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ANTONIN CHAMBOLLE, DANIELE DE GENNARO
AND MASSIMILIANO MORINI

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DISCRETE-TO-CONTINUUM CRYSTALLINE CURVATURE FLOWS

ANTONIN CHAMBOLLE, DANIELE DE GENNARO AND MASSIMILIANO MORINI

We consider here a fully discrete variant of the implicit variational scheme for mean curvature flow, see Almgren et al. (1993) and Luckhaus and Sturzenhecker (1995), in a setting where the flow is governed by a crystalline surface tension defined by the limit of pairwise interactions energy on the discrete grid. The algorithm is based on a new discrete distance from the evolving sets, which prevents the occurrence of the spatial drift and pinning phenomena identified in Misiats and Yip (2016) and Braides et al. (2010) in a similar discrete framework. We provide the first rigorous convergence result holding in any dimension, for any initial set and for a large class of purely crystalline anisotropies, in which the spatial discretization mesh can be of the same order or coarser than the time step.

1. Introduction

We analyze a space- and time-discrete approximation of crystalline mean curvature flows of the form

$$V(x,t) = -\phi(\nu_{E(t)}(x))\kappa^{\phi}_{E(t)}(x), \quad x \in \partial E(t), \quad t \geq 0, \tag{1-1}$$

for a class of crystalline norms ϕ. We recall that an anisotropy ϕ is said to be crystalline if and only if $\{\phi \leq 1\}$ is a polytope (or, equivalently, ϕ is the support function of a polytope). Moreover, in the current paper we restrict ourselves to the case where $\{\phi \leq 1\}$ is a zonotope with rational generators [McMullen 1971; Braides and Chambolle 2024]. Here $V(x,t)$ stands for the (outer) normal velocity of the boundary $\partial E(t)$ at x, ϕ is a crystalline norm on $\mathbb{R}^N$ representing the surface tension, $\kappa^{\phi}_{E(t)}$ is the crystalline mean curvature of $\partial E(t)$ associated to ϕ, and $\nu_{E(t)}$ is the outer unit normal to $\partial E(t)$. The evolution law (1-1) has been considered to describe some phenomena in materials science and crystal growth; see, e.g., [Gurtin 1993; Taylor 1978]. Our main result is a convergence result of the discrete approximation to the continuous evolution, as the time and space steps go to zero, even in the somewhat surprising case where the space step is greater or equal to the time-step.

From the mathematical point of view, the lack of regularity of the differential operator involved in the definition of the crystalline curvature (see [Bellettini et al. 2001; Bellettini and Paolini 1996]) is the main reason why the well-posedness of the crystalline mean curvature flow in every dimension has been a long-standing open problem. After some partial results (see for instance [Almgren and Taylor 1995; Angenent and Gurtin 1989; Bellettini et al. 2006; Caselles and Chambolle 2006; Giga and Giga 2001; Giga et al. 1998; 2014]), important breakthroughs have been obtained simultaneously in [Giga and Požár 2016; 2018; 2020], where a suitable crystalline theory of viscosity solutions was developed, and with a different approach in [Chambolle et al. 2017; 2019a; 2019b], where a new notion of distributional solutions was proposed.

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Let us focus on the definition of distributional solutions, referring to the nice review [Giga and Požár 2022] for further information on viscosity solutions to (1-1): we just note that the two notions are equivalent in the setting of [Chambolle et al. 2019a, Remark 6.1]. The exact definition of distributional solutions will be recalled in Definition 2.1, but when ϕ is smooth it can be motivated as follows: it is known (see for instance [Soner 1993] for the isotropic case) that $E(t)$ evolves according to (1-1) if and only if the signed distance function $d(\cdot, t) := \mathrm{sd}^{\phi^\circ}_{E(t)}$ to $\partial E(t)$ induced by the polar norm ϕ°,[1] satisfies

$$\partial_t d \geq \operatorname{div}(\nabla\phi(\nabla d)) \quad \text{in } \{d > 0\}, \tag{1-2}$$

$$\partial_t d \leq \operatorname{div}(\nabla\phi(\nabla d)) \quad \text{in } \{d < 0\} \tag{1-3}$$

in the viscosity sense. The idea of the new definition introduced in [Chambolle et al. 2017] is to reinterpret the equations above in the distributional sense. In particular, note that replacing $\nabla\phi(\nabla u)$ by a vector field $z \in L^\infty(\{d > 0\}; \mathbb{R}^N)$ such that $z(x) \in \partial\phi(\nabla d)$ for a.e. x, where $\partial\phi$ denotes the subdifferential of ϕ, means equations (1-2) and (1-3) make sense even when ϕ is crystalline. The corresponding notion of super- and subsolutions admits a comparison principle, which yields uniqueness of the motion up to fattening. Existence is obtained either by a variant of the minimizing movements scheme of [Almgren et al. 1993; Luckhaus and Sturzenhecker 1995] in the spirit of [Chambolle 2004], which consists in building a discrete-in-time evolution obtained by a recursive minimization procedure [Chambolle et al. 2017; 2019a], or by approximation with smooth anisotropies [Chambolle et al. 2019b]. We observe that the convergence of such time-discrete approaches to a motion characterized by (1-2)–(1-3) in the *viscosity sense* was shown in [Ishii 2014], including in the two-dimensional crystalline setting, while convergence in a distributional sense was established in [Caselles and Chambolle 2006] in the convex case only. Briefly, given a time step $h > 0$ and an initial closed set $E_0 =: E^{h,0}$, one defines $E^{h,k+1} = \{u^{h,k+1} \leq 0\}$, where $u^{h,k+1}$ is defined as the minimizer of a so-called "Rudin–Osher–Fatemi" [Rudin et al. 1992] problem:

$$u^{h,k+1} \in \operatorname{argmin}\left\{\int_{\mathbb{R}^N} \phi(Du) + \frac{1}{2h}\int_{\mathbb{R}^N} |u - \mathrm{sd}^{\phi^\circ}_{E^{h,k}}|^2\right\}. \tag{1-4}$$

The idea of the present work is to combine this discretization in time with a simultaneous discretization in space for the particular class of purely crystalline anisotropies ϕ of the form

$$\phi(\nu) = \sum_{i \in \mathcal{E}} \beta(i)|i \cdot \nu|, \tag{1-5}$$

where $\beta(i) > 0$ and $\mathcal{E} \subseteq \mathbb{Z}^N \setminus \{0\}$ is a finite set of generators such that $\operatorname{Span}\mathcal{E} = \mathbb{R}^N$. These kinds of convex polytopes are known in the literature as *rational zonotopes*. The class of rational zonotopes is dense in the class of symmetric convex sets if $N = 2$, while for $N \geq 3$ it is nowhere dense. This fact is due to the strong symmetry properties of zonotopes, as every facet of a zonotope is itself a zonotope [McMullen 1971]. Note however that the Euclidean ball may be approximated by rational zonotopes in every dimension.

We now specify the discrete setting we are interested in, referring the reader to [Braides and Solci 2021] for a more thorough introduction to related topics. We consider an ε-spaced square lattice $\varepsilon\mathbb{Z}^N$

[1] The norm is defined by $\phi^\circ(x) = \sup_{\phi(\nu)\leq 1} \nu \cdot x$ and satisfies $\phi(x) = \sup_{\phi^\circ(x)\leq 1} \nu \cdot x$.

and discrete functions $u : \varepsilon\mathbb{Z}^N \to \mathbb{R}$, and we define $u_i := u(i)$. We observe that we could also consider a general finite-dimensional Bravais lattice, at the expense of more tedious notation. A natural discrete version of total variation-like energies are those appearing in Ising systems, namely energies of the form

$$TV^{\varepsilon}_{\beta}(v) := \varepsilon^{N-1} \sum_{i,j\in\varepsilon\mathbb{Z}^N} \beta(i/\varepsilon - j/\varepsilon)|v_i - v_j|, \tag{1-6}$$

where β is as in (1-5), extended to 0 in $\mathbb{Z}^N \setminus \mathcal{E}$. Under the hypotheses above on β, the functionals TV^{ε}_{β} are shown to Γ-converge[2] as $\varepsilon \to 0$ to the total variation functional

$$TV_{\phi}(v) = \int_{\mathbb{R}^N} \phi(Dv),$$

where ϕ is as in (1-5); see, e.g., [Chambolle and Kreutz 2023]. It is thus natural to define a minimizing movements scheme based on TV^{ε}_{β} which is the discrete counterpart of the minimizing procedure (1-4) as follows: given $E_0 \subseteq \mathbb{R}^N$, we define $E^0_{\varepsilon,h} = \{i \in \varepsilon\mathbb{Z}^N \mid (i + [0,\varepsilon)^N) \cap E_0 \neq \varnothing\}$, and for every $k \in \mathbb{N}$ we let $u^{k+1}_{\varepsilon,h}$ be such that

$$u^{k+1}_{\varepsilon,h} \in \operatorname{argmin}\left\{ TV^{\varepsilon}_{\beta}(v) + \frac{1}{2h} \sum_{i\in\varepsilon\mathbb{Z}^N} |v_i - (\mathrm{sd}^k_{\varepsilon,h})_i|^2 \;\middle|\; v : \varepsilon\mathbb{Z}^N \to \mathbb{R} \right\}, \tag{1-7}$$

where $\mathrm{sd}^k_{\varepsilon,h}$ denotes a suitable signed ϕ°-distance function to $E^k_{\varepsilon,h}$ defined on $\varepsilon\mathbb{Z}^N$. (Actually, the energy in (1-7) is infinite and we would rather consider the Euler–Lagrange equation of the problem.) Then, one sets $E^{k+1}_{\varepsilon,h} := \{u^{k+1}_{\varepsilon,h} \leq 0\}$.

The idea is to study the asymptotic behavior of the discrete evolutions $E^k_{\varepsilon,h}$ as both $\varepsilon, h \to 0$. A similar analysis has been performed in [Braides et al. 2010], in the planar case, for $\phi = \|\cdot\|_1$ and $\mathrm{sd}^k_{\varepsilon,h}$ the continuous signed distance function from the discrete sets $E^k_{\varepsilon,h}$ restricted to the lattice $\varepsilon\mathbb{Z}^N$; see also [Misiats and Yip 2016; Braides et al. 2016; Braides and Scilla 2013; Braides and Solci 2016; Malusa and Novaga 2018; Scilla 2020] for further related results. With this choice, if $\varepsilon \gg h$ it is easy to see that the dissipation-like term in (1-7)

$$\frac{1}{2h} \sum_{i\in\varepsilon\mathbb{Z}^N} |v_i - (\mathrm{sd}^{k+1}_{\varepsilon,h})_i|^2$$

forces the functions $u^k_{\varepsilon,h}$ to be constant as k varies, therefore producing *pinning* on the moving interfaces. Moreover, when the two scales ε, h are going to zero at the same speed it is shown in [Braides et al. 2010] that a direct implementation of the standard scheme, with the choice above for the distance, introduces a systematic error of order $\varepsilon = h$ at each step, which accumulates and produces a drift in the limiting evolution. As a result, low curvature shapes remain pinned, while sets with higher curvature evolve with a law which is a nonlinear modification of the crystalline curvature flow (1-1). Thus, the evolution law (1-1) can be approximated with the scheme of [Braides et al. 2010] only if $\varepsilon \ll h$. In [Misiats and Yip 2016], similar results are derived, still in dimension 2, for the isotropic (Euclidean) mean curvature flow.

[2] Note that we do not need to assume that the lattice generated by $\{e_k\}_{k=1,\dots,m}$ is $\mathbb{Z}^N$, which is necessary to ensure the equicoercivity of the discrete functionals.

We show in our main result, Theorem 5.2, that with a new appropriate definition of the distance $\mathrm{sd}^k_{\varepsilon,h}$ we can recover in the limit ε, $h \to 0$ the actual distributional solution to (1-1) for every initial set $E_0 \subseteq \mathbb{R}^N$, for every purely crystalline anisotropy ϕ of the form (1-5) with rational coefficients, in any dimension and irrespective of relative size of the space and time steps. In fact, the assumption of the rational character of β can be removed in the regime $\varepsilon \leq O(h)$. To the best of our knowledge this is the first general rigorous convergence result for a fully discrete scheme without restrictions on the dimension, on the initial sets and in which the spatial mesh is allowed to be of the same order or even coarser than the time step.

Let us further comment on the analysis carried out in [Braides et al. 2010] in the planar case; see also [Braides and Solci 2021] for many more references on the topic. One important change between these older results and ours is that we consider distributional solutions to the crystalline mean curvature flow (1-1) instead of relying on the characterization of the motion via ODEs, which dates back to [Almgren and Taylor 1995; Angenent and Gurtin 1989]. The latter notion of solutions is indeed suited only for planar evolutions, thus the limitation $N = 2$ in the past works. With the ODE definition and for $\phi = \|\cdot\|_1$, the authors of [Braides et al. 2010] precisely prove the following results: if $\varepsilon \ll h$ then the limiting motion is consistent with (1-1), while if $h \ll \varepsilon$ pinning happens for any nonempty initial data. As already mentioned, in the critical case $\varepsilon = h$, the limit planar motion is not driven by (1-1) but instead by a slightly modified nonlinear crystalline mean curvature flow, and pinning may happen for some particular (low curvature) initial data. This striking difference with our result may be (vaguely) justified by the following remark: While in [Braides et al. 2010] the focus is on discrete sets, we rather evolve, in accordance with the definition of distributional solutions, the *signed distance functions* to the boundaries. In this way we can effectively achieve a subpixel precision in our approximation, as $u_{\varepsilon,h}$ and the signed distance function carry more information than the evolving level set $\{u_{\varepsilon,h}(t) \leq 0\}$. Our new definition of the interpolated signed distance is detailed in Section 4.

The consistency result in this paper validates the numerical experiments which we carry out in Section 6 to illustrate our results. These experiments are derived from previous experiments in [Chambolle and Darbon 2009], which however used a different redistancing operation for which no consistency was proven. Numerical schemes based on the variational approach [Almgren et al. 1993; Luckhaus and Sturzenhecker 1995] have been introduced for crystal growth [Almgren 1993]. Since then, there have been many attempts to implement implicit schemes based on this approach for isotropic and anisotropic curvature flows in various settings [Chambolle 2004; Eto et al. 2012; Oberman et al. 2011; Požár 2018; Eto and Giga 2024]. We are however not aware of a formal convergence proof for these schemes in the fully discrete setting that does not rely on the consistency of the spatial discretization with respect to the time-discrete scheme (and hence, assuming $\varepsilon \ll h$, even if in practice these implementations seem very robust).

Many other techniques have been considered to simulate crystalline flows after [Taylor 1991; 1993]; see, e.g., [Girão 1995; Girão and Kohn 1996; Dziuk 1999] for the evolution of planar curves and [Novaga and Paolini 1999; Paolini and Pasquarelli 2000] for higher-dimensional algorithms.

Let us conclude this introduction with two comments. The first one concerns the hypothesis that ϕ is purely crystalline. It seems quite technical as it implies that the associated interaction function β (in the sense of (1-5)) has finite range. While this is not necessary to carry out the existence part for the discrete minimizing movements scheme, it is essential for building a calibration which yields a bound on the speed of Wulff shapes; see Appendix A. In practice, since the closed Wulff shape $\mathcal{W} := \{\phi^\circ \le 1\}$ is a finite Minkowski sum of (rational) segments (which is called a *zonotope*), we can effectively handcraft a calibration along the directions identified by these segments. It is a remarkable difference between this discrete setting and the continuous one, where instead the vector field $x/\phi^\circ(x)$ in $\mathbb{R}^N$ is the right calibration *for any* anisotropy ϕ.

The second one is on possible generalizations of the present analysis to more general evolution laws than (1-1). The more general evolution law which is shown to admit a unique distributional solution is

$$V(x,t) = \psi(\nu_{E(t)}(x))(-\kappa^{\phi}_{E(t)}(x) + f(x,t)), \quad x \in \partial E(t), \;\; t \ge 0, \tag{1-8}$$

where ψ is a norm (usually referred to as the *mobility*) and f is a forcing term; see [Chambolle et al. 2017; 2019a]. We expect most of the present analysis to be valid even if $\psi \neq \phi$, under suitable compatibility assumptions on ψ (see the same two works for details), and it should not be difficult to consider a driving force f as long as it is Lipschitz in space and globally bounded; see [Chambolle et al. 2019a] again.

The paper is organized as follows: In Section 2, we recall the definition of distributional crystalline curvature flows from [Chambolle et al. 2017; 2019a]. Then, we study the discrete "Rudin–Osher–Fatemi" problem and its Euler–Lagrange equation in Section 3. In Section 4, we introduce the discrete minimizing movement scheme, with our particular definition of the signed distance function. We study in detail the properties of these distances, then in Section 4.3 we analyze the particular case of an initial Wulff shape. In the continuous setting, it is well known that under the law (1-1) it decreases in a self-similar way with a speed proportional to the inverse of its radius. We show an estimate bounding the decay of the discrete Wulff shapes; it relies on the delicate construction of a calibration z for the Rudin–Osher–Fatemi problem with datum ϕ°, detailed in Appendix A.

Our main result — which is that, in the limit $\varepsilon, h \to 0$, the motion defined in Section 4 converges to a crystalline flow — is stated, and proved, in Section 5. We implemented the discrete scheme in two dimensions and show some numerical simulations in Section 6. Some technical results are collected in the appendices.

2. Distributional crystalline curvature flows

We recall the distributional formulation for the crystalline mean curvature motion of sets evolving with normal velocity (1-1) introduced in [Chambolle et al. 2017]; see also [Chambolle et al. 2019a]. Here and in what follows ϕ is any norm, ϕ° denotes the polar (or dual) norm of ϕ and, given a closed set $F \subseteq \mathbb{R}^N$, $\mathrm{dist}^{\phi^\circ}(\,\cdot\,, F)$ stands for the ϕ°-distance function from F defined by

$$\mathrm{dist}^{\phi^\circ}(x, F) := \min\{\phi^\circ(x-y) \mid y \in F\}.$$

Analogously, for any E, F closed, we set

$$\mathrm{dist}^{\phi^\circ}(E, F) := \min\{\phi^\circ(x-y) \mid x \in E,\; y \in F\}.$$

We recall that a sequence of closed sets $(E_k)_{k\geq 1}$ in $\mathbb{R}^N$ converges to a closed set E in the *Kuratowski sense* if the following conditions are satisfied:

(1) If $x_k \in E_k$ for each k, any limit point of $\{x_k\}$ belongs to E.

(2) For all $x \in E$ there exists a sequence $\{x_k\}$ such that $x_k \in E_k$ for each k and $x_k \to x$.

We will write in this case

$$E_k \xrightarrow{\mathcal{K}} E.$$

One can easily verify that $E_k \xrightarrow{\mathcal{K}} E$ if and only if (for any norm ψ) $\operatorname{dist}^\psi(\,\cdot\,, E_k) \to \operatorname{dist}^\psi(\,\cdot\,, E)$ locally uniformly in $\mathbb{R}^N$. Hence, by the Ascoli–Arzelà theorem, we have that any sequence of closed sets admits a converging subsequence in the Kuratowski sense (possibly to $\varnothing$, when $\operatorname{dist}^\psi(\,\cdot\,, E_k) \to +\infty$).

Definition 2.1. Let $E_0 \subseteq \mathbb{R}^N$ be a closed set. Let E be a closed set in $\mathbb{R}^N \times [0, +\infty)$, and for each $t \geq 0$ define $E(t) := \{x \in \mathbb{R}^N \mid (x, t) \in E\}$. We say that E is a *superflow* for (1-1) with initial datum E_0 if the following conditions are satisfied:

(a) $E(0) \subseteq E_0$.

(b) $E(s) \xrightarrow{\mathcal{K}} E(t)$ as $s \nearrow t$ for all $t > 0$.

(c) If $E(t) = \varnothing$ for some $t \geq 0$, then $E(s) = \varnothing$ for all $s > t$.

(d) Set $T^* := \inf\{t > 0 \mid E(s) = \varnothing \text{ for } s \geq t\}$ and

$$d(x, t) := \operatorname{dist}^{\phi^\circ}(x, E(t)) \quad \text{for all } (x, t) \in \mathbb{R}^N \times (0, T^*) \setminus E.$$

Then,

$$\partial_t d \geq \operatorname{div} z \tag{2-1}$$

in the distributional sense in $\mathbb{R}^N \times (0, T^*) \setminus E$ for a suitable $z \in L^\infty(\mathbb{R}^N \times (0, T^*))$ such that $z \in \partial\phi(\nabla d)$ a.e., $\operatorname{div} z$ is a Radon measure in $\mathbb{R}^N \times (0, T^*) \setminus E$, and

$$(\operatorname{div} z)^+ \in L^\infty(\{(x, t) \in \mathbb{R}^N \times (0, T^*) \mid d(x, t) \geq \delta\})$$

for every $\delta \in (0, 1)$.

