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DIMENSION-FREE L^p ESTIMATES FOR HIGHER-ORDER MAXIMAL RIESZ TRANSFORMS IN TERMS OF THE RIESZ TRANSFORMS

MACIEJ KUCHARSKI, BŁAŻEJ WRÓBEL AND JACEK ZIENKIEWICZ

We prove a dimension-free $L^p(\mathbb{R}^d)$ estimate, $1<p<\infty$, for the vector of higher-order maximal Riesz transforms in terms of the corresponding Riesz transforms. This implies a dimension-free $L^p(\mathbb{R}^d)$ estimate for the vector of maximal Riesz transforms in terms of the input function. We also give explicit estimates for the dependencies of the constants on p when the order is fixed. Analogous dimension-free estimates are also obtained for single higher-order Riesz transforms with an improved estimate of the constants.

1. Introduction

Fix a positive integer k and denote by $\mathcal{H}_k=\mathcal{H}_k^d$ the space of spherical harmonics of degree k on the Euclidean sphere S^{d-1}. Throughout the paper we identify $P\in\mathcal{H}_k$ with the corresponding solid spherical harmonic. Via this identification $P\in\mathcal{H}_k$ is a harmonic polynomial on $\mathbb{R}^d$ which is homogeneous of degree k, i.e., satisfies $P(x)=|x|^kP(x/|x|),\ x\in\mathbb{R}^d$.

For $P\in\mathcal{H}_k$ the Riesz transform $R=R_P$ is defined by the kernel

$$K_P(x)=K(x)=\gamma_k\frac{P(x)}{|x|^{d+k}}\quad\text{with}\ \gamma_k=\frac{\Gamma\big(\frac{1}{2}(k+d)\big)}{\pi^{d/2}\Gamma\big(\frac{1}{2}k\big)};\tag{1-1}$$

more precisely,

$$R_Pf(x)=\lim_{t\to0^+}R_P^tf(x),\quad\text{where}\ R_P^tf(x)=\gamma_k\int_{|y|>t}\frac{P(y)}{|y|^{d+k}}f(x-y)\,dy.\tag{1-2}$$

The operator R_P^t is called the truncated Riesz transform. In the particular case of $k=1$ and $P_j(x)=x_j$ the operators $R_{P_j},\ j=1,\dots,d$, coincide with the classical first-order Riesz transforms. It is well known, see [Stein 1970, p. 73], that the Fourier multiplier associated with the Riesz transform R_P equals

$$m_P(\xi)=(-i)^kP(\xi/|\xi|),\quad \xi\in\mathbb{R}^d.\tag{1-3}$$

By the above formula m_P is bounded and Plancherel's theorem implies the $L^2(\mathbb{R}^d)$ boundedness of R_P. The $L^p(\mathbb{R}^d)$ boundedness of the single Riesz transforms R_P for $1<p<\infty$ follows from the Calderón–Zygmund method of rotations [1956].

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The systematic study of the dimension-free L^p bounds for the Riesz transforms began in the seminal paper of E. M. Stein [1983]. There he proved a dimension-free ℓ^2 vector-valued estimate for the vector of the first-order Riesz transforms

$$\left\| \left(\sum_{j=1}^{d} |R_j f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)} \leqslant C_p \|f\|_{L^p(\mathbb{R}^d)}, \quad 1 < p < \infty. \tag{1-4}$$

In the inequality above, the R_j, $j = 1, \dots, d$, denote the first-order Riesz transforms defined via (1-2) with $P_j(x) = x_j$ and the constant C_p is independent of the dimension d.

Stein's result has been extended to many other settings. The analogue of the dimension-free inequality (1-4) has been also proved for higher-order Riesz transforms; see [Duoandikoetxea and Rubio de Francia 1985, théorème 2]. The optimal constant C_p in (1-4) remains unknown when $d \geqslant 2$; however the best results to date given in [Bañuelos and Wang 1995] (see also [Dragičević and Volberg 2006]) established the correct order of the dependence on p. We note that the explicit values of $L^p(\mathbb{R}^d)$ norms of the single first-order Riesz transforms R_j, $j = 1, \dots, d$, were obtained by Iwaniec and Martin [1996] based on the method of rotations.

In this paper we study the relation between R_P and the maximal Riesz transform defined by

$$R_P^* f(x) = \sup_{t>0} |R_P^t f(x)|.$$

Clearly, we have the pointwise inequality $R_P f(x) \leqslant R_P^* f(x)$. In a series of papers [Mateu and Verdera 2006, Theorem 1] (first-order Riesz transforms), [Mateu et al. 2010, Section 4] (odd-order higher Riesz transforms), and [Mateu et al. 2011, Section 2] (even-order higher Riesz transforms), J. Mateu, J. Orobitg, C. Pérez, and J. Verdera proved that also a reverse inequality holds in the $L^p(\mathbb{R}^d)$ norm. Namely, together the results of [Mateu and Verdera 2006; Mateu et al. 2010; 2011] imply that for each $1 < p < \infty$ there exists a constant $C(p, k, d)$ such that

$$\|R_P^* f\|_{L^p(\mathbb{R}^d)} \leqslant C(p, k, d) \|R_P f\|_{L^p(\mathbb{R}^d)} \tag{1-5}$$

for all $f \in L^p(\mathbb{R}^d)$. As a matter of fact, the estimate (1-5) has been proved in [Mateu and Verdera 2006; Mateu et al. 2010; 2011] for more general singular integral operators with even kernels [Mateu et al. 2011] or with odd kernels [Mateu et al. 2010]. However, even for the higher-order Riesz transforms, the values of $C(p, k, d)$ that follow from these papers grow exponentially with the dimension. In view of [Janakiraman 2004], the question about an improved rate arises naturally.

The first step towards a dimension-free estimate of the constant $C(p, k, d)$ in (1-5) has been made by the first and the second authors, who proved that when $p = 2$ in (1-5) one may take an explicit dimension-free constant $C(2, 1, d) \leqslant 2 \cdot 10^8$; see [Kucharski and Wróbel 2023, Theorem 1.1]. The arguments applied in [Kucharski and Wróbel 2023] relied on Fourier transform estimates together with square function techniques developed by Bourgain [1986], and Bourgain, Mirek, Stein, and Wróbel [Bourgain et al. 2018; 2021], for studying dimension-free estimates for maximal functions associated with symmetric convex bodies.

Recently Liu, Melentijević, and Zhu [Liu et al. 2024] extended the results of [Kucharski and Wróbel 2023] and proved that $C(p, 1, d) \leqslant (2 + 1/\sqrt{2})^{2/p}$ for $p \geq 2$. An important ingredient of their argument is the positivity of the transition kernels M_1^t (see (1-7)), which is not at all clear in [Kucharski and Wróbel 2023].

In this paper we prove that the dimension-free estimate of the form (1-5) and its vector-valued generalization hold for Riesz transforms of arbitrary order k and for all $1 < p < \infty$. The main result of our paper is the following square function estimate of the vector of maximal Riesz transforms in terms of the Riesz transforms.

Theorem 1.1. *Take $p \in (1, \infty)$ and let $k \leqslant d$ be a positive integer. Let $\mathcal{P}_k$ be a subset of $\mathcal{H}_k$. Then there is a constant $A(p, k)$ independent of the dimension d and such that*

$$\Big\| \Big(\sum_{P \in \mathcal{P}_k} |R_P^* f|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^d)} \leqslant A(p, k) \Big\| \Big(\sum_{P \in \mathcal{P}_k} |R_P f|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^d)},$$

where $f \in L^p(\mathbb{R}^d)$. Moreover, for fixed k we have

$$A(p, k) = O(p^{5/2+k/2}) \quad \text{as } p \to \infty \qquad \text{and} \qquad A(p, k) = O((p-1)^{-5/2-k/2}) \quad \text{as } p \to 1.$$

In particular, if $\mathcal{P}_k$ contains one element P, then Theorem 1.1 immediately gives

$$\|R_P^* f\|_{L^p(\mathbb{R}^d)} \leqslant A(p, k) \|R_P f\|_{L^p(\mathbb{R}^d)}.$$

In this case however, we can slightly improve the constant $A(p, k)$. This is due to the fact that in the proof of Theorem 1.2 below we do not need to apply Khintchine's inequalities twice, which is an important ingredient in the proof of Theorem 1.1.

Theorem 1.2. *Take $p \in (1, \infty)$ and let $k \leqslant d$ be a positive integer. Let P be a spherical harmonic of degree k. Then there is a constant $B(p, k)$ independent of the dimension d and such that*

$$\|R_P^* f\|_{L^p(\mathbb{R}^d)} \leqslant B(p, k) \|R_P f\|_{L^p(\mathbb{R}^d)},$$

where $f \in L^p(\mathbb{R}^d)$. Moreover, for fixed k we have

$$B(p, k) = O(p^{2+k/2}) \quad \text{as } p \to \infty \qquad \text{and} \qquad B(p, k) = O((p-1)^{-2-k/2}) \quad \text{as } p \to 1.$$

Our last main result follows from a combination of Theorem 1.1 with a result of Duoandikoetxea and Rubio de Francia [1985, théorème 2]. Denote by $a(d, k)$ the dimension of $\mathcal{H}_k$ and let $\{Y_j\}_{j=1,\dots,a(d,k)}$ be an orthogonal basis of $\mathcal{H}_k$ normalized by the condition

$$\frac{1}{\sigma(S^{d-1})} \int_{S^{d-1}} |Y_j(\theta)|^2 \, d\sigma(\theta) = \frac{1}{a(d, k)};$$

here $d\sigma$ denotes the (unnormalized) spherical measure.

Corollary 1.3. *Take $p \in (1, \infty)$ and let $k \leqslant d$ be a positive integer. Then there is a constant $G(p, k)$ independent of the dimension d and such that*

$$\Big\| \Big(\sum_{j=1}^{a(d,k)} |R_{Y_j}^* f|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^d)} \leqslant G(p, k) \|f\|_{L^p(\mathbb{R}^d)},$$

where $f \in L^p(\mathbb{R}^d)$. Moreover, for fixed and odd k we have

$$G(p,k) = O(p^{7/2+k/2}) \quad \textit{as } p \to \infty \qquad \textit{and} \qquad G(p,k) = O((p-1)^{-7/2-k}) \quad \textit{as } p \to 1,$$

and for even k we have

$$G(p,k) = O(p^{9/2+k/2}) \quad \textit{as } p \to \infty \qquad \textit{and} \qquad G(p,k) = O((p-1)^{-9/2-k}) \quad \textit{as } p \to 1.$$

We finish this section with two remarks.

Remark 1. Corollary 1.3 seems interesting by itself. In the particular case of $k = 1$ it is a direct maximal function counterpart of Stein's inequality (1-4), namely

$$\left\| \left(\sum_{j=1}^{d} |R_j^* f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)} \leqslant C_p \|f\|_{L^p(\mathbb{R}^d)}, \quad 1 < p < \infty. \tag{1-6}$$

It is unclear to us if one can prove Corollary 1.3 or even inequality (1-6) without using Theorem 1.1 as an intermediate step.

Remark 2. We do not know what the sharp rates of the constants $A(p,k)$, $B(p,k)$ and $G(p,k)$ are in terms of p. However, the results of [Mateu et al. 2010; 2011; Mateu and Verdera 2006] suggest that

$$B(p,k) = O(\max(1,(p-1)^{-2})) \quad \text{and} \quad B(p,k) = O(\max(1,(p-1)^{-1}))$$

might be the optimal rates for fixed odd and even k, respectively. In fact, such estimates do follow from these papers at the price of involving upfront constants which depend on the dimension. When $k = 1$, then from [Liu et al. 2024] we indeed have $B(p,1) = O(1)$ as $p \to \infty$. The same holds for $B(p,2)$ because in this case the transition kernel M_2^t (see (1-8)) coincides with the centered Hardy–Littlewood averaging operator over the balls.

We are also unaware what the optimal constants $H(p,k,d)$ in the inequalities for truncated maximal Riesz transforms

$$\|R_P^* f\|_{L^p(\mathbb{R}^d)} \leqslant H(p,k,d) \|f\|_{L^p(\mathbb{R}^d)}$$

are even in the case $k = 1$. In view of the above comment about $B(p,1)$ and known optimal bounds for the Riesz transforms from [Iwaniec and Martin 1996], it might be possible to achieve $H(p,1,d) = O(\max(p,(p-1)^{-3}))$. However, even if this is true, it would not be optimal in dimension $d = 1$ for $p \to 1$. Then the maximal Riesz transform is the maximal Hilbert transform and the sharp order of its $L^p(\mathbb{R})$ norm is $\max(p,(p-1)^{-1})$.

1.1. *Structure of the paper and our methods.*

The proofs of Theorems 1.1 and 1.2. require four main ingredients.

First, we need a factorization of the truncated Riesz transform $R_P^t = M_k^t(R_P)$. Here, M_k^t, $t > 0$, is a family of radial Fourier multiplier operators. In the case $k = 1$ this factorization on the multiplier level has been one of the key steps in establishing the main results of [Kucharski and Wróbel 2023]. In particular, the operator M_1^t considered here coincides with M^t defined in [Kucharski and Wróbel

2023, equation (3.5)]. For general values of k the factorization on a kernel level is implicit in [Mateu and Verdera 2006, Section 2] ($k = 1$), [Mateu et al. 2011, Section 2] (k even), and [Mateu et al. 2010, Section 4] (k odd). In this paper we utilize the factorization on an operator level via an explicit formula for the operator M_k^t in terms of the Riesz transforms R_P and the truncated Riesz transforms R_P^t. Note that for the first-order Riesz transforms ($k = 1$) the formulas $R_j^t = M_1^t(R_j)$, $j = 1, \ldots, d$, together with the identity $I = -\sum_{j=1}^d (R_j)^2$ imply that

$$M_1^t = -\sum_{j=1}^d M_1^t (R_j)^2 = -\sum_{j=1}^d R_j^t R_j. \tag{1-7}$$

It seems that formula (1-7) and its variants for $k > 1$ have not been considered before. Yet, they are invaluable when one is interested in dimension-free $L^p(\mathbb{R}^d)$ bounds for M_k^t. Details of the factorization procedure are given in Section 2.

