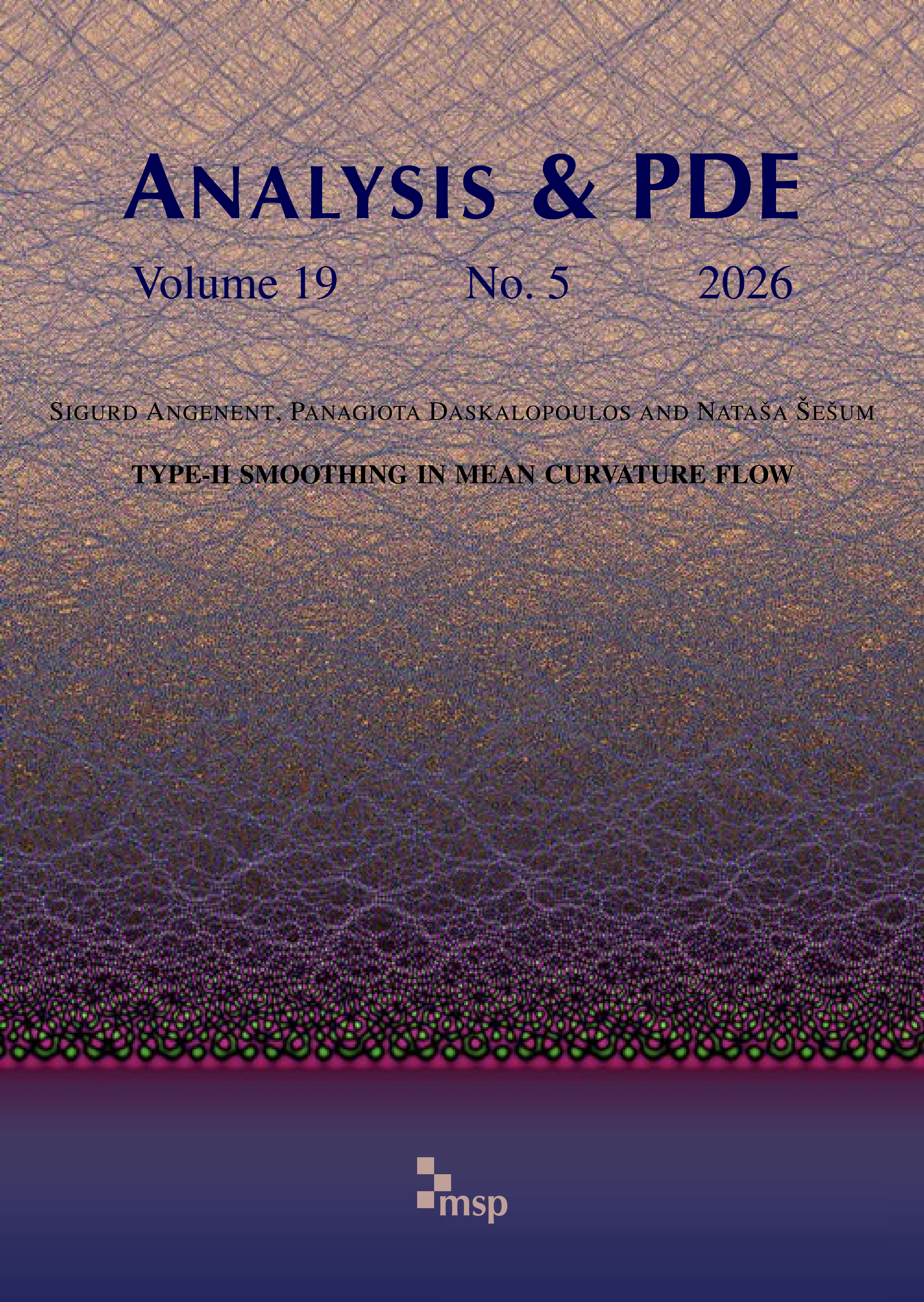

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TYPE-II SMOOTHING IN MEAN CURVATURE FLOW

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Velázquez (1994) constructed a smooth $O(4) \times O(4)$ invariant mean curvature flow that forms a type-II singularity at the origin in spacetime. Stolarski (2023) showed that the mean curvature on this solution is uniformly bounded. In a work in preparation, Angenent, Ilmanen, and Velázquez (Angenent et al.) also provided formal asymptotic expansions for a possible smooth continuation of the solution after the singularity.

Here we prove short-time existence of Velázquez' formal continuation, and we verify that the mean curvature is also uniformly bounded on the continuation. Combined with the earlier results of Velázquez and Stolarski we therefore show that there exists a solution $\{M_t^7 \subset \mathbb{R}^8 \mid -t_0 < t < t_0\}$ that has an isolated singularity at the origin $0 \in \mathbb{R}^8$ and at $t = 0$; moreover, the mean curvature is uniformly bounded on this solution, even though the second fundamental form is unbounded near the singularity.

1. Introduction

We say that a family of hypersurfaces $\{M_t\}_{t\in[0,T)} \subset \mathbb{R}^{n+1}$ moves by the mean curvature flow if

$$\frac{\partial \vec{F}}{\partial t} = \vec{H}, \tag{MCF}$$

where $\vec{H}(\,\cdot\,,t)$ is the mean curvature vector of the hypersurface M_t, and $\vec{F}(\,\cdot\,,t) : M \to M_t \subset \mathbb{R}^{n+1}$ is a smooth family of parametrizations of the moving hypersurface. In the case of closed hypersurfaces, Huisken [1984] showed the norm of the second fundamental form blows up at finite time $T < \infty$; that is

$$\limsup_{t\to T} \max_{M_t} |A|(\,\cdot\,,t) = \infty.$$

Very often, even in a complete, noncompact setting, mean curvature flow (MCF) develops a singularity at a finite time $T < \infty$. It is very natural to ask whether the mean curvature also needs to blow up at a finite time singularity, or equivalently, whether a uniform bound on $|\vec{H}|$ for all $t \in [0, T)$ guarantees the existence of smooth solution past time T.

For mean convex flows it is well known [Huisken 1984] that the mean curvature bounds the second fundamental form A, i.e., $|A|/|\vec{H}|$ attains its maximum at $t = 0$ and therefore is uniformly bounded. This implies that for mean convex flows the mean curvature is never bounded near a singularity. Dropping the assumption of mean convexity, it was shown by Le, Lin, and Sesum [Le and Sesum 2011a; 2011b; Lin and Sesum 2016] and by Xu, Ye, and Zhao [Xu et al. 2011] that for mean curvature flow of closed

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hypersurfaces the mean curvature needs to blow up at the first singular time, given some extra assumptions, such as having only type-I singularities or being close to a sphere in the L^2 sense. More recently, Li and Wang [2019] showed, using a quite involved argument that in the case of closed surfaces in $\mathbb{R}^3$ the mean curvature always blows up at the first singular time. The question of boundedness of the mean curvature on a singular mean curvature flow is therefore completely settled in the case of compact surfaces in $\mathbb{R}^3$, and for hypersurfaces in higher dimensions under a variety of extra assumptions.

For $n \geq 4$, Velázquez [1994] constructed $O(n) \times O(n)$ symmetric solutions of dimension $N = 2n - 1$ that converge to the Simons cone at parabolic scales around the singularity, and converge to a smooth minimal surface desingularizing Simons cone at the scale at which the norm of the second fundamental form blows up at the origin. Using formal asymptotic expansions, Angenent, Ilmanen, and Velázquez [Angenent et al.] also suggested a way in which the solution $\{M_t\}$ might be continued smoothly after the singularity, i.e., for $t > 0$.

These complete noncompact solutions were expected to provide examples of higher-dimensional mean curvature flow with the property that the mean curvature stays bounded at the first singular time. Stolarski [2023] used precise asymptotics of these solutions together with sophisticated blow up arguments to rigorously prove that this is indeed the case for $t < 0$; i.e., he showed that before the singularity forms the mean curvature on some of Velázquez' solutions is uniformly bounded. (To be precise, he requires the parameter k that appears in Velázquez' solutions to be even and not less than 4.)

Here we consider the case $n = 4$, i.e., the case of 7-dimensional hypersurfaces in $\mathbb{R}^8$. We first prove existence and regularity of Velázquez' formal extension of the Velázquez–Stolarski solutions, and we thereby obtain a solution $\{M_t \subset \mathbb{R}^8 \mid -t_0 < t < t_0\}$ of MCF that is smooth everywhere except at the origin $(0, 0) \in \mathbb{R}^8 \times (-t_0, t_0)$ in spacetime, and whose *mean curvature is uniformly bounded* even though its *second fundamental form blows up near* $(0, 0)$. In particular, we show that the singular hypersurface $M_0 = \lim_{t \nearrow 0} M_t$ that remains after the Velázquez–Stolarski solution forms its singularity can be used as initial data for MCF, and that at least one of the ensuing solutions has uniformly bounded mean curvature.

Stolarski [2023] indicates he expects his result to be true for closed mean curvature flow that can be obtained by compactifying Velázquez examples, but it still remains open. Another question that remains completely open is what happens in dimensions $3 \leq N \leq 6$ where neither an example of a singular solution with bounded mean curvature nor a theorem proving the impossibility of such an example are known.

1.1. ***Outline.*** In this paper we consider an $O(4) \times O(4)$ symmetric hypersurface M_0 defined by the profile function

$$u = u_0(x),$$

where $u_0 : (0, \infty) \to \mathbb{R}$ is a smooth function that near the origin satisfies

$$u_0(x) = x + K_0 x^{2(k-1)} + o(x^{2(k-1)}), \quad x \searrow 0, \tag{1.1.1}$$

for some integer

$$k \geq 4$$

and some constant $K_0 > 0$. We will also assume that for all $x > 0$ one has

$$0 \le u_0'(x) \le C_0, \quad |u_0''(x)| \le C_0, \quad |u_0''(x)| \le C_0 x^{2k-4} \tag{1.1.2}$$

for some constant $C_0 > 0$. The last assumption implies, after integration, that for all $x > 0$ one has

$$|u_0'(x) - 1| \le C x^{2k-3} \tag{1.1.3}$$

for some constant $C > 0$, depending on C_0. This implies that for x small enough we have $u_0'(x) \ge \frac{1}{2}$. By rescaling we may assume that

$$u_0'(x) \ge c > 0 \quad \text{for } x \in [0, 1]. \tag{1.1.4}$$

It turns out that such a function $u_0(x)$ is the final-time profile near a singularity $(0, 0)$ of the $O(4) \times O(4)$ MCF solution M_t, $-t_1 < t < 0$, for some small $t_1 < 0$, which was constructed by Velázquez [1994]. Under the assumption that k is even, Stolarski [2023] showed that the Velázquez solution has bounded mean curvature at the singularity, i.e., the mean curvature of M_t remains bounded as $t \to 0^-$ near $(0, 0)$.

Our goal in this paper is to show that the MCF starting at M_0 can be continued for $0 < t < t_0$, for some $t_0 > 0$ small, with a smooth solution M_t, $t \in (0, t_0)$, which is $O(4) \times O(4)$ symmetric. Furthermore, the mean curvature of M_t as $t \to 0^+$ will *remain uniformly bounded* despite the fact that M_0 is singular at $x = 0$.

The solution M_t will be defined by a profile function $u : (0, \infty) \times (0, t_0) \to (0, \infty)$ that satisfies the initial value problem

$$u_t = \frac{u_{xx}}{1 + u_x^2} + \frac{3}{x} u_x - \frac{3}{u}, \tag{1.1.5a}$$

$$\lim_{x \to 0} u_x(x, t) = 0, \tag{1.1.5b}$$

$$\lim_{t \to 0} u(x, t) = u_0(x). \tag{1.1.5c}$$

Note the condition $\lim_{x \to 0} u_x(x, t) = 0$ ensures that $u(x, t)$ defines an $O(4) \times O(4)$ hypersurface M_t that is smooth at the origin and hence everywhere.

We will prove the following:

Main Theorem. *Assume that M_0 is an $O(4) \times O(4)$ symmetric hypersurface defined by the profile function $u_0 : [0, \infty) \to \mathbb{R}$ which is smooth for $x > 0$ and at $x = 0$ satisfies condition (1.1.1) for some $k > 3$. Then, there exists $t_0 > 0$ and a C^∞-smooth $O(4) \times O(4)$ symmetric MCF solution M_t, $0 < t \le t_0$, defined by a profile function $u : (0, \infty) \times (0, t_0] \to (0, \infty)$ which satisfies the initial value problem (1.1.5a)–(1.1.5c). Furthermore the mean curvature $H(x, t)$ of the hypersurface M_t satisfies*

$$\sup_{(x,t) \in [0,\infty) \times (0,a]} |H(x, t)| < +\infty$$

for some $0 < a \le t_0$. In particular, $H(x, t)$ is uniformly bounded near the origin as $t \to 0^+$ despite the fact that the mean curvature of M_0 is undefined at the origin.

As a corollary of the main theorem and the results in [Stolarski 2023] we have the following result.

Corollary 1.1.1. *There exists an* $O(4)\times O(4)$ *symmetric complete noncompact mean curvature flow solution* $\{M_t\}_{t\in(-t_0,t_0)}$ *such that* M_t *is smooth for all* $t\in(-t_0,t_0)\backslash\{0\}$, *has a type-II singularity at the origin and at time* $t=0$, *and has uniformly bounded mean curvature away from* $t=0$. *More precisely, there exists a uniform constant* C, *so that*

$$\sup_{(x,t)\in[0,\infty)\times(-t_0,t_0)\backslash\{0\}} |H(x,t)|\le C.$$

The short-time existence of a smooth MCF solution starting at M_0 follows by standard quasilinear parabolic PDE theory. The challenge here is to establish the *uniform bound* on $H(\,\cdot\,,t)$ near the singularity $(0,0)$. For this purpose we will construct sharp upper and lower barriers which will capture the exact behavior of the profile function $u(x,t)$ of our solution M_t as $(x,t)\to(0,0)$. This will be done in Section 3. In Section 4 we will then construct the profile function $u(x,t)$, namely a solution of the initial boundary value problem (1.1.5a)–(1.1.5c). The boundary condition $u_x(0,t)=0$ and the fact that $u>0$ will guarantee that $u(x,t)$ defines a smooth MCF solution M_t which is $O(4)\times O(4)$ symmetric. In Section 5 we will show that $H(x,t)$ remains bounded as $t\to0$. The barrier construction in Section 3 is based on the formal asymptotic expansion of the profile solution $u(x,t)$ as $(x,t)\to(0,0)$. For the convenience of the reader we will start by giving this expansion in the next section.

2. Formal asymptotic expansion of $u(x,t)$

We start with the construction by Angenent, Ilmanen, and Velázquez [Angenent et al.] of a formal asymptotic expansion of the profile solution $u(x,t)$ for small $t>0$. This construction motivates our choice of barriers in different regions later in order to rigorously prove the existence of a mean curvature flow past the singular time with the following properties. Our solution before the singularity at $t=0$ coincides with the solution constructed by Velázquez [1994], it continues as a smooth solution for $t\in(0,t_1)$, for some $t_1>0$, and has uniformly bounded mean curvature for all times $t<0$, for which it exists, and all $t\in(0,t_1)$.

2.1. *Outer variables.*

We can approximate any smooth solution for small $t>0$ by using the Taylor expansion $u(x,t)=u(x,0)+tu_t(x,0)+o(t)$. In view of the PDE (1.1.5a) this implies that any solution $u(x,t)$ must satisfy

$$u(x,t)=u_0(x)+t\left\{\frac{u_0''(x)}{1+u_0'(x)^2}+\frac{3}{x}u_0'(x)-\frac{3}{u_0(x)}\right\}+o(t),\qquad t\to0. \tag{2.1.1}$$

We will see that under our assumptions (1.1.1)–(1.1.3) on the initial data, the expansion (2.1.1) holds if $x^2\gg t$. To describe possible solutions for $x^2\sim t$ we introduce a new set of coordinates, the intermediate variables.

2.2. *Intermediate variables.*

Consider the function $v(y,\tau)$ defined by

$$u(x,t)=\sqrt{t}\,v\left(\frac{x}{\sqrt{t}},\log t\right). \tag{2.2.1}$$

It satisfies

$$v_\tau = \frac{v_{yy}}{1+v_y^2} + \left(\frac{3}{y} + \frac{y}{2}\right)v_y - \frac{v}{2} - \frac{3}{v}. \tag{2.2.2}$$

Assuming that $v(y,\tau)$ is close to the cone, we set

$$v(y,\tau) = y + f(y,\tau)$$

and compute the equation for f,

$$f_\tau = \mathcal{L}f + \mathcal{N}[f], \tag{2.2.3}$$

where $\mathcal{L}$ is the linear differential operator

$$\mathcal{L}f := \frac{1}{2}f_{yy} + \left(\frac{3}{y} + \frac{y}{2}\right)f_y + \left(\frac{3}{y^2} - \frac{1}{2}\right)f, \tag{2.2.4}$$

and where

$$\mathcal{N}[f] := -3\frac{f^2}{y^2(y+f)} - \frac{2+f_y}{1+(1+f_y)^2}f_y f_{yy} \tag{2.2.5}$$

collects the nonlinear terms in the equation for f.

If we assume that the nonlinear terms are much smaller than the linear terms, then f should be approximated by a solution of the linear equation $f_\tau = \mathcal{L}f$. The outer approximation $u(x,t) = u_0(x) + \mathcal{O}(t)$ together with the assumption that the initial function satisfies $u(x,0) = x + K_0 x^{2(k-1)} + \cdots$ leads to

$$v(y,\tau) = y + K_0 e^{(k-3/2)\tau} y^{2(k-1)} + \cdots \tag{2.2.6}$$

for $y \gg e^{-\tau/2}$. This prompts us to look for approximate solutions of the form

$$v(y,\tau) = y + K_1 e^{(k-3/2)\tau}\varphi_k(y), \tag{2.2.7}$$

where φ_k is a solution of the differential equation

$$\mathcal{L}\varphi_k = \left(k - \tfrac{3}{2}\right)\varphi_k.$$

It turns out that there are positive and convex solutions of this equation that are defined for all $y > 0$. Their asymptotic behavior for small and large values of y is given by

$$\varphi_k(y) = \frac{1+o(1)}{y^2}, \quad y \to 0, \qquad \varphi_k(y) = \frac{1+o(1)}{(2k+1)!!}y^{2k-2}, \quad y \to \infty.$$

In Section A.1 we present some more details regarding the eigenfunctions φ_k.

This implies that our intermediate solution $v(y,\tau)$ from (2.2.7) is given by

$$v(y,\tau) = y + K_1 e^{(k-3/2)\tau}\frac{y^{2(k-1)}}{(2k+1)!!} + \cdots$$

when y is large.[1] Comparing with (2.2.6) we see that K_0 and K_1 are related by

$$K_1 = K_0(2k+1)!!. \tag{2.2.8}$$

---

[1] Notation: $(2k+1)!! = 1 \cdot 3 \cdot 5 \cdots (2k-1) \cdot (2k+1)$.

2.3. ***Inner variables.*** One can only expect the intermediate approximation to hold if the nonlinear terms are small compared with the linear terms. Since the linear terms are all of order $\sim f/y^2$ and the nonlinear terms are of order f^2/y^3, we see that the nonlinear terms are dominated by the linear terms if $|f/y| \ll 1$.

When y is small we have $f(y,\tau) \sim e^{(k-3/2)\tau} y^{-2}$, so $|f/y| \ll 1$ holds if

$$e^{(k-3/2)\tau} y^{-3} \ll 1, \quad \text{i.e., } y \gg e^{(k/3-1/2)\tau} = e^{\gamma\tau},$$

where we write

$$\gamma = \frac{k}{3} - \frac{1}{2}.$$

In the original (x,t) coordinates we have $y = e^{\gamma\tau}$ exactly if $x = t^{k/3}$.

This leads us to introduce the new variable

$$z = ye^{-\gamma\tau} = xt^{-k/3}$$

and a new function $w(z,\tau)$ defined by

$$v(y,\tau) = e^{\gamma\tau} w(ye^{-\gamma\tau},\tau). \tag{2.3.1}$$

Equation (2.2.2) is equivalent to

$$\frac{w_{zz}}{1+w_z^2} + \frac{3}{z}w_z - \frac{3}{w} = e^{2\gamma\tau}\left\{w_\tau + \frac{k}{3}(w - zw_z)\right\}. \tag{2.3.2}$$

For $\tau \to -\infty$ we assume the terms on the right vanish, so it is natural to look for an approximate solution of the form

$$w(z,\tau;K_2) = K_2 W\left(\frac{z}{K_2}\right) + \text{correction terms}, \tag{2.3.3}$$

where $W(z)$ is Alencar's solution[2] of the minimal surface equation

$$\frac{W''(z)}{1+W'(z)^2} + \frac{3}{z}W'(z) - \frac{3}{W(z)} = 0. \tag{2.3.4}$$

By scaling invariance of the minimal surface equation, $KW(z/K)$, with $K > 0$ an arbitrary constant, is always a solution of (2.3.4) if W is one. We choose W so that it is normalized by

$$W(z) = z + \frac{1}{z^2} + o(z^{-2}), \quad z \to \infty. \tag{2.3.5}$$

The matching condition for the inner solution $w(z,\tau) = K_2 W(z/K_2) + \cdots$ with the intermediate solution $v(y,\tau) = y + K_1 e^{(k-3/2)\tau}\varphi_k(y) + \cdots$ is then

$$w(z,\tau) \approx e^{-\gamma\tau} v(e^{\gamma\tau}z,\tau);$$

---

[2] See Section A.3 for a discussion of this solution. Alencar [1993] analyzed the ODEs that appear when one considers $\mathrm{SO}(m) \times \mathrm{SO}(m)$ invariant minimal surfaces of this type, although he mostly considered the cases $m = 2, 3$ in that first paper. Velázquez [1994] dealt with the case $m \geq 4$, and later Alencar, Barros, Palmas, Reyes, and Santos gave a complete classification in [Alencar et al. 2005]. Much earlier, Hardt and Simon [1985] proved a general existence theorem for smooth minimal hypersurfaces that accompany strictly minimizing cones, without assuming any kind of symmetry.

i.e.,

$$z+\frac{K_2^3}{z^2}+\cdots=z+K_1\frac{e^{(k-3/2)\tau}e^{-3\gamma\tau}}{z^2}+\cdots=z+\frac{K_1}{z^2}+\cdots.$$

Hence the constants K_1 and K_2 are related by

$$K_2^3=K_1=K_0(2k+1)!!, \tag{2.3.6}$$

and our approximate inner solution is given by

$$w(z,\tau)=K_1^{1/3}W(K_1^{-1/3}z).$$

3. Barriers

3.1. ***The three regions.*** Our goal in this section is to construct upper and lower barriers for

$$u_t=\frac{u_{xx}}{1+u_x^2}+\frac{3}{x}u_x-\frac{3}{u} \tag{1.1.5a}$$

that are valid for all $x\in(0,+\infty)$ and $0<t\le t_0$, for some small enough $t_0>0$.

To do this we modify the approximate solutions from Section 2 in each of the three regions and glue the resulting locally defined barriers into one set of globally defined upper and lower barriers.

First we define *the three regions*. In what follows we regard the three regions as subsets of spacetime and use the different sets of coordinates (x,t), (y,τ), and (z,τ) on spacetime to describe them.

- For any given $M>0$ we define the *outer region* to be

  $$\mathcal{O}_M=\{(x,t)\mid x\ge M\sqrt{t},\ 0<t<M^{-2}\}.$$

  We will assume that $M>1$.

- For any $R>0$ and $\tau_*\in\mathbb{R}$ we define the *intermediate region* to be

  $$\mathcal{M}_{R,\tau_*}=\{(y,\tau)\mid Re^{\gamma\tau}\le y\le e^{-\tau/2},\ \tau\le\tau_*\}.$$

  Since $y=x/\sqrt{t}=xe^{-\tau/2}$, the intermediate region is defined up to $x=1$; hence the intermediate and outer regions clearly overlap.

- Finally, we declare the *inner region* to be

  $$\mathcal{I}_{Z,\tau_*}=\{(z,\tau)\mid 0\le z\le Z,\ \tau\le\tau_*\}.$$

  Since $z=e^{-\gamma\tau}y$, we see that the intermediate and inner regions overlap if $Z>R$.

In Section 4 we will construct a nested sequence of barriers

$$U^-_{\delta_{n-1}}<U^-_{\delta_n}<U^+_{\delta_n}<U^+_{\delta_{n-1}},$$

where $\delta_n=2^{-n}\delta_0$ for some $\delta_0>0$. These barriers will be defined for all $\tau\le\tau_{\delta_n}$, where $\tau_{\delta_n}\to-\infty$ as $\delta_n\to0$. As a result we will see that we need to take $Z=Z_{\delta_n}$ and $\tau^*=\tau_{\delta_n}$ in the definitions of the intermediate and inner regions above. In addition we will see that $Z_{\delta_n}\to+\infty$ as $\delta_n\to0$.

3.2. *Fixing the parameters.* From here on we fix the parameters $k > 3$ and $K_0 > 0$, and we let K_1, K_2 be defined by (2.3.6). In all our estimates c and C will be generic constants that can depend *only* on k, K_0, K_1, and K_2. We use C in upper bounds, and c in lower bounds.

