\tilde{e}_1, \ldots, \tilde{e}_m$ denotes the canonical base of $\mathbb{R}^m$ and $y \in \sum_{j=1}^{r}[-\tilde{e}_j, \tilde{e}_j] + \sum_{j=r+1}^{m} \varepsilon_j \tilde{e}_j \subseteq \partial Q^{(m)}$. By the choice (A-5) and since $k > r$, one deduces that

$$y - \frac{\varepsilon_k}{\ell\beta(e_k)}\tilde{e}_k \in Q^{(m)};$$

thus $i - \varepsilon_k e_k \in \overline{\mathcal{W}}_\ell$. Since then $e_k \nparallel F$ for any facet containing i, we must have $\phi^\circ(i - \varepsilon_k e_k) < \ell$. By convexity one easily concludes that $\phi^\circ(i + \varepsilon_k e_k) > \ell$, which shows (i). □

We now define a calibration z_{ij} for every $(i, j) \in (\{\phi^\circ > \varepsilon\ell_0\} \cap \varepsilon\mathbb{Z}^N) \times \varepsilon\mathbb{Z}^N$. Fix $i \in \varepsilon\mathbb{Z}^N$ with $\phi^\circ(i) > \varepsilon\ell_0$. In the following we write $i \sim j$ if $(i - j)/\varepsilon \in \mathcal{E}$. We start by defining

$$z_{ij} = \begin{cases} 0 & \text{if } j \not\sim i, \\ -\beta(e_k) & \text{if } j = i \pm \varepsilon e_k \text{ and } \phi^\circ(j) > \phi^\circ(i), \\ \beta(e_k) & \text{if } j = i \pm \varepsilon e_k \text{ and } \phi^\circ(j) < \phi^\circ(i). \end{cases} \tag{A-6}$$

In particular, this definition covers case (i) in Lemma A.1. Assume then that there exists $j \sim i$ with $\phi^\circ(j) = \phi^\circ(i)$ and $(j - i)/\varepsilon = e_k \in \mathcal{E}$. Since $i \in \varepsilon\mathbb{Z}^N$ and $e_k \in \mathcal{E}$ fall into case (ii) of Lemma A.1, there exists an interval $[-\underline{n}, \bar{n}] \cap \mathbb{Z}$ for $\underline{n}, \bar{n} \in \mathbb{N}$ such that

$$(i + \varepsilon\mathbb{Z}e_k) \cap \partial W^{\phi^\circ}_{\phi^\circ(i)} = i + ([-\underline{n}, \bar{n}] \cap \mathbb{Z})\varepsilon e_k,$$

and moreover

$$\#([-\underline{n}, \bar{n}] \cap \mathbb{Z}) \geq 2[\phi^\circ(i)/(\varepsilon\ell_0)]. \tag{A-7}$$

Thus, we define z_{ij} as a linear interpolation of the values assumed at the extremal points of $i + [-\underline{n}, \bar{n}]\varepsilon e_k$:

$$\begin{aligned} z_{i+t\varepsilon e_k, i+(t+1)\varepsilon e_k} &:= \beta(e_k)\Big(1 - 2\frac{t + \underline{n} + 1}{\underline{n} + \bar{n} + 1}\Big) \quad \text{for all } t \in [-\underline{n} - 1, \bar{n}] \cap \mathbb{Z}, \\ z_{i+t\varepsilon e_k, i+(t-1)\varepsilon e_k} &:= \beta(e_k)\Big(1 - 2\frac{-t + \underline{n} + 1}{\underline{n} + \bar{n} + 1}\Big) \quad \text{for all } t \in [-\underline{n}, \bar{n} + 1] \cap \mathbb{Z}. \end{aligned} \tag{A-8}$$

By definition one easily sees that

$$|z_{ij}| \leq \alpha^\varepsilon_{ij}, \quad z_{ij}(\phi^\circ(i) - \phi^\circ(j)) = \alpha^\varepsilon_{ij}|\phi^\circ(i) - \phi^\circ(j)|. \tag{A-9}$$

We now bound the divergence $(D^*_\varepsilon z)_i$. Assume that $\phi^\circ(i+\varepsilon e_k)=\phi^\circ(i)$ or that $\phi^\circ(i-\varepsilon e_k)=\phi^\circ(i)$. Then by definition (A-8) and by (A-7) one deduces

$$z_{i,i+\varepsilon e_k}+z_{i,i-\varepsilon e_k}-z_{i+\varepsilon e_k,i}-z_{i-\varepsilon e_k,i}=-\frac{4\beta(e_k)}{\underline{n}+\bar{n}+1}\geq-\frac{2\beta(e_k)}{[\phi^\circ(i)/(\varepsilon\ell_0)]}\geq-\frac{C\varepsilon}{\phi^\circ(i)}, \tag{A-10}$$

and similarly if $\phi^\circ(i-\varepsilon e_k)=\phi^\circ(i)$. If instead $\phi^\circ(i\pm\varepsilon e_k)\neq\phi^\circ(i)$ and $\phi^\circ(i\pm\varepsilon e_k)\geq\varepsilon\ell_0$, one sees that

$$z_{i,i+\varepsilon e_k}+z_{i,i-\varepsilon e_k}=0 \quad\text{and}\quad z_{i+\varepsilon e_k,i}+z_{i-\varepsilon e_k,i}=0. \tag{A-11}$$

Combining (A-10) and (A-11) and recalling (4-2) we conclude that if $\phi^\circ(i)\geq\ell_1\varepsilon$ then

$$h(D^*_\varepsilon z)_i\geq-\frac{c_\phi h}{\phi^\circ(i)} \tag{A-12}$$

for a suitable positive constant c_ϕ depending on ϕ.

We now illustrate a procedure that allows us to extend the calibration above to $\varepsilon\mathbb{Z}^N\times\varepsilon\mathbb{Z}^N$. We set $C>1$, a sufficiently big constant, and define a function $v\in X_\varepsilon$ as

$$v:=\begin{cases}\phi^\circ+\frac{Ch}{\phi^\circ} & \text{on } \{\phi^\circ\geq C(\sqrt{h}\vee\varepsilon)\}\cap\varepsilon\mathbb{Z}^N,\\ C(\sqrt{h}\vee\varepsilon)+\frac{h}{\sqrt{h}\vee\varepsilon} & \text{on } \{\phi^\circ< C(\sqrt{h}\vee\varepsilon)\}\cap\varepsilon\mathbb{Z}^N.\end{cases} \tag{A-13}$$

A calibration $w\in Y_\varepsilon$ can be defined for $i,j\in\varepsilon\mathbb{Z}^N$ as

$$w_{ij}:=\begin{cases}z_{ij} & \text{if } \phi^\circ(i)\geq 2\sqrt{C}(\sqrt{h}\vee\varepsilon),\\ -\alpha^\varepsilon_{ij} & \text{if } \phi^\circ(i)< 2\sqrt{C}(\sqrt{h}\vee\varepsilon).\end{cases} \tag{A-14}$$

Since $x\mapsto x+Chx^{-1}$ is strictly monotone in the region $\{x\geq\sqrt{Ch}\}$, we can employ (A-9) to prove that, for every $i,j\in\varepsilon\mathbb{Z}^N$ with $\phi^\circ(i)\geq C(\sqrt{h}\vee\varepsilon)$,

$$w_{ij}(v_i-v_j)=\alpha^\varepsilon_{ij}|v_i-v_j|,\quad |w_{ij}|\leq\alpha^\varepsilon_{ij}. \tag{A-15}$$

Moreover, taking C large enough ensures that, whenever $j\sim i$, we have

$$\begin{aligned}\phi^\circ(i)\leq 2\sqrt{C}(\sqrt{h}\vee\varepsilon) &\implies \phi^\circ(j)\leq C(\sqrt{h}\vee\varepsilon),\\ \phi^\circ(i)\geq 2\sqrt{C}(\sqrt{h}\vee\varepsilon) &\implies \phi^\circ(j)\geq \sqrt{C}(\sqrt{h}\vee\varepsilon).\end{aligned} \tag{A-16}$$

Thus, equation (A-15) can be directly checked in the case $\phi^\circ(i)\leq 2\sqrt{C}(\sqrt{h}\vee\varepsilon)$ using the definition (A-14).

Note now that definition (A-14) implies $D^*_\varepsilon w=0$ in the region $\{\phi^\circ<2\sqrt{C}(\sqrt{h}\vee\varepsilon)\}$; thus, we assume $\phi^\circ(i)\geq 2\sqrt{C}(\sqrt{h}\vee\varepsilon)$ and estimate $(D^*_\varepsilon w)_i$. If $\phi^\circ(i-\varepsilon e_k)<2\sqrt{C}(\sqrt{h}\vee\varepsilon)$, by convexity $\phi^\circ(i+\varepsilon e_k)>2\sqrt{C}(\sqrt{h}\vee\varepsilon)$. Thus, by definition (A-14) we get

$$z_{i,i+\varepsilon e_k}-z_{i+\varepsilon e_k,i}+z_{i,i-\varepsilon e_k}-z_{i-\varepsilon e_k,i}=-\beta(e_k)-\beta(e_k)+\beta(e_k)-(-\beta(e_k))=0.$$

The symmetric case is analogous. On the other hand, if every $j\sim i$ is in $\{\phi^\circ\geq 2\sqrt{C}(\sqrt{h}\vee\varepsilon)\}$, equation (A-12) holds. Therefore, we have shown

$$hD^*_\varepsilon w\geq-\frac{c_\phi h}{\phi^\circ}\chi_{\{\phi^\circ\geq\sqrt{C}(\sqrt{h}\vee\varepsilon)\}}. \tag{A-17}$$

By a direct computation, using (A-17) and assuming that $C > c_\phi$, we see that the pair (v, w) defined above satisfies

$$\begin{cases} hD_\varepsilon^* w + v \geq \phi^\circ, \\ w_{ij}(v_i - v_j) = \alpha_{ij}^\varepsilon |v_i - v_j|, \ |w_{ij}| \leq \alpha_{ij}^\varepsilon. \end{cases}$$

Recalling the comparison result in Theorem 3.3, we conclude that the solution u to (3-4) satisfies $u \leq v$ in $\varepsilon\mathbb{Z}^N$.

Appendix B: A remark on the inf/sup-convolution formulas (4-5)

In this section we show that, in some particular cases, the "inf" and "sup" in the definitions in (4-5) can be replaced by "min" and "max", and that this minimization/maximization procedure can be made in a fixed neighborhood of the point considered. Yet, our proof also shows that this neighborhood can become very large, depending on the weights of the interaction, and it seems that we cannot expect in general cases that the minimum and maximum are actually reached.

We introduce the following condition. There exists $\ell_\phi > 0$ such that, for every $\varepsilon_k \in \{0, \pm 1\}$ for $k = 1, \ldots, m$, there exists $\ell \leq \ell_\phi$ such that

$$\ell \sum_{k=1}^{m} 2\beta(e_k)\varepsilon_k e_k \in \mathbb{Z}^N. \tag{B-1}$$

Note that this condition is satisfied if $\beta(e_k)/\beta(e_{k'}) \in \mathbb{Q}$ for all $k, k' = 1, \ldots, m$.

Lemma B.1. *There exists $\ell_1 > 0$ with the following property: for any $i \in \varepsilon\mathbb{Z}^N$ with $\phi^\circ(i) \geq \varepsilon\ell_1$ there exists $j \in \varepsilon\mathbb{Z}^N \setminus \{0\}$ with $\phi^\circ(j) < \phi^\circ(i)$ satisfying*

$$\phi^\circ(i) \geq \phi^\circ(j) + \phi^\circ(i - j) - c_\phi\varepsilon. \tag{B-2}$$

If (B-1) *holds, for any $i \in \varepsilon\mathbb{Z}^N$ with $\phi^\circ(i) \geq 2\varepsilon\ell_1$ there exists $j \in (\mathcal{W}_{\varepsilon\ell_1} \setminus \{0\}) \cap \varepsilon\mathbb{Z}^N$ such that*

$$\phi^\circ(i) = \phi^\circ(j) + \phi^\circ(i - j). \tag{B-3}$$

Moreover, for every $R \in (2\varepsilon\ell_1, \phi^\circ(i))$ there exists $j \in \mathcal{W}_R \setminus \mathcal{W}_{R-2\varepsilon\ell_1}$ such that (B-3) *holds.*

Proof. By scaling we prove the result in the case $\varepsilon = 1$. Given $i \in \mathbb{Z}^N \setminus \{0\}$, inequality (B-2) follows easily choosing $\ell_1 \geq 2$, and considering $\sigma i \in \mathbb{R}^N \setminus \{0\}$ for an appropriate $\sigma \in (0, 1)$ and $j \in \mathbb{Z}^N$ such that $\sigma i \in (j + [0, 1]^N)$.

