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QUASILINEAR WAVE EQUATIONS ON ASYMPTOTICALLY FLAT SPACETIMES WITH APPLICATIONS TO KERR BLACK HOLES

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We prove global existence and decay for small-data solutions to a class of quasilinear wave equations on a wide variety of asymptotically flat spacetime backgrounds, allowing in particular for the presence of horizons, ergoregions and trapped null geodesics, and including as a special case the Schwarzschild and very slowly rotating $|a| \ll M$ Kerr family of black holes in general relativity. There are two distinguishing aspects of our approach. The first aspect is its dyadically localised nature: The nontrivial part of the analysis is reduced entirely to time-translation-invariant r^p-weighted estimates, in the spirit of Dafermos and Rodnianski (2010b), to be applied on dyadic time-slabs which for large r are outgoing. Global existence and decay then both immediately follow by elementary iteration on consecutive such time-slabs, without further global bootstrap. The second, and more fundamental, aspect is our direct use of a "black box" linear inhomogeneous energy estimate on exactly stationary metrics, together with a novel but elementary physical-space top-order identity that need not capture the structure of trapping and is robust to perturbation. In the specific example of Kerr black holes, the required linear inhomogeneous estimate can then be quoted directly from the literature (Dafermos et al. (2016)), while the additional top-order physical-space identity can be shown easily in many cases (we include in the Appendix a proof for the Kerr case $|a| \ll M$, which can in fact be understood in this context simply as a perturbation of Schwarzschild). In particular, the approach circumvents the need either for producing a purely physical-space identity capturing trapping or for a careful analysis of the commutation properties of frequency projections with the wave operator of time-dependent metrics.

MSC2020: 35L05, 83C05.
Keywords: nonlinear wave equations, black holes, general relativity nonlinear stability.

1. Introduction

We consider here quasilinear equations of the form

$$\Box_{g(\psi,x)}\psi = N(\partial\psi, \psi, x) \tag{1-1}$$

on a 4-dimensional manifold $\mathcal{M}$, where $g(\psi, x)$ and $N(\partial\psi, \psi, x)$ are appropriate nonlinear terms and where $g(0, x) = g_0(x)$ defines a stationary asymptotically flat Lorentzian metric on $\mathcal{M}$ foliated by a suitable family of hypersurfaces $\Sigma(\tau)$, for $\tau \geq 0$, which for large r are outgoing null. In this paper, we prove the following:

Theorem. *For equations (1-1), under appropriate assumptions on the background spacetime* $(\mathcal{M}, g_0)$ *and on the nonlinearities* $g(\psi, x)$ *and* $N(\partial\psi, \psi, x)$ *to be described below, we have the following*:

- *Global existence of small data solutions*: *solutions* ψ *arising from smooth initial data on* Σ_0, *sufficiently small as measured in a suitable weighted Sobolev energy, exist globally on* $\mathcal{M}$.
- *Orbital stability*: *under the above assumptions, the above weighted energy flux through* $\Sigma(\tau)$ *is uniformly bounded by a constant times its initial value.*
- *Asymptotic stability*: *under the above assumptions, a suitable lower-order unweighted energy flux through* $\Sigma(\tau)$ *decays inverse polynomially in* τ *(implying also pointwise inverse polynomial decay for* ψ*).*

One possible motivation for studying (1-1) is as an illustrative model problem for issues relating to the nonlinear stability of the spacetimes $(\mathcal{M}, g_0)$, when these themselves are solutions to the celebrated Einstein equations of general relativity. An important example of such spacetimes $(\mathcal{M}, g_0)$ allowed by our theorem is provided by the Schwarzschild family of metrics for $M > 0$, and the more general Kerr family (parametrised by a and M) in the very slowly rotating regime $|a| \ll M$. For a discussion of these spacetimes and the stability problem in general relativity, see [Dafermos et al. 2021].

Among other features, the Schwarzschild and Kerr spacetimes exhibit *trapped null geodesics*, along which energy can concentrate over large timescales. Moreover, these are *asymptotically flat* spacetimes, meaning that linear waves ψ decay like $\sim r^{-1}$ along outgoing null cones. This then constrains the decay of ψ in the near region to also be at best only inverse polynomial. (In the above black hole examples, this slow decay in the near region is already a linear effect due to scattering off of far-away curvature associated to the nontrivial spacetime mass at infinity; in Minkowski space, this slow decay in the near region arises due to purely nonlinear scattering effects, even for compactly supported initial data.) It is the combination of these two specific analytical difficulties — trapped null geodesics and the relatively slow decay in the near region necessitated by asymptotic flatness — of the black hole stability problem (and related problems) which we wish to capture with our assumptions in the present paper. It is for this reason that we include both the nonlinear term implicit in the expression $\Box_{g(\psi,x)}\psi$ on the left-hand side of (1-1) (the "quasilinearity"), as this is the most dangerous term in the vicinity of trapping, as well as the nonlinear term $N(\partial\psi, \psi, x)$ on the right-hand side (the "semilinearity"), as this term models the true null structure at infinity of the Einstein equations, when the latter are written in appropriate geometric gauges. To avoid inessential complications, we will in fact assume that $g(\psi, x) = g_0$ for large r and that $N(\partial\psi, \psi, x)$ satisfies a generalised version of the null condition [Klainerman 1986].

In the case where $(\mathcal{M}, g_0)$ is Minkowski space, a version of the above theorem follows from classical work of [Klainerman 1986; Christodoulou 1986], and there have been many further amplifications over the years, particularly in the context of the obstacle problem; see, e.g., [Metcalfe and Sogge 2005]. Concerning specifically the Schwarzschild and very slowly rotating $|a| \ll M$ Kerr setting, versions of the above theorem have been shown previously in various special cases; see already [Luk 2013] in the semilinear and [Lindblad and Tohaneanu 2018; 2020] in the quasilinear case, as well as the recent [Lindblad and Tohaneanu 2024] (the latter three works concerning slightly different classes of equations, but satisfying the weak null condition [Lindblad and Rodnianski 2003]). For axisymmetric solutions to certain semilinear equations on Kerr in the full subextremal range $|a| < M$, see [Stogin 2017]. See also [Pasqualotto 2019] for a related physically motivated quasilinear problem and [Hintz and Vasy 2016; Mavrogiannis 2024] for the study of (1-1) on the nonasymptotically flat Schwarzschild–de Sitter and Kerr–de Sitter black hole backgrounds, where the cosmological constant Λ is positive and decay of ψ is in fact exponential. For work specifically connected to the related problem of stability of these spacetimes themselves under the Einstein equations, see already Section 1.4.4.

In comparison to previous related work, there are two distinguishing features of the present approach.

The first distinguishing feature is our purely dyadic framework: Rather than relying on a global bootstrap based on time-weighted norms, the argument is entirely reduced to r^p-weighted but *time-translation-invariant* estimates to be applied in spacetime slabs of time length L which are outgoing null for large r. Global existence (and decay) is then inferred by proceeding iteratively in time in consecutive slabs of length $L = 2^i$, in the spirit of [Dafermos and Rodnianski 2010b]. No bootstrap is necessary for the iteration itself, and to estimate the i-th slab, no information is necessary to remember about the past other than information on the data of the slab itself. In this sense, the argument is truly dyadically localised. This yields a more streamlined proof even restricted to the semilinear case. (In fact, for both the semilinear and quasilinear cases, if $(\mathcal{M}, g_0)$ is a suitably small perturbation of Minkowski space, the time-translation-invariant estimate can be applied directly without iteration, and the approach proves global existence without explicit reference to time decay, cf. the recent [Facci and Metcalfe 2022]. In our formalism we shall always assume g_0 to be stationary, though we note that results can also be obtained on nonstationary perturbations of Minkowski space [Yang 2013].)

The second, and more fundamental, feature is our direct use of a "black box" linear inhomogeneous energy estimate (which in applications captures both complicated trapping phenomena as well as low-frequency obstructions) on the exactly stationary background, together with a physical-space top-order identity which may be applied directly to the quasilinear equation (1-1) and is in fact completely insensitive to trapping (and, when it holds, robust to perturbation of the metric g_0). In the example of Kerr, in fact for the full subextremal case $|a| < M$, our black box assumption follows directly from [Dafermos et al. 2016], while we show explicitly how to retrieve the additional top-order physical-space estimate in the case of $|a| \ll M$ (which in this context, can in fact be thought of simply as a perturbation of Schwarzschild) in Appendix A. The correct formulation of the companion estimate to be used in connection with [Dafermos et al. 2016] for the full subextremal case $|a| < M$ will appear elsewhere. (We emphasise that it is the phenomenon of *superradiance* which is the primary difficulty in producing this identity, not the complicated structure of trapping per se; for instance, the necessary top-order physical-space identity can be shown for general stationary spacetimes without horizons or ergospheres, irrespective of the structure of trapping.) The approach thus does not depend on whether or not there exists a purely physical-space based identity capturing trapping (something very fragile!) nor does it require a careful analysis of the commutation properties of frequency projections with the wave operator $\Box_{g(\psi,x)}$ of the time-dependent metric $g(\psi, x)$ corresponding to the actual solution ψ (something quite technical, which was successfully done, however, for instance in [Lindblad and Tohaneanu 2020]).

We emphasise that the role of the companion physical-space estimate is purely in order to deal with quasilinear terms. When our method is restricted to the semilinear case, i.e., when $g(\psi, x) = g_0$ identically in (1-1), we have that the black box linear inhomogeneous energy estimate can be applied on its own. In particular, in view of [Dafermos et al. 2016], our main theorem immediately applies to semilinear equations on Kerr in the full subextremal range $|a| < M$.

We note that the direct appeal to a linear "black box" statement is similar in spirit to [Metcalfe and Sogge 2005] for instance, where results on quasilinear equations outside obstacles in Minkowski space were studied under an exponential decay assumption concerning the linear problem.

The remainder of this introduction is structured as follows: In Section 1.1 below, we will discuss in more detail the assumptions we make on the background $(\mathcal{M}, g_0)$, introducing both the black box linear inhomogeneous estimate and the additional physical-space identity, followed by the assumptions on the nonlinearity in Section 1.2. We will then sketch our proof of the above theorem in Section 1.3, introducing our purely dyadic approach. Finally, we shall end in Section 1.4 with a general discussion, giving various extensions of the method, describing in more detail the relation with the nonlinear stability problem for black holes and comparing in particular with the case of backgrounds modelled on Schwarzschild–de Sitter and Kerr–de Sitter spacetimes and with the recent work [Mavrogiannis 2024].

1.1. ***Assumptions on $(\mathcal{M}, g_0)$: "black box" estimates and physical-space identities.*** We will make assumptions directly on the geometry of $(\mathcal{M}, g_0)$ and on properties of the linear inhomogeneous wave equation

$$\Box_{g_0}\psi = F \tag{1-2}$$

on the fixed background g_0.

1.1.1. *Geometric assumptions on* $(\mathcal{M}, g_0)$. The purely geometric assumptions on $(\mathcal{M}, g_0)$ will include assumptions concerning a suitable notion of asymptotic flatness, that of stationarity (existence of a Killing field T which is timelike near infinity), a smooth T-invariant strictly positive function $r \geq r_0$, which behaves like the Euclidean area-radius as $r \to \infty$, and the existence of a T-translation-invariant foliation of $\mathcal{M}$ by $\Sigma(\tau)$ for $\tau \geq 0$ hypersurfaces which are spacelike for $r \leq R$ and "outgoing" null for $r \geq R$. (In the case of Minkowski space, which will be the most basic example, we note that the function r will only coincide with the usual radial coordinate for large values.)

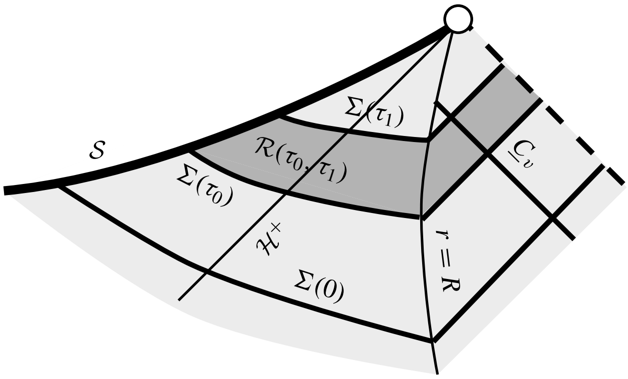

Figure 1. The underlying spacetime $(\mathcal{M}, g_0)$ in the case $\mathcal{S} \neq \varnothing$ with hypersurfaces and regions depicted.

To encompass the black hole cases of interest, we will allow for a possibly empty *space-like* future boundary $\mathcal{S} = \{r = r_0\}$ of $\mathcal{M}$, in which case we will also assume the presence of a Killing horizon $\mathcal{H}^+$ at $r = r_{\text{Killing}}$ for an $r_0 < r_{\text{Killing}}$, whose surface gravity will be required to be positive and whose Killing generator Z need not coincide with T (in which case however Z will be required to lie in the span of T and an additional assumed Killing vector field Ω_1). We will furthermore require that, for $r > r_{\text{Killing}}$, T (or more generally the span of T and Ω_1, if the latter is assumed Killing) is timelike. This will incorporate Schwarzschild and Kerr black holes in the full subextremal range $|a| < M$.

With minor modifications, we could also allow $\mathcal{S}$ to be a time-like boundary along which suitable boundary conditions are imposed, and this would allow one to consider waves outside of obstacles, cf. [Metcalfe and Sogge 2005].

We will denote by $\mathcal{R}(\tau_0, \tau_1)$ spacetime "slabs" $\mathcal{R}(\tau_0, \tau_1) = \bigcup_{\tau_0 \le \tau \le \tau_1} \Sigma(\tau) = J^-(\Sigma(\tau_1)) \cap J^+(\Sigma(\tau_0))$. The region $r \ge R$ will also be foliated by T-translation-invariant "ingoing" null hypersurfaces $\underline{C}_v$.

Refer to Figure 1 and already to Section 2 for a detailed discussion.

1.1.2. *Assumptions on* $\Box_{g_0}\psi = F$. In this section, we discuss the assumptions which we make at the level of equation (1-2).

Basic degenerate integrated local energy estimate. Our fundamental "black box" assumption at the level of solutions of the inhomogeneous linear equation (1-2) will be the validity of an integrated local energy estimate

$$\mathcal{F}(v), \quad \mathcal{E}(\tau) + c\int_{\tau_0}^{\tau_1} \overset{(-1-\delta)}{\chi\mathcal{E}'}(\tau')\,d\tau' + c\int_{\tau_0}^{\tau_1} \underset{-1}{\mathcal{E}'}(\tau')\,d\tau' \\ \le \lambda\mathcal{E}(\tau_0) + C\int_{\mathcal{R}(\tau_0,\tau_1)} |(V_0^\mu\partial_\mu\psi + w_0\psi)F| + C\int_{\mathcal{R}(\tau_0,\tau_1)} F^2, \tag{1-3}$$

where $\mathcal{E}(\tau)$ denotes an energy flux through the hypersurface $\Sigma(\tau)$, $\mathcal{F}(v)$ denotes an energy flux through $\underline{C}_v \cap \mathcal{R}(\tau_0, \tau_1)$, $\lambda \ge 1$, $C > 0$, $c > 0$ are constants, and V_0 and w are a fixed vector field and function, respectively, and $\tau_0 \le \tau \le \tau_1$ are arbitrary. In Minkowski space, such estimates go back to Morawetz [1968].

The energy flux $\mathcal{E}(\tau)$ will control all first-order derivatives of ψ on the space-like part of $\Sigma(\tau)$ while it will only control all tangential derivatives on the null parts of $\Sigma(\tau)$; similarly, $\mathcal{F}(v)$ will control all tangential derivatives on $\underline{C}_v$. The fluxes will also control a zeroth-order term with weight r^{-2}. The two additional terms on the left-hand side are as follows: The quantity ${}^{\chi}\overset{(-1-\delta)}{\mathcal{E}'}(\tau')$ denotes an integral over $\Sigma(\tau')$ controlling all first-order derivatives of ψ, whose density is multiplied by $\chi r^{-1-\delta}$, where $0 \leq \chi \leq 1$ denotes a function which is allowed to degenerate but must satisfy $\chi = 1$ in a region to be discussed below, and where $\delta > 0$ can be taken to be an arbitrarily small constant which will be fixed throughout. The quantity $\underset{-1}{\mathcal{E}}'(\tau')$ denotes simply the zeroth-order quantity

$$\underset{-1}{\mathcal{E}}'(\tau) := \int_{\Sigma(\tau)} r^{-3-\delta}\psi^2.$$

The subscript -1 indicates that it is of order 1 less in differentiability than the other energies considered. We recall that, in our setup, what we define to be r satisfies $r \geq r_0 > 0$, and thus r weights are only relevant as $r \to \infty$.

In the case where $\mathcal{M}$ has no space-like boundary, i.e., $\mathcal{S} = \varnothing$, and T is globally timelike, our only assumption on χ will be that $\chi = 1$ for large r. In the case where $\mathcal{S} \neq \varnothing$, we will require that $\chi = 1$ for $r \leq r_2$ for some $r_2 > r_{\text{Killing}}$. See already Section 3.2 for the detailed assumptions.

Let us note that estimate (1-3) with χ satisfying the above properties has indeed been shown in [Dafermos et al. 2016] in the Kerr case for the full subextremal range of parameters $|a| < M$. (See already Theorem D.1.) One essential feature in deriving (1-3) in Kerr is the fact that the trapped null geodesics of g_0, corresponding to photons in bound orbits around the black hole, are unstable in a suitable sense. We note that one can show in general that estimate (1-3) in the above form *cannot* hold in the presence of stable trapping. (Thus, (1-3) in particular constrains properties of geodesic flow which can be expressed purely geometrically in terms of g_0.) Similarly, the possibility of an estimate (1-3) with $\chi = 1$ in a neighbourhood of $\mathcal{H}^+$ is related to the nondegenerate property of the horizon of subextremal Kerr and its celebrated local *red-shift effect*. See [Dafermos and Rodnianski 2013].

Physical-space energy identities. One way to try to prove (1-3) is through a physical-space energy *identity* arising from integrating the divergence identity

$$\nabla^\mu J^{V,w,q,\varpi}_\mu[\psi] = K^{V,w,q}[\psi] + (V^\mu \partial_\mu \psi) F + w\psi F \tag{1-4}$$

for well-chosen currents $J^{V,w,q,\varpi}[\psi, g_0]$ (associated to a vector field V, a scalar function w, a 1-form q and a 2-form ϖ) with (degenerate) coercivity properties for the arising bulk and boundary terms. In this case, if one defines

$$\mathfrak{E}(\tau) = \int_{\Sigma(\tau)} J^{V,w,q,\varpi}_\mu[\psi] n^\mu_{\Sigma(\tau)}, \qquad \mathfrak{F}(v) = \int_{\underline{C}_v \cap \mathcal{R}(\tau_0,\tau_1)} J^{V,w,q,\varpi}_\mu[\psi] n^\mu_{\underline{C}_v},$$

then one has $\mathfrak{E}(\tau) \sim \mathcal{E}(\tau)$, $\mathfrak{F}(v) \sim \mathcal{F}(v)$, and one can reexpress (1-3) with $\mathfrak{E}(\tau)$ and $\mathfrak{E}(\tau_0)$ replacing $\mathcal{E}(\tau)$ and $\mathcal{E}(\tau_0)$, respectively, but with λ in (1-3) now equal to 1. This is precisely the situation in the Schwarzschild case, where appropriate coercive purely physical-space identities of the form (1-4) can be deduced from [Dafermos and Rodnianski 2007a; Marzuola et al. 2010].

If (1-3) indeed holds as a consequence of the coercivity properties of a divergence identity (1-4), then this situation appears very advantageous for nonlinear applications. The advantage of such coercive identities (1-4) is that they are robust and easily amenable to *direct* application to the quasilinear (1-1). This is because the latter can be viewed as an inhomogeneous wave equation *with respect to the wave operator* $\Box_{g(\psi,x)}$ *corresponding to the solution itself*, and both the existence of energy identities like (1-4) and their coercivity properties are stable to passing from g_0 to $g(\psi, x)$, i.e., still hold for $J^{V,w,q,\varpi}[g(\psi, x), \psi]$.

If one only has estimate (1-3) as a "black box", i.e., not proven via a physical-space identity (1-4), one may still of course apply the estimate to (1-1) by rewriting (1-1) as an inhomogeneous equation with respect to $\Box_{g_0}$:

$$\Box_{g_0}\psi = (\Box_{g_0} - \Box_{g(\psi,x)})\psi + N(\partial\psi, \psi, x). \tag{1-5}$$

Applied in this manner, however, the resulting estimate *loses derivatives* in view of the quasilinear second-order terms on the right-hand side of (1-5). (Note in contrast that in the purely semilinear case, where $g(\psi, x) = g_0$, there is no such loss in estimating (1-5), and we may base our argument entirely on (1-3). See already Section 1.4.1.) At first glance, this loss would appear to present a fundamental difficulty.

An auxiliary physical-space estimate. Here we come to the key observation of the present approach: to avoid the loss of derivatives described above, one does *not* need that (1-3) be proven via a physical-space identity (1-4); one only needs a much weaker (from the coercivity point of view) estimate arising from (1-4) which need not imply (1-3) but can be used *in conjunction* with (1-3).

Specifically, our main assumption is that, in addition to (1-3), one has a physical-space identity (1-4) with (a) coercive boundary terms and with (b) bulk terms which are only assumed to be nonnegative *at highest order*, i.e., up to lower-order "error" terms. Importantly, however, these allowed lower-order error terms must moreover be supported entirely in the region where the degeneration function χ of (1-3) satisfies $\chi = 1$, i.e., where the estimate of (1-3) is in fact nondegenerate. The identity (1-4) applied to (1-2), on its own, gives rise thus to an estimate of the form

$$\mathfrak{F}(v), \quad \mathfrak{E}(\tau) + c\int_{\tau_0}^{\tau_1} {}^{\rho}\overset{(-1-\delta)}{\mathcal{E}'}(\tau')\,d\tau' + c\int_{\tau_0}^{\tau_1} {}^{\rho}\underset{-1}{\mathcal{E}'}(\tau')\,d\tau' \\ \leq \mathfrak{E}(\tau_0) + A\int_{\tau_0}^{\tau_1} \overset{\xi}{\underset{-1}{\mathcal{E}'}}(\tau')\,d\tau' + \int_{\mathcal{R}(\tau_0,\tau_1)} |(V^{\mu}\partial_{\mu}\psi + w\psi)F| \tag{1-6}$$

for all $\tau \in [\tau_0, \tau_1]$ and sufficiently large v, where ${}^{\rho}\overset{(-1-\delta)}{\mathcal{E}'}$ is defined similarly to ${}^{\chi}\overset{(-1-\delta)}{\mathcal{E}'}$ but with degeneration function ρ which in general vanishes on a bigger set than χ. The zeroth-order term ${}^{\rho}\underset{-1}{\mathcal{E}'}$ on the left-hand side also degenerates, unlike the analogous term in (1-3). The presence of the term multiplying the new constant A on the right-hand side is necessary in view of the fact that nonnegativity is only assumed for the highest-order terms. Here, $\overset{\xi}{\underset{-1}{\mathcal{E}'}}$ is a zeroth-order energy whose density is supported only in the support of a function $\xi(r)$:

$$\overset{\xi}{\underset{-1}{\mathcal{E}'}}(\tau) := \int_{\Sigma(\tau)} \xi(r)\psi^2.$$

Importantly, in view of our comments above, we will require that ξ vanishes identically where χ of (1-3) degenerates, i.e., $\xi = 0$ where $\chi \neq 1$. We will also require ξ to vanish for large r and, in the case where $\mathcal{S} \neq \varnothing$, in the region $r \leq r_2$.

In addition, we will make on ρ the same asymptotic nonvanishing assumption that we made previously on χ; namely that $\rho = 1$ for large r and, in the presence of a nonempty space-like boundary $\mathcal{S} \neq \varnothing$, that $\rho = 1$ in $r \leq r_2$. See already Section 3.4.3 for the detailed assumptions and Figure 2 therein for a depiction of the supports of χ, ρ and ξ.

We note that, in the case where $\mathcal{S} = \varnothing$ and T is globally timelike, one can in fact easily construct a current satisfying (a) and (b) and leading to (1-6) *irrespective of the structure of trapping and the validity of an estimate of the form* (1-3). We will show in Appendix A that, for the very slowly rotating Kerr case $|a| \ll M$, although there is no globally time-like T, there is a similar elementary construction, based essentially only on the fact that superradiance is effectively governed by a small parameter and the ergoregion is confined to a part of spacetime which can be understood using only the red-shift effect (cf. the original treatment of the boundedness problem on Kerr [Dafermos and Rodnianski 2011]). In general, however, the main difficulty in proving (1-6) is ensuring the boundary coercivity (a).

Though on its own estimate (1-6) is clearly weaker than (1-3), its robustness to direct application to (1-1) allows one to avoid the above loss of derivatives, when (1-6) is used in conjunction with (1-3) applied to (1-5). Before turning to describe how this is done in the context of the proof, we will in fact have to slightly strengthen our assumed estimates (1-3) and (1-6) as follows.

Extending to r^p-weighted estimates. First of all, we will need to extend (1-3) and (1-6) to r^p-weighted estimates, for $0 \leq p < 2$, of the type first introduced in [Dafermos and Rodnianski 2010b]:

$$\overset{(p)}{\mathcal{E}}(\tau) + \overset{(p)}{\mathcal{F}}(v) + \int_{\tau_0}^{\tau_1} {}^{\chi}\overset{(p-1)}{\mathcal{E}'}(\tau')\,d\tau' + \int_{\tau_0}^{\tau_1} \overset{(p-1)}{\underset{-1}{\mathcal{E}'}}(\tau')\,d\tau' \lesssim \overset{(p)}{\mathcal{E}}(\tau_0) + \text{inhomogeneous terms}, \tag{1-7}$$

$$\overset{(p)}{\mathfrak{F}}(v), \quad \overset{(p)}{\mathfrak{E}}(\tau) + c\int_{\tau_0}^{\tau_1} {}^{\rho}\overset{(p-1)}{\mathcal{E}'}(\tau')\,d\tau' + c\int_{\tau_0}^{\tau_1} {}^{\rho}\underset{-1}{\mathcal{E}'}(\tau')\,d\tau'$$
$$\leq \overset{(p)}{\mathfrak{E}}(\tau_0) + A\int_{\tau_0}^{\tau_1} \overset{\xi}{\underset{-1}{\mathcal{E}'}}(\tau')\,d\tau' + \text{inhomogeneous terms}. \tag{1-8}$$

In the above, $\overset{(p)}{\mathcal{E}}$ (resp. ${}^{\chi}\overset{(p-1)}{\mathcal{E}'}$, etc.) are r^p (resp. r^{p-1}, etc.) weighted versions of $\mathcal{E}$ (resp. ${}^{\chi}\overset{(-1-\delta)}{\mathcal{E}'}$, etc.), while $\overset{(p)}{\mathfrak{E}}$ and $\overset{(p)}{\mathfrak{F}}$ satisfy $\overset{(p)}{\mathfrak{E}} \sim \overset{(p)}{\mathcal{E}}$ and $\overset{(p)}{\mathfrak{F}} \sim \overset{(p)}{\mathcal{F}}$ but can moreover be represented as the flux term of a current,

$$\overset{(p)}{\mathfrak{E}}(\tau) = \int_{\Sigma(\tau)} \overset{(p)}{J}{}^{V,w,q,\varpi}_{\mu}[\psi]n^{\mu}, \quad \overset{(p)}{\mathfrak{F}}(v) = \int_{\underline{C}_v} \overset{(p)}{J}{}^{V,w,q,\varpi}_{\mu}[\psi]n^{\mu},$$

and we in fact assume that (1-8), just as (1-6), is the result of a pointwise coercive energy identity (1-4) associated to the current $\overset{(p)}{J}$.

To obtain (1-7) and (1-8), given (1-3) and (1-6), it suffices to assume the existence of currents defined only in the region $r \geq \widetilde{R}$ with suitable far-away coercivity properties. For maximal generality, we will here simply directly postulate the existence of such currents as an additional assumption. See already

Section 3.5 for the precise formulation. This assumption can be shown to follow from suitable pointwise asymptotically flat assumptions on the metric g_0 and holds of course in all our examples of interest, where the currents can easily be explicitly constructed. See Appendix B, and also [Moschidis 2016], for an even more general setting.

Extending to higher-order estimates. Secondly, we will need, through suitable commutations, to extend (1-3), (1-6), (1-7) and (1-8) to higher-order statements. We will distinguish the two energies

$$\overset{(p)}{\underset{k}{\mathcal{E}}} \quad \text{and} \quad \overset{(p)}{\underset{k}{\mathfrak{E}}}$$

at k-th order.

The energy $\overset{(p)}{\underset{k}{\mathcal{E}}}$ is the sum of the usual energy, with r^p weights, $0 \le p < 2$, and with commutation vector fields $\widetilde{\mathfrak{D}}^k$ up to order k, *spanning the entire tangent space*. The set will include the vector fields Ω_i, $i = 1, \dots, 3$, which for large r correspond to the usual rotational vector fields. (Note that the latter will be the only commutation vector fields which are r-weighted.) This is a fundamental energy with respect to which initial data can be measured and with respect to which suitable Sobolev inequalities hold.

The energy $\overset{(p)}{\underset{k}{\mathfrak{E}}}$, on the other hand, denotes the precise energy flux of the current leading to (1-4) applied moreover with a new set $\mathfrak{D}^k$ of commutation vector fields, which will include T (and Ω_1, if this is assumed to be Killing), but for which *all non-Killing vectors are cut off to vanish in a suitable region of finite r*. (See already Sections 3.6.1–3.6.4 for the definition of these commutation vector fields and energies.) Nonetheless, by our geometric assumptions and elliptic estimates, we will have that the energies

$$\overset{(p)}{\underset{k}{\mathfrak{E}}} \sim \overset{(p)}{\underset{k}{\mathcal{E}}} \tag{1-9}$$

are equivalent for solutions of $\Box_{g_0}\psi = 0$.

To extend our estimates to higher order, we will need some properties of the commutation errors arising from $[\Box_{g_0}, \mathfrak{D}^k]$. Let us note that, in the case where $\mathcal{S} = \varnothing$ and T is globally timelike, these errors are only supported in the asymptotic region of large r and can be controlled if the metric is suitably asymptotically flat. For maximum generality, we will formulate the relevant asymptotic assumption directly in terms of pointwise decay bounds for these commutators. This will include all examples of interest. See already Section 3.6.2.

In the case where $\mathcal{S} \neq \varnothing$, there will be additional commutation errors $[\Box_{g_0}, \mathfrak{D}^k]$ arising in a neighbourhood of the boundary $\mathcal{S}$ including the Killing horizon $\mathcal{H}^+$. Key to their control is the assumption of positive surface gravity of $\mathcal{H}^+$, discussed in Section 1.1.1, and the inclusion among the collection $\mathfrak{D}$ of a well-chosen vector field Y which is null and transversal to $\mathcal{H}^+$. As first shown in [Dafermos and Rodnianski 2013], commutation by Y generates a term with a good sign at $\mathcal{H}^+$ (see already Proposition 3.6.1 of Section 3.6.3), and this fact, together with elliptic estimates in $r > r_{\text{Killing}}$, an enhanced red-shift estimate near $\mathcal{H}^+$ and enhanced positivity in the black hole interior up to $\mathcal{S}$ (see already Propositions 3.4.1 and 3.4.2 of Section 3.4.5), can be used to absorb the errors arising from commutation and to extend the estimates to higher order. We note that the above-mentioned good sign is yet another manifestation of the local red-shift at the horizon $\mathcal{H}^+$ associated to the positivity of the surface gravity. To understand how all commutation errors can indeed be absorbed, see already Section 3.6.6.

The final extended estimates on (1-2) resulting from (1-3) and (1-6) take the form

$$\overset{(p)}{\underset{k}{\mathcal{E}}}(\tau)+\overset{(p)}{\underset{k}{\mathcal{F}}}+\int_{\tau_0}^{\tau_1}{}^{\chi}\overset{(p-1)}{\underset{k}{\mathcal{E}'}}(\tau')\,d\tau'+\int_{\tau_0}^{\tau_1}\overset{(p-1)}{\underset{k-1}{\mathcal{E}'}}(\tau')\,d\tau'\lesssim\overset{(p)}{\mathcal{E}}(\tau_0)+\text{inhomogeneous terms},\tag{1-10}$$

$$\begin{aligned}\overset{(p)}{\underset{k}{\mathfrak{F}}}(v),\quad \overset{(p)}{\underset{k}{\mathfrak{E}}}(\tau)+c\int_{\tau_0}^{\tau_1}{}^{\rho}\overset{(p-1)}{\underset{k}{\mathcal{E}'}}(\tau')\,d\tau'+c\int_{\tau_0}^{\tau_1}{}^{\rho}\overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau')\,d\tau'&\\ \leq\overset{(p)}{\underset{k}{\mathfrak{E}}}(\tau_0)+A\int_{\tau_0}^{\tau_1}\overset{\xi}{\underset{k-1}{\mathcal{E}'}}(\tau')\,d\tau'+\text{inhomogeneous terms},&\end{aligned}\tag{1-11}$$

where again (1-11) derives from the pointwise coercivity properties of the energy identity (1-4) associated to a current $\overset{(p)}{\underset{k}{J}}$, whose flux terms are precisely given by $\overset{(p)}{\underset{k}{\mathfrak{E}}}$ and $\overset{(p)}{\underset{k}{\mathfrak{F}}}$. See already Section 3.6.8 for the precise form of the estimates and inhomogeneous terms.

1.2. ***Assumptions on the nonlinearities $g(\psi, x)$ and $N(\partial\psi, \psi, x)$.*** We now turn to the nonlinear equation (1-1). In formulating assumptions on the allowed nonlinearities, we are here largely motivated by geometric formulations of the Einstein equations, as in [Dafermos et al. 2021], which produce "almost decoupled" equations similar to (1-1), which satisfy an appropriate form of the null condition. We emphasise, however, that the resulting equations in such a formulation do not constitute a pure system of wave equations in the form (1-1), and the quasilinear structure is mitigated through coupling with transport equations. Thus, in this context, one should really only think of (1-1) as an indicative model equation. For more specific discussion of the Einstein equations, see already Section 1.4.4.

In addressing the combined difficulties of the interaction of quasilinear terms, trapped null geodesics, and merely polynomial decay at the level of model scalar equations of the form (1-1), the most natural simplest setting is to require that the *quasilinear* term $g(\psi, x)-g_0(x)$ be supported entirely in the region $r\leq R$, and to impose on the *semilinear* term $N(\partial\psi, \psi, x)$ a generalised version of the null condition [Klainerman 1986]. This also allows us to use the exact null hypersurfaces of the background $(\mathcal{M}, g_0)$ without additional complications, making the null hierarchical structure of the estimates clearer. (In the case of the Einstein equations, in our framework, these would be replaced by null hypersurfaces of the actual dynamic spacetime $(\mathcal{M}, g)$, for instance associated to a double-null gauge.) We note, however, that use of exact null hypersurfaces is in no way fundamental; one can always replace these with suitable hyperboloidal hypersurfaces, for instance, using the setup of [Moschidis 2016].

For maximum generality, rather than attempt a general algebraic definition of the null condition for the semilinear terms $N(\partial\psi, \psi, x)$ of (1-1), we will formalise a "null condition assumption" in the form of a time-translation-invariant r^p-weighted estimate for the inhomogeneous terms of (1-10)–(1-11) when applied to (1-1), restricted entirely to the asymptotic region $r\geq R$.

To describe the condition, it will be convenient to introduce "master" energies

$${}^{\rho}\overset{(p)}{\underset{k}{\mathcal{X}}}(\tau_0, \tau_1):=\sup_v\overset{(0)}{\underset{k}{\mathcal{F}}}(v)+\sup_{\tau_0\leq\tau\leq\tau_1}\overset{(p)}{\underset{k}{\mathcal{E}}}(\tau)+\int_{\tau_0}^{\tau_1}\Big({}^{\rho}\overset{(p-1)}{\underset{k}{\mathcal{E}'}}(\tau)+{}^{\rho}\overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau)\Big)\,d\tau\tag{1-12}$$

for all $\delta\leq p\leq 2-\delta$. In the case $p=0$, which is anomalous, we will define

$$\begin{aligned}{}^{\rho}\overset{(0)}{\underset{k}{\mathcal{X}}}(\tau_0, \tau_1)&:=\sup_v\overset{(0)}{\underset{k}{\mathcal{F}}}(v)+\sup_{\tau_0\leq\tau\leq\tau_1}\overset{(0)}{\mathcal{E}}(\tau)+\int_{\tau_0}^{\tau_1}\Big({}^{\rho}\overset{(-1-\delta)}{\underset{k}{\mathcal{E}'}}(\tau)+{}^{\rho}\overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau)\Big)\,d\tau,\\ {}^{\rho}\overset{(0+)}{\underset{k}{\mathcal{X}}}(\tau_0, \tau_1)&:=\sup_v\overset{(0)}{\underset{k}{\mathcal{F}}}(v)+\sup_{\tau_0\leq\tau\leq\tau_1}\overset{(0)}{\mathcal{E}}(\tau)+\int_{\tau_0}^{\tau_1}\Big({}^{\rho}\overset{(\delta-1)}{\underset{k}{\mathcal{E}'}}(\tau')+{}^{\rho}\overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau')\Big)\,d\tau.\end{aligned}\tag{1-13}$$

We may define similar master energies ${}^{\chi}\overset{(p)}{\underset{k}{\mathcal{X}}}$, etc., with χ replacing ρ, and where extra nondegenerate lower-order terms $\overset{(p-1)}{\underset{k}{\mathcal{E}'}}$ and $\overset{(-1-\delta)}{\underset{k-1}{\mathcal{E}'}}$ are added to the integrands on the right-hand side of (1-12) and (1-13), respectively, and finally $\overset{(p)}{\underset{k}{\mathcal{X}}}$, where ρ is removed. Let us note that

$$ {}^{\rho}\overset{(p)}{\underset{k}{\mathcal{X}}} \lesssim {}^{\chi}\overset{(p)}{\underset{k}{\mathcal{X}}} \lesssim \overset{(p)}{\underset{k}{\mathcal{X}}}, \quad \overset{(p)}{\underset{k-1}{\mathcal{X}}} \lesssim {}^{\chi}\overset{(p)}{\underset{k}{\mathcal{X}}}. $$

Specifically, our null condition assumption is that in any spacetime slab $\mathcal{R}(\tau_0, \tau_1)$, the far-away (i.e., $r \geq R$) contribution of the inhomogeneous terms arising from $N(\partial\psi, \psi, x)$ in the estimates (1-10) and (1-11) can be bounded by the expressions

$$ {}^{\rho}\overset{(p)}{\underset{k}{\mathcal{X}}}(\tau_0,\tau_1)\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}(\tau_0,\tau_1)} + \sqrt{{}^{\rho}\overset{(p)}{\underset{k}{\mathcal{X}}}(\tau_0,\tau_1)}\sqrt{{}^{\rho}\overset{(0)}{\underset{k}{\mathcal{X}}}(\tau_0,\tau_1)}\sqrt{\overset{(p)}{\underset{\ll k}{\mathcal{X}}}(\tau_0,\tau_1)} \quad \text{and} \quad {}^{\rho}\overset{(0+)}{\underset{k}{\mathcal{X}}}(\tau_0,\tau_1)\sqrt{\overset{(0+)}{\underset{\ll k}{\mathcal{X}}}(\tau_0,\tau_1)} \tag{1-14} $$

for $\delta \leq p \leq 2-\delta$ and $p = 0$, respectively, and where all energies may in fact be restricted to the region $r \geq \frac{8}{9}R$, where $\rho = 1$. (Note how the $p = 0$-weighted estimate is anomalous, in that the nonlinear terms require the boundedness of a bulk term associated to the energy identity of a higher ($p > 0$) weight so as to be estimated.) See already Section 4.7 for the precise formulation of the assumption.

The assumption that one can bound the relevant inhomogeneous terms arising from $N(\partial\psi, \psi, x)$ by (1-14) isolates precisely that aspect of the usual null condition which is relevant for global existence in our method. We will then show in Appendix C that our assumption indeed holds in particular for the usual nonlinearities allowed by the classical null condition [Klainerman 1986] and includes also the class of semilinear terms $N(\partial\psi, \psi, x)$ on Kerr studied previously in [Luk 2013].

1.3. ***The dyadically localised approach to global existence: sketch of the proof of the main theorem.*** We now turn to the proof of the main theorem and the other novel aspect of the present work, namely the replacement of a global bootstrap built on global decay-in-time estimates with dyadic iteration in consecutive spacetime slabs based entirely on dyadically localised, r^p-weighted — but time-translation-invariant — estimates.

We sketch the argument here. For details, see already Section 6. In our discussion below, we will focus directly on our main case of interest, referred to later in the paper as case (iii), where the black box inhomogeneous estimate (1-3) has nontrivial degeneration and indeed does not arise from a physical-space identity, and thus must be used in conjunction with the auxiliary identity (1-6). We emphasise, however, that the dyadic approach described here is already novel in the case, referred to later in the paper as case (ii), where the black box estimate (1-3) again has degeneration but is actually a consequence of a physical-space identity (as in Schwarzschild; see the discussion following (1-4)). Finally, in the case where the black box estimate (1-3) derives from a physical-space identity and is moreover nondegenerate (i.e., $\chi = 1$ identically), referred to in the paper as case (i), then dyadic iteration is in fact unnecessary, and the approach reduces to a completely elementary time-translation-invariant estimate. (This latter case applies for instance when g_0 is a small stationary perturbation of Minkowski space.) *In the paper, we will provide self-contained treatments of all three cases so the reader can compare with the more traditional approach.*

1.3.1. *The r^p-weighted estimate hierarchy on a spacetime slab of time length L.* In the dyadic approach, the key element is an appropriate time-translation-invariant r^p-weighted hierarchy on a slab of time length L. The hierarchy will be formed by combining the r^p and higher k-order estimates associated to (1-3) and (1-6) with various choices of p weights and differentiability-order k. The estimates will depend only on a time-translation-invariant bootstrap assumption, to be retrieved within the slab, concerning only lower-order energy quantities.

Specifically, from the assumptions of Sections 1.1 and 1.2, on any spacetime slab $\mathcal{R}(\tau_0, \tau_0 + L)$ of time length $L \geq 1$, we obtain, for $p = 2 - \delta$ and $p = 1$, and for $0 < \delta < \frac{1}{10}$, the following hierarchy of estimates:

$$\overset{(p)}{\underset{k}{\mathfrak{F}}}(v),\quad \overset{(p)}{\underset{k}{\mathfrak{E}}}(\tau),\quad {}^{\rho}\overset{(p)}{\underset{k}{\mathcal{X}}}(\tau_0,\tau) \leq \overset{(p)}{\underset{k}{\mathfrak{E}}}(\tau_0) + \boxed{A\int_{\tau_0}^{\tau} \overset{\xi}{\underset{k-1}{\mathcal{E}'}}} + C\,{}^{\rho}\overset{(p)}{\underset{k}{\mathcal{X}}}(\tau_0,\tau)\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}(\tau_0,\tau)} + C\sqrt{{}^{\rho}\overset{(p)}{\underset{k}{\mathcal{X}}}(\tau_0,\tau)}\sqrt{{}^{\rho}\overset{(0)}{\underset{k}{\mathcal{X}}}(\tau_0,\tau)}\sqrt{\overset{(p)}{\underset{\ll k}{\mathcal{X}}}(\tau_0,\tau)} + \boxed{C\sup_{\tau_0\leq\tau'\leq\tau}\overset{(0)}{\mathcal{E}}(\tau')\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}(\tau_0,\tau)}\sqrt{L}}, \tag{1-15}$$

$${}^{\chi}\overset{(p)}{\underset{k-1}{\mathcal{X}}}(\tau_0,\tau) \lesssim \overset{(p)}{\underset{k-1}{\mathcal{E}}}(\tau_0) + {}^{\rho}\overset{(p)}{\underset{k-1}{\mathcal{X}}}(\tau_0,\tau)\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}(\tau_0,\tau)} + \sqrt{{}^{\rho}\overset{(p)}{\underset{k-1}{\mathcal{X}}}(\tau_0,\tau)}\sqrt{{}^{\rho}\overset{(0)}{\underset{k-1}{\mathcal{X}}}(\tau_0,\tau)}\sqrt{\overset{(p)}{\underset{\ll k}{\mathcal{X}}}(\tau_0,\tau)} + \boxed{\sup_{\tau_0\leq\tau'\leq\tau}\overset{(0)}{\mathcal{E}}(\tau')}\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}(\tau_0,\tau)}\sqrt{L}. \tag{1-16}$$

The estimates are in fact contingent on an appropriate time-translation-invariant bootstrap assumption on the lower-order energy

$$\overset{(p)}{\underset{\ll k}{\mathcal{X}}}(\tau_0,\tau) \lesssim \varepsilon \tag{1-17}$$

for $p = 0$. The hierarchy (1-15)–(1-16) descends also to $p = 0$, but where

$${}^{\rho}\overset{(0+)}{\underset{k}{\mathcal{X}}}(\tau_0,\tau)\sqrt{\overset{(0+)}{\underset{\ll k}{\mathcal{X}}}(\tau_0,\tau)} \quad\text{and}\quad {}^{\rho}\overset{(0+)}{\underset{k-1}{\mathcal{X}}}(\tau_0,\tau)\sqrt{\overset{(0+)}{\underset{\ll k}{\mathcal{X}}}(\tau_0,\tau)}$$

replace the sum of the third-to-last and second-to-last terms of (1-15) and (1-16), respectively. (This anomaly, referred to already immediately after (1-14), is a fundamental aspect of the estimates.) See already Section 6.3.1.

Estimate (1-15) arises from applying the physical-space identity associated to the current $\overset{(p)}{\underset{k}{J}}$, which in the linear case led to (1-11), *directly* to the k-times commuted equation (1-1), i.e., it arises from the divergence identity (1-4) associated to the current $\overset{(p)}{\underset{k}{J}}[g(\psi, x), \psi]$, where $g(\psi, x)$ replaces g_0 covariantly in the definition of $\overset{(p)}{\underset{k}{J}}$. (The energies $\overset{(p)}{\underset{k}{\mathfrak{E}}}$ and $\overset{(p)}{\underset{k}{\mathfrak{F}}}$ in (1-15) now in fact denote the exact fluxes arising from these currents; in view of the bootstrap assumption (1-17), one can again show the equivalence (1-9) for solutions of (1-1).) As discussed already in Section 1.1.2, it follows that (1-15) does not "lose derivatives", as is clear from examining the order of differentiability of the terms on the right-hand side. We remark that the first boxed term in (1-15) reflects the analogous term in (1-11). On the right-hand side of (1-15), we recognise the bound for the far-away contribution of the nonlinear term (1-14) from our assumption capturing the null condition, while the second boxed term in (1-15) is necessary to control the nonlinear terms on the set where ρ degenerates, for there, this boxed term cannot be absorbed in the spacetime bulk term on the left-hand side. (The bad explicit dependence on the length L arises because one must take the supremum over τ for the energy of the highest-order terms there.)

Estimate (1-16), on the other hand, arises from applying the "black box" estimate (1-10) to the nonlinear equation written in the form (1-5), i.e., now thought of simply as an inhomogeneous equation with respect to the background g_0. The boxed term in (1-16), which is k-th order, reflects the loss of derivatives due to the quasilinearity discussed already in Section 1.1.2.

The estimates can in principle close because, in view of our restrictions on the support of ξ, we have the fundamental relation

$$\underset{k-1}{\overset{\xi}{\mathcal{E}}}{}' \lesssim {}^{\chi}\underset{k-1}{\overset{(-1-\delta)}{\mathcal{E}}}{}' + \underset{k-2}{\overset{(-1-\delta)}{\mathcal{E}}}{}', \tag{1-18}$$

and thus we may bound the term

$$A\int_{\tau_0}^{\tau} \underset{k-1}{\overset{\xi}{\mathcal{E}}}{}'(\tau')\,d\tau' \lesssim {}^{\chi}\underset{k-1}{\overset{(0)}{\mathcal{X}}}(\tau_0,\tau),$$

which is in turn bounded by the term on the left-hand side of (1-16) for any p.

1.3.2. *Global existence in a slab.* The first task is to show global existence in a given slab $\mathcal{R}(\tau_0, \tau_0+L)$ for arbitrary $L \geq 1$, given suitable (L-dependent) smallness assumptions at $\Sigma(\tau_0)$.