The second ingredient we need is an averaging procedure. It turns out that a useful analogue of (1-7) is not directly available for Riesz transforms of orders higher than 1. The reason behind it is the fact that not all compositions of first-order Riesz transforms are higher-order Riesz transforms according to our definition. For instance, in the case $k = 3$ the multiplier symbol of $(R_1)^3 = R_1 R_1 R_1$ on $L^2(\mathbb{R}^2)$ equals $-i\xi_1^3/|\xi|^3$ and $P(\xi) = -i\xi_1^3$ is not a spherical harmonic. However, the formula

$$I = -\sum_{j_1=1}^d \sum_{j_2=1}^d \sum_{j_3=1}^d (R_{j_1})^2 (R_{j_2})^2 (R_{j_3})^2$$

includes squares of all compositions of Riesz transforms including $(R_1)^6 = ((R_1)^3)^2$. Therefore the above formula does not directly lead to an expression of M_k^t in terms of R_P^t and R_P. To overcome this problem we average over the special orthogonal group $\mathrm{SO}(d)$. Then we obtain

$$M_k^t f(x) = C(d,k) \int_{\mathrm{SO}(d)} \sum_{j \in I} (R_j^t R_j)_U f(x)\, d\mu(U); \tag{1-8}$$

see Proposition 3.1. Here T_U is the conjugation of an operator T by $U \in \mathrm{SO}(d)$, see (3-1), $d\mu$ denotes the normalized Haar measure on $\mathrm{SO}(d)$, while $C(d,k)$ is a constant. The symbol I denotes the set of multi-indices $j = (j_1, \ldots, j_k)$ with increasing components while R_j^t and R_j are the truncated Riesz transforms and the Riesz transforms (1-2) corresponding to the monomials $P_j(x) = x_{j_1} \cdots x_{j_d}$. Note that since $j \in I$ the polynomials P_j are spherical harmonics and thus the operators R_j are indeed higher-order Riesz transforms. In view of (1-8), if we demonstrate that $C(d,k)$ is bounded by a universal constant, we are left with estimating the maximal function corresponding to $\sum_{j \in I} R_j^t R_j$. The reduction via the averaging procedure is described in detail in Section 3. It is noteworthy that in order for the averaging approach to work it is essential that for each order k the multiplier symbols of M_k^t are radial functions.

The third main ingredient of our argument is an extension to $\mathbb{C}^d$ followed by the complex method of rotations of Iwaniec and Martin [1996]. We use the complex method of rotations to estimate the maximal function $\widetilde{R}^*$ corresponding to

$$\widetilde{R}^t := \sum_{j\in I} \widetilde{R}_j^t \widetilde{R}_j. \tag{1-9}$$

Here $\widetilde{R}_j^t$ and $\widetilde{R}_j$ denote extensions to $\mathbb{C}^d$ of the truncated Riesz transform R_j^t and the Riesz transform R_j. The definition of $\widetilde{R}_j$ can be given on the multiplier level according to the scheme from [Iwaniec and Martin 1996]. We note, however, that the truncated extended operator $\widetilde{R}_j^t$ needs to be defined differently — on a kernel level. In the context of dimension-free estimates for Riesz transforms the real method of rotations has been employed by Duoandikoetxea and Rubio de Francia [1985]. However, as it can be applied only to operators with odd kernels, for the general case we need the complex version. The method of rotations itself is preceded by a number of other ingredients. In particular we need L^p vector-valued estimates for the maximal directional truncated k-th power of the complex Hilbert transform, see Proposition 4.3, and for the vector of higher-order Riesz transforms, see Proposition 4.4. En route to obtain these results we also need Khintchine's inequalities and specific computations. All of it reflects the size of the constants $A(p,k)$ in Theorem 1.1 and $B(p,k)$ in Theorem 1.2. The extension procedure and the application of the complex method of rotations are described in detail in Section 4.

The last ingredient is a restriction procedure. This allows us to deduce the estimates for R^* on $\mathbb{R}^d$ from the estimates for $\widetilde{R}^*$ on $\mathbb{C}^d$. The restriction of the complex Riesz transforms $\widetilde{R}_j$ in (1-9) can be done on the multiplier level as in [Iwaniec and Martin 1996, Chapter 4]. However, in order to restrict $\widetilde{R}_j^t$ and the maximal function $\widetilde{R}^*$ we need to work on the kernel level. A problem that we encounter here is that the resulting restricted operator of $\widetilde{R}^*$ is not the same as the desired maximal operator R^*. Therefore we need to investigate their difference and estimate it appropriately. The restriction procedure is described in Section 5.

At the first reading it might be helpful to skip the explicit values of constants in terms of k and p and only focus on these constants being independent of the dimension d. An interested reader may trace the exact dependencies of the constants in terms of k and p in the paper.

1.2. ***Notation.*** We finish the introduction with a description of the notation and conventions used in the rest of the paper.

(1) The letters d and k stand for the dimension and for the order of the Riesz transforms, respectively. In particular we always have $k \leqslant d$, even if this is not stated explicitly.

(2) The symbol $\mathbb{N}$ represents the set of positive integers. Throughout the paper we assume that $k \in \mathbb{N}$. We write $\mathbb{Q}_+$ for the set of positive rational numbers.

(3) By $[d]$ we denote the set $\{1, \dots, d\}$ of positive integers up to d.

(4) For an exponent $p \in [1,\infty]$ we let q be its conjugate exponent satisfying

$$1 = \frac{1}{p} + \frac{1}{q}.$$

When $p \in (1,\infty)$ we set

$$p^* := \max(p, (p-1)^{-1}).$$

(5) We abbreviate $L^p(\mathbb{R}^d)$ to L^p and $\|\cdot\|_{L^p}$ to $\|\cdot\|_p$. For a sublinear operator T on L^p we denote by $\|T\|_{p\to p}$ its norm. We let $\mathcal{S}(\mathbb{R}^d)=\mathcal{S}$ be the space of Schwartz functions on $\mathbb{R}^d$. Slightly abusing the notation we say that a sublinear operator T is bounded on L^p if it is bounded on $\mathcal{S}$ in the L^p norm.

(6) For $k\in\mathbb{N}$ we let $\mathcal{D}(k)$ be the linear span of $\{R_P(f): P\in\mathcal{H}_k,\ f\in\mathcal{S}\}$. Since R_P is bounded on L^p for $1<p<\infty$, the space $\mathcal{D}(k)$ is then a subspace of each of the L^p spaces.

(7) For a Banach space E the symbol $L^p(\mathbb{R}^d;E)$ stands for the space of weakly measurable functions $f:\mathbb{R}^d\to E$ with the norm $\|f\|_{L^p(\mathbb{R}^d;E)}=\big(\int_{\mathbb{R}^d}\|f(x)\|_E^p\,dx\big)^{1/p}$. Similarly, for a finite set F, by $\ell^p(F;E)$ we denote the Banach space of E-valued sequences $\{f_s\}_{s\in F}$ with the norm $\|f\|_{\ell^p(F;E)}=\big(\sum_{s\in F}\|f_s\|_E^p\big)^{1/p}$.

(8) The symbol C_Δ stands for a constant that possibly depends on $\Delta>0$. We write C without a subscript when the constant is universal in the sense that it may depend only on k but not on the dimension d nor on any other quantity.

(9) For two quantities X and Y we write $X\lesssim_\Delta Y$ if $X\leqslant C_\Delta Y$ for some constant $C_\Delta>0$ that depends only on Δ. We abbreviate $X\lesssim Y$ when C is a universal constant. We also write $X\sim Y$ if both $X\lesssim Y$ and $Y\lesssim X$ hold simultaneously. By $X\lesssim^\Delta Y$ we mean that $X\leqslant C^\Delta Y$ with a universal constant C. Note that in this case $X^{1/\Delta}\lesssim Y^{1/\Delta}$.

(10) The symbol S^{d-1} stands for the $(d-1)$-dimensional unit sphere in $\mathbb{R}^d$ and by ω we denote the uniform measure on S^{d-1} normalized by the condition $\omega(S^{d-1})=1$. We also write

$$S_{d-1}=\frac{2\pi^{d/2}}{\Gamma\big(\frac{1}{2}d\big)} \tag{1-10}$$

to denote the unnormalized surface area of S^{d-1}. We write ζ for the uniform measure on S^{2d-1} normalized by the condition $\zeta(S^{2d-1})=1$.

(11) We let

$$\gamma_k=\gamma_{k,d}:=\frac{\Gamma\big(\frac{1}{2}(k+d)\big)}{\pi^{d/2}\Gamma\big(\frac{1}{2}k\big)}\quad\text{and}\quad\widetilde{\gamma}_k=\gamma_{k,2d}=\frac{\Gamma\big(d+\frac{1}{2}k\big)}{\pi^{d}\Gamma\big(\frac{1}{2}k\big)}. \tag{1-11}$$

(12) The Fourier transform is defined for $f\in L^1$ and $\xi\in\mathbb{R}^d$ by the formula

$$\mathcal{F}f(\xi)=\hat{f}(\xi)=\int_{\mathbb{R}^d}f(x)e^{-2\pi i x\cdot\xi}\,dx.$$

(13) The Gamma function is defined for $s>0$ by the formula

$$\Gamma(s)=\int_0^\infty t^{s-1}e^{-t}\,dt.$$

We shall use Stirling's approximation for $\Gamma(s)$:

$$\Gamma(s)\sim\sqrt{2\pi}\,s^{s-1/2}e^{-s},\qquad s\to\infty. \tag{1-12}$$

A useful consequence of (1-12) is the formula

$$\Gamma(s+\alpha)\sim s^\alpha\Gamma(s),\qquad s\to\infty, \tag{1-13}$$

which is valid for each fixed $\alpha\geq 0$.

(14) We will also need the formula

$$2\int_0^\infty \frac{r^{d-1}}{(1+r^2)^{d+\alpha}}\,dr = B\big(\tfrac{1}{2}d,\,\tfrac{1}{2}d+\alpha\big) = \frac{\Gamma\big(\tfrac{1}{2}d\big)\Gamma\big(\tfrac{1}{2}d+\alpha\big)}{\Gamma(d+\alpha)}, \tag{1-14}$$

valid for $\alpha \geq 0$. This follows from the change of variables $r^2 \to r$ followed by formulas for Euler's beta function $B(a, b)$ from [Olver et al. 2010, 5.12.1, 5.12.3].

2. Factorization

The goal of this section is to show that a factorization formula for R_P^t in terms of R_P is feasible. Proposition 2.1 below is implicit in [Mateu et al. 2010, Section 4; 2011, pp. 1435–1436].

Proposition 2.1. *Let $k \in \mathbb{N}$. Then there exists a family of operators M_k^t, $t > 0$, which are bounded on L^p, $1 < p < \infty$, and such that for all $P \in \mathcal{H}_k$ we have*

$$R_P^t f = M_k^t(R_P f), \tag{2-1}$$

where $f \in L^p$. Each M_k^t is a convolution operator with a radial convolution kernel b_k^t.

Proof. We consider separately the cases of k odd or even starting with k odd.

Let

$$c_d = \frac{\Gamma\big(\tfrac{1}{2}(d-1)\big)}{2\pi^{d/2}\Gamma\big(\tfrac{1}{2}\big)}, \qquad N = \tfrac{1}{2}(k-1),$$

and denote by B the open Euclidean ball of radius 1 in $\mathbb{R}^d$. In [Mateu et al. 2010, pp. 3674–3675] it is justified that the function

$$b(x) = b_{k,d}(x) := \sum_{j=1}^{d} R_j[y_j \cdot h(y)](x), \tag{2-2}$$

where

$$h(y) = c_d(1-d)\frac{1}{|y|^{d+1}}\mathbb{1}_{B^c}(y) + (\beta_1 + \beta_2|y|^2 + \cdots + \beta_N|y|^{2N-2})\mathbb{1}_B(y),$$

satisfies the formula

$$R_P(b)(x) = K_P(x)\mathbb{1}_{B^c}. \tag{2-3}$$

Here $\beta_1, \ldots, \beta_N$ are constants which depend only on k and d and whose exact value is irrelevant for our considerations, and K_P and R_P have been defined in (1-1) and (1-2), respectively. The important point is that (2-3) remains true for any $P \in \mathcal{H}_k$.

Denote by H the radial profile of the Fourier transform of h, i.e., $H(|\xi|) = \hat{h}(\xi)$ for $\xi \in \mathbb{R}^d$. By taking the Fourier transform of (2-2) it is straightforward to see that b is a radial function. This follows since the multiplier symbol of R_j is $-i\xi_j/|\xi|$ and

$$\widehat{(y_j h(y))}(\xi) = \frac{\xi_j}{-2\pi i|\xi|}\,H'(|\xi|),$$

so that

$$\mathcal{F}b(\xi)=\sum_{j=1}^{d}\frac{\xi_j^2}{2\pi|\xi|^2}\cdot H'(|\xi|)=\frac{1}{2\pi}H'(|\xi|)$$

is indeed radial and so is b.

Let $b^t(x)=b^t_k(x):=t^{-d}b(x/t)$ be the L^1 dilation of b; clearly b^t is still radial. The dilation invariance of R_P together with (2-3) leads us to the expression

$$K_P(x)\mathbb{1}_{B^c}(x/t)=R_P(b^t)(x). \tag{2-4}$$

Let M^t_k be the convolution operator

$$M^t_k f(x)=b^t * f(x).$$

It follows from [Mateu et al. 2010, Section 4] that M^t_k is bounded on L^p spaces whenever $1<p<\infty$. Moreover, in view of (2-4) we see that

$$R^t_P f=R_P(b^t)*f=b^t*R_P(f)=M^t_k(R_P f).$$

It remains to consider k even. Let $N=k/2$. From (10) and (12) in [Mateu et al. 2011, pp. 1435–1436] it follows that the function

$$b(x)=b_{k,d}(x):=\big(\alpha_0+\alpha_1|x|^2+\cdots+\alpha_{N-1}|x|^{2(N-1)}\big)\mathbb{1}_B(x)$$

satisfies the formula

$$R_P(b)(x)=K_P(x)\mathbb{1}_{B^c}(x). \tag{2-5}$$

Here $\alpha_1,\dots,\alpha_{N-1}$ are constants which depend only on k and d and whose exact value is irrelevant for our considerations. As in the case of odd k, the important point is that (2-5) remains true for any $P\in\mathcal{H}_k$.