We now assume (B-1) and denote by ℓ_ϕ the radius associated to ϕ. We then choose $\ell_1 = \ell_\phi$. Let us fix $i \in \mathbb{Z}^N$ with $\phi^\circ(i) = \ell \geq 2\ell_1$. By (A-2) there exist $r > 0$, ε_k, λ_k with $|\varepsilon_k| = 1$ and $|\lambda_k| < 1$ such that

$$i = \ell\left(\sum_{k=1}^{r} 2\beta(e_k)\varepsilon_k e_k + \sum_{k=r+1}^{m} \lambda_k 2\beta(e_k)e_k\right).$$

Let us set the point

$$v = \sum_{k=1}^{r} 2\beta(e_k)\varepsilon_k e_k \in \partial\mathcal{W}_1$$

and define the function sign by $\operatorname{sign}(x) = x/|x|$ if $x \neq 0$ and 0 otherwise. For any $\ell' \leq \ell_\phi$ we rewrite i as

$$i = \ell'\Big(v + \sum_{k=r+1}^{m} 2\beta(e_k)\operatorname{sign}(\lambda_k)e_k\Big) + (\ell-\ell')\Big(v + \sum_{k=r+1}^{m} 2\beta(e_k)\Big(\frac{\ell}{\ell-\ell'}\lambda_k - \frac{\ell'}{\ell-\ell'}\operatorname{sign}(\lambda_k)\Big)e_k\Big)$$
$$=: \ell' w + (\ell-\ell')\Big(v + \sum_{k=r+1}^{m} 2\beta(e_k)\lambda'_k e_k\Big).$$

Notice that, since $\ell \geq 2\ell'$ and $|\lambda_k| \leq 1$, we have $|\lambda'_k| \leq 1$; thus, by formula (A-2) we get

$$v + \sum_{k=r+1}^{m} 2\beta(e_k)\lambda'_k e_k \in \partial \mathcal{W}_1,$$

and therefore $\phi^\circ(i - \ell' w) = \ell - \ell'$. We conclude by noting that from the hypothesis (B-1) we can choose $\ell' \leq \ell_1$ so that $\ell' w \in \mathbb{Z}^N$, which implies (B-3) since $\phi^\circ(\ell' w) = \ell'$.

We now prove the last assertion. Since $\phi^\circ(i) \geq 2\ell_1$, by the previous result there exists $j_0 \in (\mathcal{W}_{\ell_1} \setminus \{0\})$ so that $\phi^\circ(i) = \phi^\circ(j_0) + \phi^\circ(i - j_0)$. Now, if $R - 2\ell_1 \leq \phi^\circ(j_0)$ we conclude. If not, then $\phi^\circ(i - j_0) \geq 2\ell_1$ by (B-3), and thus we can find $k_0 \in (\mathcal{W}_{\ell_1} \setminus \{0\})$ so that

$$\phi^\circ(i - j_0) = \phi^\circ(k_0) + \phi^\circ(i - j_0 - k_0). \tag{B-4}$$

Writing $j_1 = j_0 + k_0$, on one hand (B-4) implies

$$\phi^\circ(i) = \phi^\circ(j_0) + \phi^\circ(j_1 - j_0) + \phi^\circ(i - j_1) \geq \phi^\circ(j_1) + \phi^\circ(i - j_1), \tag{B-5}$$

thus equality holds instead. If $\phi^\circ(j_1) \geq R - 2\ell_1$ we conclude; if not (B-5) yields $\phi^\circ(i - j_1) \geq 2\ell_1$ and we can iterate. Recalling that $\phi^\circ \geq c_\phi > 0$ on $\varepsilon\mathbb{Z}^N \setminus \{0\}$, it is clear that after a finite number of iterations the process stops, and one can check that the required properties are satisfied. □

By the previous lemma it is easy to prove the following result.

Corollary B.2. *Assume that* (B-1) *holds. Let $u \in X$ be a* $(1, \phi)$*-Lipschitz function, and let ℓ_1 be as in Lemma B.1. Then, for all $i \in \varepsilon\mathbb{Z}^N$*

$$\sup_{j \in \{u \geq 0\}} \{u_j - \phi^\circ(i - j)\} = \max_{j \in \{u \geq 0\}} \{u_j - \phi^\circ(i - j)\}.$$

In addition, if $i \in \{u \leq 0\}$, the maximum is reached at a point in $(\{u \leq 0\} + \mathcal{W}_{2\varepsilon\ell_1}) \cap \varepsilon\mathbb{Z}^N$.

Proof. It is enough to consider $i \in \{u < 0\} \cap \varepsilon\mathbb{Z}^N$. Let us write $F = (\{u \leq 0\} + \mathcal{W}_{2\varepsilon\ell_1}) \cap \{u > 0\}$. Firstly, by a variant of the argument by iteration employed in the proof of Lemma B.1, one can prove that

$$\sup_{j \in \{u \geq 0\}} \{u_j - \phi^\circ(i - j)\} = \sup_{j \in F}\{u_j - \phi^\circ(i - j)\}. \tag{B-6}$$

On the other hand, take a point $j_0 \in \{u > 0\}$. If $j \in F$ satisfies $u_j - \phi^\circ(i - j) \geq u_{j_0} - \phi^\circ(i - j_0)$, since $u \leq 2\varepsilon\ell_1$ in F (as u is $(1, \phi^\circ)$-Lipschitz) we obtain

$$2\varepsilon\ell_1 + \phi^\circ(i - j_0) \geq \phi^\circ(i - j),$$

which implies that the supremum in (B-6) is indeed a maximum. □

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A BILINEAR FRACTIONAL INTEGRAL OPERATOR FOR EULER–RIESZ SYSTEMS

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We establish a uniform estimate for a bilinear fractional integral operator via restricted weak-type endpoint estimates and Marcinkiewicz interpolation. This estimate is crucial in the integrability analysis of a tensor-valued bilinear fractional integral operator associated with Euler–Riesz systems modeling mean-field interactions induced by a singular kernel. The tensorial operator arises from a reformulation of the Euler–Riesz system that yields a gain in integrability for finite-energy solutions through compensated integrability. Additionally, for smooth periodic solutions of the reformulated system, we derive a stability result.

1. Introduction

We consider the following Euler–Riesz system for $t \geq 0$ and $x \in \mathbb{R}^d$ (with $d \in \mathbb{N}$),

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0, \\ \partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla \rho^\gamma + \rho \nabla K_\alpha * \rho = 0, \end{cases} \tag{1-1}$$

where $\rho : [0, \infty) \times \mathbb{R}^d \to [0, \infty)$ denotes a density, $u : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ stands for the velocity and the exponent γ is greater than 1. The kernel K_α is given by

$$K_\alpha(x) = \frac{1}{d-\alpha} |x|^{\alpha - d} \tag{1-2}$$

with $0 < \alpha < d$; the term $\rho \nabla K_\alpha * \rho$ describes the nonlocal repelling interaction of particles. Smooth solutions (ρ, u) of (1-1) decaying sufficiently fast at infinity satisfy the conservation of energy and mass identities:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} \frac{1}{2} \rho |u|^2 + \frac{1}{\gamma - 1} \rho^\gamma + \frac{1}{2} \rho (K_\alpha * \rho) \, \mathrm{d}x = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} \rho \, \mathrm{d}x = 0. \tag{1-3}$$

This, in particular, yields an a priori estimate for weak solutions, which further implies the regularity $\rho \in L^\infty((0, \infty); L^1 \cap L^\gamma(\mathbb{R}^d))$ for the density.

In this work, we exploit an intriguing connection between harmonic analysis and the theory of Euler–Riesz systems hinging on the study of a bilinear fractional integral operator. The approach is based on a reformulation of the interaction term in divergence form, as seen in (1-4) below, in conjunction with uniform bounds for an associated bilinear fractional integral operator that are established here. This reformulation is advantageous for three reasons: (i) On the one hand, all the terms of the equations are written in divergence form, allowing the derivatives to be absorbed by the test functions in a weak

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formulation. (ii) The harmonic analysis estimates lead to integrability properties of the nonlocal interaction term. (iii) Finally, for finite-energy solutions, it provides a higher integrability estimate for the density in space-time, achieved by applying the compensated integrability theory for divergence-free positive symmetric tensors [Serre 2018; 2019; 2023] to the setting of Euler–Riesz systems.

To illustrate, note that the only term of (1-1) that is not in divergence form is the interaction term $\rho \nabla K_\alpha * \rho$. Inspired by a calculation in [Serre 2019] and exploiting the symmetry of the kernel K_α, one reaches the identity

$$\rho \nabla K_\alpha * \rho = \nabla \cdot S_\alpha(\rho), \tag{1-4}$$

where $S_\alpha(\rho)$ is a tensor defined by

$$S_\alpha(\rho)(t,x) = \frac{1}{2}\int_0^1 \int_{\mathbb{R}^d} \rho(t, x+(\theta-1)y)\rho(t, x+\theta y)|y|^{\alpha-d-2} y \otimes y \,\mathrm{d}y\,\mathrm{d}\theta. \tag{1-5}$$

Identity (1-4) is derived in the Appendix and yields a reformulation of (1-1) in which the equations are expressed as a divergence-free condition of a tensor that fits into the compensated integrability framework of [Serre 2018]. This reformulation leads, in turn, to a higher integrability estimate for finite-energy solutions thereby improving on the integrability provided by the energy identity; see Theorem 3.2.

To analyze $S_\alpha(\rho)$, we consider a bilinear fractional integral operator I_α^θ defined for nonnegative measurable functions f and g on $\mathbb{R}^d$ by

$$I_\alpha^\theta(f,g)(x) = \int_{\mathbb{R}^d} f(x+(\theta-1)y)g(x+\theta y)|y|^{\alpha-d}\,\mathrm{d}y \tag{1-6}$$

with $0 < \alpha < d$ and $0 \le \theta \le 1$. The main result of this work provides a uniform bound in θ for I_α^θ with assumptions similar in style to the classical Hardy–Littlewood–Sobolev (HLS) inequality; see Theorem 2.1.

From the natural integrability of ρ induced by the energy identity, one may deduce using the classical HLS inequality that the term $\rho\nabla K_\alpha * \rho$ belongs to L^1 in space whenever $1 < \alpha < d$. By contrast, when employing the formulation (1-4) via the tensor $S_\alpha(\rho)$, one improves the range to $0 < \alpha < d$. This observation underlines the importance of the reformulation of (1-1) through identity (1-4).

The paper is organized as follows. In Section 2 we state the main theorem of this work and describe the associated results. In Section 3 we explain how the theory of compensated integrability leads to a higher integrability estimate for finite-energy solutions of the Euler–Riesz system. Section 4 contains the proof of the main theorem and its corollary, Proposition 2.3, which yields an integrability result for the tensor given by (1-5). Finally, in Section 5, we establish a stability result for smooth periodic solutions of the reformulated Euler–Riesz system via the relative energy method.

2. Description of results

The main theorem of this work provides a uniform estimate for the bilinear fractional operator I_α^θ given by (1-6); see Theorem 2.1 below.

Fractional integral operators have been of great importance in harmonic analysis for several decades; however, in recent years, their bilinear analogues have also attracted research attention. In particular, an operator B_α, with $0 < \alpha < d$, acting on nonnegative measurable functions of $\mathbb{R}^d$ as

$$B_\alpha(f, g)(x) = \int_{\mathbb{R}^d} f(x-y)g(x+y)|y|^{\alpha-d}\,\mathrm{d}y$$

was first considered in [Grafakos 1992] and later in [Kenig and Stein 1999; Grafakos and Kalton 2001], in which optimal boundedness properties between Lebesgue spaces were established. It has subsequently been studied extensively by several authors; we refer to [Ding and Lin 2002; Moen 2014; Li and Sun 2016; Hoang and Moen 2018; Hatano and Sawano 2019; Komori-Furuya 2020; He and Yan 2021] for estimates concerning B_α (and related versions) on a variety of spaces. While the operator I_α^θ is quite similar to B_α when the dependence on the parameter θ is ignored, its study becomes more intricate when seeking estimates that are uniform in the auxiliary parameter θ.

