ANALYSIS & PDE

Volume 19 No. 4 2026

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ANALYSIS AND PDE
Vol. 19 (2026), No. 4, pp. 659–720
DOI: 10.2140/apde.2026.19.659

A SHARP STABILITY CRITERION FOR EULER EQUATIONS VIA SPARSENESS

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We introduce sparse versions of function spaces that are relevant to characterize the solutions of Euler equations without concentration. The standard Sobolev space H^{-1} is given a sparse structure that allows measuring the degree of compactness of embeddings into H^{-1} and provides new quantitative general criteria for H^{-1}-stability. Indices of sparseness are defined, and function spaces whose indices have prescribed decay are constructed, resulting in an improvement of the classical H^{-1}-stability results: sparse stability. The analysis relies on the introduction of sparse Riesz–Morrey–Tadmor spaces, that are characterized via maximal operators and new sparse domination theorems, together with extrapolation techniques. Our methods also yield improvements on recent results on the conservation of energy of physically realizable solutions of 2D-Euler.

1. Preamble

The classical Euler equations for incompressible fluid flow are given by

$$\begin{cases} u_t + u\cdot\nabla u = -\nabla p, \\ \operatorname{div} u = 0, \\ \text{initial and boundary conditions,} \end{cases} \tag{1-1}$$

where $u = (u_1, \dots, u_n)$ is the *velocity* field and p is the (scalar) *pressure*. Although the Euler equations have been studied for more than two and half centuries, many important open problems remain unanswered.

MSC2020: 42B35, 42B37, 46B70, 76B03.
Keywords: H^{-1}-stability, Euler equations, energy conservation, sparseness, Morrey spaces, approximate solutions, physical solutions, extrapolation spaces.

In particular, while it is easy to see that smooth solutions of (1-1) conserve kinetic energy, the existence of weak solutions that conserve energy or the uniqueness of weak solutions are more subtle issues.

1.1. ***H^{-1}-stability for approximate solutions of Euler equations.*** Research on conservation of energy has been considerably influenced by the work of DiPerna and Majda [1987a; 1987b; 1988]. These authors introduced the key concept of *approximate solutions* $\{u^\varepsilon\}_{\varepsilon>0}$ (see Definition 16) that weakly converge to u. If $u^\varepsilon \to u$ strongly in L^2, then u is a weak solution to (1-1). Otherwise the energy concentrates on sets. Despite this, u may be still an Euler solution due to the presence of subtle cancellations. This is the so-called *concentration-cancellation phenomenon*.

In their foundational papers, Lopes Filho, Nussenzveig Lopes, and Tadmor [Lopes Filho et al. 2000] and Tadmor [2001] develop H^{-1}-stability[1] (see Definition 19) into a very powerful unifying framework to study lack of concentrations in approximate solutions. To be more precise, these authors obtained the following result.

Theorem 1 [Lopes Filho et al. 2000]. *Suppose that $\{u^\varepsilon\}_{\varepsilon>0}$ is an H^{-1}-stable approximate family of Euler solutions. Then $\{u^\varepsilon\}_{\varepsilon>0}$ converges strongly (possibly passing to a subfamily) to a weak solution of the Euler equation u in $L^\infty([0,T]; L^2_{\text{loc}}(\mathbb{R}^n;\mathbb{R}^n))$.*

The implementation of H^{-1}-stability depends on having at one's disposal sharp criteria to characterize the compact sets of H^{-1} and, in particular, leverage this knowledge to decide which function spaces, among those relevant in the description of physical phenomena connected with the Euler equations, embed compactly into H^{-1}. In this direction, the H^{-1}-criteria, as it applies to rearrangement invariant spaces, was extensively developed in [Lopes Filho et al. 2000], recovering and extending earlier results from [DiPerna and Majda 1987a; Lions 1996].

As shown in [DiPerna and Majda 1987a], solutions to 2D Euler equations when the initial vorticity is supported in a curve play a central role in fluid dynamics. These solutions are called *vortex sheets* and their regularity can be naturally measured in terms of the Morrey spaces $M^{p,\alpha}$ (see (2-2) below).

Remark. We use standard notation: If X is a function space on $\mathbb{R}^n$, we let X_c be the subspace of compactly supported functions; and we let X_{loc} be the set of functions f such that $f\mathbf{1}_{Q_0} \in X$, for all cubes Q_0. We write

$$X_c \hookrightarrow H^{-1}_{\text{loc}}(\mathbb{R}^n) \quad \text{or} \quad X_c \xrightarrow{\text{compactly}} H^{-1}_{\text{loc}}(\mathbb{R}^n)$$

if for all Q_0

$$X(Q_0) \hookrightarrow H^{-1}(\mathbb{R}^n) \quad \text{or} \quad X(Q_0) \xrightarrow{\text{compactly}} H^{-1}(\mathbb{R}^n),$$

respectively, where $X(Q_0) = \{f \in X : \operatorname{supp} f \subset Q_0\}$ with $\|f\|_{X(Q_0)} = \|f\mathbf{1}_{Q_0}\|_X$.

The H^{-1}-stability theory for Morrey spaces is also treated in [Lopes Filho et al. 2000], and relies on a compactness result[2] due independently to DeVore and Tao (a proof was given in [Lopes Filho et al. 2000, Theorem 4.2]),

$$M^{p,\alpha}_c(\mathbb{R}^n) \xrightarrow{\text{compactly}} H^{-1}_{\text{loc}}(\mathbb{R}^n), \tag{1-2}$$

[1] More precisely, H^{-1}_{loc}-stability.

[2] The same statement holds, mutatis mutandis, for the Morrey space of measures [Lopes Filho et al. 2000, Theorem 4.3].

provided that one of the following conditions is satisfied:

(a) $$p > \frac{n}{2},$$

(b) $$p = \frac{n}{2} \quad \text{and} \quad \alpha > 1.$$

Once in possession of these statements, Theorem 1 can be applied to establish that for families of approximate solutions, with uniformly bounded vorticities in $M^{p,\alpha}(\mathbb{R}^n)$, one can extract convergent subsequences to a solution of the Euler equation (1-1), without concentration. In the special case $n = 2$, $p = 1$ and $\alpha > 1$, this result[3] was first obtained by DiPerna and Majda [1987a, Theorem 3.1] using tools from elliptic theory. On the other hand, the case $n = 3$ and $p = \frac{3}{2}$ is connected with the work of Giga and Miyakawa [1989] on well-posedness of 3D Navier–Stokes equations with initial singular data such as vortex filaments.

At present time, the picture is not completely understood for all the values of the parameters involved in (1-2). Specifically in 2D, it is known that for $p = 1$ and $\alpha = \frac{1}{2}$, (1-2) does not hold (see [DiPerna and Majda 1987a; Majda 1993]). To the best of our knowledge, it remains an open problem to decide whether (1-2) with $p = 1$ still holds for $\alpha \in \left(\frac{1}{2}, 1\right]$ (the so-called "gap problem"), leaving open the existence of solutions without concentrations in $M^{1,\alpha}$. Similar types of gaps also appear when dealing with higher dimensions.

In an effort to understand the nature of these gaps, and their impact on the convergence of approximate solutions of the Euler equations, Tadmor [2001] introduced the finer scale of RMT spaces, $R_{p,q} \log^{\alpha}$, that sharpen (1-2); see Definition 20.

1.2. ***Tadmor's refinement of H^{-1}-stability.*** It is shown in [Tadmor 2001] that RMT spaces "interpolate the compactness gap" in the sense that

$$R_{p,2} \log^{\alpha}(\mathbb{R}^n)_c \xrightarrow{\text{compactly}} H^{-1}_{\text{loc}}(\mathbb{R}^n) \tag{1-3}$$

provided that one of the following conditions is satisfied:

(a) $$p > \frac{2n}{n+2},$$

(b) $$p = \frac{2n}{n+2} \quad \text{and (crucially)} \quad \alpha > \tfrac{1}{2}.$$

The $R_{p,2} \log^{\alpha}$ scale is sharp, with respect to the H^{-1}- stability, in the sense that for approximate solutions, with vorticities uniformly bounded in $R_{2n/(n+2),2} \log^{\alpha}$, $\alpha > \frac{1}{2}$, we can extract solutions without concentration, while for $\alpha \in \left(0, \frac{1}{2}\right]$ there is a weak limit solution (i.e., a concentration-cancellation effect). On the other hand, the original gap problem for Morrey spaces $M^{1,\alpha}$, is apparently not resolved in this fashion, since[4] (see [Tadmor 2001, page 519 and the discussion after (3.5)])

$$R_{1,2} \log^{\alpha}(\mathbb{R}^2) \subset M^{1,\alpha}(\mathbb{R}^2).$$

[3]The original statement from [DiPerna and Majda 1987a] involves a certain additional assumption on weak decay at infinity of vorticities.

[4]In other words, Tadmor's scale requires a stronger regularity condition than Morrey regularity on the set of vorticities to achieve compactness.

1.3. ***Paving the way to sparseness.*** The presence of logarithms in (1-3) (and (1-2)) is very natural and is connected with some implicit *extrapolation* constructions that are needed since $R_{1,2}(\mathbb{R}^2)$ (or more generally, $R_{2n/(n+2),2}(\mathbb{R}^n)$) is not suitable for H^{-1}-stability. In fact, we have (see (4-9))

$$R_{\frac{2n}{n+2},2}(\mathbb{R}^n) \not\subseteq H^{-1}(\mathbb{R}^n). \tag{1-4}$$

To overcome this obstacle, in this paper we propose a different methodology based on *sparseness*.

1.4. ***A new framework for H^{-1}-stability: Sparse stability.*** The main goal of this paper is to reformulate H^{-1}-stability applying the theory of sparse spaces, that we recently introduced in [Domínguez and Milman 2021]. In a nutshell, we show that the "defect" of RMT spaces exhibited by (1-4) can be overcome if the geometry of testing cubes in the definition of these spaces is changed. More precisely, let $SR_{2n/(n+2),2}$ the space that is obtained by replacing pairwise disjoint cubes in $R_{2n/(n+2),2}$ by sparse[5] families of cubes (see Definition 4 below). Then, somewhat informally, the following surprising formula holds (see Theorem 6 for the precise statement):

$$SR_{\frac{2n}{n+2},2}(\mathbb{R}^n) = H^{-1}(\mathbb{R}^n). \tag{1-5}$$

Armed with formula (1-5) we provide a sparse structure to H^{-1}, which we exploit to develop new methods to characterize compact sets in H^{-1}. In particular, we introduce *indices of sparseness*, associated to function spaces, that measure the degree of compactness into H^{-1}. Conversely, given a *decay* Ψ, i.e., a positive decreasing function on $[0, \infty)$ satisfying

$$\lim_{t\to\infty} \Psi(t) = 0, \tag{1-6}$$

we construct *sparse spaces* S_Ψ, whose sparse indices have the prescribed decay Ψ. This leads to the introduction of the concept of Ψ-*sparse stability* for approximate solutions; see Definition 11. As a consequence, we create a refined scale that exhausts the classical H^{-1}-stability in the following sense (compare with Theorem 1).

Theorem 2. *Let $\{u^\varepsilon\}_{\varepsilon>0}$ be a family of approximate solutions. The following are equivalent*:

(i) $\{u^\varepsilon\}_{\varepsilon>0}$ *is H^{-1}-stable*,

(ii) $\{u^\varepsilon\}_{\varepsilon>0}$ *is sparse stable.*

As a consequence, *if* (ii) *holds then* (*possibly passing to a subfamily*) $u^\varepsilon \to u$ *strongly in* L^2, *where* u *is a solution of* (1-1).

We show that sparse stability not only provides a simplified approach to all previously known existence results from [DiPerna and Majda 1987a], [Lopes Filho et al. 2000] and [Tadmor 2001] but, more importantly, it leads to the sharpening of the classical results.

We next detour to present in detail the construction of sparse spaces (including their connection with negative Sobolev spaces; see (1-5)), sparse indices and sparse stability.

[5] Loosely speaking, sparse cubes are not necessarily disjoint but possible overlappings can be controlled in a sharp fashion; see Definition 3.

1.5. ***Negative Sobolev spaces via sparse RMT spaces.*** The norms of many familiar spaces in analysis are defined in terms of coverings by disjoint cubes or "packings" (e.g., spaces like BMO, John–Nirenberg spaces, Morrey spaces, Campanato spaces, Brudnyi spaces, Lipschitz spaces, Garsia–Rodemich spaces, ...). In [Domínguez and Milman 2021], we initiated the analysis of "sparse versions" of classical spaces, obtained modifying the requirements on the coverings: we replaced the usual packings of cubes by the slightly bigger class of "sparse coverings". We briefly recall the definition of sparse family of cubes.

Let Q_0 be a (not necessarily dyadic) cube in $\mathbb{R}^n$ of sidelength $\ell > 0$ and corner $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, i.e.,

$$Q_0 = [x_1, x_1 + \ell] \times \cdots \times [x_n, x_n + \ell].$$

A (dyadic) child of Q_0 is any of the 2^n cubes obtained by partitioning Q_0 by n median hyperplanes (i.e., the hyperplanes parallel to the faces of Q_0 and dividing each edge into 2 equal parts). Iterating this process, from Q_0 to its children, then to the children of the children, ..., we construct the lattice $\mathcal{D}(Q_0)$ of dyadic subcubes in Q_0.

Definition 3 (sparse cubes). A (countable) family $(Q_i)_{i\in I} \subset \mathcal{D}(Q_0)$ is called *η-sparse*,[6] $\eta \in (0, 1)$, if for every Q_i there exists a measurable subset E_{Q_i} such that

(i) the sets E_{Q_i} are pairwise disjoint,

(ii) $\eta|Q_i| \le |E_{Q_i}|$.

We let $S(Q_0)$ be the collection of all sparse families of dyadic cubes in Q_0. Analogously, one can introduce $S(\mathbb{R}^n)$, the set formed by all sparse families of dyadic cubes in $\mathbb{R}^n$.

In particular, the sparse spaces $SR_{p,q}\log^\alpha(Q_0)$ are constructed modifying standard RMT spaces (see (2-1)) by replacing families of packings, "$(Q_i)_{i\in I} \in \Pi(Q_0)$", by sparse families, "$(Q_i)_{i\in I} \in S(Q_0)$".

Definition 4 (sparse RMT spaces). Let $1 \le p, q \le \infty$ and $\alpha \in \mathbb{R}$. The *sparse RMT space $SR_{p,q}\log^\alpha(Q_0)$* is formed by all those $f \in L^1(Q_0)$ such that

$$\|f\|_{SR_{p,q}\log^\alpha(Q_0)} = \sup_{(Q_i)_{i\in I}\in S(Q_0)} \left\{ \sum_{i\in I} \left[\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'}} \int_{Q_i} |f| \right]^q \right\}^{1/q} < \infty. \tag{1-7}$$

In particular, we let $SR_{p,q}(Q_0) = SR_{p,q}\log^0(Q_0)$. The corresponding spaces on $\mathbb{R}^n$ are introduced analogously.

Remark 5. The definition of $SR_{p,q}\log^\alpha$ is well adapted to work with signed measures $\omega \in BM^+$. Indeed, we simply replace $\int_{Q_i} |f|$ in (1-7) by $\omega(Q_i)$.

Since we trivially have $\Pi(Q_0) \subset S(Q_0)$, it follows that

$$SR_{p,q}\log^\alpha(Q_0) \subset R_{p,q}\log^\alpha(Q_0).$$

[6] In this paper the parameter η will not play a role, so in what follows we shall let $\eta = \frac{1}{2}$.

However, in general, the sparse spaces are different[7] from their parent spaces. In our context, the differences manifest themselves through the behavior of the maximal operators M_{λ,Q_0} (see (3-1)). The theory developed in Section 3 will play a crucial role in our analysis.

Suppose that $1 \leq p < q < \infty$; then a special case of Proposition 25 below shows that there exists $\{Q_i\}_{i\in I} \in S(Q_0)$ and a constant c depending only on p and q such that

$$M_{n(\frac{1}{p}-\frac{1}{q}),Q_0} f(x) \leq c \sum_{i\in I}\biggl(\frac{1}{|Q_i|^{1/p'+1/q}}\int_{Q_i}|f(y)|\,dy\biggr)\mathbf{1}_{Q_i}(x). \tag{1-8}$$

In the literature the process of constructing such coverings is referred as *sparse domination*; see [Lerner and Nazarov 2019; Hytönen 2021]. From (1-8) and more or less standard arguments, we obtain the following remarkable result connecting $SR_{p,q}(\mathbb{R}^n)$ and classical Sobolev spaces $H_q^{-\lambda}(\mathbb{R}^n)$, $\lambda \in (0,n)$ (see (4-1)).

Theorem 6. *Let* $1 \leq p < q < \infty$. *If* $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, $f \geq 0$ *a.e., then*

$$\|f\|_{SR_{p,q}(\mathbb{R}^n)} \approx \|f\|_{H_q^{-n(\frac{1}{p}-\frac{1}{q})}(\mathbb{R}^n)}.$$

In general

$$SR_{p,q}(\mathbb{R}^n) \hookrightarrow H_q^{-n(\frac{1}{p}-\frac{1}{q})}(\mathbb{R}^n).$$

In particular, the canonical choice of parameters $p = \dfrac{2n}{n+2}$, $n \geq 2$, *and* $q = 2$ *gives*[8]

$$\|f\|_{SR_{\frac{2n}{n+2},2}(\mathbb{R}^n)} \approx \|f\|_{H^{-1}(\mathbb{R}^n)} \quad \textit{if } f \geq 0. \tag{1-9}$$

Consequently

$$SR_{\frac{2n}{n+2},2}(\mathbb{R}^n) \hookrightarrow H^{-1}(\mathbb{R}^n). \tag{1-10}$$

1.6. *Sparse indices.* Since we deal with local problems, most of the analysis will be carried out on cubes, but similar constructions are also possible in $\mathbb{R}^n$. Let $Q_0 \subset \mathbb{R}^n$, $n \geq 2$, be a fixed cube and let $\mathcal{Q} \in S(Q_0)$. For $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, let $\mathbb{D}_{\leq k;Q_0} := \{Q : Q \in \mathcal{D}(Q_0) \text{ with sidelength } \ell(Q) \leq 2^{-k}\ell(Q_0)\}$, $\mathbb{D}_{\leq k;Q_0}(\mathcal{Q}) := \mathbb{D}_{\leq k;Q_0} \cap \mathcal{Q}$. When there is no danger of confusion, we use the simplified notation $\mathbb{D}_{\leq k}$ and $\mathbb{D}_{\leq k}(\mathcal{Q})$.

Definition 7 (sparse indices). (i) The *sparse indices of* $f \in L^1(Q_0)$ are defined by[9]

$$s_N(f) = \sup_{\mathcal{Q}\in S(Q_0)}\biggl[\sum_{Q\in\mathbb{D}_{\leq N-1}(\mathcal{Q})}\biggl(|Q|^{\frac{1}{n}-\frac{1}{2}}\int_Q|f|\biggr)^2\biggr]^{\frac{1}{2}}, \quad N \in \mathbb{N}. \tag{1-11}$$

[7] Note that $SR_{p,\infty}\log^{\alpha}(Q_0) = R_{p,\infty}\log^{\alpha}(Q_0) = M^{p,\alpha}(Q_0)$; see Remark 21.

[8] The role of sparseness is crucial here. In particular, for the classical space $R_{2n/(n+2),2}$, this approach fails dramatically (see (1-4)).

[9] Sparse indices may depend on the given cube Q_0. However, since this dependance will not play a role in our arguments, it will be safely omitted in the corresponding notation.

X	upper estimate for $s_N(X)$
$L^p, \quad p > \frac{2n}{n+2}$	$2^{-N(\frac{2+n}{2n}-\frac{1}{\min\{2,p\}})n}$
$M^{p,\alpha}, \quad p > \frac{n}{2}, \quad \alpha \in \mathbb{R}$	$2^{-N(\frac{2}{n}-\frac{1}{p})\frac{n}{2}}N^{-\frac{\alpha}{2}}$
$M^{\frac{n}{2},\alpha}, \quad \alpha > 1$	$N^{\frac{1-\alpha}{2}}$
$R_{p,2}\log^{\alpha}, \quad p > \frac{2n}{n+2}, \quad \alpha \in \mathbb{R}$	$2^{-N(\frac{2+n}{2n}-\frac{1}{p})n}N^{-\alpha}$
$R_{\frac{2n}{n+2},2}\log^{\alpha}, \quad \alpha > \frac{1}{2}$	$N^{\frac{1}{2}-\alpha}$

Table 1. Sparse indices for some classical function spaces.

(ii) Let X be a function space, $X \subset L^1_{\mathrm{loc}}(\mathbb{R}^n)$. The *sparse indices of* $X(Q_0)$ are defined by

$$s_N(X) = \sup_{\|f\|_{X(Q_0)} \le 1} s_N(f). \tag{1-12}$$

Remark 8. The definitions above can be extended in a natural way to the setting of measures with distinguished sign.

Compactness of embeddings into H^{-1} can be characterized in terms of sparse indices. Specifically, we have the following result.

Theorem 9. *Let X be a function space*[10] $X \subset L^1_{\mathrm{loc},+}(\mathbb{R}^n)$, $n \ge 2$, (*or more generally*, $X \subset BM_c^+$). *Then*:[11]

(i) $s_1(X) < \infty \quad \Leftrightarrow \quad X_c \hookrightarrow H^{-1}_{\mathrm{loc}}(\mathbb{R}^n)$.

(ii) $\lim_{N\to\infty} s_N(X) = 0 \quad \Leftrightarrow \quad X_c \overset{\text{compactly}}{\hookrightarrow} H^{-1}_{\mathrm{loc}}(\mathbb{R}^n)$.

The proof of this result is given in Section 4.3.

Sparse indices provide a very satisfactory criteria for H^{-1}-stability (see Theorem 1).

Corollary 10. *Let X be a function space $X \subset L^1_{\mathrm{loc},+}(\mathbb{R}^n)$, $n \ge 2$, (or more generally, $X \subset BM_c^+$) such that*

$$\lim_{N\to\infty} s_N(X) = 0. \tag{1-13}$$

Suppose that $\{u^\varepsilon\}_{\varepsilon>0}$ is an approximate family of Euler solutions with related set of vorticities $\{\omega^\varepsilon\}_{\varepsilon>0}$ uniformly bounded in $C((0,T);X)$. Then (passing to a subfamily if necessary) $u^\varepsilon \to u$ strongly in $L^\infty([0,T];L^2_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n))$, where u is a solution to (1-1).

The new indices pose a challenge: can we compute them? In Section 5 we provide the explicit calculation of sparse indices for classical spaces like Lebesgue, Morrey, and RMT spaces. The results are presented in Table 1. These computations, combined with Theorem 9, give a unified proof of (1-2) and (1-3).

[10] As usual, $L^1_{\mathrm{loc},+}(\mathbb{R}^n) = \{f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : f \ge 0 \text{ a.e.}\}$. The additional assumption $f \ge 0$ is not restrictive since $s_N(f) = s_N(|f|)$.

[11] Note that $s_1(X) = \sup_{N\in\mathbb{N}} s_N(X)$.

Furthermore, understanding the rates of decay of the sparse indices will allow us to measure the degree of H^{-1}-compactness, and pave the way to extend the known results as we now explain.

1.7. ***Sparse spaces.*** So far, given a function space X, we analyzed the decay of its sparse indices $s_N(X)$ (see (1-12)) in order to guarantee H^{-1}-stability; see Corollary 10. However, note that the definition of sparse indices $s_N(f)$ (see (1-11)) is independent of any particular space X. This simple observation leads to the following question: can we use "reverse engineering" to create new function spaces whose sparse indices have prescribed decay? A natural construction associated with this idea can be described as follows.

Definition 11 (sparse spaces). Let Ψ be a decay (see (1-6)). The *sparse space* $S_\Psi(Q_0)$ is formed by all $f \in L^1(Q_0)$ such that

$$\|f\|_{S_\Psi(Q_0)} = \sup_{N\in\mathbb{N}} \frac{s_N(f)}{\Psi(N)} < \infty. \tag{1-14}$$

The counterparts on $\mathbb{R}^n$ as well as for positive measures can be introduced analogously.

Note, parenthetically, the superficial similarity with the constructions of Yudovich spaces and extrapolation spaces in [Domínguez and Milman 2024]. As we shall soon see this connection goes deeper and, moreover, some concrete calculations can be effected which lead to the introduction of new Euler relevant function spaces.

From (1-14), we clearly have

$$s_N(S_\Psi(Q_0)) \le \Psi(N),$$

therefore, by Theorem 9,

$$S_\Psi(\mathbb{R}^n)_c \xhookrightarrow{\text{compactly}} H^{-1}_{\text{loc}}(\mathbb{R}^n). \tag{1-15}$$

In particular, using sparse embeddings we can formulate a new H^{-1}-criteria.

Theorem 12 (H^{-1}-stability via sparse embeddings). *Suppose that*

$$X_c \hookrightarrow S_\Psi(\mathbb{R}^n)_c \tag{1-16}$$

holds for some decay Ψ, then X is H^{-1}-stable, in the sense that

$$X_c \xhookrightarrow{\text{compactly}} H^{-1}_{\text{loc}}(\mathbb{R}^n).$$

In particular, suppose that $\{u^\varepsilon\}_{\varepsilon>0}$ is a family of approximate solutions of the Euler equations, such that the related set of vorticities $\{\omega^\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in X. Then there exists a subfamily of $\{u^\varepsilon\}_{\varepsilon>0}$ which converges strongly to a weak Euler solution in $L^\infty([0,T]; L^2_{\text{loc}}(\mathbb{R}^n))$.

The proof of this result is an immediate consequence of (1-15) and Theorem 1.

Next we go a step further and show that assumption (1-16) in Theorem 12 is in fact necessary to establish H^{-1}-stability. To do this, we need to introduce the natural generalization[12] (say, function space-free) of (1-16) to approximate solutions: *sparse stability*. As already anticipated in Theorem 2 (see

[12] Recall that H^{-1}-stability does not involve any function space X, but only approximate solutions; see Definition 19.

Section 6 for its proof), this new concept provides us with a remarkable characterization of H^{-1}-stability in terms of sparseness.

As usual, let $\mathbb{A}^n$ be the set of all antisymmetric matrices of order n with real entries.

Definition 13 (sparse stability). We say that a family $\{u^\varepsilon\}_{\varepsilon>0}$ of approximate solutions of the Euler equation is *sparse stable* if there exists a decay Ψ such that the corresponding set of vorticities $\{\omega^\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in[13] $C(0, T; S_\Psi(\mathbb{R}^n; \mathbb{A}^n))$. In particular, $\{u^\varepsilon\}_{\varepsilon>0}$ satisfies the *admissible sparse stability* property if Ψ is an admissible[14] decay.

Applying sparse stability, it is thus possible, at least theoretically, to improve all the known H^{-1}-stability results of [DiPerna and Majda 1987a; Lopes Filho et al. 2000; Tadmor 2001] (see Sections 1.1 and 1.2). However, to make the implied extensions meaningful, we need to exhibit concrete instantiations. In fact, we obtain significant improvements on the classical results and we show that our constructions unexpectedly connect with the theory of Yudovich spaces [1995], Vishik spaces [1999] and more specifically with the extrapolation spaces of [Jawerth and Milman 1991; Domínguez and Milman 2024].

1.8. ***New extrapolation spaces guaranteeing strong convergence to Euler solutions.*** As already mentioned in Section 1.3, extrapolation constructions seem to be implicit in (1-2)–(1-3). In Sections 7 and 8 we confirm this belief and show how the extrapolation theory of Jawerth and Milman [1991] (more precisely, the updated account given recently in [Domínguez and Milman 2024]) can be successfully applied to construct concrete examples of function spaces with prescribed sparse decay. In particular, these new spaces strictly contain the limiting spaces involved in (1-2) and (1-3), but are still H^{-1}-stable. As a consequence, we are able to extend the existence results for vortex sheets of [DiPerna and Majda 1987a; Tadmor 2001] to larger classes of vorticities.

1.8.1. *Sharpening Morrey regularity of DiPerna–Majda.* We introduce the distributional space $V_\Psi(\mathbb{R}^n)$ given by (see Definition 34)

$$\sum_{j=N}^{\infty} 2^{-2j} \|\Delta_j f\|_{L^\infty(\mathbb{R}^n)} \lesssim \Psi(N)^2, \quad N \in \mathbb{N}_0.$$

These spaces may be considered as "dual" counterparts of classical Vishik spaces proposed in [Vishik 1999] in connection with uniqueness issues for Euler flows; see Remark 35 for further explanations. Applying the set of techniques explained in previous sections, we show sparse stability[15] and nonconcentration phenomenon in V_Ψ; see Theorem 37. A crucial point in our arguments is that V_Ψ can be characterized as an extrapolation space of classical Besov spaces (see Theorem 39). In particular, for the special decay $\Psi(t) = t^{(1-\alpha)/2}$, $\alpha > 1$, we have (see Theorem 38)

$$M^{\frac{n}{2},\alpha}(\mathbb{R}^n) \hookrightarrow V_\Psi(\mathbb{R}^n).$$

[13] In what follows, we will use the simplified notation $S_\Psi(\mathbb{R}^n)$ rather than $S_\Psi(\mathbb{R}^n; \mathbb{A}^n)$.