Using (2-5) we proceed as in the proof in the case when k. Let $b^t(x)=b^t_k(x):=t^{-d}b(x/t)$ be the L^1 dilation of b. Since b is clearly radial the same is true of b^t. Let M^t_k be the convolution operator

$$M^t_k f(x)=b^t * f(x).$$

It follows from [Mateu et al. 2011, Section 2] that M^t_k is bounded on L^p spaces whenever $1<p<\infty$. Moreover, in view of (2-5) we see that

$$R^t_P f=R_P(b^t)*f=b^t*R_P(f)=M^t_k(R_P f). \qquad\square$$

As a corollary of Proposition 2.1 we see that in order to justify Theorems 1.1 and 1.2 it suffices to control vector- and scalar-valued maximal functions corresponding to the operators M^t_k. Note that for $f\in\mathcal{S}$, $P\in\mathcal{H}_k$, and $x\in\mathbb{R}^d$, the mapping $t\mapsto R^t_P f(x)$ is continuous on $(0,\infty)$ and thus, by (2-1), so is $t\mapsto M^t_k(R_P f)(x)$. Consequently, for $f\in\mathcal{D}(k)$ (see (6) in the notation section) we have

$$\sup_{t>0}|M^t_k f(x)|=\sup_{t\in\mathbb{Q}_+}|M^t_k f(x)|.$$

In particular $\sup_{t>0}|M_k^t f(x)|$ is measurable for such f, although possibly infinite for some x. Define

$$M_k^* f(x) = \sup_{t\in\mathbb{Q}_+} |M_k^t f(x)|. \tag{2-6}$$

Proposition 2.1 reduces our task to proving the following two theorems.

Theorem 2.2. *Fix $k\in\mathbb{N}$. For each $p\in(1,\infty)$ there is a constant $A(p,k)$ independent of the dimension d and such that for any $S\in\mathbb{N}$ we have*

$$\left\| \left(\sum_{s=1}^{S} |M_k^* f_s|^2 \right)^{1/2} \right\|_p \leqslant A(p,k) \left\| \left(\sum_{s=1}^{S} |f_s|^2 \right)^{1/2} \right\|_p,$$

where $f_1,\dots,f_S\in L^p$. Furthermore $A(p,k)$ satisfies $A(p,k)\lesssim_k (p^*)^{5/2+k/2}$.

Theorem 2.3. *Fix $k\in\mathbb{N}$. For each $p\in(1,\infty)$ there is a constant $B(p,k)$ independent of the dimension d and such that*

$$\|M_k^* f\|_p \leqslant B(p,k)\|f\|_p$$

whenever $f\in L^p$. Moreover $B(p,k)$ satisfies $B(p,k)\lesssim_k (p^*)^{2+k/2}$.

3. Averaging

In this section we describe the averaging procedure. The averaging procedure will allow us to pass from M_k^* to another maximal operator that is better suited for applications in Sections 4 and 5. Before moving on, we establish some notation. We define the set I of multi-indices with increasing components as

$$I = \left\{ j\in\{1,\dots,d\}^k : j_i < j_l \text{ for } i<l \right\}.$$

For a multi-index $j=(j_1,\dots,j_k)\in I$ we write

$$P_j(x) = x_j := x_{j_1}\cdots x_{j_k}$$

and denote by R_j the Riesz transform R_{P_j} associated with the monomial P_j. The truncated transform R_j^t and the maximal transform R_j^* are defined analogously. We also abbreviate $K_j(x)=K_{P_j}(x)$ and $K_j^t(x)=K_{P_j}^t(x)$.

The averaging procedure will provide an expression for M_k^t in terms of the Riesz transforms R_j and R_j^t postulated in (1-8). For $f\in L^p$, $1<p<\infty$, let

$$R^t f := \sum_{j\in I} R_j^t R_j f \quad\text{and}\quad R^* f := \sup_{t\in\mathbb{Q}_+} |R^t f|.$$

Note that both R^t and R^* are well defined on all L^p spaces. Indeed, R_j^t and R_j are bounded on L^p and the supremum in the definition of R^* runs over a countable set thus defining a measurable function.

Let $\mathrm{SO}(d)$ be the special orthogonal group in dimension d. Since it is compact, it has a bi-invariant Haar measure μ such that $\mu(\mathrm{SO}(d))=1$. For $U\in\mathrm{SO}(d)$ and a sublinear operator T on L^2 we denote by

T_U the conjugation by U, i.e., the operator acting via

$$T_U f(x) = T(f(U^{-1}\cdot))(Ux). \tag{3-1}$$

Proposition 3.1. *Fix $k \in \mathbb{N}$. Then there is a constant $C(d,k) \in \mathbb{R}$ such that*

$$M_k^t f(x) = C(d,k) \int_{\mathrm{SO}(d)} [(R^t)_U f](x)\, d\mu(U) \tag{3-2}$$

for all $t > 0$ and $f \in L^p$. Moreover, $|C(d,k)|$ has an estimate from above by a constant that depends only on k but not on the dimension d, so that

$$\left(\sum_{s=1}^{S} |M_k^* f_s(x)|^2\right)^{1/2} \lesssim \int_{\mathrm{SO}(d)} \left(\sum_{s=1}^{S} |[(R^*)_U f_s](x)|^2\right)^{1/2} d\mu(U) \tag{3-3}$$

for $S \in \mathbb{N}$ and $f_1, \dots, f_S \in L^p$.

Proof. Let A be the operator

$$A = \sum_{j \in I} (R_j)^2, \tag{3-4}$$

which by (1-3) means that its multiplier symbol equals

$$a(\xi) = (-i)^{2k} \sum_{j\in I} \frac{\xi_j^2}{|\xi|^{2k}} = (-1)^k \sum_{j \in I} \frac{\xi_j^2}{|\xi|^{2k}}.$$

Let $\tilde{A}$ be the operator with the multiplier symbol

$$\tilde{a}(\xi) := \int_{\mathrm{SO}(d)} a(U\xi)\, d\mu(U) = (-1)^k \sum_{j\in I} \int_{\mathrm{SO}(d)} \frac{((U\xi)_j)^2}{|\xi|^{2k}}\, d\mu(U). \tag{3-5}$$

Then $\tilde{a}$ is radial, and since it is homogeneous of order 0, it is constant.

Let now m^t be the multiplier symbol of M_k^t. Then, from Proposition 2.1 we see that $m^t = \hat{b}^t$ is radial, so that

$$m^t(\xi) = \tilde{a}^{-1}\tilde{a} m^t(\xi) = \tilde{a}^{-1} \int_{\mathrm{SO}(d)} m^t(\xi) a(U\xi)\, d\mu(U) = \tilde{a}^{-1} \int_{\mathrm{SO}(d)} m^t(U\xi) a(U\xi)\, d\mu(U).$$

Using properties of the Fourier transform the above equality implies that

$$M_k^t f(x) = \tilde{a}^{-1} \int_{\mathrm{SO}(d)} [(M_k^t A)_U](f)(x)\, d\mu(U).$$

Recalling (3-4) we apply (2-1) from Proposition 2.1 and obtain

$$M_k^t A = \sum_{j\in I} M_k^t R_j R_j = \sum_{j \in I} R_j^t R_j = R^t;$$

here an application of (2-1) is allowed since each R_j corresponds to the monomial x_j which is in $\mathcal{H}_k$ when $j \in I$. In summary, we justified that

$$M_k^t f(x) = \tilde{a}^{-1} \int_{\mathrm{SO}(d)} [(R^t)_U](f)(x)\, d\mu(U), \qquad f \in \mathcal{D}(k), \tag{3-6}$$

which is (3-2) with $C(d,k) = \tilde{a}^{-1}$.

Now we need to prove that

$$|C(d,k)| = |\tilde{a}|^{-1} \sim 1 \tag{3-7}$$

uniformly in the dimension d. Note that each of the integrals on the right-hand side of (3-5) has the same value independent of $j \in I$, so that

$$\tilde{a}(\xi) = (-1)^k |I| \int_{\mathrm{SO}(d)} \frac{((U\xi)_{(1,\dots,k)})^2}{|\xi|^{2k}}\, d\mu(U);$$

here $|I|$ stands for the number of elements in I. Since $\tilde{a}$ is constant, we can integrate it over S^{d-1} with probabilistic measure and change the order of integration to get

$$\tilde{a} = (-1)^k |I| \int_{S^{d-1}} \int_{\mathrm{SO}(d)} (U\omega)^2_{(1,\dots,k)}\, d\mu(U)\, d\omega = (-1)^k |I| \int_{\mathrm{SO}(d)} \int_{S^{d-1}} (U\omega)^2_{(1,\dots,k)}\, d\omega\, d\mu(U).$$

Now notice that the inner integral does not depend on U, which means that

$$\tilde{a} = (-1)^k |I| \int_{S^{d-1}} \omega_1^2 \cdots \omega_k^2\, d\omega. \tag{3-8}$$

Since k is fixed, by an elementary argument we get $|I| = \binom{d}{k} \sim d^k$. Thus it remains to show that

$$\int_{S^{d-1}} \omega_1^2 \cdots \omega_k^2\, d\omega \sim d^{-k}. \tag{3-9}$$

Formula (3-9) is given in [Sýkora 2005, (10)]. It can be also easily computed by the method from [Hörmander 2003, Chapter 3.4]; for the sake of completeness we provide a brief argument. Consider the integral $J = \int_{\mathbb{R}^d} x_1^2 \cdots x_k^2 e^{-|x|^2}\, dx$. Since J is a product of the one-dimensional integrals we calculate $J = \Gamma\big(\frac{3}{2}\big)^k \Gamma\big(\frac{1}{2}\big)^{d-k}$, while using polar coordinates gives $J = S_{d-1} \int_{S^{d-1}} \omega_1^2 \cdots \omega_k^2\, d\omega \int_0^\infty r^{2k+d-1} e^{-r^2}\, dr$, where S_{d-1} is defined by (1-10). Altogether we have justified that

$$\int_{S^{d-1}} \omega_1^2 \cdots \omega_k^2\, d\omega \sim \frac{\Gamma\big(\frac{1}{2}\big)^{d-k}}{S_{d-1}\Gamma\big(k + \frac{1}{2}d\big)}.$$

Since k is fixed and d is arbitrarily large, using (1-10), Stirling's formula for the Γ function (1-12) and the known identity $\Gamma\big(\frac{1}{2}\big) = \sqrt{\pi}$ we obtain

$$\int_{S^{d-1}} \omega_1^2 \cdots \omega_k^2\, d\omega \sim \frac{\sqrt{k + \frac{1}{2}d}\,(d/(2e))^{d/2}}{\sqrt{\frac{1}{2}d}\,((k+d/2)/e)^{k+d/2}} \sim \frac{e^{-d/2}}{e^{-k-d/2}} \left(\frac{k+d/2}{d/2}\right)^{-d/2} (k+d/2)^{-k} \sim d^{-k}.$$

This gives (3-9) and concludes the proof of (3-7).

It remains to justify (3-3). This follows from (2-6), (3-6), and (3-7), together with the norm inequality

$$\left\| \int_{\mathrm{SO}(d)} F_{s,t}(U)\, d\mu(U) \right\|_X \leqslant \int_{\mathrm{SO}(d)} \|F_{s,t}(U)\|_X \, d\mu(U)$$

on the Banach space $X = \ell^2(\{1, \ldots, S\}; \ell^\infty(\mathbb{Q}_+))$, with $F_{s,t}(U) = (R^t)_U(f_s)(x)$ and x being fixed. □

Since conjugation by $U \in \mathrm{SO}(d)$ is an isometry on all L^p spaces, in view of $\mu(\mathrm{SO}(d)) = 1$ and Minkowski's integral inequality, equation (3-3) of Proposition 3.1 allows us to deduce Theorems 2.2 and 2.3 from the two theorems below.

Theorem 3.2. *Fix $k \in \mathbb{N}$. For each $p \in (1, \infty)$ there is a constant $A(p, k)$ independent of the dimension d and such that for any $S \in \mathbb{N}$ we have*

$$\left\| \left(\sum_{s=1}^{S} |R^* f_s|^2 \right)^{1/2} \right\|_p \lesssim A(p,k) \left\| \left(\sum_{s=1}^{S} |f_s|^2 \right)^{1/2} \right\|_p,$$

where $f_1, \ldots, f_S \in L^p$. Moreover, $A(p, k)$ satisfies $A(p, k) \lesssim_k (p^*)^{5/2+k/2}$.

Theorem 3.3. *Fix $k \in \mathbb{N}$. For each $p \in (1, \infty)$ there is a constant $B(p, k)$ independent of the dimension d and such that*

$$\|R^* f\|_p \lesssim B(p,k) \|f\|_p$$

whenever $f \in L^p$. Moreover, $B(p, k)$ satisfies $B(p, k) \lesssim_k (p^*)^{2+k/2}$.

4. Extension to $\mathbb{C}^d$ and the complex method of rotations

Here we extend the operators R^t acting on $L^p(\mathbb{R}^d)$ to the operators $\widetilde{R}^t$ acting on $L^p(\mathbb{C}^d)$. Then we apply the complex method of rotations of Iwaniec and Martin [1996] to $\widetilde{R}^t$.

Let $P \in \mathcal{H}_k$. For $z = (x_1 + iy_1, \ldots, x_d + iy_d),\ x \in \mathbb{R}^d,\ y \in \mathbb{R}^d$ we denote

$$\widetilde{K}_P(z) = \widetilde{\gamma}_k \frac{P(z)}{|z|^{2d+k}} \quad \text{with } \widetilde{\gamma}_k = \frac{\Gamma(d+k/2)}{\pi^d \Gamma(k/2)}, \tag{4-1}$$

and define, for $f \in \mathcal{S}(\mathbb{C}^d)$,

$$\widetilde{R}_P f(z) = \lim_{t \to 0} \widetilde{R}^t_P f(z), \quad \text{where } \widetilde{R}^t_P f(z) = \widetilde{\gamma}_k \int_{w \in \mathbb{C}^d : |w| > t} \frac{P(w)}{|w|^{2d+k}} f(z-w)\, dw. \tag{4-2}$$

Iwaniec and Martin [1996] considered the extension on the multiplier level whereas here we need to write it on the kernel level. This makes no difference for the operator $\widetilde{R}_P$. However, the multiplier symbol corresponding to the truncated transform $\widetilde{R}^t_P$ does not have a simple formula, thus writing the extension on a kernel level seems the only reasonable option here.