These types of operators have sparked significant interest primarily due to the singular nature of their integrands, but also due to their proximity to Hilbert transforms. Notable examples include the linear fractional integral operator (also known as the Riesz potential) and the linear Hilbert transform. In our case, the bilinear operator I_α^θ is related to a bilinear Hilbert transform H^θ given by

$$H^\theta(f, g)(x) = \text{p.v.} \int_{\mathbb{R}} f(x+(\theta-1)t)g(x+\theta t)\frac{\mathrm{d}t}{t}.$$

Uniform bounds in θ for this transform can be deduced by direct application of the results obtained in [Grafakos and Li 2004; Li 2006]. Other boundedness results for similar bilinear Hilbert transforms can be found in [Lacey and Thiele 1997; 1999].

2.1. *Main result.*

Theorem 2.1. *Let $d \in \mathbb{N}$ be the dimension, $0 < \alpha < d$, and p, q, r be integrability exponents satisfying*

$$1 < p, q < \frac{d}{\alpha} \quad \textit{and} \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + \frac{\alpha}{d}.$$

Then there is a constant $C = C(\alpha, d, p, q) > 0$ independent of θ such that for all $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$ we have

$$\|I_\alpha^\theta(f, g)\|_{L^r(\mathbb{R}^d)} \le C\|f\|_{L^p(\mathbb{R}^d)}\|g\|_{L^q(\mathbb{R}^d)}. \tag{2-1}$$

Whenever $\left(\frac{1}{p}, \frac{1}{q}\right)$ lies in the interior of the square with vertices $\left(\frac{\alpha}{d}, \frac{\alpha}{d}\right)$, $\left(\frac{\alpha}{d}, 1\right)$, $\left(1, \frac{\alpha}{d}\right)$, and $(1, 1)$, the operator I_α^θ is bounded from $L^p(\mathbb{R}^d) \times L^q(\mathbb{R}^d)$ to $L^r(\mathbb{R}^d)$ *uniformly in* θ when $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + \frac{\alpha}{d}$. If one ignores the uniform bounds in θ, then I_α^θ is bounded from $L^p(\mathbb{R}^d) \times L^q(\mathbb{R}^d)$ to $L^r(\mathbb{R}^d)$ when the pair $\left(\frac{1}{p}, \frac{1}{q}\right)$ lies in the interior of the pentagon with vertices $\left(0, \frac{\alpha}{d}\right)$, $(0, 1)$, $(1, 1)$ $(1, 0)$, and $\left(\frac{\alpha}{d}, 0\right)$. See Figure 1.

The proof of Theorem 2.1 relies on a bilinear version of the Marcinkiewicz interpolation method, where, from a finite set of restricted weak-type estimates, one deduces strong-type estimates; see Proposition 4.6. A more general version was established in [Grafakos and Kalton 2001; Grafakos et al. 2012] for multilinear

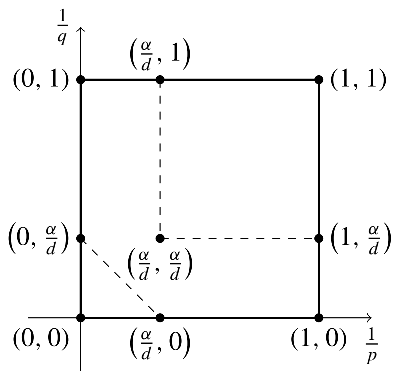

Figure 1. Region of boundedness.

operators. In particular, in [Grafakos and Kalton 2001], this method is deduced as a corollary of a Boyd interpolation theorem in the framework of quasinormed rearrangement-invariant spaces.

Finally, we would like to point out that the largest possible region in which uniform estimates hold for I_α^θ is, in fact, the open square with vertices $\left(\frac{\alpha}{d}, \frac{\alpha}{d}\right)$, $\left(\frac{\alpha}{d}, 1\right)$, $\left(1, \frac{\alpha}{d}\right)$, and $(1, 1)$. By interpolation, it suffices to check that uniform bounds fail on the boundary of this square. To verify this, let us assume that a uniform bound

$$\sup_{0\le\theta\le 1} \|I_\alpha^\theta(f, g)\|_{L^r(\mathbb{R}^d)} \le C\|f\|_{L^p(\mathbb{R}^d)}\|g\|_{L^{d/\alpha}(\mathbb{R}^d)}$$

holds on the horizontal dotted line, that is, when $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}+\frac{\alpha}{d}$ and $q=\frac{d}{\alpha}$, for some positive constant $C = C(\alpha, d, p)$. In this case, we must have $p = r$. By Fatou's lemma, it follows that

$$\Big\|\liminf_{\theta\to 1} I_\alpha^\theta(f, g)\Big\|_{L^p(\mathbb{R}^d)} \le C\|f\|_{L^p(\mathbb{R}^d)}\|g\|_{L^{d/\alpha}(\mathbb{R}^d)};$$

hence, for all Schwartz functions f and g we must have

$$\|f I_\alpha(g)\|_{L^p(\mathbb{R}^d)} \le C\|f\|_{L^p(\mathbb{R}^d)}\|g\|_{L^{d/\alpha}(\mathbb{R}^d)}, \tag{2-2}$$

where I_α is the linear fractional integral operator

$$I_\alpha(g)(x) = \int_{\mathbb{R}^d} g(x+y)|y|^{\alpha-d}\,\mathrm{d}y = \int_{\mathbb{R}^d} g(x-y)|y|^{\alpha-d}\,\mathrm{d}y.$$

Now inserting

$$f(x) = f_{\epsilon,x_0}(x) = \left(\frac{p}{\epsilon}\right)^{\frac{d}{2p}} e^{-\frac{\pi}{\epsilon}|x-x_0|^2}$$

in (2-2) and letting $\epsilon \to 0$, we obtain

$$|I_\alpha(g)(x_0)| \le C\|g\|_{L^{d/\alpha}(\mathbb{R}^d)}$$

for all $x_0 \in \mathbb{R}^d$. This would imply that I_α maps $L^{d/\alpha}(\mathbb{R}^d)$ to $L^\infty(\mathbb{R}^d)$, a fact known to be false; see [Grafakos 2024, Example 5.1.4]. An analogous argument (letting $\theta \to 0$) indicates that a uniform bound also cannot hold on the vertical dotted line of Figure 1.

2.2. ***Connections with the HLS inequality.*** We recall the HLS inequality [Lieb and Loss 2001].

Proposition 2.2. *Let $p, q > 1$ and $0 < \alpha < d$ satisfy*

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{\alpha}{d}.$$

If $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$, then

$$\left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x)|x-y|^{\alpha-d} g(y)\,\mathrm{d}x\,\mathrm{d}y \right| \le C \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)} \tag{2-3}$$

for some $C = C(\alpha, d, p) > 0$.

Note that the assumptions on the integrability exponents p and q in Theorem 2.1 with $r = 1$ and Proposition 2.2 are exactly the same. Indeed, the assumptions of Proposition 2.2 imply that $p, q < \frac{d}{\alpha}$. To check this fact, suppose, without loss of generality, that $p > 1$ and $q \ge \frac{d}{\alpha}$. Then $\frac{1}{p} + \frac{1}{q} < 1 + \frac{\alpha}{d}$, which is a contradiction.

Furthermore, the HLS inequality can be used to infer the L^1 boundedness of the operator I_α^θ since, for nonnegative measurable functions f and g, appropriate changes of variables yield

$$\|I_\alpha^\theta(f, g)\|_{L^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x)|x-y|^{\alpha-d} g(y)\,\mathrm{d}x\,\mathrm{d}y.$$

Thus, the uniform estimate (2-1) is nontrivial and of independent interest when $r \neq 1$.

2.3. ***A tensorial bilinear fractional integral operator.*** Consider a bilinear version of the tensor $S_\alpha(\rho)$; that is, define a tensorial bilinear fractional integral operator J_α for nonnegative measurable functions f and g on $\mathbb{R}^d$ by

$$J_\alpha(f, g)(x) = \int_0^1 \int_{\mathbb{R}^d} f(x + (\theta - 1)y) g(x + \theta y) |y|^{\alpha-d-2} y \otimes y\,\mathrm{d}y\,\mathrm{d}\theta. \tag{2-4}$$

Note that $S_\alpha(\rho) = \frac{1}{2} J_\alpha(\rho, \rho)$. Additionally, identity (1-4) can be written in terms of J_α as

$$f \nabla K_\alpha * f = \nabla \cdot \left(\tfrac{1}{2} J_\alpha(f, f)\right). \tag{2-5}$$

As a consequence of Theorem 2.1, for the operator J_α we obtain the following result:

Proposition 2.3. *Let $0 < \alpha < d$ and p, q, r be integrability exponents satisfying*

$$1 < p, q < \frac{d}{\alpha}, \quad r \ge 1, \quad \text{and} \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + \frac{\alpha}{d}.$$

Then for all $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$ we have

$$\|J_\alpha(f, g)\|_{L^r(\mathbb{R}^d)} \le C \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)} \tag{2-6}$$

for some $C = C(\alpha, d, p, q) > 0$.

The previous proposition leads to an integrability result for $S_\alpha(\rho)$. Indeed, if

$$\rho \in L^1(\mathbb{R}^d) \cap L^\gamma(\mathbb{R}^d),$$

as implied by the a priori bounds (1-3), and $1 < 2dr/(d+\alpha r) \le \gamma$ for some $r \ge 1$, then $S_\alpha(\rho) \in L^r(\mathbb{R}^d)$ and there exists a constant $C = C(\alpha, d, r) > 0$ such that

$$\|S_\alpha(\rho)\|_{L^r(\mathbb{R}^d)} \le C\|\rho\|^2_{L^p(\mathbb{R}^d)}, \tag{2-7}$$

where $p = 2dr/(d+\alpha r)$. Note that by interpolation, the right-hand side of (2-7) is controlled by the norm $\|\rho\|_{L^1(\mathbb{R}^d)} + \|\rho\|_{L^\gamma(\mathbb{R}^d)}$.

2.4. ***Reformulation of the Euler–Riesz system.*** Consider the Euler–Riesz system (1-1) supplemented with initial data ρ_0 and u_0. This system comprises a continuity equation for the conservation of mass and a second equation that ensures the conservation of momentum. These equations govern the dynamics of a compressible fluid with density ρ and linear velocity u, subject to pressure and interaction forces. The pressure function is given by $p(\rho) = \rho^\gamma$, with $\gamma > 1$ being the adiabatic exponent, and the interaction forces are modeled through the kernel K_α given by (1-2). For $d \ge 3$ and $\alpha = 2$, we recover the Euler–Poisson equations, as in that case, the interaction kernel K_2 is the Newtonian kernel. For existence theories on Euler–Riesz systems, we refer to [Choi and Jeong 2022; Danchin and Ducomet 2022; Carrillo et al. 2025].

As observed in the introduction, smooth solutions of (1-1) satisfy a priori bounds of conservation of energy and mass. Given the adiabatic exponent, it is reasonable to consider solutions such that ρ belongs to $L^1 \cap L^\gamma$ in space. A natural question is whether this integrability can be improved by exploiting the structure of the equations. For finite-energy solutions this can be accomplished by compensated integrability. Specifically, for a finite-energy solution (ρ, u) of (1-1), one can prove that for each $T > 0$

$$\text{if } \ \rho \in L^\infty(0, T; L^1(\mathbb{R}^d) \cap L^\gamma(\mathbb{R}^d)) \ \text{ then } \ \rho \in L^{\gamma+\frac{1}{d}}((0, T) \times \mathbb{R}^d). \tag{2-8}$$

The first step towards (2-8) is to rewrite system (1-1) as a space-time divergence-free condition for an appropriate tensor. This is made possible through identity (1-4). This reformulates system (1-1) into a divergence-free positive symmetric tensor form, fitting in the compensated integrability theory of [Serre 2018; 2019; 2023], thereby yielding the integrability improvement (2-8); see Section 3. We refer to [Guerra et al. 2024] for an extension of this theory, and to [LeFloch and Westdickenberg 2007] where a higher integrability estimate is obtained for one-dimensional finite-energy solutions of an isentropic Euler system using a different methodology.