We say that A, an open set in $\mathbb{R}^N \times [0, +\infty)$, is a *subflow* for (1-1) with initial datum E_0 if $\mathbb{R}^N \times [0, +\infty) \setminus A$ is a superflow for (1-1) with initial datum $\mathbb{R}^N \setminus \operatorname{int}(E_0)$.

Finally, we say that E, a closed set in $\mathbb{R}^N \times [0, +\infty)$, is a *weak flow* for (1-1) with initial datum E_0 if it is a superflow and if $\operatorname{int}(E)$ is a subflow,[3] both with initial datum E_0.

In [Chambolle et al. 2017] the next crucial inclusion principle between sub- and superflows is proven.

Theorem 2.2. *Let E be a superflow with initial datum E_0 and F be a subflow with initial datum F_0 in the sense of Definition 2.1. Assume that* $\operatorname{dist}^{\phi^\circ}(E^0, \mathbb{R}^N \setminus F^0) =: \Delta > 0$. *Then,*

$$\operatorname{dist}^{\phi^\circ}(E(t), \mathbb{R}^N \setminus F(t)) \geq \Delta \quad \textit{for all } t \geq 0$$

(*with the convention that* $\operatorname{dist}^{\phi^\circ}(G, \varnothing) = \operatorname{dist}^{\phi^\circ}(\varnothing, G) = +\infty$ *for any* G).

[3]Here we are taking the interior with respect to $\mathbb{R}^N \times [0, +\infty)$.

We also recall the corresponding notion of sub- and supersolutions to the level set flow associated with (1-1). In what follows $\mathrm{UC}(\mathbb{R}^N)$ stands for the space of uniformly continuous functions on $\mathbb{R}^N$.

Definition 2.3 (level set subsolutions and supersolutions). Let $u_0 \in \mathrm{UC}(\mathbb{R}^N)$. A lower-semicontinuous function $u : \mathbb{R}^N \times [0, +\infty) \to \mathbb{R}$ is called a *level set superflow* for (1-1), with initial datum u_0, if $u(\,\cdot\,, 0) \geq u_0$ and if for a.e. $\lambda \in \mathbb{R}$ the closed sublevel set $\{u(\,\cdot\,, t) \leq \lambda\}$ is a superflow for (1-1) in the sense of Definition 2.1, with initial datum $\{u_0 \leq \lambda\}$.

An upper-semicontinuous function $u : \mathbb{R}^N \times [0, +\infty) \to \mathbb{R}$ is called a *level set subflow* for (1-1), with initial datum u_0, if $-u$ is a level set superflow in the previous sense, with initial datum $-u_0$.

Finally, a continuous function $u : \mathbb{R}^N \times [0, +\infty) \to \mathbb{R}$ is called a *level set flow* for (1-1) if it is both a level set sub- and superflow.

Using Theorem 2.2, it is not difficult to deduce the following parabolic comparison principle between level set sub- and superflows, which yields in particular the uniqueness of level set flows (in the sense of Definition 2.3); see [Chambolle et al. 2019a].

Theorem 2.4. *Let $u_0, v_0 \in \mathrm{UC}(\mathbb{R}^N)$ and let u and v be a level set subflow starting from u_0 and a level set superflow starting from v_0, respectively. If $u_0 \leq v_0$, then $u \leq v$.*

We finally recall that in [Chambolle et al. 2017] (see also [Chambolle et al. 2019a]) the existence of level set flows is established by implementing a level-by-level minimizing movements scheme. This in turn yields existence and uniqueness (up to fattening) for weak flows. This is made precise in the following statement; see [Chambolle et al. 2017, Corollary 4.6; Chambolle et al. 2019a, Theorem 4.8].

Theorem 2.5. *Let $u_0 \in \mathrm{UC}(\mathbb{R}^N)$. Then the following hold*:

(i) *There exists a unique level set flow u in the sense of Definition 2.3 starting from u_0.*

(ii) *For all $\lambda \in \mathbb{R}$ the sets $\{(x, t) \mid u(x, t) \leq \lambda\}$ and $\{(x, t) \mid u(x, t) < \lambda\}$ are the maximal superflow and minimal subflow with initial datum $\{u_0 \leq \lambda\}$, respectively.*

(iii) *For all but countably many $\lambda \in \mathbb{R}$, the fattening phenomenon does not occur*; *that is*,

$$\begin{aligned} \{(x, t) \mid u(x, t) < \lambda\} &= \mathrm{int}(\{(x, t) \mid u(x, t) \leq \lambda\}), \\ \mathrm{cl}(\{(x, t) \mid u(x, t) < \lambda\}) &= \{(x, t) \mid u(x, t) \leq \lambda\}, \end{aligned} \tag{2-2}$$

where interior and closure are relative to space-time.

For all such λ, $\{(x, t) \mid u(x, t) \leq \lambda\}$ is the unique weak flow in the sense of Definition 2.1, starting from $\{u_0 \leq \lambda\}$.

The aim of this paper is to show that the convergence to the continuum level set flow also holds when the Euler implicit time discretization is combined with a suitable spatial discretization procedure.

3. The discrete "Rudin–Osher–Fatemi" problem

In this section, we describe our discrete setting. We then introduce and analyze the discrete variant (1-7) of the Rudin–Osher–Fatemi (ROF) problem (1-4).

3.1. ***Discrete function spaces and operators.*** For $\varepsilon > 0$, we define the function spaces $X_\varepsilon = \mathbb{R}^{\varepsilon\mathbb{Z}^N}$ and $Y_\varepsilon = \mathbb{R}^{\varepsilon\mathbb{Z}^N \times \varepsilon\mathbb{Z}^N}$. Given a function $u \in X_\varepsilon$ and a discrete "vector field" $z \in Y_\varepsilon$, with a slight abuse of notation we will write $u_i = u(i)$ and $z_{ij} = z(i, j),\ i, j \in \varepsilon\mathbb{Z}^N$. The discrete gradient $D_\varepsilon : X_\varepsilon \to Y_\varepsilon$ is defined, for $u \in X_\varepsilon$, as

$$(D_\varepsilon u)_{ij} = \frac{u_i - u_j}{\varepsilon}.$$

We denote its adjoint operator by $D^*_\varepsilon : Y_\varepsilon \to X_\varepsilon$, which is namely the operator that, for $\eta \in X_\varepsilon$ compactly supported and for $z \in Y_\varepsilon$, is defined as

$$\sum_i (D^*_\varepsilon z)_i \eta_i := \sum_{ij} z_{ij} (D_\varepsilon \eta)_{ij} = \sum_{ij} z_{ij} \frac{\eta_i - \eta_j}{\varepsilon},$$

where the indexes, here and throughout the paper, range over $\varepsilon\mathbb{Z}^N$ if not otherwise stated. In particular, taking $\eta = \chi_{\{i\}}$, one finds that

$$(D^*_\varepsilon z)_i = \sum_j \frac{z_{ij} - z_{ji}}{\varepsilon}, \tag{3-1}$$

which can be seen as a discrete divergence operator.

3.2. ***Discrete ROF problem.*** In this subsection we consider the discrete anisotropic ROF problem associated with the discrete total variation functional. Without loss of generality, we consider $\varepsilon = 1$ in this subsection, and define $X := X_1,\ Y := Y_1$ and $D := D_1$. Given a nonnegative $\beta \in X$, which will be called the *interaction function*, satisfying

$$\sum_{i \in \mathbb{Z}^N} \beta(i) =: c_\beta < +\infty, \tag{3-2}$$

we set $\alpha_{ij} = \beta(i - j)$ and, for any $u \in X$, we define

$$TV(u) = \sum_{i,j \in \mathbb{Z}^N} \alpha_{ij} |u_i - u_j| = \sum_{i,j} \alpha_{ij} |(Du)_{i,j}|. \tag{3-3}$$

We also consider the discrete perimeter $\mathcal{P}$ defined for every $E \subseteq \mathbb{Z}^N$ as

$$\mathcal{P}(E) := TV(\chi^E) = \sum_{i,j \in \mathbb{Z}^N} \alpha_{ij} |\chi^E_i - \chi^E_j|.$$

We also consider a suitable localization of the perimeter: namely, for any set $A \subseteq \mathbb{R}^N$, we define

$$\mathcal{P}(E; A) = \sum_{i \in A \cap \mathbb{Z}^N \text{ or } j \in A \cap \mathbb{Z}^N} \alpha_{ij} |\chi^E_i - \chi^E_j|.$$

Note that the quantities above may well be infinite.

Then, given $g \in X$, we consider the following problem: find a pair $(u, z) \in X \times Y$ such that

$$\begin{cases} D^* z + u = g, \\ z_{ij}(u_i - u_j) = \alpha_{ij} |u_i - u_j|,\ |z_{ij}| \le \alpha_{ij} \quad \text{for all } i, j \in \mathbb{Z}^N. \end{cases} \tag{3-4}$$

The equation above is the Euler–Lagrange equation of the discrete ROF functional

$$\mathrm{ROF}_g(v) = TV(v) + \frac{1}{2}\sum_{i\in\mathbb{Z}^N}(v_i - g_i)^2. \tag{3-5}$$

However, (3-4) makes sense also for those g such that $\mathrm{ROF}_g \equiv +\infty$. That (3-4) is the first-order condition for optimality in (3-5) follows from standard convex analysis: the idea is that, since

$$TV(v) = \sup\{\langle z, Dv\rangle \mid |z_{i,j}| \le \alpha_{i,j}\ \forall (i, j)\},$$

the subgradients $\partial TV(v)$ of TV at v are precisely given by the vectors D^*z for those z which realize the supremum in this expression. Then, for g with bounded support (such that there is at least some u with finite energy), (3-4) requires that $0 \in \partial\mathrm{ROF}_g(u)$, which by definition is the condition for the minimality of u.

We will also consider the following geometric minimization problem. Given $g \in X$, find

$$\min_{F\subseteq\mathbb{Z}^N} \mathcal{P}(F) + \sum_{i\in\mathbb{Z}^N}\chi_i^F g_i. \tag{3-6}$$

In order to deal with unbounded sets, possibly with infinite perimeter, we will consider the following notion of global minimality with respect to compactly supported perturbations:

Definition 3.1. A set $E \subseteq \mathbb{Z}^N$ is a global minimizer for the problem (3-6) if for every $R > 0$

$$\mathcal{P}(E; B_R) + \sum_{|i|<R}\chi_i^E g_i \le \mathcal{P}(F; B_R) + \sum_{|i|<R}\chi_i^F g_i \tag{3-7}$$

for every $F \subseteq \mathbb{Z}^N$ such that $F \triangle E \subseteq B_R$. Here $B_R = \{x \in \mathbb{R}^N \mid |x| < R\}$ is the open ball of radius R centered at the origin.

Proposition 3.2. *Let $g, g' \in X$ be such that $g' - g \ge \delta > 0$. Let E and E' be two global minimizers of problem* (3-7), *in the sense of Definition 3.1, corresponding to g and g', respectively. Then, $E' \subseteq E$.*

Proof. Let us define in the following $\chi := \chi^{E_s}$ and $\chi' := \chi^{E'_s}$. For a given $R > 0$ we define the competitor sets $F = (E_s \setminus B_R) \cup ((E'_s \cup E_s) \cap B_R)$ and $F' = (E'_s \setminus B_R) \cup ((E'_s \cap E_s) \cap B_R)$. By minimality of E_s and E'_s in B_R one has

$$\sum_{|i|<R \text{ or } |j|<R} \alpha_{ij}|\chi'_i - \chi'_j| + \sum_{|i|<R} g'_i(\chi'_i - \chi'_i \wedge \chi_i)$$
$$\le \sum_{\substack{|i|<R\\|j|<R}} \alpha_{ij}|\chi'_i\wedge\chi_i - \chi'_j\wedge\chi_j| + \sum_{\substack{|i|<R\\|j|\ge R}} (\alpha_{ij}+\alpha_{ji})|\chi'_i\wedge\chi_i - \chi'_j|, \tag{3-8}$$

$$\sum_{|i|<R \text{ or } |j|<R} \alpha_{ij}|\chi_i - \chi_j| + \sum_{|i|<R} g_i(\chi_i - \chi'_i \vee \chi_i)$$
$$\le \sum_{\substack{|i|<R\\|j|<R}} \alpha_{ij}|\chi'_i\vee\chi_i - \chi'_j\vee\chi_j| + \sum_{\substack{|i|<R\\|j|\ge R}} (\alpha_{ij}+\alpha_{ji})|\chi'_i\vee\chi_i - \chi_j|. \tag{3-9}$$

Using the inequality[4] $|a\wedge b-c\wedge d|+|a\vee b-c\vee d|\le|a-c|+|b-d|$ and summing together (3-8) and (3-9) we obtain

$$\sum_{\substack{|i|<R\\|j|\ge R}}(\alpha_{ij}+\alpha_{ji})(|\chi_i-\chi_j|+|\chi_i'-\chi_j'|)+2\sum_{|i|<R}(g_i'-g_i)(\chi_i'-\chi_i)^+ \le \sum_{\substack{|i|<R\\|j|\ge R}}(\alpha_{ij}+\alpha_{ji})(|\chi_i'\wedge\chi_i-\chi_j'|+|\chi_i'\vee\chi_i-\chi_j|). \tag{3-10}$$

We then remark that $|\chi_i'\wedge\chi_i-\chi_j'|\le|\chi_i'\wedge\chi_i-\chi_i'|+|\chi_i'-\chi_j'|=(\chi_i'-\chi_i)^++|\chi_i'-\chi_j'|$ and analogously $|\chi_i'\vee\chi_i-\chi_j|\le(\chi_i'-\chi_i)^++|\chi_i-\chi_j|$. Therefore, (3-10) implies

$$\sum_{|i|<R}(g_i'-g_i)(\chi_i'-\chi_i)^+\le\sum_{|i|<R}(\chi_i'-\chi_i)^+\sum_{|j|\ge R}(\alpha_{ij}+\alpha_{ji}). \tag{3-11}$$

Fix now $R_\delta>0$ such that

$$\sum_{|k|\ge R_\delta}\beta(k)\le\tfrac14\delta,$$

and define $V_R:=\sum_{|i|<R}(\chi_i'-\chi_i)^+$. Assuming $R>R_\delta$, for every $\ell<R$, we use (3-11) and $g+\delta\le g'$ to get

$$\begin{aligned}\delta V_R&\le\sum_{|i|<\ell}(\chi_i'-\chi_i)^+\sum_{|j|\ge R}(\alpha_{ij}+\alpha_{ji})+2c_\beta\sum_{\ell\le|i|<R}(\chi_i'-\chi_i)^+\\&\le2\sum_{|i|<\ell}(\chi_i'-\chi_i)^+\sum_{|k|\ge R-\ell}\beta(k)+2c_\beta(V_R-V_\ell).\end{aligned} \tag{3-12}$$

Therefore, choosing $\ell=R-R_\delta$ in (3-12), we obtain

$$\tfrac12\delta V_R\le2c_\beta(V_R-V_{R-R_\delta}), \tag{3-13}$$

which implies that for every $k,\ell\in\mathbb{N}$

$$V_{kR_\delta}\le\left(1-\frac{\delta}{4c_\beta}\right)^\ell V_{(k+\ell)R_\delta}. \tag{3-14}$$

Letting $\ell\to+\infty$, since $V_{(k+\ell)R_\delta}=O(\ell^N)$, we infer that $V_{kR_\delta}=0$ for every $k\in\mathbb{N}$. In particular, this implies that $(\chi'-\chi)^+=0$, i.e., $\chi'\le\chi$. □

We will prove the following theorem.

Theorem 3.3. *Given $g\in X$ there exists a unique function $u^g\in X$ and there exists a discrete vector field $z\in Y$ such that (u^g,z) is a solution of* (3-4). *Moreover, the following comparison principle holds: if $g\le g'$ then $u^g\le u^{g'}$. Finally, for any $R>0$ and $s\in\mathbb{R}$, the sublevel set $E_s:=\{i\in\mathbb{Z}^N\mid u_i^g\le s\}$ is a global minimizer (in the sense of Definition 3.1) for* (3-6) *with g replaced by $g-s$.*

[4]Indeed, if $a\ge b$ and $c\ge d$, this is an equality, while if $a>b$ and $c<d$, one deduces that $b-d<a-d<a-c$, $b-d<b-c<a-c$ so that there exists $t\in(0,1)$ with $a-d=t(b-d)+(1-t)(a-c)$, $b-c=(1-t)(b-d)+t(a-c)$: the conclusion follows by convexity of $|\cdot|$.

Proof. **Step 1.** (existence). For every $n \in \mathbb{N}$ set $g^n := g\chi^{B_n}$ and note that $g^n \in \ell^2(\mathbb{Z}^N)$. Therefore, by standard methods and by strict convexity, the functional (3-5), with g replaced by g^n, admits a unique minimizer u^n and, as previously observed, the optimality condition is the existence of a discrete field z^n such that (u^n, z^n) solves (3-4) (with g^n in place of g). Note that, for any $k \in \mathbb{Z}^N$, by (3-4),

$$|u^n_k| \le |g^n_k| + |(D^*z)_k| \le |g_k| + c_\beta \quad \text{for every } n \in \mathbb{N}, \tag{3-15}$$

where the last inequality follows from the definition (3-1) and from $|z_{ij}| \le \alpha_{ij}$ and $|g^n| \le |g|$. Now, it is clear that we can extract a subsequence n_k and find (u, z) such that $u_i^{n_k} \to u_i$ and $z_{ij}^{n_k} \to z_{ij}$ as $k \to +\infty$. Clearly we have that $|z_{ij}| \le \alpha_{ij}$ and $z_{ij}(u_i - u_j) = \alpha_{ij}|u_i - u_j|$, and it is immediate to check that (u, z) satisfies (3-4).

Step 2. (minimality of the sublevel sets). Let $R > 0$, $s \in \mathbb{R}$ and let $F \subseteq \mathbb{Z}^N$ such that $E_s \triangle F \Subset B_R$. We first remark that $\alpha_{ij}|\chi_i^{E_s} - \chi_j^{E_s}| = -z_{ij}(\chi_i^{E_s} - \chi_j^{E_s})$, which follows easily from the definition of E_s and $z_{ij}(u_i - u_j) = \alpha_{ij}|u_i - u_j|$.

We set $I_R := \{(i, j) \in \mathbb{Z}^N \times \mathbb{Z}^N \mid |i| < R \text{ or } |j| < R\}$ and compute

$$\begin{aligned}
\mathcal{P}(F; B_R) - \mathcal{P}(E_s; B_R) &= \sum_{(i,j)\in I_R} \alpha_{ij}|\chi_i^F - \chi_j^F| - \sum_{(i,j)\in I_R} \alpha_{ij}|\chi_i^{E_s} - \chi_j^{E_s}| \\
&\ge -\sum_{(i,j)\in I_R} z_{ij}(\chi_i^F - \chi_j^F) + \sum_{(i,j)\in I_R} z_{ij}(\chi_i^{E_s} - \chi_j^{E_s}) \\
&= \sum_{(i,j)\in I_R} z_{ij}(\chi_i^{E_s} - \chi_i^F - (\chi_j^{E_s} - \chi_j^F)) \\
&= \sum_{ij} z_{ij}(\chi_i^{E_s} - \chi_i^F - (\chi_j^{E_s} - \chi_j^F)),
\end{aligned} \tag{3-16}$$

where in the last equality we used the fact that $\chi_i^{E_s} = \chi_i^F$ if $|i| \ge R$. Noting that the function $\chi^{E_s} - \chi^F$ is compactly supported, we may use it as a test function for (3-4). Therefore, from (3-16) we deduce

$$\begin{aligned}
\mathcal{P}(F; B_R) - \mathcal{P}(E_s; B_R) &\ge \sum_{ij} z_{ij}(\chi_i^{E_s} - \chi_i^F - (\chi_j^{E_s} - \chi_j^F)) \\
&= \sum_i (\chi_i^{E_s} - \chi_i^F)(g_i - u_i) \ge \sum_{i\in E_s\setminus F}(g_i - s) - \sum_{i \in F\setminus E_s}(g_i - s),
\end{aligned}$$

which shows the minimality of E_s.