3.3. *Barriers in the outer region.*

Lemma 3.3.1. *For sufficiently large $M > 0$ the functions*

$$u^{\pm}(x,t) = u_0(x) \pm Mt \min\{1, x^{2k-4}\} \tag{3.3.1}$$

are super- or subsolutions in the outer region $\mathcal{O}_M$.

Proof. We only consider the upper barrier u^+. Similar arguments apply to the lower barrier.

When $x > 1$ we have $u^+(x,t) = u_0(x) + Mt$, so that for $t \in (0, M^{-2})$ one has $u^+(x,t) \geq \inf_{x\geq 1} u_0(x) =: c$. This implies

$$\left| \frac{u^+_{xx}}{1+(u^+_x)^2} + \frac{3}{x}u^+_x - \frac{3}{u^+} \right| \leq C$$

for all $x \geq 1$ and $t \leq M^{-2}$. Here C does not depend on M. On the other hand $u^+_t = M$, so for large enough M we get

$$u^+_t \geq \frac{u^+_{xx}}{1+(u^+_x)^2} + \frac{3}{x}u^+_x - \frac{3}{u^+},$$

i.e., u^+ is an upper barrier for $x \geq 1$.

If $x \geq M\sqrt{t}$ and $x \leq 1$, we have $u^+(x,t) = u_0(x) + Mtx^{2k-4}$, so that

$$|u^+_{xx}| \leq |u_{0,xx}| + CMtx^{2k-6} \leq Cx^{2k-4} + CMtx^{2k-6} \leq Cx^{2k-4}.$$

Similar estimates hold for $u^+_x - 1$ and $u^+(x,t) - x$, namely,

$$x^2|u^+_{xx}| + x|u^+_x - 1| + |u^+ - x| \leq Cx^{2k-2}.$$

Hence

$$\frac{|u^+_{xx}|}{1+(u^+_x)^2} \leq Cx^{2k-4},$$

and also

$$\left| \frac{3}{x}u^+_x - \frac{3}{u^+} \right| \leq \frac{3}{x}|u^+_x - 1| + 3\frac{|u^+ - x|}{xu^+} \leq Cx^{2k-4}.$$

Together we get

$$\left| \frac{u^+_{xx}}{1+(u^+_x)^2} + \frac{3}{x}u^+_x - \frac{3}{u^+} \right| \leq Cx^{2k-4},$$

where C does not depend on M. On the other hand, $u^+_t = Mx^{2k-4}$. Hence, it now follows that $u_0(x) + Mtx^{2k-4}$ is an upper barrier if M is large enough.

Finally we observe that at the point $x = 1$ the function $u^+(x,t)$ has a concave corner, so that $u^+(x,t) = u_0(x) + Mt\min\{1, x^{2k-4}\}$ is indeed an upper barrier for all $x \geq M\sqrt{t}$, $t < M^{-2}$.

Similar arguments show that

$$u^-(x,t) = u_0(x) - Mt\min\{1, x^{2k-4}\}$$

is a lower barrier in the same region. The only difference is that one now uses for $x > 1$, $t \in (0, M^{-2})$ the lower bound

$$u^-(x,t) \geq \inf_{x\geq 1} u_0(x) - Mt \geq \tfrac{1}{2}c,$$

for M sufficiently large, where $c := \inf_{x\geq 1} u_0(x)$. ☐

3.4. ***Barriers in the intermediate region.*** We model the upper and lower barriers in the intermediate region on the approximate solution $v(y,\tau) = y + f(y,\tau)$ from Section 2.2, where f is assumed to be a small function that satisfies (2.2.3), i.e., $f_\tau = \mathcal{L}f + \mathcal{N}[f]$. A function f defines an upper barrier for (2.2.3) in $\mathcal{M}_{R,\tau_*}$ if

$$f_\tau - \mathcal{L}f \geq \mathcal{N}[f] \tag{3.4.1}$$

holds throughout $\mathcal{M}_{R,\tau_*}$. For a lower barrier the reverse inequality must hold.

It turns out that the approximate solution $f_0(y,\tau) = Ke^{3\gamma\tau}\varphi_k(y)$ is neither a sub- nor supersolution for any choice of the constant K. To obtain barriers we therefore add a small correction term $f_1(y,\tau)$. While the resulting function $f_0(y,\tau) + f_1(y,\tau)$ does provide a barrier, it does not match the barrier we construct later in the inner region. To remedy this we add a second correction term $f_2(y,\tau)$. The resulting barriers $f_0 + f_1 + f_2$ will contain a small parameter $\delta > 0$. By choosing $\delta > 0$ smaller we get more accurate barriers, but we also have to reduce the time interval $-\infty < \tau \leq \tau_\delta$ on which they are defined. In the end this will allow us to prove convergence as $\tau \to -\infty$ of the actual solution that we construct using our barriers.

Our construction uses an auxiliary function $g : (0,\infty) \to \mathbb{R}$, which is the solution of the boundary value problem

$$\begin{cases} 6\gamma g(y) - \mathcal{L}g(y) = y^{-7} + y^{4k-7}, & 0 < y < \infty, \\ g(y) = -\frac{1}{3}y^{-5} + o(y^{-5}), & y \to 0, \\ g(y) = y^{4k-7} + o(y^{4k-7}), & y \to \infty. \end{cases} \tag{3.4.2}$$

The choice of forcing term in the equation for g above will become apparent in what follows. In Section A.2 we prove:

Lemma 3.4.1. *Equations* (3.4.2) *have a solution* $g : (0,\infty) \to \mathbb{R}$.

In fact, the proof in Section A.2 shows that there is a one-parameter family of solutions g. We choose one of these, for example the one for which the constant B from the proof in Section A.2 vanishes.

Assuming that Lemma 3.4.1 holds, we look for barriers in the family of functions

$$v_\delta^\pm(y,\tau) = y + f_\delta^\pm(y,\tau), \tag{3.4.3}$$

where

$$f_\delta^\pm(y,\tau) = f_0^\pm(y,\tau,\delta) \pm \{f_1(y,\tau) + f_2(y,\tau)\} \tag{3.4.4}$$

and

$$\begin{aligned} f_0^{\pm}(y,\tau,\delta) &= (K_1 \pm \delta)e^{3\gamma\tau}\varphi_k(y), \\ f_1(y,\tau) &= BK_1^2 e^{6\gamma\tau} g(y), \\ f_2(y,\tau) &= e^{(p+1)\gamma\tau} y^{-p}. \end{aligned} \tag{3.4.5}$$

Here, as in Section 3.2, we have $K_1 = (2k+1)!!\, K_0$, while $B, \delta > 0$ and $p \in (2,3)$ are parameters.

Proposition 3.4.2. *There exist B_*, R_*, and τ_* that only depend on k, K_0 such that for all $\delta \in \big(0, \frac{1}{2}K_1\big)$, $p \in (2,3)$, the functions $f_\delta^{\pm}$ defined in (3.4.4)–(3.4.5) are upper and lower barriers for (2.2.3) in the intermediate region $\mathcal{M}_{R_*,\tau_*}$. It follows that the functions $v_\delta^{\pm}$ defined in (3.4.3) are upper and lower barriers for (2.2.2) in $\mathcal{M}_{R_*,\tau_*}$.*

We begin with two lemmas that will simplify the proof of Proposition 3.4.2.

Lemma 3.4.3. *Wherever $f(y,\tau) \ge 0$ holds, one has*

$$|\mathcal{N}[f]| \le \frac{3}{y^3}[f]_2^2,$$

where, by definition, for any function $F(y,\tau)$ we define

$$[F]_2(y,\tau) := |F(y,\tau)| + |yF_y(y,\tau)| + |y^2F_{yy}(y,\tau)|. \tag{3.4.6}$$

Proof. Using $2|1+x| \le 1+(1+x)^2$ one finds for all $x \in \mathbb{R}$

$$\left|\frac{2+x}{1+(1+x)^2}\right| \le \frac{1}{1+(1+x)^2} + \frac{|1+x|}{1+(1+x)^2} \le \frac{3}{2}.$$

Using $f(y,\tau) \ge 0$ this implies

$$\begin{aligned} |\mathcal{N}[f]| &= \left|\frac{-3f^2}{y^2(y+f)} - \frac{2+f_y}{1+(1+f_y)^2} f_y f_{yy}\right| \\ &\le 3\frac{f^2}{y^3} + \frac{3}{2}|f_y f_{yy}| \le \frac{3}{y^3}\{f^2 + |yf_y||y^2 f_{yy}|\} \le \frac{3}{y^3}[f]_2^2. \end{aligned}$$

□

Lemma 3.4.4. *For any $B > 0$ there exist $R(B) > 0$ and $\tau(B) \in \mathbb{R}$ such that if $0 < \delta < \frac{1}{2}K_1$, then $f_\delta^{\pm}$ as defined in (3.4.4)–(3.4.5) satisfies*

$$f_\delta^{\pm}(y,\tau) > 0$$

and

$$|\mathcal{N}[f_\delta^{\pm}]| \le C_* e^{6\gamma\tau}(y^{-7} + y^{4k-7})$$

in the intermediate region $R(B)e^{\gamma\tau} \le y \le e^{-\tau/2}$, $\tau \le \tau(B)$.

As promised in Section 3.2, the constant C_* only depends on the constants k, K_0 but not on B.

Proof. Recall the notation from (3.4.6). The explicit expression (A.1.2) for φ_k implies

$$[\varphi_k]_2 \le Cy^{-2}(1+y^{2k}),$$

and the construction of the auxiliary function g implies

$$[g]_2 \le Cy^{-5}(1+y^{4k-2}).$$

We also have for all $y > 0$

$$[y^{-p}]_2 = y^{-p} + py^{-p} + p(p+1)y^{-p} = (p+1)^2 y^{-p} < 16y^{-p}$$

because $2 < p < 3$. Hence the three terms f_j in (3.4.5) that add up to $f_\delta^\pm$ satisfy

$$\begin{aligned}
[f_0]_2 &\le Ce^{3\gamma\tau} y^{-2}(1+y^{2k}),\\
[f_1]_2 &\le CBe^{6\gamma\tau} y^{-5}(1+y^{4k-2}),\\
[f_2]_2 &\le Ce^{(p+1)\gamma\tau} y^{-p},
\end{aligned}$$

assuming that $0 < \delta \le \frac{1}{2}K_1$.

If $Re^{\gamma\tau} \le y \le e^{-\tau/2}$, then we can estimate $f_\delta^\pm$ as

$$\begin{aligned}
[f_\delta^\pm]_2 &\le C\frac{e^{3\gamma\tau}}{y^2}(1+y^{2k}) + CB\frac{e^{6\gamma\tau}}{y^5}(1+y^{4k-2}) + C\frac{e^{(p+1)\gamma\tau}}{y^p}\\
&\le C\frac{e^{3\gamma\tau}}{y^2}(1+y^{2k})\left\{1 + B\frac{e^{3\gamma\tau}}{y^3} + Be^{3\gamma\tau}y^{2k-5} + \frac{e^{(p-2)\gamma\tau}}{y^{p-2}}\right\}\\
&\le C\frac{e^{3\gamma\tau}}{y^2}(1+y^{2k})\{1 + BR^{-3} + Be^{\tau} + R^{-(p-2)}\},
\end{aligned}$$

where in estimating the third term in the bracket we used $3\gamma = k - \frac{3}{2}$. Thus, if we require

$$R \ge \max\{1, B^{1/3}\} \quad \text{and} \quad \tau \le \tau(B) := -\log B, \tag{3.4.7}$$

then $1 + BR^{-3} + Be^{\tau} + R^{-(p-2)} \le 4$ and so

$$[f_\delta^\pm]_2 \le Ce^{3\gamma\tau} y^{-2}(1+y^{2k}).$$

Combined with Lemma 3.4.3 this yields

$$|\mathcal{N}[f_\delta^\pm]| \le \frac{3}{y^3} Ce^{6\gamma\tau} y^{-4}(1+y^{2k})^2 \le \widetilde{C}e^{6\gamma\tau} y^{-7}(1+y^{4k})$$

in the intermediate region, provided that we verify $f_\delta^\pm \ge 0$ when $Re^{\gamma\tau} \le y \le e^{-\tau/2}$.

To prove $f_\delta^\pm \ge 0$ in the intermediate region we recall the assumption $\delta < \frac{1}{2}K_1$, which implies

$$f_\delta^\pm(y,\tau) \ge \tfrac{1}{2}K_1 e^{3\gamma\tau}\varphi_k(y) - \{BK_1^2 e^{6\gamma\tau}|g(y)| + e^{(p+1)\gamma\tau}y^{-p}\}.$$

Use the lower bound $\varphi_k(y) \ge cy^{-2}(1+y^{2k})$ and the upper bound $|g(y)| \le Cy^{-5}(1+y^{4k-2})$ to arrive at

$$f_\delta^\pm(y,\tau) \ge c\frac{e^{3\gamma\tau}}{y^2}(1+y^{2k}) - \left\{CB\frac{e^{6\gamma\tau}}{y^5}(1+y^{4k-2}) + \frac{e^{(p+1)\gamma\tau}}{y^p}\right\},$$

which, because $(1+xy)/(1+x) \le 1+y$ for all $x, y \ge 0$, implies

$$\frac{y^2 e^{-3\gamma\tau}}{c(1+y^{2k})} f_\delta^{\pm}(y,\tau) \ge 1 - CB\frac{e^{3\gamma\tau}}{y^3}(1+y^{2k-2}) - \frac{1}{c(1+y^{2k})}\frac{e^{(p-2)\gamma\tau}}{y^{p-2}}.$$

In the region $Re^{\gamma\tau} \le y \le e^{-\tau/2}$ we get

$$\frac{y^2 e^{-3\gamma\tau}}{c(1+y^{2k})} f_\delta^{\pm}(y,\tau) \ge 1 - \frac{CB}{R^3} - CBe^{\tau} - \frac{1}{cR^{p-2}}.$$

We adjust our choice of $R(B)$, $\tau(B)$ in (3.4.7) to

$$R(B) = \widetilde{C}\max\{1, B^{1/3}\}, \qquad \tau(B) = -\log(\widetilde{C}B) \tag{3.4.8}$$

for large enough $\widetilde{C} \ge 1$. Then, for $y \ge R(B)$ and $\tau \le \tau(B)$, we have

$$\frac{y^2 e^{-3\gamma\tau}}{c(1+y^{2k})} f_\delta^{\pm}(y,\tau) \ge \frac{1}{2} > 0,$$

and thus $f_\delta^{\pm}(y,\tau) > 0$. □

Proof of Proposition 3.4.2. We consider the case of upper barriers, where we have

$$(\partial_\tau - \mathcal{L}) f_\delta^+ = (\partial_\tau - \mathcal{L}) f_0^+ + (\partial_\tau - \mathcal{L}) f_1 + (\partial_\tau - \mathcal{L}) f_2. \tag{3.4.9}$$

The first term vanishes because $f_0^{\pm}$ is a solution of the linear equation $f_\tau = \mathcal{L}f$. For the last term in (3.4.9) we note that for any $r \in \mathbb{R}$ one has

$$\mathcal{L}[y^r] = \tfrac{1}{2}(r+2)(r+3)y^{r-2} + \tfrac{1}{2}(r-1)y^r.$$

Hence, if $p \in (2,3)$ then $\mathcal{L}[y^{-p}] < 0$ for all $y > 0$. It follows that

$$(\partial_\tau - \mathcal{L}) f_2 > \partial_\tau f_2 = (p+1)\gamma f_2 > 0.$$

The middle term in (3.4.9) satisfies

$$(\partial_\tau - \mathcal{L}) f_1 = BK_1^2 e^{6\gamma\tau}(6\gamma g - \mathcal{L}g) = BK_1^2 e^{6\gamma\tau}(y^{-7} + y^{4k-7}).$$

If we choose $B_* = C_* K_1^{-2}$ where C_* is the constant from Lemma 3.4.4, and if we set $R_* = R(B_*)$ and $\tau_* = \tau(B_*)$ according to (3.4.8), then we clearly have $(\partial_\tau - \mathcal{L}) f_\delta^+ > \mathcal{N}[f_\delta^+]$ in the intermediate region $\mathcal{M}_{R_*,\tau_*}$.

We conclude that f_δ^+ is an upper barrier, i.e., (3.4.1) holds. With minor modifications this argument also shows that f_δ^- is a lower barrier. □

We next show that the barriers $f_\delta^{\pm}$ form a nested sequence, in the sense of the lemma below. The nesting of barriers will allow us to construct a solution that is bounded by all barriers at once and will enable us to prove the convergence of our solution in the inner region to the Alencar minimal surface, as $\tau \to -\infty$.

Lemma 3.4.5. *The constant R_* from Proposition 3.4.2 can be chosen so that*

$$f_\delta^-(y,\tau) < f_{\delta/2}^-(y,\tau) < f_{\delta/2}^+(y,\tau) < f_\delta^+(y,\tau) \tag{3.4.10}$$

for all (y,τ) with $R_* e^{\gamma\tau} \le y$.

Proof. We can write the barrier functions $f_\delta^\pm$ as

$$f_\delta^\pm(y,\tau) = K_1 e^{3\gamma\tau}\varphi_k(y) \pm \{\delta e^{3\gamma\tau}\varphi_k(y) + B_* K_1^2 e^{6\gamma\tau} g(y) + e^{(p+1)\gamma\tau} y^{-p}\}.$$

Since $\varphi_k(y) > 0$ for all $y > 0$, it is immediately clear that

$$f_\delta^-(y,\tau) < f_{\delta/2}^-(y,\tau) \quad \text{and} \quad f_{\delta/2}^+(y,\tau) < f_\delta^+(y,\tau)$$

for all y, τ.

To prove the middle inequality we note that $f_{\delta/2}^-(y,\tau) < f_{\delta/2}^+(y,\tau)$ holds if and only if

$$\frac{\delta}{2} e^{3\gamma\tau}\varphi_k(y) + B_* K_1^2 e^{6\gamma\tau} g(y) + e^{(p+1)\gamma\tau} y^{-p} > 0,$$

which, in view of $\varphi_k(y) > 0$, will certainly hold if

$$B_* K_1^2 e^{6\gamma\tau} g(y) + e^{(p+1)\gamma\tau} y^{-p} > 0. \tag{3.4.11}$$

Since $g(y) > 0$ for large $y > 0$, there is a constant $C_g > 0$ such that $g(y) \ge -C_g y^{-5}$ for all $y > 0$. Hence (3.4.11) follows from

$$e^{(p+1)\gamma\tau} y^{-p} - C_g B_* K_1^2 e^{6\gamma\tau} y^{-5} > 0, \quad \text{i.e., } y e^{-\gamma\tau} > (C_g B_* K_1^2)^{1/(5-p)}. \qquad \square$$

3.5. *Barriers in the inner region.* In this section we present a family of sub- and supersolutions to (2.3.2) for $w(z,\tau)$ in the inner region $0 \le z \le Z$.

We recall our notation from Section 2.3 where $W(z)$ denotes the unique Alencar solution to (2.3.4), normalized so that

$$W(z) = z + \frac{1}{z^2} + \frac{\Gamma}{z^3} + \mathcal{O}(z^{-5}), \quad z \to \infty, \tag{3.5.1}$$

holds for a certain constant $\Gamma \in \mathbb{R}$.

Lemma 3.5.1. *For all $z > 0$ one has $W_K(z) > z W_K'(z)$.*

Proof. The inequality is invariant under rescaling, so we may assume $K = 1$. The asymptotics (3.5.1) show that $W(z) - z W_z(z) \to 0$ as $z \to \infty$. On the other hand, convexity of W implies $(W - z W_z)_z = -z W_{zz} < 0$ for all $z > 0$. Hence $W(z) - z W_z(z) > \lim_{Z\to\infty} W(Z) - Z W_z(Z) = 0$ for all $z \ge 0$. $\square$

Lemma 3.5.2. *For any $K > 0$ the function $w^+(z,\tau) = W_K(z)$ is a supersolution of (2.3.2) on $[0,\infty)\times\mathbb{R}$.*

Proof. The function w^+ satisfies $w_\tau^+ = 0$ and

$$\frac{w_{zz}^+}{1 + (w_z^+)^2} + \frac{3}{z} w_z^+ - \frac{3}{w^+} = 0.$$

From Lemma 3.5.1 we have $w^+ - zw^+_z > 0$, and thus

$$e^{2\gamma\tau}\left(w^+_\tau + \frac{k}{3}(w^+ - zw^+_z)\right) > \frac{w^+_{zz}}{1+(w^+_z)^2} + \frac{3}{z}w^+_z - \frac{3}{w^+},$$

as claimed. $\square$

Lemma 3.5.3. *There exist* $D_* > 0$, $\zeta > 0$ *such that for all* $K \in \left(\tfrac{1}{2}K_2, 2K_2\right)$ *and* $D \geq D_*$ *there is a* $\tau_*(D)$ *such that*

$$w^-(z,\tau) := W_K(z) + De^{2\gamma\tau}$$

is a subsolution of (2.3.2) *for* $0 \leq z \leq \zeta e^{-\gamma\tau}$, $\tau \leq \tau_*(D)$.

Proof. Choose

$$\tau_*(D) \leq \frac{1}{2\gamma}\log\frac{W_K(0)}{D}.$$

Then $\tau \leq \tau_*(D)$ and $z \geq 0$ implies

$$De^{2\gamma\tau} \leq W_K(0) \leq W_K(z),$$

so that

$$W_K(z) \leq w^-(z,\tau) \leq 2W_K(z).$$

If we substitute $w = w^-$ in (2.3.2) and use $2\gamma + \frac{1}{3}k = k-1$, then on one hand

$$e^{2\gamma\tau}\left(w^-_\tau + \frac{k}{3}(w^- - zw^-_z)\right) = e^{2\gamma\tau}\left((k-1)De^{2\gamma\tau} + \frac{k}{3}(W_K - zW'_K)\right)$$

and on the other hand

$$\frac{w^-_{zz}}{1+(w^-_z)2} + \frac{3}{z}w^-_z - \frac{3}{w^-} = \frac{W''_K}{1+(W'_K)^2} + \frac{3}{z}W'_K - \frac{3}{w^-} = \frac{3}{W_K} - \frac{3}{w^-} = \frac{3De^{2\gamma\tau}}{W_K w^-}.$$

Hence w^- is a subsolution if

$$\frac{3D}{W_K(z)w^-(z,\tau)} > (k-1)De^{2\gamma\tau} + \frac{k}{3}(W_K(z) - zW'_K(z)). \tag{3.5.2}$$

Since $W_K \leq w^- \leq 2W_K \leq C(1+z)$, there is a constant C_1 such that the terms on the left are bounded from below by

$$\frac{3D}{W_K(z)w^-(z,\tau)} \geq \frac{C_1 D}{(1+z)^2}.$$

The terms on the right in (3.5.2) satisfy

$$(k-1)e^{2\gamma\tau} \leq C_2\frac{\zeta^2}{(1+z)^2}$$

in the region $1+z \leq \zeta e^{-\gamma\tau}$ and, due to the asymptotic expansion of $W_K(z)$ as $z \to \infty$ (which follows from (3.5.1)), they also satisfy

$$W_K(z) - zW'_K(z) \leq \frac{C_3}{(1+z)^2} \quad \text{for all } z \geq 0.$$

Hence

$$(k-1)De^{2\gamma\tau}+\frac{k}{3}(W_K(z)-zW_K'(z))\leq\frac{C_2\zeta^2D+C_3}{(1+z)^2}.$$

Choose $\zeta<\sqrt{C_1/(2C_2)}$, and choose D large enough that $C_3<\frac{1}{2}C_1D$. Then we have

$$(k-1)De^{2\gamma\tau}+\frac{k}{3}(W_K(z)-zW_K'(z))<\frac{C_1D}{(1+z)^2}\leq\frac{3D}{W_K(z)w(z,\tau)},$$

which implies (3.5.2) and thus that w^- is a lower barrier in the region $1+z\leq\zeta e^{-\gamma\tau}$. Choose τ_* so that $\zeta e^{-\gamma\tau_*}\geq 2$. Then $1+z\leq\zeta e^{-\gamma\tau}$ holds for all $z\leq 1$ and $\tau\leq\tau_*$, while for $z\geq 1$ it follows from $2z\leq\zeta e^{-\gamma\tau}$ that $1+z\leq\zeta e^{-\gamma\tau}$.

Thus w^- is a lower barrier in the region $z\leq\frac{1}{2}\zeta e^{-\gamma\tau}$, $\tau\leq\tau_*$. □

3.6. ***Matching outer and intermediate barriers.*** We show that upper and lower barriers constructed in the inner, the intermediate, and the outer regions match in the overlapping region. We begin here with the overlap of the outer and intermediate regions.