Next, we explore a possible weak formulation for the Euler–Riesz system (1-1). For the continuity equation take

$$\int_0^\infty \int_{\mathbb{R}^d} \rho \partial_t \varphi + \rho u \cdot \nabla \varphi \, \mathrm{d}x \, \mathrm{d}t + \int_{\mathbb{R}^d} \rho_0 \varphi_0 \, \mathrm{d}x = 0,$$

where $\varphi \in C^1_c([0, \infty) \times \mathbb{R}^d)$ is a test function with $\varphi|_{t=0} = \varphi_0$. For the momentum equation we have two options, according to identity (1-4). Let $\xi \in C^1_c([0, \infty) \times \mathbb{R}^d; \mathbb{R}^d)$ be a test function with $\xi|_{t=0} = \xi_0$.

Using the left-hand side of (1-4), we get

$$\int_0^\infty\int_{\mathbb{R}^d} \rho u\cdot\partial_t\xi + (\rho u\otimes u + \rho^\gamma I_d) : \nabla\xi - \rho\nabla(K_\alpha * \rho)\cdot\xi \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^d} \rho_0 u_0\cdot\xi_0\,\mathrm{d}x = 0, \tag{2-9}$$

whereas, using the right-hand side of (1-4), we have

$$\int_0^\infty\int_{\mathbb{R}^d} \rho u\cdot\partial_t\xi + (\rho u\otimes u + \rho^\gamma I_d + S_\alpha(\rho)) : \nabla\xi \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^d} \rho_0 u_0\cdot\xi_0\,\mathrm{d}x = 0, \tag{2-10}$$

where I_d is the $d\times d$ identity matrix, and for two square matrices $A=(a_{ij})$ and $B=(b_{ij})$, we define $A : B = \sum_{i,j} a_{ij}b_{ij}$.

Assume that $\rho\in L^1\cap L^\gamma(\mathbb{R}^d)$. Using the HLS inequality we obtain the following:

(i) If $1<\alpha<d$ and $\gamma\ge q = 2d/(d+\alpha-1)$, then $\rho\nabla K_\alpha * \rho \in L^1(\mathbb{R}^d)$ since

$$\|\rho\nabla K_\alpha * \rho\|_{L^1(\mathbb{R}^d)} \le C(\alpha,d)\|\rho\|^2_{L^q(\mathbb{R}^d)}.$$

(ii) If $0<\alpha<d$ and $\gamma\ge p = 2d/(d+\alpha)$, then $S_\alpha(\rho)\in L^1(\mathbb{R}^d)$ since

$$\|S_\alpha(\rho)\|_{L^1(\mathbb{R}^d)} \le C(\alpha,d)\|\rho\|^2_{L^p(\mathbb{R}^d)}.$$

The second formulation is preferable as it is well-defined for a larger range of the parameters α and γ.

3. Compensated integrability

In this section we provide a proof for (2-8), which is one of the reasons for having considered the tensor $S_\alpha(\rho)$ and subsequently the bilinear fractional integral operator I_α^θ.

3.1. ***A divergence-free positive symmetric tensor.*** First, we write system (1-1) as a space-time divergence-free condition for an appropriate tensor. Thanks to (1-4), system (1-1) can be reformulated into

$$\nabla_{t,x}\cdot A_\alpha(\rho,u) = 0, \tag{3-1}$$

where the $(1{+}d)$-tensor $A_\alpha(\rho,u)$ is given by

$$A_\alpha(\rho,u) = \begin{bmatrix} \rho & (\rho u)^\top \\ \rho u & \rho u\otimes u + p(\rho)I_d + S_\alpha(\rho)\end{bmatrix}. \tag{3-2}$$

Next, we deduce some basic properties of the tensor $S_\alpha(\rho)$ given by (1-5) that are relevant for the subsequent analysis.

Proposition 3.1. *The tensor $S_\alpha(\rho)$ is symmetric, positive semidefinite and*

$$\det(p(\rho)I_d + S_\alpha(\rho)) \ge \begin{cases} p(\rho)^d, \\ \det S_\alpha(\rho). \end{cases}$$

Proof. It is clear that $S_\alpha(\rho)$ is symmetric since $y\otimes y$ is symmetric. Moreover, given a vector $v=v(x)$,

$$v^\top S_\alpha(\rho)v = \frac{1}{2}\int_{\mathbb{R}^d}\int_0^1 \rho(x+(\theta-1)y)\rho(x+\theta y)|y|^{\alpha-d-2}(y\cdot v)^2\,\mathrm{d}\theta\,\mathrm{d}y \ge 0;$$

hence $S_\alpha(\rho)$ is positive semidefinite. Therefore, there exist nonnegative eigenvalues $\lambda_1, \dots, \lambda_d$ and their respective eigenvectors $v_1, \dots, v_d$; that is, $S_\alpha(\rho)v_i = \lambda_i v_i$. Then $p(\rho) + \lambda_i$ is an eigenvalue of $p(\rho)I_d + S_\alpha(\rho)$ since $(p(\rho)I_d + S_\alpha(\rho))v_i = (p(\rho) + \lambda_i)v_i$. Hence, given that $\lambda_i \geq 0$,

$$\det(p(\rho)I_d + S_\alpha(\rho)) = \prod_{i=1}^{d}(p(\rho) + \lambda_i) \geq \begin{cases} \prod_{i=1}^{d} p(\rho) = p(\rho)^d, \\ \prod_{i=1}^{d} \lambda_i = \det S_\alpha(\rho). \end{cases} \qquad \square$$

It follows that the tensor $A_\alpha(\rho, u)$ is symmetric and positive semidefinite. To check the latter, let $w = (w_0, \tilde{w})$, with w_0 being a scalar and $\tilde{w}$ a d-dimensional vector, and note that

$$\begin{aligned} w^\top A w &= \begin{bmatrix} w_0 & \tilde{w}^\top \end{bmatrix} \begin{bmatrix} \rho & (\rho u)^\top \\ \rho u & \rho u \otimes u + p(\rho)I_d + S_\alpha(\rho) \end{bmatrix} \begin{bmatrix} w_0 \\ \tilde{w} \end{bmatrix} \\ &= \rho(w_0 + u \cdot \tilde{w})^2 + p(\rho)|\tilde{w}|^2 + \tilde{w}^\top S_\alpha(\rho)\tilde{w} \\ &\geq 0, \end{aligned}$$

where in the last step we used the fact that $S_\alpha(\rho)$ is positive semidefinite.

Consequently $A_\alpha(\rho, u)$ is a divergence-free positive symmetric tensor.

3.2. ***Higher integrability for finite-energy solutions.*** Assume that (ρ, u) is a solution of (3-1) with finite mass and energy and such that $A_\alpha(\rho, u)$ belongs to $L^1((0, T) \times \mathbb{R}^d) \cap L^{1+1/d}_{\mathrm{loc}}((0, T) \times \mathbb{R}^d)$ for each $T > 0$. Note that, by the conservation of mass and energy, it suffices to prescribe initial data (ρ_0, u_0) with finite mass and energy. We apply [Serre 2018, Theorem 2.3] to the tensor $A_\alpha(\rho, u)$, along the same lines of the proof of [Serre 2018, Theorem 3.1].

Set

$$\begin{aligned} \Sigma &= (0, T) \times B_R, \\ B_R &= \{x \in \mathbb{R}^d : |x| < R\}, \\ \partial\Sigma &= (\{0\} \times B_R) \cup ((0, T) \times \partial B_R) \cup (\{T\} \times B_R). \end{aligned}$$

We have the estimate

$$\int_0^T \int_{B_R} (\det A_\alpha(\rho, u))^{\frac{1}{d}} \,\mathrm{d}x\,\mathrm{d}t \leq c_d \|A_\alpha(\rho, u)\nu\|_{L^1(\partial\Sigma)}^{1+\frac{1}{d}}, \tag{3-3}$$

where ν is the outward normal vector to the boundary of Σ, given by

$$\nu = \begin{cases} (-1, 0_d) & \text{on } \{0\} \times B_R, \\ z = (0, x/|x|) & \text{on } (0, T) \times B_R, \\ (1, 0_d) & \text{on } \{T\} \times B_R. \end{cases}$$

Hence

$$\|A_\alpha(\rho, u)\nu\|_{L^1(\partial\Sigma)} = \int_{B_R} |(\rho, \rho u)|_{t=0} + |(\rho, \rho u)|_{t=T} \,\mathrm{d}x + \psi(R),$$

where

$$\psi(R) = \int_0^T \int_{\partial B_R} |A_\alpha(\rho, u)z| \,\mathrm{d}x\,\mathrm{d}t.$$

Since $A_\alpha(\rho, u)$ is integrable it follows that $\psi \in L^1(0,\infty)$. Indeed,

$$\begin{aligned}\int_0^\infty |\psi(R)|\,\mathrm{d}R &\le \int_0^\infty\int_0^T\int_{\partial B_R} |A_\alpha(\rho,u)|\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}R\\ &= \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}R}\int_0^T\int_{B_R} |A_\alpha(\rho,u)|\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}R\\ &= \|A_\alpha(\rho,u)\|_{L^1((0,T)\times\mathbb{R}^d)}.\end{aligned}$$

Therefore, there exists a sequence $R_n \to \infty$ such that $\psi(R_n) \to 0$. Considering this limit in (3-3), and using the conservation of mass and momentum, gives

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^d} (\det A_\alpha(\rho,u))^{\frac{1}{d}}\,\mathrm{d}x\,\mathrm{d}t &\le c_d\bigg(\int_{\mathbb{R}^d} |(\rho,\rho u)|_{t=0} + |(\rho,\rho u)|_{t=T}\,\mathrm{d}x\bigg)^{1+\frac{1}{d}}\\ &= 2c_d\bigg(\int_{\mathbb{R}^d} \sqrt{\rho_0^2+\rho_0^2|u_0|^2}\,\mathrm{d}x\bigg)^{1+\frac{1}{d}}\\ &\le 2c_d\bigg(\int_{\mathbb{R}^d} \rho_0+\rho_0|u_0|\,\mathrm{d}x\bigg)^{1+\frac{1}{d}},\end{aligned}$$

where in the last inequality we used that $\sqrt{a^2+b^2} \le a+b$ for $a, b \ge 0$.

Now, using Proposition 3.1,

$$\begin{aligned}\det A_\alpha(\rho,u) &= \rho\det\Big(\rho u\otimes u + p(\rho)I_d + S_\alpha(\rho) - \rho u\frac{1}{\rho}\rho u^\top\Big)\\ &= \rho\det(p(\rho)I_d + S_\alpha(\rho)) \ge \rho p(\rho)^d.\end{aligned}$$

Consequently,

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^d} \rho^{\frac{1}{d}} p(\rho)\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_{\mathbb{R}^d} (\rho p(\rho)^d)^{\frac{1}{d}}\,\mathrm{d}x\,\mathrm{d}t\\ &\le \int_0^T\int_{\mathbb{R}^d} (\det A_\alpha(\rho,u))^{\frac{1}{d}}\,\mathrm{d}x\,\mathrm{d}t\\ &\le 2c_d\bigg(\int_{\mathbb{R}^d} \frac{3}{2}\rho_0 + \frac{1}{2}\rho_0|u_0|^2\,\mathrm{d}x\bigg)^{1+\frac{1}{d}}\\ &\le 2c_d\bigg(\frac{3}{2}\int_{\mathbb{R}^d}\rho_0\,\mathrm{d}x + \int_{\mathbb{R}^d} \frac{1}{2}\rho_0|u_0|^2 + h(\rho_0) + \frac{1}{2}\rho_0 K*\rho_0\,\mathrm{d}x\bigg)^{1+\frac{1}{d}}\\ &< \infty,\end{aligned}$$

which establishes the desired higher integrability estimate given that $p(\rho) = \rho^\gamma$.

Letting $T \to \infty$ we obtain:

Theorem 3.2. *Finite-energy solutions of the Euler–Riesz system* (1-1) *with* $0 < \alpha < d$ *and repelling potentials satisfy the a priori estimate*

$$\rho \in L^{\gamma+\frac{1}{d}}((0,\infty)\times\mathbb{R}^d). \tag{3-4}$$

The use of the compensated integrability theory for divergence-free positive symmetric tensors requires the potential to be repulsive.