Step 3. (comparison and uniqueness for (3-4)). Assume $g \le g'$, and let (u, z) and (u', z') be two corresponding solutions for (3-4). Let $s > s'$, and recall that by Step 2 $\{u' \le s'\}$ and $\{u \le s\}$ are global minimizers for (3-6) according to Definition 3.1, with g replaced by $g' - s'$ and $g - s$, respectively. Since $g' - s' - (g - s) \ge s - s' > 0$, from Proposition 3.2 we obtain $\{u' \le s'\} \subseteq \{u \le s\}$. By the arbitrariness of s and s' we conclude that $u \le u'$. □

Remark 3.4. We remark that, given $g \in X$, clearly $u^{-g} = -u^g$.

4. The minimizing movements scheme

In this section we provide a combined spatial and time discretization of the flow (1-1) for a particular class of norms ϕ and show the convergence of the scheme to the continuum flow. In what follows, we consider $\{e_1, \dots, e_m\} \subseteq \mathbb{Z}^N$, a finite number of integer vectors spanning the whole $\mathbb{R}^N$, and set $\mathcal{E} = \{\pm e_k\}_{k=1}^m$. We let $\beta \in X$ be a nonnegative function such that

$$\beta(-i) = \beta(i) \quad \text{and} \quad \beta(i) > 0 \ \text{ if and only if } \ i \in \mathcal{E}.$$

One can naturally associate an anisotropy ϕ with the function β by setting

$$\phi(v) = \sum_{i\in\mathcal{E}} \beta(i)|i \cdot v| = \sum_{k=1}^{m} 2\beta(e_k)|v \cdot e_k|. \tag{4-1}$$

Note that, in particular,

$$\#\{k \in \mathbb{Z}^N \mid \beta(k) \neq 0\} < +\infty. \tag{4-2}$$

We recall that the ϕ-perimeter associated with (4-1),

$$P_\phi(E) = \int_{\partial^* E} \phi(\nu_E)\, d\mathcal{H}^{N-1},$$

(defined for every $E \subseteq \mathbb{R}^N$ of finite perimeter) is the Γ-limit (in a suitable sense) as $\varepsilon \to 0$ of the scaled discrete perimeters

$$\mathcal{P}^\varepsilon(E) := \varepsilon^{N-1} \sum_{i,j\in\varepsilon\mathbb{Z}^N} \alpha^\varepsilon_{ij} |\chi^E_i - \chi^E_j| = \varepsilon^N \sum_{i,j\in\varepsilon\mathbb{Z}^N} \alpha^\varepsilon_{i,j} |(D_\varepsilon \chi^E)_{i,j}|$$

defined for all $E \subseteq \varepsilon\mathbb{Z}^N$; see for instance [Braides and Chambolle 2024]. Here we have set

$$\alpha^\varepsilon_{ij} := \beta\Big(\frac{i}{\varepsilon} - \frac{j}{\varepsilon}\Big). \tag{4-3}$$

Given ϕ a norm on $\mathbb{R}^N$ and a closed set $E \notin \{\varnothing, \mathbb{R}^N\}$, we denote by $\mathrm{sd}^{\phi^\circ}_E$ the signed ϕ°-distance function from E and define it as

$$\mathrm{sd}^{\phi^\circ}_E(x) := \min_{y\in E} \phi^\circ(x-y) - \min_{y\notin E} \phi^\circ(x-y).$$

We also set $\mathrm{sd}^{\phi^\circ}_\varnothing \equiv +\infty$ and $\mathrm{sd}^{\phi^\circ}_{\mathbb{R}^N} \equiv -\infty$. We write

$$C_\phi = \min_{i\in\mathbb{Z}^N\setminus\{0\}} \phi^\circ(i) > 0 \tag{4-4}$$

and define the ϕ-Wulff shape $\mathcal{W}_R(x)$ of radius $R > 0$ and center $x \in \mathbb{R}^N$ as $\mathcal{W}_R(x) = \{y \in \mathbb{R}^N \mid \phi^\circ(x-y) \le R\}$.

4.1. ***A discrete redistancing operator.*** In this subsection we introduce a discrete proxy for the signed distance function to a set and study some of its properties.

Given $u \in X_\varepsilon$ we define the operators $d_\pm^{\varepsilon,\phi^\circ}, \mathrm{sd}_\pm^{\varepsilon,\phi^\circ}, \mathrm{sd}^{\varepsilon,\phi^\circ} : X_\varepsilon \to X_\varepsilon$ in the following way: letting $E = \{i \in \varepsilon\mathbb{Z}^N \mid u_i \le 0\}$, we first set

$$\begin{aligned} (d_-^{\varepsilon,\phi^\circ}(u))_i &= \sup_{j\in\{u\ge 0\}} \{u_j - \phi^\circ(i-j)\}, \\ (\mathrm{sd}_-^{\varepsilon,\phi^\circ}(u))_i &= \inf_{j\in\{u\le 0\}} \{(d_-^{\varepsilon,\phi^\circ}(u))_j + \phi^\circ(i-j)\}, \\ (d_+^{\varepsilon,\phi^\circ}(u))_i &= \inf_{j\in\{u\le 0\}} \{u_j + \phi^\circ(i-j)\}, \\ (\mathrm{sd}_+^{\varepsilon,\phi^\circ}(u))_i &= \sup_{j\in\{u\ge 0\}} \{(d_+^{\varepsilon,\phi^\circ}(u))_j - \phi^\circ(i-j)\}, \\ (\mathrm{sd}^{\varepsilon,\phi^\circ}(u))_i &= \tfrac{1}{2}(\mathrm{sd}_+^{\varepsilon,\phi^\circ}(u))_i + \tfrac{1}{2}(\mathrm{sd}_-^{\varepsilon,\phi^\circ}(u))_i. \end{aligned} \tag{4-5}$$

Note that $d_+^{\varepsilon,\phi^\circ}(u) = -d_-^{\varepsilon,\phi^\circ}(-u)$ and $\mathrm{sd}_+^{\varepsilon,\phi^\circ}(u) = -\mathrm{sd}_-^{\varepsilon,\phi^\circ}(-u)$.

We will say that $f \in X_\varepsilon$ is (L, ϕ°)-Lipschitz if for all $i, j \in \varepsilon\mathbb{Z}^N$ we have $|f_i - f_j| \le L\phi^\circ(i-j)$.

Remark 4.1. We assume in what follows that u is $(1, \phi^\circ)$-Lipschitz. Then, concerning $d_-^{\varepsilon,\phi^\circ}$ and $\mathrm{sd}_-^{\varepsilon,\phi^\circ}$, we remark that

$$d_-^{\varepsilon,\phi^\circ}(u) = \min\{f \in X_\varepsilon \mid f \ge u \text{ in } \{u \ge 0\},\ f \text{ is } (1,\phi^\circ)\text{-Lipschitz}\}, \tag{4-6}$$

and analogously

$$\mathrm{sd}_-^{\varepsilon,\phi^\circ}(u) = \max\{f \in X_\varepsilon \mid f \le d_-^{\varepsilon,\phi^\circ}(u) \text{ in } \{u \le 0\},\ f \text{ is } (1,\phi^\circ)\text{-Lipschitz}\}. \tag{4-7}$$

Correspondingly we have

$$\begin{aligned} d_+^{\varepsilon,\phi^\circ}(u) &= \max\{f \in X_\varepsilon \mid f \le u \text{ in } \{u \le 0\},\ f \text{ is } (1,\phi^\circ)\text{-Lipschitz}\}, \\ \mathrm{sd}_+^{\varepsilon,\phi^\circ}(u) &= \min\{f \in X_\varepsilon \mid f \ge d_+^{\varepsilon,\phi^\circ}(u) \text{ in } \{u \ge 0\},\ f \text{ is } (1,\phi^\circ)\text{-Lipschitz}\}. \end{aligned} \tag{4-8}$$

In particular, the functions $d_\pm^{\varepsilon,\phi^\circ}(u)$, $\mathrm{sd}_\pm^{\varepsilon,\phi^\circ}(u)$, $\mathrm{sd}^{\varepsilon,\phi^\circ}(u)$ are also $(1, \phi^\circ)$-Lipschitz. Let us show (4-6), the other identities being analogous. To this aim, denote by $\hat{d}$ the function defined by the right-hand side of (4-6). Since $d_-^{\varepsilon,\phi^\circ}(u)$ is the pointwise supremum of $(1, \phi^\circ)$-Lipschitz functions, we clearly have that $d_-^{\varepsilon,\phi^\circ}(u)$ is itself $(1, \phi^\circ)$-Lipschitz. Moreover, testing with $j = i$ in the definition of $d_-^{\varepsilon,\phi^\circ}(u)$, we get $d_-^{\varepsilon,\phi^\circ}(u) \ge u$ in $\{u \ge 0\}$. Thus, we infer $\hat{d} \le d_-^{\varepsilon,\phi^\circ}(u)$. For the opposite inequality, let f be any function as in the minimization problem on the right-hand side of (4-6). Then for any $i \in \varepsilon\mathbb{Z}^N$ and $j \in \{u \ge 0\}$ we have

$$f_i \ge f_j - \phi^\circ(i-j) \ge u_j - \phi^\circ(i-j).$$

By maximizing with respect to $j \in \{u \ge 0\}$, we get $f \ge d_-^{\varepsilon,\phi^\circ}(u)$ and in turn, by the arbitrariness of f, $\hat{d} \ge d_-^{\varepsilon,\phi^\circ}(u)$, which concludes the proof of (4-6)

Since the functions $d_\pm^{\varepsilon,\phi^\circ}(u)$, $\mathrm{sd}_\pm^{\varepsilon,\phi^\circ}(u)$, $\mathrm{sd}^{\varepsilon,\phi^\circ}(u)$ are $(1, \phi^\circ)$-Lipschitz, from (4-6) it follows that

$$d_-^{\varepsilon,\phi^\circ}(u) \le u \ \text{ in } \varepsilon\mathbb{Z}^N, \quad d_-^{\varepsilon,\phi^\circ}(u) = u \ \text{ in } \{u \ge 0\}, \tag{4-9}$$

while (4-7) implies that

$$\mathrm{sd}_-^{\varepsilon,\phi^\circ}(u) \ge d_-^{\varepsilon,\phi^\circ}(u) \ \text{ in } \varepsilon\mathbb{Z}^N, \quad \mathrm{sd}_-^{\varepsilon,\phi^\circ}(u) = d_-^{\varepsilon,\phi^\circ}(u) \ \text{ in } \{u \le 0\}. \tag{4-10}$$

Reasoning in the same way, we see that

$$\begin{aligned} d_+^{\varepsilon,\phi^\circ}(u) &\ge u && \text{in } \varepsilon\mathbb{Z}^N, & d_+^{\varepsilon,\phi^\circ}(u) &= u && \text{in } \{u \le 0\}, \\ \mathrm{sd}_+^{\varepsilon,\phi^\circ}(u) &\le d_+^{\varepsilon,\phi^\circ}(u) && \text{in } \varepsilon\mathbb{Z}^N, & \mathrm{sd}_+^{\varepsilon,\phi^\circ}(u) &= d_+^{\varepsilon,\phi^\circ}(u) && \text{in } \{u \ge 0\}. \end{aligned} \tag{4-11}$$

In particular we conclude

$$\mathrm{sd}^{\varepsilon,\phi^\circ}(u) \ge u \ \text{ in } \{u \ge 0\}, \quad \mathrm{sd}^{\varepsilon,\phi^\circ}(u) \le u \ \text{ in } \{u \le 0\}. \tag{4-12}$$

Note that (4-12) implies $\{\mathrm{sd}_+^{\varepsilon,\phi^\circ}(u) \ge 0\} \supseteq \{u \ge 0\}$, and (4-11) yields $\{\mathrm{sd}_+^{\varepsilon,\phi^\circ}(u) < 0\} \supseteq \{u < 0\}$; thus $\{\mathrm{sd}_+^{\varepsilon,\phi^\circ}(u) \ge 0\} = \{u \ge 0\}$ (and analogously for $\mathrm{sd}_-^{\varepsilon,\phi^\circ}$). Similarly, one shows that $\{\mathrm{sd}_\pm^{\varepsilon,\phi^\circ}(u) \le 0\} = \{u \le 0\}$. In particular, if the level set 0 of u is "fat", then this is preserved by these discrete "signed distance functions". Further properties of these discrete signed distance functions are presented in Lemma 4.3 below and in Remark 4.9

Moreover, it follows directly from the definition of $d_\pm^{\varepsilon,\phi^\circ}(u)$, $\mathrm{sd}_\pm^{\varepsilon,\phi^\circ}(u)$ that the function $\mathrm{sd}^{\varepsilon,\phi^\circ}(u)$ is invariant under integer translations, meaning that, for any $i, \tau \in \varepsilon\mathbb{Z}^N$,

$$\big(\mathrm{sd}^{\varepsilon,\phi^\circ}(u(\cdot + \tau))\big)_i = (\mathrm{sd}^{\varepsilon,\phi^\circ}(u))_{i+\tau}. \tag{4-13}$$

We now show that the redistancing operator $\mathrm{sd}^\varepsilon(u)$ is indeed a discrete approximation of the signed distance function to the 0-sublevel set of the function u.

Given a set $E \subseteq \varepsilon\mathbb{Z}^N$, we will denote with $\widehat{E} \subseteq \mathbb{R}^N$ the closed set defined by

$$\widehat{E} := E + [0, \varepsilon]^N.$$

Lemma 4.2. *Given a $(1, \phi^\circ)$-Lipschitz function $u \in X_\varepsilon$, we have*

$$\sup_{\varepsilon\mathbb{Z}^N \setminus E} |\mathrm{sd}_\pm^{\varepsilon,\phi^\circ}(u) - \mathrm{sd}_{\widehat{E}}^{\phi^\circ}| \le c_\phi \varepsilon \tag{4-14}$$

for a suitable positive constant c_ϕ, where $E = \{i \in \varepsilon\mathbb{Z}^N \mid u_i \le 0\}$. Moreover,

$$\mathrm{sd}_\pm^{\varepsilon,\phi^\circ}(u) \ge \mathrm{sd}_{\widehat{E}}^{\phi^\circ} - c_\phi\varepsilon \quad \text{in } \varepsilon\mathbb{Z}^N. \tag{4-15}$$

Proof. In this proof we let c_ϕ denote a positive constant which depends on ϕ and that may change from line to line and also within the same line.

We start by introducing a slightly modified definition of the discrete signed distance $\mathrm{sd}^{\varepsilon,\phi^\circ}(u)$. Namely, setting

$$\begin{aligned} \partial_\varepsilon^+ E &:= \{i \in \varepsilon\mathbb{Z}^N \setminus E \mid \exists j \in E \text{ with } \|i - j\|_\infty = \varepsilon\}, \\ \partial_\varepsilon^- E &:= \{i \in E \mid \exists j \in \varepsilon\mathbb{Z}^N \setminus E \text{ with } \|i - j\|_\infty = \varepsilon\}, \end{aligned} \tag{4-16}$$

we define

$$\tilde{d}_i = \begin{cases} \inf\{u_j + \phi^\circ(i - j) \mid j \in \partial_\varepsilon^- E\} & \text{for } i \in \varepsilon\mathbb{Z}^N \setminus E, \\ \sup\{u_j - \phi^\circ(i - j) \mid j \in \partial_\varepsilon^+ E\} & \text{for } i \in E. \end{cases} \tag{4-17}$$

We start by showing that

$$\begin{aligned} \mathrm{sd}_{\pm}^{\varepsilon,\phi^\circ}(u) &\geq \tilde{d} \quad \text{in } E, \\ \mathrm{sd}_{\pm}^{\varepsilon,\phi^\circ}(u) &\leq \tilde{d} \quad \text{in } \varepsilon\mathbb{Z}^N \setminus E. \end{aligned} \tag{4-18}$$

Indeed, we note that for every $i \in E$ we have

$$(\mathrm{sd}_{-}^{\varepsilon,\phi^\circ}(u))_i = (d_{-}^{\varepsilon,\phi^\circ}(u))_i = \sup_{j\in\{u\geq 0\}} \{u_j - \phi^\circ(i-j)\} \geq \sup_{j\in\partial_\varepsilon^+ E} \{u_j - \phi^\circ(i-j)\} = \tilde{d}_i.$$

On the other hand, recalling that $d_{-}^{\varepsilon,\phi^\circ}(u) \leq u$ in E, for every $i \in \varepsilon\mathbb{Z}^N \setminus E$ we see

$$(\mathrm{sd}_{-}^{\varepsilon,\phi^\circ}(u))_i = \inf_{j\in\{u\leq 0\}} \{(d_{-}^{\varepsilon,\phi^\circ}(u))_j + \phi^\circ(i-j)\} \leq \inf_{j\in\partial_\varepsilon^- E} \{u_j + \phi^\circ(i-j)\} = \tilde{d}_i.$$

Reasoning analogously we show the same inequalities between $\mathrm{sd}_{+}^{\varepsilon,\phi^\circ}$ and $\tilde{d}$ and thus prove (4-18).

Next, we prove

$$\sup_{\varepsilon\mathbb{Z}^N} |\tilde{d} - \mathrm{sd}_{\widehat{E}}^{\phi^\circ}| \leq c_\phi \varepsilon. \tag{4-19}$$

Recall that by definition (4-16), since $u \leq 0$ in E and $u > 0$ in $\varepsilon\mathbb{Z}^N \setminus E$ and since u is $(1, \phi^\circ)$-Lipschitz, we have

$$|u_j| \leq c_\phi \varepsilon \quad \text{for } j \in \partial_\varepsilon^\pm E.$$

Then, for every $i \in \varepsilon\mathbb{Z}^N \setminus E$, we have

$$\tilde{d}_i = \inf_{j\in\partial_\varepsilon^- E} \{u_j + \phi^\circ(i-j)\} \geq \inf_{j\in\partial_\varepsilon^- E} \phi^\circ(i-j) - c_\phi\varepsilon \geq \mathrm{sd}_{\widehat{E}}^{\phi^\circ}(i) - c_\phi\varepsilon. \tag{4-20}$$

On the other hand, by definition of $\mathrm{sd}_{\widehat{E}}^{\phi^\circ}$ there exists $x \in \partial\widehat{E}$ such that $\mathrm{sd}_{\widehat{E}}^{\phi^\circ}(i) = \phi^\circ(i-x)$. Let $k \in \varepsilon\mathbb{Z}^N$ be the closest point to x in $\partial_\varepsilon^- E$. We have

$$\mathrm{sd}_{\widehat{E}}^{\phi^\circ}(i) = \phi^\circ(i-x) \geq \phi^\circ(i-k) - c_\phi\varepsilon \geq \phi^\circ(i-k) + u_k - c_\phi\varepsilon \geq \tilde{d}_i - c_\phi\varepsilon. \tag{4-21}$$

Finally, equations (4-20) and (4-21) imply (4-19) outside E. The other case is analogous.

We now finally prove (4-14) outside E. From (4-18) and (4-19) we have

$$d_{-}^{\varepsilon,\phi^\circ}(u) = \mathrm{sd}_{-}^{\varepsilon,\phi^\circ}(u) \geq \tilde{d} \geq \mathrm{sd}_{\widehat{E}}^{\phi^\circ} - c_\phi\varepsilon \quad \text{in } E.$$

In particular, $\mathrm{sd}_{\widehat{E}}^{\phi^\circ} - c_\phi\varepsilon$ is an admissible competitor in (4-7), thus $\mathrm{sd}_{-}^{\varepsilon,\phi^\circ}(u) \geq \mathrm{sd}_{\widehat{E}}^{\phi^\circ} - c_\phi\varepsilon$ in $\varepsilon\mathbb{Z}^N$. On the other hand, in $\varepsilon\mathbb{Z}^N \setminus E$ we have (4-18); thus we conclude (4-14) for $\mathrm{sd}_{-}^{\varepsilon,\phi^\circ}(u)$. Concerning $\mathrm{sd}_{+}^{\varepsilon,\phi^\circ}(u)$, we note that by Remark 4.1 and the equation above we have

$$u \geq \mathrm{sd}_{-}^{\varepsilon,\phi^\circ}(u) \geq \mathrm{sd}_{\widehat{E}}^{\phi^\circ} - c_\phi\varepsilon \quad \text{in } E.$$

The function $\mathrm{sd}_{\widehat{E}}^{\phi^\circ} - c_\phi\varepsilon$ is therefore admissible in (4-8). Thus by maximality

$$d_{+}^{\varepsilon,\phi^\circ}(u) \geq \mathrm{sd}_{\widehat{E}}^{\phi^\circ} - c_\phi\varepsilon.$$

Since $\mathrm{sd}_{+}^{\varepsilon,\phi^\circ}(u) = d_{+}^{\varepsilon,\phi^\circ}(u)$ in $\varepsilon\mathbb{Z}^N \setminus E$, we conclude (4-14), taking also into account again (4-18) and (4-19). Finally, (4-15) follows by combining (4-14), (4-18) and (4-19). □

We conclude the subsection with some further properties of the operator $\mathrm{sd}^{\varepsilon,\phi^\circ}$.