We first note that the estimate (1-15) alone can be used to easily show *local* existence, by which we mean existence in an entire smaller slab of the form $\mathcal{R}(\tau_0, \tau_0+\epsilon)$, provided that some $\underset{k}{\overset{(p)}{\mathcal{E}}} \lesssim \varepsilon_0$ for sufficiently high k. This is in fact true also for $p=0$. (Note that this "local" result already uses nontrivially the null condition assumption of Section 1.2. This is because our foliation $\Sigma(\tau)$ is outgoing null for $r \geq R$. Equations (1-1) not satisfying some version of the null condition can fail to yield solutions in $\mathcal{R}(\tau_0, \tau_0+\epsilon)$ for any $\epsilon > 0$, no matter how small the data are.) Moreover, we show that smallness of a suitably high $\underset{k}{\overset{(p)}{\mathcal{E}}}$ norm defines a continuation criterion.

In view of the above, for global existence, it suffices to show that, under suitable assumptions at $\tau = \tau_0$, the quantity $\underset{k}{\overset{(p)}{\mathcal{E}}}$, for some $p \in \{0\} \cup [\delta, 2-\delta]$ and suitably high k, remains globally small in the slab. Examining the hierarchy, for global existence in a slab $\mathcal{R}(\tau_0, \tau_0+L)$ of length L, it seems necessary to have at least

$$\underset{\ll k}{\overset{(0)}{\mathcal{X}}}(\tau_0,\tau) \lesssim \varepsilon L^{-1}. \tag{1-19}$$

Indeed, in view of the nonlinear terms, the L^{-1} factor in (1-19) represents a natural threshold, as $s=1$ represents the minimum value of s for which $\sqrt{L^{-s}}\sqrt{L} \lesssim 1$, allowing the quantities in a fixed slab to be easily bounded by their initial values.

Assuming

$$\underset{k}{\overset{(1)}{\mathcal{E}}}(\tau_0) \lesssim \varepsilon_0, \quad \underset{k-2}{\overset{(0)}{\mathcal{E}}}(\tau_0) \lesssim L^{-1}\varepsilon_0, \tag{1-20}$$

which of course imply, for $\varepsilon_0 \ll \varepsilon$, both (1-17) (for $p=1$) and (1-19) at $\tau = \tau_0$, an easy continuity argument, with (1-17) and (1-19) taken as bootstrap assumptions, shows that if estimates (1-20) hold for $\tau = \tau_0$, then the solution indeed exists globally in the entire slab $\mathcal{R}(\tau_0, \tau_0+L)$.

See already Proposition 6.3.2 for the details of the proof. Note that our above appeal to (1-17) and (1-19) is in fact the only instance that a bootstrap assumption must be used in the proof of our main theorem.

1.3.3. *Improved estimates and the pigeonhole principle.* Once global existence within a slab has been established, one can improve a posteriori the estimates given stronger assumptions on initial data. Anticipating dyadic iteration, one seeks a set of initial estimates at $\tau = \tau_0$ with the property that they are *exactly recoverable* at the top of the slab $\tau_0 + L$, with suitable redefinition of ε_0 and with L replaced by $2L$.

Re-examining the hierarchy (1-15)–(1-16), we show that, for an appropriate large constant α and any $L \geq 1$, $\tau_0 \geq 1$, for $\hat{\epsilon}_0$ sufficiently small and given for example the stronger estimates

$$\underset{k}{\overset{(1)}{\mathfrak{E}}}(\tau_0) \leq \hat{\varepsilon}_0, \quad \underset{k-2}{\overset{(2-\delta)}{\mathfrak{E}}}(\tau_0) \leq \hat{\varepsilon}_0, \quad \underset{k-2}{\overset{(0)}{\mathfrak{E}}}(\tau_0) \leq \hat{\varepsilon}_0 \alpha L^{-1}, \quad \underset{k-4}{\overset{(1)}{\mathfrak{E}}}(\tau_0) \leq \hat{\varepsilon}_0 \alpha L^{-1+\delta}, \quad \underset{k-6}{\overset{(0)}{\mathfrak{E}}}(\tau_0) \leq \hat{\varepsilon}_0 \alpha^2 L^{-2+\delta}, \tag{1-21}$$

we have that (1-21) holds also at the final time $\tau_0 + L$, where τ_0, L and $\hat{\varepsilon}_0$ are now however replaced by

$$\tau_0 + L, \quad 2L \quad \text{and} \quad \hat{\varepsilon}_0(1 + \alpha L^{-\frac{1}{4}}), \tag{1-22}$$

respectively. Note that the statement that the last three inequalities of (1-21) hold at $\tau_0 + L$ with $2L$ in place of L is a *stronger* smallness assumption than that assumed for these quantities at τ_0, signifying that these quantities have in fact decayed, and relies on a pigeonhole argument as in [Dafermos and Rodnianski 2010b], based on the fundamental relation that, for $p - 1 \geq 0$, the *bulk* integrand $\underset{k-2}{\overset{(p-1)}{\mathcal{E}'}}$ of the p-weighted identity (1-16) at order $k-1$ (when integrated, included in the $\chi \underset{k-2}{\overset{(p-1)}{\mathcal{X}}}$ master energy) controls the *boundary* term of the $(p-1)$-weighted identity at arbitrary order k, denoted without the prime:

$$\underset{k}{\overset{(p-1)}{\mathcal{E}'}} \gtrsim \underset{k}{\overset{(p-1)}{\mathcal{E}}}. \tag{1-23}$$

Briefly, the pigeonhole principle is applied as follows: The integral of the left-hand side of (1-23) for $p = 1$ and k replaced by $k-2$ can be shown to be bounded $\lesssim \hat{\varepsilon}_0$, whence we obtain the existence of a $\tau = \tau''$ slice for which the right-hand side $\underset{k-2}{\overset{(0)}{\mathcal{E}}}(\tau'')$ is bounded $\lesssim \hat{\varepsilon}_0 L^{-1}$, where the L^{-1} factor arises from the length of the interval. We can then propagate this to $\tau_0 + L$ and use that $\alpha \gg 1$ to arrange that we have the precise new version of the third inequality of (1-21) at $\tau_0 + L$, i.e., for $\underset{k-2}{\overset{(0)}{\mathfrak{E}}}(\tau_0 + L)$. Similarly, the integral of the left-hand side of (1-23) for $p = 2 - \delta$ and k replaced by $k-4$ can be shown to be bounded $\lesssim \hat{\varepsilon}_0$, whence we obtain the existence of a $\tau = \tau''$ slice for which the right-hand side $\underset{k-4}{\overset{(1-\delta)}{\mathcal{E}}}(\tau'')$ is bounded $\lesssim \hat{\varepsilon}_0 L^{-1}$, the L^{-1} factor again arising from the length of the interval. Interpolating with the boundedness statement $\underset{k-4}{\overset{(2-\delta)}{\mathcal{E}}}(\tau'') \lesssim \hat{\varepsilon}_0$, we obtain that on this slice

$$\underset{k-4}{\overset{(1)}{\mathcal{E}}}(\tau'') \lesssim \hat{\varepsilon}_0 L^{-1+\delta}.$$

Again, we can then propagate this to $\tau_0 + L$ and use that $\alpha \gg 1$ to arrange that we have the precise new version of the fourth inequality of (1-21) at $\tau_0 + L$, i.e., for $\underset{k-4}{\overset{(1)}{\mathfrak{E}}}(\tau_0 + L)$. Finally, a similar argument, applying (1-23) for $p = 1$ and $k-6$, yields the new version of the last inequality of (1-21).

Note that, to obtain the new versions of the first two inequalities of (1-21) at $\tau_0 + L$, which refer to quantities that do not decay with respect to their initial values, it is important that the constant appearing in the first term on the right-hand side of (1-15) is indeed 1.

1.3.4. *Iteration over consecutive spacetime slabs of dyadic time length $L_i = 2^i$.* Let us assume that our initial data on $\tau_0 := 1$ satisfy

$$\overset{(1)}{\underset{k}{\mathcal{E}}}(\tau_0) + \overset{(2-\delta)}{\underset{k-2}{\mathcal{E}}}(\tau_0) \leq \varepsilon_0$$

for sufficiently small ε_0. It follows that (1-21) is satisfied for $\tau_0 = 1$, $L := L_0 = 1$ and appropriate $\hat{\varepsilon}_0$.

Examining closely (1-21) and (1-22), we see that we have now obtained a series of estimates which, in addition to being sufficient to obtain existence in $\mathcal{R}(\tau_0, \tau_0 + L)$, are moreover preserved under dyadic iteration. We simply iterate the above statement on consecutive spacetime slabs of dyadic time length $L_i = 2^i$, defining $\tau_{i+1} = \tau_i + L_i = 2^i$ and $\hat{\varepsilon}_{i+1} = \hat{\varepsilon}_i(1 + \alpha L_i^{-1/4})$. Note that $\hat{\varepsilon}_{i+1} \lesssim \hat{\varepsilon}_0 \lesssim \varepsilon_0$ for all i.

This immediately yields global existence in $\mathcal{R}(\tau_0, \infty) = \bigcup_{i \geq 0} \mathcal{R}(\tau_i, \tau_{i+1})$, the top-order boundedness statement

$$\overset{(1)}{\underset{k}{\mathcal{E}}}(\tau) + \overset{(2-\delta)}{\underset{k-2}{\mathcal{E}}}(\tau) \lesssim \varepsilon_0 \tag{1-24}$$

and the decay estimate

$$\overset{(0)}{\underset{k-6}{\mathcal{E}}}(\tau_i) \lesssim \varepsilon_0 \tau^{-2+\delta}.$$

Thus, our result is a true orbital stability statement at highest order in addition to yielding asymptotic stability.

1.4. *Discussion.*

We end with some additional discussion.

1.4.1. *The semilinear case.* We have already remarked that in the semilinear case, i.e., the case where $g(\psi, x) = g_0$ identically and the equation takes the form

$$\Box_{g_0} \psi = N(\partial\psi, \psi, x), \tag{1-25}$$

there is no loss of derivatives when estimating (1-25) as a linear inhomogeneous equation. Thus, one can base the entire argument directly on the black box estimate (1-3), without the need for the auxiliary physical-space based estimate (1-6), provided of course that the appropriate asymptotic flatness and commutation assumptions are made so that (1-3) can be extended to the higher-order weighted (1-10). See already Remark 5.2 for the modified statement appropriate in this case and a guide to its proof. In particular, our theorem applies directly to the purely semilinear version of (1-1) on Kerr in the full subextremal range $|a| < M$.

1.4.2. *Additional examples and extensions.* We collect here some additional examples to which our results apply, as well as various natural future extensions.

Other spacetimes $(\mathcal{M}, g_0)$ satisfying our assumptions. We have already specifically mentioned the examples where $(\mathcal{M}, g_0)$ is Minkowski space, Schwarzschild and Kerr, but let us remark that our black box assumption (1-3) holds in fact for the full subextremal Kerr–Newman family [Civin 2015]. We also note that combining the use of an energy current construction similar to Appendix A with [Wunsch and Zworski 2011], one can show that (1-3) holds for *arbitrary* stationary perturbations $(\mathcal{M}, g_0)$ of Schwarzschild, provided that they are close to Schwarzschild in a suitably regular norm. Since we have already remarked that the physical-space based estimate (1-6) holds also for such spacetimes, in fact under much weaker regularity assumptions, this means that our main theorem indeed applies in this case. Thus, for very slowly rotating $|a| \ll M$ Kerr spacetimes, one sees that the special additional structure of Kerr (like the various manifestations of Carter separability, the Killing tensor, etc.) has no significance for the problem, and the spacetimes can just be understood as a very small stationary perturbation of Schwarzschild with a Killing horizon. *It is only in the case $|a| \sim M$ where true Kerr properties are of any relevance.* A treatment of this case, reducing directly to the black box estimate proven in [Dafermos et al. 2016], will appear elsewhere.

The axisymmetric case. In the case where we assume that the metric g_0 admits an additional Killing field Ω_1, one may also restrict to nonlinear equations of the form (1-1) which moreover preserve this symmetry, i.e., with the property that if the data are preserved by Ω_1, then so is the solution. Let us call such equations and solutions axisymmetric. Under the geometric assumptions of this paper, for axisymmetric equations (1-1) and restricted to axisymmetric solutions, one may in fact easily derive the existence of the necessary auxiliary physical-space identity yielding (1-6), just as in Schwarzschild. This in particular applies to Kerr–Newman in the full subextremal case. Thus, in view of [Dafermos et al. 2016; Civin 2015] which obtain (1-3), the main theorem of the paper applies also in these settings. We leave the details as an exercise for the reader. Note that in this case, if one prefers, one may quote the original [Dafermos and Rodnianski 2010a] for (1-3) for axisymmetric solutions in place of the more general (1-3) obtained for all solutions in [Dafermos et al. 2016].

Potentials. Our method applies also if a suitably decaying T-independent potential $\mathcal{V}(x)$ is added to (1-1), i.e., for

$$\Box_{g(\psi,x)}\psi - \mathcal{V}(x)\psi = N(\partial\psi, \psi, x),$$

provided that both the black box assumption (1-3) and the physical-space based (1-6) hold for solutions of the inhomogeneous version of the new linearised equation $\Box_{g_0}\psi - \mathcal{V}(x)\psi = F$ in place of (1-2). We note that, in the case where T is globally timelike and $\mathcal{V}$ decays suitably and is nonnegative, one can always obtain also the analogue of (1-6). The same statement applies for any such potential in the slowly rotating Kerr case $|a| \ll M$.

Systems. For notational convenience, we have restricted here to scalar equations. The considerations generalise readily to systems, for which a generalised null condition can again be formulated in terms of bounds of the nonlinear terms by (1-14); this now includes many examples. See also the discussion of the weak null condition in Section 1.4.5 below.

The obstacle problem. We have already remarked that, with a mild adaptation of the setup, one can also consider the case where the boundary $\mathcal{S}$ is in fact timelike, imposing now also suitable boundary conditions on $\mathcal{S}$. This is for instance the setting for the classical obstacle problem. The analogue of (1-3) has indeed been proven in the nontrapping case (with $\chi = 1$ identically, i.e., without degeneration) but also, with suitable degeneration functions χ, for many examples with nontrivial hyperbolic trapping; see for example [Lafontaine 2022]. We note again that the additional physical-space estimate (1-6) can always be retrieved in this case. Thus, this situation can also be incorporated in our framework.

Extremal black holes. Extremal black holes do not satisfy our assumptions on $(\mathcal{M}, g_0)$, already because the stationary vector field T will not be tangential to the boundary $\mathcal{S}$, but more seriously because χ must degenerate at the black hole horizon [Sbierski 2015], where T will not be timelike. Nonetheless, a version of (1-3) has been proven in the extremal Reissner–Nordström case [Aretakis 2011] (and in the extremal Kerr case, under the assumption of axisymmetry [Aretakis 2012]), with an additional hierarchy of degeneration associated to the horizon, and nonlinear stability for semilinear equations has been proven [Angelopoulos et al. 2020], where, however, an additional null structural condition is required at the horizon, in full analogy with the situation at null infinity. It would be interesting to reformulate this latter work in the dyadic setup used here. This is related to the nonlinear stability problem of extremal black holes. See the discussion in [Dafermos et al. 2021] and Conjecture IV.2 therein.

The asymptotically AdS case. Finally, although our assumptions are modelled on the asymptotically flat setting, one could also try to reformulate things in the asymptotically anti-de Sitter case, appropriate for solutions of the Einstein equations with negative cosmological constant $\Lambda < 0$ (for a discussion of the $\Lambda > 0$ case see already Section 1.4.6 below). Here, one must also impose boundary conditions *at infinity*, since infinity itself can be thought of as an asymptotic *time-like* boundary. We note that, under reflecting boundary conditions, there exist periodic (and thus nondecaying) solutions of the wave equation on pure AdS, while general solutions on the Schwarzschild–anti de Sitter and Kerr–anti de Sitter case [Holzegel and Smulevici 2013; 2014], again under such boundary conditions, decay only inverse logarithmically. (This is due to stable photon orbits repeatedly reflecting off of infinity only to return later having been repelled centrifugally by the black hole.) Based on this lack of decay, pure AdS has in fact been proven to be nonlinearly *unstable* under reflecting boundary conditions as a solution to various Einstein-matter systems, where the problem can be studied already under spherical symmetry [Moschidis 2023]. Thus, if one hopes to show nonlinear stability for an equation of the type (1-1) on asymptotically AdS backgrounds, it is more natural to consider *dissipative* boundary conditions like those considered in [Holzegel et al. 2020]. Note that on Kerr–AdS, one expects to indeed satisfy the analogue of (1-6) with the help of the Hawking–Reall Killing vector field, provided that the parameters satisfy the Hawking–Reall bound. Thus, given a version of (1-3), nonlinear stability for a suitable class of equations (1-1) should be tractable for all such Kerr–AdS parameters.

1.4.3. *The dyadic approach vs. the traditional approach.* The dyadic approach followed here is of course closely related to the more traditional approach. Our L-weighted smallness translates easily into t-decay assumptions, and if one wishes one can rewrite the argument using a bootstrap with t-decaying norms. We believe, however, that the dyadic localisation of the argument both makes the proof more modular and may serve to better identify possible future refinements with respect to regularity and decay. One sees clearly, for instance, that the L^{-1} smallness assumptions necessary for global existence *within a slab* are manifestly weaker than the $L^{-2+\delta}$ smallness necessary to improve and iterate (in fact one may weaken this to $L^{-1-\epsilon}$). Since bootstrap is only used to show existence *within a slab*, this already simplifies the argument considerably. We also note that the $L^{-1-\epsilon}$ threshold corresponds to pointwise decay $t^{-1/2-\epsilon/2}$, considerably weaker than the $t^{-1-\epsilon}$ "improved decay" which is often invoked for global existence and stability for (1-1). Proving sharper decay is of course an extremely important problem in itself (see [Angelopoulos et al. 2018; 2023; Hintz 2022; Kehrberger 2022]) but is not necessary (or even particularly helpful) for nonlinear stability.

Let us also point out that for problems with gauge invariance, like the Einstein equations, dyadic localisation also provides a convenient opportunity to refresh the gauge. This suggests an alternative approach to the global teleological normalisations of gauge done in [Dafermos et al. 2019b; 2021].

1.4.4. *Applications to the Teukolsky equation and derived equations and to the nonlinear stability of black holes.* As discussed in Section 1.2, one motivation for the precise assumptions on the nonlinearities of (1-1) made here is viewing these as a model for understanding the Einstein equations, when the latter are considered under suitable geometrically defined gauges. Such gauges were first exploited analytically in the monumental proof of the nonlinear stability of Minkowski space [Christodoulou and Klainerman 1993].

The geometric-gauge approach to black hole stability was taken up in [Dafermos et al. 2019b], where the problem was studied in double-null gauge and the full linear stability of Schwarzschild was first proven. (See [Christodoulou 2009] for an introduction to double-null gauge, including a discussion of the characteristic initial value problem and a derivation of all equations.) The setup is of course intimately connected to the formalism of Newman and Penrose [1962]. In the system resulting from linearising the reduced Einstein equations in double-null gauge around Schwarzschild, the gauge-invariant quantities are determined by the extremal curvature components α and $\underline{\alpha}$, which each satisfy the Teukolsky equation, first derived in this setting by [Bardeen and Press 1973] (and generalised to Kerr in [Teukolsky 1973]). The transport equations satisfied by the residual gauge-dependent quantities, on the other hand, are always already *linearly* coupled to the gauge-invariant ones. To analyse first the gauge-invariant quantities, [Dafermos et al. 2019b] introduced a pair of novel physical-space quantities P and $\underline{P}$, related to α and $\underline{\alpha}$ by second-order weighted null differential operators, but satisfying the more tractable Regge–Wheeler equation (an equation first derived in a different context in [Regge and Wheeler 1957]), which, unlike the Teukolsky equation, could be understood by the exact same methods as the linear wave equation. In particular, an analogue of (1-3) was proven, via a physical-space identity, for P and $\underline{P}$, and this led to boundedness and decay through the hierarchical structure of the system, first for the quantities P and $\underline{P}$ themselves, then for the original gauge-independent quantities α and $\underline{\alpha}$, and then, after teleological normalisation of the double-null gauge, for all residual gauge-dependent quantities as well. (For a complete scattering theory for this system, see [Masaood 2022; 2024]. For other approaches to the gauge in Schwarzschild under linear theory, see [Benomio 2024] and the references discussed in Section 1.4.5 below.) The origin of this relation between Teukolsky and Regge–Wheeler goes back to the fixed frequency transformation theory of [Chandrasekhar 1975]. On the basis of this linear theory, the full nonlinear stability of the Schwarzschild family, without symmetry, was proven in our [Dafermos et al. 2021]. (For previous nonlinear results for Schwarzschild under symmetry assumptions, see [Christodoulou 1987; Dafermos and Rodnianski 2005] in the case of the Einstein-scalar field system under spherical symmetry, and [Holzegel 2010; Klainerman and Szeftel 2020] for vacuum, the former in the higher-dimensional case under biaxial Bianchi symmetry, and the latter under polarised axisymmetry, reducing to 2+1 dimensions, the first such result beyond 1+1-dimensional reductions.) Note that since Schwarzschild is a codimension-3 subfamily of Kerr, nonlinear asymptotic stability of Schwarzschild refers to constructing the full (teleologically defined) codimension-3 set of initial data which asymptote to Schwarzschild in the future.

A generalisation of the quantities P and $\underline{P}$ of [Dafermos et al. 2019b] to the slowly rotating Kerr case $|a| \ll M$ was given independently by [Dafermos et al. 2019a; Ma 2020]. The equations satisfied by these quantities are now however (weakly) coupled to the quantities satisfying the Teukolsky equation. The works [Dafermos et al. 2019a; Ma 2020] both show an analogue of (1-3) for this system, not via a physical-space identity however but based on the framework for frequency-localised estimates on Kerr introduced in [Dafermos and Rodnianski 2010a]. These results were followed by full linear stability statements for the gauge-dependent quantities in various gauges [Andersson et al. 2025; Häfner et al. 2021] (the work [Häfner et al. 2021] considers in fact harmonic gauge; cf. the discussion in Section 1.4.5). The estimates of [Dafermos et al. 2019a; Ma 2020] have recently been reformulated by [Giorgi et al. 2024] in the language of the higher-order physical-space commutation by the Carter tensor first introduced in [Andersson and Blue 2015]. The work [Giorgi et al. 2024] then uses this, among other ingredients, in the context of the formalism of [Giorgi et al. 2020] to obtain nonlinear stability results for the very slowly rotating Kerr case $|a| \ll M$, completing an impressive series of preprints. For generalisations to the Einstein–Maxwell system, see [Giorgi 2020; 2021; Apetroaie 2023].

The full subextremal Kerr case $|a| < M$ is more subtle, as it cannot be understood as a small perturbation of Schwarzschild. Remarkably, however, an analogue of (1-3) for this system has been obtained in the full subextremal case [Shlapentokh-Rothman and da Costa 2020; 2023] using the already highly nontrivial mode stability results [Whiting 1989; Shlapentokh-Rothman 2015; Teixeira da Costa 2020] (see also [Andersson et al. 2017]) but, also, nontrivial new high-frequency structure with no apparent precise analogue in the context of the wave equation. To exploit frequency analysis, the proof of [Shlapentokh-Rothman and da Costa 2020] uses a version of the frequency localisation framework of [Dafermos et al. 2016]. (In connection with the Teukolsky equation, we also note [Ma and Zhang 2023; Millet 2023] for precise asymptotics in the special case of smooth compactly supported data. For a discussion of what are the appropriate initial assumptions on data for the Teukolsky equation in various physical settings, see [Gajic and Kehrberger 2022].)

In view of the method introduced in the present paper, it should be clear that there is absolutely nothing to fear in these types of frequency localisations for nonlinear applications, provided that they are indeed used to prove a spacetime-localised statement which can be expressed in the form (1-3). The results of [Dafermos et al. 2019a; Ma 2020] and [Shlapentokh-Rothman and da Costa 2020] can thus in principle be used *directly* for the nonlinear problem, in fact, as “black box” results. Thus, in our view, given the breakthrough of [Shlapentokh-Rothman and da Costa 2020], the technical complications in extending the nonlinear Schwarzschild analysis of [Dafermos et al. 2021] to the full subextremal case $|a| < M$ of Kerr may not be as severe as one might naively have thought.

1.4.5. *Harmonic gauge and the weak null condition.* In our discussion in Section 1.4.4 we have focussed above primarily on “geometric” gauges, but, as is well known, another way to relate (1-1) to the Einstein equations is via the harmonic gauge condition (see for instance [Lindblad and Rodnianski 2010], where the nonlinear stability of Minkowski space is proven in this gauge). The resulting reduced equations (for $\psi^{\mu\nu} = g^{\mu\nu} - g_0^{\mu\nu}$) produce additional complicated linear terms and moreover fail to satisfy the null condition, although they do satisfy the so-called *weak null condition* introduced in [Lindblad and Rodnianski 2003]. Note that a linear stability result has been given for this system in the Schwarzschild case [Johnson 2019] (see also [Hung 2018]) and, as mentioned earlier, also on very slowly rotating Kerr $|a| \ll M$ in [Häfner et al. 2021].

In order to model the Einstein equations in harmonic gauge, classes of equations (1-1) satisfying the weak null condition have been studied in the recent [Lindblad and Tohaneanu 2018; 2020; 2024]. Though we do not here implement our method in this latter setting, nonetheless, we emphasise that our analysis can in principle be extended to equations or systems satisfying the weak null condition, following [Keir 2018], under an appropriate black box assumption for the linearisation.

1.4.6. *Comparison with the $\Lambda > 0$ case.* Finally, it is interesting to compare with the $\Lambda > 0$ case. Here, the analogue of the Schwarzschild and Kerr black holes are the Schwarzschild–de Sitter and Kerr–de Sitter family. See [Dafermos and Rodnianski 2013] for a discussion of their geometry. The study of decay for semilinear and quasilinear equations of the type (1-1) on Kerr–de Sitter was pioneered by Hintz and Vasy [2016], eventually leading to their groundbreaking proof of the nonlinear stability of very slowly rotating ($|a| \ll M,\ \Lambda$) Kerr–de Sitter in harmonic gauge. The proof appeals to extensive machinery from microlocal analysis, with an elaborate compactification of spacetime, and with a Nash–Moser iteration. For a more recent approach replacing Nash–Moser iteration with a global bootstrap, see [Fang 2021].

Returning to the model equation (1-1), an elementary new method was recently introduced by Mavrogiannis [Mavrogiannis 2024; 2022] to treat nonlinear stability on Schwarzschild–de Sitter and Kerr–de Sitter backgrounds. In [Mavrogiannis 2024], the entire analysis is reduced to a "black box" estimate for the linear problem on time-slabs of some *fixed* length L. The required black box linear estimate, however, is not just the analogue of the (degenerate) integrated local energy estimate (1-3) (first proven in this context in [Dafermos and Rodnianski 2007b]), but a higher-order refinement of (1-3) which is *relatively nondegenerate*, i.e., where the bulk and boundary term have identical degeneration function and thus again are comparable. This is possible through a commutation with a globally defined well-chosen vector field orthogonal to the photon sphere and vanishing at the horizons. This is an analogue of an energy originally constructed in the Schwarzschild case in [Holzegel and Kauffman 2020]. (In the Kerr–de Sitter case, both the degenerate integrated local energy decay statement and the accompanying relatively nondegenerate statement are now proven using frequency localisation in a framework similar to [Dafermos et al. 2016].) Exponential decay is then a derived statement which follows from iterating a suitable estimate on consecutive slabs, each now of fixed length L.

In comparison to the asymptotically flat case studied in the present paper, it is noteworthy that in the Kerr–de Sitter case one does *not* require an additional nontrivial physical-space-based ingredient, analogous to (1-6), other than this new, relatively nondegenerate version of the black box linear estimate (1-3). From the time-slab-localised point of view of the two works, the reason for the difference is clear: since estimates in [Mavrogiannis 2024] have been reduced to a fixed time scale L, the role of (1-6) is essentially provided by *Cauchy stability*, a completely soft statement.

From this point of view, the difference between the Schwarzschild and Kerr case on the one hand and their de Sitter analogues on the other is more than simply one between polynomial and exponential decay, but is one between *dyadically localised* and *truly local*. In this approach, it is this fundamentally local nature of the analysis in the de Sitter case that renders the *nonlinear* aspects of stability problems on such de Sitter backgrounds to be essentially soft.

1.5. ***Outline of the paper.*** We end with an outline of the remainder of the paper.

In Section 2 we shall describe the geometric assumptions on the background spacetime, followed in Section 3 by the assumptions on properties of the linear inhomogeneous equation (1-2). We shall then introduce in Section 4 the class of nonlinear equations (1-1) that we shall consider, stating in particular a local well-posedness theorem and continuation criterion. We shall give the precise statement of the main theorem in Section 5. The proof will then be carried out in Section 6.

The paper also contains four appendices. In Appendix A, we show how to obtain (1-6) from a physical-space identity in the very slowly rotating Kerr case. In Appendix B, we will show how to define the far-away currents encoding the r^p method necessary to extend the estimates (1-3) and (1-6) to (1-7) and (1-8), respectively. In Appendix C, we show that our null condition assumption encoded by the bounds (1-14) indeed includes the classical null condition [Klainerman 1986] and also the more general class of semilinear terms considered on Kerr in [Luk 2013]. In Appendix D, we show explicitly how to obtain the inhomogeneous estimate (1-3) in the Kerr case from the homogeneous estimates of [Dafermos et al. 2016]. Together, these appendices show that our main theorem holds in particular for a wide class of equations of the form (1-1) on very slowly rotating $|a| \ll M$ Kerr backgrounds (and, for the semilinear case $g = g_0$, in the full subextremal range $|a| < M$).

2. Geometric assumptions on the background spacetime

We consider a manifold $\mathcal{M}$ with a background metric g_0 satisfying certain assumptions. In this section, we collect the assumptions which concern directly the geometry of $(\mathcal{M}, g_0)$. The assumptions here will not be the most general possible but are sufficient to include the main examples of interest. They can be easily further generalised in various directions if desired. Some of the assumptions are in principle redundant, but we have not attempted to derive them from the most minimal considerations. We note already that the assumptions of this section are modelled on (and are satisfied by — see Section 2.7!) the basic cases of Minkowski space and the (extended) exterior regions of Schwarzschild and subextremal $|a| < M$ Kerr spacetimes.

2.1. ***Underlying differential structure and the positive function r.*** For definiteness, we consider the underlying differential structure of $\mathcal{M}$ to be given by $\mathbb{R}^4_{(x^0,x^1,x^2,x^3)}$ or alternatively by the manifold with boundary $\mathbb{R}^4 \setminus \{|x^1|^2 + |x^2|^2 + |x^3|^2 < r_0^2\}$. (The black hole examples in fact correspond to the latter case.) In this latter case, let us define

$$r := \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}. \tag{2-1}$$

In the former case where the underlying manifold is $\mathbb{R}^4$, let us pick an arbitrary $r_0 > 0$ and define

$$r := f(\sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}), \tag{2-2}$$

where $f : [0, \infty) \to [r_0, \infty)$ is a smooth function with $f'(v) > 0$ such that $f(z) = \sqrt{r_0^2 + z^2}$ near $z = 0$ and $f(z) = z$ for $z \geq 2r_0$. Thus, in all cases r is a smooth positive function on $\mathcal{M}$ satisfying $r \geq r_0 > 0$. Let us also fix a large $R \geq 20r_0$.

We may also define ambient spherical coordinates in the usual way by the relation

$$(x^1, x^2, x^3) = (r \cos\varphi \sin\vartheta, r \sin\varphi \sin\vartheta, r \cos\vartheta). \tag{2-3}$$

In the case of nonempty boundary, we will denote by $\Omega_1 = \partial_\varphi$, Ω_2, and Ω_3 the standard rotation vector fields associated to the ambient spherical coordinates $(x^0, r, \vartheta, \varphi)$. These are globally regular vector fields.

In the case where the underlying structure is $\mathbb{R}^4$, the above vector fields would only be regular for $r > r_0$. In this case then, we will use the notation Ω_i, $i = 1, \ldots, 3$, for the above standard vector fields multiplied by $\omega(r)$, for a smooth cut-off function satisfying $\omega(r) = 1$ for $r \geq \frac{1}{2}R$ and $\omega(r) = 0$ in a neighbourhood of r_0. These are again globally regular vector fields.

2.2. ***The Lorentzian metric g_0, time orientation, and the causality of the boundary.*** We assume that g_0 is a time-oriented Lorentzian metric on $\mathcal{M}_0$ such that the coordinate vector field ∂_{x^0} is future-directed timelike for $r \geq \frac{1}{6}R$.

In the case where $\mathcal{M}$ is a manifold with boundary $\mathcal{S} = \{r = r_0\}$, we assume that the boundary $\mathcal{S}$ is spacelike and the future-directed normal points out of the spacetime.

2.3. ***The stationary Killing field T.*** Denoting now by T the coordinate vector field

$$T = \partial_{x^0}$$

with respect to the ambient coordinates (x^0, x^1, x^2, x^3), we assume that T is Killing with respect to the metric g_0. It follows by the previous assumptions that the ambient function r above is T-invariant and that T is future-directed timelike in the region $r \geq \frac{1}{6}R$. Let us further assume that $g(T, T) \to -1$ as $r \to \infty$.

We will denote by ϕ_τ the 1-parameter group of diffeomorphisms generated by T.

In the case where $\mathcal{S} = \emptyset$, it is natural to already assume that T is globally timelike, in view of the Friedman instability [Moschidis 2018] and its evanescent analogue [Keir 2020] which applies in the marginal case, both of which would be incompatible with assumptions we will make later on concerning the behaviour of waves.

In the case where $\mathcal{S} \neq \emptyset$, we notice that T cannot be globally timelike as T is tangential to and thus spacelike on $\mathcal{S} = \{r = r_0\}$. In this case we will need to assume the existence of a Killing horizon.

2.4. ***The Killing horizon $\mathcal{H}^+$.*** If $\mathcal{S} \neq \emptyset$, we will assume the following further properties:

- We assume that there exists an r_{Killing} such that the hypersurface $\mathcal{H}^+ := \{r = r_{\text{Killing}}\}$ is a Killing horizon with future-directed null generator $Z = T$ or, more generally, with future-directed null generator Z in the span of T and Ω_1, in which case we will assume that Ω_1 is globally Killing. We will denote by Z the globally defined Killing field given by this combination of T and Ω_1.

- We will assume furthermore that $\mathcal{H}^+$ has strictly positive surface gravity, i.e., $\nabla_Z Z = \kappa(\vartheta, \varphi) Z$ for some $\kappa > 0$, and we will assume that T, or more generally the span of T and Ω_1, is timelike for $r > r_{\text{Killing}}$.
- Finally, in the case $\mathcal{S} \neq \varnothing$, we will assume that the vector field ∇r is future-directed timelike in the region $r < r_{\text{Killing}}$. Note that ∇r on $\mathcal{H}^+ = \{r = r_{\text{Killing}}\}$ will be null and in the direction of Z.

2.5. *Foliations, subregions, and volume forms.*

2.5.1. *The foliation* $\Sigma(\tau)$. We will assume $\mathcal{M}$ admits a hypersurface Σ_0 with 2-dimensional corner at $\Sigma_0 \cap \{r = R\}$ such that $\Sigma_0 \cap \{r \leq R\}$ is strictly spacelike, $\Sigma_0 \cap \{r \geq R\}$ is null, and Σ_0 is transversal to T.

We assume that, for $r' \geq \frac{1}{2}R$, Σ_0 is transversal to the hypersurface $\{r = r'\}$ and that Ω_1, Ω_2, Ω_3 are tangent to $\Sigma_0 \cap \left\{r \geq \frac{1}{2}R\right\}$. In particular, the 2-dimensional space $T_p(\Sigma_0 \cap \{r \leq R\}) \cap T_p(\Sigma_0 \cap \{r \geq R\})$ should coincide with the span of these vectors.

If $\mathcal{S} \neq \varnothing$, we assume that Σ_0 is transversal to the hypersurface $\mathcal{S}$ and $\Sigma_0 \cap \mathcal{S}$ is diffeomorphic to the 2-sphere. We assume finally that the vector field Z of Section 2.4 is orthogonal to $\Sigma_0 \cap \mathcal{H}^+$, while Ω_1, Ω_2, Ω_3 are tangent to $\Sigma_0 \cap \mathcal{H}^+$.

We assume that Σ_0 separates $\mathcal{M}$ into two connected components and that Σ_0 is a past Cauchy hypersurface for $J^+(\Sigma_0)$.

Writing $\Sigma(\tau) := (\phi_\tau)_*(\Sigma_0)$, we assume

$$J^+(\Sigma_0) = \bigcup_{\tau \geq 0} \Sigma(\tau),$$

where J^+ denotes causal future in $\mathcal{M}$ with respect to the metric g_0.

Clearly $\{\Sigma(\tau)\}_{\tau \in \mathbb{R}}$ defines a foliation of $\mathcal{M}$ and thus defines globally on $\mathcal{M}$ a Lipschitz function τ, which is smooth separately on $r \leq R$ and $r \geq R$.

Note that by our assumptions above, for $r \geq \frac{1}{2}R$, we have that $\tau = \tau(x^0, r)$ and the triple (r, ϑ, φ) represent a smooth coordinate system on $\Sigma(\tau) \cap \{r \geq R\}$ (modulo the spherical degeneration).

We will denote by L a smooth choice of the future-directed null generator of $\Sigma_0 \cap \{r \geq R\}$ normalised to satisfy the constraint $g(L, T) \sim -1$. By translation invariance, this extends to a smooth vector field on all of $\{r \geq R\}$ in the direction of the null generator of $\Sigma(\tau)$. (Note that we also use L to denote a general length parameter; in practice these two notations will not interfere with one another.)

2.5.2. *The regions* $\mathcal{R}(\tau_0, \tau_1)$. Let us define

$$\mathcal{R}(\tau_0, \tau_1) := \bigcup_{\tau_0 \tau \leq \tau_1} \Sigma(\tau).$$

We shall refer to such regions as spacetime slabs. We will also use the notation

$$\mathcal{R}(\tau_0, \infty) := \bigcup_{\tau \geq \tau_0} \Sigma(\tau).$$

Note that $\mathcal{R}(\tau_0, \infty) = J^+(\Sigma_0)$.

We shall define

$$\mathcal{S}(\tau_0, \tau_1) := \mathcal{S} \cap \mathcal{R}(\tau_0, \tau_1).$$

2.5.3. *The ingoing cones $\underline{C}_v$ and truncated regions.* We also assume that the region $r \geq R$ is foliated by translation-invariant "ingoing" null cones $\underline{C}_v$ parametrised by a smooth function v defined on $r \geq R$, increasing towards the future, which may moreover be expressed as $v(x^0, r)$. In particular, the vector fields Ω_i are tangent to $\underline{C}_v$. We again define a smooth future-directed null generator $\underline{L}$ of $\underline{C}_v$, normalised by the constraint $g(\underline{L}, T) \sim -1$ and translation-invariant; this defines a smooth vector field on $r \geq R$.

Let us assume moreover that $g(\underline{L}, L) \sim -1$ globally in $r \geq R$.

Note that, under the above assumptions, r is constant on $\underline{C}_v \cap \Sigma(\tau)$ and the future boundary of $\underline{C}_v$ is the set $\{r = R\} \cap \Sigma(\tau(v))$, where this relation defined $\tau(v)$.

If $\tau_0 \leq \tau_1 \leq \tau(v)$, we shall define

$$\mathcal{R}(\tau_0, \tau_1, v) := \mathcal{R}(\tau_0, \tau_1) \setminus \bigcup_{\tilde{v} > v} \underline{C}_{\tilde{v}}$$

and

$$\Sigma(\tau, v) := \Sigma(\tau) \setminus \bigcup_{\tilde{v} > v} \underline{C}_{\tilde{v}}.$$

The spacetime region $\mathcal{R}(\tau_0, \tau_1, v)$ is a compact subset of spacetime with past boundary $\Sigma(\tau_0, v)$ and future boundary $\mathcal{S}(\tau_0, \tau_1) \cup \Sigma(\tau_1, v) \cup \underline{C}_v$.

2.5.4. *Comparison of volume forms.* We will assume that, in the region $r \geq R$, writing the volume form for $(\mathcal{M}, g)$ as

$$dV_{\mathcal{M}} = \upsilon(r, \vartheta, \varphi)\, d\tau r^2\, dr \sin\vartheta\, d\vartheta\, d\varphi,$$

for $\Sigma(\tau) \cap \{r \geq R\}$, with the choice L for the normal, as

$$dV_{\Sigma(\tau)\cap\{r\geq R\}} := \tilde{\upsilon}(r, \vartheta, \varphi) r^2\, dr \sin\vartheta\, d\vartheta\, d\varphi,$$

and for $\underline{C}_v$, with the choice $\underline{L}$ for the normal, as

$$dV_{\underline{C}_v} := \tilde{\tilde{\upsilon}}(r, \vartheta, \varphi) r^2\, dr \sin\vartheta\, d\vartheta\, d\varphi,$$

we then have

$$\upsilon \sim \tilde{\upsilon} \sim \tilde{\tilde{\upsilon}} \sim 1.$$

With this assumption, it follows by the coarea formula and compactness that, globally, the volume form of $(\mathcal{M}, g_0)$ is related to the volume form of $\Sigma(\tau)$,

$$dV_{\mathcal{M}} \sim d\tau\, dV_{\Sigma(\tau)},$$

where $\sim$ is interpreted for 4-forms in the obvious sense.

Note that when volume forms are omitted from integrals, the above induced volume forms from the metric g_0 will be understood, unless otherwise noted.

2.6. ***Other vector fields.*** It will be useful to extend the vector fields defined above to a global set of vector fields which span the tangent space $T_x\mathcal{M}$ for all $x \in \mathcal{M}$.

2.6.1. *The global extensions of the vector fields L, $\underline{L}$, and Ω_i.* For notational convenience, in the case where $\mathcal{S} = \varnothing$, let us define $\Omega_4 = (1-\omega(r))\partial_{x^1}$, $\Omega_5 = (1-\omega(r))\partial_{x^2}$, $\Omega_6 = (1-\omega(r))\partial_{x^3}$, where ω is as in Section 2.1, and let us define L and $\underline{L}$ to be translation-invariant extensions of the vector fields defined already with the property that L, $\underline{L}$ and $\Omega_1, \dots, \Omega_6$ span the tangent space globally. Note that this is easy to satisfy in general. We are moreover not requiring that L and $\underline{L}$ be null, and the Ω_i, $i = 1, \dots, 6$, are not required to be tangential to the ambient spheres for $r \le \frac{1}{2}R$. (See already Section 2.7.1.)

In the case where $\mathcal{S} \neq \varnothing$, we will define L and $\underline{L}$ to be smooth ϕ_τ-invariant extensions of L and $\underline{L}$ to $r_0 \le r \le R$, with the property that L, $\underline{L}$ everywhere span the same plane spanned by the coordinate vector fields ∂_{x^0} and ∂_r of ambient spherical coordinates $(x^0, r, \vartheta, \varphi)$. This again can easily seen to be possible. Note again that we are not requiring these vector fields to be globally null.

2.6.2. *The notation $|\nabla\!\!\!/\psi|^2$.* We define the notation

$$|\nabla\!\!\!/\psi|^2 := \sum_i r^{-2}|\Omega_i\psi|^2.$$

We note that, under our assumptions, the above expression is comparable to the induced gradient squared on the space-like spheres $\Sigma(\tau) \cap \{r = r'\}$ in the region $r \ge \frac{1}{2}R$.

In view, however, of our spanning assumptions, in both the case $\mathcal{S} = \varnothing$ and $\mathcal{S} \neq \varnothing$, the expression

$$|L\psi|^2 + |\underline{L}\psi|^2 + |\nabla\!\!\!/\psi|^2$$

will always be a translation-invariant coercive expression on first derivatives of ψ, similar (by compactness) to any other such coercive expression in $r \le R$, while in the region $r \ge R$ it will have the right weights in r so as to be a suitable reference point for natural energy expressions.

2.6.3. *The 1-forms ϱ^L, $\varrho^{\underline{L}}$, ϱ^1, ϱ^2, and ϱ^3.* We will need to introduce a set of weighted spanning 1-forms on the sphere so as to properly measure weighted boundedness of forms. Define 1-forms ϱ^L, $\varrho^{\underline{L}}$, and ϱ^1, ϱ^2, ϱ^3 on the far region $\mathcal{M} \cap \{r \ge R\}$ as follows.

The region $\mathcal{M} \cap \{r \ge R\}$ can be written as the product manifold $\mathcal{M} \cap \{r \ge R\} = \mathbb{R} \times [R, \infty) \times S^2$. Let $\pi : \mathcal{M} \cap \{r \ge R\} \to \mathbb{R} \times [R, \infty)$ denote the canonical projection. Let ϱ^L, $\varrho^{\underline{L}}$ be defined by $\varrho^L = \pi^*\tilde{\varrho}^L$ and $\varrho^{\underline{L}} = \pi^*\tilde{\varrho}^{\underline{L}}$, where $\tilde{\varrho}^L$ and $\tilde{\varrho}^{\underline{L}}$ are the dual coframes of $\pi_* L$ and $\pi_*\underline{L}$, respectively, on $\mathbb{R} \times [R, \infty)$. It follows that

$$\varrho^L(L) = \varrho^{\underline{L}}(\underline{L}) = 1, \quad \varrho^L(\underline{L}) = \varrho^{\underline{L}}(L) = \varrho^L(\Omega_i) = \varrho^{\underline{L}}(\Omega_i) = 0, \ \ i = 1, 2, 3.$$

For all smooth functions ψ, one can then write

$$d\psi = L\psi\, \varrho^L + \underline{L}\psi\, \varrho^{\underline{L}} + d\!\!\!/\psi,$$

where $d\!\!\!/\psi(L) = d\!\!\!/\psi(\underline{L}) = 0$.

For each $i = 1, 2, 3$, there is a polar coordinate system (ϑ^i, φ^i) on S^2, associated to the corresponding rotation vector field Ω_i, with the property that, in the $(x^0, r, \vartheta^i, \varphi^i)$ coordinate system for $\mathcal{M} \cap \{r \ge R\}$, one has $\partial_{\varphi^i} = \Omega_i$. Such a (ϑ^i, φ^i) coordinate system on S^2 is unique up to a choice of meridian. Define then, for $i = 1, 2, 3$,

$$\varrho^1 = r \sin\vartheta^1\, d\varphi^1, \quad \varrho^2 = r \sin\vartheta^2\, d\varphi^2, \quad \varrho^3 = r \sin\vartheta^3\, d\varphi^3.$$

There exist (nonunique, but universal) bounded functions $a^i{}_j(\vartheta, \varphi)$ for $i, j = 1, 2, 3$ such that, for any smooth function $\psi : \mathcal{M} \cap \{r \geq R\} \to \mathbb{R}$,

$$d\psi = L\psi \varrho^L + \underline{L}\psi \varrho^{\underline{L}} + r^{-1} a^i{}_j(\vartheta, \varphi) \Omega_i \psi \varrho^j. \tag{2-4}$$

2.6.4. *The vector field Y.* In the case $\mathcal{S} \neq \varnothing$, we may define Y to be a ϕ_τ-invariant vector field such that: Y is future-directed null on $\mathcal{H}^+$, transversal to $\mathcal{H}^+$, and orthogonal to $\mathcal{H}^+ \cap \Sigma_0$; Y is supported in $r \leq r_1 + \frac{1}{2}(r_2 - r_1)$ for some $r_2 > r_1 > r_{\text{Killing}}$; and Z, Y, and Ω_i span the tangent space in $r \leq r_1 + \frac{1}{4}(r_2 - r_1)$.

The existence of such a vector field in a neighbourhood of a Killing horizon follows from [Dafermos and Rodnianski 2013].

In the case where $\mathcal{S} = \varnothing$, since we are assuming that T is globally timelike, we may simply set $Y = 0$.

2.7. ***Examples: Minkowski, Schwarzschild and Kerr.*** We note that Schwarzschild and Kerr in the full subextremal black hole range of parameters $|a| < M$ satisfy the assumptions of this section with an appropriate definition of the underlying differential structure.

