ANALYSIS & PDE

Volume 19 No. 4 2026

IN-JEE JEONG AND SUNG-JIN OH

ILL-POSEDNESS FOR DISPERSIVE EQUATIONS: DEGENERATE DISPERSION AND THE TAKEUCHI–MIZOHATA CONDITION

msp

ANALYSIS AND PDE
Vol. 19 (2026), No. 4, pp. 721–782
DOI: 10.2140/apde.2026.19.721

ILL-POSEDNESS FOR DISPERSIVE EQUATIONS: DEGENERATE DISPERSION AND THE TAKEUCHII–MIZOHATA CONDITION

IN-JEE JEONG AND SUNG-JIN OH

We provide a unified viewpoint on two ill-posedness mechanisms for dispersive equations in one spatial dimension, namely degenerate dispersion and (the failure of) the Takeuchi–Mizohata condition. Our approach is based on a robust energy- and duality-based method introduced in an earlier work of the authors in the setting of Hall-magnetohydrodynamics. Concretely, the main results in this paper concern strong ill-posedness of the Cauchy problem (e.g., nonexistence and unboundedness of the solution map) in high-regularity Sobolev spaces for various quasilinear degenerate Schrödinger- and KdV-type equations, including the Hunter–Smothers equation, $K(m,n)$ models of Rosenau–Hyman, and the inviscid surface growth model. The mechanism behind these results may be understood in terms of the combination of two effects: degenerate dispersion — which is a property of the principal term in the presence of degenerating coefficients — and the evolution of the amplitude governed by the Takeuchi–Mizohata condition — which concerns the subprincipal term. We also demonstrate how the same techniques yield a more quantitative version of the classical L^2-ill-posedness result by Mizohata for linear variable-coefficient Schrödinger equations with failed Takeuchi–Mizohata condition.

1. Introduction

1.1. ***Quasilinear degenerate dispersive equations.*** We study the issue of ill-posedness of the Cauchy problem for various quasilinear dispersive equations in one spatial dimension in the presence of *degenerate dispersion*. We consider both Schrödinger- and KdV-type equations. Examples of Schrödinger-type equations we treat include, for instance, the equation of Hunter and Smothers [2019]

$$i\partial_t\phi + \partial_x(|\phi|^2\partial_x\phi) = 0, \tag{1-1}$$

which was derived from the equation of Majda, Rosales and Schonbek [Majda et al. 1988] describing the resonant reflection of sound waves off a sawtooth entropy wave, as well as its Hamiltonian variant

MSC2020: 35A01, 35Q53, 35Q55.
Keywords: degenerate dispersive, ill-posedness, Takeuchi–Mizohata condition, wave packets.

considered by Germain, Harrop-Griffith and Marzuola [Germain et al. 2020; Harrop-Griffiths and Marzuola 2022]

$$i\partial_t\phi + \bar{\phi}\partial_x(\phi\partial_x\phi) - \mu_0|\phi|^2\phi = 0, \quad \mu_0 = -1, 0, 1, \tag{1-2}$$

where $\phi : \mathbb{R}_+ \times \mathbb{T} \to \mathbb{C}$. A degenerate Schrödinger-type equation similar in form to (1-1) and (1-2) (but with different lower-order terms) also arose earlier in the work of Rosenau and Schochet [2005] in the study of compact breathers (see also Remark 1.3 below). In the KdV case, our results cover the $K(m, n)$ equation of Rosenau and Hyman [1993] with $n = 2$, i.e.,

$$\partial_t u + \left(\frac{1}{m}u^m\right)_x + \left(\frac{1}{2}u^2\right)_{xxx} = 0, \quad m \text{ a nonnegative integer}, \tag{1-3}$$

which has been studied extensively in connection with the remarkable nonlinear phenomenon of the existence of *compactons* (solitons with compact spatial support) [Rosenau 1994; 2005; 2006; Rosenau and Hyman 1993; Zilburg and Rosenau 2017; 2018] (see [Rosenau and Zilburg 2018] for a recent review), as well as the inviscid surface growth model (see [Blömker and Romito 2009; 2012; Choi and Yang 2021; Ożański and Robinson 2019] for the full surface growth model, with the dissipation $-\nu h_{xxxx}$ on the right-hand side)

$$\partial_t h + ((h_x)^2)_{xx} = 0, \tag{1-4}$$

where $u, h : \mathbb{R}_+ \times \mathbb{T} \to \mathbb{R}$. Indeed, degenerate KdV-type equations similar in form to (1-3) appear in various subjects including sedimentation models [Zumbrun 1999; Betancourt et al. 2011], the shoreline problem in water waves [Lannes and Métivier 2018] and magma dynamics [Simpson et al. 2007; 2008], to name a few. A more extensive list of references on degenerate KdV equations can be found in [Germain et al. 2019; 2020].

In each of these equations, observe that the highest-order term is nonlinear — more specifically, quadratic or cubic — in the solution. Vanishing of the solution, therefore, leads to some kind of "degeneracy" of the highest-order term, which in turn gives rise to delicate issues in the (local) well-posedness of the associated Cauchy problem.

Indeed, for initial data that are uniformly bounded away from 0 (a property henceforth referred to as *nondegeneracy*), one expects local well-posedness in high-regularity L^2-based Sobolev spaces $H^s(\mathbb{T})$. For example, in the case of (1-1), for a sufficiently regular solution ϕ, one has the conservation of the L^2 norm:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{T}} |\phi|^2 \,\mathrm{d}x\right) = 0.$$

Obtaining higher-regularity a priori estimates is a much more nontrivial task. One can observe the following bound for $n \geq 1$ (in operator notation) at each t:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{T}} |(\partial_x|\phi|^2\partial_x)^n\phi|^2 \,\mathrm{d}x\right) \lesssim_n \|\phi\|_{H^{2n}}^{4n+2}.$$

Furthermore, as long as the solution stays nondegenerate at t, in the sense that $\inf_x |\phi(x, t)|^2 > c$ for some $c > 0$, a standard argument involving the ellipticity of $(\partial_x|\phi(x, t)|^2\partial_x)^n$ allows us to bound $\|\phi(\cdot, t)\|_{H^{2n}}^2$ by the integral on the left-hand side up to errors of the form $O(\|\phi(\cdot, t)\|_{H^{2n}}^{4n+2})$. Putting these together, one can establish a short-time H^{2n} a priori estimate for the solution ϕ with nondegenerate initial data. However, in the case of *degenerate* initial data (i.e., those without a uniform bound away from 0), the above scheme for a short-time H^{2n} a priori estimate with $n \geq 1$ clearly breaks down.

In this paper, we show that this failure of proof of higher-derivative a priori estimates is, in fact, a manifestation of genuine ill-posedness in standard function spaces. Despite the formal conservation of the L^2 norm, we demonstrate that all of the equations above are rather strongly ill-posed — in the sense of nonexistence of solutions and unboundedness of the data-to-solution map in suitable set-ups — in a neighborhood of degenerate initial data (e.g., zero data) in high-regularity spaces ($C^{k-1,1}$, Sobolev or Hölder spaces).

1.2. *Main results for quasilinear degenerate dispersive equations.*

1.2.1. *Results for Schrödinger-type equations.* To treat the Hunter–Smothers equation (1-1) and the Hamiltonian equation (1-2) simultaneously, we shall consider the general equation

$$i\partial_t\phi + |\phi|^2\partial_{xx}\phi + \alpha_1\phi|\partial_x\phi|^2 + \beta_1\bar{\phi}(\partial_x\phi)^2 + \mu_1|\phi|^2\phi = 0, \tag{1-5}$$

where $\phi : \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{C}$, $\alpha_1, \beta_1 \in \mathbb{R}$ and $\mu_1 \in \mathbb{C}$. Indeed, the case $\alpha_1 = \beta_1 = 1$ and $\mu_1 = 0$ corresponds to (1-1), while the case $\alpha_1 = 0$ and $\beta_1 = 1$ corresponds to (1-2).

For the statement of the main results, we need to introduce the following exponents. Given $\alpha_1, \beta_1 \in \mathbb{R}$, we introduce the exponent

$$\sigma_c = -\left(\frac{\alpha_1}{2} + \beta_1 - 1\right) \tag{1-6}$$

and let s_c be the smallest integer greater than 1 and $\sigma_c - \frac{1}{2}$, i.e.,

$$s_c = \max\left\{2, \left\lfloor \sigma_c - \tfrac{1}{2} \right\rfloor + 1\right\}. \tag{1-7}$$

Note that $\sigma_c = -\frac{1}{2}$ and $s_c = 2$ for (1-1), while $\sigma_c = 0$ and $s_c = 2$ for (1-2). For the significance of σ_c and $\left\lfloor \sigma_c - \frac{1}{2} \right\rfloor + 1$, see Remark 1.4 and Section 1.3. We note already that the lower bound $s_c \geq 2$ is a technical byproduct of our proof, which we have not attempted to optimize.

Our first result is unboundedness of the solution map (i.e., norm inflation) in C^{s_c} near any solution with a linear degeneracy.

Theorem 1.1 (unboundedness of the solution map near a linearly degenerate solution). *Assume that there exists a solution $f \in L^\infty([0, \delta]; C^{s_c+1,1}(\mathbb{T}))$ to* (1-5) *with some $\delta > 0$ such that $f(t = 0) = f_0$ is linearly degenerate*; *that is, there exists $x_0 \in \mathbb{T}$ with $f(x_0) = 0$ and $f_0'(x_0) \neq 0$.*

Then, for any $\epsilon > 0$, $s_0 \geq s_c$, and $0 < \delta' \leq \delta$, we can find $\tilde{\phi}_0 \in C^\infty(\mathbb{T})$ such that $\|\tilde{\phi}_0\|_{C^{s_0}} \leq \epsilon$ and one of the following holds:

- *there exists **no** $L^\infty([0, \delta']; C^{s_c}(\mathbb{T}))$ solution to* (1-5) *with initial data $f_0 + \tilde{\phi}_0$*; *or*
- ***any** $L^\infty([0, \delta']; C^{s_c}(\mathbb{T}))$ solution ϕ with $\phi(t = 0) = f_0 + \tilde{\phi}_0$ satisfies*

$$\sup_{0<t<\delta'} \|\phi(t, \cdot) - f(t, \cdot)\|_{C^{s_c}(\mathbb{T})} > c_0(\delta')^{-1/2},$$

with some $c_0 > 0$ depending only on f.

While the solutions considered in this theorem and below are assumed to be only L^∞ in time, it is immediate from the equation and the high spatial regularity (i.e., $s_c \geq 2$ in the present case, and $s_c \geq 3$ in the KdV-type case below) that they are in fact (at least) continuous as a function of (t, x). Hence, there is no ambiguity in the notion of the initial data (i.e., the restriction to $\{t = 0\}$) for such solutions.

We remark that the norm inflation assertion immediately implies the inflation of any norm that controls C^{s_c}, such as H^σ with $\sigma > s_c + \frac{1}{2}$. In fact, our proof readily extends to norm inflation in H^σ for any $\sigma > \sigma_c$ in the second alternative, which is expected to be sharp according to Remark 1.4 and Section 1.3 below; see Remark 2.10 for further details. We also remark that the statement of Theorem 1.1 should extend over to the case of solutions with orders of degeneracy other than 1. For simplicity, however, we restrict ourselves to the linearly degenerate case, which is "critical" in some sense; see Section 1.3.

Our second result is the nonexistence of a regular local-in-time solution in arbitrarily high-regularity C^{s_0} spaces.

Theorem 1.2 (nonexistence of regular local-in-time solution). *For any $\epsilon > 0$ and $s_0 \geq s_c + 2$, there exists an initial data $\phi_0 \in C^\infty(\mathbb{T})$ satisfying $\|\phi_0\|_{C^{s_0}} < \epsilon$ for which there is no corresponding solution to* (1-5) *belonging to $L^\infty([0, \delta]; C^{s_c+2}(\mathbb{T}))$ with any $\delta > 0$.*

As an immediate corollary of the above, we have that (1-1) and (1-2) are ill-posed in the strongest sense of Hadamard in function spaces which contain C^∞ and control the C^4 norm (where $4 = s_c + 2$): there exists C^∞ initial data without a local solution in $L^\infty_t W^{s,p}$ with $s - \frac{1}{p} > 4$ and $L^\infty_t C^{k,\alpha}$ with $k + \alpha \geq 4$.

Remark 1.3. We give a few simple remarks regarding the above.

- In all of the above, the physical domain could be taken to be $\mathbb{R}$ instead of $\mathbb{T}$.
- As one may expect, the lower-order term $\mu_1|\phi|^2\phi$ in (1-5) does not play any essential role.
- In [Rosenau and Schochet 2005], the following equation (with $\mu = 1$) was studied:

$$i\partial_t w + \tfrac{3}{8}(\partial_x(|\partial_x w|^2 \partial_x w) + \mu |w|^2 w) = 0. \tag{1-8}$$

If the coefficient μ in front of the lower-order term is zero, then observe that $\phi := \partial_x w$ obeys exactly an equation of the form (1-5), to which our theorems apply. In view of the preceding remark, we expect our method to be readily extendible to (1-8) for $\mu = 1$ as well.

Remark 1.4 (exponents σ_c, s_c, Takeuchi–Mizohata condition and degenerate dispersion). Observe that the ill-posedness results, Theorems 1.1 and 1.2, hold for all possible coefficients α_1, β_1 in front of subprincipal terms, although these possibly *alter* the exponents σ_c and s_c. Heuristically, σ_c is the expected critical L^2-Sobolev regularity exponent above which the linearization of (1-5) around a regular linearly degenerate solution is ill-posed. In fact, the negativity of σ_c already signals L^2-ill-posedness of the linearized equation by the classical Takeuchi–Mizohata condition [Mizohata 1985, Chapter VII]! Even if σ_c is positive, it turns out to be L^∞-ill-posed after taking k many derivatives with $k > \sigma_c - \frac{1}{2}$. This consideration motivates the exponent s_c and our ill-posedness results. We shall elaborate on this remark in Section 1.3.

1.2.2. *Results for KdV-type equations.* To unify our treatment of KdV-type equations, we consider the general equation

$$\partial_t u + u u_{xxx} + \alpha_1 u_x u_{xx} + \frac{\mu_1}{m}(u^m)_x = 0, \tag{1-9}$$

where $u : \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{R}$, $\alpha_1 \in \mathbb{R}$, $\mu_1 \in \mathbb{R}$ and m is an integer greater than or equal to 2. Note that there is no need to separately consider the case $m = 1$, as then this term can be easily removed by the change of variables $(t, x) = (t', x' + \mu_1 t')$.

Note that $\alpha_1 = 3$ and $\mu_1 = 1$ corresponds to the $K(m, 2)$ equation (1-3). The inviscid surface growth model (1-4) reduces to the case $\alpha_1 = 3$ and $\mu_1 = 0$ after making the change of variables $u = \sqrt{2} h_x$.

In the present case, the role of linear degeneracy in the Schrödinger case is played by cubic degeneracy, see Section 1.3. As before, we introduce the constant

$$\sigma_c = -\big(\alpha_1 - \tfrac{3}{2}\big)$$

and the regularity exponent

$$s_c = \max\big\{5, \big\lfloor \sigma_c - \tfrac{1}{2} \big\rfloor + 1\big\}.$$

Here, σ_c is the critical L^2-Sobolev regularity exponent above which the linearization of (1-9) around a regular cubically degenerate solution is ill-posed (see Section 1.3). The linearized equation is L^∞-ill-posed after taking k many derivatives with $k > \sigma_c - \frac{1}{2}$; this motivates the exponent s_c. The lower bound $s_c \geq 5$ is again a nonoptimal technical byproduct of our proof; see Proposition 3.2 for where it is used.

Theorem 1.5 (unboundedness of the solution map near a cubically degenerate solution). *Assume that there exists a solution $f \in L^\infty([0, \delta]; C^{s-1,1}(I))$ of* (1-9) *with some $\delta > 0$ and $I = [a, b]$, such that the initial data f_0 is positive on $I \setminus \partial I$, vanishes cubically on ∂I and $f_0 \in C^{s_0-1,1}$, where $s_c \leq s \leq s_0$. Then, for any $\epsilon > 0$, $s \leq m_0 \leq s_0$, and $0 < \delta' \leq \delta$, we can find $\phi_0 \in C^\infty(\mathbb{T}; \mathbb{R})$ such that* $\operatorname{supp} \phi_0 \subseteq I \setminus \partial I$, $\|\phi_0\|_{C^{m_0}} \leq \epsilon$, *and one of the following holds*:

- *there exists no solution to* (1-9) *with initial data $f_0 + \phi_0$ that belongs to $L^\infty([0, \delta']; C^{s-1,1}(I))$; or*
- *any solution u with $u(0) = f_0 + \phi_0$ and belonging to $L^\infty([0, \delta']; C^{s-1,1}(I))$ satisfies, for every $s_c \leq s' \leq 2\big\lfloor \frac{1}{2} s \big\rfloor$,*

$$\sup_{0<t<\delta'} \|u(t, \cdot) - f(t, \cdot)\|_{C^{s'}(I)} > (\delta')^{-1/2}.$$

That $s' \leq 2\big\lfloor \frac{1}{2} s \big\rfloor$ is not essential and is expected to be replaceable by $s' \leq s$; however, it is assumed here to simplify the proof (see the proof of Theorem 1.5 below).

As in the Schrödinger case, the statement of Theorem 1.5 should extend over to the case of solutions with orders of degeneracy other than 3, provided that s_c is modified suitably. We however focus on the cubic degeneracy case for simplicity, which is "critical"; see Section 1.3.

The nonexistence result for (1-9) is as follows.

Theorem 1.6 (nonexistence of regular local-in-time solutions). *For any $\epsilon > 0$ and $s_c \leq s \leq s_0$, where s is an even integer, there exists an initial data $u_0 \in C^\infty(\mathbb{T})$ satisfying $\|u_0\|_{C^{s_0}} < \epsilon$ for which there is no corresponding solution to* (1-9) *belonging to $L^\infty([0, \delta]; C^s(\mathbb{T}))$ for any $\delta > 0$.*

That s is an even integer is not essential but is assumed here to simplify the proof (see the proof of Theorem 1.6 below).

Remark 1.7. We now give a few simple remarks regarding the above.

- As in the Schrödinger case, the physical domain could be taken to be $\mathbb{R}$ instead of $\mathbb{T}$. Also, the lower-order term $(u^m)_x$, for any $m \geq 2$, does not play any essential role in the proof of ill-posedness of (1-9).

- Our proof easily extends to norm inflation in H^σ for any $\sigma > \sigma_c$ in the second alternative in Theorem 1.5. Moreover, in contrast to the Schrödinger case, we may also easily extend Theorem 1.6 to the nonexistence of solutions in H^σ for any $\sigma > \max\{\sigma_c, 5 + \frac{1}{2}\}$ (see Remark 1.15 for why the situations are different). These numerologies are expected to be sharp, as we shall discuss in Section 1.3 below. We refer the reader to Remark 2.10 for more details on this modification (which is for norm inflation in the Schrödinger case, but the overall idea is the same).

- We expect our results to generalize to $K(m, n)$ with $n > 2$, as well as $\mathcal{C}(m, a, b)$ equations [Rosenau 2006] with $n := a + b > 2$, by considering degeneracies of order $\frac{3}{n-1}$ (which are critical).

Remark 1.8 (comparison with the work of Ambrose, Simpson, Wright and Yang [Ambrose et al. 2012]). In the pioneering paper [Ambrose et al. 2012], the ill-posedness of $u_t = uu_{xxx}$ in the (fairly low-regularity) Sobolev space H^2 has been proved based on the construction of a compactly supported H^2 (but not smooth) self-similar solution A. However, the existence of such a solution is specific to the equation $u_t = uu_{xxx}$, and the proof does not extend to the more general class of equations (1-9), nor to higher-regularity Sobolev spaces, as in our results. Our approach is distinct from that of [Ambrose et al. 2012]: it does not involve self-similar solutions, but is rather based on appropriate smooth wave packet-type approximate solutions traveling towards the degeneracy; see Section 1.3. While our results (Theorems 1.5 and 1.6) do not cover Sobolev regularities as low as H^2 due to technical reasons, our heuristics suggest that our ill-posedness mechanism should extend to H^σ with $\sigma > \sigma_c = \frac{3}{2}$.

Nevertheless, we point out that a key heuristic consideration of our approach, namely, the combined effect of degenerate dispersion and subprincipal terms, can already be found in [Ambrose et al. 2012], albeit with a different viewpoint.

Remark 1.9 (comparison with the works of Germain, Harrop-Griffith and Marzuola [Germain et al. 2019] and Harrop-Griffith and Marzuola [2022]). For solutions to (1-1) and (1-2) with degenerate initial data (i.e., initial data with a zero), our proof identifies and exploits, in a nonlinear fashion, a mechanism by which energies in low frequencies are transferred to high frequencies at arbitrarily fast rates, where the frequencies are defined with respect to the original variable x. We emphasize, however, that it does *not* rule out the possibility of well-posedness in regularity classes adapted to the degeneracies of the initial data, by working with a renormalized variable and/or suitable weights. Indeed, such positive results have been proved in the interesting works of Germain, Harrop-Griffith and Marzuola [Germain et al. 2019] for a KdV-type quasilinear dispersive equation, and Harrop-Griffith and Marzuola [2022] for (1-2), where Lagrangian-type coordinates adapted to the solution were used to formulate the function spaces.

1.3. ***Key mechanism*: *degenerate dispersion and the Takeuchi–Mizohata condition.*** The nonlinear ill-posedness results in this paper are firmly based on a detailed and quantitative understanding of ill-posedness for the linearized equation around a background solution f whose initial data contains a degeneracy. For simplicity, in this subsection we shall assume that the linearization takes the form

$$\begin{cases} \partial_t u - i\partial_x(a\partial_x u) - ib\partial_x u = \text{(lower-order)} & \text{in the Schrödinger case,} \\ \partial_t u + \frac{1}{2}(\partial_x^3(au) + a\partial_x^3 u) + b\partial_x^2 u = \text{(lower-order)} & \text{in the KdV case,} \end{cases} \tag{1-10}$$

where $a = a(x)$ is real-valued in both cases and $b = b(x)$ is also real-valued in the KdV case.[1]

Remark 1.10 (on time independence of the coefficients in (1-10)). While we assumed that $a(x)$ and $b(x)$ are time independent, the actual linearization of (1-5) and (1-9) on a dynamic background solution $f(t, x)$ would, of course, have time-dependent coefficients. Nevertheless, the timescale of the ill-posedness mechanism is arbitrarily short, and hence we may effectively approximate these coefficients by the initial values for the purpose of our discussion.

It is conceptually useful to distinguish two intertwined mechanisms for ill-posedness, *degenerate dispersion* and *Takeuchi–Mizohata instability*, which can be seen from the principal and subprincipal terms, respectively. Both phenomena must be taken into account to obtain a comprehensive picture of the ill-posedness of (1-5) and (1-9) in the presence of a degeneracy in the initial data (and, more concretely, to explain the relevance of the exponents σ_c and s_c).

(1) *Principal term*: *dynamics of bicharacteristics*. The ill-posedness of (1-10) from degenerate dispersion can be most easily described at the level of the *bicharacteristic ODE system* associated with the principal symbol p of the spatial part of (1-10), which is given by

$$\begin{cases} \dot{X} = \partial_\xi p(X, \Xi), \\ \dot{\Xi} = -\partial_x p(X, \Xi), \end{cases} \tag{1-11}$$

where $p(x, \xi)$ equals $-a(x)\xi^2$ in the Schrödinger case and $-a(x)\xi^3$ in the KdV case. By geometric optics, the trajectory $(X(t), \Xi(t))$ describes (at least on sufficiently short timescales) wave packets concentrated near $X(t)$ in the physical space and $\Xi(t)$ in the frequency space; see (1-15) and (1-16) below for further discussion. If (1-11) admits the growth of $|\Xi|$ by a definite factor (e.g., 2) in arbitrarily short timescales, we would have a strong indication of ill-posedness of (1-10) in high-regularity Sobolev spaces. In turn, such a growth may come from some degeneracy of p in X — this phenomenon is what we shall refer to as *degenerate dispersion*.

To be concrete, let us assume that the dynamics is given in $(x, \xi) \in \mathbb{R} \times \mathbb{R}$ and the coefficient a in p is of the degenerate form $a(x) \approx Ax^n$ $(n > 0)$ for $|x|$ small, so that

$$p(x, \xi) \approx -Ax^n\xi^m \quad \text{for } |x| \text{ small and } |\xi| \text{ large.} \tag{1-12}$$

[1] For Schrödinger-type problems, we regard first-order terms of the form $\tilde{b}(x)\partial_x \bar{u}$ as (lower-order), as it can be removed by a suitable change of the dependent variable; see the introduction of ψ in Section 2.3.2 below.

Note that m equals 2 in the Schrödinger case and 3 in the KdV case. The associated bicharacteristic ODE system is

$$\begin{cases} \dot{X} \approx -AmX^n \Xi^{m-1}, \\ \dot{\Xi} \approx AnX^{n-1}\Xi^m. \end{cases} \tag{1-13}$$

In view of the fact that the group velocity $\dot{X}$ vanishes at the point $x = 0$ (since $n > 0$), we shall say that p is *degenerate* at $x = 0$.

We shall now describe the ill-posedness mechanism of degenerate dispersion in this concrete case. (This analysis can be found in the introduction of [Germain et al. 2019] as well.) With a change of the time variable, we may take $A = 1$. Assume, for the sake of this heuristic discussion, that the $\approx$ above are exact equalities. Consider the solution to (1-13) with initial conditions (X_0, Ξ_0), where $0 < X_0 \ll 1$ and $\Xi_0 \gg 1$. Then, appealing to the fact that $X^n \Xi^m$ is conserved in time, we have

$$\Xi(t) = \Xi_0(1 + (n-m)X_0^{n-1}\Xi_0^{m-1}t)^{m/(m-n)}$$

for $m \neq n$. When $m = n$, which will be referred to as the *critical* case, we have instead

$$\Xi(t) = \Xi_0 \exp(mX_0^{m-1}\Xi_0^{m-1}t). \tag{1-14}$$

In all cases, the frequency magnitude doubles (i.e., $|\Xi(\tau_2)| = 2|\Xi_0|$) at time $\tau_2 \simeq |\Xi_0|^{1-m}|X_0|^{1-n}$. If the order of p is greater than 1 (i.e., $m > 1$), the doubling time τ_2 may be taken to be arbitrarily small by choosing $|\Xi_0|$ large, as we desired. Such an arbitrarily fast growth of Ξ suggests that high derivatives of the solution following this bicharacteristic flow would also grow arbitrarily fast — this is what we shall refer to as *ill-posedness via degenerate dispersion*.

Finally, let us connect the above model case to the equations considered in this work. Recall that the principal coefficient a in the linearized operator is determined by the background solution f, where $a = |f|^2$ for (1-5) and $a = f$ for (1-9). Since the relevant frequency doubling timescale τ_2 is arbitrarily small, it is reasonable to make the approximation $f \approx f_0$. Assuming that f_0 is degenerate at $x = 0$, in the sense that $|f_0|$ vanishes to some finite order at 0, we arrive at the ansatz $a(x) \approx Ax^n$ for some $A \neq 0$ and $n > 0$.

Remark 1.11 (critical degeneracy). In this work, for simplicity, we shall consider only background solutions with critical degeneracy $n = m$. The heuristics suggest, however, that a similar arbitrary fast growth of $|\Xi|$ is expected for any order of degeneracy $n > 0$. The techniques in this paper should be generalizable to these cases.

(2) *Subprincipal term*: *evolution of wave packet amplitude and the Takeuchi–Mizohata condition*. While subprincipal terms do not enter in the dynamics of bicharacteristics, they need to be considered in order to fully understand the well- and ill-posedness issues for (1-10). In fact, the subprincipal term may already cause ill-posedness in L^2 even when the principal term is *nondegenerate*! This phenomenon is captured by the classical *Takeuchi–Mizohata condition* (after the works [Takeuchi 1980; Mizohata 1981] in the Schrödinger case); see (1-19) below.

To understand this phenomenon, it is instructive to delve a little deeper into the construction of wave packet (approximate) solutions for (1-10). Consider the ansatz[2] $u = \boldsymbol{a}(t,x)e^{i\boldsymbol{\Phi}(t,x)}$ with the following properties: (i) $\boldsymbol{\Phi}(t,x)$ is real-valued, $\partial_x\boldsymbol{\Phi}(0,x) = \Xi_0$ on the support of $\boldsymbol{a}(0,x)$, and (ii) $\boldsymbol{a}(t,x)$ is complex-valued, and $\boldsymbol{a}(0,x)$ is a smooth bump function adapted to a small ball centered at X_0. With the expectation that the $\partial_x\boldsymbol{\Phi}(t,x)$ stays large compared to the characteristic frequencies of $\boldsymbol{a}$, a and b, we may write

$$e^{-i\boldsymbol{\Phi}}(\partial_t - i\partial_x a(x)\partial_x - ib\partial_x)(\boldsymbol{a}e^{i\boldsymbol{\Phi}}) = i(\partial_t\boldsymbol{\Phi} + a(\partial_x\boldsymbol{\Phi})^2)\boldsymbol{a} + \partial_t\boldsymbol{a} + 2a\partial_x\boldsymbol{\Phi}\partial_x\boldsymbol{a} + \left(\partial_x a + b + a\frac{\partial_x^2\boldsymbol{\Phi}}{\partial_x\boldsymbol{\Phi}}\right)\partial_x\boldsymbol{\Phi}\boldsymbol{a} + \cdots$$

in the Schrödinger case (where we omitted terms that do not involve $\partial_x\boldsymbol{\Phi}$) and

$$e^{-i\boldsymbol{\Phi}}\big(\partial_t + \tfrac{1}{2}(\partial_x^3 a + a\partial_x^3) + b\partial_x^2\big)(\boldsymbol{a}e^{i\boldsymbol{\Phi}}) = i(\partial_t\boldsymbol{\Phi} - a(\partial_x\boldsymbol{\Phi})^3)\boldsymbol{a} + \partial_t\boldsymbol{a} - 3a(\partial_x\boldsymbol{\Phi})^2\partial_x\boldsymbol{a} - \left(b + \tfrac{3}{2}\partial_x a - 3a\frac{\partial_x^2\boldsymbol{\Phi}}{\partial_x\boldsymbol{\Phi}}\right)(\partial_x\boldsymbol{\Phi})^2\boldsymbol{a} + \cdots$$

in the KdV case (where we omitted terms of order 0 and 1 in $\partial_x\boldsymbol{\Phi}$). To eliminate the main terms on the right-hand sides, we are led to impose the following classical *Hamilton–Jacobi* and *transport equations* for $\boldsymbol{\Phi}$ and $\boldsymbol{a}$:

$$\begin{cases}\partial_t\boldsymbol{\Phi} + a(\partial_x\boldsymbol{\Phi})^2 = 0,\\ \partial_t\boldsymbol{a} + 2a\partial_x\boldsymbol{\Phi}\partial_x\boldsymbol{a} + \partial_x(a\partial_x\boldsymbol{\Phi})\boldsymbol{a} = -b\partial_x\boldsymbol{\Phi}\boldsymbol{a}\end{cases} \quad \text{in the Schrödinger case,} \tag{1-15}$$

$$\begin{cases}\partial_t\boldsymbol{\Phi} - a(\partial_x\boldsymbol{\Phi})^3 = 0,\\ \partial_t\boldsymbol{a} - 3a(\partial_x\boldsymbol{\Phi})^2\partial_x\boldsymbol{a} - \tfrac{3}{2}\partial_x(a(\partial_x\boldsymbol{\Phi})^2)\boldsymbol{a} = b(\partial_x\boldsymbol{\Phi})^2\boldsymbol{a}\end{cases} \quad \text{in the KdV case.} \tag{1-16}$$

Observe that $(X(t), \Xi(t))$ solving (1-11) are precisely the bicharacteristics for the above equations in the method of characteristics [Evans 2010, Chapter 3], which explains the relevance of (1-11). Moreover, the transport equations show how b influences the evolution of the amplitude $\boldsymbol{a}$. In fact, we may easily check that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{a}\|_{L^2}^2 = \begin{cases}-\langle \operatorname{Re} b\,\partial_x\boldsymbol{\Phi}\boldsymbol{a}, \boldsymbol{a}\rangle & \text{in the Schrödinger case,}\\ \langle b(\partial_x\boldsymbol{\Phi})^2\boldsymbol{a}, \boldsymbol{a}\rangle & \text{in the KdV case,}\end{cases} \tag{1-17}$$

which clearly demonstrates how b influences the evolution of the L^2 norm (here $\langle\cdot,\cdot\rangle$ is the standard L^2-inner product).

We are now ready to give a heuristic derivation of the Takeuchi–Mizohata conditions. By the method of characteristics, we expect, at least for a short time, that $\partial_x\boldsymbol{\Phi}(t, X(t)) = \Xi(t)$ and $\boldsymbol{a}$ remains a bump function adapted to a ball centered at $X(t)$. Hence, on $\operatorname{supp}\boldsymbol{a}$, we expect

$$-\langle \operatorname{Re} b\,\partial_x\boldsymbol{\Phi}\boldsymbol{a}, \boldsymbol{a}\rangle \approx -\operatorname{Re} b(X(t))\,\Xi(t)\|\boldsymbol{a}\|_{L^2}^2 = -\frac{\operatorname{Re} b(X(t))}{2a(X(t))}\dot{X}(t)\|\boldsymbol{a}\|_{L^2}^2$$

[2] In the KdV case, we can take the real or imaginary part of $\boldsymbol{a}e^{i\boldsymbol{\Phi}}$ at the end to obtain a real-valued wave packet.

in the Schrödinger case and

$$\langle b(\partial_x \boldsymbol{\Phi})^2 \boldsymbol{a}, \boldsymbol{a}\rangle \approx b(X(t))\Xi(t)^2\|\boldsymbol{a}\|_{L^2}^2 = -\frac{b(X(t))}{3a(X(t))}\dot{X}(t)\|\boldsymbol{a}\|_{L^2}^2$$

in the KdV case, where we used (1-11) for the last equalities. Using (1-17) and $\|u\|_{L^2} = \|\boldsymbol{a}\|_{L^2}$, we arrive at the expectations

$$\|u(t)\|_{L^2} \simeq \begin{cases} \exp\bigl(-\int_{X_0}^{X(t)} (\operatorname{Re} b)/(2a)\,\mathrm{d}x\bigr)\|u(t=0)\|_{L^2} & \text{in the Schrödinger case,} \\ \exp\bigl(\int_{X(t)}^{X_0} b/(3a)\,\mathrm{d}x\bigr)\|u(t=0)\|_{L^2} & \text{in the KdV case.} \end{cases} \tag{1-18}$$

The *Takeuchi–Mizohata conditions* (see [Takeuchi 1980; Mizohata 1981] in the Schrödinger case; see [Akhunov 2014; Ambrose and Wright 2013] and Remark 1.13 for the KdV case) are simply sufficient conditions for the forward-in-time boundedness of $\|\boldsymbol{a}\|_{L^2}$ read off from (1-18):

$$\sup_{x_0<x_1}\left|\int_{x_0}^{x_1} \frac{\operatorname{Re} b}{2a}\,\mathrm{d}x\right| < +\infty \quad \text{in the Schrödinger case,} \tag{1-19}$$

$$\sup_{x_0<x_1}\int_{x_0}^{x_1} \frac{b}{3a}\,\mathrm{d}x < +\infty \quad \text{in the KdV case.} \tag{1-20}$$

Conversely, the failure of the Takeuchi–Mizohata conditions (1-19) and (1-20) signals arbitrarily fast growth (i.e., norm inflation) of the L^2 norm of u, since $X(t)$ may travel arbitrarily far from X_0 in any fixed duration of time if Ξ_0 is large. In this paper, we shall refer to this norm inflation (or ill-posedness) mechanism as the *Takeuchi–Mizohata instability*. Below, we shall consider the interaction of degenerate dispersion and the Takeuchi–Mizohata instability, which provides us with a detailed heuristic understanding of the ill-posedness properties of the linearization of (1-5) and (1-9) in the presence of a (critical) degeneracy in the initial data.