We start with an $M>0$ large enough that the functions

$$u^\pm(x,t)=u_0(x)\pm Mt\min\{1,x^{2k-4}\}$$

are sub- and supersolutions of (1.1.5a) in the outer region $\mathcal{O}_M$ (see Lemma 3.3.1). In order to match the outer barriers with the barriers in the intermediate region, we express the outer barriers $u=u^\pm(x,t)$ in the intermediate variables (v,y,τ):

$$v^\pm_{\text{out}}(y,\tau):=e^{-\tau/2}u^\pm(e^{\tau/2}y,e^\tau).$$

In (3.3.1) we defined $u^\pm(x,t)=u_0(x)\pm Mtx^{2k-4}$ for $0<x\leq 1$. If we write the assumption (1.1.1) on the initial data in the form

$$u_0(x)=x+(K_0+\epsilon_0(x))x^{2k-2},\tag{3.6.1}$$

where $\epsilon_0:(0,\infty)\to\mathbb{R}$ satisfies $\lim_{x\to 0}\epsilon_0(x)=0$, then we get the following expression for the outer barriers in the intermediate variables:

$$v^\pm_{\text{out}}(y,\tau)=y+(K_0+\epsilon_0(ye^{\tau/2}))e^{3\gamma\tau}y^{2k-2}\pm Me^{3\gamma\tau}y^{2k-4}.\tag{3.6.2}$$

The outer barriers only contain the parameter M and thus do not depend on other parameters such as δ and B that appeared in the barriers we constructed for the intermediate and inner regions.

We now consider the intermediate barriers, continuing to use the conventions from Section 3.2 which relate the constants K_0, K_1, etc.

In Proposition 3.4.2 we found B_*, R_*, and τ_*, such that for any $\delta\in\left(0,\frac{1}{2}K_1\right)$ and $p\in(2,3)$ the functions

$$v^\pm_\delta(y,\tau)=y+(K_1\pm\delta)e^{3\gamma\tau}\varphi_k(y)\pm\{e^{(p+1)\gamma\tau}y^{-p}+B_*K_1^2e^{6\gamma\tau}g(y)\}$$

are upper and lower barriers in the intermediate region $\mathcal{M}_{R_*,\tau_*}=\{R_*e^{\gamma\tau}\leq y\leq e^{-\tau/2},\ \tau\leq\tau_*\}$.

To compare $v_{\rm out}^{\pm}$ and $v_{\delta}^{\pm}$ we rewrite them as

$$e^{-3\gamma\tau}(v_{\rm out}^{\pm}(y,\tau)-y)=(K_0+\epsilon_0(ye^{\tau/2}))y^{2k-2}\pm My^{2k-4},$$
$$e^{-3\gamma\tau}(v_{\delta}^{\pm}(y,\tau)-y)=(K_1\pm\delta)\varphi_k(y)\pm e^{(p-2)\gamma\tau}y^{-p}\pm B_*K_1^2e^{3\gamma\tau}g(y).$$

We now let $\tau\to-\infty$ and conclude that

$$\begin{cases} e^{-3\gamma\tau}(v_{\rm out}^{\pm}(y,\tau)-y)\to K_0y^{2k-2}\pm My^{2k-4}, \\ e^{-3\gamma\tau}(v_{\delta}^{\pm}(y,\tau)-y)\to (K_1\pm\delta)\varphi_k(y), \end{cases} \tag{3.6.3}$$

uniformly for bounded y.

The explicit expression (A.1.2) for φ_k implies

$$\varphi_k(y)=\frac{y^{2k-2}}{(2k+1)!!}+c(y)y^{2k-4},$$

where

$$c(y)=c_0+\frac{c_1}{y^2}+\cdots+\frac{c_{k-1}}{y^{2k-2}},\qquad c_j=\frac{\binom{k}{j+1}}{(2(k-j)-1)!!}.$$

Substitute this expression for φ_k in (3.6.3) and keep in mind that $K_1=(2k+1)!!\,K_0$. Then

$$e^{-3\gamma\tau}(v_{\rm out}^{\pm}(y,\tau)-v_{\delta}^{\pm}(y,\tau))\to \pm y^{2k-4}\left\{-\frac{\delta y^2}{(2k+1)!!}+M-c(y)\right\}.$$

The function $c(y)$ is clearly bounded for $y\ge 1$, so if M is sufficiently large, one can neglect $c(y)$ and conclude that $v_{\rm out}^{\pm}(y,\tau)-v_{\delta}^{\pm}(y,\tau)$ changes sign when

$$\frac{\delta y^2}{(2k+1)!!}=M-c(y)\approx M.$$

To make this more precise we introduce

$$Y_\delta:=2\sqrt{(2k+1)!!\,M/\delta} \tag{3.6.4}$$

and compare the barriers $v_{\rm out}^{\pm}(y,\tau)$ and $v_{\delta}^{\pm}(y,\tau)$ at the endpoints $y_\delta(\tau)\in\left(\frac14 Y_\delta,Y_\delta\right)$.

Lemma 3.6.1. *For any $\delta>0$ there is a $\tau_\delta\in\mathbb{R}$ such that for all $\tau\le\tau_\delta$ one has*

$$v_{\rm out}^{+}\left(\tfrac14 Y_\delta,\tau\right)>v_{\delta}^{+}\left(\tfrac14 Y_\delta,\tau\right)\quad\textit{and}\quad v_{\rm out}^{-}\left(\tfrac14 Y_\delta,\tau\right)<v_{\delta}^{-}\left(\tfrac14 Y_\delta,\tau\right).$$

Moreover, we also have

$$v_{\rm out}^{+}(Y_\delta,\tau)<v_{\delta}^{+}(Y_\delta,\tau)\quad\textit{and}\quad v_{\rm out}^{-}(Y_\delta,\tau)>v_{\delta}^{-}(Y_\delta,\tau)$$

for all $\tau\le\tau_\delta$.

Proof. We only consider the upper barriers, the other case being nearly identical.

We have found that as $\tau\to-\infty$

$$e^{-3\gamma\tau}\left(v_{\rm out}^{+}\left(\tfrac14 Y_\delta,\tau\right)-v_{\delta}^{+}\left(\tfrac14 Y_\delta,\tau\right)\right)\to\left(\tfrac14 Y_\delta\right)^{2k-4}\left\{-\tfrac14 M+M-c\left(\tfrac14 Y_\delta\right)\right\}.$$

Since $c(y)$ is bounded for $y\ge 1$, given any large M we will still have

$$\tfrac34 M-c\left(\tfrac14 Y_\delta\right)>0.$$

Hence

$$\lim_{\tau\to-\infty} e^{-3\gamma\tau}\big(v_{\mathrm{out}}^{+}\big(\tfrac{1}{4}Y_\delta,\tau\big)-v_\delta^{+}\big(\tfrac{1}{4}Y_\delta,\tau\big)\big)>0,$$

which implies that for $-\tau$ sufficiently large one has $v_{\mathrm{out}}^{+}\big(\tfrac{1}{4}Y_\delta,\tau\big)>v_\delta^{+}\big(\tfrac{1}{4}Y_\delta,\tau\big)$, as claimed.

If on the other hand we compare v_{out}^{+} and v_δ^{+} at $y=Y_\delta$, then we find that for $\tau\to-\infty$

$$e^{-3\gamma\tau}(v_{\mathrm{out}}^{+}(Y_\delta,\tau)-v_\delta^{+}(Y_\delta,\tau))\to Y_\delta^{2k-4}\{-4M+M-c(Y_\delta)\}=-Y_\delta^{2k-4}\{3M+c(Y_\delta)\}.$$

Since $c(y)$ is bounded for $y\geq 1$, it follows that for M large enough we indeed have $v_{\mathrm{out}}^{+}(Y_\delta,\tau)<v_\delta^{+}(Y_\delta,\tau)$, as $\tau\to-\infty$. □

3.7. ***Matching intermediate and inner barriers.*** For any $\delta\in\big(0,\tfrac{1}{2}K_1\big)$, $p\in(2,3)$ and $B=B_*$ the barriers $v_\delta^{\pm}(y,\tau)=y+f_\delta^{\pm}(y,\tau)$ constructed above are defined in the intermediate region

$$\mathcal{M}_{R_*,\tau_*}=\{R_*e^{2\gamma\tau}\leq y\leq e^{-\tau/2},\ \tau\leq\tau_*\}.$$

If we assume that $Z>2R_*$, then it follows that $v_\delta^{\pm}(y,\tau)$ are defined in parts of the inner region $\mathcal{I}_{Z,\tau_*}=\{(z,\tau)\mid 0\leq z\leq Z,\ \tau\leq\tau_*\}$. Define

$$w_{\mathrm{md}}^{\pm}(z,\tau):=e^{-\gamma\tau}v_\delta^{\pm}(e^{\gamma\tau}z,\tau).$$

Then

$$w_{\mathrm{md}}^{\pm}(z,\tau)=z+\frac{K_1\pm\delta}{z^2}(1+\epsilon_1(z,\tau))\pm\frac{1}{z^p}\pm\frac{B_*K_1^2}{z^5}(1+\epsilon_2(z,\tau)),$$

where $\epsilon_i(z,\tau)$ are generic functions for which $\epsilon_i(z,\tau)\to 0$ as $\tau\to-\infty$, uniformly for $0\leq z\leq Z$. In particular, for all $z\in[0,Z]$ we have

$$\lim_{\tau\to-\infty} w_{\mathrm{md}}^{\pm}(z,\tau)=z+\frac{K_1}{z^2}\pm\left\{\frac{\delta}{z^2}+\frac{1}{z^p}+\frac{B_*K_1^2}{z^5}\right\}. \tag{3.7.1}$$

We will now use Lemmas 3.5.2 and 3.5.3 to match $w_{\mathrm{md}}^{\pm}(z,\tau)$ with appropriately chosen barriers $w_\delta^{\pm}(z,\tau)$ in the inner region $0\leq z\leq Z$. For suitable δ-dependent constants $K_2^{\pm}\in\big(\tfrac{1}{2}K_2,2K_2\big)$, with $(K_2)^3=K_1$, we consider

$$w_\delta^{+}(z,\tau):=W_{K_2^{+}}(z),\quad w_\delta^{-}(z,\tau):=W_{K_2^{-}}(z)+De^{2\gamma\tau},$$

where D depends on K_2^{-} and Z as described in Lemma 3.5.3.

It follows from Lemmas 3.5.2 and 3.5.3 that, for each $K_2^{+}>0$ and $K_2^{-}>0$, w_δ^{+} and w_δ^{-} are the upper barrier and the lower barrier, respectively, for (2.3.2) in the inner region. Furthermore the asymptotics at infinity of the Alencar solution in (3.5.1) imply that

$$\lim_{\tau\to-\infty} w_\delta^{\pm}(z,\tau)=z+\frac{(K_2^{\pm})^3}{z^2}+\frac{\Gamma(K_2^{\pm})^4}{z^3}+\mathcal{O}(z^{-5}),\quad z\gg 1.$$

Comparing the asymptotic expansions of $w_{\mathrm{md}}^{\pm}$ and $w_\delta^{\pm}$ we see that they match when $(K_2^{\pm})^3=K_1\pm\delta$. However with this choice the barriers $w_{\mathrm{md}}^{\pm}$ and $w_\delta^{\pm}$ may not intersect. For this reason we choose the constants $K_2^{\pm}$ such that

$$(K_2^{\pm})^3=K_1\pm 2\delta.$$

With this choice we then have

$$\lim_{\tau\to-\infty} w_\delta^\pm(z,\tau) = z + \frac{K_1 \pm 2\delta}{z^2} + \frac{\Gamma(K_1 \pm 2\delta)^{4/3}}{z^3} + \mathcal{O}(z^{-5}), \quad z \gg 1. \tag{3.7.2}$$

Lemma 3.7.1. *Let* $p \in (2,3)$ *be given, and let* $B = B_k$ *as in Proposition 3.4.2. Then there exist* $\bar\delta > 0$ *and* $R = R(B)$ *so that for any* $\delta \in (0, \bar\delta)$ *and* $\tau \le \tau_\delta$ *the barriers* $w_\delta^\pm$ *and* $w_{\mathrm{md}}^\pm$ *cross in the interval* $\left(\frac12 Z_\delta, Z_\delta\right)$, *where* $Z_\delta := \frac43 \delta^{-1/(p-2)}$, *in the sense that*

$$w_{\mathrm{md}}^+\left(\tfrac12 Z_\delta, \tau\right) > w_\delta^+\left(\tfrac12 Z_\delta, \tau\right) \quad \textit{and} \quad w_{\mathrm{md}}^-\left(\tfrac12 Z_\delta, \tau\right) < w_\delta^-\left(\tfrac12 Z_\delta, \tau\right),$$

and

$$w_{\mathrm{md}}^+(Z_\delta, \tau) < w_\delta^+(Z_\delta, \tau) \quad \textit{and} \quad w_{\mathrm{md}}^-(Z_\delta, \tau) > w_\delta^-(Z_\delta, \tau).$$

Proof. We only consider the upper barriers, the other case being nearly identical. Proposition 3.4.2 asserts that for $\delta < \frac12 K_1$ the function $w_{\mathrm{md}}^+(z,\tau)$ is an upper barrier in the intermediate region $R_* \le z \le e^{-(k/3)\tau}$ and it satisfies (3.7.1) with this choice of constants; that is

$$\lim_{\tau\to-\infty} w_{\mathrm{md}}^+(z,\tau) = z + \frac{K_1+\delta}{z^2} + \frac{1}{z^p} + \mathcal{O}(z^{-5}), \quad z \to \infty,$$

where the $\mathcal{O}(z^{-5})$ term is uniform in $\delta \in \left(0, \frac12 K_1\right)$. We have also seen that

$$\lim_{\tau\to-\infty} w_\delta^+(z,\tau) = z + \frac{K_1+2\delta}{z^2} + \mathcal{O}(z^{-3}), \quad z \to \infty,$$

where $\mathcal{O}(z^{-3})$ is again uniform in δ. Therefore

$$\lim_{\tau\to-\infty} w_\delta^+(z,\tau) - w_{\mathrm{md}}^+(z,\tau) = \frac{\delta}{z^2} - \frac{1}{z^p} + O(z^{-3}), \quad z \to \infty.$$

Consider $Z_\delta := \frac43 \delta^{-1/(p-2)}$. For small enough $\delta > 0$ one has $Z_\delta \ge 2R_*$, so that $w_\delta^\pm(z,\tau)$ and $w_{\mathrm{md}}^\pm(z,\tau)$ are defined for all $z \ge \frac12 Z_\delta$ and all $\tau \le \tau_*$. We evaluate these differences at $z = Z_\delta$ and $z = \frac12 Z_\delta$. Eliminating δ by using $\delta = \left(\frac34 Z_\delta\right)^{-(p-2)}$ we find

$$\lim_{\tau\to-\infty} w_\delta^+(Z_\delta,\tau) - w_{\mathrm{md}}^+(Z_\delta,\tau) = \left(\left(\tfrac43\right)^{p-2} - 1\right) Z_\delta^{-p} + \mathcal{O}(Z_\delta^{-3}).$$

For small enough $\delta > 0$, Z_δ is large, and thus the first term dominates the second. This implies that for small $\delta > 0$ there is a $\tau_\delta < 0$ such that

$$w_\delta^+(Z_\delta,\tau) - w_{\mathrm{md}}^+(Z_\delta,\tau) > 0$$

for all $\tau \le \tau_\delta$. Similarly, we have

$$\lim_{\tau\to-\infty} w_\delta^+\left(\tfrac12 Z_\delta,\tau\right) - w_{\mathrm{md}}^+\left(\tfrac12 Z_\delta,\tau\right) = \left(\left(\tfrac23\right)^{p-2} - 1\right) 2^p Z_\delta^{-p} + \mathcal{O}(Z_\delta^{-3}).$$

This implies that if $\delta > 0$ is small then there is a $\tau_\delta < 0$ such that

$$w_\delta^+\left(\tfrac12 Z_\delta,\tau\right) - w_{\mathrm{md}}^+\left(\tfrac12 Z_\delta,\tau\right) < 0$$

for all $\tau \le \tau_\delta$. □

3.8. ***A summary of our construction so far.*** The initial data u_0 determines two constants $k \geq 4$ and K_0. Throughout the paper we let $K_1 = (2k+1)!!\, K_0$ and $K_2 = K_1^{1/3}$.

In Section 3.3 we chose a constant $M > 0$ so that Lemma 3.3.1 holds and constructed upper and lower barriers $u^{\pm}(x,t)$ in the outer region $\mathcal{O}_M$.

For any small enough $\delta > 0$ we then constructed a family of barriers $v_\delta^{\pm}$ in the intermediate region defined by $R_* e^{\gamma\tau} \leq y \leq e^{-\tau/2}$, $\tau \leq \tau_\delta$. Here Propositions 3.4.2 and 3.4.5 specify R_*, while τ_δ is determined when we match the intermediate and inner barriers in Lemma 3.6.1.

For small $\delta > 0$ we then considered the inner region

$$\mathcal{I}_{Z_\delta,\tau_\delta} = \{(z,\tau) \mid 0 \leq z \leq Z_\delta,\ \tau \leq \tau_\delta\}$$

with $Z_\delta := \frac{4}{3}\delta^{-1/(p-2)}$ and where τ_δ is as above. Since $\delta > 0$ is small and R_* does not depend on δ, we have $\delta < \left(\frac{3}{2}R_*\right)^{2-p}$, which implies $Z_\delta > 2R_*$. Hence the intermediate and inner regions overlap at least on $\frac{1}{2}Z_\delta \leq z \leq Z_\delta$.

Lemma 3.5.2 with K_2^+ satisfying $(K_2^+)^3 = K_1 + 2\delta$ defines the upper barrier w_δ^+ in the inner region $\mathcal{I}_{Z_\delta,\tau_\delta}$, and Lemma 3.5.3 with K_2^- satisfying $(K_2^-)^3 = K_1 - 2\delta$ defines the constant $D = D(K_2^-)$ and the lower barrier w_δ^- in $\mathcal{I}_{Z_\delta,\tau_\delta}$.

3.9. ***The upper and lower barriers $U_\delta^+(x,t)$ and $U_\delta^-(x,t)$.*** In the previous subsections, we constructed upper barriers $u^+(x,t)$, $v_\delta^+(y,\tau)$, $w_\delta^+(z,\tau)$ and lower barriers $u^-(x,t)$, $v_\delta^-(y,\tau)$, $w_\delta^-(z,\tau)$ in the outer, intermediate, and inner regions, respectively, and showed that they are correctly ordered in the overlaps between the three regions. These barriers exist for all $0 < t \leq t_\delta$ or equivalently $-\infty < \tau \leq \tau_\delta$. We now define the global barrier $U_\delta^+(x,t)$ by taking the minimum of the upper barriers when all are expressed in the unrescaled (x,t) variables. More precisely, we define

$$U_\delta^+(x,t) = \begin{cases} u^+(x,t), & y \geq Y_\delta, \\ \min\{u^+(x,t),\, t^{1/2} v_\delta^+(y,\log t)\}, & \frac{1}{4}Y_\delta \leq y \leq Y_\delta, \\ t^{1/2} v_\delta^+(y,\log t), & Z_\delta \leq z \text{ and } y \leq \frac{1}{4}Y_\delta, \\ \min\{t^{1/2} v_\delta^+(y,\log t),\, t^{k/3} w_\delta^+(z,\log t)\}, & \frac{1}{2}Z_\delta \leq z \leq Z_\delta, \\ t^{k/3} w_\delta^+(z,\log t), & 0 \leq z \leq \frac{1}{2}Z_\delta, \end{cases} \tag{3.9.1}$$

where, as before, $y = xt^{-1/2}$ and $z = xt^{-k/3}$. Lemmas 3.7.1 and 3.6.1 imply that U_δ^+ is a weak supersolution of (1.1.5a) and, similarly,

$$U_\delta^-(x,t) = \begin{cases} u^-(x,t), & y \geq Y_\delta, \\ \max\{u^-(x,t),\, t^{1/2} v_\delta^-(y,\log t)\}, & \frac{1}{4}Y_\delta \leq y \leq Y_\delta, \\ t^{1/2} v_\delta^-(y,\log t), & Z_\delta \leq z \text{ and } y \leq \frac{1}{4}Y_\delta, \\ \max\{t^{1/2} v_\delta^-(y,\log t),\, t^{k/3} w_\delta^-(z,\log t)\}, & \frac{1}{2}Z_\delta \leq z \leq Z_\delta, \\ t^{k/3} w_\delta^-(z,\log t), & 0 \leq z \leq \frac{1}{2}Z_\delta, \end{cases} \tag{3.9.2}$$

is a weak subsolution of (1.1.5a). This is summarized in the following proposition.

Proposition 3.9.1. *There exist a number* $\delta_0 > 0$ *and a sequence of times* $t_n \searrow 0$ *such that the functions* $U^{\pm}_{\delta_n}(x,t)$ *given in* (3.9.1) *and* (3.9.2), *with* $\delta_n = 2^{-n}\delta_0$, *define weak super- and subsolutions of* (1.1.5a) *for all* $0 < t \le t_n$.

Moreover, one has

$$U^-_{\delta_n}(x,t) \le U^-_{\delta_{n+1}}(x,t) < U^+_{\delta_{n+1}}(x,t) \le U^+_{\delta_n}(x,t) \tag{3.9.3}$$

for all $x > 0$ *and* $0 < t \le t_{n+1}$.

Proof. The fact that $U^{\pm}_{\delta_n}(x,t)$, $0 < t \le t_n$, define weak super- and subsolutions of (1.1.5a) follows from Lemma 3.3.1, Proposition 3.4.2, Lemmas 3.5.2 and 3.5.3, and the matching of our barriers in Sections 3.6 and 3.7.

For (3.9.3), we recall that our barriers $u^{\pm}(x,t)$ in the outer region do not depend on δ; hence they are ordered in their common domain and furthermore it is clear that $u^-(x,t) < u^+(x,t)$. In Proposition 3.4.2 we proved (3.4.10), which implies that (3.9.3) holds in the intermediate region for $0 < t \le t_{n+1}$. To finish the proof of (3.9.3) it is sufficient to show that for any $\delta \le \delta_0$ the inequalities

$$w^-_{\delta}(z,\tau) < w^-_{\delta/2}(z,\tau) < w^+_{\delta/2}(z,\tau) < w^+_{\delta}(z,\tau) \tag{3.9.4}$$

hold for all $0 \le z \le Z_\delta$, $\tau \le \tau_\delta$. This follows from the definition of $w^{\pm}_{\delta}(z,\tau)$ in Section 3.7 by observing that the rescaled Alencar solutions $W_K(z) := K\,W(z/K)$ are ordered for $K > 0$; that is,

$$\kappa < \bar\kappa \implies W_\kappa(z) < W_{\bar\kappa}(z) \quad \text{for all } z \in [0,+\infty). \tag{3.9.5}$$

To see this, recall the inequality $W - zW_z > 0$, $z \ge 0$, which is a consequence of the convexity of W and was shown in Lemma 3.5.1. This inequality implies that

$$\frac{d}{d\kappa}W_\kappa(z) = \frac{d}{d\kappa}\left(\kappa W\left(\frac{z}{\kappa}\right)\right) = W\left(\frac{z}{\kappa}\right) - \frac{z}{\kappa}W'\left(\frac{z}{\kappa}\right) > 0; \tag{3.9.6}$$

i.e., $\kappa \to W_\kappa(z)$ is monotone increasing in κ. We conclude that (3.9.4), holds which finishes the proof of (3.9.3) and the proof of the proposition. □

4. Existence of a smooth solution

4.1. ***Outline of the existence proof.*** In this section we return to the $O(4)\times O(4)$ symmetric hypersurface M_0 with profile function $u_0 : [0,\infty) \to \mathbb{R}$. Recall that u_0 is smooth for $x > 0$ and satisfies conditions (1.1.1) and (1.1.2) for some fixed $k > 3$ and some constant $C_0 > 0$. In Proposition 3.9.1 we constructed sequences of nested upper and lower barriers for (1.1.5a). We will show in this section how to use them to prove the existence of a smooth solution $u(x,t)$ to the initial value problem (1.1.5a)–(1.1.5c) defined for all $0 < t \le t_0$, for some $t_0 > 0$. Our main result in this section is as follows.

Theorem 4.1.1 (existence of a smooth solution). *Assume that* M_0 *is an* $O(4)\times O(4)$ *symmetric hypersurface defined by a profile function* $u_0 : [0,\infty) \to \mathbb{R}$ *which is smooth for* $x > 0$ *and satisfies conditions* (1.1.1)–(1.1.2). *Then there exists* $t_0 > 0$ *and a* C^∞*-smooth* $O(4)\times O(4)$ *symmetric MCF solution* M_t,

$0 < t \le t_0$, *defined by a profile function* $u : (0,\infty) \times (0, t_0] \to (0,\infty)$ *which satisfies the initial value problem* (1.1.5a)–(1.1.5c). *Furthermore,* $u(x,t)$ *satisfies*

$$U^-_{\delta_n}(x,t) \le u(x,t) \le U^+_{\delta_n}(x,t), \quad (x,t) \in [0,\infty) \times (0, t_n), \tag{4.1.1}$$

where $\delta_n = 2^{-n}\delta_0$ *and* $U^{\pm}_{\delta_n}(x,t)$, *for* $t \in (0, t_n)$, *are the upper and lower barriers constructed in Proposition 3.9.1.*

It follows from (4.1.1) *that*

$$\lim_{t \searrow 0} t^{-k/3} u(t^{k/3} z, t) = W_{K_2}(z), \tag{4.1.2}$$

uniformly for bounded $z \ge 0$.