[14] see Definition 36(i).

[15] In fact, sparse numbers of V_Ψ behave like Ψ.

Furthermore, this result is sharp, i.e., we give a constructive method to produce functions in $V_\Psi(\mathbb{R}^n)$ but not in $M^{n/2,\alpha}(\mathbb{R}^n)$. As a by-product, we get a nontrivial improvement of (1-2).

1.8.2. *Sharpening Tadmor regularity.* The results stated in Section 1.8.1 for Morrey spaces admit counterparts for RMT spaces. In this setting, the role of V_Ψ is played by the new space $T_\Psi(\mathbb{R}^n)$ (see Definition 44), which admits the following nice characterization in terms of Fourier integrals:

$$\int_{|\xi|>2^N} (1+|\xi|^2)^{-1} |\hat{f}(\xi)|^2 \, d\xi \lesssim \Psi(N)^2, \quad N \in \mathbb{N}_0.$$

Then we establish sparse stability and nonconcentration phenomenon in T_Ψ; see Theorem 47. In particular, for the special decay $\Psi(t) = t^{1/2-\alpha}$, $\alpha > \frac{1}{2}$, we have (see Theorem 48)

$$R_{\frac{2n}{n+2},2} \log^\alpha(\mathbb{R}^n) \hookrightarrow T_\Psi(\mathbb{R}^n).$$

Again, this result is sharp. As a consequence, we improve Tadmor's embedding (1-3).

1.9. ***Energy conservation for physically realizable solutions via sparse stability.*** In Section 10 we show that our methods are sufficiently robust to provide criteria for the preservation of energy by *physically realizable solutions*[16] of 2D Euler equations on the two-dimensional torus $\mathbb{T}^2$. Indeed, extending $L^p(\mathbb{T}^2)$-results,[17] $p > 1$, obtained by Cheskidov, Lopes Filho, Nussenzveig Lopes, and Shvydkoy [Cheskidov et al. 2016] (see also [Ciampa et al. 2021] for the case on the whole plane $\mathbb{R}^2$), we show that our framework can be used to provide conditions for physically realizable solutions to conserve energy.

Theorem 14. *Let u be a physically realizable weak solution of the 2D Euler equations with a physical realization $\{u^\varepsilon\}_{\varepsilon>0}$ satisfying admissible sparse stability. Then u is conservative, i.e., $\|u(t)\|_{L^2(\mathbb{T}^2)} = \|u_0\|_{L^2(\mathbb{T}^2)}$.*

Very recently, Lanthaler, Mishra, and Parés-Pulido [Lanthaler et al. 2021] proposed an interesting approach to energy conservation based on the so-called structure functions (i.e., the L^2-modulus of smoothness) of $\{u^\varepsilon\}_{\varepsilon>0}$. On the other hand, Theorem 14 relies on the sparse indices of $\{\omega^\varepsilon\}_{\varepsilon>0}$. Switching from u^ε to ω^ε has important advantages from the point of view of applications, as it is illustrated by the following.

Corollary 15. *Let X be a function space $X \subset L^1(\mathbb{T}^2)$ (or more generally, $X \subset BM^+$) with sparse indices $s_N(X)$ satisfying* (1-13) *and the admissibility condition* (*see Definition 36*)*. Let $\{u^\varepsilon\}_{\varepsilon>0}$ be a physical realization of the Euler solution u. If $\{\omega^\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in X, then u is conservative.*

This result follows immediately from Theorem 14 from the fact that $X \hookrightarrow S_\Psi(\mathbb{T}^2)$, where $\Psi(N) = s_N(X)$. In particular, Corollary 15 can be applied to all the classical function spaces exhibited in Table 1, as well as the new spaces $X = V_\Psi$ and $X = T_\Psi$.

[16] Roughly speaking, physically realizable solutions are weak solutions of Euler equations that can be obtained as weak limits of vanishing viscosity; see Definition 61.

[17] For general L^p solutions with $p \geq \frac{3}{2}$, conservation of energy can be derived from the well-known Besov-type criterion of Cheskidov, Constantin, Friedlander and Shvydkoy [Cheskidov et al. 2008]; see also [Cheskidov et al. 2016, Theorem 1] for an alternative and streamlined proof.

1.10. ***Brief interlude: some references.*** Existence and uniqueness of weak solutions for the 2D Euler equations are well established. In particular, we mention the concentration-cancellation result by Delort [1991] (resp. Vecchi and Wu [1993]) proving existence of weak solutions for initial vorticities in $BM_c^+ \cap H^{-1}$ (resp. in $L_c^1 \cap H^{-1}$); existence and uniqueness results of weak solutions for initial bounded vorticities (resp. vorticities in Yudovich spaces) were established by Yudovich [1963] (resp. [Yudovich 1995]), and the corresponding results for vorticities in BMO (and related spaces) obtained by Vishik [1999]. For vorticities in L^p the uniqueness problem remains open, although substantial progress has been achieved recently. Relaxing assumptions in the sense of forced 2D Euler equations, Vishik [2018a; 2018b] (see also [Albritton et al. 2024; Castro et al. 2025]) established nonuniqueness for vorticities in L^p, $p < \infty$. More recently, using a newly developed refined version of the convex integration technique, the nonuniqueness of weak solutions under vorticity in $L^{1+\varepsilon}$, with ε sufficiently small, was shown by Bruè, Colombo and Kumar [Bruè et al. 2024].

We close this introduction stating our belief that, given the central role of negative Sobolev spaces in PDEs, our methods could find applications elsewhere.

Notation. Given two normed spaces X and Y, the symbol $X \hookrightarrow Y$ means that the identity operator from X into Y is continuous. Given two positive quantities A and B, we write $A \lesssim B$ if there is a constant $C > 0$ such that $A \le CB$. We also use $A \approx B$ if $A \lesssim B$ and $A \gtrsim B$. For $a \in \mathbb{R}$, $a_- = \min\{a, 0\}$, p' denotes the dual exponent of p given by $1/p + 1/p' = 1$ and $|E|$ is the (Lebesgue) measure of a measurable set E.

2. Background

2.1. ***Approximate solutions and H^{-1}-stability.*** For convenience of the reader, we recall the well-known concepts of approximate solutions of Euler equations and their H^{-1}-stability, as introduced in [DiPerna and Majda 1987a] and [Lopes Filho et al. 2000], respectively.

Definition 16. A family of velocity vector fields $\{u^\varepsilon(\cdot, t)\}_{\varepsilon>0}$, $t \in [0, T]$, defines an *approximate solution* of (1-1) if for some $L > 1$, it is uniformly bounded[18] in

$$L^\infty([0, T]; L_c^2(\mathbb{R}^n; \mathbb{R}^n)) \cap \mathrm{Lip}((0, T); H_{\mathrm{loc}}^{-L}(\mathbb{R}^n; \mathbb{R}^n)),$$

with $\operatorname{div} u^\varepsilon = 0$ (in the distributional sense), and is weakly consistent with (1-1), in the sense that[19]

$$\int_0^T \int_{\mathbb{R}^n} \varphi_t \cdot u^\varepsilon + (D\varphi\, u^\varepsilon) \cdot u^\varepsilon \, dx \, dt + \int_{\mathbb{R}^n} \varphi(x, 0) \cdot u^\varepsilon(x, 0) \, dx \to 0 \quad \text{as } \varepsilon \to 0$$

for every test field $\varphi \in C_c^\infty([0, T) \times \mathbb{R}^n; \mathbb{R}^n)$ with $\operatorname{div} \varphi = 0$. Here, $D\varphi$ is the Jacobian matrix of φ.

Remark 17. If the family is constant, $u^\varepsilon \equiv u$ for all $\varepsilon > 0$, then u is in fact a classical *weak solution* to (1-1).

[18] The uniform bound in $\mathrm{Lip}((0, T); H_{\mathrm{loc}}^{-L}(\mathbb{R}^n; \mathbb{R}^n))$ is a technical assumption in order to guarantee that initial vector fields $u^\varepsilon(\cdot, 0)$ are well-defined. In practice, this follows easily from the uniform energy bound $L^\infty([0, T]; L_c^2(\mathbb{R}^n; \mathbb{R}^n))$; see [DiPerna and Majda 1987a].

[19] The weak formulation related to domains with boundary is analogous to that of $\mathbb{R}^n$, taking into account the additional boundary condition $u^\varepsilon \cdot n = 0$ (in the trace sense).

Remark 18. There are standard methodologies to construct approximation solution families, e.g., through mollification of initial data, Navier–Stokes approximate solutions (also known as vanishing viscosity method), vortex blob approximations, discrete methods, ...

Definition 19 (H^{-1}-stability). We say that a family $\{u^\varepsilon\}_{\varepsilon>0}$ of approximate solutions of the Euler equation is *H^{-1}-stable* if the corresponding set of vorticities $\{\omega^\varepsilon = \operatorname{curl} u^\varepsilon\}_{\varepsilon>0}$ (i.e., $\omega^\varepsilon_{i,j} = (u^\varepsilon_i)_{x_j} - (u^\varepsilon_j)_{x_i}$ for $i, j = 1, \dots, n$) is precompact in $C((0,T); H^{-1}_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{A}^n))$.

2.2. ***Riesz–Morrey–Tadmor spaces.*** Let $\Pi(Q_0)$ be the set of families of packings[20] $(Q_i)_{i\in I}$, with $Q_i \in \mathcal{D}(Q_0)$.

Definition 20 [Tadmor 2001]. The *Riesz–Morrey–Tadmor spaces*[21] (RMT spaces, in short) $R_{p,q} \log^\alpha(Q_0)$, $1 \le p, q \le \infty$, $\alpha \in \mathbb{R}$, are defined through the condition

$$\|f\|_{R_{p,q}\log^\alpha(Q_0)} = \sup_{(Q_i)_{i\in I}\in\Pi(Q_0)} \left\{ \sum_{i\in I} \left[\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'}} \int_{Q_i} |f| \right]^q \right\}^{\frac{1}{q}} < \infty. \tag{2-1}$$

The corresponding spaces on $\mathbb{R}^n$ are defined analogously.

Remark 21. In particular, *Morrey spaces* are part of this scale. Let $1 \le p \le \infty$, $\alpha \in \mathbb{R}$. The Morrey space $M^{p,\alpha}(Q_0)$ is defined by

$$\|f\|_{M^{p,\alpha}(Q_0)} = \sup_{Q\in\mathcal{D}(Q_0)} \frac{(1-(\log|Q|)_-)^\alpha}{|Q|^{1/p'}} \int_Q |f| < \infty. \tag{2-2}$$

Consequently, $M^{p,\alpha}(Q_0) = R_{p,\infty} \log^\alpha(Q_0)$.

Remark 22. A similar comment to Remark 5 also applies to $R_{p,q} \log^\alpha$ and $M^{p,\alpha}$.

3. Characterization of sparse RMT spaces via maximal operators

Let $0 \le \lambda < n$ and $\alpha \in \mathbb{R}$. For $f \in L^1(Q_0)$, consider the maximal operator

$$M_{\lambda,\alpha,Q_0} f(x) = \sup_{\substack{Q\in\mathcal{D}(Q_0)\\ x\in Q}} |Q|^{\frac{\lambda}{n}-1} (1-(\log|Q|)_-)^\alpha \int_Q |f(y)|\, dy, \quad x \in Q_0. \tag{3-1}$$

In the absence of logarithmic weight (i.e., $\alpha = 0$), we simply write M_{λ,Q_0}. In addition, if $\lambda = 0$ then one recovers the classical (dyadic) maximal operator M_{Q_0}.

In this section we show that, under some natural conditions, the sparse $SR_{p,q} \log^\alpha$ spaces (see Definition 4) admit simple characterizations in terms of maximal operators (3-1). This is in sharp contrast with the parent spaces $R_{p,q} \log^\alpha$.

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[20]Families of pairwise disjoint cubes.

[21]Our notation differs from [Tadmor 2001] where the space $R_{p,q} \log^\alpha(Q_0)$ is instead denoted by $V^{pq}(\log V)^\alpha(Q_0)$ (or simply by $V^{pq,\alpha}(Q_0)$). The reason behind this change of notation comes from the Riesz theorem (see (4-7)).

Theorem 23. *Suppose that* p, q, α *satisfy*

$$1 \le p \le q < \infty \quad \text{and} \quad \alpha \in \mathbb{R} \quad (\alpha \le 0 \text{ if } p = q). \tag{3-2}$$

Then

$$SR_{p,q} \log^{\alpha}(Q_0) = M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_0} L^q(Q_0).$$

More precisely,

$$\|f\|_{SR_{p,q}\log^{\alpha}(Q_0)} \approx \|M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_0} f\|_{L^q(Q_0)}, \tag{3-3}$$

where the hidden constants of equivalence are independent of f *and* Q_0.

Remark 24. When $p = q$, the restriction $\alpha \le 0$ is necessary to avoid trivial cases. To be more precise, if $\alpha > 0$ and $p = q$, then

$$\|M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_0} f\|_{L^q(Q_0)} < \infty \implies f = 0 \quad \text{a.e. on } Q_0.$$

Indeed, since $M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_0} f(x) = M_{0,\alpha,Q_0} f(x) < \infty$ a.e. $x \in Q_0$, we have

$$\frac{1}{|Q|}\int_Q |f| \le (1-(\log|Q|)_-)^{-\alpha} M_{0,\alpha,Q_0} f(x) \tag{3-4}$$

for every $Q \in \mathcal{D}(Q_0)$, with $x \in Q$, and $|Q|$ sufficiently small. Taking limits on both sides of (3-4) as $|Q| \to 0$, and applying the Lebesgue differentiation theorem, we conclude that $f(x) = 0$ for every Lebesgue point x.

Sparse domination principles underlie the characterizations of sparse spaces via maximal functions.

Proposition 25. *Suppose that* p, q *and* α *satisfy* (3-2), *and let* $f \in L^1(Q_0)$. *Then there exists a family* $(Q_i)_{i\in I} \in S(Q_0)$ (*depending on* f *and the parameters* p, q *and* α) *such that, for almost every* $x \in Q_0$,

$$M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_0} f(x) \le 2\max\left\{1, e^{\frac{1}{p}-\frac{1}{q}-\alpha}\left(\frac{pq\alpha}{q-p}\right)^{\alpha}\right\} \sum_{i\in I}\left(\frac{(1-(\log|Q_i|)_-)^{\alpha}}{|Q_i|^{1/p'+1/q}}\int_{Q_i}|f(y)|\,dy\right)\mathbf{1}_{Q_i}(x) \tag{3-5}$$

Remark 26. As usual, if $p = q$ the constant $\max\{1, e^{1/p-1/q-\alpha}(pq\alpha/(q-p))^{\alpha}\}$ in (3-5) should be interpreted to be equal to 1.

Proof of Proposition 25. The desired decomposition will be obtained by a standard process of exhaustion, whereby for each cube of the starting decomposition we shall apply the process again and again. We set up the selection process by letting $\mathcal{Q}_f = \mathcal{Q}_{f,p,q,\alpha}$ be the collection of $Q \in \mathcal{D}(Q_0)$ such that the following condition is satisfied:

$$\frac{(1-(\log|Q|)_-)^{\alpha}}{|Q|^{1/p'+1/q}}\int_Q |f(y)|\,dy \ge 2\max\left\{1, e^{\lambda-\alpha}\left(\frac{\alpha}{\lambda}\right)^{\alpha}\right\}\frac{(1-(\log|Q_0|)_-)^{\alpha}}{|Q_0|^{1/p'+1/q}}\int_{Q_0}|f(y)|\,dy, \tag{3-6}$$

where $\lambda := \frac{1}{p} - \frac{1}{q}$. If the collection $\mathcal{Q}_f$ is empty then we let $E_{Q_0} = \{Q_0\}$, and we readily verify that (3-5) holds. Otherwise we continue the process selecting $(Q_i)_{i\in\mathbb{N}}$, the family of maximal dyadic cubes in $\mathcal{Q}_f$. By construction, the selected family $(Q_i)_{i\in\mathbb{N}}$ is pairwise disjoint and, therefore, for almost every $x \in Q_0$,

$$\begin{aligned} M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_0} f(x) &= M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_0} f(x) \mathbf{1}_{Q_0 \setminus \bigcup_{i=1}^\infty Q_i}(x) + \sum_{i=1}^\infty M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_0} f(x) \mathbf{1}_{Q_i}(x) \\ &=: (\mathrm{A}) + (\mathrm{B}). \end{aligned}$$

Next we estimate each of the terms (A) and (B) separately.

Estimate (A). We claim that, for $x \notin \bigcup_{i=1}^\infty Q_i$,

$$M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_0} f(x) \le 2 \max\Big\{1, e^{\lambda-\alpha}\Big(\frac{\alpha}{\lambda}\Big)^\alpha\Big\} \frac{(1-(\log|Q_0|)_-)^\alpha}{|Q_0|^{1/p'+1/q}} \int_{Q_0} |f(y)|\,dy. \tag{3-7}$$

Indeed, suppose, to the contrary, that for some $x \notin \bigcup_{i=1}^\infty Q_i$, (3-7) does not hold. Then, by the definition of $M_{n(1/p-1/q),\alpha,Q_0}$ (see (3-1)), there exists a dyadic cube $Q \subset Q_0$, such that $x \in Q$, and $Q \in \mathcal{Q}_f$. Consequently, there exists a maximal cube Q_i such that $Q \subset Q_i$, but this leads to a contradiction since $x \notin Q_i$. Therefore, for $x \notin \bigcup_{i=1}^\infty Q_i$, we have

$$(\mathrm{A}) \le 2 \max\Big\{1, e^{\lambda-\alpha}\Big(\frac{\alpha}{\lambda}\Big)^\alpha\Big\} \frac{(1-(\log|Q_0|)_-)^\alpha}{|Q_0|^{1/p'+1/q}} \int_{Q_0} |f(y)|\,dy.$$

Moreover, from $(Q_i)_{i\in\mathbb{N}} \subset \mathcal{Q}_f$ (see (3-6)) we see that

$$\varphi(|Q_i|)|Q_i|^{-1} \int_{Q_i} |f(y)|\,dy \ge 2 \max\Big\{1, e^{\lambda-\alpha}\Big(\frac{\alpha}{\lambda}\Big)^\alpha\Big\} \varphi(|Q_0|)|Q_0|^{-1} \int_{Q_0} |f(y)|\,dy, \tag{3-8}$$

where

$$\varphi(t) := t^{\frac{1}{p}-\frac{1}{q}} (1-(\log t)_-)^\alpha, \quad t > 0.$$

We distinguish two possible cases.

(I) Suppose first that $\alpha \le \lambda$. Routine computations show, under this assumption, that φ is a nondecreasing function. It follows from (3-8) that

$$\begin{aligned} \sum_{i=1}^\infty |Q_i| &\le \frac{1}{2\max\{1, e^{\lambda-\alpha}(\alpha/\lambda)^\alpha\}} \frac{|Q_0|}{\varphi(|Q_0|)\int_{Q_0}|f(y)|\,dy} \sum_{i=1}^\infty \varphi(|Q_i|) \int_{Q_i} |f(y)|\,dy \\ &\le \frac{1}{2\max\{1, e^{\lambda-\alpha}(\alpha/\lambda)^\alpha\}} \frac{|Q_0|}{\int_{Q_0}|f(y)|\,dy} \sum_{i=1}^\infty \int_{Q_i} |f(y)|\,dy \le \frac{|Q_0|}{2}. \end{aligned}$$

Note that, in the first step of above computations, we assume that f is not identically zero (almost everywhere) on Q_0; otherwise the desired result (3-5) holds trivially. Therefore, if we assign to the cube Q_0 the set

$$E_{Q_0} := Q_0 \Big\backslash \bigcup_{i=1}^\infty Q_i, \tag{3-9}$$

then

$$\frac{|Q_0|}{2} \le |E_{Q_0}|, \tag{3-10}$$

i.e., the sparseness condition given in Definition 3(ii) holds.

(II) Suppose now that $\alpha > \lambda$. Let

$$\psi(t) := \begin{cases} t^{\lambda}\,(1-\log t)^{\alpha} & \text{if } t \in (0, e^{1-\alpha/\lambda}), \\ e^{\lambda-\alpha}(\alpha/\lambda)^{\alpha} & \text{if } t \in [e^{1-\alpha/\lambda}, \infty). \end{cases}$$

It is plain that ψ is a nondecreasing function such that, moreover, $\varphi(t) \le \psi(t)$ for $t > 0$. By (3-8), we have

$$\begin{aligned}
\sum_{i=1}^{\infty} |Q_i| &\le \frac{1}{2\max\{1, e^{\lambda-\alpha}(\alpha/\lambda)^{\alpha}\}} \frac{|Q_0|}{\varphi(|Q_0|)\int_{Q_0}|f(y)|\,dy} \sum_{i=1}^{\infty} \varphi(|Q_i|) \int_{Q_i} |f(y)|\,dy \\
&\le \frac{1}{2\max\{1, e^{\lambda-\alpha}(\alpha/\lambda)^{\alpha}\}} \frac{|Q_0|}{\varphi(|Q_0|)\int_{Q_0}|f(y)|\,dy} \sum_{i=1}^{\infty} \psi(|Q_i|) \int_{Q_i} |f(y)|\,dy \\
&\le \frac{1}{2\max\{1, e^{\lambda-\alpha}(\alpha/\lambda)^{\alpha}\}} \frac{|Q_0|\psi(|Q_0|)}{\varphi(|Q_0|)\int_{Q_0}|f(y)|\,dy} \sum_{i=1}^{\infty} \int_{Q_i} |f(y)|\,dy \\
&\le \frac{|Q_0|}{2\max\{1, e^{\lambda-\alpha}(\alpha/\lambda)^{\alpha}\}} \frac{\psi(|Q_0|)}{\varphi(|Q_0|)}.
\end{aligned}$$

Furthermore, using the estimate

$$\frac{\psi(|Q_0|)}{\varphi(|Q_0|)} \le \max\Big\{1, e^{\lambda-\alpha}\Big(\frac{\alpha}{\lambda}\Big)^{\alpha}\Big\},$$

we obtain

$$\sum_{i=1}^{\infty} |Q_i| \le \frac{|Q_0|}{2}.$$

Hence the set E_{Q_0} defined by (3-9) satisfies the required sparseness condition (3-10).

Estimate (B). We will show that the procedure used to estimate (A) can be iterated to estimate each term of the sum (B). Fix $i \in \mathbb{N}$. Observe that for $x \in Q_i$, the maximality of the Q_i's and the nesting property of dyadic cubes, yield

$$M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_0} f(x) = M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_i} f(x). \tag{3-11}$$

Indeed, we only need to prove that the right-hand side is $\ge$ the left-hand side. Consider Q a generic dyadic cube such that $x \in Q \subset Q_0$. In particular, $Q \cap Q_i \neq \varnothing$. Now there are two possible situations. Firstly, if $Q \subseteq Q_i$, the cube Q enters in the competition for computing both, the left- and right-hand sides of (3-11), which is consistent with what we wish to prove. Assume now that $Q_i \subset Q$. In this case, since Q_i is a maximal element of $\mathcal{Q}_f$, we must have that $Q \notin \mathcal{Q}_f$. Therefore (see (3-6))

$$\frac{(1-(\log|Q|)_-)^{\alpha}}{|Q|^{1/p'+1/q}} \int_Q |f(y)|\,dy < 2\max\Big\{1, e^{\lambda-\alpha}\Big(\frac{\alpha}{\lambda}\Big)^{\alpha}\Big\} \frac{(1-(\log|Q_0|)_-)^{\alpha}}{|Q_0|^{1/p'+1/q}} \int_{Q_0} |f(y)|\,dy. \tag{3-12}$$

On the other hand, since $Q_i \in \mathcal{Q}_f$, it follows that

$$\begin{aligned}
2\max\Big\{1, e^{\lambda-\alpha}\Big(\frac{\alpha}{\lambda}\Big)^{\alpha}\Big\} \frac{(1-(\log|Q_0|)_-)^{\alpha}}{|Q_0|^{1/p'+1/q}} \int_{Q_0} |f(y)|\,dy &\le \frac{(1-(\log|Q_i|)_-)^{\alpha}}{|Q_i|^{1/p'+1/q}} \int_{Q_i} |f(y)|\,dy \\
&\le M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_i} f(x).
\end{aligned} \tag{3-13}$$

Putting together (3-12) and (3-13),

$$\frac{(1-(\log|Q|)_-)^\alpha}{|Q|^{1/p'+1/q}}\int_Q |f(y)|\,dy < M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_i} f(x),$$

and taking now the supremum over all possible dyadic cubes $Q \subset Q_0$ with $x \in Q$, we arrive at the desired upper estimate $\leq$ in (3-11).

By (3-11), we can write (B) as

$$(\mathrm{B}) = \sum_{i=1}^\infty M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_i} f(x)\mathbf{1}_{Q_i}(x). \tag{3-14}$$

The proof can be now completed applying the procedure used to estimate (A) to each of the terms that appear on the right-hand side of (3-14). □

Proof of Theorem 23. Let $f \in SR_{p,q}\log^\alpha(Q_0)$. In light of Proposition 25, there exists $(Q_i)_{i\in I} \in S(Q_0)$ (depending, in particular, on f) such that the estimate (3-5) holds. Then, taking L^q-norms on both sides of this estimate, we find

$$\|M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_0} f\|_{L^q(Q_0)} \lesssim \left\| \sum_{i\in I}\left(\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'+1/q}}\int_{Q_i}|f(y)|\,dy\right)\mathbf{1}_{Q_i}\right\|_{L^q(Q_0)}.$$

To estimate the right-hand side, we shall use duality, the properties of sparseness and the Hardy–Littlewood maximal theorem (recall that $q<\infty$). This requires a number of elementary manipulations, but to facilitate the reading we present all the steps,

$$\begin{aligned}
&\|M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_0} f\|_{L^q(Q_0)}\\
&\quad\lesssim \sup_{\|g\|_{L^{q'}(Q_0)}\leq 1}\int_{Q_0}\left(\sum_{i\in I}\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'+1/q}}\int_{Q_i}|f(y)|\,dy\,\mathbf{1}_{Q_i}(x)\right)|g(x)|\,dx\\
&\quad= \sup_{\|g\|_{L^{q'}(Q_0)}\leq 1}\sum_{i\in I}\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'+1/q}}\int_{Q_i}|f(y)|\,dy\int_{Q_i}|g(x)|\,dx\\
&\quad\lesssim \sup_{\|g\|_{L^{q'}(Q_0)}\leq 1}\sum_{i\in I}\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'+1/q}}\frac{|E_{Q_i}|}{|Q_i|}\int_{Q_i}|f(y)|\,dy\int_{Q_i}|g(x)|\,dx\\
&\quad= \sup_{\|g\|_{L^{q'}(Q_0)}\leq 1}\sum_{i\in I}\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'+1/q}}\int_{Q_i}|f(y)|\,dy\int_{E_{Q_i}}\left(\frac{1}{|Q_i|}\int_{Q_i}|g(u)|\,du\right)dx\\
&\quad\leq \sup_{\|g\|_{L^{q'}(Q_0)}\leq 1}\sum_{i\in I}\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'+1/q}}\int_{Q_i}|f(y)|\,dy\int_{E_{Q_i}}M_{Q_0}g(x)\,dx\\
&\quad= \sup_{\|g\|_{L^{q'}(Q_0)}\leq 1}\int_{Q_0}\left(\sum_{i\in I}\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'+1/q}}\int_{Q_i}|f(y)|\,dy\,\mathbf{1}_{E_{Q_i}}(x)\right)M_{Q_0}g(x)\,dx\\
&\quad\leq \sup_{\|g\|_{L^{q'}(Q_0)}\leq 1}\|M_{Q_0}g\|_{L^{q'}(Q_0)}\left\|\sum_{i\in I}\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'+1/q}}\int_{Q_i}|f(y)|\,dy\,\mathbf{1}_{E_{Q_i}}\right\|_{L^q(Q_0)}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim q \Big\| \sum_{i\in I} \frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'+1/q}} \int_{Q_i} |f(y)|\,dy\,\mathbf{1}_{E_{Q_i}} \Big\|_{L^q(Q_0)} \\
&= q \bigg(\sum_{i\in I} \bigg(\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'+1/q}} \int_{Q_i} |f(y)|\,dy \bigg)^q |E_{Q_i}| \bigg)^{1/q} \\
&\le q \bigg(\sum_{i\in I} \bigg(\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'}} \int_{Q_i} |f(y)|\,dy \bigg)^q \bigg)^{1/q} \\
&\le q \|f\|_{SR_{p,q}\log^\alpha(Q_0)}.
\end{aligned}
$$

Conversely, for any $(Q_i)_{i\in I} \in S(Q_0)$, we have (recalling the sparseness condition in Definition 3(ii))

$$
\begin{aligned}
\sum_{i\in I} \bigg(\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'}} \int_{Q_i} |f(y)|\,dy \bigg)^q &\le 2 \sum_{i\in I} \bigg(\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'+1/q}} \int_{Q_i} |f(y)|\,dy \bigg)^q |E_{Q_i}| \\
&= 2 \int_{Q_0} \sum_{i\in I} \bigg(\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'+1/q}} \int_{Q_i} |f(y)|\,dy \bigg)^q \mathbf{1}_{E_{Q_i}}(x)\,dx \\
&\le 2 \|M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_0} f\|^q_{L^q(Q_0)}.
\end{aligned}
$$

Taking the supremum over all $(Q_i)_{i\in I} \in S(Q_0)$, we arrive at

$$
\|f\|^q_{SR_{p,q}\log^\alpha(Q_0)} \le 2 \|M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_0} f\|^q_{L^q(Q_0)}. \qquad \square
$$

Remark 27. Note that, since the Hardy–Littlewood maximal function is not bounded on L^1, the above proof does not work if $q=\infty$. However, Theorem 23 holds trivially if $q=\infty$ (for any value of $\alpha\in\mathbb{R}$) since

$$
\|f\|_{SR_{p,\infty}\log^\alpha(Q_0)} = \|f\|_{M^{p,\alpha}(Q_0)} = \|M_{\frac{n}{p},\alpha,Q_0} f\|_{L^\infty(Q_0)};
$$

see (2-2) and (3-1).