Remark 1.12 (rigorous results on Takeuchi–Mizohata-type conditions). The necessity of (1-19) for the L^2-well-posedness of (1-10) in the Schrödinger case has been known since the early works [Takeuchi 1980; Mizohata 1981]; see also [Akhunov 2014] for the KdV case. On the other hand, whether such a condition alone is sufficient for L^2 boundedness in general is less clear, especially in higher dimensions. Nevertheless, some strengthened form of the Takeuchi–Mizohata condition underlies many works on the well-posedness of the Cauchy problem for linear and even nonlinear Schrödinger- and KdV-type equations; see, e.g., [Akhunov 2014; Akhunov et al. 2019; Ambrose and Wright 2013; Harrop-Griffiths 2015a; 2015b: Kenig et al. 1998; 2004; Marzuola et al. 2012; 2014; 2021; Mizohata 1985].

Remark 1.13 (role of $\operatorname{sgn} b$ in the KdV case). Observe that the absolute value is needed in the Schrödinger case (1-19) since $X(t)$ may travel in both directions, while it is not necessary in the KdV case (1-20) since $X(t)$ is *always* decreasing if a is positive (resp. increasing if a is negative) according to (1-11). In particular, in the KdV case, (1-20) is always satisfied if $b < 0$, and even when b has some positive parts, it is possible that the Takeuchi–Mizohata condition is still satisfied (e.g., when b oscillates). This phenomenon has been explored by Ambrose and Wright [2013], who prove well-posedness of some variable coefficient linear KdV-type equations in the periodic setting in the presence of the positive part of b (referred to as "antidiffusion" in that paper).

Remark 1.14. While our main focus is the interaction of Takeuchi–Mizohata instability with degenerate dispersion, the method developed in this paper also provides a new and effective way to rigorously establish the necessity of (1-19) and (1-20) for the L^2-well-posedness of (1-10). We refer the reader to Section 1.5 and the Appendix for sample results in the Schrödinger case for $a = 1$ (but in arbitrarily dimensions).

(3) *Combined effect of degenerate dispersion and Takeuchi–Mizohata instability*. We are now ready to discuss the combined effect of the principal and subprincipal terms in (1-10) obtained by linearizing around a background solution f with a degeneracy. Keeping Remarks 1.10 and 1.11 in mind, we consider the linearization of (1-5) and (1-9) around $f(t, x) = x$ and x^3 (for $|x|$ small), respectively. Then we arrive at (1-10) with

$$\begin{aligned} a(x) &= x^2, \quad b(x) = 3\big(\tfrac{1}{2}\alpha_1 + \beta_1 - 1\big)x && \text{in the Schrödinger case,} \\ a(x) &= x^3, \quad b(x) = 3\big(\alpha_1 - \tfrac{3}{2}\big)x^2 && \text{in the KdV case.} \end{aligned}$$

Recall from the above that we are considering bicharacteristics $(X(t), \Xi(t))$ with $X_0 > 0$ and $X(t)$ traveling to the degeneracy 0 in both cases. Wave packets corresponding to such bicharacteristics shall be called *degenerating wave packets*.

The relevant Takeuchi–Mizohata condition (see (1-21) below with $\sigma = 0$) for $\|u\|_{L^2}$ may or may not hold, meaning that degenerating wave packets may or may not remain bounded in L^2. Nevertheless, it *always fails for high enough derivatives*, which is consistent with the heuristic $\Xi(t) \to \infty$! Indeed, observe that commutation of (1-10) with ∂_x^σ leads to a similar equation for $\partial_x^\sigma u$ but with the following coefficients:

$$\begin{aligned} a(x) &= x^2, \quad b(x) = 3\big(\tfrac{1}{2}\alpha_1 + \beta_1 - 1 + \sigma\big)x && \text{in the Schrödinger case,} \\ a(x) &= x^3, \quad b(x) = 3\big(\alpha_1 - \tfrac{3}{2} + \sigma\big)x^2 && \text{in the KdV case.} \end{aligned}$$

In view of $0 < X(t) < X_0$, the Takeuchi–Mizohata condition for boundedness of $\|u\|_{H^\sigma}$ is

$$\begin{aligned} \sup_{0<x_0<x_1\ll 1} \int_{x_0}^{x_1} \big(\tfrac{1}{2}\alpha_1 + \beta_1 - 1 + \sigma\big)\frac{\mathrm{d}x}{x} &< +\infty \quad \text{in the Schrödinger case,} \\ \sup_{0<x_0<x_1\ll 1} \int_{x_0}^{x_1} \big(\alpha_1 - \tfrac{3}{2} + \sigma\big)\frac{\mathrm{d}x}{x} &< +\infty \quad \text{in the KdV case,} \end{aligned} \tag{1-21}$$

which fails exactly when $\sigma > \sigma_c$ in both cases. Moreover, the preceding heuristic analysis suggests that the H^σ norm of the degenerating wave packet grows if $\sigma > \sigma_c$, stays constant if $\sigma = \sigma_c$, and decays if $\sigma < \sigma_c$. This consideration explains the relevance of the exponent σ_c.

Working directly with the transport equations for $\boldsymbol{a}$ in (1-15)–(1-16) in place of (1-17), we may also see that, for $\tilde{s}_c = \sigma_c + \frac{1}{2}$, the $W^{s,\infty}$ norm of the wave packet grows if $s > \tilde{s}_c$, stays constant if $s = \tilde{s}_c$, and decays if $s < \tilde{s}_c$. This consideration motivates the integer exponent s_c in our results.

1.4. ***Discussion of the proof.*** Our discussion so far has been rather formal; deriving actual nonlinear ill-posedness in standard function spaces requires more ideas. Our main technical contribution in this work is developing a robust scheme for establishing quantitative ill-posedness, which is not only able to deduce strong ill-posedness in quasilinear cases but also yields much stronger statements for linear equations. The scheme largely consists of three parts: (1) construction of degenerating wave packets for the linearized equation, (2) duality testing argument and (3) incorporation of the nonlinearity.

(1) *Degenerating wave packets.* We first describe the ideas for construction of a degenerating wave packet. Compared to the heuristic discussion above, the actual construction of such an approximate solution $\tilde{u}$ to (1-10) has to (i) allow for time-dependent coefficients $a = a(t, x)$ and $b = b(t, x)$ (as the background solution may depend on time, see Part (3) below), and (ii) solve the linearized equation up to an equation error $\boldsymbol{\epsilon}_{\tilde{u}}$ of size $\mathcal{O}(\Xi_0^{m-1-\delta})$ (in a suitable norm) for some $\delta > 0$ (here, m equals 2 for Schrödinger and 3 for KdV). Property (ii) is necessary to justify the approximation on a longer timescale than Ξ_0^{1-m}, which is the instability timescale; see (1-14).

Our idea is to make appropriate changes of the independent and dependent variables from (x, u) to (y, v) to reduce the problem to the constant coefficient case, for which the construction is standard. For time-dependent coefficients $a = a(t, x)$ and $b = b(t, x)$, the transformation $(x, u) \mapsto (y, \check{v})$ is of the form

$$\mathrm{d}x = (a(t, x))^{1/m}\,\mathrm{d}y, \quad u = (w\check{w})^{-1}\check{v},$$

where

$$w^{-1}\,\partial_x w = \frac{\operatorname{Re} b}{ma}, \quad \check{w}^{-1}\,\partial_x \check{w} = \frac{\partial_x a}{2ma}.$$

Roughly speaking, the Takeuchi–Mizohata instability is renormalized by the conjugation of the dependent variable by the weight w, in the sense that $v(t, x) := wu(t, x)$ solves (1-10) with $b = 0$ (with possibly different lower-order terms). Similarly, degenerate dispersion is renormalized by the change of variables $x \mapsto y$ accompanied with the conjugation of the dependent variable by the weight $\check{w}$, in the sense that $\check{v}(t, y)$ solves the constant coefficient problem $(\partial_t + i(i\partial_y)^m)\check{v} =$ (lower-order terms). Now, starting from a standard wave packet for the constant coefficient problem traveling towards the degeneracy and returning to original variables, we obtain a degenerating wave packet $\tilde{u}$ with the desired properties.

In order to make the above heuristic discussion precise, there are several more factors to consider. For instance, we need to make sure that the contribution of $\partial_t a$, $\partial_t b$ are indeed acceptable, which ultimately relies on the estimates we have on the time derivative of the background solution $f(t, x)$ in applications; see Part (3) below. In the Schrödinger case, we need the following two additional ideas: (a) an extra change of dependent variables to treat terms of the form $\tilde{b}\partial_x \bar{u}$, and (b) an extra phase rotation $e^{i\lambda S}$ for the wave packet $\check{v}$ to treat terms of the form $-(\operatorname{Im} b + \partial_t y)\partial_y \check{v}$, both of which are potentially problematic for achieving Property (ii). For more details, see the proofs of Propositions 2.7 and 3.3 for details.

Remark 1.15 (numerologies in the Schrödinger vs. KdV cases). In order to justify the properties of $\tilde{u}$ needed for the proof of the H^s or C^s norm growth, (b) above forces the technical restriction that $f \in C^{s+1,1}$ with $s \geq 2$ in the Schrödinger case, while $f \in C^{s-1,1}$ with $s \geq 4$ is sufficient in the KdV case; compare the degeneration bounds in Propositions 2.7 and 3.3. This point explains the different numerologies in Theorems 1.1 and 1.2 in the Schrödinger case versus Theorems 1.5 and 1.6 in the KdV case.

(2) *Modified energy estimate and duality testing argument*. In order to upgrade the norm growth for a degenerating wave packet $\tilde{u}$ to an actual solution u to (1-10), we adapt the *energy estimate and duality method* introduced in our previous work [Jeong and Oh 2022] on Hall-magnetohydrodynamics (Hall-MHD). Here we shall briefly explain the argument, in the simplest setup.

Given a degenerating wave packet $\tilde{u}$ for (1-10), denote by u any[3] solution to (1-10) with $u(t=0)=\tilde{u}_0$, where $\tilde{u}_0 := \tilde{u}(t=0)$. In view of the aforementioned fact that $v = wu$ solves (1-10) with $b = 0$, the following *modified energy estimates* should hold (at least when u is sufficiently regular):

$$\|wu\|_{L_t^\infty([0,t_0];L^2)} \lesssim \|wu_0\|_{L^2}, \quad \|w\tilde{u}\|_{L_t^\infty([0,t_0];L^2)} \lesssim \|w\tilde{u}_0\|_{L^2}, \tag{1-22}$$

where $0 < t_0 < 1$. By the same token, the following *generalized* (*bilinear*) *energy estimate* should also hold:

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\langle wu, w\tilde{u}\rangle\right| \lesssim \|w\boldsymbol{\epsilon}_u\|_{L^2}\|w\tilde{u}\|_{L^2} + \|wu\|_{L^2}\|w\boldsymbol{\epsilon}_{\tilde{u}}\|_{L^2}.$$

(We remark that $\partial_t w$ also arises, but in applications we shall have $|\partial_t w| \lesssim w$.) Here, $\boldsymbol{\epsilon}_u$ and $\boldsymbol{\epsilon}_{\tilde{u}}$ are the errors associated with u and $\tilde{u}$ viewed as approximate solutions to (1-10). Then, as long as the error terms are bounded, we may deduce that $\langle wu, w\tilde{u}\rangle \simeq \langle wu_0, w\tilde{u}_0\rangle = \|w\tilde{u}_0\|_{L^2}^2$, which allows us to obtain behavior of u in various norms by simply estimating the degenerating wave packet $\tilde{u}$ and using duality.

In actual applications, the errors often contain derivatives and hence $\|w\boldsymbol{\epsilon}_{\tilde{u}}\|_{L^2}$ (resp. $\|w\boldsymbol{\epsilon}_u\|_{L^2}$) may diverge as $|\Xi_0| \to \infty$. Nevertheless, in view of the fact the instability time-scale is $\simeq |\Xi_0|^{1-m}$, for the above argument to work it suffices to have $\int_0^{t_0}\|w\boldsymbol{\epsilon}_{\tilde{u}}\|_{L^2} \lesssim 1$ $\big($resp. $\int_0^{t_0}\|w\boldsymbol{\epsilon}_u\|_{L^2} \lesssim 1\big)$ for $t_0 > |\Xi_0|^{1-m+\delta}$. For $\tilde{u}$, this is precisely Property (ii) in the preceding discussion. For an actual solution u to (1-10), this follows from the fact that $\boldsymbol{\epsilon}_u$ does not contain principal nor subprincipal terms (except $\tilde{b}\partial_x\bar{u}$ in the Schrödinger case, which may be eliminated using integration by parts).

(3) *Incorporation of the nonlinearity*. The ideas discussed so far explain how to prove the ill-posedness of (1-10) that arises from linearizing (1-5) and (1-9) around a regular solution $f(t, x)$ whose initial data has a critical degeneracy at $x = 0$. As in [Jeong and Oh 2022], the nonlinear norm inflation results (Theorems 1.1 and 1.5) are derived by assuming the existence of a nonlinear perturbation u around f (i.e., $f + u$ solves the nonlinear equation) without the instability behavior, then applying the above argument. Moreover, the nonlinear nonexistence results (Theorems 1.2 and 1.6) are proved by superposition of infinitely many configurations exhibiting norm inflation (with unbounded rates of growth), with disjoint supports in physical space. We refer to [Jeong and Oh 2022, Section 1.6] for a more detailed summary of the ideas involved, and to Sections 2.5, 2.6, 3.5 and 3.6 for details. A key new feature of the present paper compared to [Jeong and Oh 2022], however, is that the background solution f need not be stationary solutions, and are given as a part of the contradiction hypothesis in the proof of the nonexistence theorems.

1.5. ***Revisiting L^2-ill-posedness à la Takeuchi–Mizohata.*** In view of the extensive appearance of the Takeuchi–Mizohata instability in this paper, it is perhaps not surprising that our techniques also apply to the original setting considered by Takeuchi and Mizohata of L^2-ill-posedness of linear nondegenerate

[3] Note that it is a priori possible that uniqueness of the Cauchy problem for (1-10) fails. Nevertheless, the method is still applicable and establishes the norm growth of every solution u with the same initial data satisfying (1-22).

Schrödinger-type equations. In the Appendix, we provide a few results concerning the Takeuchi–Mizohata condition obtained through our approach. In particular, we recover the following result of Mizohata [1985, §VII.2]:

Proposition 1.16. *Consider the linear first-order perturbation of the Schrödinger equation on* $\mathbb{R}^d$

$$i\partial_t u + \Delta u + b^j(x)\partial_j u = 0, \tag{1-23}$$

where $b \in C^{1,1}(\mathbb{R}^d)$. *Suppose that the Takeuchi–Mizohata condition for* (1-23) *fails, i.e.,*

$$\sup\left\{ \int_0^T \operatorname{Re} b^j(x - 2s\omega)\omega_j \,\mathrm{d}s : x \in \mathbb{R}^d,\ \omega \in \mathbb{S}^{d-1},\ T > 0 \right\} = +\infty. \tag{1-24}$$

Then, for any $\delta > 0$, *every solution map* $L^2 \to L^\infty_t([0,\delta]; L^2)$ *for* (1-23), *if it exists, is unbounded.*

Note that Proposition 1.16 clearly implies the result proved in [Mizohata 1985, §VII.2], namely, the impossibility of having a solution map for the inhomogeneous equation

$$i\partial_t u + \Delta u + b^j(x)\partial_j u = f \tag{1-25}$$

satisfying

$$\|u\|_{L^\infty([0,\delta];L^2)} \le C_0(\|u_0\|_{L^2} + \|f\|_{L^1([0,\delta];L^2)})$$

for some $C_0 < +\infty$. In fact, via Duhamel's principle, this result is equivalent to Proposition 1.16. Nonetheless, our techniques generalize easily to other situations when such an equivalence is not obvious, e.g., when b depends on time.

More interestingly, we also provide some new unconditional quantitative lower bounds for (1-23), which are valid way past the trivial $O\left(\frac{1}{\lambda}\right)$ timescale (where λ is the initial characteristic frequency), up to a time when the L^2 norm may grow at a *quantitative* rate depending on λ; see Propositions A.1 and A.3. These results should be contrasted with the proofs of Proposition 1.16 and [Mizohata 1985, §VII.2], which rely on *qualitative* contradiction arguments up to $O\left(\frac{1}{\lambda}\right)$ timescales.

Organization of the paper. The Schrödinger- and KdV-type equations are treated respectively in Sections 2 and 3. In the Appendix, we prove Proposition 1.16 and related results for the linear nondegenerate Schrödinger-type equation (1-23).

2. Schrödinger-type equations

This section is organized as follows. To motivate our approach, we analyze in Section 2.1 a model problem (2-2) derived from (1-1). In Section 2.2, we study the properties of linearly degenerate solutions — typically denoted by f — and in Section 2.3, we construct degenerating wave packets for the linearized equation around f. In Section 2.4, we establish a modified and generalized (bilinear) energy estimates for the perturbation (solving the nonlinear difference equation) around f. Finally, in Sections 2.5 and 2.6, we prove Theorems 1.1 and 1.2, respectively.

2.1. ***Degenerating wave packets for model linear equation.*** To motivate what is to follow, consider the case (1-1) (i.e., (1-5) with $\alpha_1 = \beta_1 = 1$ and $\mu_1 = 0$), which we recall here for convenience:

$$i\partial_t\phi + \partial_x(|\phi|^2\,\partial_x\phi) = 0. \tag{1-1}$$

We note that when the domain is taken to be $\mathbb{R}$, $f(t,x) = xe^{2it}$ — which degenerates (i.e., vanishes) linearly at 0 — is *formally* a solution to (1-1).[4] Indeed, even in the absence of any well-posedness results for the Cauchy problem, it is not difficult to show that any hypothetical smooth solution to (1-1) with initial data $f_0(x)$ that equals x in some neighborhood of the origin should approximate xe^{2it} uniformly for small $|x|$ and $|t|$. To illustrate our ill-posedness mechanism for initial data close to $f(0,x)$, we consider the linearization of (1-1) around the background solution $f(t,x) = xe^{2it}$. To wit, by plugging in the ansatz

$$\phi(t,x) = xe^{2it} + \tilde{\phi}(t,x) \tag{2-1}$$

into (1-1) and dropping quadratic or higher terms in $\tilde{\phi}$, we obtain the linearization of (1-1) around the explicit solution xe^{2it}:

$$i\partial_t\tilde{\phi} + \partial_x(x^2\,\partial_x\tilde{\phi}) + 2\partial_x(x\operatorname{Re}(\tilde{\phi})) + (e^{4it}-1)\partial_x(x\bar{\tilde{\phi}}) = 0.$$

Freezing the coefficients of the linearized equation at $t = 0$ and dropping zeroth-order terms in $\tilde{\phi}$, which can be readily incorporated into our ill-posedness proof if desired (see Remark 2.3), we arrive at the *model linear equation*:

$$i\partial_t\tilde{\phi} + \mathcal{L}[\tilde{\phi}] = 0, \quad \mathcal{L}[\,\cdot\,] = \partial_x(x^2\,\partial_x(\,\cdot\,)) + 2x\,\partial_x\operatorname{Re}(\,\cdot\,). \tag{2-2}$$

The goal of this section is to sketch a proof of the fact that this model linear equation is *ill-posed*; see Proposition 2.1.

An important observation regarding the operator $\mathcal{L}$ is that for any sufficiently regular v, we have the estimate

$$|\langle |x|^{1/2} i\mathcal{L}[v], |x|^{1/2}v\rangle| \le C\,\||x|^{1/2}v\|_{L^2}^2. \tag{2-3}$$

To see this, we first expand $\mathcal{L}$ and perform an integration by parts to get

$$\begin{aligned}
&\langle |x|^{1/2} i\mathcal{L}[v], |x|^{1/2}v\rangle \\
&\quad= \int \operatorname{Re}[(i\partial_x(x^2\,\partial_x v) + ix\,\partial_x v + x\,\partial_x\bar{v})(\operatorname{sgn} x)x\bar{v}]\,\mathrm{d}x \\
&\quad= \int \operatorname{Re}[-i(\operatorname{sgn} x)x^3\,\partial_x v\,\partial_x\bar{v} - i(\operatorname{sgn} x)(x^2\,\partial_x v)(\partial_x x)\bar{v} + i(\operatorname{sgn} x)x^2\,\partial_x v\bar{v} + i(\operatorname{sgn} x)x^2\,\partial_x\bar{v}\bar{v}]\,\mathrm{d}x,
\end{aligned}$$

where the contribution of $\partial_x \operatorname{sgn} x$ is zero thanks to the vanishing integrand at $x = 0$. Inside $\operatorname{Re}[\,\cdot\, s]$, the first term vanishes since it is purely imaginary, and the second and third terms exactly cancel (which

[4] In this section, we take the domain to be $\mathbb{R}$ rather than $\mathbb{T}$.

dictates the power $\frac{1}{2}$ in (2-3)). For the fourth term, we write $\partial_x \bar{v}\bar{v} = \frac{1}{2}\partial_x(\bar{v}^2)$ and perform another integration by parts to obtain

$$\langle |x|^{1/2} i\mathcal{L}[v], |x|^{1/2} v\rangle = \int \mathrm{Re}\big[-i\tfrac{1}{2}(\operatorname{sgn} x)(\partial_x x^2)\bar{v}^2\big]\,\mathrm{d}x,$$

whose absolute value is clearly estimated by $\||x|^{1/2}v\|_{L^2}^2$, as desired.

Equation (2-3) suggests that the correct way to measure regularity for solutions of (2-2) is to use $|x|^{1/2}$-weighted spaces.[5] To this end, we set $\|v\|_{L^2_w} = \||x|^{1/2}v\|_{L^2}$. We are now ready to state the main result of this section.

Proposition 2.1. *Equation* (2-2) *is ill-posed in L^2. More specifically, for any profile $g_0 \in C^\infty\big(\frac{1}{2}, 1\big)$,* ***any*** *$L^\infty_t L^2_w$ solution $\tilde{\phi}_{(\lambda)}$ to* (2-2) *with initial data*

$$\tilde{\phi}_{(\lambda),0}(x) = g_0(x)\exp(i\lambda \ln|x|), \quad \lambda < 0,$$

satisfies the growth

$$\|\tilde{\phi}_{(\lambda)}\|_{L^2}(t) \geq c_0 \exp(|\lambda| t) \quad \textit{for any } 0 < t < T,$$

with constants $c_0, T > 0$ depending only on g_0.

Remark 2.2. By an $L^\infty_t L^2_w$ solution to (2-2), we mean a weak solution $\tilde{\phi}$ which satisfies the bound

$$\|\tilde{\phi}\|_{L^2_w}(t) \leq \exp(Ct)\|\tilde{\phi}_0\|_{L^2_w},$$

where $C > 0$ is the constant from (2-3) and attains the initial data in the weak sense. Existence of an $L^\infty_t L^2_w$ solution given an L^2_w initial data follows from a standard argument involving the Aubin–Lions lemma (see [Jeong and Oh 2022, Appendix A] for instance). Note that $\|\tilde{\phi}_{(\lambda),0}\|_{L^2}, \|\tilde{\phi}_{(\lambda),0}\|_{L^2_w} \lesssim 1$ uniformly in λ. While we cannot rule out the possibility of nonuniqueness, the above result applies to *all* $L^\infty_t L^2_w$ solutions.

In following the proof, the reader may find Parts (1) (degenerating wave packets) and (2) (modified energy estimate and duality testing argument) in Section 1.4 expanded in detail. See also the remarks following the proof, which discuss additional ideas that go into the proof of Theorems 1.1 and 1.2.

Proof. We demonstrate how to construct approximate solutions to (2-2), from which Proposition 2.1 naturally follows. To begin with, we make a change of variable $y = \ln x$ for $x \geq 0$. Then using $x\,\partial_x = \partial_y$, (2-2) transforms into

$$i\partial_t\tilde{\phi} + \partial_{yy}\tilde{\phi} + \partial_y\tilde{\phi} + 2\partial_y \mathrm{Re}(\tilde{\phi}) = 0.$$

Defining $\varphi = e^y\tilde{\phi}$,

$$i\partial_t\varphi + \partial_{yy}\varphi + \partial_y\bar{\varphi} - 2\varphi - \bar{\varphi} = 0.$$

We then introduce

$$\psi = \varphi + \mathcal{A}\bar{\varphi}, \tag{2-4}$$

[5]As we shall see below, the exponents 2 and $\frac{1}{2}$ in xe^{2it} and $|x|^{1/2}$, respectively, should be replaced by appropriate constants for (1-5) in general.

where $\mathcal{A}=\frac{1}{2}\partial_y^{-1}$ is (formally) an operator of order -1. Then

$$\begin{aligned} i\partial_t\psi &= -\partial_{yy}\varphi+\mathcal{A}\,\partial_{yy}\bar\varphi-\partial_y\bar\varphi+\mathcal{A}\,\partial_y\varphi-(-2\varphi-\bar\varphi+2\mathcal{A}\bar\varphi+\mathcal{A}\varphi)\\ &= -\partial_{yy}\psi+\mathcal{A}\,\partial_{yy}\bar\varphi+\partial_{yy}\mathcal{A}\bar\varphi-\partial_y\bar\varphi+\mathcal{A}\,\partial_y\varphi-(-2\varphi-\bar\varphi+2\mathcal{A}\bar\varphi+\mathcal{A}\varphi).\end{aligned}$$

Then we see that

$$i\partial_t\psi+\partial_{yy}\psi=\mathcal{A}\,\partial_y\varphi-(-2\varphi-\bar\varphi+2\mathcal{A}\bar\varphi+\mathcal{A}\varphi).$$

Since the right-hand side is of order zero, it suggests that a degenerating wave packet φ may be constructed by taking ψ to be an approximate wave packet solution to the one-dimensional Schrödinger equation, and then going back to φ. More precisely, take

$$\psi^{\mathrm{app}}_{(\lambda)}(t,y)=\exp(i\lambda y-i\lambda^2 t)a_0(y-2\lambda t), \tag{2-5}$$

where we fix a_0 to be C^∞-smooth and supported in $\{-2<y<-1\}$. We need to take $\lambda<0$, so that the support of $\psi^{\mathrm{app}}(t,\cdot)$ is confined to $\{y<-1\}$ for all $t\ge 0$. To invert (2-4), we wish to take $\varphi\approx\psi-\mathcal{A}\bar\psi$. Since $\mathcal{A}=\frac{1}{2}\partial_y^{-1}$ acts like $-\frac{1}{2i\lambda}$ on $\bar\psi$, we are motivated to take

$$\varphi^{\mathrm{app}}(t,y)=\psi^{\mathrm{app}}_{(\lambda)}(t,y)+\frac{1}{2i\lambda}\overline{\psi^{\mathrm{app}}_{(\lambda)}}(t,y) \tag{2-6}$$

and then set

$$\tilde\phi^{\mathrm{app}}=e^{-y}\varphi^{\mathrm{app}}=e^{-y}\biggl(e^{i\lambda(y-\lambda t)}a_0(y-2\lambda t)+\frac{1}{2i\lambda}e^{-i\lambda(y-\lambda t)}\overline{a_0}(y-2\lambda t)\biggr).$$

Returning to the x-coordinates and defining the error by $\boldsymbol{\epsilon}_{\tilde\phi}=[i\partial_t+\mathcal{L}]\tilde\phi^{\mathrm{app}}$, we have

$$\|\boldsymbol{\epsilon}_{\tilde\phi}(t)\|_{L^2_w}\lesssim\|a_0\|_{H^2_x},\quad t\ge 0, \tag{2-7}$$

uniformly in λ. In this sense, $\tilde\phi^{\mathrm{app}}$ is an approximate solution of (2-2). Moreover, $\tilde\phi^{\mathrm{app}}$ itself satisfies the bound $\|\tilde\phi^{\mathrm{app}}(t)\|_{L^2_w}\lesssim\|a_0\|_{L^2_x}$. The last key property is degeneration: with a weight higher than $|x|^{1/2}$, $\tilde\phi^{\mathrm{app}}(t)$ decays in the $O(|\lambda|^{-1})$-timescale: for example, with the weight $|x|$, we have

$$\||x|\tilde\phi^{\mathrm{app}}(t,x)\|_{L^2_x}\lesssim e^{-|\lambda|t}\|a_0\|_{L^2_x}. \tag{2-8}$$

Interpolating (2-8) with the L^2_w-estimate shows that $\|\tilde\phi^{\mathrm{app}}(t,\cdot)\|_{L^2}\gtrsim e^{|\lambda|t}$. Now, let $\tilde\phi$ be an $L^\infty_t L^2_w$ solution of (2-2). Then, with a direct computation, we have the *generalized energy estimate* for the weighted L^2-estimate (see Section 2.4.2)

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle|x|^{1/2}\tilde\phi,|x|^{1/2}\tilde\phi^{\mathrm{app}}\rangle=\langle|x|^{1/2}\tilde\phi,|x|^{1/2}\boldsymbol{\epsilon}_{\tilde\phi}\rangle,$$

which gives, together with (2-7),

$$\mathrm{Re}\langle|x|^{1/2}\tilde\phi,|x|^{1/2}\tilde\phi^{\mathrm{app}}\rangle(t)\ge\mathrm{Re}\langle|x|^{1/2}\tilde\phi,|x|^{1/2}\tilde\phi^{\mathrm{app}}\rangle(t=0)-Ct\|\tilde\phi\|_{L^\infty_t L^2_w}\|a_0\|_{H^2_x}.$$

At the initial time, by choosing a_0 in a way depending only on g_0, we can guarantee that

$$\mathrm{Re}\langle|x|^{1/2}\tilde\phi,|x|^{1/2}\tilde\phi^{\mathrm{app}}\rangle(t=0)\ge\tfrac{1}{2}\|\tilde\phi_0\|_{L^2_w}\|\tilde\phi_0^{\mathrm{app}}\|_{L^2_w}.$$

Then, for $0 < t < C\|a_0\|_{H^2_x}/(4\|\tilde{\phi}_0\|_{L^2_w})$, we obtain with (2-8) that

$$\frac{1}{C}e^{-|\lambda|t}\|a_0\|_{L^2}\|\tilde{\phi}(t)\|_{L^2} \geq \operatorname{Re}\langle\tilde{\phi}, |x|\tilde{\phi}^{\mathrm{app}}\rangle(t) = \operatorname{Re}\langle|x|^{1/2}\tilde{\phi}, |x|^{1/2}\tilde{\phi}^{\mathrm{app}}\rangle(t) \geq \tfrac{1}{4}\|\tilde{\phi}_0\|_{L^2_w}\|\tilde{\phi}_0^{\mathrm{app}}\|_{L^2_w},$$

which gives the claimed exponential growth of $\|\tilde{\phi}(t)\|_{L^2}$. □

Remark 2.3. (1) *Ill-posedness of the linearization of* (1-1). A small modification of the above proof gives an analogous ill-posedness result for the linearization of (1-1) around xe^{2it}, which takes the form

$$i\partial_t\phi + \mathcal{L}\phi + (e^{4it}-1)x\partial_x\bar{\tilde{\phi}} = (\text{zeroth-order in } \tilde{\phi}).$$

Note that the additional zeroth-order terms in $\tilde{\phi}$ do not affect argument in any way; the main modification is due to the presence of the additional term $(e^{4it}-1)x\partial_x\bar{\tilde{\phi}}$. Specifically, to cancel the contribution of $\partial_y\bar{\varphi}$ (where y and $\varphi = e^y\tilde{\phi}$ are as before), the operator in (2-4) needs to be modified to $\mathcal{A} = \frac{1}{2}e^{4it}\partial_y^{-1}$, which in turn motivates the modified ansatz

$$\varphi^{\mathrm{app}}(t, y) = \psi^{\mathrm{app}}_{(\lambda)}(t, y) + \frac{e^{4it}}{2i\lambda}\overline{\psi^{\mathrm{app}}_{(\lambda)}}(t, y),$$

with $\psi^{\mathrm{app}}_{(\lambda)}(t, y)$ as before. The remainder of the proof proceeds similarly as before; we leave the details to the interested reader.

(2) *Ill-posedness in H^m for $m > 0$.* In fact, another small modification of the above proof shows that (2-2) is ill-posed in H^m for $m > 0$. More precisely, we have the growth

$$\|\partial_x^m\tilde{\phi}_{(\lambda)}\|_{L^2}(t) \geq c_0\exp((1+2m)|\lambda|t) \quad \text{for any } m \geq 0 \text{ and } 0 < t < T,$$

with $c_0, T > 0$ depending only on g_0 and m.

We now sketch the needed modification; see Section 2.5 for the complete proof. We would like to modify the last part of the proof of Proposition 2.1 using "differentiation by parts": under the assumption that $\partial_x^{-m}(|x|\tilde{\phi}^{\mathrm{app}}) \in L^2$,

$$\|\partial_x^m\tilde{\phi}\|_{L^2}\|\partial_x^{-m}(|x|\tilde{\phi}^{\mathrm{app}})\|_{L^2} \geq \operatorname{Re}\langle\partial_x^m\tilde{\phi}, \partial_x^{-m}(|x|\tilde{\phi}^{\mathrm{app}})\rangle(t) = \operatorname{Re}\langle|x|^{1/2}\tilde{\phi}, |x|^{1/2}\tilde{\phi}^{\mathrm{app}}\rangle(t).$$

Now the point is that $\partial_x^{-1} = x\partial_y^{-1} = e^y\partial_y^{-1}$ and $y \simeq -2|\lambda|t$ on the support of $\tilde{\phi}^{\mathrm{app}}(t)$, which gives a faster rate of degeneration $\|\partial_x^{-m}(|x|\tilde{\phi}^{\mathrm{app}})\|_{L^2} \lesssim \exp(-(2m+1)|\lambda|t)$. This gives the claimed lower bound for $\|\partial_x^m\tilde{\phi}\|_{L^2}$. In general, there could be some low frequency part of $|x|\tilde{\phi}^{\mathrm{app}}$ which does not degenerate, and for this reason we introduce a decomposition of $|x|\tilde{\phi}^{\mathrm{app}}$ into high and low frequency parts in the actual proof in Section 2.5.