Lemma 4.3. *Given $u \in X_\varepsilon$ that is $(1, \phi^\circ)$-Lipschitz, we have*

$$\mathrm{sd}^{\varepsilon,\phi^\circ}(-u) = -\mathrm{sd}^{\varepsilon,\phi^\circ}(u). \tag{4-22}$$

Furthermore, if $u_1, u_2 \in X_\varepsilon$ are $(1, \phi^\circ)$-Lipschitz and $u_1 \le u_2$ then

$$\mathrm{sd}^{\varepsilon,\phi^\circ}(u_1) \le \mathrm{sd}^{\varepsilon,\phi^\circ}(u_2). \tag{4-23}$$

Finally, for any $s > 0$ and $u \in X_\varepsilon$ that is $(1, \phi^\circ)$-Lipschitz, we have

$$\mathrm{sd}^{\varepsilon,\phi^\circ}(u - s) \le \mathrm{sd}^{\varepsilon,\phi^\circ}(u) - s. \tag{4-24}$$

Proof. For every $i \in \varepsilon\mathbb{Z}^N$ we have

$$(d_-^{\varepsilon,\phi^\circ}(-u))_i = \max_{j\in\{(-u)\ge 0\}}\{-u_j - \phi^\circ(i-j)\} = -\min_{j\in\{u\le 0\}}\{u_j + \phi^\circ(i-j)\} = -(d_+^{\varepsilon,\phi^\circ}(u))_i.$$

In turn,

$$(\mathrm{sd}_-^{\varepsilon,\phi^\circ}(-u))_i = \min_{j\in\{(-u)\le 0\}}\{(d_-^{\varepsilon,\phi^\circ}(-u))_j + \phi^\circ(i-j)\} = -\max_{j\in\{u\ge 0\}}\{(d_+^{\varepsilon,\phi^\circ}(u))_j - \phi^\circ(i-j)\} = -(\mathrm{sd}_+^{\varepsilon,\phi^\circ}(u))_i.$$

Reasoning in the same way for $d_+^{\varepsilon,\phi^\circ}$ and $\mathrm{sd}_+^{\varepsilon,\phi^\circ}$ we arrive at

$$\mathrm{sd}_\pm^{\varepsilon,\phi^\circ}(-u) = -\mathrm{sd}_\mp^{\varepsilon,\phi^\circ}(u) \tag{4-25}$$

and thus $\mathrm{sd}^{\varepsilon,\phi^\circ}(-u) = -\mathrm{sd}^{\varepsilon,\phi^\circ}(u)$. The monotonicity property (4-23) follows easily from the definitions in (4-5). The proofs of the other results also follow from the definitions in (4-5); we present only the one concerning (4-24). Fix $s > 0$, and let $u \in X_\varepsilon$ be a $(1, \phi^\circ)$-Lipschitz function. By definition of $d_-^{\varepsilon,\phi^\circ}(u)$ we have

$$(d_-^{\varepsilon,\phi^\circ}(u))_i = \sup_{j\in\{u\ge 0\}}\{u_j - \phi^\circ(i-j)\} \ge s + \sup_{j\in\{u\ge s\}}\{(u_j - s) - \phi^\circ(i-j)\} = (d_-^{\varepsilon,\phi^\circ}(u-s))_i + s.$$

Analogously,

$$\begin{aligned}(\mathrm{sd}_-^{\varepsilon,\phi^\circ}(u))_i &= \inf_{j\in\{u\le 0\}}\{(d_-^{\varepsilon,\phi^\circ}(u))_j + \phi^\circ(i-j)\}\\ &\ge s + \inf_{j\in\{u\le s\}}\{(d_-^{\varepsilon,\phi^\circ}(u-s))_j + \phi^\circ(i-j)\} = s + (\mathrm{sd}_+^{\varepsilon,\phi^\circ}(u-s))_i.\end{aligned}$$

Since the proofs for $d_+^{\varepsilon,\phi^\circ}(u)$ and $\mathrm{sd}_+^{\varepsilon,\phi^\circ}(u)$ are analogous, we conclude. □

4.2. ***The discrete scheme.*** We now describe our minimizing movements scheme, discretized in both time and space. A particularity of our scheme is that, in practice, it evolves the distance function to a set rather than the set itself. In particular, at the discrete level, it may depend on the initialization (even if in the limit the flow is geometric and only depends on the initial set).

Recalling (4-3), we rescale (3-4) on the lattice $\varepsilon\mathbb{Z}^N$ in the following way: We recall that $X_\varepsilon = \mathbb{R}^{\varepsilon\mathbb{Z}^N}$ and $Y_\varepsilon = \mathbb{R}^{\varepsilon\mathbb{Z}^N\times\varepsilon\mathbb{Z}^N}$. Given $g \in X_\varepsilon$ and a time step $h > 0$, the problem (3-4) now becomes to find

$(u, z) \in X_\varepsilon \times Y_\varepsilon$ satisfying

$$\begin{cases} hD^*_\varepsilon z + u = g & \text{on } \varepsilon\mathbb{Z}^N, \\ z_{ij}(u_i - u_j) = \alpha^\varepsilon_{ij}|u_i - u_j|, \ |z_{ij}| \le \alpha^\varepsilon_{ij}, \end{cases} \tag{4-26}$$

where $D^*_\varepsilon z$ is defined in (3-1). For ease of notation we assume $\varepsilon = \varepsilon(h)$, with $\varepsilon \to 0$ as $h \to 0$, and we will specify the dependence on h only.

Let $E_0 \subseteq \mathbb{R}^N$ be a closed set. We define $E^{h,0} := \{i \in \varepsilon\mathbb{Z}^N \mid (i + [0, \varepsilon)^N) \cap E_0 \neq \varnothing\}$. We note that

$$\widehat{E}^{h,0} \to E_0, \quad E^{h,0} \to E_0 \tag{4-27}$$

as $h \to 0$ in the Kuratowski sense, where with a slight abuse of notation we write $\widehat{E}^{h,0}$ to denote the set $E^{h,0} + [0, \varepsilon]^N$.

Given a closed set $E_0 \subseteq \mathbb{R}^N$ with $E_0 \notin \{\varnothing, \mathbb{R}^N\}$, we consider $u^{h,0}$, a $(1, \phi^\circ)$-Lipschitz function on $\varepsilon\mathbb{Z}^N$ which is negative inside $E^{h,0}$ and positive outside. For instance, we set

$$u^{h,0} := \tfrac{1}{2}C_\phi\varepsilon(1 - \chi_{E^{h,0}}) - \tfrac{1}{2}C_\phi\varepsilon\chi_{E^{h,0}},$$

where C_ϕ is defined in (4-4), so that $u^{h,0}$ is $(1, \phi^\circ)$-Lipschitz. Let us set $(z^{h,0})_{ij} = 0$ for all $i, j \in \varepsilon\mathbb{Z}^N$. Then, as long as $E^{h,k} \notin \{\varnothing, \mathbb{R}^N\}$, we can iteratively define $u^{h,k+1}$, $z^{h,k+1}$ for $k \in \mathbb{N}$ by solving (4-26) with $g = \mathrm{sd}^{\varepsilon,\phi^\circ}(u^{h,k})$; i.e.,

$$\begin{cases} hD^*_\varepsilon z^{h,k+1} + u^{h,k+1} = \mathrm{sd}^{\varepsilon,\phi^\circ}(u^{h,k}) & \text{on } \varepsilon\mathbb{Z}^N, \\ z^{h,k+1}_{ij}(u^{h,k+1}_i - u^{h,k+1}_j) = \alpha^\varepsilon_{ij}|u^{h,k+1}_i - u^{h,k+1}_j|, \ |z^{h,k+1}_{ij}| \le \alpha^\varepsilon_{ij}. \end{cases} \tag{4-28}$$

We recall that the redistancing operator $\mathrm{sd}^{\varepsilon,\phi^\circ}$ has been introduced in the previous section. We then set

$$E^{h,k+1} = \{i \in \varepsilon\mathbb{Z}^N \mid u^{h,k+1}_i \le 0\}.$$

If either $E^{h,k} = \varnothing$ or $E^{h,k} = \mathbb{R}^N$, we define $E^{h,k+1} = E^{h,k}$. We denote by T^*_h the first discrete time hk such that $E^{h,k} = \varnothing$, if any; otherwise we let $T^*_h = +\infty$. Analogously, we set T'^*_h to be the first discrete time hk such that $E^{h,k} = \mathbb{R}^N$, if any; otherwise we let $T'^*_h = +\infty$.

For ease of notation we will set

$$\begin{gathered} E^h(t) := E^{h,[t/h]} \subseteq \varepsilon\mathbb{Z}^N, \quad d^h(t) := \mathrm{sd}^{\varepsilon,\phi^\circ}(u^{h,[t/h]}) \in X_\varepsilon, \quad u^h(t) := u^{h,[t/h]} \in X_\varepsilon, \\ z^h(t) := z^{h,[t/h]} \in Y_\varepsilon, \quad \hat{d}^h(\,\cdot\,, t) := \mathrm{sd}^{\phi^\circ}_{\widehat{E}^h(t)} \in \mathrm{Lip}(\mathbb{R}^N), \end{gathered} \tag{4-29}$$

where again, with a slight abuse of notation, $\widehat{E}^h(t)$ stands for $\widehat{E^h(t)}$. Note that in the definition of $\hat{d}^h(\,\cdot\,, t)$ we are possibly using the convention $\mathrm{sd}^{\phi^\circ}_\varnothing \equiv +\infty$ and $\mathrm{sd}^{\phi^\circ}_{\mathbb{R}^N} \equiv -\infty$. Note also that $z^h(t)$ is well defined only for $0 \le t < \min\{T^*_h, T'^*_h\}$; however, if needed, we can set $z^h(t) = 0$ for $t \ge \min\{T^*_h, T'^*_h\}$.

Remark 4.4. If u is the solution of (4-26) with (L, ϕ°)-Lipschitz datum g, by standard arguments, based on the comparison principle and translation invariance, one can show that u satisfies the same Lipschitz bound of g. Indeed, given $j \in \varepsilon\mathbb{Z}^N$, the function $u(\,\cdot - j) \pm L\phi^\circ(j)$ solves (4-26) with datum $g(\,\cdot - j) \pm L\phi^\circ(j)$. By comparison one concludes, as $g(\,\cdot - j) - L\phi^\circ(j) \le g(\,\cdot\,) \le g(\,\cdot - j) + L\phi^\circ(j)$.

Lemma 4.5. *Let u^h, E^h and d^h be defined as in* (4-29)*. Then, for every $t \geq 0$, $d^h(t)$ is $(1, \phi^\circ)$-Lipschitz and satisfies*

$$\begin{cases} u^h(t) \leq d^h(t) & \text{in } \varepsilon\mathbb{Z}^N \setminus E^h(t), \\ u^h(t) \geq d^h(t) & \text{in } E^h(t). \end{cases} \tag{4-30}$$

Proof. It follows from Remarks 4.1 and 4.4. □

Remark 4.6 (evolution of the complement). Let $E^h(t)$ and $u^h(t)$ be as in (4-29). We note that, if $F_0 \subseteq \mathbb{R}^N$ is a closed set such that $F^{h,0} = \varepsilon\mathbb{Z}^N \setminus E^{h,0}$, then the discrete evolution starting from F_0 coincides with $\{u^h(t) \geq 0\}$ for every $t \geq 0$. Indeed, denoting by v^h the discrete evolution starting from F_0, by definition $v^{h,0} = -u^{h,0}$. Thus recalling (4-22) we have

$$\mathrm{sd}^{\varepsilon,\phi^\circ}(v^{h,0}) = -\mathrm{sd}^{\varepsilon,\phi^\circ}(u^{h,0})$$

and, by uniqueness for (4-26), it follows that $v^h(h) = -u^h(h)$. Then we can iterate to conclude.

Remark 4.7 (comparison principle). Let E_0 and F_0 be closed sets in $\mathbb{R}^N$ such that $E^{h,0} \subseteq F^{h,0}$ (note that this condition is satisfied if $E_0 \subseteq F_0$). Let $E^h(t)$ and $F^h(t)$ be the corresponding discrete evolutions, and let $u^h(t)$ and $v^h(t)$ be the associated functions as in (4-29). Then, for every $t \geq 0$, we have $E^h(t) \subseteq F^h(t)$. This follows easily by iteration from the monotonicity property (4-23) and from the comparison principle for (4-26). One in fact could also consider the "open" discrete evolution given by

$$\mathring{E}^h(t) := \{u^h(t) < 0\} \quad \text{and} \quad \mathring{F}^h(t) := \{v^h(t) < 0\}.$$

Then, by the same argument one also has that $\mathring{E}^h(t) \subseteq \mathring{F}^h(t)$.

Remark 4.8 (avoidance principle). Let $E_0, F_0 \subseteq \mathbb{R}^N$ be closed sets such that $E^{h,0} \cap F^{h,0} = \varnothing$ (which is, for example, implied by $\mathrm{dist}(E_0, F_0) > c_\phi \varepsilon$ for a suitable $c_\phi > 0$). Let E^h, u^h and $\mathring{F}^h(t)$, v^h be the closed and open discrete evolutions starting from E_0 and F_0, respectively (where the open discrete evolution has been defined in Remark 4.7). Then,

$$\mathring{F}^h(t) \subseteq \varepsilon\mathbb{Z}^N \setminus E^h(t).$$

Indeed, $F^{h,0} \subseteq \varepsilon\mathbb{Z}^N \setminus E^{h,0}$ implies that $-u^{h,0} \leq v^{h,0}$, and thus by (4-22) and (4-23)

$$-\mathrm{sd}^{\varepsilon,\phi^\circ}(u^{h,0}) = \mathrm{sd}^{\varepsilon,\phi^\circ}(-u^{h,0}) \leq \mathrm{sd}^{\varepsilon,\phi^\circ}(v^{h,0}).$$

By the comparison principle for (4-26) and iterating one sees that $-u^h(t) \leq v^h(t)$ for all $t \geq 0$, which implies

$$\mathring{F}^h(t) = \{v^h(t) < 0\} \subseteq \{u^h(t) > 0\} = \varepsilon\mathbb{Z}^N \setminus E^h(t).$$

Remark 4.9. We conclude this subsection by observing that we could have made different choices of the distance function without affecting the final convergence result. In definition (4-5) we could have set

$$\begin{aligned} (d^<(u))_i &= \inf_{j \in \{u<0\}} \{u_j + \phi^\circ(i-j)\}, & (d^\leq(u))_i &= \inf_{j \in \{u \leq 0\}} \{u_j + \phi^\circ(i-j)\}, \\ (\mathrm{sd}^<(u))_i &= \sup_{j \in \{u \geq 0\}} \{(d^<(u))_j - \phi^\circ(i-j)\}, & (\mathrm{sd}^\leq(u))_i &= \sup_{j \in \{u>0\}} \{(d^<(u))_j - \phi^\circ(i-j)\}. \end{aligned} \tag{4-31}$$

One can see that $\mathrm{sd}^{\leq}(u)$ mimics the signed distance function to the boundary of $\{u \leq 0\}$ while $\mathrm{sd}^{<}(u)$ mimics the signed distance function to the boundary of $\{u < 0\}$. Defining the algorithm as in (4-28) but with $\mathrm{sd}^{<}$, $\mathrm{sd}^{\leq}$ replacing $\mathrm{sd}^{\varepsilon,\phi^\circ}$, adapting our proof one can conclude the same convergence result. Let us further comment on the relation between $\mathrm{sd}^{\varepsilon,\phi^\circ}$, $\mathrm{sd}^{\leq}$, $\mathrm{sd}^{<}$. One can prove that, for any $(1,\phi^\circ)$-Lipschitz function $u \in X_\varepsilon$,

$$\mathrm{sd}^{\leq}(u) \leq \mathrm{sd}_-^{\varepsilon,\phi^\circ}(u) \leq \mathrm{sd}_+^{\varepsilon,\phi^\circ}(u) \leq \mathrm{sd}^{<}(u). \tag{4-32}$$

Thus, between the many possible choices we could have performed in (4-5), it turns out that $\mathrm{sd}^{<}$ is the "maximal" one, while $\mathrm{sd}^{\leq}$ is the "minimal". Indeed, let us show that $\mathrm{sd}_-^{\varepsilon,\phi^\circ}(u) \leq \mathrm{sd}_+^{\varepsilon,\phi^\circ}(u)$. By definition (4-5) and equations (4-9) and (4-11), for every $i \in \{u \geq 0\}$ we have

$$(\mathrm{sd}_-^{\varepsilon,\phi^\circ}(u))_i = \inf_{j\in\{u\leq 0\}}\{(d_-^{\varepsilon,\phi^\circ}(u))_j + \phi^\circ(i-j)\} \leq \inf_{j\in\{u\leq 0\}}\{u_j + \phi^\circ(i-j)\} = (\mathrm{sd}_+^{\varepsilon,\phi^\circ}(u))_i.$$

Reasoning analogously, for every $i \in \{u \leq 0\}$ we have

$$(\mathrm{sd}_+^{\varepsilon,\phi^\circ}(u))_i = \sup_{j\in\{u\geq 0\}}\{(d_+^{\varepsilon,\phi^\circ}(u))_j - \phi^\circ(i-j)\} \geq \sup_{j\in\{u\geq 0\}}\{u_j - \phi^\circ(i-j)\} = (\mathrm{sd}_-^{\varepsilon,\phi^\circ}(u))_i.$$

Furthermore, for any two $(1,\phi^\circ)$-Lipschitz functions $u, u' \in X_\varepsilon$, if $u \leq u' - s$ for $s > 0$ then

$$\mathrm{sd}^{<}(u) \leq \mathrm{sd}^{\leq}(u') - s.$$

In particular, this implies that, for any $(1,\phi^\circ)$-Lipschitz function $u \in X_\varepsilon$ and $s' > s$,

$$\mathrm{sd}^{\varepsilon,\phi^\circ}(u-s) \leq \mathrm{sd}^{\varepsilon,\phi^\circ}(u-s') + s' - s.$$

Fix $u_0 \in X_\varepsilon$, a $(1,\phi^\circ)$-Lipschitz function. Using the properties above and standard arguments, one can see that for all but countably many $s \in \mathbb{R}$ the discrete evolutions starting from $\{u_0 \leq s\}$ and corresponding to the three possible choices of distances in (4-32) coincide.

4.3. ***Discrete evolution of Wulff shapes.*** In this section we provide some control on the evolution speed of discrete Wulff shapes. The first result estimates the solution of (4-26) for the distance to the Wulff shape.

Lemma 4.10. *There exists a constant $C = C(\phi) > 0$ with the following property: if u is the solution of* (4-26) *with $g = \phi^\circ$, then $u \leq \phi^h$, where $\phi^h \in X_\varepsilon$ is defined as*

$$\phi_i^h := \begin{cases} \phi^\circ(i) + \frac{Ch}{\phi^\circ(i)} & \text{if } \phi^\circ(i) \geq C(\sqrt{h} \vee \varepsilon), \\ C(\sqrt{h}\vee\varepsilon) + \frac{Ch}{\sqrt{h}\vee\varepsilon} & \text{otherwise.} \end{cases} \tag{4-33}$$

The proof of Lemma 4.10, based on the construction of a calibration, is postponed until Appendix A. We now prove a useful lemma used to estimate the redistancing step in our algorithm for functions of the form (4-33).