2.7.1. *Minkowski.* For the Minkowski case, we consider the underlying manifold to be $\mathbb{R}^4$, i.e., without boundary, and we define the metric to be the familiar expression

$$g_0 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2.$$

We define $r_0 := 1$, which will determine the function r. Let us distinguish this function from $\tilde{r} := \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$ which we may think of as a function $\tilde{r}(r)$. Fixing any $R \geq 20$, we note that $u = t - \tilde{r}$, $v = t + \tilde{r}$ are null coordinates. We may define $L = \partial_u$, $\underline{L} = \partial_v$, with respect to coordinates $(u, v, \vartheta, \varphi)$, in the region $r \geq 2$.

We may take

$$\Sigma_0 = (\{t = 0\} \cap \{r \leq R\}) \cup (\{v = \tilde{r}(R)\} \cap \{r \geq R\})$$

and $\underline{C}_v$ defined by the level sets of v.

Note that $r/\tilde{r} \sim 1$ for $r \geq r_0$ and that the spacetime volume form is $\tilde{r}^2 \sin\vartheta \, d\tilde{r} \, d\tau \, d\vartheta \, d\varphi$, while the volume form on $\Sigma_0 \cap \{r \geq R\}$ and $\underline{C}_v$ is $\tilde{r}^2 \sin\vartheta \, d\tilde{r} \, d\vartheta \, d\varphi$ with our choice of L and $\underline{L}$.

Note also, in this case, we define $\Omega_4 = \eta(r)\partial_{x^1}$, $\Omega_5 = \eta(r)\partial_{x^2}$, and $\Omega_6 = \eta(r)\partial_{x^3}$, where η is a cutoff vanishing with $\eta = 1$ for $r \leq \frac{1}{4}$ and $\eta = 0$ for $r \geq \frac{1}{2}$, and we can extend L and $\underline{L}$ simply by $L = (1 - \eta)\partial_u + \eta \partial_v$ and $\underline{L} = (1 - \eta)\partial_v$. (We may then define the cutoff ω of Section 2.1, so that $\omega = 1$ for $r \geq \frac{1}{4}$ and $\omega = 0$ for $r \leq \frac{1}{8}$.) We have of course $Y = 0$ in this case.

2.7.2. *Schwarzschild.* For the Schwarzschild case, given a real parameter $M > 0$, we may define $r_0 := (2 - \delta_1)M$ for sufficiently small $0 < \delta_1 \ll M$, denote x^0 by t^*, let r be defined by (1-1) and standard spherical coordinates (ϑ, φ) on $\mathbb{R}^3$ by (2-3), and define the metric g_M to be

$$-\left(1 - \frac{2M}{r}\right)(dt^*)^2 + \frac{4M}{r} \, dr \, dt^* + \left(1 + \frac{2M}{r}\right) dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta \, d\varphi^2).$$

We define the vector fields Ω_i to be the standard Killing vector fields associated to spherical symmetry, and we define $T = \partial_{t^*}$ to be the coordinate vector field with respect to the above coordinates.

Note that $r = 2M$ is a Killing horizon with null generator T and positive surface gravity, and the hypersurfaces $r = r'$ for $r' \in [r_0, 2M)$ are indeed spacelike.

We may define the function t in the region $r > 2M$ by

$$t = t^* - 2M \log(r - 2M).$$

We note that, in the coordinates (t, r, θ, ϕ), the metric takes the familiar form

$$-\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta \, d\varphi^2).$$

We may define $r^* = r + 2M \log(r - 2M)$, and we may define coordinates u and v in $r > 2M$ by $u = t - r^*$ and $v = t + r^*$.

We may fix $R \geq 20r_0$ and define now $L = \partial_v$ and $\underline{L} = \partial_u$ to be the coordinate vector fields with respect to (u, v, θ, ϕ) coordinates in $r \geq R$. We may extend these to be globally defined linearly independent translation-invariant vector fields in the span of the coordinate vector fields ∂_{t*} and ∂_r.

We then may define

$$\Sigma_0 = (\{t^* = 0\} \cap \{r \leq R\}) \cup (\{v = R\} \cap \{r \geq R\}),$$

which satisfies all the required transversality properties, etc.

Finally, we may define for instance Y to be a translation-invariant vector field in the span of ∂_{t*} and ∂_r, which is null and future-directed and satisfies $g(\partial_{t*}, Y) = -2$ at $r = 2M$, such that Y is supported entirely in $r \leq r_1 + \frac{1}{2}(r_2 - r_1)$, with $r_1 := (2 + \delta_1)M$.

2.7.3. *Kerr.* To put Kerr in our preferred form, one uses a combination of Kerr star coordinates (based on Boyer–Lindquist coordinates) and double-null coordinates. We leave the details to the reader but note the following.

Given subextremal parameters $|a| < M$ and defining

$$r_+ = M + \sqrt{M^2 - a^2},$$

we set $r_0 = r_+ - \delta_1$ for a small δ_1.

One may define the ambient differential structure such that the r of (2-1) will coincide with the Boyer–Lindquist r of Appendix A in the region $r \leq \frac{1}{3}R$, while for $r \geq \frac{1}{2}R$ it will coincide with the coordinate r_* of Section C.2. The coordinate v coincides with the double-null v of Section C.2, and $\Sigma(\tau) \cap \{r \geq R\}$ will be a hypersurface of constant u, where again u is as defined in Section C.2.

Further, one may set things up so that $\Omega_1 = \partial_\varphi$ is the axisymmetric Killing field.

Let us note finally that extremal Kerr (corresponding to $|a| = M$) cannot in fact be put into our preferred form already because of our requirement that T be tangential to $\mathcal{S}$ and thus spacelike. Note also that the Killing horizon of extremal Kerr has zero surface gravity, which would also contradict our assumptions of Section 2.4.

2.8. ***Table of r-parameters.*** We collect finally a list of important r-values in increasing order in Table 1. Some will only be introduced later in the paper. The parameters r_{Killing}, r_1, r_2 only occur if $\mathcal{S} \neq \varnothing$.

r_0	$r \geq r_0$; moreover, if $\mathcal{S} \neq \varnothing$, then $\mathcal{S} = \{r = r_0\}$
r_{Killing}	$\mathcal{H}^+ = \{r = r_{\text{Killing}}\}$ a Killing horizon with positive surface gravity; span of T and Ω_1 is timelike for $r > r_{\text{Killing}}$
r_1	parameter related to the vector field Y if $\mathcal{S} \neq \varnothing$
$r_1 + \frac{1}{4}(r_2 - r_1)$	$Z,\ Y,\ \Omega_i$ span the tangent space for $r_0 \leq r_1 + \frac{1}{4}(r_2 - r_1)$
$r_1 + \frac{1}{2}(r_2 - r_1)$	commutation vector fields $\mathfrak{D}$ all Killing for $r_1 + \frac{1}{2}(r_2 - r_1) \leq r \leq \frac{1}{2}R$
r_2	$\rho = 1,\ \chi = 1$ for $r \leq r_2$
$\frac{1}{6}R$	T timelike for $r \geq \frac{1}{6}R$
$\frac{1}{4}R$	$\rho = 1,\ \chi = 1$ for $r \geq \frac{1}{4}R$
$\frac{1}{2}R$	commutation vector fields $\mathfrak{D}$ all Killing for $r_1 + \frac{1}{2}(r_2 - r_1) \leq r \leq \frac{1}{2}R$; $g = g_0$ for $r \geq \frac{1}{2}R$
$\frac{8}{9}R$	the generalised null condition assumption concerns $r \geq \frac{8}{9}R$
R	$\Sigma(\tau) \cap \{r \geq R\}$ is null, $\underline{C}_v \subset \{r \geq R\}$
$\widetilde{R}$	parameter related to positivity properties of far-away currents

Table 1. Important r-values in this paper.

3. Assumed identities and estimates for $\Box_{g_0}\psi = F$

Our fundamental assumptions in this paper are connected with the behaviour of solutions of the linear inhomogeneous equation (1-2) on the exactly stationary background g_0.

3.1. ***Constants and parameters.*** Before stating assumptions, we make a remark concerning constants and parameters.

Given a spacetime $(\mathcal{M}, g_0)$ satisfying the assumptions of Section 2, we will consider the parameters of Section 2.8 as fixed. Let us also fix once and for all a

$$0 < \delta < \tfrac{1}{10}. \tag{3-1}$$

We will use k to denote integers ≥ 0 which will parametrise number of derivatives.

In inequalities, we will denote by C and c generic positive constants depending only on (a) $(\mathcal{M}, g_0)$ (with the choice of r-parameters), (b) the above choice of δ, and (c) if there is k-dependence in the relevant statement, also on k. (We use C for large constants and c for small constants.)

For nonnegative quantities A and B, the notation

$$A \lesssim B$$

means $A \leq CB$, while

$$A \sim B$$

means $cB \leq A \leq CB$.

The reader should be prepared to distinguish between $\leq$ and $\lesssim$, as both will appear!

For a discussion of additional smallness parameters depending also on the nonlinearity, see already Section 4.2.

3.2. ***Basic degenerate integrated local energy estimate.*** As discussed in the introduction, our basic assumption will be that of a (degenerate) spacetime-localised integrated local energy decay statement for the inhomogeneous equation (1-2).

The statement is that a certain energy flux on $\Sigma(\tau)$ plus a nonnegative bulk are controlled by an initial energy flux on $\Sigma(\tau_0)$ and a spacetime integral over the slab $\mathcal{R}(\tau_0, \tau)$ relating to the inhomogeneous term F. The controlled bulk integral is allowed to degenerate where a certain degeneration function $\chi = \chi(r)$ vanishes, but it is assumed to control a zeroth-order term without degeneration (with decaying weight at infinity).

Let us thus define

$$\chi : [r_0, \infty) \to [0, 1] \tag{3-2}$$

to be a function such that $\chi = 1$ for $r \geq \frac{1}{4}R$ and $r \leq r_2$.

The assumed estimate is

$$\sup_{v:\tau\leq\tau(v)} \overset{(0)}{\mathcal{F}}(v, \tau_0, \tau), \quad \overset{(0)}{\mathcal{E}}(\tau) + c\overset{(0)}{\mathcal{E}}_{\mathcal{S}}(\tau) + c\int_{\tau_0}^{\tau} {}^{\chi}\overset{(-1-\delta)}{\mathcal{E}'}(\tau')\,d\tau' + c\int_{\tau_0}^{\tau} \underset{-1}{\mathcal{E}'}(\tau')\,d\tau'$$
$$\leq \lambda\overset{(0)}{\mathcal{E}}(\tau_0) + C\int_{\mathcal{R}(\tau_0,\tau)} |(V_0^{\mu}\partial_{\mu}\psi + w_0\psi)F| + C\int_{\mathcal{R}(\tau_0,\tau)} F^2 \tag{3-3}$$

for some constants $\lambda \geq 1,\ C \geq 1,\ 0 < c < 1$, and where V_0 is a fixed vector field and w_0 is a fixed function satisfying

$$|g_0(V_0, L)| \lesssim 1, \quad |g_0(V_0, \underline{L})| \lesssim 1, \quad \sum |g_0(V_0, \Omega_i)|^2 \lesssim 1, \quad |w_0| \lesssim r^{-1}. \tag{3-4}$$

Here, the unprimed energies are defined by

$$\overset{(0)}{\mathcal{E}}(\tau) := \int_{\Sigma(\tau)} |L\psi|^2 + |\slashed{\nabla}\psi|^2 + \iota_{r\leq R}|\underline{L}\psi|^2 + r^{-2}\psi^2, \tag{3-5}$$

$$\overset{(0)}{\mathcal{E}}_{\mathcal{S}}(\tau) := \int_{\mathcal{S}(\tau_0,\tau)} |L\psi|^2 + |\slashed{\nabla}\psi|^2 + |\underline{L}\psi|^2 + \psi^2, \tag{3-6}$$

$$\overset{(0)}{\mathcal{F}}(\tau_0, \tau, v) := \int_{\underline{C}_v\cap\mathcal{R}(\tau_0,\tau)} |\underline{L}\psi|^2 + |\slashed{\nabla}\psi|^2 + r^{-2}\psi^2, \tag{3-7}$$

while the primed energies are defined by

$$\begin{aligned} {}^{\chi}\overset{(-1-\delta)}{\mathcal{E}'}(\tau) &:= \int_{\Sigma(\tau)} r^{-1-\delta}\chi(r)(|L\psi|^2 + |\underline{L}\psi|^2 + |\slashed{\nabla}\psi|^2), \\ \underset{-1}{\mathcal{E}'}(\tau) &:= \int_{\Sigma(\tau)} r^{-3-\delta}\psi^2. \end{aligned} \tag{3-8}$$

(The prime $'$ notation will be used in general to denote energies that naturally appear in bulk terms.) The estimate (3-3) is to hold for all smooth ψ such that the right-hand side of (3-3) is finite. In the above, we already see the δ fixed in (3-1). For future reference let us also define the quantity

$$\overset{(-1-\delta)}{\mathcal{E}'}(\tau) := \int_{\Sigma(\tau)} r^{-1-\delta}(|L\psi|^2 + |\underline{L}\psi|^2 + |\nabla\!\!\!/\psi|^2) + r^{-3-\delta}\psi^2. \tag{3-9}$$

Note that if $\chi = 1$ identically, then ${}^{\chi}\overset{(-1-\delta)}{\mathcal{E}'} + \underset{-1}{\mathcal{E}'} = \overset{(-1-\delta)}{\mathcal{E}'}$. We note that all expressions defined above are T-invariant.

We distinguish the constants C and λ in (3-3) to highlight the significance of the case $\lambda = 1$, when (3-3) is derived from a suitable energy identity and the energies are replaced by exact fluxes. See already Section 3.4.1.

Note that in cases (i) and (ii), we shall see that (3-3) holds, and in fact one may drop the $\int F^2$ term on the right-hand side.

In fact, in all of the above cases more precise estimates than (3-3) are available with respect to what can actually be controlled by the right-hand side; we shall not need to make use of this here.

Again, the assumptions are motivated by our model cases of Minkowski space, Schwarzschild, and subextremal Kerr black hole exteriors. The estimate (3-3) indeed holds in the case of Kerr in the full subextremal range $|a| < M$ (this is the statement of Theorem D.1 of Appendix D). We note, however, the following remark.

Remark 3.2.1. In our arguments in subsequent sections, we will in fact only use the statement that follows from (3-3) by partitioning the middle term on the right-hand side into the regions $r \le R$ and $r \ge R$, and applying Cauchy–Schwarz to the former, replacing thus the full middle term with the expression

$$\sqrt{\int_{\mathcal{R}(\tau_0,\tau_1)\cap\{r\le R\}} |L\psi|^2 + |\underline{L}\psi|^2 + |\nabla\!\!\!/\psi|^2 + r^{-2}|\psi|^2}\sqrt{\int_{\mathcal{R}(\tau_0,\tau_1)\cap\{r\le R\}} F^2} + \int_{\mathcal{R}(\tau_0,\tau)\cap\{r\ge R\}} |(V_0^{\mu}\partial_\mu\psi + w_0\psi)F|. \tag{3-10}$$

We could thus alternatively consider this weaker statement, i.e., (3-3) but with (3-10) replacing the middle term on the right-hand side, as our main assumption. We prefer, however, to keep our general assumption in the form of (3-3) because it is more compact and we do not know of a physical example where this weaker assumption holds but our original assumption does not. On the other hand, it is often easier to prove the weaker assumption directly than to prove (3-3). This is indeed the case for Kerr in the full subextremal range $|a| < M$, where in Appendix D we will in fact only give a proof of this weaker version.

3.3. ***Physical-space identities on a general Lorentzian metric.*** As explained in the introduction, for nonlinear applications to (1-1), it is most convenient when (3-3) is the result of a *physical-space energy* identity (1-4) for solutions ψ of (1-2). Thus, we will recall some general properties of such identities here.

3.3.1. *Definition of energy currents.* Physical-space energy identities can be associated to a quadruple (V, w, q, ϖ), where V^μ is a vector field on $\mathcal{M}$, w is a scalar function, q_μ is a 1-form, and $\varpi_{\mu\nu}$ is a 2-form. Given such a quadruple, a general Lorentzian metric g, and a suitably regular function ψ, we define

$$J_\mu^{V,w,q,\varpi}[g,\psi] := T_{\mu\nu}[g,\psi]V^\nu + w\psi\partial_\mu\psi + q_\mu\psi^2 + *d(\psi^2\varpi)_\mu, \tag{3-11}$$

$$K^{V,w,q}[g,\psi] := \pi^V_{\mu\nu}[g]T^{\mu\nu}[g,\psi] + \nabla^\mu w\psi\partial_\mu\psi + w\nabla^\mu\psi\partial_\mu\psi + \nabla^\mu q_\mu\psi^2 + 2\psi q_\mu g^{\mu\nu}\partial_\nu\psi, \tag{3-12}$$

$$H^{V,w}[\psi] := V^\mu\partial_\mu\psi + w\psi, \tag{3-13}$$

where

$$T_{\mu\nu}[g,\psi] = \partial_\mu\psi\partial_\nu\psi - \tfrac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\psi\partial_\beta\psi, \quad \pi^X_{\mu\nu}[g] = \tfrac{1}{2}(\mathcal{L}_X g)_{\mu\nu} = \tfrac{1}{2}(\nabla_\mu X_\nu + \nabla_\nu X_\mu),$$

and where $*\colon \Lambda^3\mathcal{M} \to \Lambda^1\mathcal{M}$ denotes the Hodge star operator.

Let us note already that it is often more natural to parametrise choices of currents in a slightly different way as twisted currents [Holzegel and Warnick 2014]; for instance, given a scalar function, one may define a twisted energy momentum tensor $\widetilde{T}_{\mu\nu}[g,\psi]$ according to (A-8) and then consider for instance currents of the form $\widetilde{J}^V_\mu[g,\psi] := \widetilde{T}_{\mu\nu}[g,\psi]V^\nu$, etc. Such a current can always be rewritten as (3-11) for some w, q, ϖ.

We note that the additional ϖ component in (3-11) arises naturally for twisted currents and is useful in generating positive zeroth-order flux terms on the boundary.

3.3.2. *The divergence identity.* With the above definitions, one can compute the following identity for a general function ψ:

$$\nabla^\mu_g J_\mu^{V,w,q,\varpi}[g,\psi] = K^{V,w,q}[g,\psi] + H^{V,w}[\psi]\Box_g\psi. \tag{3-14}$$

In particular, for solutions of the covariant wave equation $\Box_g\psi = 0$, one obtains a divergence relation between the currents J and K which both depend only on the 1-jet of ψ. See [Christodoulou 2000] for a discussion of the classification of currents with this property. Note that ϖ does not contribute to the bulk current K and neither ϖ nor q contribute to H, which moreover is independent of the metric g.

The significance of identity (3-14) is that it can be integrated in a spacetime region bounded by homologous hypersurfaces to obtain a relation between boundary fluxes of J and a bulk integral of K. We give the form of this relation below in the special case of the region $\mathcal{R}(\tau_0,\tau_1,v) \subset \mathcal{M}$.

3.3.3. *The integrated identity on* $\mathcal{R}(\tau_0,\tau_1,v)$. For a solution of (1-2), identity (3-14) upon integration in $\mathcal{R}(\tau_0,\tau_1,v)$ yields

$$\int_{\Sigma(\tau_1,v)} J[\psi]\cdot n + \int_{\mathcal{S}(\tau_0,\tau_1)} J[\psi]\cdot n + \int_{\underline{C}_v(\tau_0,\tau_1)} J[\psi]\cdot n + \int_{\mathcal{R}(\tau_0,\tau_1,v)} K[\psi]$$
$$= \int_{\Sigma(\tau_0,v)} J[\psi]\cdot n - \int_{\mathcal{R}(\tau_0,\tau_1,v)} H[\psi]F, \tag{3-15}$$

where the normals and volume forms are with respect to the metric g_0 according to the convention of Section 2.5.4.

Identity (3-15) will be useful when it satisfies suitable *positivity properties for its bulk and boundary terms*.

3.4. ***Assumed unweighted first-order physical-space identities: cases*** **(i)–(iii).** As discussed already in the introduction, we shall *not* in general require that (3-3) is the result of an integrated divergence identity (3-15) associated to currents with a pointwise coercivity property. We shall, however, require, *in addition* to assuming (3-3), the existence of currents generating an identity (3-15) with much weaker nonnegativity properties, properties which in particular are insensitive to the presence and structure of possible trapping.

It is indeed useful to see first, however, how the existence of a physical-space proof of (3-3) via an identity of type (3-15) simplifies the considerations. Thus, we shall distinguish two simpler cases, to be called case (i) and (ii), to be discussed in Sections 3.4.1 and 3.4.2 below, where indeed (3-3) is proven via an identity (3-15), (with case (i) corresponding to the even simpler setting where there is no degeneration at all in the coercivity).

The most general case, case (iii), which represents the main goal of this paper, will be discussed in Section 3.4.3. We will introduce some helpful common notation in Section 3.4.4.

Finally, in Section 3.4.5, for the case $\mathcal{S}\neq\varnothing$, we will provide a family of currents with enhanced red-shift control at $\mathcal{H}^+$ and in the black hole interior, parametrised by a parameter ς, which will be useful for obtaining higher-order estimates in Section 3.6.

3.4.1. *Case* (i). The simplest case to consider is when (3-3) indeed follows from a physical-space energy identity (3-15), and when moreover there is in fact no degeneration in the estimate (3-3), i.e., the function χ of (3-2) satisfies $\chi=1$ identically.

That is to say, we assume that there exists a T-invariant quadruple (V,w,q,ϖ), where V is a vector field, w is a scalar function, q is a 1-form, and ϖ is a 2-form, satisfying

$$\begin{gathered}
|g_0(V,L)|\lesssim 1,\quad |g_0(V,\underline{L})|\lesssim 1,\quad \sum|g_0(V,\Omega_i)|^2\lesssim 1,\\
|w|\lesssim r^{-1},\quad |q_\mu L^\mu|\lesssim r^{-2},\quad |q_\mu \underline{L}^\mu|\lesssim r^{-2},\\
|(*d\varpi)_\mu L^\mu|\lesssim r^{-2},\quad |(*d\varpi)_\mu \underline{L}^\mu|\lesssim r^{-2},\\
|*(\varrho^L\wedge\varpi)_\mu L^\mu|\lesssim r^{-1},\quad |*(\varrho^L\wedge\varpi)_\mu \underline{L}^\mu|\lesssim r^{-1},\\
|*(\varrho^{\underline{L}}\wedge\varpi)_\mu L^\mu|\lesssim r^{-1},\quad |*(\varrho^{\underline{L}}\wedge\varpi)_\mu \underline{L}^\mu|\lesssim r^{-1},\\
|*(\varrho^i\wedge\varpi)_\mu L^\mu|\lesssim r^{-1},\quad |*(\varrho^i\wedge\varpi)_\mu \underline{L}^\mu|\lesssim r^{-1},\qquad i=1,2,3,
\end{gathered}\tag{3-16}$$

such that, defining $J^{V,w,q,\varpi}[\psi]$ and $K^{V,w,q}$ by (3-11)–(3-12), the energy identity (3-15) corresponding to these currents has the following properties: (a) the boundary terms of (3-15) on $\Sigma(\tau)$ are coercive, (b) the remaining boundary terms are nonnegative, and (c) the bulk term K is nonnegative and coercive, with no degeneration, except "at infinity", where standard derivatives are in general only controlled with weight $r^{-1-\delta}$.

More precisely, we assume the pointwise bulk coercivity relation

$$K^{V,w,q}[\psi]\gtrsim r^{-1-\delta}((L\psi)^2+(\underline{L}\psi)^2+|\nabla\!\!\!/\psi|^2)+r^{-3-\delta}\psi^2\tag{3-17}$$

and pointwise boundary coercivity relations

$$\begin{aligned} J^{V,w,q,\varpi}_\mu[\psi]n^\mu_{\Sigma(\tau)} &\gtrsim (L\psi)^2+|\nabla\!\!\!/\psi|^2+\iota_{r\le R}(\underline{L}\psi)^2+r^{-2}\psi^2,\\ J^{V,w,q,\varpi}_\mu[\psi]n^\mu_{\underline{C}_v} &\gtrsim (\underline{L}\psi)^2+|\nabla\!\!\!/\psi|^2+r^{-2}\psi^2,\\ J^{V,w,q,\varpi}_\mu[\psi]n^\mu_{\mathcal{S}} &\gtrsim ((L\psi)^2+(\underline{L}\psi)^2+|\nabla\!\!\!/\psi|^2+\psi^2). \end{aligned} \tag{3-18}$$

Here the normals are of course taken with respect to the metric g_0.

Let us note that the boundary coercivity statement on $\Sigma(\tau)$ can only possibly hold if V is timelike on $\Sigma(\tau)$.

Defining

$$\begin{gathered} \overset{\scriptscriptstyle(0)}{\mathfrak{E}}(\tau):=\int_{\Sigma(\tau)} J^{V,w,q,\varpi}_\mu[\psi]n^\mu_{\Sigma(\tau)},\qquad \overset{\scriptscriptstyle(0)}{\mathfrak{E}}_{\mathcal{S}}(\tau):=\int_{\mathcal{S}} J^{V,w,q,\varpi}_\mu[\psi]n^\mu_{\mathcal{S}},\\ \overset{\scriptscriptstyle(0)}{\mathfrak{F}}(v,\tau_0,\tau_1):=\int_{\underline{C}_v\cap\mathcal{R}(\tau_0,\tau_1)} J^{V,w,q,\varpi}_\mu[\psi]n^\mu_{\underline{C}_v}, \end{gathered} \tag{3-19}$$

it follows from (3-18) that

$$\overset{\scriptscriptstyle(0)}{\mathcal{E}}(\tau)\lesssim\overset{\scriptscriptstyle(0)}{\mathfrak{E}}(\tau),\qquad \overset{\scriptscriptstyle(0)}{\mathcal{E}}_{\mathcal{S}}(\tau)\lesssim\overset{\scriptscriptstyle(0)}{\mathfrak{E}}_{\mathcal{S}}(\tau),\qquad \overset{\scriptscriptstyle(0)}{\mathcal{F}}(v,\tau_0,\tau_1)\lesssim\overset{\scriptscriptstyle(0)}{\mathfrak{F}}(v,\tau_0,\tau_1). \tag{3-20}$$

From (3-16), it follows that, in addition to the coercivity (3-18), we have the corresponding boundedness

$$\begin{aligned} J^{V,w,q,\varpi}_\mu[\psi]n^\mu_{\Sigma(\tau)} &\lesssim (L\psi)^2+|\nabla\!\!\!/\psi|^2+\iota_{r\le R}(\underline{L}\psi)^2+r^{-2}\psi^2,\\ J^{V,w,q,\varpi}_\mu[\psi]n^\mu_{\underline{C}_v} &\lesssim (\underline{L}\psi)^2+|\nabla\!\!\!/\psi|^2+r^{-2}\psi^2,\\ J^{V,w,q,\varpi}_\mu[\psi]n^\mu_{\mathcal{S}} &\lesssim (L\psi)^2+(\underline{L}\psi)^2+|\nabla\!\!\!/\psi|^2+\psi^2, \end{aligned} \tag{3-21}$$

and thus

$$\overset{\scriptscriptstyle(0)}{\mathcal{E}}(\tau)\sim\overset{\scriptscriptstyle(0)}{\mathfrak{E}}_r(\tau),\qquad \overset{\scriptscriptstyle(0)}{\mathcal{E}}_{\mathcal{S}}(\tau)\sim\overset{\scriptscriptstyle(0)}{\mathfrak{E}}_{\mathcal{S}}(\tau),\qquad \overset{\scriptscriptstyle(0)}{\mathcal{F}}(v,\tau_0,\tau_1)\sim\overset{\scriptscriptstyle(0)}{\mathfrak{F}}(v,\tau_0,\tau_1). \tag{3-22}$$

Note that, to estimate the boundary terms arising from the 2-form ϖ, we have used (2-4) and the fact that

$$*\,d(\psi^2\varpi)=2\psi*(d\psi\wedge\varpi)+\psi^2*d\varpi.$$

Under the above assumptions, in the notation of Section 3.2 (recall now (3-9)), the identity (3-15) gives rise to the estimate

$$\sup_{v:\tau\le\tau(v)}\overset{\scriptscriptstyle(0)}{\mathfrak{F}}(v,\tau_0,\tau),\quad \overset{\scriptscriptstyle(0)}{\mathfrak{E}}(\tau)+\overset{\scriptscriptstyle(0)}{\mathfrak{E}}_{\mathcal{S}}(\tau)+c\int_{\tau_0}^{\tau_1}\overset{\scriptscriptstyle(-1-\delta)}{\mathcal{E}'}(\tau')\,d\tau'\le\overset{\scriptscriptstyle(0)}{\mathfrak{E}}(\tau_0)+\int_{\mathcal{R}(\tau_0,\tau_1)}|H[\psi]F|. \tag{3-23}$$

We note that, in view of (3-20), (3-22), and the fact that we may express

$$H[\psi]=V^\mu\partial_\mu\psi+w\psi,$$

this gives (3-3) for some $\lambda\ge 1$ and without degeneration, i.e., with $\chi=1$, and with $V_0=V$ and $w_0=w$, and where including the final $\int F^2$ term in (3-3) is here unnecessary. The point of expressing the estimate in terms of the fraktur energies (3-19) is that (3-23) is a sharper statement than (3-3), corresponding to $\lambda=1$, which will be useful for us.

Let us note immediately that Minkowski space itself, but also suitably small stationary perturbations of the Minkowski metric, satisfy the assumptions of this section (see Appendix B). More generally, given a metric g_0 as in Section 2 and a T-invariant quadruple (V, w, q, ϖ) satisfying (3-16), whose energy currents $J^{V,w,q,\varpi}[g_0, \psi]$, $K^{V,w,q}[g_0, \psi]$ satisfy the above coercivity properties (3-17), (3-18), and (3-21), it is clear that $J^{V,w,q,\varpi}[g_\epsilon, \psi]$, $K^{V,w,q}[g_\epsilon, \psi]$ retain the coercivity properties (3-17), (3-18), and (3-21) for any stationary perturbation g_ϵ of g_0 satisfying the assumptions of Section 2, sufficiently close to g, such that $g = g_\epsilon$ in $r \geq R$. Thus, we see that when, as in the present section, estimate (3-3) is proven via (3-23) and there is no degeneration, i.e., $\chi = 1$, then the estimate (3-3) can immediately be inferred to be stable to suitably small perturbations of the metric g_0.

Unfortunately, however, for most of our examples of spacetimes $(\mathcal{M}, g_0)$ of interest, it turns out that the assumptions of the present section cannot in fact hold. Specifically, if $(\mathcal{M}, g_0)$ contains trapped null geodesics, then one can show that (3-3) cannot hold with $\chi = 1$ identically, and thus no quadruple (V, w, q, ϖ) as above can exist satisfying (3-17); see [Sbierski 2015]. In particular, the assumptions of this section do not encompass the black hole cases of interest like Schwarzschild or Kerr.

3.4.2. *Case* (ii). The next simplest case is when estimate (3-3) again follows from the coercivity properties of a suitable physical-space energy identity (3-15), but where the degeneration function χ of (3-2) is now nontrivial, potentially vanishing on some set.

More precisely, we assume that there exists a T-invariant quadruple (V, w, q, ϖ), again bounded in the sense of (3-16), but now satisfying the following relaxed coercivity properties: defining currents $J^{V,w,q,\varpi}$, $K^{V,w,q}$ by (3-11)–(3-12), we again assume the boundary coercivity properties (3-18) on $J^{V,w,q,\varpi}$, but we weaken the bulk coercivity assumption on $K^{V,w,q}$ to

$$K^{V,w,q}[\psi] \gtrsim \chi(r) r^{-1-\delta}((L\psi)^2 + (\underline{L}\psi)^2 + |\nabla\!\!\!/\psi|^2) + r^{-3-\delta}\psi^2, \tag{3-24}$$

where χ is the function (3-2) in Section 3.2.

We define the fraktur energies again by (3-19), and we note that again we have (3-20) and, in view of (3-16), also (3-21) and thus (3-22). Under the above assumptions, identity (3-15) gives rise to the estimate

$$\sup_{v:\tau\leq\tau(v)} \overset{(0)}{\mathfrak{F}}(v, \tau_0, \tau), \quad \overset{(0)}{\mathfrak{E}}(\tau) + \overset{(0)}{\mathfrak{E}}_{\mathcal{S}}(\tau) + c\int_{\tau_0}^{\tau} \chi \overset{(-1-\delta)}{\mathcal{E}'}(\tau')\,d\tau' + c\int_{\tau_0}^{\tau} \underset{-1}{\mathcal{E}'}(\tau')\,d\tau' \leq \overset{(0)}{\mathfrak{E}}(\tau_0) + \int_{\mathcal{R}(\tau_0,\tau)} |H[\psi]F|. \tag{3-25}$$

We note again that, in view of (3-20) and (3-22), estimate (3-25) indeed implies (3-3), for some $\lambda \geq 1$, with $V_0 = V$ and $w_0 = w$ and where including the final $\int F^2$ term in (3-3) is here unnecessary. As with (3-23) of case (i), the point of expressing the estimate in terms of the fraktur energies (3-19) is that (3-23) is a sharper statement than (3-3), corresponding to $\lambda = 1$, which will be useful for us.

We note that a current-defining quadruple (V, w, q, ϖ) satisfying the properties of this section indeed exists for the Schwarzschild metric and can be constructed from the considerations in [Dafermos and Rodnianski 2007a; Marzuola et al. 2010]. Note that a prerequisite for even the degenerate bulk coercivity property (3-24) is that any trapped null geodesics be "unstable" in a suitable sense (cf. the

Schwarzschild–AdS case with reflective boundary conditions at infinity, where there exist stably trapped null geodesics [Holzegel and Smulevici 2014]).

In the Kerr case for all $|a| \neq 0$, even though all trapped null geodesics are again unstable, and even though estimate (3-3) is true (by [Dafermos et al. 2016]), one can show that no quadruple (V, w, q, ϖ) can give rise to currents satisfying the coercivity properties of this section; see [Alinhac 2009]. (For a higher-order current defined using second-order operators which gives an analogue of the coercivity properties here in the $|a| \ll M$ case, see however [Andersson and Blue 2015].)

One of the main motivations of this paper is to show that, from a suitable point of view, not only is a purely physical-space proof of (3-3) unnecessary for nonlinear applications, but such a proof would only result in a minor and inessential simplification of the argument. We now turn to our main case of interest, case (iii).

3.4.3. *Case* (iii). The most general case we wish to consider in this paper is where the estimate (3-3) is assumed as a "black box", i.e., it is *not* necessarily the consequence of the coercivity properties of some more fundamental physical-space identity (3-15). We note for instance that (3-3) indeed holds when $(\mathcal{M}, g_0)$ is Kerr, in fact for the full subextremal range $|a| < M$ of parameters [Dafermos et al. 2016], but as discussed above, it does not arise (see [Alinhac 2009]) from a current as in case (ii).

In this most general case, however, in addition to assuming (3-3), we will still make a further assumption on the existence of an auxiliary pair of currents $J^{V,w,q,\varpi}$, $K^{V,w,q}$ whose energy identity will *not* in general imply (3-3) but rather will be used *in combination* with (3-3). This auxiliary current will have the following properties: The bulk K current will be nonnegative in a neighbourhood of the set $\{\chi \neq 1\}$, where χ is the function (3-2) of Section 3.2 appearing in (3-3). In nontrivial applications, K will often vanish identically in this region, making it completely insensitive to the possible presence and nature of trapping. Where $\chi = 1$, on the other hand, the bulk K current will be assumed nonnegative only modulo lower-order terms, provided these lower-order terms are supported in the region $r_2 \leq r \leq \frac{1}{2}R$. Finally, for $r \geq R$, the bulk current K will control the terms familiar from cases (i) and (ii).

We now lay out the assumptions in detail.

We will define functions

$$\rho : [r_0, \infty) \to [0, 1], \qquad \xi : [r_0, \infty) \to [0, 1] \tag{3-26}$$

such that $\rho = 1$ for $r \geq \frac{1}{4}R$ and $r \leq r_2$ and such that

$$\xi = 0 \text{ in } \{\chi \neq 1\} \cup \{r \leq r_2\} \cup \left\{r \geq \tfrac{1}{4}R\right\}, \tag{3-27}$$

where χ is the function (3-2) in Section 3.2 appearing in the assumed estimate (3-3). (In nontrivial applications, ρ will vanish identically in a neighbourhood containing the set where $\chi \neq 1$.) The supports of the various functions are indicated in Figure 2. We note finally that we lose no generality in the present paper in always taking ρ, χ, ξ to be indicator functions of appropriate sets, but the sharpest estimates one could prove would often involve degeneration on sets of positive codimension.

We assume the existence of a T-invariant quadruple (V, w, q, ϖ) satisfying the boundedness properties (3-16) such that the associated current $J^{V,w,q,\varpi}[\psi]$ still satisfies (3-18) but where the bulk coercivity

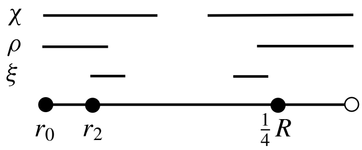

Figure 2. The supports of χ, ρ and ξ.

assumption (3-24) on $K^{V,w,q}[\psi]$ is now further relaxed to

$$K^{V,w,q}[\psi]+\tilde{A}\xi(r)\psi^2 \gtrsim \rho(r)r^{-1-\delta}((L\psi)^2+(\underline{L}\psi)^2+|\nabla\!\!\!/\,\psi|^2)+\rho(r)r^{-3-\delta}\psi^2, \tag{3-28}$$

where $\tilde{A}\geq 0$ is a possibly large constant.

With this current, we define the fraktur energies again by (3-19), and we again have (3-20) and, in view of (3-16), also (3-21) and thus (3-22).

In view of the relaxed coercivity properties, the identity (3-15) gives rise to

$$\sup_{v:\tau\leq\tau(v)} \overset{(0)}{\mathfrak{F}}(v,\tau_0,\tau),\quad \overset{(0)}{\mathfrak{E}}(\tau)+\overset{(0)}{\mathfrak{E}}_{\mathcal{S}}(\tau)+c\int_{\tau_0}^{\tau} {}^{\rho}\overset{(-1-\delta)}{\mathcal{E}'}(\tau')+{}^{\rho}\underset{-1}{\mathcal{E}'}(\tau')\,d\tau' \\ \leq \overset{(0)}{\mathfrak{E}}(\tau_0)+A\int_{\tau_0}^{\tau'} \overset{\xi}{\underset{-1}{\mathcal{E}'}}(\tau')\,d\tau'+\int_{\mathcal{R}(\tau_0,\tau)}|H[\psi]F| \tag{3-29}$$

for an $A\geq 0$, where ${}^{\rho}\overset{(-1-\delta)}{\mathcal{E}'}(\tau)$ is defined analogously to ${}^{\chi}\overset{(-1-\delta)}{\mathcal{E}'}(\tau)$; i.e.,

$${}^{\rho}\overset{(-1-\delta)}{\mathcal{E}'}(\tau):=\int_{\Sigma(\tau)} r^{-1-\delta}\rho(r)(|L\psi|^2+|\underline{L}\psi|^2+|\nabla\!\!\!/\,\psi|^2),\qquad {}^{\rho}\underset{-1}{\mathcal{E}'}(\tau):=\int_{\Sigma(\tau)} r^{-3-\delta}\rho(r)\psi^2,$$

and

$$\overset{\xi}{\underset{-1}{\mathcal{E}'}}(\tau):=\int_{\Sigma(\tau)}\xi(r)\psi^2. \tag{3-30}$$

Again, we note that it is expressing things with respect to the fraktur energies which allows the constant to be exactly 1 in the above estimate (3-29), and this fact will be useful for us.

The existence of currents $J^{V,w,q,\varpi}$, $K^{V,w,q}$ satisfying the above assumptions can indeed be shown for Kerr in the range $|a|\ll M$ (and in fact for general stationary suitably small perturbations of Schwarzschild satisfying the assumptions of Section 2 and appropriate assumptions at infinity). See Appendix A. Note that estimate (3-29) is manifestly weaker than (3-3). The point, as discussed in the introduction, is that, being derived from the (relaxed) coercivity properties of (3-28) applied to (3-14), estimate (3-29), or more properly the identity (3-14) itself, can be applied directly to (1-1). See already Section 4.3.1.

3.4.4. *Summary of the unweighted assumptions for cases* (i), (ii), *and* (iii). So as to not have to always refer to separate formulas in the distinct cases (i), (ii), and (iii), we summarise the assumptions in a way which can be subsequently interpreted for all cases simultaneously.

In case (i), we set $\tilde{\rho}=\rho=\chi=1$ and $A=\tilde{A}=0$.

In case (ii), we set χ to be the function (3-2) appearing in both (3-3) and (3-24), and we set $\rho=\chi$, $\tilde{\rho}=1$, and $A=\tilde{A}=0$.

Finally, in case (iii), we set χ to be the function (3-2) of Section 3.2 appearing in (3-3), we let ρ and ξ be the functions (3-26) and the constants A and $\tilde{A}$ be as in Section 3.4.3, and we set $\tilde{\rho} = \rho$.

Our assumptions, applicable for all cases, are (a) that (3-3) holds and (b) that there exist T-invariant (V, w, q, ϖ) satisfying the bounds (3-16) such that, defining the currents $J^{V,w,q,\varpi}[\psi]$, $K^{V,w,q}[\psi]$, and $H^{V,w}[\psi]$ by (3-11)–(3-13), we have the bulk coercivity property

$$K^{V,w,q}[\psi] + \tilde{A}\xi(r)\psi^2 \gtrsim \rho(r) r^{-1-\delta}((L\psi)^2 + (\underline{L}\psi)^2 + |\nabla\!\!\!/\psi|^2 + r^{-3}\psi^2) + \tilde{\rho}(r) r^{-3-\delta}\psi^2 \tag{3-31}$$

and the boundary coercivity properties

$$\begin{aligned} J^{V,w,q,\varpi}_\mu[\psi] n^\mu_{\Sigma(\tau)} &\gtrsim (L\psi)^2 + |\nabla\!\!\!/\psi|^2 + \iota_{r\le R}(\underline{L}\psi)^2 + r^{-2}\psi^2, \\ J^{V,w,q,\varpi}_\mu[\psi] n^\mu_{\underline{C}_v} &\gtrsim (\underline{L}\psi)^2 + |\nabla\!\!\!/\psi|^2 + r^{-2}\psi^2, \\ J^{V,w,q,\varpi}_\mu[\psi] n^\mu_{\mathcal{S}} &\gtrsim (L\psi)^2 + (\underline{L}\psi)^2 + |\nabla\!\!\!/\psi|^2 + \psi^2. \end{aligned} \tag{3-32}$$

With these currents, we define the fraktur energies again by (3-19), and we again have (3-20) and, in view of (3-16), also (3-21) and thus (3-22).

The energy identity (3-15) gives rise then to

$$\sup_{v:\tau\le\tau(v)} \overset{(0)}{\mathfrak{F}}(v,\tau_0,\tau), \quad \overset{(0)}{\mathfrak{E}}(\tau) + \overset{(0)}{\mathfrak{E}}_{\mathcal{S}}(\tau) + c\int_{\tau_0}^{\tau} {}^{\rho}\overset{(-1-\delta)}{\mathcal{E}'}(\tau')\,d\tau' + c\int_{\tau_0}^{\tau} {}^{\tilde{\rho}}\underset{-1}{\mathcal{E}'}(\tau')\,d\tau' \\ \le \overset{(0)}{\mathfrak{E}}(\tau_0) + A\int_{\tau_0}^{\tau} \underset{-1}{\overset{\xi}{\mathcal{E}'}}(\tau')\,d\tau' + \int_{\mathcal{R}(\tau_0,\tau)} |H[\psi]F|, \tag{3-33}$$

where we define

$${}^{\tilde{\rho}}\underset{-1}{\mathcal{E}'}(\tau) := \int_{\Sigma(\tau)} \tilde{\rho}(r) r^{-3-\delta}\psi^2.$$

We emphasise again that in cases (i) and (ii) the statement that (3-3) holds need not be taken as an independent assumption, as (3-3) in fact follows from (3-33) with the above definitions in these two cases.

3.4.5. *An enhanced red-shift current and enhanced positivity in the black hole interior.* In the case $\mathcal{S} \ne \varnothing$, for the purpose of higher-order estimates to be considered in Section 3.6, we will need to enhance the positivity near $\mathcal{H}^+$.

We first state the following proposition which allows us to introduce an arbitrary largeness factor ς in front of some of the terms of our coercivity estimate.

Proposition 3.4.1. *Under the assumptions of Section 3.4.4, there exists a constant $c > 0$ such that the following holds*:

Given arbitrary $\varsigma \ge 1$, there exist parameters $r_0 \le r_0'(\varsigma) < r_{\text{Killing}} < r_1(\varsigma) \le r_1$, a translation-invariant vector field V_ς', and a 1-form q_ς' such that, defining $J^{V_\varsigma',w,q_\varsigma',\varpi}[\psi]$, $K^{V_\varsigma',w,q_\varsigma'}[\psi]$, $H^{V_\varsigma',w}[\psi]$ as in Section 3.4.4, we have

$$K^{V_\varsigma',w,q_\varsigma'}[\psi] \ge c(Y\psi)^2 + c\varsigma\left(\sum_{i=1}^{3}(\Omega_i\psi)^2 + (T\psi)^2 + \psi^2\right) \tag{3-34}$$

in $r_0'(\varsigma) \le r \le r_1(\varsigma)$ and all properties of Section 3.4.4 hold for these currents with implicit constants in (3-16), (3-31), (3-32), (3-21), which may be taken independently of ς. Finally, $V_\varsigma' = V$, $q_\varsigma' = q$ for $r \ge r_1$.

Proof. This is clear by examining the proof of Theorem 7.1 of [Dafermos and Rodnianski 2013]. □

Using the above and the time-like character of ∇r in the "black hole interior" region $r < r_{\text{Killing}}$, we may further modify our current to obtain the following enhanced positivity in $r_0 \le r \le r_1(\varsigma)$, in particular, all the way up to $\mathcal{S}$, at the expense of a lower-order term. The resulting modified current will be used for higher-order estimates.

Proposition 3.4.2. *Under the assumptions of Section 3.4.4, there exist constants $C > 0$ and $c > 0$ such that the following holds:*

Given arbitrary $\varsigma \ge 1$, let V'_ς and q'_ς be as given in Proposition 3.4.1. Then there exists a translation-invariant quadruple $(V_\varsigma,\ w_\varsigma,\ q_\varsigma,\ \varpi_\varsigma)$, with $V_\varsigma = V'_\varsigma,\ w_\varsigma = w,\ q_\varsigma = q',\ \varpi_\varsigma = \varpi$ for $r \ge r_{\text{Killing}}$, and a positive function $\lambda_\varsigma(r)$ such that, defining $J^{V_\varsigma, w_\varsigma, q_\varsigma, \varpi_\varsigma}[\psi],\ K^{V_\varsigma, w_\varsigma, q_\varsigma}[\psi],\ H^{V_\varsigma, w_\varsigma}[\psi]$ as in Section 3.4.4, the enhanced positivity (modulo lower-order terms)

$$K^{V_\varsigma, w_\varsigma, q_\varsigma}[\psi] \ge c\varsigma\lambda_\varsigma(r)\left(\sum_{i=1}^{3}(\Omega_i\psi)^2 + (T\psi)^2 + |r_{\text{Killing}} - r|(Y\psi)^2\right) + c\lambda_\varsigma(r)(Y\psi)^2 - C\varsigma\lambda_\varsigma(r)\psi^2 \tag{3-35}$$

holds in $r_0 \le r \le r_{\text{Killing}}$.