Remark 2.4 (additional ideas in the proof of Theorems 1.1 and 1.2). For a general equation of the form (1-5), we do not have access to a stationary solution with a linear degeneracy in general (furthermore, we shall also require that f_0 be compactly supported, which rules out xe^{2it}, too). Hence, we shall carry out the above analysis (degenerating wave packet construction, modified energy estimate and duality), where the background solution f is merely a regular (most likely) *time-dependent* solution to (1-5), which has compactly supported initial data f_0 with a linear degeneracy, in place of xe^{2it}.

Theorem 1.1 is proved by considering a perturbation $f+\tilde{\phi}$ of such an f, and arguing that if $f+\tilde{\phi}$ exists as a regular solution (i.e., if we are in the second case in Theorem 1.1), then the above growth mechanism for $\tilde{\phi}$ can be justified. To prove Theorem 1.2, we consider initial data $\tilde{\phi}_0$ consisting of a superposition of an infinitude of configurations as above (i.e., $\sum_k (f_{k,0}+\tilde{\phi}_{k,0})$, where $f_{k,0}$ has a linear degeneracy and $\tilde{\phi}_{k,0}$ is a degenerating wave packet adapted to $f_{k,0}$) with unbounded rates (i.e., the initial frequencies of the degenerating wave packets are unbounded), disjoint supports (i.e., $\{\operatorname{supp} f_{k,0} \cup \operatorname{supp} \phi_{k,0}\}_k$ is pairwise disjoint), yet with an ϵ-small C^{m_0} norm. Then we perform a contradiction argument: if a regular solution ϕ to such initial data exists, then we may justify the growth mechanism (as in Proposition 2.1), which is absurd. For details, see Sections 2.5 and 2.6 below.

2.2. *Properties of a regular linearly degenerate solution.* We shall assume that there exists a smooth solution to (1-5) which is linearly degenerate and analyze its properties. To be precise, we will let $f : [0, \delta] \times [-x_1, x_1] \to \mathbb{C}$ be a $L^\infty([0, \delta]; C^{3,1}([x_0 - x_1, x_0 + x_1]))$ solution to (1-5) with some $x_1, \delta > 0$ satisfying

$$f_0 \in C^{3,1}([x_0 - x_1, x_0 + x_1]), \quad f_0(x_0) = 0, \quad |f_0'(x_0)| > 0$$

at $t = 0$ for some $x_0 \in \mathbb{T}$.

Owing to the symmetries of (1-5) (translation and phase rotation), as well as its behavior under the transformation $\phi \mapsto c\phi$, we may assume without loss of generality that $x_0 = 0$ and $f_0'(0) = 1$. Then, from the equation it is easy to see that, on the time interval $[0, \delta]$,

$$f(t, 0) = 0$$

and

$$i\frac{\mathrm{d}}{\mathrm{d}t} f'(t, 0) = -(\alpha_1 + \beta_1)|f'(t, 0)|^2 f'(t, 0),$$

which implies in particular that

$$|f'(t, 0)| = 1 \quad \text{and} \quad |f(t, x)| = x + O(|x|^2) \text{ uniformly in } t.$$

More generally, we have the following lemma.

Lemma 2.5. *Let $s \geq 2$ be an integer, and let $f \in C_t([0, \delta]; C^{s-1,1}(\mathbb{T}))$ be a solution to* (1-5). *Then*:

(1) *The zero set of $f(t, x)$ is preserved in time, i.e., $a \in \mathbb{T}$ is a zero of $f(0, x)$ if and only if it is a zero for $f(t, x)$ for all $t \in [0, \delta]$.*

(2) *Let $a \in \mathbb{T}$ be a zero of $f(0, x)$. Then $\{\partial_x^k f(t, a)\}_{k=0}^{s-1}$ is determined by the initial data at $x = a$, i.e., $\{\partial_x^k f(0, a)\}_{k=0}^{s-1}$.*

Here, the important point is that, thanks to the regularity assumption, $f(0, x)$ vanishes at least linearly at each zero $x = a$, which is *critical* for (1-5) in the senses discussed in Section 1.3.

Proof. By the regularity assumption (in particular, that $s \geq 2$), it follows from (1-5) that $|\partial_t f(t, x)| \leq C|f(t, x)|$; hence the first statement follows. To prove the second statement, consider a zero a of $f(0, x)$. Without any loss of generality, we may assume that $a = 0$. By the assumption and Taylor expansion, we have

$$f(t,x)=\sum_{k=1}^{s-1}\frac{1}{k!}\partial_x^k f(t,0)x^k+O(|x|^s),$$

$$f_x(t,x)=\sum_{k=0}^{s-2}\frac{1}{k!}\partial_x^{k+1} f(t,0)x^k+O(|x|^{s-1}),$$

$$f_{xx}(t,x)=\sum_{k=0}^{s-3}\frac{1}{k!}\partial_x^{k+2} f(t,0)x^k+O(|x|^{s-2}),$$

where the implicit constants depend only on $\|f\|_{L_t^\infty C_x^{s-1,1}}$ (and in particular are independent of (t,x)). Plugging this into (1-5) and matching the coefficients of x^k for $k=1,\dots,s-1$, we formally obtain a determined system of first-order ODEs for $\{\partial_x^k f(t,0)\}_{k=1}^{s-1}$; here, the fact that $f(t,x)$ vanishes at least linearly is crucially used to ensure that no $\partial_x^k f(t,0)$ with $k>s-1$ arises. Indeed, these ODEs may be justified in the sense of distributions by testing (1-5) against a test function of the form $\eta(t)(-1)^k\partial_x^k\chi_\epsilon(x)$, where $\eta\in C_c^\infty(0,\delta)$, $\chi\in C_c^\infty\big(-\frac12,\frac12\big)$ with $\int\chi=1$, and $\chi_\epsilon(x)=\epsilon^{-1}\chi(\epsilon^{-1}x)$. By the uniqueness of this ODE system, the desired statement follows. □

From now on, given f_0 which are linearly degenerate at $x=0$ and $f_x(0,0)$ positive real, we are going to take $0<x_1<1$ smaller if necessary, so that

$$\Big(\sup_{x\in[-x_1,x_1]}|f_{xx}(0,x)|\Big)x_1<\tfrac12 f_x(0,0). \tag{2-9}$$

In particular, we have

$$\tfrac12 f_x(0,0)<|f_x(0,x)|<2f_x(0,0)\quad \text{for all } x\in[-x_1,x_1].$$

Proposition 2.6. *Let* $f\in L_t^\infty([0,\delta];C^{3,1}([-x_1,x_1])$ *be a solution to* (1-5), *and set* $M=\|f\|_{L^\infty([0,\delta];C^{3,1})}$. *Then*, *we have the pointwise bounds*

$$|f(t,x)|\le|f_0(x)|\exp(CM^2t) \tag{2-10}$$

and

$$|\partial_t(|f(t,x)|^2)|\le CM\exp(CM^2\delta)(1+(f_x(0,0))^{-1}M)^3(|f_0(x)|^3+tM^3|f_0(x)|^2) \tag{2-11}$$

for all $t\in[0,\delta]$ *and* $|x|\le x_1$.

Proof. We first note directly from (1-5) that $\big|\partial_t|f(t,x)|\big|\le C\|f\|^2_{L_t^\infty C_x^{1,1}}|f(t,x)|$ holds, which gives (2-10). Now note that $f\in L^\infty([0,\delta];C^{3,1}([-x_1,x_1]))$ implies, via (1-5), that

$$|\partial_t f(t,x)|\le C|f(t,x)|\|f\|^2_{L_t^\infty C_x^{1,1}},\quad |\partial_{tt} f(t,x)|\le C|f(t,x)|^2\|f\|^3_{L_t^\infty C_x^{3,1}}. \tag{2-12}$$

Then, the Taylor expansion in time of $f(t,x)$ gives

$$|f(t,x)|^2=|f_0(x)|^2+2\,\mathrm{Re}\bigg(\overline{f_0(x)}\int_0^t(\partial_t f)(t',x)\,dt'\bigg)+\bigg|\int_0^t(\partial_t f)(t',x)\,dt'\bigg|^2.$$

Taking the time derivative,

$$\partial_t(|f(t,x)|^2) = 2\,\mathrm{Re}(\overline{f_0(x)}(\partial_t f)(t,x)) + 2\,\mathrm{Re}\biggl(\overline{(\partial_t f)(t,x)} \int_0^t (\partial_t f)(t',x)\,\mathrm{d}t'\biggr). \tag{2-13}$$

Using (2-10), the last term in (2-13) is bounded by

$$C\|f\|^4_{L^\infty_t C^{3,1}_x} |f(t,x)| \int_0^t |f(t',x)|\,\mathrm{d}t' \le CM^4 t \exp(CM^2\delta)|f_0(x)|^2.$$

For the other term in the right-hand side of (2-13), we further rewrite it as

$$2\,\mathrm{Re}(\overline{f_0(x)}(\partial_t f)(t,x)) = 2\,\mathrm{Re}(\overline{f_0(x)}(\partial_t f)(0,x)) + 2\,\mathrm{Re}\biggl(\overline{f_0(x)} \int_0^t (\partial_{tt} f)(t',x)\,\mathrm{d}t'\biggr)$$

and note that the last term is bounded using (2-12) by $C\exp(CM^2\delta)M^3 t|f_0(x)|^3$. On the other hand, the first term on the right-hand side equals

$$\begin{aligned}\mathrm{Im}\bigl(\overline{f_0(x)}\bigl(|f_0(x)|^2\partial_{xx} f_0(x) + \alpha_1 f_0(x)|\partial_x f_0(x)|^2 + \beta_1 \overline{f_0(x)}(\partial_x f_0(x))^2 + \mu_1|f_0(x)|^2 f_0(x)\bigr)\bigr)&\\ = \beta_1\,\mathrm{Im}\bigl(\overline{f_0(x)}^2(\partial_x f_0(x))^2\bigr) + O(\|f_0\|_{C^{1,1}})|f_0(x)|^3&,\end{aligned}$$

and we see that the leading term in the Taylor expansion of $\overline{f_0(x)}^2(\partial_x f_0(x))^2$ is purely real, with remainder bounded by

$$C(f_x(0,0)^3|x|^3\|f_0\|_{\dot C^{1,1}} + f_x(0,0)^2|x|^4\|f_0\|^2_{\dot C^{1,1}} + f_x(0,0)^3|x|^5\|f_0\|^2_{\dot C^{1,1}}) \le C(1+f_x(0,0)^{-1}M)^3|f_0(x)|^3,$$

where we have used $|x| < x_1$ and the smallness of x_1 from (2-9). Collecting the bounds, we obtain the proposition. □

2.3. ***Degenerating wave packets for the linearized equation.*** In this subsection, our goal is to construct approximate solutions, called degenerating wave packets, for the linearization of (1-5) around a (possibly hypothetical) regular linearly degenerate solution, which possess the desired degeneration property that is responsible for the ill-posedness of (1-5); see Proposition 2.7 below.

2.3.1. *Properties of degenerating wave packets.* Given a smooth solution f to (1-5), let us write $\phi = f + \tilde\phi$, where ϕ is another smooth solution to (1-5). The equation for $\tilde\phi$ is given by

$$i\partial_t\tilde\phi + |f|^2\,\partial_{xx}\tilde\phi + \alpha_1 f(\overline{\partial_x f}\,\partial_x\tilde\phi + \partial_x f\,\partial_x\bar{\tilde\phi}) + 2\beta_1\bar f\,\partial_x f\,\partial_x\tilde\phi + V_f\tilde\phi + W_f\bar{\tilde\phi} = Q_f[\tilde\phi], \tag{2-14}$$

with

$$\begin{aligned} V_f &= \bar f\,\partial_{xx} f + \alpha_1|\partial_x f|^2 + 2\mu_1|f|^2,\\ W_f &= f\,\partial_{xx} f + \beta_1(\partial_x f)^2 + \mu_1 f^2,\\ Q_f[\tilde\phi] &= -(\bar f\tilde\phi + f\bar{\tilde\phi})\partial_{xx}\tilde\phi - \alpha_1\tilde\phi(\partial_x\bar f\,\partial_x\tilde\phi + \partial_x f\,\partial_x\bar{\tilde\phi}) - \alpha_1 f|\partial_x\tilde\phi|^2 - 2\beta_1\bar{\tilde\phi}\,\partial_x f\,\partial_x\tilde\phi - \beta_1\bar f(\partial_x\tilde\phi)^2\\ &\quad - |\tilde\phi|^2\,\partial_{xx}\tilde\phi - \alpha_1\tilde\phi|\partial_x\tilde\phi|^2 - \beta_1\bar{\tilde\phi}(\partial_x\tilde\phi)^2 - \mu_0|\tilde\phi|^2\tilde\phi - 2\mu_1 f|\tilde\phi|^2 - \mu_1\bar f(\tilde\phi)^2. \end{aligned} \tag{2-15}$$

Note that $Q_f[\tilde{\phi}]$ is at least quadratic in $\tilde{\phi}$ and its derivatives. Dropping the right-hand side, we obtain the linearized equation around f:

$$i\partial_t\tilde{\phi} + |f|^2\,\partial_{xx}\tilde{\phi} + \alpha_1 f(\overline{\partial_x f}\,\partial_x\tilde{\phi} + \partial_x f\,\partial_x\bar{\tilde{\phi}}) + 2\beta_1\bar{f}\,\partial_x f\,\partial_x\tilde{\phi} + V_f\tilde{\phi} + W_f\bar{\tilde{\phi}} = 0. \tag{2-16}$$

We now state the key proposition of this section, which shows properties of degenerating wave packets for (2-16). Given a positive number L, we introduce the notation

$$\|g\|_{W^{s,p}_{(L)}} = \sum_{j=0}^{s}\|(L\partial_x)^j g\|_{L^p}$$

and write $H^s_{(L)}$ when $p = 2$.

Proposition 2.7. *Let $f \in L^\infty([0,\delta]; C^{s_0-1,1}([0,x_1]))$ be a solution to* (1-5) *with $s_0 \ge 4$ satisfying*

$$f(0,0) = 0, \qquad f'(0,0) = A \tag{2-17}$$

for some $A > 0$. By taking $x_1 < 1$ small if necessary, assume that (2-9) *holds. Then, to any $\lambda \le -1$ and a C^∞-smooth complex-valued profile g_0 supported in $\left(\tfrac{1}{2}x_1, x_1\right)$, we may associate a function $\tilde{\phi}^{\mathrm{app}}_{(\lambda)} = \tilde{\phi}^{\mathrm{app}}_{(\lambda)}[g_0, f]$ defined in $[0,\delta]\times\mathbb{R}$ satisfying the following properties*:

- *Linearity*: *the map $g_0 \mapsto \tilde{\phi}^{\mathrm{app}}_{(\lambda)}$ is* (*real*) *linear*;
- *Support property*: $\mathrm{supp}(\tilde{\phi}^{\mathrm{app}}_{(\lambda)}[g_0]) \subset (0, e^{-|\lambda|A^2 t}x_1)$;
- *Initial data*: *for any $1 \le p \le \infty$,*

$$\frac{1}{C}\|g_0\|_{L^2} \le A^{\sigma_c-1}\||f|^{-\sigma_c}\tilde{\phi}^{\mathrm{app}}(0,x)\|_{L^2} \le C\|g_0\|_{L^2},$$
$$A^{\sigma_c-1}\||f|^{\sigma_c}\tilde{\phi}^{\mathrm{app}}(0,x)\|_{L^p} \le Cx_1^{1/p-1/2}\|g_0\|_{L^p};$$

- *Regularity*: *for $0 \le n \le s_0-2$, we have*

$$\||f|^{-\sigma_c}(|f|\,\partial_x)^n\tilde{\phi}^{\mathrm{app}}_{(\lambda)}(t,x)\|_{L^2} \le C_{f,\delta}A^{-\sigma_c+n+1}|\lambda|^n\|g_0\|_{H^n_{(x_1)}}, \quad t \le \min\{A^{-2}|\lambda|^{-1/2}, \delta\}; \tag{2-18}$$

- *Degeneration*: *for any $1 \le p \le 2$, $0 \le s \le s_0-2$, and $\gamma' \ge -s-\frac{1}{p}+\frac{1}{2}$, we have*

$$|f|^{-\sigma_c+\gamma'}\tilde{\phi}^{\mathrm{app}}_{(\lambda)} = \partial_x^s\left(\frac{|f|^{\gamma'+s-1/2}}{i^s\lambda^s(1+|f|\,\partial_x S)^s}\psi^{\mathrm{app}}_{(\lambda)}\right) + |f|^{-\sigma_c+\gamma'}\tilde{\phi}^{\mathrm{small}}_{(\lambda)} \tag{2-19}$$

for some $\psi^{\mathrm{app}}_{(\lambda)}$, $\tilde{\phi}^{\mathrm{small}}_{(\lambda)}$, and S, where $\psi^{\mathrm{app}}_{(\lambda)}$ is independent of p, s and γ', and

$$\left\|\frac{|f|^{\gamma'+s-1/2}}{\lambda^s(1+|f|\,\partial_x S)^s}\psi^{\mathrm{app}}_{(\lambda)}(t,x)\right\|_{L^p} \le C^{1+\gamma'}_{f,\delta}A^{\gamma'+s+1/2}|\lambda|^{-s} \times \exp\!\left(-2|\lambda|\left(\gamma'+s+\tfrac{1}{p}-\tfrac{1}{2}\right)A^2 t\right)\|g_0\|_{L^2}, \tag{2-20}$$

$$\||f|^{-\sigma_c}\tilde{\phi}^{\mathrm{small}}_{(\lambda)}(t,x)\|_{L^2} \le C_{f,\delta}A^{-\sigma_c+1}|\lambda|^{-1}\|g_0\|_{H^s_{(x_1)}} \tag{2-21}$$

for $t \le \min\{A^{-2}|\lambda|^{-1/2}, \delta\}$, after taking $\delta > 0$ smaller in a way that $\delta\|f\|^2_{L^\infty([0,\delta];C^{1,1})}$ is small in terms of $A^{-1}\|f\|_{L^\infty([0,\delta];C^{3,1})}$;

- Error estimate: *defining the error* $\boldsymbol{\epsilon}[\tilde{\phi}^{\mathrm{app}}_{(\lambda)}]$ *by the left-hand side of* (2-16) *with* $\tilde{\phi} = \tilde{\phi}^{\mathrm{app}}_{(\lambda)}$, *we have the estimate*

$$\| |f|^{-\sigma_c} \boldsymbol{\epsilon}[\tilde{\phi}^{\mathrm{app}}_{(\lambda)}](t) \|_{L^2} \le C_{f,\delta} A^{-\sigma_c+3} \|g_0\|_{H^2_{(x_1)}}, \quad t \le \min\{A^{-2}|\lambda|^{-1/2}, \delta\}. \tag{2-22}$$

In the above estimates, the constant $C_{f,\delta}$ *satisfies*

$$C_{f,\delta} \le C_0 (1 + A^{-1} \|f\|_{L^\infty_t C^{s_0-1,1}})^{N_0} \exp(C_0 \|f\|^2_{L^\infty_t C^{s_0-1,1}} \delta) \tag{2-23}$$

for some C_0, $N_0 > 0$ *depending on* α_1, β_1, μ_1 *and* s_0 *but not on* f *and* x_1.

We fix $A = 1$ and prove Proposition 2.7 in the remainder of this subsection. In the general case, given f we can define $\tilde{f}(t,x) := A^{-1} f(A^{-2}t, x)$ which is another $L^\infty_t C^{s-1,1}$ solution to (1-5) satisfying $\tilde{f}_x(0,0) = 1$. Then, we simply define

$$\tilde{\phi}^{\mathrm{app}}_{(\lambda)}[g_0, f](t,x) := \tilde{\phi}^{\mathrm{app}}_{(\lambda)}[Ag_0, \tilde{f}](A^2 t, x)$$

and verify the claimed properties of $\tilde{\phi}^{\mathrm{app}}_{(\lambda)}[g_0, f]$ using those for $\tilde{\phi}^{\mathrm{app}}_{(\lambda)}[Ag_0, \tilde{f}]$. In the proof, it will be seen that $|f|\,\partial_x S$ remains invariant under this rescaling.

2.3.2. *Renormalization and wave packet construction.* With $x_1 > 0$ given in Proposition 2.7, we define the variable y for $t \in [0,\delta]$ and $x \in (0, x_1]$ by

$$y(t,x) = -\int_x^{x_1} \frac{1}{|f(t,x')|}\, \mathrm{d}x' \le 0.$$

For each $t \ge 0$, the inverse of $x \mapsto y(t,x)$ is denoted by $x = x(t,y)$. From $|f(t,x)| = x + O(|x|^2)$, we have

$$y(t,x) - \ln \frac{x}{x_1} = B(t,x), \quad x(t,y) = x_1 e^{y-B}, \quad |B(t,x)| \le C x_1 \|f\|_{L^\infty_t C^{1,1}}. \tag{2-24}$$

Using $|f|\,\partial_x = \partial_y$, we rewrite (2-16) in (t,y)-coordinates:

$$i\partial_t \tilde{\phi} + ih\,\partial_y \tilde{\phi} + \partial_y^2 \tilde{\phi} + \frac{\alpha_1 f\, \overline{\partial_y f} + 2\beta_1 \bar{f}\,\partial_y f - |f|\,\partial_y |f|}{|f|^2} \partial_y \tilde{\phi} + \alpha_1 \frac{f\,\partial_y f}{|f|^2} \partial_y \bar{\tilde{\phi}} + V_f \tilde{\phi} + W_f \bar{\tilde{\phi}} = 0. \tag{2-25}$$

Here, we have introduced $h(t,y) = \partial_t y$ so that $\partial_t \tilde{\phi}(t,x) = \partial_t \tilde{\phi}(t,y) + h(t,y)\partial_y \tilde{\phi}(t,y)$. Now defining

$$G(t,y) = \left(-\sigma_c + \tfrac{1}{2}\right) \ln|f|(t,y), \quad (\partial_y G)(t,y) = \left(-\sigma_c + \tfrac{1}{2}\right) \frac{\mathrm{Re}(\bar{f}\,\partial_y f)}{|f|^2}(t,y)$$

and introducing the conjugation $\varphi = e^G \tilde{\phi}$, we obtain $\big($recall from (1-6) that $\sigma_c = -\left(\tfrac{1}{2}\alpha_1 + \beta_1 - 1\right)\big)$

$$i\partial_t \varphi + \partial_{yy}\varphi + \alpha_1 \frac{f\,\partial_y f}{|f|^2} \overline{\partial_y \varphi} + \left((-\alpha_1 + 2\beta_1) \frac{\mathrm{Im}(\bar{f}\,\partial_y f)}{|f|^2} + h \right) i\partial_y \varphi = \mathcal{B}_0[\varphi] \tag{2-26}$$

with

$$\begin{aligned} \mathcal{B}_0[\varphi] = {} & i(\partial_t G)\varphi + (\partial_{yy} G + (\partial_y G)^2)\varphi \\ & + \alpha_1 \frac{f\,\partial_y f}{|f|^2} \partial_y G \bar{\varphi} + \left((-\alpha_1 + 2\beta_1) \frac{\mathrm{Im}(\bar{f}\,\partial_y f)}{|f|^2} + h \right) i(\partial_y G)\varphi - V_f \varphi - W_f \bar{\varphi}. \end{aligned}$$

Note that the terms in $\mathcal{B}_0$ do not contain derivatives of φ. To handle the term containing $\overline{\partial_y \varphi}$ in the left-hand side of (2-26), we make yet another change of variables: introducing formally

$$\psi = \varphi + \frac{\alpha_1}{2} \frac{f\,\partial_y f}{|f|^2} \partial_y^{-1} \bar{\varphi},$$

we have that (2-26) turns into

$$i\partial_t \psi + \partial_{yy}\psi + \left((-\alpha_1 + 2\beta_1) \frac{\mathrm{Im}(\bar{f}\,\partial_y f)}{|f|^2} + h \right) i\partial_y \psi = \cdots, \tag{2-27}$$

where the terms on the right-hand side do not contain any derivatives of ψ. Indeed, introducing the shorthand

$$\mathcal{A}\bar{\varphi} = \frac{\alpha_1}{2} \frac{f\,\partial_y f}{|f|^2} \partial_y^{-1} \bar{\varphi}$$

and omitting any zeroth-order terms in φ, we have the formal computation

$$\begin{aligned} i\partial_t \psi &= -\partial_{yy}\varphi + i[\partial_t, \mathcal{A}]\bar{\varphi} + \mathcal{A}\,\partial_{yy}\bar{\varphi} - \alpha_1 \frac{f\,\partial_y f}{|f|^2} \overline{\partial_y \varphi} - \left((-\alpha_1 + 2\beta_1) \frac{\mathrm{Im}(\bar{f}\,\partial_y f)}{|f|^2} + h \right) i\partial_y \varphi + \cdots \\ &= -\partial_{yy}\psi + \partial_{yy}\mathcal{A}\bar{\varphi} + \mathcal{A}\,\partial_{yy}\bar{\varphi} - \alpha_1 \frac{f\,\partial_y f}{|f|^2} \overline{\partial_y \varphi} - \left((-\alpha_1 + 2\beta_1) \frac{\mathrm{Im}(\bar{f}\,\partial_y f)}{|f|^2} + h \right) i\partial_y \psi + \cdots \\ &= -\partial_{yy}\psi - \left((-\alpha_1 + 2\beta_1) \frac{\mathrm{Im}(\bar{f}\,\partial_y f)}{|f|^2} + h \right) i\partial_y \psi + \cdots. \end{aligned}$$

Motivated by this computation, we construct a wave packet approximate solution for (2-26) by starting with a wave packet for the preceding equation for ψ, then coming back to φ. More precisely, given $g_0(x)$ as in Proposition 2.7, we take

$$a_0(y) = x_1^{1/2} g_0(x(0, y)),$$

which is supported in $y \in \left(-\frac{1}{2}\ln 2, 0\right)$ by (2-24). For each $\lambda < 0$, we define

$$\psi^{\mathrm{app}}_{(\lambda)}(t, y) := e^{i\lambda(y - \lambda t)} a_{(\lambda)}(t, y), \tag{2-28}$$

where $a_{(\lambda)}(t, y)$ is the unique solution to

$$\partial_t a_{(\lambda)} + 2\lambda\,\partial_y a_{(\lambda)} = \frac{\lambda}{i} \left((-\alpha_1 + 2\beta_1) \frac{\mathrm{Im}(\bar{f}\,\partial_y f)}{|f|^2} + h \right) a_{(\lambda)} \tag{2-29}$$

with initial data $a_{(\lambda)}(0, y) = a_0(y)$. The function $\psi^{\mathrm{app}}_{(\lambda)}$ defined via (2-28) and (2-29) turns out to be a suitable approximate solution to (2-27) (more precisely, (2-38) holds). Next, given ψ^{app}, set

$$\varphi^{\mathrm{app}}_{(\lambda)} = \psi^{\mathrm{app}}_{(\lambda)} + \frac{\alpha_1}{2i\lambda} \frac{f\,\partial_y f}{|f|^2} \overline{\psi^{\mathrm{app}}_{(\lambda)}}, \tag{2-30}$$

which will be shown to be a suitable approximate solution to (2-26) (for more details, see the end of the proof of Proposition 2.7). Finally, the degenerating wave packet is defined by

$$\tilde{\phi}^{\mathrm{app}}_{(\lambda)}[g_0, f] = e^{-G} \varphi^{\mathrm{app}}_{(\lambda)} = |f|^{\sigma_c - 1/2} \varphi^{\mathrm{app}}_{(\lambda)}. \tag{2-31}$$

2.3.3. *Proof of Proposition 2.7.* Now that we have defined the wave packet solution, let us proceed to confirm the properties stated in Proposition 2.7.

Linearity and support property. From the definition, linearity is clear. Furthermore, note from (2-31), (2-30) and (2-28) that the support of $\tilde{\phi}^{\mathrm{app}}_{(\lambda)}(t,\cdot)$ coincides with that of $a_{(\lambda)}(t,\cdot)$. (From now on, we shall refrain from writing out the subscript λ.) On the other hand, note the following formulae for a:

$$a(t,y) = e^{i\lambda S(t,y)} a_0(y-2\lambda t), \tag{2-32}$$

$$S(t,y) = \int_0^t \Big((-\alpha_1 + 2\beta_1) \frac{\operatorname{Im}(\bar{f}\,\partial_y f)}{|f|^2} + h \Big)(t', y - 2\lambda(t-t'))\,\mathrm{d}t'. \tag{2-33}$$

Since $\lambda < 0$, the support of $a(t,\cdot)$ is contained in the interval $\big(-\frac{1}{2}\ln 2 + 2\lambda t, 2\lambda t\big) \subseteq (-\infty, 0)$ for $t \geq 0$, which verifies the support property of $\tilde{\phi}^{\mathrm{app}}$ via (2-24).

Regularity estimates. To begin with, we obtain estimates on $h := \partial_t y$. Recalling (2-13), we have

$$h = -\int_x^{x_1} \partial_t \Big(\frac{1}{|f(t,x')|} \Big)\,\mathrm{d}x' = \int_x^{x_1} \frac{\partial_t(|f|^2)}{|f|^3}\,\mathrm{d}x', \qquad \partial_y h = |f| \partial_x h = -\frac{\partial_t(|f|^2)}{|f|^2}.$$

Applying (2-10) and (2-11), we obtain the pointwise estimates

$$|h| \leq C_{f,\delta} \Big(1 + t \ln \frac{1}{x} \Big) x_1, \qquad |\partial_y h| \leq C_{f,\delta}(x+t). \tag{2-34}$$

We now estimate a. Observing that the right-hand side in (2-29) is purely imaginary,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|a|^2(t,y) = -2\lambda \operatorname{Re}(\partial_y a \bar{a})(t,y), \qquad \text{which gives } \frac{\mathrm{d}}{\mathrm{d}t}\|a\|^2_{L^2(\mathrm{d}y)} = 0.$$

In what follows, we use the notation $L^2(\mathrm{d}y)$ to denote the L^2 norm taken with respect to the y variable, to avoid confusion with the corresponding norm in the original x variable. Similarly, we use the notation $H^1(\mathrm{d}y)$ and so on. Now, taking a y-derivative and then integrating in y, we see that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_y a\|^2_{L^2(\mathrm{d}y)} \leq C_{f,\delta}|\lambda|(e^{-|\lambda| t} + t)\|a\|_{L^2(\mathrm{d}y)}\|\partial_y a\|_{L^2(\mathrm{d}y)},$$

where we have used

$$\Big| \partial_y \Big((-\alpha_1 + 2\beta_1)\frac{\operatorname{Im}(\bar{f}\,\partial_y f)}{|f|^2} + h \Big) \Big| \leq C_{f,\delta}(x(t,y) + t) \leq C_{f,\delta}(\exp(-|\lambda| t) + t) \tag{2-35}$$

on the support of $a(t,\cdot)$. This estimate follows from (2-34) and

$$\Big| \partial_y \frac{\operatorname{Im}(\bar{f}\,\partial_y f)}{|f|^2} \Big| \leq \Big| \partial_x \partial_y \frac{\operatorname{Im}(\bar{f}\,\partial_y f)}{|f|^2} \Big| x.$$

Therefore, by integrating in time, we obtain

$$\|\partial_y a(t)\|_{L^2(\mathrm{d}y)} \leq C_{f,\delta}\|a_0\|_{H^1(\mathrm{d}y)}$$

uniformly in λ, for (t,λ) satisfying $t \le |\lambda|^{-1/2}$. A similar argument applies to the estimate of $\partial_y^k a$, as long as $k \le s_0 - 2$; one can proceed by an induction in k, using the bound

$$\left|\partial_y^k\left((-\alpha_1 + 2\beta_1)\frac{\operatorname{Im}(\bar{f}\,\partial_y f)}{|f|^2} + h\right)\right| \le C_{f,\delta}(x(t,y)+t) \le C_{f,\delta}(\exp(-|\lambda|t)+t)$$

on the support of $a(t,\cdot)$. The estimate for $|\partial_y^k h|$ readily follows from the explicit decomposition $\partial_y h = h_1 + t h_2$, where h_1 and h_2 are $L_t^\infty C^{s_0-2,1}$-smooth functions defined by

$$h_1(t,x) = 2|f|^{-2}\operatorname{Re}(\overline{f_0(x)}(\partial_t f)(t,x)), \quad h_2(t,x) = 2|f|^{-2}\operatorname{Re}\left(\overline{(\partial_t f)(t,x)}\frac{1}{t}\int_0^t (\partial_t f)(t',x)\,\mathrm{d}t'\right).$$

Hence we conclude

$$\|a(t)\|_{H^k(\mathrm{d}y)} \le C_{f,\delta}\|a_0\|_{H^k(\mathrm{d}y)}, \quad 0 \le t \le \min\{|\lambda|^{-1/2}, \delta\}. \tag{2-36}$$

In what follows, we shall restrict the variable t to $[0, \min\{|\lambda|^{-1/2}, \delta\}]$.

Initial data and regularity estimates. At the initial time, from $a_0(y) = x_1^{1/2} g_0(x(0,y))$ we have that

$$\int |a_0(y)|^2\,\mathrm{d}y = \int x_1 |f_0(x)|^{-1} |g_0(x)|^2\,\mathrm{d}x,$$

and we note that the right-hand side is equivalent up to constants with $\|g_0\|_{L_x^2}^2$. This gives the claimed initial data estimate in the case $p = 2$, and the case of general p can be proved similarly. Next, with $\partial_y = |f_0(x)|\partial_x$ at the initial time, we note the bound

$$|\partial_y^k a_0(x)| \le C_k x_1^{1/2}\left(\sum_{j=1}^{k} \|f_0\|_{C^{k-2,1}}^{k-j} |f_0(x)|^j |\partial_x^j g_0(x)|\right),$$

which gives

$$\|a_0\|_{H^k(\mathrm{d}y)} \le C_k(1+\|f_0\|_{C^{k-2,1}})^{k-1}\|g_0\|_{H^k_{(x_1)}}, \quad k \le s_0 - 1. \tag{2-37}$$

Let us now check the regularity estimate (2-18) in the case $n = 0$: using $|f|^{-\sigma_c+1/2} = e^G$,

$$\begin{aligned}\||f|^{-\sigma_c}\tilde{\phi}_{(\lambda)}^{\mathrm{app}}(t,x)\|_{L^2}^2 &= \int_0^{x_1} |\tilde{\phi}_{(\lambda)}^{\mathrm{app}}(t,x)|^2 |f(t,x)|^{-2\sigma_c}\,\mathrm{d}x = \int_{-\infty}^0 |\tilde{\phi}_{(\lambda)}^{\mathrm{app}}(t,y)|^2 |f(t,y)|^{-2\sigma_c+1}\,\mathrm{d}y \\ &= \int_{-\infty}^0 |\varphi_{(\lambda)}^{\mathrm{app}}(t,y)|^2\,\mathrm{d}y \le C(1+|\lambda|^{-1})\int_{-\infty}^0 |\psi_{(\lambda)}^{\mathrm{app}}(t,y)|^2\,\mathrm{d}y \\ &\le C\|a(t)\|_{L^2(\mathrm{d}y)}^2 \le C\|a_0\|_{L^2(\mathrm{d}y)} \le C\|g_0\|_{L^2}.\end{aligned}$$

The cases $1 \le n \le s_0 - 2$ can be handled similarly, using (2-36) and (2-37).