Formulas (4-1) and (4-2) lead us to define the extension of R^t by

$$\widetilde{R}^t = \widetilde{R}^t_k := \sum_{j \in I} \widetilde{R}^t_j \widetilde{R}_j. \tag{4-3}$$

Using the complex method of rotations [Iwaniec and Martin 1996, Section 6] we will prove $L^p(\mathbb{C}^d)$ estimates for

$$\widetilde{R}^* f(z) = \sup_{t\in\mathbb{Q}_+} |\widetilde{R}^t f(z)|.$$

Theorem 4.1. *Fix $k \in \mathbb{N}$. For each $p \in (1, \infty)$ there is a constant $A(p, k)$ independent of the dimension d and such that for any $S \in \mathbb{N}$ we have*

$$\Big\| \Big(\sum_{s=1}^{S} |\widetilde{R}^* f_s|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{C}^d)} \leqslant A(p, k) \Big\| \Big(\sum_{s=1}^{S} |f_s|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{C}^d)}$$

whenever $f_1, \dots, f_S \in L^p(\mathbb{C}^d)$. Moreover, $A(p, k)$ satisfies $A(p, k) \lesssim_k (p^*)^{5/2+k/2}$.

Theorem 4.2. *Fix $k \in \mathbb{N}$. For each $p \in (1, \infty)$ there is a constant $B(p, k)$ independent of the dimension d and such that*

$$\|\widetilde{R}^* f\|_{L^p(\mathbb{C}^d)} \leqslant B(p, k) \|f\|_{L^p(\mathbb{C}^d)}$$

whenever $f \in L^p(\mathbb{C}^d)$. Moreover, $B(p, k)$ satisfies $B(p, k) \lesssim_k (p^*)^{2+k/2}$.

The reminder of this section will be devoted to the proofs of Theorems 4.1 and 4.2. From these results we shall obtain Theorems 2.2 and 2.3 provided we develop a restriction procedure from $\mathbb{C}^d$ to $\mathbb{R}^d$. As we already remarked this is not straightforward, since the restriction of the complex truncated Riesz transform is not the real truncated Riesz transform. Details of the restriction and estimates for the resulting operators are given in Section 5.

We now focus on the proofs of Theorems 4.1 and 4.2. Let $P \in \mathcal{H}_k$. Note that

$$2\pi \int_{\mathbb{C}^d} F(w)\, dw = \int_{S^{2d-1}} \int_{\mathbb{C}} F(\lambda\theta) |\lambda|^{2d-2}\, d\lambda\, d\theta,$$

where $F \in \mathcal{S}(\mathbb{C}^d)$ and $d\theta$ stands for the spherical measure on S^{2d-1} normalized by the condition $\theta(S^{2d-1}) = S_{2d-1}$. Take $f \in \mathcal{S}(\mathbb{C}^d)$. Applying the above identity with

$$F(w) = \widetilde{\gamma}_k \frac{P(w)}{|w|^{2d+k}} \mathbb{1}_{|w|\geq t} f(z - w)$$

gives

$$\begin{aligned} \widetilde{R}^t_P f(z) &= \widetilde{\gamma}_k \int_{\mathbb{C}^d} \frac{P(w)}{|w|^{2d+k}} \mathbb{1}_{|w|\geq t} f(z - w)\, dw \\ &= \frac{\widetilde{\gamma}_k}{2\pi} \int_{S^{2d-1}} \int_{\mathbb{C}} \frac{P(\lambda\theta)}{|\lambda|^{2d+k}} \mathbb{1}_{|\lambda|\geq t} f(z - \lambda\theta) |\lambda|^{2d-2}\, d\lambda\, d\theta \\ &= \frac{\widetilde{\gamma}_k}{2\pi} \int_{S^{2d-1}} P(\theta) \int_{\mathbb{C}} \Big(\frac{\lambda}{|\lambda|} \Big)^k \frac{f(z - \lambda\theta)}{|\lambda|^2} \mathbb{1}_{|\lambda|\geq t}\, d\lambda\, d\theta, \end{aligned}$$

where in the last equality above we used the k-homogeneity of P. This means that we got

$$\widetilde{R}^t_P f(z) = \frac{\widetilde{\gamma}_k}{2\pi} \int_{S^{2d-1}} P(\theta) H^t_{\theta,k} f(z)\, d\theta, \tag{4-4}$$

where

$$H^t_{\theta,k} f(z) = H^t_\theta f(z) := \int_{\mathbb{C}} \left(\frac{\lambda}{|\lambda|}\right)^k \frac{f(z-\lambda\theta)}{|\lambda|^2} \mathbb{1}_{|\lambda|\ge t}(\lambda)\, d\lambda$$

is the truncated directional k-th power of the complex Hilbert transform.

Identity (4-4) can be written in terms of the probabilistic spherical measure $d\zeta$ on S^{2d-1} in the following way:

$$\widetilde{R}^t_P f(z) = \frac{\Gamma\left(d+\frac{1}{2}k\right)}{\pi\Gamma(d)\Gamma\left(\frac{1}{2}k\right)} \int_{S^{2d-1}} P(\zeta) H^t_\zeta f(z)\, d\zeta. \tag{4-5}$$

The limiting case of (4-5) is then

$$\widetilde{R}_P f(z) = \frac{\Gamma\left(d+\frac{1}{2}k\right)}{\pi\Gamma(d)\Gamma\left(\frac{1}{2}k\right)} \int_{S^{2d-1}} P(\zeta) H_\zeta f(z)\, d\zeta, \tag{4-6}$$

where

$$H_\zeta f = H_{\zeta,k} f = \text{p.v.} \int_{\mathbb{C}} \left(\frac{\lambda}{|\lambda|}\right)^k \frac{f(z-\lambda\zeta)}{|\lambda|^2}\, d\lambda$$

is the directional k-th power of the complex Hilbert transform.

Identities (4-5) and (4-6) were initially established for $f \in \mathcal{S}(\mathbb{C}^d)$. However, a density argument based on the $L^p(\mathbb{C}^d)$ boundedness of H^t_ζ and H_ζ allows us to write these identities for all $f \in L^p(\mathbb{C}^d)$. For further reference we note that when k is fixed, then by (1-13) we have

$$\frac{\Gamma\left(d+\frac{1}{2}k\right)}{\pi\Gamma(d)\Gamma\left(\frac{1}{2}k\right)} \sim d^{k/2}. \tag{4-7}$$

In the proofs of Theorems 3.2 and 3.3 we shall need boundedness properties of the maximal operator

$$H^*_\zeta f(z) = H^*_{\zeta,k} f(z) := \sup_{t\in\mathbb{Q}_+} |H^t_\zeta f(z)|$$

associated to H^t_ζ.

Proposition 4.3. *For each* $1 < p < \infty$ *we have*

$$\left\| \left(\sum_{s=1}^S |H^*_\zeta f_s|^2 \right)^{1/2} \right\|_{L^p(\mathbb{C}^d)} \lesssim p^* \left\| \left(\sum_{s=1}^S |f_s|^2 \right)^{1/2} \right\|_{L^p(\mathbb{C}^d)}$$

uniformly in $\zeta \in S^{2d-1}$ *and the dimension* d.

The proof of Proposition 4.3 is standard and therefore we omit it here. For the convenience of the reader we include the proof in the Appendix.

We will also need vector-valued estimates for $\{\widetilde{R}_j(f_s)\},\ j \in I,\ s = 1, \dots, d$.

Proposition 4.4. *Fix $k \in \mathbb{N}$. Then for each $1 < p < \infty$ we have*

$$\Big\| \Big(\sum_{s=1}^{S} \sum_{j \in I} |\widetilde{R}_j f_s|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{C}^d)} \lesssim_k p^* p^{1/2} q^{(k+1)/2} \Big\| \Big(\sum_{s=1}^{S} |f_s|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{C}^d)}, \tag{4-8}$$

$$\Big\| \Big(\sum_{j \in I} |\widetilde{R}_j f|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{C}^d)} \lesssim_k p^* q^{k/2} \|f\|_{L^p(\mathbb{C}^d)} \tag{4-9}$$

uniformly in the dimension d.

Proposition 4.4 can be proved by an iterative application of its $k = 1$ case together with Khintchine's inequalities. However, such an approach produces worse constants than those in (4-8), (4-9). An important ingredient in the proof are properties of the functions

$$\zeta_j = (x_{j_1} + i y_{j_1}) \cdots (x_{j_k} + i y_{j_k}).$$

Note that $\zeta_j,\ j \in I$, are orthogonal with respect to the inner product on S^{2d-1}. Moreover,

$$\int_{S^{2d-1}} |\zeta_j|^2 \, d\zeta \lesssim d^{-k}. \tag{4-10}$$

Indeed, all the integrals on the left-hand side of (4-10) are equal for $j \in I$ and thus

$$\int_{S^{2d-1}} |\zeta_j|^2 \, d\zeta = \frac{1}{|I|} \int_{S^{2d-1}} \sum_{j \in I} |\zeta_j|^2 \, d\zeta \le \frac{1}{|I|} \int_{S^{2d-1}} \sum_{j \in [d]^k} |\zeta_j|^2 \, d\zeta = \frac{1}{|I|} \int_{S^{2d-1}} |\zeta|^{2k} \, d\zeta \lesssim d^{-k}$$

since $|I| \approx d^k$.

We justify (4-8) and (4-9) separately, starting with the latter.

Proof of (4-9). Take numbers $\lambda_j(f, z) = \lambda_j(z),\ j \in I$, such that

$$\Big(\sum_{j \in I} |\widetilde{R}_j f(z)|^2 \Big)^{1/2} = \sum_{j \in I} \lambda_j(z) \widetilde{R}_j f(z), \quad \sum_{j \in I} \lambda_j^2(z) = 1.$$

Using (4-6) and (4-7) followed by Hölder's inequality we obtain

$$\begin{aligned} \Big\| \Big(\sum_{j \in I} |\widetilde{R}_j f|^2 \Big)^{1/2} \Big\|_p^p &= \int_{\mathbb{C}^d} \Big| \sum_{j \in I} \lambda_j(z) \widetilde{R}_j f(z) \Big|^p dz \\ &\lesssim^p d^{kp/2} \int_{\mathbb{C}^d} \Big| \int_{S^{2d-1}} \sum_{j \in I} \lambda_j(z) \zeta_j H_\zeta f(z) \, d\zeta \Big|^p dz \\ &\leqslant d^{kp/2} \int_{\mathbb{C}^d} \Big(\int_{S^{2d-1}} \Big| \sum_{j \in I} \lambda_j(z) \zeta_j \Big|^q d\zeta \Big)^{p/q} \int_{S^{2d-1}} |H_\zeta f(z)|^p \, d\zeta \, dz. \end{aligned} \tag{4-11}$$

Now we deal with the first inner integral in (4-11). Since $\zeta_j \in \mathcal{H}_k^{2d}$ for $j \in I$, for fixed z the function $\zeta \mapsto \sum_{j \in I} \zeta_j \lambda_j(z)$ also belongs to $\mathcal{H}_k^{2d}$. Using [Duoandikoetxea 1987, Theorem 1], orthogonality of the functions $\zeta_j,\ j \in I$, inequality (4-10), and the formula $\sum_{j\in I} \lambda_j(z)^2 = 1$, we get

$$\begin{aligned}\biggl(\int_{S^{2d-1}} \Bigl|\sum_{j\in I} \lambda_j(z)\zeta_j\Bigr|^q \,d\zeta\biggr)^{1/q} &\lesssim q^{k/2}\biggl(\int_{S^{2d-1}} \Bigl|\sum_{j\in I} \lambda_j(z)\zeta_j\Bigr|^2 \,d\zeta\biggr)^{1/2} \\ &= q^{k/2}\biggl(\int_{S^{2d-1}} \sum_{j\in I} \lambda_j(z)^2|\zeta_j|^2 \,d\zeta\biggr)^{1/2} \\ &\lesssim q^{k/2}\biggl(d^{-k}\sum_{j\in I} \lambda_j(z)^2\biggr)^{1/2} \le q^{k/2}d^{-k/2}. \end{aligned} \tag{4-12}$$

Applying (4-12) and coming back to (4-11) we obtain

$$\biggl\|\biggl(\sum_{j\in I} |\widetilde{R}_j f|^2\biggr)^{1/2}\biggr\|_p \lesssim q^{k/2}\biggl(\int_{S^{2d-1}} \|H_\zeta f\|^p_{L^p(\mathbb{C}^d)} \,d\zeta\biggr)^{1/p}.$$

Now Proposition 4.3 completes the proof of (4-9). □

We are now ready to prove (4-8). This is similar to the proof of (4-9) with an addition of Khintchine's inequalities. For $s = 1, 2, \ldots$ we let $\{r_s\}$ be the Rademacher functions; see [Grafakos 2014, Appendix C]. These form an orthonormal set on $L^2([0, 1])$. Moreover we have Khintchine's inequalities [Grafakos 2014, Appendix C.2],

$$\biggl\|\sum_{j=1}^{\infty} a_j r_j\biggr\|_{L^p([0,1])} \lesssim p^{1/2}\biggl(\sum_{j=1}^{\infty} |a_j|^2\biggr)^{1/2} \tag{4-13}$$

and

$$\biggl(\sum_{j=1}^{\infty} |a_j|^2\biggr)^{1/2} \lesssim \biggl\|\sum_{j=1}^{\infty} a_j r_j\biggr\|_{L^p([0,1])}, \tag{4-14}$$

for any complex sequence $(a_s)_{s=1}^{\infty}$ and $1 \leqslant p < \infty$. The explicit bounds on constants in (4-13) and (4-14) follow from explicit values of the optimal constants established by Haagerup [1981] together with Stirling's formula (1-12).