3.1. ***Spaces defined on the whole space.*** The analogue of Theorem 23 for $SR_{p,q}\log^\alpha(\mathbb{R}^n)$ can be now formulated in terms of the (dyadic) maximal function

$$
M_{\lambda,\alpha} f(x) = \sup_{\substack{Q\in\mathcal{D}(\mathbb{R}^n)\\ x\in Q}} |Q|^{\frac{\lambda}{n}-1} (1-(\log|Q|)_-)^\alpha \int_Q |f(y)|\,dy, \quad x\in\mathbb{R}^n, \tag{3-15}
$$

where $\lambda\in[0,n)$ and[22] $\alpha\in\mathbb{R}$. For $q\in[1,\infty)$, we define

$$
M_{\lambda,\alpha} L^q(\mathbb{R}^n) = \{ f\in L^1_{\text{loc}}(\mathbb{R}^n) : \|M_{\lambda,\alpha} f\|_{L^q(\mathbb{R}^n)} < \infty \}.
$$

Theorem 28. *Suppose that* p,q,α *satisfy* (3-2). *Then*

$$
SR_{p,q}\log^\alpha(\mathbb{R}^n) = M_{n(\frac{1}{p}-\frac{1}{q}),\alpha} L^q(\mathbb{R}^n).
$$

[22] In the absence of the log-parameter (i.e., $\alpha=0$), we simply write M_λ instead of $M_{\lambda,0}$. If, in addition, $\lambda=0$ then we get back the classical Hardy–Littlewood maximal function M.

Proof. To facilitate the reading we have divided the proof into four steps, which we now outline. The general goal is to extend the local estimate (3-3) to a global one. For this purpose in Step 1 we construct a suitable nested sequence of cubes Q_k such that $\bigcup_k Q_k = \mathbb{R}^n$ and invoke (3-3) for each Q_k. The quantities involved in (3-3) are local maximal operators and local sparse RMT functionals related to each cube Q_k. In Steps 2 and 3 we develop the asymptotic analysis that will enable us to take limits[23] when $k \to \infty$ in Step 4, and in this manner effect the required transference from local to global estimates for sparse RMT functionals.

Step 1. Consider the sequence of (not dyadic) cubes

$$Q_k := [-2^k, 2^k]^n, \quad k \in \mathbb{N}.$$

According to Theorem 23, with equivalence constants independent of f and k,

$$\|f\mathbf{1}_{Q_k}\|_{SR_{p,q}\log^\alpha(Q_k)} \approx \left\|M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_k}(f\mathbf{1}_{Q_k})\right\|_{L^q(Q_k)}. \tag{3-16}$$

Step 2. We claim that, for every $k \in \mathbb{N}$ and $x \in Q_k$,

$$M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_k}(f\mathbf{1}_{Q_k})(x) \approx M_{n(\frac{1}{p}-\frac{1}{q}),\alpha}(f\mathbf{1}_{Q_k})(x), \tag{3-17}$$

(see (3-1) and (3-15)). Indeed, the estimate $\lesssim$ follows from the simple observation that

$$\mathcal{D}(Q_k)\backslash\{Q_k\} \subset \mathcal{D}(\mathbb{R}^n),$$

and the fact that the first (dyadic) generation of Q_k, say $\{Q^1_{k,l} : l = 1, \dots, 2^n\}$, gives a pairwise disjoint decomposition of Q_k. In particular,

$$Q_k = \bigcup_{l=1}^{2^n} Q^1_{k,l}$$

and $|Q^1_{k,l}| = 2^{-n}|Q_k|$. Hence, given any $x \in Q_k$,

$$\begin{aligned} |Q_k|^{\frac{1}{p}-\frac{1}{q}-1} \int_{Q_k} |f(y)|\,dy &= 2^{n(\frac{1}{p}-\frac{1}{q}-1)} \sum_{l=1}^{2^n} |Q^1_{k,l}|^{\frac{1}{p}-\frac{1}{q}-1} \int_{Q^1_{k,l}} |f(y)|\,dy \\ &\le 2^{n(\frac{1}{p}-\frac{1}{q})} M_{n(\frac{1}{p}-\frac{1}{q}),\alpha}(f\mathbf{1}_{Q_k})(x). \end{aligned}$$

Accordingly,

$$\begin{aligned} &M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_k}(f\mathbf{1}_{Q_k})(x) \\ &\quad\le \sup_{\substack{Q\in\mathcal{D}(Q_k)\backslash\{Q_k\} \\ x\in Q}} |Q|^{\frac{1}{p}-\frac{1}{q}-1}(1-(\log|Q|)_-)^\alpha \int_{Q\cap Q_k} |f(y)|\,dy + |Q_k|^{\frac{1}{p}-\frac{1}{q}-1} \int_{Q_k} |f(y)|\,dy \\ &\quad\lesssim \sup_{\substack{Q\in\mathcal{D}(\mathbb{R}^n) \\ x\in Q}} |Q|^{\frac{1}{p}-\frac{1}{q}-1}(1-(\log|Q|)_-)^\alpha \int_Q |f(y)|\mathbf{1}_{Q_k}(y)\,dy + M_{n(\frac{1}{p}-\frac{1}{q}),\alpha}(f\mathbf{1}_{Q_k})(x) \\ &\quad\approx M_{n(\frac{1}{p}-\frac{1}{q}),\alpha}(f\mathbf{1}_{Q_k})(x). \end{aligned}$$

[23] In particular, to justify the passage to the limit requires estimates that are independent of Q_k.

Next we focus on the estimate $\gtrsim$ in (3-17). Consider $x \in Q_k$, and $Q \in \mathcal{D}(\mathbb{R}^n)$ with $Q \ni x$ and moreover $Q \not\subset Q_k$ (indeed, if $Q \subset Q_k$ then $Q \in \mathcal{D}(Q_k)$). We cannot assert that $Q_k \subset Q$ (recall that Q_k is not dyadic), but what is certainly true is that there exists $Q_k^1 \in \mathcal{D}(Q_k)$, in the first dyadic generation (i.e., $2\,\ell(Q_k^1) = \ell(Q_k)$), such that $Q_k^1 \subset Q$ (because $\mathcal{D}(Q_k)\backslash\{Q_k\} \subset \mathcal{D}(\mathbb{R}^n)$ and $x \in Q_k$). Note that, in particular, $|Q| \geq |Q_k^1| = \ell(Q_k^1)^n = 2^{-n}\ell(Q_k)^n = 2^{-n}|Q_k| = 2^{kn} > 1$. Since $\frac{1}{p} - \frac{1}{q} - 1 < 0$, we have

$$\begin{aligned} |Q|^{\frac{1}{p}-\frac{1}{q}-1}(1-(\log|Q|)_-)^{\alpha} &= |Q|^{\frac{1}{p}-\frac{1}{q}-1} \leq |Q_k^1|^{\frac{1}{p}-\frac{1}{q}-1} = 2^{n(1+\frac{1}{q}-\frac{1}{p})}|Q_k|^{\frac{1}{p}-\frac{1}{q}-1} \\ &= 2^{n(1+\frac{1}{q}-\frac{1}{p})}|Q_k|^{\frac{1}{p}-\frac{1}{q}-1}(1-(\log|Q_k|)_-)^{\alpha}, \end{aligned}$$

which yields

$$\begin{aligned} |Q|^{\frac{1}{p}-\frac{1}{q}-1}(1-(\log|Q|)_-)^{\alpha} \int_{Q\cap Q_k} |f(y)|\,dy & \\ &\leq 2^{n(1+\frac{1}{q}-\frac{1}{p})}|Q_k|^{\frac{1}{p}-\frac{1}{q}-1}(1-(\log|Q_k|)_-)^{\alpha} \int_{Q_k} |f(y)|\,dy \\ &\leq 2^{n(1+\frac{1}{q}-\frac{1}{p})} M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_k}(f\mathbf{1}_{Q_k})(x). \end{aligned}$$

Consequently,

$$M_{n(\frac{1}{p}-\frac{1}{q}),\alpha}(f\mathbf{1}_{Q_k})(x) \leq 2^{n(1+\frac{1}{q}-\frac{1}{p})} M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_k}(f\mathbf{1}_{Q_k})(x).$$

This completes the proof of (3-17).

On the other hand, since

$$Q_k \subset Q_{k+1} \quad \text{and} \quad \bigcup_{k\in\mathbb{N}} Q_k = \mathbb{R}^n, \tag{3-18}$$

we have

$$\lim_{k\to\infty} M_{n(\frac{1}{p}-\frac{1}{q}),\alpha}(f\mathbf{1}_{Q_k})(x)\mathbf{1}_{Q_k}(x) = M_{n(\frac{1}{p}-\frac{1}{q}),\alpha}f(x), \quad x \in \mathbb{R}^n. \tag{3-19}$$

Indeed, given any fixed $x \in \mathbb{R}^n$, we have

$$M_{n(\frac{1}{p}-\frac{1}{q}),\alpha}(f\mathbf{1}_{Q_k})(x)\mathbf{1}_{Q_k}(x) \leq M_{n(\frac{1}{p}-\frac{1}{q}),\alpha}f(x), \quad \text{for all } k \in \mathbb{N},$$

and (see (3-18))

$$M_{n(\frac{1}{p}-\frac{1}{q}),\alpha}(f\mathbf{1}_{Q_k})(x)\mathbf{1}_{Q_k}(x) \leq M_{n(\frac{1}{p}-\frac{1}{q}),\alpha}(f\mathbf{1}_{Q_{k+1}})(x)\mathbf{1}_{Q_{k+1}}(x). \tag{3-20}$$

By the monotone convergence theorem for sequences of real numbers, we derive

$$\begin{aligned} \lim_{k\to\infty} M_{n(\frac{1}{p}-\frac{1}{q}),\alpha}(f\mathbf{1}_{Q_k})(x)\mathbf{1}_{Q_k}(x) &= \sup_{k\in\mathbb{N}} M_{n(\frac{1}{p}-\frac{1}{q}),\alpha}(f\mathbf{1}_{Q_k})(x) \\ &= \sup_{\substack{Q\in\mathcal{D}(\mathbb{R}^n)\\ x\in Q}} |Q|^{\frac{1}{p}-\frac{1}{q}-1}(1-(\log|Q|)_-)^{\alpha} \sup_{k\in\mathbb{N}} \int_{Q\cap Q_k} |f(y)|\,dy \\ &= M_{n(\frac{1}{p}-\frac{1}{q}),\alpha}f(x), \end{aligned}$$

where we have used (3-18) in the last step.

It follows from (3-17) that

$$\big\| M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_k}(f\mathbf{1}_{Q_k})\big\|_{L^q(Q_k)} \approx \big\| M_{n(\frac{1}{p}-\frac{1}{q}),\alpha}(f\mathbf{1}_{Q_k})\mathbf{1}_{Q_k}\big\|_{L^q(\mathbb{R}^n)},$$

uniformly with respect to k. Consequently, applying the monotone convergence theorem (see (3-19) and (3-20)):

$$\lim_{k\to\infty}\big\| M_{n(\frac{1}{p}-\frac{1}{q}),\alpha,Q_k}(f\mathbf{1}_{Q_k})\big\|_{L^q(Q_k)} \approx \| M_{n(\frac{1}{p}-\frac{1}{q}),\alpha} f\|_{L^q(\mathbb{R}^n)}. \tag{3-21}$$

Step 3. Next we deal with the left-hand side of (3-16). We claim that

$$\|f\mathbf{1}_{Q_k}\|_{SR_{p,q}\log^\alpha(Q_k)} \approx \|f\mathbf{1}_{Q_k}\|_{SR_{p,q}\log^\alpha(\mathbb{R}^n)} \tag{3-22}$$

uniformly with respect to k and f.

The estimate $\lesssim$ can be obtained as follows. Given any $(Q_i)_{i\in I}\in S(Q_k)$, there are two possible scenarios. (I) $Q_i\neq Q_k$ for every $i\in I$. Then $(Q_i)_{i\in I}\in S(\mathbb{R}^n)$, since $\mathcal{D}(Q_k)\backslash\{Q_k\}\subset\mathcal{D}(\mathbb{R}^n)$. Clearly, this implies $\|f\mathbf{1}_{Q_k}\|_{SR_{p,q}\log^\alpha(Q_k)}\le\|f\mathbf{1}_{Q_k}\|_{SR_{p,q}\log^\alpha(\mathbb{R}^n)}$. (II) Suppose now that there is $i_0\in I$ such that $Q_{i_0}=Q_k$. In particular, $(Q_i)_{i\in I\backslash\{i_0\}}\in S(\mathbb{R}^n)$ and $Q_i\subset Q_k$ for $i\in I\backslash\{i_0\}$. Now, the first dyadic decomposition of Q_k (i.e., $\{Q^1_{k,l}:l=1,\dots,2^n\}$) is formed by pairwise disjoint cubes in $\mathcal{D}(\mathbb{R}^n)$ (so, in particular, $\{Q^1_{k,l}:l=1,\dots,2^n\}\in S(\mathbb{R}^n)$) with $|Q^1_{k,l}|=2^{-n}|Q_k|$. Hence, in this case, we can split the sum related to the $SR_{p,q}\log^\alpha(Q_k)$-norm as

$$\begin{aligned}
\sum_{i\in I}&\bigg(\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'}}\int_{Q_i}|f(y)|\,dy\bigg)^q\\
&=\sum_{\substack{i\in I\\ i\neq i_0}}\bigg(\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'}}\int_{Q_i}|f(y)|\,dy\bigg)^q+\bigg(\frac{1}{|Q_k|^{1/p'}}\int_{Q_k}|f(y)|\,dy\bigg)^q\\
&\le\|f\mathbf{1}_{Q_k}\|^q_{SR_{p,q}\log^\alpha(\mathbb{R}^n)}+\frac{1}{2^{nq/p'}}\bigg(\sum_{l=1}^{2^n}\frac{1}{|Q^1_{k,l}|^{1/p'}}\int_{Q^1_{k,l}}|f(y)|\,dy\bigg)^q\\
&\lesssim\|f\mathbf{1}_{Q_k}\|^q_{SR_{p,q}\log^\alpha(\mathbb{R}^n)}+\sum_{l=1}^{2^n}\bigg(\frac{1}{|Q^1_{k,l}|^{1/p'}}\int_{Q^1_{k,l}}|f(y)|\,dy\bigg)^q\\
&\lesssim\|f\mathbf{1}_{Q_k}\|^q_{SR_{p,q}\log^\alpha(\mathbb{R}^n)}.
\end{aligned}$$

Therefore, taking the supremum over all possible $(Q_i)_{i\in I}\in S(Q_k)$, we achieve

$$\|f\mathbf{1}_{Q_k}\|_{SR_{p,q}\log^\alpha(Q_k)}\lesssim\|f\mathbf{1}_{Q_k}\|_{SR_{p,q}\log^\alpha(\mathbb{R}^n)},$$

i.e., the estimate $\lesssim$ in (3-22) is shown.

To deal with the converse estimate, for any $\mathcal{Q}=(Q_i)_{i\in I}\in S(\mathbb{R}^n)$, we consider the index set

$$I_k:=\{i\in I:Q_i\subset Q_k\}.$$

Therefore we can split

$$\begin{aligned}\sum_{i\in I}&\biggl(\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'}}\int_{Q_i}|f(y)|\mathbf{1}_{Q_k}(y)\,dy\biggr)^q\\&=\sum_{i\in I_k}\biggl(\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'}}\int_{Q_i}|f(y)|\,dy\biggr)^q+\sum_{i\in I\backslash I_k}\biggl(\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'}}\int_{Q_i\cap Q_k}|f(y)|\,dy\biggr)^q\\&=:R_1+R_2.\end{aligned}\tag{3-23}$$

Note that $(Q_i)_{i\in I_k}\in S(Q_k)$ (since $(Q_i)_{i\in I_k}\in S(\mathbb{R}^n)\cap\mathcal{D}(Q_k)$). Accordingly

$$R_1\le\|f\mathbf{1}_{Q_k}\|^q_{SR_{p,q}\log^\alpha(Q_k)}.\tag{3-24}$$

Concerning R_2, we argue as follows. Let $i\in I\backslash I_k$, i.e., $Q_i\not\subset Q_k$. Assume further that $Q_i\cap Q_k\neq\varnothing$. Note that Q_k is not a dyadic cube in $\mathbb{R}^n$, but its first dyadic generation $\{Q^1_{k,l}:l=1,\dots,2^n\}$ is formed by dyadic cubes in $\mathbb{R}^n$. Since Q_k can be expressed as the disjoint union of the cubes $Q^1_{k,l}$, we can assert that there exists a unique $l(i)$ such that $Q_i\cap Q^1_{k,l(i)}\neq\varnothing$. By the structure of dyadic cubes in $\mathbb{R}^n$, we have either $Q_i\subset Q^1_{k,l(i)}$ or $Q^1_{k,l(i)}\subset Q_i$. The former is not possible; otherwise, $Q_i\subset Q_k$ but $i\notin I_k$. Hence $Q^1_{k,l(i)}\subset Q_i$, therefore

$$\int_{Q_i\cap Q_k}|f(y)|\,dy=\int_{Q^1_{k,l(i)}}|f(y)|\,dy.\tag{3-25}$$

For $l\in\{1,\dots,2^n\}$, we define

$$\mathcal{Q}_{k,l}:=\{Q_i:i\in I\backslash I_k\text{ and }Q^1_{k,l}\subset Q_i\}.$$

The above argument leads to

$$(Q_i)_{i\in I\backslash I_k}=\bigcup_{l=1}^{2^n}\mathcal{Q}_{k,l}.\tag{3-26}$$

Moreover, since the Q_i's are dyadic cubes in $\mathbb{R}^n$, the elements of $\mathcal{Q}_{k,l}=\{Q_1,Q_2,\dots\}$ can be ordered in such a way that $Q^1_{k,l}\subset Q_1\subset Q_2\subset\cdots$ We cannot exclude the possibility that some of the cubes in $\mathcal{Q}_{k,l}$ coincide, but the number of these cubes is uniformly bounded by the sparse constant η in Definition 3. Therefore, by (3-25) and (3-26),

$$\begin{aligned}R_2&\le\sum_{i\in I\backslash I_k}\biggl(\frac{1}{|Q_i|^{1/p'}}\int_{Q^1_{k,l(i)}}|f(y)|\,dy\biggr)^q=\sum_{l=1}^{2^n}\biggl(\int_{Q^1_{k,l}}|f(y)|\,dy\biggr)^q\sum_{Q_i\in\mathcal{Q}_{k,l}}\frac{1}{|Q_i|^{q/p'}}\\&\lesssim\sum_{l=1}^{2^n}\biggl(\int_{Q^1_{k,l}}|f(y)|\,dy\biggr)^q\sum_{j=k}^{\infty}2^{-jnq/p'}\approx\sum_{l=1}^{2^n}\biggl(\int_{Q^1_{k,l}}|f(y)|\,dy\biggr)^q2^{-knq/p'}\\&=\sum_{l=1}^{2^n}\biggl(\frac{1}{|Q^1_{k,l}|^{1/p'}}\int_{Q^1_{k,l}}|f(y)|\,dy\biggr)^q\le\|f\mathbf{1}_{Q_k}\|^q_{SR_{p,q}\log^\alpha(Q_k)}.\end{aligned}\tag{3-27}$$

Combining (3-23), (3-24) and (3-27),

$$\sum_{i\in I}\left(\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'}}\int_{Q_i}|f(y)|\mathbf{1}_{Q_k}(y)\,dy\right)^q \lesssim \|f\mathbf{1}_{Q_k}\|^q_{SR_{p,q}\log^\alpha(Q_k)}$$

for all $(Q_i)_{i\in I}\in S(\mathbb{R}^n)$. In particular,

$$\|f\mathbf{1}_{Q_k}\|_{SR_{p,q}\log^\alpha(\mathbb{R}^n)} \lesssim \|f\mathbf{1}_{Q_k}\|_{SR_{p,q}\log^\alpha(Q_k)},$$

completing the proof of (3-22).

By the lattice property of sparse RMT spaces, we have (recall (3-18))

$$\|f\mathbf{1}_{Q_k}\|_{SR_{p,q}\log^\alpha(\mathbb{R}^n)} \le \|f\mathbf{1}_{Q_{k+1}}\|_{SR_{p,q}\log^\alpha(\mathbb{R}^n)}$$

and

$$\|f\mathbf{1}_{Q_k}\|_{SR_{p,q}\log^\alpha(\mathbb{R}^n)} \le \|f\|_{SR_{p,q}\log^\alpha(\mathbb{R}^n)}, \quad k\in\mathbb{N}.$$

Hence, applying the monotone convergence theorem and the dominated convergence theorem, we obtain

$$\begin{aligned}\lim_{k\to\infty}\|f\mathbf{1}_{Q_k}\|_{SR_{p,q}\log^\alpha(\mathbb{R}^n)} &= \sup_{k\in\mathbb{N}}\|f\mathbf{1}_{Q_k}\|_{SR_{p,q}\log^\alpha(\mathbb{R}^n)}\\ &= \sup_{(Q_i)_{i\in I}\in S(\mathbb{R}^n)}\ \sup_{k\in\mathbb{N}}\left\{\sum_{i\in I}\left[\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'}}\int_{Q_i\cap Q_k}|f|\right]^q\right\}^{\frac{1}{q}}\\ &= \sup_{(Q_i)_{i\in I}\in S(\mathbb{R}^n)}\ \lim_{k\to\infty}\left\{\sum_{i\in I}\left[\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'}}\int_{Q_i\cap Q_k}|f|\right]^q\right\}^{\frac{1}{q}}\\ &= \sup_{(Q_i)_{i\in I}\in S(\mathbb{R}^n)}\left\{\sum_{i\in I}\left[\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{1/p'}}\int_{Q_i}|f|\right]^q\right\}^{\frac{1}{q}}.\end{aligned}$$

In other words, we have shown that

$$\lim_{k\to\infty}\|f\mathbf{1}_{Q_k}\|_{SR_{p,q}\log^\alpha(\mathbb{R}^n)} = \|f\|_{SR_{p,q}\log^\alpha(\mathbb{R}^n)}. \tag{3-28}$$

Step 4. Finally, taking limits on both sides of (3-16) as $k\to\infty$, and invoking (3-21), (3-22) and (3-28), we achieve the desired estimate

$$\|f\|_{SR_{p,q}\log^\alpha(\mathbb{R}^n)} \approx \|M_{n(\frac{1}{p}-\frac{1}{q}),\alpha}f\|_{L^q(\mathbb{R}^n)}.$$ □

4. A sparse approach to H^{-1}-stability

As already mentioned in Section 1.5, one of the main features of the theory of sparse function spaces lies in the fact that, unlike their classical parent spaces, they often admit complete explicit characterizations. Indeed, Theorem 6 provides us with the following surprising (informal) characterization: *$SR_{p,q}$ spaces can be identified with negative Sobolev spaces*. Before we give the proof of this result, we introduce some basic notation.

Consider the *Riesz potential operators* I_λ, $\lambda \in (0,n)$, formally defined, for $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, by

$$I_\lambda f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\lambda}}\,dy, \quad x \in \mathbb{R}^n.$$

For $1 < q < \infty$, we let

$$H_q^{-\lambda}(\mathbb{R}^n) := \{ f \in L^1_{\text{loc}}(\mathbb{R}^n) : \|f\|_{H_q^{-\lambda}} = \|I_\lambda f\|_{L^q(\mathbb{R}^n)} < \infty \}, \tag{4-1}$$

the *Riesz potential space*, and its associated lattice

$$\mathcal{H}_q^{-\lambda}(\mathbb{R}^n) := \{ f \in L^1_{\text{loc}}(\mathbb{R}^n) : \|f\|_{\mathcal{H}_q^{-\lambda}} = \|I_\lambda(|f|)\|_{L^q(\mathbb{R}^n)} < \infty \}. \tag{4-2}$$

It is plain that

$$\mathcal{H}_q^{-\lambda}(\mathbb{R}^n) \subset H_q^{-\lambda}(\mathbb{R}^n).$$

Furthermore, as it is customary, we shall suppress the subindex $q = 2$ and simply write

$$\mathcal{H}^{-\lambda}(\mathbb{R}^n) := \mathcal{H}_2^{-\lambda}(\mathbb{R}^n) \quad \text{and} \quad H^{-\lambda}(\mathbb{R}^n) := H_2^{-\lambda}(\mathbb{R}^n). \tag{4-3}$$

4.1. ***Proof of Theorem 6.*** In order to be able to use a result of Muckenhoupt and Wheeden [1974] we introduce the fractional maximal[24] operator, defined for $f \in L^1_{\text{loc}}(\mathbb{R}^n)$,

$$\mathcal{M}_\lambda f(x) := \sup_{x \in Q} |Q|^{\frac{\lambda}{n}-1} \int_Q |f(y)|\,dy, \quad x \in \mathbb{R}^n,$$

where the supremum runs over all (not necessarily dyadic) cubes Q in $\mathbb{R}^n$, with $x \in Q$. It is plain (see (3-15)) that $M_{\lambda,0} f(x) \le \mathcal{M}_\lambda f(x)$, and, although this pointwise inequality cannot be reversed, it is well-known that by the $\frac{1}{3}$-translation trick (see [Christ 1988]) we have the equivalence

$$\|M_{\lambda,0} f\|_{L^q(\mathbb{R}^n)} \approx \|\mathcal{M}_\lambda f\|_{L^q(\mathbb{R}^n)}. \tag{4-4}$$

Putting together Theorem 28, with $\alpha = 0$ and $\lambda = n\big(\frac{1}{p} - \frac{1}{q}\big)$, and (4-4), we get

$$\|f\|_{SR_{p,q}(\mathbb{R}^n)} \approx \|\mathcal{M}_{n(\frac{1}{p}-\frac{1}{q})} f\|_{L^q(\mathbb{R}^n)}. \tag{4-5}$$

For the maximal operator $\mathcal{M}_\lambda$ we have the trivial estimate

$$\mathcal{M}_\lambda f(x) \le c_n I_\lambda(|f|)(x), \quad x \in \mathbb{R}^n,$$

where c_n depends only on n. In fact, via the Muckenhoupt–Wheeden theorem [1974, Theorem 1], we achieve, for $0 < q < \infty$,

$$\|\mathcal{M}_\lambda f\|_{L^q(\mathbb{R}^n)} \approx \|I_\lambda(|f|)\|_{L^q(\mathbb{R}^n)}. \tag{4-6}$$

Combining (4-2), (4-5) and (4-6) we arrive at

$$\|f\|_{SR_{p,q}(\mathbb{R}^n)} \approx \|f\|_{\mathcal{H}_q^{-n(\frac{1}{p}-\frac{1}{q})}(\mathbb{R}^n)},$$

as we wished to show. □

[24] Compare with the dyadic local version M_{λ,Q_0} defined in (3-1).