Degeneration estimate. Next, we check the degeneration property (2-19). To simplify the notation, we introduce the notation

$$H = q\,O_k(a_0) \quad \Longleftrightarrow \quad \sup_{t\in[0,\min\{\lambda^{-1/2},\delta\}]} \left\| |f|^{1/2}\frac{H}{q}\right\|_{L^2(\mathrm{d}y)} \le C_{f,\delta}\|a_0\|_{H^k(\mathrm{d}y)}.$$

Note that $\| |f|^{1/2}(\,\cdot\,)\|_{L^2(\mathrm{d}y)} = \|\cdot\|_{L^2(\mathrm{d}x)}$ for each t. The terms that are abbreviated as $\frac{1}{\lambda}O_k(a_0)$ (for $k \le s$) will constitute $|f|^{-\sigma_c}\tilde{\phi}^{\mathrm{small}}_{(\lambda)}$; the desired estimate (2-21) would be an immediate consequence of the L^2 norm estimate embedded in the $O_k(\,\cdot\,)$ notation. Recalling the definitions of $\tilde{\phi}^{\mathrm{app}}_{(\lambda)}$, $\varphi^{\mathrm{app}}_{(\lambda)}$, and $\psi^{\mathrm{app}}_{(\lambda)}$, and arguing as in the proof of the regularity estimate, we have

$$|f|^{-\sigma_c+\gamma'}\tilde{\phi}^{\mathrm{app}}_{(\lambda)} = |f(t,y)|^{\gamma'-1/2}\psi^{\mathrm{app}}_{(\lambda)} + \frac{|f|^{\gamma'}}{\lambda}O_0(a_0).$$

For the first term, we have

$$\begin{aligned}
\| |f(t,x)|^{\gamma'-1/2}\psi^{\mathrm{app}}_{(\lambda)}(t,x)\|^p_{L^p} &\le C\int_{-\infty}^0 |\psi^{\mathrm{app}}_{(\lambda)}(t,y)|^p |f(t,y)|^{p\gamma'-p/2+1}\,\mathrm{d}y \\
&\le C\biggl(\int_{-\infty}^0 |\psi^{\mathrm{app}}_{(\lambda)}(t,y)|^2\,\mathrm{d}y\biggr)^{p/2}\biggl(\int_{\operatorname{supp}\psi^{\mathrm{app}}_{(\lambda)}(t,\cdot)} |f(t,y)|^{\frac{p}{1-p/2}\gamma'+1}\,\mathrm{d}y\biggr)^{1-p/2} \\
&\le C\|a_0\|^p_{L^2(\mathrm{d}y)}\biggl(\int_{\operatorname{supp}\psi^{\mathrm{app}}_{(\lambda)}(t,\cdot)} |f(t,y)|^{\frac{p}{1-p/2}\gamma'+1}\,\mathrm{d}y\biggr)^{p(1/p-1/2)},
\end{aligned}$$

so it remains to estimate the last factor. Note that, since $|f|^{-1}\partial_y|f| = \partial_x|f| = 1 + O(x)$ and $x \le x_1 e^{y/2}$ for $y \in (-\infty, 0)$, we have

$$|f(t,y)| \le Ce^y \quad \text{for } y \in (-\infty, 0).$$

Using the support property $\operatorname{supp}\psi^{\mathrm{app}}_{(\lambda)}(t,\cdot) \subseteq (-\infty, -2|\lambda|t)$, we see that

$$\biggl(\int_{\operatorname{supp}\psi^{\mathrm{app}}_{(\lambda)}(t,\cdot)} |f(t,y)|^{\frac{p}{1-p/2}\gamma'+1}\,\mathrm{d}y\biggr)^{1/p-1/2} \lesssim \exp\bigl(-2|\lambda|\bigl(\gamma' + \tfrac{1}{p} - \tfrac{1}{2}\bigr)t\bigr).$$

Hence the desired estimate (2-19) in the case $s = 0$ now follows.

To treat the cases $s > 0$, we begin by recalling that $\psi^{\mathrm{app}}_{(\lambda)} = \psi^{\mathrm{app}} = \exp\bigl(i\lambda(y - \lambda t + S(t,y))\bigr)a_0(y - 2\lambda t)$. Note the identity

$$\exp(i\lambda(y - \lambda t + S)) = \frac{|f|}{i\lambda(1+\partial_y S)}\biggl(\frac{1}{|f|}\partial_y\biggr)\exp(i\lambda(y - \lambda t + S)).$$

For the expression $\partial_y S$ in the denominator, recalling (2-33) and (2-35), we have

$$|\partial_y S| \le C_{f,\delta}tx,$$

and in particular we note that $1 + \partial_y S \ge \frac{1}{2}$ when t is sufficiently small, which can be arranged by taking $\delta > 0$ smaller. Commuting $\frac{1}{|f|}\partial_y$ (which equals ∂_x in the (t,x)-coordinates) outside, we have

$$|f(t,y)|^{\gamma'-1/2}\psi^{\mathrm{app}}_{(\lambda)} = \frac{1}{|f|}\partial_y\biggl(\frac{|f(t,y)|^{\gamma'+1-1/2}}{i\lambda(1+\partial_y S)}\psi^{\mathrm{app}}_{(\lambda)}\biggr) + \frac{|f|^{\gamma'}}{\lambda}O_1(a_0).$$

By arguing as in the case of $s = 0$, the expression inside the parentheses can be shown to obey the degeneration bound (2-20). The cases $s > 1$ are handled similarly.

Error estimate. To begin with, at the level of $\psi^{\mathrm{app}}_{(\lambda)}$, the point of choosing $a(t, y)$ as the solution of (2-29) is to have

$$i\partial_t\psi^{\mathrm{app}}_{(\lambda)} + \partial_{yy}\psi^{\mathrm{app}}_{(\lambda)} + \left((-\alpha_1+2\beta_1)\frac{\mathrm{Im}(\bar f\,\partial_y f)}{|f|^2}+h\right)i\partial_y\psi^{\mathrm{app}}_{(\lambda)} = O_2(a_0), \tag{2-38}$$

which can be checked with a direct computation using (2-27). We now see that $\varphi^{\mathrm{app}}_{(\lambda)}$ is an approximate solution to (2-26), which is motivated by the following heuristics: recalling (2-28), we have

$$\varphi^{\mathrm{app}}_{(\lambda)} \simeq \psi^{\mathrm{app}}_{(\lambda)} - \frac{\alpha_1}{2}\frac{f\,\partial_y f}{|f|^2}\partial_y^{-1}\overline{\psi^{\mathrm{app}}_{(\lambda)}} \simeq \psi^{\mathrm{app}}_{(\lambda)} + \frac{\alpha_1}{2i\lambda}\frac{f\,\partial_y f}{|f|^2}\overline{\psi^{\mathrm{app}}_{(\lambda)}}.$$

To this end, (2-38) gives

$$-i\overline{\partial_t\psi^{\mathrm{app}}_{(\lambda)}} + \overline{\partial_{yy}\psi^{\mathrm{app}}_{(\lambda)}} - \left((-\alpha_1+2\beta_1)\frac{\mathrm{Im}(\bar f\,\partial_y f)}{|f|^2}+h\right)i\overline{\partial_y\psi^{\mathrm{app}}_{(\lambda)}} = O_2(a_0)$$

and from this it is not difficult to see that

$$\frac{1}{2i\lambda}[i\partial_t+\partial_{yy}]\overline{\psi^{\mathrm{app}}_{(\lambda)}} = \frac{1}{i\lambda}\partial_{yy}\overline{\psi^{\mathrm{app}}_{(\lambda)}} + O_2(a_0) = -\partial_y\overline{\psi^{\mathrm{app}}_{(\lambda)}} + O_2(a_0), \tag{2-39}$$

so that

$$(i\partial_t+\partial_{yy})\left(\frac{\alpha_1}{2i\lambda}\frac{f\,\partial_y f}{|f|^2}\overline{\psi^{\mathrm{app}}_{(\lambda)}}\right) = -\alpha_1\frac{f\,\partial_y f}{|f|^2}\partial_y\overline{\psi^{\mathrm{app}}_{(\lambda)}} + O_2(a_0).$$

Using (2-38), (2-39), and

$$\left((-\alpha_1+2\beta_1)\frac{\mathrm{Im}(\bar f\,\partial_y f)}{|f|^2}+h\right)i\partial_y\left(\frac{\alpha_1}{2i\lambda}\frac{f\,\partial_y f}{|f|^2}\overline{\psi^{\mathrm{app}}_{(\lambda)}}\right) = O_2(a_0), \quad \frac{\alpha_1^2 f\,\partial_y f}{2i\lambda|f|^2}\partial_y\left(\overline{\frac{f\,\partial_y f}{|f|^2}}\psi^{\mathrm{app}}_{(\lambda)}\right) = O_2(a_0),$$

we simplify

$$\begin{aligned}
&\left[i\partial_t+\partial_{yy}+\left((-\alpha_1+2\beta_1)\frac{\mathrm{Im}(\bar f\,\partial_y f)}{|f|^2}+h\right)i\partial_y\right]\varphi^{\mathrm{app}}_{(\lambda)} + \alpha_1\frac{f\,\partial_y f}{|f|^2}\overline{\partial_y\varphi^{\mathrm{app}}_{(\lambda)}}\\
&\quad= \left[i\partial_t+\partial_{yy}+\left((-\alpha_1+2\beta_1)\frac{\mathrm{Im}(\bar f\,\partial_y f)}{|f|^2}+h\right)i\partial_y\right]\left(\psi^{\mathrm{app}}_{(\lambda)} + \frac{\alpha_1}{2i\lambda}\frac{f\,\partial_y f}{|f|^2}\overline{\psi^{\mathrm{app}}_{(\lambda)}}\right)\\
&\qquad\qquad + \alpha_1\frac{f\,\partial_y f}{|f|^2}\overline{\partial_y\psi^{\mathrm{app}}_{(\lambda)}} - \frac{\alpha_1^2 f\,\partial_y f}{2i\lambda|f|^2}\partial_y\left(\overline{\frac{f\,\partial_y f}{|f|^2}}\psi^{\mathrm{app}}_{(\lambda)}\right)\\
&\quad= -\alpha_1\frac{f\,\partial_y f}{|f|^2}\partial_y\overline{\psi^{\mathrm{app}}_{(\lambda)}} + \alpha_1\frac{f\,\partial_y f}{|f|^2}\partial_y\overline{\psi^{\mathrm{app}}_{(\lambda)}} + O_2(a_0) = O_2(a_0).
\end{aligned}$$

Moreover, it is easy to see that $\mathcal{B}_0[\varphi^{\mathrm{app}}_{(\lambda)}] = O_2(a_0)$, and finally the error estimate (2-22) follows from

$$\left[i\partial_t+\partial_{yy}+\left((-\alpha_1+2\beta_1)\frac{\mathrm{Im}(\bar f\,\partial_y f)}{|f|^2}+h\right)i\partial_y\right]\varphi^{\mathrm{app}}_{(\lambda)} + \alpha_1\frac{f\,\partial_y f}{|f|^2}\overline{\partial_y\varphi^{\mathrm{app}}_{(\lambda)}} - \mathcal{B}_0[\varphi^{\mathrm{app}}_{(\lambda)}] = O_2(a_0)$$

and (2-37). This completes the proof of Proposition 2.7. ☐

2.4. ***Modified and generalized energy estimates.*** In Sections 2.4.1 and 2.4.2, we establish the modified and generalized (or bilinear) energy estimates that we shall need in the proofs of Theorems 1.1 and 1.2, respectively.

2.4.1. *Modified energy estimate.* Assume that f and $\phi = f + \tilde{\phi}$ are solutions to (1-1) on some time interval. Then, recall that $\tilde{\phi}$ solves

$$i\partial_t\tilde{\phi} + |f|^2\,\partial_{xx}\tilde{\phi} + \alpha_1 f(\overline{\partial_x f}\,\partial_x\tilde{\phi} + \partial_x f\,\partial_x\bar{\tilde{\phi}}) + 2\beta_1\bar{f}\,\partial_x f\,\partial_x\tilde{\phi} + V_f\tilde{\phi} + W_f\bar{\tilde{\phi}} = Q_f[\tilde{\phi}], \tag{2-40}$$

where V_f, W_f and $Q_f[\,\cdot\,]$ are defined in (2-15). To deal with solutions of (2-40), it turns out that the following time-dependent Hermitian product and norm are very natural (which will be referred to as the modified energy): given some f, we define

$$\langle v, u\rangle_{L^2_f}(t) := \int |f(t,\cdot)|^{-2\sigma_c} v(t,\cdot)\overline{u(t,\cdot)}\,\mathrm{d}x, \qquad \|v\|^2_{L^2_f}(t) := \int |f(t,\cdot)|^{-2\sigma_c}|v(t,\cdot)|^2\,\mathrm{d}x.$$

Regarding this modified energy, we have the following estimate.

Proposition 2.8. *Let $f \in L^\infty([0,\delta]; C^{s_c-1,1}))$ be a solution to (1-5) and $\tilde{\phi} \in L^\infty([0,\delta']; C^{s_c-1,1}))$ be a solution to (2-40) for some $0 < \delta' \le \delta$. When $\sigma_c \ge \frac{1}{2}$, assume furthermore that at every zero a of f_0, we have $\partial_x f_0(a) \neq 0$ and $\tilde{\phi}_0(x)$ vanishes up to order $\lfloor \sigma_c - \frac{1}{2} \rfloor$ at a. Then, on $t \in [0,\delta']$, we have*

$$\|\tilde{\phi}\|_{L^2_f}(t) \le \|\tilde{\phi}\|_{L^2_f}(0)\exp(C(\|f\|^2_{L^\infty_t C^{1,1}} + \|\tilde{\phi}\|^2_{L^\infty_t C^{1,1}})t), \tag{2-41}$$

where σ_c is as in (1-6) and $C > 0$ is an absolute constant.

Proof. We first present a formal computation without worrying about the finiteness of the modified energy and the validity of integration by parts, and discuss its justification below. We begin with

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{\phi}\|^2_{L^2_f}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int |\tilde{\phi}|^2|f|^{-2\sigma_c}\,\mathrm{d}x = \int |\tilde{\phi}|^2\,\partial_t(|f|^{-2\sigma_c})\,\mathrm{d}x + \int \partial_t(|\tilde{\phi}|^2)|f|^{-2\sigma_c}\,\mathrm{d}x.$$

The term involving $\partial_t(|f|^{-2\sigma_c})$ in the right-hand side can be bounded using the pointwise inequality

$$\left|\frac{\partial_t|f|}{|f|}\right| \lesssim \|f\|^2_{L^\infty_t C^{1,1}}.$$

To handle the second term, we write

$$\begin{aligned}\partial_t|\tilde{\phi}|^2 = \operatorname{Re}(i|f|^2\,\partial_{xx}\tilde{\phi}\bar{\tilde{\phi}}) + \alpha_1\operatorname{Re}(if(\partial_x\bar{f}\,\partial_x\tilde{\phi} + \partial_x f\,\partial_x\bar{\tilde{\phi}})\bar{\tilde{\phi}}) + 2\beta_1\operatorname{Re}(i\bar{f}\,\partial_x f\,\partial_x\tilde{\phi}\bar{\tilde{\phi}})\\ + \operatorname{Re}(iV_f\tilde{\phi}\bar{\tilde{\phi}}) + \operatorname{Re}(iW_f(\bar{\tilde{\phi}})^2) - \operatorname{Re}(iQ_f[\tilde{\phi}]\bar{\tilde{\phi}}).\end{aligned} \tag{2-42}$$

We multiply both sides by $|f|^{-2\sigma_c}$ and integrate in x. From the first term on the right-hand side, we obtain, after an integration by parts,

$$\begin{aligned}\int \operatorname{Re}(i|f|^2\,\partial_{xx}\tilde{\phi}\bar{\tilde{\phi}})|f|^{-2\sigma_c}\,\mathrm{d}x &= -\int |f|^{2-2\sigma_c}\operatorname{Re}(i\partial_x\tilde{\phi}\,\partial_x\bar{\tilde{\phi}})\,\mathrm{d}x - (2-2\sigma_c)\int |f|^{1-2\sigma_c}\,\partial_x|f|\operatorname{Re}(i\partial_x\tilde{\phi}\bar{\tilde{\phi}})\\ &= -(2-2\sigma_c)\int |f|^{-2\sigma_c}\operatorname{Re}(\bar{f}\,\partial_x f)\operatorname{Re}(i\partial_x\tilde{\phi}\bar{\tilde{\phi}}).\end{aligned} \tag{2-43}$$

From the second and third terms on the right-hand side of (2-42), we have

$$\begin{aligned}
&\int [\alpha_1 \operatorname{Re}(if(\partial_x \bar f\, \partial_x \tilde\phi + \partial_x f\, \partial_x \bar{\tilde\phi})\bar{\tilde\phi}) + 2\beta_1 \operatorname{Re}(i \bar f\, \partial_x f\, \partial_x \tilde\phi \bar{\tilde\phi})]|f|^{-2\sigma_c}\, \mathrm{d}x \\
&= (\alpha_1 + 2\beta_1) \int |f|^{-2\sigma_c} \operatorname{Re}(\bar f\, \partial_x f) \operatorname{Re}(i \partial_x \tilde\phi \bar{\tilde\phi})\, \mathrm{d}x + \frac{\alpha_1 - 2\beta_1}{2} \int |f|^{-2\sigma_c} \operatorname{Im}(\bar f\, \partial_x f) \partial_x |\tilde\phi|^2\, \mathrm{d}x \\
&\qquad + \frac{\alpha_1}{2} \int |f|^{-2\sigma_c} \operatorname{Re}(if\, \partial_x f\, \partial_x (\bar{\tilde\phi})^2)\, \mathrm{d}x \\
&= (\alpha_1 + 2\beta_1) \int |f|^{-2\sigma_c} \operatorname{Re}(\bar f\, \partial_x f) \operatorname{Re}(i \partial_x \tilde\phi \bar{\tilde\phi})\, \mathrm{d}x - \frac{\alpha_1 - 2\beta_1}{2} \int \big(\partial_x (|f|^{-2\sigma_c} \operatorname{Im}(\bar f\, \partial_x f))\big) |\tilde\phi|^2\, \mathrm{d}x \\
&\qquad - \frac{\alpha_1}{4} \int (i \partial_x (|f|^{-2\sigma_c} f\, \partial_x f))(\bar{\tilde\phi})^2\, \mathrm{d}x + \frac{\alpha_1}{4} \int (i \partial_x (|f|^{-2\sigma_c} \overline{f\, \partial_x f}))(\tilde\phi)^2\, \mathrm{d}x.
\end{aligned}$$

By our choice of σ_c in (1-6), the first term on the right-hand side cancels exactly with (2-43). The remaining terms are estimated from the above by $C\|f\|^2_{L^\infty_t C^{1,1}} \|\tilde\phi\|^2_{L^2_f}$. Next, it is easy to see that

$$\left| \int \operatorname{Re}(i V_f \tilde\phi \bar{\tilde\phi}) |f|^{-2\sigma_c}\, \mathrm{d}x \right| + \left| \int \operatorname{Re}(i W_f \bar{\tilde\phi} \bar{\tilde\phi}) |f|^{-2\sigma_c}\, \mathrm{d}x \right| \lesssim \|f\|^2_{L^\infty_t C^{1,1}} \|\tilde\phi\|^2_{L^2_f}.$$

It remains to estimate $\int \operatorname{Re}(i Q_f[\tilde\phi] \bar{\tilde\phi}) |f|^{-2\sigma_c}\, \mathrm{d}x$. The contribution of any term with at least one factor of $\tilde\phi$ (without any derivatives) may be easily estimated by $(\|f\|^2_{C^{1,1}} + \|\tilde\phi\|^2_{C^{1,1}}) \int |\tilde\phi|^2 |f|^{-2\sigma_c}\, \mathrm{d}x$. Recalling the expression for Q_f from (2-15), we may estimate

$$\left| \int \operatorname{Re}(i Q_f[\tilde\phi] \bar{\tilde\phi}) |f|^{-2\sigma_c}\, \mathrm{d}x \right| \lesssim (\|f\|_{C^{1,1}} \|\tilde\phi\|_{C^{1,1}} + \|\tilde\phi\|^2_{C^{1,1}}) \|\tilde\phi\|^2_{L^2_f} + \left(\int |\partial_x \tilde\phi|^4 |f|^{2-2\sigma_c} \right)^{1/2} \|\tilde\phi\|_{L^2_f}.$$

Integrating by parts and using Hölder's inequality, we have

$$\begin{aligned}
\int (\partial_x \tilde\phi)^4 |f|^{2-2\sigma_c} &= \int \tilde\phi (\partial_x \tilde\phi)^2 (-3 \partial_{xx} \tilde\phi |f| - (2 - 2\sigma_c) \partial_x \tilde\phi \partial_x |f|) |f| |f|^{-2\sigma_c}\, \mathrm{d}x \\
&\le C \|\tilde\phi\|_{C^{1,1}} \|f\|_{C^{0,1}} \|\tilde\phi\|_{L^2_f} \left(\int (\partial_x \tilde\phi)^4 |f|^{2-2\sigma_c}\, \mathrm{d}x \right)^{1/2}.
\end{aligned}$$

Hence

$$\left| \int \operatorname{Re}(i Q_f[\tilde\phi] \bar{\tilde\phi}) |f|^{-2\sigma_c}\, \mathrm{d}x \right| \lesssim (\|f\|_{C^{1,1}} \|\tilde\phi\|_{C^{1,1}} + \|\tilde\phi\|^2_{C^{1,1}}) \|\tilde\phi\|^2_{L^2_f}.$$

Collecting all the terms, we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\tilde\phi\|^2_{L^2_f} \lesssim (\|f\|^2_{L^\infty_t C^{1,1}} + \|\tilde\phi\|^2_{L^\infty_t C^{1,1}}) \|\tilde\phi\|^2_{L^2_f}.$$

Integrating in time gives the desired conclusion.

We now sketch the observations needed to make the above computation rigorous. Note that, in order for (2-41) to be nontrivial, the right-hand side must be finite, i.e., $\|\tilde\phi\|_{L^2_f}(t=0) = \||f_0|^{-\sigma_c} \tilde\phi_0\|_{L^2} < +\infty$. When $\sigma_c \ge \frac{1}{2}$, this implies the vanishing of $\tilde\phi_0$ at each zero a of f (which is isolated by the assumption in this case) up to order $\lfloor \sigma_c - \frac{1}{2} \rfloor$. Applying Lemma 2.5 to the $L^\infty_t([0,\delta]; C^{s_c-1,1}))$ solutions f and $f + \tilde\phi$, it follows that the zero set of $f(t,x)$, as well as the nonvanishing of $f'(t,a)$ and the vanishing of $\tilde\phi(t,x)$

up to order $\lfloor \sigma_c - \frac{1}{2} \rfloor$ at each zero a of f, is preserved in $t \in [0, \delta]$. As a consequence, $\|\tilde{\phi}\|_{L^2_f} < +\infty$ for every $t \in [0, \delta]$ as well. Using the vanishing properties of f and $\tilde{\phi}$ (the latter is needed only when $\sigma_c \geq \frac{1}{2}$), the above computation can then be justified. □

2.4.2. *Generalized (bilinear) energy estimate.* We proceed to prove the generalized energy estimate.

Proposition 2.9. *Let $\tilde{\phi}$ be a solution of*

$$[i\partial_t + \mathcal{L}_f]\tilde{\phi} = Q_f[\tilde{\phi}],$$

where $[i\partial_t + \mathcal{L}_f]\tilde{\phi}$ denotes the left-hand side of (2-14), *and let $\tilde{\phi}^{\mathrm{app}} = \tilde{\phi}^{\mathrm{app}}[g_0, f]$ be the degenerating wave packet constructed in Proposition 2.7. Then, we have the following estimate on $t \in [0, \min\{|\lambda|^{-1/2}, \delta\}]$:*

$$\left|\frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Re}\langle\tilde{\phi}, \tilde{\phi}^{\mathrm{app}}\rangle_{L^2_f}\right| \leq (C(\|f\|^2_{L^\infty_t C^{1,1}_x} + \|\tilde{\phi}\|^2_{L^\infty_t C^{1,1}_x})\|\tilde{\phi}^{\mathrm{app}}\|_{L^2_f} + C_{f,\delta} A^{-\sigma_c+3}\|g_0\|_{H^2_{(x_1)}})\|\tilde{\phi}\|_{L^2_f}. \tag{2-44}$$

Proof. In the proof, the time variable t will be restricted to the interval $[0, \min\{|\lambda|^{-1/2}, \delta\}]$. Before we proceed, let us recall that $\mathcal{L}_f$ is given by

$$\begin{aligned}\mathcal{L}_f[\tilde{\phi}] &:= |f|^2\partial_{xx}\tilde{\phi} + \alpha_1 f(\overline{\partial_x f}\partial_x\tilde{\phi} + \partial_x f \partial_x\bar{\tilde{\phi}}) + 2\beta_1 \bar{f}\partial_x f \partial_x \tilde{\phi} + V_f\tilde{\phi} + W_f\bar{\tilde{\phi}} \\ &= |f|^2\partial_{xx}\tilde{\phi} + (\alpha_1 + 2\beta_1)\operatorname{Re}(\bar{f}\partial_x f)\partial_x\tilde{\phi} + (-\alpha_1 + 2\beta_1) i \operatorname{Im}(\bar{f}\partial_x f)\partial_x\tilde{\phi} + \alpha_1 f \partial_x f \partial_x\bar{\tilde{\phi}} + V_f\tilde{\phi} + W_f\bar{\tilde{\phi}}\end{aligned}$$

and that $\tilde{\phi}^{\mathrm{app}}$ satisfies $[i\partial_t + \mathcal{L}_f]\tilde{\phi}^{\mathrm{app}} = \boldsymbol{\epsilon}_{\tilde{\phi}}$. We compute[6]

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Re}\langle\tilde{\phi}, \tilde{\phi}^{\mathrm{app}}\rangle_{L^2_f} = \operatorname{Re}\biggl(\int -2\sigma_c|f|^{-2\sigma_c-1}\partial_t|f|\tilde{\phi}\overline{\tilde{\phi}^{\mathrm{app}}} \\ + \int i|f|^{-2\sigma_c}(\mathcal{L}_f[\tilde{\phi}] - Q_f[\tilde{\phi}])\overline{\tilde{\phi}^{\mathrm{app}}} - \int i|f|^{-2\sigma_c}\tilde{\phi}(\overline{\mathcal{L}_f[\tilde{\phi}^{\mathrm{app}}] - \boldsymbol{\epsilon}_{\tilde{\phi}}})\biggr).\end{aligned}$$

Using the estimates for $|\partial_t|f||$, $Q_f[\tilde{\phi}]$, and $\boldsymbol{\epsilon}_{\tilde{\phi}}$, we can bound

$$\left|\operatorname{Re}\int |f|^{-2\sigma_c-1}\partial_t|f|\tilde{\phi}\overline{\tilde{\phi}^{\mathrm{app}}}\right| \lesssim \|f\|^2_{L^\infty_t C^{1,1}}\|\tilde{\phi}\|_{L^2_f}\|\tilde{\phi}^{\mathrm{app}}\|_{L^2_f},$$

$$\left|\operatorname{Re}\int i|f|^{-2\sigma_c} Q_f[\tilde{\phi}]\overline{\tilde{\phi}^{\mathrm{app}}}\right| \lesssim (\|f\|^2_{L^\infty_t C^{1,1}} + \|\tilde{\phi}\|^2_{L^\infty_t C^{1,1}})\|\tilde{\phi}\|_{L^2_f}\|\tilde{\phi}^{\mathrm{app}}\|_{L^2_f}$$

and

$$\left|\operatorname{Re}\int i|f|^{-2\sigma_c}\tilde{\phi}\overline{\boldsymbol{\epsilon}_{\tilde{\phi}}}\right| \leq C_{f,\delta}\|\tilde{\phi}^{\mathrm{app}}\|_{L^2_f} A^{-\sigma_c+3}\|g_0\|_{H^2_{(x_1)}}.$$

We now consider the remaining expression

$$\int |f|^{-2\sigma_c}\operatorname{Re}(i\mathcal{L}_f[\tilde{\phi}]\overline{\tilde{\phi}^{\mathrm{app}}})\,\mathrm{d}x - \int |f|^{-2\sigma_c}\operatorname{Re}(i\tilde{\phi}\overline{\mathcal{L}_f[\tilde{\phi}^{\mathrm{app}}]})\,\mathrm{d}x.$$

[6] Here, since $\tilde{\phi}^{\mathrm{app}}$ is smooth and compactly supported away from the zeroes of f at each t, there are no issues whatsoever in justifying the computation that follows.

For the contribution of the principal term $|f|^2\,\partial_{xx}$, we obtain

$$\begin{aligned}
&\int |f|^{-2\sigma_c}\operatorname{Re}(i|f|^2\,\partial_{xx}\tilde{\phi}\overline{\tilde{\phi}^{\mathrm{app}}})\,\mathrm{d}x-\int |f|^{-2\sigma_c}\operatorname{Re}(i\tilde{\phi}\overline{|f|^2\,\partial_{xx}\tilde{\phi}^{\mathrm{app}}})\,\mathrm{d}x\\
&\quad=-\int |f|^{2-2\sigma_c}\operatorname{Re}(i\partial_x\phi\,\overline{\partial_x\tilde{\phi}^{\mathrm{app}}})\,\mathrm{d}x-\int(2-2\sigma_c)|f|^{1-2\sigma_c}\,\partial_x|f|\operatorname{Re}(i\partial_x\phi\overline{\tilde{\phi}^{\mathrm{app}}})\,\mathrm{d}x\\
&\qquad\qquad+\int |f|^{2-2\sigma_c}\operatorname{Re}(i\partial_x\phi\,\overline{\partial_x\tilde{\phi}^{\mathrm{app}}})\,\mathrm{d}x+\int(2-2\sigma_c)|f|^{1-2\sigma_c}\,\partial_x|f|\operatorname{Re}(i\phi\,\overline{\partial_x\tilde{\phi}^{\mathrm{app}}})\,\mathrm{d}x\\
&\quad=-\int(2-2\sigma_c)|f|^{1-2\sigma_c}\,\partial_x|f|(\operatorname{Re}(i\partial_x\phi\overline{\tilde{\phi}^{\mathrm{app}}})-\operatorname{Re}(i\phi\,\overline{\partial_x\tilde{\phi}^{\mathrm{app}}}))\,\mathrm{d}x=:\mathrm{I}.
\end{aligned}$$

Since $|f|\,\partial_x|f|=\operatorname{Re}(\bar{f}\,\partial_x f)$, this term cancels with some of the first-order terms, i.e.,

$$\begin{aligned}
&\int(\alpha_1+2\beta_1)|f|^{-2\sigma_c}\operatorname{Re}(\bar{f}\,\partial_x f)\operatorname{Re}(i\partial_x\tilde{\phi}\overline{\tilde{\phi}^{\mathrm{app}}})\,\mathrm{d}x\\
&\qquad\qquad-\int(\alpha_1+2\beta_1)|f|^{-2\sigma_c}\operatorname{Re}(\bar{f}\,\partial_x f)\operatorname{Re}(i\tilde{\phi}\,\overline{\partial_x\tilde{\phi}^{\mathrm{app}}})\,\mathrm{d}x=-\mathrm{I}.
\end{aligned}$$

For the remaining first-order terms, we have, after integrating by parts,

$$\begin{aligned}
&-\int(-\alpha_1+2\beta_1)|f|^{-2\sigma_c}\operatorname{Im}(\bar{f}\,\partial_x f)\operatorname{Re}(\partial_x\tilde{\phi}\overline{\tilde{\phi}^{\mathrm{app}}})\,\mathrm{d}x-\int(-\alpha_1+2\beta_1)|f|^{-2\sigma_c}\operatorname{Im}(\bar{f}\,\partial_x f)\operatorname{Re}(\tilde{\phi}\,\overline{\partial_x\tilde{\phi}^{\mathrm{app}}})\,\mathrm{d}x\\
&\qquad\qquad=(-\alpha_1+2\beta_1)\int(\partial_x(|f|^{-2\sigma_c}\operatorname{Im}(\bar{f}\,\partial_x f)))\operatorname{Re}(\tilde{\phi}\overline{\tilde{\phi}^{\mathrm{app}}})\,\mathrm{d}x,
\end{aligned}$$

$$\begin{aligned}
&\int\alpha_1|f|^{-2\sigma_c}\operatorname{Re}(if\partial_x f\partial_x\bar{\tilde{\phi}}\overline{\tilde{\phi}^{\mathrm{app}}})\,\mathrm{d}x-\int\alpha_1|f|^{-2\sigma_c}\operatorname{Re}(i\phi\overline{f\partial_x f\partial_x\bar{\tilde{\phi}}^{\mathrm{app}}})\,\mathrm{d}x\\
&=\frac{\alpha_1}{2}\int|f|^{-2\sigma_c}(if\partial_x f\partial_x\bar{\tilde{\phi}}\overline{\tilde{\phi}^{\mathrm{app}}}-i\overline{f\partial_x f}\partial_x\tilde{\phi}\tilde{\phi}^{\mathrm{app}})\,\mathrm{d}x-\frac{\alpha_1}{2}\int|f|^{-2\sigma_c}(i\overline{f\partial_x f}\phi\partial_x\tilde{\phi}^{\mathrm{app}}-if\partial_x f\bar{\phi}\partial_x\overline{\tilde{\phi}^{\mathrm{app}}})\,\mathrm{d}x\\
&=-\frac{\alpha_1}{2}\int(\partial_x(i|f|^{-2\sigma_c}f\partial_x f))\bar{\tilde{\phi}}\overline{\tilde{\phi}^{\mathrm{app}}}\,\mathrm{d}x+\frac{\alpha_1}{2}\int(\partial_x(i|f|^{-2\sigma_c}\overline{f\partial_x f}))\tilde{\phi}\tilde{\phi}^{\mathrm{app}}\,\mathrm{d}x.
\end{aligned}$$

Both expressions may be bounded from above by $C\|f\|^2_{C^{1,1}}\|\tilde{\phi}\|_{L^2_f}\|\tilde{\phi}^{\mathrm{app}}\|_{L^2_f}$. Finally, for the zeroth-order terms, we easily have

$$\begin{aligned}
&\left|\int|f|^{-2\sigma_c}\operatorname{Re}(i(V_f\tilde{\phi}+W_f\bar{\tilde{\phi}})\overline{\tilde{\phi}^{\mathrm{app}}})\,\mathrm{d}x-\int|f|^{-2\sigma_c}\operatorname{Re}(i\tilde{\phi}\overline{(V_f\tilde{\phi}^{\mathrm{app}}+W_f\overline{\tilde{\phi}^{\mathrm{app}}})})\,\mathrm{d}x\right|\\
&\qquad\qquad\lesssim\|f\|^2_{C^{1,1}}\|\tilde{\phi}\|_{L^2_f}\|\tilde{\phi}^{\mathrm{app}}\|_{L^2_f}.
\end{aligned}$$

This gives (2-44), which concludes the proposition. □

2.5. ***Proof of Theorem 1.1.*** We are now in a position to conclude the proof of Theorem 1.1 for equation (1-5). To begin with, let f satisfy the assumptions of the theorem with $f_0=f(t=0)$. We may assume that $f_0(0)=0$ and $f_0'(0)=:A>0$ by translation and phase rotation if necessary. We also fix x_1 as in Proposition 2.7.