Moreover, all boundedness and coercivity properties of the currents of Section 3.4.4 hold for these currents as well with constants independent of ς in the region $r \ge r_{\text{Killing}}$, while in the region $r < r_{\text{Killing}}$ the properties of Section 3.4.4 again hold for these currents but with (3-31) replaced by (3-35), and with coercivity bounds that now however depend in general on ς. In particular, we have

$$|H^{V_\varsigma, w_\varsigma}[\psi]| \le C\lambda_\varsigma(r)\left(|Y\psi| + \sum_{i=1}^{3}|\Omega_i\psi| + |T\psi|\right) + C\lambda_\varsigma(r)\psi^2. \tag{3-36}$$

Proof. Given ς, let $r'_0(\varsigma)$ be as in Proposition 3.4.1. Let us define $\lambda_\varsigma(r)$ to be a smooth positive function such that $\lambda_\varsigma(r) = 1$ for $r \ge r_{\text{Killing}}$,

$$\lambda_\varsigma(r) = e^{-\varsigma(r_{\text{Killing}} - r)} \tag{3-37}$$

for $\{r_0 \le r \le r_{\text{Killing}}\} \cap (\{\varsigma \ge (r_{\text{Killing}} - r)^{-1}\} \cup \{r \le r'_0(\varsigma)\})$, and

$$0 \le \frac{d\lambda_\varsigma(r)}{dr} \le 2\varsigma\lambda_\varsigma(r).$$

We define now

$$V_\varsigma := \lambda_\varsigma(r)V'_\varsigma, \quad w_\varsigma := \lambda_\varsigma(r)w, \quad q_\varsigma := \lambda_\varsigma(r)q - *(d\lambda_\varsigma \wedge \varpi), \quad \varpi_\varsigma = \lambda_\varsigma(r)\varpi.$$

We note that, under these definitions, $J^{V_\varsigma, w_\varsigma, q_\varsigma, \varpi_\varsigma}[\psi] = \lambda_\varsigma(r)J^{V'_\varsigma, w, q'_\varsigma, \varpi}[\psi]$, and thus the positivity properties of the boundary currents are preserved but with constants that now depend on ς.

We have that

$$\nabla^\mu V_\varsigma^\nu = \nabla^\mu(\lambda_\varsigma(r))V_\varsigma'^{\nu} + \lambda_\varsigma(r)\nabla^\mu V_\varsigma'^{\nu} = \frac{d\lambda_\varsigma(r)}{dr}\nabla^\mu r V_\varsigma'^{\nu} + \lambda_\varsigma(r)\nabla^\mu V_\varsigma'^{\nu},$$

and thus

$$
\begin{aligned}
K^{V_\varsigma^\nu}[\psi] + w_\varsigma \nabla^\mu\psi\partial_\mu\psi &= T_{\mu\nu}\tfrac{1}{2}(\nabla^\mu V_\varsigma^\nu + \nabla^\nu V_\varsigma^\mu) + w_\varsigma \nabla^\mu\psi\partial_\mu\psi \\
&= \frac{d\lambda_\varsigma(r)}{dr} T_{\mu\nu}\nabla^\mu r V_\varsigma'^{\nu} + \lambda_\varsigma(r)\big(T_{\mu\nu}\tfrac{1}{2}(\nabla^\mu V_\varsigma'^{\nu} + \nabla^\nu V_\varsigma'^{\mu}) + w\nabla^\mu\psi\partial_\mu\psi\big) \\
&\geq c\frac{d\lambda_\varsigma(r)}{dr}\bigg(\sum_{i=1}^{3}(\Omega_i\psi)^2 + (T\psi)^2 + |r_{\text{Killing}} - r|(Y\psi)^2\bigg) \\
&\qquad\qquad + c\lambda_\varsigma(r)\bigg((Y\psi)^2 + \sum_{i=1}^{3}(\Omega_i\psi)^2 + (T\psi)^2\bigg) \\
&\geq c\varsigma\lambda_\varsigma(r)\bigg(\sum_{i=1}^{3}(\Omega_i\psi)^2 + (T\psi)^2 + |r_{\text{Killing}} - r|(Y\psi)^2\bigg) + c\lambda_\varsigma(r)(Y\psi)^2.
\end{aligned}
$$

For the first inequality, we are using the fact that ∇r is a smooth vector field which is future-directed null on $\mathcal{H}^+$ and future-directed timelike in $r < r_{\text{Killing}}$ by our assumptions from Section 2.4, as well as the fact that

$$
T_{\mu\nu}[\psi]\tfrac{1}{2}(\nabla^\mu V_\varsigma'^{\nu} + \nabla^\nu V_\varsigma'^{\mu}) + w\nabla^\mu\psi\partial_\mu\psi \geq c\bigg((Y\psi)^2 + \sum_{i=1}^{3}(\Omega_i\psi)^2 + (T\psi)^2\bigg), \tag{3-38}
$$

which follows because coercivity of the current $K^{V_\varsigma', w, q_\varsigma'}$ asserted in (3-34) requires in particular coercivity of its highest-order terms. Note that the second inequality above clearly follows from the first wherever (3-37) holds. On the other hand, wherever (3-37) does not hold, we again obtain the second inequality in view of the definition of $\lambda_\varsigma(r)$ since, in this region, the enhanced positivity (3-34) applies, allowing us to put extra ς factors on the second two terms of (3-38) as well as the bound $\varsigma(r_{\text{Killing}} - r) \leq 1$.

On the other hand,

$$
\begin{aligned}
&|K^{w_\varsigma, q_\varsigma}[\psi] - w_\varsigma\nabla^\mu\psi\partial_\mu\psi| \\
&\quad = |\nabla^\mu w_\varsigma \psi\partial_\mu\psi + \nabla^\mu(q_\varsigma)_\mu\psi^2 + 2\psi(q_\varsigma)_\mu g^{\mu\nu}\partial_\nu\psi| \\
&\quad \leq C\bigg(\frac{d\lambda_\varsigma(r)}{dr} + \lambda_\varsigma(r)\bigg)|\psi|\bigg(|Y\psi| + \sum_{i=1}^{3}|\Omega_i\psi| + |T\psi|\bigg) \\
&\qquad\qquad + C\bigg(\frac{d\lambda_\varsigma(r)}{dr} + \lambda_\varsigma(r)\bigg)\psi^2 + C\bigg(\frac{d\lambda_\varsigma(r)}{dr} + C\lambda_\varsigma(r)\bigg)|\psi|\bigg(|Y\psi| + \sum_{i=1}^{3}|\Omega_i\psi| + |T\psi|\bigg),
\end{aligned}
$$

whence we deduce

$$
|K^{w_\varsigma, q_\varsigma}[\psi - w_\varsigma\nabla^\mu\psi\partial_\mu\psi]| \leq \tfrac{1}{2}(K^{V_\varsigma^\nu}[\psi] + w_\varsigma\nabla^\mu\psi\partial_\mu\psi) + C\varsigma\lambda_\varsigma(r)\psi^2.
$$

Further,

$$
|H^{V_\varsigma, w_\varsigma}[\psi]| = |V_\varsigma^\nu\partial_\nu\psi| = |\lambda_\varsigma(r)V_\varsigma'^{\nu}\psi + w_\varsigma(r)\psi| \leq C\lambda_\varsigma(r)\bigg(|Y\psi| + \sum_{i=1}^{3}|\Omega_i\psi| + |T\psi|\bigg) + C\lambda_\varsigma(r)|\psi|,
$$

giving (3-36). The statement (3-35) now follows since we have

$$
K^{V_\varsigma, w_\varsigma, q_\varsigma} = K^{V_\varsigma} + w_\varsigma\nabla^\mu\psi\partial_\mu\psi + K^{w_\varsigma, q_\varsigma} - w_\varsigma\nabla^\mu\psi\partial_\mu\psi
$$

and the above bounds. □

3.5. ***The r^p hierarchy.*** For nonlinear applications, we will need to extend our estimates to suitable weighted estimates satisfying the r^p hierarchy [Dafermos and Rodnianski 2010b].

In a suitable framework, very general assumptions on stationary metrics g_0 allowing one to apply the r^p hierarchy are contained in [Moschidis 2016]. Here, let us note that this class was shown in particular to contain Minkowski space, Schwarzschild, and Kerr in the full range of parameters.

So as not to translate to the setup of [Moschidis 2016], however, and rather than formulate the most general asymptotically flat assumptions which we will allow in terms of g_0, it will be convenient to make the assumptions *directly* in terms of coercivity properties of appropriate currents defined in a region $r \geq \widetilde{R} \geq R$ for some large $\widetilde{R}$.

To give our precise assumption, let us first define $\overset{(p)}{V}{}^{\circ}_{\mathrm{far}} := r^p L$ and let $\overset{(p)}{w}{}^{\circ}_{\mathrm{far}}$, $\overset{(p)}{q}{}^{\circ}_{\mathrm{far}}$, $\overset{(p)}{\varpi}{}^{\circ}_{\mathrm{far}}$ be as defined in Section B.2. We assume then that, for any $\delta \leq p \leq 2-\delta$, there exists a T-invariant quadruple $(\overset{(p)}{V}_{\mathrm{far}}, \overset{(p)}{w}_{\mathrm{far}}, \overset{(p)}{q}_{\mathrm{far}}, \overset{(p)}{\varpi}_{\mathrm{far}})$, with $\overset{(p)}{V}_{\mathrm{far}} = \overset{(p)}{V}{}^{\circ}_{\mathrm{far}} + \widetilde{V}_{\mathrm{far}}$ a vector field, $\overset{(p)}{w}_{\mathrm{far}} = \overset{(p)}{w}{}^{\circ}_{\mathrm{far}} + \tilde{w}_{\mathrm{far}}$ a scalar function, $\overset{(p)}{q}_{\mathrm{far}} = \overset{(p)}{q}{}^{\circ}_{\mathrm{far}} + \tilde{q}_{\mathrm{far}}$ a 1-form, and $\overset{(p)}{\varpi}_{\mathrm{far}} = \overset{(p)}{\varpi}{}^{\circ}_{\mathrm{far}} + \tilde{\varpi}_{\mathrm{far}}$ a 2-form, defined on $r \geq \widetilde{R}$ and satisfying

$$\begin{gathered} |g(\widetilde{V}_{\mathrm{far}}, L)| \lesssim 1, \quad |g(\widetilde{V}_{\mathrm{far}}, \underline{L})| \lesssim 1, \quad \sum |g(\widetilde{V}_{\mathrm{far}}, \Omega_i)|^2 \lesssim 1, \\ |\tilde{w}_{\mathrm{far}}| \lesssim r^{-1}, \quad |L^{\mu} \tilde{q}_{\mathrm{far}\,\mu}| \lesssim r^{-2}, \quad |\underline{L}^{\mu} \tilde{q}_{\mathrm{far}\,\mu}| \lesssim r^{-2}, \\ |(* \, d\tilde{\varpi}_{\mathrm{far}})_{\mu} L^{\mu}| \lesssim r^{-2}, \quad |(* \, d\tilde{\varpi}_{\mathrm{far}})_{\mu} \underline{L}^{\mu}| \lesssim r^{-2}, \\ |* (\varrho^{L} \wedge \tilde{\varpi}_{\mathrm{far}})_{\mu} L^{\mu}| \lesssim r^{-1}, \quad |* (\varrho^{L} \wedge \tilde{\varpi}_{\mathrm{far}})_{\mu} \underline{L}^{\mu}| \lesssim r^{-1}, \\ |* (\varrho^{\underline{L}} \wedge \tilde{\varpi}_{\mathrm{far}})_{\mu} L^{\mu}| \lesssim r^{-1}, \quad |* (\varrho^{\underline{L}} \wedge \tilde{\varpi}_{\mathrm{far}})_{\mu} \underline{L}^{\mu}| \lesssim r^{-1}, \\ |* (\varrho^{i} \wedge \tilde{\varpi}_{\mathrm{far}})_{\mu} L^{\mu}| \lesssim r^{-1}, \quad |* (\varrho^{i} \wedge \tilde{\varpi}_{\mathrm{far}})_{\mu} \underline{L}^{\mu}| \lesssim r^{-1}, \qquad i = 1, 2, 3, \end{gathered} \tag{3-39}$$

such that, defining the associated currents $\overset{(p)}{J}_{\mathrm{far}}$, $\overset{(p)}{K}_{\mathrm{far}}$ by (3-11), (3-12), these satisfy the weighted bulk coercivity property

$$\overset{(p)}{K}_{\mathrm{far}}[\psi] \gtrsim r^{p-1}((r^{-1}L(r\psi))^2 + (L\psi)^2 + |\nabla\!\!\!/\,\psi|^2) + r^{-1-\delta}(\underline{L}\psi)^2 + r^{p-3}\psi^2 \tag{3-40}$$

and the weighted boundary coercivity properties

$$\begin{aligned} \overset{(p)}{J}_{\mathrm{far}\,\mu}[\psi] n^{\mu}_{\Sigma(\tau)} &\gtrsim r^{p}(r^{-1}L(r\psi))^2 + r^{\frac{p}{2}}(L\psi)^2 + |\nabla\!\!\!/\,\psi|^2 + r^{\frac{p}{2}-2}\psi^2, \\ \overset{(p)}{J}_{\mathrm{far}\,\mu}[\psi] n^{\mu}_{\underline{C}_v} &\gtrsim (\underline{L}\psi)^2 + r^{p}|\nabla\!\!\!/\,\psi|^2 + r^{p-2}\psi^2. \end{aligned} \tag{3-41}$$

We note that the $r^{p-1}(r^{-1}L(r\psi))^2$ term is redundant in (3-40) as it can be estimated pointwise from $r^{p-1}(L\psi)^2$ and $r^{p-3}\psi^2$. We retain it to compare with the boundary term (3-41) where it is necessary to retain explicitly the $(r^{-1}L(r\psi))^2$ term, as it is not controlled by the terms $r^{p/2}(L\psi)^2$ and $r^{p/2-2}\psi^2$.

See Appendix B for the construction of such a current on Minkowski space and a broad class of spacetimes with suitable asymptotic flatness assumptions at infinity, including Schwarzschild and Kerr in the full subextremal range $|a| < M$.

Let us note immediately that, given such currents satisfying the far-away coercivity assumptions (3-40) and (3-41) and given (V, w, q, ϖ) as in Section 3.4.4, by defining a suitable cut-off function $\zeta(r)$ with $\zeta = 0$ for $r \leq \widetilde{R}$ and $\zeta(r) = 1$ for $r \geq \widetilde{R} + 1$, introducing a small fixed parameter $e > 0$, and defining the

T-invariant vector field, functions and forms

$$\overset{(p)}{V} = V + e\zeta(r)\overset{(p)}{V}_{\text{far}}, \quad \overset{(p)}{w} = w + e\zeta\overset{(p)}{w}_{\text{far}}, \quad \overset{(p)}{q} = q + e\zeta\overset{(p)}{q}_{\text{far}}, \quad \overset{(p)}{\varpi} = \varpi + e\zeta\overset{(p)}{\varpi}_{\text{far}}, \tag{3-42}$$

one sees immediately that the associated currents $\overset{(p)}{J}$, $\overset{(p)}{K}$ satisfy the global (relaxed) weighted bulk coercivity assumptions

$$\overset{(p)}{K}[\psi]+\tilde{A}\xi(r)\psi^2 \gtrsim r^{p-1}\rho(r)((r^{-1}L(r\psi))^2+(L\psi)^2+|\nabla\!\!\!/\psi|^2)+r^{-1-\delta}\rho(r)(\underline{L}\psi)^2+r^{p-3}\tilde{\rho}(r)\psi^2 \tag{3-43}$$

and the global weighted boundary coercivity assumptions

$$\begin{aligned} \overset{(p)}{J}_\mu[\psi]n^\mu_{\Sigma(\tau)} &\gtrsim r^p(r^{-1}L(r\psi))^2 + r^{\frac{p}{2}}(L\psi)^2 + |\nabla\!\!\!/\psi|^2 + \iota_{r\le R}(\underline{L}\psi)^2 + r^{\frac{p}{2}-2}\psi^2, \\ \overset{(p)}{J}_\mu[\psi]n^\mu_{\underline{C}_v} &\gtrsim (\underline{L}\psi)^2 + r^p|\nabla\!\!\!/\psi|^2 + r^{p-2}\psi^2, \\ \overset{(p)}{J}_\mu[\psi]n^\mu_{\mathcal{S}} &\gtrsim (L\psi)^2 + (\underline{L}\psi)^2 + |\nabla\!\!\!/\psi|^2 + \psi^2, \end{aligned} \tag{3-44}$$

as one can absorb the error terms in the region $\widetilde{R} \le r \le \widetilde{R}+1$ arising from the cutoff by terms controlled by the coercivity relations (3-31) and (3-32), provided e is suitably chosen. We define also the associated $\overset{(p)}{H} = \overset{(p)}{V}{}^\mu\partial_\mu\psi + \overset{(p)}{w}$, so that the divergence identity (3-14) holds with $\overset{(p)}{J}$, $\overset{(p)}{K}$, and $\overset{(p)}{H}$.

From the bounds (3-39) and (3-44), we see that we have in fact

$$\begin{aligned} \overset{(p)}{J}_\mu[\psi]n^\mu_{\Sigma(\tau)} &\sim r^p(r^{-1}L(r\psi))^2 + r^{\frac{p}{2}}(L\psi)^2 + |\nabla\!\!\!/\psi|^2 + \iota_{r\le R}(\underline{L}\psi)^2 + r^{\frac{p}{2}-2}\psi^2, \\ \overset{(p)}{J}_\mu[\psi]n^\mu_{\underline{C}_v} &\sim (\underline{L}\psi)^2 + r^p|\nabla\!\!\!/\psi|^2 + r^{p-2}\psi^2, \\ \overset{(p)}{J}_\mu[\psi]n^\mu_{\mathcal{S}} &\sim (L\psi)^2 + (\underline{L}\psi)^2 + |\nabla\!\!\!/\psi|^2 + \psi^2. \end{aligned} \tag{3-45}$$

To state the resulting weighted versions of estimate (3-3) let us introduce some notation. For $\delta \le p \le 2-\delta$ we define

$$\overset{(p)}{\mathcal{E}}(\tau) := \overset{(0)}{\mathcal{E}}(\tau) + \int_{\Sigma(\tau)\cap\{r\ge R\}} r^p(r^{-1}L(r\psi))^2 + r^{\frac{p}{2}}(L\psi)^2 + r^{\frac{p}{2}-2}\psi^2, \tag{3-46}$$

$$\overset{(p)}{\mathcal{F}}(v,\tau_0,\tau) := \overset{(0)}{\mathcal{F}}(v,\tau_0,\tau) + \int_{\underline{C}_v\cap\mathcal{R}(\tau_0,\tau)} r^p|\nabla\!\!\!/\psi|^2 + r^{p-2}\psi^2, \tag{3-47}$$

$$\overset{(p-1)}{\mathcal{E}'}(\tau) := \overset{(-1-\delta)}{\mathcal{E}'}(\tau) + \int_{\Sigma(\tau)\cap\{r\ge R\}} r^{p-1}((r^{-1}L(r\psi))^2 + (L\psi)^2 + |\nabla\!\!\!/\psi|^2) + r^{p-3}\psi^2. \tag{3-48}$$

We also define versions of (3-48) where the first term is replaced by ${}^{\chi}\overset{(-1-\delta)}{\mathcal{E}'}(\tau)$ or ${}^{\rho}\overset{(-1-\delta)}{\mathcal{E}'}(\tau)$, respectively. We will call these ${}^{\chi}\overset{(p-1)}{\mathcal{E}'}(\tau)$ and ${}^{\rho}\overset{(p-1)}{\mathcal{E}'}(\tau)$. Note that these latter two expressions do not control a zeroth-order term in the region $r < R$.

Note the following properties. For $2-\delta \ge p \ge \delta$, we have

$$\overset{(p)}{\mathcal{E}} \gtrsim \overset{(p')}{\mathcal{E}},\ \overset{(p)}{\mathcal{F}} \gtrsim \overset{(p')}{\mathcal{F}} \text{ for } p \ge p' \ge \delta \text{ or } p'=0, \quad \overset{(p-1)}{\mathcal{E}'} \gtrsim \overset{(p-1)}{\mathcal{E}} \text{ for } p \ge 1+\delta, \quad \overset{(p-1)}{\mathcal{E}'} \gtrsim \overset{(0)}{\mathcal{E}} \text{ for } p \ge 1. \tag{3-49}$$

Let us also define the fluxes

$$\overset{(p)}{\mathfrak{E}}(\tau) := \int_{\Sigma(\tau)} \overset{(p)}{J}_\mu[\psi]n^\mu_{\Sigma(\tau)}, \quad \overset{(p)}{\mathfrak{E}}_{\mathcal{S}}(\tau) := \int_{\mathcal{S}(\tau_0,\tau)} \overset{(p)}{J}_\mu[\psi]n^\mu_{\mathcal{S}}, \quad \overset{(p)}{\mathfrak{F}}(v,\tau_0,\tau) := \int_{\underline{C}_v\cap\mathcal{R}(\tau_0,\tau)} \overset{(p)}{J}_\mu[\psi]n^\mu_{\underline{C}_v}. \tag{3-50}$$

We note that, by (3-45), it follows that

$$\overset{(p)}{\mathfrak{E}}(\tau) \sim \overset{(p)}{\mathcal{E}}(\tau), \quad \overset{(p)}{\mathfrak{E}}_{\mathcal{S}}(\tau_0, \tau) = \overset{(0)}{\mathfrak{E}}_{\mathcal{S}}(\tau_0, \tau) \sim \overset{(0)}{\mathcal{E}}_{\mathcal{S}}(\tau_0, \tau), \tag{3-51}$$

$$\overset{(p)}{\mathfrak{F}}(v, \tau_0, \tau) \sim \overset{(p)}{\mathcal{F}}(v, \tau_0, \tau). \tag{3-52}$$

We may now state the main result of this section.

Proposition 3.5.1. *Let $(\mathcal{M}, g_0)$ satisfy the assumptions of Sections 2 and 3.4.4, and let (V, w, q, ϖ) be as in Section 3.4.4. (In particular, the estimates (3-3) and (3-33) are assumed to hold.) Assume for all $0 < \delta \le p \le 2-\delta$ the existence of a quadruple $(\overset{(p)}{V}_{\text{far}}, \overset{(p)}{w}_{\text{far}}, \overset{(p)}{q}_{\text{far}}, \overset{(p)}{\varpi}_{\text{far}})$ as above satisfying (3-39) and the far-away coercivity properties (3-40) and (3-41).*

Then, for all $0 < \delta \le p \le 2-\delta$, the estimate

$$\sup_{v:\tau\le\tau(v)} \overset{(p)}{\mathfrak{F}}(v, \tau_0, \tau), \quad \overset{(p)}{\mathfrak{E}}(\tau) + \overset{(p)}{\mathfrak{E}}_{\mathcal{S}}(\tau_0, \tau) + c\int_{\tau_0}^{\tau} {}^{\rho}\overset{(p-1)}{\mathcal{E}'}(\tau')\,d\tau' + c\int_{\tau_0}^{\tau} {}^{\tilde{\rho}}\underset{-1}{\mathcal{E}'}(\tau')\,d\tau'$$
$$\le \overset{(p)}{\mathfrak{E}}(\tau_0) + A\int_{\tau_0}^{\tau} \underset{-1}{\overset{\xi}{\mathcal{E}'}}(\tau') + \int_{\mathcal{R}(\tau_0,\tau)} |\overset{(p)}{H}[\psi]F| + C\int_{\mathcal{R}(\tau_0,\tau)} F^2 \tag{3-53}$$

as well as the estimate

$$\sup_{v:\tau\le\tau(v)} \overset{(p)}{\mathcal{F}}(v, \tau_0, \tau) + \overset{(p)}{\mathcal{E}}(\tau) + \overset{(0)}{\mathcal{E}}_{\mathcal{S}}(\tau_0, \tau) + \int_{\tau_0}^{\tau} {}^{\chi}\overset{(p-1)}{\mathcal{E}'}(\tau')\,d\tau' + \int_{\tau_0}^{\tau} \underset{-1}{\mathcal{E}'}(\tau')\,d\tau'$$
$$\lesssim \overset{(p)}{\mathcal{E}}(\tau_0) + \int_{\mathcal{R}(\tau_0,\tau)} (|r^p r^{-1} L(r\psi)| + |\widetilde{V}_p^{\mu}\partial_\mu\psi| + |\tilde{w}_p\psi|)|F| + \int_{\mathcal{R}(\tau_0,\tau)} F^2 \tag{3-54}$$

hold for all $\tau_0 \le \tau$, where $\widetilde{V}_p$ is a fixed vector field and $\tilde{w}_p$ is a fixed function satisfying

$$|g(\widetilde{V}_p, L)| \lesssim 1, \quad |g(\widetilde{V}_p, \underline{L})| \lesssim 1, \quad \sum |g(\widetilde{V}_p, \Omega_i)|^2 \lesssim 1, \quad |\tilde{w}_p| \lesssim r^{-1}. \tag{3-55}$$

Remark 3.5.2. In the case where we replace the middle term of (3-3) with (3-10), we should add the first term of (3-10) to the right-hand side of (3-54).

Proof. The estimate (3-53) follows immediately from the energy identity (3-15) corresponding to the current $\overset{(p)}{J}$ in view of the properties assumed.

Note that in cases (i) or (ii), estimate (3-53) already implies (3-54) in view also of (3-39). In general, to obtain (3-54), we add a large multiple of estimate (3-3) to (3-53). This allows us to absorb the term multiplying A on the right-hand side of (3-53) in view of the trivial relation

$$\underset{-1}{\overset{\xi}{\mathcal{E}'}} \lesssim \underset{-1}{\mathcal{E}'}. \qquad \square$$

For consistency, we will in what follows often denote the quadruple (V, w, q, ϖ) of Section 3.4.4 as $\overset{(p)}{V}, \overset{(p)}{w}, \overset{(p)}{q}, \overset{(p)}{\varpi}$, and the currents as $\overset{(0)}{J}, \overset{(0)}{K}, \overset{(0)}{H}$.

Finally, in view of Proposition 3.4.2, given $\varsigma \ge 1$, we may define versions of the above currents where $V_\varsigma, w_\varsigma, q_\varsigma, \varpi_\varsigma$ replace V, w, q, ϖ, respectively, in definition (3-42). We will denote these currents as

$$\overset{(p)}{J}_\varsigma, \quad \overset{(p)}{K}_\varsigma, \quad \overset{(p)}{H}_\varsigma.$$

All boundedness and coercivity inequalities will continue to hold with constants independent of ς in the region $r \geq r_{\text{Killing}}$, while in the region $r < r_{\text{Killing}}$, the resulting $\overset{(p)}{K}_\varsigma$ will satisfy the enhanced positivity property (3-35), at the expense of lower-order terms. (In the region $r < r_{\text{Killing}}$, however, we emphasise the dependence of the coercivity constants on ς.)

3.6. ***Higher-order estimates.*** As is well known, in applications to nonlinear problems, one must be able to prove higher-order estimates in order for the estimates to close.

3.6.1. *The commutation vector fields $\mathfrak{D}$ and the auxiliary $\widetilde{\mathfrak{D}}$.* Let $\zeta(r)$ denote a smooth cut-off function such that $\zeta = 1$ for $r \geq \frac{3}{4}R$ and $\zeta = 0$ for $r \leq \frac{1}{2}R$. If $\mathcal{S} = \varnothing$, let $\hat{\zeta} = 0$. Otherwise, let $\hat{\zeta}(r)$ denote a smooth cut-off function such that $\hat{\zeta} = 1$ for $r \leq r_1 + \frac{1}{4}(r_2 - r_1)$ and $\hat{\zeta} = 0$ for $r \geq r_1 + \frac{1}{2}(r_2 - r_1)$. If Ω_1 is Killing, we define $\upsilon = 1$, otherwise we set $\upsilon = 0$. (More generally, we may take $\upsilon = 0$ if T coincides with the Killing generator and is timelike in $r > r_{\text{Killing}}$.)

We define

$$\mathfrak{D}_1 = T, \quad \mathfrak{D}_2 = \upsilon\Omega_1, \quad \mathfrak{D}_3 = Y, \quad \mathfrak{D}_4 = \zeta L, \quad \mathfrak{D}_5 = \zeta \underline{L}, \quad \mathfrak{D}_{5+i} = (\zeta + \hat{\zeta})\Omega_i, \quad i = 1, \ldots, 3. \tag{3-56}$$

We will write $\boldsymbol{k} = (k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8)$ and $|\boldsymbol{k}| = \sum_{i=1}^{8} k_i$, and we will denote by $\mathfrak{D}^{\boldsymbol{k}}\psi$ the expression

$$\mathfrak{D}^{\boldsymbol{k}}\psi = \mathfrak{D}_1^{k_1}\mathfrak{D}_2^{k_2}\cdots\mathfrak{D}_8^{k_8}\psi. \tag{3-57}$$

Note that the vector fields (3-56) span the tangent space for $r \geq \frac{3}{4}R$ and, if $\mathcal{S} \neq \varnothing$, in $r \leq r_1 + \frac{1}{4}(r_2 - r_1)$, but not in general for $r_1 + \frac{1}{4}(r_2 - r_1) \leq r \leq \frac{3}{4}R$. It is also useful to have a collection of operators that do. We thus define $\widetilde{\mathfrak{D}}^{\boldsymbol{k}}$ to denote commutation strings of operators from the collection

$$\widetilde{\mathfrak{D}}_1 = L, \quad \widetilde{\mathfrak{D}}_2 = \underline{L}, \quad \widetilde{\mathfrak{D}}_{2+i} = \Omega_i, \quad i = 1, \ldots, 6; \tag{3-58}$$

i.e., we define

$$\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi = \widetilde{\mathfrak{D}}_1^{k_1}\widetilde{\mathfrak{D}}_2^{k_2}\cdots\widetilde{\mathfrak{D}}_8^{k_8}\psi. \tag{3-59}$$

(Recall that the additional Ω_i, $i = 4, 5, 6$, were introduced in the case $\mathcal{S} = \varnothing$ to ensure the existence of a convenient globally defined spanning set of vectors. In the case $\mathcal{S} \neq \varnothing$, we may understand the above formula with $\Omega_4 = \Omega_5 = \Omega_6 = 0$.)

3.6.2. *Assumption on commutation errors in $r \geq R_k$.* As is to be expected, in order to obtain higher-order estimates on (3-3), we will need to strengthen our asymptotic flatness assumption to a higher-order statement on g_0. Again, instead of formulating sufficient conditions in terms of the decay properties of g_0 for large r, it will be convenient here to directly assume exactly the statement we shall need in terms of decay properties of the coefficients appearing in $[\mathfrak{D}^{\boldsymbol{k}}, \Box_{g_0}]\psi$.

Our assumption is thus simply the following: for all $k \geq 1$, there exists an R_k such that the following pointwise bound for $p = 0$ and for $\delta \leq p \leq 2 - \delta$ holds in $r \geq R_k$:

$$\left| \sum_{|\boldsymbol{k}|=k} \overset{(p)}{H}[\mathfrak{D}^{\boldsymbol{k}}\psi][\mathfrak{D}^{\boldsymbol{k}}, \Box_{g_0}]\psi \right| \leq \frac{1}{12} \sum_{|\boldsymbol{k}|=k} \overset{(p)}{K}[\mathfrak{D}^{\boldsymbol{k}}\psi] + C \sum_{|\boldsymbol{k}|\leq k-1} \overset{(p)}{K}[\mathfrak{D}^{\boldsymbol{k}}\psi] \quad \text{for all smooth functions } \psi. \tag{3-60}$$

Again, we emphasise that, by our conventions of Section 3.1, the constant C on the right-hand side of (3-60) in general depends on k. (In actuality, we need only assume (3-60) for all $k \le k_{\text{asympt}}$ for some sufficiently large k_{asympt}, but the statement of our theorem will then be restricted to such k.)

Assumption (3-60) is easily seen to be satisfied in all examples of Section 2.7. We note that the k-dependence of R_k is in general necessary, even for Minkowski space, as we must take $R_k \to \infty$ as $k \to \infty$.

3.6.3. *The red-shift commutation.* We recall the following:

Proposition 3.6.1 [Dafermos and Rodnianski 2013]. *Let Y be the vector field of Section 2.6.4, and let $\mathcal{H}^+$ be a Killing horizon with positive surface gravity as assumed in Section 2.4 such that the generator Z lies in the span of T and Ω_1. Then, along $\mathcal{H}^+$, we have*

$$[Y^k, \Box_{g_0}]\psi = \kappa_k(\vartheta, \varphi)(Y^{k+1}\psi) + Y^k\Box_{g_0}\psi + \sum_{\substack{|\boldsymbol{k}|\le k+1\\ k_3<k+1}} \alpha_{\boldsymbol{k}}(\vartheta, \phi)(\mathfrak{D}^{\boldsymbol{k}}\psi)$$

for a $\kappa_k \ge c > 0$.

We have the following:

Corollary 3.6.2. *Let $\overset{(p)}{H}$, $\overset{(p)}{H}_\varsigma$ be as defined in Section 3.5. Then there exist constants $c > 0$, $C > 0$ (independent of ς) such that, along $\mathcal{H}^+$, we have*

$$\overset{(p)}{H}[Y^k\psi][Y^k, \Box_{g_0}]\psi \ge c(Y^{k+1}\psi)^2 - C \sum_{\substack{|\boldsymbol{k}|\le k+1\\ k_3\ne k+1}} (\mathfrak{D}^{\boldsymbol{k}}\psi)^2. \tag{3-61}$$

Again, we emphasise that, by our conventions of Section 3.1, the constants c and C on the right-hand side of (3-61) in general depend on k.

3.6.4. *Divergence identity for the higher-order master currents.* We define the positive signature function for $1 \le |\boldsymbol{k}| \le k$,

$$\sigma(\boldsymbol{k}, k) = \begin{cases} \sigma_{12}(k) & \text{if } k_1 + k_2 = |\boldsymbol{k}|,\\ 1 & \text{otherwise,} \end{cases} \tag{3-62}$$

where σ_{12} will be chosen later such that moreover $\sigma_{12} \ge 1$.

We define an additional signature function $\varsigma(k)$, for $k \ge 1$ also to be determined later, which will be used to select the parameter of Proposition 3.4.2 to be used in the currents for higher-order estimates.

Let us finally fix a positive function

$$\varkappa(\boldsymbol{k}, k) = \varkappa(|\boldsymbol{k}|, k) = (\varkappa_0(k))^{1-|\boldsymbol{k}|} \tag{3-63}$$

for a $\varkappa_0(k) \ge 1$ to be determined later.

Given the above commutation vector fields (3-56) and the notation (3-57), we may now define currents

$$\begin{aligned} \underset{k}{\overset{(p)}{J}}[\psi] &:= \varkappa(0, k)\overset{(p)}{J}[\psi] + \sum_{1\le|\boldsymbol{k}|\le k} \varkappa(\boldsymbol{k}, k)\sigma(\boldsymbol{k}, k)\overset{(p)}{J}_{\varsigma(k)}[\mathfrak{D}^{\boldsymbol{k}}\psi],\\ \underset{k}{\overset{(p)}{K}}[\psi] &:= \varkappa(0, k)\overset{(p)}{K}[\psi] + \sum_{1\le|\boldsymbol{k}|\le k} \varkappa(\boldsymbol{k}, k)\sigma(\boldsymbol{k}, k)\overset{(p)}{K}_{\varsigma(k)}[\mathfrak{D}^{\boldsymbol{k}}\psi] \end{aligned} \tag{3-64}$$

and, given a collection $G_k = \{G_{\boldsymbol{k}}\}_{|\boldsymbol{k}|\le k}$ of scalar functions, the current

$$\overset{(p)}{\underset{k}{H}}[\psi]\cdot G_k := \varkappa_0(k)\overset{(p)}{H}[\psi]G_0 + \sum_{1\le|\boldsymbol{k}|\le k} \varkappa(\boldsymbol{k},k)\sigma(\boldsymbol{k},k)\overset{(p)}{H}_{\varsigma(k)}[\mathfrak{D}^{\boldsymbol{k}}\psi]G_{\boldsymbol{k}}, \tag{3-65}$$

where $\overset{(p)}{J}$, $\overset{(p)}{K}$ and $\overset{(p)}{H}$, as well as $\overset{(p)}{J}_\varsigma$, $\overset{(p)}{K}_\varsigma$ and $\overset{(p)}{H}_\varsigma$, are as defined in Section 3.5.

If ψ satisfies the inhomogeneous equation (3-3), then $\mathfrak{D}^{\boldsymbol{k}}\psi$ satisfies the equation

$$\Box_{g_0}(\mathfrak{D}^{\boldsymbol{k}}\psi) = [\mathfrak{D}^{\boldsymbol{k}}, \Box_{g_0}]\psi + \mathfrak{D}^{\boldsymbol{k}}F,$$

and the currents satisfy

$$\nabla^\mu \overset{(p)}{\underset{k}{J}}_\mu[\psi] = \overset{(p)}{\underset{k}{K}}[\psi] + \overset{(p)}{\underset{k}{H}}[\psi]\cdot\{[\mathfrak{D}^{\boldsymbol{k}}, \Box_{g_0}]\psi\} + \overset{(p)}{\underset{k}{H}}[\psi]\cdot\{\mathfrak{D}^{\boldsymbol{k}}F\}, \tag{3-66}$$

which can again be integrated in the region $\mathcal{R}(\tau_0,\tau_1,v)$ to yield

$$\int_{\Sigma(\tau_1,v)} \overset{(p)}{\underset{k}{J}}[\psi]\cdot n + \int_{\mathcal{S}(\tau_0,\tau_1)} \overset{(p)}{\underset{k}{J}}[\psi]\cdot n + \int_{\underline{C}_v(\tau_0,\tau_1)} \overset{(p)}{\underset{k}{J}}[\psi]\cdot n + \int_{\mathcal{R}(\tau_0,\tau_1,v)} \overset{(p)}{\underset{k}{K}}[\psi] \\ = \int_{\Sigma(\tau_0,v)} \overset{(p)}{\underset{k}{J}}[\psi]\cdot n - \int_{\mathcal{R}(\tau_0,\tau_1,v)} \overset{(p)}{\underset{k}{H}}[\psi]\cdot\{[\mathfrak{D}^{\boldsymbol{k}}, \Box_{g_0}]\psi\} - \int_{\mathcal{R}(\tau_0,\tau_1,v)} \overset{(p)}{\underset{k}{H}}[\psi]\cdot\{\mathfrak{D}^{\boldsymbol{k}}F\}. \tag{3-67}$$

Note that

$$\overset{(p)}{\underset{k}{H}}[\psi]\cdot\{[\mathfrak{D}^{\boldsymbol{k}}, \Box_{g_0}]\psi\} = 0 \tag{3-68}$$

in $r_1 + \frac{1}{2}(r_2 - r_1) \le r \le \frac{1}{2}R$.

Finally, if

$$\varkappa_0(k) \gg \varsigma(k) \tag{3-69}$$

is sufficiently large, then we notice that $\overset{(p)}{\underset{k}{K}}[\psi] \ge 0$ even for $r \le r_{\text{Killing}}$, and in fact, in $r_0 \le r \le r_1(\varsigma)$, we have

$$\overset{(p)}{\underset{k}{K}}[\psi] \ge c\varkappa_0(k)\overset{(p)}{K}[\psi] + c\sum_{1\le|\boldsymbol{k}|\le k}\varkappa(\boldsymbol{k},k)\varsigma\lambda_\varsigma(r)\left(\sum_{i=1}^{3}(\Omega_i\mathfrak{D}^{\boldsymbol{k}}\psi)^2 + (T\mathfrak{D}^{\boldsymbol{k}}\psi)^2 + |r_{\text{Killing}} - r|(Y\mathfrak{D}^{\boldsymbol{k}}\psi)^2\right) \\ + \varkappa(\boldsymbol{k},k)\lambda_\varsigma(r)(Y\mathfrak{D}^{\boldsymbol{k}}\psi)^2, \tag{3-70}$$

where we have absorbed the lower-order term in (3-35) with the wrong sign by the largeness of

$$\varkappa_0(k) = \varkappa(|\boldsymbol{k}|,k)^{-1}\varkappa(|\boldsymbol{k}|-1,k)$$

and the positivity of $\overset{(p)}{K}[\psi]$, and we have dropped the factor $\sigma(\boldsymbol{k},k)$ using that $\sigma \ge 1$. We will always assume (3-69) in what follows so that (3-70) holds.

Defining

$$\overset{(p)}{\underset{k}{\mathfrak{E}}}(\tau) := \int_{\Sigma(\tau)} \overset{(p)}{\underset{k}{J}}_\mu[\psi]n^\mu_{\Sigma(\tau)}, \quad \overset{(p)}{\underset{k}{\mathfrak{E}}}_{\mathcal{S}}(\tau_0,\tau) := \int_{\mathcal{S}(\tau_0,\tau)} \overset{(p)}{\underset{k}{J}}_\mu[\psi]n^\mu_{\mathcal{S}}, \\ \overset{(p)}{\underset{k}{\mathfrak{F}}}(v,\tau_0,\tau) := \int_{\underline{C}_v\cap\mathcal{R}(\tau_0,\tau)} \overset{(p)}{\underset{k}{J}}_\mu[\psi]n^\mu_{\underline{C}_v}, \tag{3-71}$$

note that by construction

$$\overset{(p)}{\underset{k}{\mathfrak{E}}}(\tau) \geq 0, \quad \overset{(p)}{\underset{k}{\mathfrak{E}}}_{\mathcal{S}}(\tau_0, \tau) \geq 0, \quad \overset{(p)}{\underset{k}{\mathfrak{F}}}(v, \tau_0, \tau) \geq 0,$$

and thus, from (3-67), we have in particular

$$\sup_{v:\tau_1 \leq \tau(v)} \overset{(p)}{\underset{k}{\mathfrak{F}}}(v, \tau_0, \tau_1) + \overset{(p)}{\underset{k}{\mathfrak{E}}}(\tau) + \overset{(p)}{\underset{k}{\mathfrak{E}}}_{\mathcal{S}}(\tau_0, \tau) + \int_{\mathcal{R}(\tau_0,\tau_1)} \overset{(p)}{\underset{k}{K}}[\psi]$$
$$\leq \overset{(p)}{\underset{k}{\mathfrak{E}}}(\tau_0) - \int_{\mathcal{R}(\tau_0,\tau_1)\cap\{r \leq r_2\}} \overset{(p)}{\underset{k}{H}}[\psi] \cdot \{[\mathfrak{D}^{\boldsymbol{k}}, \Box_{g_0}]\psi\} + \int_{\mathcal{R}(\tau_0,\tau_1)\cap\{r \geq R/2\}} |\overset{(p)}{\underset{k}{H}}[\psi] \cdot \{[\mathfrak{D}^{\boldsymbol{k}}, \Box_{g_0}]\psi\}|$$
$$+ \int_{\mathcal{R}(\tau_0,\tau_1)} |\overset{(p)}{\underset{k}{H}}[\psi] \cdot \{\mathfrak{D}^{\boldsymbol{k}} F\}|, \quad \text{(3-72)}$$

provided the terms on the right-hand side of the above inequalities are suitably integrable.

3.6.5. *Elliptic estimates for $\Box_{g_0}\psi = F$.* Since in the region $r \leq \frac{3}{4}R$ our commutation operators $\mathfrak{D}^{\boldsymbol{k}}$ do not necessarily span the tangent space, we will need to also invoke elliptic estimates. These will allow one to estimate, for solutions of $\Box_{g_0}\psi = F$, all highest-order derivatives $\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi$ from only derivatives $\mathfrak{D}^{\boldsymbol{k}}\psi$ with respect to our commutation operators together with appropriate terms involving F. These estimates require integration over hypersurfaces $\Sigma(\tau)$ or spacetime regions $\mathcal{R}(\tau_0, \tau_1)$ and rely on the fact that the commutation operators $\mathfrak{D}_i$ span an appropriate time-like direction.

For estimates at order k, we will in fact more generally need spacetime elliptic estimates in $r \leq \frac{9}{8}R_k$, despite the fact that the $\mathfrak{D}^{\boldsymbol{k}}$ span the tangent space as long as $r \geq \frac{3}{4}R$, in order to absorb commutation error terms where the formula (3-60) does not yet apply.

We have the following proposition.

Proposition 3.6.3. *Let $(\mathcal{M}, g_0)$ satisfy the assumptions of Section 2. Let ψ be a solution of the inhomogeneous equation (4-4) in $\mathcal{R}(\tau_0, \tau_1)$, and let $\tau_0 \leq \tau' \leq \tau_1$. Then, for all $k \geq 1$ and for all $r_{\text{Killing}} < r'_- < r' < r'' < r''_+ \leq R$,*

$$\int_{\Sigma(\tau')\cap\{r' \leq r \leq r''\}} \sum_{|\boldsymbol{k}| \leq k+1} (\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2$$
$$\lesssim \int_{\Sigma(\tau')\cap\{r'_- \leq r \leq r''_+\}} \sum_{\substack{1 \leq |\boldsymbol{k}| \leq k+1 \\ k_1+k_2 \geq |\boldsymbol{k}|-1}} (\mathfrak{D}^{\boldsymbol{k}}\psi)^2 + \sum_{|\boldsymbol{k}| \leq 1} (\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2 + \sum_{|\boldsymbol{k}| \leq k-1} (\widetilde{\mathfrak{D}}^{\boldsymbol{k}} F)^2, \quad \text{(3-73)}$$

where here $\lesssim$ depends on the choice of $r'_- < r' < r'' < r''_+$. We also have the estimate, for all $r' \leq r_1$,

$$\int_{\Sigma(\tau')\cap\{r \geq r'\}} \sum_{|\boldsymbol{k}| \leq k+1} (\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2 \lesssim \int_{\Sigma(\tau')\cap\{r \geq r'\}} \sum_{|\boldsymbol{k}| \leq k+1} (\mathfrak{D}^{\boldsymbol{k}}\psi)^2 + \sum_{|\boldsymbol{k}| \leq k-1} (\widetilde{\mathfrak{D}}^{\boldsymbol{k}} F)^2. \quad \text{(3-74)}$$

Note that the analogous statements to (3-73), (3-74) with integration on $\mathcal{R}(\tau_0, \tau_1) \cap \{r' \leq r \leq r''\}$, etc., follow immediately in view of the coarea formula.

In fact, even without assuming $r''_+ \leq R$, we still have the spacetime elliptic estimate

$$\int_{\mathcal{R}(\tau_0,\tau_1)\cap\{r'\leq r\leq r''\}} \sum_{|\boldsymbol{k}|\leq k+1} (\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2 \lesssim \int_{\mathcal{R}(\tau_0,\tau_1)\cap\{r'_-\leq r\leq r''_+\}} \sum_{\substack{1\leq|\boldsymbol{k}|\leq k+1\\ k_1+k_2\geq|\boldsymbol{k}|-1}} (\mathfrak{D}^{\boldsymbol{k}}\psi)^2 + \sum_{|\boldsymbol{k}|\leq 1} (\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2 + \sum_{|\boldsymbol{k}|\leq k-1} (\widetilde{\mathfrak{D}}^{\boldsymbol{k}}F)^2, \tag{3-75}$$

where again $\lesssim$ depends on the choice of $r'_- < r' < r'' < r''_+$.

Proof. Estimate (3-73) is a standard elliptic estimate, using the fact that the span of T, Ω_1 always contains a time-like direction in the region $r \geq r'_- > r_{\text{Killing}}$. Estimate (3-74) then follows from the previous in view of the fact that the $\mathfrak{D}^k$ span the tangent space for $r \leq r_1$ and $r \geq \frac{1}{2}R$.

As we have remarked, in the case $r''_+ \leq R$, estimate (3-75) follows from (3-73) by the coarea formula. It can be obtained more generally, even if $R \leq r''$ or $R \leq r''_+$, by a suitable integration by parts argument. We sketch this here for the case $k = 1$. Let us consider the most difficult case where $R < r'' < r''_+$. One first does the usual elliptic estimates on $\Sigma(\tau) \cap \{r \leq R\}$, then integrates over τ to estimate the left-hand side of (3-75) integrated over $\mathcal{R}(\tau_0, \tau_1) \cap \{r \leq R\}$ from the right-hand side, but with an additional boundary term on $r = R$. Squaring the inhomogeneous wave equation in $r \geq R$ and integrating by parts along the null cone $\Sigma(\tau) \cap \{r \geq R\}$, one may again bound the left-hand side of (3-75), integrated over $\mathcal{R}(\tau_0, \tau_1) \cap \{R \leq r \leq r''\}$, from the right-hand side and an additional boundary term on $r = R$. These boundary terms can be arranged to cancel modulo a term of the form

$$\int_{\mathcal{R}(\tau_0,\tau_1)\cap\{r=R\}} T\psi \,\not\Delta\psi,$$

where $\not\Delta$ denotes the Laplacian on $\Sigma(\tau) \cap \{r = R\}$, and this term can in turn be related to a bulk integral which can again be controlled by the right-hand side of (3-75). □

3.6.6. *Global control of the commutation errors.* We may now give the final statement allowing for the global control of the commutation error terms in the identity (3-66).