Our next result refers to the limiting case $p = q$ in Theorem 6 and it can be viewed as the sparse counterpart of the Riesz's theorem (see [Domínguez and Milman 2021, p. 1062]),

$$R_{p,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n). \tag{4-7}$$

Theorem 29. *Let* $1 < p < \infty$. *Then*

$$SR_{p,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n).$$

Proof. This is an immediate consequence of Theorem 28 (with $p = q$ and $\alpha = 0$) and the classical Hardy–Littlewood maximal theorem:

$$\|Mf\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}, \quad p > 1. \qquad \square$$

4.2. ***On the difference between*** $R_{1,2}(\mathbb{R}^2)$ ***and*** $SR_{1,2}(\mathbb{R}^2)$**.** In view of Theorem 29 and (4-7),

$$SR_{p,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n) = R_{p,p}(\mathbb{R}^n), \quad 1 < p < \infty.$$

One may be tempted to think that $SR_{p,q}(\mathbb{R}^n) = R_{p,q}(\mathbb{R}^n)$ for general values of p and q. However, this is far from being true. Next we concentrate on the most relevant case for the purposes of this paper, i.e., we will show that the embedding

$$SR_{1,2}(\mathbb{R}^2) \subset R_{1,2}(\mathbb{R}^2)$$

is strict, in the sense that,

$$SR_{1,2}(\mathbb{R}^2) \neq R_{1,2}(\mathbb{R}^2). \tag{4-8}$$

As a by-product (see (1-9)),

$$R_{1,2}(\mathbb{R}^2) \not\subseteq H^{-1}(\mathbb{R}^2). \tag{4-9}$$

We shall use an elementary but indirect method. It is well known (see, e.g., [Lions 1996, p. 141]) that the largest rearrangement invariant space embedded in $H^{-1}_{\mathrm{loc}}(\mathbb{R}^2)$ is the *Lorentz space* defined for a given cube Q_0[25] by

$$L^{(1,2)}(Q_0) := \left\{ f : \|f\|_{L^{(1,2)}(Q_0)} = \left[\int_0^1 (t f^{**}_{Q_0}(t))^2 \frac{dt}{t} \right]^{\frac{1}{2}} < \infty \right\}.$$

Here, we use standard notation: $f^*_{Q_0}$ is the nonincreasing rearrangement of a measurable function f restricted to Q_0, more precisely, $f^*_{Q_0}$ is the generalized inverse of the distribution function

$$\lambda_f(\alpha) = \left| \{x \in Q_0 : |f(x)| \geq \alpha\} \right|, \quad \alpha > 0,$$

and $f^{**}_{Q_0}$ is the maximal function given by $f^{**}_{Q_0}(t) = t^{-1} \int_0^t f^*_{Q_0}(s)\, ds$. When there is no danger of confusion, we shall simply drop Q_0 and use the notation f^* and f^{**} rather than $f^*_{Q_0}$ and $f^{**}_{Q_0}$, respectively.

[25] Without loss of generality, we may assume that $|Q_0| = 1$.

Using that $t \mapsto t f^{**}(t)$ is an increasing function, we have, for every $u \in (0,1)$,

$$\|f\|_{L^{(1,2)}(Q_0)} \geq \left[\int_u^1 (t f^{**}(t))^2 \frac{dt}{t}\right]^{\frac{1}{2}} \geq (-\log u)^{\frac{1}{2}} u f^{**}(u).$$

It follows that

$$L^{(1,2)}(Q_0) \subset L^{(1,\infty)}(\log L)^{\frac{1}{2}}(Q_0),$$

where $L^{(1,\infty)}(\log L)^{\frac{1}{2}}(Q_0)$ is the *Lorentz–Zygmund space* defined by[26]

$$\|f\|_{L^{(1,\infty)}(\log L)^{\frac{1}{2}}(Q_0)} := \sup_{0<t<1} t(1-\log t)^{\frac{1}{2}} f^{**}(t).$$

Then

$$L^{(1,\infty)}(\log L)^{\frac{1}{2}}(Q_0) \not\subseteq H^{-1}_{\text{loc}}(\mathbb{R}^2),$$

which in turn yields (see Theorem 6)

$$L^{(1,\infty)}(\log L)^{\frac{1}{2}}(Q_0) \not\subseteq SR_{1,2,\text{loc}}(\mathbb{R}^2). \tag{4-10}$$

On the other hand, by the Hardy–Littlewood inequality for rearrangements (see, e.g., [Bennett and Sharpley 1988, Lemma 2.1, p. 44]) and (4-10),

$$\begin{aligned}
\|f\|^2_{R_{1,2}(Q_0)} &= \sup_{(Q_i)_{i\in I}\in\Pi(Q_0)} \sum_{i\in I}\left(\int_{Q_i}|f|\right)^2 \\
&\leq \sup_{(Q_i)_{i\in I}\in\Pi(Q_0)} \sum_{i\in I}\left(\int_0^{|Q_i|} f^*\right)\left(\int_{Q_i}|f|\right) \\
&= \sup_{(Q_i)_{i\in I}\in\Pi(Q_0)} \sum_{i\in I}|Q_i| f^{**}(|Q_i|)\int_{Q_i}|f| \\
&\leq \|f\|_{L^{(1,\infty)}(\log L)^{\frac{1}{2}}(Q_0)} \sup_{(Q_i)\in\Pi(Q_0)} \sum_{i\in I}(1-\log|Q_i|)^{-\frac{1}{2}}\int_{Q_i}|f| \\
&\leq \|f\|_{L^{(1,\infty)}(\log L)^{\frac{1}{2}}(Q_0)} \sup_{(Q_i)_{i\in I}\in\Pi(Q_0)} \sum_{i\in I}\int_{Q_i}|f| \\
&\leq \|f\|_{L^{(1,\infty)}(\log L)^{\frac{1}{2}}(Q_0)}\|f\|_{L^1(Q_0)} \\
&\leq \|f\|^2_{L^{(1,\infty)}(\log L)^{\frac{1}{2}}(Q_0)}.
\end{aligned}$$

It follows that

$$L^{(1,\infty)}(\log L)^{\frac{1}{2}}(Q_0) \subset R_{1,2,\text{loc}}(\mathbb{R}^2). \tag{4-11}$$

Consequently, (4-8) now follows from (4-10) and (4-11).

[26] A basic reference to Lorentz–Zygmund spaces is [Bennett and Rudnick 1980].

4.3. ***Proof of Theorem 9.*** (i) Let Q_0 be a cube. Recalling that

$$\|f\|_{SR_{\frac{2n}{n+2},2}(Q_0)} = \sup_{(Q_i)_{i\in I}\in S(Q_0)} \left\{ \sum_{i\in I} \left(|Q_i|^{\frac{1}{n}-\frac{1}{2}} \int_{Q_i} |f| \right)^2 \right\}^{\frac{1}{2}}, \tag{4-12}$$

we observe $s_1(f) = \|f\|_{SR_{\frac{2n}{n+2},2}(Q_0)}$ (see (1-11)). Then

$$s_1(X) = \sup_{\|f\|_{X(Q_0)}\le 1} \|f\|_{SR_{\frac{2n}{n+2},2}(Q_0)} < \infty \quad \iff \quad X(Q_0) \hookrightarrow SR_{\frac{2n}{n+2},2}(Q_0).$$

The desired assertions follow immediately from (1-9) and (1-10).

(ii) We need to introduce some notation: Given $\mathcal{Q} = (Q_i)_{i\in I} \in S(Q_0)$, note that $\mathcal{Q}$ can be split as $\mathcal{Q} = \bigcup_{k=0}^{\infty} \mathbb{D}_{k;Q_0}(\mathcal{Q})$, where

$$\mathbb{D}_{k;Q_0} := \{Q \in \mathcal{D}(Q_0) : \ell(Q) = 2^{-k}\ell(Q_0)\}, \quad \mathbb{D}_{k;Q_0}(\mathcal{Q}) := \mathbb{D}_{k;Q_0} \cap \mathcal{Q}.$$

When there is no danger of confusion, we use the simplified notation $\mathbb{D}_k$ and $\mathbb{D}_k(\mathcal{Q})$. By construction, $\mathbb{D}_k(\mathcal{Q})$ is formed by pairwise disjoint cubes (i.e., $\mathbb{D}_k(\mathcal{Q}) \subset \Pi(Q_0)$), and for $Q_i \in \mathbb{D}_k(\mathcal{Q})$, we have $|Q_i| = 2^{-kn}|Q_0|$.

Assume that $\lim_{N\to\infty} s_N(X) = 0$, i.e., given any $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$, such that for all $N > N_0$,

$$\sup_{\|f\|_{X(Q_0)}\le 1} s_N(f) \le \varepsilon. \tag{4-13}$$

Let $\mathcal{Q} = (Q_i)_{i\in I} \in S(Q_0)$ and let $f \in X(Q_0)$ be such that $\|f\|_{X(Q_0)} \le 1$. Then

$$\left\{ \sum_{i\in I} \left(|Q_i|^{\frac{1}{n}-\frac{1}{2}} \int_{Q_i} |f| \right)^2 \right\}^{\frac{1}{2}} = \left\{ \sum_{k=0}^{\infty} \sum_{i\in I: Q_i\in\mathbb{D}_k(\mathcal{Q})} \left(|Q_i|^{\frac{1}{n}-\frac{1}{2}} \int_{Q_i} |f| \right)^2 \right\}^{\frac{1}{2}} \le \mathrm{I} + \mathrm{II}, \tag{4-14}$$

where

$$\mathrm{I} := \left\{ \sum_{k=0}^{N_0} \sum_{i\in I: Q_i\in\mathbb{D}_k(\mathcal{Q})} \left(|Q_i|^{\frac{1}{n}-\frac{1}{2}} \int_{Q_i} |f| \right)^2 \right\}^{\frac{1}{2}}$$

and

$$\mathrm{II} := \left\{ \sum_{k=N_0+1}^{\infty} \sum_{i\in I: Q_i\in\mathbb{D}_k(\mathcal{Q})} \left(|Q_i|^{\frac{1}{n}-\frac{1}{2}} \int_{Q_i} |f| \right)^2 \right\}^{\frac{1}{2}}.$$

It follows from (1-11) and (4-13) that

$$\mathrm{II} = \left\{ \sum_{i\in I: Q_i\in\mathbb{D}_{\le N_0+1}(\mathcal{Q})} \left(|Q_i|^{\frac{1}{n}-\frac{1}{2}} \int_{Q_i} |f| \right)^2 \right\}^{\frac{1}{2}} \le s_{N_0+2}(f) \le \varepsilon. \tag{4-15}$$

On the other hand, we obviously have

$$\mathrm{I} \le \left\{ \sum_{k=0}^{N_0} \sum_{Q\in\mathbb{D}_k} \left(|Q|^{\frac{1}{n}-\frac{1}{2}} \int_{Q} |f| \right)^2 \right\}^{\frac{1}{2}}. \tag{4-16}$$

Combining (4-14), (4-15) and (4-16), we find that, for all $(Q_i)_{i\in I}\in S(Q_0)$ and for all f in the unit ball of $X(Q_0)$,

$$\left\{\sum_{i\in I}\left(|Q_i|^{\frac{1}{n}-\frac{1}{2}}\int_{Q_i}|f|\right)^2\right\}^{\frac{1}{2}}\le\left\{\sum_{k=0}^{N_0}\sum_{Q\in\mathbb{D}_k}\left(|Q|^{\frac{1}{n}-\frac{1}{2}}\int_{Q}|f|\right)^2\right\}^{\frac{1}{2}}+\varepsilon.$$

Therefore, by (4-12),

$$\|f\|_{SR_{\frac{2n}{n+2},2}(Q_0)}\le\left\{\sum_{k=0}^{N_0}\sum_{Q\in\mathbb{D}_k}\left(|Q|^{\frac{1}{n}-\frac{1}{2}}\int_{Q}|f|\right)^2\right\}^{\frac{1}{2}}+\varepsilon. \tag{4-17}$$

Let $\mathbb{D}_{\ge N_0;Q_0}=\mathbb{D}_{\ge N_0}=\bigcup_{k=0}^{N_0}\mathbb{D}_k$; then the cardinality of $\mathbb{D}_{\ge N_0}$ is

$$L:=\frac{2^{n(N_0+1)}-1}{2^n-1}.$$

Consider the linear operator

$$T:f\in X(Q_0)\mapsto\left(|Q|^{\frac{1}{n}-\frac{1}{2}}\int_Q f\right)_{Q\in\mathbb{D}_{\ge N_0}}\in\ell_2^L.$$

It is easy to see that T is well-defined: if $f\in X(Q_0)$ (and hence $f\ge 0$) then

$$\begin{aligned}\|Tf\|_{\ell_2^L}&=\left\{\sum_{k=0}^{N_0}\sum_{Q\in\mathbb{D}_k}\left(|Q|^{\frac{1}{n}-\frac{1}{2}}\int_{Q}|f|\right)^2\right\}^{\frac{1}{2}}\\&=|Q_0|^{\frac{1}{n}-\frac{1}{2}}\left\{\sum_{k=0}^{N_0}2^{-kn(\frac{1}{n}-\frac{1}{2})2}\sum_{Q\in\mathbb{D}_k}\left(\int_Q|f|\right)^2\right\}^{\frac{1}{2}}\\&\le|Q_0|^{\frac{1}{n}-\frac{1}{2}}\left\{\sum_{k=0}^{N_0}2^{-kn(\frac{1}{n}-\frac{1}{2})2}\left(\sum_{Q\in\mathbb{D}_k}\int_Q|f|\right)^2\right\}^{\frac{1}{2}}\\&=|Q_0|^{\frac{1}{n}-\frac{1}{2}}\left\{\sum_{k=0}^{N_0}2^{-kn(\frac{1}{n}-\frac{1}{2})2}\right\}^{\frac{1}{2}}\|f\|_{L^1(Q_0)}\\&\lesssim 2^{N_0(\frac{n}{2}-1)}\|f\|_{X(Q_0)}.\end{aligned}$$

Furthermore, T is compact, since it is a finite rank operator. We can equivalently rewrite (4-17) in terms of T as follows: for every $f\in X(Q_0)$, $\|f\|_{X(Q_0)}\le 1$,

$$\|f\|_{SR_{\frac{2n}{n+2},2}(Q_0)}\le\|Tf\|_{\ell_2^L}+\varepsilon. \tag{4-18}$$

Let $\{f_l\}_{l\in\mathbb{N}}$ be a bounded sequence in $X(Q_0)$ (without loss we may assume that, for all l, $\|f_l\|_X\le\frac{1}{2}$). The compactness of $T:X(Q_0)\to\ell_2^L$ guarantees (modulo passing to a subsequence) that $\{Tf_l\}_{l\in\mathbb{N}}$ is convergent in ℓ_2^L. Accordingly, there exists l_0 such that

$$\|Tf_l - Tf_{l'}\|_{\ell_2^L} \le \varepsilon, \quad \text{if } l, l' \ge l_0.$$

Therefore, by (4-18),

$$\|f_l - f_{l'}\|_{SR_{\frac{2n}{n+2},2}(Q_0)} \le 2\varepsilon.$$

Consequently, from (1-10) we see that $\{f_l\}_{l\in\mathbb{N}}$ is a Cauchy sequence in H^{-1}.

Next we show the converse statement, i.e., if[27] $X_c \xrightarrow{\text{compactly}} \mathcal{H}^{-1}_{\text{loc}}(\mathbb{R}^n)$ then

$$\lim_{N\to\infty} s_N(X) = 0. \tag{4-19}$$

By assumption $U_{X(Q_0)}$, the closure of the unit ball of $X(Q_0)$, is a compact set in $\mathcal{H}^{-1}(\mathbb{R}^n)$. In particular, for any $\delta > 0$ there exist $f_1, \dots, f_L \in U_{X(Q_0)}$ such that

$$U_{X(Q_0)} \subset \bigcup_{l=1}^{L} B\Big(f_l, \frac{\delta}{2}\Big),$$

where $B(f_l, \delta/2)$ denotes the ball in $\mathcal{H}^{-1}$ centered at f_l and radius $\delta/2$. Hence, for any $f \in X(Q_0)$, $\|f\|_{X(Q_0)} \le 1$, there exists $l \in \{1, \dots, L\}$ such that

$$\|f - f_l\|_{\mathcal{H}^{-1}(\mathbb{R}^n)} < \frac{\delta}{2}.$$

As a consequence (see (1-9))

$$\begin{aligned} s_N(f) \le s_N(f - f_l) + s_N(f_l) \le s_1(f - f_l) + s_N(f_l) &\lesssim \|f - f_l\|_{\mathcal{H}^{-1}(\mathbb{R}^n)} + s_N(f_l) \\ &< \frac{\delta}{2} + \sup_{l\in\{1,\dots,L\}} s_N(f_l). \end{aligned} \tag{4-20}$$

Assume momentarily that

$$\lim_{N\to\infty} s_N(\omega) = 0 \quad \text{for every } \omega \in \mathcal{H}^{-1}(\mathbb{R}^n). \tag{4-21}$$

In particular, we have, for N sufficiently large depending only on δ,

$$\sup_{l\in\{1,\dots,L\}} s_N(f_l) \le \frac{\delta}{2}.$$

Inserting this estimate into (4-20), we conclude that (4-19) holds.

To complete the proof, it remains to show (4-21): Fix $\chi \in C^\infty(\mathbb{R}^n)$ with[28] $\operatorname{supp} \chi \subset B(0,1)$ and $\chi \ge 0$. For every dyadic cube $Q_{jm} \in \mathbb{D}_j$, we let[29]

$$\chi_{jm}(f) := (\chi_{jm}, f) = \int_{\mathbb{R}^n} \chi_{jm}(x) f(x)\, dx, \quad m \in \mathbb{Z}^n, \tag{4-22}$$

[27] Recall that $X \subset L^1_{\text{loc},+}(\mathbb{R}^n)$; see (4-3).

[28] One may think that $\chi(x) = e^{-1/(1-|x|^2)} \mathbf{1}_{B(0,1)}(x)$.

[29] $\chi_{jm}(f)$ should be adequately interpreted in the distributional sense.

where $\chi_{jm}(x) := 2^{jn}\chi(2^j x - m)$. Without loss of generality, we may assume that $\operatorname{supp}\chi_{jm} \subset dQ_{jm}$ for a fixed constant $d > 1$ and

$$\inf_{x\in Q_{jm}} \chi_{jm}(x) \gtrsim 2^{jn}. \tag{4-23}$$

Then, we have

$$\begin{aligned}
s_N(\omega) &= \sup_{Q\in S(\mathbb{R}^n)} \left[\sum_{k=N-1}^{\infty} \sum_{i\in I: Q_i\in \mathbb{D}_k(Q)} \left(|Q_i|^{\frac{1}{n}-\frac{1}{2}} \int_{Q_i} d\omega \right)^2 \right]^{\frac{1}{2}} \\
&\le \left[\sum_{k=N-1}^{\infty} \sum_{Q\in \mathbb{D}_k} \left(|Q|^{\frac{1}{n}-\frac{1}{2}} \int_{Q} d\omega \right)^2 \right]^{\frac{1}{2}} \\
&= \left[\sum_{k=N-1}^{\infty} \sum_{Q\in \mathbb{D}_k} \left(2^{-kn(\frac{1}{n}+\frac{1}{2})} \int_{Q} 2^{kn}\, d\omega \right)^2 \right]^{\frac{1}{2}} \\
&\le \left[\sum_{k=N-1}^{\infty} 2^{k(-1-\frac{n}{2})2} \sum_{Q\in \mathbb{D}_k} \left(\int_{Q} \chi_Q(x)\, d\omega \right)^2 \right]^{\frac{1}{2}}.
\end{aligned} \tag{4-24}$$

Note that the last step is true because both $\chi_Q \ge 0$ and $\omega \ge 0$. Furthermore, using well-known estimates of function spaces in terms of local means (see, e.g., [Triebel 2008, Theorem 1.15]), we get

$$\left[\sum_{k=0}^{\infty} 2^{k(-1-\frac{n}{2})2} \sum_{Q\in \mathbb{D}_k} \left(\int_{Q} \chi_Q(x)\, d\omega \right)^2 \right]^{\frac{1}{2}} \lesssim \|\omega\|_{H^{-1}(\mathbb{R}^n)}.$$

In particular, this implies

$$\lim_{N\to\infty} \left[\sum_{k=N-1}^{\infty} 2^{k(-1-\frac{n}{2})2} \sum_{Q\in \mathbb{D}_k} \left(\int_{Q} \chi_Q(x)\, d\omega \right)^2 \right]^{\frac{1}{2}} = 0$$

provided that $\omega \in H^{-1}(\mathbb{R}^n) \cap BM_c^+$ and (see (4-24))

$$\lim_{N\to\infty} s_N(\omega) = 0.$$

This shows the desired result (4-21). $\square$

5. Computability of sparse indices

In this section we estimate the sparse indices for familiar scales of spaces.

Proposition 30 (sparse indices for L^p). *Let $n \ge 2$ and $p > \frac{2n}{n+2}$. Then, for every $N \in \mathbb{N}$,*

$$s_N(L^p) \lesssim 2^{-Nn(\frac{2+n}{2n} - \frac{1}{\min\{2,p\}})}. \tag{5-1}$$

Proof. We can estimate $s_N(f)$ (see (1-11)) as follows: Let $\mathcal{Q} = (Q_i)_{i\in I} \in S(Q_0)$. By Hölder's inequality we have

$$\begin{aligned}\sum_{Q\in\mathbb{D}_{\leq N-1}(\mathcal{Q})}\Big(|Q|^{\frac{1}{n}-\frac{1}{2}}\int_Q|f|\Big)^2 &= \sum_{k=N-1}^{\infty}\sum_{i\in I:Q_i\in\mathbb{D}_k(\mathcal{Q})}\Big(|Q_i|^{\frac{1}{n}-\frac{1}{2}}\int_{Q_i}|f|\Big)^2\\ &\leq \sum_{k=N-1}^{\infty}\sum_{i\in I:Q_i\in\mathbb{D}_k(\mathcal{Q})}|Q_i|^{\frac{2}{n}-\frac{2}{p}+1}\Big(\int_{Q_i}|f|^p\Big)^{\frac{2}{p}}\\ &= |Q_0|^{(\frac{2+n}{2n}-\frac{1}{p})2}\sum_{k=N-1}^{\infty}2^{-kn(\frac{2+n}{2n}-\frac{1}{p})2}\sum_{i\in I:Q_i\in\mathbb{D}_k}\Big(\int_{Q_i}|f|^p\Big)^{\frac{2}{p}}. \end{aligned}\tag{5-2}$$

We distinguish two possible cases. First, assume that $p \leq 2$. Then

$$\begin{aligned}\sum_{k=N-1}^{\infty}2^{-kn(\frac{2+n}{2n}-\frac{1}{p})2}\sum_{i\in I:Q_i\in\mathbb{D}_k}\Big(\int_{Q_i}|f|^p\Big)^{\frac{2}{p}} &\leq \sum_{k=N-1}^{\infty}2^{-kn(\frac{2+n}{2n}-\frac{1}{p})2}\Big(\sum_{i\in I:Q_i\in\mathbb{D}_k}\int_{Q_i}|f|^p\Big)^{\frac{2}{p}}\\ &\leq \|f\|^2_{L^p(Q_0)}\sum_{k=N-1}^{\infty}2^{-kn(\frac{2+n}{2n}-\frac{1}{p})2}\\ &\leq c_n\Big(\frac{2+n}{2n}-\frac{1}{p}\Big)^{-1}2^{-Nn(\frac{2+n}{2n}-\frac{1}{p})2}\|f\|^2_{L^p(Q_0)}. \end{aligned}\tag{5-3}$$

On the other hand, if $p > 2$ then, by Hölder's inequality,

$$\begin{aligned}\sum_{k=N-1}^{\infty}2^{-kn(\frac{2+n}{2n}-\frac{1}{p})2}\sum_{i\in I:Q_i\in\mathbb{D}_k}\Big(\int_{Q_i}|f|^p\Big)^{\frac{2}{p}} &\leq \sum_{k=N-1}^{\infty}2^{-kn(\frac{2+n}{2n}-\frac{1}{2})2}\Big(\sum_{i\in I:Q_i\in\mathbb{D}_k}\int_{Q_i}|f|^p\Big)^{\frac{2}{p}}\\ &\leq \|f\|^2_{L^p(Q_0)}\sum_{k=N-1}^{\infty}2^{-kn(\frac{2+n}{2n}-\frac{1}{2})2}\\ &\leq c_n 2^{-Nn(\frac{2+n}{2n}-\frac{1}{2})2}\|f\|^2_{L^p(Q_0)}. \end{aligned}\tag{5-4}$$

Combining (5-2)–(5-4) (and noting that all estimates are uniform with respect to $\mathcal{Q}$), we obtain

$$s_N(f) \lesssim 2^{-Nn(\frac{2+n}{2n}-\frac{1}{\min\{2,p\}})}\|f\|_{L^p(Q_0)}.$$

Taking the supremum over all $f \in L^p(Q_0)$, $\|f\|_{L^p(Q_0)} \leq 1$, we achieve the desired estimate (5-1). □

Proposition 31 (sparse indices for $M^{p,\alpha}$). *Let $n \geq 2$. If N is sufficiently large*[30] *then*

$$s_N(M^{p,\alpha}) \lesssim \begin{cases} 2^{-N(\frac{2}{n}-\frac{1}{p})\frac{n}{2}}N^{-\frac{\alpha}{2}} & \text{if } p > \frac{n}{2},\ \alpha\in\mathbb{R},\\ N^{\frac{-\alpha+1}{2}} & \text{if } p = \frac{n}{2},\ \alpha > 1.\end{cases}$$

[30] To be more precise, $2^{(N-1)n} > |Q_0|$. This assumption is not restrictive since we are only interested in the asymptotic behavior of indices. For the sake of completeness, we mention that $s_N(M^{p,\alpha})$ with $2^{(N-1)n} \leq |Q_0|$ can be also computed using the same ideas, but now the log-parameter α does not play any role.

Proof. By definition (see (2-2)),

$$\int_Q |f| \le |Q|^{\frac{1}{p'}}(1-(\log|Q|)_-)^{-\alpha}\,\|f\|_{M^{p,\alpha}(Q_0)} \quad \text{for all } Q\in\mathcal{D}(Q_0).$$

Therefore, for any $\mathcal{Q}=(Q_i)_{i\in I}\in S(Q_0)$,

$$\begin{aligned}
\sum_{Q\in\mathbb{D}_{\le N-1}(\mathcal{Q})}\Big(|Q|^{\frac{1}{n}-\frac{1}{2}}\int_Q|f|\Big)^2
&= \sum_{k=N-1}^{\infty}\sum_{i\in I:Q_i\in\mathbb{D}_k(\mathcal{Q})}\Big(|Q_i|^{\frac{1}{n}-\frac{1}{2}}\int_{Q_i}|f|\Big)^2\\
&\le \|f\|_{M^{p,\alpha}(Q_0)}\sum_{k=N-1}^{\infty}\sum_{i\in I:Q_i\in\mathbb{D}_k(\mathcal{Q})}|Q_i|^{\frac{2}{n}-\frac{1}{p}}(1-(\log|Q_i|)_-)^{-\alpha}\int_{Q_i}|f|\\
&\approx \|f\|_{M^{p,\alpha}(Q_0)}\sum_{k=N-1}^{\infty}2^{-kn(\frac{2}{n}-\frac{1}{p})}k^{-\alpha}\sum_{i\in I:Q_i\in\mathbb{D}_k(\mathcal{Q})}\int_{Q_i}|f|\\
&\le \|f\|_{M^{p,\alpha}(Q_0)}\|f\|_{L^1(Q_0)}\sum_{k=N-1}^{\infty}2^{-kn(\frac{2}{n}-\frac{1}{p})}k^{-\alpha}\\
&\le \|f\|^2_{M^{p,\alpha}(Q_0)}\sum_{k=N-1}^{\infty}2^{-kn(\frac{2}{n}-\frac{1}{p})}k^{-\alpha}. \qquad (5\text{-}5)
\end{aligned}$$

Furthermore

$$\sum_{k=N-1}^{\infty}2^{-kn(\frac{2}{n}-\frac{1}{p})}k^{-\alpha}\approx\begin{cases}2^{-Nn(\frac{2}{n}-\frac{1}{p})}N^{-\alpha} & \text{if } p>\frac{n}{2},\ \alpha\in\mathbb{R},\\ N^{-\alpha+1} & \text{if } p=\frac{n}{2},\ \alpha>1.\end{cases} \qquad (5\text{-}6)$$

The desired result follows then from (5-5) and (5-6). ☐

Proposition 32 (sparse indices for RMT spaces). *Let $n\ge 2$. If N is sufficiently large then*

$$s_N(R_{p,2}\log^\alpha)\lesssim\begin{cases}2^{-Nn(\frac{n+2}{2n}-\frac{1}{p})}N^{-\alpha} & \text{if } p>\frac{2n}{n+2},\ \alpha\in\mathbb{R},\\ N^{-\alpha+\frac{1}{2}} & \text{if } p=\frac{2n}{n+2},\ \alpha>\frac{1}{2}.\end{cases}$$

Proof. Let $f\in R_{p,2}\log^\alpha(Q_0)$ and $\mathcal{Q}=(Q_i)_{i\in I}\in S(Q_0)$. Since $\mathbb{D}_{k;Q_0}(\mathcal{Q})\subset\mathbb{D}_{k;Q_0}\subset\Pi(Q_0)$, we get

$$\begin{aligned}
\sum_{Q\in\mathbb{D}_{\le N-1}(\mathcal{Q})}\Big(|Q|^{\frac{1}{n}-\frac{1}{2}}\int_Q|f|\Big)^2 &= \sum_{k=N-1}^{\infty}\sum_{i\in I:Q_i\in\mathbb{D}_k(\mathcal{Q})}\Big(|Q_i|^{\frac{1}{n}-\frac{1}{2}}\int_{Q_i}|f|\Big)^2\\
&\lesssim \sum_{k=N-1}^{\infty}2^{-kn2(\frac{n+2}{2n}-\frac{1}{p})}(1+k)^{-2\alpha}\sum_{Q\in\mathbb{D}_k}\Big(\frac{(1-(\log|Q|)_-)^\alpha}{|Q|^{1/p'}}\int_Q|f|\Big)^2\\
&\le \|f\|^2_{R_{p,2}\log^\alpha(Q_0)}\sum_{k=N-1}^{\infty}2^{-kn2(\frac{n+2}{2n}-\frac{1}{p})}(1+k)^{-2\alpha}.
\end{aligned}$$

Since the previous estimates are uniform with respect to $\mathcal{Q}$, we derive

$$s_N(f)^2 \lesssim \mathcal{I}_N \, \|f\|^2_{R_{p,2}\log^\alpha(Q_0)}, \tag{5-7}$$

where

$$\mathcal{I}_N := \sum_{k=N-1}^{\infty} 2^{-kn2(\frac{n+2}{2n}-\frac{1}{p})}(1+k)^{-2\alpha}.$$

Observe that

$$\mathcal{I}_N \approx \begin{cases} 2^{-Nn2(\frac{n+2}{2n}-\frac{1}{p})} N^{-2\alpha} & \text{if } p > \frac{2n}{n+2},\ \alpha \in \mathbb{R}, \\ N^{-2\alpha+1} & \text{if } p = \frac{2n}{n+2},\ \alpha > \frac{1}{2}. \end{cases} \tag{5-8}$$

Plugging (5-8) into (5-7) and taking the supremum over all $f \in R_{p,2}\log^\alpha$, $\|f\|_{R_{p,2}\log^\alpha} \leq 1$, we arrive at the desired estimate for $s_N(R_{p,2}\log^\alpha)$. □

Remark 33. The proof of Proposition 32 gives a slightly stronger result using the refined class $CR_{p,q}\log^\alpha$. These spaces are defined using congruent cubes, by[31]

$$\|f\|_{CR_{p,q}\log^\alpha(Q_0)} := \sup_{k\in\mathbb{N}_0} \sup_{(Q_i)_{i\in I}\in\mathbb{D}_{k;Q_0}} \left\{ \sum_{i\in I} \left[\frac{(1-(\log|Q_i|)_-)^\alpha}{|Q_i|^{\frac{1}{p'}}} \int_{Q_i} |f| \right]^q \right\}^{\frac{1}{q}} < \infty.$$

Clearly $\|f\|_{CR_{p,q}\log^\alpha(Q_0)} \leq \|f\|_{R_{p,q}\log^\alpha(Q_0)}$ and hence

$$s_N(R_{p,q}\log^\alpha(Q_0)) \leq s_N(CR_{p,q}\log^\alpha(Q_0)).$$

Then Proposition 32 with $CR_{p,2}\log^\alpha$ also holds.