Now let $\epsilon > 0$, $s_0 \geq s_c$ and $0 < \delta' \leq \delta$ be given. We take some $\lambda \leq -1$ and g_0 satisfying the assumptions of Proposition 2.7, and define $\tilde{\phi}_0$ by

$$\tilde{\phi}_0 = \epsilon c(s_0)|\lambda|^{-s_0}\tilde{\phi}^{\mathrm{app}}_{(\lambda)}(t=0)[g_0;\, f].$$

Here, $\tilde{\phi}^{\mathrm{app}}_{(\lambda)} = \tilde{\phi}^{\mathrm{app}}_{(\lambda)}[g_0;\, f]$ is the degenerating wave packet constructed in Proposition 2.7 using g_0. It is not difficult to check that $\tilde{\phi}^{\mathrm{app}}_{(\lambda)}(t=0) \in C^\infty_c$ and $\|\tilde{\phi}^{\mathrm{app}}_{(\lambda)}(t=0)\|_{C^{s_0}} \lesssim_{s_0} |\lambda|^{s_0}$. Hence, by taking a sufficiently small $c(s_0) > 0$, we can ensure that $\|\tilde{\phi}_0\|_{C^{s_0}} \leq \epsilon$ uniformly for all $\lambda \leq -1$, as required by the statement of the theorem. We observe that

$$\mathrm{Re}\langle \tilde{\phi}_0, \tilde{\phi}^{\mathrm{app}}_{(\lambda)}(t=0)\rangle_{L^2_f} \geq c_0\|\tilde{\phi}_0\|_{L^2_f}\|\tilde{\phi}^{\mathrm{app}}(t=0)\|_{L^2_f} \tag{2-45}$$

for some $c_0 > 0$ independent of λ. To proceed, let us assume that the first option in the theorem does not hold; namely, there exists a solution ϕ to (1-5) satisfying $\|\phi - f\|_{L^\infty([0,\delta'];C^{s_c})} < +\infty$ and $\phi(t=0) = f_0 + \tilde{\phi}_0$. On $[0, \delta']$, we write $\tilde{\phi} = \phi - f$ and set

$$M_2 = \sup_{t\in[0,\delta']}(\|f(t)\|_{C^{1,1}} + \|\tilde{\phi}(t)\|_{C^{1,1}}).$$

We shall now establish the claimed norm inflation statement for $\tilde{\phi}$ by taking $|\lambda|$ sufficiently large but in a way depending only on f and δ'.

On the time interval $[0, \delta']$, using Proposition 2.8 and (2-18) we obtain that

$$\|\tilde{\phi}(t)\|_{L^2_f} \leq \exp(CM_2^2 t)\|\tilde{\phi}_0\|_{L^2_f}, \quad \|\tilde{\phi}^{\mathrm{app}}(t)\|_{L^2_f} \leq C_{f,\delta}A^{-\sigma_c+1}\|g_0\|_{L^2}.$$

In particular, we note that $\|\tilde{\phi}_0\|_{L^2_f} < +\infty$ since $\tilde{\phi}_0$ is supported away from the zeroes of f, and as discussed in Section 2.4.1, $\tilde{\phi}(t)$ vanishes sufficiently fast at the zeroes of f (ultimately due to Lemma 2.5) so that $\|\tilde{\phi}(t)\|_{L^2_f}$ is well-defined and obeys the above bound. Applying (2-44), integrating in time on the interval $[0, \min\{\delta', cM_2^{-2}, A^{-2}|\lambda|^{-1/2}\}]$ for a sufficiently small $c > 0$ and using (2-45), we have

$$\mathrm{Re}\langle \tilde{\phi}(t), \tilde{\phi}^{\mathrm{app}}(t)\rangle_{L^2_f} \geq \tfrac{1}{2}c_0\|\tilde{\phi}_0\|_{L^2_f}\|g_0\|_{L^2} \quad \text{for } |t| \leq \min\{\delta', cM_2^{-2}, A^{-2}|\lambda|^{-1/2}\}. \tag{2-46}$$

Next, applying (2-19)–(2-21) with $\gamma' = -\sigma_c$ and $s = s_c$, we have

$$\begin{aligned}
\mathrm{Re}\langle \tilde{\phi}(t), \tilde{\phi}^{\mathrm{app}}(t)\rangle_{L^2_f} &- C_{f,\delta}A^{-\sigma_c+1}|\lambda|^{-1}\|\tilde{\phi}_0\|_{L^2_f}\|g_0\|_{H^{|s_c|}_{(x_1)}} \\
&\leq \mathrm{Re}\left\langle \tilde{\phi}(t), \partial_x^{s_c}\left(\frac{|f|^{-\sigma_c+s_c-1/2}}{i^{s_c}\lambda^{s_c}(1+|f|\partial_x S)^{s_c}}\psi^{\mathrm{app}}(t)\right)\right\rangle \\
&\leq \|\partial_x^{s_c}\tilde{\phi}(t)\|_{L^\infty}|\lambda|^{-s_c}\|(1+|f|\partial_x S)^{-s_c}|f|^{s_c-\sigma_c-1/2}\psi^{\mathrm{app}}(t)\|_{L^1} \\
&\leq C_{f,\delta}^{1-\sigma_c}A^{s_c-\sigma_c+1/2}|\lambda|^{-s_c}\exp\bigl(-2|\lambda|\bigl(s_c-\sigma_c+\tfrac{1}{2}\bigr)A^2 t\bigr)\|\partial_x^{s_c}\tilde{\phi}(t)\|_{L^\infty}\|g_0\|_{L^2}.
\end{aligned}$$

Taking $|\lambda|$ sufficiently large, we may ensure that

$$C_{f,\delta}A^{-\sigma_c+1}|\lambda|^{-1}\|g_0\|_{H^{|s_c|}_{(x_1)}} \leq \tfrac{1}{4}c_0\|g_0\|_{L^2} \quad \text{and} \quad A^{-2}|\lambda|^{-1/2} < \delta', \tag{2-47}$$

which gives, after combining the previous two inequalities with (2-46),

$$\tfrac{1}{4}c_0 C_{f,\delta}^{-\sigma_c+1}A^{-(s_c-\sigma_c+1/2)}|\lambda|^{s_c}\exp\bigl(2|\lambda|\bigl(s_c-\sigma_c+\tfrac{1}{2}\bigr)A^2 t\bigr)\|\tilde{\phi}_0\|_{L^2_f} \leq \|\partial_x^{s_c}\tilde{\phi}(t)\|_{L^\infty}$$

for $|t| \le \min\{cM_2^{-2}, A^{-2}|\lambda|^{-1/2}\}$. For each $|\lambda|$ satisfying (2-47) there are two cases; either (i) $cM_2^{-2} < A^{-2}|\lambda|^{-1/2}$ or (ii) $cM_2^{-2} \ge A^{-2}|\lambda|^{-1/2}$. In the case (i), we obtain that $M_2 \gtrsim_A |\lambda|^{1/4} \gtrsim_A (\delta')^{-1/2}$ using (2-47). Here, we could have assumed that $|\lambda|$ is sufficiently large from the beginning so that $\sup_{t\in[0,\delta']}\|f(t)\|_{C^{1,1}} \ll_A |\lambda|^{1/4}$. Then, $M_2 \simeq \sup_{t\in[0,\delta']}\|\tilde{\phi}(t)\|_{C^{1,1}}$ and the desired norm inflation follows simply from our assumption in (1-7) that $s_c \ge 2$. In the case (ii), we simply take $t = A^{-2}|\lambda|^{-1/2}$ in (2-46), which gives the claimed norm inflation (actually, in this case we obtain a much stronger growth in terms of $1/\delta'$) using that $s_c > \sigma_c - \frac{1}{2}$. This finishes the proof of Theorem 1.1. □

Remark 2.10. At the end of the above proof, observe that we could have followed the same argument but have used (2-19)–(2-21) with $\gamma' = -\sigma_c$, $s = \sigma$ and $p = 2$ to derive

$$\tfrac{1}{4}c_0 C_{f,\delta}^{-\sigma_c+1} A^{-(\sigma-\sigma_c)}|\lambda|^{s_c} \exp(2|\lambda|(\sigma-\sigma_c)A^2 t)\|\tilde{\phi}_0\|_{L^2_f} \le \|\tilde{\phi}(t)\|_{H^\sigma}$$

for $t \le \min\{cM_2^{-2}, A^{-2}|\lambda|^{-1/2}\}$. This can be used to prove the inflation of the H^σ norm for any $\sigma > \sigma_c$ in the second alternative of Theorem 1.1.

2.6. ***Proof of Theorem 1.2.*** Let us divide the proof of Theorem 1.2 into several steps.

Choice of background solution. Towards a contradiction, we shall assume that there exist $\epsilon > 0$ and $s_0 \ge s_c + 2$ such that, for any $\phi_0 \in C^\infty(\mathbb{T})$ satisfying $\|\phi_0\|_{C^{s_0}} < \epsilon$, there exist $\delta = \delta(\phi_0) > 0$ and a solution $\phi \in L^\infty([0,\delta]; C^{s_c+1,1})$ to (1-5) with initial data $\phi(t=0) = \phi_0$.

Under this assumption, let us fix a function $\mathring{f}_0 \in C^\infty(\mathbb{T})$ which is supported in $\left(-\frac{1}{2}, \frac{1}{2}\right)$ and $\mathring{f}_0(x) = x$ in $\left[-\frac{1}{4}, \frac{1}{4}\right]$. We then set

$$f_0 := \sum_{k=k_0}^{\infty} f_{k,0} := \sum_{k=k_0}^{\infty} A_k 2^{-k} \mathring{f}_0(2^k(x-x_k)), \quad A_k = 2^{-k^2}, \ \ x_k = 2^{-k/2},$$

where $k_0 = k_0(s_0, \epsilon, \mathring{f}_0) \ge 1$ is taken sufficiently large to achieve $\|f_0\|_{C^{s_0}} < \frac{1}{2}\epsilon$. It is not difficult to see that $f_0 \in C^\infty(\mathbb{T})$. Furthermore, since the supports of $f_{k,0}$ are disjoint from each other, for each $k \ge k_0$, we may choose $\chi_k \in C^\infty(\mathbb{T})$ to be a cutoff function satisfying $\chi_k = 1$ on $\operatorname{supp}(f_{k,0})$ and $\chi_k = 0$ on $\operatorname{supp}(f_{k',0})$ for any $k' \ne k$. From the contradiction hypothesis, we have a solution $f(t) \in L^\infty([0,\delta]; C^{s_c+1,1})$ to (1-5) with initial data f_0 for some $\delta > 0$. The estimate $|\partial_t f| \lesssim |f|$ from (2-12) shows that $\operatorname{supp}(f(t)) = \operatorname{supp}(f_0)$ on $[0,\delta]$, and since χ_k equals either 0 or 1 on $\operatorname{supp}(f(t)) = \operatorname{supp}(f_0)$, we have that

$$\chi_k = \chi_k^3, \quad \partial_x \chi_k = 0$$

on $\operatorname{supp}(f(t))$ for any $k \ge k_0$. Using these observations, it follows, for each $k \ge k_0$, that $f_k := \chi_k f$ is again a solution to (1-5) with initial data $f_k(t=0) = f_{k,0}$. Furthermore, the $L^\infty([0,\delta]; C^{s_c+1,1})$ norm of f_k is bounded uniformly in k.

In the following, as in the above, we use the notation

$$\langle a, b\rangle_f(t) := \int_{\mathbb{T}} |f(t,x)|^{-2\sigma_c} a(t,x)\overline{b(t,x)}\,\mathrm{d}x, \quad \|a\|_{L^2_f}^2 := \langle a, a\rangle_f.$$

Let us also use the shorthand $\|a_0\|_{L^2_{f_0}} = \|a\|_{L^2_f}(t=0)$.

Choice of wave packet solutions. We now fix some nonzero function $g_0 \in C^\infty$ supported in $\left(\frac{1}{8}, \frac{1}{4}\right)$ and take $g_k(x) := 2^{k/2} g_0(2^k(x - x_k))$. For some sequence $\{\lambda_k\}_{k \geq k_0}$ to be determined (for now, we take $-\lambda_k \geq A_k^{10}$), we consider the sequence of wave packet solutions

$$\tilde{\phi}_k^{\mathrm{app}} := \tilde{\phi}_{(\lambda_k)}^{\mathrm{app}}[g_k;\, f_k],$$

where $\tilde{\phi}_{(\lambda_k)}^{\mathrm{app}}[g_k;\, f_k]$ is the wave packet solution from Proposition 2.7 with data g_k, λ_k, adapted to the linearly degenerate solution f_k, with $A = A_k$ and $x_1 = 2^{-k-2}$. We define the corresponding error by

$$[i\partial_t + \mathcal{L}_{f_k}]\tilde{\phi}_k^{\mathrm{app}} = \boldsymbol{\epsilon}_k, \tag{2-48}$$

where the operator $[i\partial_t + \mathcal{L}_{f_k}]$ is obtained from (2-14) by replacing f with f_k. Applying Proposition 2.7, we obtain the following bounds: with $\delta_k := \min\{\delta,\, A_k^{-2}|\lambda_k|^{-1/2}\} = A_k^{-2}|\lambda_k|^{-1/2}$ (by our choice of $-\lambda_k$ in the above),

- $\|\tilde{\phi}_k^{\mathrm{app}}\|_{L^\infty([0,\delta_k];L^2_f)} \lesssim C_{f_k,\delta_k}\|\tilde{\phi}_k^{\mathrm{app}}(t=0)\|_{L^2_{f_0}} \lesssim C_{f_k,\delta_k} A_k^{-\sigma_c+1}\|g_0\|_{L^2}$;
- $\|\boldsymbol{\epsilon}_k\|_{L^\infty([0,\delta_k],L^2_f)} \lesssim C_{f_k,\delta_k} A_k^{-\sigma_c+3}\|g_k\|_{H^2_{(2^{-k-2})}} \lesssim C_{f_k,\delta_k} A_k^{-\sigma_c+3}\|g_0\|_{H^2}$;

and

$$|f|^{-2\sigma_c}\tilde{\phi}_k^{\mathrm{app}} = \partial_x^{s_c}\left(\frac{|f|^{-\sigma_c+s_c-1/2}}{(i\lambda_k)^{s_c}(1+|f|\,\partial_x S)^{s_c}}\psi_k^{\mathrm{app}}\right) + |f|^{-2\sigma_c}\tilde{\phi}_k^{\mathrm{small}}$$

with

$$\left\|\frac{|f|^{-\sigma_c+s_c-1/2}}{(i\lambda_k)^{s_c}(1+|f|\,\partial_x S)^{s_c}}\psi_k^{\mathrm{app}}\right\|_{L^1} \lesssim C_{f_k,\delta_k}^{1-\sigma_c} A_k^{-\sigma_c+s_c+1/2}|\lambda_k|^{-s_c}\exp\bigl(-2|\lambda_k|\bigl(-\sigma_c+s_c+\tfrac{1}{2}\bigr)A_k t\bigr)\|g_0\|_{L^2}$$

for $0 \leq t \leq \delta_k$ and

$$\begin{aligned}\|\tilde{\phi}_k^{\mathrm{small}}\|_{L^\infty([0,\delta_k];L^2_f)} &\lesssim C_{f_k,\delta_k} A_k^{-\sigma_c+1}|\lambda_k|^{-1}\|g_k\|_{H^{s_c}_{(2^{-k-2})}}\\ &\lesssim C_{f_k,\delta_k} A_k^{-\sigma_c+1}|\lambda_k|^{-1}\|g_0\|_{H^{s_c}}.\end{aligned}$$

From (2-23) we see that

$$C_{f_k,\delta_k} \lesssim (1 + A_k^{-1}M)^{N_0}\exp(C_0 M^2\delta_k), \quad M = \sup_{t\in[0,\delta]}\|f(t,\cdot)\|_{C^{s_c+1,1}},$$

where the implicit constant and N_0 depends on $g_0, \alpha_1, \beta_1, \mu_1, s_c$, but not on k and λ_k. Then, simply using $\delta_k \leq \delta$ and recalling $A_k = 2^{-k^2}$, we see that $C_{f_k,\delta_k} \lesssim 2^{N_0 k^2}$ holds, where the implicit constant depends further on M and δ but not on k and λ_k. In turn, this gives an upper bound on the constants in the estimates above; for instance

$$C_{f_k,\delta_k}^{1-\sigma_c} A_k^{-\sigma_c+s_c+1/2} \lesssim 2^{N_1 k^2}$$

with some $N_1 > 0$ depending additionally on σ_c and s_c. *In the following, we shall write* $\lesssim$ *as long as the implicit constant does not depend on* k *and* λ_k.

Choice of initial data. We now take

$$\tilde{\phi}_0(x) = \sum_{k=k_0}^{\infty} \tilde{\phi}_{k,0}(x) := \sum_{k=k_0}^{\infty} \exp(-|\lambda_k|^{1/4}) \tilde{\phi}_k^{\mathrm{app}}(t=0, x), \tag{2-49}$$

which belongs to $C^\infty(\mathbb{T})$. By taking k_0 even larger if necessary, we can guarantee that $\|\tilde{\phi}_0\|_{C^m} < \frac{1}{2}\epsilon$. Then we set $\phi_0 = f_0 + \tilde{\phi}_0$, which satisfies $\|\phi_0\|_{C^m} < \epsilon$. Again from the contradiction hypothesis, we have a $L_t^\infty C^{s_c+1,1}$ solution $\phi(t)$ to (1-5) with initial data ϕ_0 on some time interval $[0, \delta']$. We may assume that $0 < \delta' \le \delta$ and define

$$\tilde{\phi}(t) := \phi(t) - f(t), \quad \tilde{\phi}_k(t) := \chi_k \tilde{\phi}(t)$$

for all $k \ge k_0$. We have that $\sum_{k=k_0}^{\infty} \tilde{\phi}_k = \tilde{\phi}$; this follows from $\partial_t |f + \tilde{\phi}| \lesssim |f + \tilde{\phi}|$ and the uniform pointwise estimate $|f + \tilde{\phi}|(t, x) \lesssim |f_0 + \tilde{\phi}_0|(x) \lesssim |f_0|(x)$. Then we see that $\tilde{\phi}_k$ solves

$$[i\partial_t + \mathcal{L}_{f_k}]\tilde{\phi}_k = Q_{f_k}[\tilde{\phi}_k],$$

(which is (2-40) with f and $\tilde{\phi}$ replaced with f_k and $\tilde{\phi}_k$, respectively). We note that the $L^\infty([0, \delta']; C^{s_0+1,1})$ norm is uniformly bounded for $\{f_k\}_{k\ge k_0}$ and $\{\tilde{\phi}_k\}_{k \ge k_0}$. Therefore, from Proposition 2.8, we obtain the estimate

$$\|\tilde{\phi}_k\|_{L^\infty([0,\delta']; L^2_f)} \lesssim \|\tilde{\phi}_{k,0}\|_{L^2_{f_0}} \tag{2-50}$$

uniformly in $k \ge k_0$. Now, combining this with the generalized energy estimate (2-44) for $\tilde{\phi}_k$ and $\tilde{\phi}_k^{\mathrm{app}}$, we obtain that

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{Re}\langle \tilde{\phi}_k, \tilde{\phi}_k^{\mathrm{app}} \rangle_f \right| \lesssim 2^{N_1 k^2} \|\tilde{\phi}_{k,0}\|_{L^2_{f_0}} \|g_0\|_{H^2} \tag{2-51}$$

for $t \in [0, \delta_k]$. We shall now take $|\lambda_k|$ larger so that $\delta_k = A_k^{-2} |\lambda_k|^{-1/2}$ satisfies $2^{N_1 k^2} \delta_k$ is very small with respect to the implicit constants in (2-50) and (2-51). Then, since at $t = 0$ we have

$$\mathrm{Re}\langle \tilde{\phi}_k, \tilde{\phi}_k^{\mathrm{app}} \rangle_f (t=0) \ge \tfrac{1}{4} \|\tilde{\phi}_{k,0}\|_{L^2_{f_0}} \|\tilde{\phi}_k^{\mathrm{app}}(t=0)\|_{L^2_{f_0}},$$

by integrating (2-51) in time from $t = 0$ to δ_k, we obtain

$$\mathrm{Re}\langle \tilde{\phi}_k, \tilde{\phi}_k^{\mathrm{app}} \rangle_f (\delta_k) \ge \tfrac{1}{8} \|\tilde{\phi}_{k,0}\|_{L^2_{f_0}} \|\tilde{\phi}_{k,0}^{\mathrm{app}}\|_{L^2_{f_0}}. \tag{2-52}$$

At $t = \delta_k$ we write

$$\mathrm{Re}\langle \tilde{\phi}_k, \tilde{\phi}_k^{\mathrm{app}} \rangle_f (\delta_k) = (-1)^{s_c} \mathrm{Re}\left\langle \partial_x^{s_c} \tilde{\phi}_k, \frac{|f|^{-\sigma_c + s_c - 1/2}}{(i\lambda_k)^{s_c}(1 + |f| \partial_x S)^{s_c}} \psi_k^{\mathrm{app}} \right\rangle + \mathrm{Re}\langle \tilde{\phi}_k, \tilde{\phi}_k^{\mathrm{small}} \rangle_f,$$

and then combining the estimates of the right-hand side with (2-52), we get

$$\|\tilde{\phi}_{k,0}\|_{L^2_{f_0}} \|\tilde{\phi}_{k,0}^{\mathrm{app}}\|_{L^2_{f_0}} \lesssim 2^{N_1 k^2} \big(\|\partial_x^{s_c} \tilde{\phi}_k\|_{L^\infty} |\lambda_k|^{-s_c} \exp\big(-2|\lambda_k|^{1/2}\big(-\sigma_c + s_c + \tfrac{1}{2}\big)\big) \|g_0\|_{L^2} + |\lambda_k|^{-1} \|g_0\|_{H^{s_c}} \big).$$

By taking $|\lambda_k|$ even larger if necessary, we can guarantee that $|\lambda_k|^{-1} 2^{N_1 k^2} \|g_0\|_{H^{s_c}} \ll \|\tilde{\phi}_{k,0}^{\mathrm{app}}\|_{L^2_{f_0}}$ holds ($\ll$ is defined in terms of the implicit constant in the previous inequality), so that we deduce

$$\|\tilde{\phi}_{k,0}\|_{L^2_{f_0}} \|\tilde{\phi}_{k,0}^{\mathrm{app}}\|_{L^2_{f_0}} \lesssim 2^{N_1 k^2} \|\partial_x^{s_c} \tilde{\phi}_k\|_{L^\infty} |\lambda_k|^{-s_c} \exp\big(-2|\lambda_k|^{1/2}\big(-\sigma_c + s_c + \tfrac{1}{2}\big)\big) \|g_0\|_{L^2},$$

and then recalling the form of $\tilde{\phi}_{k,0}$ from (2-49),

$$\|\partial_x^{s_c}\tilde{\phi}_k(t=\delta_k)\|_{L^\infty} \gtrsim 2^{-N_1k^2}|\lambda_k|^{s_c}\exp\bigl(2|\lambda_k|^{1/2}\bigl(-\sigma_c+s_c+\tfrac{1}{2}\bigr)-|\lambda_k|^{1/4}\bigr).$$

Note that the left-hand side is bounded by

$$\|f\|_{L^\infty([0,\delta'];C^{s_c+1,1})}+\|\phi\|_{L^\infty([0,\delta'];C^{s_c+1,1})}$$

for all k sufficiently large. This is a contradiction since the right-hand side diverges as $k\to\infty$. The proof is now complete. □

3. KdV-type equations

This section is organized as follows. After setting up some pieces of notation in Section 3.1, we study the properties of regular cubically degenerate solutions — typically denoted by f — in Section 3.2. Then in Section 3.3, we carry out the key construction of degenerating wave packets for the linearized equation around f, and in Section 3.4, we establish a modified energy estimate for the perturbation (solving the nonlinear difference equation) around f. Finally, in Sections 3.5 and 3.6, we prove Theorems 1.5 and 1.6, respectively.

3.1. ***Preliminaries.*** We introduce the following quantity defined for a $C^{2,1}$ function f on an interval I:

$$\|f\|_{Y(I)}=\|f^{-2/3}\,\partial_x f\|^3_{L^\infty(I)}+\|f^{-1/3}\,\partial_{xx}f\|^{3/2}_{L^\infty(I)}+\|f\|_{L^\infty(I)}+\|\partial_{xxx}f\|_{L^\infty(I)}.$$

We shall write $f\in Y(I)$ if $\|f\|_{Y(I)}$ is finite. This quantity is appropriate to handle solutions with degeneracies of order at least 3. For convenience we set

$$\|f\|_{\widetilde{C}^{k,\alpha}(I)}=\|f\|_{C^{k,\alpha}(I)}+\|f\|_{Y(I)}.$$

For f depending on time, we say $f\in L^\infty([0,\delta];\widetilde{C}^{k,\alpha}(I))$ (resp. $f\in L^\infty([0,\delta];Y(I))$ if

$$\|f\|_{L^\infty([0,\delta];\widetilde{C}^{k,\alpha}(I))}:=\sup_{t\in[0,\delta]}\|f(t)\|_{\widetilde{C}^{k,\alpha}(I)}<+\infty\quad\bigl(\text{resp. }\|f\|_{L^\infty([0,\delta];Y(I))}:=\sup_{t\in[0,\delta]}\|f(t)\|_{Y(I)}<+\infty\bigr).$$

It is easy to see using the Taylor expansion that any $C^{3,\alpha}$ function which vanishes cubically at its zeroes must belong to Y. However, *propagation* of Y-boundedness for (1-9) in general requires higher regularity, e.g., $C^{4,1}$ (see Proposition 3.2).

For later use, we introduce the notation

$$\langle a,b\rangle_f(t):=\int_I f(t,x)^{-2\sigma_c/3}a(t,x)b(t,x)\,\mathrm{d}x,\qquad \|a\|^2_{L^2_f}(t):=\langle a,a\rangle_f(t). \tag{3-1}$$

For the motivation behind the power $f^{-2\sigma_c/3}$, see Section 3.4.

3.2. ***Properties of a regular cubically degenerate solution.*** We first discuss a few basic properties of a regular cubically degenerate solution f to (1-9), which shall serve as the background for our ill-posedness mechanism.

Under the assumption $f\in L^\infty_t\widetilde{C}^{3,\alpha}(I)$ with any $\alpha>0$, we can propagate the information that f vanishes cubically on an endpoint of I and compute the coefficient.

Lemma 3.1. *Let $f \in L^\infty([0,\delta]; \widetilde{C}^{3,\alpha}(I))$ be a solution of* (1-9) *with initial data f_0 that is positive on $I \setminus \partial I$ and vanishes to order at least* 3 *on each point in ∂I, where $0 < \alpha \le 1$. Then the following statements hold*:

(1) *The zeroes and the sign of $f(t,x)$ are preserved in time*, *i.e.*, *$f(t,x)$ vanishes on ∂I and $f(t,x) > 0$ for $x \in I \setminus \partial I$ for all $t \in [0,\delta]$.*

(2) *Let $I = [a,b]$. Then*, *the set of t-dependent functions $\{\partial_x^k f(t,a)\}_{k=0}^3$ for $t \in [0,\delta]$ is determined by the initial data at $x = a$*, *i.e.*, *$\{\partial_x^k f(0,a)\}_{k=0}^3$. In particular*,

$$f(t,x) = (\beta(t)(x-a))^3 + O(\|f\|_{L_t^\infty([0,\delta];C^{3,\alpha}(I))}|x-a|^{3+\alpha}), \quad x \to a^+, \tag{3-2}$$

where $\beta(t)$ is the solution of

$$\dot\beta(t) = -(2+6\alpha_1)\beta^4(t), \quad 6\beta^3(0) = f_{0,xxx}(a), \tag{3-3}$$

and the implicit constant in $O(\,\cdot\,)$ is universal. The same statement applies to $b \in \partial I$.

Proof. Since we are assuming that $f(t,\cdot) \in C^3$, from (1-9), we have

$$|\partial_t f| \le C(|f_{xxx}||f| + |f_x||f_{xx}| + |f_x||f|^{m-1}).$$

From the assumption that $f(t,\cdot) \in Y$, we have the pointwise estimate

$$|\partial_t f| \le C(\|f\|_Y + \|f\|_{C^3}(1 + \|f\|_{L^\infty}^{m-2}))|f|.$$

This shows that

$$f_0(x)\exp(-Ct\|f\|_{L_t^\infty \tilde{C}^{3,\alpha}}(1+\|f\|_{L^\infty}^{m-2})) \le f(t,x) \le f_0(x)\exp(Ct\|f\|_{L_t^\infty \tilde{C}^{3,\alpha}}(1+\|f\|_{L^\infty}^{m-2})) \tag{3-4}$$

for any $x \in I$, which proves the first statement. The second statement follows from simply evaluating equation (1-9) at $x = a, b$ and carrying out a minor modification of the proof of Lemma 2.5. Here, the fact that f vanishes at least cubically at $x = a$ ensures that no $\partial_x^k f(t,a)$ with $k > 3$ occurs in the ODEs for the Taylor coefficients. We omit the details. ☐

Before we proceed further, let us note that the assumptions of Theorem 1.5 on the solution f are automatically satisfied for any sufficiently smooth solutions of (1-9).

Proposition 3.2. *Consider an interval $I = [a,b] \subseteq \mathbb{T}$. Let $f_0 \in C^{4,1}(\mathbb{T})$ satisfy $f_0 > 0$ on $I \setminus \{a\}$* (*resp. $I \setminus \{b\}$*) *and vanishes at least cubically at a* (*resp. b*), *so that $f_0 \in Y(I)$. Then there exists $\delta > 0$ depending on $\|f_0\|_{Y(I)}$ such that*, *if f is a solution to* (1-9) *with initial data f_0 satisfying $f \in L^\infty([0,\delta]; C^{4,1}(\mathbb{T}))$*, *then f satisfies $f|_I \in L^\infty([0,\delta]; Y(I))$ with the bound $\|f\|_{L^\infty([0,\delta];Y(I))} \le CC_0\exp(CM\delta)$.*

Furthermore, *for this value of δ*, *let u be another solution to* (1-9) *belonging to $L^\infty([0,\delta]; C^{4,1}(\mathbb{T}))$ with initial data u_0 satisfying $u_0 \in Y(I)$ and*

$$|u_0(x)| \le C_1 f_0(x) \tag{3-5}$$

for some $C_1 > 0$ uniformly for $x \in I$. Then, *for some $0 < \delta' \le \delta$ depending only on $\|u_0\|_Y$ and $\|f_0\|_Y$*,

$$|u(t,x)| \le C_1(1 + CC_0 t)\exp(CMt) f(t,x), \quad t \in [0,\delta'], \tag{3-6}$$

uniformly for $x \in I$, *where $C_0 = C_0(\|f_0\|_Y, \|u_0\|_Y)$ and $M = M(\|f\|_{L_t^\infty C^{4,1}}, \|u\|_{L_t^\infty C^{4,1}})$.*

Proof. Without loss of generality, we consider the case $f_0 > 0$ on $I \setminus \{a\}$ with f_0 vanishing at least cubically at a. We compute that

$$\begin{aligned}\partial_t f &= -\mu_1 f^{m-1} f_x - \alpha_1 f_x f_{xx} - f f_{xxx},\\ \partial_t f_x &= -\mu_1 (f^{m-1} f_x)_x - \alpha_1 (f_{xx})^2 - (\alpha_1+1) f_x f_{xxx} - f f_{xxxx},\\ \partial_t f_{xx} &= -\mu_1 (f^{m-1} f_x)_{xx} - (3\alpha_1+1) f_{xx} f_{xxx} - (\alpha_1+2) f_x f_{xxxx} - f f_{xxxxx}.\end{aligned}$$

Upon $f \in L_t^\infty C^{4,1}$, we have the pointwise estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}(|f|^2 + |f_x|^3 + |f_{xx}|^6) \le CM(|f|^2 + |f_x|^3 + |f_{xx}|^6),$$

where we introduce the shorthand

$$M = 1 + \|f\|^4_{L_t^\infty C^{4,1}} + \|f\|^{6m-2}_{L_t^\infty C^{2,1}}$$

for simplicity. By Gronwall's inequality, we have the pointwise estimate

$$(|f|^2 + |f_x|^3 + |f_{xx}|^6)(t,x) \le \exp(CMt)(|f_0|^2 + |f_{0,x}|^3 + |f_{0,xx}|^6)(x). \tag{3-7}$$

From the assumptions on the initial data, we have that

$$|f_{0,x}(x)|^3 + |f_{0,xx}(x)|^6 \le C_0^2 (f_0(x))^2 \tag{3-8}$$

holds pointwise on I, with some $C_0 > 0$ depending only on $\|f_0\|_Y$. Returning to (3-7) and applying Young's inequality, we deduce the pointwise bound

$$|f_x(t,x) f_{xx}(t,x)| \le CC_0 \exp(CMt) f_0(x)$$

for all $x \in I$. In turn, using this bound in the equation for $\partial_t f$, we obtain for all $x \in I$ that

$$|\partial_t f(t,x)| \le CM|f(t,x)| + CC_0 \exp(CMt) f_0(x). \tag{3-9}$$

Dividing by f_0 and applying Gronwall's inequality to $f/f_0 - 1$, we obtain

$$\left|\frac{f}{f_0} - 1\right| \le CC_0 t \exp(CMt)$$

which, after some simplification, implies

$$f(t,x) \le (1 + CC_0 t) \exp(CMt) f_0(x) \tag{3-10}$$

as well as

$$f(t,x) \ge (1 - CC_0 t) \exp(-CMt) f_0(x). \tag{3-11}$$

This guarantees that $f \in L_t^\infty([0,\delta]; Y)$, provided that δ is sufficiently small depending on $C_0 = C_0(\|f_0\|_Y)$.