Proposition 3.6.4. *Under the assumptions of Section 3.6.2, we may choose our weight functions $\sigma_{12}(k)$ in the definition (3-62), $\varkappa_0(k)$ in the definition (3-63), and $\varsigma(k)$ such that the following holds:*

For all $k \geq 1$, there exists $r_{\text{Killing}} < r_1(k) \leq r_1$ such that, for all ψ satisfying the inhomogeneous equation (4-4) in $\mathcal{R}(\tau_0, \tau_1)$, the following estimates hold:

$$-\overset{(p)}{\underset{k}{H}}[\psi]\cdot\{[\mathfrak{D}^{\boldsymbol{k}}, \Box_{g_0}]\psi\} \leq \tfrac{1}{3}\overset{(p)}{\underset{k}{K}}[\psi], \qquad r_0 \leq r \leq r_1(k), \tag{3-76}$$

$$\int_{\mathcal{R}(\tau_0,\tau_1)\cap\{r_1(k)\leq r_2\}} |\overset{(p)}{\underset{k}{H}}[\psi]\cdot\{[\mathfrak{D}^{\boldsymbol{k}}, \Box_{g_0}]\psi\}| \leq \int_{\mathcal{R}(\tau_0,\tau_1)\cap\{r_{\text{Killing}}\leq r\leq r_2\}} \tfrac{1}{3}\overset{(p)}{\underset{k}{K}}[\psi] + C\sum_{|\boldsymbol{k}|\leq k-1} |\widetilde{\mathfrak{D}}^{\boldsymbol{k}}F|^2, \tag{3-77}$$

$$\overset{(p)}{\underset{k}{H}}[\psi]\cdot\{[\mathfrak{D}^{\boldsymbol{k}}, \Box_{g_0}]\psi\} = 0, \qquad r_1 + \tfrac{1}{2}(r_2 - r_1) \leq r \leq \tfrac{1}{2}R, \tag{3-78}$$

$$\int_{\mathcal{R}(\tau_0,\tau_1)\cap\{R/2\leq r\leq R_k\}} |\overset{(p)}{\underset{k}{H}}[\psi]\cdot\{[\mathfrak{D}^{\boldsymbol{k}}, \Box_{g_0}]\psi\}| \leq \int_{\mathcal{R}(\tau_0,\tau_1)\cap\{R/4\leq r\leq 9R_k/8\}} \tfrac{1}{3}\overset{(p)}{\underset{k}{K}}[\psi] + C\sum_{|\boldsymbol{k}|\leq k-1} |\widetilde{\mathfrak{D}}^{\boldsymbol{k}}F|^2, \tag{3-79}$$

$$|\overset{(p)}{\underset{k}{H}}[\psi]\cdot\{[\mathfrak{D}^{\boldsymbol{k}}, \Box_{g_0}]\psi\}| \leq \tfrac{1}{10}\overset{(p)}{\underset{k}{K}}[\psi], \qquad r \geq R_k. \tag{3-80}$$

Proof. We first prove (3-80). Here we will use (3-60). We have that the left-hand side of (3-80) is bounded in $r \geq R_k$ by

$$\begin{aligned}
&\Big|\overset{(p)}{\underset{k}{H}}[\psi]\cdot\{[\mathfrak{D}^{\boldsymbol{k}},\Box_{g_0}]\psi\}_{|\boldsymbol{k}|\leq k}\Big| \\
&\qquad = \Big|\sum_{|\boldsymbol{k}|\leq k}\varkappa(\boldsymbol{k},k)\overset{(p)}{H}[\mathfrak{D}^{\boldsymbol{k}}\psi][\mathfrak{D}^{\boldsymbol{k}},\Box_{g_0}]\Big| \\
&\qquad \leq \frac{1}{12}\sum_{|\boldsymbol{k}|\leq k}\sigma(\boldsymbol{k},k)\varkappa(\boldsymbol{k},k)\overset{(p)}{K}[\mathfrak{D}^{\boldsymbol{k}}\psi]+C\sum_{|\boldsymbol{k}|\leq k-1}\varkappa(|\boldsymbol{k}|+1,k)\sigma(\boldsymbol{k},k)\overset{(p)}{K}[\mathfrak{D}^{\boldsymbol{k}}\psi] \\
&\qquad \leq \frac{1}{12}\sum_{|\boldsymbol{k}|\leq k}\sigma(\boldsymbol{k},k)\varkappa(\boldsymbol{k},k)\overset{(p)}{K}[\mathfrak{D}^{\boldsymbol{k}}\psi]+C\sum_{|\boldsymbol{k}|\leq k-1}(\varkappa(|\boldsymbol{k}|+1,k)\varkappa^{-1}(|\boldsymbol{k}|,k))\varkappa(|\boldsymbol{k}|,k)\sigma(\boldsymbol{k},k)\overset{(p)}{K}[\mathfrak{D}^{\boldsymbol{k}}\psi] \\
&\qquad \leq \tfrac{1}{12}\overset{(p)}{\underset{k}{K}}[\psi]+\tfrac{1}{100}\overset{(p)}{\underset{k}{K}}[\psi]\leq\tfrac{1}{10}\overset{(p)}{\underset{k}{K}}[\psi]
\end{aligned}$$

provided that $\varkappa(\boldsymbol{k},k)$ is defined so that

$$\varkappa_0^{-1}(k)\ll 1. \tag{3-81}$$

To prove (3-76), we note that, in the region $r_0\leq r\leq r_1$, we may estimate

$$\begin{aligned}
&-\overset{(p)}{\underset{k}{H}}[\psi]\cdot\{[\mathfrak{D}^{\boldsymbol{k}},\Box_{g_0}]\psi\} \\
&\quad = -\sum_{\substack{1\leq|\boldsymbol{k}|\leq k\\ k_3<k}}\varkappa(|\boldsymbol{k}|,k)\overset{(p)}{H}_{\varsigma(k)}[\mathfrak{D}^{\boldsymbol{k}}\psi][\mathfrak{D}^{\boldsymbol{k}},\Box_{g_0}]\psi-\varkappa(k,k)\overset{(p)}{H}_{\varsigma(k)}[Y^k\psi][Y^k,\Box_{g_0}]\psi \\
&\quad \leq C\varsigma(k)^{\frac12}\lambda_{\varsigma(k)}(r)\sum_{1\leq|\boldsymbol{k}|\leq k}\varkappa(|\boldsymbol{k}|,k)\Big((T\mathfrak{D}^{\boldsymbol{k}}\psi)^2+\sum_{i=1}^{3}(\Omega_i\mathfrak{D}^{\boldsymbol{k}}\psi)^2+(\mathfrak{D}^{\boldsymbol{k}}\psi)^2\Big) \\
&\qquad\quad +C\varsigma(k)^{-\frac12}\lambda_{\varsigma(k)}(r)\sum_{1\leq|\boldsymbol{k}|\leq k+1}\varkappa(|\boldsymbol{k}|-1,k)(\mathfrak{D}^{\boldsymbol{k}}\psi)^2-c\lambda_{\varsigma(k)}(r)\varkappa(k,k)(Y^{k+1}\psi)^2 \\
&\qquad\quad +C\lambda_{\varsigma(k)}(r)|r-r_{\mathrm{Killing}}|\varkappa(k,k)\sum_{1\leq|\boldsymbol{k}|\leq k+1}(\mathfrak{D}^{\boldsymbol{k}}\psi)^2.
\end{aligned}\tag{3-82}$$

Here, we have applied (3-61) from Corollary 3.6.2 and the bounds on $\overset{(p)}{H}_{\varsigma(k)}$ following from (3-36) of Proposition 3.4.2. Note the dependence on $\varsigma(k)$ through $\lambda_{\varsigma(k)}(r)$, and note that the terms in the commutation identity not present on $\mathcal{H}^+$ appear with an extra $(r-r_{\mathrm{Killing}})$ factor. (We emphasise again the conventions from Section 3.1 that constants C, c also depends on k!)

We now have that, given $\varsigma=\varsigma(k)$, from (3-70), it follows that in the region $r_0\leq r\leq r_1(\varsigma)$, provided that (3-69) holds, we have

$$\begin{aligned}
\varsigma(k)^{\frac12}\lambda_{\varsigma(k)}(r)\sum_{1\leq|\boldsymbol{k}|\leq k}\varkappa(|\boldsymbol{k}|,k)\Big((T\mathfrak{D}^{\boldsymbol{k}}\psi)^2+\sum_{i=1}^{3}(\Omega_i\mathfrak{D}^{\boldsymbol{k}}\psi)^2+(\mathfrak{D}^{\boldsymbol{k}}\psi)^2\Big)&\leq C\varsigma^{-\frac12}(k)\overset{(p)}{\underset{k}{K}}[\psi]. \\
\varsigma^{-\frac12}(k)\lambda_{\varsigma(k)}(r)\sum_{1\leq|\boldsymbol{k}|\leq k+1}\varkappa(|\boldsymbol{k}|-1)(\mathfrak{D}^{\boldsymbol{k}}\psi)^2&\leq C\varsigma^{-\frac12}(k)\overset{(p)}{\underset{k}{K}}[\psi].
\end{aligned}\tag{3-83}$$

Finally, again provided (3-69) holds and we further restrict $r_1(\varsigma)$ such that $0 < r_1(\varsigma) - r_{\text{Killing}} \le \varsigma^{-1}$, we have

$$\lambda_{\varsigma(k)}(r)|r - r_{\text{Killing}}|\varkappa(k,k) \sum_{|\boldsymbol{k}|\le k+1} (\mathfrak{D}^{\boldsymbol{k}}\psi)^2 \le C\varsigma^{-1}(k)\overset{(p)}{\underset{k}{K}}[\psi].$$

It follows that, for $r_0 \le r \le r_1(\varsigma)$, we have

$$-\overset{(p)}{\underset{k}{H}}[\psi]\cdot\{[\mathfrak{D}^{\boldsymbol{k}}, \Box_{g_0}]\psi\} \le C\varsigma^{-\frac{1}{2}}(k)\overset{(p)}{\underset{k}{K}}[\psi]. \tag{3-84}$$

Now we fix $\varsigma = \varsigma(k)$ to be sufficiently large such that $C\varsigma^{-1/2}(k) \le \frac{1}{3}$. This yields (3-76).

Since $\varsigma(k)$ is now fixed, we also have $r_1(k) := r_1(\varsigma(k))$, and, according to our conventions, the dependence on these parameters may now be absorbed into the constants C, c, etc.

To prove (3-77), we estimate

$$\begin{aligned}
\sum_{|\boldsymbol{k}|\le k}\int_{\mathcal{R}(\tau_0,\tau_1)\cap\{r_1(k)\le r\le r_2\}} &|\overset{(p)}{\underset{k}{H}}[\psi]\cdot\{[\mathfrak{D}^{\boldsymbol{k}}, \Box_{g_0}]\psi\}| \\
&\le C\int_{\mathcal{R}(\tau_0,\tau_1)\cap\{r_1(k)\le r\le r_2\}} \sum_{|\boldsymbol{k}|\le k+1} \varkappa(|\boldsymbol{k}|-1,k)(\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2 \\
&\le C\int_{\mathcal{R}(\tau_0,\tau_1)\cap\{r_{\text{Killing}}\le r\le r_2\}} \sum_{\substack{1\le|\boldsymbol{k}|\le k+1\\ k_1+k_2\ge|\boldsymbol{k}|-1}} \varkappa(|\boldsymbol{k}|-1,k)(\mathfrak{D}^{\boldsymbol{k}}\psi)^2 + \sum_{|\boldsymbol{k}|\le 1}(\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2 + C\sum_{|\boldsymbol{k}|\le k-1}(\widetilde{\mathfrak{D}}^{\boldsymbol{k}}F)^2 \\
&\le C\int_{\mathcal{R}(\tau_0,\tau_1)\cap\{r_{\text{Killing}}\le r\le r_2\}} \sigma_{12}^{-1}(k)\overset{(p)}{\underset{k}{K}}[\psi] + C\varkappa^{-1}(0,k)\overset{(p)}{\underset{k}{K}}[\psi] + C\sum_{|\boldsymbol{k}|\le k-1}(\widetilde{\mathfrak{D}}^{\boldsymbol{k}}F)^2.
\end{aligned}$$

(We emphasise again the conventions from Section 3.1 that the constants C also depends on k.) We have used the spacetime version of estimate (3-73) of Proposition 3.6.3 in the above (i.e., (3-75)), with

$$r'_- := r_{\text{Killing}} + \tfrac{1}{2}(r_1(k) - r_{\text{Killing}}), \quad r''_+ := r_1 + \tfrac{3}{4}(r_2 - r_1), \quad r' = r_1(k), \quad r'' = r_1 + \tfrac{1}{2}(r_2 - r_1),$$

where we also use that the integrand on the left-hand side is nonzero only in $r' \le r \le r''$.

Requiring now $\sigma_{12}(k) \gg 1$ to be sufficiently large and $\varkappa(0,k) = \varkappa_0(k) \gg 1$ to be sufficiently large, we obtain (3-77).

The proof of (3-79) is analogous to (3-77) and also constrains $\sigma_{12}(k)$, $\varkappa_0(k)$ to be sufficiently large. This finally fixes $\sigma_{12}(k)$ and $\varkappa_0(k)$. Note here we use (3-75) with $r'' = R_k$ and $r''_+ = \frac{9}{8}R_k$ and in general we may have $\frac{9}{8}R_k \ge R$.

Identity (3-78) follows immediately from the definition of the commutation vector fields $\mathfrak{D}$. □

In what follows, we shall consider ς, σ, and $\varkappa$ fixed so as to satisfy the above Proposition. Thus, from now on, dependence of constants on ς, σ, and $\varkappa$ will be absorbed in the C and $\lesssim$ notation. We emphasise again that, according to our conventions from Section 3.1, all our constants c, C will in general depend on k.

We have the following immediate corollary of Proposition 3.6.4.

Corollary 3.6.5. *Let $k \geq 1$, and let ψ be a solution of the inhomogeneous equation (4-4) in $\mathcal{R}(\tau_0, \tau_1)$. Then we have the global (one-sided) bound*

$$\int_{\mathcal{R}(\tau_0,\tau_1)\cap\{r\leq r_2\}} -\overset{(p)}{\underset{k}{H}}[\psi]\cdot\{[\mathfrak{D}^{\boldsymbol{k}},\Box_{g_0}]\psi\} + \int_{\mathcal{R}(\tau_0,\tau_1)\cap\{r\geq r_2\}} |\overset{(p)}{\underset{k}{H}}[\psi]\cdot\{[\mathfrak{D}^{\boldsymbol{k}},\Box_{g_0}]\psi\}| \leq \int_{\mathcal{R}(\tau_0,\tau_1)\cap\{r\leq r_2\}\cap\{r\geq R/4\}} \tfrac{1}{2}\overset{(p)}{\underset{k}{K}}[\psi] + C\sum_{|\boldsymbol{k}|\leq k-1}|\widetilde{\mathfrak{D}}^{\boldsymbol{k}}F|^2. \tag{3-85}$$

Proof. Note that the presence of $\frac{1}{2}$ instead of $\frac{1}{3}$ is due to the fact that both estimates (3-79) and (3-80) borrow from the bulk of the region $r \geq R_k$. Note that restricted to the region $r \geq R$, if desired, we may of course replace the $\widetilde{\mathfrak{D}}$ commutation on the last term on the right-hand side by $\mathfrak{D}$. □

3.6.7. *The higher-order energy notation.* We are now ready to derive the higher-order weighted estimates which will allow us in particular to address nonlinear problems. We will turn to the estimates themselves in Section 3.6.8. In the meantime, let us introduce some notation below.

We proceed to summarise the definitions: given $k \geq 1$, $1 \leq \tau_0 \leq \tau$, v, and a spacetime function ψ, we define the following energies.

In general, our fundamental energies without degeneration functions χ or ρ will be defined with $\widetilde{\mathfrak{D}}^{\boldsymbol{k}}$ as commutation vector fields, i.e., they will contain all derivatives at the relevant order. Specifically, we define first the unweighted energies:

$$\overset{(0)}{\mathcal{E}}(\tau) := \sum_{|\boldsymbol{k}|\leq k}\int_{\Sigma(\tau)} (L\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2 + |\nabla\!\!\!/\,\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi|^2 + \iota_{r\leq R}(\underline{L}\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2 + r^{-2}(\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2, \tag{3-86}$$

$$\overset{(0)}{\mathcal{E}}_{\mathcal{S}}(\tau_0,\tau) := \sum_{|\boldsymbol{k}|\leq k}\int_{\mathcal{S}(\tau_0,\tau)} (L\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2 + |\nabla\!\!\!/\,\psi|^2 + (\underline{L}\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2 + (\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2, \tag{3-87}$$

$$\overset{(0)}{\underset{k}{\mathcal{F}}}(v,\tau_0,\tau) := \sum_{|\boldsymbol{k}|\leq k}\int_{\underline{C}_v\cap\mathcal{R}(\tau_0,\tau)} (\underline{L}\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2 + (\nabla\!\!\!/\,\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2 + r^{-2}(\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2, \tag{3-88}$$

$$\overset{(-1-\delta)}{\underset{k}{\mathcal{E}'}}(\tau) := \sum_{|\boldsymbol{k}|\leq k}\int_{\Sigma(\tau)} r^{-1-\delta}((L\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2 + (\underline{L}\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2 + |\nabla\!\!\!/\,\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi|^2) + r^{-3-\delta}(\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2. \tag{3-89}$$

We now define (in analogy with (3-46)–(3-48)) the higher-order p-weighted energies for $\delta \leq p \leq 2-\delta$:

$$\overset{(p)}{\underset{k}{\mathcal{E}}}(\tau) := \overset{(0)}{\mathcal{E}}(\tau) + \sum_{|\boldsymbol{k}|\leq k}\int_{\Sigma(\tau)\cap\{r\geq R\}} r^p(r^{-1}L(r\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi))^2 + r^{\frac{p}{2}}(L\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2 + r^{\frac{p}{2}-2}(\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2, \tag{3-90}$$

$$\overset{(p)}{\underset{k}{\mathcal{F}}}(v,\tau_0,\tau) := \overset{(0)}{\underset{k}{\mathcal{F}}}(v) + \sum_{|\boldsymbol{k}|\leq k}\int_{\underline{C}_v\cap\mathcal{R}(\tau_0,\tau)} r^p|\nabla\!\!\!/\,\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi|^2 + r^{p-2}(\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2, \tag{3-91}$$

$$\overset{(p-1)}{\underset{k}{\mathcal{E}'}}(\tau) := \overset{(-1-\delta)}{\underset{k}{\mathcal{E}'}}(\tau) + \sum_{|\boldsymbol{k}|\leq k}\int_{\Sigma(\tau)} r^{p-1}((r^{-1}L(r\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi))^2 + (L\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2 + |\nabla\!\!\!/\,\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi|^2) + r^{p-3}(\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2. \tag{3-92}$$

Note the following important relation:

$$\overset{(p)}{\underset{k}{\mathcal{E}}} \gtrsim \overset{(p')}{\underset{k}{\mathcal{E}}},\ \overset{(p)}{\underset{k}{\mathcal{F}}} \gtrsim \overset{(p')}{\underset{k}{\mathcal{F}}} \text{ for } p \geq p' \geq \delta \text{ or } p'=0, \quad \overset{(p-1)}{\underset{k}{\mathcal{E}'}} \gtrsim \overset{(p-1)}{\underset{k}{\mathcal{E}}} \text{ for } p \geq 1+\delta, \quad \overset{(p-1)}{\underset{k}{\mathcal{E}'}} \gtrsim \overset{(0)}{\mathcal{E}} \text{ for } p \geq 1, \tag{3-93}$$

representing the higher-order analogue of (3-49)

In contrast, we define the higher-order energies carrying the degeneration functions χ, ρ and $\tilde{\rho}$ in terms of the $\mathfrak{D}$ commutators:

$$^{\chi}\overset{(-1-\delta)}{\underset{k}{\mathcal{E}'}}(\tau) := \sum_{|\boldsymbol{k}|\leq k} \int_{\Sigma(\tau)} r^{-1-\delta}\chi(r)((L\mathfrak{D}^k\psi)^2 + (\underline{L}\mathfrak{D}^k\psi)^2 + |\nabla\!\!\!/\,\mathfrak{D}^k\psi|^2), \tag{3-94}$$

$$^{\rho}\overset{(-1-\delta)}{\underset{k}{\mathcal{E}'}}(\tau) := \sum_{|\boldsymbol{k}|\leq k} \int_{\Sigma(\tau)} r^{-1-\delta}\rho(r)((L\mathfrak{D}^k\psi)^2 + (\underline{L}\mathfrak{D}^k\psi)^2 + |\nabla\!\!\!/\,\mathfrak{D}^k\psi|^2), \tag{3-95}$$

$$^{\tilde{\rho}}\overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau) := \sum_{|\boldsymbol{k}|\leq k} \int_{\Sigma(\tau)} \tilde{\rho}(r) r^{-3-\delta}(\mathfrak{D}^k\psi)^2, \tag{3-96}$$

and then

$$\begin{aligned} ^{\chi}\overset{(p-1)}{\underset{k}{\mathcal{E}'}}(\tau) &:= {}^{\chi}\overset{(-1-\delta)}{\underset{k}{\mathcal{E}'}}(\tau) \\ &\quad + \sum_{|\boldsymbol{k}|\leq k} \int_{\Sigma(\tau)\cap\{r\geq R\}} r^{p-1}((r^{-1}L(r\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi))^2 + (L\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2 + |\nabla\!\!\!/\,\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi|^2) + r^{p-3}(\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2, \end{aligned} \tag{3-97}$$

$$\begin{aligned} ^{\rho}\overset{(p-1)}{\underset{k}{\mathcal{E}'}}(\tau) &:= {}^{\rho}\overset{(-1-\delta)}{\underset{k}{\mathcal{E}'}}(\tau) \\ &\quad + \sum_{|\boldsymbol{k}|\leq k} \int_{\Sigma(\tau)\cap\{r\geq R\}} r^{p-1}((r^{-1}L(r\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi))^2 + (L\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2 + |\nabla\!\!\!/\,\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi|^2) + r^{p-3}(\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2. \end{aligned} \tag{3-98}$$

(Note that in the integrals over $r \geq R$, it does not matter whether we use $\widetilde{\mathfrak{D}}$ or $\mathfrak{D}$ as these would here define directly comparable energies.)

In analogy with (3-30), we define

$$\overset{\xi}{\underset{k-1}{\mathcal{E}'}}(\tau) := \sum_{|\boldsymbol{k}|\leq k} \int_{\Sigma(\tau)} \xi(r)(\mathfrak{D}^k\psi)^2. \tag{3-99}$$

We note the fundamental relation

$$\overset{\xi}{\underset{k-1}{\mathcal{E}'}} \lesssim {}^{\chi}\overset{(-1-\delta)}{\underset{k-1}{\mathcal{E}'}} + \overset{(-1-\delta)}{\underset{k-2}{\mathcal{E}'}} \tag{3-100}$$

which follows from our constraints on the support of ξ and the degeneration of χ.

Proposition 3.6.6. *For a general smooth function ψ, we have the following relations between energies*:

$$\overset{(p)}{\underset{k}{\mathfrak{E}}}(\tau) \lesssim \overset{(p)}{\underset{k}{\mathcal{E}}}(\tau), \tag{3-101}$$

$$\overset{(p)}{\underset{k}{\mathfrak{E}}}_{\mathcal{S}}(\tau) = \overset{(0)}{\underset{k}{\mathfrak{E}}}_{\mathcal{S}}(\tau) \sim \overset{(p)}{\underset{k}{\mathcal{E}}}_{\mathcal{S}}(\tau), \quad \overset{(p)}{\underset{k}{\mathfrak{F}}}(v,\tau_0,\tau) \sim \overset{(p)}{\underset{k}{\mathcal{F}}}(v,\tau_0,\tau), \tag{3-102}$$

$$\begin{aligned} \int_{\mathcal{R}(\tau_0,\tau_1)} \overset{(p)}{\underset{k}{K}}[\psi] + \sum_{|\boldsymbol{k}|\leq k} \tilde{A}\xi(r)(\mathfrak{D}^k\psi)^2 &\gtrsim \int_{\tau_0}^{\tau_1} {}^{\rho}\overset{(p-1)}{\underset{k}{\mathcal{E}'}}(\tau')\,d\tau' + \int_{\tau_0}^{\tau_1} {}^{\tilde{\rho}}\overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau')\,d\tau', \quad 2-\delta \geq p \geq \delta, \\ \int_{\mathcal{R}(\tau_0,\tau_1)} \overset{(p)}{\underset{k}{K}}[\psi] + \sum_{|\boldsymbol{k}|\leq k} \tilde{A}\xi(r)(\mathfrak{D}^k\psi)^2 &\gtrsim \int_{\tau_0}^{\tau_1} {}^{\rho}\overset{(-1-\delta)}{\underset{k}{\mathcal{E}'}}(\tau')\,d\tau' + \int_{\tau_0}^{\tau_1} {}^{\tilde{\rho}}\overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau')\,d\tau', \quad p=0. \end{aligned} \tag{3-103}$$

For ψ a solution of the inhomogeneous equation $\Box_{g_0}\psi = F$, we have

$$\overset{(p)}{\underset{k}{\mathcal{E}}}(\tau) \lesssim \overset{(p)}{\underset{k}{\mathfrak{E}}}(\tau) + \int_{\Sigma(\tau)\cap\{r\le R\}} \sum_{|\boldsymbol{k}|\le k-1} (\widetilde{\mathfrak{D}}^{\boldsymbol{k}} F)^2. \tag{3-104}$$

If $\chi = 1$ and $\tilde{\rho} = 1$ identically (as in case (i)*), then*

$$\overset{(p-1)}{\underset{k}{\mathcal{E}'}}(\tau) \lesssim {}^{\chi}\overset{(p-1)}{\underset{k}{\mathcal{E}'}}(\tau) + {}^{\tilde{\rho}}\overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau) + \int_{\Sigma(\tau)\cap\{r\le R\}} \sum_{|\boldsymbol{k}|\le k-1} (\widetilde{\mathfrak{D}}^{\boldsymbol{k}} F)^2, \quad 2-\delta \ge p \ge \delta, \tag{3-105}$$

$$\overset{(-1-\delta)}{\underset{k}{\mathcal{E}'}}(\tau) \lesssim {}^{\chi}\overset{(-1-\delta)}{\underset{k}{\mathcal{E}'}}(\tau) + {}^{\tilde{\rho}}\overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau) + \int_{\Sigma(\tau)\cap\{r\le R\}} \sum_{|\boldsymbol{k}|\le k-1} (\widetilde{\mathfrak{D}}^{\boldsymbol{k}} F)^2, \tag{3-106}$$

and if $\tilde{\rho} = 1$ identically (i.e., as in cases (i) *and* (ii)*), then*

$$\overset{(p-1)}{\underset{k-1}{\mathcal{E}'}}(\tau) \lesssim {}^{\rho}\overset{(p-1)}{\underset{k}{\mathcal{E}'}}(\tau) + {}^{\tilde{\rho}}\overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau) + \int_{\Sigma(\tau)\cap\{r\le R\}} \sum_{|\boldsymbol{k}|\le k-2} (\widetilde{\mathfrak{D}}^{\boldsymbol{k}} F)^2, \quad 2-\delta \ge p \ge \delta, \tag{3-107}$$

$$\overset{(-1-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau) \lesssim {}^{\rho}\overset{(-1-\delta)}{\underset{k}{\mathcal{E}'}}(\tau) + {}^{\tilde{\rho}}\overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau) + \int_{\Sigma(\tau)\cap\{r\le R\}} \sum_{|\boldsymbol{k}|\le k-2} (\widetilde{\mathfrak{D}}^{\boldsymbol{k}} F)^2. \tag{3-108}$$

Proof. The relations (3-101)–(3-103) follow immediately from our coercivity and boundedness assumptions on the currents, while the inequalities (3-104)–(3-108) follow easily from the elliptic estimate (3-74) of Proposition 3.6.3. □

We note the following immediate corollary of the above proposition:

Corollary 3.6.7. *Let ψ be a solution of*

$$\Box_{g_0}\psi = 0 \quad \textit{in } \mathcal{R}(\tau_0, \tau_1).$$

Then calligraphic and fraktur energies on $\Sigma(\tau)$ are equivalent:

$$\overset{(p)}{\underset{k}{\mathcal{E}}}(\tau) \sim \overset{(p)}{\underset{k}{\mathfrak{E}}}(\tau).$$

If $\chi = 1$ and $\tilde{\rho} = 1$ identically (as in case (i)*), then*

$$\overset{(p-1)}{\underset{k}{\mathcal{E}'}}(\tau) \sim {}^{\chi}\overset{(p-1)}{\underset{k}{\mathcal{E}'}}(\tau) + {}^{\tilde{\rho}}\overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau),$$
$$\overset{(-1-\delta)}{\underset{k}{\mathcal{E}'}}(\tau) \sim {}^{\chi}\overset{(-1-\delta)}{\underset{k}{\mathcal{E}'}}(\tau) + {}^{\tilde{\rho}}\overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau).$$

If $\tilde{\rho} = 1$ identically (i.e., as in cases (i) *and* (ii)*), then*

$$\overset{(p-1)}{\underset{k-1}{\mathcal{E}'}}(\tau) \lesssim {}^{\rho}\overset{(p-1)}{\underset{k}{\mathcal{E}'}}(\tau) + {}^{\tilde{\rho}}\overset{(-1-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau),$$
$$\overset{(-1-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau) \lesssim {}^{\rho}\overset{(-1-\delta)}{\underset{k}{\mathcal{E}'}}(\tau) + {}^{\tilde{\rho}}\overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau).$$

3.6.8. *The final higher-order estimates.* We conclude with our higher-order version of Proposition 3.5.1.

Proposition 3.6.8. *Under the assumptions of Proposition 3.5.1, let us assume the additional assumptions of Section 3.6.2.*

Fix $k \geq 1$. Then, for all $0 < \delta \leq p \leq 2-\delta$ and for all $\tau_0 \leq \tau \leq \tau_1$, we have the following statement: Let ψ be a solution of the inhomogeneous equation (4-4) in $\mathcal{R}(\tau_0, \tau_1)$. Then we have

$$\sup_{v:\tau\leq\tau(v)} \overset{(p)}{\mathfrak{F}}(v, \tau_0, \tau), \quad \overset{(p)}{\underset{k}{\mathfrak{E}}}(\tau) + \overset{(0)}{\underset{k}{\mathfrak{E}}}_{\mathcal{S}}(\tau_0, \tau) + c\int_{\tau_0}^{\tau} {}^{\rho}\overset{(p-1)}{\underset{k}{\mathcal{E}'}}(\tau')\,d\tau' + c\int_{\tau_0}^{\tau} {}^{\tilde{\rho}}\overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau')\,d\tau'$$
$$\leq \overset{(p)}{\underset{k}{\mathfrak{E}}}(\tau_0) + A\int_{\tau_0}^{\tau} \overset{\xi}{\underset{k-1}{\mathcal{E}'}}(\tau') + \int_{\mathcal{R}(\tau_0,\tau)} |\overset{(p)}{\underset{k}{H}}[\psi]\cdot\{\mathfrak{D}^{\boldsymbol{k}}F\}| + C\int_{\mathcal{R}(\tau_0,\tau)} \sum_{|\boldsymbol{k}|\leq k} (\mathfrak{D}^{\boldsymbol{k}}F)^2$$
$$+ C\int_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq R\}} \sum_{|\boldsymbol{k}|\leq k-1} (\widetilde{\mathfrak{D}}^{\boldsymbol{k}}F)^2 \quad (3\text{-}109)$$

as well as the estimate

$$\sup_{v:\tau\leq\tau(v)} \overset{(p)}{\underset{k}{\mathcal{F}}}(v, \tau_0, \tau) + \overset{(p)}{\underset{k}{\mathcal{E}}}(\tau) + \overset{(0)}{\mathcal{E}}_{\mathcal{S}}(\tau_0, \tau) + \int_{\tau_0}^{\tau} {}^{\chi}\overset{(p-1)}{\underset{k}{\mathcal{E}'}}(\tau')\,d\tau' + \int_{\tau_0}^{\tau} \overset{(p-1)}{\underset{k-1}{\mathcal{E}'}}(\tau')\,d\tau'$$
$$\lesssim \overset{(p)}{\underset{k}{\mathcal{E}}}(\tau_0) + \int_{\mathcal{R}(\tau_0,\tau)} \sum_{|\boldsymbol{k}|\leq k} (|V_p^{\mu}(\mathfrak{D}^{\boldsymbol{k}}\psi)_{\mu}| + |w_p\mathfrak{D}^{\boldsymbol{k}}\psi|)|\mathfrak{D}^{\boldsymbol{k}}F| + \int_{\mathcal{R}(\tau_0,\tau)} \sum_{|\boldsymbol{k}|\leq k} (\mathfrak{D}^{\boldsymbol{k}}F)^2$$
$$+ C\int_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq R\}} \sum_{|\boldsymbol{k}|\leq k-1} (\widetilde{\mathfrak{D}}^{\boldsymbol{k}}F)^2 + \int_{\Sigma(\tau)\cap\{r\leq R\}} \sum_{|\boldsymbol{k}|\leq k-1} (\widetilde{\mathfrak{D}}^{\boldsymbol{k}}F)^2. \quad (3\text{-}110)$$

For $p = 0$, identical statements hold with $p-1$ replaced by $-1-\delta$.

Remark 3.6.9. In the case where we replace the middle term of (3-3) with (3-10), we should add

$$\sum_{|\boldsymbol{k}|\leq k} \sqrt{\int_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq R\}} |L\mathfrak{D}^{\boldsymbol{k}}\psi|^2 + |\underline{L}\mathfrak{D}^{\boldsymbol{k}}\psi|^2 + |\nabla\!\!\!\!/\,\mathfrak{D}^{\boldsymbol{k}}\psi|^2 + r^{-2}|\mathfrak{D}^{\boldsymbol{k}}\psi|^2} \sqrt{\int_{\mathcal{R}(\tau_0,\tau_1)\cap\{r\leq R\}} (\mathfrak{D}^{\boldsymbol{k}}F)^2}$$

to the right-hand side of (3-110).

Proof. We first prove (3-109). This follows from (3-72) and applying Proposition 3.6.4 (in the form of Corollary 3.6.5) and the properties of Section 3.6.7.

To prove (3-110), let us first commute the equation by $T^{\tilde{k}}$ (and $\Omega_1^{\tilde{k}}$, if assumed Killing) and apply the black box estimate (3-54) to $T^{\tilde{k}}\psi$ (and $\Omega_1^{\tilde{k}}\psi$) for $\tilde{k} \leq k$. This gives

$$\sup_{v} \overset{(p)}{\mathcal{F}}[T^{\tilde{k}}\psi](v, \tau_0, \tau) + \overset{(p)}{\mathcal{E}}[T^{\tilde{k}}\psi](\tau) + \overset{(p)}{\mathcal{E}}_{\mathcal{S}}[T^{\tilde{k}}\psi](\tau) + \int_{\tau_0}^{\tau} {}^{\chi}\overset{(p-1)}{\mathcal{E}'}[T^{\tilde{k}}\psi](\tau')\,d\tau + \int_{\tau_0}^{\tau} \underset{-1}{\mathcal{E}'}[T^{\tilde{k}}\psi](\tau')\,d\tau'$$
$$\lesssim \overset{(p)}{\mathcal{E}}[T^{\tilde{k}}\psi](\tau_0) + \int_{\mathcal{R}(\tau_0,\tau)} (|r^p r^{-1}L(rT^{\tilde{k}}\psi)| + |\widetilde{V}_p^{\mu}\partial_{\mu}T^{\tilde{k}}\psi| + |\tilde{w}_p\psi|)|T^{\tilde{k}}F| + \int_{\mathcal{R}(\tau_0,\tau)} (T^{\tilde{k}}F)^2, \quad (3\text{-}111)$$

and similarly with $\Omega_1^{\tilde{k}}\psi$.

Now note that, in the support of ξ, all $\mathfrak{D}$ vanish except for T (and Ω_1), whence we manifestly have

$$A\int_{\tau_0}^{\tau}\overset{\xi}{\underset{k-1}{\mathcal{E}}}'(\tau')\,d\tau' \lesssim \sum_{\tilde{k}\le k}\int_{\tau_0}^{\tau}\underset{-1}{\mathcal{E}}'[T^{\tilde{k}}\psi](\tau')+\underset{-1}{\mathcal{E}}'[\upsilon\Omega_1^{\tilde{k}}\psi](\tau'),$$

where we recall that $\upsilon=1$ only if Ω_1 is Killing. It follows that the left-hand side of (3-109) is bounded by the right-hand side of (3-110).

Now apply the black box estimate (3-54) to $\mathfrak{D}^{\boldsymbol{k}}\psi$ and note that we can bound the terms arising from $[\Box_{g_0},\mathfrak{D}^{\boldsymbol{k}}]\psi$ by $c\int_{\tau_0}^{\tau}{}^{\rho}\overset{(p-1)}{\underset{k}{\mathcal{E}}}'(\tau')+\overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}}}'(\tau')\,d\tau'$ and the spacetime term involving $\widetilde{\mathcal{D}}^{\boldsymbol{k}}F$ on the right-hand side of (3-110).

We thus have that (3-111) holds with $\mathfrak{D}^{\boldsymbol{k}}\psi$ in place of $T^k\psi$.

We apply our elliptic estimate of Proposition 3.6.6 to bound $\overset{(p)}{\underset{k}{\mathcal{E}}}(\tau)$ from $\overset{(p)}{\underset{k}{\mathfrak{E}}}(\tau)$, generating the last term on the right-hand side of (3-110).

Finally, by (3-74) of Proposition 3.6.3, we may estimate

$$\int_{\tau_0}^{\tau}\overset{(p-1)}{\underset{k-1}{\mathcal{E}}}'(\tau')\,d\tau' \lesssim \int_{\tau_0}^{\tau}\Big({}^{\rho}\overset{(p-1)}{\underset{k-1}{\mathcal{E}}}'(\tau')+\overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}}}'(\tau')\Big)\,d\tau' + \int_{\mathcal{R}(\tau_0,\tau)\cap\{r\le R\}}\sum_{|\boldsymbol{k}|\le k-1}|\widetilde{\mathfrak{D}}^{\boldsymbol{k}}F|^2. \qquad \square$$

Remark 3.6.10. Note that for convenience we have used (3-109) to obtain (3-110). This was because our precise commutation assumptions were stated with respect to the global currents. If one does not assume estimate (3-33), and thus one does not have (3-109), one may still obtain (3-110) under the assumption of asymptotic flatness of Sections 3.5 and 3.6.2, where we insert simply the far-away currents $\overset{(p)}{J}_{\text{far}}$, $\overset{(p)}{K}_{\text{far}}$ in (3-60). (We also use that, from the assumptions of Section 2, we can still define currents as in Section 3.4.5 giving enhanced positivity near $\mathcal{H}^+$ and in the black hole interior.) Thus, in particular, estimate (3-110) holds in the Kerr case in the full subextremal range $|a|<M$.

3.6.9. *Sobolev inequalities and interpolation of p-weighted energies.* We end this section recording two easy statements about the energies we have defined.

In anticipation of studying nonlinear equations, one will need to estimate lower-order pointwise quantities from higher-order energies by Sobolev inequalities. We have the following:

Proposition 3.6.11. *Let ψ be a smooth function on a neighbourhood of $\Sigma(\tau)$. Then we have*

$$\sum_{|\boldsymbol{k}|\le k-3}\ \sup_{x\in\Sigma(\tau)\cap\{r\le R\}}(\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi(x))^2\lesssim\min\Big\{\overset{(-1-\delta)}{\underset{k}{\mathcal{E}}}'(\tau),\overset{(0)}{\mathcal{E}}(\tau)\Big\}. \tag{3-112}$$

Proof. We note that the left-hand side of (3-112) is in fact bounded by the energy restricted to $\Sigma\cap\{r\le R\}$, which is why we may choose either of the two quantities on the right-hand side (3-112). $\square$

Weighted Sobolev inequalities will also hold globally on $\Sigma(\tau)$, and in practice these are used to estimate nonlinearities in the region near infinity. Because this use is incorporated in our assumption on the null condition (see already Section 4.7), we shall not need to state such inequalities, although, in practice, they will appear in the context of verifying the assumptions of Section 4.7. See already Appendix C.

Finally, we have the following easy interpolation result.

Proposition 3.6.12. *For δ as fixed in (3-1), one has the following interpolation inequalities*:

$$\underset{k}{\overset{(1)}{\mathcal{E}}}(\tau) \lesssim \big(\underset{k}{\overset{(1-\delta)}{\mathcal{E}}}(\tau)\big)^{1-\delta}\big(\underset{k}{\overset{(2-\delta)}{\mathcal{E}}}(\tau)\big)^{\delta}, \tag{3-113}$$

$$\underset{k}{\overset{(\delta-1)}{\mathcal{E}'}}(\tau) \lesssim \big(\underset{k}{\overset{(-1-\delta)}{\mathcal{E}'}}(\tau)\big)^{\frac{1-\delta}{1+\delta}}\big(\overset{(0)}{\mathcal{E}'}(\tau)\big)^{\frac{2\delta}{1+\delta}}, \tag{3-114}$$

$$\int_{\tau_0}^{\tau_0+1/2}\underset{k-1}{\overset{(0)}{\mathcal{E}}}(\tau) \lesssim \sqrt{\int_{\tau_0}^{\tau_0+1/2}\overset{(0)}{\mathcal{E}}(\tau)}\sqrt{\int_{\tau_0}^{\tau_0+1/2}\underset{k-2}{\overset{(0)}{\mathcal{E}}}(\tau)}. \tag{3-115}$$

Proof. The proof of these inequalities is standard and is left to the reader. □

4. Quasilinear equations: preliminaries, the null condition, and local existence

We assume throughout that $(\mathcal{M}, g_0)$ satisfies the assumptions of Sections 2 and 3 (for cases (i), (ii) or (iii)). We will introduce in this section the class of quasilinear equations to be considered in this paper, and derive some preliminary results which will be used in the proof of our main theorem.

4.1. ***The class of equations.*** We will consider solutions ψ to quasilinear equations of the form

$$\Box_{g(\psi,x)}\psi = N^{\mu\nu}(\psi, x)\partial_\mu\psi\,\partial_\nu\psi, \tag{4-1}$$

where

$$g : \mathbb{R}\times\mathcal{M} \to T^*\mathcal{M}\otimes T^*\mathcal{M} \quad \text{and} \quad N : \mathbb{R}\times\mathcal{M} \to T\mathcal{M}\otimes T\mathcal{M} \tag{4-2}$$

are such that $\pi\circ g(\psi, x) = x$ and $\pi\circ N(\psi, x) = x$, where

$$\pi : T\mathcal{M}\otimes T\mathcal{M} \to \mathcal{M} \quad \text{and} \quad \pi : T^*\mathcal{M}\otimes T^*\mathcal{M} \to \mathcal{M}$$

denote the canonical projections, and such that $g(0, x) = g_0(x)$ for all x, while $g(\cdot\,, x) = g_0(x)$ for $r(x) \geq \frac{1}{2}R$, and N and g are smooth maps.

We will assume moreover that, for each k,

$$\partial_{\boldsymbol{x}}^{\boldsymbol{k}}\partial_\xi^s g^{\alpha\beta}(\xi, x) \quad \text{and} \quad \partial_{\boldsymbol{x}}^{\boldsymbol{k}}\partial_\xi^s N^{\alpha\beta}(\xi, x)$$

are uniformly bounded for all $|\boldsymbol{k}| + s \leq k$, all $r \leq R$, and $|\xi| \leq 1$. Here $\partial_{\boldsymbol{x}}^{\boldsymbol{k}}$ denote multi-indices with respect to the ambient Cartesian coordinates of Section 2.1.

For $N^{\mu\nu}(\psi, x)$, we will eventually need in addition to assume some version of the null condition. We will only introduce this in Section 4.7 (see already Assumption 4.7.1). Let us note that our assumption on the support of $g - g_0$ is merely so as not to deal with formulations of the null condition for the quasilinear part.

We will sometimes view the equation in (4-1) as an inhomogeneous equation on a fixed g_0 background, i.e., we will write it as

$$\Box_{g_0}\psi = N^{\mu\nu}(\psi, x)\partial_\mu\psi\,\partial_\nu\psi + (\Box_{g_0} - \Box_{g(\psi,x)})\psi, \tag{4-3}$$

and similarly for its commuted versions. We may thus apply estimates to the inhomogeneous equation

$$\Box_{g_0}\psi = F. \tag{4-4}$$

4.2. ***Smallness parameters.*** Starting in this section, we shall introduce smallness parameters (i.e., parameters related to making smallness assumptions on solutions), denoted using the symbol ε and various subscripts, e.g., $\varepsilon_{\mathrm{prelim}}$, $\varepsilon_{\mathrm{local}}$. Unless otherwise noted, these will depend only on $(\mathcal{M}, g_0)$, on the nonlinearities of (4-1) defined by (4-2) and, in general, on k, if there is k dependence in the statement.

When these smallness parameters depend on an additional quantity, this will be indicated in parenthesis; e.g., $\hat{\varepsilon}_{\mathrm{slab}}(\alpha)$.

Note finally that our convention on constants denoted C, c, etc. remains the same as set in Section 3.1, i.e., these will *not* depend on (4-2).

4.3. ***Energy currents for $\Box_{g(\psi,x)}$ and the stability of coercivity properties.*** Let ψ denote a solution of (4-1) on a domain $\mathcal{R}(\tau_0, \tau_1)$. Because we shall use energy identities connected with $\Box_{g(\psi,x)}$, we shall require certain basic smallness assumptions on ψ which ensure that the causal nature of relevant hypersurfaces is retained and that induced volume forms of g and g_0 are comparable.

In the sections to follow, the main smallness parameter we will consider will be $\varepsilon_{\mathrm{prelim}} > 0$. We emphasise that, according to our conventions of Section 4.2, $\varepsilon_{\mathrm{prelim}} > 0$ will in general depend on the nonlinearity (4-2) (and also on k). The parameter $\varepsilon_{\mathrm{prelim}} > 0$ can be taken as fixed everywhere in the paper, but note that its smallness is constrained in multiple propositions whose preambles refer to its existence.

As we shall see, the propositions in this section will always refer to solutions satisfying

$$\sum_{|\boldsymbol{k}|\leq 1} (\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2 \leq \sqrt{\varepsilon} \tag{4-5}$$

in $r \leq R$ for $0 < \varepsilon \leq \varepsilon_{\mathrm{prelim}}$. (We recall here that, for $r(x) \geq R$, we have $g(\psi, x) = g_0$.) Eventually, (4-5) will be the consequence of a stronger estimate (see already (4-23)).

4.3.1. *Currents for $\Box_{g(x,\psi)}\psi = F$ and the stability of coercivity properties.* Let us for the moment consider more generally the equation

$$\Box_{g(\psi,x)}\psi = F \tag{4-6}$$

for arbitrary F, where ψ satisfies (4-5) for sufficiently small ε.

We may define now the currents

$$\overset{(p)}{\underset{k}{J}}[g(\psi, x), \psi], \qquad \overset{(p)}{\underset{k}{K}}[g(\psi, x), \psi] \tag{4-7}$$

again by the expressions (3-64), where the constituent (3-11), (3-12) are defined with the same quadruples (3-42) as before, but now with $g = g(\psi, x)$ replacing g_0. These currents satisfy (3-15), with respect now to the normals and volume forms of the metric g, and where $\overset{(p)}{\underset{k}{H}}[\psi]$ is defined by (3-13). (Notice that the definition of $\overset{(p)}{\underset{k}{H}}[\psi]$ does not depend on the metric.) In the region $r \geq R$ of course, all currents coincide with their g_0 versions since $g = g_0$ in this region.

We have the following proposition.

Proposition 4.3.1. *There exists an $\varepsilon_{\text{prelim}} > 0$ such that the following statement holds*:

Let ψ be a solution of (4-6) in $\mathcal{R}(\tau_0, \tau_1)$ satisfying (4-5) for $0 < \varepsilon \leq \varepsilon_{\text{prelim}}$. Then $x \mapsto g(\psi(x), x)$ defines a Lorentzian metric on $\mathcal{R}(\tau_0, \tau_1)$, and the identity (3-15) holds in $\mathcal{R}(\tau_0, \tau_1, v)$ for all v such that $\tau_1 \leq \tau(v)$, where the coefficients, normals and volume forms are $\varepsilon^{1/4}$ close to those of the currents (4-7) corresponding to $\Box_{g_0}\phi = F$.