6. Proof of Theorem 2

(ii) ⇒ (i). Assume that $\{u^\varepsilon\}_{\varepsilon>0}$ is sparse stable, i.e., $\{\omega^\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $S_\Psi(\mathbb{R}^n)$ for some decay Ψ. It follows from (1-15) that $\{\omega^\varepsilon\}_{\varepsilon>0}$ is a precompact set in $H^{-1}_{\text{loc}}(\mathbb{R}^n)$, i.e., $\{u^\varepsilon\}_{\varepsilon>0}$ is H^{-1}-stable.

(i) ⇒ (ii). Define

$$\Psi(N) := \sup_{\varepsilon>0} s_N(\omega^\varepsilon). \tag{6-1}$$

It is clear that Ψ is decreasing. Furthermore $s_N(\omega^\varepsilon) \leq \Psi(N)$, which yields that for all $\varepsilon > 0$,

$$\|\omega^\varepsilon\|_{S_\Psi(\mathbb{R}^n)} = \sup_{N\in\mathbb{N}} \frac{s_N(\omega^\varepsilon)}{\Psi(N)} \leq 1,$$

i.e., $\{\omega^\varepsilon\}_{\varepsilon>0}$ is bounded in $S_\Psi(\mathbb{R}^n)$.

[31] Note that $CR_{p,q}\log^\alpha(Q_0)$ can be equivalently introduced as the set of all $f \in L^1(Q_0)$ such that, for every $k \geq 0$,

$$\left\{ \sum_{Q_i\in\mathbb{D}_{k;Q_0}} \left(\int_{Q_i} |f| \right)^q \right\}^{1/q} \lesssim \begin{cases} 2^{kn/p'}(1+k)^{-\alpha} & \text{if } 2^k \geq \ell(Q_0), \\ 2^{kn/p'} & \text{if } 2^k \leq \ell(Q_0). \end{cases}$$

It remains to show that $\Psi(N) \to 0$ as $N \to \infty$. The proof follows closely the one of (4-19). On account of (i), the set $W = \overline{\{\omega^\varepsilon\}_{\varepsilon>0}} \subset BM_c^+$ is compact (in $\mathcal{H}^{-1}(\mathbb{R}^n)$). In particular, for any $\delta > 0$ there exist $\omega_1, \dots, \omega_L \in W$ such that

$$W \subset \bigcup_{l=1}^{L} B\Big(\omega_l, \frac{\delta}{2}\Big).$$

As a by-product, for any $\varepsilon > 0$ we can find $l \in \{1, \dots, L\}$ such that

$$\|\omega^\varepsilon - \omega_l\|_{\mathcal{H}^{-1}(\mathbb{R}^n)} < \frac{\delta}{2}. \tag{6-2}$$

Therefore

$$s_N(\omega^\varepsilon) \le s_N(\omega^\varepsilon - \omega_l) + s_N(\omega_l) \lesssim \|\omega^\varepsilon - \omega_l\|_{\mathcal{H}^{-1}(\mathbb{R}^n)} + s_N(\omega_l),$$

where the hidden equivalence constant is independent of ε. As a consequence (see (6-1) and (6-2))

$$\Psi(N) \lesssim \frac{\delta}{2} + \sup_{l\in\{1,\dots,L\}} s_N(\omega_l).$$

Then, by (4-21), $\lim_{N\to\infty} \Psi(N) = 0$. □

7. Sharpening Morrey regularity of DiPerna–Majda via V_Ψ

As already mentioned in Section 1.1, a famous 2D result due to DiPerna and Majda [1987a] asserts strong convergence of approximate solutions with initial vortex sheet satisfying Morrey regularity $M^{1,\alpha}(\mathbb{R}^2)$ with $\alpha > 1$. A proof can be obtained from the compactness assertion (see (1-2))

$$M_c^{\frac{n}{2},\alpha}(\mathbb{R}^n) \xhookrightarrow{\text{compactly}} H^{-1}_{\text{loc}}(\mathbb{R}^n), \quad \alpha > 1,\ n \ge 2. \tag{7-1}$$

The goal of this section is to show that these results can be further improved using the sparse techniques developed in previous sections combined with extrapolation techniques.

7.1. V_Ψ-spaces. Given a decay function Ψ, in this section we construct a new Besov-type space V_Ψ, whose sparse indices are controlled by Ψ.

Definition 34. Let $V_\Psi(\mathbb{R}^n)$ be the space formed by all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that[32]

$$\|f\|_{V_\Psi(\mathbb{R}^n)} := \sup_{N\in\mathbb{N}_0} \frac{1}{\Psi(N)^2} \sum_{j=N}^{\infty} 2^{-2j} \|\Delta_j f\|_{L^\infty(\mathbb{R}^n)} < \infty.$$

Let $V_\Psi^+(\mathbb{R}^n) = V_\Psi(\mathbb{R}^n) \cap BM_c^+$.

Remark 35. The construction of the space V_Ψ is in some sense "dual" to the one used to define the classical Vishik space[33] B_Γ, where $\Gamma : [0,\infty) \to (0,\infty)$ is an increasing function with $\lim_{t\to\infty} \Gamma(t) = \infty$

[32] As usual, $\{\Delta_j\}_{j\in\mathbb{N}_0}$ refers to standard (inhomogeneous) Littlewood–Paley operators on $\mathbb{R}^n$.

[33] See [Domínguez and Milman 2024].

(a "growth function") and the norm of B_Γ is given by

$$\|f\|_{B_\Gamma(\mathbb{R}^n)} := \sup_{N\in\mathbb{N}_0} \frac{1}{\Gamma(N)} \sum_{j=0}^{N} \|\Delta_j f\|_{L^\infty(\mathbb{R}^n)}. \tag{7-2}$$

Note that we could have elements $f \in B_\Gamma(\mathbb{R}^n)$ such that $\sum_{j=0}^{\infty} \|\Delta_j f\|_{L^\infty(\mathbb{R}^n)} = \infty$ (i.e., f does not belong to the Besov space[34] $B^0_{\infty,1}(\mathbb{R}^n)$), as long as the growth of the corresponding partial sums is controlled by Γ. On the other hand, $V_\Psi(\mathbb{R}^n)$ is formed by elements $f \in B^{-2}_{\infty,1}(\mathbb{R}^n)$, the classical Besov space of negative order, equipped with

$$\|f\|_{B^{-2}_{\infty,1}(\mathbb{R}^n)} = \sum_{j=0}^{\infty} 2^{-2j} \|\Delta_j f\|_{L^\infty(\mathbb{R}^n)}, \tag{7-3}$$

such that the remainder of the corresponding series in (7-3) has a prescribed decay given by $\Psi(N)^2$. The connection of these spaces becomes apparent through the use of stream functions. Let $\omega \in V_\Psi(\mathbb{R}^n)$, and let ψ be a stream function, i.e., $\Delta\psi = \omega$. Using Fourier multipliers one can show that

$$\omega \in V_\Psi(\mathbb{R}^n) \quad \iff \quad \sup_{N\in\mathbb{N}_0} \frac{1}{\Psi(N)^2} \sum_{j=N}^{\infty} \|\Delta_j \psi\|_{L^\infty(\mathbb{R}^n)} < \infty.$$

In other words, the space $V_\Psi(\mathbb{R}^n)$ is formed by vorticities ω with corresponding stream functions ψ satisfying the "dual" of the Vishik condition (7-2).

7.2. ***V_Ψ-regularity of Euler flows.*** In this section, we restrict ourselves to the following sufficiently rich class of decays.

Definition 36 (admissible/doubling decays). Let Ψ be a decay. We say that Ψ is

(i) *admissible*[35] provided that

$$\sum_{r=0}^{N} (2^r \Psi(r))^2 \lesssim (2^N \Psi(N))^2, \quad N \in \mathbb{N}_0;$$

(ii) *doubling* provided that $\Psi(ct) \gtrsim \Psi(t)$ for some $c > 1$.

We are now ready to state the main result of this section.

Theorem 37. *Let Ψ be an admissible doubling decay. Then*:

(i) $V^+_\Psi(\mathbb{R}^n)_c \hookrightarrow S_\Psi(\mathbb{R}^n)_c$. *As a consequence* (*see* (1-15)), $V^+_\Psi(\mathbb{R}^n)_c \xhookrightarrow{\text{compactly}} H^{-1}_{\text{loc}}(\mathbb{R}^n)$.

[34] Recall that the Besov spaces $B^s_{p,q}(\mathbb{R}^n)$, $s \in \mathbb{R}$, $p, q \in [1,\infty]$, are endowed with the norm

$$\|f\|_{B^s_{p,q}(\mathbb{R}^n)} = \left(\sum_{j=0}^{\infty} 2^{jsq} \|\Delta_j f\|^q_{L^p(\mathbb{R}^n)}\right)^{1/q}.$$

See, e.g., [Stein 1970; Bennett and Sharpley 1988; Triebel 2008].

[35] This is a very weak assumption on the monotonicity properties of Ψ. Basic examples of admissible decays are $\Psi(t) = t^{-\lambda}$, $\Psi(t) = (\log t)^{-\lambda}$, where $\lambda > 0$, (or more generally, concatenations of logarithms) and their products. Exponential decays $\Psi(t) = 2^{-Ct}$, $C \geq 1$, are excluded. However, this is not restrictive since $V_{2^{-Ct}}(\mathbb{R}^n) \hookrightarrow V_{t^{-\lambda}}(\mathbb{R}^n)$.

(ii) *Let* $\{u^\varepsilon\}_{\varepsilon>0}$ *be a family of approximate solutions to Euler equations* (1-1) *such that the related family of vorticities* $\{\omega^\varepsilon\}_{\varepsilon>0}$ *is uniformly bounded in* $L^\infty([0,T]; V_\Psi^+(\mathbb{R}^n)_c)$. *Then* $\{u^\varepsilon\}_{\varepsilon>0}$ *has a strong limit* u *in* $L^\infty([0,T]; L^2_{\mathrm{loc}}(\mathbb{R}^n))$, *where* u *is a solution with no concentrations.*

Specializing the previous result with $\Psi(t) = t^{(1-\alpha)/2}$, we are able to improve[36] (7-1) in the following sense.

Theorem 38. *Assume that* $\alpha > 1$. *Then*

$$M^{\frac{n}{2},\alpha}(\mathbb{R}^n) \hookrightarrow V_\Psi(\mathbb{R}^n).$$

Furthermore, this embedding is strict in the sense that $M^{\frac{n}{2},\alpha}(\mathbb{R}^n) \neq V_\Psi(\mathbb{R}^n)$.

The proofs of these results are based on extrapolation methods developed in the following section.

7.3. *Extrapolation characterization of* V_Ψ. Let (A_0, A_1) be an interpolation pair[37] of Banach spaces. Recall that the *K-functional* relative to (A_0, A_1) is defined by

$$K(t, f; A_0, A_1) = \|f\|_{A_0+tA_1} = \inf_{f=f_0+f_1} (\|f_0\|_{A_0} + t\|f_1\|_{A_1})$$

for $t > 0$ and $f \in A_0 + A_1$.

Theorem 39. *Suppose that* Ψ *is an admissible doubling decay. Then*

$$\|f\|_{V_\Psi(\mathbb{R}^n)} \approx \sup_{t\in(0,1)} \frac{K(t, f; B^{-2}_{\infty,1}(\mathbb{R}^n), B^0_{\infty,1}(\mathbb{R}^n))}{\Psi(-\log t)^2}. \tag{7-4}$$

Remark 40. This result shows that the V_Ψ spaces can be described as extrapolation spaces[38] relative to the classical Besov pair $(B^{-2}_{\infty,1}, B^0_{\infty,1})$, a fact that will be very useful later, since it enables the transfer of fundamental properties of the classical Besov spaces to V_Ψ.

Remark 41. The assumption that Ψ is doubling is necessary in order to ensure that the right-hand side of (7-4) is nontrivial. For instance, for the admissible decay $\Psi(t) = 2^{-Ct}$, $C \in \left(\frac{1}{2}, 1\right)$,

$$\sup_{t\in(0,1)} \frac{K(t, f; B^{-2}_{\infty,1}(\mathbb{R}^n), B^0_{\infty,1}(\mathbb{R}^n))}{t^{2C}} < \infty \iff f = 0.$$

Proof of Theorem 39. We use the retraction method of interpolation theory. Recall that $\ell^s_1(L^\infty(\mathbb{R}^n))$, $s \in \mathbb{R}$, is the vector-valued sequence space equipped with the norm

$$\|\{f_j\}_{j\in\mathbb{N}_0}\|_{\ell^s_1(L^\infty(\mathbb{R}^n))} = \sum_{j=0}^\infty 2^{js}\|f_j\|_{L^\infty(\mathbb{R}^n)}.$$

[36] We only focus on the most interesting case $p = n/2$, but similar improvements can also be obtained in the noncritical regime $p > n/2$.

[37] Loosely speaking, $A_0 + A_1$ makes sense.

[38] with respect to the so-called Σ-method of extrapolation (see [Jawerth and Milman 1991]).

It is well known (see [Domínguez and Milman 2024, Appendix A1]) that $B^s_{\infty,1}(\mathbb{R}^n)$ is a retract of $\ell^s_1(L^\infty(\mathbb{R}^n))$ via

$$f \mapsto \{\Delta_j f\}_{j\in\mathbb{N}_0}.$$

In particular this yields

$$K(t, f; B^{-2}_{\infty,1}(\mathbb{R}^n), B^s_{\infty,1}(\mathbb{R}^n)) \approx K\big(t, \{\Delta_j f\}_{j\in\mathbb{N}_0}; \ell^{-2}_1(L^\infty(\mathbb{R}^n)), \ell^s_1(L^\infty(\mathbb{R}^n))\big).$$

Furthermore, a well-known estimate for K-functionals asserts

$$K\big(t, \{f_j\}_{j\in\mathbb{N}_0}; \ell^{-2}_1(L^\infty(\mathbb{R}^n)), \ell^s_1(L^\infty(\mathbb{R}^n))\big) \approx \sum_{j=0}^\infty 2^{js} \min\{2^{(-2-s)j}, t\} \|f_j\|_{L^\infty(\mathbb{R}^n)}.$$

Hence (letting $s = 0$)

$$\begin{aligned} K\big(2^{-2N}, f; B^{-2}_{\infty,1}(\mathbb{R}^n), B^0_{\infty,1}(\mathbb{R}^n)\big) &\approx \sum_{j=0}^\infty \min\{2^{-2j}, 2^{-2N}\} \|\Delta_j f\|_{L^\infty(\mathbb{R}^d)} \\ &= 2^{-2N} \sum_{j=0}^N \|\Delta_j f\|_{L^\infty(\mathbb{R}^n)} + \sum_{j=N+1}^\infty 2^{-2j} \|\Delta_j f\|_{L^\infty(\mathbb{R}^n)}. \end{aligned}$$

This implies that

$$\sup_{N\in\mathbb{N}_0} \frac{K\big(2^{-2N}, f; B^{-2}_{\infty,1}(\mathbb{R}^n), B^s_{\infty,1}(\mathbb{R}^n)\big)}{\Psi(N)^2} \approx \mathcal{A} + \|f\|_{V_\Psi(\mathbb{R}^n)}, \tag{7-5}$$

where

$$\mathcal{A} := \sup_{N\in\mathbb{N}_0} \frac{2^{-2N} \sum_{j=0}^N \|\Delta_j f\|_{L^\infty(\mathbb{R}^n)}}{\Psi(N)^2}.$$

Furthermore, we claim that

$$\mathcal{A} \lesssim \|f\|_{V_\Psi(\mathbb{R}^n)}. \tag{7-6}$$

Indeed,

$$\begin{aligned} \sum_{j=0}^N \|\Delta_j f\|_{L^\infty(\mathbb{R}^n)} &= \sum_{j=0}^N 2^{2j} \Psi(j)^2 \frac{2^{-2j}}{\Psi(j)^2} \|\Delta_j f\|_{L^\infty(\mathbb{R}^n)} \\ &\le \Big(\sup_{j\in\mathbb{N}_0} \frac{2^{-2j}}{\Psi(j)^2} \|\Delta_j f\|_{L^\infty(\mathbb{R}^n)}\Big) \sum_{j=0}^N 2^{2j} \Psi(j)^2 \\ &\lesssim 2^{2N} \Psi(N)^2 \|f\|_{V_\Psi(\mathbb{R}^n)} \quad \text{(by the admissibility of } \Psi\text{)}, \end{aligned}$$

which yields the desired estimate (7-6).

Combining (7-5) and (7-6), we see that

$$\sup_{N\in\mathbb{N}_0} \frac{K\big(2^{-2N}, f; B^{-2}_{\infty,1}(\mathbb{R}^n), B^s_{\infty,1}(\mathbb{R}^n)\big)}{\Psi(N)^2} \approx \|f\|_{V_\Psi(\mathbb{R}^n)}.$$

Note that, by basic monotonicity properties of the expressions involved (recall that Ψ is doubling),

$$\sup_{t\in(0,1)} \frac{K(t, f; B^{-2}_{\infty,1}(\mathbb{R}^n), B^0_{\infty,1}(\mathbb{R}^n))}{\Psi(-\log t)^2}$$

$$\approx \sup_{N\in\mathbb{N}_0} K\big(2^{-2N}, f; B^{-2}_{\infty,1}(\mathbb{R}^n), B^0_{\infty,1}(\mathbb{R}^n)\big) \sup_{t\in(2^{-2(N+1)},2^{-2N})} \frac{1}{\Psi(-\log t)^2}$$

$$\approx \sup_{N\in\mathbb{N}_0} \frac{K\big(2^{-2N}, f; B^{-2}_{\infty,1}(\mathbb{R}^n), B^0_{\infty,1}(\mathbb{R}^n)\big)}{\Psi(N)^2}. \qquad \square$$

Remark 42. The proof above shows that $B^0_{\infty,1}(\mathbb{R}^n)$ plays an auxiliary role in Theorem 39, in the sense that the same result is obtained if we replace it by $B^s_{\infty,1}(\mathbb{R}^n)$, for any $s > -2$. More precisely, let $s > -2$, and replace the admissibility condition given in Definition 36(i) by

$$\sum_{r=0}^{N} 2^{(2+s)r}\Psi(r)^2 \lesssim 2^{(2+s)N}\Psi(N)^2, \quad N \in \mathbb{N}_0.$$

Then using the same methodology we can readily show that

$$\|f\|_{V_\Psi(\mathbb{R}^n)} \approx \sup_{t\in(0,1)} \frac{K(t, f; B^{-2}_{\infty,1}(\mathbb{R}^n), B^s_{\infty,1}(\mathbb{R}^n))}{\Psi(-\log t)^2}.$$

7.4. ***Proof of Theorem 37.*** The proof relies strongly on the extrapolation description of V_Ψ. In particular, Theorem 39 will be applied to decompose functions in V_Ψ in terms of wavelets (see Proposition 43 below).

Let $\{\Upsilon^G_{Nl} : N \in \mathbb{N}_0,\ G \in G^N,\ l \in \mathbb{Z}^n\}$ be an orthonormal wavelet basis in $L^2(\mathbb{R}^n)$.

Remark. We briefly recall that orthonormal wavelet bases may be constructed in a standard way from two compactly supported (Daubechies) wavelets $\psi_F \in C^1(\mathbb{R})$ (*father wavelet*) and $\psi_M \in C^1(\mathbb{R})$ (*mother wavelet*) satisfying certain moment conditions. More precisely

$$\Upsilon^G_{Nl}(x) = 2^{Nn/2}\prod_{r=1}^{n} \psi_{G_r}(2^N x_r - l_r).$$

Here $G^0 = \{F, M\}^n$ and $G^N = \{F, M\}^{n*}$, $N \in \mathbb{N}$, where $*$ indicates that at least one of the components of $G \in G^N$ must be an M. The role played by the tensor index $G \in G^N$ is auxiliary (note that $\operatorname{card} G^N \approx 1$). To simplify the exposition, the index G may be safely removed from our computations.

Proposition 43. *Suppose that Ψ is an admissible doubling decay. Then, $f \in V_\Psi(\mathbb{R}^n)$ if and only if*

$$f = \sum_{\substack{N\in\mathbb{N}_0,\, G\in G^N\\ l\in\mathbb{Z}^n}} \lambda^G_{Nl} 2^{-Nn/2}\Upsilon^G_{Nl}, \quad \{\lambda^G_{Nl}\} \in v_\Psi \tag{7-7}$$

(*unconditional convergence in the sense of $\mathcal{S}'(\mathbb{R}^n)$*), *where*

$$\|\{\lambda^G_{Nl}\}\|_{v_\Psi} := \sup_{N\in\mathbb{N}_0} \frac{1}{\Psi(N)^2}\sum_{k=N}^{\infty} 2^{-2k} \sup_{G\in G^k,\, l\in\mathbb{Z}^n} |\lambda^G_{kl}| < \infty.$$

The representation of f is unique, and the coefficients λ_{Nl}^{G} are determined by

$$\lambda_{Nl}^{G} = 2^{Nn/2}(f, \Upsilon_{Nl}^{G}), \tag{7-8}$$

and the operator

$$I : f \mapsto \{\lambda_{Nl}^{G}\} \tag{7-9}$$

defines an isomorphism of $V_\Psi(\mathbb{R}^n)$ onto v_Ψ. Furthermore

$$\|f\|_{V_\Psi(\mathbb{R}^n)} \approx \|\{\lambda_{Nl}\}\|_{v_\Psi}. \tag{7-10}$$

Proof. By the classical wavelet theory for Besov spaces [Triebel 2008, Theorem 1.20], the operator I given by (7-9) acts as an isomorphism

$$I : B^{-2}_{\infty,1}(\mathbb{R}^n) \to \ell_1^{-2}(\ell_\infty) \quad \text{and} \quad I : B^{0}_{\infty,1}(\mathbb{R}^n) \to \ell_1(\ell_\infty). \tag{7-11}$$

Here, as usual, $\ell_1^s(\ell_\infty)$, $s \in \mathbb{R}$, is the mixed sequence space formed by all those $\{\lambda_{Nl}\}$ such that

$$\|\{\lambda_{Nl}\}\|_{\ell_1^s(\ell_\infty)} = \sum_{N=0}^{\infty} 2^{Ns} \sup_{l \in \mathbb{Z}^n} |\lambda_{Nl}| < \infty.$$

We also let $\ell_1(\ell_\infty) = \ell_1^0(\ell_\infty)$.

It follows from (7-11) that

$$K\big(t, f; B^{-2}_{\infty,1}(\mathbb{R}^n), B^{0}_{\infty,1}(\mathbb{R}^n)\big) \approx K\big(t, \{\lambda_{Nl}\}; \ell_1^{-2}(\ell_\infty), \ell_1(\ell_\infty)\big).$$

Consequently, combining with Theorem 39 yields

$$\|f\|_{V_\Psi(\mathbb{R}^n)} \approx \sup_{t \in (0,1)} \frac{K\big(t, \{\lambda_{Nl}\}; \ell_1^{-2}(\ell_\infty), \ell_1(\ell_\infty)\big)}{\Psi(-\log t)^2}. \tag{7-12}$$

At this point the method of proof developed in Theorem 39 can be applied line by line (ℓ_∞ now playing the role previously played by $L^\infty(\mathbb{R}^n)$) to show that

$$\sup_{t \in (0,1)} \frac{K\big(t, \{\lambda_{Nl}\}; \ell_1^{-2}(\ell_\infty), \ell_1(\ell_\infty)\big)}{\Psi(-\log t)^2} \approx \|\{\lambda_{Nl}\}\|_{v_\Psi}. \tag{7-13}$$

Combining (7-12) and (7-13), we obtain (7-10).