Other methodological approaches for obtaining improved integrability estimates exist in the literature, starting with [LeFloch and Westdickenberg 2007], pertaining to one-dimensional spherically symmetric Euler equations. Additionally, in a recent and interesting work, Carrillo, Charles, Chen and Yuan [Carrillo et al. 2025] establish the existence of global finite-energy weak solutions for the axisymmetric version of the Euler–Riesz system (1-1). Their approach uses compensated compactness for radially symmetric solutions of Euler–Riesz systems and is based on an integrability estimate for the density [Carrillo et al. 2025, Lemma 3.12]. When compared to (3-4), their estimate corresponds to the case $d = 1$, but it applies to potentials that are either repulsive or attractive with a wide range of power or logarithmic interactions.

Remark 3.3. The special case $\alpha = 2$ with $d \geq 3$ corresponds to the Euler–Poisson system

$$\begin{aligned} \partial_t \rho + \nabla \cdot (\rho u) &= 0, \\ \partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) &= -\nabla \rho^\gamma - \rho \nabla \varphi, \\ -\Delta \varphi &= \rho, \end{aligned} \tag{3-5}$$

commonly used in models for electrically charged fluids.

In this case, as the potential is given as the solution of Poisson's equation, one could also write

$$\rho \nabla \varphi = \nabla \cdot \big(\tfrac{1}{2} |\nabla \varphi|^2 I_d - \nabla \varphi \otimes \nabla \varphi \big),$$

however, the tensor being applied by the divergence on the right-hand side is not positive semidefinite, and therefore it does not fit into the theory of compensated integrability.

4. Bilinear harmonic analysis

The aim of this section is to prove Theorem 2.1. Since all the operators involved are positive, we assume that all the considered functions are nonnegative. Given that all the Lebesgue spaces in this section are over $\mathbb{R}^d$, we shorten the notation of $L^p(\mathbb{R}^d)$ to L^p.

4.1. ***An auxiliary operator I^θ.*** For $0 \leq \theta \leq 1$, let I^θ be the bilinear operator defined for (nonnegative) measurable functions f and g on $\mathbb{R}^d$ by

$$I^\theta(f, g)(x) = \int_{|y| \leq 1} f(x + (\theta - 1)y) g(x + \theta y) \, \mathrm{d}y.$$

Lemma 4.1. *The operator I^θ maps $L^1 \times L^1$ to $L^{1/2}$ uniformly in θ. Precisely,*

$$\|I^\theta(f, g)\|_{L^{1/2}} \leq C \|f\|_{L^1} \|g\|_{L^1} \tag{4-1}$$

with $C = 3^d 5^{2d}$.

Proof. We first prove (4-1) with $C = 3^d$ for integrable functions f and g supported in cubes with sides of length 1 parallel to the axes. Let $Q_0 = [0, 1]^d$ and, for each $k \in \mathbb{Z}^d$, let $Q_k = k + Q_0$ denote the cube with side length 1 whose sides are parallel to the axes and whose lower left corner is k. For $k = (k_1, \ldots, k_d)$

and $l = (l_1, \dots, l_d)$ in $\mathbb{Z}^d$, assume that f is supported in Q_k and that g is supported in Q_l. Under these conditions, we claim that $I^\theta(f, g)$ is supported in a cube Q of side length 3. Indeed, for each $i = 1, \dots, d$, the inequalities

$$k_i \le x_i + (\theta - 1)y_i \le k_i + 1, \quad l_i \le x_i + \theta y_i \le l_i + 1$$

together with $|y| \le 1$ and $0 \le \theta \le 1$ imply that

$$k_i - 1 \le x_i \le k_i + 2, \quad l_i - 1 \le x_i \le l_i + 2,$$

which establishes the claim. Thus, for these f and g, the Cauchy–Schwarz inequality gives

$$\begin{aligned} \|I^\theta(f, g)\|_{L^{1/2}} = \left(\int_{\mathbb{R}^d} \chi_Q |I^\theta(f, g)|^{\frac{1}{2}} \, \mathrm{d}x \right)^2 &\le 3^d \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x + (\theta - 1)y) g(x + \theta y) \, \mathrm{d}y \, \mathrm{d}x \\ &\le 3^d \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(z - y) g(z) \, \mathrm{d}z \, \mathrm{d}y \\ &\le 3^d \|f\|_{L^1} \|g\|_{L^1}. \end{aligned}$$

Now that we have established (4-1) for all integrable f and g supported in cubes with side length 1, we proceed to the general case. For each k and m in $\mathbb{Z}^d$, set $f_k = \chi_{Q_k} f$ and $g_m = \chi_{Q_m} g$.

Given $x \in \mathbb{R}^d$, we claim that if $I^\theta(f_k, g_m)(x) \ne 0$, then each m_i satisfies $k_i \le m_i \le k_i + 2$. Indeed, under the hypothesis $I^\theta(f_k, g_m)(x) \ne 0$, we have that $x + (\theta - 1)y \in Q_k$ and $x + \theta y \in Q_m$, and so the conditions

$$k_i \le x_i + (\theta - 1)y_i \le k_i + 1$$

and

$$m_i \le x_i + \theta y_i \le m_i + 1$$

hold for each $i = 1, \dots, d$. Since $|y| \le 1$, the conditions above imply that

$$k_i \le x_i + \theta y_i - y_i \le x_i + \theta y_i + 1 \le m_i + 2$$

and

$$m_i \le x_i + \theta y_i = x_i + (\theta - 1)y_i + y_i \le k_i + 2,$$

which establishes the claim. So, for any fixed $k \in \mathbb{Z}^d$, if $I^\theta(f_k, g_m)(x) \ne 0$, then $m = k + l$, where $l \in [-2, 2]^d \cap \mathbb{Z}^d = F$. Note that F contains at most 5^d elements.

We have

$$|I^\theta(f, g)|^{\frac{1}{2}} \le \sum_{k \in \mathbb{Z}^d} \sum_{m \in \mathbb{Z}^d} |I^\theta(f_k, g_m)|^{\frac{1}{2}} = \sum_{l \in F} \sum_{k \in \mathbb{Z}^d} |I^\theta(f_k, g_{k+l})|^{\frac{1}{2}}$$

and so, using the fact that (4-1) with $C = 3^d$ holds for the functions f_k and g_{k+l}, it follows that

$$\|I^\theta(f, g)\|_{L^{1/2}} \le \left(\sum_{l \in F} \sum_{k \in \mathbb{Z}^d} \|I^\theta(f_k, g_{k+l})\|_{L^{1/2}}^{\frac{1}{2}} \right)^2 \le 3^d \left(\sum_{l \in F} \sum_{k \in \mathbb{Z}^d} \|f_k\|_{L^1}^{\frac{1}{2}} \|g_{k+l}\|_{L^1}^{\frac{1}{2}} \right)^2.$$

Finally, applying the Cauchy–Schwarz inequality to the last term above yields

$$\begin{aligned}\|I^\theta(f,g)\|_{L^{1/2}} &\le 3^d\biggl(\sum_{l\in F}\biggl(\sum_{k\in\mathbb{Z}^d}\|f_k\|_{L^1}\biggr)^{\frac12}\biggl(\sum_{k\in\mathbb{Z}^d}\|g_{k+l}\|_{L^1}\biggr)^{\frac12}\biggr)^2\\ &\le 3^d\biggl(\sum_{l\in F}\|f\|_{L^1}^{\frac12}\|g\|_{L^1}^{\frac12}\biggr)^2 \le 3^d5^{2d}\|f\|_{L^1}\|g\|_{L^1},\end{aligned}$$

which concludes the proof. □

4.2. ***A dilated version of I^θ.*** In this section we consider a dilated version of I^θ, denoted by I_j^θ, for $j\in\mathbb{Z}$. This is defined as

$$I_j^\theta(f,g)(x)=\int_{|y|\le 2^j} f(x+(\theta-1)y)g(x+\theta y)\,\mathrm{d}y.$$

Lemma 4.2. *The operator I_j^θ maps $L^1\times L^1$ to L^1 uniformly in θ and in j. Precisely,*

$$\|I_j^\theta(f,g)\|_{L^1}\le\|f\|_{L^1}\|g\|_{L^1}. \tag{4-2}$$

Proof. Let $f,g\in L^1$. By Fubini's theorem and the change of variables $x+(\theta-1)y=z$ we have

$$\|I_j^\theta(f,g)\|_{L^1}=\int_{|y|\le 2^j}\int_{\mathbb{R}^d} f(z)g(z+y)\,\mathrm{d}z\,\mathrm{d}y.$$

Using Fubini's theorem once more, together with the change of variables $z+y=w$, it follows that

$$\|I_j^\theta(f,g)\|_{L^1}=\int_{\mathbb{R}^d} f(z)\int_{|w-z|\le 2^j} g(w)\,\mathrm{d}w\,\mathrm{d}z\le\|f\|_{L^1}\|g\|_{L^1}.$$ □

Lemma 4.3. *The operator I_j^θ maps $L^1\times L^1$ to $L^{1/2}$ uniformly in θ. Precisely,*

$$\|I_j^\theta(f,g)\|_{L^{1/2}}\le 2^{dj}3^d5^{2d}\|f\|_{L^1}\|g\|_{L^1}. \tag{4-3}$$

Proof. This is a consequence of (4-1) via a dilation argument which we include for convenience.

We have

$$\begin{aligned}\|I_j^\theta(f,g)\|_{L^{1/2}} &= \biggl(\int_{\mathbb{R}^d}|I_j^\theta(f,g)(2^jx)|^{\frac12}2^{dj}\,\mathrm{d}x\biggr)^2\\ &= 2^{2dj}\biggl(\int_{\mathbb{R}^d}\biggl(\int_{|y|\le 2^j} f(2^jx+(\theta-1)y)g(2^jx+\theta y)\,\mathrm{d}y\biggr)^{\frac12}\mathrm{d}x\biggr)^2\\ &= 2^{2dj}\biggl(\int_{\mathbb{R}^d}\biggl(\int_{|y|\le 1} f\bigl(2^j(x+(\theta-1)y)\bigr)g(2^j(x+\theta y))2^{dj}\,\mathrm{d}y\biggr)^{\frac12}\mathrm{d}x\biggr)^2\\ &= 2^{3dj}\|I^\theta(f_j,g_j)\|_{L^{1/2}},\end{aligned}$$

where $f_j(x)=f(2^jx)$ and $g_j(x)=g(2^jx)$.

Using (4-1), it follows that

$$\begin{aligned}\|I_j^\theta(f,g)\|_{L^{1/2}} &\le 2^{3dj}3^d5^{2d}\int_{\mathbb{R}^d} f_j(x)\,\mathrm{d}x\int_{\mathbb{R}^d} g_j(x)\,\mathrm{d}x\\ &= 2^{3dj}3^d5^{2d}\int_{\mathbb{R}^d} f(x)2^{-dj}\,\mathrm{d}x\int_{\mathbb{R}^d} g(x)2^{-dj}\,\mathrm{d}x = 2^{dj}3^d5^{2d}\|f\|_{L^1}\|g\|_{L^1}. \qquad \square\end{aligned}$$

Lemma 4.4. *There exists $c>0$, depending only on d, such that for all measurable sets $E, A, B \subseteq \mathbb{R}^d$ we have*

$$\left(\int_E |I_j^\theta(\chi_A,\chi_B)|^{\frac{1}{2}}\,\mathrm{d}x\right)^2 \le \begin{cases} c|A||B|\min\{2^{dj},|E|\}, & (4\text{-}4)\\ c|A||E|\min\{2^{dj},|B|\}, & (4\text{-}5)\\ c|B||E|\min\{2^{dj},|A|\}, & (4\text{-}6)\end{cases}$$

and

$$\int_E |I_j^\theta(\chi_A,\chi_B)|\,\mathrm{d}x \le c\min\{2^{dj}|E|,|A||B|\}. \tag{4-7}$$

Proof. First we prove estimate (4-4). Using (4-3) with $f=\chi_A$ and $g=\chi_B$ we have that

$$\left(\int_E |I_j^\theta(\chi_A,\chi_B)|^{\frac{1}{2}}\,\mathrm{d}x\right)^2 \le \left(\int_{\mathbb{R}^d} |I_j^\theta(\chi_A,\chi_B)|^{\frac{1}{2}}\,\mathrm{d}x\right)^2 \le 2^{dj}3^d5^{2d}|A||B|.$$

On the other hand, by (4-2) and the Cauchy–Schwarz inequality we have

$$\left(\int_E |I_j^\theta(\chi_A,\chi_B)|^{\frac{1}{2}}\,\mathrm{d}x\right)^2 = \left(\int_{\mathbb{R}^2} \chi_E|I_j^\theta(\chi_A,\chi_B)|^{\frac{1}{2}}\,\mathrm{d}x\right)^2 \le |E||A||B| \le 3^d5^{2d}|E||A||B|.$$

Estimate (4-4) follows from combining the two estimates above.