Since equation (1.1.5a) is singular at $u = 0$, we cannot directly apply one of the standard short-time existence results to obtain our solution $u(x,t)$. Instead, we will construct it as the limit of a sequence of approximating solutions $u_n(x,t)$, each of which is defined on some time interval starting at a carefully chosen initial time s_n, where $s_n \searrow 0$. We will define the approximating solutions u_n by choosing their initial times s_n and values $u_n(x, s_n)$ in such a way that they satisfy

$$U^-_{\delta_n}(x,s_n) \le u_n(x,s_n) \le U^+_{\delta_n}(x,s_n) \quad \text{for all } x \ge 0, \tag{4.1.3}$$

where $\delta_n := 2^{-n}\delta_0$ and where $U^{\pm}_{\delta_n}(\,\cdot\,, t)$ are the barriers constructed in Proposition 3.9.1.

The barrier $U^-_{\delta_n}$ is bounded away from $u = 0$, and this allows us to invoke a classical short-time existence theorem for the quasilinear parabolic initial value problem (1.1.5a)–(1.1.5b). The short-time existence theorem guarantees that our solution exists for $s_n \le t < \bar{t}_n$, i.e., until some time $\bar{t}_n > s_n$. This time may exceed the life time t_n of the barriers $U^{\pm}_{\delta_n}$. In fact, by finding *a priori* estimates for the solutions $u_n(x,t)$ we will show that there is an n_0 such that for all $n \ge n_0$ we have $\bar{t}_n > t_{n_0}$, and that we can extract a convergent subsequence $u_{n_j}(x,t)$ whose limit $u(x,t)$ is a solution of the full initial value problem (1.1.5a)–(1.1.5c), and which is defined for $x \ge 0$ and $0 \le t \le t_{n_0}$.

The first a priori estimate we derive for the u_n follows directly from the maximum principle applied to the barriers $U^{\pm}_{\delta_n}$. Since the barriers are ordered by (3.9.3), the a priori bound (4.1.3) implies that for all n_0, $n \ge n_0$ and $x \ge 0$ one has

$$U^-_{\delta_{n_0}}(x,s_n) \le U^-_{\delta_n}(x,s_n) \le u_n(x,s_n) \le U^+_{\delta_n}(x,s_n) \le U^+_{\delta_{n_0}}(x,s_n). \tag{4.1.4}$$

The maximum principle tells us that for all $n \ge n_0$ and $x \ge 0$ one has

$$U^-_{\delta_{n_0}}(x,t) \le u_n(x,t) \le U^+_{\delta_{n_0}}(x,t) \tag{4.1.5}$$

for all $t \ge s_n$ at which $U^{\pm}_{\delta_{n_0}}(x,t)$ and $u_n(x,t)$ are defined, i.e., for $s_n \le t < \min\{\bar{t}_n, t_{n_0}\}$.

Thereafter we establish a priori estimates for the higher-order derivatives of the u_n. We conclude this work in the next Section 5 by showing that the mean curvatures $H_n(x,t)$ of the evolving surfaces corresponding to the approximating solutions $u_n(x,t)$ are uniformly bounded for all x, n, t, and hence that the mean curvature of the limit solution $u(x,t)$ also is uniformly bounded.

The simplest choice for the initial value for u_n would be to simply set $u_n(x, s_n) = U^-_{\delta_n}(x, s_n)$, but this function is not necessarily smooth in the overlaps between inner, intermediate, and outer regions, and this complicates the estimation of the higher derivatives of u_n. Furthermore, to prove that the mean curvatures $H_n(x, t)$ are uniformly bounded, it will be important to have $H_n(x, s_n) = 0$ on $0 \le x \le \epsilon s_n^{1/2}$ for some small fixed $\epsilon > 0$. For these reasons we will construct $u_n(x, s_n)$ by smoothly gluing the lower barrier $U^-_{\delta_n}(x, s_n)$ to an Alencar surface in the inner region $x \le \epsilon s_n^{1/2}$. Let us now turn to the details of this construction.

4.2. ***Short time existence and the comparison principle.*** Equation (1.1.5a) for $u(x, t)$ has a singular term at $x = 0$ which is there because we consider radially symmetric solutions only. To derive short-time existence from existing results, it is more convenient to consider the more general case of hypersurfaces that are only partially symmetric, i.e., with $\{1\} \times O(4)$ rather than $O(4) \times O(4)$ symmetry. For any positive function $r : \mathbb{R}^4 \times [0, t_0) \to \mathbb{R}$ we consider the family of hypersurfaces parametrized by

$$F : \mathbb{R}^4 \times S^3 \times [0, t_0) \to \mathbb{R}^8,$$

where

$$F(x, \Omega, t) = (x, r(x, t)\Omega).$$

A direct computation shows that F evolves by MCF if and only if r satisfies

$$r_t = g^{ij}(Dr) r_{x_i x_j} - \frac{3}{r}, \tag{4.2.1}$$

in which

$$g_{ij}(p) = \delta_{ij} + p_i p_j, \quad g^{ij}(p) = \delta_{ij} - \frac{p_i p_j}{1 + |p|^2}.$$

As long as Dr is uniformly bounded, (4.2.1) is a uniformly parabolic quasilinear equation. The solutions that interest us are not bounded, so we choose a reference function $R : \mathbb{R}^4 \to \mathbb{R}$ that is uniformly bounded from below, has uniformly bounded derivatives up to third order, and for which $R(x) - u_0(\|x\|)$ is uniformly bounded.

All initial data we prescribe in the following sections are bounded perturbations of $R(x)$. We therefore consider solutions of the form $r(x, t) = R(x) + a(x, t)$ and derive the equation for a:

$$a_t = g^{ij}(DR + Da) a_{x_i x_j} + g^{ij}(DR + Da) R_{x_i x_j} - \frac{3}{R + a}. \tag{4.2.2}$$

Since we assume that DR and $D^2 R$ are uniformly bounded, this equation is uniformly parabolic, as long as Da is bounded. By assumption $D^m R$ with $m \le 3$ are all uniformly bounded, so (4.2.2) is of the form

$$a_t = A_{ij}(x, Da) a_{x_i x_j} + B(x, a, Da),$$

where A_{ij} are uniformly parabolic, and where the functions A_{ij}, B are C^1 in $x \in \mathbb{R}^4$ and real analytic in (a, Da).

This implies the existence of a short-time solution $a(x, t)$ for any initial $a(x, 0)$ with $a(\cdot, 0) \in C^{1,\alpha}(\mathbb{R}^4)$ and for which $\inf_x R(x) + a(x, 0) > 0$. The classical theory for quasilinear parabolic equations [Ladyženskaja et al. 1968, §VI.1] implies that as long as $\sup_x |a(x, t)|$ and $\sup_x |Da(x, t)|$ are bounded, and as long as $\inf_x R(x) + a(x,t)$ has a positive lower bound, one can show that $Da(\,\cdot\,, t)$ is uniformly Hölder continuous. This in turn implies higher derivative bounds, and hence that the solution can be extended to a larger time interval.

For such solutions the standard comparison principle also holds: if $a_\pm : \mathbb{R}^4 \times [0, t_0) \to \mathbb{R}$ are two solutions with $Da_\pm$ bounded, for which $a_-(x, 0) \le a_+(x, 0)$ holds for all $x \in \mathbb{R}^4$, then $a_-(x, t) \le a_+(x, t)$ for all $x \in \mathbb{R}^4$ and $t < t_0$.

4.3. ***The approximating sequence of solutions u_n with $n \ge n_0$.*** For a fixed small $\epsilon > 0$ (independent of n) we choose functions Ψ, ψ_n with

$$\psi_n(x) = \Psi\Big(\frac{x}{\epsilon\sqrt{s_n}}\Big), \qquad \Psi \in C^\infty(\mathbb{R}), \qquad \Psi(\xi) = \begin{cases} 1, & 0 \le \xi \le 1, \\ 0, & \xi \ge 2. \end{cases}$$

We define

$$u_{0n}(x) := \psi_n(x) s_n^{k/3} W_{K_2}(x s_n^{-k/3}) + (1 - \psi_n(x)) U^-_{\delta_n}(x, s_n) \tag{4.3.1}$$

and let $u_n : (0, \infty) \times [s_n, \bar{t}_n) \to (0, \infty)$ be the solution to the initial value problem (1.1.5a)–(1.1.5c) with initial data $u_n(\,\cdot\,, s_n) = u_{0n}(x)$ instead of $u_0(x)$.

We will only consider the initial data for sufficiently large n; i.e., we choose an $n_0 \in \mathbb{N}$ and only consider those solutions u_n with $n \ge n_0$. Throughout this section "for all n" will mean "for all $n \ge n_0$," and in each lemma we assume that n_0 has been chosen large enough for the statement to hold.

In Corollary 4.8.2 we verify that our chosen initial data are caught between the barriers, as in (4.1.1). Before doing that we establish some derivative bounds for $u_{0n}(x)$.

Lemma 4.3.1 (monotonicity and derivative bounds). *For large enough n_0 and any $n \ge n_0$ there is an $s_n \in (0, t_n)$ such that the sequence $\{s_n : n \ge n_0\}$ is decreasing and such that $u_n(x, s_n)$ satisfies the following estimates for all n:*

(i) *The function $x \mapsto u_n(x, s_n)$ is locally Lipschitz and*

$$0 \le (u_n)_x(x, s_n) \le C_1 \tag{4.3.2}$$

for almost all $x > 0$, for some $C_1 > 0$

(ii) *The function $x \mapsto u_n(x, s_n)$ is C^3 on the interval $0 \le x \le M s_n^{1/2}$, where for $j = 2, 3$, and all n, one has*

$$(1 + s_n^{-k/3} x)^{j+2} |\partial_x^j u_n(x, s_n)| \le C s_n^{-(j-1)k/3}. \tag{4.3.3}$$

We present the proof in the following Sections 4.4–4.7. Along the way we finally choose the initial times $s_n \searrow 0$, and we use generic constants C that only depend on the various parameters defining the barriers, and the fixed small parameter ϵ, but not on n.

4.4. ***Proof of the first derivative bound (4.3.2).*** We have

$$(u_n)_x(x, s_n) = \psi_n' s_n^{k/3} W_{K_2}(x s_n^{k/3}) - \psi_n' U^-_{\delta_n} + \psi_n W'_{K_2}(x s_n^{-k/3}) + (1 - \psi_n)(U^-_{\delta_n})'. \tag{4.4.1}$$

We estimate these terms one by one.

The terms in (4.4.1) involving ψ_n' vanish outside the interval $\epsilon s_n^{1/2} \le x \le 2\epsilon s_n^{1/2}$. Thus we have

$$|\psi_n'(x) s_n^{k/3} W_{K_2}(x s_n^{-k/3})| \le \max_{x\ge 0} |\psi_n'(x)| \cdot \max_{x \le 2\epsilon s_n^{1/2}} |s_n^{k/3} W_{K_2}(x s_n^{-k/3})| \le C s_n^{-1/2} \cdot C s_n^{1/2} \le C,$$

where we have estimated $W_{K_2}(z) \le C(1+z)$ for all $z \ge 0$.

To estimate the other term involving $\psi_n'(x)$ we recall that in the region $z \ge Z_{\delta_n},\ y \le \frac{1}{4} Y_{\delta_n}$ the definition (3.9.2) implies that U_δ^- is given by

$$U_\delta^-(x,t) = t^{1/2} v_\delta^-(x t^{-1/2}, \log t).$$

Hence, if we choose $s_n > 0$ so small that $\epsilon s_n^{-\gamma} > Z_{\delta_n} = \frac{4}{3}\delta_n^{-1/(p-2)}$ then in the region $\epsilon s_n^{1/2} \le x \le 2\epsilon s_n^{1/2}$ we have

$$U_{\delta_n}^-(x, s_n) = s_n^{1/2} v_{\delta_n}(x s_n^{-1/2}, \log s_n),$$

and thus also

$$|\psi_n'| \, |U_{\delta_n}^-| \le C s_n^{-1/2} |s_n^{1/2} v_{\delta_n}(x s_n^{-1/2}, \log s_n)| \le C v_{\delta_n}(y, \log s_n),$$

where $y = x/\sqrt{s_n}$ lies in the interval $[\epsilon, 2\epsilon]$. This implies that $\psi_n'(x) U_{\delta_n}^-(x, s_n)$ is uniformly bounded.

To estimate the third term we recall that $0 \le W_{K_2}'(z) \le 1$, which implies

$$|\psi_n(x) W_{K_2}'(x s_n^{-k/3})| \le \psi_n(x) \le 1.$$

Finally, the term $(1-\psi_n)(U_{\delta_n}^-)'$ vanishes for $x \le \epsilon\sqrt{s_n}$. For $x \ge \epsilon\sqrt{s_n}$ we have

$$U_{\delta_n}^-(x, s_n) = \begin{cases} \sqrt{s_n}\, v_{\delta_n}^-\big(\frac{x}{\sqrt{s_n}}, \log s_n\big), & x \le \frac{1}{4} Y_{\delta_n}\sqrt{s_n}, \\ \max\big\{\sqrt{s_n}\, v_{\delta_n}^-\big(\frac{x}{\sqrt{s_n}}, \log s_n\big),\ u^-(x, s_n)\big\}, & \frac{1}{4} Y_{\delta_n}\sqrt{s_n} \le x \le Y_{\delta_n}\sqrt{s_n}, \\ u^-(x, s_n), & x \ge Y_{\delta_n}\sqrt{s_n}, \end{cases}$$

with $Y_{\delta_n} = 2\sqrt{(2k+1)!!\, M/\delta_n}$ as in Lemma 3.6.1.

It follows that $x \mapsto U_{\delta_n}^-(x, s_n)$ is a Lipschitz continuous function whose derivative is almost everywhere given by $(v_{\delta_n}^-)_y$ or $u_x^-(x, s_n)$. If $y = x/\sqrt{s_n} \in [\epsilon, Y_{\delta_n}]$ then

$$(v_{\delta_n})_y(y, \log s_n) = 1 + (K_1^- - \delta_n) s_n^{3\gamma} \varphi_k'(y) - B K_1^2 s_n^{6\gamma} g'(y) + p \frac{s_n^{(p+1)\gamma}}{y^{p+1}} \le C,$$

for a uniform constant C, independent of n and for $n \ge n_0$, sufficiently big.

On the other hand, $u^-(x, s_n) = u_0(x) - M s_n \min\{1, x^{2k-4}\}$. For $x \ge 1$ we have $u_x^-(x, s_n) = u_0'(x)$, which is uniformly bounded by the assumption (1.1.2), while for $x < 1$ we have $u_x^-(x, s_n) = u_0'(x) - (2k-4) M s_n x^{2k-5}$, which is also uniformly bounded because we assume $k \ge 4$.

Combining all these estimates together with (4.4.1) yields the uniform Lipschitz bound on u_n.

4.5. ***Proof of the second derivative estimate (4.3.3).*** We will show

$$|(u_n)_{xx}(x, s_n)| \le C s_n^{-k/3} (1 + x s_n^{-k/3})^{-4} \tag{4.5.1}$$

for all $x \in [0, M\sqrt{s_n}]$.

Writing $z = xs_n^{-k/3}$, we estimate the terms on the right-hand side of

$$(u_n)_{xx} = \psi_n'' s_n^{k/3} W_{K_2}(z) + 2\psi_n' W_{K_2}'(z) + \psi_n W_{K_2}''(z) s_n^{-k/3} + (1-\psi_n)(U^-_{\delta_n})_{xx} - 2\psi_n'(U^-_{\delta_n})_x - \psi_n'' U^-_{\delta_n}. \quad (4.5.2)$$

For $0 \le x \le \epsilon s_n^{1/2}$ we have

$$(u_n)_{xx}(x, s_n) = s_n^{-k/3} W_{K_2}''(z).$$

The asymptotic expansion (3.5.1) for W implies that for all $z \ge 0$

$$0 \le W_{K_2}''(z) \le C(1+z)^{-4}.$$

Hence (4.5.1) holds for $x \le \epsilon s_n^{1/2}$.

If $2\epsilon s_n^{1/2} \le x \le M s_n^{1/2}$, i.e., if $2\epsilon \le y \le M$, then $u_n(x, s_n) = s_n^{-1/2} v^-_{\delta_n}(y, \log s_n)$ and thus, using the definition (3.4.3) for $v^-_{\delta_n}$, we find for $2\epsilon \le y \le M$

$$(u_n)_{xx}(x, s_n) = s_n^{-1/2}(v^-_{\delta_n})_{yy}(y, \log s_n) \le C s_n^{-1/2} \frac{s_n^{3\gamma}}{y^4} \le \frac{C s_n^{-k/3}}{(1 + xs_n^{-k/3})^4}.$$

Finally, if $\epsilon s_n^{1/2} \le x \le 2\epsilon s_n^{1/2}$, then similarly to the previous two cases we get

$$|\psi_n W_{K_2}''(z) s_n^{-k/3} + (1-\psi_n)(U^-_{\delta_n})_{xx}| \le C s_n^{-k/3}(1 + xs_n^{-k/3})^{-4}.$$

To bound the remaining terms in (4.5.2) it is enough to estimate

$$2|\psi_n'||W_{K_2}'(z) - (U_{\delta_n})_x| + |\psi_n''||s_n^{k/3} W_{K_2}(z) - U^-_{\delta_n}|.$$

Both ψ_n' and ψ_n'' vanish unless $\epsilon s_n^{1/2} \le x \le 2\epsilon s_n^{1/2}$. In this region one has $xs_n^{-k/3} \ge 1$, and thus our desired upper bound satisfies

$$\frac{1}{C} s_n^{k-2} \le s_n^{-k/3}(1 + xs_n^{-k/3})^{-4} \le C s_n^{k-2}.$$

By the asymptotic expansion (3.5.1) of the Alencar solution W for large z, we have $W_{K_2}(z) = z + \mathcal{O}(z^{-2})$ and $W_{K_2}'(z) = 1 + \mathcal{O}(z^{-3})$. When $\epsilon s_n^{1/2} \le x \le 2\epsilon s_n^{1/2}$ this implies

$$\begin{aligned} s_n^{k/3} W_{K_2}(xs_n^{-k/3}) - x &= \mathcal{O}(s_n^k x^{-2}) = \mathcal{O}(s_n^{k-1}), \\ W_{K_2}'(xs_n^{-k/3}) - 1 &= \mathcal{O}(s_n^k x^{-3}) = \mathcal{O}(s_n^{k-3/2}). \end{aligned} \quad (4.5.3)$$

In the region $\epsilon s_n^{1/2} \le x \le 2\epsilon s_n^{1/2}$ we have, by definition, and by the asymptotic expansions of the terms f_0^-, f_1, f_2 in (3.4.5),

$$\begin{aligned} U^-_{\delta_n}(x, s_n) &= s_n^{1/2} v^-_{\delta_n}(y, \log s_n) \qquad \text{where } y = xs_n^{-1/2} \\ &= s_n^{1/2} y + s_n^{1/2} \mathcal{O}(s_n^{k-3/2} y^{-2}) i \\ &= x + \mathcal{O}(s_n^k x^{-2}). \end{aligned} \quad (4.5.4)$$

This expansion may be differentiated with respect to x, resulting in

$$|(U^-_{\delta_n})_x - 1| \le C s_n^k x^{-3} \le C s_n^{k-3/2}. \quad (4.5.5)$$

The bounds $|\psi_n'| = \mathcal{O}(s_n^{-1/2})$ and $|\psi_n''| = \mathcal{O}(s_n^{-1})$ now lead to

$$|\psi_n''|\,|s_n^{k/3}W_{K_2}(xs_n^{-k/3}) - U_{\delta_n}^-| \le Cs_n^{-1}s_n^{k-1} = Cs_n^{k-2} \le \frac{Cs_n^{-k/3}}{(1+xs_n^{-k/3})^4}$$

and also

$$|\psi_n'|\,|W_{K_2}'(xs_n^{-k/3}) - (U_{\delta_n}^-)_x| \le Cs_n^{-1/2}s_n^{k-3/2} \le \frac{\bar{C}s_n^{-k/3}}{(1+xs_n^{-k/3})^4}.$$

This concludes the proof of the stated weighted C^2 estimate for u_n at time $t = s_n$.

4.6. ***Proof of the third-order derivative bound (4.3.3).*** We outline the arguments, which are similar to those for the second derivative estimate.

For $0 \le x \le \epsilon s_n^{1/2}$ the definition (4.3.1) of $u_{0n}(x) = u_n(x, s_n)$ directly implies

$$|(u_n)_{xxx}(x, s_n)| = |W_{K_2}'''(z)|s_n^{-2k/3}, \quad \text{where again } z = xs_n^{-k/3}.$$

Using the asymptotic expansion for $W(z)$ as $z \to \infty$ one then verifies the third derivative estimate for $x \le \epsilon s_n^{1/2}$.

If $2\epsilon s_n^{1/2} \le x \le Ms_n^{1/2}$, i.e., if $2\epsilon \le y \le M$, then

$$(u_n)_{xxx}(x, s_n) = (U_{\delta_n}^-)_{xxx}(x, s_n) = s_n^{-1}(v_{\delta_n}^-)_{yyy}(y, \log s_n),$$

and the estimate follows from the explicit expression (3.4.3) for $v_{\delta_n}^-(y, \tau)$.

If $\epsilon s_n^{1/2} \le x \le 2\epsilon s_n^{1/2}$, then u_n is given by

$$u_n(x, s_n) = s_n^{k/3}W_{K_2}(z) + \psi_n(x)\{s_n^{k/3}W_{K_2}(z) - U_{\delta_n}^-(x, s_n)\}, \quad z = xs_n^{-k/3}.$$

The third derivative of the first term can be estimated exactly as in the region $x \le \epsilon s_n^{1/2}$. After differentiating the second term three times one ends up with terms of the form

$$\psi_n^{(3-\ell)}(x)\Big(\frac{\partial}{\partial x}\Big)^{\ell}\{s_n^{k/3}W_{K_2}(z) - U_{\delta_n}^-(x, s_n)\}, \quad 0 \le \ell \le 3.$$

Using the asymptotic descriptions we have for W and $U_{\delta_n}^-$, and taking care to cancel the leading terms in these descriptions when $\ell \in \{0, 1\}$, we get the third derivative bounds in (4.3.3). The estimates are similar to the first and second-order estimates.

4.7. ***Proof that $x \mapsto u_n(x, s_n)$ is nondecreasing.*** We consider four regions: the region $0 \le x \le \epsilon s_n^{1/2}$, the region $\epsilon s_n^{1/2} \le x \le 2\epsilon s_n^{1/2}$ where we glue the inner and intermediate barriers, the intermediate region $2\epsilon s_n^{1/2} \le x \le 1$, and finally the region $x \ge 1$.

In the region $0 < x \le \epsilon s_n^{1/2}$ we have $u_n(x, s_n) = s_n^{k/3}W_{K_2}(xs_n^{-k/3})$, which is an increasing function of x because W is increasing.

In the region $\epsilon s_n^{1/2} \le x \le 2\epsilon s_n^{1/2}$, we have

$$(u_n)_x(x, s_n) = \psi_n'(x)(s_n^{k/3}W_{K_2}(xs_n^{-k/3}) - U_{\delta_n}^-(x, s_n)) + \psi_n(x)W_{K_2}'(xs_n^{-k/3}) + (1-\psi_n(x))(U_{\delta_n}^-)_x(x, s_n).$$

Using (4.5.3), (4.5.4), as well as $|\psi_n'(x)| \le Cs_n^{-1/2}$, we estimate the first term above by

$$|\psi_n'(x)|\,|s_n^{k/3}W_{K_2}(xs_n^{-k/3}) - U_{\delta_n}^-(x, s_n)| \le C|\psi_n'(x)|s_n^{k-1} \le Cs_n^{k-3/2}.$$

Furthermore, (4.5.3) and (4.5.5) imply

$$|W_{K_2}'(xs_n^{-k/3}) - 1| + |(U_{\delta_n}^-)_x(x, s_n) - 1| \le Cs_n^{k-3/2}.$$

It follows that

$$|(u_n)_x(x, s_n) - 1| \le Cs_n^{k-3/2}$$

throughout the region $\epsilon s_n^{1/2} \le x \le 2\epsilon s_n^{1/2}$. Since $s_n \to 0$ and $k \ge 4$, so $k - \frac{3}{2} > 0$, we see that for large enough n the function $x \mapsto u_n(x, s_n)$ is strictly increasing when $\epsilon s_n^{1/2} \le x \le 2\epsilon s_n^{1/2}$.