For the second statement, we note that the assumption (3-5) implies

$$|u_0(x)|^2 + |u_{0,x}(x)|^3 + |u_{0,xx}(x)|^6 \le C_1^2 (1 + C\|u_0\|_Y) |f_0(x)|^2$$

for some absolute constant $C > 0$. With this bound, we may apply the above argument to u instead of f and obtain the bound

$$|u(t,x)| \le (1 + CC_0 t)\exp(CMt)|u_0(x)| \le C_1(1 + C_0 t)\exp(CMt) f_0(x), \quad t \in [0, \delta'],$$

for some $0 < \delta' \le \delta$ depending only on $C_0 = C_0(\|u_0\|_Y)$. Here, $M = 1 + \|u\|^4_{L^\infty_t C^{4,1}} + \|u\|^{6m-2}_{L^\infty_t C^{2,1}}$. Using this bound together with (3-11), we obtain the desired estimate (3-6), by taking δ' smaller in a way depending only on $\|u_0\|_Y$ and $\|f_0\|_Y$ if necessary. This finishes the proof. □

3.3. ***Degenerating wave packets for the linearized equation.*** In this subsection, our goal is to construct degenerating wave packets for the linearization of (1-9) around a (possibly hypothetical) regular cubic degenerate solution; see Proposition 3.3.

3.3.1. *Linearized equation and degenerating wave packets.* In the following, we fix some function f that satisfies all the assumptions from Theorem 1.5 and further assume for simplicity that the interval is given by $I = [0, b]$ for some $b > 0$. We fix some $0 < x_1 < b$ such that

$$\tfrac{1}{2} f_{0,xxx}(x) < f_{0,xxx}(0) < 2 f_{0,xxx}(x) \quad \text{for all } x \in [0, x_1]. \tag{3-12}$$

We now write $u = f + \phi$, where u is a solution to (1-9). Then, we have that ϕ must solve

$$\partial_t \phi + \mathcal{L}_f \phi = Q[\phi], \tag{3-13}$$

with

$$\mathcal{L}_f \phi = f\phi_{xxx} + \alpha_1 f_x \phi_{xx} + (\alpha_1 f_{xx} + \mu_1 f^{m-1})\phi_x + (f_{xxx} + (m-1)\mu_1 f^{m-2} f_x)\phi, \tag{3-14}$$

$$Q[\phi] = -\phi\phi_{xxx} - \alpha_1 \phi_x \phi_{xx} - \frac{\mu_1}{m}((f+\phi)^m - f^m - m f^{m-1}\phi)_x. \tag{3-15}$$

We are concerned with constructing wave packets to the linearized equation

$$\partial_t \phi + \mathcal{L}_f \phi = 0. \tag{3-16}$$

Recall the notation $\|g\|_{W^{s,p}_{(L)}} = \sum_{j=0}^{s} \|(L\partial_x)^j g\|_{L^p(\mathrm{d}x)}$ and $H^s_{(L)} = W^{s,2}_{(L)}$ from the previous section. Our aim is to prove the following result.

Proposition 3.3. *Let $f \in L^\infty_t([0,\delta]; \widetilde{C}^{s-1,1}(I))$ be a solution to* (1-9) *with initial data f_0 satisfying $f_0 > 0$ on $I \setminus \{0\}$ and vanishing cubically at* 0 *and $f_0 \in \widetilde{C}^{s_0-1,1}(I)$, where $4 \le s \le s_0$. Let $A = \frac{1}{6} f_{0,xxx}(0)$ and fix $0 < x_1 < 1$ so that* (3-12) *holds. Then, given $\lambda \in \mathbb{N}$ and $g_0 \in C^\infty_c$ supported in $\left(\frac{1}{2}x_1, x_1\right)$, we may associate a function $\phi^{\mathrm{app}}_{(\lambda)}[g_0, f]$ defined in $[0,\delta] \times I$ satisfying the following properties*:

- <u>*Linearity*</u>: *the map $g_0 \mapsto \phi^{\mathrm{app}}_{(\lambda)}[g_0, f]$ is linear*;
- <u>*Support property*</u>: $\operatorname{supp}(\phi^{\mathrm{app}}_{(\lambda)}[g_0, f](t,\cdot)) \subset (0, x_1) \cap \left(0, C_{\tilde{f}} x_1 \exp(-3\beta(t) A^{2/3} \lambda^2 t)\right)$;
- <u>*Initial data estimates*</u>: *for $0 \le n \le s_0$ and $1 \le p \le \infty$, we have*

$$\frac{1}{C_{\tilde{f}_0}}(\|g_0\|_{L^2} - \lambda^{-1}\|g_0\|_{H^1_{(x_1)}}) \le \|f_0^{-\sigma_c/3}(x)\phi^{\mathrm{app}}_{\lambda}(0,x)\|_{L^2} \le C_{\tilde{f}_0}\|g_0\|_{L^2}, \tag{3-17}$$

$$\|f_0^{-\sigma_c/3}(x)(A^{-1/3} f_0^{1/3}(x)\partial_x)^n \phi^{\mathrm{app}}_{(\lambda)}(0,x)\|_{L^p} \le C_{\tilde{f}_0} x_1^{1/p-1/2} |\lambda|^n \|g_0\|_{W^{n,p}_{(x_1)}}; \tag{3-18}$$

- *Regularity*: *for* $t \in [0, \delta]$ *and* $0 \le n \le s$,

$$\| f^{-\sigma_c/3}(A^{-1/3} f^{1/3}(t,x)\partial_x)^n \phi^{\mathrm{app}}_{(\lambda)}(t,x)\|_{L^2} \le C_{\tilde f} |\lambda|^n \|g_0\|_{H^n_{(x_1)}}; \tag{3-19}$$

- *Degeneration*: *for any* $1 \le p \le 2$, *a nonnegative even integer* $s' \le s$, *and* $\gamma' \ge -s' - \frac{1}{p} + \frac{1}{2}$, *we have*

$$f^{(-\sigma_c+\gamma')/3}\phi^{\mathrm{app}}_{(\lambda)} = \partial_x^{s'}\left(\frac{f^{(-\sigma_c+\gamma'+s')/3}}{(-1)^{s'/2}A^{s'/3}\lambda^{s'}}\phi^{\mathrm{app}}_{(\lambda)}\right) + f^{(-\sigma_c+\gamma')/3}\phi^{\mathrm{small}}_{(\lambda)}, \tag{3-20}$$

where, for $0 \le j \le 1$ *and* $t \in [0, \delta]$, *we have*

$$\begin{aligned} \|\partial_x^j(A^{-s'/3}\lambda^{-s'} f^{(-\sigma_c+\gamma'+s')/3}\phi^{\mathrm{app}}_{(\lambda)})(t,x)\|_{L^p} &\le C_{\tilde f}^{1+\gamma'} x_1^{(\gamma'+(s'-j)+1/p-1/2)} A^{\gamma'/3}\lambda^{-(s'-j)} \\ &\quad \times \exp\bigl(-3\beta(t)A^{2/3}\lambda^2\bigl(\gamma'+(s'-j)+\tfrac{1}{p}-\tfrac{1}{2}\bigr)t\bigr)\|g_0\|_{H^1_{(x_1)}}, \end{aligned} \tag{3-21}$$

$$\| f^{-\sigma_c/3}\tilde\phi^{\mathrm{small}}_{(\lambda)}(t,x)\|_{L^2} \le C_{\tilde f}\lambda^{-1}\|g_0\|_{H^{s'}_{(x_1)}}; \tag{3-22}$$

- *Error estimate*: *letting*

$$\boldsymbol{\epsilon}[\phi^{\mathrm{app}}_{(\lambda)}] = (\partial_t + \mathcal{L}_f)\phi^{\mathrm{app}}_{(\lambda)},$$

for $t \in [0, \delta]$, *we have*

$$\| f^{-\sigma_c/3}\boldsymbol{\epsilon}[\phi^{\mathrm{app}}_{(\lambda)}](t,x)\|_{L^2} \le C_{\tilde f}(1 + \|f\|^{m-2}_{L^\infty_t C^{0,1}})(1 + |\lambda|^2 t)|\lambda|\|g_0\|_{H^3_{(x_1)}}. \tag{3-23}$$

In the above properties, each constant referred to as $C_{\tilde f}$ *(resp.* $C_{\tilde f_0}$*) obeys the estimate*

$$C_{\tilde f} \le C_s \exp(N_s A^{-1}\|f\|_{L^\infty_t \widetilde{C}^{s-1,1}}) \quad (\textit{resp. } C_{\tilde f_0} \le C_s \exp(N_s A^{-1}\|f_0\|_{\widetilde{C}^{s_0-1,1}}))$$

for some $C_s > 0$ *and* $N_s \in \mathbb{N}$ *independent of* f *and* x_1 *(but possibly dependent on* s*).*

When f or g_0 are clear from the context, we shall often simply omit them in $\phi^{\mathrm{app}}_{(\lambda)}[g_0, f]$.

3.3.2. *Renormalization and conjugation.* For the construction, we introduce the normalization

$$\tilde f(t,x) := \frac{f(t,x)}{A}.$$

By this normalization, we have $\tilde f(0,x) = x^3 + o_f(x^3)$. Using $\tilde f$, we define for $t \in [0, \delta]$ and $x \in (0, x_1]$

$$y(t,x) = -\int_x^{x_1} \frac{1}{\tilde f(t,x')^{1/3}}\,\mathrm{d}x' \le 0.$$

Note that $\partial_x y > 0$. Then, we compute from $\partial_y = \tilde f^{1/3}\,\partial_x$ that

$$\tilde f^{2/3}\,\partial_{xx} = \partial_{yy} - \tfrac{1}{3} f^{-1} f_y\,\partial_y, \qquad \tilde f\,\partial_{xxx} = \partial_{yyy} - f^{-1} f_y\,\partial_{yy} + \bigl(-\tfrac{1}{3} f^{-1} f_{yy} + \tfrac{5}{9} f^{-2}(f_y)^2\bigr)\partial_y.$$

Furthermore, in the time derivative of ϕ,

$$\partial_t\phi(t,x) = \partial_t\phi(t,y) + (\partial_t y)\partial_y\phi(t,y),$$

and we set $q := \partial_t y$ for simplicity. Note that in the (t, x)-coordinates, we have

$$A^{-1}q = \frac{1}{3}\int_x^{x_1} \frac{-\tilde{f}\tilde{f}_{xxx} - \alpha_1 \tilde{f}_x \tilde{f}_{xx} - \frac{\mu_1}{m}(f^{m-2}\tilde{f}^2)_x}{\tilde{f}(t,x')^{4/3}}\,\mathrm{d}x'. \tag{3-24}$$

Then, in the (t, y)-coordinates, (3-16) transforms into

$$\begin{aligned} &A^{-1}\partial_t\phi + \phi_{yyy} + (\alpha_1 - 1)\tilde{f}^{-1}\tilde{f}_y\phi_{yy} \\ &\quad = -A^{-1}q\phi_y + \big(\big(\tfrac{1}{3} - \alpha_1\big)\tilde{f}^{-1}\tilde{f}_{yy} + \big(-\tfrac{5}{9} + \tfrac{2}{3}\alpha_1\big)\tilde{f}^{-2}(\tilde{f}_y)^2 + \mu_1 f^{m-2}\tilde{f}^{2/3}\big)\phi_y \\ &\qquad\qquad - ((\tilde{f}^{-1/3}\,\partial_y)^3\tilde{f} + (m-1)\mu_1 f^{m-3} f_y \tilde{f}^{2/3})\phi. \end{aligned} \tag{3-25}$$

We shall regard the expressions on the right-hand side of (3-25) as error terms. To remove the last term on the left-hand side, we introduce the conjugated variable

$$\varphi = e^{-G}\phi, \tag{3-26}$$

where G shall be determined below. We compute

$$\begin{aligned} \partial_y\phi &= e^G(\partial_y\varphi + \partial_y G\varphi), \\ \partial_y^2\phi &= e^G(\partial_y^2\varphi + 2\partial_y G\,\partial_y\varphi + (\partial_y^2 G + (\partial_y G)^2)\varphi), \\ \partial_y^3\phi &= e^G(\partial_y^3\varphi + 3\partial_y G\,\partial_y^2\varphi + (3\partial_y^2 G + 3(\partial_y G)^2)\partial_y\varphi + (\partial_y^3 G + 3\partial_y G\,\partial_y^2 G + (\partial_y G)^3)\varphi). \end{aligned}$$

Hence, the left-hand side of (3-25), after factoring out e^G, becomes

$$\begin{aligned} &A^{-1}\partial_t\varphi + \varphi_{yyy} + (3G_y + (\alpha_1 - 1)f^{-1}f_y)\varphi_{yy} + (3G_{yy} + 3G_y^2 + 2(\alpha_1 - 1)G_y f^{-1}f_y)\varphi_y \\ &\qquad + (A^{-1}G_t + G_{yyy} + 3G_y G_{yy} + G_y^3 + (\alpha_1 - 1)(G_{yy} + G_y^2)f^{-1}f_y)\varphi. \end{aligned}$$

The right-hand side, after factoring out e^G, becomes

$$\begin{aligned} &-A^{-1}q\varphi_y + \big(\big(\tfrac{1}{3} - \alpha_1\big)f^{-1}f_{yy} + \big(-\tfrac{5}{9} + \tfrac{2}{3}\alpha_1\big)f^{-2}(f_y)^2 + \mu_1 A^{-1}f^{m-1}\tilde{f}^{-1/3}\big)\varphi_y \\ &\qquad - \Big(q - \big(\big(\tfrac{1}{3} - \alpha_1\big)f^{-1}f_{yy} + \big(-\tfrac{5}{9} + \tfrac{2}{3}\alpha_1\big)f^{-2}(f_y)^2 + \mu_1 A^{-1}f^{m-1}\tilde{f}^{-1/3}\big)\Big)G_y\varphi \\ &\qquad\qquad - ((f^{-1/3}\partial_y)^3 f + (m-1)\mu_1 A^{-1}f^{m-2}f_y\tilde{f}^{-1/3})\varphi. \end{aligned}$$

To remove the second-order term, we are motivated to choose

$$G_y = -\frac{\alpha_1 - 1}{3}f^{-1}f_y = \frac{\sigma_c - \frac{1}{2}}{3}f^{-1}f_y.$$

Noting that $f^{-1}f_y = (\ln f)_y$, we see that $G(t, y) = \big(\sigma_c - \frac{1}{2}\big)\frac{1}{3}\ln f(t, y) + C$ for some choice of C. We choose $C = \frac{1}{6}\ln A$ so that

$$e^{G(t,y)} = f^{\sigma_c/3}\tilde{f}^{-1/6}, \tag{3-27}$$

in view of the weight $f^{-2\sigma_c/3}$ in the modified energy estimate we shall prove later (see also the definition of L^2_f in Section 3.1).

In conclusion, (3-25), after factoring out e^G, may be rewritten as

$$\begin{aligned} &A^{-1}\partial_t\varphi+\varphi_{yyy}\\ &=(-A^{-1}q+C_{1,1}f^{-1}f_{yy}+C_{1,2}f^{-2}(f_y)^2+\mu_1A^{-1}f^{m-1}\tilde f^{-1/3})\varphi_y\\ &\quad+(-A^{-1}G_t+C_{0,1}f^{-1}f_{yyy}+C_{0,2}f^{-2}f_yf_{yy}+C_{0,3}f^{-3}(f_y)^3+C_{0,4}\mu_1A^{-1}f^{m-2}f_y\tilde f^{-1/3})\varphi, \end{aligned} \tag{3-28}$$

where $C_{j,k}\in\mathbb{R}$ are constants that depend on α_1, μ_1 and m.

3.3.3. *Specification of the wave packet and the proof of Proposition 3.3.* Given $g_0(x)$, we set $h_0(y)=x_1^{1/2}g_0(x(0,y))$, and for each $\lambda\in\mathbb{N}$, we first take $\varphi^{\mathrm{app}}_{(\lambda)}[g_0,f]$ to be the standard wave packet for the Airy equation with time rescaled by A and with frequency λ, i.e.,

$$\varphi^{\mathrm{app}}_{(\lambda)}[g_0,f](t,y)=\mathrm{Re}(e^{i\lambda(y+A\lambda^2t)}h_0(y+3A\lambda^2t))=\cos(\lambda(y+A\lambda^2t))h_0(y+3A\lambda^2t). \tag{3-29}$$

Then, we define the degenerating wave packet $\phi^{\mathrm{app}}_{(\lambda)}[g_0,f]$ by $e^G\varphi^{\mathrm{app}}_{(\lambda)}[g_0,f]$. Explicitly, we have

$$\phi^{\mathrm{app}}_{(\lambda)}[g_0,f]=f(t,y)^{\sigma_c/3}\tilde f(t,y)^{-1/6}\cos(\lambda(y+A\lambda^2t))h_0(y+3A\lambda^2t). \tag{3-30}$$

Proof of Proposition 3.3. Now that we have specified the construction of $\phi^{\mathrm{app}}_{(\lambda)}[g_0,f]$, we verify its properties claimed in Proposition 3.3. In what follows, the dependence of constants on f, A and x_1 has been made explicit. Moreover, we shall use the notation $C_{\tilde f}$ introduced in Proposition 3.3.

Linearity and support property. To begin with, the linearity property is clear. To prove the support property, we first note from (3-2) and the positivity of $\tilde f$ that $y<0$ implies $x<x_1$, and vice versa. Note also that

$$y(t,x)=\frac{1}{\tilde\beta(t)}\Big(\ln\frac{x}{x_1}+B(t,x)\Big),$$

where

$$\tilde\beta(t):=\frac{\beta(t)}{A^{1/3}},\quad |B(t,x)|\le Cx_1\|\tilde f\|_{L^\infty_tC^{3,1}}.$$

Here, $\beta(t)^3=\frac{1}{6}f_{xxx}(t,0)$ as in Lemma 3.1; observe that $\tilde\beta(0)=1$ by definition. The above formula for y gives

$$x(t,y)=x_1e^{\tilde\beta(t)y-B}, \tag{3-31}$$

from which the rest of the support property follows.

From (3-31), it follows that

$$\begin{aligned} \tilde f(t,y)&=x_1^3\tilde\beta^3(t)e^{3\tilde\beta(t)y-3B}(1+O(x_1e^{\tilde\beta(t)y-B}\|\tilde f\|_{L^\infty_tC^{3,1}}))\\ &\le\exp(C\|f\|_{L^\infty_t\tilde C^{3,1}})x_1^3\tilde\beta^3(t)e^{3\tilde\beta(t)y}. \end{aligned} \tag{3-32}$$

Using the control of $\|f\|_{L^\infty_tY}$, we furthermore have

$$|f_y|\le C\|\tilde f\|^{1/3}_{L^\infty_tY}|f|,\quad |f_{yy}|\le C\|\tilde f\|^{2/3}_{L^\infty_tY}|f|,\quad |f_{yyy}|\le C\|\tilde f\|_{L^\infty_tY}|f|. \tag{3-33}$$

For higher derivatives, it is straightforward to verify by induction that

$$|\partial_y^k f| \le C_k \|\tilde{f}\|_{L_t^\infty \widetilde{C}^{k-1,1}}^{k/3} |f| \quad \text{for } k \ge 4. \tag{3-34}$$

We furthermore claim that, for any integer $0 \le k \le s+1$,

$$\|h_0\|_{H^k(\mathrm{d}y)} \le C_k (1 + \|\tilde{f}\|_{L_t^\infty \widetilde{C}^{s-1,1}}^{(k-1)/3}) \|g_0\|_{H^k_{(x_1)}}. \tag{3-35}$$

Indeed, arguing via induction in a similar fashion as above, we may verify that, for any $0 \le k \le s+1$,

$$\sum_{j'=0}^{k} |\partial_y^{j'} h_0| \le x_1^{1/2} \biggl(|g_0| + C_k \sum_{j'=1}^{k} \sum_{j=1}^{j'} \|\tilde{f}_0\|_{L_t^\infty \widetilde{C}^{s-1,1}}^{(j'-j)/3} \tilde{f}_0^{j/3} |\partial_x^j g_0| \biggr).$$

Note furthermore that, by (3-12), $C^{-1} x_1^3 \le \tilde{f}(x) \le C x_1^3$ on $\operatorname{supp} g_0$. Taking the $L^2(\mathrm{d}y)$ norm of both sides and changing variables, we are led to (3-35).

3.3.4. *Initial data and regularity estimates.* Let us now verify the initial data and regularity estimates. We begin by noting that

$$\begin{aligned} \int f^{-2\sigma_c/3} \phi^{\mathrm{app}}(t,x)^2 \, \mathrm{d}x &= \int (f^{-\sigma_c/3} \tilde{f}^{1/6} \phi^{\mathrm{app}}(t,y))^2 \, \mathrm{d}y \\ &= \int \varphi^{\mathrm{app}}(t,y)^2 \, \mathrm{d}y = \|h_0\|_{L^2(\mathrm{d}y)}^2, \end{aligned}$$

from which the regularity estimate in the case $n = 0$ follows. Moreover, from this identity it is clear that

$$\|f^{-\sigma_c/3} \phi^{\mathrm{app}}(0,x)\|_{L^2} \le C_{\tilde{f}_0} \|g_0\|_{L^2}.$$

To obtain the claimed lower bound, first note that

$$\|\cos(\lambda y) h_0\|_{L^2(\mathrm{d}y)}^2 = \int \frac{1}{\lambda} \partial_y (\sin(2\lambda y)) h_0^2(y) \, \mathrm{d}y + \frac{1}{2} \|h_0\|_{L^2(\mathrm{d}y)}^2,$$

and then one can integrate by parts in the first term on the right-hand side, with $\|h_0\|_{L^2(\mathrm{d}y)} \gtrsim_{\tilde{f}_0} \|g_0\|_{L^2}$.

Next, when $n = 1$, we note that

$$\begin{aligned} \partial_y \phi^{\mathrm{app}}(t,x) &= \operatorname{Re}(i\lambda f(t,y)^{\sigma_c/3} \tilde{f}^{1/6} e^{i\lambda(y + A\lambda^2 t)} h_0(y + 3A\lambda^2 t)) \\ &\quad + \frac{\sigma_c - \frac{1}{2}}{3} \frac{\partial_y f}{f} f(t,y)^{\sigma_c/3} \tilde{f}^{1/6} \operatorname{Re}(e^{i\lambda(y + A\lambda^2 t)} h_0(y + 3A\lambda^2 t)) \\ &\quad + f(t,y)^{\sigma_c/3} \tilde{f}^{1/6} \operatorname{Re}(e^{i\lambda(y + A\lambda^2 t)} \partial_y h_0(y + 3A\lambda^2 t)). \end{aligned}$$

From this expression, the regularity estimate follows by the earlier computation; the power $|\lambda|$ arises from the first term, the second term is estimated using (3-33), and the need for the $H^1_{(x_1)}$ norm of g_0 is due to the third term. The case of higher n can be handled similarly; we omit the details. Lastly, the initial data estimate can be proved simply by taking $t = 0$; see the computations below for $s' = 0$.

Degeneration property. When $s' = 0$, we simply set $\phi^{\text{small}}_{(\lambda)} = 0$. Arguing as in the proof of the regularity property, we have

$$
\begin{aligned}
&\| f(t,x)^{(-\sigma_c+\gamma')/3} \phi^{\text{app}}_{(\lambda)}(t,x)\|^p_{L^p(\mathrm{d}x)} \\
&\quad = \int |\varphi^{\text{app}}_{(\lambda)}(t,y)|^p f(t,y)^{p\gamma'/3} \tilde f(t,y)^{\frac{-p/2+1}{3}}\,\mathrm{d}y \\
&\quad \le A^{p\gamma'/3}\Big(\int |\varphi^{\text{app}}_{(\lambda)}(t,y)|^2\,\mathrm{d}y\Big)^{p/2}\Big(\int_{\operatorname{supp}\varphi^{\text{app}}_{(\lambda)}(t,\cdot)} \tilde f(t,y)^{\frac{\frac{1}{1/p-1/2}\gamma'+1}{3}}\,\mathrm{d}y\Big)^{p(1/p-1/2)} \\
&\quad \le \|h_0\|^p_{L^2} A^{p\gamma'/3}(C_{\tilde f} x_1\tilde\beta)^{p(\gamma'+1/p-1/2)}\Big(\int_{-\infty}^{-3A\lambda^2 t}\exp\Big(\Big(\frac{1}{1/p-1/2}\gamma'+1\Big)\tilde\beta(t)y\Big)\,\mathrm{d}y\Big)^{p(1/p-1/2)} \\
&\quad \le \|h_0\|^p_{L^2}(C_{\tilde f}x_1)^{p(\gamma'+1/p-1/2)}(A^{1/3}\tilde\beta)^{p\gamma'}\exp\big(-3p\tilde\beta(t)A\lambda^2\big(\gamma'+\tfrac1p-\tfrac12\big)t\big),
\end{aligned}
$$

where we have simply used (3-32) to bound $\tilde f(t,y)$. This proves (3-21) in the case $s = 0$.

To handle the case $s > 0$, it is convenient to introduce the following notation (as in the Schrödinger case): given some function $r = r(t,y)$,

$$
H = r\,O_k(h_0) \quad\Longleftrightarrow\quad \sup_{t\in[0,\delta]}\Big\| \tilde f^{1/6}\frac{H}{r}\Big\|_{L^2(\mathrm{d}y)} \le C_{\tilde f}\|h_0\|_{H^k(\mathrm{d}y)}.
$$

In this case, note that $\|\tilde f^{1/6}(\,\cdot\,)\|_{L^2(\mathrm{d}y)} = \|\cdot\|_{L^2(\mathrm{d}x)}$ for each t. We shall also freely use (3-35) to relate the right-hand side with $\|g_0\|_{H^k_{(L)}}$. In what follows, the expression abbreviated as $\frac{1}{\lambda}O_k(h_0)$ constitutes $f^{-\sigma_c/3}\phi^{\text{small}}_{(\lambda)}$; the desired estimate (3-22) would be an immediate consequence of the L^2 boundedness property embedded in the $O_k(\,\cdot\,)$ notation.

We treat the case $s = 2$. We begin with the identity

$$
\cos(\lambda(y + A\lambda^2 t)) = -\frac{f^{2/3}}{A^{2/3}\lambda^2}(\tilde f^{-1/3}\partial_y)^2\cos(\lambda(y+A\lambda^2 t)) - \frac{1}{3\lambda}f^{-1}\partial_y f\,\sin(\lambda(y+A\lambda^2 t)).
$$

Plugging this identity into the expression (3-30) for $\phi^{\text{app}}_{(\lambda)}[g_0, f]$ and commuting $(\tilde f^{-1/3}\partial_y)^2$ (which equals ∂_x^2 in the (t,x)-coordinates) outside, we have

$$
f(t,y)^{(-\sigma_c+\gamma')/3}\phi^{\text{app}}_{(\lambda)} = (\tilde f^{-1/3}\partial_y)^2\Big(\frac{f(t,y)^{(-\sigma_c+\gamma'+2)/3}}{(-1)A^{2/3}\lambda^2}\phi^{\text{app}}_{(\lambda)}\Big) + \frac{f(t,y)^{\gamma'/3}}{\lambda}O_2(h_0).
$$

Arguing as in the case $s = 0$, the expression inside the parentheses can be shown to obey the degeneration bound (3-21). The cases $s > 2$ are handled similarly.

Error bound. We begin by noticing that, by our construction, we have

$$
\|f^{-\sigma_c/3}\epsilon[\phi^{\text{app}}_{(\lambda)}]\|_{L^2(\mathrm{d}x)} \le \|(\partial_t + \partial_{yyy})\varphi^{\text{app}}_{(\lambda)}\|_{L^2(\mathrm{d}y)} + \|(\text{RHS of (3-28)})\|_{L^2(\mathrm{d}y)}.
$$

The first term is the error for the standard wave packet for the Airy equation with frequency λ; it is easily bounded by $C|\lambda|\|h_0\|_{H^3_y}$, which is acceptable. Now, it only remains to estimate the $L^2(\mathrm{d}y)$ norm of each term on the right-hand side of (3-28). The worst contribution turns out to be $-q\varphi_y$, which we turn to first.

By (3-32), (3-33), and the definition of q, it follows that

$$|A^{-1}q(t,y)| \le C_{\tilde f}(1+\|f\|^{m-2}_{L^\infty_t C^{0,1}}) \int_y^0 \mathrm{d}y' \le C_{\tilde f}(1+\|f\|^{m-2}_{L^\infty_t C^{0,1}})|y|. \tag{3-36}$$

By the support property of $\varphi^{\mathrm{app}}_{(\lambda)}$, we have

$$\|A^{-1}q(\varphi^{\mathrm{app}}_{(\lambda)})_y\|_{L^2} \le C_{\tilde f}(1+\|f\|^{m-2}_{L^\infty_t C^{0,1}})(1+A\lambda^2 t)|\lambda|\|h_0\|_{H^1(\mathrm{d}y)},$$

which is acceptable. The remaining terms on the right-hand side of (3-28) involving φ_y are bounded by $C_{\tilde f}(1+\|f\|^{m-2}_{L^\infty_t C^{0,1}})|\lambda|\|h_0\|_{H^1(\mathrm{d}y)}$, which are strictly better. Next, since $|A^{-1}\partial_t f| \le C_{\tilde f}(1+\|f\|^{m-2}_{L^\infty_t C^{0,1}})|f|$ (as in the estimate for q), we have

$$|A^{-1}\partial_t G| \le C_{\tilde f}(1+\|f\|^{m-2}_{L^\infty_t C^{0,1}}).$$

Using this bound, as well as (3-32) and (3-33), the terms on the right-hand side of (3-28) involving φ are bounded by $C_{\tilde f}(1+\|f\|^{m-2}_{L^\infty_t C^{0,1}})\|\varphi\|_{L^2(\mathrm{d}y)}$, which is good. This completes the proof of (3-23). □

3.4. *Modified energy estimate for the perturbation.* Recall the equation satisfied by ϕ:

$$\begin{aligned}\partial_t\phi + f\phi_{xxx} + \alpha_1 f_x\phi_{xx} + (\alpha_1 f_{xx} + \mu_1 f^{m-1})\phi_x + (f_{xxx} + (m-1)\mu_1 f^{m-2} f_x)\phi \\ = -\phi\phi_{xxx} - \alpha_1\phi_x\phi_{xx} - \mu_1((f+\phi)^m - f^m - mf^{m-1}\phi)_x.\end{aligned} \tag{3-37}$$

Regarding a solution ϕ of the above and recalling the notation $\|\cdot\|_{L^2_f}$ from (3-1), we have the modified energy estimate

$$\|\phi\|^2_{L^2_f}(t) = \int_I \phi^2(t,x) f(t,x)^{-2\sigma_c/3}\,\mathrm{d}x, \quad \|\phi_0\|^2_{L^2_{f_0}} = \int_I \phi^2(0,x) f(0,x)^{-2\sigma_c/3}\,\mathrm{d}x$$

assuming that f is defined on I.

Proposition 3.4. *Assume that f is a solution to* (1-9) *satisfying $f \in L^\infty([0,\delta];\widetilde{C}^{3,\alpha}(I))$ with initial data f_0 that is positive on $I\setminus\partial I$ and vanishes to order at least* 3 *on each point in ∂I. Moreover, assume that $\phi \in L^\infty([0,\delta]; C^{3,\alpha}(I))$ is a solution to* (3-37) *satisfying*

$$f+\phi \in L^\infty([0,\delta];\widetilde{C}^{3,\alpha}(I)), \quad f^{-1}(f+\phi) \in L^\infty([0,\delta]; L^\infty(I)).$$

Then we have the estimate

$$\|\phi\|_{L^2_f}(t) \le \exp(C_{f,f+\phi}t)\|\phi_0\|_{L^2_{f_0}}$$

for $t\in[0,\delta]$, where $\phi_0(x)=\phi(0,x)$ and

$$C_{f,f+\phi} \le C \sup_{t\in[0,\delta]} \bigl(\|f\|_Y + (1+\|f^{-1}(f+\phi)\|^{1/2}_{L^\infty})\|f+\phi\|_Y + (\|f\|_{C^{0,1}} + \|f+\phi\|_{C^{0,1}})^{m-1}\bigr), \tag{3-38}$$

with $C>0$ an absolute constant.

Proof. In what follows, we shall simply present a formal computation without worrying about the validity of the expressions and manipulations. Also, all integrals are taken over I. As in Section 2.4.1, the assumption $|\phi_0(x)| \le C|f_0(x)|$ and the finiteness of the right-hand side $\|\phi_0\|_{L^2_{f_0}} < +\infty$ would imply, via

Lemma 3.1, the vanishing property of $\phi(t,\cdot)$ on ∂I that is necessary to justify the computation; we shall leave the routine details to the reader.