Next, we turn our attention to estimates (4-5) and (4-6). We only give a proof of the former due to their symmetrical nature. First, we use the Cauchy–Schwarz inequality as above to obtain

$$\left(\int_E |I_j^\theta(\chi_A,\chi_B)|^{\frac{1}{2}}\,\mathrm{d}x\right)^2 \le |E|\int_{\mathbb{R}^d} I_j^\theta(\chi_A,\chi_B)\,\mathrm{d}x.$$

There are two ways to estimate the integral on the right-hand side of the previous inequality. One way is by $|A||B|$ (using (4-2)), and the other is as follows (using that $\chi_B \le 1$):

$$\int_{\mathbb{R}^d} I_j^\theta(\chi_A,\chi_B)\,\mathrm{d}x \le \int_{\mathbb{R}^d}\int_{|y|\le 2^j} \chi_A(x+(\theta-1)y)\,\mathrm{d}y\,\mathrm{d}x = \int_{|y|\le 2^j}\int_{\mathbb{R}^d} \chi_A(x+(\theta-1)y)\,\mathrm{d}x\,\mathrm{d}y \le \nu_d 2^{dj}|A|,$$

where ν_d denotes the measure of the unit ball in $\mathbb{R}^d$. This proves (4-5).

In order to prove (4-7), we observe that

$$I_j^\theta(\chi_A,\chi_B) \le \nu_d 2^{dj}$$

from which it follows that

$$\int_E |I_j^\theta(\chi_A,\chi_B)|\,\mathrm{d}x \le \nu_d 2^{dj}|E|.$$

The previous inequality together with

$$\int_E |I_j^\theta(\chi_A, \chi_B)|\,\mathrm{d}x \le \int_{\mathbb{R}^d} |I_j^\theta(\chi_A, \chi_B)|\,\mathrm{d}x \le |A|\,|B|$$

yields the desired estimate. □

4.3. ***Bilinear Marcinkiewicz interpolation.*** Recall the definition of weak Lebesgue spaces. For $0 < r < \infty$, the weak L^r space, denoted by $L^{r,\infty}$, is the space of all measurable functions f on $\mathbb{R}^d$ such that

$$\|f\|_{L^{r,\infty}} := \sup_{\lambda>0} \lambda \Big|\{x \in \mathbb{R}^d : |f(x)| > \lambda\}\Big|^{\frac{1}{r}} < \infty. \tag{4-8}$$

The map $\|\cdot\|_{L^{r,\infty}}$ is a quasinorm, and we have (see [Grafakos 2014a])

$$\|f\|_{L^{r,\infty}} \le \sup_{0<|E|<\infty} |E|^{-\frac{1}{s}+\frac{1}{r}} \left(\int_E |f|^s\,\mathrm{d}x\right)^{\frac{1}{s}}, \tag{4-9}$$

where $0 < s < r$ and the supremum is taken over measurable sets $E \subseteq \mathbb{R}^d$ of finite measure.

Definition 4.5. Let $0 < p, q, r \le \infty$. A bilinear operator U acting on measurable functions is said to be of restricted weak type (p, q, r) (with constant $c > 0$) if

$$\|U(\chi_A, \chi_B)\|_{L^{r,\infty}} \le c|A|^{\frac{1}{p}}|B|^{\frac{1}{q}} \tag{4-10}$$

for all measurable sets A and B of finite measure.

The next proposition, a version of the multilinear Marcinkiewicz interpolation, is the main step towards establishing Theorem 2.1. It yields strong-type bounds for bilinear operators, assuming only a finite set of restricted weak-type estimates. For a proof of the general case, see [Grafakos et al. 2012] or [Grafakos 2014b, Theorem 7.2.2 and Corollary 7.2.4].

Proposition 4.6. *Let $0 < p_i, q_i, r_i \le \infty$ for $i = 1, 2, 3$. Suppose that the points*

$$\Big(\frac{1}{p_1}, \frac{1}{q_1}\Big), \quad \Big(\frac{1}{p_2}, \frac{1}{q_2}\Big), \quad \Big(\frac{1}{p_3}, \frac{1}{q_3}\Big)$$

do not lie on the same line in $\mathbb{R}^2$. Let $0 < \theta_1, \theta_2, \theta_3 < 1$ satisfy $\theta_1 + \theta_2 + \theta_3 = 1$, and define $0 < p, q, r \le \infty$ by

$$\Big(\frac{1}{p}, \frac{1}{q}, \frac{1}{r}\Big) = \sum_{i=1}^{3} \theta_i \Big(\frac{1}{p_i}, \frac{1}{q_i}, \frac{1}{r_i}\Big).$$

Assume that

$$\frac{1}{r} \le \frac{1}{p} + \frac{1}{q}.$$

Let U be a bilinear operator that is of restricted weak-type (p_i, q_i, r_i) (with constant $c_i > 0$) for all $i = 1, 2, 3$. Then there is a constant $C > 0$ depending only on p_i, q_i, r_i, and θ_i $(i = 1, 2, 3)$ such that

$$\|U(f, g)\|_{L^r} \le C c_1^{\theta_1} c_2^{\theta_2} c_3^{\theta_3} \|f\|_{L^p} \|g\|_{L^q}$$

for all functions $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$.

We note that the conclusion of Proposition 4.6 is also valid in the interior of the convex hull of four (or more) points at which initial restricted weak-type estimates are known. The reason is that any polygon can be written as a union of triangles.

4.4. ***Proof of Theorem 2.1.*** Turning our attention to the bilinear fractional integral operator I_α^θ defined by (1-6), we note that by a dilation argument, if it maps $L^p \times L^q$ to L^r, then necessarily

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{r}+\frac{\alpha}{d}. \tag{4-11}$$

Moreover, Theorem 2.1 affirms that I_α^θ is bounded uniformly in θ from $L^p \times L^q$ to L^r when (p, q) lies in the open square with vertices $(1, 1)$, $\bigl(1, \frac{d}{\alpha}\bigr)$, $\bigl(\frac{d}{\alpha}, 1\bigr)$, $\bigl(\frac{d}{\alpha}, \frac{d}{\alpha}\bigr)$ and (4-11) holds. In this case, we have

(i) if $(p, q) = (1, 1)$, then $r = \frac{d}{2d-\alpha}$,

(ii) if $(p, q) = \bigl(1, \frac{d}{\alpha}\bigr)$, then $r = 1$,

(iii) if $(p, q) = \bigl(\frac{d}{\alpha}, 1\bigr)$, then $r = 1$,

(iv) if $(p, q) = \bigl(\frac{d}{\alpha}, \frac{d}{\alpha}\bigr)$, then $r = \frac{d}{\alpha}$.

Set

$$(p_1, q_1, r_1) = \Bigl(1, 1, \frac{d}{2d-\alpha}\Bigr), \quad (p_2, q_2, r_2) = \Bigl(1, \frac{d}{\alpha}, 1\Bigr),$$
$$(p_3, q_3, r_3) = \Bigl(\frac{d}{\alpha}, 1, 1\Bigr), \qquad (p_4, q_4, r_4) = \Bigl(\frac{d}{\alpha}, \frac{d}{\alpha}, \frac{d}{\alpha}\Bigr).$$

To establish Theorem 2.1 it suffices to prove that I_α^θ is of restricted weak type (p_i, q_i, r_i) (with constant c_i that is independent of θ) for $i = 1, 2, 3, 4$. Then, the result follows by bilinear Marcinkiewicz interpolation, Proposition 4.6.

That is, we need to prove the estimates

$$\|I_\alpha^\theta(\chi_A, \chi_B)\|_{L^{d/(2d-\alpha),\infty}} \le c_1|A||B|, \tag{4-12}$$

$$\|I_\alpha^\theta(\chi_A, \chi_B)\|_{L^{1,\infty}} \le c_2|A||B|^{\frac{\alpha}{d}}, \tag{4-13}$$

$$\|I_\alpha^\theta(\chi_A, \chi_B)\|_{L^{1,\infty}} \le c_3|A|^{\frac{\alpha}{d}}|B|, \tag{4-14}$$

$$\|I_\alpha^\theta(\chi_A, \chi_B)\|_{L^{d/\alpha,\infty}} \le c_4|A|^{\frac{\alpha}{d}}|B|^{\frac{\alpha}{d}}, \tag{4-15}$$

uniformly in θ, for all measurable sets A and B of finite measure.

In the proofs of (4-12)–(4-15) we utilize the following lemma.

Lemma 4.7. *Let $d \in \mathbb{N}$, $0 < \alpha < d$ and $a, b > 0$. There exists $c = c(d, \alpha) > 0$ such that*

$$\Bigl(\sum_{j\in\mathbb{Z}} 2^{\frac{(\alpha-d)j}{2}}(\min\{2^{dj}, a\})^{\frac{1}{2}}\Bigr)^2 \le c a^{\frac{\alpha}{d}} \tag{4-16}$$

and

$$\sum_{j\in\mathbb{Z}} 2^{(\alpha-d)j}\min\{2^{dj}a, b\} \le c a \Bigl(\frac{b}{a}\Bigr)^{\frac{\alpha}{d}}. \tag{4-17}$$

Proof. We only prove (4-16) as the other one is similar. Let $m = \max\{j \in \mathbb{Z} : 2^{dj} < a\}$. Then

$$\begin{aligned}\sum_{j\in\mathbb{Z}} 2^{\frac{(\alpha-d)j}{2}} (\min\{2^{dj}, a\})^{\frac{1}{2}} &= \sum_{j=-\infty}^{m} 2^{\frac{\alpha j}{2}} + \sum_{j=m+1}^{\infty} 2^{\frac{(\alpha-d)j}{2}} a^{\frac{1}{2}} \\ &= \sum_{k=0}^{\infty} 2^{-\frac{\alpha(k-m)}{2}} + \sum_{i=0}^{\infty} 2^{\frac{(\alpha-d)}{2}(i+m+1)} a^{\frac{1}{2}} \\ &= \left(\sum_{k=0}^{\infty} 2^{-\frac{\alpha k}{2}}\right) 2^{\frac{\alpha m}{2}} + \left(\sum_{i=0}^{\infty} 2^{\frac{(\alpha-d)i}{2}}\right) 2^{\frac{(\alpha-d)}{2}(m+1)} a^{\frac{1}{2}}.\end{aligned}$$

The desired inequality is achieved by noting that

$$2^{\frac{\alpha m}{2}} < a^{\frac{\alpha}{2d}} \quad \text{and} \quad 2^{\frac{(\alpha-d)}{2}(m+1)} \le a^{\frac{\alpha-d}{2d}}. \qquad \square$$

Proof of Theorem 2.1. First, note that $\mathbb{R}^d$ can be expressed as the union of annuli,

$$\mathbb{R}^d = \bigcup_{j\in\mathbb{Z}} (B(2^j) \setminus B(2^{j-1})),$$

where $B(R)$ denotes the open ball in $\mathbb{R}^d$ centered at the origin with radius R.

Therefore,

$$\begin{aligned} I_\alpha^\theta(f, g)(x) &\le \sum_{j\in\mathbb{Z}} \int_{2^{j-1}\le|y|\le 2^j} f(x+(\theta-1)y)g(x+\theta y)|y|^{\alpha-d}\,\mathrm{d}y \\ &\le \sum_{j\in\mathbb{Z}} 2^{d-\alpha} \int_{2^{j-1}\le|y|\le 2^j} f(x+(\theta-1)y)g(x+\theta y)2^{(\alpha-d)j}\,\mathrm{d}y \\ &\le 2^{d-\alpha} \sum_{j\in\mathbb{Z}} 2^{(\alpha-d)j} I_j^\theta(f, g)(x).\end{aligned}$$

Let A, B be measurable sets of $\mathbb{R}^d$ of finite measure. In what follows, the positive constant C might change from line to line, but it will always be independent of θ.