Next, in the region $2\epsilon\sqrt{s_n} \le x \le 1$ we have

$$u_n(x, s_n) = U_{\delta_n}^-(x, s_n) = \max\{s_n^{1/2}v_{\delta_n}^-(xs_n^{-1/2}, \log s_n),\, u^-(x, s_n)\}$$

if $xs_n^{-1/2} \le Y_{\delta_n}$, and $u_n(x, s_n) = u^-(x, s_n)$ otherwise. It is easy to see that $x \mapsto u^-(x, s_n)$ is an increasing function. Concerning $v_{\delta_n}^-(y, \log s_n)$ we recall definition (3.4.3), i.e.,

$$v_{\delta_n}^-(y, \log s_n) = y + (K_1 - \delta)s_n^{3\gamma}\varphi_k(y) - BK_1^2 s_n^{6\gamma} g(y) - s_n^{(p+1)\gamma} y^{-p}.$$

If we choose s_n small enough, then the last three terms will be uniformly small in C^1 on the fixed interval $2\epsilon \le y \le Y_{\delta_n}$ compared to the leading term y, so that $y \mapsto v_{\delta_n}(y, \log s_n)$ is also increasing on the interval $2\epsilon \le y \le Y_{\delta_n}$. It follows that $x \mapsto u_n(x, s_n)$ is increasing on $2\epsilon s_n^{1/2} \le x \le 1$.

The very last situation we must consider is where $x \ge 1$. In this case (1.1.2) implies

$$(U_{\delta_n}^-)_x(x, s_n) = u_0'(x) \ge 0.$$

Since we have covered all cases, the proof of monotonicity of $x \mapsto u_n(x, s_n)$ is complete.

4.8. ***Proof of* (4.1.3).** We turn to the proof that the initial data $u_n(x, s_n)$ is sandwiched between the two barriers $U_{\delta_n}^{\pm}$, as in (4.1.3).

Lemma 4.8.1. *If n_0 is large enough then, for each $n \ge n_0$, we can choose $s_n \in (0, t_n)$ small enough that*

$$U_{\delta_n}^-(x, s_n) \le s_n^{k/3}W_{K_2}(xs_n^{-k/3}) \le U_{\delta_n}^+(x, s_n) \tag{4.8.1}$$

for $0 \le x \le 2\epsilon s_n^{1/2}$.

Proof. In this proof we write $y = xs_n^{-1/2}$ and $z = xs_n^{-k/3}$.

In the region $0 \le y \le 2\epsilon$ the barriers $U_{\delta_n}^{\pm}$ as defined in (3.9.1), (3.9.2) are given by

$$\begin{aligned} U_{\delta_n}^+(x, s_n) &= \min\{s_n^{1/2}v_{\delta_n}^+(y, \log s_n),\, s_n^{k/3}W_{K_2^+(n)}(z)\},\\ U_{\delta_n}^-(x, s_n) &= \max\{s_n^{1/2}v_{\delta_n}^-(y, \log s_n),\, s_n^{k/3}W_{K_2^-(n)}(z) + Ds_n^{k-1}\}, \end{aligned}$$

where $K_2^{\pm}(n) = (K_2^3 \pm 2\delta_n)^{1/3}$ (see Section 3.8).

In Lemma 3.7.1 we defined $Z_n := Z_{\delta_n} = \frac{4}{3}\delta_n^{-1/(p-2)}$ and showed that the functions whose max/min define $U^{\pm}_{\delta_n}$ cross in the interval $\frac{1}{2}Z_n \le z \le Z_n$. To prove (4.8.1) we therefore must show

$$s_n^{k/3} W_{K_2^-(n)}(z) + D s_n^{k-1} \le s_n^{k/3} W_{K_2}(z) \le s_n^{k/3} W_{K_2^+(n)}(z) \tag{4.8.2}$$

if $0 \le z \le Z_n$, and

$$s_n^{1/2} v^-_{\delta_n}(y, \log s_n) \le s_n^{k/3} W_{K_2}(z) \le s_n^{1/2} v^+_{\delta_n}(y, \log s_n) \tag{4.8.3}$$

if $z \ge \frac{1}{2}Z_n$ and $y \le 2\epsilon$.

Since $\kappa \mapsto W_\kappa(z) = \kappa W(z/\kappa)$ is strictly increasing (see (3.9.5)), from $K^+_{2,n} = (K_2^3 + 2\delta_n)^{1/3} > K_2$ it follows that $W_{K_2}(z) \le W_{K_2^+(n)}(z)$ holds for all $z \ge 0$. Thus the second inequality in (4.8.2) holds.

The first inequality in (4.8.2) is equivalent to

$$W_{K_2}(z) - W_{K_2^-(n)}(z) \ge D s_n^{2k/3-1} \quad \text{for all } z \le Z_n.$$

By integrating

$$\frac{\partial}{\partial\kappa}\frac{\partial}{\partial z} W_\kappa(z) = -\frac{z}{\kappa^2} W''\Big(\frac{z}{\kappa}\Big) < 0$$

from $\kappa = K_2^-(n)$ to K_2 we see that $W_{K_2}(z) - W_{K_2^-(n)}(z)$ is a decreasing function of z. We therefore must guarantee

$$W_{K_2}(Z_n) - W_{K_2^-(n)}(Z_n) \ge D s_n^{2k/3-1}.$$

This holds for each n provided we choose $s_n \in (0, t_n)$ small enough.

We now consider (4.8.3), which is equivalent to

$$s_n^{-\gamma} v^-_{\delta_n}(s_n^\gamma z, \log s_n) \le W_{K_2}(z) \le s_n^{-\gamma} v^+_{\delta_n}(s_n^\gamma z, \log s_n), \tag{4.8.4}$$

and we must establish these inequalities for $\frac{1}{2}Z_n \le z \le 2\epsilon s_n^{-\gamma}$. Both inequalities can be proved in the same way, and we focus on the one involving $v^-_{\delta_n}$.

Keeping in mind that $K_2 = K_1^3$, the asymptotics (3.5.1) for the Alencar function W imply that there is a constant C such that

$$z + K_1 z^{-2} - C z^{-3} \le W_{K_2}(z) \le z + K_1 z^{-2} + C z^{-3} \tag{4.8.5}$$

for $z \ge 1$. On the other hand, the definition (3.4.3) of v^-_δ implies

$$\begin{aligned} s_n^{-\gamma} v^-_{\delta_n}(s_n^\gamma z, \log s_n) &= z + (K_1 - \delta_n) s_n^{2\gamma} \varphi_k(s_n^\gamma z) - z^{-(p-1)} - B K_1^2 s_n^{5\gamma} g(s_n^{-\gamma} z) \\ &= z + K_1 s_n^{2\gamma} \varphi_k(s_n^\gamma z) - \{\delta_n s_n^{2\gamma} \varphi_k(s_n^\gamma z) + z^{-(p-1)}\} - B K_1^2 s_n^{5\gamma} g(s_n^{-\gamma} z). \end{aligned}$$

For $y \le 2\epsilon$ we have

$$|\varphi_k(y) - y^{-2}| \le C \quad \text{and} \quad |g(y)| \le C y^{-5}.$$

Hence

$$s_n^{-\gamma} v^-_{\delta_n}(s_n^\gamma z, \log s_n) \le z + K_1 z^{-2} - \{\delta_n z^{-2} + z^{-(p-1)}\} + C(s_n^{2\gamma} + z^{-5}), \tag{4.8.6}$$

where C is the same for all sufficiently large $n \in \mathbb{N}$ and for $1 \le z \le 2\epsilon s_n^{-\gamma}$.

If $z \geq 1$ then $z^{-5} \leq z^{-3}$, so (4.8.5) and (4.8.6) together lead to

$$W_{K_2}(z) - s_n^{-\gamma} v_{\delta_n}^-(s_n^\gamma z, \log s_n) \geq \delta_n z^{-2} - C s_n^{2\gamma} + z^{-(p-1)} - C z^{-3}. \tag{4.8.7}$$

Now choose s_n small enough that $s_n < (\delta_n Z_n / C)^{1/2\gamma}$. Then for all $z \geq Z_n$ one has

$$\delta_n z^{-2} - C s_n^{2\gamma} \geq \delta_n Z_n^{-2} - C s_n^{2\gamma} > 0.$$

If we also require n to be large enough that $Z_n > C^{1/(4-p)}$, then we have for all $z \geq Z_n$

$$z^{-(p-1)} - C z^{-3} \geq (z^{4-p} - C) z^{-3} \geq (Z_n^{4-p} - C) z^{-3} > 0.$$

Applying the last two inequalities to (4.8.7) we conclude that the first inequality in (4.8.4) holds. A slight modification of these arguments also proves the second inequality in (4.8.4). □

Corollary 4.8.2. *If for each $n \geq n_0$ we choose $s_n \in (0, t_n)$ as in Lemma 4.8.1, then (4.1.3) holds, i.e., $U_{\delta_n}^-(x, s_n) \leq u_n(x, s_n) \leq U_{\delta_n}^+(x, s_n)$ for all $x \geq 0$.*

Proof. If $x \geq 2\epsilon s_n^{1/2}$ then $u_n(x, s_n) = U_{\delta_n}^-(x, s_n)$ and there is nothing to prove.

If $0 \leq x \leq 2\epsilon s_n^{1/2}$, then $u_n(x, s_n)$ is a convex combination of $U_{\delta_n}^-(x, s_n)$ and $s_n^{k/3} W_{K_2}(s_n^{-k/3} x)$. We have just shown that this second function lies between the barriers so the convex combination u_n also lies between the barriers $U_{\delta_n}^\pm$. □

4.9. *Monotonicity and uniform C^1 bound for $u_n(x,t)$.*

In the following lemma we show that the initial uniform C^1 bound $\|u_n(\cdot, s_n)\|_{C^1} \leq C$ persists for as long as each $u_n(x,t)$ exists, provided that n is sufficiently large.

Lemma 4.9.1. *If C_1 is the upper bound for $(u_n)_x(x, s_n)$ from Lemma 4.3.1 then for sufficiently large n we have $0 \leq (u_n)_x(x,t) \leq C_1$ for all $(x,t) \in [0,\infty) \times [s_n, \bar{t}_n)$.*

In order to prove this lemma we will apply the maximum principle to the evolution equation of $(u_n)_x$. For this we first need the following observation.

Lemma 4.9.2. *Let M be the same constant as in Lemma 3.3.1. There is an $\alpha > 0$ such that for all sufficiently large n one has $U_{\delta_n}^-(x,t) \geq x$ for all $x \in [0,\alpha]$ and all $t \in (0, t_n)$.*

Proof. In the part of the outer region where $M\sqrt{t} \leq x \leq 1$ we have $t \leq M^{-2} x^2$, so that

$$\begin{aligned} U_{\delta_n}^-(x,t) &= u_0(x) - M t x^{2(k-2)} \\ &= x + (K_1 + o(1)) x^{2(k-1)} - M t x^{2(k-2)} \qquad (x \to 0) \\ &\geq x + (K_1 - M^{-1} + o(1)) x^{2(k-1)} \qquad (x \to 0). \end{aligned}$$

If we choose $M > 2/K_1$, then there is an $\alpha > 0$ such that $K_1 - M^{-1} + o(1) > 0$ and hence such that $U_{\delta_n}^-(x,t) > x$ holds when $M\sqrt{t} \leq x \leq \alpha$.

In the intermediate region the lower barrier is given by $t^{1/2} v_{\delta_n}^-(t^{-1/2} x, \log t)$, where in the rescaled variables (y, τ) we have $v_{\delta_n}^-(y,\tau) = y + f_{\delta_n}^-(y,\tau)$. Lemma 3.4.4 tells us that $f_{\delta_n}^-(y,\tau) \geq 0$, so in the intermediate region we have $v_{\delta_n}^-(y,\tau) \geq y$ and hence $U_{\delta_n}^-(x,t) \geq x$.

Finally, in the inner region we have

$$U^-_{\delta_n}(x,t) = t^{k/3} w^-_n(t^{-k/3}x, \log t)$$

and, according to the definition in Lemma 3.5.3,

$$w^-_n(z,\tau) = W_{K^-_2}(z) + De^{2\gamma\tau} > W_{K^-_2}(z) > z$$

because $W_\kappa(z) > z$ for all $z \geq 0$. This implies $U^-_{\delta_n}(x,t) \geq x$ in the inner region as well. ☐

Proof of Lemma 4.9.1. If u_n is one of the approximating solutions of (1.1.5a), then by differentiating in x we find that $\eta := (u_n)_x$ satisfies

$$\eta_t = \mathcal{M}_n[\eta] - Q_n(x,t)\eta, \tag{4.9.1}$$

where

$$\mathcal{M}_n[\eta] := \frac{\eta_{xx}}{1+(u_n)^2_x} + \frac{3}{x}\eta_x \quad \text{and} \quad Q_n(x,t) := \frac{2(u_n)^2_{xx}}{(1+(u_n)^2_x)^2} - \frac{3}{u_n^2} + \frac{3}{x^2}.$$

Lemma 4.9.2 says that $u_n(x,t) \geq U^-_{\delta_n}(x,t) \geq x$, so $Q_n(x,t) \geq 0$.

If the domain of η were bounded we could directly apply the maximum principle and conclude that η is bounded by its initial values. Since the domain is not bounded, we consider $\Omega(x,t) := x^{-1} + \kappa e^t x^2$ in the domain $x > 0$, $0 \leq t \leq 1$. (Without loss of generality we assume that $\bar{t}_n \leq 1$ for all n.) In this region Ω satisfies

$$\begin{aligned}\Omega_t - \mathcal{M}_n[\Omega] + Q_n(x,t)\Omega &\geq \kappa e^t x^2 - \frac{2x^{-3}}{1+(u_n)^2_x} + 3x^{-3} - \frac{2\kappa e^t}{1+(u_n)^2_x} - 6\kappa e^t \\ &\geq \kappa e^t x^2 - 2x^{-3} + 3x^{-3} - 2\kappa e^t - 6\kappa e^t \\ &\geq \kappa e^t x^2 + x^{-3} - 8\kappa e^t \\ &\geq \kappa(x^2 - 8e) + x^{-3}.\end{aligned}$$

If we choose $\kappa > 0$ sufficiently small, then the left-hand side is positive for all $x > 0$ and $t \in [0, 1]$.

For any $\epsilon > 0$ we therefore have

$$\Big(\frac{\partial}{\partial t} - \mathcal{M}_n + Q_n\Big)(\eta + \epsilon\Omega) > 0 \quad \text{in } (0,\infty) \times [s_n, \bar{t}_n).$$

Furthermore $\eta + \epsilon\Omega \to \infty$ as $x \to \{0,\infty\}$, so the maximum principle implies that $\eta + \epsilon\Omega$ attains its minimum at the initial time $t = s_n$. Since $0 \leq u_{n,x}(x, s_n) \leq C_1$ (by Lemma 4.3.1), we find that $\eta(x,t) + \epsilon\Omega(x,t) \geq 0$ for all $\epsilon > 0$, which implies that $u_{n,x}(x,t) = \eta(x,t) \geq 0$ for all $x > 0$ and $t \in [s_n, \bar{t}_n)$.

By considering $\eta - \epsilon\Omega$ for arbitrary $\epsilon > 0$ we similarly conclude that η is bounded by its largest initial value, i.e., $(u_n)_x(x,t) = \eta(x,t) \leq C_1$ for all $x > 0$ and $t \in [s_n, \bar{t}_n)$. This finishes the proof of Lemma 4.9.1. ☐

Corollary 4.9.3. *Let $u_n(x,t)$ be a solution to the initial value problem (1.1.5a)–(1.1.5c) with initial data $u_n(x, s_n)$ as above, and let $n \geq n_0$, where n_0 is large enough that all previous results hold. Then the solution $u_n(x,t)$, which exists for all $t \in [s_n, \bar{t}_n)$, satisfies $U^-_{\delta_{n_0}}(x,t) \leq u_n(x,t) \leq U^+_{\delta_{n_0}}(x,t)$ and $0 \leq (u_n)_x \leq C_1$ for all $x \geq 0$ and all $t \in [s_n, \min\{\bar{t}_n, t_{n_0}\})$, where C_1 is as in Lemma 4.3.1.*

Proof. We have shown that $(u_n)_x$ is uniformly bounded, that $u_n \geq U^-_{\delta_n}$ has a positive lower bound, and that $u_n(x,t) - u_0(x)$ is uniformly bounded (because $U^\pm_{\delta_n} - u_0$ is bounded). The discussion in Section 4.2 and (4.1.4) then shows that the maximum principle can be applied to conclude that the solution u_n remains between the barriers $U^\pm_{\delta_{n_0}}$ for as long as both u_n and $U^\pm_{\delta_{n_0}}$ are defined. ☐

4.10. ***Uniform lower bound for $\bar{t}_n$.*** Each of the approximating solutions u_n exists at least until time $\bar{t}_n$. We now argue that if n_0 is large enough, then $\bar{t}_n > t_{n_0}$ for all $n \geq n_0$.

We have already verified for all $x \geq 0$ and $t \in [s_n, \min\{\bar{t}_n, t_{n_0}\}]$ that the solution $u_n(x,t)$ remains between the barriers $U^\pm_{\delta_{n_0}}(x,t)$ and that its derivative $(u_n)_x(x,t)$ is uniformly bounded. Standard estimates for quasilinear parabolic equations applied to (4.2.1) or (4.2.2) then imply that higher derivatives of u_n also are uniformly bounded. If we had $\bar{t}_n \leq t_{n_0}$, then $\lim_{t\nearrow\bar{t}_n} u(x,t)$ would exist, and we could extend the solution to a larger time interval. Therefore $\bar{t}_n$ would not be the maximal time of existence for the solution u_n after all.

4.11. ***Proof of the main existence Theorem 4.1.1.*** We have constructed the sequence of solutions u_n and have established a priori bounds for its derivatives, which imply that there is a subsequence u_{n_j} that converges locally uniformly to a function $u : [0,\infty) \times (0, t_{n_0}] \to \mathbb{R}$. The derivative bounds for the approximating solutions u_n imply that u_n, $u_{n,x}$, $u_{n,xx}$, and $u_{n,t}$ also converge locally uniformly, and that the limit u is a solution of (1.1.5a).

We now verify that u also satisfies the initial and boundary conditions (1.1.5b), (1.1.5c), as well as the asymptotic description (4.1.2) of the inner region.

4.11.1. *The initial condition.* Let n_0 be large enough that all previous results in this section hold. Then all solutions u_{n_j} are caught between the barriers $U^\pm_{n_0}$, so the limit also lies between $U^\pm_{n_0}$. In the outer region, defined by $x \geq M\sqrt{t}$, the lower (upper) barriers are defined in (3.3.1) to be the maximum (minimum) of $u^\pm(x,t) = u_0(x) \pm Mt \min\{1, x^{2k-4}\}$ and the barriers defined in the intermediate region. This implies that for $x \geq M\sqrt{t}$ we have

$$u_0(x,t) - Mt\max\{1, x^{2k-4}\} \leq u(x,t) \leq u_0(x,t) + Mt\max\{1, x^{2k-4}\}.$$

Therefore $\lim_{t\searrow 0} u(x,t) = u_0(x)$ uniformly for all $x > 0$.

4.11.2. *Boundary condition.* The solutions $u_n(x,t)$ all satisfy $u_{n,x}(0,t) = 0$. They converge in C^1 to $u(x,t)$, so we have $u_x(0,t) = 0$ for all $t \in (0, t_{n_0}]$.

4.11.3. *Asymptotics in the inner region.* To finish the proof of the theorem, we will show that

$$\lim_{\tau\to-\infty} w(z,\tau) = W_{K_2}(z),$$

uniformly on compact sets in z. This follows almost immediately from (4.1.1) and the definition of our barriers $\tilde{u}^\pm_n(x,t)$ in the inner region. Using the definitions $w^-_n(z,\tau) = W_{K_2^-(n)}(z) + De^{\gamma\tau}$ and $w^-_n(z,\tau) = W_{K_2^+(n)}(z)$ from Section 3.5, (4.1.1) implies $w^-_n(z,\tau) \leq w(z,\tau) \leq w^+_n(z,\tau)$, and hence

$$W_{K_2^-(n)}(z) + De^{\gamma\tau_n} \leq w(z,\tau) \leq W_{K_2^+(n)}(z) \tag{4.11.1}$$

for all $z \in [0, Z_{\delta_n}]$, and $\tau \leq \tau_n := \log t_n$.

Since $Z_{\delta_n} := \frac{4}{3}\delta_n^{-1/(p-2)} \to +\infty$ and $K_2^{\pm}(n) = (K_2^3 \pm 2\delta_n)^{1/3} \to K_2$ as $n \to +\infty$, (4.11.1) holds on $[0, Z] \times (-\infty, \tau_n)$ for any $Z > 0$, provided n is sufficiently large. The rescaled Alencar solution $W_K(z) = K\,W(z/K)$ depends continuously on K, so after taking the limit $n \to \infty$ in (4.11.1) we conclude that $\lim_{\tau\to 0} w(z, \tau) = W_{K_2}(z)$, uniformly on any bounded interval $0 \le z \le Z$, as claimed in Theorem 4.1.1.

5. Uniform L^∞ bound on the mean curvature

5.1. *Bounding H.* In Theorem 4.1.1 we showed the short-time existence of an $O(4) \times O(4)$ symmetric MCF solution $\mathcal{M}_t$, $0 < t \le t_0$, which is smooth for $t > 0$ and defined by a profile function

$$u : [0, +\infty) \times (0, t_0] \to \mathbb{R}$$

which satisfies the initial value problem (1.1.5a)–(1.1.5c) for the given initial data $u_0(x)$. In this section we will show that the mean curvature of $\mathcal{M}_t$ is uniformly bounded on $[0, +\infty) \times (0, t_0]$ despite the fact that the initial data u_0 is singular at the origin. The life time of the solution is $t_0 = t_{n_0}$ for some large enough n_0.

Theorem 5.1.1. *Let $\mathcal{M}_t$, $0 < t \le t_0$, be the $O(4) \times O(4)$ symmetric MCF solution constructed in Theorem 4.1.1. Then*

$$\sup_{0<t\le t_0} \sup_{\mathcal{M}_t} |H| < \infty. \tag{5.1.1}$$

To prove this theorem we will first show, using a direct argument, that $H(x, t)$ is uniformly bounded in the outer region $x \ge M\sqrt{t}$, $0 < t \le t_0$. Then, using an argument by contradiction that is strongly inspired by the approach of Stolarski [2023], we will show that $H(x, t)$ is uniformly bounded in the remaining region $x \le M\sqrt{t}$, $0 < t \le t_0$.

5.2. *Bounding $H(x, t)$ in the outer region.* Assuming that t_0 is sufficiently small we show in this section that (5.1.1) holds in the part of the outer region where $x \gg \sqrt{t}$ and $0 < t \le t_0$.

Lemma 5.2.1. *Let Y_{δ_0} be as in (3.6.4). There exist $t_0 > 0$ and a uniform constant $C > 0$ so that for all (x, t) with $t \in (0, t_0)$ and $x \ge Y_{\delta_0}\sqrt{t}$ one has*

$$|H(x, t)| \le C. \tag{5.2.1}$$

Proof. We fix (x_1, t_1) with $0 < t_1 < t_0$ and $x_1 \ge Y_{\delta_0}\sqrt{t_1}$. We first deal with the case when $x_1 \in \left(0, \frac{1}{2}\right)$.

By (4.1.1) the solution u lies between our upper and lower barriers $U_{\delta_0}^{\pm}$ constructed in Proposition 3.9.1. In the region $x \ge \frac{1}{4}Y_{\delta_0}\sqrt{t}$ we have $U_{\delta_0}^{\pm}(x, t) = u^{\pm}(x, t)$, and hence

$$u^-(x, t) \le u(x, t) \le u^+(x, t), \quad \tfrac{1}{4}Y_{\delta_0}\sqrt{t} \le x \le 1, \tag{5.2.2}$$

and, by definition (3.3.1) of $u^{\pm}$,

$$|u(x, t) - u_0(x)| \le M t x^{2k-4}, \quad \tfrac{1}{4}Y_{\delta_0}\sqrt{t} \le x \le 1.$$

Rescaling the solution $u(x,t)$, we consider

$$U(\xi,s)=x_1^{-1}u(x_1\xi,t_1+x_1^2s)\quad\text{for } (\xi,s)\in\mathcal{Q}:=\Big(\frac{1}{4},2\Big)\times\Big(-\frac{t_1}{x_1^2},0\Big),$$

which satisfies

$$U_\tau=\frac{U_{\xi\xi}}{1+U_\xi^2}+\frac{3}{\xi}U_\xi-\frac{3}{U}. \tag{5.2.3}$$

If $(\xi,s)\in\mathcal{Q}$, then $x=x_1\xi$ and $t=t_1+x_1^2s$ satisfy $\frac{1}{4}Y_{\delta_0}\sqrt{t}\le x\le 1$, so that (5.2.2) applies, and so that

$$\begin{aligned}|U(\xi,s)-x_1^{-1}u_0(x_1\xi)|&\le M(t_1+x_1^2s)x_1^{2k-5}\xi^{2k-4}\le CMt_1x_1^{2k-5}\\&\le\frac{CM}{Y_{\delta_0}^2}x_1^{2k-3}\qquad\text{since } x_1\ge Y_{\delta_0}\sqrt{t_1}.\end{aligned}$$

The initial profile u_0 satisfies $x\le u_0(x)\le x+Cx^{2k-2}$ for $0<x<2$, and thus

$$|x_1^{-1}u_0(x_1\xi)-\xi|\le Cx_1^{2k-3}\quad\text{for } \xi\in\big(\tfrac{1}{4},2\big).$$

The last two inequalities together imply that

$$|U(\xi,s)-\xi|\le Cx_1^{2k-3} \tag{5.2.4}$$

holds on $\mathcal{Q}$. Therefore the function

$$F(\xi,s):=\frac{U(\xi,s)-\xi}{x_1^{2k-3}},$$

which satisfies the equation

$$F_s=\frac{F_{\xi\xi}}{1+U_\xi^2}+\frac{3}{\xi}F_\xi+\frac{3}{\xi U(\xi,s)}F, \tag{5.2.5}$$

is bounded on $\mathcal{Q}$ by $|F(\xi,s)|\le C$ for some constant C that does not depend on (x_1,t_1).