Lemma 4.11. *Let $R \geq \delta > 0$, and set*

$$u := (\phi^\circ - R) \vee \left(\tfrac{1}{2}\delta - R\right).$$

Then, for ε small enough depending on δ we have

$$\mathrm{sd}^{\varepsilon,\phi^\circ}(u) \le \phi^\circ - R + \hat{c}\varepsilon \quad \text{in } \varepsilon\mathbb{Z}^N \tag{4-34}$$

for a suitable positive constant $\hat{c}$, depending on ϕ. Furthermore, if we assume (B-1), *we have*

$$\mathrm{sd}^{\varepsilon,\phi^\circ}(u) \le \phi^\circ - R \quad \text{in } \varepsilon\mathbb{Z}^N. \tag{4-35}$$

Proof. By (4-32), it is sufficient to prove the claim for $\mathrm{sd}_+^{\varepsilon,\phi^\circ}$. We start by showing that $d_+^{\varepsilon,\phi^\circ}(u) = u$ and noting that by (4-11) it suffices to prove $d_+^{\varepsilon,\phi^\circ}(u) \le u$ in $\{u \ge 0\} = \{\phi^\circ \ge R\}$. Assuming (B-1), given $i \in \{u \ge 0\}$ we note that $\phi^\circ(i) \ge R$; thus by Lemma B.1 there exists $j \in \mathcal{W}_R \setminus \mathcal{W}_{R-2\varepsilon\ell_1}$ satisfying

$$\phi^\circ(j) + \phi^\circ(i-j) = \phi^\circ(i).$$

Taking $\varepsilon = \varepsilon(\delta)$ we can ensure that $R - 2\varepsilon\ell_1 \ge \frac{1}{2}\delta$, so that $j \in (\mathcal{W}_R \setminus \mathcal{W}_{\delta/2}) \cap \varepsilon\mathbb{Z}^N$. By definition (4-5) and the equation above we conclude that

$$d_+^{\varepsilon,\phi^\circ}(u) \le u_j + \phi^\circ(i-j) = \phi^\circ(j) - R + \phi^\circ(i-j) = \phi^\circ(i) - R,$$

and hence we have shown that $d_+^{\varepsilon,\phi^\circ}(u) = u$. Finally, from definition (4-5) and since $d_+^{\varepsilon,\phi^\circ}(u) = u = \phi^\circ - R$ on $\{u \ge 0\}$, we conclude by the triangular inequality that $\mathrm{sd}_+^{\varepsilon,\phi^\circ}(u) \le \phi^\circ - R$. All in all, we have obtained (4-35).

If instead (B-1) does not hold, using the first part of Lemma B.1 and reasoning as above, one concludes that

$$\mathrm{sd}_+^{\varepsilon,\phi^\circ}(u) \le \phi^\circ - R + \hat{c}\varepsilon$$

for a positive constant $\hat{c}$, and then the conclusion follows. □

Combining the two results above we can provide a bound on the evolution speed of Wulff shapes in the algorithm (4-28).

Proposition 4.12. *Assume either $\varepsilon \le O(h)$ or that* (B-1) *holds. For every $\delta > 0$ there exist positive constants ε_0, h_0, c_0 depending on δ with the following property: if $R \ge \delta$, $\varepsilon \le \varepsilon_0$ and $h \le h_0$, then the discrete evolution of $\mathcal{W}_R$ defined in* (4-28), *denoted $\mathcal{W}^h(t)$, satisfies*

$$\mathcal{W}^h(t) \supseteq \mathcal{W}_{R-c_0(t+\varepsilon)} \cap \varepsilon\mathbb{Z}^N \tag{4-36}$$

as long as $R - c_0(t+\varepsilon) \ge \frac{1}{2}\delta$.

Proof. Let $\mathring{\mathcal{W}}^h(t)$ be the open discrete evolution (see Remark 4.7) starting from the closure of $\mathcal{W}_R$ for some $R > 0$ and let $v^h(t)$ be the associated function as in the third equation in (4-29). Using the definition of $v^{h,0}$, (4-10) and the first definition in (4-5), it is easy to see that

$$(\mathrm{sd}_-^{\varepsilon,\phi^\circ}(v^{h,0}))_0 = (d_-^{\varepsilon,\phi^\circ}(v^{h,0}))_0 \le -R + c_\phi\varepsilon. \tag{4-37}$$

On the other hand, consider $i \in \{v^{h,0} \ge 0\}$ and let $x' \in \partial\mathcal{W}_R$ be such that

$$\phi^\circ(i - x') = \phi^\circ(i) - \phi^\circ(x') = \phi^\circ(i) - R.$$

Since there exists $j' \in \{v^{h,0} \le 0\}$ such that $\phi^\circ(j'-x') \le c_\phi \varepsilon$, then by the triangular inequality

$$\phi^\circ(i-j') \le \phi^\circ(i) - R + c_\phi \varepsilon.$$

Thus, using again definition (4-5), we get

$$(d_+^{\varepsilon,\phi^\circ}(v^{h,0}))_i \le \inf_{j\in\{v^{h,0}\le 0\}} \phi^\circ(i-j) \le \phi^\circ(i) - R + c_\phi\varepsilon,$$

which implies

$$(\mathrm{sd}_+^{\varepsilon,\phi^\circ}(v^{h,0}))_0 \le \sup_{j\in\{v^{h,0}\ge 0\}} (d_+^{h,0}(v^{h,0}))_j - \phi^\circ(j) \le -R + c_\phi\varepsilon. \tag{4-38}$$

Therefore, since $\mathrm{sd}^{\varepsilon,\phi^\circ}(v^{h,0})$ is a $(1,\phi^\circ)$-Lipschitz function, from (4-37) and (4-38) we get that

$$\mathrm{sd}^{\varepsilon,\phi^\circ}(v^{h,0}) \le \phi^\circ - R + c_\phi\varepsilon \quad \text{in } \varepsilon\mathbb{Z}^N.$$

By comparison and Lemma 4.10 we obtain

$$v^h(h) \le \phi^h - R + c_\phi\varepsilon, \tag{4-39}$$

where $\phi^h \in X_\varepsilon$ is defined in (4-33). Considering $R \ge \delta$ and $h = h(\delta)$, $\varepsilon = \varepsilon(\delta)$ small enough, the equation above implies that

$$v^h(h) \le (\phi^\circ - R + c_0 h + c_\phi\varepsilon) \vee \left(\tfrac{1}{2}\delta - R\right), \tag{4-40}$$

where $c_0 = 4C/\delta$, with C the same as in (4-33). Assume first (B-1). From Lemma 4.11, with R replaced by $R - c_0 h - c_\phi\varepsilon$, we get

$$\mathrm{sd}^{\varepsilon,\phi^\circ}(v^h(h)) \le \phi^\circ - R + c_0 h + c_\phi\varepsilon, \tag{4-41}$$

and therefore by comparison and Lemma 4.10 we get

$$v^h(2h) \le \phi^h - R + c_0 h + c_\phi\varepsilon,$$

which, reasoning as above, implies for $\varepsilon(\delta)$ and $h(\delta)$ small

$$v^h(2h) \le (\phi^\circ - R + 2c_0 h + c_\phi\varepsilon) \vee \left(\tfrac{1}{2}\delta - R\right).$$

Hence we can iterate the argument to conclude that

$$v^h(t) \le (\phi^\circ - R + c_0 t + c_\phi\varepsilon) \vee \left(\tfrac{1}{2}\delta - R\right) \tag{4-42}$$

as long as $R - c_0 t - c_\phi\varepsilon \ge \frac{1}{2}\delta$ and ε, h are sufficiently small. In particular, this implies (4-36) (possibly changing the value of c_0).

If instead (B-1) does not hold and $\varepsilon \le O(h)$, we obtain (4-39) and (4-40) in the same way. Then, using the first part of Lemma 4.11 we get

$$\mathrm{sd}^{\varepsilon,\phi^\circ}(v^h(h)) \le \phi^\circ - R + c_0 h + \hat{c}\varepsilon + c_\phi\varepsilon, \tag{4-43}$$

and then iterating we get

$$v^h(kh) \le (\phi^\circ - R + kc_0 h + k\hat{c}\varepsilon + c_\phi\varepsilon) \vee \left(\tfrac{1}{2}\delta - R\right).$$

Hence, recalling that $\varepsilon \le O(h)$, we conclude (4-42) and (4-36) as long as $R - c_0 t - c_\phi\varepsilon \ge \frac{1}{2}\delta$, with ε, h sufficiently small and possibly changing the value of c_0. □

As a corollary of the previous result, we deduce an estimate of the evolution of the distance function $\hat{d}^h$ at a distance from the evolving boundary, which we show next.

Corollary 4.13. *Let $E_0 \subseteq \mathbb{R}^N$ be a closed set, and consider the discrete evolution defined in* (4-29). *Assume either that $\varepsilon \le O(h)$ or that* (B-1) *holds. Then, for every $\delta > 0$ there exist $c_0 = c_0(\delta) > 0$, $h_0 = h_0(\delta) > 0$ and $\varepsilon_0 = \varepsilon_0(\delta)$ such that the following holds. If $\hat{d}^h(x,t) \ge \delta$, then, for $s \ge t$,*

$$\hat{d}^h(x,s) \ge \hat{d}^h(x,t) - c_0(s - t + \varepsilon + h) \tag{4-44}$$

provided $0 < h \le h_0$, $0 < \varepsilon < \varepsilon_0$ and as long as $\hat{d}^h(x,t) - c_0(s-t+\varepsilon+h) \ge \frac{1}{2}\delta$. Similarly, if $\hat{d}^h(x,t) \le -\delta$, then, for $s \ge t$,

$$\hat{d}^h(x,s) \le \hat{d}^h(x,t) + c_0(s - t + \varepsilon + h) \tag{4-45}$$

provided $0 < h \le h_0$ and as long as $\hat{d}^h(x,t) + c_0(s - t + \varepsilon + h) \le -\frac{1}{2}\delta$.

Proof. As usual, in this proof we denote by c_ϕ a positive constant depending on ϕ whose value may change from line to line and also within the same line.

Assume $\hat{d}^h(x,t) \ge \delta$. Without loss of generality we may assume $t \in [0, T_h^*)$ so that $\hat{d}^h(x,t)$ is finite. Denote by $x_\varepsilon \in \varepsilon\mathbb{Z}^N$ an element of the lattice such that $x \in x_\varepsilon + [0,\varepsilon)^N$. Note that there exists a constant $c_\phi > 0$ such that, setting $R := \hat{d}^h(x,t) - c_\phi\varepsilon$, one has $(\mathcal{W}_R(x_\varepsilon))^{h,0} \cap E^h(t) = \varnothing$ and $R > \frac{1}{2}\delta$ (if ε, h are sufficiently small, depending on δ). By the avoidance principle stated in Remark 4.8, we deduce that the open discrete evolution of $\mathcal{W}_R(x_\varepsilon)$, which we denote by $F(\tau)$, lies outside $E^h([t/h]h + \tau)$ for all $\tau \ge 0$. By Proposition 4.12 we deduce

$$F(\tau) \supseteq \mathcal{W}_{R - c_0(\tau+\varepsilon)}(x_\varepsilon) \cap \varepsilon\mathbb{Z}^N \tag{4-46}$$

provided that $R - c_0(\tau + \varepsilon) \ge \frac{1}{2}\delta$. Note that in particular

$$\mathcal{W}_{R - c_0(\tau+h+\varepsilon)}(x_\varepsilon) \cap \varepsilon\mathbb{Z}^N \subseteq \varepsilon\mathbb{Z}^N \setminus E^h(t+\tau)$$

as long as $R - c_0(\tau + h + \varepsilon) \ge \frac{1}{2}\delta$. In turn, we get

$$\hat{d}^h(x_\varepsilon, t+\tau) \ge R - c_0(\tau + h + \varepsilon) \tag{4-47}$$

provided $R - c_0(\tau + h + \varepsilon) \ge \frac{1}{2}\delta$ (for a possibly larger value of c_0). Recalling the definition of R and x_ε and possibly increasing the value of c_0, we infer

$$\hat{d}^h(x, t+\tau) \ge \hat{d}^h(x,t) - c_0(\tau + h + \varepsilon) \tag{4-48}$$

as long as $\hat{d}^h(x,t) - c_0(\tau + h + \varepsilon) \ge \delta$. The case $\hat{d}^h(x,t) \le -\delta$ is analogous. □

5. Convergence of the scheme

We now are ready to study the convergence of the scheme as $\varepsilon \to 0,\ h \to 0$. Recall that we assumed that $\varepsilon = \varepsilon(h)$ goes to 0 as $h \to 0$. In this section we assume that either $\varepsilon \le O(h)$ or that (B-1) holds. Let $E^h(\cdot)$ be the discrete evolution defined in (4-29), and recall that $\widehat{E}^h(\cdot) = E^h(\cdot) + [0,\varepsilon]^N$. We introduce the closed space-time tubes

$$\overline{E}^h := \mathrm{cl}(\{(x,t) \in \mathbb{R}^N \times [0,+\infty) : x \in \widehat{E}^h(t)\}), \tag{5-1}$$

where the closure is in space-time. Then, there exist A and E, open and closed (respectively) subsets of $\mathbb{R}^N \times [0, +\infty)$, with $A \subseteq E$, and a subsequence $h_k \to 0$ such that

$$\bar{E}^{h_k} \xrightarrow{\mathcal{K}} E \quad \text{and} \quad \mathbb{R}^N \times [0, +\infty) \setminus \operatorname{int}(\bar{E}^{h_k}) \xrightarrow{\mathcal{K}} \mathbb{R}^N \times [0, +\infty) \setminus A,$$

where taking the interior and Kuratowski convergence are meant in space-time. Let $E(t)$ and $A(t)$ be the t-time slice of E and A, respectively.

Note that if $E(t) = \varnothing$ for some $t \geq 0$, then (4-44) implies $E(s) = \varnothing$ for all $s \geq t$ so that we can define, as in Definition 2.1, the extinction time T^* of E. In the same fashion one can define the extinction time T'^* of $\mathbb{R}^N \times [0, +\infty) \setminus A$ (notice that at least one of T^* and T'^* is $+\infty$). Possibly extracting a further (not relabeled) subsequence and arguing exactly as in [Chambolle et al. 2017, Proof of Proposition 4.4] (and relying on the bounds (4-44) and (4-45)), one can in fact show the following result.

Proposition 5.1. *There exists a countable set $\mathcal{N} \subseteq (0, +\infty)$ such that $\hat{d}^{h_k}(\,\cdot\,, t)^+ \to \operatorname{dist}^{\phi^\circ}(\,\cdot\,, E(t))$ and $\hat{d}^{h_k}(\,\cdot\,, t)^- \to \operatorname{dist}^{\phi^\circ}(\,\cdot\,, \mathbb{R}^N \setminus A(t))$ locally uniformly for all $t \in (0, +\infty) \setminus \mathcal{N}$. Moreover, E and $\mathbb{R}^N \times [0, +\infty) \setminus A$ satisfy the continuity properties* (b) *and* (c) *of Definition 2.1. In addition, if $T^* > 0$, then $\{\hat{d}^{h_k}\}$ is locally uniformly bounded in $\mathbb{R}^N \times (0, T^*) \setminus E$, and analogously $\{\hat{d}^{h_k}\}$ is locally uniformly bounded in $\mathbb{R}^N \times (0, T'^*) \cap A$ if $T'^* > 0$. Finally, $E(0) = E_0$ and $A(0) = \operatorname{int}(E_0)$.*

Theorem 5.2. *The set E is a superflow in the sense of Definition 2.1 with initial datum E_0, while A is a subflow with initial datum E_0.*

The proof of this result follows the main lines of the proof of [Chambolle et al. 2017, Theorem 4.5]. One important difference with respect to the local, continuous setting is that the variable z^{h_k} is defined on the *edges* (i, j) between the vertices $i, j \in \varepsilon\mathbb{Z}^N$, and it is therefore unclear how to pass to the limit in this variable to obtain the limiting vector field $z(x, t)$. In order to do so, we associate with the discrete vector field $z^h_{ij}(t) \in Y_\varepsilon$ a vector field $\mathbf{z}^h(\,\cdot\,, t)$ in $\mathbb{R}^N$ defined as

$$\mathbf{z}^h(x, t) := \frac{1}{\varepsilon} \sum_{j \in \varepsilon\mathbb{Z}^N} z^h_{ij}(t)(i - j), \tag{5-2}$$

where $i \in \varepsilon\mathbb{Z}^N$ is such that $x \in i + [0, \varepsilon)^N$. Recall that we can take $z^h_{ij}(t)$ and thus $\mathbf{z}^h(\,\cdot\,, t)$ identically zero for $t \geq \min\{T^*_h, T'^*_h\}$. First, we show the following:

Lemma 5.3. *The vector field $\mathbf{z}^h$ satisfies*

$$\phi^\circ(\mathbf{z}^h) \leq 1. \tag{5-3}$$

Proof. Take $v \neq 0$ in $\mathbb{R}^N$. Recalling that $\phi(v) = \sum_{\ell \in \mathbb{Z}^N} \beta(\ell)|v \cdot \ell|$, one has, for any $x \in \mathbb{R}^N$ and $i \in \varepsilon\mathbb{Z}^N$ such that $x \in i + [0, \varepsilon)^N$,

$$\mathbf{z}^h(x, t) \cdot v = \frac{1}{\varepsilon} \sum_{j \in \varepsilon\mathbb{Z}^N} z^h_{ij}(t)(i - j) \cdot v = \sum_{\ell \in \mathbb{Z}^N} z^h_{i, i+\varepsilon\ell}(t)\ell \cdot v \leq \phi(v), \tag{5-4}$$

where we used that $|z^h_{i,i+\varepsilon\ell}(t)| \leq \beta(\ell)$. $\square$

Hence, being globally bounded, this vector field is weakly-$*$ compact in $L^\infty(\mathbb{R}^N \times (0,T);\mathbb{R}^N)$ for any $T>0$. The following lemma establishes a relationship between the divergence of its limits and the limits of the discrete divergences of z^h.

Lemma 5.4. *Assume that $z^{h_k} \overset{*}{\rightharpoonup} z$ in $L^\infty(\mathbb{R}^N \times (0,T);\mathbb{R}^N)$ along a subsequence $h_k \to 0$. Then, for every $\varphi \in C^\infty(\mathbb{R}^N \times (0,T))$ and $\eta \in C_c^\infty(\mathbb{R}^N \times (0,T))$, we have*

$$\lim_{k\to\infty}\left(\varepsilon_k^N \int \sum_{i,j\in\varepsilon_k\mathbb{Z}^N} z_{ij}^{h_k}(t)\eta(i,t)\frac{\varphi(i,t)-\varphi(j,t)}{\varepsilon_k}\,\mathrm{d}t\right) = \iint \eta z\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t.$$

Proof. Let $\varphi \in C^\infty(\mathbb{R}^N \times (0,T))$ and $\eta \in C_c^\infty(\mathbb{R}^N \times (0,T))$, and write $S(t)=\operatorname{supp}(\eta(t))$ and $Q_k:=[0,\varepsilon_k)^N$. We have

$$\varepsilon_k^N \sum_{i,j\in\varepsilon_k\mathbb{Z}^N} z_{ij}^{h_k}(t)\eta(i,t)\frac{\varphi(i,t)-\varphi(j,t)}{\varepsilon_k} = \varepsilon_k^N \sum_{i,j\in\varepsilon_k\mathbb{Z}^N} \frac{z_{ij}^{h_k}(t)}{\varepsilon_k}\eta(i,t)\nabla\varphi(x_{ij})\cdot(i-j), \tag{5-5}$$

where x_{ij} belongs to the segment between i and j. Furthermore we have

$$\begin{aligned}
&\left|\varepsilon_k^N \sum_{i,j\in\varepsilon_k\mathbb{Z}^N} z_{ij}^{h_k}(t)\eta(i,t)\frac{\varphi(i,t)-\varphi(j,t)}{\varepsilon_k} - \sum_{i,j\in\varepsilon_k\mathbb{Z}^N}\frac{z_{ij}^{h_k}(t)}{\varepsilon_k}\int_{i+Q_k}\eta\nabla\varphi\cdot(i-j)\,\mathrm{d}x\right| \\
&\qquad\le \sum_{i,j\in\varepsilon_k\mathbb{Z}^N}\frac{\alpha_{ij}^{\varepsilon_k}}{\varepsilon_k}|\eta(i,t)|\int_{i+Q_k}|(\nabla\varphi(x_{ij},t)-\nabla\varphi(x,t))\cdot(i-j)|\,\mathrm{d}x + O(\varepsilon_k^N) \\
&\qquad\le 2\|\eta\|_\infty \sum_{i\in S(t)\cap\varepsilon_k\mathbb{Z}^N}\sum_{j\in\varepsilon_k\mathbb{Z}^N}\frac{\alpha_{ij}^{\varepsilon_k}}{\varepsilon_k}\int_{i+Q_k}|(\nabla\varphi(x_{ij},t)-\nabla\varphi(x,t))\cdot(i-j)|\,\mathrm{d}x + O(\varepsilon_k^N) \\
&\qquad\le c\varepsilon_k^N \sum_{i\in S(t)\cap\varepsilon_k\mathbb{Z}^N}\sum_{j\in\varepsilon_k\mathbb{Z}^N}\frac{\alpha_{ij}^{\varepsilon_k}}{\varepsilon_k}|i-j|^2 + O(\varepsilon_k^N) \\
&\qquad= c\varepsilon_k^{N+1}\sum_{\substack{i\in\mathbb{Z}^N\\ \varepsilon_k i\in S(t)}}\sum_{j\in\mathbb{Z}^N}\alpha_{ij}|i-j|^2 + O(\varepsilon_k^N) \\
&\qquad\le c\varepsilon_k^{N+1}\left(\sum_{\ell\in\mathbb{Z}^N}\beta(\ell)|\ell|^2\right)(\#S(t)\cap\varepsilon_k\mathbb{Z}^N) + O(\varepsilon_k^N) \\
&\qquad\le c\varepsilon_k\sum_{\ell\in\mathbb{Z}^N}\beta(\ell)|\ell|^2 + O(\varepsilon_k^N),
\end{aligned} \tag{5-6}$$

where in the second line we used the Lipschitz property of η and (4-2), while in the fourth line we used the Lipschitz property of $\nabla\varphi$ and $|x_{ij}-x|\le(1+\sqrt{N})|i-j|$ for $i\ne j$ and $x\in i+Q_k$, and finally in the last line we used that $\#(S(t)\cap\varepsilon\mathbb{Z}^N)=O(\varepsilon_k^{-N})$, which holds locally uniformly in time. Moreover, note that the estimate provided above is uniform as t varies in compact subsets of $(0,T)$. Recalling (4-2), we conclude by integrating in time and sending $k\to\infty$. □

At this point, we may proceed with the proof of Theorem 5.2.