To prove the proposition, we compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\phi\|_{L^2_f}^2=\frac{\mathrm{d}}{\mathrm{d}t}\int \phi^2 f(t,x)^{-2\sigma_c/3}\,\mathrm{d}x=\int 2\phi\,\partial_t\phi f(t,x)^{-2\sigma_c/3}\,\mathrm{d}x+\int \phi^2\,\partial_t(f(t,x)^{-2\sigma_c/3})\,\mathrm{d}x,$$

and the last term can be bounded as in the proof of (3-4); we have

$$\left|\int \phi^2\partial_t(f(t,x)^{-2\sigma_c/3})\,\mathrm{d}x\right|\le C\|f\|_Y\|\phi\|_{L^2_f}^2.$$

We decompose the other term in the right-hand side as follows, up to a factor of 2:

$$\begin{aligned}
\mathrm{I}&=-\int \phi(f\phi_{xxx}+\phi f_{xxx}+\alpha_1 f_x\phi_{xx}+\alpha_1 f_{xx}\phi_x)f^{-2\sigma_c/3}\,\mathrm{d}x,\\
\mathrm{II}&=-\mu_1\int \phi((m-1)\phi f^{m-2}f_x+f^{m-1}\phi_x)f^{-2\sigma_c/3}\,\mathrm{d}x,\\
\mathrm{III}&=-\int \phi Q[\phi]f^{-2\sigma_c/3}\,\mathrm{d}x.
\end{aligned}$$

To estimate I, we observe the following chain of inequalities:

$$\begin{gathered}
\left|\int \phi^2 f_{xxx}f^{-2\sigma_c/3}\,\mathrm{d}x\right|\le C\|f\|_Y\|\phi\|_{L^2_f}^2,\\
\left|\int \alpha_1\phi\phi_x f_{xx}f^{-2\sigma_c/3}\,\mathrm{d}x\right|=\left|\frac{\alpha_1}{2}\int \phi^2\partial_x(f_{xx}f^{-2\sigma_c/3})\,\mathrm{d}x\right|,\\
\int \alpha_1\phi\phi_{xx}f_x f^{-2\sigma_c/3}\,\mathrm{d}x=-\alpha_1\int(\phi_x)^2 f_x f^{-2\sigma_c/3}\,\mathrm{d}x+\frac{\alpha_1}{2}\int \phi^2\partial_{xx}(f_x f^{-2\sigma_c/3})\,\mathrm{d}x,
\end{gathered}$$

and lastly

$$\begin{aligned}
\int \phi\phi_{xxx}ff^{-2\sigma_c/3}\,\mathrm{d}x&=-\int \phi_{xx}\phi_x ff^{-2\sigma_c/3}-\phi_{xx}\phi\partial_x(ff^{-2\sigma_c/3})\,\mathrm{d}x\\
&=\frac{3}{2}\int(\phi_x)^2\partial_x(ff^{-2\sigma_c/3})\,\mathrm{d}x-\frac{1}{2}\int \phi^2\partial_{xxx}(ff^{-2\sigma_c/3})\,\mathrm{d}x.
\end{aligned}$$

From

$$\partial_x(ff^{-2\sigma_c/3})=\left(1-\tfrac{2}{3}\sigma_c\right)f^{-2\sigma_c/3}f_x=\tfrac{2}{3}\alpha_1 f^{-2\sigma_c/3}f_x,$$

we obtain a cancellation of terms involving $(\phi_x)^2$ and then we observe

$$|\partial_x(f_{xx}f^{-2\sigma_c/3})|+|\partial_{xx}(f_x f^{-2\sigma_c/3})|+|\partial_{xxx}(ff^{-2\sigma_c/3})|\le C\|f\|_Y f^{-2\sigma_c/3}$$

to conclude the estimate

$$|\mathrm{I}|\le C\|f\|_Y\|\phi\|_{L^2_f}^2.$$

Next, to treat II we simply integrate by parts:

$$|\mathrm{II}| = \left| \mu_1 \int \phi^2 \left((m-1) f^{m-2} f_x - \frac{-\frac{2}{3}\sigma_c + m - 1}{2} f^{m-2} f_x \right) f^{-2\sigma_c/3} \, \mathrm{d}x \right| \le C \|f\|_{C^{0,1}}^{m-1} \|\phi\|_{L^2_f}^2 .$$

Finally, we turn to III. Observe that we may use $\|\phi\|_{C^{3,\alpha}}$ since it is controlled by $\|f\|_{C^{3,\alpha}} + \|f+\phi\|_{C^{3,\alpha}}$; similarly for $\|\phi\|_{C^{0,1}}$. Recall the expression for $Q[\phi]$ given in (3-15). We first estimate

$$\left| \int \phi(-\phi\phi_{xxx}) f^{-2\sigma_c/3} \, \mathrm{d}x \right| \le C \|\phi\|_{C^{2,1}} \|\phi\|_{L^2_f}^2$$

and

$$\begin{aligned} &\left| \int \phi(-\alpha_1 \phi_x \phi_{xx}) f^{-2\sigma_c/3} \, \mathrm{d}x \right| \\ &\le \left| \int \frac{\alpha_1}{2} \phi^2 \phi_{xxx} f^{-2\sigma_c/3} \, \mathrm{d}x \right| + \left| \int \frac{\alpha_1 \sigma_c}{3} \phi^2 (\phi + f)_{xx} f^{(-2\sigma_c - 3)/3} f_x \, \mathrm{d}x \right| + \left| \int \frac{\alpha_1 \sigma_c}{3} \phi^2 f_{xx} f^{(-2\sigma_c - 3)/3} f_x \, \mathrm{d}x \right| \\ &\le \left| \int \frac{\alpha_1}{2} \phi^2 \phi_{xxx} f^{-2\sigma_c/3} \, \mathrm{d}x \right| + \left| \int \frac{\alpha_1 \sigma_c}{3} \phi^2 (\phi + f)_{xx} f^{(-2\sigma_c - 3)/3} f_x \, \mathrm{d}x \right| + \left| \int \frac{\alpha_1 \sigma_c}{3} \phi^2 f_{xx} f^{(-2\sigma_c - 3)/3} f_x \, \mathrm{d}x \right| \\ &\le C (\|\phi\|_{C^{2,1}} + \|f^{-1}(f+\phi)\|_{L^\infty}^{1/3} \|f+\phi\|_Y^{2/3} \|f\|_Y^{1/3} + \|f\|_Y) \|\phi\|_{L^2_f}^2 . \end{aligned}$$

The remaining terms in $Q[\phi]$ are easier to treat, and collecting the estimates, we conclude

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} \|\phi\|_{L^2_f}^2 \right| \le C_{f, f+\phi} \|\phi\|_{L^2_f}^2$$

for some $C_{f,f+\phi}$ satisfying (3-38). The proposition follows by integrating in time. □

3.5. ***Generalized energy estimate and the proof of Theorem 1.5.*** Let f satisfy the assumptions of Theorem 1.5. Without loss of generality, we may assume that $a = 0$ and $0 < \epsilon \le 1$. For simplicity, we shall focus on the case $\beta(0) = \left(\frac{1}{6} f_{0,xxx}(0)\right)^{1/3} = 1$, the general case being analogous. Fix $0 < x_1 < b$ so that (3-12) holds. Fix $g_0 \in C_c^\infty$ supported in $\left(\frac{1}{2}x_1, x_1\right)$ with normalization $\|g_0\|_{L^2} = 1$. In what follows, *we shall suppress the dependence of constants on f and g_0, in addition to α_1, μ_1 and m as before.* Also, *we write $C(M)$ for a positive strictly increasing function of $M \in (0, \infty)$ such that $C(M) \to \infty$ as $M \to \infty$, which may vary from line to line.*

Let $\phi^{\mathrm{app}}_{(\lambda)} = \phi^{\mathrm{app}}_{(\lambda)}[g_0, f]$ according to Proposition 3.3. We shall take

$$\phi_0 = c_0 \epsilon \lambda^{-m_0} x_1^{1/2} \phi^{\mathrm{app}}_{(\lambda)}(0),$$

where c_0 is chosen so that $\|\phi_0\|_{C^{m_0}} \le \epsilon$ using (3-19) and λ is to be determined below. Furthermore, by the normalization $\|g_0\|_{L^2} = 1$, we have

$$\frac{1}{C_0} |\lambda|^{-m_0} \le \|\phi_0\|_{L^2_{f_0}} \le C_0 |\lambda|^{-m_0}, \quad \langle \phi_0, \phi^{\mathrm{app}}_{(\lambda)}(0) \rangle_{f_0} \ge \frac{1}{C_0} \|\phi_0\|_{L^2_{f_0}} \tag{3-39}$$

for some constant $C_0 > 0$ (which, in fact, depends on $\|f\|_{\tilde{C}^{s_0-1,1}}$). At this point, it is easy to ensure that (3-5) is satisfied with $C_1 = 2$, where $u_0 = f_0 + \phi_0$.

Fix also $0 < \delta' \le \delta$. To prove the theorem, we assume that the first alternative does not hold, i.e., there exists a solution $f+\phi \in L^\infty([0,\delta']; C^{s-1,1}(I))$ to (1-9). By Proposition 3.2 (and since $s \ge 5$), there exists $0 < t_0 \le \delta'$ depending only on $\|f(0)\|_Y$ and $\|(f+\phi)(0)\|_Y$ such that $f,\ f+\phi \in L^\infty([0,t_0]; Y(I))$ and $f^{-1}(f+\phi) \in L^\infty([0,t_0]; L^\infty(I))$. Moreover, by the same proposition, we have

$$\sup_{0<t<t_0} \big(\|f(t)\|_{\widetilde{C}^{4,1}(I)} + \|(f+\phi)(t)\|_{\widetilde{C}^{4,1}(I)} + \|f^{-1}(f+\phi)(t)\|_{L^\infty(I)}\big) \le C(M_5), \tag{3-40}$$

where $M_5 := \|\phi\|_{L^\infty([0,t_0];C^{4,1}(I))}$. (Here, we remind the reader of our convention of omitting the dependence on f in this proof.) By Proposition 3.4 and the preceding bound, we have

$$\|\phi(t)\|_{L^2_f} \le \exp(C_1(M_5)t)\|\phi_0\|_{L^2_{f_0}} \tag{3-41}$$

for some positive strictly increasing function $C_1(\,\cdot\,)$ that diverges at infinity. We emphasize that this function is *independent* of λ, although $M_5 = \|\phi\|_{L^\infty([0,t_0];C^{4,1}(I))}$ might be dependent on λ.

Now using that ϕ is a solution to (3-37) and $(\partial_t + \mathcal{L}_f)\phi^{\mathrm{app}}_{(\lambda)} = \boldsymbol{\epsilon}[\phi^{\mathrm{app}}_{(\lambda)}]$, we compute that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}&\langle \phi, \phi^{\mathrm{app}}_{(\lambda)}\rangle_f \\ &= -\langle \phi, \mathcal{L}_f[\phi^{\mathrm{app}}_{(\lambda)}]\rangle_f + \langle \phi, \boldsymbol{\epsilon}[\phi^{\mathrm{app}}_{(\lambda)}]\rangle_f - \langle \mathcal{L}_f[\phi], \phi^{\mathrm{app}}_{(\lambda)}\rangle_f + \langle Q_f[\phi], \phi^{\mathrm{app}}_{(\lambda)}\rangle_f - \tfrac{2}{3}\sigma_c\langle f^{-1}\partial_t f\phi, \phi^{\mathrm{app}}_{(\lambda)}\rangle_f.\end{aligned}$$

We first uncover some cancellations between the two terms involving the linearized operator $\mathcal{L}_f$, which resemble those in the proof of Proposition 3.4. We write

$$\begin{aligned}-\langle \phi, &\mathcal{L}_f[\phi^{\mathrm{app}}_{(\lambda)}]\rangle_f - \langle \mathcal{L}_f[\phi], \phi^{\mathrm{app}}_{(\lambda)}\rangle_f \\ &= -\int \phi(f\phi^{\mathrm{app}}_{(\lambda)xxx} + \alpha_1 f_x\phi^{\mathrm{app}}_{(\lambda)xx} + \alpha_1 f_{xx}\phi^{\mathrm{app}}_{(\lambda)x} + f_{xxx}\phi^{\mathrm{app}}_{(\lambda)})f^{-2\sigma_c/3}\,\mathrm{d}x \\ &\qquad - \mu_1\int \phi(f^{m-1}\phi^{\mathrm{app}}_{(\lambda)x} + (m-1)f^{m-2}f_x\phi^{\mathrm{app}}_{(\lambda)})f^{-2\sigma_c/3}\,\mathrm{d}x \\ &\qquad\qquad - \int (f\phi_{xxx} + \alpha_1 f_x\phi_{xx} + \alpha_1 f_{xx}\phi_x + f_{xxx}\phi)\phi^{\mathrm{app}}_{(\lambda)} f^{-2\sigma_c/3}\,\mathrm{d}x \\ &\qquad\qquad\qquad - \mu_1\int (f^{m-1}\phi_x + (m-1)f^{m-2}f_x\phi)\phi^{\mathrm{app}}_{(\lambda)} f^{-2\sigma_c/3}\,\mathrm{d}x \\ &=: \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}.\end{aligned}$$

We begin with $\mathrm{I}+\mathrm{III}$. The zeroth-order terms (in both ϕ and $\phi^{\mathrm{app}}_{(\lambda)}$) are not dangerous, but we need to perform some integration by parts for the higher-order terms. For the third-order terms, we have

$$\begin{aligned}-\int &(\phi\phi^{\mathrm{app}}_{(\lambda)xxx} + \phi_{xxx}\phi^{\mathrm{app}}_{(\lambda)})f^{1-2\sigma_c/3}\,\mathrm{d}x \\ &= \int (\phi_x\phi^{\mathrm{app}}_{(\lambda)xx} + \phi_{xx}\phi^{\mathrm{app}}_{(\lambda)x})f^{1-2\sigma_c/3}\,\mathrm{d}x + \big(1-\tfrac{2}{3}\sigma_c\big)\int (\phi\phi^{\mathrm{app}}_{(\lambda)xx} + \phi_{xx}\phi^{\mathrm{app}})f^{-1}f_x f^{1-2\sigma_c/3}\,\mathrm{d}x \\ &= -3\big(1-\tfrac{2}{3}\sigma_c\big)\int \phi_x\phi^{\mathrm{app}}_{(\lambda)x} f_x f^{-2\sigma_c/3}\,\mathrm{d}x - \big(1-\tfrac{2}{3}\sigma_c\big)\int (\phi\phi^{\mathrm{app}}_{(\lambda)x} + \phi_x\phi^{\mathrm{app}})(f_x f^{-2\sigma_c/3})_x\,\mathrm{d}x \\ &= -3\big(1-\tfrac{2}{3}\sigma_c\big)\int \phi_x\phi^{\mathrm{app}}_{(\lambda)x} f_x f^{-2\sigma_c/3}\,\mathrm{d}x + \big(1-\tfrac{2}{3}\sigma_c\big)\int \phi\phi^{\mathrm{app}}_{(\lambda)}(f_x f^{-2\sigma_c/3})_{xx}\,\mathrm{d}x;\end{aligned}$$

for the second-order terms, we have

$$-\int (\alpha_1 \phi \phi^{\mathrm{app}}_{(\lambda)xx} f_x + \alpha_1 \phi_{xx} \phi^{\mathrm{app}}_{(\lambda)} f_x) f^{-2\sigma_c/3}\, \mathrm{d}x$$
$$= 2\alpha_1 \int \phi_x \phi^{\mathrm{app}}_{(\lambda)x} f_x f^{-2\sigma_c/3}\, \mathrm{d}x + \alpha_1 \int (\phi \phi^{\mathrm{app}}_{(\lambda)x} + \phi_x \phi^{\mathrm{app}}_{(\lambda)})(f_x f^{-2\sigma_c/3})_x$$
$$= 2\alpha_1 \int \phi_x \phi^{\mathrm{app}}_{(\lambda)x} f_x f^{-2\sigma_c/3}\, \mathrm{d}x - \alpha_1 \int \phi \phi^{\mathrm{app}}_{(\lambda)} (f_x f^{-2\sigma_c/3})_{xx};$$

and for the first-order terms, we have

$$-\int (\alpha_1 \phi \phi^{\mathrm{app}}_{(\lambda)x} f_{xx} + \alpha_1 \phi_x \phi^{\mathrm{app}}_{(\lambda)} f_{xx}) f^{-2\sigma_c/3}\, \mathrm{d}x = \alpha_1 \int \phi \phi^{\mathrm{app}}_{(\lambda)} (f_{xx} f^{-2\sigma_c/3})_x\, \mathrm{d}x.$$

In particular, since $3\big(1 - \frac{2}{3}\sigma_c\big) = 2\alpha_1$, integrals that involve $\phi_x \phi^{\mathrm{app}}_{(\lambda)x}$ cancel and we are left with

$$|\mathrm{I} + \mathrm{III}| \le C \| f \|_Y \|\phi\|_{L^2_f} \|\phi^{\mathrm{app}}_{(\lambda)}\|_{L^2_f}.$$

Next, $\mathrm{II} + \mathrm{IV}$ consist of first- and zeroth-order terms, where the former may be treated as above and the latter are already acceptable. We have

$$|\mathrm{II} + \mathrm{IV}| \le C \| f \|^{m-1}_{C^{0,1}} \|\phi\|_{L^2_f} \|\phi^{\mathrm{app}}_{(\lambda)}\|_{L^2_f}.$$

For the remaining terms in $\frac{\mathrm{d}}{\mathrm{d}t}\langle \phi, \phi^{\mathrm{app}}_{(\lambda)}\rangle_f$, we have

$$\|\epsilon[\phi^{\mathrm{app}}_{(\lambda)}]\|_{L^2_f} \le C(1 + \lambda^2 t)|\lambda|,$$
$$\|Q[\phi]\|_{L^2_f} \le C\|\phi\|_{L^2_f},$$
$$\|f^{-1}\partial_t f\|_{L^\infty} \le C.$$

We conclude that

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} \langle \phi, \phi^{\mathrm{app}}_{(\lambda)}\rangle_f \right| \le C(1 + (1 + \lambda^2 t)|\lambda|)\|\phi\|_{L^2_f} \le C(1 + (1 + \lambda^2 t)|\lambda|) \exp(C_1(M_5)t) \frac{\|\phi_0\|_{L^2}}{4C_0}. \tag{3-42}$$

Integrating this estimate in time and using (3-39), we have

$$\langle \phi, \phi^{\mathrm{app}}_{(\lambda)}\rangle_f(t) \ge \frac{3}{4C_0}\|\phi_0\|_{L^2} \quad \text{for } |t| \le \min\{t_0, C_1(M_5)^{-1}, c|\lambda|^{-3/2}\} \tag{3-43}$$

for some $c > 0$.

To proceed, let m be the smallest even integer greater than or equal to s', and define $j = m - s'$. Applying Proposition 3.3 with $\gamma' = -\sigma_c$ and $s' = m$, we have

$$\langle \phi, \phi^{\mathrm{app}}_{(\lambda)}\rangle_f \le \int \phi \partial_x^{s'+j} \left(\frac{f^{(-2\sigma_c + s' + j)/3}}{(-1)^{s_0+j} \lambda^{s_0+j}} \phi^{\mathrm{app}}_{(\lambda)} \right) \mathrm{d}x + \|\phi\|_{L^2_f} \|\phi^{\mathrm{small}}_{(\lambda)}\|_{L^2_f}. \tag{3-44}$$

By (3-22) and (3-41), the last term may be bounded as follows for some $C_2 > 0$:

$$\begin{aligned} \|\phi\|_{L^2_f} \|\phi^{\mathrm{small}}_{(\lambda)}\|_{L^2_f} &\le C_2 \lambda^{-1} \exp(C_1(M_5)t) \frac{1}{4C_0} \|\phi_0\|_{L^2} \\ &\le \frac{1}{4C_0}\|\phi_0\|_{L^2} \qquad \text{if } |t| \le C_1(M_5)^{-1} \text{ and } |\lambda| > C_2. \end{aligned} \tag{3-45}$$

We are now ready to conclude the proof of the theorem. For each λ, there are two possible cases: (i) $C_1(M_5)^{-1} \le c|\lambda|^{-3/2}$, or (ii) $C_1(M_5)^{-1} > c|\lambda|^{-3/2}$. In case (i), we have $c^{-1}|\lambda|^{3/2} \le C_1(M_5)$, so $M_5 > (\delta')^{-1/2}$ if $|\lambda|$ is chosen large enough depending on $C_1(\cdot)$ and δ'. Since $s' \ge s_c \ge 5$, the desired norm inflation follows. Hence, it only remains to consider case (ii). Then, by (3-43), (3-44) and (3-45), we have

$$\frac{1}{2C_0}\|\phi_0\|_{L^2_f} \le \int \phi \partial_x^{s'+j}\left(\frac{f^{(-2\sigma_c+s'+j)/3}}{(-1)^{s_0+j}\lambda^{s_0+j}}\phi^{\mathrm{app}}_{(\lambda)}\right)\mathrm{d}x \quad \text{for } |t| \le \min\{t_0, c|\lambda|^{-3/2}\}.$$

Using duality and applying (3-21), we arrive at

$$\begin{aligned}\frac{1}{2C_0}\|\phi_0\|_{L^2_f} &\le \|\partial_x^{s'}\phi\|_{L^\infty}\|\partial_x^{j}(\lambda^{-s_0-j}f^{(-2\sigma_c+s'+j)/3}\phi^{\mathrm{app}}_{(\lambda)})\|_{L^1}\\ &\le C\lambda^{-s'}\exp\bigl(-3\beta(t)\lambda^2\bigl(-\sigma_c+s'+\tfrac12\bigr)t\bigr)\|\partial_x^{s'}\phi\|_{L^\infty}.\end{aligned}$$

Rearranging the factors, we finally arrive at

$$\|\partial_x^{s'}\phi(t)\|_{L^\infty} \ge \frac{1}{C}\lambda^{s'-m_0}\exp\bigl(3\beta(t)\lambda^2\bigl(-\sigma_c+s'+\tfrac12\bigr)t\bigr) \quad \text{for } 0 \le t \le \min\{t_0, c|\lambda|^{-3/2}\}.$$

Fix $t = c|\lambda|^{-3/2}$; by taking λ sufficiently large, we may clearly ensure that $t \le t_0$. Since $s' \ge s_c > \sigma_c - \frac12$ and $\lambda^2 t = c|\lambda|^{1/2}$, by taking λ larger we may also ensure that the right-hand side is at least $(\delta')^{-1/2}$ for each s'. This completes the proof of Theorem 1.5. □

3.6. ***Proof of Theorem 1.6.*** We are now in a position to complete the proof of Theorem 1.6. As we shall see, the argument is parallel to that for Theorem 1.2 in the Schrödinger case.

Let $s_0 \ge s \ge s_c$ and $\epsilon > 0$ be given as in the statement of Theorem 1.6. Suppose, for contradiction, that for every $u_0 \in C^\infty(\mathbb{T})$ satisfying $\|u_0\|_{C^{s_0}} < \epsilon$ there exists $\delta = \delta(u_0) > 0$ and a corresponding solution u to (1-9) belonging to $L^\infty([0,\delta]; C^s(\mathbb{T}))$.

We shall fix a function $\mathring{f}_0 \in C^\infty(\mathbb{T})$ supported in $[-4x_1, 4x_1]$ which satisfies $\mathring{f}_0(x) = x^3$ in $[-x_1, x_1]$, $\mathring{f}_0 > 0$ on $(0, 2x_1)$ and the cubic vanishing property at $2x_1$ for some small $0 < x_1 < \frac{1}{100}$; in what follows, we omit the dependence of constants on $\mathring{f}_0$. Then, we take

$$f_0 := \sum_{k=k_0}^{\infty} f_{k,0} := \sum_{k=k_0}^{\infty} A_k 2^{-3k}\mathring{f}_0(2^k(x-x_k)), \quad x_k = 2^{-k/2},\ \ A_k = 2^{-k^2},$$

where k_0 shall be fixed below. We see that $f_0 \in C^\infty$ and $\|f_0\|_{C^{s_0}} < \frac12\epsilon$ provided that k_0 is sufficiently large depending on s_0 and ϵ. Moreover, we can check that there exists a constant $C_0 = C_0(f_0) > 0$ such that (3-8) holds for f_0. For each $k \in \mathbb{N}$, we take cutoff functions χ_k which are equal to 1 on the support of $f_{k,0}$ and vanish on the support of $f_{k',0}$ for all $k' \ne k$.

Now, let $f \in L^\infty([0,\delta]; C^s(\mathbb{T}))$ be a solution to (1-9) with $f(t=0) = f_0$. Using the equation and $C^{4,1}$-regularity, we have the pointwise estimate $|\partial_t f| \lesssim |f_0| + |f|$, which guarantees that the support of $f(t,\cdot)$ is preserved in time. For simplicity, we set $M_f = \|f\|_{L^\infty([0,\delta];C^{4,1})}$ and replace δ with $\min\{\delta, c\}$ with some small constant $c = c(M_f) > 0$, so that we have uniformly

$$\tfrac12 f_0(x) \le f(t,x) \le \tfrac32 f_0(x), \quad t \in [0,\delta], \tag{3-46}$$

whenever $f_0(x) > 0$. The existence of such a constant c follows from (3-10) and (3-11).

Since χ_k is either 0 or 1 on $\mathrm{supp}(f_0) = \mathrm{supp}(f(t,\cdot))$, we have $\partial_x \chi_k \equiv 0$ on the support of f and $\chi_k f = \chi_k^2 f$. From these observations, it follows that $\chi_k f =: f_k$ provides a solution to (1-9) for any $k \geq k_0$ with initial data $f_{k,0}$. Let $I_k = [x_k, x_k + 2^{-k} 2x_1]$, and observe that

$$\|f_{k,0}\|_{Y(I_k)} \leq A_k(\|\mathring{f}_0\|_{Y([0,2x_1])} + 2^{-3k}\|\mathring{f}_0\|_{L^\infty([0,2x_1])}) \to 0 \quad \text{as } k \to \infty. \tag{3-47}$$

Taking k_0 larger and δ smaller if necessary, by Proposition 3.2, we have, for every $k \geq k_0$,

$$\|f_k\|_{L^\infty([0,\delta];Y(I_k))} \leq C(M_f).$$

Let us fix a C^∞-smooth profile g_0 supported in $\left(\frac{1}{2}x_1, x_1\right)$ and normalized in L^2; in what follows, we omit the dependence of constants on g_0. We take $g_k(x) = 2^{k/2} g_0(2^k(x - x_k))$. For a strictly increasing sequence $\{\lambda_k\}_{k \geq k_0}$ ($\lambda_k \gg 1$) to be determined, we consider the wave packets

$$\phi_k^{\mathrm{app}}(t,x) := \phi_{(\lambda_k)}^{\mathrm{app}}[g_k, f_k],$$

where $\phi_{(\lambda_k)}^{\mathrm{app}}[g_k, f_k]$ denotes the wave packet constructed in Proposition 3.3 using the solution f_k with profile g_k and frequency λ_k. We define the corresponding error by

$$[\partial_t + \mathcal{L}_{f_k}]\phi_k^{\mathrm{app}} = \boldsymbol{\epsilon}_{\phi_k}, \tag{3-48}$$

where $\mathcal{L}_{f_k}$ is simply (3-14) with f replaced with f_k. Recall also the definition of the L^2_f norm from (3-1), and observe that $f = f_k$ on the support of ϕ_k^{app}. Applying Proposition 3.3, we obtain the following properties of ϕ_k^{app} for all $t \in [0,\delta]$:

- $\|\phi_k^{\mathrm{app}}(t,x)\|_{L^2_f} \leq C_{\tilde{f}_k}\|\phi_{k,0}^{\mathrm{app}}\|_{L^2_{f_k}} \leq C_{\tilde{f}_k}\|g_0\|_{L^2}$,
- $\|(\tilde{f}^{1/3}\partial_x)^n \phi_k^{\mathrm{app}}(t,x)\|_{L^2_f} \leq C_{\tilde{f}_k}\|\phi_{k,0}^{\mathrm{app}}\|_{L^2_{f_k}} \leq C_{\tilde{f}_k}$,
- $\|\boldsymbol{\epsilon}_{\phi_k}(t,x)\|_{L^2_f} \leq C_{\tilde{f}_k}(1 + A_k^{m-2})\lambda_k(1 + \lambda_k^2 t)$,

and, since s is even, we have

$$f^{-2\sigma_c/3}\phi_k^{\mathrm{app}} = \partial_x^s\left(\frac{f^{(-2\sigma_c+s)/3}}{(-1)^{s/2}A_k^{s/3}\lambda_k^s}\phi_k^{\mathrm{app}}\right) + f^{-2\sigma_c/3}\phi_k^{\mathrm{small}}, \tag{3-49}$$

with

- $\|(A_k^{-s/3}\lambda^{-s} f^{(-2\sigma_c+s)/3}\phi_k^{\mathrm{app}})(t,\cdot)\|_{L^1} \leq C_{\tilde{f}_k} A_k^{-\sigma_c/3}\lambda_k^{-s}\exp\left(-3\beta_k(t)A_k^{2/3}\lambda_k^2\left(-\sigma_c + s + \frac{1}{2}\right)t\right)$,
- $\|f^{-\sigma_c/3}\tilde{\phi}_k^{\mathrm{small}}(t,\cdot)\|_{L^2} \leq C_{\tilde{f}_k}\lambda_k^{-1}$,

where $\beta_k(t)$ is the solution to (3-3) with f replaced by f_k. Observe that $C_{\tilde{f}_k}$ depends on M_f, k and s, but *not* on λ_k. In view of (3-17), note that λ_k should be sufficiently large depending on k and g_0 to ensure that the first inequality in the first item holds. Define

$$\phi_0(x) := \sum_{k=k_0}^{\infty}\phi_{k,0}(x) := \sum_{k=k_0}^{\infty}\exp(-\lambda_k^{1/8})\phi_{k,0}^{\mathrm{app}}(x).$$

By ensuring some growth of λ_k (e.g., $\lambda_k \geq 2^k$) and by taking k_0 even larger if necessary, we can guarantee that $\|\phi_0\|_{C^s} < \frac{1}{2}\epsilon$, so that $u_0 := f_0 + \phi_0$ is C^∞-smooth and satisfies $\|u_0\|_{C^{s_0}} < \epsilon$. From the contradiction hypothesis, we have a $L_t^\infty C^s$ solution $u(t, x)$ to (1-9) with initial data u_0 on some time interval $[0, \delta']$. By shrinking either δ or δ', we may assume that $0 < \delta' = \delta$. We now set

$$\phi(t) := u(t) - f(t), \quad \phi_k(t) := \chi_k \phi(t)$$

for all $k \geq k_0$. Moreover, taking k_0 larger if necessary, we can easily arrange that $f_0(x)$ and $u_0(x)$ are uniformly comparable; for all x,

$$\tfrac{7}{8} u_0(x) \leq f_0(x) \leq \tfrac{9}{8} u_0(x).$$

From the conservation of the support in time, we have that $\sum_{k=k_0}^{\infty} \phi_k = \phi$. We now introduce

$$M = 1 + \sup_{t \in [0,\delta]} (\|f(t)\|_{C^s} + \|\phi(t)\|_{C^s}) \geq M_f, \tag{3-50}$$

which is finite by the contradiction hypothesis, and further replace δ with $\min\{\delta, c\}$, where $c = c(M) > 0$ is large enough that we have

$$\tfrac{1}{2} u_0(x) \leq u(t, x) \leq \tfrac{3}{2} u_0(x), \quad t \in [0, \delta], \tag{3-51}$$

whenever $u_0(x) > 0$. Note also that, by a computation similar to (3-47), we have $\|u_0\|_{Y(I_k)} \to 0$ as $k \to \infty$. Choosing k_0 sufficiently large and δ small enough, by Proposition 3.2, we have, for every $k \geq k_0$,

$$\|u_k\|_{L^\infty([0,\delta]; Y(I_k))} \leq C(M).$$

We see that $\chi_k u$ is a solution to (1-9) and it follows that ϕ_k solves

$$[\partial_t + \mathcal{L}_{f_k}]\phi_k = Q_{f_k}[\phi_k],$$

where Q_{f_k} is simply (3-15) with f replaced with f_k.

From Proposition 3.4, we have the modified energy estimate for ϕ_k,

$$\|\phi_k(t)\|_{L^2_f(I_k)} \leq C(M) \|\phi_{k,0}\|_{L^2_{f_0}(I_k)}.$$

Proceeding as in the proof of (3-42), we obtain

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} \langle \phi_k, \phi_k^{\mathrm{app}} \rangle_f \right| \leq C(M, k, s)(1 + \lambda_k(1 + \lambda_k^2 t)) \|\phi_{k,0}\|_{L^2_{f_0}(I_k)}, \quad t \in [0, \delta]. \tag{3-52}$$

We shall take $t_k := \lambda_k^{-5/3} \ll 1$ and make sure that k_0 is large enough so that $t_k \leq \delta$. Integrating (3-52) from $t = 0$ to t_k,

$$\langle \phi_k, \phi_k^{\mathrm{app}} \rangle_f(t_k) \geq \big(1 - C(M, k, s)(1 + \lambda_k(1 + \lambda_k^2 t_k))t_k\big) \|\phi_{k,0}\|_{L^2_{f_0}} \geq \tfrac{1}{2} \|\phi_{k,0}\|_{L^2_{f_0}},$$

by taking λ_k larger if necessary. On the other hand, we write

$$\langle \phi_k, \phi_k^{\mathrm{app}} \rangle_f(t_k) = \frac{1}{(-1)^{s/2} A_k^{s/3} \lambda_k^s} \langle \phi_k, \partial_x^s (f^{(-2\sigma_c + s)/3} \phi_k^{\mathrm{app}}) \rangle(t_k) + \langle f^{-\sigma_c/3} \phi_k, f^{-\sigma_c/3} \phi_k^{\mathrm{small}} \rangle(t_k).$$

Using the above estimates for ϕ_k and ϕ_k^{small} at $t = t_k$, for λ_k sufficiently large, the last term on the right-hand side is bounded by $\frac{1}{4}\|\phi_{k,0}\|_{L^2_{f_0}}$. We may therefore obtain

$$\begin{aligned}\tfrac{1}{4}\|\phi_{k,0}\|_{L^2_{f_0}} &\le A_k^{-s/3}\lambda_k^{-s}\|\partial_x^s\phi_k(t_k)\|_{L^\infty}\|f^{(-2\sigma_c+s)/3}\phi_k^{\mathrm{app}}(t_k)\|_{L^1}\\ &\le C(M,k,s)A_k^{-\sigma_c/3}\lambda_k^{-s}\exp\bigl(-3\beta_k(t)A_k^{2/3}\bigl(-\sigma_c+s+\tfrac{1}{2}\bigr)\lambda_k^{2-5/3}\bigr)\|\partial_x^s\phi_k(t_k)\|_{L^\infty}.\end{aligned}$$

Recalling that $\|\phi_{k,0}\|_{L^2_{f_0}} \ge c(k)\exp(-\lambda_k^{1/8})$, we arrive at the lower bound

$$\|\phi_k(t_k)\|_{C^s} \ge c(M,k,s)\lambda_k^s\exp\bigl(3\beta_k(t)A_k^{2/3}\bigl(-\sigma_c+s+\tfrac{1}{2}\bigr)\lambda_k^{1/3}-\lambda_k^{-1/8}\bigr).$$

Finally choosing λ_k to be sufficiently large, we may guarantee that

$$M \ge \sup_{t\in[0,\delta']}\|\phi(t)\|_{C^s} \ge \|\phi_k(t_k)\|_{C^s} \to \infty \quad \text{as } k\to\infty.$$

This contradicts the finiteness of M in (3-50), which completes the proof of Theorem 1.6. □

Appendix: Takeuchi–Mizohata ill-posedness via duality

In this appendix, we show how an application of the duality (or generalized energy) argument from [Jeong and Oh 2022] and this paper leads to simple proofs of quantitative ill-posedness results for first-order perturbations of the free Schrödinger equation related to the Takeuchi–Mizohata condition, including Proposition 1.16 (see Section A.2).