Claim 5.2.2. *U and $1+U_\xi^2$ are Hölder continuous on*

$$\mathcal{Q}'=\Big(\frac{1}{2},\frac{3}{2}\Big)\times\Big(-\frac{t_1}{x_1^2},0\Big],$$

uniformly in (x_1,t_1).

Proof of Claim 5.2.2. By (5.2.4) we have $\|U\|_{C^0(\mathcal{Q})}\le C$ for a uniform constant C, independent of (x_1,t_1), where $x_1\in\big(0,\frac{1}{2}\big)$. Furthermore, in $\mathcal{Q}$ we also have

$$|U_\xi(\xi,s)|=|u_x(x_1\xi,t_1+x_1^2s)|\le C, \tag{5.2.6}$$

where C is a uniform constant, independent of (x_1,t_1). This follows by Lemma 4.9.1 and the fact that $u_n(x,t)$ smoothly converges as $n\to\infty$ to $u(x,t)$ for all $x>0$ and $t\in(0,t_1]$. Since $U(\xi,s)$ satisfies the uniformly parabolic equation (5.2.3), standard regularity theory applied to (5.2.3) implies that there exists a uniform constant C, independent of (x_1,t_1), so that $|U_{\xi\xi}(\xi,s)|\le C$ in $\mathcal{Q}'$. This implies U and $1+U_\xi^2$ are uniformly Hölder continuous functions on $\mathcal{Q}'$ as claimed. ☐

Interior parabolic regularity for (5.2.5) then implies that F, F_ξ, and $F_{\xi\xi}$ are uniformly bounded (and even Hölder continuous) on $\mathcal{Q}'$. We conclude that for some constant C that does not depend on (x_1, t_1) we have

$$|F_s(1,0)| \le C.$$

In terms of the original solution $u(x,t)$ this then implies

$$|u_t(x_1,t_1)| \le C x_1^{2k-4} \le C,$$

where we have used $k \ge 4$ and $x_1 \le \frac{1}{2}$ in the last step. We conclude that $|H(x_1,t_1)| \le |u_t(x_1,t_1)|$ is uniformly bounded whenever $Y_{\delta_0}\sqrt{t_1} \le x_1 \le \frac{1}{2}$ and $0 < t_1 < t_0$.

To deal with the case where $x_1 \ge \frac{1}{2}$ we recall that the single variable PDE (1.1.5a) for $u_n(x,t)$ can be interpreted as a parabolic equation in more variables (4.2.1) or (4.2.2). As discussed in Section 4.2, the uniform lower bound $u_n(x,t) \ge U^-_{\delta_0}(x,t)$ combined with the estimate $0 \le (u_n)_x(x,t) \le C_1$ allows one to invoke classical parabolic estimates which imply that $(u_n)_{xx}(x,t)$ is uniformly bounded for all $x \ge \frac{1}{2}$, sufficiently large n, and all $t \in (s_n, \bar{t}_n)$. This then implies that the mean curvature on the limiting solution is bounded when $x_1 \ge \frac{1}{2}$.

Combining the two cases $x_1 \in \left(0, \frac{1}{2}\right)$ and $x_1 \ge \frac{1}{2}$ leads to (5.2.1), which finishes the proof of the proposition. □

5.3. ***Second-order derivative bounds for $x \le M\sqrt{t}$.*** Before we bound $H(x,t)$ in the intermediate and inner regions, we will establish the following weighted C^2 bound, which is crucial for our purposes, for our approximating sequence of solutions $u_n(x,t)$ defined in Section 4.

Lemma 5.3.1. *There exists n_0 sufficiently large and a constant C independent of n so that for all $n \ge n_0$ the bound*

$$|(u_n)_{xx}(x,t)| \le C t^{-k/3}(1 + t^{-k/3}x)^{-4} \tag{5.3.1}$$

holds for all $0 \le x \le M\sqrt{t}$, $t \in [s_n, t_0]$.

Proof. The proof follows from scaling and standard regularity theory for linear and quasilinear parabolic equations. We repeatedly use the first-order derivative bound $0 \le (u_n)_x(x,t) \le C_1$ from Lemma 4.9.1, as well as the initial derivative bounds (4.3.3)

$$|\partial^j u_n(x,s_n)| \le C s_n^{-(j-1)k/3}(1 + s_n^{-k/3}x)^{-(j+2)}, \quad j = 2, 3,$$

which were shown in Lemma 4.3.1.

Since our solutions $u_n(x,t)$ scale differently in the intermediate and inner regions, we need to treat the cases $x \in [2Rt^{k/3}, Mt^{1/2}]$ and $x \in [0, 2Rt^{k/3}]$ separately. We will choose R in the proof of Case 1 below to be a sufficiently large constant which is independent of n. Then for this choice of R we will show that Case 2 holds. In both cases we will assume that $n \ge n_0$ and $s_n \le t \le t_0$, and n_0 will be chosen to be sufficiently large and t_0 will be chosen to be sufficiently small, uniformly in n.

We start by fixing $n \ge n_0$ and a point (x_1, t_1), where $0 \le x_1 \le M\sqrt{t_1}$, $t_1 \in [s_n, t_0]$.

Case 1: Assume $x_1 \in [2Rt_1^{k/3}, Mt_1^{1/2}]$, where R is a sufficiently large constant. Similarly to the proof of Lemma 5.2.1, we consider the rescaling

$$U_n(\xi, s) = x_1^{-1} u_n(x_1 \xi, t_1 + x_1^2 s)$$

which satisfies the equation

$$(U_n)_s = \frac{(U_n)_{\xi\xi}}{1 + U_{n\xi}^2} + \frac{3}{\xi}(U_n)_\xi - \frac{3}{U_n} \tag{5.3.2}$$

in the region

$$\mathcal{Q}_n = \left\{ (\xi, s) : \frac{1}{2} < \xi < \frac{3}{2}, \ -\frac{t_1 - s_n}{x_1^2} < s \le \frac{t_0 - t_1}{x_1^2} \right\}.$$

We subdivide into the *two cases*

$$\frac{t_1 - s_n}{x_1^2} > \frac{1}{2M^2} \quad \text{and} \quad \frac{t_1 - s_n}{x_1^2} \le \frac{1}{2M^2}.$$

Case 1a: If $(t_1 - s_n)/x_1^2 > 1/(2M^2)$, then the parabolic square

$$\mathcal{Q}'_M = \left\{ (\xi, s) : \frac{1}{2} < \xi < \frac{3}{2}, \ -\frac{1}{2M^2} < s \le 0 \right\}$$

has fixed size (independent of (x_1, t_1) and n) and satisfies $\mathcal{Q}'_M \subset \mathcal{Q}_n$. We will restrict to $\mathcal{Q}'_M$.

For any $(\xi, s) \in \mathcal{Q}'_M$ we have $x := x_1 \xi \in [Rt_1^{k/3}, 2Mt_1^{1/2}]$ and $t := t_1 + x_1^2 s \in \left[\frac{1}{2}t_1, t_1\right]$. In particular, we have $y := xt^{-1/2} \in [Rt_1^{\gamma}, 2\sqrt{2}M]$, i.e., (x, t) lies in the intermediate region, a fact that will be used momentarily.

To obtain the desired bound on $u_{xx}(x_1, t_1)$, we will bound $U_{\xi\xi}(1, 0)$ by applying interior parabolic regularity estimates to the function $U_n(\xi, s) - \xi$ defined in $\mathcal{Q}'_M$.

We first estimate the L^∞ norm of this function on $\mathcal{Q}'_M$ by bounding $|u_n(x, t) - x|$ for $x = x_1 \xi$ and $t = t_1 + x_1^2 s$, where $(\xi, s) \in \mathcal{Q}'_M$.

By (4.1.1) the solution u_n lies between our upper and lower barriers constructed in Proposition 3.9.1. Hence

$$|u_n(x, t) - x| \le \max\{|U_{\delta_0}^+(x, t) - x|, \ |U_{\delta_0}^-(x, t) - x|\} \tag{5.3.3}$$

for all $n \ge n_0$ sufficiently large. Using the definition of our barriers $U_{\delta_0}^\pm(x, t)$ (see (3.9.1) and (3.9.2)) the difference $|U_{\delta_0}^\pm(x, t) - x|$ for $n \ge n_0$ is bounded by $t^{1/2}|f_{\delta_{n_0}}^\pm(xt^{-1/2}, t)|$; $f_{\delta_{n_0}}^\pm$ was defined in (3.4.4). The latter can be bounded by $2K_1 t^{k-1} \varphi_k(xt^{-1/2})$, provided that t_0 is sufficiently small. This follows from the definition of $f_{\delta_{n_0}}^\pm$ and our estimates in Section 3.4, after expressing these estimates in the (x, t) variables using (2.2.1). Since $\varphi_k(y) \le C_k(y^{2k-2} + y^{-2})$ with $y := xt^{-1/2} \in [Rt_1^\gamma, 2\sqrt{2}M]$ and $t \in \left[\frac{1}{2}t_1, t_1\right]$, we get

$$\max\{|U_{\delta_0}^+(x, t) - x|, \ |U_{\delta_0}^-(x, t) - x|\} \le Ct^{k-1}(xt^{-1/2})^{-2} \le Cx_1^{-2}t^k \tag{5.3.4}$$

for some constant C (depending only on k and M) which is uniform in (x_1, t_1) and n. Combining (5.3.3) and (5.3.4) while using $t = t_1 + x_1^2 s \le t_1$ yields

$$|U_n(\xi, s) - \xi| \le Cx_1^{-3}t_1^k \quad \text{in } \mathcal{Q}'_M. \tag{5.3.5}$$

It follows that the function

$$F_n(\xi, s) := x_1^3 t_1^{-k}(U_n(\xi, s) - \xi)$$

which satisfies the equation

$$(F_n)_s = \frac{(F_n)_{\xi\xi}}{1 + U_{n\xi}^2} + \frac{3}{\xi}(F_n)_\xi + \frac{3}{\xi U_n(\xi, s)} F_n \tag{5.3.6}$$

is uniformly bounded in the parabolic cube $\mathcal{Q}'_M$, namely $\|F_n\|_{C^0(\mathcal{Q}'_M)} \le C$, where the constant C is independent of (x_1, t_1) and n.

Claim 5.3.2. *U_n and $1 + U_{n\xi}^2$ are Hölder continuous on the parabolic cube*

$$\mathcal{Q}''_M = \left\{(\xi, s) : \frac{1}{4} < \xi < \frac{5}{4}, -\frac{1}{4M^2} < s \le 0\right\} \subset \mathcal{Q}'_M,$$

uniformly in (x_1, t_1) and n. Furthermore, $\frac{1}{4} \le U_n(\xi, s) \le 2$ for all $(\xi, s) \in \mathcal{Q}'_M$.

Proof. Since $x_1 \ge R t_1^{k/3}$, by (5.3.5) we have that $|U_n(\xi, s) - \xi| \le C R^{-3}$, and since the constant C doesn't depend on R, we may choose R large enough that $\frac{1}{4} \le U_n(\xi, s) \le 2$ for all $(\xi, s) \in \mathcal{Q}'_M$. In addition Lemma 4.9.1 implies that $|U_{n\xi}(\xi, s)| = |(u_n)_x(x_1\xi, t_1 + x_1^2 s)| \le C_1$ in $\mathcal{Q}'_M$. It follows that $U_n(\xi, s)$ satisfies in $\mathcal{Q}'_M$ a uniformly parabolic equation (5.3.2) with bounded coefficients, and therefore standard interior (in spacetime) regularity theory applied to the quasilinear equation (5.3.2) implies the existence of a uniform constant C, independent of (x_1, t_1) and n, so that $|U_{n\xi\xi}(\xi, s)| \le C$ in $\mathcal{Q}''_M \subset \mathcal{Q}'_M$. All the above give us that U_n and $1 + U_{n\xi}^2$ are uniformly Hölder continuous functions on $\mathcal{Q}''_M$ as claimed. □

Claim 5.3.2 implies that equation (5.3.6) is uniformly parabolic in $\mathcal{Q}''_M$ and its coefficients are Hölder continuous (uniformly in (x_1, t_1) and n). Interior (in spacetime) Schauder theory applied to (5.3.6) in $\mathcal{Q}''_M$ bounds $|(F_n)_{\xi\xi}(1, 0)|$ in terms of $\|F_n\|_{C^0(\mathcal{Q}''_M)}$, concluding that $|(F_n)_{\xi\xi}(1, 0)| \le C$ for a uniform constant C. Equivalently, $|(U_n)_{\xi\xi}(1, 0)| \le C x_1^{-3} t_1^k$, and converting back to the original solution gives the bound $|(u_n)_{xx}(x_1, t_1)| \le C x_1^{-4} t_1^k$. In the considered region we have $x_1 t^{-k/3} \ge R$, therefore $t_1^k x_1^{-4} = t_1^{-k/3}(t_1^{-k/3} x_1)^{-4} \le C t_1^{-k/3}(1 + x_1 t_1^{-k/3})^{-4}$ (where C depends on R). We conclude that the desired bound (5.3.1) holds when $x_1 \in [2R t_1^{k/3}, M t_1^{1/2}]$ and $(t_1 - s_n)/x_1^2 > 1/(2M^2)$.

Case 1b: If $(t_1 - s_n)/x_1^2 \le 1/(2M^2)$, then $x_1 \le M t_1^{1/2}$ implies that $t_1 - s_n \le x_1^2/(2M^2) \le \frac{1}{2} t_1$, and hence in this case $t_1 \in [s_n, 2s_n]$. This in turn gives $x_1 \le M\sqrt{2s_n}$, implying in particular that

$$\frac{t_0 - t_1}{x_1^2} \ge \frac{t_0 - 2s_n}{2M^2 s_n} \ge 1,$$

provided that $n \ge n_0$ with n_0 sufficiently large. Hence the cube

$$\mathcal{Q}'_n = \left\{(\xi, s) : \frac{1}{2} < \xi < \frac{3}{2}, -\frac{t_1 - s_n}{x_1^2} < s \le -\frac{t_1 - s_n}{x_1^2} + 1\right\}$$

has fixed size and satisfies $\mathcal{Q}'_n \subset \mathcal{Q}_n$. The difference between this and the previous case is that the cube $\mathcal{Q}'_n$ starts at $s = -(t_1 - s_n)/x_1^2$ corresponding to the initial time $t = s_n$ for the solution $u_n(x, t)$. This means that our estimates need to include bounds on the initial data $u_n(x, s_n)$.

As in the previous case, we will begin by bounding $|U_n(\xi,s)-\xi|$ in $\mathcal{Q}'_n$. For any $(\xi,s)\in\mathcal{Q}'_n$ we have $x:=x_1\xi\in[Rt_1^{k/3},2M\sqrt{t_1}]\subset[Rs_n^{k/3},2M\sqrt{2s_n}]$ (using $t_1\in[s_n,2s_n]$) and $t:=t_1+x_1^2s\in[s_n,(2M^2+2)s_n]$ (using $x_1\le Mt_1^{1/2}$). Hence $y:=xt^{-1/2}\in[R/(\sqrt{2}M)s_n^{\gamma},2\sqrt{2}M]$, which shows that the point (x,t) belongs to the intermediate region. Now similar arguments as in Case 1a imply that bounds (5.3.3) and (5.3.4) hold (with s_n instead of t_1). We conclude that $|u_n(x,t)-x|\le Cx_1^{-2}s_n^{3\gamma+3/2}$ holds at $x=x_1\xi,\ t:=t_1+s\xi_1^2$ for any $(\xi,s)\in\mathcal{Q}'_n$, where C is independent of (x_1,t_1) and n. In terms of $U_n(\xi,s)$ we obtain

$$|U_n(\xi,s)-\xi|\le Cx_1^{-3}s_n^{3\gamma+3/2}\le Cx_1^{-3}t_1^k\quad\text{in }\mathcal{Q}'_n. \tag{5.3.7}$$

Claim 5.3.3. *U_n and $1+U_{n\xi}^2$ are Hölder continuous on the parabolic cube*

$$\mathcal{Q}''_n:=\left\{(\xi,s):\frac{3}{4}<\xi<\frac{5}{4},\ -\frac{t_1-s_n}{x_1^2}<s\le-\frac{t_1-s_n}{x_1^2}+1\right\}\subset\mathcal{Q}'_n,$$

uniformly in (x_1,t_1) and n. Furthermore, $\frac{1}{4}\le U_n(\xi,s)\le 2$ for all $(\xi,s)\in\mathcal{Q}'_n$.

Proof. Similarly to Claim 5.3.2, the bounds (5.3.7) and Lemma 4.9.1 imply that on $\mathcal{Q}'_n$ we have $\frac{1}{4}\le U_n\le 2$ and $0\le U_{n\xi}\le C_1$. In addition, for $j=2,3$ we have

$$\sup_{1/2\le\xi\le3/2}\left|\partial_\xi^jU_n\left(\xi,-\frac{t_1-s_n}{x_1^2}\right)\right|\le x_1^{j-1}\sup_{x_1/2\le x\le3x_1/2}|\partial_x^ju_n(x,s_n)|\le Cx_1^{-3}s_n^k\le C, \tag{5.3.8}$$

where we used (4.3.3) and our assumption $x_1\ge2Rt_1^{k/3}$ combined with $t_1\in[s_n,2s_n]$. In all the above bounds C is a uniform constant, independent of (x_1,t_1) and n. Since $U_n(\xi,s)$ satisfies a uniformly parabolic equation (5.3.2) in $\mathcal{Q}'_n$, standard interior (in space) theory for quasilinear equations applied to (5.3.2) yields the C^2 bound $\|U_{n\xi\xi}\|_{C^2(\mathcal{Q}''_n)}\le C$ (and even a $C^{2,1}$ bound), where C is a constant that depends only on

$$\|U_n\|_{C^0(\mathcal{Q}'_n)}\quad\text{and}\quad\left\|U_n\left(\cdot,-\frac{t_1-s_n}{x_1^2}\right)\right\|_{C^3([\xi/2,3\xi/2])},$$

therefore C is uniform in (x_1,t_1) and n, since these bounds are as well. We conclude that U_n and $1+U_{n\xi}^2$ are uniformly Hölder continuous functions on $\mathcal{Q}''_n$, finishing the proof of the claim. □

Consider the function $F_n(\xi,s):=x_1^3t_1^{-k}(U_n(\xi,s)-\xi)$ on $\mathcal{Q}''_n$ which satisfies (5.3.6) and the uniform bound $\|F_n\|_{C^0(\mathcal{Q}''_n)}\le C$, where C is independent of (x_1,t_1) and n. Claim 5.3.3 implies that $F_n(\xi,s)$ satisfies a uniformly parabolic equation (5.3.6) on $\mathcal{Q}''_n$ with coefficients which are uniformly Hölder continuous. Therefore, standard interior (in space) Schauder estimates applied to (5.3.2) on the cube $\mathcal{Q}''_n$ imply that $|(F_n)_{\xi\xi}(1,0)|$ can be bounded in terms of

$$\|F_n\|_{C^0(\mathcal{Q}''_n)}\quad\text{and}\quad\left\|F_n\left(\cdot,-\frac{t_1-s_n}{x_1^2}\right)\right\|_{C^{2,1}([3/4,5/4])}.$$

We have just seen that $\|F_n\|_{C^0(\mathcal{Q}''_n)}\le C$. We will next show the bound

$$\left\|F_n\left(\cdot,-\frac{t_1-s_n}{x_1^2}\right)\right\|_{C^3([3/4,5/4])}\le C.$$

First, (5.3.8) and the definition of F_n give

$$\left|\partial_\xi^j F_n\left(\xi, -\frac{t_1 - s_n}{x_1^2}\right)\right| = x_1^3 t_1^{-k} |\partial_\xi^j U_n(\xi, s)| \le C t_1^{-k} s_n^k \le C$$

for $j = 2, 3$ and all $\xi \in \left[\frac{3}{4}, \frac{5}{4}\right]$. The bound for $j = 1$ follows similarly from $0 \le (u_n)_x(x, s_n) \le C$. In all the above bounds C is independent of (x_1, t_1) and n.

We conclude that $|(F_n)_{\xi\xi}(1, 0)| \le C$, where C is independent of (x_1, t_1) and n, and, similarly to Case 1a, the desired bound (5.3.1) holds for $x_1 \in [2Rt_1^{k/3}, Mt_1^{1/2}]$ and $(t_1 - s_n)/x_1^2 \le 1/(2M^2)$. This completes the argument in Case 1b.

Case 2: Suppose next that $x_1 \in [0, Rt_1^{k/3}]$, that is (x_1, t_1) belongs to the tip region. Here R is a large fixed constant, chosen as in Case 1. In this case we will not scale around x_1 but around the origin, and we will show

$$\sup_{x \in [0, Rt_1^{k/3}]} |(u_n)_{xx}(x, t_1)| \le C t_1^{-k/3}, \quad 0 < t_1 \le t_0, \tag{5.3.9}$$

for a uniform constant C independent of n and t_1 (C may depend on R). This estimate is equivalent to (5.3.1) because in the considered region one has $x_1 t_1^{-k/3} \le R$.

To this end we set $\alpha := \frac{1}{3}k \ge 1$ for simplicity and introduce the rescaled function

$$U_n(\xi, s) = t_1^{-\alpha} u_n(t_1^\alpha \xi, t_1 + t_1^{2\alpha} s), \tag{5.3.10}$$

which satisfies (5.3.2) in the region

$$\mathcal{Q}_n = \left\{(\xi, s) : 0 \le \xi \le 2R,\ -\frac{t_1 - s_n}{t_1^{2\alpha}} < s \le \frac{t_0 - t_1}{t_1^{2\alpha}}\right\}.$$

Bound (5.3.9) is equivalent to

$$\sup_{\xi \in [0, R]} |(U_n)_{\xi\xi}(\xi, 0)| \le C \tag{5.3.11}$$

and will follow by applying standard regularity theory to (5.3.2) in an appropriate cube $\mathcal{Q}_n' \subset \mathcal{Q}_n$.

First, one needs to bound U_n on $\mathcal{Q}_n'$ from above and below away from zero. To this end, observe that (4.1.5), (3.9.1)–(3.9.2) and the definition of the inner region barriers in Section 3.5 give

$$t^\alpha W_{K_2^-(n_0)}(xt^{-\alpha}) + De^{2\gamma \log t} \le u_n(x, t) \le t^\alpha W_{K_2^-(n_0)}(xt^{-\alpha}) \tag{5.3.12}$$

for all $n \ge n_0$ sufficiently large and all $x \in [0, Zt^\alpha]$ (for any $Z > 0$) and $t \le t_0$. Here $D > 0$, and thus we can drop the small term $De^{2\gamma \log t}$. The above estimate when expressed in terms of $U_n(\xi, s)$ gives

$$\vartheta_n(s) W_{K_2^-(n_0)}\left(\frac{\xi}{\vartheta_n(s)}\right) \le U_n(\xi, s) \le \vartheta(s) W_{K_2^+(n_0)}\left(\frac{\xi}{\vartheta_n(s)}\right), \tag{5.3.13}$$

where $\vartheta_n(s) := t^\alpha t_1^{-\alpha} = (1 + t_1^{2\alpha - 1} s)^\alpha$. Note that in order to obtain (5.3.13) from (5.3.12) we need to have $\xi/\vartheta_n(s) \le Z$ for all $(\xi, s) \in \mathcal{Q}_n'$ and for some $Z > 0$ which is independent of $(\xi, s) \in \mathcal{Q}_n'$. This will be checked below. We need to consider two cases, $(t_1 - s_n)t_1^{-2\alpha} > 1$ and $(t_1 - s_n)t_1^{-2\alpha} \le 1$, and choose $\mathcal{Q}_n'$ appropriately.

Case 2a: If $(t_1 - s_n)t_1^{-2\alpha} > 1$, then we restrict to the parabolic cube of fixed size

$$\mathcal{Q}' = \{(\xi, s) : 0 \le \xi \le 2R,\ -1 < s \le 0\}$$

(independent of t_1 and n), which obviously satisfies $\mathcal{Q}' \subset \mathcal{Q}_n$. We will restrict to $\mathcal{Q}'$, where $s \in (-1, 0]$ implies the bounds $\vartheta_n(s) \ge (1 - t_1^{2\alpha-1})^\alpha \ge \frac{1}{2}$ and $\vartheta_n(s) \le 1$ (for the former use $t_1 \le t_0$, where t_0 can be chosen sufficiently small).