Proof of Theorem 5.2. As usual, in this proof we denote by c_ϕ a positive constant depending on ϕ whose value may change from line to line and also within the same line.

We only show that E is a superflow, as the subflow property of A can be proven analogously. Points (a), (b) and (c) of Definition 2.1 follow from Proposition 5.1. We are left with showing (d). Without loss of generality we may assume $T^* > 0$ (which follows from Corollary 4.13 if the initial set is not trivial). Note also that by Proposition 5.1 we have $\liminf_k T^*_{h_k} \geq T^*$.

Step 1. (proof of (2-1)). For $(x,t) \in \mathbb{R}^N \times (0,T^*) \setminus E$ we set $d(x,t) := \mathrm{dist}^{\phi^\circ}(\,\cdot\,, E(t))$. By Lemma 4.2 and Proposition 5.1 we have

$$\sup_{\varepsilon_k \mathbb{Z}^N \cap K} |d^{h_k}(t) - d(\,\cdot\,,t)| \to 0 \text{ as } k \to \infty \text{ for } t \in (0,T^*) \setminus \mathcal{N} \text{ and for any compact } K \subseteq \mathbb{R}^N \setminus E(t). \tag{5-7}$$

Moreover, d^{h_k} and d are locally uniformly bounded in $\mathbb{R}^N \times (0,T^*) \setminus E$. Set $z^{h_k}(\,\cdot\,,t) := 0$ for $t > T^*_{h_k}$ if $T^*_{h_k} < T^*$. Extracting a further subsequence, if needed, and recalling Lemma 5.3, we may assume that z^{h_k} converges weakly-$*$ in $L^\infty(\mathbb{R}^N \times (0,T^*); \mathbb{R}^N)$ to some vector-field z satisfying

$$\phi^\circ(z) \leq 1 \tag{5-8}$$

almost everywhere. Recall that by (4-30) we have $u^{h_k}(t) \leq d^{h_k}(t)$ in $\varepsilon_k \mathbb{Z}^N \setminus E^{h_k}(t)$, i.e., in the region where $d^{h_k}(t)$ is nonnegative. Combining with (4-28) (and recalling (4-29)) we infer that for $t < T^*_{h_k}$

$$-D^*_{\varepsilon_k} z^{h_k}(t+h_k) \leq \frac{d^{h_k}(t+h_k) - d^{h_k}(t)}{h_k} \quad \text{in } \varepsilon_k \mathbb{Z}^N \setminus E^{h_k}(t). \tag{5-9}$$

Consider a nonnegative test function $\varphi \in C^\infty_c((\mathbb{R}^N \times (0,T^*)) \setminus E)$. If k is large enough, then the distance of the support of φ from $\bar{E}^{h_k}$ is bounded away from zero. In particular, d^{h_k} is finite and positive on $\operatorname{supp}\varphi$. We deduce from (5-9) that

$$\begin{aligned}
&\varepsilon_k^N \int \sum_{i \in \varepsilon_k \mathbb{Z}^N} \varphi(i,t) \biggl(\frac{d_i^{h_k}(t+h_k) - d_i^{h_k}(t)}{h_k} + (D^*_{\varepsilon_k} z^{h_k}(t+h_k))_i \biggr) \mathrm{d}t \\
&\quad = -\varepsilon_k^N \int \sum_{i \in \varepsilon_k \mathbb{Z}^N} \frac{\varphi(i,t) - \varphi(i,t-h_k)}{h_k} d_i^{h_k}(t)\,\mathrm{d}t + \varepsilon_k^N \int \sum_{i,j \in \varepsilon_k \mathbb{Z}^N} \frac{z_{ij}^{h_k}(t+h_k) - z_{ji}^{h_k}(t+h_k)}{h_k} \varphi(i,t)\,\mathrm{d}t \\
&\quad = -\varepsilon_k^N \int \sum_{i \in \varepsilon_k \mathbb{Z}^N} \frac{\varphi(i,t) - \varphi(i,t-h_k)}{h_k} d_i^{h_k}(t)\,\mathrm{d}t + \varepsilon_k^N \int \sum_{i,j \in \varepsilon_k \mathbb{Z}^N} z_{ij}^{h_k}(t+h_k) \frac{\varphi(i,t) - \varphi(j,t)}{h_k}\,\mathrm{d}t \\
&\quad \geq 0.
\end{aligned} \tag{5-10}$$

It is easy to check that the first integral in (5-10) converges to $-\iint d\,\partial_t \varphi\,\mathrm{d}x\,\mathrm{d}t$ as $k \to \infty$ thanks to (5-7) and since d^{h_k} and d are uniformly bounded. Recalling that z^{h_k} converges weakly-$*$ in $L^\infty(\mathbb{R}^N \times (0,T^*))$ to z, we use Lemma 5.4 to conclude that the second integral in (5-10) converges to $\iint z \cdot \nabla\varphi\,\mathrm{d}x\,\mathrm{d}t$. We thus conclude (2-1).

Step 2. (convergence of u^{h_k} to d). Firstly, we establish an upper bound for $-D^*_{\varepsilon_k} z_{h_k}$ away from E^{h_k}. We start by noting that definition (4-5) implies

$$\mathrm{sd}^{\varepsilon,\phi^\circ}(u) \le \tfrac{1}{2}((d_-^{\varepsilon,\phi^\circ}(u))_j + u_\ell + \phi^\circ(\cdot - j) + \phi^\circ(\cdot - \ell)) \quad \text{in } \varepsilon\mathbb{Z}^N \setminus \{u \le 0\} \tag{5-11}$$

for every $(1, \phi^\circ)$-Lipschitz function $u \in X_\varepsilon$ and $j, \ell \in \{u \le 0\}$. Therefore, specifying the inequality above for $u^{h_k}(t)$, by the comparison principle and Lemma 4.10 we conclude

$$u_i^{h_k}(t+h_k) \le \tfrac{1}{2}\big(\phi^{h_k}_{i-j} + \phi^{h_k}_{i-\ell} + \big(d_-^{\varepsilon,\phi^\circ}(u^{h_k}(t))_j\big) + u_\ell^{h_k}(t)\big) \quad \text{for all } i \in \varepsilon_k\mathbb{Z}^N \setminus E^{h_k}(t), \tag{5-12}$$

where $j, \ell \in E^{h_k}(t)$. If $\hat{d}^{h_k}(i,t) \ge R > 0$, recalling the definition of ϕ^h, we get

$$u_i^{h_k}(t+h_k) \le \tfrac{1}{2}\big(\phi^\circ(i-j) + \phi^\circ(i-\ell) + (d_-^{\varepsilon,\phi^\circ}(u^{h_k}(t))_j) + u_\ell^{h_k}(t)\big) + \frac{Ch_k}{R - c_\phi \varepsilon} \tag{5-13}$$

for all $i \in \varepsilon_k\mathbb{Z}^N \setminus E^{h_k}(t)$. Taking the infimum in j, ℓ over $E^{h_k}(t)$ in (5-13) and using again (4-5) and (4-11), we conclude

$$u_i^{h_k}(t+h_k) \le d_i^{h_k}(t) + h_k \frac{C}{R - c_\phi \varepsilon_k} \le d_i^{h_k}(t) + h_k \frac{C}{R} \tag{5-14}$$

provided h_k and ε_k are small enough depending on R, and for a possibly larger value of C. As a consequence of (5-14), we obtain

$$-D^*_{\varepsilon_k} z^{h_k}(t+h_k) \le \frac{C}{R} \quad \text{in } \{\hat{d}^{h_k}(\cdot, t) \ge R\} \cap \varepsilon_k\mathbb{Z}^N. \tag{5-15}$$

Using again Lemma 5.4 and the convergences of E_{h_k} and d_{h_k}, it follows that

$$\operatorname{div} z \le \frac{C}{R} \quad \text{in } \{(x,t) \in \mathbb{R}^N \times (0, T^*) \mid d(x,t) > R\} \tag{5-16}$$

in the sense of distributions. Hence we have that $\operatorname{div} z$ is a Radon measure in $\mathbb{R}^N \times (0, T^*) \setminus E$ and $(\operatorname{div} z)^+ \in L^\infty(\{(x,t) \in \mathbb{R}^N \times (0,T^*) \mid d(x,t) \ge \delta\})$ for every $\delta > 0$.

On the other hand, note that for every $i \in \varepsilon_k\mathbb{Z}^N$ we have

$$d^{h_k}(t) \ge d_i^{h_k}(t) - \phi^\circ(\cdot - i). \tag{5-17}$$

Thus, by Lemma 4.10 and by comparison as before, we get

$$u_i^{h_k}(t+h_k) \ge d_i^{h_k}(t) - \phi_0^{h_k} = d_i^{h_k}(t) - (C+1)\sqrt{h_k}.$$

Combining the above inequality with (5-14), we deduce for all $t \in (0, T^*) \setminus \mathcal{N}$ and any $\delta > 0$ that

$$\sup_{\{\hat{d}_{h_k}(\cdot,t) \ge \delta\} \cap \varepsilon_k\mathbb{Z}^N} |u^{h_k}(t+h_k) - d^{h_k}(t)| \le \sqrt{h_k}(C+2)$$

provided that k is large enough. In particular, recalling also (5-7), we deduce that

$$\sup_{\varepsilon_k\mathbb{Z}^N \cap K} |u^{h_k}(t) - d(\cdot, t)| \to 0 \text{ as } k \to \infty \text{ for } t \in (0,T^*) \setminus \mathcal{N} \text{ and for any compact } K \subseteq \mathbb{R}^N \setminus E(t), \tag{5-18}$$

also with the sequence $\{u^{h_k}\}$ locally (in space and time) uniformly bounded.

Step 3. (the subdifferential inclusion). It remains to show that

$$z \in \partial\phi(\nabla d) \quad \text{a.e. in } \mathbb{R}^N \times (0, T^*) \setminus E. \tag{5-19}$$

Recall that $\xi \in \partial\phi(\eta)$ if and only if $\xi \in \{v \mid \phi^\circ(v) \le 1,\ v \cdot \eta \ge \phi(\eta)\}$. Since one inequality has been proved in (5-8), we show the other one. Consider a test function $\eta \ge 0,\ \eta \in C_c^\infty((\mathbb{R}^N \times (0, T^*)) \setminus E)$. Let $\sigma > 0$, and set $d_\sigma \in C^\infty(\mathbb{R}^N \times (0, T^*))$, as $d_\sigma = d * \rho_\sigma$, where the ρ_σ are space-time mollifiers. Obviously

$$\begin{aligned}\sum_{i,j\in\varepsilon_k\mathbb{Z}^N} z_{ij}^{h_k}(t)\eta(i,t)(u_i^{h_k}(t)-u_j^{h_k}(t)) &= \sum_{i,j\in\varepsilon_k\mathbb{Z}^N} z_{ij}^{h_k}(t)\eta(i,t)(d_\sigma(i,t)-d_\sigma(j,t)) \\ &\quad + \sum_{i,j\in\varepsilon_k\mathbb{Z}^N} z_{ij}^{h_k}(t)\eta(i,t)(u_i^{h_k}(t)-d_\sigma(i,t)-u_j^{h_k}(t)+d_\sigma(j,t)).\end{aligned} \tag{5-20}$$

In turn, Lemma 5.4 implies that

$$\lim_{k\to\infty} \varepsilon_k^N \int \Biggl(\sum_{i,j\in\varepsilon_k\mathbb{Z}^N} z_{ij}^{h_k}(t)\eta(i,t)\frac{d_\sigma(i,t)-d_\sigma(j,t)}{\varepsilon_k}\Biggr)\,\mathrm{d}t = \iint z\cdot\nabla d_\sigma\,\eta\,\mathrm{d}x\,\mathrm{d}t. \tag{5-21}$$

Let us thus show that

$$\lim_{\sigma\to 0}\lim_{k\to\infty} \varepsilon_k^N \int \sum_{i,j\in\varepsilon_k\mathbb{Z}^N} \Biggl(z_{ij}^{h_k}(t)\eta(i,t)\frac{u_i^{h_k}(t)-d_\sigma(i,t)-u_j^{h_k}(t)-d_\sigma(j,t)}{\varepsilon_k}\Biggr)\,\mathrm{d}t = 0. \tag{5-22}$$

We set for every $t \in (0, T_h^*)$ and $\sigma > 0$

$$\begin{aligned} m_{k,\sigma}(t) &:= \min_{i\in\mathrm{supp}(\eta)\cap\varepsilon_k\mathbb{Z}^N} (u_i^{h_k}(t) - d_\sigma(i,t)), \\ M_{k,\sigma}(t) &:= \max_{i\in\mathrm{supp}(\eta)\cap\varepsilon_k\mathbb{Z}^N} (u_i^{h_k}(t) - d_\sigma(i,t)). \end{aligned}$$

The convergence (5-18) implies that these quantities are uniformly bounded and

$$\begin{aligned} \lim_{\sigma\to 0}\lim_{k\to+\infty} m_{k,\sigma}(t) &= 0, \\ \lim_{\sigma\to 0}\lim_{k\to+\infty} M_{k,\sigma}(t) &= 0 \end{aligned} \tag{5-23}$$

uniformly for all $t \notin \mathcal{N}$. For all times $t \in (0, T^*) \setminus \mathcal{N}$

$$\begin{aligned} &\varepsilon_k^N \sum_{i,j\in\varepsilon_k\mathbb{Z}^N} z_{ij}^{h_k}(t)\eta(i,t)\frac{u_i^{h_k}(t)-d_\sigma(i,t)-u_j^{h_k}(t)+d_\sigma(j,t)}{\varepsilon_k} \\ &\quad= \varepsilon_k^N \sum_{i,j\in\varepsilon_k\mathbb{Z}^N} z_{ij}^{h_k}(t)\eta(i,t)\frac{(u_i^{h_k}(t)-d_\sigma(i,t)-m_{k,\sigma}(t))-(u_j^{h_k}(t)-d_\sigma(j,t)-m_{k,\sigma}(t))}{\varepsilon_k} \\ &\quad= \varepsilon_k^N \sum_{i\in\varepsilon_k\mathbb{Z}^N} (u_i^{h_k}(t)-d_\sigma(i,t)-m_{k,\sigma}(t)) \sum_{j\in\varepsilon_k\mathbb{Z}^N} \Biggl(\frac{z_{ij}^{h_k}(t)-z_{ji}^{h_k}(t)}{\varepsilon_k}\eta(i,t) + z_{ji}^{h_k}(t)\frac{\eta(i,t)-\eta(j,t)}{\varepsilon_k}\Biggr). \end{aligned} \tag{5-24}$$

For k large enough, since the support of η is at positive distance from E, by the bound (5-15) one has $D^*_{\varepsilon_k} z^{h_k}(t) \geq -c(\delta)$ on the support for h_k small enough. Thus

$$\varepsilon_k^N \sum_{i\in\varepsilon_k\mathbb{Z}^N} (u_i^{h_k}(t) - d_\sigma(i,t) - m_{k,\sigma}(t))\eta(i,t) \sum_{j\in\varepsilon_k\mathbb{Z}^N} \frac{z_{ij}^{h_k}(t) - z_{ji}^{h_k}(t)}{\varepsilon_k} \geq -c(\delta)\varepsilon_k^N \sum_{i\in\varepsilon_k\mathbb{Z}^N} (u_i^{h_k}(t) - d_\sigma(i,t) - m_{k,\sigma}(t))\eta(i,t).$$

Recalling that

$$\#(\mathrm{supp}(\eta) \cap \varepsilon_k\mathbb{Z}^N) = O(h_k^{-N})$$

uniformly in time, by uniform convergence and (5-18) we conclude that

$$\lim_{\sigma\to 0} \liminf_{k\to\infty} \varepsilon_k^N \int \sum_{i\in\varepsilon_k\mathbb{Z}^N} (u_i^{h_k}(t) - d_\sigma(i,t) - m_{\varepsilon,k}(t))\eta(i,t) \sum_{j\in\varepsilon_k\mathbb{Z}^N} \frac{z_{ij}^{h_k}(t) - z_{ji}^{h_k}(t)}{\varepsilon_k}\,\mathrm{d}t \geq 0. \tag{5-25}$$

The other term in (5-24) can be estimated using the Lipschitz constant of η as

$$\left|\int \sum_{i,j\in\varepsilon_k\mathbb{Z}^N} \varepsilon_k^N (u_i^{h_k}(t) - d_\sigma(i,t) - m_{\varepsilon,k}(t)) z_{ji}^{h_k}(t) \frac{\eta(i,t) - \eta(j,t)}{\varepsilon_k}\,\mathrm{d}t\right| \leq \|\nabla\eta\|_\infty \varepsilon_k^N \int \sum_{i,j\in\varepsilon_k\mathbb{Z}^N} (u_i^{h_k}(t) - d_\sigma(i,t) - m_{\varepsilon,k}(t))\alpha_{ji}^{h_k} \frac{|i-j|}{\varepsilon_k}\,\mathrm{d}t \to 0,$$

letting first $k \to +\infty$ and then $\sigma \to 0$, thanks to (5-18) and (5-23). Note that reasoning as in (5-22) but using $M_{\varepsilon,k}(t)$ instead of $m_{\varepsilon,k}(t)$, one proves that

$$\begin{aligned} &\lim_{\sigma\to 0} \limsup_{k\to\infty} \varepsilon_k^N \int \Bigg(\sum_{i\in\varepsilon_k\mathbb{Z}^N} (u_i^{h_k}(t) - d_\sigma(i,t) - M_{\varepsilon,k}(t))\eta(i,t) \sum_{j\in\varepsilon_k\mathbb{Z}^N} \frac{z_{ij}^{h_k}(t) - z_{ji}^{h_k}(t)}{\varepsilon_k} \Bigg)\,\mathrm{d}t \leq 0, \\ &\lim_{\varepsilon\to 0} \lim_{k\to\infty} \varepsilon_k^N \int \Bigg| \sum_{i,j\in\varepsilon_k\mathbb{Z}^N} (u_i^{h_k}(t) - d_\sigma(i,t) - M_{\varepsilon,k}(t)) z_{ji}^{h_k}(t) \frac{\eta(i,t) - \eta(j,t)}{\varepsilon_k} \Bigg|\,\mathrm{d}t = 0. \end{aligned} \tag{5-26}$$

Combining (5-24), (5-25) and (5-26), we conclude (5-22).