Proof of (4-8). Take numbers $\lambda_{j,s}(z, \{f_s\}) = \lambda_{j,s}(z),\ j \in I,\ s = 1, \ldots, S$, such that

$$\biggl(\sum_{j\in I}\sum_{s=1}^{S} |\widetilde{R}_j f_s(z)|^2\biggr)^{1/2} = \sum_{s=1}^{S}\sum_{j\in I} \lambda_{j,s}(z)\widetilde{R}_j f_s(z), \qquad \sum_{s=1}^{S}\sum_{j\in I} \lambda^2_{j,s}(z) = 1. \tag{4-15}$$

Using (4-15), (4-6), and (4-7) we obtain

$$\begin{aligned}\biggl\|\biggl(\sum_{j\in I}\sum_{s=1}^{S} |\widetilde{R}_j f_s|^2\biggr)^{1/2}\biggr\|_p^p &= \int_{\mathbb{C}^d} \Bigl|\sum_{s=1}^{S}\sum_{j\in I} \lambda_{j,s}(z)\widetilde{R}_j f_s(z)\Bigr|^p \,dz \\ &\lesssim^p d^{kp/2}\int_{\mathbb{C}^d} \Bigl|\int_{S^{2d-1}} \sum_{s=1}^{S}\sum_{j\in I} \lambda_{j,s}(z)\zeta_j H_\zeta f_s(z)\,d\zeta\Bigr|^p \,dz. \end{aligned} \tag{4-16}$$

Orthogonality of the Rademacher functions $\{r_s\}$ and Hölder's inequality imply

$$\begin{aligned}
&d^{kp/2}\int_{\mathbb{C}^d}\left|\int_{S^{2d-1}}\sum_{s=1}^{S}\sum_{j\in I}\lambda_{j,s}(z)\zeta_j H_\zeta f_s(z)\,d\zeta\right|^p dz\\
&= d^{kp/2}\int_{\mathbb{C}^d}\left|\int_{S^{2d-1}}\int_0^1\Big(\sum_{s=1}^{S}\sum_{j\in I}r_s(\xi)\lambda_{j,s}(z)\zeta_j\Big)\Big(\sum_{s=1}^{S}r_s(\xi)H_\zeta f_s(z)\Big)\,d\xi\,d\zeta\right|^p dz\\
&\le d^{kp/2}\int_{\mathbb{C}^d}\left(\int_{S^{2d-1}}\int_0^1\left|\sum_{s=1}^{S}\sum_{j\in I}r_s(\xi)\lambda_{j,s}(z)\zeta_j\right|^q d\xi\,d\zeta\right)^{p/q}\\
&\qquad\qquad\times\int_{S^{2d-1}}\int_0^1\left|\sum_{s=1}^{S}r_s(\xi)H_\zeta f_s(z)\right|^p d\xi\,d\zeta\,dz. \qquad (4\text{-}17)
\end{aligned}$$

Let

$$Q_{S,q}(z):=\left(\int_{S^{2d-1}}\int_0^1\left|\sum_{s=1}^{S}\sum_{j\in I}r_s(\xi)\lambda_{j,s}(z)\zeta_j\right|^q d\xi\,d\zeta\right)^{1/q}.$$

Then, coming back to (4-16) and applying Khintchine's inequality (4-13) to the second factor in the last inequality in (4-17), we reach

$$\left\|\left(\sum_{j\in I}\sum_{s=1}^{S}|\widetilde{R}_j f_s|^2\right)^{1/2}\right\|_p^p \lesssim^p p^{p/2}d^{kp/2}\|Q_{S,q}\|^p_{L^\infty(\mathbb{C}^d)}\int_{S^{2d-1}}\int_{\mathbb{C}^d}\left(\sum_{s=1}^{S}|H_\zeta f_s(z)|^2\right)^{p/2}dz\,d\zeta.$$

Thus, Proposition 4.3 implies

$$\left\|\left(\sum_{j\in I}\sum_{s=1}^{S}|\widetilde{R}_j f_s|^2\right)^{1/2}\right\|_p \lesssim p^* p^{1/2}d^{k/2}\|Q_{S,q}\|_{L^\infty(\mathbb{C}^d)}\left\|\left(\sum_{s=1}^{S}|f_s|^2\right)^{1/2}\right\|_{L^p(\mathbb{C}^d)}.$$

Therefore, the proof of (4-8) will be completed if we justify that

$$\|Q_{S,q}\|_{L^\infty(\mathbb{C}^d)}\lesssim q^{(k+1)/2}d^{-k/2}. \qquad (4\text{-}18)$$

The proof of (4-18) splits into two cases.

If $q\geqslant 2$, we proceed similarly as in (4-12). Namely we apply Khintchine's inequality (4-13), Minkowski's inequality and [Duoandikoetxea 1987, Theorem 1], obtaining

$$\begin{aligned}
(Q_{S,q}(z))^q &\lesssim^q q^{q/2}\int_{S^{2d-1}}\left(\sum_{s=1}^{S}\left|\sum_{j\in I}\lambda_{j,s}(z)\zeta_j\right|^2\right)^{q/2}d\zeta \leqslant q^{q/2}\left(\sum_{s=1}^{S}\left(\int_{S^{2d-1}}\left|\sum_{j\in I}\lambda_{j,s}(z)\zeta_j\right|^q d\zeta\right)^{2/q}\right)^{q/2}\\
&\lesssim^q q^{q/2}q^{kq/2}\left(\sum_{s=1}^{S}\int_{S^{2d-1}}\left|\sum_{j\in I}\lambda_{j,s}(z)\zeta_j\right|^2 d\zeta\right)^{q/2},
\end{aligned}$$

uniformly in $z\in\mathbb{C}^d$. Here an application of [Duoandikoetxea 1987, Theorem 1] is justified since $\zeta_j\in\mathcal{H}_k^{2d}$ for $j\in I$ and thus also the sum $\sum_{j\in I}\lambda_{j,s}(z)\zeta_j$ belongs to $\mathcal{H}_k^{2d}$ for each fixed $z\in\mathbb{C}^d$. Now, using the orthogonality of ζ_j, $j\in I$, inequality (4-10) and the formula $\sum_{s=1}^S\sum_{j\in I}\lambda_{j,s}^2(z)=1$ we see that

$$(Q_{S,q}(z))^q\lesssim^q q^{q/2}q^{kq/2}\biggl(\sum_{s=1}^S\int_{S^{2d-1}}\sum_{j\in I}\lambda_{j,s}(z)^2|\zeta_j|^2\,d\zeta\biggr)^{q/2}=q^{q/2}q^{kq/2}\biggl(d^{-k}\sum_{s=1}^S\sum_{j\in I}\lambda_{j,s}(z)^2\biggr)^{q/2}$$
$$\lesssim q^{q/2}q^{kq/2}d^{-kq/2}.$$

Therefore, (4-18) is justified in the case $q\geqslant 2$.

If on the other hand $1<q<2$, an application of Hölder's inequality together with (4-18) in the case $q=2$ shows that

$$Q_{S,q}(z)\le Q_{S,2}(z)\lesssim d^{-k/2}.$$

This completes the proof of (4-18) and thus also the proof of (4-8) from Proposition 4.4. □

We are now ready to prove Theorems 4.1 and 4.2. In both the proofs we shall need the formula

$$\widetilde{R}^t f(z)=\frac{\Gamma\bigl(d+\tfrac12 k\bigr)}{\pi\Gamma(d)\Gamma\bigl(\tfrac12 k\bigr)}\int_{S^{2d-1}}H_\zeta^t\Bigl[\sum_{j\in I}\zeta_j\widetilde{R}_j f\Bigr](z)\,d\zeta, \tag{4-19}$$

which follows from (4-3) and (4-5). We start with the proof of Theorem 4.2.

Proof of Theorem 4.2. Using (4-19) and (4-7) we see that

$$|\widetilde{R}^* f(z)|\lesssim d^{k/2}\int_{S^{2d-1}}H_\zeta^*\Bigl[\sum_{j\in I}\zeta_j\widetilde{R}_j f\Bigr](z)\,d\zeta,\qquad z\in\mathbb{C}^d.$$

Hence, Minkowski's integral inequality followed by Proposition 4.3 shows that

$$\|\widetilde{R}^* f\|_{L^p(\mathbb{C}^d)}\lesssim p^* d^{k/2}\int_{S^{2d-1}}\Bigl\|\sum_{j\in I}\zeta_j\widetilde{R}_j f\Bigr\|_{L^p(\mathbb{C}^d)}\,d\zeta.$$

Using Hölder's inequality and Fubini's theorem we obtain

$$\|\widetilde{R}^* f\|_{L^p(\mathbb{C}^d)}\lesssim p^* d^{k/2}\biggl(\int_{\mathbb{C}^d}\int_{S^{2d-1}}\Bigl|\sum_{j\in I}\zeta_j\widetilde{R}_j f(z)\Bigr|^p\,d\zeta\,dz\biggr)^{1/p}. \tag{4-20}$$

Since for fixed z the function $\zeta\mapsto\sum_{j\in I}\zeta_j\widetilde{R}_j f(z)$ belongs to $\mathcal{H}_k^{2d}$, applying [Duoandikoetxea 1987, Theorem 1] we obtain

$$\biggl(\int_{S^{2d-1}}\Bigl|\sum_{j\in I}\zeta_j\widetilde{R}_j f(z)\Bigr|^p\,d\zeta\biggr)^{1/p}\lesssim p^{k/2}\biggl(\int_{S^{2d-1}}\Bigl|\sum_{j\in I}\zeta_j\widetilde{R}_j f(z)\Bigr|^2\,d\zeta\biggr)^{1/2}.$$

Using orthogonality and (4-10) we thus see that

$$\biggl(\int_{S^{2d-1}}\Bigl|\sum_{j\in I}\zeta_j\widetilde{R}_j f(z)\Bigr|^p\,d\zeta\biggr)^{1/p}\lesssim d^{-k/2}\,p^{k/2}\biggl(\sum_{j\in I}|\widetilde{R}_j f(z)|^2\biggr)^{1/2}, \tag{4-21}$$

which, together with (4-20), leads to

$$\|\widetilde{R}^* f\|_{L^p(\mathbb{C}^d)} \lesssim p^* p^{k/2} \Big\| \Big(\sum_{j\in I} |\widetilde{R}_j f|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{C}^d)}.$$

Thus, (4-9) from Proposition 4.4 completes the proof of Theorem 4.2. □

We finish this section with the proof of Theorem 4.1.

Proof of Theorem 4.1. Using (4-19), (4-7), and Minkowski's integral inequality on the space

$$\ell^2(\{1,\dots,S\}; L^\infty(\mathbb{Q}_+))$$

we see that

$$\Big(\sum_{s=1}^S |\widetilde{R}^* f_s(z)|^2 \Big)^{1/2} \lesssim d^{k/2} \int_{S^{2d-1}} \Big(\sum_{s=1}^S \Big(H_\zeta^* \Big[\sum_{j\in I} \zeta_j \widetilde{R}_j f_s \Big](z) \Big)^2 \Big)^{1/2} d\zeta, \quad z \in \mathbb{C}^d.$$

Thus, another application of Minkowski's integral inequality followed by Proposition 4.3 gives

$$\Big\| \Big(\sum_{s=1}^S |\widetilde{R}^* f_s|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{C}^d)} \lesssim p^* d^{k/2} \int_{S^{2d-1}} \Big\| \Big(\sum_{s=1}^S \Big| \sum_{j\in I} \zeta_j \widetilde{R}_j f_s \Big|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{C}^d)} d\zeta.$$

Using Khintchine's inequality (4-14) followed by Hölder's inequality on S^{2d-1} we see that

$$\begin{aligned} \Big\| \Big(\sum_{s=1}^S |\widetilde{R}^* f_s|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{C}^d)} &\lesssim p^* d^{k/2} \int_{S^{2d-1}} \Big(\int_{\mathbb{C}^d} \int_0^1 \Big| \sum_{s=1}^S r_s(\xi) \sum_{j\in I} \zeta_j \widetilde{R}_j f_s(z) \Big|^p d\xi\, dz \Big)^{1/p} d\zeta \\ &\lesssim p^* d^{k/2} \Big(\int_{\mathbb{C}^d} \int_0^1 \int_{S^{2d-1}} \Big| \sum_{j\in I} \zeta_j \widetilde{R}_j \Big[\sum_{s=1}^S r_s(\xi) f_s(z) \Big] \Big|^p d\zeta\, d\xi\, dz \Big)^{1/p}. \end{aligned}$$

Finally, (4-21) followed by (4-9) from Proposition 4.4 and Khintchine's inequality (4-13) gives

$$\begin{aligned} \Big\| \Big(\sum_{s=1}^S |\widetilde{R}^* f_s|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{C}^d)} &\lesssim p^* p^{k/2} \Big(\int_{\mathbb{C}^d} \int_0^1 \Big(\sum_{j\in I} \Big| \widetilde{R}_j \Big[\sum_{s=1}^S r_s(\xi) f_s(z) \Big] \Big|^2 \Big)^{p/2} d\xi\, dz \Big)^{1/p} \\ &\lesssim (p^*)^{2+k/2} \Big(\int_{\mathbb{C}^d} \int_0^1 \Big| \sum_{s=1}^S r_s(\xi) f_s(z) \Big|^p d\xi\, dz \Big)^{1/p} \\ &\lesssim (p^*)^{5/2+k/2} \Big(\int_{\mathbb{C}^d} \Big(\sum_{s=1}^S |f_s|^2 \Big)^{p/2} dz \Big)^{1/p}. \end{aligned}$$

□

5. Restriction to the initial Riesz transforms

The purpose of this section is twofold. Firstly, we restrict the maximal operator $\widetilde{R}^*$ acting on $L^p(\mathbb{C}^d)$ to a maximal operator $\mathcal{R}^*$ acting on $L^p(\mathbb{R}^d)$. This is done in a way which preserves estimates for the norms.

However, the restricted maximal operator $\mathcal{R}^*$ is not the same as R^*. Therefore, we need to estimate their difference, which is done in the second part of Section 5.

5.1. ***Bounding the restriction $\mathcal{R}^*$ of $\widetilde{R}^*$.*** In Theorems 4.1 and 4.2, we proved dimension-free estimates for the operator $\widetilde{R}^*$ acting on $L^p(\mathbb{C}^d)$. An approach similar to [Iwaniec and Martin 1996, Chapter 4] leads to dimension-free estimates for the restriction of this operator to $L^p(\mathbb{R}^d)$ which we now describe.