For $f \in V_\Psi(\mathbb{R}^n)$, the convergence of the wavelet expansion (7-7), as well as the uniqueness of wavelet coefficients given by (7-8), is guaranteed by the fact that $V_\Psi(\mathbb{R}^n) \hookrightarrow B^{-2}_{\infty,1}(\mathbb{R}^n)$ (see Remark 35) since the corresponding assertions are valid for classical Besov spaces. □

Proof of Theorem 37. (i) Without loss of generality, we may assume that $Q_0 = (0,1)^n$. Let $L \in \mathbb{N}_0$ and $\mathcal{Q} = (Q_i)_{i \in I} \in S(Q_0)$. If $f \geq 0$ is compactly supported in Q_0, then

$$\bigg[\sum_{Q \in \mathbb{D}_{\leq L-1}(\mathcal{Q})} \bigg(|Q|^{\frac{1}{n}-\frac{1}{2}} \int_Q |f|\bigg)^2\bigg]^{\frac{1}{2}} = \bigg[\sum_{k=L-1}^{\infty} \sum_{i \in I : Q_i \in \mathbb{D}_k(\mathcal{Q})} \bigg(|Q_i|^{\frac{1}{n}-\frac{1}{2}} \int_{Q_i} f\bigg)^2\bigg]^{\frac{1}{2}}$$

$$\le \|f\|_{L^1(Q_0)}^{\frac{1}{2}} \Bigg[\sum_{k=L-1}^{\infty} 2^{-kn(\frac{1}{n}-\frac{1}{2})2} \sup_{Q\in\mathbb{D}_k} \int_Q f \Bigg]^{\frac{1}{2}}$$

$$\le \|f\|_{L^1(Q_0)}^{\frac{1}{2}} \Psi(L) \sup_{N\in\mathbb{N}_0} \Bigg[\frac{1}{\Psi(N)^2} \sum_{k=N-1}^{\infty} 2^{k(-2+n)} \sup_{Q\in\mathbb{D}_k} \int_Q f \Bigg]^{\frac{1}{2}}$$

$$\le \Psi(L) \sup_{N\in\mathbb{N}_0} \frac{1}{\Psi(N)^2} \sum_{k=N-1}^{\infty} 2^{k(-2+n)} \sup_{Q\in\mathbb{D}_k} \int_Q f.$$

As a by-product (see (1-14))

$$\|f\|_{S_\Psi(Q_0)} = \sup_{L\in\mathbb{N}} \frac{s_L(f)}{\Psi(L)} \le \sup_{N\in\mathbb{N}_0} \frac{1}{\Psi(N)^2} \sum_{k=N-1}^{\infty} 2^{k(-2+n)} \sup_{Q\in\mathbb{D}_k} \int_Q f.$$

Hence the desired embedding $(V_\Psi^+(\mathbb{R}^n))_c \hookrightarrow (S_\Psi(\mathbb{R}^n))_c$ follows if we show that

$$\sup_{N\in\mathbb{N}_0} \frac{1}{\Psi(N)^2} \sum_{k=N-1}^{\infty} 2^{k(-2+n)} \sup_{Q\in\mathbb{D}_k} \int_Q f \lesssim \|f\|_{V_\Psi(\mathbb{R}^n)}. \tag{7-14}$$

Let $f \in V_\Psi(\mathbb{R}^n)$. According to Proposition 43, f can be expressed as

$$f = \sum_{r=0}^{\infty} \sum_{l\in\mathbb{Z}^n} \lambda_{rl} 2^{-rn/2} \Upsilon_{rl}, \tag{7-15}$$

where λ_{rl} is given by (7-8). The wavelets Υ_{rl} can be chosen such that

$$\operatorname{supp} \Upsilon_{rl} \subset cQ_{rl} = c(2^{-r}l + 2^{-r}Q_0), \tag{7-16}$$

$$|\Upsilon_{rl}(x)| \lesssim 2^{rn/2}, \quad r \in \mathbb{N}_0,\ l \in \mathbb{Z}^n, \tag{7-17}$$

and there exists $A \in \mathbb{N},\ A > 2$, satisfying

$$\int_{\mathbb{R}^n} x^\beta \Upsilon_{rl}(x)\, dx = 0, \quad |\beta| < A,\ r \in \mathbb{N},\ l \in \mathbb{Z}^n. \tag{7-18}$$

We are going to compute $\chi_{jm}(f)$ given by (4-22). For every $j \in \mathbb{N}_0$, we can split f as (see (7-15))

$$f = f_j + f^j := \sum_{r=0}^{j} \sum_{l\in\mathbb{Z}^n} \lambda_{rl} 2^{-rn/2} \Upsilon_{rl} + \sum_{r=j+1}^{\infty} \sum_{l\in\mathbb{Z}^n} \lambda_{rl} 2^{-rn/2} \Upsilon_{rl}. \tag{7-19}$$

Then, for $m \in \mathbb{Z}^n$,

$$\chi_{jm}(f) = \chi_{jm}(f_j) + \chi_{jm}(f^j). \tag{7-20}$$

Next, we estimate each of these terms separately.

We first estimate $\chi_{jm}(f_j)$. For $r \le j$, we let (recall that $\operatorname{supp} \chi_{jm} \subset dQ_{jm}$)

$$\ell_r^j(m) := \{l \in \mathbb{Z}^n : dQ_{jm} \cap cQ_{rl} \ne \varnothing\}. \tag{7-21}$$

Obviously (see (7-16))

$$\chi_{jm}(\Upsilon_{rl}) = 0 \quad \text{for } l \notin \ell_r^j(m). \tag{7-22}$$

On the other hand, if $l \in \ell_r^j(m)$ then (noting that $|\chi_{jm}(x)| \lesssim 2^{jn}$)

$$|\chi_{jm}(\Upsilon_{rl})| \le \int_{dQ_{jm}} |\chi_{jm}(x)||\Upsilon_{rl}(x)|\,dx \lesssim 2^{jn} \int_{dQ_{jm}} |\Upsilon_{rl}(x)|\,dx \lesssim 2^{rn/2}, \tag{7-23}$$

where we have used the property (7-17) in the last step. In light of (7-22) and (7-23), we derive

$$\begin{aligned} |\chi_{jm}(f_j)| \le \sum_{r=0}^{j} \sum_{l\in\mathbb{Z}^n} |\lambda_{rl}| 2^{-rn/2} |\chi_{jm}(\Upsilon_{rl})| &= \sum_{r=0}^{j} \sum_{l\in\ell_r^j(m)} |\lambda_{rl}| 2^{-rn/2} |\chi_{jm}(\Upsilon_{rl})| \\ &\lesssim \sum_{r=0}^{j} \sum_{l\in\ell_r^j(m)} |\lambda_{rl}|. \end{aligned} \tag{7-24}$$

Note that since $\operatorname{card} \ell_r^j(m) \approx 1$, if $0 \le r \le j$, from (7-24) it follows that

$$|\chi_{jm}(f_j)| \lesssim \sum_{r=0}^{j} \sup_{l\in\mathbb{Z}^n} |\lambda_{rl}|. \tag{7-25}$$

Next we deal with $\chi_{jm}(f^j)$. Let $r > j$ and $l \in \ell_r^j(m)$ (see (7-21)). Using the Taylor expansion of χ_{jm} around $2^{-r}l$ and using the cancellation conditions (7-18), one can show that

$$\begin{aligned} |\chi_{jm}(\Upsilon_{rl})| &= \left| \int_{\mathbb{R}^n} \chi_{jm}(x) \Upsilon_{rl}(x)\,dx \right| \\ &\le \sum_{|\gamma|=A} \sup_{x\in\mathbb{R}^n} |D^\gamma \chi_{jm}(x)| \int_{\mathbb{R}^n} |\Upsilon_{rl}(x)||x - 2^{-r} l|^A\,dx \\ &\lesssim 2^{j(n+A)} \int_{\mathbb{R}^n} 2^{rn/2} |\psi(2^r x - l)| 2^{-rA} |2^r x - l|^A\,dx \\ &= 2^{(j-r)(n+A)} 2^{rn/2} \int_{\mathbb{R}^n} |\psi(x)||x|^A\,dx \\ &\approx 2^{(j-r)(n+A)} 2^{rn/2}. \end{aligned}$$

Using this estimate we obtain

$$|\chi_{jm}(f^j)| \le \sum_{r=j}^{\infty} \sum_{l\in\ell_r^j(m)} |\lambda_{rl}| 2^{-rn/2} |\chi_{jm}(\Upsilon_{rl})| \lesssim \sum_{r=j}^{\infty} 2^{(j-r)(n+A)} \sum_{l\in\ell_r^j(m)} |\lambda_{rl}|. \tag{7-26}$$

Note that $\operatorname{card} \ell_r^j(m) \approx 2^{n(r-j)}$ if $r \ge j$. Hence, by (7-26),

$$|\chi_{jm}(f^j)| \lesssim \sum_{r=j}^{\infty} 2^{(j-r)(n+A)} \sup_{l\in\mathbb{Z}^n} |\lambda_{rl}| 2^{n(r-j)} = 2^{jA} \sum_{r=j}^{\infty} 2^{-rA} \sup_{l\in\mathbb{Z}^n} |\lambda_{rl}|. \tag{7-27}$$

Putting together (7-20), (7-25) and (7-27),

$$|\chi_{jm}(f)| \le |\chi_{jm}(f_j)| + |\chi_{jm}(f^j)| \lesssim \sum_{r=0}^{j} \sup_{l\in\mathbb{Z}^n} |\lambda_{rl}| + 2^{jA} \sum_{r=j}^{\infty} 2^{-rA} \sup_{l\in\mathbb{Z}^n} |\lambda_{rl}|$$

uniformly with respect to $m \in \mathbb{Z}^n$. Consequently,

$$\sup_{m\in\mathbb{Z}^n} |\chi_{jm}(f)| \lesssim \sum_{r=0}^{j} \sup_{l\in\mathbb{Z}^n} |\lambda_{rl}| + 2^{jA} \sum_{r=j}^{\infty} 2^{-rA} \sup_{l\in\mathbb{Z}^n} |\lambda_{rl}|. \tag{7-28}$$

Let $N \in \mathbb{N}_0$. From (7-28), changing the order of summation and using $A > 2$, we get

$$\begin{aligned}\sum_{j=N}^{\infty} 2^{-2j} \sup_{m\in\mathbb{Z}^n} |\chi_{jm}(f)| &\lesssim \sum_{j=N}^{\infty} 2^{-2j} \sum_{r=0}^{j} \sup_{l\in\mathbb{Z}^n} |\lambda_{rl}| + \sum_{j=N}^{\infty} 2^{(A-2)j} \sum_{r=j}^{\infty} 2^{-rA} \sup_{l\in\mathbb{Z}^n} |\lambda_{rl}| \\ &\lesssim 2^{-2N} \sum_{r=0}^{N} \sup_{l\in\mathbb{Z}^n} |\lambda_{rl}| + \sum_{r=N}^{\infty} 2^{-2r} \sup_{l\in\mathbb{Z}^n} |\lambda_{rl}|.\end{aligned}$$

It follows that

$$\begin{aligned}\sup_{N\in\mathbb{N}_0} \frac{1}{\Psi(N)^2} \sum_{j=N}^{\infty} 2^{-2j} \sup_{m\in\mathbb{Z}^n} |\chi_{jm}(f)| &\lesssim \sup_{N\in\mathbb{N}_0} \frac{2^{-2N}}{\Psi(N)^2} \sum_{r=0}^{N} \sup_{l\in\mathbb{Z}^n} |\lambda_{rl}| \\ &\quad + \sup_{N\in\mathbb{N}_0} \frac{1}{\Psi(N)^2} \sum_{r=N}^{\infty} 2^{-2r} \sup_{l\in\mathbb{Z}^n} |\lambda_{rl}| \\ &=: \mathcal{I} + \mathcal{II}.\end{aligned} \tag{7-29}$$

Next we show $\mathcal{I} \lesssim \mathcal{II}$. Indeed, by condition (i) from Definition 36,

$$\begin{aligned}\sum_{r=0}^{N} \sup_{l\in\mathbb{Z}^n} |\lambda_{rl}| = \sum_{r=0}^{N} 2^{2r}\Psi(r)^2 \frac{2^{-2r}}{\Psi(r)^2} \sup_{l\in\mathbb{Z}^n} |\lambda_{rl}| &\le \left(\sup_{M\in\mathbb{N}_0} \left(\frac{2^{-2M}}{\Psi(M)^2} \sup_{l\in\mathbb{Z}^n} |\lambda_{Ml}| \right) \right) \sum_{r=0}^{N} 2^{2r}\Psi(r)^2 \\ &\lesssim \left(\sup_{M\in\mathbb{N}_0} \left(\frac{2^{-2M}}{\Psi(M)^2} \sup_{l\in\mathbb{Z}^n} |\lambda_{Ml}| \right) \right) 2^{2N}\Psi(N)^2.\end{aligned}$$

Consequently,

$$\frac{2^{-2N}}{\Psi(N)^2} \sum_{r=0}^{N} \sup_{l\in\mathbb{Z}^n} |\lambda_{rl}| \lesssim \sup_{M\in\mathbb{N}_0} \left(\frac{1}{\Psi(M)^2} \sum_{r=M}^{\infty} 2^{-2r} \sup_{l\in\mathbb{Z}^n} |\lambda_{rl}| \right) = \mathcal{II}.$$

Now, taking supremum over all $N \in \mathbb{N}_0$, we arrive at

$$\mathcal{I} \lesssim \mathcal{II}.$$

Consequently (7-29) reads as

$$\sup_{N\in\mathbb{N}_0} \frac{1}{\Psi(N)^2} \sum_{j=N}^{\infty} 2^{-2j} \sup_{m\in\mathbb{Z}^n} |\chi_{jm}(f)| \lesssim \mathcal{II}. \tag{7-30}$$

Since $f \geq 0$ and (4-23) holds, we have (using the notation $\chi_Q = \chi_{jm}$ if $Q = Q_{jm} \in \mathbb{D}_j$)

$$\sup_{N \in \mathbb{N}_0} \frac{1}{\Psi(N)^2} \sum_{k=N-1}^{\infty} 2^{k(-2+n)} \sup_{Q \in \mathbb{D}_k} \int_Q f \lesssim \sup_{N \in \mathbb{N}_0} \frac{1}{\Psi(N)^2} \sum_{k=N-1}^{\infty} 2^{-2k} \sup_{Q \in \mathbb{D}_k} \chi_Q(f).$$

Consequently by (7-29)–(7-30),

$$\sup_{N \in \mathbb{N}_0} \frac{1}{\Psi(N)^2} \sum_{k=N-1}^{\infty} 2^{k(-2+n)} \sup_{Q \in \mathbb{D}_k} \int_Q f \lesssim \sup_{N \in \mathbb{N}_0} \frac{1}{\Psi(N)^2} \sum_{r=N}^{\infty} 2^{-2r} \sup_{l \in \mathbb{Z}^n} |\lambda_{rl}|,$$

which combined with Proposition 43 yields

$$\sup_{N \in \mathbb{N}_0} \frac{1}{\Psi(N)^2} \sum_{k=N-1}^{\infty} 2^{k(-2+n)} \sup_{Q \in \mathbb{D}_k} \int_Q f \lesssim \|f\|_{V_\Psi(\mathbb{R}^n)},$$

concluding the proof of (7-14).

(ii) Combine (i) with Theorem 12. □

7.5. *Proof of Theorem 38.* Let $\Psi(t) = t^{(1-\alpha)/2}$, $\alpha > 1$. It follows from (7-8), (7-16) and (7-17) that

$$\begin{aligned}
\sum_{k=N}^{\infty} 2^{-2k} \sup_{l \in \mathbb{Z}^n} |\lambda_{kl}| &\lesssim \sum_{k=N}^{\infty} 2^{-2k} \sup_{l \in \mathbb{Z}^n} 2^{kn/2} \int_{cQ_{kl}} |f||\Upsilon_{kl}| \\
&\lesssim \sum_{k=N}^{\infty} 2^{-2k} \sup_{l \in \mathbb{Z}^n} 2^{kn} \int_{cQ_{kl}} |f| \\
&\lesssim \|f\|_{M^{n/2,\alpha}(\mathbb{R}^n)} \sum_{k=N}^{\infty} 2^{-2k} \sup_{l \in \mathbb{Z}^n} 2^{kn} |Q_{kl}|^{1-\frac{2}{n}} (1 - (\log |Q_{kl}|)_-)^{-\alpha} \\
&\approx \|f\|_{M^{n/2,\alpha}(\mathbb{R}^n)} \sum_{k=N}^{\infty} (1+k)^{-\alpha} \\
&\approx N^{-\alpha+1} \|f\|_{M^{n/2,\alpha}(\mathbb{R}^n)} = \Psi(N)^2 \|f\|_{M^{n/2,\alpha}(\mathbb{R}^n)}.
\end{aligned}$$

Taking the supremum over all $N \in \mathbb{N}_0$ and invoking Proposition 43, we get

$$M^{\frac{n}{2},\alpha}(\mathbb{R}^n) \hookrightarrow V_\Psi(\mathbb{R}^n) \tag{7-31}$$

as desired.

To show that (7-31) is strict, we apply again Proposition 43, and use the fact that $\alpha > 1$, to derive

$$\|2^{(2-\frac{n}{2})K} \Upsilon_{K(0,\dots,0)}\|_{V_\Psi(\mathbb{R}^n)} \approx \sup_{0 \leq N \leq K} \frac{1}{N^{-\alpha+1}} = \frac{1}{K^{-\alpha+1}}. \tag{7-32}$$

On the other hand, we have

$$\|\Upsilon_{K(0,\dots,0)}\|_{M^{n/2,\alpha}(\mathbb{R}^n)} = \sup_Q \frac{(1 - (\log |Q|)_-)^\alpha}{|Q|^{(n-2)/n}} \int_Q |\Upsilon_{K,(0,\dots,0)}|$$

$$\begin{aligned} &\geq \frac{(1-(\log|Q_{K(0,\dots,0)}|)_-)^\alpha}{|Q_{K(0,\dots,0)}|^{(n-2)/n}} \int_{Q_{K(0,\dots,0)}} |\Upsilon_{K,(0,\dots,0)}| \\ &\approx \frac{K^\alpha}{2^{-K(n-2)}} \int_{Q_{K(0,\dots,0)}} |\Upsilon_{K,(0,\dots,0)}| \\ &\gtrsim \frac{K^\alpha}{2^{-K(n-2)}} 2^{-Kn} 2^{Kn/2} = 2^{K(\frac{n}{2}-2)} K^\alpha, \end{aligned}$$

where we have used the wavelet properties in the penultimate step. Consequently,

$$\|2^{(2-\frac{n}{2})K} \Upsilon_{K(0,\dots,0)}\|_{M^{n/2,\alpha}(\mathbb{R}^n)} \gtrsim K^\alpha. \tag{7-33}$$

We now argue by contradiction. Suppose that, to the contrary,

$$V_\Psi(\mathbb{R}^n) \hookrightarrow M^{\frac{n}{2},\alpha}(\mathbb{R}^n).$$

In particular, we have, uniformly with respect to K,

$$\|2^{(2-\frac{n}{2})K} \Upsilon_{K(0,\dots,0)}\|_{M^{n/2,\alpha}(\mathbb{R}^n)} \lesssim \|2^{(2-\frac{n}{2})K} \Upsilon_{K(0,\dots,0)}\|_{V_\Psi(\mathbb{R}^n)}.$$

Combining the last inequality with (7-32) and (7-33) yields

$$K^\alpha \lesssim K^{\alpha-1},$$

and letting $K \to \infty$ we arrive at a contradiction. □

8. Sharpening Tadmor's regularity via T_Ψ

As already mentioned in Section 1.2, Tadmor [2001] proposed an approach, based on $R_{p,2}\log^\alpha$-spaces, guaranteeing existence of Euler solutions with no concentration. In particular, in the distinguished 2D case, the author was able to improve the Morrey regularity of vortex sheets obtained in [DiPerna and Majda 1987a] from $\alpha = 1$ to the borderline regularity $\alpha = \frac{1}{2}$. This is an application of the H^{-1}-stability method since (see (1-3))

$$R_{2n/(n+2),2}\log^\alpha(\mathbb{R}^n)_c \xrightarrow{\text{compactly}} H^{-1}_{\text{loc}}(\mathbb{R}^n), \quad \alpha > \tfrac{1}{2}. \tag{8-1}$$

The goal of this section is to show that the results from [Tadmor 2001] admit improvements in terms of new scales of extrapolation spaces (T_Ψ-spaces; see Definition 44 below) and the method of sparse stability developed in previous sections.

8.1. ***T_Ψ-spaces.*** To motivate the constructions that follow it is instructive to compare the scalings of the spaces involved in the critical embeddings (1-2). For a function space $X(\mathbb{R}^n)$, let the scaling parameter of X be the number i_X such for all $\lambda > 0$, and all $\|f\|_X = 1$, we have $\lambda^{-i_X}\|f(\lambda\,\cdot)\|_X = 1$. For the Morrey spaces $M^p(\mathbb{R}^n)$, for $\lambda > 0$, and $\|f\|_{M^p(\mathbb{R}^n)} = 1$, we have $\|f(\lambda\,\cdot)\|_{M^p(\mathbb{R}^n)} = \lambda^{-n/p}$ so that $i_{M^p(\mathbb{R}^n)} = -n/p$. Likewise, for $H^{-1}(\mathbb{R}^n)$, $i_{H^{-1}(\mathbb{R}^n)} = -1-n/2$. Comparing scaling parameters in the critical case $p = n/2$, we see that $i_{M^{n/2}(\mathbb{R}^n)} = i_{H^{-1}(\mathbb{R}^n)}$ only when $n = 2$, in which case the common value of the parameter is -2. This suggests to seek for an alternative to (1-2) where the involved spaces have the same scaling parameter as H^{-1} (i.e., $-1-n/2$). With this in mind, we propose a new space T_Ψ, a variant of V_Ψ introduced in Definition 34, which is obtained by measuring the dyadic frequencies $\Delta_j f$ in the L^2-norm rather than the L^∞-norm.

Definition 44. Let $T_\Psi(\mathbb{R}^n)$ be the space formed by all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|^2_{T_\Psi(\mathbb{R}^n)} := \sup_{N\in\mathbb{N}_0} \frac{1}{\Psi(N)^2} \sum_{j=N}^{\infty} 2^{-2j} \|\Delta_j f\|^2_{L^2(\mathbb{R}^n)} < \infty.$$

Let $T^+_\Psi(\mathbb{R}^n) = T_\Psi(\mathbb{R}^n) \cap BM^+_c$.

Remark 45. Similarly as in Remark 35, the space T_Ψ can be seen as a dual counterpart of classical Vishik spaces involving the Besov space $B^{-1}_{2,2}$. Note that $B^{-1}_{2,2}$ can be identified with the inhomogeneous version of H^{-1}.

Remark 46. The space T_Ψ admits a somewhat simpler characterization in terms of Fourier transforms. Namely

$$\|f\|^2_{T_\Psi(\mathbb{R}^n)} \approx \sup_{N\in\mathbb{N}_0} \frac{1}{\Psi(N)^2} \int_{|\xi|>2^{N-1}} (1+|\xi|^2)^{-1} |\hat{f}(\xi)|^2 \, d\xi.$$

Indeed, this is a consequence of Plancherel's and Fubini's theorem, together with basic properties of[39] $\{\varphi_j\}_{j\in\mathbb{N}_0}$,

$$\begin{aligned}\|f\|^2_{T_\Psi(\mathbb{R}^n)} &\approx \sup_{N\in\mathbb{N}_0} \frac{1}{\Psi(N)^2} \int_{\mathbb{R}^n} \sum_{j=N}^{\infty} [2^{-j}\varphi_j(\xi)]^2 |\hat{f}(\xi)|^2 \, d\xi \\ &\approx \sup_{N\in\mathbb{N}_0} \frac{1}{\Psi(N)^2} \int_{\mathbb{R}^n} (1+|\xi|^2)^{-1} \mathbf{1}_{(2^{N-1},\infty)}(|\xi|) |\hat{f}(\xi)|^2 \, d\xi.\end{aligned}$$

8.2. *T_Ψ-regularity of Euler flows.* We state now the main results of this section.

Theorem 47. *Let Ψ be an admissible doubling decay. Then*:

(i) $T^+_\Psi(\mathbb{R}^n)_c \hookrightarrow S_\Psi(\mathbb{R}^n)_c$. *As a consequence* (*see* (1-15)), $T^+_\Psi(\mathbb{R}^n)_c \xrightarrow{\text{compactly}} H^{-1}_{\text{loc}}(\mathbb{R}^n)$.

(ii) *Let $\{u^\varepsilon\}_{\varepsilon>0}$ be a family of approximate solutions to Euler equations* (1-1) *such that the related family of vorticities $\{\omega^\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $L^\infty([0,T]; T^+_\Psi(\mathbb{R}^n)_c)$. Then $\{u^\varepsilon\}_{\varepsilon>0}$ has a strong limit u in $L^\infty([0,T]; L^2_{\text{loc}}(\mathbb{R}^n))$, where u is a solution with no concentrations.*

Specializing the previous result with $\Psi_\alpha(t) = t^{-\alpha+1/2}$, we are able to improve (8-1) in the following sense.

Theorem 48. *Assume that $\alpha > \frac{1}{2}$. Then*

$$R_{2n/(n+2),2} \log^\alpha(\mathbb{R}^n) \hookrightarrow T_{\Psi_\alpha}(\mathbb{R}^n).$$

Furthermore, this embedding is strict in the sense that $R_{2n/(n+2),2} \log^\alpha(\mathbb{R}^n) \neq T_{\Psi_\alpha}(\mathbb{R}^n)$.

[39] Here φ_j denotes the Fourier multiplier associated with Δ_j, i.e., $\widehat{\Delta_j f}(\xi) = \varphi_j(\xi)\hat{f}(\xi)$.

8.3. ***Extrapolation characterization of T_Ψ.*** The next result represents the T_Ψ spaces as extrapolation spaces for the pair (H^{-1}, L^2). Since the proof follows closely the one for Theorem 39, we shall leave the details to the interested reader.

Theorem 49. *Suppose that Ψ is an admissible doubling decay. Then*[40]

$$\|f\|_{T_\Psi(\mathbb{R}^n)} \approx \sup_{t\in(0,1)} \frac{K(t, f; H^{-1}(\mathbb{R}^n), L^2(\mathbb{R}^n))}{\Psi(-\log t)}. \tag{8-2}$$

Remark 50. A variant of Remark 42 also holds for the $T_\Psi(\mathbb{R}^n)$ spaces. Indeed, suppose that the admissibility condition stated Definition 36(i) is replaced by

$$\sum_{r=0}^{N} (2^{(1+s)r}\Psi(r))^2 \lesssim (2^{(1+s)N}\Psi(N))^2.$$

Let $H^s(\mathbb{R}^n)$ be the standard (fractional) Sobolev space endowed with the norm

$$\|f\|_{H^s(\mathbb{R}^n)} = \|(I-\Delta)^{s/2} f\|_{L^2(\mathbb{R}^n)},$$

then formula (8-2) holds if we replace the pair $(H^{-1}(\mathbb{R}^n), L^2(\mathbb{R}^n))$ by $(H^{-1}(\mathbb{R}^n), H^s(\mathbb{R}^n))$, where $s > -1$.

8.4. ***Proof of Theorem 47.*** For the proof we will use the following analogue of Proposition 43, that can be obtained mutatis mutandis and we therefore leave its proof to the interested reader.

Proposition 51. *Let $\{\Upsilon^G_{Nl} : N \in \mathbb{N}_0,\ G \in G^N,\ l \in \mathbb{Z}^n\}$ be a wavelet system satisfying* (7-16)–(7-18) *with*[41] $A > 1$. *Assume that Ψ is an admissible doubling decay. Then, $f \in T_\Psi(\mathbb{R}^n)$ if and only if*

$$f = \sum_{\substack{N\in\mathbb{N}_0,\, G\in G^N\\ l\in\mathbb{Z}^n}} \lambda^G_{Nl} 2^{-Nn/2} \Upsilon^G_{Nl}, \quad \{\lambda^G_{Nl}\} \in t_\Psi \tag{8-3}$$

(*unconditional convergence in the sense of $\mathcal{S}'(\mathbb{R}^n)$*), *where*

$$\|\{\lambda^G_{Nl}\}\|^2_{t_\Psi} := \sup_{N\in\mathbb{N}_0} \frac{1}{\Psi(N)^2} \sum_{k=N}^{\infty} 2^{k(-2-n)} \sum_{G\in G^k,\, l\in\mathbb{Z}^n} |\lambda^G_{kl}|^2 < \infty. \tag{8-4}$$

This representation is unique in the sense that the coefficients λ^G_{Nl} are determined by (7-8) *and the operator I given by* (7-9) *defines an isomorphism from $T_\Psi(\mathbb{R}^n)$ onto t_Ψ. Furthermore*

$$\|f\|_{T_\Psi(\mathbb{R}^n)} \approx \|\{\lambda^{N,G}_l\}\|_{t_\Psi}. \tag{8-5}$$

[40] Since T_Ψ is not homogeneous, the space $H^{-1}(\mathbb{R}^n)$ in (8-2) should be adequately interpreted from the context as the inhomogeneous counterpart of (4-3), equipped with the norm $\|f\|_{H^{-1}(\mathbb{R}^n)} = \|(I-\Delta)^{-1/2} f\|_{L^2(\mathbb{R}^n)}$. Recall the well-known fact that $\|(-\Delta)^{-1/2} f\|_{L^2(\mathbb{R}^n)} \approx \|I_1 f\|_{L^2(\mathbb{R}^n)}$.

[41] The explanation behind $A > 1$ comes from Theorem 49 and well-known wavelet assumptions on H^{-1}.

Proof of Theorem 47. (i) Let $L \in \mathbb{N}_0$. Given $f \geq 0$ compactly supported on Q_0 (without loss of generality, we may assume $Q_0 = (0,1)^n$), we observe that

$$s_L(f) = \sup_{\mathcal{Q} \in S(Q_0)} \left[\sum_{k=L}^{\infty} 2^{k(-2+n)} \sum_{i \in I : Q_i \in \mathbb{D}_k(\mathcal{Q})} \left(\int_{Q_i} f \right)^2 \right]^{1/2}.$$

Using Proposition 51 we will show that

$$\sup_{L \in \mathbb{N}_0} \frac{1}{\Psi(L)} \left[\sum_{k=L}^{\infty} 2^{k(-2+n)} \sum_{i \in I : Q_i \in \mathbb{D}_k(\mathcal{Q})} \left(\int_{Q_i} f \right)^2 \right]^{1/2} \lesssim \|f\|_{T_\Psi(\mathbb{R}^n)}, \tag{8-6}$$

with a constant independent of the sparse family $\mathcal{Q}$. Then the desired (local) embedding

$$\|f\|_{S_\Psi(Q_0)} \lesssim \|f\|_{T_\Psi(\mathbb{R}^n)}$$

follows readily.