Integrating in time (5-20) and combining (5-21) and (5-22), since the $\nabla d_\sigma = \rho_\sigma * \nabla d$ go to ∇d pointwise a.e. and are uniformly bounded in $L^\infty(\mathbb{R}^N \times (0,T^*); \mathbb{R}^N)$, we have

$$\lim_{k\to\infty} \varepsilon_k^N \int \Bigg(\sum_{i,j\in\varepsilon_k\mathbb{Z}^N} \eta(i,t) z_{ij}^{h_k}(t) \frac{u_i^{h_k}(t) - u_j^{h_k}(t)}{\varepsilon_k} \Bigg)\,\mathrm{d}t = \iint z\cdot\nabla d\,\eta\,\mathrm{d}x\,\mathrm{d}t.$$

The convergence above can be paired with the lower semicontinuity of the Γ-convergence of the discrete total variations (which follows from an adaptation of classical arguments; we suggest the reader consult, e.g., [Chambolle and Kreutz 2023]) and $z^{\varepsilon}_{ij}(u^{\varepsilon}_i - u^{\varepsilon}_j) = \alpha^{\varepsilon}_{ij}|u^{\varepsilon}_i - u^{\varepsilon}_j|$ to obtain

$$\begin{aligned}\iint \phi(\nabla d)\eta &\le \liminf_{k\to\infty} \varepsilon_k^N \int\Bigg(\sum_{i,j\in\varepsilon_k\mathbb{Z}^N} \eta(i,t)\alpha^{h_k}_{ij}\frac{|u^{h_k}_i(t) - u^{h_k}_j(t)|}{\varepsilon_k}\Bigg)\,\mathrm{d}t \\ &= \liminf_{k\to\infty} \varepsilon_k^N \int\Bigg(\sum_{i,j\in\varepsilon_k\mathbb{Z}^N} \eta(i,t)z^{h_k}_{ij}(t)\frac{u^{h_k}_i(t) - u^{h_k}_j(t)}{\varepsilon_k}\Bigg)\,\mathrm{d}t = \iint z\cdot\nabla d\,\eta,\end{aligned}$$

which shows that $\phi(\nabla d) = z\cdot\nabla d$ a.e. on the support of η, from which we deduce (5-19). □

We conclude this section by observing that the discrete scheme converges to the unique weak flow (in the sense of Definition 2.1) starting from E_0 for "generic" initial data E_0, i.e., whenever fattening does not occur. More precisely, we have the following corollary.

Corollary 5.5. *Let $u_0 \in \mathrm{UC}(\mathbb{R}^N)$, and for every $\lambda\in\mathbb{R}$ let $\bar{E}^h_\lambda$ be the closed space-time tube of the h-discrete evolution starting from $\{u_0\le\lambda\}$, i.e., as in* (5-1) *with $E_0 = \{u_0\le\lambda\}$. Then, there exists a countable set $\mathcal{N}$ such that for all $\lambda\in\mathbb{R}^N\setminus\mathcal{N}$*

$$\bar{E}^h_\lambda \xrightarrow{\mathcal{K}} E_\lambda \quad \text{in } \mathbb{R}^N\times[0,+\infty)$$

as $h\to 0$, where E_λ is the unique weak flow in the sense of Definition 2.1 starting from $\{u_0\le\lambda\}$.

Proof. It follows by combining Theorems 5.2 and 2.5. □

6. Numerical experiments

We show some numerical experiments to illustrate our results in dimension 2 (an implementation in three dimensions is currently being developed). We follow the implementation described in [Chambolle and Darbon 2009] (see also [Chambolle and Darbon 2012]), except that now the distance is properly computed using the inf/sup-convolution formulas (4-5). The (exact) numerical resolution of the discrete ROF functional is computed using the parametric maximum flow algorithm of Hochbaum [2001; 2013], implemented upon the maxflow/mincut implementation of Boykov and Kolmogorov [2004]. This particular algorithm has the advantage to provide an *exact* solution of the ROF problem, up to computer precision. Other implementations of the algorithm yielding approximate minimizers have been considered for instance in [Chambolle 2004; Oberman et al. 2011]: of course they work in practice and allow one to address more (an)isotropies than the current work, yet the joint convergence as $\varepsilon = h\to 0$ is not clear in these contexts. For numerical speedup, the infimum and supremum of the definitions in (4-5) are computed only in a neighborhood of fixed size and not on the whole grid. We expect this to yield, in general, an error of order $C\varepsilon$ with C getting smaller as the width of the strip increases; however, we observe that Corollary B.2 justifies this restriction (showing that $C = 0$) in some cases, notably the case $\phi = \|\cdot\|_{\ell^1}$, $\phi^\circ = \|\cdot\|_{\ell^\infty}$; see Figure 1, left, for which the sup/inf are in fact min/max which are reached very close to the evolving boundary (as one can chose $\ell_1 = 1$ in Lemma B.1). Similarly, the ROF minimization is only performed in a neighborhood of the boundary (one can show that this does not affect the solution in a smaller neighborhood, hence the overall error is the same as when computing the distance in a strip only).

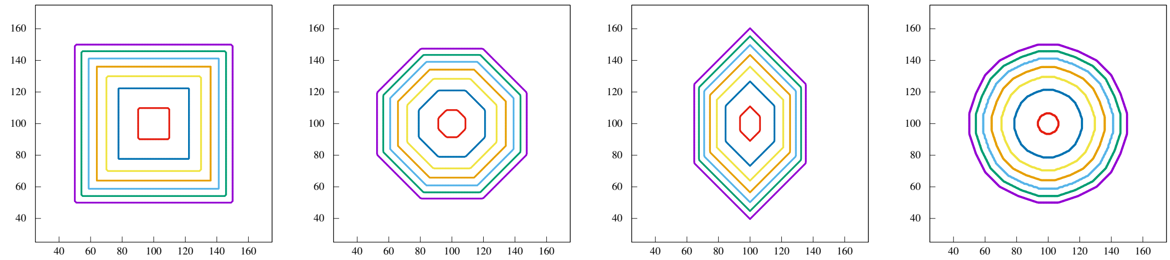

Figure 1. Wulff shapes of initial radius $R_0 = 50$ evolved at times $t = 0, 200, 400, \dots, 1200$ for four different anisotropies: square, octagon, diamond and "almost isotropic".

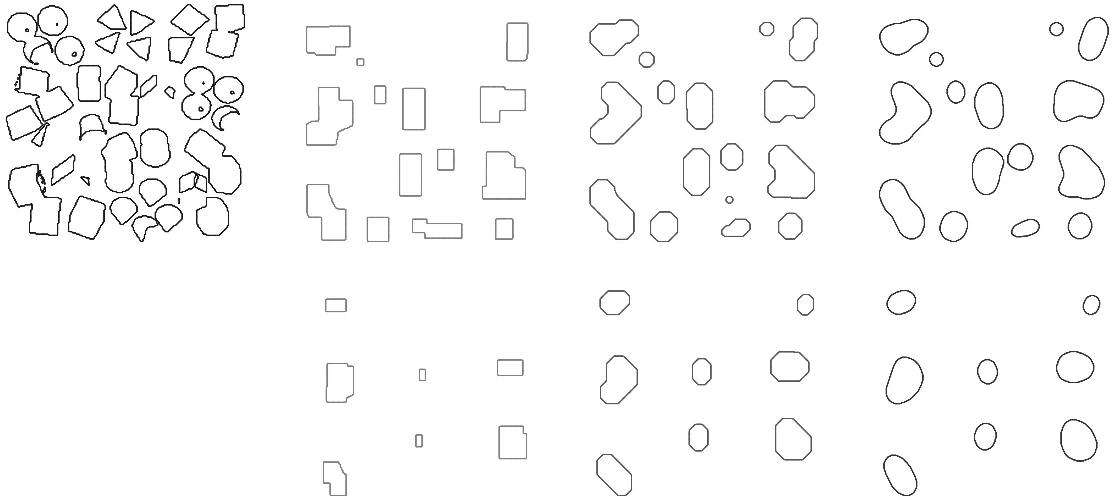

Figure 2. An initial datum and evolutions for square, octagonal and "almost isotropic" anisotropies, at two different times.

The code is available at https://plmlab.math.cnrs.fr/chambolle/discretecrystals/ (implemented in `C/C++` and running on GNU/linux with `gcc`).

Figure 2 shows three examples of flows from the same starting set, composed of random shapes. The anisotropies are square (nearest neighbors interactions), octagonal (next nearest neighbors, weighted so that the corresponding Wulff shape is a regular octagon), and "almost isotropic", which is generated by the interactions in the 12 directions $e \in \{(0, \pm 1), (\pm 1, 0), (\pm 1, \pm 2), (\pm 1, \pm 3)\}$ weighted so that the Wulff shape is a polygon with 24 facets of equal lengths. This is obtained by setting the weights in the discrete total variation as $0.131/\text{length}(e)$ for each direction e, so that the total perimeter of the unit Wulff shape is $24 \times (2 \times 0.131) \approx 2\pi$, in the hope that the corresponding crystalline curvature will be close to the Euclidean one.

Then, we estimate the decay of the radius of an initial Wulff shape $\mathcal{W}_{R_0} = \{\phi \le R_0\}$ along the evolution, up to extinction. In our experiment, $R_0 = 50$. It is well known that the solution is the Wulff shape of radius $R(t) = \sqrt{R_0^2 - 2(N-1)t}$ (where here $N = 2$). The evolutions are depicted in Figure 1. We use the same anisotropies as in Figure 2, with additionally a "diamond" Wulff shape generated by the directions

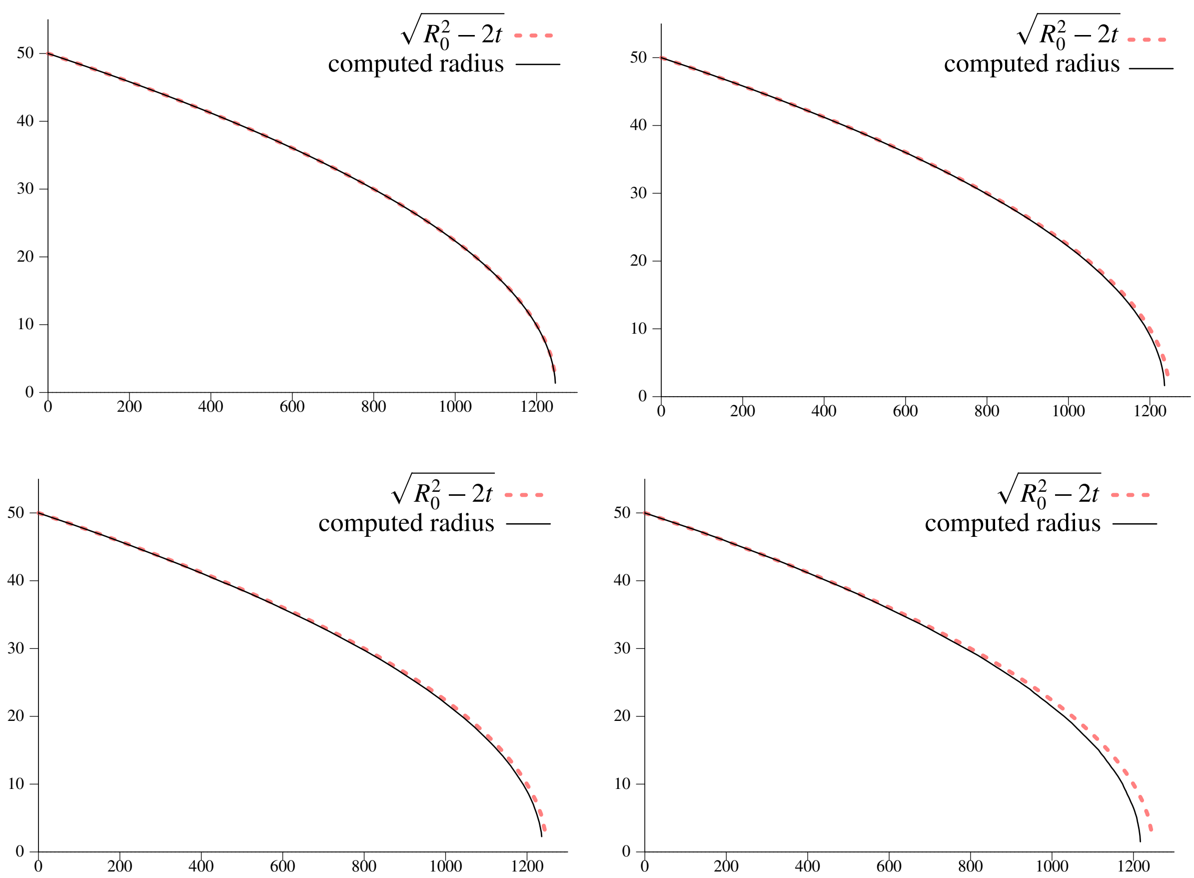

Figure 3. Top: Evolution of the radius for the square (left) and octogonal (right) anisotropies. Bottom: Evolution of the radius for the diamond (left) and "almost isotropic" (right) anisotropies.

$(0, \pm 1)$, $(\pm 1, \pm 2)$ and with sides of equal lengths. In all cases, the weights have been calibrated so that the perimeters of the Wulff shapes are $6.28 \approx 2\pi$.

The plots in Figure 3 show that the decay of the radii is remarkably close to the theoretical prediction, even if this is less precise when more directions of interactions are involved, near extinction. This might be due in part to the fact that the computation of the distance through truncated variants of (4-5) becomes less precise.

Finally, we perform the same experiment with varying ε and h. We observe that the results look remarkably close even if, at low resolution, the error becomes huge when the size of the Wulff shape is of the order of the discretization. Figure 4 shows the shapes. Observe that the shape at time $t = 49$ is only computed for $\varepsilon = 0.1$ and $h = 0.1$ (the shape vanishes before for the two other experiments). On the other hand, this computation took more than one hour, while the case $\varepsilon = 1$ took less than a minute and the case $\varepsilon = 0.1$ and $h = 0.5$ a bit less than an hour. Figure 5 shows the decay of the radii, which should be $\sqrt{R_0^2 - 2t}$ for $R_0 = 10$ and $t \in [0, 50]$.

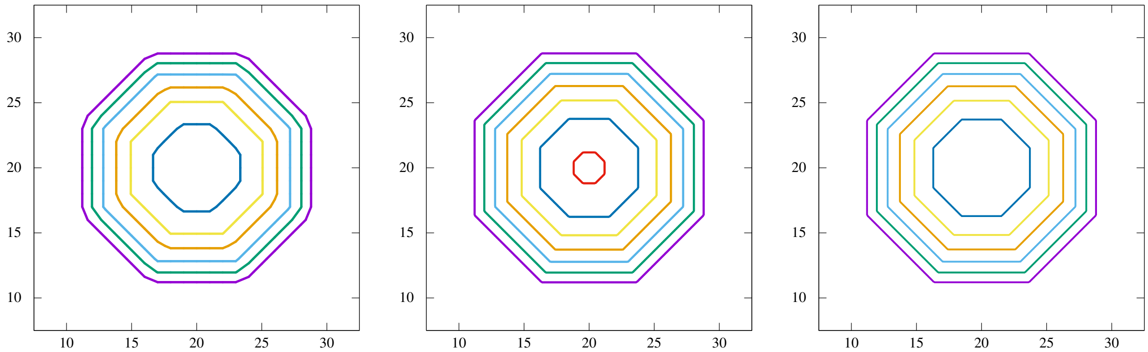

Figure 4. Evolution of an initial octagon with $R_0 = 10$ at times 0, 7, 14, ... (left: $\varepsilon = 1$, $h = 0.1$; middle: $\varepsilon = 0.1$, $h = 0.1$; right: $\varepsilon = 0.1$, $h = 0.5$).

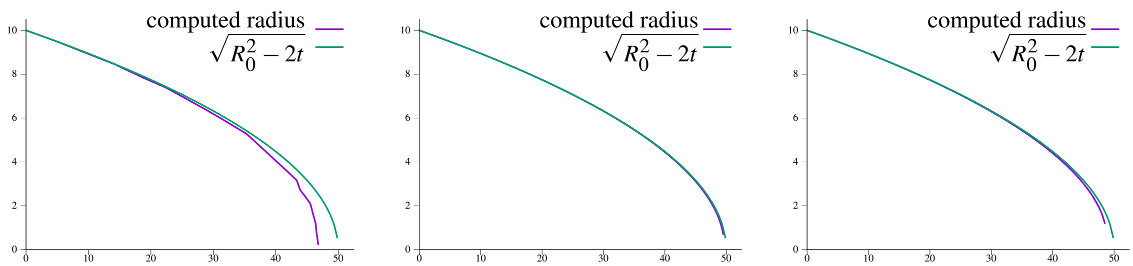

Figure 5. Evolution of the radius for an initial octagon with $R_0 = 10$ until the vanishing time $t = 50$ (left: $\varepsilon = 1$, $h = 0.1$; middle: $\varepsilon = 0.1$, $h = 0.1$; right: $\varepsilon = 0.1$, $h = 0.5$).

Appendix A: Proof of Lemma 4.10

We build here a supersolution to Problem (4-26) when $g = \phi^\circ$. Let us first recall some notation and results concerning zonotopes; see, e.g., [McMullen 1971]. Recall that $\mathcal{E} = \{\pm e_k\}_{k=1}^m \subseteq \mathbb{Z}^N$, where, without loss of generality, the vectors $e_1, \dots, e_m$ span the whole $\mathbb{R}^N$. Given a nonnegative interaction function $\beta \in X$, we assume that $\beta = 0$ on $\mathbb{Z}^N \setminus \mathcal{E}$ and that $\beta(-i) = \beta(i)$ for every $i \in \mathbb{Z}^N$. The anisotropy ϕ associated to β, as defined in (1-5), is such that its 1-Wulff shape $\mathcal{W}_1 \subseteq \mathbb{R}^N$ is a zonotope, which can be expressed as the Minkowski sum

$$\mathcal{W}_1 = \sum_{e \in \mathcal{E}} \beta(e)[-e, e] = \sum_{k=1}^{m} 2\beta(e_k)[-e_k, e_k],$$

where $[-e, e] \subseteq \mathbb{R}$ denotes the closed segment from $-e$ to e. Alternatively, one can define the zonotope $\mathcal{W}_1$ as the image of a cube under an affine map. Indeed,

$$\mathcal{W}_1 = V(Q^{(m)}), \tag{A-1}$$

where $V = (2\beta(e_1)e_1, \dots, 2\beta(e_m)e_m) \in \mathbb{R}^{N \times m}$ and $Q^{(m)} = [-1, 1]^m$. Since the set $\mathcal{E}$ is uniquely defined up to sign changes, the matrix V is also uniquely determined up to permutations of columns or sign changes.

Note that by definition of zonotope any element $x \in \mathcal{W}_\ell$ for $\ell > 0$ can be written as

$$x = \ell \sum_{k=1}^{m} 2\beta(e_k)\lambda_k e_k$$

for suitable coefficients $|\lambda_k| \leq 1$. We note that (the closure of) a facet F (of nonzero dimension) of the zonotope $\mathcal{W}_\ell$ can be described in the form

$$F = \ell \sum_{j=1}^{r} 2\beta(e_{\sigma(j)})[-e_{\sigma(j)}, e_{\sigma(j)}] + \ell \sum_{j=r+1}^{m} 2\beta(e_{\sigma(j)})\varepsilon_{\sigma(j)} e_{\sigma(j)}, \tag{A-2}$$

where σ is a permutation of $\{1, \dots, m\}$, $1 \leq r \leq m$ and $|\varepsilon_j| = 1$. Moreover (see [McMullen 1971, p. 206] for details), the vectors $e_{\sigma(1)}, \dots, e_{\sigma(r)}$ uniquely identify

$$\{e \in \mathcal{E} \mid e \parallel F\},$$

and r is uniquely defined as the number of vectors in the family $\mathcal{E}$ which are parallel to the facet F. Analogously, any vertex v of the zonotope $\mathcal{W}_\ell$ is of the form

$$v = \ell \sum_{j=1}^{m} 2\beta(e_{\sigma(j)})\varepsilon_{\sigma(j)} e_{\sigma(j)}, \tag{A-3}$$

where $\varepsilon_j \in \{\pm 1\}$ for every $j = 1, \dots, m$ and σ is a permutation of $\{1, \dots, m\}$. Note however that not every point of this form is a vertex of the zonotope.

Lemma A.1. *There exists $\ell_0 > 0$ such that, for every $\varepsilon > 0$ and every $\ell \geq \ell_0$, if $i \in \varepsilon\mathbb{Z}^N$ belongs to $\partial\mathcal{W}_{\varepsilon\ell}$, then for each $k \in \{1, \dots, m\}$ one of the following holds*:

(i) *Neither $i + \varepsilon e_k$ nor $i - \varepsilon e_k$ belong to $\partial\mathcal{W}_{\varepsilon\ell}$. In this case either $\phi^\circ(i + \varepsilon e_k) > \phi^\circ(i) > \phi^\circ(i - \varepsilon e_k)$ or $\phi^\circ(i - \varepsilon e_k) > \phi^\circ(i) > \phi^\circ(i + \varepsilon e_k)$.*

(ii) *One of $i \pm \varepsilon e_k$ belongs to $\partial\mathcal{W}_{\varepsilon\ell}$. In this case $\phi^\circ(i \pm \varepsilon e_k) \geq \ell$ and*

$$\#((i + \varepsilon\mathbb{Z}e_k) \cap \partial\mathcal{W}_{\varepsilon\ell}) \geq 2[\ell/\ell_0]. \tag{A-4}$$

Proof. By scaling, it suffices to prove the result in the case $\varepsilon = 1$. We take ℓ_0 such that

$$\ell_0 \geq \max_{k=1,\dots,m} \frac{1}{2\beta(e_k)} \tag{A-5}$$

and remark that $\ell_0 \in (0, +\infty)$. Note that the choice (A-5) implies for every $j = 1, \dots, m$ that

$$|[-2\ell\beta(e_j)e_j, 2\ell\beta(e_j)e_j]| = 4\ell\beta(e_j)|e_j| \geq 2\frac{\ell}{\ell_0}|e_j|.$$

We then fix $i \in \partial\mathcal{W}_\ell \cap \mathbb{Z}^N$ and $e_k \in \mathcal{E}$. We have to distinguish two cases.