Using $\xi\vartheta_n^{-1} \le 4R$ and $\frac{1}{2} \le \vartheta_n \le 1$, we conclude from (5.3.13) that there exists a uniform in n and t_1 constant $C > 0$ (depending on $\inf_{z\in[0,4R]} W_{K_2^-(n)}(z)$ and $\sup_{z\in[0,4R]} W_{K_2^+(n)}(z)$) such that

$$0 < C^{-1} \le U_n(\xi, s) \le C \quad \text{for all } (\xi, s) \in \mathcal{Q}'. \tag{5.3.14}$$

Furthermore, arguing as in (5.2.6), we note that the definition (5.3.10) of U_n implies

$$U_{n\xi}(\xi, s) = u_{nx}(t_1^\alpha, t_1 + t_1^{2\alpha} s),$$

so that $\|U_{n\xi}\|_{C^0(\mathcal{Q}')} \le \sup |u_{nx}| \le C$, where C does not depend on n or t_1. Standard interior (in spacetime) regularity theory applied to (5.3.2) implies that there exists a uniform constant C, independent of n and t_1, so that $\sup_{\xi\in[0,R]} |(U_n)_{\xi\xi}(\xi, 0)| \le C$; that is (5.3.11) holds. In terms of the original solution $u_n(x, t)$ this implies the desired bound (5.3.9) in the case $(t_1 - s_n)t_1^{-2\alpha} > 1$, with $\alpha = \frac{1}{3}k$.

Case 2b: Finally, if $(t_1 - s_n)t_1^{-2\alpha} \le 1$, then since $t_1 \le t_0$ is small and $\alpha \ge 1$, we have $t_1 \le s_n + t_1^{2\alpha} \le s_n + \frac{1}{2}t_1$; that is $t_1 \in [s_n, 2s_n]$. In this case we restrict to the parabolic cube of fixed size

$$\mathcal{Q}_n' = \left\{(\xi, s) : 0 \le \xi \le 2R,\ -\frac{t_1 - s_n}{t_1^{2\alpha}} < s \le -\frac{t_1 - s_n}{t_1^{2\alpha}} + 1\right\},$$

which contains the point $(1, 0)$ and satisfies $\mathcal{Q}_n' \subset \mathcal{Q}_n$. Since $0 < (t_1 - s_n)/t_1^{2\alpha} \le 1$, for any $(\xi, s) \in \mathcal{Q}_n'$ we have $s \in [-1, 1]$; thus $\vartheta_n := (1 + t_1^{2\alpha-1}s)^\alpha$ satisfies the bounds $\frac{1}{2} \le \vartheta_n(s) \le \frac{3}{2}$ for all $t_1 \le t_0$ with t_0 sufficiently small.

Claim 5.3.4. *The bounds* $0 < C^{-1} \le U_n(\xi, s) \le C$ *and* $|(U_n)_\xi(\xi, s)| \le C$ *hold on* $\mathcal{Q}_n'$. *Furthermore,*

$$\left\| U_n\left(\,\cdot\,, -\frac{t_1 - s_n}{t_1^{2\alpha}}\right)\right\|_{C^3([0,2R])} \le C.$$

In all these bounds C is a uniform constant independent of n and t_1.

Proof. Since $\frac{1}{2} \le \vartheta_n(s) \le \frac{3}{2}$, similarly to Case 2a we can apply (5.3.13) to obtain $0 < C^{-1} \le U_n(\xi, s) \le C$ in $\mathcal{Q}_n'$. Also, similarly to the previous cases, $0 \le (U_n)_\xi(\xi, s) \le C_1$ in $\mathcal{Q}_n'$ follows from Lemma 4.9.1. For the third bound it is sufficient to just estimate second- and third-order derivatives. To this end we use (4.3.3) which implies that $|\partial_x^j u_n(x, s_n)| \le C s_n^{-(j-1)k/3}$ for $j = 2, 3$ and for all $x \in [0, 2Rt_1^{k/3}]$ (recall that $t_1 \sim s_n$).

In terms of U_n we get $|\partial_\xi^j U_n(\xi, -(t_1 - s_n)/t_1^{2\alpha})| \le C$ for $j = 2, 3$ and for all $\xi \in [0, 2R]$. The above bounds imply that $\|U_n(\,\cdot\,, -(t_1 - s_n)/t_1^{2\alpha})\|_{C^3([0,2R])} \le C$. In all these bounds the constant C is uniform, independent of n and t_1. □

The previous claim and standard interior (in space) regularity theory applied to (5.3.2) on the cube $\mathcal{Q}_n'$ implies that $\sup_{0\le\xi\le R}|(U_n)_{\xi\xi}(\xi,0)|$ (even $\|U_n(\,\cdot\,,0)\|_{C^{2,1}([0,R])}$) can be bounded in terms of $\|U_n\|_{C^0(\mathcal{Q}_n')}$ and $\|U_n(\,\cdot\,,-(t_1-s_n)/t_1^{2\alpha})\|_{C^3([0,2R])}$, and thus both are bounded by a constant C which is uniform in t_1 and n. We conclude that (5.3.11) holds, which expressed in terms of $u_n(x,t)$ gives that (5.3.9) holds in the last case where $(t_1-s_n)t_1^{-2\alpha}>1$, with $\alpha=\frac{k}{3}$.

Combining Cases 1a–1b and Cases 2a–2b concludes the proof that the desired bound (5.3.1) holds for all (x,t) satisfying $0\le x\le M\sqrt{t},\ t\in[s_n,t_0]$ and all $n\ge n_0$, provided n_0 is sufficiently large and $t_0>0$ is sufficiently small. □

5.4. ***Bounding H in the intermediate and inner regions.*** We will now show that $H(x,t)$ is bounded in the region $x\le M\sqrt{t},\ 0<t\le t_0$. Instead of showing that H is bounded, we will prove that

$$h(x,t):=u_t=H\sqrt{1+u_x^2}$$

is bounded. Since u_x is uniformly bounded (Lemma 4.9.1), the bounds for h and H are equivalent. Arguments in this section have been inspired by arguments from [Stolarski 2023].

The PDE for u implies that $h=u_t$ satisfies

$$h_t=\frac{\partial}{\partial x}\left(\frac{h_x}{1+u_x^2}\right)+\frac{3}{x}h_x+\frac{3}{u^2}h.$$

For $n\ge n_0$, define $h_n(x,t):=\partial_t u_n(x,t)$, where $u_n:[0,\infty)\times[s_n,t_0]\to\mathbb{R}$ is our approximating sequence of solutions from the proof of Theorem 4.1.1 in Section 4. We choose a fixed $m\in(2,3)$ and set

$$\Lambda_n=\max\{(1+t^{-k/3}x)^m|h_n(x,t)|:0\le x\le M\sqrt{t},\ t\in[s_n,t_0]\}.$$

We claim the following holds.

Lemma 5.4.1. *We have* $\sup_n\Lambda_n<\infty$.

This lemma implies that $|h_n(x,t)|$ is uniformly bounded and hence that $H_n=h_n/\sqrt{1+u_x^2}$ is also uniformly bounded. Since the bound is uniform in n, by passing to the limit as $n\to+\infty$ we will then obtain that the mean curvature $H(x,t)$ of our solution is bounded for $0\le x\le M\sqrt{t},\ 0\le t\le t_0$.

5.5. ***Choice of the blow-up sequences.*** For the proof of Lemma 5.4.1 we argue by contradiction and assume that $\sup_n\Lambda_n=\infty$. Then we can pass to a subsequence so that we may assume without loss of generality that

$$\lim_{n\to\infty}\Lambda_n=+\infty. \tag{5.5.1}$$

Our goal in this section is *to contradict* (5.5.1).

The bound (5.3.1) for u_n implies the same bound for h_n; namely, we have

$$|h_n(x,t)|\lesssim t^{-k/3}(1+t^{-k/3}x)^{-4},\quad x\le M\sqrt{t},\quad t\in[s_n,t_0]. \tag{5.5.2}$$

The quantity $(1+t^{-k/3}x)^m|h_n(x,t)|$ attains its maximum in the region $\{(x,t)\mid 0\le x\le M\sqrt{t},\ s_n\le t\le t_0\}$, so we can choose $T_n\in[s_n,t_0]$ and $a_n\in[0,M\sqrt{T_n}]$ such that

$$|h(a_n,T_n)|=\Lambda_n(1+T_n^{-k/3}a_n)^{-m}. \tag{5.5.3}$$

The inequality (5.5.2) implies

$$T_n^{k/3}(1+T_n^{-k/3}a_n)^{4-m}\lesssim\Lambda_n^{-1},$$

and thus

$$\max\{T_n^{k/3},\,T_n^{(m-3)k/3}a_n^{4-m}\}\lesssim\Lambda_n^{-1}.$$

Since $\Lambda_n\to\infty$, we find that $T_n\to 0$ and also

$$a_n\ll T_n^{\frac{3-m}{4-m}\frac{k}{3}}.$$

At this point we use our assumption that $k>3$ and choose m close enough to $m=2$ that the exponent of T_n satisfies $\frac{3-m}{4-m}\frac{k}{3}>\frac{1}{2}$, which then implies

$$a_n\ll T_n^{1/2}. \tag{5.5.4}$$

To complete the proof we distinguish between two cases $a_n\lesssim T_n^{k/3}$ and $T_n^{k/3}\ll a_n\ll T_n^{1/2}$, depending on where the maximum a_n is attained.

5.6. ***Case 1*****:** $\boldsymbol{a_n\lesssim T_n^{k/3}}$**.** We choose the scale $\alpha_n=T_n^{k/3}$ and form the blow-up sequences

$$\bar u_n(\xi,s)=\alpha_n^{-1}u_n(\xi\alpha_n,T_n+s\alpha_n^2), \tag{5.6.1}$$

$$\bar h_n(\xi,s)=\Lambda_n^{-1}h_n(\xi\alpha_n,T_n+s\alpha_n^2). \tag{5.6.2}$$

These functions are defined for

$$\xi>0\quad\text{and}\quad -S_n\le s\le 0,\qquad \text{where } S_n=\frac{T_n-s_n}{\alpha_n^2},$$

and they satisfy the equations

$$\frac{\partial\bar u_n}{\partial s}=\frac{\bar u_{n\xi\xi}}{1+\bar u_{n\xi}^2}+\frac{3}{\xi}\bar u_{n\xi}-\frac{3}{\bar u_n}, \tag{5.6.3}$$

$$\frac{\partial\bar h_n}{\partial s}=\frac{\partial}{\partial\xi}\left(\frac{\bar h_{n\xi}}{1+\bar u_{n\xi}^2}\right)+\frac{3}{\xi}\bar h_{n\xi}+\frac{3}{\bar u_n^2}\bar h_n. \tag{5.6.4}$$

We use (5.6.1) with $\alpha_n=T_n^{k/3}$ and the definition of the inner region rescaling $w_n(z,\tau)$ of $u_n(x,t)$, i.e.,

$$u_n(x,t)=t^{k/3}w_n(t^{-k/3}x,\log t),$$

with $t=T_n+T_n^{2k/3}s$ to express $\bar u_n(\xi,s)$ in terms of $w_n(z,\tau)$. We get

$$\bar u_n(\xi,s)=\vartheta_n(s)w_n\left(\frac{\xi}{\vartheta_n(s)},\log t\right),$$

where

$$\vartheta_n(s):=t^{k/3}T_n^{-k/3}=(T_n+T_n^{2k/3}s)^{k/3}T_n^{-k/3}=(1+T_n^{2k/3-1}s)^{k/3}.$$

Since $T_n \to 0$, we have $\vartheta_n(s) \to 1$ uniformly for bounded s, and thus

$$\log t = \log T_n + \frac{3}{k} \log \vartheta_n(s) \to -\infty,$$

uniformly for bounded s. Similarly to the last statement of Theorem 4.1.1 we claim the following.

Claim 5.6.1. *$\bar{u}_n(\xi, s) \to W_{K_2}(\xi)$ in C^∞_{loc}.*

Proof. For every fixed $\xi > 0$ there exists an n_0 such that for all $n \geq n_0$ we have

$$\vartheta_n w_n^- \biggl(\frac{\xi}{\vartheta_n(s)}, \log t \biggr) \leq \bar{u}_n(\xi, s) \leq \vartheta_n w_n^+ \biggl(\frac{\xi}{\vartheta_n(s)}, \log t \biggr),$$

where $\log t = \log T_n + \frac{3}{k} \log \vartheta_n(s)$ and w_n^- and w_n^+ are the lower and the upper barriers in the inner region, respectively. See Lemmas 3.5.2 and 3.5.3. This implies

$$\vartheta_n W_{K_2^-(n)} \biggl(\frac{\xi}{\vartheta_n} \biggr) + D(T_n \vartheta_n^{3/k})^{2\gamma} \leq \bar{u}_n(\xi, s) \leq \vartheta_n W_{K_2^+(n)} \biggl(\frac{\xi}{\vartheta_n} \biggr),$$

where we recall that $(K_2^\pm(n))^3 = K_2^3 \pm \delta_n$. Since $\lim_{n\to\infty} T_n = 0$, $\lim_{n\to\infty} \vartheta_n = 1$, and $\lim_{n\to\infty} K_2^\pm(n) = K_2$, we conclude that $\bar{u}_n(\xi, s) \to W_{K_2}(\xi)$ uniformly for bounded $\xi \geq 0$ and bounded s.

Furthermore, since $(\bar{u}_n)_{\xi\xi}(\xi, s) = \vartheta_n(s)^{-1}(w_n)_{zz}(z, \tau)$ is uniformly bounded for bounded ξ and s, it follows that $\bar{u}_{n\xi}$ also converges locally uniformly. After bootstrapping the nondegenerate parabolic equation (5.6.3) for $\bar{u}_n$ we find that $\bar{u}_n(\xi, s) \to W_{K_2}(\xi)$ in C^∞_{loc}. □

Recall next that by the definition of Λ_n we have

$$|\bar{h}_n(\xi, s)| \leq (1 + T_n^{k/3} \xi (T_n + T_n^{2k/3} s)^{-k/3})^{-m} = (1 + \xi (1 + T_n^{2k/3-1} s)^{-k/3})^{-m}.$$

For $s \leq 0$ and $\xi > 0$ this implies

$$|\bar{h}_n(\xi, s)| \leq \frac{1}{(1+\xi)^m}.$$

Lemma 5.6.2. *Let $\Phi(\xi) = W(\xi) - \xi W'(\xi)$, where $W(\xi)$ is a solution to (2.3.4). Then for any $S_* > 0$ there is a $\kappa_* > 0$ such that $e^{\kappa_* s}\Phi(\xi)$ is a supersolution for (5.6.4) in the region $-\min\{S_n, S_*\} \leq s \leq 0$, $0 < \xi < \frac{1}{2} M T_n^{1/2-k/3}$, where $S_n = (T_n - s_n)/\alpha_n^2$.*

Proof. Expanding the derivative in (5.6.4) leads to

$$\bar{h}_{ns} = \mathcal{M}_n(\bar{h}_n) := \frac{\bar{h}_{n\xi\xi}}{1 + \bar{u}_{n\xi}^2} + \biggl\{ \frac{3}{\xi} - \frac{2\bar{u}_{n\xi}\bar{u}_{n\xi\xi}}{(1 + \bar{u}_{n\xi}^2)^2} \biggr\} \bar{h}_{n\xi} + \frac{3}{\bar{u}_n^2} \bar{h}_n. \tag{5.6.5}$$

In order to estimate $\mathcal{M}_n[\Phi]$ we write the right-hand side as

$$\mathcal{M}_n[\bar{h}_n] = \mathcal{M}_\infty[\bar{h}_n] + \mathcal{R}_n[\bar{h}_n]$$

with

$$\mathcal{M}_\infty[\eta] := \frac{\eta_{\xi\xi}}{1 + W'(\xi)^2} + \biggl\{ \frac{3}{\xi} - \frac{2W'(\xi)W''(\xi)}{(1 + W'(\xi)^2)^2} \biggr\} \eta_\xi + \frac{3}{W(\xi)^2} \eta \tag{5.6.6}$$

and where the remainder is given by

$$\mathcal{R}_n[\eta] = a_n(\xi, s)\eta_{\xi\xi} + b_n(\xi, s)\eta_\xi + c_n(\xi, s)\eta$$

for certain coefficients a_n, b_n, c_n which one obtains by subtracting (5.6.6) and (5.6.5).

We now argue that a_n, b_n, c_n are uniformly bounded if s remains bounded. The estimate (5.3.1) says $|u_{nxx}| \lesssim t^{-k/3}(1 + t^{-k/3}x)^{-4}$ for small $t > 0$ and for $0 < x < M\sqrt{t}$. By definition (5.6.1) of $\bar{u}_n$ this implies that for $\xi < MT_n^{1/2-k/3}\sqrt{1 + T_n^{2k/3-1}s}$ one has

$$\begin{aligned} |\bar{u}_{n\xi\xi}(\xi, s)| &= T_n^{k/3}|u_{nxx}(T_n^{k/3}\xi, T_n + T_n^{2k/3}s)| \\ &\lesssim (1 + T_n^{2k/3-1}s)^{-k/3}(1 + (1 + T_n^{2k/3-1}s)^{-k/3}\xi)^{-4}. \end{aligned}$$

Under the assumption that $-\min\{S_n, S_*\} \le s \le 0$, and because $T_n \to 0$, we may conclude

$$|\bar{u}_{n\xi\xi}| \lesssim (1 + \xi)^{-4} \quad \text{for } -\min\{S_n, S_*\} \le s \le 0, \;\; 0 < \xi < \tfrac{1}{2}MT_n^{1/2-k/3}.$$

This implies that $\bar{u}_{n\xi\xi}$ is uniformly bounded for $0 < \xi < \frac{1}{2}MT_n^{1/2-k/3}$.

Since $\bar{u}_{n\xi}(\xi, s) = u_{nx}(T_n^{k/3}\xi, T_n + T_n^{2k/3}s)$, we have $0 \le \bar{u}_{n\xi} \le C_1$ (by Lemma 4.9.1), so that $\bar{u}_{n\xi}$ is uniformly bounded.

Finally, by comparing $\bar{u}_n$ with the lower barrier $U_{\delta_0}^-$ from (3.9.2) we have

$$\begin{aligned} \bar{u}_n(\xi, s) \ge \bar{u}_n(0, s) &= T_n^{-k/3}u_n(0, T_n + T_n^{2k/3}s) \\ &\ge T_n^{-k/3}U_{\delta_0}^-(0, T_n + T_n^{2k/3}s) && \text{use (3.9.2)} \\ &= T_n^{-k/3}(T_n + T_n^{2k/3}s)^{k/3}w_{\delta_0}^-(0, \log(T_n + T_n^{2k/3}s)) \\ &\ge (1 + T_n^{2k/3-1}s)^{k/3}W_{K_2/2}(0). \end{aligned}$$

The assumption $-\min\{S_n, S_*\} \le s \le 0$ together with $T_n \to 0$ and $W_K(0) = KW(0) = K$ then implies

$$\bar{u}_n(\xi, s) \ge \tfrac{1}{4}K_2 \quad \text{for all } \xi > 0 \text{ and } -\min\{S_n, S_*\} \le s \le 0.$$

This implies that $\bar{u}_n(\xi, s)^{-2}$ is uniformly bounded, and hence that the coefficients a_n, b_n, c_n are uniformly bounded.

Therefore there is a $\kappa > 0$ such that

$$|\mathcal{R}_n[\eta]| \le \kappa(|\eta_{\xi\xi}| + |\eta_\xi| + |\eta|)$$

whenever $-\min\{S_n, S_*\} \le s \le 0$ and $0 \le \xi \le \frac{1}{2}MT_n^{1/2-k/3}$. Since

$$\mathcal{M}_\infty[\Phi] = 0 \quad \text{and} \quad |\Phi''(x)| + |\Phi'(x)| \lesssim \Phi(x),$$

we find that

$$\mathcal{M}_n[\Phi] \le C\kappa\Phi.$$

If we define $\kappa_* = C\kappa$, then we have found that $e^{\kappa s}\Phi(x)$ is an upper barrier for $\bar{h}_{ns} = \mathcal{M}_n[\bar{h}_n]$. □

Lemma 5.6.3. $S_n \to \infty$.

Proof. We argue by contradiction. Assume that there is a subsequence of S_n along which the limit is finite. Without loss of generality we can take this to be S_n itself, that is assume that

$$S_n = \frac{T_n - s_n}{\alpha_n^2} \le \bar{S} < +\infty \quad \text{for all } n.$$

This implies that $T_n \le s_n + \bar{S}\alpha_n^2 = s_n + \bar{S}T_n^{2k/3}$. Since $T_n \to 0$ and $k > 3$, we then conclude that $T_n \le 2s_n$ for $n \gg 1$.

We will now apply the maximum principle to $\bar{h}_n$ in the region

$$-S_n \le s \le 0, \quad 0 \le \xi \le \epsilon T_n^{-(k/3-1/2)}.$$

Observe first that the definition (4.3.1) of our initial data $u_n(x, s_n)$ is such that the surface coincides with an Alencar surface in the region $y = o(1)$, i.e., for $x \le \epsilon\sqrt{s_n}$ with ϵ as in Section 4.3. This implies that $h_n(x, s_n) = 0$ for $x \le \epsilon\sqrt{s_n}$. Using $T_n \le 2s_n$ for $n \gg 1$, we conclude that by taking $n \gg 1$ and ϵ sufficiently small we can guarantee that $\bar{h}_n(\xi, -S_n) = \Lambda_n^{-1} h_n(\alpha_n\xi, s_n) = 0$ for $\xi \le \epsilon\alpha_n^{-1}T_n^{1/2} = \epsilon T_n^{-(k/3-1/2)}$. At the end of this region, where $\xi = \epsilon T_n^{1/2-k/3}$, we have

$$|\bar{h}_n(\xi, s)| \le (1+\xi)^{-m} = (1+\xi)^{-2}(1+\xi)^{-(m-2)} \lesssim T_n^{(m-2)(k/3-1/2)}\Phi(\xi).$$

Here we have used the expansion (A.3.1) combined with the fact that $\Phi(\xi) = W(\xi) - \xi W'(\xi) > 0$ (Lemma A.3.1) to conclude that $\Phi(\xi) \gtrsim (1+\xi)^{-2}$. Choosing κ_* as in Lemma 5.6.2, we see that for suitably large $\widetilde{C}$ the function

$$\widetilde{C}T_n^{(m-2)(k/3-1/2)}e^{\kappa_* s}\Phi(\xi)$$

is an upper bound for both $\bar{h}_n(\xi, s)$ and $-\bar{h}_n(\xi, s)$ in the region $-S_n \le s \le 0$, $\xi \le \epsilon T_n^{1/2-k/3}$ and for all n.

Finally, at $s = 0$ this implies

$$|\bar{h}_n(\xi, 0)| \lesssim T_n^{(m-2)(k/3-1/2)} \to 0 \quad \text{as } n \to \infty.$$

This cannot be, because $\max_\xi |\bar{h}_n(\xi, 0)| = 1$, thus showing that $S_n \to \infty$. ☐

We can now complete the blow up argument, at least in the case where $a_n \lesssim T_n^{k/3}$. Since $S_n \to \infty$, we can pass to another subsequence along which $\bar{h}_n$ converges in C^∞_{loc} to an ancient solution $\bar{h}$ of

$$\bar{h}_s = \frac{\partial}{\partial\xi}\left(\frac{\bar{h}_\xi}{1+W'(\xi)^2}\right) + \frac{3}{\xi}\bar{h}_\xi + \frac{3}{W(\xi)^2}\bar{h}. \tag{5.6.7}$$

The ancient solution $\bar{h}$ satisfies the bound

$$|\bar{h}(\xi, s)| \le (1+\xi)^{-m}, \quad \xi \ge 0, \ s \le 0.$$

By the definition of a_n (see (5.5.3)) the function $(1+\xi)^m|\bar{h}_n(\xi, s)|$ attains its maximum at $\xi_n = a_n T_n^{-k/3}$. We assumed here that $a_n \lesssim T_n^{k/3}$, so we may assume also that $\xi_n \to \bar{\xi}$ for some finite $\bar{\xi} \ge 0$. Thus we have

$$\bar{h}(\bar{\xi}, 0) = (1+\bar{\xi})^{-m}. \tag{5.6.8}$$

To complete the proof we compare this ancient solution with the stationary solution $\Phi(\xi)=W(\xi)-\xi W'(\xi)$. By the asymptotic expansion of the Alencar solution we have

$$\Phi(\xi)=(\Gamma_1+o(1))\xi^{-2}, \quad \xi\to\infty,$$

for some constant $\Gamma_1>0$.