To elaborate, for $x \in \mathbb{R}^d$ and $t > 0$ we define the restricted kernel $\mathcal{K}_j^t(x)$ by

$$\mathcal{K}_j^t(x) = \begin{cases} \widetilde{\gamma}_k S_{d-1} \dfrac{x_j}{|x|^{d+k}} \displaystyle\int_{\sqrt{t^2/|x|^2-1}}^{\infty} \frac{r^{d-1}}{(1+r^2)^{d+k/2}}\, dr & \text{for } |x| < t, \\ K_j^t(x) & \text{for } |x| \geq t. \end{cases} \tag{5-1}$$

Recall that K_j^t is the truncation of the kernel K_j given by (1-1) when $P_j(x) = x_{j_1} \cdots x_{j_k}$, $j \in I$. A short computation based on (1-10), (1-11), and (1-14) gives, for $x \neq 0$,

$$\begin{aligned} \lim_{t\to 0^+} \widetilde{\gamma}_k S_{d-1} \frac{x_j}{|x|^{d+k}} \int_{\sqrt{t^2/|x|^2-1}}^{\infty} \frac{r^{d-1}}{(1+r^2)^{d+k/2}}\, dr &= \frac{\Gamma\left(d+\frac{1}{2}k\right)}{\pi^{d/2}\Gamma\left(\frac{1}{2}k\right)\Gamma\left(\frac{1}{2}d\right)} \int_0^{\infty} \frac{2r^{d-1}}{(1+r^2)^{d+k/2}}\, dr \cdot \frac{x_j}{|x|^{d+k}} \\ &= \gamma_k \frac{x_j}{|x|^{d+k}} = K_j(x). \end{aligned} \tag{5-2}$$

For $f \in L^p(\mathbb{R}^d)$ we let $\mathcal{R}_j^t f = f * \mathcal{K}_j^t$ and define

$$\mathcal{R}^t f = \sum_{j\in I} \mathcal{R}_j^t R_j f \quad \text{and} \quad \mathcal{R}^* f = \sup_{t\in\mathbb{Q}_+} |\mathcal{R}^t f|,$$

where R_j is as in (1-2) with $P(x) = P_j(x) = x_{j_1} \cdots x_{j_k}$.

A transference argument leads to the two results below. The proofs of Theorems 5.1 and 5.2 are based on ideas from [Iwaniec and Martin 1996, Section 4], but extra difficulties arise. These complications stem from the fact that we need to restrict compositions of singular integral operators instead of just one singular integral operator. Furthermore, useful formulas for the multiplier symbols of $\widetilde{R}_j^t$ or $\mathcal{R}_j^t$ are not available.

Theorem 5.1. *Fix $k \in \mathbb{N}$. For each $p \in (1, \infty)$ there is a constant $A(p, k)$ independent of the dimension d and such that for any $S \in \mathbb{N}$ we have*

$$\left\| \left(\sum_{s=1}^{S} |\mathcal{R}^* f_s|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)} \leqslant A(p, k) \left\| \left(\sum_{s=1}^{S} |f_s|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)}$$

whenever $f_1, \ldots, f_S \in L^p(\mathbb{R}^d)$. Moreover, $A(p, k)$ satisfies $A(p, k) \lesssim_k (p^*)^{5/2+k/2}$.

Theorem 5.2. *Fix $k \in \mathbb{N}$. For each $p \in (1, \infty)$ there is a constant $B(p, k)$ independent of the dimension d and such that*

$$\|\mathcal{R}^* f\|_{L^p(\mathbb{R}^d)} \leqslant B(p, k) \|f\|_{L^p(\mathbb{R}^d)}$$

whenever $f \in L^p(\mathbb{R}^d)$. Moreover, $B(p, k)$ satisfies $B(p, k) \lesssim_k (p^*)^{2+k/2}$.

The restriction procedure from Theorems 4.1 and 4.2 to Theorems 5.1 and 5.2 will result in the kernels $\widetilde{K}_j$ and $\widetilde{K}_j^t$ defined in (4-1) being integrated over their imaginary component iy in $\mathbb{R}^d$. This is the origin of the kernel $\mathcal{K}_j^t$ as the next lemma justifies.

Lemma 5.3. *For each $t>0$ and $x\in\mathbb{R}^d$,*

$$\int_{\mathbb{R}^d} \widetilde{K}_j^t(x+iy)\,dy = \mathcal{K}_j^t(x). \tag{5-3}$$

Proof. To justify (5-3) consider two cases: $|x|\ge t$ and $|x|<t$. In the first case, integrating in polar coordinates in $\mathbb{R}^d$ and noting that $\int_{S^{d-1}} P_j(x+ir\omega)\,d\omega = P_j(x)$,

$$\begin{aligned}\int_{\mathbb{R}^d} \widetilde{K}_j^t(x+iy)\,dy &= \int_{y\in\mathbb{R}^d:\,|x+iy|\ge t} \widetilde{\gamma}_k \frac{P_j(x+iy)}{|x+iy|^{2d+k}}\,dy = \int_{\mathbb{R}^d} \widetilde{\gamma}_k \frac{P_j(x+iy)}{|x+iy|^{2d+k}}\,dy\\ &= \widetilde{\gamma}_k S_{d-1} P_j(x) \int_0^\infty \frac{r^{d-1}}{(|x|^2+r^2)^{d+k/2}}\,dr = \widetilde{\gamma}_k S_{d-1} \frac{P_j(x)}{|x|^{d+k}} \int_0^\infty \frac{r^{d-1}}{(1+r^2)^{d+k/2}}\,dr\\ &= K_j(x) = K_j^t(x) = \mathcal{K}_j^t(x).\end{aligned}$$

In the fourth equality above we used the change of variable $r\to r|x|$ and then we used (5-2). Similarly, in the second case $|x|<t$ we obtain

$$\int_{y\in\mathbb{R}^d:\,|x+iy|\ge t} \widetilde{\gamma}_k \frac{P_j(x+iy)}{|x+iy|^{2d+k}}\,dy = \widetilde{\gamma}_k S_{d-1} P_j(x) \int_{\sqrt{t^2-|x|^2}}^\infty \frac{r^{d-1}}{(|x|^2+r^2)^{d+k/2}}\,dr = \mathcal{K}_j^t(x),$$

where in the second equality we used the change of variable $r\to r|x|$. Thus (5-3) is justified. ☐

We present only the proof of Theorem 5.1. The proof of Theorem 5.2 is similar. We merely need a simpler duality argument instead of (5-4) below and an application of Theorem 4.2 instead of Theorem 4.1.

Proof of Theorem 5.1. By Lebesgue's monotone convergence theorem we may restrict the supremum in the definition of $\mathcal{R}^*$ to a finite set of positive numbers $\{t_1,\dots,t_N\}$, as long as our final estimate is independent of N. Further, a density argument shows that it suffices to consider $f_1,\dots,f_S\in\mathcal{S}(\mathbb{R}^d)$.

For $F:\mathbb{C}^d\to\mathbb{C}$ and $u>0$ we let $(\delta_u F)(x+iy)=F(x+iuy)$ and define

$$\widetilde{R}^{t,u}(F)(x+iy) := (\delta_{u^{-1}}\circ \widetilde{R}^t\circ\delta_u)(F)(x+iy) = \widetilde{R}^t(\delta_u F)(x+iu^{-1}y).$$

Using Theorem 4.1 it is straightforward to see that

$$\bigg\| \bigg(\sum_{s=1}^S \sup_{n\in\{1,\dots,N\}} |\widetilde{R}^{t_n,u} F_s|^2 \bigg)^{1/2} \bigg\|_{L^p(\mathbb{C}^d)} \leqslant A(p,k) \bigg\| \bigg(\sum_{s=1}^S |F_s|^2 \bigg)^{1/2} \bigg\|_{L^p(\mathbb{C}^d)}.$$

Note that by duality between the spaces $L^p(\mathbb{C}^d;E_\infty)$ and $L^q(\mathbb{C}^d;E_1)$, where

$$E_\infty = \ell^2\big(\{1,\dots,S\};\ell^\infty(\{t_1,\dots,t_N\})\big) \quad\text{and}\quad E_1=\ell^2\big(\{1,\dots,S\};\ell^1(\{t_1,\dots,t_N\})\big),$$

the above inequality can be rewritten in the equivalent form

$$\bigg| \sum_{s=1}^S \sum_{n=1}^N \langle \widetilde{R}^{t_n,u} F_s, G_{n,s}\rangle_{L^2(\mathbb{C}^d)} \bigg| \leqslant A(p,k) \bigg\| \bigg(\sum_{s=1}^S |F_s|^2 \bigg)^{1/2} \bigg\|_{L^p(\mathbb{C}^d)} \bigg\| \bigg(\sum_{s=1}^S \bigg(\sum_{n=1}^N |G_{n,s}| \bigg)^2 \bigg)^{1/2} \bigg\|_{L^q(\mathbb{C}^d)}, \tag{5-4}$$

where $G_{n,s}\in L^q(\mathbb{C}^d,E_1)$.

Let $\eta \in \mathcal{S}(\mathbb{R}^d)$ be a fixed function such that $\|\eta\|_{L^p(\mathbb{R}^d)} = 1$ and take $f \in \mathcal{S}(\mathbb{R}^d)$. Defining

$$F(x+iy) := (f \otimes \eta)(x, y) = f(x) \cdot \eta(y), \quad x, y \in \mathbb{R}^d,$$

we claim that

$$\lim_{u \to 0^+} \langle \widetilde{R}^{t,u} F, G \rangle_{L^2(\mathbb{C}^d)} = \langle \mathcal{R}^t(f) \otimes \eta, G \rangle_{L^2(\mathbb{C}^d)} \tag{5-5}$$

for any function $G \in \mathcal{S}(\mathbb{C}^d)$ and all $t > 0$.

Assume for a moment that the claim holds. Fix $\varepsilon \in (0, 1)$ and let $\psi \in \mathcal{S}(\mathbb{R}^d)$ be a function of $L^q(\mathbb{R}^d)$ norm 1 and such that $|\langle \eta, \psi \rangle_{L^2(\mathbb{R}^d)}| \geqslant (1 - \varepsilon)$. Take $f_s \in \mathcal{S}(\mathbb{R}^d)$ and $g_{n,s} \in \mathcal{S}(\mathbb{R}^d)$ for $n = 1, \dots, N$, $s = 1, \dots, S$. Then, substituting $F_s = f_s \otimes \eta$ and $G_{n,s} = g_{n,s} \otimes \psi$ in (5-4) we have

$$\begin{aligned}\Bigl|\sum_{s=1}^{S}\sum_{n=1}^{N} \langle \widetilde{R}^{t_n,u}(f_s \otimes \eta), g_{n,s} \otimes \psi \rangle_{L^2(\mathbb{C}^d)}\Bigr| \\ \leqslant A(p,k) \Bigl\| \Bigl(\sum_{s=1}^{S} |f_s \otimes \eta|^2 \Bigr)^{1/2} \Bigr\|_{L^p(\mathbb{C}^d)} \Bigl\| \Bigl(\sum_{s=1}^{S} \Bigl(\sum_{n=1}^{N} |g_{n,s} \otimes \psi| \Bigr)^2 \Bigr)^{1/2} \Bigr\|_{L^q(\mathbb{C}^d)}.\end{aligned}$$

At this point the claim (5-5) implies

$$\begin{aligned}\Bigl|\sum_{s=1}^{S}\sum_{n=1}^{N} \langle \mathcal{R}^{t_n} f_s, g_{n,s} \rangle_{L^2(\mathbb{R}^d)}\Bigr| |\langle \eta, \psi \rangle_{L^2(\mathbb{R}^d)}| \\ \leqslant A(p,k) \Bigl\| \Bigl(\sum_{s=1}^{S} |f_s|^2 \Bigr)^{1/2} \Bigr\|_{L^p(\mathbb{R}^d)} \Bigl\| \Bigl(\sum_{s=1}^{S} \Bigl(\sum_{n=1}^{N} |g_{n,s}| \Bigr)^2 \Bigr)^{1/2} \Bigr\|_{L^q(\mathbb{R}^d)}.\end{aligned}$$

Now, using duality between the spaces $L^p(\mathbb{R}^d; E_\infty)$ and $L^q(\mathbb{R}^d; E_1)$ together with the density of Schwartz functions in $L^q(\mathbb{R}^d)$ we conclude that

$$(1-\varepsilon) \Bigl\| \Bigl(\sum_{s=1}^{S} \sup_{n \in \{1,\dots,N\}} |\mathcal{R}^{t_n} f_s|^2 \Bigr)^{1/2} \Bigr\|_{L^p(\mathbb{R}^d)} \leqslant A(p,k) \Bigl\| \Bigl(\sum_{s=1}^{S} |f_s|^2 \Bigr)^{1/2} \Bigr\|_{L^p(\mathbb{R}^d)}.$$

Since $\varepsilon \in (0, 1)$ was arbitrary this completes the proof of Theorem 5.1.

It remains to verify the claim (5-5). Since $\widetilde{R}^t = \sum_{j \in I} \widetilde{R}_j^t \widetilde{R}_j$ it is easy to see that

$$\widetilde{R}^{t,u} F = \sum_{j \in I} \widetilde{R}_j^{t,u} \widetilde{R}_j^u F,$$

where, for $F = f \otimes \eta$, we let

$$\widetilde{R}_j^{t,u}(F)(x+iy) = \widetilde{R}_j^t(\delta_u F)(x + iu^{-1}y), \qquad \widetilde{R}_j^u(F)(x+iy) = \widetilde{R}_j(\delta_u F)(x + iu^{-1}y).$$

Thus, it is enough to justify that

$$\lim_{u \to 0^+} \langle \widetilde{R}_j^{t,u} \widetilde{R}_j^u F, G \rangle_{L^2(\mathbb{C}^d)} = \langle (\mathcal{R}_j^t R_j f) \otimes \eta, G \rangle_{L^2(\mathbb{C}^d)} \tag{5-6}$$

for $j \in I$, $t > 0$, and $G \in \mathcal{S}(\mathbb{C}^d)$.

Fix $j \in I$ and $t > 0$ and denote by m^t and m the multiplier symbols on $\mathbb{C}^d$ corresponding to the operators $\widetilde{R}_j^t$ and $\widetilde{R}_j$, respectively. Then $\delta_u(m^t)$ and $\delta_u(m)$ are the multiplier symbols corresponding to the operators $\widetilde{R}_j^{t,u}$ and $\widetilde{R}_j^{u}$, respectively. Thus, identifying $\mathbb{C}^d$ with $\mathbb{R}^{2d}$, taking the Fourier transform on $\mathbb{R}^{2d}$, and using Plancherel's theorem, we see that

$$\langle \widetilde{R}_j^{t,u}\widetilde{R}_j^{u}F, G\rangle_{L^2(\mathbb{C}^d)} = \big\langle \delta_u(m)\delta_u(m^t)\mathcal{F}[F], \mathcal{F}[G]\big\rangle_{L^2(\mathbb{C}^d)}. \tag{5-7}$$

By formula (1-3) (applied on $\mathbb{R}^{2d}$) and definitions (4-1), (4-2) for $P_j(z) := z_j = z_{j_1}\cdots z_{j_k}$, we have

$$\delta_u(m)(\xi,\tau) = (-i)^k\frac{P_j(\xi+iu\tau)}{|\xi+iu\tau|^k}$$

for $\xi,\tau\in\mathbb{R}^d$. Hence, for $\xi\neq 0$ and $\tau\in\mathbb{R}^d$ it holds that

$$\lim_{u\to 0^+} m(\xi,u\tau) = m(\xi,0) = (-i)^k\frac{P_j(\xi)}{|\xi|^k}.$$

Another application of (1-3) (this time on $\mathbb{R}^d$) shows that the function $m_0(\xi) := m(\xi,0)$ is the multiplier symbol of the operator R_j acting on $L^2(\mathbb{R}^d)$.