To prove the restricted estimate (4-12) we use (4-4), (4-9) with $s = \frac{1}{2}$, and (4-16) as follows:

$$\begin{aligned} \|I_\alpha^\theta(\chi_A, \chi_B)\|_{L^{d/(2d-\alpha),\infty}} &\le C \sup_{0<|E|<\infty} |E|^{-2+\frac{2d-\alpha}{d}} \left(\int_E \left|\sum_{j\in\mathbb{Z}} 2^{(\alpha-d)j} I_j^\theta(\chi_A, \chi_B)\right|^{\frac{1}{2}} \mathrm{d}x\right)^2 \\ &\le C \sup_{0<|E|<\infty} |E|^{-\frac{\alpha}{d}} \left(\sum_{j\in\mathbb{Z}} 2^{\frac{(\alpha-d)j}{2}} \int_E I_j^\theta(\chi_A, \chi_B)^{\frac{1}{2}}\,\mathrm{d}x\right)^2 \\ &\le C|A||B| \sup_{0<|E|<\infty} |E|^{-\frac{\alpha}{d}} \left(\sum_{j\in\mathbb{Z}} 2^{\frac{(\alpha-d)j}{2}} (\min\{2^{dj}, |E|\})^{\frac{1}{2}}\right)^2 \\ &\le C|A||B|.\end{aligned}$$

Next we prove (4-13). Here we use (4-5), (4-9) with $s = \frac{1}{2}$, and (4-16) as follows:

$$\begin{aligned}\|I^\theta_\alpha(\chi_A, \chi_B)\|_{L^{1,\infty}} &\le C \sup_{0<|E|<\infty} |E|^{-2+1}\biggl(\int_E \biggl|\sum_{j\in\mathbb{Z}} 2^{(\alpha-d)j} I^\theta_j(\chi_A, \chi_B)\biggr|^{\frac{1}{2}} \mathrm{d}x\biggr)^2 \\ &\le C \sup_{0<|E|<\infty} |E|^{-1}\biggl(\sum_{j\in\mathbb{Z}} 2^{\frac{(\alpha-d)j}{2}} \int_E I^\theta_j(\chi_A, \chi_B)^{\frac{1}{2}} \mathrm{d}x\biggr)^2 \\ &\le C \sup_{0<|E|<\infty} |E|^{-1}|A||E|\biggl(\sum_{j\in\mathbb{Z}} 2^{\frac{(\alpha-d)j}{2}} (\min\{2^{dj}, |B|\})^{\frac{1}{2}}\biggr)^2 \\ &\le C|A||B|^{\frac{\alpha}{d}}.\end{aligned}$$

The estimate (4-14) is based on (4-6) and is deduced similarly as the one above.

Finally, we turn to (4-15). Here we use (4-7), (4-9) with $s = 1$, and (4-17) as follows:

$$\begin{aligned}\|I^\theta_\alpha(\chi_A, \chi_B)\|_{L^{d/\alpha,\infty}} &\le C \sup_{0<|E|<\infty} |E|^{-1+\frac{\alpha}{d}} \int_E \biggl|\sum_{j\in\mathbb{Z}} 2^{(\alpha-d)j} I^\theta_j(\chi_A, \chi_B)\biggr| \mathrm{d}x \\ &\le C \sup_{0<|E|<\infty} |E|^{-1+\frac{\alpha}{d}} \sum_{j\in\mathbb{Z}} 2^{(\alpha-d)j} \int_E I^\theta_j(\chi_A, \chi_B)\, \mathrm{d}x \\ &\le C \sup_{0<|E|<\infty} |E|^{-1+\frac{\alpha}{d}} \sum_{j\in\mathbb{Z}} 2^{(\alpha-d)j} \min\{2^{dj}|E|, |A||B|\} \\ &\le C \sup_{0<|E|<\infty} |E|^{-1+\frac{\alpha}{d}}|E|\biggl(\frac{|A||B|}{|E|}\biggr)^{\frac{\alpha}{d}} \\ &= C|A|^{\frac{\alpha}{d}}|B|^{\frac{\alpha}{d}}.\end{aligned}$$

This completes the proof of the theorem. □

4.5. ***Proof of Proposition 2.3.*** First, we observe that the tensor $J_\alpha(f, g)$ is pointwise bounded by $\int_0^1 I^\theta_\alpha(f, g)\,\mathrm{d}\theta$. Indeed, for any $x \in \mathbb{R}^d$ and $v(x) \in \mathbb{R}^d \setminus \{0\}$ we have

$$\begin{aligned}|J_\alpha(f, g)(x)v(x)| &= \biggl|\int_0^1\int_{\mathbb{R}^d} f(x+(\theta-1)y)g(x+\theta y)|y|^{\alpha-d-2}(y\cdot v(x))y\,\mathrm{d}y\,\mathrm{d}\theta\biggr| \\ &\le \int_0^1\int_{\mathbb{R}^d} f(x+(\theta-1)y)g(x+\theta y)|y|^{\alpha-d-2}|y\cdot v(x)||y|\,\mathrm{d}y\,\mathrm{d}\theta \\ &\le \int_0^1 I^\theta_\alpha(f, g)(x)\,\mathrm{d}\theta|v(x)|,\end{aligned}$$

and hence

$$\begin{aligned}|J_\alpha(f, g)(x)| &= \sup_{v(x)\neq 0} \frac{|J_\alpha(f, g)(x)v(x)|}{|v(x)|} \\ &\le \int_0^1 I^\theta_\alpha(f, g)(x)\,\mathrm{d}\theta.\end{aligned}$$

Therefore, using Jensen's inequality we deduce that

$$\begin{aligned}\|J_\alpha(f,g)\|_{L^r}^r &\le \int_{\mathbb{R}^d} \left| \int_0^1 I_\alpha^\theta(f,g)\,\mathrm{d}\theta \right|^r \mathrm{d}x \\ &\le \int_{\mathbb{R}^d}\int_0^1 |I_\alpha^\theta(f,g)|^r \,\mathrm{d}\theta\,\mathrm{d}x.\end{aligned}$$

Now, we use Fubini's theorem to obtain

$$\begin{aligned}\|J_\alpha(f,g)\|_{L^r}^r &\le \int_0^1\int_{\mathbb{R}^d} |I_\alpha^\theta(f,g)|^r \,\mathrm{d}x\,\mathrm{d}\theta \\ &= \int_0^1 \|I_\alpha^\theta(f,g)\|_{L^r}^r \,\mathrm{d}\theta\end{aligned}$$

from which the desired result follows upon applying Theorem 2.1.

5. Stability for Euler–Riesz systems

In this section, we establish a stability result for smooth solutions of an Euler–Riesz system with periodic boundary conditions, written according to identity (1-4). Two smooth solutions are compared using the relative energy functional. Using the abstract formalism developed in [Giesselmann et al. 2017], we derive an identity that describes the time evolution of the relative energy. The right-hand side of the relative energy identity is controlled with the help of the HLS inequality and then Gronwall's lemma provides a stability result.

Stability results of this type have been obtained for similar systems of equations, where one of the considered solutions is assumed to be merely a weak or even measure-valued solution, yielding a weak-strong uniqueness or measure-valued versus strong uniqueness principle; see [Alves 2024; Alves et al. 2024; Carrillo et al. 2024; Lattanzio and Tzavaras 2017]. The result obtained here can be phrased in the language of weak-strong stability, but we avoid doing that and we refer to [Alves et al. 2024] for details of such a formulation.

Let $T > 0$ and denote by $\mathbb{T}^d$ the d-dimensional open cube $\left(-\frac{1}{2}, \frac{1}{2}\right)^d$. Consider the Euler–Riesz system in $(0, T) \times \mathbb{T}^d$, expressed using the abstract functional framework developed in [Giesselmann et al. 2017],

$$\begin{cases} \partial_t \rho + \nabla\cdot(\rho u) = 0, \\ \partial_t(\rho u) + \nabla\cdot(\rho u\otimes u) + \rho\nabla \frac{\delta\mathcal{E}}{\delta\rho}(\rho) = 0, \\ \rho|_{t=0} = \rho_0, \quad u|_{t=0} = u_0, \end{cases} \tag{5-1}$$

with the potential energy functional $\mathcal{E}$ defined as

$$\mathcal{E}(\rho) = \int_{\mathbb{T}^d} h(\rho) + \kappa\frac{1}{2}\rho(K_\alpha * \rho)\,\mathrm{d}x,$$

where $h(\rho) = \frac{1}{\gamma-1}\rho^\gamma$ is the internal energy function, K_α is the kernel given by (1-2), and the constant κ represents the interaction strength and for this section it is allowed to take positive and negative values. The size of $|\kappa|$ will be restricted to ensure that the relative energy is nonnegative.

The density ρ and velocity u are assumed to be periodic in space with unit period.

5.1. ***Relative energy identity.*** The functional derivative $\frac{\delta \mathcal{E}}{\delta \rho}$ is given by

$$\frac{\delta \mathcal{E}}{\delta \rho}(\rho) = h'(\rho) + \kappa K_\alpha * \rho, \tag{5-2}$$

which can be computed through the formula

$$\left\langle \frac{\delta \mathcal{E}}{\delta \rho}(\rho), \varphi \right\rangle = \int_{\mathbb{T}^d} \frac{\delta \mathcal{E}}{\delta \rho}(\rho)\varphi \, \mathrm{d}x := \lim_{\delta \to 0} \frac{\mathcal{E}(\rho + \delta\varphi) - \mathcal{E}(\rho)}{\delta},$$

where φ is an arbitrary test function.

Furthermore, using (1-4), we have

$$\rho \nabla \frac{\delta \mathcal{E}}{\delta \rho}(\rho) = \nabla \cdot R_\alpha(\rho), \tag{5-3}$$

where $R_\alpha(\rho) = p(\rho) I_d + \kappa S_\alpha(\rho)$, with $p(\rho) = \rho^\gamma$ being the pressure function. Note that the calculations in the Appendix that lead to identity (1-4) are valid if one replaces $\mathbb{R}^d$ by $\mathbb{T}^d$ due to the symmetrical assumption on the torus.

The relative potential energy functional $\mathcal{E}(\cdot \,|\, \cdot)$ is defined as

$$\begin{aligned} \mathcal{E}(\rho \,|\, \bar\rho) &= \mathcal{E}(\rho) - \mathcal{E}(\bar\rho) - \left\langle \frac{\delta \mathcal{E}}{\delta \rho}(\bar\rho), \rho - \bar\rho \right\rangle \\ &= \int_{\mathbb{T}^d} h(\rho \,|\, \bar\rho) + \kappa \frac{1}{2}(\rho - \bar\rho)(K_\alpha * (\rho - \bar\rho)) \, \mathrm{d}x, \end{aligned}$$

where $h(\rho \,|\, \bar\rho) = h(\rho) - h(\bar\rho) - h'(\bar\rho)(\rho - \bar\rho)$.