Let χ be a smooth cut-off function introduced in the proof of Theorem 9, and define the corresponding coefficients $\chi_{jm}(f)$ via (4-22). According to (7-19), (7-20), (7-24) and (7-26), these coefficients can be estimated as

$$|\chi_{jm}(f)| \lesssim \sum_{r=0}^{j} \sum_{l \in \ell_r^j(m)} |\lambda_{rl}| + \sum_{r=j}^{\infty} 2^{(j-r)(n+A)} \sum_{l \in \ell_r^j(m)} |\lambda_{rl}|, \tag{8-7}$$

where $\ell_r^j(m)$, which was introduced in (7-21), satisfies

$$\operatorname{card} \ell_r^j(m) \approx \begin{cases} 2^{n(r-j)} & \text{if } r \geq j, \\ 1 & \text{if } r \leq j. \end{cases} \tag{8-8}$$

Using Hölder's inequality and (8-8),

$$\sum_{l \in \ell_r^j(m)} |\lambda_{rl}| \lesssim \left(\sum_{l \in \ell_r^j(m)} |\lambda_{rl}|^2 \right)^{1/2} \times \begin{cases} 2^{n(r-j)/2} & \text{if } r \geq j, \\ 1 & \text{if } r \leq j, \end{cases}$$

and inserting this into (8-7), we achieve

$$|\chi_{jm}(f)| \lesssim \sum_{r=0}^{j} \left(\sum_{l \in \ell_r^j(m)} |\lambda_{rl}|^2 \right)^{1/2} + 2^{j(A+\frac{n}{2})} \sum_{r=j}^{\infty} 2^{-r(A+\frac{n}{2})} \left(\sum_{l \in \ell_r^j(m)} |\lambda_{rl}|^2 \right)^{1/2}. \tag{8-9}$$

Let $\varepsilon \in (0, \min\{1, A-1\})$ (recall that $A > 1$ is fixed). By Hölder's inequality, the two terms given in the right-hand side of (8-9) can be bounded by

$$\sum_{r=0}^{j} \left(\sum_{l \in \ell_r^j(m)} |\lambda_{rl}|^2 \right)^{1/2} \lesssim 2^{j\varepsilon} \left(\sum_{r=0}^{j} 2^{-r\varepsilon 2} \sum_{l \in \ell_r^j(m)} |\lambda_{rl}|^2 \right)^{1/2}$$

and

$$\sum_{r=j}^{\infty} 2^{-r(A+\frac{n}{2})} \left(\sum_{l \in \ell_r^j(m)} |\lambda_{rl}|^2 \right)^{1/2} \lesssim 2^{-j\varepsilon} \left(\sum_{r=j}^{\infty} 2^{-r(A+\frac{n}{2}-\varepsilon)2} \sum_{l \in \ell_r^j(m)} |\lambda_{rl}|^2 \right)^{1/2}.$$

Hence

$$|\chi_{jm}(f)|^2 \lesssim 2^{j\varepsilon 2} \sum_{r=0}^{j} 2^{-r\varepsilon 2} \sum_{l\in \ell_r^j(m)} |\lambda_{rl}|^2 + 2^{j(A+\frac{n}{2}-\varepsilon)2} \sum_{r=j}^{\infty} 2^{-r(A+\frac{n}{2}-\varepsilon)2} \sum_{l\in \ell_r^j(m)} |\lambda_{rl}|^2$$

and summing up over all $m \in \mathbb{Z}^n$, we have

$$\sum_{m\in\mathbb{Z}^n} |\chi_{jm}(f)|^2 \lesssim 2^{j\varepsilon 2} \sum_{r=0}^{j} 2^{-r\varepsilon 2} \sum_{m\in\mathbb{Z}^n} \sum_{l\in \ell_r^j(m)} |\lambda_{rl}|^2 + 2^{j(A+\frac{n}{2}-\varepsilon)2} \sum_{r=j}^{\infty} 2^{-r(A+\frac{n}{2}-\varepsilon)2} \sum_{m\in\mathbb{Z}^n} \sum_{l\in \ell_r^j(m)} |\lambda_{rl}|^2. \quad (8\text{-}10)$$

Furthermore, we remark that

$$\operatorname{card}\{m \in \mathbb{Z}^n : l \in \ell_r^j(m)\} \approx \begin{cases} 2^{n(j-r)} & \text{if } r \le j, \\ 1 & \text{if } r \ge j. \end{cases}$$

Using this information and changing the order of summation in (8-10), we arrive at

$$\sum_{m\in\mathbb{Z}^n} |\chi_{jm}(f)|^2 \lesssim 2^{j\varepsilon 2} \sum_{r=0}^{j} 2^{-r\varepsilon 2} 2^{n(j-r)} \sum_{l\in\mathbb{Z}^n} |\lambda_{rl}|^2 + 2^{j(A+\frac{n}{2}-\varepsilon)2} \sum_{r=j}^{\infty} 2^{-r(A+\frac{n}{2}-\varepsilon)2} \sum_{l\in\mathbb{Z}^n} |\lambda_{rl}|^2,$$

where the equivalence constant is independent of j. Summing up the last estimate over all $j \ge L$ and using Fubini's theorem (recall that $\varepsilon < \min\{1, A-1\}$), we have

$$\begin{aligned}
\sum_{j=L}^{\infty} 2^{j(-2-n)} \sum_{m\in\mathbb{Z}^n} |\chi_{jm}(f)|^2
&\lesssim \sum_{j=L}^{\infty} 2^{j(-1+\varepsilon)2} \sum_{r=0}^{j} 2^{-r(\varepsilon 2+n)} \sum_{l\in\mathbb{Z}^n} |\lambda_{rl}|^2 + \sum_{j=L}^{\infty} 2^{j2(-1+A-\varepsilon)} \sum_{r=j}^{\infty} 2^{-r(A+\frac{n}{2}-\varepsilon)2} \sum_{l\in\mathbb{Z}^n} |\lambda_{rl}|^2 \\
&\lesssim 2^{L(-1+\varepsilon)2} \sum_{r=0}^{L} 2^{-r(\varepsilon 2+n)} \sum_{l\in\mathbb{Z}^n} |\lambda_{rl}|^2 + \sum_{r=L}^{\infty} 2^{-r(\varepsilon 2+n)} \sum_{l\in\mathbb{Z}^n} |\lambda_{rl}|^2 \sum_{j=r}^{\infty} 2^{j(-1+\varepsilon)2} \\
&\qquad + \sum_{r=L}^{\infty} 2^{-r(A+\frac{n}{2}-\varepsilon)2} \sum_{l\in\mathbb{Z}^n} |\lambda_{rl}|^2 \sum_{j=L}^{r} 2^{j2(-1+A-\varepsilon)} \\
&\lesssim 2^{L(-1+\varepsilon)2} \sum_{r=0}^{L} 2^{-r(\varepsilon 2+n)} \sum_{l\in\mathbb{Z}^n} |\lambda_{rl}|^2 + \sum_{r=L}^{\infty} 2^{-r(2+n)} \sum_{l\in\mathbb{Z}^n} |\lambda_{rl}|^2.
\end{aligned}$$

Hence

$$\sup_{L\in\mathbb{N}_0} \frac{1}{\Psi(L)} \left[\sum_{j=L}^{\infty} 2^{j(-2-n)} \sum_{m\in\mathbb{Z}^n} |\chi_{jm}(f)|^2 \right]^{1/2} \lesssim \mathcal{I} + \mathcal{II}, \quad (8\text{-}11)$$

where

$$\mathcal{I} := \sup_{L\in\mathbb{N}_0} \frac{2^{L(-1+\varepsilon)}}{\Psi(L)} \left[\sum_{r=0}^{L} 2^{-r(\varepsilon 2+n)} \sum_{l\in\mathbb{Z}^n} |\lambda_{rl}|^2 \right]^{1/2}$$

and

$$\mathcal{II} := \sup_{L\in\mathbb{N}_0} \frac{1}{\Psi(L)}\bigg[\sum_{r=L}^{\infty} 2^{-r(2+n)} \sum_{l\in\mathbb{Z}^n} |\lambda_{rl}|^2\bigg]^{1/2}.$$

We have

$$\begin{aligned}\bigg[\sum_{r=0}^{L} 2^{-r(\varepsilon 2+n)} \sum_{l\in\mathbb{Z}^n} |\lambda_{rl}|^2\bigg]^{1/2} &\leq \bigg[\sum_{r=0}^{L} 2^{r(1-\varepsilon)2}\Psi(r)^2\bigg]^{1/2} \sup_{M\in\mathbb{N}_0} \frac{2^{-M(1+\frac{n}{2})}}{\Psi(M)}\bigg[\sum_{l\in\mathbb{Z}^n} |\lambda_{Ml}|^2\bigg]^{1/2} \\ &\leq \bigg[\sum_{r=0}^{L} 2^{r(1-\varepsilon)2}\Psi(r)^2\bigg]^{1/2} \mathcal{II}. \end{aligned} \tag{8-12}$$

Furthermore, the following estimate holds:

$$\bigg[\sum_{r=0}^{L} 2^{r(1-\varepsilon)2}\Psi(r)^2\bigg]^{1/2} \lesssim 2^{L(1-\varepsilon)}\Psi(L). \tag{8-13}$$

Indeed, by monotonicity properties, a simple change of variables and the doubling property of Ψ (see (ii) in Definition 36),

$$\begin{aligned}\sum_{r=0}^{L} 2^{r(1-\varepsilon)2}\Psi(r)^2 \approx \int_0^L 2^{t(1-\varepsilon)2}\Psi(t)^2\,dt &\approx \int_0^{L(1-\varepsilon)} 2^{t2}\Psi\Big(\frac{t}{1-\varepsilon}\Big)^2\,dt \\ &\approx \int_0^{L(1-\varepsilon)} 2^{t2}\Psi(t)^2\,dt \approx \sum_{r=0}^{\lfloor L(1-\varepsilon)\rfloor} 2^{r2}\Psi(r)^2 \\ &\lesssim 2^{L(1-\varepsilon)2}\Psi(L)^2,\end{aligned}$$

where[42] the last step follows from (i) in Definition 36. This proves (8-13). Applying now (8-13) in (8-12), we find

$$\bigg[\sum_{r=0}^{L} 2^{-r(\varepsilon 2+n)} \sum_{l\in\mathbb{Z}^n} |\lambda_{rl}|^2\bigg]^{1/2} \lesssim 2^{L(1-\varepsilon)}\Psi(L)\,\mathcal{II},$$

i.e., we have shown that $\mathcal{I} \lesssim \mathcal{II}$. As a consequence (see (8-11))

$$\sup_{L\in\mathbb{N}_0} \frac{1}{\Psi(L)}\bigg[\sum_{j=L}^{\infty} 2^{j(-2-n)} \sum_{m\in\mathbb{Z}^n} |\chi_{jm}(f)|^2\bigg]^{1/2} \lesssim \mathcal{II},$$

or equivalently (see (8-4))

$$\sup_{L\in\mathbb{N}_0} \frac{1}{\Psi(L)}\bigg[\sum_{j=L}^{\infty} 2^{j(-2-n)} \sum_{m\in\mathbb{Z}^n} |\chi_{jm}(f)|^2\bigg]^{1/2} \lesssim \|\{\lambda_{rl}\}\|_{t_\Psi}.$$

[42]As usual, $\lfloor x\rfloor$ denotes the integer part of $x \in \mathbb{R}$.

Consequently, invoking Proposition 51,

$$\sup_{L\in\mathbb{N}_0}\frac{1}{\Psi(L)}\biggl[\sum_{j=L}^{\infty}2^{j(-2-n)}\sum_{m\in\mathbb{Z}^n}|\chi_{jm}(f)|^2\biggr]^{1/2}\lesssim\|f\|_{T_\Psi(\mathbb{R}^n)}. \tag{8-14}$$

On the other hand, the assumption $f\geq 0$ and (4-23) enable us to derive

$$\begin{aligned}\sup_{L\in\mathbb{N}_0}\frac{1}{\Psi(L)}\biggl[\sum_{k=L}^{\infty}2^{k(-2+n)}\sum_{i\in I:Q_i\in\mathbb{D}_k(\mathcal{Q})}\biggl(\int_{Q_i}f\biggr)^2\biggr]^{1/2}&\\ =\sup_{L\in\mathbb{N}_0}\frac{1}{\Psi(L)}\biggl[\sum_{k=L}^{\infty}2^{k(-2-n)}\sum_{i\in I:Q_i\in\mathbb{D}_k(\mathcal{Q})}\biggl(\int_{Q_i}2^{kn}f\biggr)^2\biggr]^{1/2}&\\ \lesssim\sup_{L\in\mathbb{N}_0}\frac{1}{\Psi(L)}\biggl[\sum_{k=L}^{\infty}2^{k(-2-n)}\sum_{i\in I:Q_i\in\mathbb{D}_k(\mathcal{Q})}\chi_{Q_i}(f)^2\biggr]^{1/2}&\\ \lesssim\|f\|_{T_\Psi(\mathbb{R}^n)},&\end{aligned}$$

where in the last step we used (8-14). This concludes the proof of (8-6) and hence (i).

(ii) Invoking Theorem 12, statement (ii) is a consequence of (i). □

8.5. ***Proof of Theorem 48.*** To avoid unnecessary technicalities, we assume, without loss of generality, that the constant c in (7-16) is equal to 1. From (7-8) and (7-17), we find

$$\begin{aligned}\sum_{k=N}^{\infty}2^{k(-2-n)}\sum_{G\in G^k,\,l\in\mathbb{Z}^n}|\lambda_{kl}^G|^2&\lesssim\sum_{k=N}^{\infty}2^{k(-2+n)}\sum_{l\in\mathbb{Z}^n}\biggl(\int_{Q_{kl}}|f|\biggr)^2\\ &\approx\sum_{k=N}^{\infty}2^{k(-2+n)}2^{-k(-2+n)}k^{-2\alpha}\sum_{l\in\mathbb{Z}^n}\biggl(\frac{\bigl|\log|Q_{kl}|\bigr|^\alpha}{|Q_{kl}|^{1/(2n/(n+2))'}}\int_{Q_{kl}}|f|\biggr)^2\\ &\leq\|f\|^2_{R_{2n/(n+2),2}\log^\alpha(\mathbb{R}^n)}\sum_{k=N}^{\infty}k^{-2\alpha}\\ &\approx N^{-2\alpha+1}\|f\|^2_{R_{2n/(n+2),2}\log^\alpha(\mathbb{R}^n)}.\end{aligned}$$

Consequently,

$$\sup_{N\in\mathbb{N}_0}\frac{1}{\Psi_\alpha(N)^2}\sum_{k=N}^{\infty}2^{k(-2-n)}\sum_{G\in G^k,\,l\in\mathbb{Z}^n}|\lambda_{kl}^G|^2\lesssim\|f\|^2_{R_{2n/(n+2),2}\log^\alpha(\mathbb{R}^n)},$$

where $\Psi_\alpha(t)=t^{-\alpha+1/2}$. Then by (8-4) and (8-5) it follows that

$$R_{\frac{2n}{n+2},2}\log^\alpha(\mathbb{R}^n)\hookrightarrow T_{\Psi_\alpha}(\mathbb{R}^n),\quad \Psi_\alpha(t)=t^{-\alpha+\frac{1}{2}}. \tag{8-15}$$

To show that the embedding (8-15) is strict we argue by contradiction. Suppose to the contrary that

$$R_{\frac{2n}{n+2},2}\log^\alpha(\mathbb{R}^n)=T_{\Psi_\alpha}(\mathbb{R}^n).$$

In particular (for a fixed G)

$$\|2^K \Upsilon^G_{K(0,\dots,0)}\|_{T_{\Psi_\alpha}(\mathbb{R}^n)} \approx \|2^K \Upsilon^G_{K(0,\dots,0)}\|_{R_{2n/(n+2),2}\log^\alpha(\mathbb{R}^n)} \quad \text{for every } K \in \mathbb{N}. \tag{8-16}$$

Using Proposition 51, we compute

$$\|2^K \Upsilon^G_{K(0,\dots,0)}\|_{T_{\Psi_\alpha}(\mathbb{R}^n)} \approx \sup_{N\le K} \frac{1}{N^{-\alpha+1/2}} = \frac{1}{K^{-\alpha+1/2}},$$

which combined with (8-16) results in

$$\begin{aligned} \frac{1}{K^{-\alpha+1/2}} &\approx \|2^K \Upsilon^G_{K(0,\dots,0)}\|_{R_{2n/(n+2),2}\log^\alpha(\mathbb{R}^n)} \\ &\gtrsim \frac{\bigl|\log|Q_{K(0,\dots,0)}|\bigr|^\alpha}{|Q_{K(0,\dots,0)}|^{(n-2)/(2n)}} \int_{Q_{K(0,\dots,0)}} 2^K |\Upsilon^G_{K(0,\dots,0)}| \\ &\approx \frac{K^\alpha}{2^{-Kn/2}} \int_{Q_{K(0,\dots,0)}} |\Upsilon^G_{K(0,\dots,0)}| \\ &\gtrsim \frac{K^\alpha}{2^{-Kn/2}} 2^{Kn/2} |Q_{K(0,\dots,0)}| = K^\alpha. \end{aligned}$$

Taking the limit as $K \to \infty$ we arrive at a contradiction. □

Next we seek a strategy to establish a priori T_Ψ-bounds as required in Theorem 47(ii). In particular, we shall focus on the critical case $\alpha = \frac{1}{2}$ in the prototypical choice $\Psi_\alpha(t) = t^{-\alpha+1/2}$ (see Theorem 48). This case presents several intrinsic difficulties. In particular, according to Definition 44,

$$T_1(\mathbb{R}^n) = H^{-1}(\mathbb{R}^n) \quad \text{if } \Psi_{1/2}(t) = 1, \tag{8-17}$$

so we shall redefine T_{Ψ_α} to avoid trivial statements. In order to do this, we first observe that, for $\alpha > \frac{1}{2}$, the following characterization of T_{Ψ_α} holds (see Proposition 51):

$$\|f\|^2_{T_{\Psi_\alpha}(\mathbb{R}^n)} \approx \sup_{N\in\mathbb{N}_0} \frac{1}{N^{-2\alpha+1}} \sum_{k=N}^\infty 2^{k(-2-n)} \sum_{G\in G^k,\, l\in\mathbb{Z}^n} |\lambda^G_{kl}|^2. \tag{8-18}$$

Putting (at least formally) $\alpha = \frac{1}{2}$ in the previous characterization, we would arrive at the wavelet counterpart of (8-17), i.e.,

$$\|f\|^2_{T_1(\mathbb{R}^n)} \approx \sum_{k=0}^\infty 2^{k(-2-n)} \sum_{G\in G^k,\, l\in\mathbb{Z}^n} |\lambda^G_{kl}|^2 \approx \|f\|^2_{H^{-1}(\mathbb{R}^n)},$$

where the last equivalence corresponds to the classical wavelet characterization of $H^{-1}(\mathbb{R}^n)$. By basic monotonicity properties, (8-18) can be rewritten as

$$\|f\|^2_{T_{\Psi_\alpha}(\mathbb{R}^n)} \approx \sup_{j\in\mathbb{N}_0} \frac{1}{2^{j(-2\alpha+1)}} \sum_{k=2^j-1}^{2^{j+1}-2} 2^{k(-2-n)} \sum_{G\in G^k,\, l\in\mathbb{Z}^n} |\lambda^G_{kl}|^2 \tag{8-19}$$

if $\alpha > \frac{1}{2}$. When compared with the standard characterization (8-18) of $T_{\Psi_\alpha}(\mathbb{R}^n)$, the alternative characterization given by (8-19) has the important advantage of providing us with nontrivial spaces (i.e., different from H^{-1}) in the critical case $\alpha = \frac{1}{2}$, and leads to the following:

Definition 52. The space $T_1(\mathbb{R}^n)$ is formed by all those $f \in \mathcal{S}'(\mathbb{R}^n)$ with wavelet decomposition (8-3) such that

$$\|f\|^2_{T_1(\mathbb{R}^n)} = \sup_{j\in\mathbb{N}_0} \sum_{k=2^j-1}^{2^{j+1}-2} 2^{k(-2-n)} \sum_{G\in G^k,\, l\in\mathbb{Z}^n} |\lambda^G_{kl}|^2 < \infty,$$

where $\{\lambda^G_{kl}\}$ is the sequence of wavelet coefficients of f given by (7-8).

Proposition 53. *Let $\{u^\varepsilon\}_{\varepsilon>0}$ be a family of approximate solutions of the 2D Euler equations with corresponding initial vorticities $\{\omega^\varepsilon_0\}_{\varepsilon>0}$ of positive sign. Then, for every $\varepsilon > 0$ and $t > 0$,*

$$\|\omega^\varepsilon(t)\|_{T_1(\mathbb{R}^2)_c} \lesssim 1. \tag{8-20}$$

Proof. Consider the pseudoenergy

$$H(\omega) = -\frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log|x-y|\omega(x)\omega(y)\,dx\,dy.$$

For every fixed $k \in \mathbb{N}_0$, we can express $H(\omega^\varepsilon(t))$ as

$$\begin{aligned} H(\omega^\varepsilon(t)) = -\frac{1}{2\pi}\sum_{l\in\mathbb{Z}^2}\int_{Q_{kl}}\int_{Q_{kl}} \log|x-y|\omega^\varepsilon(x,t)\omega^\varepsilon(y,t)\,dx\,dy \\ -\frac{1}{2\pi}\sum_{l\neq m}\int_{Q_{kl}}\int_{Q_{km}} \log|x-y|\omega^\varepsilon(x,t)\omega^\varepsilon(y,t)\,dx\,dy \\ =: H_{si,k}(\omega^\varepsilon(t)) + H_{ie,k}(\omega^\varepsilon(t)). \end{aligned} \tag{8-21}$$

Here, $H_{si,k}$ and $H_{ie,k}$ refer to the self-induced part and the interaction energy at the dyadic level k, respectively.

Since $(\log|x-y|)_+ \le 2(|x|^2+|y|^2)$ (apply the parallelogram rule!), we can estimate $H_{ie,k}$ as

$$\begin{aligned} -H_{ie,k}(\omega^\varepsilon(t)) &\le \frac{1}{\pi}\sum_{l\neq m}\int_{Q_{kl}}\int_{Q_{km}} (|x|^2+|y|^2)\omega^\varepsilon(x,t)\omega^\varepsilon(y,t)\,dx\,dy \\ &\le \frac{2}{\pi} I_0(\omega^\varepsilon(t))I_2(\omega^\varepsilon(t)), \end{aligned} \tag{8-22}$$

where

$$I_0(\omega) := \int_{\mathbb{R}^2}\omega(x)\,dx, \quad I_2(\omega) := \int_{\mathbb{R}^2}|x|^2\omega(x)\,dx.$$

Let $j \in \mathbb{N}_0$. By the positivity assumption of ω^ε and using (7-8), (7-16) and (7-17), we have

$$\sum_{k=2^j-1}^{2^{j+1}-2} 2^{-4k}\sum_{l\in\mathbb{Z}^2} |\lambda_{kl}(\omega^\varepsilon(t))|^2 \le \sum_{k=2^j-1}^{2^{j+1}-2}\sum_{l\in\mathbb{Z}^2}\left(\int_{Q_{kl}}|\omega^\varepsilon(x,t)|\,dx\right)^2$$

$$= \sum_{k=2^j-1}^{2^{j+1}-2} \sum_{l\in\mathbb{Z}^2} \int_{Q_{kl}} \int_{Q_{kl}} \omega^\varepsilon(x,t)\omega^\varepsilon(y,t)\,dx\,dy$$
$$\lesssim \sum_{k=2^j-1}^{2^{j+1}-2} \frac{1}{k} \sum_{l\in\mathbb{Z}^2} \int_{Q_{kl}} \int_{Q_{kl}} \big|\log|x-y|\big|\omega^\varepsilon(x,t)\omega^\varepsilon(y,t)\,dx\,dy$$
$$\approx \sum_{k=2^j-1}^{2^{j+1}-2} \frac{1}{k} H_{si,k}(\omega^\varepsilon(t)).$$

Then, by (8-21) and (8-22),

$$\begin{aligned}\sum_{k=2^j-1}^{2^{j+1}-2} 2^{-4k} \sum_{l\in\mathbb{Z}^2} |\lambda_{kl}(\omega^\varepsilon(t))|^2 &\lesssim \sum_{k=2^j-1}^{2^{j+1}-2} \frac{1}{k}\big[H(\omega^\varepsilon(t)) - H_{ie,k}(\omega^\varepsilon(t))\big] \\ &\le \Big[H(\omega^\varepsilon(t)) + \frac{2}{\pi} I_0(\omega^\varepsilon(t)) I_2(\omega^\varepsilon(t))\Big] \sum_{k=2^j-1}^{2^{j+1}-2} \frac{1}{k} \\ &\lesssim H(\omega^\varepsilon(t)) + I_0(\omega^\varepsilon(t)) I_2(\omega^\varepsilon(t)).\end{aligned}$$

Taking now the supremum over all j, we arrive at

$$\|\omega^\varepsilon(t)\|^2_{T_{1/2}(\mathbb{R}^2)} \lesssim H(\omega^\varepsilon(t)) + I_0(\omega^\varepsilon(t)) I_2(\omega^\varepsilon(t)) \lesssim 1,$$

where the last step follows from well-known conservation laws and ε-independence bounds for the quantities H, I_0 and I_2; see [Majda 1993, Section 3]. □

Remark 54. An alternative proof of Proposition 53 can be obtained from RMT spaces. To be more precise, we claim that

$$R_{1,2} \log^{\frac{1}{2}}(\mathbb{R}^2) \hookrightarrow T_1(\mathbb{R}^2). \tag{8-23}$$

Assuming momentarily the validity of this embedding, (8-20) follows from the a priori estimates given in [Tadmor 2001, Lemma 4.1] that assert (under the same assumptions of Proposition 53)

$$\|\omega^\varepsilon(t)\|_{R_{1,2}\log^{1/2}(\mathbb{R}^2)_c} \lesssim 1.$$

Next we show (8-23): for every $j \in \mathbb{N}_0$, we have

$$\begin{aligned}\sum_{k=2^j-1}^{2^{j+1}-2} 2^{-4k} \sum_{l\in\mathbb{Z}^2} |\lambda_{kl}|^2 &\le \sum_{k=2^j-1}^{2^{j+1}-2} \sum_{l\in\mathbb{Z}^2} \Big(\int_{Q_{kl}} |f|\Big)^2 \\ &\lesssim \|f\|^2_{R_{1,2}\log^{1/2}(\mathbb{R}^2)} \sum_{k=2^j-1}^{2^{j+1}-2} k^{-1} \approx \|f\|^2_{R_{1,2}\log^{1/2}(\mathbb{R}^2)}.\end{aligned}$$

This gives the desired result (8-23).

Remark 55. A priori estimates in the spirit of Proposition 53 but now for the V_Ψ-spaces introduced in Section 7 also hold. In particular, similar to in Definition 52, the distributional space $V_1(\mathbb{R}^2)$ is endowed with[43]

$$\|f\|_{V_1(\mathbb{R}^n)} = \sup_{j\in\mathbb{N}_0} \sum_{k=2^j-1}^{2^{j+1}-2} 2^{-2k}(1+k)^{-1/2} \sup_{G\in G^k,\, l\in\mathbb{Z}^n} |\lambda_{kl}^G|,$$

where $\{\lambda_{kl}^G\}$ is the sequence of wavelet coefficients of f given by (7-8); see Proposition 43. If $\{u^\varepsilon\}_{\varepsilon>0}$ is a family of approximate solutions of the 2D Euler equations with corresponding initial vorticities $\{\omega_0^\varepsilon\}_{\varepsilon>0}$ of positive sign then, for every $\varepsilon>0$ and $t>0$,

$$\|\omega^\varepsilon(t)\|_{V_1(\mathbb{R}^2)_c} \lesssim 1.$$

The proof of this result is similar but easier than the one given for Proposition 53.

9. Comparison between V_Ψ and T_Ψ

In view of the results obtained in Sections 7 and 8, it is natural to compare[44] the spaces $V_\Psi(\mathbb{R}^2)$ and $T_\Psi(\mathbb{R}^2)$ for a fixed decay Ψ. In this section we show that neither space contains the other. More precisely, we construct explicit examples of functions showing that $T_\Psi(\mathbb{R}^2)\setminus V_\Psi(\mathbb{R}^2)\neq\varnothing$ and $V_\Psi(\mathbb{R}^2)\setminus T_\Psi(\mathbb{R}^2)\neq\varnothing$.