Since the operators $\widetilde{R}_j^t$ and $\widetilde{R}_j$ are both bounded on $L^2(\mathbb{C}^d)$ the functions $\delta_u(m)$ and $\delta_u(m^t)$ are in $L^\infty(\mathbb{C}^d)$, uniformly in $u>0$. Thus, coming back to (5-7) and using Lebesgue's dominated convergence theorem we see that

$$\lim_{u\to 0^+}\langle \widetilde{R}_j^{t,u}\widetilde{R}_j^{u}F, G\rangle_{L^2(\mathbb{C}^d)} = \lim_{u\to 0^+}\big\langle \delta_u(m^t)\mathcal{F}[F], \bar{m}_0\mathcal{F}[G]\big\rangle_{L^2(\mathbb{C}^d)},$$

provided the limit on the right-hand side exists. By definition of m_0 applying again Plancherel's theorem we obtain

$$\lim_{u\to 0^+}\langle \widetilde{R}_j^{t,u}\widetilde{R}_j^{u}F, G\rangle_{L^2(\mathbb{C}^d)} = \lim_{u\to 0^+}\langle \widetilde{R}_j^{t,u}F, (R_j\otimes I)^*G\rangle_{L^2(\mathbb{C}^d)}, \tag{5-8}$$

provided the limit on the right-hand side exists. In the above formula $R_j\otimes I$ denotes the operator R_j acting only on the $\mathbb{R}^d$ coordinates of a function defined on $\mathbb{C}^d$ and the adjoint is taken with respect to the inner product on $L^2(\mathbb{C}^d)$. Now, if we justify that

$$\lim_{u\to 0^+}\big\langle \widetilde{R}_j^{t,u}F, (R_j\otimes I)^*G\big\rangle_{L^2(\mathbb{C}^d)} = \big\langle \mathcal{R}_j^t(f)\otimes\eta, (R_j\otimes I)^*G\big\rangle_{L^2(\mathbb{C}^d)} \tag{5-9}$$

and use the formula

$$\big\langle \mathcal{R}_j^t(f)\otimes\eta, (R_j\otimes I)^*G\big\rangle_{L^2(\mathbb{C}^d)} = \langle (\mathcal{R}_j^t R_j f)\otimes\eta, G\rangle_{L^2(\mathbb{C}^d)}$$

together with (5-8), then we will complete the proof of the claim (5-6).

Since the operators $\widetilde{R}_j^{t,u}$ are uniformly bounded on $L^2(\mathbb{C}^d)$ with respect to $u>0$ to prove (5-9) it suffices to show that

$$\lim_{u\to 0^+}\langle \widetilde{R}_j^{t,u}F, \widetilde{G}\rangle_{L^2(\mathbb{C}^d)} = \langle \mathcal{R}_j^t(f)\otimes\eta, \widetilde{G}\rangle_{L^2(\mathbb{C}^d)}, \tag{5-10}$$

where $\widetilde{G}\in\mathcal{S}(\mathbb{C}^d)$. For $z = x+iy,\ x,y\in\mathbb{R}^d$, we have

$$\begin{aligned}
\widetilde{R}_j^{t,u}(F)(z) &= \widetilde{R}_j^{t,u}(f\otimes\eta)(z) = u^{-d}\delta_{u^{-1}}(\widetilde{K}_j^t)*(f\otimes\eta)(z)\\
&= \int_{\mathbb{R}^d} f(x-x')\int_{y'\in\mathbb{R}^d:\,|x'+iu^{-1}y'|\geq t}\widetilde{\gamma}_k u^{-d}\frac{P_j(x'+iu^{-1}y')}{|x'+iu^{-1}y'|^{2d+k}}\eta(y-y')\,dy'\,dx'\\
&= \int_{\mathbb{R}^d}\int_{y'\in\mathbb{R}^d:\,|x'+iy'|\geq t} f(x-x')\widetilde{\gamma}_k\frac{P_j(x'+iy')}{|x'+iy'|^{2d+k}}\eta(y-uy')\,dy'\,dx'.
\end{aligned} \tag{5-11}$$

Moreover, a computation shows that for fixed $x \in \mathbb{R}^d$ and $t > 0$ it holds that

$$f(x-x')\widetilde{\gamma}_k \frac{P_j(x'+iy')}{|x'+iy'|^{2d+k}} \mathbb{1}_{|x'+iy'|\geq t} \in L^1(\mathbb{C}^d) \tag{5-12}$$

uniformly in $x \in \mathbb{R}^d$. Hence, taking the limit as $u \to 0^+$ in (5-11) and using Lebesgue's dominated convergence theorem followed by Lemma 5.3 we obtain

$$\begin{aligned}\lim_{u\to 0^+} \widetilde{R}_j^{t,u}(F)(z) &= \eta(y)\int_{\mathbb{R}^d} f(x-x') \int_{y'\in\mathbb{R}^d\,:\,|x'+iy'|\geq t} \widetilde{\gamma}_k \frac{P_j(x'+iy')}{|x'+iy'|^{2d+k}}\, dy'\, dx' \\ &= \eta(y)\int_{\mathbb{R}^d} f(x-x')\mathcal{K}_j^t(x')\,dx' = \eta(y)\mathcal{R}_j^t f(x) = (\mathcal{R}_j^t(f)\otimes\eta)(x,y) \end{aligned}\tag{5-13}$$

for $x, y \in \mathbb{R}^d$. Moreover, another application of (5-12) shows that $\widetilde{R}_j^{t,u}(F) \in L^\infty(\mathbb{C}^d)$, uniformly in $u > 0$. Now, since $\widetilde{G} \in \mathcal{S}(\mathbb{C}^d)$ using again Lebesgue's dominated convergence theorem followed by (5-13) we reach

$$\lim_{u\to 0^+} \langle \widetilde{R}_j^{t,u} F, \widetilde{G}\rangle_{L^2(\mathbb{C}^d)} = \big\langle \lim_{u\to 0^+} \widetilde{R}_j^{t,u} F, \widetilde{G}\big\rangle_{L^2(\mathbb{C}^d)} = \langle \mathcal{R}_j^t(f)\otimes\eta, \widetilde{G}\rangle_{L^2(\mathbb{C}^d)},$$

This justifies (5-10), and hence also the claim (5-6). The proof of Theorem 5.1 is thus complete. □

5.2. ***Bounding the difference between R^t and $\mathcal{R}^t$.*** Define the difference kernels on $\mathbb{R}^d$ by

$$E_j^t(x) := K_j^t(x) - \mathcal{K}_j^t(x). \tag{5-14}$$

Recall that by definitions (1-1) of K_j^t and (5-1) of $\mathcal{K}_j^t$ we have $E_j^t(x) = -\mathcal{K}_j^t(x)$ if $|x| < t$ and $E_j^t(x) = 0$ if $|x| \geq t$. We let D_j be the operator on $L^p(\mathbb{R})$ given by $D_j^t f = f * E_j^t$ and define

$$D^t f = \sum_{j\in I} D_j^t R_j f, \qquad D^* f = \sup_{t\in\mathbb{Q}_+} |D^t f|.$$

Clearly,

$$R^t = \mathcal{R}^t + D^t,$$

so using Theorems 5.1 and 5.2 we reduce Theorems 3.2 and 3.3 to the following two statements.

Theorem 5.4. *Fix $k \in \mathbb{N}$. For each $p \in (1,\infty)$ there is a constant $A(p,k)$ independent of the dimension d and such that for any $S \in \mathbb{N}$ we have*

$$\Big\| \Big(\sum_{s=1}^{S} |D^* f_s|^2\Big)^{1/2} \Big\|_{L^p(\mathbb{R}^d)} \leqslant A(p,k) \Big\| \Big(\sum_{s=1}^{S} |f_s|^2\Big)^{1/2} \Big\|_{L^p(\mathbb{R}^d)}$$

whenever $f_1, \dots, f_S \in L^p(\mathbb{R}^d)$. Moreover, $A(p,k)$ satisfies $A(p,k) \lesssim_k (p^*)^{5/2+k/2}$.

Theorem 5.5. *Fix $k \in \mathbb{N}$. For each $p \in (1,\infty)$ there is a constant $B(p,k)$ independent of the dimension d and such that*

$$\|D^* f\|_{L^p(\mathbb{R}^d)} \leqslant B(p,k)\|f\|_{L^p(\mathbb{R}^d)}$$

whenever $f \in L^p(\mathbb{R}^d)$. Moreover, $B(p,k)$ satisfies $B(p,k) \lesssim_k (p^*)^{2+k/2}$.

The proofs of the above two theorems will follow the scheme of the proofs of Theorems 4.1 and 4.2. The main difference lies in the application of the real method of rotations. The reason for taking complex extensions (4-3) of the operators R^t is due to the fact that the real method of rotations is only applicable to singular integrals with odd kernels. Using this method for k odd one may express R^t as an integral of directional truncated Hilbert transforms. In the case of the operator D^t the cancellations are not important. We can use the real method of rotations to estimate D^* by an integral of one-dimensional directional Hardy–Littlewood maximal functions.

For $t > 0$ we let I^t be the function on $(0, \infty)$ given by

$$I^t(r) = \mathbb{1}_{(0,t)}(r) \int_{\sqrt{t^2/r^2-1}}^{\infty} \frac{s^{d-1}}{(1+s^2)^{d+k/2}}\, ds, \qquad r > 0. \tag{5-15}$$

Using the definitions (5-1) and (5-14) and integrating in polar coordinates in $\mathbb{R}^d$ we obtain

$$\begin{aligned} -D_j^t f(x) &= \int_{\mathbb{R}^d} \widetilde{\gamma}_k S_{d-1} \frac{y_j}{|y|^{d+k}} I^t(|y|) f(x-y)\, dy \\ &= \widetilde{\gamma}_k S_{d-1}^2 \int_0^t \int_{S^{d-1}} \frac{\omega_j}{r} I^t(r) f(x - r\omega)\, d\omega\, dr \\ &= \gamma_k S_{d-1} \int_{S^{d-1}} \omega_j \mathcal{H}_\omega^t f(x)\, d\omega = \frac{2\Gamma\big(\frac{1}{2}(k+d)\big)}{\Gamma\big(\frac{1}{2}k\big)\Gamma\big(\frac{1}{2}d\big)} \int_{S^{d-1}} \omega_j \mathcal{H}_\omega^t f(x)\, d\omega, \end{aligned} \tag{5-16}$$

where

$$\mathcal{H}_\omega^t f(x) = \frac{\widetilde{\gamma}_k}{\gamma_k} S_{d-1} \int_0^t I^t(r) \frac{f(x-r\omega)}{r}\, dr. \tag{5-17}$$

Let now $\mathcal{H}_\omega^* f(x) = \sup_{t \in \mathbb{Q}_+} |\mathcal{H}_\omega^t f(x)|$. The next proposition serves as a replacement for Proposition 4.3.

Proposition 5.6. *For each $1 < p < \infty$ we have*

$$\bigg\| \bigg(\sum_{s=1}^S |\mathcal{H}_\omega^* f_s|^2 \bigg)^{1/2} \bigg\|_{L^p(\mathbb{R}^d)} \lesssim p^* \bigg\| \bigg(\sum_{s=1}^S |f_s|^2 \bigg)^{1/2} \bigg\|_{L^p(\mathbb{R}^d)} \tag{5-18}$$

uniformly in $\omega \in S^{d-1}$ and the dimension d.

Proof. For $\omega \in S^{d-1}$ and $t > 0$ we let

$$\mathcal{M}_\omega^t f(x) = \frac{1}{t} \int_{-t}^t |f(x - r\omega)|\, dr \quad \text{and} \quad \mathcal{M}_\omega^* f(x) = \sup_{t>0} |\mathcal{M}_\omega^t f(x)|$$

be the directional Hardy–Littlewood averaging operator and the directional Hardy–Littlewood maximal function. Using Fubini's theorem and one-dimensional estimates for the Hardy–Littlewood maximal function, see, e.g., [Grafakos 2014, Theorem 5.6.6], we obtain

$$\bigg\| \bigg(\sum_{s=1}^S |\mathcal{M}_\omega^* f_s|^2 \bigg)^{1/2} \bigg\|_{L^p(\mathbb{R}^d)} \lesssim p^* \bigg\| \bigg(\sum_{s=1}^S |f_s|^2 \bigg)^{1/2} \bigg\|_{L^p(\mathbb{R}^d)},$$

uniformly in $\omega \in S^{d-1}$. Thus, to prove (5-18) it suffices to show the pointwise estimate

$$\mathcal{H}_\omega^t f(x) \lesssim \mathcal{M}_\omega^t f(x)$$

uniformly in $x \in \mathbb{R}^d$, $\omega \in S_{d-1}$, with in-explicit constants independent of the dimension.