In particular, there exist constants C, c such that, for $p = 0$ or $\delta \leq p \leq 2-\delta$, the corresponding coercivity properties (3-45) are retained for the currents (4-7), while (3-43) is replaced by

$$\overset{(p)}{K}[g(\psi,x),\psi] + \tilde{A}\xi(r)\psi^2 \geq cr^{p-1}\rho(r)((L\psi)^2 + |\nabla\!\!\!/\psi|^2) + cr^{-1-\delta}\rho(r)(\underline{L}\psi)^2 + cr^{-3}\tilde{\rho}(r)\psi^2 \\ - C\varepsilon^{\frac{1}{4}}\,\iota_{\{r\leq R\}\cap\{\rho\leq C\varepsilon^{1/4}<1/2\}}((L\psi)^2 + (\underline{L}\psi)^2 + |\nabla\!\!\!/\psi|^2 + \psi^2). \quad (4\text{-}8)$$

In particular, we still have (3-43) in $r \geq R$ and also in $\{r \leq R\} \cap \{\rho \geq C\varepsilon^{1/4}\}$.

Given $\varsigma = \varsigma(k)$ as fixed previously, we have that the above applies also for the currents $\overset{(p)}{J}_{\varsigma}[g(\psi,x),\psi]$, $\overset{(p)}{K}_{\varsigma}[g(\psi,x),\psi]$ in the region $r \geq r_{\text{Killing}}$, while in the region $r \leq r_1(\varsigma)$ the coercivity properties of Propositions 3.4.1 and 3.4.2 hold for these currents.

Proof. This is clear from the properties of tensor identities. Note how the smallness constraint on $\varepsilon_{\text{prelim}}$ provided by this proposition depends of course on the map g of (4-2) and is needed even simply to ensure that g is Lorentzian and that the inverse metric to be well defined. □

Note that, since $\rho = \tilde{\rho} = 1$ in case (i), the bound (4-8) implies that the full coercivity applies in that case. (Note that if ρ is a step function valued in $\{0, 1\}$, then $\{\rho \geq C\varepsilon^{1/4}\} = \left\{\rho \geq \frac{1}{2}\right\} = \{\rho = 1\}$ and $\{\rho \leq C\varepsilon^{1/4}\} = \{\rho = 0\}$.)

Corollary 4.3.2. *Under the assumption of Proposition 4.3.1, we have the coercivity statement*

$$\overset{(p)}{\underset{k}{K}}[g(\psi,x),\psi] \geq \tfrac{8}{9}\overset{(p)}{\underset{k}{K}}[g_0,\psi]$$

in $\{r \leq r_2\} \cup \left\{r \geq \frac{1}{4}R\right\}$ and an analogue of the coercivity statement of (3-85) holds in the form

$$\int_{\mathcal{R}(\tau_0,\tau_1)\cap\{r\leq r_2\}} -\overset{(p)}{\underset{k}{H}}[\psi]\cdot\{[\mathfrak{D}^{\boldsymbol{k}},\Box_{g_0}]\psi\} + \int_{\mathcal{R}(\tau_0,\tau_1)\cap\{r\geq r_2\}} |\overset{(p)}{\underset{k}{H}}[\psi]\cdot\{[\mathfrak{D}^{\boldsymbol{k}},\Box_{g_0}]\psi\}| \\ \leq \int_{\mathcal{R}(\tau_0,\tau_1)\cap\{r\leq r_2\}\cap\{r\geq R/4\}} \tfrac{2}{3}\overset{(p)}{\underset{k}{K}}[g(\psi,x),\psi] + C\sum_{|\boldsymbol{k}|\leq k-1}(\widetilde{\mathfrak{D}}^{\boldsymbol{k}}(F + (\Box_{g_0} - \Box_{g(\psi,x)})\psi))^2. \quad (4\text{-}9)$$

We have moreover the following version of (3-103):

$$\begin{aligned} &\int_{\mathcal{R}(\tau_0,\tau_1)} \overset{(p)}{\underset{k}{K}}[g(\psi,x),\psi] + \sum_{|\boldsymbol{k}|\leq k}\tilde{A}\xi(r)(\mathfrak{D}^{\boldsymbol{k}}\psi)^2 \gtrsim \int_{\tau_0}^{\tau_1} {}^{\rho}\overset{(p-1)}{\underset{k}{\mathcal{E}'}}(\tau')\,d\tau' + \int_{\tau_0}^{\tau_1} {}^{\tilde{\rho}}\overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau')\,d\tau', \quad 2-\delta \geq p \geq \delta, \\ &\int_{\mathcal{R}(\tau_0,\tau_1)} \overset{(p)}{\underset{k}{K}}[g(\psi,x),\psi] + \sum_{|\boldsymbol{k}|\leq k}\tilde{A}\xi(r)(\mathfrak{D}^{\boldsymbol{k}}\psi)^2 \gtrsim \int_{\tau_0}^{\tau_1} {}^{\rho}\overset{(-1-\delta)}{\underset{k}{\mathcal{E}'}}(\tau')\,d\tau' + \int_{\tau_0}^{\tau_1} {}^{\tilde{\rho}}\overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau')\,d\tau', \quad p = 0. \end{aligned} \quad (4\text{-}10)$$

Proof. Note that we have retained g_0 on the left-hand side of (4-9), so this is simply a statement about the stability of coercivity properties of the first term on the right-hand side. □

4.3.2. *Relations of energy fluxes.* In view of the above we will define a new set of energy quantities, $\overset{(0)}{\underset{k}{\mathfrak{F}}}(g)$, $\overset{(p)}{\underset{k}{\mathfrak{E}}}(g)$, etc., defined in parallel with those of $\overset{(0)}{\underset{k}{\mathfrak{F}}}$, $\overset{(p)}{\underset{k}{\mathfrak{E}}}$ of Section 3.6.7 (to be denoted below as $\overset{(0)}{\underset{k}{\mathfrak{F}}}(g_0)$, $\overset{(p)}{\underset{k}{\mathfrak{E}}}(g_0)$), but where the flux is defined with respect to the divergence identity with respect to $g = g(\psi, x)$, i.e., we define

$$\begin{aligned} \overset{(p)}{\underset{k}{\mathfrak{E}}}(g,\tau) &:= \int_{\Sigma(\tau)} \overset{(p)}{\underset{k}{J}}{}_\mu[g,\psi] n(g)^\mu_{\Sigma(\tau)}, \quad \overset{(p)}{\underset{k}{\mathfrak{E}}}{}_{\mathcal{S}}(g,\tau) := \int_{\mathcal{S}} \overset{(p)}{\underset{k}{J}}{}_\mu[g,\psi] n(g)^\mu_{\mathcal{S}}, \\ \overset{(p)}{\underset{k}{\mathfrak{F}}}(g,v,\tau_0,\tau) &:= \int_{\underline{C}_v \cap \mathcal{R}(\tau_0,\tau)} \overset{(p)}{\underset{k}{J}}{}_\mu[g,\psi] n(g)^\mu_{\underline{C}_v}, \quad \tau_0 \le \tau \le \tau(v), \end{aligned} \tag{4-11}$$

where the omitted volume forms above are here understood with respect to $g = g(\psi, x)$.

Corollary 4.3.3. *We have that, under the assumptions of Proposition 4.3.1, for all $\tau_0 \le \tau \le \tau_1$ and v such that $\tau_1 \le \tau(v)$,*

$$\overset{(p)}{\underset{k}{\mathfrak{E}}}(g,\tau) \sim \overset{(p)}{\underset{k}{\mathfrak{E}}}(g_0,\tau) \lesssim \overset{(p)}{\underset{k}{\mathcal{E}}}(\tau), \quad \overset{(0)}{\mathfrak{E}}_{\mathcal{S}}(g,\tau) = \overset{(p)}{\mathfrak{E}}_{\mathcal{S}}(g,\tau) \sim \overset{(p)}{\underset{k}{\mathfrak{E}}}{}_{\mathcal{S}}(g_0,\tau) = \overset{(0)}{\underset{k}{\mathfrak{E}}}{}_{\mathcal{S}}(g_0,\tau) \sim \overset{(0)}{\mathcal{E}}_{\mathcal{S}}(\tau),$$
$$\overset{(p)}{\underset{k}{\mathfrak{F}}}(g,v,\tau_0,\tau) = \overset{(p)}{\underset{k}{\mathfrak{F}}}(g_0,v,\tau_0,\tau) \sim \overset{(p)}{\underset{k}{\mathcal{F}}}(v,\tau_0,\tau),$$

and in fact

$$\begin{aligned} &\overset{(0)}{\underset{k}{\mathfrak{E}}}(g) \le (1 + C\varepsilon^{\frac14}) \overset{(0)}{\underset{k}{\mathfrak{E}}}(g_0), \quad \overset{(0)}{\underset{k}{\mathfrak{E}}}(g_0) \le (1 + C\varepsilon^{\frac14}) \overset{(0)}{\underset{k}{\mathfrak{E}}}(g), \\ &\overset{(p)}{\underset{k}{\mathcal{E}}}(\tau) \lesssim \overset{(p)}{\underset{k}{\mathfrak{E}}}[g](\tau) + \int_{\Sigma(\tau)\cap\{r_1 \le r \le R\}} \sum_{|\boldsymbol{k}| \le k-1} (\widetilde{\mathfrak{D}}^{\boldsymbol{k}}(F + (\Box_{g_0} - \Box_{g(\psi,x)})\psi))^2. \end{aligned} \tag{4-12}$$

As always, the significance of expressing the above with respect to the energies $\overset{(p)}{\underset{k}{\mathfrak{E}}}(g)$, etc., is that the constants in our nonlinear inequalities will be exactly 1.

Note that, in all integrals below with volume form omitted, we will continue to use the volume form induced by g_0 unless otherwise noted. (We shall only use the volume form induced by $g(\psi, x)$ when applying the divergence identity involving (4-11). The corresponding volume forms are of course equivalent under the assumptions of Proposition 4.3.1, but again one must distinguish where appropriate so as to obtain the exact constant.)

4.4. ***Higher-order estimates for the quasilinear equation.*** Using the above, we can finally obtain the following:

Proposition 4.4.1. *For all $k \ge 1$, there exists an $\varepsilon_{\text{prelim}} > 0$ such that the following statement holds:*

Let $0 < \delta \le p \le 2 - \delta$, $\tau_0 \le \tau_1$, and let ψ be a solution of (4-1) in $\mathcal{R}(\tau_0, \tau_1)$ satisfying (4-5) for $0 < \varepsilon \le \varepsilon_{\text{prelim}}$. Then, for all $\tau \in [\tau_0, \tau_1]$, we have

$$\begin{aligned} &\sup_{v:\tau \le \tau(v)} \overset{(p)}{\underset{k}{\mathfrak{F}}}(v,\tau_0,\tau), \quad \overset{(p)}{\underset{k}{\mathfrak{E}}}(\tau) + \overset{(p)}{\underset{k}{\mathfrak{E}}}{}_{\mathcal{S}}(\tau_0,\tau) + c\int_{\tau_0}^{\tau} {}^{\rho}\overset{(p-1)}{\underset{k}{\mathcal{E}'}}(\tau')\,d\tau' + c\int_{\tau_0}^{\tau} {}^{\tilde\rho}\overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau')\,d\tau' \\ &\quad \le \overset{(p)}{\underset{k}{\mathfrak{E}}}(\tau_0) + A\int_{\tau_0}^{\tau} \overset{\xi}{\underset{k-1}{\mathcal{E}'}}(\tau')\,d\tau' \\ &\qquad + \int_{\mathcal{R}(\tau_0,\tau)} \left|\overset{(p)}{\underset{k}{H}}[\psi] \cdot \{\mathfrak{D}^{\boldsymbol{k}}(N^{\alpha\beta}(\psi,x)\partial_\alpha\psi\partial_\beta\psi)\}\right| \end{aligned} \tag{4-13}$$

$$+\int_{\mathcal{R}(\tau_0,\tau)\cap\{r\le R\}} \Big|\overset{(p)}{\underset{k}{H}}[\psi]\cdot\{[\Box_{g(\psi,x)}-\Box_{g_0},\mathfrak{D}^{\boldsymbol{k}}]\psi\}\Big| \tag{4-14}$$

$$+C\int_{\mathcal{R}(\tau_0,\tau)}\sum_{|\boldsymbol{k}|\le k}(\mathfrak{D}^{\boldsymbol{k}}(N^{\alpha\beta}(\psi,x)\partial_\alpha\psi\partial_\beta\psi))^2+C\int_{\mathcal{R}(\tau_0,\tau)\cap\{r\le R\}}\sum_{|\boldsymbol{k}|\le k}([\Box_{g(\psi,x)}-\Box_{g_0},\mathfrak{D}^{\boldsymbol{k}}]\psi)^2$$

$$+C\int_{\mathcal{R}(\tau_0,\tau)\cap\{r\le R\}}\sum_{|\boldsymbol{k}|\le k-1}(\widetilde{\mathfrak{D}}^{\boldsymbol{k}}(N^{\alpha\beta}(\psi,x)\partial_\alpha\psi\,\partial_\beta\psi+(\Box_{g_0}-\Box_{g(\psi,x)})\psi))^2 \tag{4-15}$$

$$+C\boxed{\varepsilon^{\frac14}}\int_{\mathcal{R}(\tau_0,\tau)\cap\{r\le R\}\cap\{\rho\le C\sqrt{\varepsilon}\}}\sum_{|\boldsymbol{k}|\le k}((L(\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi))^2+(\underline{L}(\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi))^2+|\nabla\!\!\!/(\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)|^2)+\psi^2, \tag{4-16}$$

as well as the estimate

$$\sup_{v:\tau\le\tau(v)}\overset{(p)}{\underset{k-1}{\mathcal{F}}}(v,\tau_0,\tau)+\overset{(p)}{\underset{k-1}{\mathcal{E}}}(\tau)+\overset{(0)}{\underset{k-1}{\mathcal{E}}}_{\mathcal{S}}(\tau_0,\tau)+\int_{\tau_0}^{\tau}\chi\overset{(p-1)}{\underset{k-1}{\mathcal{E}'}}(\tau')\,d\tau'+\int_{\tau_0}^{\tau}\overset{(p-1)}{\underset{k-2}{\mathcal{E}'}}(\tau')\,d\tau' \tag{4-17}$$

$$\lesssim\overset{(p)}{\underset{k-1}{\mathcal{E}}}(\tau_0)$$

$$+\int_{\mathcal{R}(\tau_0,\tau)}\sum_{|\boldsymbol{k}|\le k-1}(|V_p^\mu\partial_\mu(\mathfrak{D}^{\boldsymbol{k}}\psi)|+|w_p\mathfrak{D}^{\boldsymbol{k}}\psi|)|\mathfrak{D}^{\boldsymbol{k}}(N^{\alpha\beta}(\psi,x)\partial_\alpha\psi\,\partial_\beta\psi+(\Box_{g_0}-\Box_{g(\psi,x)})\psi)| \tag{4-18}$$

$$+\int_{\mathcal{R}(\tau_0,\tau)}\sum_{|\boldsymbol{k}|\le k-1}(\mathfrak{D}^{\boldsymbol{k}}(N^{\alpha\beta}(\psi,x)\partial_\alpha\psi\,\partial_\beta\psi+(\Box_{g_0}-\Box_{g(\psi,x)})\psi))^2 \tag{4-19}$$

$$+\int_{\mathcal{R}(\tau_0,\tau)\cap\{r\le R\}}\sum_{|\boldsymbol{k}|\le k-2}(\widetilde{\mathfrak{D}}^{\boldsymbol{k}}(N^{\alpha\beta}(\psi,x)\partial_\alpha\psi\,\partial_\beta\psi+(\Box_{g_0}-\Box_{g(\psi,x)})\psi))^2 \tag{4-20}$$

$$+\int_{\Sigma(\tau)\cap\{r\le R\}}\sum_{|\boldsymbol{k}|\le k-2}(\widetilde{\mathfrak{D}}^{\boldsymbol{k}}(N^{\alpha\beta}(\psi,x)\partial_\alpha\psi\,\partial_\beta\psi+(\Box_{g_0}-\Box_{g(\psi,x)})\psi))^2. \tag{4-21}$$

Moreover, for $p=0$, *identical statements hold with* $-1-\delta$ *replacing* $p-1$.

Remark 4.4.2. In the case where we replace the middle term of (3-3) with (3-10), we should add

$$\sum_{|\boldsymbol{k}|\le k-1}\sqrt{\int_{\mathcal{R}(\tau_0,\tau)\cap\{r\le R\}}|L\mathfrak{D}^{\boldsymbol{k}}\psi|^2+|\underline{L}\mathfrak{D}^{\boldsymbol{k}}\psi|^2+|\nabla\!\!\!/\mathfrak{D}^{\boldsymbol{k}}\psi|^2+r^{-2}|\mathfrak{D}^{\boldsymbol{k}}\psi|^2}$$
$$\times\sqrt{\int_{\mathcal{R}(\tau_0,\tau_1)\cap\{r\le R\}}\mathfrak{D}^{\boldsymbol{k}}(N^{\alpha\beta}(\psi,x)\partial_\alpha\psi\,\partial_\beta\psi+(\Box_{g_0}-\Box_{g(\psi,x)})\psi)^2}$$

to the right-hand side of (4-18); cf. Remark 3.6.9.

Proof. Note the term on line (4-14) arose by expanding the term appearing in the actual identity as follows. Write

$$\int_{\mathcal{R}(\tau_0,\tau)\cap\{r\le R\}}\overset{(p)}{\underset{k}{H}}[\psi]\cdot\{[\Box_{g(\psi,x)},\mathfrak{D}^{\boldsymbol{k}}]\psi\}$$
$$=\int_{\mathcal{R}(\tau_0,\tau)\cap\{r\le R\}}\overset{(p)}{\underset{k}{H}}[\psi]\cdot\{[\Box_{g(\psi,x)}-\Box_{g_0},\mathfrak{D}^{\boldsymbol{k}}]\psi\}+\int_{\mathcal{R}(\tau_0,\tau)\cap\{r\le R\}}\overset{(p)}{\underset{k}{H}}[\psi]\cdot\{[\Box_{g_0},\mathfrak{D}^{\boldsymbol{k}}]\psi\},$$

then bring the second term to the left-hand side to use Corollary 4.3.2 and absorb it in the bulk. (Recall that, in the region $r \geq R$, we have $g = g_0$.) The term on line (4-15) arose from our application of elliptic estimates to ψ, considering it as a solution of (4-3). The term on line (4-16) arose from the term on the last line of (4-8). We note that, restricted to $r \geq R$, we may replace $\widetilde{\mathfrak{D}}$ commutation with $\mathfrak{D}$ commutation.

The inequality (4-17)–(4-21), on the other hand, simply arises from applying the estimate (3-110) to the nonlinear equation written in the form (4-3). Note that we have applied it at one order less, i.e., with $k-1$ in place of k, in view of the fact that the right-hand side is of order k. □

4.5. ***Summed norm notation and the lower-order smallness assumption.*** In addition to the primitive assumption (4-5), in the context of our proof, we will need to introduce stronger a priori energy smallness assumptions on ψ in our region $\mathcal{R}(\tau_0, \tau_1)$ under consideration. In the context of the proof of the main theorem, this will appear as a bootstrap assumption. This will ensure that higher-order nonlinear terms are indeed absorbable.

We first explain some additional notation. We define the following summed quantities for $\delta \leq p \leq 2-\delta$:

$$\overset{(p)}{\underset{k}{\mathcal{X}}}(\tau_0, \tau_1) := \sup_{\tau' \in [\tau_0,\tau_1]} \overset{(p)}{\underset{k}{\mathcal{E}}}(\tau') + \sup_{v:\tau_1 \leq v(\tau)} \overset{(p)}{\underset{k}{\mathcal{F}}}(v, \tau_0, \tau_1) + \int_{\tau_0}^{\tau_1} \overset{(p-1)}{\underset{k}{\mathcal{E}'}}(\tau')\, d\tau',$$

$${}^{\rho}\overset{(p)}{\underset{k}{\mathcal{X}}}(\tau_0, \tau_1) := \sup_{\tau' \in [\tau_0,\tau_1]} \overset{(p)}{\underset{k}{\mathcal{E}}}(\tau') + \sup_{v:\tau_1 \leq \tau(v)} \overset{(p)}{\underset{k}{\mathcal{F}}}(v, \tau_0, \tau_1) + \int_{\tau_0}^{\tau_1} \Big({}^{\rho}\overset{(p-1)}{\underset{k}{\mathcal{E}'}}(\tau') + {}^{\tilde{\rho}}\overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau') \Big)\, d\tau',$$

$${}^{\chi}\overset{(p)}{\underset{k}{\mathcal{X}}}(\tau_0, \tau_1) := \sup_{\tau' \in [\tau_0,\tau_1]} \overset{(p)}{\underset{k}{\mathcal{E}}}(\tau') + \sup_{v:\tau_1 \leq \tau(v)} \overset{(p)}{\underset{k}{\mathcal{F}}}(v, \tau_0, \tau_1) + \int_{\tau_0}^{\tau_1} \Big({}^{\chi}\overset{(p-1)}{\underset{k}{\mathcal{E}'}}(\tau') + \overset{(p-1)}{\underset{k-1}{\mathcal{E}'}}(\tau') \Big)\, d\tau'.$$

For $p = 0$, we first define the analogous quantities, where $p-1$ is replaced by $-1-\delta$:

$$\overset{(0)}{\underset{k}{\mathcal{X}}}(\tau_0, \tau_1) := \sup_{\tau' \in [\tau_0,\tau_1]} \overset{(0)}{\mathcal{E}}(\tau') + \sup_{v:\tau_1 \leq \tau(v)} \overset{(0)}{\underset{k}{\mathcal{F}}}(v, \tau_0, \tau_1) + \int_{\tau_0}^{\tau_1} \overset{(-1-\delta)}{\underset{k}{\mathcal{E}'}}(\tau')\, d\tau',$$

$${}^{\rho}\overset{(0)}{\underset{k}{\mathcal{X}}}(\tau_0, \tau_1) := \sup_{\tau' \in [\tau_0,\tau_1]} \overset{(0)}{\mathcal{E}}(\tau') + \sup_{v:\tau_1 \leq \tau(v)} \overset{(0)}{\underset{k}{\mathcal{F}}}(v, \tau_0, \tau_1) + \int_{\tau_0}^{\tau_1} \Big({}^{\rho}\overset{(-1-\delta)}{\underset{k}{\mathcal{E}'}}(\tau') + {}^{\tilde{\rho}}\overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau') \Big)\, d\tau',$$

$${}^{\chi}\overset{(0)}{\underset{k}{\mathcal{X}}}(\tau_0, \tau_1) := \sup_{\tau' \in [\tau_0,\tau_1]} \overset{(0)}{\mathcal{E}}(\tau') + \sup_{v:\tau_1 \leq \tau(v)} \overset{(0)}{\underset{k}{\mathcal{F}}}(v, \tau_0, \tau_1) + \int_{\tau_0}^{\tau_1} \Big({}^{\chi}\overset{(-1-\delta)}{\underset{k}{\mathcal{E}'}}(\tau') + \overset{(-1-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau') \Big)\, d\tau'.$$

Because $p = 0$ is anomalous, however, we will need in addition the following stronger energies which will appear *on the right-hand side* of $p = 0$ estimates:

$$\overset{(0+)}{\underset{k}{\mathcal{X}}}(\tau_0, \tau_1) := \sup_{\tau' \in [\tau_0,\tau_1]} \overset{(0)}{\mathcal{E}}(\tau') + \sup_{v:\tau_1 \leq \tau(v)} \overset{(0)}{\underset{k}{\mathcal{F}}}(v, \tau_0, \tau_1) + \int_{\tau_0}^{\tau_1} \overset{(\delta-1)}{\underset{k}{\mathcal{E}'}}(\tau')\, d\tau',$$

$${}^{\rho}\overset{(0+)}{\underset{k}{\mathcal{X}}}(\tau_0, \tau_1) := \sup_{\tau' \in [\tau_0,\tau_1]} \overset{(0)}{\mathcal{E}}(\tau') + \sup_{v:\tau_1 \leq \tau(v)} \overset{(0)}{\underset{k}{\mathcal{F}}}(v, \tau_0, \tau_1) + \int_{\tau_0}^{\tau_1} \Big({}^{\rho}\overset{(\delta-1)}{\underset{k}{\mathcal{E}'}}(\tau') + {}^{\tilde{\rho}}\overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau') \Big)\, d\tau',$$

$${}^{\chi}\overset{(0+)}{\underset{k}{\mathcal{X}}}(\tau_0, \tau_1) := \sup_{\tau' \in [\tau_0,\tau_1]} \overset{(0)}{\mathcal{E}}(\tau') + \sup_{v:\tau_1 \leq \tau(v)} \overset{(0)}{\underset{k}{\mathcal{F}}}(v, \tau_0, \tau_1) + \int_{\tau_0}^{\tau_1} \Big({}^{\chi}\overset{(\delta-1)}{\underset{k}{\mathcal{E}'}}(\tau') + \overset{(\delta-1)}{\underset{k-1}{\mathcal{E}'}}(\tau') \Big)\, d\tau'.$$

Note the general properties that $p' \geq p,\ k' \geq k$ implies $\overset{(p')}{\underset{k'}{\mathcal{X}}} \gtrsim \overset{(p)}{\underset{k}{\mathcal{X}}}$, ${}^{\chi}\overset{(p')}{\underset{k'}{\mathcal{X}}} \gtrsim {}^{\chi}\overset{(p)}{\underset{k}{\mathcal{X}}}$, ${}^{\rho}\overset{(p')}{\underset{k'}{\mathcal{X}}} \gtrsim {}^{\rho}\overset{(p)}{\underset{k}{\mathcal{X}}}$, while

$$\overset{(p)}{\underset{k-1}{\mathcal{X}}} \lesssim {}^{\chi}\overset{(p)}{\underset{k}{\mathcal{X}}}. \tag{4-22}$$

We will use the notation $\ll k$ to denote some particular positive integer, depending on k, which may vary across different instances of our use of the notation, such that $\ll k \leq k$ and, in fact, $\ll k$ is "much less than k", provided k is sufficiently large. In particular, for all positive integers n, we assume there exists a $k(n)$ for which $k \geq k(n)$ implies $\ll k \leq k - n$.

We have the following:

Proposition 4.5.1. *Let $k \geq 4$ be sufficiently large. There exists an $\varepsilon_{\text{prelim}} > 0$ such that, for all $0 < \varepsilon \leq \varepsilon_{\text{prelim}}$, the following holds*:

Let ψ satisfy

$$\overset{(p)}{\underset{\ll k}{\mathcal{X}}} \leq \varepsilon \tag{4-23}$$

in $\mathcal{R}(\tau_0, \tau_1)$ for $p = 0$ (or some $\delta \leq p \leq 2 - \delta$), where $\ll k \geq 4$. Then the following improved version of (4-5) holds:

$$\sup_{r \leq R} \sum_{|\boldsymbol{k}| \leq 1} (\widetilde{\mathfrak{D}}^{\boldsymbol{k}} \psi)^2 \lesssim \varepsilon \leq \sqrt{\varepsilon}. \tag{4-24}$$

Proof. This follows of course immediately from the Sobolev inequality (3-112). □

In the context of the proof of the main theorem, inequality (4-23) will be introduced as a bootstrap assumption, and Proposition 4.5.1 will be applied to retrieve the assumption (4-5), which is necessary for the results of Sections 4.3–4.4. Note that, with the assumption (4-23), we may replace the boxed $\varepsilon^{1/4}$ in (4-16) with $\sqrt{\varepsilon}$.

We note that, in what follows, restrictions on k sufficiently large will always include the condition $\ll k \geq 4$.

4.5.1. *Comparability of $\mathfrak{E}$ and $\mathcal{E}$ energies.* We note first the following:

Proposition 4.5.2. *Let $k \geq 4$ be sufficiently large. There exists an $\varepsilon_{\text{prelim}} > 0$ such that the following holds*:

Let ψ be a solution of (4-1) in $\mathcal{R}(\tau_0, \tau_1)$ satisfying (4-23) for $p = 0$ and some $0 < \varepsilon \leq \varepsilon_{\text{prelim}}$. Then, setting $F = N^{\mu\nu}(\psi, x)\partial_\mu \psi \, \partial_\nu \psi + (\Box_{g_0} - \Box_{g(\psi,x)})\psi$, we have

$$\int_{\Sigma(\tau') \cap \{r \leq R\}} \sum_{|\boldsymbol{k}| \leq k-1} (\widetilde{\mathfrak{D}}^{\boldsymbol{k}} F)^2 \lesssim \min\{\overset{(0)}{\underset{\ll k}{\mathcal{E}}}(\tau'), \overset{(-1-\delta)}{\underset{\ll k}{\mathcal{E}'}}(\tau')\} \overset{(0)}{\mathcal{E}}(\tau') \lesssim \varepsilon \overset{(0)}{\mathcal{E}}(\tau'). \tag{4-25}$$

Proof. This is standard in view of our assumptions on N and $g(\psi, x)$ from Section 4.1 and can be proven using the Sobolev inequality (3-112) on $\Sigma(\tau') \cap \{r \leq R\}$. □

We may now obtain the following result, which can be viewed as a corollary of Proposition 3.6.6.

Corollary 4.5.3. *Let $k \geq 4$ be sufficiently large. Then there exists an $\varepsilon_{\text{prelim}} > 0$ such that the following statement holds*:

Let ψ be a solution of (4-1) in $\mathcal{R}(\tau_0, \tau_1)$ satisfying (4-23) for $p = 0$ and some $0 < \varepsilon \leq \varepsilon_{\text{prelim}}$. We have the analogue of Corollary 3.6.7: The calligraphic and fraktur (when applicable) energies are comparable, i.e.,

$$\overset{(p)}{\underset{k}{\mathcal{E}}}(\tau) \sim \overset{(p)}{\underset{k}{\mathfrak{E}}}(\tau)(g),$$

for all $\tau_0 \le \tau \le \tau_1$. *Moreover, if* $\chi = 1$ *and* $\tilde{\rho} = 1$ *identically* (*as in case* (i)), *then*

$$\overset{(p-1)}{\underset{k}{\mathcal{E}'}}(\tau) \sim {}^{\chi}\overset{(p-1)}{\underset{k}{\mathcal{E}'}}(\tau) + {}^{\tilde{\rho}}\overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau), \qquad \overset{(-1-\delta)}{\underset{k}{\mathcal{E}'}}(\tau) \sim {}^{\chi}\overset{(-1-\delta)}{\underset{k}{\mathcal{E}'}}(\tau) + {}^{\tilde{\rho}}\overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau)$$

and thus

$${}^{\chi}\overset{(p)}{\underset{k}{\mathcal{X}}}(\tau_0, \tau_1) \sim {}^{\rho}\overset{(p)}{\underset{k}{\mathcal{X}}}(\tau_0, \tau_1) \sim \overset{(p)}{\underset{k}{\mathcal{X}}}(\tau_0, \tau_1).$$

If $\tilde{\rho} = 1$ *identically and* $\rho = \chi$, (*i.e., as in cases* (i) *and* (ii)), *then*

$${}^{\chi}\overset{(p-1)}{\underset{k}{\mathcal{E}'}}(\tau) + \overset{(p-1)}{\underset{k-1}{\mathcal{E}'}}(\tau) \sim {}^{\rho}\overset{(p-1)}{\underset{k}{\mathcal{E}'}}(\tau) + {}^{\tilde{\rho}}\overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau), \qquad {}^{\chi}\overset{(-1-\delta)}{\underset{k}{\mathcal{E}'}}(\tau) + \overset{(-1-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau) \sim {}^{\rho}\overset{(-1-\delta)}{\underset{k}{\mathcal{E}'}}(\tau) + {}^{\tilde{\rho}}\overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau)$$

and thus

$${}^{\chi}\overset{(p)}{\underset{k}{\mathcal{X}}}(\tau_0, \tau_1) \sim {}^{\rho}\overset{(p)}{\underset{k}{\mathcal{X}}}(\tau_0, \tau_1).$$

Proof. This follows from (3-104)–(3-108), with F defined as in Proposition 4.5.2, using the estimate (4-25) to absorb the inhomogeneous term, for sufficiently small $\varepsilon_{\text{prelim}}$. □

4.6. *Estimates on the nonlinear terms in the near region.*

Proposition 4.6.1. *Let* $k \ge 4$ *be sufficiently large. There exists an* $\varepsilon_{\text{prelim}} > 0$ *such that the following holds*:

Let ψ *be a solution of* (4-1) *in* $\mathcal{R}(\tau_0, \tau_1)$ *satisfying* (4-23) *for* $p = 0$ *and some* $0 < \varepsilon \le \varepsilon_{\text{prelim}}$. *Then all integrals on lines* (4-13)–(4-16), ***restricted to*** $\boldsymbol{\mathcal{R}(\tau_0, \tau_1) \cap \{r \le R\}}$, *may be estimated by*

$$\cdots \lesssim \int_{\tau_0}^{\tau} \overset{(0)}{\mathcal{E}}(\tau') \sqrt{\overset{(-1-\delta)}{\underset{\ll k}{\mathcal{E}'}}(\tau')}\, d\tau', \tag{4-26}$$

while the integrals on lines (4-18)–(4-19), *again* ***restricted to*** $\boldsymbol{\mathcal{R}(\tau_0, \tau_1) \cap \{r \le R\}}$, *with or without the extra term of Remark 4.4.2, may be similarly estimated by*

$$\cdots \lesssim \int_{\tau_0}^{\tau} \overset{(0)}{\mathcal{E}}(\tau') \sqrt{\overset{(-1-\delta)}{\underset{\ll k}{\mathcal{E}'}}(\tau')}\, d\tau'. \tag{4-27}$$

Proof. This is standard given the assumptions in Section 4.1 and can be proven using the Sobolev inequality (3-112) on $\Sigma(\tau) \cap \{r \le R\}$; cf. the proof of (4-25) which in fact already bounds several of the terms. (Note that the boxed $\varepsilon^{1/4}$ in (4-16) may clearly now be replaced by the quantity $\sqrt{\overset{(-1-\delta)}{\underset{\ll k}{\mathcal{E}'}}(\tau')}$ within the integral.) □

We remark again that, as opposed to (4-26), the control (4-27) loses a derivative, as the energy on the left-hand side of (4-18) is of $(k-1)$-th order while the right-hand side of (4-27) is of k-th order.

4.7. *Assumptions for the nonlinearity in the far region: the "null" condition.*

We may now state our specific assumption capturing the null condition for the semilinear terms.

Rather than formulate algebraically the null condition in terms of the form of N, for maximum generality we will make our basic assumption directly at the level of an estimate for terms on the right-hand side of the inequalities of Proposition 4.4.1.

Our assumptions will only depend on the region $r \ge \frac{8}{9}R$, so we will need some notation to denote the restriction of energies, etc., to this region. Given $r_* \ge r_0$ and v, by

$$\overset{(p)}{\underset{k}{\mathcal{E}}}{}_{r_*,v}, \qquad \overset{(p)}{\underset{k}{\mathcal{X}}}{}_{r_*,v}, \qquad {}^{\rho}\overset{(p)}{\underset{k}{\mathcal{X}}}{}_{r_*,v}, \qquad \ldots$$

we shall mean the expressions defined as before, but where all domains of integration are restricted to the region $\mathcal{R}(\tau_0, \tau_1, v) \cap \{r \geq r_*\}$. We may thus define these expressions for functions

$$\psi : \mathcal{R}(\tau_0, \tau_1, v) \cap \{r \geq r_*\} \to \mathbb{R}.$$

Similarly, the above expressions without the v subscript, e.g., $\overset{(p)}{\underset{k}{\mathcal{E}}}_{r_*}$, will be defined where all domains of integration are restricted to $\mathcal{R}(\tau_0, \tau_1) \cap \{r \geq r_*\}$. These can then be defined for functions

$$\psi : \mathcal{R}(\tau_0, \tau_1) \cap \{r \geq r_*\} \to \mathbb{R}.$$

Let us note that

$$r_* \geq \tfrac{8}{9}R \quad \Longrightarrow \quad \overset{(p)}{\underset{k}{\mathcal{X}}}_{r_*,v} = {}^{\chi}\overset{(p)}{\underset{k}{\mathcal{X}}}_{r_*,v} = {}^{\rho}\overset{(p)}{\underset{k}{\mathcal{X}}}_{r_*,v}, \tag{4-28}$$

and similarly for the quantities without the v subscript.

Assumption 4.7.1 (null condition for semilinear terms). *There exists a* k_{null} *and, for all* $k \geq k_{\mathrm{null}}$, *an* $\varepsilon_{\mathrm{null}} > 0$ *such that the following holds for all* $\tau_0 \leq \tau_1$, v.

Let ψ *be a smooth function defined on* $\mathcal{R}(\tau_0, \tau_1, v) \cap \left\{r \geq \frac{8}{9}R\right\}$ *satisfying the bound*

$$\overset{(0)}{\underset{\ll k}{\mathcal{X}}}_{8R/9,v} \leq \varepsilon \tag{4-29}$$

for some $0 < \varepsilon \leq \varepsilon_{\mathrm{null}}$. *Then, for all* $\delta \leq p \leq 2-\delta$,

$$\begin{aligned}\int_{\mathcal{R}(\tau_0,\tau_1,v)\cap\{r\geq R\}} \sum_{|\boldsymbol{k}|\leq k} (|r^p r^{-1} L(r\mathfrak{D}^{\boldsymbol{k}}\psi)| + |L\mathfrak{D}^{\boldsymbol{k}}\psi| + |\underline{L}\mathfrak{D}^{\boldsymbol{k}}\psi| + |r^{-1}\nabla\!\!\!\!/\,\mathfrak{D}^{\boldsymbol{k}}\psi| + |r^{-1}\mathfrak{D}^{\boldsymbol{k}}\psi|) \\ \times |\mathfrak{D}^{\boldsymbol{k}}(N^{\alpha\beta}(\psi, x)\partial_\alpha\psi\, \partial_\beta\psi)| + (\mathfrak{D}^{\boldsymbol{k}}(N^{\alpha\beta}(\psi, x)\partial_\alpha\psi\, \partial_\beta\psi))^2 \\ \lesssim \overset{(p)}{\underset{k}{\mathcal{X}}}_{8R/9,v}(\tau_0, \tau_1)\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}_{8R/9,v}(\tau_0, \tau_1)} + \sqrt{\overset{(p)}{\underset{k}{\mathcal{X}}}_{8R/9,v}(\tau_0, \tau_1)}\sqrt{\overset{(0)}{\underset{k}{\mathcal{X}}}_{8R/9,v}(\tau_0, \tau_1)}\sqrt{\overset{(p)}{\underset{\ll k}{\mathcal{X}}}_{8R/9,v}(\tau_0, \tau_1)},\end{aligned} \tag{4-30}$$

while, corresponding to $p = 0$, *we have*

$$\begin{aligned}\int_{\mathcal{R}(\tau_0,\tau_1,v)\cap\{r\geq R\}} \sum_{|\boldsymbol{k}|\leq k} (|r^{-1} L(r\mathfrak{D}^{\boldsymbol{k}}\psi)| + |L\mathfrak{D}^{\boldsymbol{k}}\psi| + |\underline{L}\mathfrak{D}^{\boldsymbol{k}}\psi| + |r^{-1}\nabla\!\!\!\!/\,\mathfrak{D}^{\boldsymbol{k}}\psi| + |r^{-1}\mathfrak{D}^{\boldsymbol{k}}\psi|) \\ \times |\mathfrak{D}^{\boldsymbol{k}}(N^{\alpha\beta}(\psi, x)\partial_\alpha\psi\, \partial_\beta\psi)| + (\mathfrak{D}^{\boldsymbol{k}}(N^{\alpha\beta}(\psi, x)\partial_\alpha\psi\, \partial_\beta\psi))^2 \\ \lesssim \overset{(0+)}{\underset{k}{\mathcal{X}}}_{8R/9,v}(\tau_0, \tau_1)\sqrt{\overset{(0+)}{\underset{\ll k}{\mathcal{X}}}_{8R/9,v}(\tau_0, \tau_1)}.\end{aligned} \tag{4-31}$$

We note the following.

Proposition 4.7.2. *Assumption 4.7.1 holds for equations (4-1) when* g_0 *is Minkowski and the semilinear terms* N *satisfy the classical null condition of Klainerman [1986], and more generally when* g_0 *is the Kerr metric and the semilinear terms* N *belong to the class considered by Luk [2013].*

Proof. See Appendix C. □

Remark 4.7.3. Let us note that, for ψ defined on the slab $\mathcal{R}(\tau_1, \tau_2) \cap \left\{r \geq \frac{8}{9}R\right\}$, from the trivial bound $\overset{(p)}{\underset{k}{\mathcal{X}}}_{r_*,v} \leq \overset{(p)}{\underset{k}{\mathcal{X}}}_{r_*}$, we may drop the v subscripts. For ψ defined globally on the slab $\mathcal{R}(\tau_1, \tau_2)$, in view of (4-28) and the trivial bounds ${}^{\rho}\overset{(p)}{\underset{k}{\mathcal{X}}}_{r_*} \leq {}^{\rho}\overset{(p)}{\underset{k}{\mathcal{X}}}$, etc., we may replace the $\frac{8}{9}M$, v subscripts in all the $\overset{(p)}{\underset{k}{\mathcal{X}}}$, etc., energies on the right-hand side with the ρ superscript, i.e., replace $\overset{(p)}{\underset{k}{\mathcal{X}}}_{8M/9,v}$, etc., with ${}^{\rho}\overset{(p)}{\underset{k}{\mathcal{X}}}$, etc. Also, for such ψ, we may clearly replace (4-29) with assumption (4-23).

4.8. ***The final estimates.*** Putting everything together we have the following:

Proposition 4.8.1. *Consider* $(\mathcal{M}, g_0)$ *satisfying the assumptions of Sections* 2 *and* 3 (*for cases* (i), (ii), *or* (iii)) *and equation* (4-1) *satisfying the assumptions of Sections* 4.1 *and* 4.7.

Then, for all $k \geq 4$ *sufficiently large, there exist constants* $C > 0$ (*also implicit in the* $\lesssim$ *below*), $c > 0$, *and an* $\varepsilon_{\mathrm{prelim}} > 0$, *such that the following is true*:

Let $\tau_0 \leq \tau_1$, *and let* ψ *be a solution of* (4-1) *in* $\mathcal{R}(\tau_0, \tau_1)$ *satisfying* (4-23) *with* $p = 0$ *for some* $0 < \varepsilon \leq \varepsilon_{\mathrm{prelim}}$. *Then, for* $\delta \leq p \leq 2 - \delta$, *one has the estimates*

$$\begin{aligned}\sup_{v:\tau\leq\tau(v)} \overset{(p)}{\mathfrak{F}}(v, \tau_0, \tau), \quad c^{\rho}\underset{k}{\overset{(p)}{\mathcal{X}}}(\tau_0, \tau_1), \quad \underset{k}{\overset{(p)}{\mathfrak{E}}}(\tau_1) &\\ \leq \underset{k}{\overset{(p)}{\mathfrak{E}}}(\tau_0) + A\int_{\tau_0}^{\tau_1} \underset{k-1}{\overset{\xi}{\mathcal{E}}}{}'(\tau')\,d\tau' + C\underset{k}{\overset{(p)}{\mathcal{X}}}_{8R/9}(\tau_0, \tau_1)\sqrt{\underset{\ll k}{\overset{(0)}{\mathcal{X}}}_{8R/9}(\tau_0, \tau_1)} &\\ + C\sqrt{\underset{k}{\overset{(p)}{\mathcal{X}}}_{8R/9}(\tau_0, \tau_1)}\sqrt{\underset{k}{\overset{(0)}{\mathcal{X}}}_{8R/9}(\tau_0, \tau_1)}\sqrt{\underset{\ll k}{\overset{(p)}{\mathcal{X}}}_{8R/9}(\tau_0, \tau_1)} + C\int_{\tau_0}^{\tau_1} \overset{(0)}{\mathcal{E}}(\tau')\sqrt{\underset{\ll k}{\overset{(-1-\delta)}{\mathcal{E}'}}(\tau')}\,d\tau', &\quad (4\text{-}32)\end{aligned}$$

$$\begin{aligned}\chi\underset{k-1}{\overset{(p)}{\mathcal{X}}}(\tau_0, \tau_1) \lesssim \underset{k-1}{\overset{(p)}{\mathcal{E}}}(\tau_0) + \underset{k-1}{\overset{(p)}{\mathcal{X}}}_{8R/9}(\tau_0, \tau_1)\sqrt{\underset{\ll k}{\overset{(0)}{\mathcal{X}}}_{8R/9}(\tau_0, \tau_1)} + \sqrt{\underset{k-1}{\overset{(p)}{\mathcal{X}}}_{8R/9}(\tau_0, \tau_1)}\sqrt{\underset{k-1}{\overset{(0)}{\mathcal{X}}}_{8R/9}(\tau_0, \tau_1)}\sqrt{\underset{\ll k}{\overset{(p)}{\mathcal{X}}}_{8R/9}(\tau_0, \tau_1)} &\\ + \int_{\tau_0}^{\tau_1} \overset{(0)}{\mathcal{E}}(\tau')\sqrt{\underset{\ll k}{\overset{(-1-\delta)}{\mathcal{E}'}}(\tau')}\,d\tau', &\quad (4\text{-}33)\end{aligned}$$

while, for $p = 0$, *one has the estimates*

$$\begin{aligned}\sup_{v:\tau\leq\tau(v)} \overset{(0)}{\mathfrak{F}}(v, \tau_0, \tau), \quad c^{\rho}\underset{k}{\overset{(0)}{\mathcal{X}}}(\tau_0, \tau_1), \quad \underset{k}{\overset{(0)}{\mathfrak{E}}}(\tau_1) &\\ \leq \underset{k}{\overset{(0)}{\mathfrak{E}}}(\tau_0) + A\int_{\tau_0}^{\tau_1} \underset{k-1}{\overset{\xi}{\mathcal{E}}}{}'(\tau')\,d\tau' + C\underset{k}{\overset{(0+)}{\mathcal{X}}}_{8R/9}(\tau_0, \tau_1)\sqrt{\underset{\ll k}{\overset{(0+)}{\mathcal{X}}}_{8R/9}(\tau_0, \tau_1)} + C\int_{\tau_0}^{\tau_1} \overset{(0)}{\mathcal{E}}(\tau')\sqrt{\underset{\ll k}{\overset{(0)}{\mathcal{E}}}(\tau')}\,d\tau', &\quad (4\text{-}34)\\ \chi\underset{k-1}{\overset{(0)}{\mathcal{X}}}(\tau_0, \tau_1) \lesssim \underset{k-1}{\overset{(0)}{\mathcal{E}}}(\tau_0) + \underset{k}{\overset{(0+)}{\mathcal{X}}}_{8R/9}(\tau_0, \tau_1)\sqrt{\underset{\ll k}{\overset{(0+)}{\mathcal{X}}}_{8R/9}(\tau_0, \tau_1)} + \int_{\tau_0}^{\tau_1} \overset{(0)}{\mathcal{E}}(\tau')\sqrt{\underset{\ll k}{\overset{(0)}{\mathcal{E}}}(\tau')}\,d\tau'. &\quad (4\text{-}35)\end{aligned}$$

Finally, in both the case $p = 0$ *and the case* $\delta \leq p \leq 2 - \delta$, *one has the alternative estimate*

$$\underset{k}{\overset{(p)}{\mathcal{X}}}(\tau_0, \tau_1) \lesssim \underset{k}{\overset{(p)}{\mathcal{E}}}(\tau_0) + (1 + \tau_1 - \tau_0)\underset{k}{\overset{(p)}{\mathcal{X}}}(\tau_0, \tau_1)\sqrt{\underset{\ll k}{\overset{(0)}{\mathcal{X}}}(\tau_0, \tau_1)}, \qquad (4\text{-}36)$$

depending on $\tau_1 - \tau_0$.

Proof. This follows from Proposition 4.4.1, Corollary 4.5.3, Proposition 4.6.1 and Assumption 4.7.1. □

Remark 4.8.2. Note how the case $p = 0$ is anomalous in that one sees "0+" on the right-hand side of (4-34)–(4-35). It is this fact which will require us to use $p > 0$ weights to close the global estimates (4-32)–(4-33), even in the case (i). On the other hand, (4-36) is sufficient to show local existence even using only the $p = 0$ weight.

Remark 4.8.3. In using the above we shall always simply replace terms like $\underset{k}{\overset{(p)}{\mathcal{X}}}_{8R/9}(\tau_0, \tau_1)$, etc., by $\rho\underset{k}{\overset{(p)}{\mathcal{X}}}(\tau_0, \tau_1)$, etc., according to Remark 4.7.3. We have kept the terms in the above form simply to highlight their origin in the estimate of Assumption 4.7.1.