Choose a large number $\ell>0$ and consider the function

$$\Psi(\xi)=\Phi(\xi)-\tfrac{1}{2}\Phi(\ell).$$

Since $\Phi(\xi)$ is a decreasing function of ξ, we have

$$\tfrac{1}{2}\Phi(\xi)\le\Psi(\xi)\le\Phi(\xi) \quad \text{for all } \xi\in[0,\ell].$$

Furthermore, it follows from $\mathcal{M}_\infty[\Phi]=0$ that

$$\mathcal{M}_\infty[\Psi](\xi)=-\frac{3\Phi(\ell)}{2W(\xi)^2}.$$

Since $W(\xi)=\xi+o(1)$ and $\Phi(\xi)\sim\xi^{-2}$ for large ξ, there is a $c>0$ such that $W(\xi)^{-2}\ge c\Phi(\xi)\ge c\Psi(\xi)$. There is also a constant $c>0$ with $\Phi(\ell)\ge c\ell^{-2}$. Therefore we get

$$\mathcal{M}_\infty[\Psi]\le -c\ell^{-2}\Psi(\xi) \quad \text{for } \xi\in[0,\ell].$$

It follows that for any s_0

$$\hat{h}(\xi,s)=e^{-c\ell^{-2}(s+s_0)}\Psi(\xi)$$

satisfies $\hat{h}_s\ge\mathcal{M}[\hat{h}]$ for $\xi\in[0,\ell]$.

We will next compare $\bar{h}$ with $\hat{h}$ in the domain $\{0<\xi<\ell,\ -s_0<s<0\}$ which will lead to a contradiction. At $\xi=\ell$ we have

$$\frac{|\bar{h}(\ell,s)|}{\hat{h}(\ell,s)}\le\frac{(1+\ell)^{-m}}{\Psi(\ell)}e^{c\ell^{-2}(s+s_0)}.$$

Using

$$\Psi(\ell)\ge\frac{1}{2}\Phi(\ell)\ge\frac{1}{C}(1+\ell)^{-2}$$

we therefore find for $-s_0\le s\le 0$

$$\frac{|\bar{h}(\ell,s)|}{\hat{h}(\ell,s)}\le C(1+\ell)^{-(m-2)}e^{c\ell^{-2}(s+s_0)}\le C(1+\ell)^{-(m-2)}e^{c\ell^{-2}s_0}.$$

Since $\Psi(\xi)\ge c(1+\xi)^{-2}$ for a uniform c, at time $-s_0$ we have

$$\frac{|\bar{h}(\xi,-s_0)|}{\hat{h}(\xi,-s_0)}\le\frac{(1+\xi)^{-m}}{\Psi(\xi)}\le C(1+\xi)^{-(m-2)}\le C.$$

To conclude our argument, for any given $\ell>0$ we choose $s_0>0$ large enough that

$$C(1+\ell)^{-(m-2)}e^{c\ell^{-2}s_0}>1.$$

Applying the maximum principle to the linear equation $h_s = \mathcal{M}_\infty[h]$ on the domain $\{0 < \xi < \ell, -s_0 < s < 0\}$, we have

$$\frac{|\bar{h}(\xi, s)|}{\hat{h}(\xi, s)} \le C(1+\ell)^{-(m-2)} e^{c\ell^{-2} s_0} \quad \text{for } 0 \le \xi \le \ell, \quad -s_0 \le s \le 0.$$

In particular,

$$\frac{|\bar{h}(\xi, 0)|}{\hat{h}(\xi, 0)} \le C(1+\ell)^{-(m-2)} e^{c\ell^{-2} s_0} \quad \text{for } 0 \le \xi \le \ell,$$

and hence, using the definition of $\hat{h}$,

$$|\bar{h}(\xi, 0)| \le C(1+\ell)^{-m} \Psi(\xi) \quad \text{for } 0 \le \xi \le \ell.$$

The constant C does not depend on ℓ, so by choosing ℓ large enough we reach a contradiction if $\bar{h}(\xi, 0) \ne 0$ for some $\xi \ge 0$ since (5.6.8) needs to hold at the same time as well.

This completes the proof of Lemma 5.4.1 in the case $a_n \lesssim T_n^{-k/3}$.

5.7. ***Case 2:*** $a_n \gg T_n^{-k/3}$**.** If we are not in Case 1, i.e., if it is not true that $a_n \lesssim T_n^{-k/3}$, then there is a subsequence along which $a_n T_n^{k/3} \to \infty$. In this case we choose our scale to be $\alpha_n = a_n$, and we define the blow-ups

$$\bar{u}_n(\xi, s) = a_n^{-1} u_n(a_n \xi, T_n + a_n^2 s), \quad \bar{h}_n(\xi, s) = \frac{h_n(a_n \xi, T_n + a_n^2 s)}{h_n(a_n, T_n)}. \tag{5.7.1}$$

These blow-ups are defined for all $\xi \ge 0$ and for

$$-S_n \le s \le 0, \quad \text{with } S_n = \frac{T_n - s_n}{a_n^2}.$$

By our intermediate region asymptotics for u_n^- and u_n^+, since $e^{(\gamma+1/2)\tau} \ll a_n \ll T_n^{1/2}$ (see (5.5.4)) and $u_n^-(x, s) \le u_n(x, s) \le u_n^+(x, s)$, we have

$$\bar{u}_n(\xi, s) \to \bar{u}_\infty(\xi) = \xi$$

uniformly for bounded $\xi \ge 0$ and s and in C^∞_{loc} for $\xi > 0$ and $s \le 0$.

Lemma 5.7.1. *For* $\bar{h}_n(\xi, s)$ *we have the pointwise bound*

$$|\bar{h}_n(\xi, s)| \le \left(1 + \frac{T_n^{k/3}}{a_n}\right)\left(1 + \frac{a_n^2 s}{T_n}\right)^{km/3} \xi^{-m} \tag{5.7.2}$$

for all ξ *with* $0 < a_n \xi \le M\sqrt{T_n + a_n^2 s}$. *In particular, for large enough* n *we also have*

$$|\bar{h}_n(\xi, s)| \le 2\xi^{-m} \tag{5.7.3}$$

for all ξ *with* $0 < a_n \xi \le M\sqrt{T_n + a_n^2 s}$ *and for bounded* s.

Proof. By definition of Λ_n, a_n, and T_n we have for all $x \le M\sqrt{t}$ and $t \in [s_n, t_0]$

$$|h_n(x, t)| \le \Lambda_n (1 + t^{-k/3} x)^{-m}, \quad |h_n(a_n, T_n)| = \Lambda_n (1 + T_n^{-k/3} a_n)^{-m}.$$

Hence

$$\left|\frac{h_n(a_n\xi, T_n + a_n^2 s)}{h_n(a_n, T_n)}\right| \le \left\{\frac{1 + T_n^{-k/3}a_n}{1 + (T_n + a_n^2 s)^{-k/3}a_n\xi}\right\}^m.$$

Discarding the "+1" in the denominator and multiplying numerator and denominator with $T_n^{k/3}a_n^{-1}$ we find

$$\left|\frac{h_n(a_n\xi, T_n + a_n^2 s)}{h_n(a_n, T_n)}\right| \le \left(\frac{T_n^{k/3}}{a_n} + 1\right)^m \left(1 + \frac{a_n^2 s}{T_n}\right)^{mk/3} \xi^{-m}.$$

This proves (5.7.2). Since $T_n^{k/3} \ll a_n \ll T_n^{1/2}$ (recall that (5.5.4) implies $a_n \ll T_n^{1/2}$), we have

$$\left(\frac{T_n^{k/3}}{a_n} + 1\right)^m \left(1 + \frac{a_n^2 s}{T_n}\right)^{mk/3} \to 1$$

uniformly for bounded s which implies (5.7.3). □

This lemma tells us we have a sequence of solutions $\bar{h}_n$ of the linear equation

$$\frac{\partial \bar{h}_n}{\partial t} = \frac{\partial}{\partial \xi}\left\{\frac{\bar{h}_{n\xi}}{1 + \bar{u}_{n\xi}^2}\right\} + \frac{3}{\xi}\frac{\partial \bar{h}_n}{\partial \xi} + \frac{3}{\bar{u}_n^2}\bar{h}_n = \frac{\bar{h}_{n\xi\xi}}{1 + \bar{u}_{n\xi}^2} + \left\{\frac{3}{\xi} - \frac{2\bar{u}_{n\xi}u_{n\xi\xi}}{(1 + \bar{u}_{n\xi}^2)^2}\right\}\frac{\partial \bar{h}_n}{\partial \xi} + \frac{3}{\bar{u}_n^2}\bar{h}_n \tag{5.7.4}$$

which satisfies the uniform bound (5.7.3) for all $n \ge n_0 \gg 1$. As before we have:

Lemma 5.7.2. $S_n \to \infty$.

Proof. Assume that S_n is bounded and, after passing to a subsequence, that we have $S_n \to S_\infty$.

The function $\bar{u}_n$ converges in C^∞_{loc} to $\bar{u}_\infty(\xi, s) = \xi$, so interior estimates for the divergence form (5.7.4) imply that $\bar{h}_n$ is locally uniformly Hölder continuous for $\xi > 0$ and $-S_n \le s \le 0$. Moreover, the construction of $u_n(\,\cdot\,, s_n)$ guarantees that $\bar{h}_n$ starts out with $\bar{h}_n(\xi, -S_n) = 0$ for all $a_n\xi \ll T_n^{1/2}$. We may therefore assume that there is a convergent subsequence $\bar{h}_n(\xi, s) \to \bar{h}(\xi, s)$, where

$$|\bar{h}(\xi, s)| \le \xi^{-m}$$

for all $\xi > 0$ and $s \in [-S_\infty, 0]$ and where $\bar{h}$ is a solution of

$$\bar{h}_s = \frac{1}{2}\bar{h}_{\xi\xi} + \frac{3}{\xi}\bar{h}_\xi + \frac{3}{\xi^2}\bar{h} := \mathcal{M}_0[\bar{h}],$$

with $\bar{h}(1, 0) = \pm 1$ and $\bar{h}(\xi, -S_\infty) = 0$ for all $\xi > 0$. The limiting function $\bar{h}$ is smooth for $\xi > 0$, $-S_\infty \le s \le 0$. We note that $\hat{h}(\xi) = \xi^{-2} + \xi^{-3}$ is a stationary solution of $\hat{h}_s = \mathcal{M}_0[\hat{h}]$, so that for any $\eta > 0$ the functions $\pm\eta\hat{h}$ provide upper and lower barriers for $\bar{h}$, provided we can show that $-\eta\hat{h} < \bar{h} < \eta\hat{h}$ as $\xi \to 0$ or $\xi \to \infty$. This boundary condition is fulfilled because $|\bar{h}(\xi, s)| \le \xi^{-m}$ with $2 < m < 3$. The maximum principle therefore implies that $|\bar{h}| \le \eta\hat{h}$ for all $\eta > 0$. Letting $\eta \to 0$ this yields $\bar{h}(\xi, s) = 0$ for all $\xi > 0$ and all $s \in [-S_\infty, 0]$. This contradicts $\bar{h}(1, 0) = \pm 1$ and shows that the sequence S_n is indeed unbounded. □

Let $\bar{h}(\xi, s)$ be a limit of the $\bar{h}_n(\xi, s)$ along some subsequence $n = n_k \nearrow \infty$. We now show that $\bar{h}(1, 0) = 0$, which contradicts the fact that $\bar{h}(1, 0) = \pm 1$ and therefore *completes the proof of Lemma 5.4.1*.

Lemma 5.7.3. $\bar{h}(1,0)=0$.

Proof. Choose a small $\epsilon>0$ and consider the function

$$k(\xi,s)=\bar{h}(\xi,s)-\epsilon\xi^{-2}-\epsilon\xi^{-3}.$$

This function is a solution of the linear equation $k_s=\mathcal{M}_0[k]$. In view of the bound $\bar{h}(\xi,s)\le\xi^{-m}$, which holds for all $\xi>0$ and $s\le 0$, we have

$$k(\xi,s)\le\xi^{-m}-\epsilon\xi^{-2}-\epsilon\xi^{-3}.$$

Since $2<m<3$, this implies that $k(\xi,s)<0$ if $\xi\le\epsilon^{1/(3-m)}$ or $\xi\ge\epsilon^{-1/(m-2)}$.

The differential operator $\mathcal{M}_0$ is a standard Sturm–Liouville operator with smooth coefficients on the interval $I_\epsilon=[\epsilon^{1/(3-m)},\epsilon^{-1/(m-2)}]$. Since ξ^{-2} is a strictly positive solution of $\mathcal{M}_0[\varphi]=0$, the principal eigenvalue λ_0 of

$$\mathcal{M}_0[\Omega(\xi)]=-\lambda_0\Omega(\xi),\quad \Omega(\epsilon^{1/(3-m)})=\Omega(\epsilon^{-1/(m-2)})=0$$

is positive, and the corresponding eigenfunction $\Omega(\xi)$ is also positive for all ξ in the interior of the interval I_ϵ. Choose $C_\epsilon>0$ so that

$$\xi^{-m}-\epsilon\xi^{-2}-\epsilon\xi^{-3}\le C_\epsilon\Omega(\xi)$$

for all $\xi\in I_\epsilon$.

For any given $s_0>0$ we then have

$$k(\xi,-s_0)\le C_\epsilon\Omega(\xi)\quad\text{for all } \xi\in I_\epsilon.$$

Moreover, $\hat{k}(\xi,s)=C_\epsilon e^{-\lambda_0(s+s_0)}\Omega(\xi)$ is a solution of $\hat{k}_s=\mathcal{M}[\hat{k}]$, so the maximum principle applied on the domain $I_\epsilon\times[-s_0,0]$ implies that at time $s=0$ we have

$$k(\xi,0)\le\hat{k}(\xi,0)=C_\epsilon e^{-\lambda_0 s_0}\Omega(\xi).$$

Since this is true for all $s_0>0$, we conclude $k(\xi,0)\le 0$. By definition of $k(\xi,s)$ this implies that $\bar{h}(\xi,0)\le\epsilon\xi^{1/(3-m)}+\epsilon\xi^{-1/(m-2)}$ for all $\xi\in I_\epsilon$. In particular, this holds for $\xi=1$ where it implies $\bar{h}(1,0)\le 2\epsilon$. This argument goes through for all $\epsilon>0$, so we find $\bar{h}(1,0)\le 0$.

Applying the whole argument once more to $\tilde{k}(\xi,s)=-\bar{h}(\xi,s)-\epsilon\xi^{1/(3-m)}-\epsilon\xi^{-1/(2-m)}$ instead, we find $-\bar{h}(1,0)\le 0$. Hence $\bar{h}(1,0)=0$, as claimed. □

The proof of Lemma 5.4.1 is now complete. We can now conclude the proof of Theorem 5.1.1.

Proof of Theorem 5.1.1. Lemma 5.4.1 implies $\sup_n\Lambda_n<\infty$. Using the definition of Λ_n this implies that $|H_n|=|h_n|/\sqrt{1+u_{nx}^2}$ is also uniformly bounded. The derivative bounds from sections Section 4.4–4.6 imply that $u_n(x,t)$ converges uniformly smoothly to $u(x,t)$ for $t\in(0,t_0]$ as $n\to\infty$. Therefore we get $|H(x,t)|=\lim_{n\to\infty}|H_n|\le C$ for all $0\le x\le M\sqrt{t}$ and $t\in(0,t_0]$. On the other hand, Lemma 5.2.1 shows that $H(x,t)$ is bounded when $x\ge Y_{\delta_0}\sqrt{t}$. By definition (3.6.4) we have $Y_{\delta_0}:=2\sqrt{(2k+1)!!\,M/\delta_0}$; if we choose M large enough then $Y_{\delta_0}<M$ so the two regions $0\le x\le M\sqrt{t}$ and $x\ge Y_{\delta_0}\sqrt{t}$ overlap. The mean curvature H is therefore uniformly bounded on the whole solution, which concludes the proof of Theorem 5.1.1. □

Appendix

A.1. ***The linear equation in the intermediate region.*** The eigenvalue equation $\mathcal{L}\varphi = \big(k-\frac{3}{2}\big)\varphi$ is

$$\frac{1}{2}\varphi_{yy} + \left(\frac{3}{y}+\frac{y}{2}\right)\varphi_y + \left(\frac{3}{y^2}-\frac{1}{2}\right)\varphi = \left(k-\frac{3}{2}\right)\varphi;$$

i.e.,

$$\varphi_{yy} + \left(\frac{6}{y}+y\right)\varphi_y + \frac{6}{y^2}\varphi = 2(k-1)\varphi.$$

Let $\varphi(y) = y^{-2}\chi_k(y)$. Then χ_k satisfies the equation

$$\chi_k'' + \left(\frac{2}{y}+y\right)\chi_k' = 2k\chi_k.$$

For every *real* $k > 0$ there is a unique solution with $\chi_k(0) = 1,\ \chi_k'(0) = 0$. This solution is monotone increasing and for large y has the expansion

$$\chi_k(y) = C_k y^{2k} + o(y^{2k}), \quad y \to \infty.$$

For $k \in \mathbb{N}$ it is given by the series expansion

$$\chi_k(y) = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!\,(2n+1)!!} y^{2n}, \tag{A.1.1}$$

where $(2n+1)!! := 1\cdot 3\cdot 5\cdot 7\cdots(2n+1)$. This defines φ_k for all real k. We will only need these functions for integer values of k, in which case χ_k is a polynomial, and $\varphi_k(y) = y^{-2}\chi_k(y)$ is given by

$$\varphi_k(y) = y^{-2}\sum_{n=0}^{k}\binom{k}{n}\frac{y^{2n}}{(2n+1)!!}. \tag{A.1.2}$$

There is a second solution $\hat{\chi}_k$ that satisfies

$$\hat{\chi}_k(y) = e^{-y^2/2+o(y^2)}, \quad y \to \infty.$$

At $y = 0$ this solution is singular:

$$\hat{\chi}_k(y) = \frac{C}{y} + \mathcal{O}(y), \quad y \to 0.$$

A.2. ***Proof of Lemma 3.4.1.*** The homogeneous equation $6\gamma\varphi - \mathcal{L}\varphi = 0$ has solutions of the form

$$\varphi = C\varphi_k^1(y) + B\psi_k^1(y), \quad C, B \in \mathbb{R},$$

where $\varphi_k^1(y)$ and ψ_k^1 are solutions with

$$\varphi_k^1(y) = \begin{cases} y^{-2}, & y \to 0, \\ \mathcal{O}(y^{4k-5}), & y \to \infty, \end{cases}$$

and

$$\psi_k^1(y) = \begin{cases} y^{-3}, & y \to 0, \\ \mathcal{O}(e^{-y^2/2+o(y^2)}), & y \to \infty. \end{cases}$$

Since $y = 0$ is a regular singular point for the differential equation $6\gamma g - \mathcal{L}g = G(y) = y^{-7} + y^{4k-7}$, we look for the solution in the form of a power series. From

$$(6\gamma - \mathcal{L})[y^r] = -\tfrac{1}{2}(r+2)(r+3)y^{r-2} + \tfrac{1}{2}(4k-7-r)y^r \tag{A.2.1}$$

it follows that (3.4.2) has a particular solution of the form

$$g_{0p}(y) = C_0 y^{-5} P_0(y^2) + C_1 y^{-3}\log(y) P_1(y^2),$$

where $P_j(y^2)$ are power series in y^2 with $P_j(0) = 1$. The logarithmic term appears because $r = -3$ is one of the characteristic exponents. The coefficient C_0 is obtained by substitution in the equation. One finds $C_0 = -\frac{1}{3}$.

Every solution φ of the homogeneous equation satisfies $\varphi = \mathcal{O}(y^{-3}) = o(g_{0p})$ as $y \to 0$, and therefore every solution g of the inhomogeneous equation satisfies

$$g = g_{0p} + \mathcal{O}(y^{-3}) = -\tfrac{1}{3}y^{-5} + \mathcal{O}(y^{-3}\log y), \quad \text{as } y \to 0. \tag{A.2.2}$$

The differential equation $6\gamma g - \mathcal{L}g = G$ has an irregular singular point at $y = \infty$, so we cannot use the power series method. Instead, we obtain a solution using sub- and supersolutions. For any $m \in \mathbb{R}$ the functions $g_\pm(y) = y^{4k-7} \pm m y^{4k-9}$ satisfy

$$(6\gamma - \mathcal{L})g_\pm = \bigl(-\tfrac{1}{2}(4k-5)(4k-4) \pm m\bigr)y^{4k-9} + \mathcal{O}(y^{4k-11}), \quad y \to \infty.$$

For $m > \frac{1}{2}(4k-5)(4k-4)$ it follows that $g_- < g_+$ are sub- and supersolutions for $6\gamma g - \mathcal{L}g = G$ on the interval $[y_0, \infty)$ if y_0 is large enough. Hence there is a particular solution $g_{\infty p}$ satisfying

$$g_{\infty p}(y) = y^{4k-7} + \mathcal{O}(y^{4k-9}), \quad y \to \infty.$$

At $y = 0$ all solutions satisfy (A.2.2), so $g_{\infty p}$ also satisfies $g_{\infty p}(y) = -\frac{1}{3}y^{-5} + \mathcal{O}(y^{-3}\log y)$. The general solution of the nonhomogeneous equation (3.4.2) is then of the form $g := g_{\infty p} + C\varphi_k^1 + B\psi_k^1$ for $C, B \in \mathbb{R}$. However, the boundary condition $g(y) = y^{4k-5} + o(y^{4k-5})$ as $y \to \infty$ requires $C = 0$. One concludes that $g_B := g_{\infty p} + B\psi_k^1$, $B \in \mathbb{R}$, is a one-parameter set of solutions to (3.4.2) which satisfies the conditions of our lemma, thus finishing the proof.

A.3. *The Alencar solution.*

Lemma A.3.1. *Let $W : [0, \infty) \to \mathbb{R}$ be the solution of*

$$\frac{W_{zz}}{1 + W_z^2} + \frac{3}{z}W_z - \frac{3}{W} = 0, \quad W(0) = 1, \quad W'(0) = 0.$$

Then $W_{zz} > 0$ and $0 < W - zW_z \le 1$ for all $z \ge 0$.

For large z the solution $W(z)$ has the expansion

$$W = z + \frac{\Gamma_2}{z^2} + \frac{\Gamma_3}{z^3} + \frac{\Gamma_5}{z^5} + \cdots \tag{A.3.1}$$

for certain coefficients $\Gamma_i \in \mathbb{R}$.

Proof. The differential equation for W has been thoroughly studied. In particular, $W_{zz} > 0$ and $W > zW_z$ were shown by Velázquez [1994, Proposition 2.2], ($B''(u) > 0$ and $G_a(r) < 0$ in his notation). Here we prove that $W(z)$ has the stated asymptotic expansion. Let

$$P = W_z \quad \text{and} \quad Q = \frac{z}{W}.$$

Then (P, Q) as a function of $\log z$ satisfy an autonomous system of differential equations,

$$\begin{cases} zP_z = 3(1+P^2)(Q-P), \\ zQ_z = P - P^2 Q. \end{cases} \tag{A.3.2}$$

This system has two fixed points with $Q \geq 0$, namely, the origin $(0, 0)$ and the point $(1, 1)$.

The origin corresponds to the boundary condition $W_z = 0,\ z = 0$, while the fixed point corresponds to the Simons cone on which $W = z$ and $W_z = 1$.

The matrix of the linearization at $(0, 0)$ is $\left(\begin{smallmatrix} 1 & 0 \\ 3 & -3 \end{smallmatrix}\right)$. Its eigenvalues are $\lambda_1 = +1$ and $\lambda_2 = -3$. The eigenvector corresponding to the unstable eigenvalue is $\left(\begin{smallmatrix} 4 \\ 3 \end{smallmatrix}\right)$. The unique orbit in the unstable manifold of the origin is the Alencar solution. It approaches the fixed point $(1, 1)$ as $z \to \infty$. The matrix of the linearization at $(1, 1)$ is $\left(\begin{smallmatrix} -1 & -1 \\ 6 & -6 \end{smallmatrix}\right)$ with eigenvalues/vectors

$$\lambda_1 = -3,\ \ \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \lambda_2 = -4,\ \ \vec{v}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

The eigenvalues are both negative and they satisfy the "no resonance" condition, i.e., neither eigenvalue is an integer multiple of the other. This implies that there is a real analytic conjugacy of the nonlinear system (A.3.2) near the fixed point $(1, 1)$ with the linearization (see the chapter on normal forms and Poincaré's theorem in [Arnold 1983]). The general solution of the linear system is

$$C_1 z^{-3}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 z^{-4}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} C_1 z^{-3} + C_2 z^{-4} \\ 2C_1 z^{-3} + 3C_2 z^{-4} \end{pmatrix}.$$

This in turn implies that all solutions of (A.3.2) that converge to $(1, 1)$ are convergent power series in z^{-3} and z^{-4}. In particular, $1/Q = W/z$ has an expansion of the form

$$\frac{W}{z} = 1 + C_3 z^{-3} + C_4 z^{-4} + C_6 z^{-6} + C_7 z^{-7} + \cdots = 1 + \sum_{l,m \geq 1} C_{l,m} z^{-3l-4m}.$$

Therefore $W(z)$ satisfies

$$W = z + C_3 z^{-2} + C_4 z^{-3} + C_6 z^{-5} + C_7 z^{-6} + \cdots = z + \sum_{l,m \geq 1} C_{l,m} z^{1-3l-4m}.$$

So if we set $\Gamma_m = C_{m+1}$ we have proved the expansion (A.3.1) ☐

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