This bound will follow if we justify that

$$\frac{\widetilde{\gamma}_k}{\gamma_k} S_{d-1} \frac{I^t(r)}{r} \lesssim \frac{1}{t}, \tag{5-19}$$

with the implicit constant being uniform in $t > 0$, $0 \le r \leqslant t$, and the dimension d. Note that for $s \geqslant (t^2/r^2 - 1)^{1/2}$ we have $\frac{1}{r} \leqslant \sqrt{s^2+1}/t$. Hence, recalling (5-15) and using (1-14), we obtain

$$\begin{aligned}\frac{\widetilde{\gamma}_k}{\gamma_k} S_{d-1} \frac{I^t(r)}{r} &\leqslant \frac{\widetilde{\gamma}_k}{\gamma_k} S_{d-1} \frac{1}{t} \int_{\sqrt{t^2/r^2-1}}^{\infty} \frac{s^{d-1}}{(1+s^2)^{d+(k-1)/2}}\, ds \\ &\leqslant S_{d-1} \frac{\widetilde{\gamma}_k}{\gamma_k} \frac{1}{t} \int_0^{\infty} \frac{s^{d-1}}{(1+s^2)^{d+(k-1)/2}}\, ds = S_{d-1} \frac{\widetilde{\gamma}_k}{\gamma_k} \frac{\Gamma\big(\frac12(d+k-1)\big)\Gamma\big(\frac12 d\big)}{2\Gamma\big(d+\frac12(k-1)\big)} \cdot \frac{1}{t}.\end{aligned}$$

Applying (1-10) and (1-11) we reach

$$\begin{aligned}S_{d-1} \frac{\widetilde{\gamma}_k}{\gamma_k} \frac{I^t(r)}{r} &\le \frac{2\pi^{d/2}}{\Gamma\big(\frac12 d\big)} \frac{\Gamma\big(d+\frac12 k\big)}{\pi^{d/2}\Gamma\big(\frac12(d+k)\big)} \frac{\Gamma\big(\frac12(d+k-1)\big)\Gamma\big(\frac12 d\big)}{2\Gamma\big(d+\frac12(k-1)\big)} \cdot \frac{1}{t} \\ &= \frac{\Gamma\big(d+\frac12 k\big)}{\Gamma\big(d+\frac12(k-1)\big)} \cdot \frac{\Gamma\big(\frac12(d+k-1)\big)}{\Gamma\big(\frac12(d+k)\big)} \cdot \frac{1}{t}.\end{aligned}$$

Since k is fixed, using (1-13) we conclude that

$$S_{d-1} \frac{\widetilde{\gamma}_k}{\gamma_k} \frac{I^t(r)}{r} \lesssim \frac{\big(d+\frac12(k-1)\big)^{1/2}}{\big(\frac12 d+\frac12(k-1)\big)^{1/2}} \cdot \frac{1}{t} \lesssim \frac{1}{t}.$$

Thus, we have completed the proof of (5-19) and hence also the proof of Proposition 5.6. □

We will also need vector-valued estimates for $\{R_j(f_s)\}$, $j \in I$, $s = 1, \dots, d$. The following proposition can be deduced from Proposition 4.4 if we proceed along the lines of [Iwaniec and Martin 1996, Section 4].

Proposition 5.7. *For each $1 < p < \infty$ we have*

$$\bigg\| \bigg(\sum_{s=1}^{S} \sum_{j \in I} |R_j f_s|^2 \bigg)^{1/2} \bigg\|_{L^p(\mathbb{R}^d)} \lesssim p^* p^{1/2} q^{(k+1)/2} \bigg\| \bigg(\sum_{s=1}^{S} |f_s|^2 \bigg)^{1/2} \bigg\|_{L^p(\mathbb{R}^d)}, \tag{5-20}$$

$$\bigg\| \bigg(\sum_{j \in I} |R_j f|^2 \bigg)^{1/2} \bigg\|_{L^p(\mathbb{R}^d)} \lesssim p^* q^{k/2} \|f\|_{L^p(\mathbb{R}^d)}, \tag{5-21}$$

uniformly in the dimension d.

Proof. In contrast to the proofs of Theorems 5.1 and 5.2, here we apply the methods from [Iwaniec and Martin 1996, Section 4] in a direct way. Therefore we shall be brief. Let $n = k = d$ and identify $\mathbb{C}^d$ with $\mathbb{R}^{2d}$.

For the proof of (5-20) we take $E = \ell^2(\{1, \dots, S\})$ and $F = \ell^2(\{1, \dots, S\} \times I)$. The operator $\boldsymbol{T}$ is defined by

$$\boldsymbol{T}(\{f_s\}_{s=1,\dots,S}) = \{\widetilde{R}_j(f_s)\}_{(s,j) \in \{1,\dots,S\} \times I}.$$

Using (1-3) for $P(z) = z_{j_1} \cdots z_{j_k}$ one can check that the restricted operator $\boldsymbol{T}_0$ is then

$$\boldsymbol{T}_0(\{f_s\}_{s=1,\dots,S}) = \{R_j(f_s)\}_{(s,j)\in\{1,\dots,S\}\times I}.$$

Hence, [Iwaniec and Martin 1996, equation (45)] together with (4-8) lead to (5-20).

The proof of (5-21) is similar. We take $E = \mathbb{C}$ and $F = \ell^2(I)$. The operators $\boldsymbol{T}$ and $\boldsymbol{T}_0$ are defined as above. The desired inequality follows from [Iwaniec and Martin 1996, equation (45)] together with (4-9). □

We are finally ready to justify Theorems 5.4 and 5.5. At this point the proofs mimic the corresponding proofs of Theorems 4.1 and 4.2. Therefore we shall be brief and only point out the differences.

Proof of Theorem 5.4. We proceed analogously to the proof of Theorem 4.1 on page 646. In particular, we replace $\mathbb{C}^d$ with $\mathbb{R}^d$, $\widetilde{R}_j^{t_n}$ with $D_j^{t_n}$ and $\widetilde{R}_j$ with R_j. The most important difference is that (5-16) replaces (4-5). This leads to the replacement of (4-19) by

$$D^t f(x) = -\frac{2\Gamma\left(\frac{1}{2}(k+d)\right)}{\Gamma\left(\frac{1}{2}k\right)\Gamma\left(\frac{1}{2}d\right)} \int_{S^{d-1}} \mathcal{H}_\omega^t\Big[\sum_{j\in I} \omega_j R_j f\Big](x)\, d\omega. \tag{5-22}$$

In the proof we also use (5-20) in place of (4-8) and Proposition 5.6 instead of Proposition 4.3. □

Proof of Theorem 5.5. We proceed analogously to the proof of Theorem 4.2 on page 645, making the replacements as in the proof of Theorem 5.4. In particular we use (5-22), (5-21), and Proposition 5.6. □

Appendix

Proof of Proposition 4.3. A (complex) rotational invariance argument reduces the inequality to its one-dimensional case:

$$\Big\|\Big(\sum_{s=1}^S |H^* f_s|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{C})} \lesssim p^* \Big\|\Big(\sum_{s=1}^S |f_s|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{C})}.$$

Here

$$H^* f(z) := \sup_{t\in\mathbb{Q}_+} |H_k^t f(z)|, \quad \text{with } H_k^t f(z) = \int_{\mathbb{C}} \Big(\frac{\lambda}{|\lambda|}\Big)^k \frac{f(z-\lambda)}{|\lambda|^2} \mathbb{1}_{|\lambda|\ge t}(\lambda)\, d\lambda,$$

is the k-th power of the complex Hilbert transform on $\mathbb{C}$.

We split the operator H^* into two parts. To this end let $\varphi : \mathbb{C} \to \mathbb{R}$ be a smooth radial function satisfying $\varphi(z) = 1$ for $|z| < 2$, and $\varphi(z) = 0$ for $|z| > 4$. Define $\varphi_t(z) = \varphi(z/t)$ and let

$$\chi_t(z) = \Big(\frac{z}{|z|}\Big)^k \frac{1}{|z|^2} \mathbb{1}_{|z|\geqslant t}$$

be the kernel of H_k^t. Then

$$H^* f(z) \leqslant \sup_{t>0} |(\varphi_t \chi_t * f)(z)| + \sup_{t>0} \big|((1-\varphi_t)\chi_t * f)(z)\big| =: H_\varphi^* f(z) + H_{1-\varphi}^* f(z) \lesssim \mathcal{M} f(z) + H_{1-\varphi}^* f(z),$$

where $\mathcal{M}$ denotes the Hardy–Littlewood maximal operator on $\mathbb{R}^2$. Since [Grafakos 2014, Theorem 5.6.6] gives us vector-valued estimates for $\mathcal{M}$, we get

$$\Big\|\Big(\sum_{s=1}^{S}|H_{\varphi}^{*}f_s|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{C})}\lesssim p^{*}\Big\|\Big(\sum_{s=1}^{S}|f_s|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{C})}.$$

The remaining ingredient is to prove

$$\Big\|\Big(\sum_{s=1}^{S}|H_{1-\varphi}^{*}f_s|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{C})}\lesssim p^{*}\Big\|\Big(\sum_{s=1}^{S}|f_s|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{C})}. \tag{A-1}$$

We will apply [Grafakos 2014, Theorem 5.6.1] with

$$\mathcal{B}_1=\ell^2(\{1,\dots,S\})\quad\text{and}\quad \mathcal{B}_2=\ell^2(\{1,\dots,S\};L^{\infty}(\mathbb{Q}_+))$$

and

$$\vec{K}(z)(u)=\big((1-\varphi_t)\chi_t(z)\cdot u_1,\dots,(1-\varphi_t)\chi_t(z)\cdot u_S\big)\in\mathcal{B}_2 \tag{A-2}$$

for any sequence $u=(u_s)_{s=1}^{S}\in\mathcal{B}_1$. Then, taking $e_s=(0,\dots,1,\dots,0)$, with 1 on the s-th coordinate, we see that the operator $\vec{T}$ defined in [Grafakos 2014, 5.6.4] satisfies

$$\vec{T}\Big(\sum_{s=1}^{S}f_se_s\Big)(z)=\big(H_{1-\varphi}^{t}f_1(z),\dots,H_{1-\varphi}^{t}f_S(z)\big) \tag{A-3}$$

and

$$\Big\|\vec{T}\Big(\sum_{s=1}^{S}f_se_s\Big)(z)\Big\|_{\mathcal{B}_2}=\Big(\sum_{s=1}^{S}|H_{1-\varphi}^{*}f_s(z)|^2\Big)^{1/2}$$

for any sequence $(f_s)_{s=1}^{S}$ of smooth functions that vanish at infinity. In order to use [Grafakos 2014, Theorem 5.6.1] we need to verify conditions (5.6.1), (5.6.2) and (5.6.3) from [Grafakos 2014] and check that $\vec{T}$ is bounded from $L^2(\mathbb{C},\mathcal{B}_1)$ to $L^2(\mathbb{C},\mathcal{B}_2)$.

Condition (5.6.1) is a straightforward consequence of (A-2). It is also not hard to verify that $\int_{\varepsilon\le|z|\le 1}\vec{K}(z)\,dz=0$, so that condition (5.6.3) is satisfied with $\vec{K}_0=0$.

We shall now justify (5.6.2). Let $\widetilde{\varphi}_t:=1-\varphi_t$ and $g_t=\widetilde{\varphi}_t\chi_t$ so that

$$g_t(z)=\widetilde{\varphi}_t(z)\frac{z^k}{|z|^{k+2}}.$$

Since

$$\|\vec{K}(z-w)-\vec{K}(z)\|_{\mathcal{B}_1\to\mathcal{B}_2}=\sup_{t>0}|g_t(z-w)-g_t(z)|,$$

we have

$$\begin{aligned}\|\vec{K}(z-w)-\vec{K}(z)\|_{\mathcal{B}_1\to\mathcal{B}_2}&\\ &=\sup_{t>0}\Big|\widetilde{\varphi}_t(z-w)\frac{(z-w)^k}{|z-w|^{k+2}}-\widetilde{\varphi}_t(z)\frac{z^k}{|z|^{k+2}}\Big|\\ &\leqslant\sup_{t>0}\Big|(\widetilde{\varphi}_t(z-w)-\widetilde{\varphi}_t(z))\frac{(z-w)^k}{|z-w|^{k+2}}\Big|+\sup_{t>0}\Big|\widetilde{\varphi}_t(z)\Big(\frac{(z-w)^k}{|z-w|^{k+2}}-\frac{z^k}{|z|^{k+2}}\Big)\Big|.\end{aligned} \tag{A-4}$$

Hence, the proof of (5.6.2) boils down to estimating the two terms in (A-4) under the assumption $|z| \geq 2|w|$. We begin with the first term. Since $|z| \geq 2|w|$ we have $|z| \approx |z-w|$. Hence, in order for the expression inside the absolute value to be nonzero, t has to be comparable to $|z|$ and $|z-w|$. In that case, using the smoothness of φ we obtain

$$\left|(\widetilde{\varphi}_t(z-w)-\widetilde{\varphi}_t(z))\frac{(z-w)^k}{|z-w|^{k+2}}\right| \lesssim \frac{|w|}{2t}\frac{1}{|z-w|^2} \approx \frac{|w|}{|z|\,|z-w|^2} \approx \frac{|w|}{|z|^3}.$$

In the second term of (A-4) we omit $\widetilde{\varphi}_t$ and get

$$\begin{aligned}\left|\frac{(z-w)^k}{|z-w|^{k+2}}-\frac{z^k}{|z|^{k+2}}\right| &\leqslant \left|\frac{(z-w)^k}{|z-w|^{k+2}}-\frac{(z-w)^k}{|z|^{k+2}}\right|+\left|\frac{(z-w)^k}{|z|^{k+2}}-\frac{z^k}{|z|^{k+2}}\right| \\ &= |z-w|^k\frac{\left||z|^{k+2}-|z-w|^{k+2}\right|}{|z-w|^{k+2}|z|^{k+2}}+\frac{1}{|z|^{k+2}}\left|(z-w)^k-z^k\right| \approx \frac{|w|}{|z|^3}.\end{aligned}$$

This means that we have proved that

$$\|\vec{K}(z-w)-\vec{K}(z)\|_{\mathcal{B}_1\to\mathcal{B}_2} \lesssim \frac{|w|}{|z|^3}$$

for $|z| \geqslant 2|w|$. Integrating this yields

$$\int_{|z|\geqslant 2|w|}\|\vec{K}(z-w)-\vec{K}(z)\|_{\mathcal{B}_1\to\mathcal{B}_2}\,dz \lesssim |w|\int_{|z|\geqslant 2|w|}\frac{1}{|z|^3}\,dz \approx 1,$$

so that condition (5.6.2) is satisfied.

It remains to justify the boundedness of $\vec{T}$ from $L^2(\mathbb{C},\mathcal{B}_1)$ to $L^2(\mathbb{C},\mathcal{B}_2)$. We have the pointwise bound

$$H^*_{1-\varphi}f(z) \lesssim \mathcal{M}f(z)+H^*f(z).$$

Therefore the desired L^2 boundedness of $\vec{T}$ is a consequence of (A-3) and the $L^2(\mathbb{C})$ boundedness of H^*. This allows us to use [Grafakos 2014, Theorem 5.6.1] and completes the proof of (A-1), and hence also the proof of Proposition 4.3. □

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