Next, we present the evolution of $\mathcal{E}(\rho \,|\, \bar\rho)$ over time, assuming that ρ and $\bar\rho$ evolve according to system (5-1). For the full details of the calculations involved, refer to [Giesselmann et al. 2017]. We have

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}(\rho \,|\, \bar\rho) = -\int_{\mathbb{T}^d} \nabla \bar u : R_\alpha(\rho \,|\, \bar\rho) \, \mathrm{d}x - \left\langle \frac{\delta \mathcal{E}}{\delta \rho}(\rho) - \frac{\delta \mathcal{E}}{\delta \rho}(\bar\rho), \nabla \cdot (\rho(u - \bar u)) \right\rangle, \tag{5-4}$$

where

$$R_\alpha(\rho \,|\, \bar\rho) = p(\rho \,|\, \bar\rho) I_d + \kappa S_\alpha(\rho \,|\, \bar\rho).$$

Now, the linear velocities satisfy

$$\begin{cases} \partial_t u + (u \cdot \nabla) u + \nabla \frac{\delta \mathcal{E}}{\delta \rho}(\rho) = 0, \\ \partial_t \bar u + (\bar u \cdot \nabla) \bar u + \nabla \frac{\delta \mathcal{E}}{\delta \rho}(\bar\rho) = 0, \end{cases}$$

from which it can be deduced that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{T}^d} \frac{1}{2} \rho |u - \bar u|^2 \, \mathrm{d}x = -\int_{\mathbb{T}^d} \nabla \bar u : \rho(u - \bar u) \otimes (u - \bar u) \, \mathrm{d}x + \left\langle \frac{\delta \mathcal{E}}{\delta \rho}(\rho) - \frac{\delta \mathcal{E}}{\delta \rho}(\bar\rho), \nabla \cdot (\rho(u - \bar u)) \right\rangle. \tag{5-5}$$

Combining (5-4) with (5-5) yields the relative total energy identity

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\mathcal{E}(\rho \,|\, \bar\rho) + \int_{\mathbb{T}^d} \frac{1}{2} \rho |u - \bar u|^2 \, \mathrm{d}x \right) = -\int_{\mathbb{T}^d} \nabla \bar u : (\rho(u - \bar u) \otimes (u - \bar u) + R_\alpha(\rho \,|\, \bar\rho)) \, \mathrm{d}x. \tag{5-6}$$

5.2. ***Stability of smooth solutions.*** Let (ρ, u) and $(\bar{\rho}, \bar{u})$ be two smooth solutions of (5-1), and suppose additionally that for $(\bar{\rho}, \bar{u})$ the density $\bar{\rho}$ is bounded away from vacuum; that is, there exist $\bar{\delta} > 0$ and $\bar{M} < \infty$ such that

$$\bar{\delta} \le \bar{\rho}(t, x) \le \bar{M} \quad \text{for } (t, x) \in [0, T) \times \mathbb{T}^d$$

and also

$$\nabla \bar{u} \in L^\infty(0, T; L^\infty(\mathbb{T}^d)).$$

The solutions (ρ, u) and $(\bar{\rho}, \bar{u})$ satisfy the relative energy identity (5-6). Let $\Psi : [0, T) \to \mathbb{R}$ denote the relative energy between this pair of solutions,

$$\begin{aligned}\Psi(t) &= \int_{\mathbb{T}^d} \frac{1}{2}\rho|u - \bar{u}|^2 \,\mathrm{d}x + \mathcal{E}(\rho \,|\, \bar{\rho}) \\ &= \int_{\mathbb{T}^d} \frac{1}{2}\rho|u - \bar{u}|^2 + h(\rho \,|\, \bar{\rho}) + \kappa \frac{1}{2}(\rho - \bar{\rho})(K_\alpha * (\rho - \bar{\rho})) \,\mathrm{d}x.\end{aligned}$$

The objective is to prove a stability estimate connecting the behavior at time $t \in (0, T)$ to the initial behavior at time zero.

Identity (5-6) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi(t) = -\int_{\mathbb{T}^d} \nabla \bar{u} : \rho(u - \bar{u}) \otimes (u - \bar{u}) \,\mathrm{d}x - \int_{\mathbb{T}^d} (\nabla \cdot \bar{u}) p(\rho \,|\, \bar{\rho}) \,\mathrm{d}x - \kappa \int_{\mathbb{T}^d} \nabla \bar{u} : S_\alpha(\rho \,|\, \bar{\rho}) \,\mathrm{d}x. \tag{5-7}$$

To use Ψ as a yardstick for comparing the two solutions, we need to show that Ψ is nonnegative. This is based on two key ingredients. First, the HLS inequality (2-3) gives

$$\|(\rho - \bar{\rho})(K_\alpha * (\rho - \bar{\rho}))\|_{L^1(\mathbb{T}^d)} \le C_0 \|\rho - \bar{\rho}\|^2_{L^p(\mathbb{T}^d)}$$

for some positive constant $C_0 = C_0(\alpha, d)$, where $p = 2d/(d + \alpha)$. This is improved by using interpolation and properties of the function $h(\rho \,|\, \bar{\rho})$ to show (see [Lattanzio and Tzavaras 2017, Lemma 3.6] for the Newtonian potential and [Alves et al. 2024, Proposition 4.2] for the general case):

Lemma 5.1. *Consider the function $h(\rho) = \frac{1}{\gamma - 1}\rho^\gamma$ with $\gamma \ge 2 - \frac{\alpha}{d}$ and $0 < \alpha < d$. Let $\rho \in L^\gamma(\mathbb{T}^d)$ be nonnegative, and let $\bar{\rho} \in L^\infty(\mathbb{T}^d)$ be bounded away from vacuum. Then, there exists a positive constant C_* such that*

$$\|(\rho - \bar{\rho})K_\alpha * (\rho - \bar{\rho})\|_{L^1(\mathbb{T}^d)} \le C_* \int_{\mathbb{T}^d} h(\rho \,|\, \bar{\rho}) \,\mathrm{d}x. \tag{5-8}$$

Choosing κ so that $0 < |\kappa| < \dfrac{2}{C_*}$ and setting $\lambda := 1 - \dfrac{|\kappa| C_*}{2} > 0$, we obtain

$$\lambda \int_{\mathbb{T}^d} h(\rho \,|\, \bar{\rho}) \,\mathrm{d}x \le \int_{\mathbb{T}^d} h(\rho \,|\, \bar{\rho}) + \kappa \frac{1}{2}(\rho - \bar{\rho})K_\alpha * (\rho - \bar{\rho}) \,\mathrm{d}x$$

from which the nonnegativity of Ψ follows.

Next, we bound the terms on the right-hand side of identity (5-7) in terms of Ψ. The first term is bounded by the relative kinetic energy and hence by Ψ. The bound for the second term is also clear as

$p(\rho\,|\,\bar\rho) = (\gamma-1)h(\rho\,|\,\bar\rho)$. Regarding the last term, we first observe that due to the quadratic nature of $S_\alpha(\rho)$ one has

$$S_\alpha(\rho\,|\,\bar\rho) = S_\alpha(\rho-\bar\rho).$$

Moreover, for any fixed time $t \in [0, T)$, the L^1-norm of $S_\alpha(\rho-\bar\rho)$ is bounded by

$$\mathcal{I} := \frac{1}{2}\int_0^1\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} |(\rho-\bar\rho)(x+(\theta-1)y)|\,|(\rho-\bar\rho)(x+\theta y)|\,|y|^{\alpha-d}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\theta,$$

where the dependency on time is omitted for simplicity. We then estimate

$$\begin{aligned}\mathcal{I} &\le \frac{1}{2}\int_0^1\int_{\mathbb{T}^d}\int_{|z|<1} |(\rho-\bar\rho)(z)|\,|(\rho-\bar\rho)(z+y)|\,|y|^{\alpha-d}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}\theta\\ &\le \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |(\rho-\bar\rho)(z)|\chi_{(-2,2)^d}(z)|(\rho-\bar\rho)(w)|\chi_{(-2,2)^d}(w)|z-w|^{\alpha-d}\,\mathrm{d}z\,\mathrm{d}w\\ &\le C(\alpha,d)\|(\rho-\bar\rho)\chi_{(-2,2)^d}\|^2_{L^p(\mathbb{R}^d)},\end{aligned}$$

where $p = 2d/(d+\alpha)$, by the HLS inequality. Finally, the periodicity in space of $\rho-\bar\rho$ implies that

$$\|(\rho-\bar\rho)\chi_{(-2,2)^d}\|_{L^p(\mathbb{R}^d)} = 4^d\|\rho-\bar\rho\|_{L^p(\mathbb{T}^d)}.$$

Hence, similarly to Lemma 5.1,

$$\int_{\mathbb{T}^d}\nabla\bar u : S_\alpha(\rho\,|\,\bar\rho)\,\mathrm{d}x \le \kappa\|\nabla\bar u\|_\infty\|S_\alpha(\rho-\bar\rho)\|_{L^1(\mathbb{T}^d)} \le C\int_{\mathbb{T}^d} h(\rho\,|\,\bar\rho)\,\mathrm{d}x \le C\Psi.$$

In summary, we have obtained the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi \le C\Psi.$$

By Gronwall's lemma, for each $t \in [0, T)$, it follows that $\Psi(t) \le e^{CT}\Psi(0)$ which, together with the strict convexity of the internal energy function h, yields the desired stability result.

A weak-strong uniqueness theorem is proved in [Alves et al. 2024, Theorem 3.1] following the general approach outlined above. The method of proof differs in the treatment of the nonlocal term, achieved here via the use of the representation formula (1-4). This provides an improvement in the range of parameters α achieving the full range $0 < \alpha < d$. By contrast, the range of γ is still restricted by $\gamma \ge 2 - \frac{\alpha}{d}$.

Appendix

Here we give a formal proof that for our symmetric kernel $K_\alpha : \mathbb{R}^d \to \mathbb{R}$, with $K_\alpha(x) = \mathcal{K}_\alpha(|x|)$, we have

$$f\nabla K_\alpha * f = \nabla\cdot S_\alpha(f) \tag{A-1}$$

for any sufficiently smooth $f : \mathbb{R}^d \to \mathbb{R}$, where the tensor $S_\alpha(f)$ is given by

$$S_\alpha(f)(x) = -\frac{1}{2}\int_{\mathbb{R}^d}\int_0^1 \mathcal{K}'_\alpha(|y|)\frac{1}{|y|}f(x+(\theta-1)y)f(x+\theta y)y\otimes y\,\mathrm{d}\theta\,\mathrm{d}y. \tag{A-2}$$

Note that for $\mathcal{K}_\alpha(|y|) = \frac{1}{d-\alpha}|y|^{\alpha-d}$ we have $\mathcal{K}'_\alpha(|y|) = -|y|^{\alpha-d-1}$, and thus the corresponding tensor is positive semidefinite for nonnegative f.

To prove (A-1), we first deduce that

$$(f\nabla K_\alpha * f)(x) = -\frac{1}{2}\int_{\mathbb{R}^d} \mathcal{K}'_\alpha(|y|)\frac{y}{|y|}\nabla_x \cdot \int_0^1 yf(x+(\theta-1)y)f(x+\theta y)\,\mathrm{d}\theta\,\mathrm{d}y. \tag{A-3}$$

Using the symmetry of the convolution one has

$$\begin{aligned}(f\nabla K_\alpha * f)(x) &= f(x)\int_{\mathbb{R}^d} \mathcal{K}'_\alpha(|y|)\frac{y}{|y|}f(x-y)\,\mathrm{d}y\\ &= -f(x)\int_{\mathbb{R}^d} \mathcal{K}'_\alpha(|y|)\frac{y}{|y|}f(x+y)\,\mathrm{d}y,\end{aligned}$$

where in the second equality we used the change of variables $y \to -y$.

Hence

$$(f\nabla K_\alpha * f)(x) = \frac{1}{2}\int_{\mathbb{R}^d} \mathcal{K}'_\alpha(|y|)\frac{y}{|y|}f(x)(f(x-y)-f(x+y))\,\mathrm{d}y.$$

Now it is claimed that

$$f(x)(f(x-y)-f(x+y)) = -\nabla_x \cdot \int_0^1 yf(x+(\theta-1)y)f(x+\theta y)\,\mathrm{d}\theta$$

from which identity (A-3) follows. Indeed,

$$\begin{aligned}&-f(x)(f(x-y)-f(x+y))\\ &\qquad= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\theta}\big(f(x+(\theta-1)y)f(x+\theta y)\big)\,\mathrm{d}\theta\\ &\qquad= \int_0^1 \big(\nabla f(x+(\theta-1)y)\cdot y\big)f(x+\theta y) + (\nabla f(x+\theta y)\cdot y)f(x+(\theta-1)y)\,\mathrm{d}\theta\\ &\qquad= \int_0^1 \nabla\big(f(x+(\theta-1)y)f(x+\theta y)\big)\,\mathrm{d}\theta\cdot y = \nabla_x\cdot\int_0^1 yf(x+(\theta-1)y)f(x+\theta y)\,\mathrm{d}\theta,\end{aligned}$$

as desired.

Consequently, componentwise one has

$$\begin{aligned}(\nabla\cdot S_\alpha(f)(x))_i &= -\frac{1}{2}\nabla_x\cdot\int_{\mathbb{R}^d}\int_0^1 \mathcal{K}'_\alpha(|y|)\frac{y_i}{|y|}yf(x+(\theta-1)y)f(x+\theta y)\,\mathrm{d}\theta\,\mathrm{d}y\\ &= -\frac{1}{2}\int_{\mathbb{R}^d}\mathcal{K}'_\alpha(|y|)\frac{y_i}{|y|}\nabla_x\cdot\int_0^1 yf(x+(\theta-1)y)f(x+\theta y)\,\mathrm{d}\theta\,\mathrm{d}y\\ &= (f\nabla K_\alpha * f(x))_i,\end{aligned}$$

which establishes identity (A-1).

We note that for a d-dimensional cube $[-a,a]^d$ centered at the origin, with $a > 0$, and periodic functions with period equal to $2a$, the formulas (A-1) and (A-2) are still valid with the integrations performed over $[-a,a]^d$.

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