Remark 4.8.4. In the case where we only assume the black box inequality (3-3) and the assumptions of asymptotic flatness of Sections 3.5 and 3.6.2 as interpreted in Remark 3.6.10 (necessary to obtain (3-110)) and, moreover, the equation (4-1) is semilinear, i.e., $g(x,\psi) = g_0$ for all $x \in \mathcal{M}$, we then obtain the statement of Proposition 4.8.1 without inequalities (4-32) and (4-34) and where (4-33) and (4-35) are replaced by

$$\chi \underset{k}{\overset{(p)}{\mathcal{X}}}(\tau_0,\tau_1) \lesssim \underset{k}{\overset{(p)}{\mathcal{E}}}(\tau_0) + \underset{k}{\overset{(p)}{\mathcal{X}}}{}_{8R/9}(\tau_0,\tau_1)\sqrt{\underset{\ll k}{\overset{(0)}{\mathcal{X}}}{}_{8R/9}(\tau_0,\tau_1)} + \sqrt{\underset{k}{\overset{(p)}{\mathcal{X}}}{}_{8R/9}(\tau_0,\tau_1)}\sqrt{\underset{k}{\overset{(0)}{\mathcal{X}}}{}_{8R/9}(\tau_0,\tau_1)}\sqrt{\underset{\ll k}{\overset{(p)}{\mathcal{X}}}{}_{8R/9}(\tau_0,\tau_1)} + \int_{\tau_0}^{\tau_1} \overset{(0)}{\mathcal{E}}(\tau')\sqrt{\underset{\ll k}{\overset{(-1-\delta)}{\mathcal{E}'}}(\tau')}\,d\tau', \tag{4-37}$$

$$\chi \underset{k}{\overset{(0)}{\mathcal{X}}}(\tau_0,\tau_1) \lesssim \overset{(0)}{\mathcal{E}}(\tau_0) + \underset{k}{\overset{(0+)}{\mathcal{X}}}{}_{8R/9}(\tau_0,\tau_1)\sqrt{\underset{\ll k}{\overset{(0+)}{\mathcal{X}}}{}_{8R/9}(\tau_0,\tau_1)} + \int_{\tau_0}^{\tau_1} \overset{(0)}{\mathcal{E}}(\tau')\sqrt{\underset{\ll k}{\overset{(0)}{\mathcal{E}}}(\tau')}\,d\tau', \tag{4-38}$$

respectively. (In particular, (4-36) also still holds as stated.)

4.9. *Local well-posedness.* Finally, we give a local well-posedness statement and an extension criterion.

Since our initial data hypersurfaces $\Sigma(\tau_0)$ are in part null, it will be convenient to assume that our initial data are smooth. We define a smooth initial data set on $\Sigma(\tau_0)$ to be a pair $(\uppsi, \uppsi')$ where $\uppsi$ is a function on $\Sigma(\tau_0)$ smooth on $\Sigma(\tau_0)\cap\{r \le R\}$ and smooth on $\Sigma(\tau_0)\cap\{r \ge R\}$ and $\uppsi'$ is a smooth function on $\Sigma(\tau_0)\cap\{r \le R\}$, such that moreover there exists a smooth function Ψ on $\mathcal{R}(\tau_0,\tau_1)$ for some τ_1 such that

$$\Psi|_{\Sigma(\tau_0)} = \uppsi, \quad n\Psi|_{\Sigma(\tau_0)\cap\{r\le R\}} = \uppsi',$$

where n denotes the normal to $\Sigma(\tau_0)\cap\{r \le R\}$.

Note that given such initial data on $\Sigma(\tau_0)$ one may define

$$\underset{k}{\overset{(p)}{\mathcal{E}}}[\uppsi,\uppsi'] \tag{4-39}$$

as follows:

Using (4-1) we may compute along $\Sigma(\tau_0)$ the $(k+1)$-jet of any smooth solution ψ of (4-1) such that

$$\psi|_{\Sigma(\tau_0)} = \uppsi, \quad n\psi|_{\Sigma(\tau_0)\cap\{r\le R\}} = \uppsi'. \tag{4-40}$$

We may define $\underset{k}{\overset{(p)}{\mathcal{E}}}[\uppsi,\uppsi']$ to equal the usual $\underset{k}{\overset{(p)}{\mathcal{E}}}(\tau_0)$ where the derivatives are interpreted in terms of this formal computation. Note of course that the expression (4-39) may well be infinite.

Alternatively, we may express this as follows. Using the above computation, we may in fact define a smooth Ψ as in the previous paragraph such that Ψ moreover satisfies (4-1) along $\Sigma(\tau_0)$ and, for all $|\boldsymbol{k}| \le k$, $\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\Psi$ satisfies along $\Sigma(\tau_0)$ the equation one obtains by commuting (4-1) by $\widetilde{\mathfrak{D}}^{\boldsymbol{k}}$. We may then simply define (4-39) by substituting this Ψ for ψ in the usual expression for $\underset{k}{\overset{(p)}{\mathcal{E}}}(\tau_0)$.

With this, we can now state our local well-posedness:

Proposition 4.9.1 (local well-posedness). *Consider $(\mathcal{M}, g_0)$ satisfying the assumptions of Sections 2 and 3 (for cases* (i), (ii), *or* (iii)) *and equation (4-1) satisfying the assumptions of Sections 4.1 and 4.7. Fix either $p = 0$ or $\delta \le p \le 2-\delta$.*

There exists a positive integer $k_{\mathrm{loc}} \geq 4$ *such that the following holds: Let* $k \geq k_{\mathrm{loc}}$. *Then there exists a positive real constant* $C > 0$ *sufficiently large, a positive real parameter* $\varepsilon_{\mathrm{loc}} > 0$ *sufficiently small, and a decreasing positive function* $\tau_{\mathrm{exist}} : (0, \varepsilon_{\mathrm{loc}}) \to \mathbb{R}$ *such that, for all smooth initial data* (ψ, ψ') *on* $\Sigma(\tau_0)$ *such that*

$$\overset{(p)}{\underset{k}{\mathcal{E}}}[\psi, \psi'] \leq \varepsilon_0 \leq \varepsilon_{\mathrm{loc}},$$

there exists a smooth solution ψ *of* (4-1) *in* $\mathcal{R}(\tau_0, \tau_1)$ *for* $\tau_1 := \tau_0 + \tau_{\mathrm{exist}}$ *which satisfies*

$$\overset{(p)}{\underset{k}{\mathcal{X}}}(\tau_0, \tau_1) \leq C\varepsilon_0.$$

Moreover, for all $\tau_0 \leq \tau \leq \tau_1$, *any other smooth* $\tilde{\psi}$ *defined on* $\mathcal{R}(\tau_0, \tau)$ *satisfying* (4-1) *in* $\mathcal{R}(\tau_0, \tau)$ *and attaining the initial data, i.e., satisfying* (4-40) *(with* $\tilde{\psi}$ *replacing* ψ*), coincides with the restriction of* ψ*; i.e.,* $\tilde{\psi} = \psi|_{\mathcal{R}(\tau_0, \tau)}$.

We have in addition the following propagation of higher-order regularity and/or higher weighted estimates. Given p, $k \geq k_{\mathrm{loc}}$, $\varepsilon_0 \leq \varepsilon_{\mathrm{loc}}$, τ_0, τ_1 *as above,* $2-\delta \geq p' \geq p$ *if* $p \geq \delta$ *and either* $2-\delta \geq p' \geq \delta$ *or* $p' = 0$ *if* $p = 0$, *and* $k' \geq k$. *Then there exists a constant* $C(k', \tau_1 - \tau_0)$ *such that, given* ψ *as above,*

$$\overset{(p')}{\underset{k'}{\mathcal{X}}}(\tau_0, \tau_1) \leq C(k', \tau_1 - \tau_0) \overset{(p')}{\underset{k'}{\mathcal{E}}}(\tau_0).$$

Proof. This can be easily proven using estimate (4-36). We leave the details to the reader. □

Note also the following easy corollary, which we will use as an extension criterion.

Corollary 4.9.2 (continuation criterion). *Fix* $p = 0$ *or* $\delta \leq p \leq 2-\delta$, *and let* $k \geq k_{\mathrm{loc}}$ *and* $\varepsilon_{\mathrm{loc}}$ *be as in Proposition 4.9.1. There exists a constant* $C > 0$ *and an* $\epsilon > 0$ *such that the following is true:*

Let $\tau_f > \tilde{\tau}_0$, *and suppose that there exists a smooth solution* ψ *of* (4-1) *on* $\bigcup_{\tilde{\tau}_0 < \tau < \tau_f} \mathcal{R}(\tilde{\tau}_0, \tau)$ *such that*

$$\overset{(p)}{\underset{k}{\mathcal{E}}}(\tau) \leq \varepsilon_{\mathrm{loc}}$$

for $k \geq k_{\mathrm{loc}}$ *and all* $\tilde{\tau}_0 \leq \tau < \tau_f$. *Then, defining* $\tau_1 := \tau_f + \epsilon$, ψ *extends uniquely to a smooth function on* $\mathcal{R}(\tilde{\tau}_0, \tau_1)$ *satisfying* (4-1) *on this set and, setting* $\tau_0 := \max\{\tau_f - \epsilon, \tilde{\tau}_0\}$, *satisfying the estimate*

$$\overset{(p)}{\underset{k}{\mathcal{X}}}(\tau_0, \tau_1) \leq C\varepsilon_{\mathrm{loc}}. \tag{4-41}$$

Moreover, for all $k' \geq k$ *and all* $2-\delta \geq p' \geq p$, *there exist constants* $C(k')$ *such that*

$$\overset{(p')}{\underset{k'}{\mathcal{X}}}(\tau_0, \tau_1) \leq C(k') \overset{(p')}{\underset{k'}{\mathcal{E}}}(\tau_0). \tag{4-42}$$

Remark 4.9.3. Recall that by our conventions the constants C, k_{loc}, $\varepsilon_{\mathrm{loc}}$ depend in addition on k. We may assume that k_{loc} is large enough that $k \geq k_{\mathrm{loc}}$ satisfies the largeness constraint of all previous propositions in this section and that $\varepsilon_{\mathrm{loc}} \leq \varepsilon_{\mathrm{prelim}}$.

Remark 4.9.4. In view of Remark 4.8.4 and the fact that one needs only (4-36), all the statements of this section also hold in the semilinear case under the relaxed assumptions described there.

5. The main theorem: global existence and stability

We may now state the main result of this paper.

Theorem 5.1 (global existence and stability). *Consider* $(\mathcal{M}, g_0)$ *satisfying the assumptions of Sections* 2 *and* 3 (*for cases* (i), (ii), *or* (iii)) *and equation* (4-1) *satisfying the assumptions of Section 4.1 and Assumption 4.7.1 of Section 4.7.*

There exists a positive integer $k_{\text{global}} \geq k_{\text{local}}$ *large enough that, given* $k \geq k_{\text{global}}$, *there exists a positive* $0 < \varepsilon_{\text{global}} < \varepsilon_{\text{loc}}$ *sufficiently small and a positive constant* $C > 0$ *sufficiently large, so that the following holds*:

Fix $\tau_0 = 1$ *and consider as in Proposition 4.9.1 initial data* (ψ, ψ') *on* $\Sigma(\tau_0)$ *for* (4-1) *satisfying*

$$\overset{(p)}{\underset{k}{\mathcal{E}}}[\psi, \psi'] \leq \varepsilon_0 \leq \varepsilon_{\text{global}},$$

$$\overset{(1)}{\underset{k}{\mathcal{E}}}[\psi, \psi'] + \overset{(2-\delta)}{\underset{k-1}{\mathcal{E}}}[\psi, \psi'] \leq \varepsilon_0 \leq \varepsilon_{\text{global}},$$

$$\overset{(1)}{\underset{k}{\mathcal{E}}}[\psi, \psi'] + \overset{(2-\delta)}{\underset{k-2}{\mathcal{E}}}[\psi, \psi'] \leq \varepsilon_0 \leq \varepsilon_{\text{global}},$$

according to case (i), (ii), *and* (iii), *respectively*, *where in the former case* $\delta \leq p \leq 2-\delta$. *Then the* ψ *given by Proposition 4.9.1 extends to a unique globally defined solution of* (4-1) *in* $\mathcal{R}(\tau_0, \infty)$ *satisfying the estimates*

$$^{\rho}\overset{(p)}{\underset{k}{\mathcal{X}}}(\tau_0, \tau) \leq C\varepsilon_0, \quad ^{\rho}\overset{(1)}{\underset{k}{\mathcal{X}}}(\tau_0, \tau) + {}^{\rho}\overset{(2-\delta)}{\underset{k-1}{\mathcal{X}}}(\tau_0, \tau) \leq C\varepsilon_0, \quad ^{\rho}\overset{(1)}{\underset{k}{\mathcal{X}}}(\tau_0, \tau) + {}^{\rho}\overset{(2-\delta)}{\underset{k-2}{\mathcal{X}}}(\tau_0, \tau) \leq C\varepsilon_0, \tag{5-1}$$

according to cases (i), (ii), (iii), *respectively*, *for all* $\tau \geq \tau_0$; *in particular*

$$\overset{(p)}{\underset{k}{\mathcal{E}}}(\tau) \leq C\varepsilon_0, \quad \overset{(1)}{\underset{k}{\mathcal{E}}}(\tau) + \overset{(2-\delta)}{\underset{k-1}{\mathcal{E}}}(\tau) \leq C\varepsilon_0, \quad \overset{(1)}{\underset{k}{\mathcal{E}}}(\tau) + \overset{(2-\delta)}{\underset{k-2}{\mathcal{E}}}(\tau) \leq C\varepsilon_0,$$

respectively.

The solution will satisfy moreover estimates (6-8) *in case* (i), *estimates* (6-62) *and* (6-64)–(6-68) *in case* (ii), *and estimates* (6-116)–(6-121) *in case* (iii).

We shall give the proof of this theorem in Section 6.

Remark 5.2. Our theorem also applies under the relaxed assumptions of Remark 4.8.4, where (4-1) is however required to be semilinear. Here we can again distinguish case (i) where $\chi = 1$ and case (ii) where χ is allowed to degenerate. In case (i), the theorem holds as stated except for (6-8), which no longer applies. Similarly, in case (ii), the theorem holds as stated except for (6-62), which no longer applies. The necessary modifications will be collected in a series of remarks in Sections 6.1 and 6.2. (See already Section 6.2.5 for the completion of case (ii).)

We reiterate (cf. Propositions 4.7.2, Theorem A.1 and Theorem D.1) that the above theorem in particular holds for equations (4-1) on the Kerr spacetime $(\mathcal{M}, g_{a,M})$ for $|a| \ll M$, with quasilinear terms as described in Section 4.1 and the general semilinear terms considered in [Luk 2013], and, by the above remark, restricted to the purely semilinear case, for the full range of parameters $|a| < M$.

6. The estimate hierarchies and global existence and decay

We proceed with the proof of Theorem 5.1 in cases (i), (ii), and (iii). These will be treated in Sections 6.1, 6.2, and 6.3 below, respectively.

Note that we will now suppress some of the cumbersome notation with the following conventions:

In this section, the fraktur energies $\mathfrak{E}$, $\mathfrak{F}$ will everywhere denote $\mathfrak{E}(g)$, $\mathfrak{F}(g)$, so we will drop explicit reference to g here. When the region $\mathcal{R}(\tau_0, \tau_1)$ is assumed, we will omit the τ_0 or (τ_0, τ_1) arguments in various energies, etc., and denote $\mathcal{X}(\tau_0, \tau_1)$ by $\mathcal{X}$.

The smallness parameter $\varepsilon_{\text{global}}$ will be determined in the context of the proof. We will in fact introduce other parameters on the way; these will be in the relation

$$\varepsilon_{\text{global}} \leq \hat{\varepsilon}_{\text{slab}} \leq \varepsilon_{\text{slab}} \leq \varepsilon_{\text{boot}} \leq \varepsilon_{\text{loc}}, \tag{6-1}$$

where parameters $\hat{\varepsilon}_{\text{slab}}$, $\varepsilon_{\text{slab}}$ will only appear for cases (ii) and (iii). The parameter $\varepsilon_{\text{boot}}$ will be used in the context of a continuity or bootstrap argument. Let us remark already that if the basic bootstrap estimate (4-23) holds for an $\varepsilon \leq \varepsilon_{\text{boot}}$, then by Remark 4.9.3 ε will satisfy the smallness requirements of all propositions of Section 4 for the regions under consideration.

6.1. *Case* (i). Case (i) is the most elementary case, where moreover only the minimal nontrivial r^p weights for $2-\delta \geq p \geq \delta$ are required on data. (For instance, one may fix $p = \delta$.) We present first the fundamental estimate in Section 6.1.1 that holds under the bootstrap assumption (4-23) and then carry out the global existence proof in Section 6.1.2.

The proof will also hold for the semilinear case with the relaxed assumptions of Remark 4.8.4 if we are in the analogue of case (i), i.e., if $\chi = 1$. The modifications necessary to treat this case are described in a series of remarks (see already Remarks 6.1.2 and 6.1.3).

6.1.1. *The fundamental estimate.* The fundamental estimate is given simply by the following:

Proposition 6.1.1. *Let $\delta \leq p \leq 2-\delta$, let k be sufficiently large, and let us assume the case* (i) *assumptions. There exists an $\varepsilon_{\text{boot}} > 0$ small enough that the following is true*:

Let ψ solve (4-1) on $\mathcal{R}(\tau_0, \tau_1)$ and satisfy moreover (4-23) with our chosen p and with $0 < \varepsilon \leq \varepsilon_{\text{boot}}$. Then we have

$$\overset{(p)}{\underset{k}{\mathcal{X}}} \lesssim \overset{(p)}{\underset{k}{\mathcal{E}}}(\tau_0), \qquad \sup_{\tau_0 \leq \tau \leq \tau_1} \overset{(p)}{\underset{k}{\mathfrak{E}}}(\tau) \leq \overset{(p)}{\underset{k}{\mathfrak{E}}}(\tau_0)(1 + C\varepsilon^{\frac{1}{2}}). \tag{6-2}$$

Proof. Let $\varepsilon_{\text{boot}} \leq \varepsilon_{\text{loc}}$. The assumption of Proposition 4.8.1 is satisfied in view of our discussion following (6-1). From estimate (4-32) of Proposition 4.8.1, since in case (i) we have $\rho = 1$, $A = 0$, $\tilde{A} = 0$, it follows that

$$\overset{(p)}{\underset{k}{\mathcal{X}}} \lesssim \overset{(p)}{\underset{k}{\mathcal{E}}}(\tau_0) + \overset{(p)}{\underset{k}{\mathcal{X}}} \sqrt{\overset{(p)}{\underset{\ll k}{\mathcal{X}}}}, \qquad \sup_{\tau_0 \leq \tau \leq \tau_1} \overset{(p)}{\underset{k}{\mathfrak{E}}}(\tau) \leq \overset{(p)}{\underset{k}{\mathfrak{E}}}(\tau_0) + C \overset{(p)}{\underset{k}{\mathcal{X}}} \sqrt{\overset{(p)}{\underset{\ll k}{\mathcal{X}}}}. \tag{6-3}$$

By our bootstrap assumption (4-23), we have

$$\overset{(p)}{\underset{\ll k}{\mathcal{X}}} \leq \varepsilon. \tag{6-4}$$

Thus, possibly requiring $\varepsilon_{\text{boot}}$ to be even smaller, we may absorb the nonlinear term in the first inequality of (6-3) to obtain the first inequality of (6-2).

To obtain the second inequality of (6-2), we remark that Corollary 4.5.3 applies to yield in particular

$$\overset{(p)}{\underset{k}{\mathcal{E}}}(\tau_0) \sim \overset{(p)}{\underset{k}{\mathfrak{E}}}(\tau_0). \tag{6-5}$$

The second inequality of (6-2) now immediately follows from the second inequality of (6-3), plugging in the first inequality of (6-2) just established, (6-5) and (6-4). □

Remark 6.1.2. Under the relaxed assumptions of Remark 4.8.4, where (4-1) is however required to be semilinear and we are in the analogue of case (i), i.e., $\chi = 1$, from (4-38) we obtain the first inequality of (6-3). Thus, the above proposition holds as stated for the first inequality of (6-2).

6.1.2. *Proof of Theorem 5.1 in case* (i). We now carry out the proof of Theorem 5.1 proper in case (i).

Let $\delta \le p \le 2-\delta$ be as in the statement, and recall the assumption

$$\overset{(p)}{\underset{k}{\mathcal{E}}}(\tau_0) \le \varepsilon_0 \le \varepsilon_{\text{global}}$$

on initial data for a sufficiently small $\varepsilon_{\text{global}}$ to be determined.

Consider the set $\mathfrak{B}$ consisting of all $\tau_f > \tau_0$ such that a solution ψ of (4-1) achieving the data exists on $\mathcal{R}(\tau_0, \tau_f)$ and such that moreover the energy bootstrap assumption (4-23) holds on $\mathcal{R}(\tau_0, \tau_1 := \tau_f)$ with our chosen p and where $0 < \varepsilon \le \varepsilon_{\text{boot}}$ is chosen to satisfy

$$1 \gg \varepsilon \gg \varepsilon_{\text{global}}. \tag{6-6}$$

(The above relation in particular constrains $\varepsilon_{\text{global}}$ to be small.) By the local well-posedness statement Proposition 4.9.1, it follows that, since $k \ge k_{\text{loc}}$ and $\varepsilon_0 \le \varepsilon_{\text{global}} \le \varepsilon_{\text{loc}}$, we have $\tau_0 + \tau_{\text{exist}} \in \mathfrak{B}$, and thus $\mathfrak{B} \neq \varnothing$. Also note that, a fortiori, if $\tau_f \in \mathfrak{B}$, then $(\tau_0, \tau_f] \subset \mathfrak{B}$ and thus $\mathfrak{B}$ is manifestly a connected subset of (τ_0, ∞).

For any $\tau_1 \in \mathfrak{B}$, Proposition 6.1.1 applies in $\mathcal{R}(\tau_0, \tau_1)$. The first estimate of (6-2) then yields

$$\overset{(p)}{\underset{k}{\mathcal{X}}} \lesssim \overset{(p)}{\underset{k}{\mathcal{E}}}(\tau_0) \lesssim \varepsilon_0. \tag{6-7}$$

If follows that, for sufficiently small $\varepsilon_{\text{glob}}$ and all $\varepsilon_0 \le \varepsilon_{\text{glob}}$, the continuation criterion Corollary 4.9.2 applies to obtain that ψ extends as a solution to (4-1) on some $\mathcal{R}(\tau_0, \tau_f + \epsilon)$, where ϵ is independent of τ_f, and that, considering now $\tau_1 := \tau_f + \epsilon$, inequality (6-7) holds also in $\mathcal{R}(\tau_0, \tau_1 := \tau_f + \epsilon)$ (with a slightly different implicit constant than that of (6-7)). (To infer this from (4-41), which was an estimate on $\overset{(p)}{\underset{k}{\mathcal{X}}}(\tau_f, \tau_f + \epsilon)$, note that

$$\overset{(p)}{\underset{k}{\mathcal{X}}}(\tau_0, \tau_1) \lesssim \overset{(p)}{\underset{k}{\mathcal{X}}}(\tau_0, \tau_f) + \overset{(p)}{\underset{k}{\mathcal{X}}}(\tau_f, \tau_f + \epsilon).)$$

Finally we note that, for ε satisfying (6-6), inequality (6-7) in $\mathcal{R}(\tau_0, \tau_1 := \tau_f + \epsilon)$ shows in particular that inequalities (4-23) hold in this set. It follows by the definition of $\mathfrak{B}$ that $\tau_f + \epsilon \in \mathfrak{B}$ and thus, given also its connectedness, the set $\mathfrak{B}$ is open. Since ϵ does not depend on τ_f, it follows that $\mathfrak{B}$ is also closed, as any limit point τ_{limit} of $\mathfrak{B}$ in (τ_0, ∞) satisfies $\tau_{\text{limit}} \le \tau_f + \epsilon$ for some $\tau_f \in \mathfrak{B}$. We have shown that $\mathfrak{B}$ is a nonempty open and closed subset of (τ_0, ∞), and thus $\mathfrak{B} = (\tau_0, \infty)$. Hence the solution ψ exists globally in $\mathcal{R}(\tau_0, \infty)$ and satisfies (6-7) in $\mathcal{R}(\tau_0, \tau_1)$ where τ_1 is now any $\tau_1 > \tau_0$. This gives (5-1).

Revisiting (6-2) we see finally that we have the more precise global estimate

$$\overset{(p)}{\underset{k}{\mathfrak{E}}}(\tau) \leq \overset{(p)}{\underset{k}{\mathfrak{E}}}(\tau_0)(1 + C\varepsilon_0^{\frac{1}{2}}). \tag{6-8}$$

This completes the proof.

Remark 6.1.3. Note that, in view of Remark 6.1.2, the above proof goes through, except for the last paragraph involving (6-8), also under the relaxed assumptions of Remark 4.8.4, where (4-1) is however required to be semilinear and we are in the analogue of case (i), i.e., $\chi = 1$. Thus, in this case, one indeed obtains the statement as given in Remark 5.2.

6.2. *Case* (ii). Case (ii) is the second simplified case which we shall consider. Unlike case (i), we shall require proving actual τ-decay above a certain threshold in order to close, and this in turn will require raising p to $p = 2 - \delta$ in our initial energy assumptions.

The logic of the proof is a simplified version of the scheme we shall use for the general case and which has already been summarised in the introduction:

(1) Rather than bootstrap directly t-weighted estimates, we formulate the fundamental estimates, depending only on the bootstrap assumption (4-23), as a hierarchy of estimates on a spacetime slab of τ length at most L. This is the content of Section 6.2.1.

(2) We shall then show by a bootstrap argument, *restricted to such a slab*, how global existence and estimates on the slab can be proven, with suitable assumptions on the initial data of the slab, which now involve L. This is the content of Section 6.2.2.

(3) Still restricted to a given slab of length L, we will show that one can in fact a posteriori improve the above estimates under suitable additional assumptions on the initial data and, using a pigeonhole argument, show moreover an improved estimate for any $\Sigma(\tau')$ slice near the top of the slab. This is the content of Section 6.2.3.

(4) Finally, global existence and τ decay now follow by iterating the above estimates on a consecutive sequence of spacetime slabs of dyadic time length $L_i = 2^i$. This is the content of Section 6.2.4.

We shall in addition give an alternative iteration proof in Section 6.2.5 which does not use the exact boundedness statement of the fraktur energies. The advantage of this alternative proof is that it allows us to treat the semilinear case with the relaxed assumptions of Remark 4.8.4. The modifications necessary to treat this case are described in a series of remarks (see already Remarks 6.2.2, 6.2.4, 6.2.7 and 6.2.8).

6.2.1. *The hierarchy of inequalities.*

Proposition 6.2.1. *Let k be sufficiently large, and let us assume the case* (ii) *assumptions. There exist constants $C > 0$, $c > 0$ and an $\varepsilon_{\text{boot}} > 0$ small enough that the following is true*:

Consider a region $\mathcal{R}(\tau_0, \tau_1)$ and a ψ solving (4-1) on $\mathcal{R}(\tau_0, \tau_1)$, satisfying moreover (4-23) with $p = 0$ and with $0 < \varepsilon \leq \varepsilon_{\text{boot}}$. Let us assume moreover that

$$\tau_1 \leq \tau_0 + L$$

for some arbitrary $L>0$. We have the following hierarchy of inequalities on $\mathcal{R}(\tau_0,\tau_1)$:

$$\overset{(2-\delta)}{\underset{k}{\mathfrak{F}}}(v,\tau_1),\ \ \overset{(2-\delta)}{\underset{k}{\mathfrak{E}}}(\tau_1),\ \ c\,{}^{\chi}\overset{(2-\delta)}{\underset{k}{\mathcal{X}}} \le \overset{(2-\delta)}{\underset{k}{\mathfrak{E}}}(\tau_0) + C\left({}^{\chi}\overset{(2-\delta)}{\underset{k}{\mathcal{X}}}\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}} + \sqrt{{}^{\chi}\overset{(2-\delta)}{\underset{k}{\mathcal{X}}}}\sqrt{{}^{\chi}\overset{(0)}{\underset{k}{\mathcal{X}}}}\sqrt{\overset{(2-\delta)}{\underset{\ll k}{\mathcal{X}}}}\right) + C\,{}^{\chi}\overset{(0)}{\underset{k}{\mathcal{X}}}\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}}\sqrt{L}, \tag{6-9}$$

$$\overset{(1)}{\underset{k}{\mathfrak{F}}}(v,\tau_1),\ \ \overset{(1)}{\underset{k}{\mathfrak{E}}}(\tau_1),\ \ c\,{}^{\chi}\overset{(1)}{\underset{k}{\mathcal{X}}} \le \overset{(1)}{\underset{k}{\mathfrak{E}}}(\tau_0) + C\,{}^{\chi}\overset{(1)}{\underset{k}{\mathcal{X}}}\sqrt{\overset{(1)}{\underset{\ll k}{\mathcal{X}}}} + C\,{}^{\chi}\overset{(0)}{\underset{k}{\mathcal{X}}}\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}}\sqrt{L}, \tag{6-10}$$

$$\overset{(0)}{\underset{k}{\mathfrak{F}}}(v,\tau_1),\ \ \overset{(0)}{\underset{k}{\mathfrak{E}}}(\tau_1),\ \ c\,{}^{\chi}\overset{(0)}{\underset{k}{\mathcal{X}}} \le \overset{(0)}{\underset{k}{\mathfrak{E}}}(\tau_0) + C\Big({}^{\chi}\overset{(0)}{\underset{k}{\mathcal{X}}} + \big({}^{\chi}\overset{(0)}{\underset{k}{\mathcal{X}}}\big)^{\frac{1-\delta}{1+\delta}}\big({}^{\chi}\overset{(1)}{\underset{k}{\mathcal{X}}}\big)^{\frac{2\delta}{1+\delta}}\Big)\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}} + \big(\overset{(0)}{\underset{\ll k}{\mathcal{X}}}\big)^{\frac{1-\delta}{1+\delta}}\big(\overset{(1)}{\underset{\ll k}{\mathcal{X}}}\big)^{\frac{2\delta}{1+\delta}}} + C\,{}^{\chi}\overset{(0)}{\underset{k}{\mathcal{X}}}\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}}\sqrt{L}. \tag{6-11}$$

Proof. Again we recall that if $\varepsilon_{\mathrm{boot}} \le \varepsilon_{\mathrm{loc}}$, then the assumption of Proposition 4.8.1 holds. The first two inequalities follow from estimate (4-32) of Proposition 4.8.1 applied to $p=2-\delta,\ p=1$ in view of the fact that $\rho=\chi$, where we have used the relations (4-22) and (4-28) to replace the far-away supported nonlinear terms with those displayed above, together with the estimate

$$\int_{\tau_0}^{\tau_1}\overset{(0)}{\mathcal{E}}(\tau')\sqrt{\overset{(-1-\delta)}{\underset{\ll k}{\mathcal{E}'}}(\tau')}\,d\tau' \lesssim \sup_{\tau_0\le\tau\le\tau_1}\overset{(0)}{\mathcal{E}}(\tau)\sqrt{\int_{\tau_0}^{\tau_1}\overset{(-1-\delta)}{\underset{\ll k}{\mathcal{E}'}}(\tau')\,d\tau'}\cdot\sqrt{L} \lesssim {}^{\chi}\overset{(0)}{\underset{k}{\mathcal{X}}}\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}}\sqrt{L}.$$

Note that for the $p=1$ estimate we retain only the less precise expression ${}^{\chi}\overset{(1)}{\underset{k}{\mathcal{X}}}\sqrt{\overset{(1)}{\underset{\ll k}{\mathcal{X}}}}$, which will be sufficient for our purposes.

The third inequality follows similarly using estimate (4-34), where we use also the interpolation inequality (3-114) to obtain

$$\overset{(0+)}{\underset{k}{\mathcal{X}}}_{8R/9}(\tau_0,\tau_1)\sqrt{\overset{(0+)}{\underset{\ll k}{\mathcal{X}}}_{8R/9}(\tau_0,\tau_1)} \lesssim \Big({}^{\chi}\overset{(0)}{\underset{k}{\mathcal{X}}} + \big({}^{\chi}\overset{(0)}{\underset{k}{\mathcal{X}}}\big)^{\frac{1-\delta}{1+\delta}}\big({}^{\chi}\overset{(1)}{\underset{k}{\mathcal{X}}}\big)^{\frac{2\delta}{1+\delta}}\Big)\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}} + \big(\overset{(0)}{\underset{\ll k}{\mathcal{X}}}\big)^{\frac{1-\delta}{1+\delta}}\big(\overset{(1)}{\underset{\ll k}{\mathcal{X}}}\big)^{\frac{2\delta}{1+\delta}}}. \qquad \square$$

Remark 6.2.2. We note that inequalities (6-9)–(6-11) imply of course

$${}^{\chi}\overset{(2-\delta)}{\underset{k}{\mathcal{X}}} \lesssim \overset{(2-\delta)}{\underset{k}{\mathcal{E}}}(\tau_0) + \left({}^{\chi}\overset{(2-\delta)}{\underset{k}{\mathcal{X}}}\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}} + \sqrt{{}^{\chi}\overset{(2-\delta)}{\underset{k}{\mathcal{X}}}}\sqrt{{}^{\chi}\overset{(0)}{\underset{k}{\mathcal{X}}}}\sqrt{\overset{(2-\delta)}{\underset{\ll k}{\mathcal{X}}}}\right) + {}^{\chi}\overset{(0)}{\underset{k}{\mathcal{X}}}\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}}\sqrt{L}, \tag{6-12}$$

$${}^{\chi}\overset{(1)}{\underset{k}{\mathcal{X}}} \lesssim \overset{(1)}{\underset{k}{\mathcal{E}}}(\tau_0) + {}^{\chi}\overset{(1)}{\underset{k}{\mathcal{X}}}\sqrt{\overset{(1)}{\underset{\ll k}{\mathcal{X}}}} + {}^{\chi}\overset{(0)}{\underset{k}{\mathcal{X}}}\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}}\sqrt{L}, \tag{6-13}$$

$${}^{\chi}\overset{(0)}{\underset{k}{\mathcal{X}}} \lesssim \overset{(0)}{\underset{k}{\mathcal{E}}}(\tau_0) + \Big({}^{\chi}\overset{(0)}{\underset{k}{\mathcal{X}}} + \big({}^{\chi}\overset{(0)}{\underset{k}{\mathcal{X}}}\big)^{\frac{1-\delta}{1+\delta}}\big({}^{\chi}\overset{(1)}{\underset{k}{\mathcal{X}}}\big)^{\frac{2\delta}{1+\delta}}\Big)\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}} + \big(\overset{(0)}{\underset{\ll k}{\mathcal{X}}}\big)^{\frac{1-\delta}{1+\delta}}\big(\overset{(1)}{\underset{\ll k}{\mathcal{X}}}\big)^{\frac{2\delta}{1+\delta}}} + {}^{\chi}\overset{(0)}{\underset{k}{\mathcal{X}}}\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}}\sqrt{L}. \tag{6-14}$$

Following the above proof but now using equations (4-37) and (4-38), we may in fact directly deduce inequalities (6-12)–(6-14) under the relaxed assumptions of Remark 4.8.4, where (4-1) is however required to be semilinear and we are in the analogue of case (ii). Thus, the analogue of Proposition 6.2.1 holds in that case where (6-9)–(6-11) are replaced by (6-12)–(6-14).

6.2.2. *Global existence in L-slabs.* The presence of positive powers of L in (6-9)–(6-11) means that our smallness assumptions must involve negative powers of L in order for the estimates to close.

Proposition 6.2.3. *Let $k-1\ge k_{\mathrm{loc}}$ be sufficiently large, and let us assume the case* (ii) *assumptions. Then there exists an $0<\varepsilon_{\mathrm{slab}}\le\varepsilon_{\mathrm{loc}}$ and a constant $C>0$ implicit in the sign $\lesssim$ below such that the following is true*:

Given arbitrary $L \geq 1$, $\tau_0 \geq 0$, $0 < \varepsilon_0 \leq \varepsilon_{\text{slab}}$ *and initial data* (ψ, ψ') *on* $\Sigma(\tau_0)$ *as in Proposition 4.9.1 satisfying moreover*

$$\overset{(1)}{\underset{k}{\mathcal{E}}}(\tau_0) \leq \varepsilon_0, \quad \overset{(0)}{\underset{k-1}{\mathcal{E}}}(\tau_0) \leq \varepsilon_0 L^{-1}, \tag{6-15}$$

we have that the unique solution of Proposition 4.9.1 achieving the data can be extended to a ψ *defined on the entire spacetime slab* $\mathcal{R}(\tau_0, \tau_0 + L)$ *satisfying (4-1) and the estimates*

$$^{\chi}\overset{(1)}{\underset{k}{\mathcal{X}}} \lesssim \varepsilon_0, \quad ^{\chi}\overset{(0)}{\underset{k-1}{\mathcal{X}}} \lesssim \varepsilon_0 L^{-1}. \tag{6-16}$$

Proof. Consider the set $\mathfrak{B} \subset (\tau_0, \tau_0 + L]$ consisting of all $\tau_0 + L \geq \tau_f \geq \tau_0$ such that a solution ψ of (4-1) achieving the data exists on $\mathcal{R}(\tau_0, \tau_f)$ and such that the bootstrap assumption (4-23) with $p = 1$ and also the additional bootstrap assumption

$$\overset{(0)}{\underset{\ll k}{\mathcal{X}}} \leq \varepsilon L^{-1} \tag{6-17}$$

hold in $\mathcal{R}(\tau_0, \tau_1 := \tau_f)$, where $0 < \varepsilon \leq \varepsilon_{\text{boot}}$ is a small constant satisfying

$$1 \gg \varepsilon \gg \varepsilon_{\text{slab}}. \tag{6-18}$$

(The above relation in particular already constrains $\varepsilon_{\text{slab}}$ to be sufficiently small.)

By the local well-posedness statement Proposition 4.9.1, it follows that, since $k - 1 \geq k_{\text{loc}}$ and $\varepsilon_0 \leq \varepsilon_{\text{slab}} \leq \varepsilon_{\text{loc}}$, we have $\tau_0 + \tau_{\text{exist}} \in \mathfrak{B}$ and thus $\mathfrak{B} \neq \varnothing$, provided that ε satisfies (6-18). Also note that, a fortiori, if $\tau_f \in \mathfrak{B}$, then $(\tau_0, \tau_f] \in \mathfrak{B}$ and thus $\mathfrak{B}$ is manifestly a connected subset of (τ_0, ∞).

For $\tau_1 := \tau_f \in \mathcal{B}$, Proposition 6.2.1 applies in $\mathcal{R}(\tau_0, \tau_1)$. From (6-10) and (6-15) we obtain

$$^{\chi}\overset{(1)}{\underset{k}{\mathcal{X}}} \lesssim \varepsilon_0 + \varepsilon^{\frac{1}{2}}\, ^{\chi}\overset{(1)}{\underset{k}{\mathcal{X}}}, \tag{6-19}$$

where we have used the bootstrap assumptions (4-23) and (6-17).

From (6-11) for $k-1$ and (6-15) we obtain

$$^{\chi}\overset{(0)}{\underset{k-1}{\mathcal{X}}} \lesssim \varepsilon_0 L^{-1} + \varepsilon^{\frac{1}{2}} \big(^{\chi}\overset{(0)}{\underset{k-1}{\mathcal{X}}}\big)^{\frac{1-\delta}{1+\delta}} \big(^{\chi}\overset{(1)}{\underset{k-1}{\mathcal{X}}}\big)^{\frac{2\delta}{1+\delta}} L^{-\frac{1}{2}\frac{1-\delta}{1+\delta}} + \varepsilon^{\frac{1}{2}}\, ^{\chi}\overset{(0)}{\underset{k-1}{\mathcal{X}}}. \tag{6-20}$$

It follows that, for ε satisfying (6-18), we obtain

$$^{\chi}\overset{(2-\delta)}{\underset{k}{\mathcal{X}}} \lesssim \varepsilon_0, \quad ^{\chi}\overset{(0)}{\underset{k-1}{\mathcal{X}}} \lesssim \varepsilon_0 L^{-\frac{1}{2}\frac{1-\delta}{1+\delta}}, \tag{6-21}$$

and plugging the second inequality of (6-21) into (6-20) we obtain

$$^{\chi}\overset{(0)}{\underset{k-1}{\mathcal{X}}} \lesssim \varepsilon_0 L^{-1} + \varepsilon^{\frac{1}{2}} \varepsilon_0 L^{-\frac{1}{2}\left(\frac{1-\delta}{1+\delta}\right)^2} L^{-\frac{1}{2}\frac{1-\delta}{1+\delta}}. \tag{6-22}$$

Now defining

$$\gamma_0 := \frac{1}{2}\left(\frac{1-\delta}{1+\delta}\right)^2 + \frac{1}{2}\frac{1-\delta}{1+\delta},$$

we have

$$^{\chi}\overset{(0)}{\underset{k-1}{\mathcal{X}}} \lesssim \varepsilon_0 L^{-\gamma_0}, \tag{6-23}$$

whence, plugging (6-23) into (6-20), we improve to

$$^{\chi}\overset{(0)}{\underset{k-1}{\mathcal{X}}} \lesssim \varepsilon_0 L^{-1} + \varepsilon^{\frac{1}{2}} \varepsilon_0 L^{-\gamma_0\left(\frac{1-\delta}{1+\delta}\right)^2} L^{-\frac{1}{2}\frac{1-\delta}{1+\delta}}. \tag{6-24}$$

Defining γ_i iteratively by

$$\gamma_i = \gamma_{i-1}\left(\frac{1-\delta}{1+\delta}\right)^2 + \frac{1}{2}\frac{1-\delta}{1+\delta}$$

and in view of the restriction (3-1), there exists a first i such that $\gamma_i \geq 1$, whence we obtain

$$^{\chi}\overset{(0)}{\underset{k-1}{\mathcal{X}}} \lesssim \varepsilon_0 L^{-1}. \tag{6-25}$$

We may now already apply our continuation criterion Corollary 4.9.2, applied with $p = 0$ and $k-1$, to assert the existence of an ϵ, independent of τ_1, such that, now defining $\tau_1 := \min\{\tau_f + \epsilon, \tau_0 + L\}$, the solution ψ extends to a smooth solution of (4-1) on $\mathcal{R}(\tau_0, \tau_1)$ and, moreover, from (4-41), that

$$^{\chi}\overset{(0)}{\underset{k-1}{\mathcal{X}}} \lesssim \varepsilon_0 L^{-1} \tag{6-26}$$

holds on $\mathcal{R}(\tau_0, \tau_1)$ and, from (4-42), that

$$^{\chi}\overset{(1)}{\underset{k}{\mathcal{X}}} \lesssim \varepsilon_0 \tag{6-27}$$

holds on $\mathcal{R}(\tau_0, \tau_1)$. It follows that (6-17) and (4-23) (with $p = 1$) hold on $\mathcal{R}(\tau_0, \tau_1)$, and thus we have $\tau_1 = \min\{\tau_f + \epsilon, \tau_0 + L\} \in \mathfrak{B}$.

Since ϵ is independent of τ_f and in view also of the connectivity of $\mathfrak{B}$, it follows that $\mathfrak{B}$ is a nonempty open and closed subset of $(\tau_0, \tau_0 + L]$ and thus $\mathfrak{B} = (\tau_0, \tau_0 + L]$, and hence the solution exists in the entire spacetime slab $\mathcal{R}(\tau_0, \tau_0 + L)$.

The estimates (6-21) and (6-25) thus hold in the entire spacetime slab. This gives (6-16). □

Remark 6.2.4. Examining the proof, it is clear that we have only used inequalities (6-12)–(6-14) and not the full (6-9)–(6-11) of Proposition 6.2.1. Thus, in view of the comments of Remark 6.2.2, the proof holds also for the semilinear case under the relaxed assumptions described in Remark 4.8.4. In general, examining the proof, we may in fact relax the assumption of the first inequality of (6-15) to the assumption

$$\overset{(1)}{\underset{k}{\mathcal{E}}}(\tau_0) \leq \varepsilon_0 L^{\beta} \tag{6-28}$$

for a sufficiently small β, in which case the first inequality of (6-16) is replaced by

$$^{\chi}\overset{(1)}{\underset{k}{\mathcal{X}}}(\tau_0) \leq \varepsilon_0 L^{\beta}. \tag{6-29}$$

This will again be useful for the semilinear case.

6.2.3. *The pigeonhole argument.* The above assumptions on initial data are sufficient for global existence in the slab but are not sufficient to iterate. For this we shall need strengthened assumptions.

Proposition 6.2.5. *Under the assumptions of Proposition 6.2.3, there exists a constant $C > 0$, implicit in the inequalities $\lesssim$ below, a parameter $\alpha_0 \gg 1$ and, for all $\alpha \geq \alpha_0$, a parameter $\hat{\varepsilon}_{\text{slab}}(\alpha)$ such that, for all $0 < \hat{\varepsilon}_0 \leq \hat{\varepsilon}_{\text{slab}}(\alpha)$, the following holds:*

Let us assume that in addition to (6-15) we have

$$\overset{(1)}{\underset{k}{\mathfrak{E}}}(\tau_0) \leq \hat{\varepsilon}_0, \quad \overset{(2-\delta)}{\underset{k-1}{\mathfrak{E}}}(\tau_0) \leq \hat{\varepsilon}_0, \quad \overset{(0)}{\underset{k-1}{\mathfrak{E}}}(\tau_0) \leq \hat{\varepsilon}_0 \alpha L^{-1}, \quad \overset{(1)}{\underset{k-2}{\mathfrak{E}}}(\tau_0) \leq \hat{\varepsilon}_0 \alpha L^{-1+\delta}, \quad \overset{(0)}{\underset{k-3}{\mathfrak{E}}}(\tau_0) \leq \hat{\varepsilon}_0 \alpha^2 L^{-2+\delta}. \tag{6-30}$$

Then the solution ψ of (4-1) on $\mathcal{R}(\tau_0, \tau_0+L)$ given by Proposition 6.2.3 satisfies the additional estimates

$$\sup_{\tau_0\le\tau\le\tau_0+L} \overset{(1)}{\mathcal{E}}_k(\tau) \le \alpha\hat{\varepsilon}_0, \tag{6-31}$$

$$\sup_{\tau_0\le\tau\le\tau_0+L} \overset{(1)}{\mathfrak{E}}_k(\tau) \le \hat{\varepsilon}_0(1+\hat{\varepsilon}_0^{\frac14}L^{\frac{-1+\delta}{2}}), \tag{6-32}$$

$$\sup_{\tau_0\le\tau\le\tau_0+L} \overset{(2-\delta)}{\mathfrak{E}}_{k-1}(\tau) \le \hat{\varepsilon}_0(1+\hat{\varepsilon}_0^{\frac14}L^{-\frac12}), \tag{6-33}$$

$$ {}^{\chi}\overset{(1)}{\mathcal{X}}_k \lesssim \hat{\varepsilon}_0, \quad {}^{\chi}\overset{(2-\delta)}{\mathcal{X}}_{k-1} \lesssim \hat{\varepsilon}_0, \quad {}^{\chi}\overset{(0)}{\mathcal{X}}_{k-1} \lesssim \hat{\varepsilon}_0\alpha L^{-1}, \quad {}^{\chi}\overset{(1)}{\mathcal{X}}_{k-2} \lesssim \hat{\varepsilon}_0\alpha L^{-1+\delta}, \quad {}^{\chi}\overset{(0)}{\mathcal{X}}_{k-3} \lesssim \hat{\varepsilon}_0\alpha^2 L^{-2+\delta}. \tag{6-34}$$

Moreover, for all times τ' with $L \ge \tau'-\tau_0 \ge \frac12 L$, we have

$$\overset{(0)}{\mathfrak{E}}_{k-1}(\tau') \le \tfrac12\hat{\varepsilon}_0\alpha L^{-1}, \tag{6-35}$$

$$\overset{(1)}{\mathfrak{E}}_{k-1}(\tau') \le \tfrac12\hat{\varepsilon}_0\alpha L^{-1+\delta}, \tag{6-36}$$

$$\overset{(0)}{\mathfrak{E}}_{k-2}(\tau') \le \tfrac14\hat{\varepsilon}_0\alpha^2 L^{-2+\delta}. \tag{6-37}$$

Proof. Note that, by the statement of the proposition, we are in particular also assuming a priori (6-15) for *some* $\varepsilon_0 \le \varepsilon_{\mathrm{slab}}$ since this is included in the assumptions of Proposition 6.2.3. In view now of Corollary 4.5.3, however, for α_0 sufficiently large, if say $\hat{\varepsilon}_{\mathrm{slab}}(\alpha) \ll \alpha^{-3}\varepsilon_{\mathrm{slab}}$, it follows from the additional assumptions (6-30) that the estimates (6-15) in fact hold with the specific constant $\varepsilon_0 := \hat{\varepsilon}_0\alpha^3$ for all $\alpha \ge \alpha_0$.