Example 56 ($T_\Psi(\mathbb{R}^2)\setminus V_\Psi(\mathbb{R}^2)\neq\varnothing$). Given a scalar sequence $\{c_N\}_{N\in\mathbb{N}_0}$, let f be given by (7-7), where

$$\lambda_{Nl} = \begin{cases} 2^{2N}c_N, & N\in\mathbb{N}_0,\ l=(0,\dots,0),\\ 0, & \text{otherwise.}\end{cases}$$

We compute the norms of f in V_Ψ and T_Ψ using Propositions 43 and 51, respectively,

$$\|f\|_{V_\Psi(\mathbb{R}^2)} \approx \sup_{N\in\mathbb{N}_0} \frac{1}{\Psi(N)^2}\sum_{k=N}^\infty |c_k|$$

and

$$\|f\|_{T_\Psi(\mathbb{R}^2)} \approx \sup_{N\in\mathbb{N}_0} \frac{1}{\Psi(N)}\biggl(\sum_{k=N}^\infty |c_k|^2\biggr)^{1/2}.$$

Therefore, if we select $\{c_N\}_{N\in\mathbb{N}_0}\in\ell_2$ such that

$$\sum_{k=N}^\infty c_k^2 \approx \Psi(N)^2, \tag{9-1}$$

then $\|f\|_{T_\Psi(\mathbb{R}^2)}\approx 1$. The existence of such sequences (even with $\approx$ replaced by $=$ in (9-1)) is guaranteed by classical results in approximation theory (see, e.g., [Timan 1963, Section 2.5]). On the other hand, since

$$\sum_{k=N}^\infty |c_k| \geq \biggl(\inf_{l\geq N}\frac{1}{|c_l|}\biggr)\sum_{k=N}^\infty |c_k|^2 \approx \frac{\Psi(N)^2}{\sup_{l\geq N}|c_l|},$$

[43] Note that the additional log-smoothness of order $\frac{1}{2}$ (i.e., $(1+k)^{-1/2}$) is now introduced in the definition of $V_1(\mathbb{R}^n)$; see also Proposition 43. This modification is rather natural in view of the classical a priori estimates for $M^{1,1/2}(\mathbb{R}^2)$ obtained in [Majda 1993, Proposition, p. 928].

[44] According to the discussion at the beginning of Section 8.1, we may restrict our attention to the 2D setting.

we have

$$\frac{1}{\Psi(N)^2}\sum_{k=N}^{\infty}|c_k| \gtrsim \frac{1}{\sup_{l\geq N}|c_l|}, \tag{9-2}$$

but since $\{c_N\}_{N\in\mathbb{N}_0}\in\ell_2$, we have $\lim_{N\to\infty}|c_N|=0$ and therefore the left-hand side of (9-2) tends to ∞ as $N\to\infty$, showing that $f\notin V_\Psi(\mathbb{R}^2)$.

Example 57 ($V_\Psi(\mathbb{R}^2)\backslash T_\Psi(\mathbb{R}^2)\neq\varnothing$). Given a scalar sequence $\{c_l\}_{l\in\mathbb{Z}^n}$, define f using (7-7) with

$$\lambda_{Nl}=\begin{cases}c_l, & N=0,\ l\in\mathbb{Z}^n,\\ 0, & \text{otherwise.}\end{cases}$$

Computing norms using Propositions 43 and 51, we find

$$\|f\|_{V_\Psi(\mathbb{R}^2)}\approx\sup_{l\in\mathbb{Z}^n}|c_l| \quad\text{and}\quad \|f\|_{T_\Psi(\mathbb{R}^2)}\approx\biggl(\sum_{l\in\mathbb{Z}^n}|c_l|^2\biggr)^{1/2}.$$

Thus, if we select $\{c_l\}_{l\in\mathbb{Z}^n}\in\ell_\infty\backslash\ell_2$ we obtain an example of $f\in V_\Psi(\mathbb{R}^2)$ but $f\notin T_\Psi(\mathbb{R}^2)$.

Remark 58. The above computations show a stronger assertion: given any decays Ψ and Φ, one can always construct $f\in V_\Psi(\mathbb{R}^2)$ such that $f\notin T_\Phi(\mathbb{R}^2)$.

The previous remark shows that for different decays $\Psi\neq\Phi$, $V_\Psi(\mathbb{R}^2)$ cannot be contained in $T_\Phi(\mathbb{R}^2)$. The next result shows that under some additional condition the reverse inclusion is possible.

Proposition 59. *Suppose that*

$$\sum_{j=N}^{\infty}\Phi(j)\lesssim\Psi(N)^2,\quad N\in\mathbb{N}_0. \tag{9-3}$$

Then

$$T_\Phi(\mathbb{R}^2)\hookrightarrow V_\Psi(\mathbb{R}^2).$$

Proof. We use Nikolskii's inequality for entire functions of exponential type (see, e.g., [Timan 1963, Section 4.9.53]) to estimate

$$\begin{aligned}\sum_{j=N}^{\infty}2^{-2j}\|\Delta_j f\|_{L^\infty(\mathbb{R}^2)} &\lesssim \sum_{j=N}^{\infty}2^{-j}\|\Delta_j f\|_{L^2(\mathbb{R}^2)}\\ &\leq\biggl(\sup_{M\in\mathbb{N}_0}\frac{2^{-M}}{\Phi(M)}\|\Delta_M f\|_{L^2(\mathbb{R}^2)}\biggr)\sum_{j=N}^{\infty}\Phi(j)\\ &\lesssim\Psi(N)^2\sup_{M\in\mathbb{N}_0}\frac{2^{-M}}{\Phi(M)}\|\Delta_M f\|_{L^2(\mathbb{R}^2)}\\ &\leq\Psi(N)^2\sup_{M\in\mathbb{N}_0}\frac{1}{\Phi(M)}\biggl(\sum_{j=M}^{\infty}2^{-2j}\|\Delta_j f\|^2_{L^2(\mathbb{R}^2)}\biggr)^{1/2}\\ &=\Psi(N)^2\,\|f\|_{T_\Phi(\mathbb{R}^2)}.\end{aligned}$$

Therefore $T_\Phi(\mathbb{R}^2)\hookrightarrow V_\Psi(\mathbb{R}^2)$. □

Remark 60. The assumption (9-3) is quite restrictive. In particular, it forces the series $\sum_{j=0}^{\infty}\Phi(j)$ to be convergent. This automatically excludes many interesting examples of spaces T_Φ such as the corresponding one to the decay $\Phi(j)=j^{-\alpha}$ with $\alpha\in(0,1]$, which are connected with R-spaces (see Theorem 48). In conclusion, the analysis of T_Φ cannot be, in general, reduced to study of the simpler spaces V_Ψ.

10. A sparse approach to energy conservation

Throughout this section, we work with the following special class of approximate solutions on $\mathbb{T}^2\equiv[0,2\pi]^2$ introduced in [Cheskidov et al. 2016, Definition 3].

Definition 61. Let $u\in C(0,T;L^2(\mathbb{T}^2))$ with $u_0\in L^2(\mathbb{T}^2)$. We say that a weak solution u of Euler equations is *physically realizable* with initial velocity u_0 provided that there exists a family $\{u^\varepsilon\}_{\varepsilon>0}$ of solutions of Navier–Stokes equations with viscosity ε, such that $u^\varepsilon\rightharpoonup u$ weakly* in $L^\infty(0,T;L^2(\mathbb{T}^2))$ and $u_0^\varepsilon\to u_0$ strongly in $L^2(\mathbb{T}^2)$. In this case, $\{u^\varepsilon\}_{\varepsilon>0}$ is called a *physical realization* of u.

Next we provide the proof of Theorem 14. In this regard, the following interpolation inequality involving sparse function spaces plays a crucial role. This result is of independent interest.

Lemma 62. *Let Q_0 be a cube in $\mathbb{R}^2$ or $Q_0=\mathbb{T}^2$, and let Ψ be an admissible decay. Assume that $f\in S_\Psi(Q_0)\cap\dot H^1(Q_0)$. Then, with absolute constants, we have*

$$\|f-f_{Q_0}\|_{L^2(Q_0)}\lesssim\frac{\Psi(-\log r)}{r}\|f\|_{S_\Psi(Q_0)}+r\|\nabla f\|_{L^2(Q_0)}\quad\textit{for all } r\in(0,1). \tag{10-1}$$

Remark 63 (see equation (11) in [Cheskidov et al. 2016]). Let us show how (10-1) can be applied to produce a classical Gagliardo–Nirenberg inequality. Let $f\in L^p$, $p\in(1,2)$. Then by Proposition 30, with decay $\Psi(t)=2^{-2t(1-1/p)}$, $L^p(Q_0)\hookrightarrow S_\Psi(Q_0)$. Applying (10-1) for this special decay, we obtain

$$\|f-f_{Q_0}\|_{L^2(Q_0)}\lesssim r^{1-\frac{2}{p}}\|f\|_{L^p(Q_0)}+r\|\nabla f\|_{L^2(Q_0)}$$

for all $r\in(0,1)$. Optimizing the right-hand side by equating both terms, i.e., selecting

$$r=\left(\frac{\|\nabla f\|_{L^2(Q_0)}}{\|f\|_{L^p(Q_0)}}\right)^{-\frac{p}{2}},$$

we find

$$\|f-f_{Q_0}\|_{L^2(Q_0)}\lesssim\|\nabla f\|_{L^2(Q_0)}^{1-\frac{p}{2}}\|f\|_{L^p(Q_0)}^{\frac{p}{2}}.$$

Proof of Lemma 62. We will use the sparse characterization of L^2 (see Theorem 29 and [Domínguez and Milman 2021]):

$$\|f-f_{Q_0}\|_{L^2(Q_0)}\approx\sup_{\mathcal{Q}=(Q_i)_{i\in I}\in S(Q_0)}\left\{\sum_{i\in I}\left(\frac{1}{|Q_i|}\int_{Q_i}|f-f_{Q_i}|\right)^2|Q_i|\right\}^{1/2} \tag{10-2}$$

and

$$\|f\|_{L^2(Q_0)}\approx\sup_{\mathcal{Q}\in S(Q_0)}\left\{\sum_{i\in I}\left(\frac{1}{|Q_i|}\int_{Q_i}|f|\right)^2|Q_i|\right\}^{1/2}. \tag{10-3}$$

To estimate the left-hand side of (10-1) we use (10-2). Let $f \in S_\Psi(Q_0)$ and $\mathcal{Q} \in S(Q_0)$. Then, for $M \in \mathbb{N}_0$ we have

$$\sum_{i\in I}\biggl(\frac{1}{|Q_i|}\int_{Q_i}|f-f_{Q_i}|\biggr)^2|Q_i| = J_1(M)+J_2(M), \tag{10-4}$$

where

$$J_1(M) := \sum_{k=0}^{M}\sum_{Q_i\in\mathbb{D}_k(\mathcal{Q})}\biggl(\frac{1}{|Q_i|}\int_{Q_i}|f-f_{Q_i}|\biggr)^2|Q_i|,$$
$$J_2(M) := \sum_{k=M+1}^{\infty}\sum_{Q_i\in\mathbb{D}_k(\mathcal{Q})}\biggl(\frac{1}{|Q_i|}\int_{Q_i}|f-f_{Q_i}|\biggr)^2|Q_i|.$$

We estimate $J_1(M)$ and $J_2(M)$. Since $f \in S_\Psi(Q_0)$, we find

$$\begin{aligned} J_1(M) \lesssim \sum_{k=0}^{M}\sum_{Q_i\in\mathbb{D}_k(\mathcal{Q})}\biggl(\frac{1}{|Q_i|}\int_{Q_i}|f|\biggr)^2|Q_i| &\approx \sum_{k=0}^{M}2^{2k}\sum_{Q_i\in\mathbb{D}_k(\mathcal{Q})}\biggl(\int_{Q_i}|f|\biggr)^2 \\ &\le \sum_{k=0}^{M}2^{2k}s_{k+1}(f)^2 \\ &\lesssim \|f\|_{S_\Psi(Q_0)}^2\sum_{k=1}^{M}2^{2k}\Psi(k)^2 \\ &\lesssim \|f\|_{S_\Psi(Q_0)}^2 2^{2M}\Psi(M)^2, \end{aligned} \tag{10-5}$$

where we have used Definition 36(i) in the last estimate.

To estimate $J_2(M)$ we will make use of the classical Poincaré inequality,

$$\int_Q|f-f_Q| \lesssim \ell(Q)\int_Q|\nabla f|. \tag{10-6}$$

Then, by (10-6) and (10-3) applied to $|\nabla f|$,

$$\begin{aligned} J_2(M) &\lesssim \sum_{k=M+1}^{\infty}\sum_{Q_i\in\mathbb{D}_k(\mathcal{Q})}\biggl(\frac{\ell(Q_i)}{|Q_i|}\int_{Q_i}|\nabla f|\biggr)^2|Q_i| \\ &\approx \sum_{k=M+1}^{\infty}2^{-k2}\sum_{Q_i\in\mathbb{D}_k(\mathcal{Q})}\biggl(\frac{1}{|Q_i|}\int_{Q_i}|\nabla f|\biggr)^2|Q_i| \\ &\lesssim \|\nabla f\|_{L^2(Q_0)}^2\sum_{k=M+1}^{\infty}2^{-k2} \quad \text{(by Hölder's inequality)} \\ &\approx \|\nabla f\|_{L^2(Q_0)}^2 2^{-M2}. \end{aligned} \tag{10-7}$$

Inserting the estimates (10-5) and (10-7) into (10-4), we achieve

$$\left\{\sum_{i\in I}\left(\frac{1}{|Q_i|}\int_{Q_i}|f-f_{Q_i}|\right)^2|Q_i|\right\}^{1/2}\lesssim\|f\|_{S_\Psi(Q_0)}2^M\Psi(M)+\|\nabla f\|_{L^2(Q_0)}2^{-M}.$$

Since this bound is independent of the sparse family $\mathcal{Q}$, we arrive at (see (10-2))

$$\|f-f_{Q_0}\|_{L^2(Q_0)}\lesssim\|f\|_{S_\Psi(Q_0)}2^M\Psi(M)+\|\nabla f\|_{L^2(Q_0)}\,2^{-M}.$$

Since Ψ is decreasing, the previous estimate can be expressed as

$$\|f-f_{Q_0}\|_{L^2(Q_0)}\lesssim\frac{\Psi(-\log r)}{r}\|f\|_{S_\Psi(Q_0)}+r\|\nabla f\|_{L^2(Q_0)}$$

for all $r\in(0,1)$. □

We are now ready to present the proof of Theorem 14. The strategy of proof is inspired by [Lanthaler et al. 2021]; we have replaced the role played there by structure functions with our decays of sparse indices. We provide full details for the sake of completeness.

Proof of Theorem 14. Let $\{u^\varepsilon\}_{\varepsilon>0}$ be a physical realization of u and let $\{\omega^\varepsilon\}_{\varepsilon>0}$ be the related vorticities. By assumption, there exists an admissible decay Ψ such that

$$M:=\sup_{\varepsilon>0}\|\omega^\varepsilon\|_{C(0,T;S_\Psi(\mathbb{T}^2))}<\infty. \tag{10-8}$$

Furthermore, ω^ε satisfies the transport equation,

$$\omega^\varepsilon_t+u^\varepsilon\cdot\nabla\omega^\varepsilon=\varepsilon\Delta\omega^\varepsilon,$$

and $\operatorname{div}u^\varepsilon=0$. Multiplying both sides of the previous equation by ω^ε and integrating on $\mathbb{T}^2$ yields

$$\frac{d}{dt}\,\|\omega^\varepsilon\|^2_{L^2(\mathbb{T}^2)}=-2\varepsilon\|\nabla\omega^\varepsilon\|^2_{L^2(\mathbb{T}^2)}.$$

Consequently, for any $\delta\in(0,T)$ and $t\in(\delta,T)$,

$$\|\omega^\varepsilon(t)\|^2_{L^2(\mathbb{T}^2)}=\|\omega^\varepsilon(\delta)\|^2_{L^2(\mathbb{T}^2)}-2\varepsilon\int_\delta^t\|\nabla\omega^\varepsilon(s)\|^2_{L^2(\mathbb{T}^2)}\,ds. \tag{10-9}$$

According to Lemma 62 (with $Q_0=\mathbb{T}^2$ and[45] $f=\omega^\varepsilon=\omega^\varepsilon(\cdot\,,t),\ t\in(0,T)$) and (10-8), there exists a universal constant $C>0$ such that

$$\|\omega^\varepsilon\|^2_{L^2(\mathbb{T}^2)}\le C\frac{\Psi(-\log r)^2}{r^2}M^2+Cr^2\|\nabla\omega^\varepsilon\|^2_{L^2(\mathbb{T}^2)}\quad\text{for all }\ r\in(0,1).$$

Integrating,

$$\int_\delta^t\|\omega^\varepsilon(s)\|^2_{L^2(\mathbb{T}^2)}\,ds\le CTM^2\frac{\Psi(-\log r)^2}{r^2}+Cr^2\int_\delta^t\|\nabla\omega^\varepsilon(s)\|^2_{L^2(\mathbb{T}^2)}\,ds \tag{10-10}$$

[45] Note that ω^ε has mean zero, i.e., $\omega^\varepsilon_{\mathbb{T}^2}=0$.

for $r \in (0,1)$. In fact, letting $\Psi(t) = \Psi(0)$ for $t < 0$, (10-10) with $r \geq 1$ follows immediately from Poincaré's inequality.[46] Next we optimize the right-hand side of (10-10), setting

$$r_0 := \log \frac{\left(\int_\delta^t \|\nabla \omega^\varepsilon(s)\|^2_{L^2(\mathbb{T}^2)}\, ds\right)^{1/4}}{\Psi(0)^{1/2}} \quad \text{and} \quad r = \frac{\Psi(r_0)^{1/2}}{\left(\int_\delta^t \|\nabla \omega^\varepsilon(s)\|^2_{L^2(\mathbb{T}^2)}\, ds\right)^{1/4}}.$$

Note that $-\log r \geq r_0$ (since Ψ is decreasing) and thus $\Psi(-\log r) \leq \Psi(r_0)$. Accordingly, it follows from (10-10) that

$$\left(\int_\delta^t \|\omega^\varepsilon(s)\|^2_{L^2(\mathbb{T}^2)}\, ds\right)^2 \leq C^2(TM^2+1)^2 \Psi\left(\log \frac{\left(\int_\delta^t \|\nabla \omega^\varepsilon(s)\|^2_{L^2(\mathbb{T}^2)}\, ds\right)^{1/4}}{\Psi(0)^{1/2}}\right)^2 \int_\delta^t \|\nabla \omega^\varepsilon(s)\|^2_{L^2(\mathbb{T}^2)}\, ds.$$

Setting $x_\varepsilon = x_\varepsilon(t) = \varepsilon \int_\delta^t \|\omega^\varepsilon(s)\|^2_{L^2(\mathbb{T}^2)}\, ds$ and $y_\varepsilon = y_\varepsilon(t) = \int_\delta^t \|\nabla \omega^\varepsilon(s)\|^2_{L^2(\mathbb{T}^2)}\, ds$, the previous estimate can be rewritten as

$$\left(\frac{x_\varepsilon}{\varepsilon}\right)^2 \leq f(y_\varepsilon), \tag{10-11}$$

where

$$f(y) = C^2(TM^2+1)^2 y \Psi\left(\log \frac{y^{1/4}}{\Psi(0)^{1/2}}\right)^2.$$

The function f satisfies

$$\sup_{y>0} \frac{f(y)}{y} = C^2(TM^2+1)^2 \sup_{y>0} \Psi\left(\log \frac{y^{1/4}}{\Psi(0)^{1/2}}\right)^2 = C^2(TM^2+1)^2 \Psi(0)^2 < \infty,$$

and (recall that $\lim_{y\to\infty} \Psi(y) = 0$)

$$\limsup_{y\to\infty} \frac{f(y)}{y} = C^2(TM^2+1)^2 \limsup_{y\to\infty} \Psi\left(\log \frac{y^{1/4}}{\Psi(0)^{1/2}}\right)^2 = 0.$$

In addition $f(0) = 0$ (note that $\lim_{y\to-\infty} \Psi(y) = \Psi(0)$). Hence [Lanthaler et al. 2021, Lemma C.1] guarantees the existence of a strictly increasing function F with $F(y) \geq f(y)$ such that the corresponding inverse function of F can be expressed as $F^{-1}(x) = \sigma(\sqrt{x})x$, where σ is a continuous increasing function with $\sigma(\sqrt{x}) \geq \sigma_0 > 0$ and $\lim_{x\to\infty} \sigma(x) = \infty$. From (10-11), we have

$$\left(\frac{x_\varepsilon}{\varepsilon}\right)^2 \leq F(y_\varepsilon),$$

and thus

$$\sigma\left(\frac{x_\varepsilon}{\varepsilon}\right)\left(\frac{x_\varepsilon}{\varepsilon}\right)^2 = F^{-1}\left(\left(\frac{x_\varepsilon}{\varepsilon}\right)^2\right) \leq y_\varepsilon$$

---

[46] By the Poincaré inequality $\|\omega^\varepsilon\|_{L^2(\mathbb{T}^2)} \lesssim \|\nabla\omega^\varepsilon\|_{L^2(\mathbb{T}^2)}$, we have, for $r \geq 1$,

$$\int_\delta^t \|\omega^\varepsilon(s)\|^2_{L^2(\mathbb{T}^2)}\, ds \lesssim \int_t^\delta \|\nabla \omega^\varepsilon(s)\|^2_{L^2(\mathbb{T}^2)}\, ds \leq r^2 \int_t^\delta \|\nabla \omega^\varepsilon(s)\|^2_{L^2(\mathbb{T}^2)}\, ds.$$

or equivalently

$$-\varepsilon^2\int_\delta^t \|\nabla\omega^\varepsilon(s)\|^2_{L^2(\mathbb{T}^2)}\,ds \le -\sigma\Big(\frac{x_\varepsilon}{\varepsilon}\Big)x_\varepsilon^2. \tag{10-12}$$

Note that (10-9) can be rewritten as

$$\frac{d}{dt}\,x_\varepsilon = \varepsilon\|\omega^\varepsilon(\delta)\|^2_{L^2(\mathbb{T}^2)} - 2\varepsilon^2\int_\delta^t \|\nabla\omega^\varepsilon(s)\|^2_{L^2(\mathbb{T}^2)}\,ds$$

and then, by (10-12) and well-known a priori L^2-estimates[47] for Navier–Stokes solutions (see [Lanthaler et al. 2021, Lemma A.2]),

$$\frac{d}{dt}\,x_\varepsilon \le \varepsilon\|\omega^\varepsilon(\delta)\|^2_{L^2(\mathbb{T}^2)} - 2\sigma\Big(\frac{x_\varepsilon}{\varepsilon}\Big)x_\varepsilon^2 \le \frac{\|u_0\|^2_{L^2(\mathbb{T}^2)}}{\delta} - 2\sigma\Big(\frac{x_\varepsilon}{\varepsilon}\Big)x_\varepsilon^2. \tag{10-13}$$

Next we show that

$$\lim_{\varepsilon\to 0}\varepsilon\int_\delta^t \|\omega^\varepsilon(s)\|^2_{L^2(\mathbb{T}^2)}\,ds = 0 \tag{10-14}$$

uniformly with respect to $t \in [\delta, T]$. For $\eta > 0$ arbitrary, consider the set

$$A_{\eta,t} := \{\varepsilon > 0 : x_\varepsilon(t) \ge \eta\}.$$

Assume first that $0 \in \bar{A}_{\eta,t}$. In particular, there exists $\{\varepsilon_l\}_{l\in\mathbb{N}} \subset A_{\eta,t}$ with $\lim_{l\to\infty}\varepsilon_l = 0$. Since σ is increasing, $\sigma(x_{\varepsilon_l}(t)/\varepsilon_l) \ge \sigma(\eta/\varepsilon_l)$ and (see (10-13))

$$\frac{d}{dt}\,x_{\varepsilon_l} \le \frac{\|u_0\|^2_{L^2(\mathbb{T}^2)}}{\delta} - 2\sigma\Big(\frac{\eta}{\varepsilon_l}\Big)\eta^2.$$

Observe that $\lim_{l\to\infty}\sigma(\eta/\varepsilon_l) = \infty$, which yields that $x_{\varepsilon_l}(t)$ is decreasing with respect to t whenever $l \ge l_0 = l_0(\eta, \sigma, \|u_0\|_{L^2(\mathbb{T}^2)}, \delta)$. Since $x_{\varepsilon_l}(\delta) = 0$ and $x_{\varepsilon_l}(t) \ge 0$, we conclude that $x_{\varepsilon_l}(t) = 0$ for all $t > \delta$ and $l \ge l_0$. Therefore there exists $\varepsilon_0 = \varepsilon_0(\eta, \sigma, \|u_0\|_{L^2(\mathbb{T}^2)}, \delta) > 0$ such that

$$x_\varepsilon(t) \le \eta \quad \text{if } \varepsilon \le \varepsilon_0. \tag{10-15}$$

On the other hand, if $0 \notin \bar{A}_{\varepsilon,t}$ then (10-15) holds trivially. Either way, we have shown that (10-14) is fulfilled.

Recall the energy formula for 2D Navier–Stokes solutions

$$\frac{d}{dt}\,\|u^\varepsilon\|^2_{L^2(\mathbb{T}^2)} = -2\varepsilon\|\omega^\varepsilon\|^2_{L^2(\mathbb{T}^2)}.$$

Integrating over $[\delta, t]$:

$$\|u^\varepsilon(t)\|^2_{L^2(\mathbb{T}^2)} - \|u^\varepsilon(\delta)\|^2_{L^2(\mathbb{T}^2)} = -2\varepsilon\int_\delta^t \|\omega^\varepsilon(s)\|^2_{L^2(\mathbb{T}^2)}\,ds.$$

In light of (10-14), we get (uniformly in t)

$$\lim_{\varepsilon\to 0}\|u^\varepsilon(t)\|^2_{L^2(\mathbb{T}^2)} - \|u^\varepsilon(\delta)\|^2_{L^2(\mathbb{T}^2)} = 0. \tag{10-16}$$

[47] Recall that $u_0 \in L^2(\mathbb{T}^2)$; see Definition 61.

Since $\{u^\varepsilon\}_{\varepsilon>0}$ is sparse stable, by virtue of Theorem 2, one can find a sequence of ε's with $\varepsilon \to 0$ such that

$$\lim_{\varepsilon\to 0} \|u^\varepsilon(t)\|^2_{L^2(\mathbb{T}^2)} = \|u(t)\|^2_{L^2(\mathbb{T}^2)}$$

for all $t \in (0, T)$. This combined with (10-16) lead to

$$\|u(t)\|_{L^2(\mathbb{T}^2)} = \|u(\delta)\|_{L^2(\mathbb{T}^2)} \quad \text{for any } t \in (\delta, T). \tag{10-17}$$

On the other hand, since $\|u(t)\|_{L^2(\mathbb{T}^2)}$ is right-continuous at $t = 0$, given any $\eta > 0$ one can find $\delta > 0$ (depending on η) such that

$$0 \le \|u(t)\|_{L^2(\mathbb{T}^2)} - \|u_0\|_{L^2(\mathbb{T}^2)} \le \eta \quad \text{for all } t \in (0, \delta].$$

Since (10-17) also holds, we have

$$0 \le \|u(t)\|_{L^2(\mathbb{T}^2)} - \|u_0\|_{L^2(\mathbb{T}^2)} \le \eta \quad \text{for all } t \in (0, T].$$

Clearly this shows that u is conservative, i.e., $\|u(t)\|_{L^2(\mathbb{T}^2)} = \|u_0\|_{L^2(\mathbb{T}^2)}$, $t \in [0, T]$. □

Added in proof

After this paper was submitted, the sparse spaces technology was successfully applied by Domínguez and D. Spector [2024] to resolve in the negative the DiPerna–Majda gap problem (see Section 1.1): $M^{1,1}$ is the borderline regularity space regarding existence of approximate solution sequences with concentrations for the 2D Euler equations. This closes the gap between lack of concentration (and hence existence of weak solutions with energy conservation) in $M^{1,\alpha}$ with $\alpha > 1$ [DiPerna and Majda 1987a] and the concentration-cancellation phenomenon in $M^{1,1/2}$ [Delort 1991; Majda 1993]. As a consequence, the sufficient conditions for the H^{-1}-compactness assertion (1-2) turn out to be also necessary. Hence the sparse methods introduced in this paper provide an optimal strategy to characterize lack of concentration/concentration-cancellation.

Acknowledgements

We are grateful to the referees for their constructive criticism and suggestions that helped us improve the quality of the paper. Óscar Domínguez is supported by the grants RYC2022-037402-I and PID2023-152878NB-I00 funded by the Spanish Ministry of Science and Innovation. Part of this research was carried out while he was a postdoc at the Institut Camille Jordan, Lyon, supported by the LABEX MILYON (ANR-10-LABX-0070) of Université de Lyon, within the program "Investissement d'Avenir" (ANR-11-IDEX-0007) operated by the French National Research Agency (ANR).

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Received 15 Dec 2023. Revised 23 Feb 2025. Accepted 8 Jun 2025.

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004, is published continuously online.

APDE peer review and production are managed by EditFlow® from MSP.

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ANALYSIS & PDE

Volume 19 No. 4 2026