Case 1. There exists a facet $F \ni i$ of $\mathcal{W}_\ell$ such that $e_k \parallel F$. By (A-2) we then see that

$$i \in 2\ell\beta(e_k)[-e_k, e_k] + j,$$

where $j \in F$. This implies in particular that $\{n \in \mathbb{Z} \mid i + ne_k \in F\}$ is an interval of $\mathbb{Z}$ containing 0. Furthermore, by the assumption (A-5), it contains at least $[2\ell|e_k|/\ell_0]$ points and we conclude (A-4). Since i and one of $i \pm e_k$ belong to $\partial \mathcal{W}_\ell$, we have that $\phi^\circ(i \pm e_k) \geq \ell$ by convexity.

Case 2. For every facet $F \ni i$ of $\mathcal{W}_\ell$, we have $e_k \not\parallel F$. Let us fix a facet $F \ni i$ and note that, by (A-2) and up to relabeling the indexes,

$$i \in \ell \sum_{j=1}^{r} 2\beta(e_j)[-e_j, e_j] + \ell \sum_{j=r+1}^{m} 2\beta(e_j)\varepsilon_j e_j,$$

with $k > r$ and $|\varepsilon_j| = 1$ for $j = r+1, \dots, m$. Recalling (A-1), we see that

$$i - \varepsilon_k e_k = \ell V\Big(y - \frac{\varepsilon_k}{\ell\beta(e_k)}\tilde{e}_k\Big),$$

where $\tilde{e}_1, \dots, \tilde{e}_m$ denotes the canonical base of $\mathbb{R}^m$ and $y \in \sum_{j=1}^{r}[-\tilde{e}_j, \tilde{e}_j] + \sum_{j=r+1}^{m} \varepsilon_j \tilde{e}_j \subseteq \partial Q^{(m)}$. By the choice (A-5) and since $k > r$, one deduces that

$$y - \frac{\varepsilon_k}{\ell\beta(e_k)}\tilde{e}_k \in Q^{(m)};$$

thus $i - \varepsilon_k e_k \in \overline{\mathcal{W}}_\ell$. Since then $e_k \not\parallel F$ for any facet containing i, we must have $\phi^\circ(i - \varepsilon_k e_k) < \ell$. By convexity one easily concludes that $\phi^\circ(i + \varepsilon_k e_k) > \ell$, which shows (i). □

We now define a calibration z_{ij} for every $(i, j) \in (\{\phi^\circ > \varepsilon\ell_0\} \cap \varepsilon\mathbb{Z}^N) \times \varepsilon\mathbb{Z}^N$. Fix $i \in \varepsilon\mathbb{Z}^N$ with $\phi^\circ(i) > \varepsilon\ell_0$. In the following we write $i \sim j$ if $(i - j)/\varepsilon \in \mathcal{E}$. We start by defining

$$z_{ij} = \begin{cases} 0 & \text{if } j \not\sim i, \\ -\beta(e_k) & \text{if } j = i \pm \varepsilon e_k \text{ and } \phi^\circ(j) > \phi^\circ(i), \\ \beta(e_k) & \text{if } j = i \pm \varepsilon e_k \text{ and } \phi^\circ(j) < \phi^\circ(i). \end{cases} \tag{A-6}$$

In particular, this definition covers case (i) in Lemma A.1. Assume then that there exists $j \sim i$ with $\phi^\circ(j) = \phi^\circ(i)$ and $(j - i)/\varepsilon = e_k \in \mathcal{E}$. Since $i \in \varepsilon\mathbb{Z}^N$ and $e_k \in \mathcal{E}$ fall into case (ii) of Lemma A.1, there exists an interval $[-\underline{n}, \bar{n}] \cap \mathbb{Z}$ for $\underline{n}, \bar{n} \in \mathbb{N}$ such that

$$(i + \varepsilon\mathbb{Z}e_k) \cap \partial W^{\phi^\circ}_{\phi^\circ(i)} = i + ([-\underline{n}, \bar{n}] \cap \mathbb{Z})\varepsilon e_k,$$

and moreover

$$\#([-\underline{n}, \bar{n}] \cap \mathbb{Z}) \geq 2[\phi^\circ(i)/(\varepsilon\ell_0)]. \tag{A-7}$$

Thus, we define z_{ij} as a linear interpolation of the values assumed at the extremal points of $i + [-\underline{n}, \bar{n}]\varepsilon e_k$:

$$\begin{aligned} z_{i+t\varepsilon e_k, i+(t+1)\varepsilon e_k} &:= \beta(e_k)\Big(1 - 2\frac{t + \underline{n} + 1}{\underline{n} + \bar{n} + 1}\Big) \quad \text{for all } t \in [-\underline{n} - 1, \bar{n}] \cap \mathbb{Z}, \\ z_{i+t\varepsilon e_k, i+(t-1)\varepsilon e_k} &:= \beta(e_k)\Big(1 - 2\frac{-t + \underline{n} + 1}{\underline{n} + \bar{n} + 1}\Big) \quad \text{for all } t \in [-\underline{n}, \bar{n} + 1] \cap \mathbb{Z}. \end{aligned} \tag{A-8}$$

By definition one easily sees that

$$|z_{ij}| \leq \alpha^\varepsilon_{ij}, \quad z_{ij}(\phi^\circ(i) - \phi^\circ(j)) = \alpha^\varepsilon_{ij}|\phi^\circ(i) - \phi^\circ(j)|. \tag{A-9}$$

We now bound the divergence $(D_\varepsilon^* z)_i$. Assume that $\phi^\circ(i+\varepsilon e_k)=\phi^\circ(i)$ or that $\phi^\circ(i-\varepsilon e_k)=\phi^\circ(i)$. Then by definition (A-8) and by (A-7) one deduces

$$z_{i,i+\varepsilon e_k}+z_{i,i-\varepsilon e_k}-z_{i+\varepsilon e_k,i}-z_{i-\varepsilon e_k,i}=-\frac{4\beta(e_k)}{\underline{n}+\bar{n}+1}\geq-\frac{2\beta(e_k)}{[\phi^\circ(i)/(\varepsilon\ell_0)]}\geq-\frac{C\varepsilon}{\phi^\circ(i)}, \tag{A-10}$$

and similarly if $\phi^\circ(i-\varepsilon e_k)=\phi^\circ(i)$. If instead $\phi^\circ(i\pm\varepsilon e_k)\neq\phi^\circ(i)$ and $\phi^\circ(i\pm\varepsilon e_k)\geq\varepsilon\ell_0$, one sees that

$$z_{i,i+\varepsilon e_k}+z_{i,i-\varepsilon e_k}=0 \quad\text{and}\quad z_{i+\varepsilon e_k,i}+z_{i-\varepsilon e_k,i}=0. \tag{A-11}$$

Combining (A-10) and (A-11) and recalling (4-2) we conclude that if $\phi^\circ(i)\geq\ell_1\varepsilon$ then

$$h(D_\varepsilon^* z)_i\geq-\frac{c_\phi h}{\phi^\circ(i)} \tag{A-12}$$

for a suitable positive constant c_ϕ depending on ϕ.

We now illustrate a procedure that allows us to extend the calibration above to $\varepsilon\mathbb{Z}^N\times\varepsilon\mathbb{Z}^N$. We set $C>1$, a sufficiently big constant, and define a function $v\in X_\varepsilon$ as

$$v:=\begin{cases}\phi^\circ+\frac{Ch}{\phi^\circ} & \text{on } \{\phi^\circ\geq C(\sqrt{h}\vee\varepsilon)\}\cap\varepsilon\mathbb{Z}^N,\\ C(\sqrt{h}\vee\varepsilon)+\frac{h}{\sqrt{h}\vee\varepsilon} & \text{on } \{\phi^\circ< C(\sqrt{h}\vee\varepsilon)\}\cap\varepsilon\mathbb{Z}^N.\end{cases} \tag{A-13}$$

A calibration $w\in Y_\varepsilon$ can be defined for $i,j\in\varepsilon\mathbb{Z}^N$ as

$$w_{ij}:=\begin{cases}z_{ij} & \text{if } \phi^\circ(i)\geq 2\sqrt{C}(\sqrt{h}\vee\varepsilon),\\ -\alpha_{ij}^\varepsilon & \text{if } \phi^\circ(i)<2\sqrt{C}(\sqrt{h}\vee\varepsilon).\end{cases} \tag{A-14}$$

Since $x\mapsto x+Chx^{-1}$ is strictly monotone in the region $\{x\geq\sqrt{Ch}\}$, we can employ (A-9) to prove that, for every $i,j\in\varepsilon\mathbb{Z}^N$ with $\phi^\circ(i)\geq C(\sqrt{h}\vee\varepsilon)$,

$$w_{ij}(v_i-v_j)=\alpha_{ij}^\varepsilon|v_i-v_j|,\quad |w_{ij}|\leq\alpha_{ij}^\varepsilon. \tag{A-15}$$

Moreover, taking C large enough ensures that, whenever $j\sim i$, we have

$$\begin{aligned}\phi^\circ(i)\leq 2\sqrt{C}(\sqrt{h}\vee\varepsilon) \quad&\Longrightarrow\quad \phi^\circ(j)\leq C(\sqrt{h}\vee\varepsilon),\\ \phi^\circ(i)\geq 2\sqrt{C}(\sqrt{h}\vee\varepsilon) \quad&\Longrightarrow\quad \phi^\circ(j)\geq \sqrt{C}(\sqrt{h}\vee\varepsilon).\end{aligned} \tag{A-16}$$

Thus, equation (A-15) can be directly checked in the case $\phi^\circ(i)\leq 2\sqrt{C}(\sqrt{h}\vee\varepsilon)$ using the definition (A-14).

Note now that definition (A-14) implies $D_\varepsilon^* w=0$ in the region $\{\phi^\circ<2\sqrt{C}(\sqrt{h}\vee\varepsilon)\}$; thus, we assume $\phi^\circ(i)\geq 2\sqrt{C}(\sqrt{h}\vee\varepsilon)$ and estimate $(D_\varepsilon^* w)_i$. If $\phi^\circ(i-\varepsilon e_k)<2\sqrt{C}(\sqrt{h}\vee\varepsilon)$, by convexity $\phi^\circ(i+\varepsilon e_k)>2\sqrt{C}(\sqrt{h}\vee\varepsilon)$. Thus, by definition (A-14) we get

$$z_{i,i+\varepsilon e_k}-z_{i+\varepsilon e_k,i}+z_{i,i-\varepsilon e_k}-z_{i-\varepsilon e_k,i}=-\beta(e_k)-\beta(e_k)+\beta(e_k)-(-\beta(e_k))=0.$$

The symmetric case is analogous. On the other hand, if every $j\sim i$ is in $\{\phi^\circ\geq 2\sqrt{C}(\sqrt{h}\vee\varepsilon)\}$, equation (A-12) holds. Therefore, we have shown

$$hD_\varepsilon^* w\geq-\frac{c_\phi h}{\phi^\circ}\chi_{\{\phi^\circ\geq\sqrt{C}(\sqrt{h}\vee\varepsilon)\}}. \tag{A-17}$$

By a direct computation, using (A-17) and assuming that $C > c_\phi$, we see that the pair (v, w) defined above satisfies

$$\begin{cases} hD_\varepsilon^* w + v \geq \phi^\circ, \\ w_{ij}(v_i - v_j) = \alpha_{ij}^\varepsilon |v_i - v_j|, \ |w_{ij}| \leq \alpha_{ij}^\varepsilon. \end{cases}$$

Recalling the comparison result in Theorem 3.3, we conclude that the solution u to (3-4) satisfies $u \leq v$ in $\varepsilon\mathbb{Z}^N$.

Appendix B: A remark on the inf/sup-convolution formulas (4-5)

In this section we show that, in some particular cases, the "inf" and "sup" in the definitions in (4-5) can be replaced by "min" and "max", and that this minimization/maximization procedure can be made in a fixed neighborhood of the point considered. Yet, our proof also shows that this neighborhood can become very large, depending on the weights of the interaction, and it seems that we cannot expect in general cases that the minimum and maximum are actually reached.

We introduce the following condition. There exists $\ell_\phi > 0$ such that, for every $\varepsilon_k \in \{0, \pm 1\}$ for $k = 1, \dots, m$, there exists $\ell \leq \ell_\phi$ such that

$$\ell \sum_{k=1}^{m} 2\beta(e_k)\varepsilon_k e_k \in \mathbb{Z}^N. \tag{B-1}$$

Note that this condition is satisfied if $\beta(e_k)/\beta(e_{k'}) \in \mathbb{Q}$ for all $k, k' = 1, \dots, m$.

Lemma B.1. *There exists $\ell_1 > 0$ with the following property: for any $i \in \varepsilon\mathbb{Z}^N$ with $\phi^\circ(i) \geq \varepsilon\ell_1$ there exists $j \in \varepsilon\mathbb{Z}^N \setminus \{0\}$ with $\phi^\circ(j) < \phi^\circ(i)$ satisfying*

$$\phi^\circ(i) \geq \phi^\circ(j) + \phi^\circ(i - j) - c_\phi\varepsilon. \tag{B-2}$$

If (B-1) *holds, for any $i \in \varepsilon\mathbb{Z}^N$ with $\phi^\circ(i) \geq 2\varepsilon\ell_1$ there exists $j \in (\mathcal{W}_{\varepsilon\ell_1} \setminus \{0\}) \cap \varepsilon\mathbb{Z}^N$ such that*

$$\phi^\circ(i) = \phi^\circ(j) + \phi^\circ(i - j). \tag{B-3}$$

Moreover, for every $R \in (2\varepsilon\ell_1, \phi^\circ(i))$ there exists $j \in \mathcal{W}_R \setminus \mathcal{W}_{R-2\varepsilon\ell_1}$ such that (B-3) *holds.*

Proof. By scaling we prove the result in the case $\varepsilon = 1$. Given $i \in \mathbb{Z}^N \setminus \{0\}$, inequality (B-2) follows easily choosing $\ell_1 \geq 2$, and considering $\sigma i \in \mathbb{R}^N \setminus \{0\}$ for an appropriate $\sigma \in (0, 1)$ and $j \in \mathbb{Z}^N$ such that $\sigma i \in (j + [0, 1]^N)$.

We now assume (B-1) and denote by ℓ_ϕ the radius associated to ϕ. We then choose $\ell_1 = \ell_\phi$. Let us fix $i \in \mathbb{Z}^N$ with $\phi^\circ(i) = \ell \geq 2\ell_1$. By (A-2) there exist $r > 0$, ε_k, λ_k with $|\varepsilon_k| = 1$ and $|\lambda_k| < 1$ such that

$$i = \ell\Bigg(\sum_{k=1}^{r} 2\beta(e_k)\varepsilon_k e_k + \sum_{k=r+1}^{m} \lambda_k 2\beta(e_k)e_k\Bigg).$$

Let us set the point

$$v = \sum_{k=1}^{r} 2\beta(e_k)\varepsilon_k e_k \in \partial\mathcal{W}_1$$

and define the function sign by $\operatorname{sign}(x) = x/|x|$ if $x \neq 0$ and 0 otherwise. For any $\ell' \leq \ell_\phi$ we rewrite i as

$$i = \ell'\Big(v + \sum_{k=r+1}^{m} 2\beta(e_k)\operatorname{sign}(\lambda_k)e_k\Big) + (\ell-\ell')\Big(v + \sum_{k=r+1}^{m} 2\beta(e_k)\Big(\frac{\ell}{\ell-\ell'}\lambda_k - \frac{\ell'}{\ell-\ell'}\operatorname{sign}(\lambda_k)\Big)e_k\Big)$$
$$=: \ell' w + (\ell-\ell')\Big(v + \sum_{k=r+1}^{m} 2\beta(e_k)\lambda'_k e_k\Big).$$

Notice that, since $\ell \geq 2\ell'$ and $|\lambda_k| \leq 1$, we have $|\lambda'_k| \leq 1$; thus, by formula (A-2) we get

$$v + \sum_{k=r+1}^{m} 2\beta(e_k)\lambda'_k e_k \in \partial \mathcal{W}_1,$$

and therefore $\phi^\circ(i - \ell' w) = \ell - \ell'$. We conclude by noting that from the hypothesis (B-1) we can choose $\ell' \leq \ell_1$ so that $\ell' w \in \mathbb{Z}^N$, which implies (B-3) since $\phi^\circ(\ell' w) = \ell'$.

We now prove the last assertion. Since $\phi^\circ(i) \geq 2\ell_1$, by the previous result there exists $j_0 \in (\mathcal{W}_{\ell_1} \setminus \{0\})$ so that $\phi^\circ(i) = \phi^\circ(j_0) + \phi^\circ(i - j_0)$. Now, if $R - 2\ell_1 \leq \phi^\circ(j_0)$ we conclude. If not, then $\phi^\circ(i - j_0) \geq 2\ell_1$ by (B-3), and thus we can find $k_0 \in (\mathcal{W}_{\ell_1} \setminus \{0\})$ so that

$$\phi^\circ(i - j_0) = \phi^\circ(k_0) + \phi^\circ(i - j_0 - k_0). \tag{B-4}$$

Writing $j_1 = j_0 + k_0$, on one hand (B-4) implies

$$\phi^\circ(i) = \phi^\circ(j_0) + \phi^\circ(j_1 - j_0) + \phi^\circ(i - j_1) \geq \phi^\circ(j_1) + \phi^\circ(i - j_1), \tag{B-5}$$

thus equality holds instead. If $\phi^\circ(j_1) \geq R - 2\ell_1$ we conclude; if not (B-5) yields $\phi^\circ(i - j_1) \geq 2\ell_1$ and we can iterate. Recalling that $\phi^\circ \geq c_\phi > 0$ on $\varepsilon\mathbb{Z}^N \setminus \{0\}$, it is clear that after a finite number of iterations the process stops, and one can check that the required properties are satisfied. □

By the previous lemma it is easy to prove the following result.

Corollary B.2. *Assume that* (B-1) *holds. Let $u \in X$ be a* $(1, \phi)$*-Lipschitz function, and let ℓ_1 be as in Lemma B.1. Then, for all $i \in \varepsilon\mathbb{Z}^N$*

$$\sup_{j \in \{u \geq 0\}} \{u_j - \phi^\circ(i - j)\} = \max_{j \in \{u \geq 0\}} \{u_j - \phi^\circ(i - j)\}.$$

In addition, if $i \in \{u \leq 0\}$, the maximum is reached at a point in $(\{u \leq 0\} + \mathcal{W}_{2\varepsilon\ell_1}) \cap \varepsilon\mathbb{Z}^N$.

Proof. It is enough to consider $i \in \{u < 0\} \cap \varepsilon\mathbb{Z}^N$. Let us write $F = (\{u \leq 0\} + \mathcal{W}_{2\varepsilon\ell_1}) \cap \{u > 0\}$. Firstly, by a variant of the argument by iteration employed in the proof of Lemma B.1, one can prove that

$$\sup_{j \in \{u \geq 0\}} \{u_j - \phi^\circ(i - j)\} = \sup_{j \in F} \{u_j - \phi^\circ(i - j)\}. \tag{B-6}$$

On the other hand, take a point $j_0 \in \{u > 0\}$. If $j \in F$ satisfies $u_j - \phi^\circ(i - j) \geq u_{j_0} - \phi^\circ(i - j_0)$, since $u \leq 2\varepsilon\ell_1$ in F (as u is $(1, \phi^\circ)$-Lipschitz) we obtain

$$2\varepsilon\ell_1 + \phi^\circ(i - j_0) \geq \phi^\circ(i - j),$$

which implies that the supremum in (B-6) is indeed a maximum. □

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ANTONIN CHAMBOLLE: `chambolle@ceremade.dauphine.fr`
CEREMADE (CNRS UMR 7534), Université Paris-Dauphine, PSL University, Paris, France

DANIELE DE GENNARO: `daniele.degennaro@unibocconi.it`
Bocconi University, Milano, Italy

MASSIMILIANO MORINI: `massimiliano.morini@unipr.it`
Dipartimento di Matematica, Università degli Studi di Parma, Parma, Italy

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