A.1. ***One-dimensional case.*** We begin with the one-dimensional first-order perturbation of the free Schrödinger equation,

$$i\partial_t u + \partial_{xx}u + b(x)\partial_x u = 0. \tag{A-1}$$

Fix $x_0 \in \mathbb{R}$. For $T > 0$ and $\mu \ge 1$, we define the weight

$$w(x) = \exp\biggl(\int_0^x \operatorname{Re}\frac{b(x')}{2}\,\mathrm{d}x'\biggr)$$

and the growth factor

$$M(T,\mu) = \inf_{(y,y_0):|y|\le\mu^{-1},|y_0|\le\mu^{-1}}\exp\biggl(\int_{x_0+y_0}^{x_0+2T+y}\operatorname{Re}\frac{b(x')}{2}\,\mathrm{d}x'\biggr).$$

Fix also $\psi_1 \in C^\infty(\mathbb{R})$ with $\operatorname{supp}\psi_1 \subseteq \{x : |x| < 1\}$ and $\|\psi_1\|_{L^2} = 1$. Given $\mu \ge 1$ (inverse spatial scale), define $\psi_{\mu,x_0} = \mu^{d/2}\psi_1(\mu(x-x_0))$. Given also $\lambda \ge 1$ (frequency, or inverse semiclassical parameter) — which in practice would be much larger than μ — define

$$\tilde{u}(t,x) = w^{-1}(x)e^{i\lambda x - i\lambda^2 t}\exp\biggl(-\int_0^{\lambda t} i\operatorname{Im} b(x-2s)\,\mathrm{d}s\biggr)\psi_{\mu,x_0}(x-2\lambda t). \tag{A-2}$$

This is a wave packet that approximately solves (A-1); see (A-7)–(A-9) below.

Proposition A.1. *Let $\tilde{u}$ be as in* (A-2), *and let u_0 satisfy*

$$\int \operatorname{Re}(u_0\overline{\tilde{u}(0)})w^2\,\mathrm{d}x = 1, \qquad \operatorname{supp} u_0 \subseteq [x_0-\mu^{-1}, x_0+\mu^{-1}].$$

Then there exists at least one corresponding solution u of (A-1) *belonging to $L^\infty_{\mathrm{loc},t}(\mathbb{R}; L^2_w)$. Assume that it furthermore satisfies $u \in L^\infty_t([0, t_f]; L^2)$ with*

$$t_f \le c\mu^{-1}, \tag{A-3}$$

where c is a constant depending only on $\|b\|_{C^{1,1}}$ and $\|\psi_1\|_{H^2}$. Then, $u(t)$ necessarily satisfies the pointwise lower bound

$$\|u(t)\|_{L^2} \ge \tfrac{1}{2}M(\lambda t, \mu)\|u_0\|_{L^2} \quad \textit{for all } 0 \le t \le t_f. \tag{A-4}$$

Proof. As discussed in Section 1.4, we consider the conjugation $v = wu$. Introducing the formally self-adjoint operator

$$\widetilde{\mathcal{L}} = \Delta + i\operatorname{Im} b(x)\partial_x + \frac{i}{2}\operatorname{Im} b_x,$$

we have the conjugation identity

$$(i\partial_t + \widetilde{\mathcal{L}})v = \Big(w^{-1}\,\partial_x^2 w + bw^{-1}\,\partial_x w + \frac{i}{2}\operatorname{Im} b_x\Big)v + w(i\partial_t + \Delta + b(x)\partial_x)u.$$

Under the assumption that $v(t) \in L^2$, we have for $t \ge 0$

$$\|v(t)\|_{L^2} \le e^{C_0 t}\|v_0\|_{L^2}, \tag{A-5}$$

where C_0 depends only on $\|b\|_{C^{1,1}}$ and $\|\psi_1\|_{H^2}$. Moreover, we also have

$$\|(i\partial_t + \widetilde{\mathcal{L}})v(t)\|_{L^2} \le C_0 e^{C_0 t}\|v_0\|_{L^2}, \tag{A-6}$$

where C_0 depends only on $\|b\|_{C^{1,1}}$ and $\|\psi_1\|_{H^2}$.

Next, we consider the standard wave packet $\tilde{v}$ for $i\partial_t + \widetilde{\mathcal{L}}$ obtained by solving (1-15) with $a = 1$, $\operatorname{Re} b = 0$, $\mathbf{\Phi}(0, x) = \lambda x$ and $\boldsymbol{a}(0, x) = \psi_{\mu,x_0}(x)$. It is given explicitly by

$$\tilde{v} = e^{i\lambda x - i\lambda^2 t}\exp\Big(-\int_0^{\lambda t} i\operatorname{Im} b(x-2s)\,\mathrm{d}s\Big)\psi_{\mu,x_0}(x-2\lambda t).$$

(Note, furthermore, that $\tilde{u} = w^{-1}\tilde{v}$.) By the definition, we clearly have, for all t,

$$\|\tilde{v}(t)\|_{L^2} = 1. \tag{A-7}$$

Moreover, by the support property of $\tilde{v}$, it follows that

$$\|w\tilde{v}(t)\|_{L^2} \le \sup_{y:|y|\le\mu^{-1}} w(x-2\lambda t + y). \tag{A-8}$$

Finally, we consider the error incurred by $\tilde{v}$. A straightforward computation gives the following:

Lemma A.2. *We have*

$$\|(i\partial_t + \widetilde{\mathcal{L}})\tilde{v}\|_{L^2} \le \widetilde{C}_0\mu^2, \tag{A-9}$$

where $\widetilde{C}_0$ depends only on $\|b\|_{C^{1,1}}$ and $\|\psi_1\|_{H^2}$.

Given the above lemma, by the self-adjointness of $\widetilde{\mathcal{L}}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle v, \tilde{v}\rangle = -\langle (i\partial_t + \widetilde{\mathcal{L}})v, i\tilde{v}\rangle + \langle iv, i(i\partial_t + \widetilde{\mathcal{L}})\tilde{v}\rangle.$$

By (A-5), (A-6), (A-7) and (A-9), we have

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\langle v, \tilde{v}\rangle\right| \le (C_0 + \widetilde{C}_0\mu^2)e^{C_0 t}\|v_0\|_{L^2}.$$

Provided that (A-3) holds with c sufficiently small compared to C_0 and $\widetilde{C}_0$, we see that

$$\langle v, \tilde{v}\rangle(t) \ge \tfrac{1}{2}\|wu_0\|_{L^2} \quad \text{for all } 0 \le t \le t_f.$$

On the one hand, by duality (i.e., Cauchy–Schwartz) and (A-8),

$$\langle v, \tilde{v}\rangle(t) = \langle u, w\tilde{v}\rangle(t) \le \sup_{y:|y|\le\mu^{-1}} w(x - 2\lambda t + y)\|u(t)\|_{L^2}.$$

On the other hand, by the support property of u_0, we have

$$\|u_0\|_{L^2} \le \sup_{y:|y|\le\mu^{-1}} w^{-1}(x_0 - 2\lambda t + y)\|wu_0\|_{L^2}.$$

Combining the preceding three inequalities, we arrive at (A-4). □

Proof of Lemma A.2. We compute, with $\psi = \psi_{\mu,x_0}$,

$$i\partial_t\tilde{v} = \lambda^2\tilde{v} + \lambda\tilde{v}\operatorname{Im} b(x - 2\lambda t) - 2i\lambda\tilde{v}\frac{\psi_x(x-2\lambda t)}{\psi(x-2\lambda t)},$$

$$\partial_x\tilde{v} = \left(i\lambda - \int_0^{\lambda t} i\operatorname{Im} b_x(x-2s)\,\mathrm{d}s + \frac{\psi_x(x-2\lambda t)}{\psi(x-2\lambda t)}\right)\tilde{v},$$

and

$$\begin{aligned}\partial_{xx}\tilde{v} = -\left(\lambda - \int_0^{\lambda t}\operatorname{Im} b_x(x-2s)\,\mathrm{d}s\right)^2\tilde{v} + 2\left(i\lambda - \int_0^{\lambda t} i\operatorname{Im} b_x(x-2s)\,\mathrm{d}s\right)\tilde{v}\frac{\psi_x(x-2\lambda t)}{\psi(x-2\lambda t)}\\ - \tilde{v}\int_0^{\lambda t} i\operatorname{Im} b_{xx}(x-2s)\,\mathrm{d}s + \tilde{v}\frac{\psi_{xx}(x-2\lambda t)}{\psi(x-2\lambda t)}.\end{aligned}$$

Then, after several direct cancellations, we have

$$\begin{aligned}(i\partial_t + \widetilde{\mathcal{L}})\tilde{v} = \lambda\left(\operatorname{Im} b(x-2\lambda t) - \operatorname{Im} b(x) + 2\int_0^{\lambda t}\operatorname{Im} b_x(x-2s)\,\mathrm{d}s\right)\tilde{v}\\ - \operatorname{Im} b\left(-\int_0^{\lambda t} i\operatorname{Im} b_x(x-2s)\,\mathrm{d}s + \frac{\psi_x(x-2\lambda t)}{\psi(x-2\lambda t)}\right)\tilde{v} + \left(\int_0^{\lambda t} i\operatorname{Im} b_x(x-2s)\,\mathrm{d}s\right)^2\tilde{v}\\ + 2\left(-\int_0^{\lambda t} i\operatorname{Im} b_x(x-2s)\,\mathrm{d}s\right)\tilde{v}\frac{\psi_x(x-2\lambda t)}{\psi(x-2\lambda t)} + \left(-\int_0^{\lambda t} i\operatorname{Im} b_{xx}(x-2s)\,\mathrm{d}s\right)\tilde{v}\\ + \frac{\psi_{xx}(x-2\lambda t)}{\psi(x-2\lambda t)}\tilde{v} + \frac{i}{2}\operatorname{Im} b_x\tilde{v}.\end{aligned}$$

Using

$$\int_0^{\lambda t} \operatorname{Im} b_x(x-2s)\,\mathrm{d}s = -\tfrac{1}{2}\operatorname{Im}(b(x-2\lambda t)-b(x))$$

we get a cancellation of remaining terms of order λ. Moreover, the same identity eliminates all integrals on the domain $[0, \lambda t]$. Using $\mu \geq 1$ to bound all the other terms by $O(\mu^2)$, the proof is complete. □

A.2. ***Multidimensional case.*** We consider the following equation on $\mathbb{R}^d$:

$$i\partial_t u + \Delta u + b^j(x)\partial_j u = 0. \tag{A-10}$$

Unfortunately, the proof of a pointwise lower bound in Proposition A.1 breaks down due to the lack of a simple physical space conjugation that removes $\operatorname{Re} b^j(x)\partial_j$. Instead, we shall prove two (conceptually) weaker statements using the duality method, including Proposition 1.16.

The first result is an unconditional *integrated* lower bound that is valid for nontrivially long (i.e., $t \gg \lambda^{-1}$) timescales. To state this result, given $x_0 \in \mathbb{R}^d$, $\omega_0 \in \mathbb{S}^{d-1}$, $\mu \geq 1$ and $T > 0$, define

$$M_{x_0,\omega_0}(T,\mu) = \inf_{y:|y|\leq \mu^{-1}} \exp\biggl(-\int_0^T \operatorname{Re} b^j(x_0+y-2s\omega_0)(\omega_0)_j\,\mathrm{d}s\biggr).$$

Fix $\psi_1 \in C^\infty(\mathbb{R}^d)$ with $\operatorname{supp}\psi_1 \subseteq \{x : |x| < 1\}$ and $\|\psi_1\|_{L^2} = 1$. Given $\mu \geq 1$, define

$$\psi_{\mu,x_0} = \mu^{d/2}\psi_1(\mu(x-x_0)).$$

Given also $\lambda \geq 1$, define (see [Mizohata 1985, §VII.2])

$$\tilde{u}(t,x) = e^{i\lambda\omega_0\cdot x - i\lambda^2 t}\exp\biggl(-\int_0^{\lambda t} b^j(x-2s\omega_0)(\omega_0)_j\,\mathrm{d}s\biggr)\psi_{\mu,x_0}(x-2\lambda\omega_0 t).$$

Proposition A.3. *Let $u \in L_t^\infty([0,t_f]; L^2)$ be a solution to equation* (A-10) *with initial data u_0 satisfying $\langle u_0, \tilde{u}(0)\rangle = 1$, where $\tilde{u}$ is determined from μ, ν and ψ_1 as above. Then as long as*

$$\mu \leq c\lambda^{1/3} \quad \textit{and} \quad t_f \leq c\lambda^{-2/3}, \tag{A-11}$$

where c is a constant depending only on $\|b\|_{C^{1,1}}$ and $\|\psi_1\|_{H^2}$, we have that $u(t)$ necessarily satisfies the averaged lower bound

$$\frac{1}{t_f}\int_0^{t_f} \frac{(1+\mu^{-1}\lambda t)^2}{(1+\mu^{-1}\lambda t_f)^2} M_{x_0,\omega_0}(\lambda t,\mu)^{-1}\|u(t)\|_{L^2}\,\mathrm{d}t \geq \tfrac{1}{6}\|u_0\|_{L^2}. \tag{A-12}$$

An example of an initial data u_0 satisfying the above hypothesis is, of course, $u_0 = \tilde{u}(0)$, in which case u is expected to behave like $\tilde{u}$. An argument similar to the proof of (A-14) shows that

$$\|\tilde{u}(t)\|_{L^2} \leq \sup_{y:|y|\leq\mu^{-1}} \exp\biggl(-\int_0^{\lambda t} \operatorname{Re} b^j(x_0+y-2s\omega_0)(\omega_0)_j\,\mathrm{d}s\biggr)\|\tilde{u}(0)\|_{L^2}.$$

Thence, provided we choose μ^{-1} to be sufficiently small depending on b, (A-12) is sharp for $\tilde{u}$ up to a constant.

Proof. We introduce $\mathcal{L}$ and its formal L^2-adjoint $\mathcal{L}^*$ (in operator notation),

$$\mathcal{L} = \Delta + b^j(x)\partial_j, \quad \mathcal{L}^* = \Delta - \partial_j \bar{b}^j(x).$$

The basis of the proof of Proposition A.3 is the generalized energy identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u_1, u_2\rangle = -\langle (i\partial_t + \mathcal{L})u_1, iu_2\rangle - \langle iu_1, (i\partial_t + \mathcal{L}^*)u_2\rangle, \tag{A-13}$$

which is a consequence of the Leibniz rule for ∂_t and $0 = \langle i\mathcal{L}u_1, u_2\rangle + \langle iu_1, i\mathcal{L}^* u_2\rangle$. A simple but important observation is that (A-13) holds even under the weak assumption $u_1 = u \in L^\infty_t([0, t_f]; L^2)$, provided that u_2 is nice enough, e.g., smooth in t, x and compactly supported in space for each fixed time.

The identity (A-13) motivates us to consider *not* a wave packet for $i\partial_t + \mathcal{L}$, but rather its adjoint $i\partial_t + \mathcal{L}^*$. Given $1 \le \mu \le \lambda$, consider

$$\tilde{u}^*(t, x) = e^{i\lambda\omega_0\cdot x - i\lambda^2 t} \exp\biggl(\int_0^{\lambda t} \bar{b}^j(x - 2s\omega_0)(\omega_0)_j \,\mathrm{d}s\biggr)\psi_{\mu,x_0}(x - 2\lambda\omega_0 t).$$

Observe that

$$\exp\biggl(\int_0^{\lambda t} \operatorname{Re}\bar{b}^j(x - 2s\omega_0)(\omega_0)_j \,\mathrm{d}s\biggr) \le M(\lambda t, \mu)^{-1} \quad \text{for } x \in \operatorname{supp}\psi_{\mu,x_0}(\cdot - 2\lambda\omega_0 t),$$

where we have introduced the abbreviation $M(\lambda t, \mu) = M_{x_0,\omega_0}(\lambda t, \mu)$. Hence, using also that $\operatorname{Re}\bar{b}^j = \operatorname{Re} b^j$, it follows that

$$\|\tilde{u}^*(t)\|_{L^2} \le M(\lambda t, \mu)^{-1}. \tag{A-14}$$

The following lemma quantifies the error $\boldsymbol{\epsilon}[\tilde{u}^*] = (i\partial_t + \Delta)\tilde{u}^* - \partial_j(\bar{b}^j(x)\tilde{u}^*)$ incurred by $\tilde{u}^*$.

Lemma A.4. *There exists a constant C_0, which depends only on $\|b\|_{C^{1,1}}$ and $\|\psi_1\|_{H^2}$, such that*

$$\|\boldsymbol{\epsilon}[\tilde{u}^*](t)\|_{L^2} \le C_0(\mu + \lambda t)^2 M(\lambda t, \mu)^{-1}. \tag{A-15}$$

We are now ready to implement the duality method. Assume for the moment that, for some $B > 0$, we have

$$\frac{1}{t_f}\biggl\| \frac{(1 + \mu^{-1}\lambda t)^2}{(1 + \mu^{-1}\lambda t_f)^2} M(\lambda t, \mu)^{-1} u \biggr\|_{L^1_t([0,t_f];L^2)} < B\|u_0\|_{L^2}. \tag{A-16}$$

By (A-13) and (A-15), we then have

$$\biggl|\frac{\mathrm{d}}{\mathrm{d}t}\langle u, \tilde{u}^*\rangle\biggr| \le C_0(\mu + \lambda t)^2 M(\lambda t, \mu)^{-1}\|u(t)\|_{L^2}.$$

Integrating in t and using the contradiction assumption, we arrive at

$$\langle u, \tilde{u}^*\rangle(t) \ge \|v_0\|_{L^2}(1 - BC_0\mu^2(1 + \mu^{-1}\lambda t_f)^2 t_f).$$

Suppose that

$$\mu \le \Bigl(\frac{1}{8BC_0}\Bigr)^{1/3}\lambda^{1/3}, \quad t_f \le \Bigl(\frac{1}{8BC_0}\Bigr)^{1/3}\lambda^{-2/3}. \tag{A-17}$$

Dividing into two cases $t_f \le \mu/\lambda$ and $t_f \ge \mu/\lambda$, it follows that

$$BC_0\mu^2(1+\mu^{-1}\lambda t_f)^2 t_f \le \tfrac{1}{2}.$$

Therefore,

$$\langle u, \tilde{u}^*\rangle(t) \ge \tfrac{1}{2}\|u_0\|_{L^2} \quad \text{for all } 0 \le t \le t_f.$$

Then, applying (A-14), we obtain the lower bound

$$\|u(t)\|_{L^2} \ge \tfrac{1}{2} M(\lambda t, \mu)\|u_0\|_{L^2}. \tag{A-18}$$

We now multiply both sides by $(1+\mu^{-1}\lambda t)^2 M(\lambda t, \mu)^{-1}$ and integrate. Since

$$\int_0^{t_f} (1+\mu^{-1}\lambda t)^2 \,\mathrm{d}t \ge \tfrac{1}{3} t_f (1+\mu^{-1}\lambda t_f)^2,$$

we arrive at

$$\frac{1}{t_f}\left\| \frac{(1+\mu^{-1}\lambda t)^2}{(1+\mu^{-1}\lambda t_f)^2} M(\lambda t, \mu)^{-1} u \right\|_{L^1_t([0,t_f];L^2)} \ge \tfrac{1}{6}\|u_0\|_{L^2}. \tag{A-19}$$

To complete the proof, we assume, for the purpose of contradiction, that (A-16) holds with $B = \frac{1}{6}$. Take $c = (3/(4C_0))^{1/3}$ in (A-11) so that (A-17) is satisfied. Then by the preceding argument, we arrive at (A-19), which is a contradiction that establishes (A-10). □

Proof of Lemma A.4. As in the proof of Lemma A.2, we compute with $\psi = \psi_{\mu,x_0}$ that

$$i\partial_t \tilde{u}^* = \lambda^2 \tilde{u}^* + i\lambda \bar{b}^j(x - 2\lambda t\omega_0)(\omega_0)_j \tilde{u}^* - 2i\lambda \frac{\omega_0 \cdot \nabla\psi}{\psi}\tilde{u}^*.$$

Introducing for simplicity

$$I_k(x) = -\int_0^{\lambda t} \partial_k \bar{b}^j(x - 2s\omega_0)(\omega_0)_j \,\mathrm{d}s, \qquad I(x) = (I_1, \ldots, I_d),$$

we have

$$\partial_k \tilde{u}^* = \left(i\lambda(\omega_0)_k - I_k + \frac{\partial_k\psi}{\psi}\right)\tilde{u}^*,$$
$$\Delta\tilde{u}^* = -\lambda^2|\omega_0|^2\tilde{u}^* - 2i\lambda\omega_0 \cdot I\tilde{u}^* + 2i\lambda\frac{\omega_0\cdot\nabla\psi}{\psi}\tilde{u}^* + \mathcal{R}[\tilde{u}^*],$$

where

$$\mathcal{R}[\tilde{u}^*] = \left(|I|^2 - \frac{2I\cdot\nabla\psi}{\psi} + \frac{|\nabla\psi|^2}{\psi^2} + \sum_k \partial_k \frac{\partial_k\psi}{\psi} - \nabla\cdot I\right)\tilde{u}^*.$$

Then, after several direct cancellations, we have

$$\begin{aligned}(i\partial_t + \widetilde{\mathcal{L}})\tilde{u}^* - \partial_j(\bar{b}^j(x)\tilde{u}^*) = (-2i\lambda\omega_0\cdot I + i\lambda\bar{b}^j(x-2\lambda t\omega_0)(\omega_0)_j - i\lambda\bar{b}^j(x)(\omega_0)_j)\tilde{u}^* \\ + \mathcal{R}[\tilde{u}^*] - \bar{b}^j(x)I_j(x)\tilde{u}^* + \frac{\bar{b}^j(x)\partial_j\psi}{\psi}\tilde{u}^* - (\partial_j\bar{b}^j(x))\tilde{u}^*\end{aligned}$$

and, as in the one-dimensional case, we use that

$$\tfrac{1}{2}(\bar{b}^j(x-2\lambda t\omega_0) - \bar{b}^j(x)) = \int_0^{\lambda t} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\bar{b}^j(x-2s\omega_0)\,\mathrm{d}s = \sum_k I_k(x)(\omega_0)_k$$

to get cancellations among the $O(\lambda)$ terms.[7] It is now not difficult to see that the remaining terms are bounded in L^2 by the right-hand side of (A-15). □

As alluded to before, the second result we shall prove using essentially the same argument is Proposition 1.16, i.e., that the failure of the Takeuchi–Mizohata condition (see (1-24)) implies *norm inflation* for (A-10).

Proof of Proposition 1.16. Assume, for contradiction, that there exists $B_0 < +\infty$ such that, for every $u_0 \in L^2$,

$$\|u\|_{L^\infty([0,\delta];L^2)} \le B_0 \|u_0\|_{L^2}. \tag{A-20}$$

By (1-24), there exists a sequence (x_n, ω_n, T_n) such that

$$M_{x_n,\omega_n}(T_n) := \exp\left(\int_0^{T_n} \operatorname{Re} b^j(x_n - 2s\omega_n)(\omega_n)_j \,\mathrm{d}x \right) \ge e^{2n}.$$

By restarting from the point $x_n + 2T\omega_n$, where $M_{x_n,\omega_n}(T) = 1$ if necessary, we may assume also that $M_{x_n,\omega_n}(T) \ge 1$ for all $0 \le T \le T_n$. Since $M_{x_n,\omega_n}(T_n, \mu) \to M_{x_n,\omega_n}(T_n)$ as $\mu \to \infty$, we may choose μ_n so that

$$M_{x_n,\omega_n}(T_n, \mu_n) \ge e^n.$$

We shall apply the argument in the proof of Proposition A.3 with the parameters

$$t_f = \frac{T_n}{\lambda_n}, \quad x_0 = x_n, \quad \omega_0 = \omega_n, \quad \mu = \mu_n, \quad \lambda = \lambda_n,$$

where λ_n shall be determined below. We denote by $\tilde{u}_n^*$ the wave packet for $i\partial_t + \widetilde{\mathcal{L}}^*$ with these parameters, and by u_n the solution to (A-10) with initial data $u_0 = \tilde{u}_n^*(0)$ satisfying (A-20). Taking λ_n to be large enough, we may guarantee that $t_f = T_n/\lambda_n \le \delta$. Then by the contradiction assumption and the bound $M(T) \ge 1$, it follows that (A-16) is satisfied for $u = u_n$ with $B = C_1 B_0$, where C_1 depends only on $\|\partial b\|_{L^\infty}$. Furthermore, choosing λ_n sufficiently large depending on B_0, C_0, C_1 and T_n, we may ensure that (A-17) holds (here, it is important that the power of λ in the second inequality is greater than -1). Thence, it follows from (A-18) and our choices of parameters that

$$\left\| u_n\left(\frac{T_n}{\lambda_n}\right) \right\|_{L^2} \ge \tfrac{1}{2} e^n \|u_n(0)\|_{L^2}.$$

Taking $n \to \infty$, we arrive at a contradiction. □

Remark A.5. We note that when $d = 1$, Proposition 1.16 is essentially a consequence of Proposition A.1, although pedantically the notion of solution is slightly different due to the presence of a conjugation when $d = 1$. On the other hand, the preceding proof applies to all $d \ge 1$.

[7] Unlike the one-dimensional case, however, we cannot eliminate the integral on the domain $[0, \lambda t]$ in I. Hence, we let the right-hand side of (A-15) depend on λt.

Acknowledgements

We thank J. Hunter for bringing equation (1-1) to our attention. We also thank Benjamin Harrop-Griffiths and Jeremy Marzuola for several insightful discussions on degenerate dispersive equations. I.-J. Jeong has been supported by the National Research Foundation of Korea (grant nos. 2022R1C1C1011051 and RS-2024-00406821). S.-J. Oh was partially supported by the Sloan Research Fellowship and the National Science Foundation CAREER Grant under NSF-DMS-1945615. I.-J. Jeong acknowledges the support from the KIAS associate member program.

References

[Akhunov 2014] T. Akhunov, "A sharp condition for the well-posedness of the linear KdV-type equation", *Proc. Amer. Math. Soc.* **142**:12 (2014), 4207–4220. MR

[Akhunov et al. 2019] T. Akhunov, D. M. Ambrose, and J. D. Wright, "Well-posedness of fully nonlinear KdV-type evolution equations", *Nonlinearity* **32**:8 (2019), 2914–2954. MR

[Ambrose and Wright 2013] D. M. Ambrose and J. D. Wright, "Dispersion vs. anti-diffusion: well-posedness in variable coefficient and quasilinear equations of KdV type", *Indiana Univ. Math. J.* **62**:4 (2013), 1237–1281. MR

[Ambrose et al. 2012] D. M. Ambrose, G. Simpson, J. D. Wright, and D. G. Yang, "Ill-posedness of degenerate dispersive equations", *Nonlinearity* **25**:9 (2012), 2655–2680. MR

[Betancourt et al. 2011] F. Betancourt, R. Bürger, K. H. Karlsen, and E. M. Tory, "On nonlocal conservation laws modelling sedimentation", *Nonlinearity* **24**:3 (2011), 855–885. MR

[Blömker and Romito 2009] D. Blömker and M. Romito, "Regularity and blow up in a surface growth model", *Dyn. Partial Differ. Equ.* **6**:3 (2009), 227–252. MR

[Blömker and Romito 2012] D. Blömker and M. Romito, "Local existence and uniqueness in the largest critical space for a surface growth model", *NoDEA* **19**:3 (2012), 365–381. MR

[Choi and Yang 2021] J. Choi and M. Yang, "New regularity criterion for suitable weak solutions of the surface growth model", *Appl. Math. Lett.* **112** (2021), art. id. 106720. MR

[Evans 2010] L. C. Evans, *Partial differential equations*, 2nd ed., Graduate Studies in Mathematics **19**, Amer. Math. Soc., Providence, RI, 2010. MR

[Germain et al. 2019] P. Germain, B. Harrop-Griffiths, and J. L. Marzuola, "Existence and uniqueness of solutions for a quasilinear KdV equation with degenerate dispersion", *Comm. Pure Appl. Math.* **72**:11 (2019), 2449–2484. MR

[Germain et al. 2020] P. Germain, B. Harrop-Griffiths, and J. L. Marzuola, "Compactons and their variational properties for degenerate KDV and NLS in dimension 1", *Quart. Appl. Math.* **78**:1 (2020), 1–32. MR

[Harrop-Griffiths 2015a] B. Harrop-Griffiths, "Large data local well-posedness for a class of KdV-type equations", *Trans. Amer. Math. Soc.* **367**:2 (2015), 755–773. MR

[Harrop-Griffiths 2015b] B. Harrop-Griffiths, "Large data local well-posedness for a class of KdV-type equations, II", *Int. Math. Res. Not.* **2015**:18 (2015), 8590–8619. MR

[Harrop-Griffiths and Marzuola 2022] B. Harrop-Griffiths and J. L. Marzuola, "Local well-posedness for a quasilinear Schrödinger equation with degenerate dispersion", *Indiana Univ. Math. J.* **71**:4 (2022), 1585–1626. MR

[Hunter and Smothers 2019] J. K. Hunter and E. B. Smothers, "On the resonant reflection of weak, nonlinear sound waves off an entropy wave", *Stud. Appl. Math.* **143**:2 (2019), 157–175. MR

[Jeong and Oh 2022] I.-J. Jeong and S.-J. Oh, "On the Cauchy problem for the Hall and electron magnetohydrodynamic equations without resistivity, I: Illposedness near degenerate stationary solutions", *Ann. PDE* **8**:2 (2022), art. id. 15. MR

[Kenig et al. 1998] C. E. Kenig, G. Ponce, and L. Vega, "Smoothing effects and local existence theory for the generalized nonlinear Schrödinger equations", *Invent. Math.* **134**:3 (1998), 489–545. MR

[Kenig et al. 2004] C. E. Kenig, G. Ponce, and L. Vega, "The Cauchy problem for quasi-linear Schrödinger equations", *Invent. Math.* **158**:2 (2004), 343–388. MR

[Lannes and Métivier 2018] D. Lannes and G. Métivier, "The shoreline problem for the one-dimensional shallow water and Green–Naghdi equations", *J. Éc. polytech. Math.* **5** (2018), 455–518. MR

[Majda et al. 1988] A. Majda, R. Rosales, and M. Schonbek, "A canonical system of integrodifferential equations arising in resonant nonlinear acoustics", *Stud. Appl. Math.* **79**:3 (1988), 205–262. MR

[Marzuola et al. 2012] J. L. Marzuola, J. Metcalfe, and D. Tataru, "Quasilinear Schrödinger equations, I: Small data and quadratic interactions", *Adv. Math.* **231**:2 (2012), 1151–1172. MR

[Marzuola et al. 2014] J. L. Marzuola, J. Metcalfe, and D. Tataru, "Quasilinear Schrödinger equations, II: Small data and cubic nonlinearities", *Kyoto J. Math.* **54**:3 (2014), 529–546. MR

[Marzuola et al. 2021] J. L. Marzuola, J. Metcalfe, and D. Tataru, "Quasilinear Schrödinger equations, III: Large data and short time", *Arch. Ration. Mech. Anal.* **242**:2 (2021), 1119–1175. MR

[Mizohata 1981] S. Mizohata, "On some Schrödinger type equations", *Proc. Japan Acad. Ser. A Math. Sci.* **57**:2 (1981), 81–84. MR

[Mizohata 1985] S. Mizohata, *On the Cauchy problem*, Notes and Reports in Mathematics in Science and Engineering **3**, Academic Press, Orlando, FL, 1985. MR

[Ożański and Robinson 2019] W. S. Ożański and J. C. Robinson, "Partial regularity for a surface growth model", *SIAM J. Math. Anal.* **51**:1 (2019), 228–255. MR

[Rosenau 1994] P. Rosenau, "Nonlinear dispersion and compact structures", *Phys. Rev. Lett.* **73**:13 (1994), 1737–1741. MR

[Rosenau 2005] P. Rosenau, "What is... a compacton?", *Notices Amer. Math. Soc.* **52**:7 (2005), 738–739. MR

[Rosenau 2006] P. Rosenau, "On a model equation of traveling and stationary compactons", *Phys. Lett. A* **356**:1 (2006), 44–50.

[Rosenau and Hyman 1993] P. Rosenau and J. M. Hyman, "Compactons: solitons with finite wavelength", *Phys. Rev. Lett.* **70**:5 (1993), 564–567.

[Rosenau and Schochet 2005] P. Rosenau and S. Schochet, "Compact and almost compact breathers: a bridge between an anharmonic lattice and its continuum limit", *Chaos* **15**:1 (2005), 015111. MR

[Rosenau and Zilburg 2018] P. Rosenau and A. Zilburg, "Compactons", *J. Phys. A* **51**:34 (2018), 343001. MR

[Simpson et al. 2007] G. Simpson, M. Spiegelman, and M. I. Weinstein, "Degenerate dispersive equations arising in the study of magma dynamics", *Nonlinearity* **20**:1 (2007), 21–49. MR

[Simpson et al. 2008] G. Simpson, M. I. Weinstein, and P. Rosenau, "On a Hamiltonian PDE arising in magma dynamics", *Discrete Contin. Dyn. Syst. Ser. B* **10**:4 (2008), 903–924. MR

[Takeuchi 1980] J. Takeuchi, "On the Cauchy problem for some non-Kowalewskian equations with distinct characteristic roots", *J. Math. Kyoto Univ.* **20**:1 (1980), 105–124. MR

[Zilburg and Rosenau 2017] A. Zilburg and P. Rosenau, "On Hamiltonian formulations of the $\mathcal{C}_1(m, a, b)$ equations", *Phys. Lett. A* **381**:18 (2017), 1557–1562. MR

[Zilburg and Rosenau 2018] A. Zilburg and P. Rosenau, "Loss of regularity in the $K(m, n)$ equations", *Nonlinearity* **31**:6 (2018), 2651–2665. MR

[Zumbrun 1999] K. Zumbrun, "On a nonlocal dispersive equation modeling particle suspensions", *Quart. Appl. Math.* **57**:3 (1999), 573–600. MR

Received 21 Dec 2023. Revised 27 May 2025. Accepted 27 Aug 2025.

IN-JEE JEONG: ijeong@kias.re.kr
School of Mathematics, Korea Institute for Advanced Study, Seoul, South Korea

SUNG-JIN OH: sjoh@math.berkeley.edu
Department of Mathematics, University of California Berkeley, Berkeley, CA, United States

mathematical sciences publishers msp

Analysis & PDE

msp.org/apde

Cover image: Eric J. Heller: “Linear Ramp”

See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2026 is US $500/year for the electronic version, and $780/year (+$75, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscriber address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004, is published continuously online.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

ANALYSIS & PDE

Volume 19 No. 4 2026