To obtain (6-34) we argue as follows. We note from the proof of Proposition 6.2.3 that we have the following system of inequalities:

$${}^{\chi}\overset{(1)}{\mathcal{X}}_k \lesssim \hat{\varepsilon}_0 + \varepsilon_0^{\frac12}\,{}^{\chi}\overset{(1)}{\mathcal{X}}_k, \tag{6-38}$$

$${}^{\chi}\overset{(2-\delta)}{\mathcal{X}}_{k-1} \lesssim \hat{\varepsilon}_0 + \varepsilon_0^{\frac12}\,{}^{\chi}\overset{(2-\delta)}{\mathcal{X}}_{k-1}, \tag{6-39}$$

$${}^{\chi}\overset{(0)}{\mathcal{X}}_{k-1} \lesssim \hat{\varepsilon}_0\alpha L^{-1} + \big({}^{\chi}\overset{(0)}{\mathcal{X}}_{k-1}\big)^{\frac{1-\delta}{1+\delta}}\big({}^{\chi}\overset{(1)}{\mathcal{X}}_{k-1}\big)^{\frac{2\delta}{1+\delta}}\sqrt{\overset{(0)}{\mathcal{X}}_{\ll k} + \big(\overset{(0)}{\mathcal{X}}_{\ll k}\big)^{\frac{1-\delta}{1+\delta}}\big(\overset{(1)}{\mathcal{X}}_{\ll k}\big)^{\frac{2\delta}{1+\delta}}} + \varepsilon_0^{\frac12}\,{}^{\chi}\overset{(0)}{\mathcal{X}}_{k-1}, \tag{6-40}$$

$${}^{\chi}\overset{(1)}{\mathcal{X}}_{k-2} \lesssim \hat{\varepsilon}_0\alpha L^{-1+\delta} + \varepsilon_0^{\frac12}\,{}^{\chi}\overset{(1)}{\mathcal{X}}_{k-2}, \tag{6-41}$$

$${}^{\chi}\overset{(0)}{\mathcal{X}}_{k-3} \lesssim \hat{\varepsilon}_0\alpha^2 L^{-2+\delta} + \big({}^{\chi}\overset{(0)}{\mathcal{X}}_{k-3}\big)^{\frac{1-\delta}{1+\delta}}\big({}^{\chi}\overset{(1)}{\mathcal{X}}_{k-3}\big)^{\frac{2\delta}{1+\delta}}\sqrt{\overset{(0)}{\mathcal{X}}_{\ll k} + \big(\overset{(0)}{\mathcal{X}}_{\ll k}\big)^{\frac{1-\delta}{1+\delta}}\big(\overset{(1)}{\mathcal{X}}_{\ll k}\big)^{\frac{2\delta}{1+\delta}}} + \varepsilon_0^{\frac12}\,{}^{\chi}\overset{(0)}{\mathcal{X}}_{k-3}, \tag{6-42}$$

where we have used also the initial data assumptions (6-30).

For $\hat{\varepsilon}_{\mathrm{slab}}(\alpha)$ sufficiently small, we have $\varepsilon_0 \ll 1$ and thus we obtain immediately that

$$\begin{gathered}
{}^{\chi}\overset{(1)}{\mathcal{X}}_k \lesssim \hat{\varepsilon}_0, \quad {}^{\chi}\overset{(2-\delta)}{\mathcal{X}}_{k-1} \lesssim \hat{\varepsilon}_0, \quad {}^{\chi}\overset{(1)}{\mathcal{X}}_{k-2} \lesssim \hat{\varepsilon}_0\alpha L^{-1+\delta},\\
{}^{\chi}\overset{(0)}{\mathcal{X}}_{k-1} \lesssim \hat{\varepsilon}_0\alpha L^{-1} + \big({}^{\chi}\overset{(0)}{\mathcal{X}}_{k-1}\big)^{\frac{1-\delta}{1+\delta}}\big({}^{\chi}\overset{(1)}{\mathcal{X}}_{k-1}\big)^{\frac{2\delta}{1+\delta}}\sqrt{\overset{(0)}{\mathcal{X}}_{\ll k} + \big(\overset{(0)}{\mathcal{X}}_{\ll k}\big)^{\frac{1-\delta}{1+\delta}}\big(\overset{(1)}{\mathcal{X}}_{\ll k}\big)^{\frac{2\delta}{1+\delta}}},\\
{}^{\chi}\overset{(0)}{\mathcal{X}}_{k-3} \lesssim \hat{\varepsilon}_0\alpha^2 L^{-2+\delta} + \big({}^{\chi}\overset{(0)}{\mathcal{X}}_{k-3}\big)^{\frac{1-\delta}{1+\delta}}\big({}^{\chi}\overset{(1)}{\mathcal{X}}_{k-3}\big)^{\frac{2\delta}{1+\delta}}\sqrt{\overset{(0)}{\mathcal{X}}_{\ll k} + \big(\overset{(0)}{\mathcal{X}}_{\ll k}\big)^{\frac{1-\delta}{1+\delta}}\big(\overset{(1)}{\mathcal{X}}_{\ll k}\big)^{\frac{2\delta}{1+\delta}}}.
\end{gathered} \tag{6-43}$$

The first two inequalities of (6-43) yield the first two inequalities of (6-34), while the third inequality of (6-43) yields the fourth inequality of (6-34). On the other hand, from the latter inequality of (6-43), we obtain

$$ {}^{\chi}\overset{(0)}{\underset{k-3}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0\alpha^2 L^{-2+\delta} + \hat{\varepsilon}_0\varepsilon_0^{\frac{1}{2}} L^{-\frac{3}{2}\frac{(1-\delta)(1+2\delta)}{1+\delta}}. \tag{6-44} $$

Note that $-\frac{3}{2}\frac{(1-\delta)(1+2\delta)}{1+\delta} > -2+\delta$. We may however improve (6-44) iteratively as follows: If

$$ {}^{\chi}\overset{(0)}{\underset{k-3}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0\alpha^2 L^{-\gamma}, $$

for $\gamma < 2-\delta$, then

$$ \overset{(0)}{\underset{\ll k}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0\alpha^2 L^{-\gamma} \lesssim \varepsilon_0 L^{-\gamma}; $$

hence, plugging this again into the final inequality of (6-43) to estimate the nonlinear term, we obtain

$$ {}^{\chi}\overset{(0)}{\underset{k-3}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0\alpha^2 L^{-2+\delta} + \hat{\varepsilon}_0\varepsilon_0^{\frac{1}{2}} L^{-\frac{3}{2}\frac{(1-\delta)(\gamma+2\delta)}{1+\delta}}. \tag{6-45} $$

Setting $\gamma_0 = \frac{3}{2}\frac{(1-\delta)(1+2\delta)}{1+\delta}$ and defining inductively $\gamma_i = \frac{3}{2}\frac{(1-\delta)(\gamma_{i-1}+2\delta)}{1+\delta}$, we have that there exists a first $i \geq 1$ such that

$$ \frac{3}{2}\frac{(1-\delta)(\gamma_i+2\delta)}{1+\delta} > 2-\delta. $$

It follows that (6-45) holds for $\gamma = \gamma_i$; hence

$$ {}^{\chi}\overset{(0)}{\underset{k-3}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0\alpha^2 L^{-2+\delta}. $$

This yields the fifth inequality of (6-34). Note that this implies

$$ \overset{(0)}{\underset{\ll k}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0\alpha^2 L^{-2+\delta} \lesssim \varepsilon_0 L^{-2+\delta}. \tag{6-46} $$

Note also that the third inequality of (6-43) implies

$$ \overset{(1)}{\underset{\ll k}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0\alpha L^{-1+\delta} \lesssim \varepsilon_0 L^{-1+\delta}. \tag{6-47} $$

Finally, plugging these bounds into the fourth inequality of (6-43) yields the third inequality of (6-34), completing the proof of (6-34).

To obtain (6-32), we recall (6-10) from Proposition 6.2.1. This gives, for sufficiently small $\hat{\varepsilon}_{\text{slab}}$,

$$ \begin{aligned} \sup_{\tau_0\le\tau\le\tau_0+L} \overset{(1)}{\underset{k}{\mathfrak{E}}}(\tau) &\le \overset{(1)}{\underset{k}{\mathfrak{E}}}(\tau_0) + C\,{}^{\chi}\overset{(1)}{\underset{k}{\mathcal{X}}}\sqrt{\overset{(1)}{\underset{\ll k}{\mathcal{X}}}} + C\,{}^{\chi}\overset{(0)}{\underset{k}{\mathcal{X}}}\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}}\sqrt{L}, \\ &\le \hat{\varepsilon}_0 + C\hat{\varepsilon}_0\varepsilon_0^{\frac{1}{2}} L^{\frac{-1+\delta}{2}} + C\hat{\varepsilon}_0\varepsilon_0^{\frac{1}{2}} L^{\frac{-2+\delta}{2}} L^{\frac{1}{2}} \le \hat{\varepsilon}_0(1+\hat{\varepsilon}_0^{\frac{1}{4}} L^{\frac{-1+\delta}{2}}), \end{aligned} $$

where we have used the first inequality of (6-30), the first inequality of (6-43), the estimate (6-47) and the estimate (6-46), and we have replaced $\varepsilon_0^{1/2}$ by $\hat{\varepsilon}_0^{1/4}$ in the penultimate line, sacrificing a quarter power to absorb the resulting α term and the constant C.

To obtain (6-33), we now recall (6-9) from Proposition 6.2.1. This gives, for sufficiently small $\hat{\varepsilon}_{\text{slab}}$,

$$ \begin{aligned} \sup_{\tau_0\le\tau\le\tau_0+L} \overset{(2-\delta)}{\underset{k-1}{\mathfrak{E}}}(\tau) &\le \overset{(2-\delta)}{\underset{k-1}{\mathfrak{E}}}(\tau_0) + C\Big({}^{\chi}\overset{(2-\delta)}{\underset{k-1}{\mathcal{X}}}\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}} + \sqrt{{}^{\chi}\overset{(2-\delta)}{\underset{k-1}{\mathcal{X}}}}\sqrt{{}^{\chi}\overset{(0)}{\underset{k-1}{\mathcal{X}}}}\sqrt{\overset{(2-\delta)}{\underset{\ll k}{\mathcal{X}}}}\Big) + C\,{}^{\chi}\overset{(0)}{\underset{k-1}{\mathcal{X}}}\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}}\sqrt{L} \\ &\le \hat{\varepsilon}_0 + C\hat{\varepsilon}_0\varepsilon_0^{\frac{1}{2}} L^{\frac{-2+\delta}{2}} + C\hat{\varepsilon}_0\alpha^{\frac{1}{2}}\varepsilon_0^{\frac{1}{2}} L^{-\frac{1}{2}} + C\hat{\varepsilon}_0\alpha\varepsilon_0^{\frac{1}{2}} L^{\frac{-2+\delta}{2}} \\ &\le \hat{\varepsilon}_0(1+\hat{\varepsilon}_0^{\frac{1}{4}} L^{-\frac{1}{2}}), \end{aligned} $$

where we have used the second inequality of (6-30), the second and third inequalities of (6-34), the estimate (6-46) and the first inequality of (6-44), and we have replaced $\varepsilon_0^{1/2}$ by $\hat{\varepsilon}_0^{1/4}$ in the penultimate line, sacrificing a quarter power to absorb the resulting α term and the constant C.

To show (6-35)–(6-37), we first apply the pigeonhole principle as in [Dafermos and Rodnianski 2010b] to the inequality

$$\int_{\tau_0}^{\tau_0+L} \big(\overset{(0)}{\underset{k-1}{\mathcal{E}}}{}'(\tau') + \overset{(1-\delta)}{\underset{k-2}{\mathcal{E}}}{}'(\tau') + \alpha^{-1}L^{1-\delta}\overset{(0)}{\underset{k-3}{\mathcal{E}}}{}'(\tau')\big)\, d\tau' \lesssim \hat{\varepsilon}_0,$$

which, upon addition, follows from the first, second and fourth inequalities of the estimate (6-34) already shown. Recalling from (3-93) that, for both $p = 2-\delta$ and $p = 1$, we have

$$\overset{(p-1)}{\underset{k-2}{\mathcal{E}}}{}' \gtrsim \overset{(p-1)}{\underset{k-2}{\mathcal{E}}}, \quad \overset{(p-1)}{\underset{k-3}{\mathcal{E}}}{}' \gtrsim \overset{(p-1)}{\underset{k-3}{\mathcal{E}}},$$

we obtain that there exists $\tau'' \in \big[\tau_0, \tau_0 + \frac{1}{2}L\big]$, whose precise value depends on the solution, such that

$$\overset{(0)}{\underset{k-1}{\mathcal{E}}}(\tau'') \lesssim \hat{\varepsilon}_0 \cdot L^{-1}, \tag{6-48}$$

$$\overset{(1-\delta)}{\underset{k-2}{\mathcal{E}}}(\tau'') \lesssim \hat{\varepsilon}_0 \cdot L^{-1}, \tag{6-49}$$

$$\overset{(0)}{\underset{k-3}{\mathcal{E}}}(\tau'') \lesssim \hat{\varepsilon}_0 \alpha L^{-1+\delta} \cdot L^{-1}. \tag{6-50}$$

Now in view of the interpolation estimate (3-113) of Proposition 3.6.12, we have

$$\begin{aligned}\overset{(1)}{\underset{k-2}{\mathcal{E}}}(\tau'') &\lesssim \big(\overset{(1-\delta)}{\underset{k-2}{\mathcal{E}}}(\tau'')\big)^{1-\delta}\big(\overset{(2-\delta)}{\underset{k-2}{\mathcal{E}}}(\tau'')\big)^{\delta} \\ &\le \big(\overset{(1-\delta)}{\underset{k-2}{\mathcal{E}}}(\tau'')\big)^{1-\delta}\big(\sup_{\tau\in[\tau_0,\tau_0+L]} \overset{(2-\delta)}{\underset{k-2}{\mathcal{E}}}(\tau)\big)^{\delta}\end{aligned}$$

and thus

$$\overset{(1)}{\underset{k-2}{\mathcal{E}}}(\tau'') \lesssim \hat{\varepsilon}_0 L^{-1+\delta}, \tag{6-51}$$

where we have used (6-49) and the estimate for $\overset{(2-\delta)}{\underset{k-2}{\mathcal{E}}}$ contained in (6-34).

Now we apply (6-10) and (6-11) again, with τ'' in place of τ_0, using (6-48), (6-51) and (6-50) to bound the initial data, to obtain that, for all $\tau_0 + L \ge \tau' \ge \tau_0 + \frac{1}{2}L$, we have

$$\overset{(0)}{\underset{k-1}{\mathfrak{E}}}(\tau') \sim \overset{(0)}{\underset{k-1}{\mathcal{E}}}(\tau') \lesssim \hat{\varepsilon}_0 L^{-1}, \tag{6-52}$$

$$\overset{(1)}{\underset{k-2}{\mathfrak{E}}}(\tau') \sim \overset{(1)}{\underset{k-2}{\mathcal{E}}}(\tau') \lesssim \hat{\varepsilon}_0 L^{-1+\delta}, \tag{6-53}$$

$$\overset{(0)}{\underset{k-3}{\mathfrak{E}}}(\tau') \sim \overset{(0)}{\underset{k-3}{\mathcal{E}}}(\tau') \lesssim \hat{\varepsilon}_0 \alpha L^{-2+\delta}. \tag{6-54}$$

Thus, in view of the requirement $\alpha \ge \alpha_0 \gg 1$, for sufficiently large α_0 we may absorb the constants implicit in $\lesssim$ by explicit constants of our choice by adding extra positive α powers to the right-hand side of (6-53) and (6-54). In this way, we obtain the specific estimates (6-36) and (6-37) for all $\tau' \ge \tau''$ and thus in particular for all $L \ge \tau' - \tau_0 \ge \frac{1}{2}L$. In the same way, we also obtain the specific constant of the estimate of (6-31) which will be convenient in our scheme. □

Let us state an alternative "relaxed" version of the above proposition.

Proposition 6.2.6. *Under the relaxed assumptions for Proposition 6.2.3 described in Remark 6.2.4, there exist a constant $C>0$, implicit in the inequalities $\lesssim$ below, a parameter $\beta>0$ sufficiently small, a parameter $\alpha_0 \gg 1$ and, for all $\alpha \ge \alpha_0$, a parameter $\hat\varepsilon_{\mathrm{slab}}(\alpha)$ such that the following holds*:

Given $0<\hat\varepsilon_0 \le \hat\varepsilon_{\mathrm{slab}}$, let us assume in addition that

$$\underset{k}{\overset{(1)}{\mathcal{E}}}(\tau_0) \le \hat\varepsilon_0 L^\beta, \quad \underset{k-1}{\overset{(2-\delta)}{\mathcal{E}}}(\tau_0) \le \hat\varepsilon_0 L^\beta, \quad \underset{k-1}{\overset{(0)}{\mathcal{E}}}(\tau_0) \le \hat\varepsilon_0 \alpha^{\frac12} L^{-1+\beta}, \\ \underset{k-2}{\overset{(1)}{\mathcal{E}}}(\tau_0) \le \hat\varepsilon_0 \alpha L^{-1+\delta+\beta}, \quad \underset{k-3}{\overset{(0)}{\mathcal{E}}}(\tau_0) \le \hat\varepsilon_0 \alpha^2 L^{-2+\delta+\beta}. \tag{6-55}$$

Then the solution ψ of (4-1) on $\mathcal{R}(\tau_0, \tau_0+L)$ given by Proposition 6.2.3 satisfies the additional estimates

$$\sup_{\tau\le\tau\le\tau_0+L} \underset{k}{\overset{(1)}{\mathcal{E}}}(\tau) \le \alpha^\beta \hat\varepsilon_0 L^\beta, \qquad \sup_{\tau_0\le\tau\le\tau_0+L} \underset{k-1}{\overset{(2-\delta)}{\mathcal{E}}}(\tau) \le \alpha^\beta \hat\varepsilon_0 L^\beta, \tag{6-56}$$

$${}^\chi\underset{k}{\overset{(1)}{\mathcal{X}}} \lesssim \hat\varepsilon_0 L^\beta, \quad {}^\chi\underset{k-1}{\overset{(2-\delta)}{\mathcal{X}}} \lesssim \hat\varepsilon_0 L^\beta, \quad {}^\chi\underset{k-1}{\overset{(0)}{\mathcal{X}}} \lesssim \hat\varepsilon_0 \alpha L^{-1+\beta}, \quad {}^\chi\underset{k-2}{\overset{(1)}{\mathcal{X}}} \lesssim \hat\varepsilon_0 \alpha L^{-1+\delta+\beta}, \quad {}^\chi\underset{k-3}{\overset{(0)}{\mathcal{X}}} \lesssim \hat\varepsilon_0 \alpha^2 L^{-2+\delta+\beta}. \tag{6-57}$$

Moreover, for all times τ' with $L \ge \tau'-\tau_0 \ge \frac12 L$, we have that

$$\underset{k-1}{\overset{(0)}{\mathcal{E}}}(\tau') \le \hat\varepsilon_0 \alpha^{2\beta} L^{-1+\beta}, \quad \underset{k-1}{\overset{(1)}{\mathcal{E}}}(\tau') \le \hat\varepsilon_0 \alpha^{2\beta} L^{-1+\delta+\beta}, \quad \underset{k-2}{\overset{(0)}{\mathcal{E}}}(\tau') \le \hat\varepsilon_0 \alpha^{1+2\beta} L^{-2+\delta+\beta}. \tag{6-58}$$

Proof. The proof is similar to that of Proposition 6.2.5 and is left to the reader. □

Remark 6.2.7. The above proposition can now immediately be seen to also hold in the semilinear case, with the relaxed assumptions described in Remark 4.8.4.

6.2.4. *The iteration*: *proof of Theorem 5.1 in case* (ii). We may now prove Theorem 5.1 in case (ii).

We shall proceed iteratively.

We define $\tau_0=1$, $L_0=1$, $L_i=2^i$, $\tau_{i+1}=\tau_i+L_i$ and fix $\alpha\ge\alpha_0$ so that the statement of Proposition 6.2.5 holds. (Note that we shall no longer track the dependence of constants and parameters on α since it is now fixed; one could simply take $\alpha=\alpha_0$.)

For $0<\varepsilon_0\le\varepsilon_{\mathrm{global}}$ and $\varepsilon_{\mathrm{global}}$ sufficiently small, since

$$\underset{k}{\overset{(1)}{\mathcal{E}}}(\tau_0) + \underset{k-1}{\overset{(2-\delta)}{\mathcal{E}}}(\tau_0) \le \varepsilon_0,$$

in view of Corollary 4.5.3, we have

$$\underset{k}{\overset{(1)}{\mathfrak{E}}}(\tau_0) \lesssim \varepsilon_0, \quad \underset{k-1}{\overset{(2-\delta)}{\mathfrak{E}}}(\tau_0) \lesssim \varepsilon_0.$$

Thus, for sufficiently small $\varepsilon_{\mathrm{global}} \ll \hat\varepsilon_{\mathrm{slab}}$, it follows that there exists an $\hat\varepsilon_0(\varepsilon_0)\sim\varepsilon_0$, satisfying moreover $2\alpha\hat\varepsilon_0 \le \varepsilon_{\mathrm{slab}}$, such that

$$\hat\varepsilon_0 \le \tfrac12 \hat\varepsilon_{\mathrm{slab}}$$

and

$$\underset{k}{\overset{(1)}{\mathfrak{E}}}(\tau_0) \le \hat\varepsilon_0, \quad \underset{k-1}{\overset{(2-\delta)}{\mathfrak{E}}}(\tau_0) \le \hat\varepsilon_0, \quad \underset{k-1}{\overset{(0)}{\mathfrak{E}}}(\tau_0) \le \hat\varepsilon_0 \alpha L^{-1}, \quad \underset{k-2}{\overset{(1)}{\mathfrak{E}}}(\tau_0) \le \hat\varepsilon_0 \alpha L^{-1+\delta}, \quad \underset{k-3}{\overset{(0)}{\mathfrak{E}}}(\tau_0) \le \hat\varepsilon_0 \alpha^2 L^{-2+\delta}.$$

Finally, with our restriction on the definition of ε_0, we may also write

$$\underset{k}{\overset{(1)}{\mathcal{E}}}(\tau_0) \le \alpha\hat\varepsilon_0.$$

In general, given $\tau_i \geq 1$ defined above, $\hat{\varepsilon}_i \leq \hat{\varepsilon}_{\text{slab}}$, and a solution ψ of (4-1) on $\mathcal{R}(\tau_0, \tau_i)$ such that

$$\underset{k}{\overset{(1)}{\mathcal{E}}}(\tau_i) \leq \alpha \hat{\varepsilon}_i \tag{6-59}$$

and

$$\underset{k}{\overset{(1)}{\mathfrak{E}}}(\tau_i) \leq \hat{\varepsilon}_i, \quad \underset{k-1}{\overset{(2-\delta)}{\mathfrak{E}}}(\tau_i) \leq \hat{\varepsilon}_i, \quad \underset{k-1}{\overset{(0)}{\mathfrak{E}}}(\tau_i) \leq \hat{\varepsilon}_i \alpha L^{-1}, \quad \underset{k-1}{\overset{(1)}{\mathfrak{E}}}(\tau_i) \leq \hat{\varepsilon}_i \alpha L_i^{-1+\delta}, \quad \underset{k-2}{\overset{(0)}{\mathfrak{E}}}(\tau_i) \leq \hat{\varepsilon}_i \alpha^2 L_i^{-2+\delta}, \tag{6-60}$$

we note that, by our restrictions on $\hat{\varepsilon}_{\text{slab}}$, the assumptions of Proposition 6.2.3 hold with $\alpha^2 \hat{\varepsilon}_i$ in place of ε_0 (here we are using (6-59) to invoke Corollary 4.5.3 to rewrite (6-60) in terms of the calligraphic energies; cf. the first lines of the proof of Proposition 6.2.5) and the assumptions of Proposition 6.2.5 then apply with $\hat{\varepsilon}_i$ in place of $\hat{\varepsilon}_0$, where both propositions are understood now with τ_i, τ_{i+1} in place of τ_0, τ_1. It follows that the solution ψ extends to a solution defined also in $\mathcal{R}_i := \mathcal{R}(\tau_i, \tau_i + L_i)$ and satisfying the estimates (6-31)–(6-34), while for $\tau' = \tau_{i+1} = \tau_i + L_i$ we have in addition (6-36)–(6-37). We have thus

$$\begin{aligned}
&\underset{k}{\overset{(1)}{\mathcal{E}}}(\tau_{i+1}) \leq \alpha \hat{\varepsilon}_i \leq \alpha \hat{\varepsilon}_{i+1},\\
&\underset{k}{\overset{(1)}{\mathfrak{E}}}(\tau_{i+1}) \leq \hat{\varepsilon}_i (1 + \hat{\varepsilon}_i^{\frac{1}{4}} L_i^{-\frac{1+\delta}{2}}) \leq \hat{\varepsilon}_{i+1},\\
&\underset{k-1}{\overset{(2-\delta)}{\mathfrak{E}}}(\tau_{i+1}) \leq \hat{\varepsilon}_i (1 + \hat{\varepsilon}_i^{\frac{1}{4}} L_i^{-\frac{1}{2}}) \leq \hat{\varepsilon}_{i+1},\\
&\underset{k-1}{\overset{(1)}{\mathfrak{E}}}(\tau_{i+1}) \leq \tfrac{1}{2} \hat{\varepsilon}_i \alpha L_i^{-1} \leq \hat{\varepsilon}_{i+1} \alpha L_{i+1}^{-1},\\
&\underset{k-2}{\overset{(1)}{\mathfrak{E}}}(\tau_{i+1}) \leq \tfrac{1}{2} \hat{\varepsilon}_i \alpha L_i^{-1+\delta} \leq \hat{\varepsilon}_{i+1} \alpha L_{i+1}^{-1+\delta},\\
&\underset{k-3}{\overset{(0)}{\mathfrak{E}}}(\tau_{i+1}) \leq \tfrac{1}{4} \hat{\varepsilon}_i \alpha^2 L_i^{-2+\delta} \leq \hat{\varepsilon}_{i+1} \alpha^2 L_{i+1}^{-2+\delta}
\end{aligned}$$

as long as

$$\hat{\varepsilon}_{i+1} := \hat{\varepsilon}_i (1 + \hat{\varepsilon}_i^{\frac{1}{4}} L_i^{-\frac{1+\delta}{2}}). \tag{6-61}$$

Now note that, for $\varepsilon_{\text{global}}$ sufficiently small, the above inductive definition (6-61) of $\hat{\varepsilon}_i$ ensures that $\hat{\varepsilon}_i \leq 2\hat{\varepsilon}_0$ for all i.

It follows that a solution exists in $\mathcal{R}(\tau_0, \infty) = \bigcup \mathcal{R}(\tau_i, \tau_i + L_i)$ and in each interval the estimates (6-34) hold.

We obtain finally that for all $\tau \geq 1$ we have

$$\underset{k}{\overset{(1)}{\mathfrak{E}}}(\tau) \leq \hat{\varepsilon}_0 + C \hat{\varepsilon}_0^{\frac{3}{2}}, \quad \underset{k-1}{\overset{(2-\delta)}{\mathfrak{E}}}(\tau) \leq \hat{\varepsilon}_0 + C \hat{\varepsilon}_0^{\frac{3}{2}}, \tag{6-62}$$

$$\int_{\tau_0}^{\tau} \chi \underset{k-1}{\overset{(1-\delta)}{\mathcal{E}'}} + \underset{k-2}{\overset{(1-\delta)}{\mathcal{E}'}} \lesssim \hat{\varepsilon}_0 \log(\tau + 1), \tag{6-63}$$

$$\underset{k-2}{\overset{(1)}{\mathcal{E}}}(\tau) \lesssim \hat{\varepsilon}_0 \tau^{-1+\delta}, \quad \underset{k-2}{\overset{(1)}{\mathcal{F}}}(v, \tau) \lesssim \hat{\varepsilon}_0 \tau^{-1+\delta}, \tag{6-64}$$

$$\int_{\tau}^{\infty} \chi \underset{k-2}{\overset{(0)}{\mathcal{E}'}} + \underset{k-3}{\overset{(0)}{\mathcal{E}'}} \lesssim \hat{\varepsilon}_0 \tau^{-1+\delta}, \tag{6-65}$$

$$\underset{k-3}{\overset{(0)}{\mathcal{E}}}(\tau) \lesssim \hat{\varepsilon}_0 \tau^{-2+\delta}, \quad \underset{k-3}{\overset{(0)}{\mathcal{F}}}(v, \tau) \lesssim \hat{\varepsilon}_0 \tau^{-2+\delta}, \tag{6-66}$$

$$\int_{\tau}^{\infty} \chi \underset{k-3}{\overset{(-1-\delta)}{\mathcal{E}'}} + \underset{k-4}{\overset{(-1-\delta)}{\mathcal{E}'}} \lesssim \hat{\varepsilon}_0 \tau^{-2+\delta}. \tag{6-67}$$

Here, by $\mathcal{F}(v, \tau)$ we denote the restriction of the flux on $\underline{C}_v$ to $J^+(\Sigma(\tau))$.

Let us note that we may improve, a posteriori, the inequality (6-63) to

$$\int_{\tau_0}^{\infty} \chi \underset{k-1}{\overset{(1-\delta)}{\mathcal{E}'}} + \underset{k-2}{\overset{(1-\delta)}{\mathcal{E}'}} \lesssim \hat{\varepsilon}_0. \tag{6-68}$$

To show (6-68), for all $\tau \geq \tau_0$, we reapply the estimates of Proposition 4.8.1 globally on $\mathcal{R}(\tau_0, \tau)$. To control the nonlinear error bulk integrals, we redecompose the domain in dyadic time slabs, apply the estimates proven, and then sum. In view of the estimates, all these error bulk integrals can be controlled by $C\hat{\varepsilon}_0^{3/2}$.

6.2.5. *Alternative proof using Proposition 6.2.6 and the semilinear case.* We note that we may prove the theorem alternatively using Proposition 6.2.6. We first fix $\alpha \geq \alpha_0$ and now define $L_i := \alpha^i$. Here our iterative assumption is

$$\underset{k}{\overset{(1)}{\mathcal{E}}}(\tau_0) \leq \hat{\varepsilon}_0 L_i^{\beta}, \quad \underset{k-1}{\overset{(2-\delta)}{\mathcal{E}}}(\tau_0) \leq \hat{\varepsilon}_0 L_i^{\beta}, \quad \underset{k-1}{\overset{(0)}{\mathcal{E}}}(\tau_0) \leq \hat{\varepsilon}_0 \alpha L_i^{-1+\beta}, \quad \underset{k-2}{\overset{(1)}{\mathcal{E}}}(\tau_0) \leq \hat{\varepsilon}_0 \alpha L_i^{-1+\delta+\beta}, \quad \underset{k-3}{\overset{(0)}{\mathcal{E}}}(\tau_0) \leq \hat{\varepsilon}_0 \alpha^2 L_i^{-2+\delta+\beta}.$$

In view of (6-56) and (6-58) and the definition of L_i, we see that this is closed under iteration. We obtain thus existence in $\mathcal{R}(\tau_0, \infty)$ and at first instance the bounds

$$\underset{k}{\overset{(1)}{\mathcal{E}}}(\tau) \lesssim \hat{\varepsilon}_0 \tau^{\beta}, \quad \underset{k-1}{\overset{(2-\delta)}{\mathcal{E}}}(\tau) \lesssim \hat{\varepsilon}_0 \tau^{\beta},$$
$$\underset{k-1}{\overset{(0)}{\mathcal{E}}}(\tau) \lesssim \hat{\varepsilon}_0 \tau^{-1+\beta}, \quad \underset{k-2}{\overset{(1)}{\mathcal{E}}}(\tau) \lesssim \hat{\varepsilon}_0 \tau^{-1+\delta+\beta}, \quad \underset{k-3}{\overset{(0)}{\mathcal{E}}}(\tau_0) \lesssim \hat{\varepsilon}_0 \tau^{-2+\delta+\beta}.$$

These bounds are of course weaker than those of (6-62)–(6-67). The τ^{β} factor terms may be, however, removed a posteriori, as in the last paragraph of Section 6.2.4, by revisiting the estimates globally on $\mathcal{R}(\tau_0, \tau)$, controlling the nonlinear error bulk integrals by redecomposing into dyadic time slabs, applying the estimates already proven, and summing.

Thus this proof in the end yields finally the same estimates as before, but where (6-62) is replaced by

$$\overset{(0)}{\mathcal{E}} \lesssim \hat{\varepsilon}_0, \quad \underset{k-2}{\overset{(1)}{\mathcal{E}}}(\tau) \lesssim \hat{\varepsilon}_0. \tag{6-69}$$

Remark 6.2.8. In view of Remark 6.2.7, this proof in particular applies to the semilinear case with the relaxed assumptions of Remark 4.8.4, giving the result for case (ii) as stated in Remark 5.2.

Remark 6.2.9. The disadvantage of this alternative proof over the proof given in Section 6.2.4 is that, to obtain the fundamental top-order orbital stability statement (6-69), one must revisit the estimates globally. We thus believe that the proof in Section 6.2.4 better reflects the dyadically localised philosophy of our method.

6.3. ***Case* (iii).** We now turn to the most general case we consider, case (iii), where we do not assume that (3-3) follows from a physical-space identity (3-15) but only assume (3-3) as a black box, together with the weaker physical-space identity described in Section 3.4.3.

This case is slightly more involved because we must combine the estimate originating from Section 3.4.3 with the estimate originating in Section 3.2. This alters a bit the numerology of the number of derivatives we must take, but the basic scheme is the same as case (ii).

6.3.1. *The hierarchy of inequalities.*

Proposition 6.3.1. *Let k be sufficiently large, and let us assume the case* (iii) *assumptions. There exist constants $C > 0$, $c > 0$ and an $\varepsilon_{\text{boot}} > 0$ small enough that the following is true*:

Consider a region $\mathcal{R}(\tau_0, \tau_1)$ and a ψ solving (4-1) *on $\mathcal{R}(\tau_0, \tau_1)$ and satisfying moreover* (4-23) *for $p = 0$ and $0 < \varepsilon \le \varepsilon_{\text{boot}}$. Let us assume moreover that*

$$\tau_1 \le \tau_0 + L$$

for some arbitrary $L > 0$. We have the following hierarchy of inequalities:

$$\overset{(2-\delta)}{\underset{k}{\mathfrak{F}}}(v, \tau_1), \quad \overset{(2-\delta)}{\underset{k}{\mathfrak{E}}}(\tau_1), \quad c\,{}^{\rho}\overset{(2-\delta)}{\underset{k}{\mathcal{X}}} \le \overset{(2-\delta)}{\underset{k}{\mathfrak{E}}}(\tau_0) + AC\,{}^{\chi}\overset{(0)}{\underset{k-1}{\mathcal{X}}} + C\Big({}^{\rho}\overset{(2-\delta)}{\underset{k}{\mathcal{X}}}\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}} + \sqrt{{}^{\rho}\overset{(2-\delta)}{\underset{k}{\mathcal{X}}}}\sqrt{{}^{\rho}\overset{(0)}{\underset{k}{\mathcal{X}}}}\sqrt{\overset{(2-\delta)}{\underset{\ll k}{\mathcal{X}}}}\Big) + C\,{}^{\rho}\overset{(0)}{\underset{k}{\mathcal{X}}}\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}}\sqrt{L}, \tag{6-70}$$

$${}^{\chi}\overset{(2-\delta)}{\underset{k-1}{\mathcal{X}}} \lesssim \overset{(2-\delta)}{\underset{k-1}{\mathcal{E}}}(\tau_0) + {}^{\rho}\overset{(2-\delta)}{\underset{k-1}{\mathcal{X}}}\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}} + \sqrt{{}^{\rho}\overset{(2-\delta)}{\underset{k-1}{\mathcal{X}}}}\sqrt{{}^{\rho}\overset{(0)}{\underset{k-1}{\mathcal{X}}}}\sqrt{\overset{(2-\delta)}{\underset{\ll k}{\mathcal{X}}}} + {}^{\rho}\overset{(0)}{\underset{k}{\mathcal{X}}}\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}}\sqrt{L}, \tag{6-71}$$

$$\overset{(1)}{\underset{k}{\mathfrak{F}}}(v, \tau_1), \quad \overset{(1)}{\underset{k}{\mathfrak{E}}}(\tau_1), \quad c\,{}^{\rho}\overset{(1)}{\underset{k}{\mathcal{X}}} \le \overset{(1)}{\underset{k}{\mathfrak{E}}}(\tau_0) + AC\,{}^{\chi}\overset{(0)}{\underset{k-1}{\mathcal{X}}} + C\,{}^{\rho}\overset{(1)}{\underset{k}{\mathcal{X}}}\sqrt{\overset{(1)}{\underset{\ll k}{\mathcal{X}}}} + C\,{}^{\rho}\overset{(0)}{\underset{k}{\mathcal{X}}}\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}}\sqrt{L}, \tag{6-72}$$

$${}^{\chi}\overset{(1)}{\underset{k-1}{\mathcal{X}}} \lesssim \overset{(1)}{\underset{k-1}{\mathcal{E}}}(\tau_0) + C\,{}^{\rho}\overset{(1)}{\underset{k-1}{\mathcal{X}}}\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}} + C\,{}^{\rho}\overset{(0)}{\underset{k}{\mathcal{X}}}\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}}\sqrt{L}, \tag{6-73}$$

$$\begin{aligned}\overset{(0)}{\underset{k}{\mathfrak{F}}}(v, \tau_1), \quad \overset{(0)}{\underset{k}{\mathfrak{E}}}(\tau_1), \quad c\,{}^{\rho}\overset{(0)}{\underset{k}{\mathcal{X}}} \le \overset{(0)}{\underset{k}{\mathfrak{E}}}(\tau_0) + AC\,{}^{\chi}\overset{(0)}{\underset{k-1}{\mathcal{X}}} + C\Big({}^{\rho}\overset{(0)}{\underset{k}{\mathcal{X}}} + \big({}^{\rho}\overset{(0)}{\underset{k}{\mathcal{X}}}\big)^{\frac{1-\delta}{1+\delta}}\big({}^{\rho}\overset{(1)}{\underset{k}{\mathcal{X}}}\big)^{\frac{2\delta}{1+\delta}}\Big)\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}} + \big(\overset{(0)}{\underset{\ll k}{\mathcal{X}}}\big)^{\frac{1-\delta}{1+\delta}}\big(\overset{(1)}{\underset{\ll k}{\mathcal{X}}}\big)^{\frac{2\delta}{1+\delta}}} \\ + C\,{}^{\rho}\overset{(0)}{\underset{k}{\mathcal{X}}}\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}}\sqrt{L},\end{aligned} \tag{6-74}$$

$$\begin{aligned}{}^{\chi}\overset{(0)}{\underset{k-1}{\mathcal{X}}} \lesssim \overset{(0)}{\underset{k-1}{\mathcal{E}}}(\tau_0) + \Big({}^{\rho}\overset{(0)}{\underset{k-1}{\mathcal{X}}} + \big({}^{\rho}\overset{(0)}{\underset{k-1}{\mathcal{X}}}\big)^{\frac{1-\delta}{1+\delta}}\big({}^{\rho}\overset{(1)}{\underset{k-1}{\mathcal{X}}}\big)^{\frac{2\delta}{1+\delta}}\Big)\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}} + \big(\overset{(0)}{\underset{\ll k}{\mathcal{X}}}\big)^{\frac{1-\delta}{1+\delta}}\big(\overset{(1)}{\underset{\ll k}{\mathcal{X}}}\big)^{\frac{2\delta}{1+\delta}}} \\ + {}^{\rho}\overset{(0)}{\underset{k}{\mathcal{X}}}\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}}\sqrt{L}.\end{aligned} \tag{6-75}$$

Proof. Again we recall that if $\varepsilon_{\text{boot}} \le \varepsilon_{\text{loc}}$, then the assumption of Proposition 4.8.1 holds. The proposition then follows from Proposition 4.8.1 as in the proof of Proposition 6.2.1, where we have used also the crucial estimate

$$A\int_{\tau_0}^{\tau_1} \overset{\xi}{\underset{k-1}{\mathcal{E}}}{}'(\tau')\,d\tau' \lesssim AC\,{}^{\chi}\overset{(0)}{\underset{k-1}{\mathcal{X}}},$$

which follows from the fundamental relation (3-100). □

6.3.2. *Global existence in L-slabs.* To show global existence in L-slabs, we require a minor modification of the assumptions of Proposition 6.2.3.

Proposition 6.3.2. *Let $k - 2 \ge k_{\text{loc}}$ be sufficiently large, and let us assume the case* (iii) *assumptions. Then there exists a positive constant $\varepsilon_{\text{slab}} \le \varepsilon_{\text{loc}}$ and a constant $C > 0$ implicit in the sign $\lesssim$ below such that the following is true*:

Given arbitrary $L \ge 1$, $\tau_0 \ge 0$, $0 < \varepsilon_0 \le \varepsilon_{\text{slab}}$ and initial data (ψ, ψ') on $\Sigma(\tau_0)$ as in Proposition 4.9.1 satisfying moreover

$$\overset{(1)}{\underset{k}{\mathcal{E}}}(\tau_0) \le \varepsilon_0, \quad \overset{(0)}{\underset{k-2}{\mathcal{E}}}(\tau_0) \le \varepsilon_0 L^{-1}, \tag{6-76}$$

we have that the unique solution of Proposition 4.9.1 achieving the data can be extended to a ψ defined on the entire spacetime slab $\mathcal{R}(\tau_0, \tau_0+L)$ satisfying (4-1) and the estimates

$$ {}^{\rho}\overset{(1)}{\underset{k}{\mathcal{X}}} + {}^{\chi}\overset{(1)}{\underset{k-1}{\mathcal{X}}} \lesssim \varepsilon_0, \quad {}^{\rho}\overset{(0)}{\underset{k-2}{\mathcal{X}}} + {}^{\chi}\overset{(0)}{\underset{k-3}{\mathcal{X}}} \lesssim \varepsilon_0 L^{-1}. \tag{6-77} $$

Proof. Consider the set $\mathfrak{B} \subset (\tau_0, \tau_0+L]$ consisting of all $\tau_0+L \geq \tau_f \geq \tau_0$ such that a solution ψ of (4-1) achieving the data exists on $\mathcal{R}(\tau_0, \tau_f)$ and such that the bootstrap assumption (4-23) (with $p=1$) and also the additional bootstrap assumption

$$ \overset{(0)}{\underset{\ll k}{\mathcal{X}}} \leq \varepsilon L^{-1} \tag{6-78} $$

hold in $\mathcal{R}(\tau_0, \tau_1 := \tau_f)$, where $0 < \varepsilon \leq \varepsilon_{\text{boot}}$ is a small constant satisfying

$$ 1 \gg \varepsilon \gg \varepsilon_{\text{slab}}. \tag{6-79} $$

(The above relation in particular already constrains $\varepsilon_{\text{slab}}$ to be sufficiently small.)

By the local well-posedness statement Proposition 4.9.1, it follows that, since $k-2 \geq k_{\text{loc}}$ and $\varepsilon_0 \leq \varepsilon_{\text{slab}} \leq \varepsilon_{\text{loc}}$, we have $\tau_0 + \tau_{\text{exist}} \subset \mathfrak{B}$ and thus $\mathfrak{B} \neq \varnothing$, provided that ε satisfies (6-79). Also note that, a fortiori, if $\tau_f \in \mathfrak{B}$, then $(\tau_0, \tau_f] \in \mathfrak{B}$ and thus $\mathfrak{B}$ is manifestly a connected subset of (τ_0, ∞).

Proposition 6.3.1 holds for $\mathcal{R}(\tau_0, \tau_1)$ with $\tau_1 := \tau_f$ for any $\tau_f \in \mathfrak{B}$. Adding the equations (6-72)–(6-75) pairwise with a suitable constant so as to absorb the term multiplying A, with $k-2$ replacing k for the latter pair, we obtain the system

$$ {}^{\rho}\overset{(1)}{\underset{k}{\mathcal{X}}} + {}^{\chi}\overset{(1)}{\underset{k-1}{\mathcal{X}}} \lesssim \overset{(1)}{\underset{k}{\mathfrak{E}}}(\tau_0) + {}^{\rho}\overset{(1)}{\underset{k}{\mathcal{X}}}\sqrt{\overset{(1)}{\underset{\ll k}{\mathcal{X}}}} + {}^{\rho}\overset{(0)}{\underset{k}{\mathcal{X}}}\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}}\sqrt{L}, \tag{6-80} $$

$$ {}^{\rho}\overset{(0)}{\underset{k-2}{\mathcal{X}}} + {}^{\chi}\overset{(0)}{\underset{k-3}{\mathcal{X}}} \lesssim \overset{(0)}{\underset{k-2}{\mathfrak{E}}}(\tau_0) + \Big({}^{\rho}\overset{(0)}{\underset{k-2}{\mathcal{X}}} + \big({}^{\rho}\overset{(0)}{\underset{k-2}{\mathcal{X}}}\big)^{\frac{1-\delta}{1+\delta}}\big({}^{\rho}\overset{(1)}{\underset{k-2}{\mathcal{X}}}\big)^{\frac{2\delta}{1+\delta}}\Big)\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}} + \big(\overset{(0)}{\underset{\ll k}{\mathcal{X}}}\big)^{\frac{1-\delta}{1+\delta}}\big(\overset{(1)}{\underset{\ll k}{\mathcal{X}}}\big)^{\frac{2\delta}{1+\delta}}} + {}^{\rho}\overset{(0)}{\underset{k-2}{\mathcal{X}}}\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}}\sqrt{L}. \tag{6-81} $$

We now obtain

$$ {}^{\rho}\overset{(1)}{\underset{k}{\mathcal{X}}} + {}^{\chi}\overset{(1)}{\underset{k-1}{\mathcal{X}}} \lesssim \varepsilon_0, \quad {}^{\rho}\overset{(0)}{\underset{k-2}{\mathcal{X}}} + {}^{\chi}\overset{(0)}{\underset{k-3}{\mathcal{X}}} \lesssim \varepsilon_0 L^{-1}. \tag{6-82} $$

We may now already apply our continuation criterion Corollary 4.9.2, applied with $p=0$ and $k-2$, to assert the existence of an ϵ, independent of τ_1, such that, now defining $\tau_1 := \min\{\tau_f + \epsilon, \tau_0 + L\}$, the solution ψ extends to a smooth solution of (4-1) on $\mathcal{R}(\tau_0, \tau_1)$ and, moreover, from (4-41), that

$$ {}^{\chi}\overset{(0)}{\underset{k-3}{\mathcal{X}}} + {}^{\rho}\overset{(0)}{\underset{k-2}{\mathcal{X}}} \lesssim \varepsilon_0 L^{-1} \tag{6-83} $$

holds on $\mathcal{R}(\tau_0, \tau_1)$ and, from (4-42), that

$$ {}^{\chi}\overset{(1)}{\underset{k}{\mathcal{X}}} \lesssim \varepsilon_0 \tag{6-84} $$

holds on $\mathcal{R}(\tau_0, \tau_1)$. It follows that (6-78) and (4-23) (with $p=1$) hold on $\mathcal{R}(\tau_0, \tau_1)$, and we thus have $\tau_1 = \min\{\tau_f + \epsilon, \tau_0 + L\} \in \mathfrak{B}$.

Since ϵ is independent of τ_f, and in view also of the connectivity of $\mathfrak{B}$, it follows that $\mathfrak{B}$ is a nonempty open and closed subset of $(\tau_0, \tau_0+L]$ and thus $\mathfrak{B} = (\tau_0, \tau_0+L]$, and hence the solution exists in the entire spacetime slab $\mathcal{R}(\tau_0, \tau_0+L)$.

The estimates (6-82) thus hold in the entire spacetime slab. This gives (6-77). □

6.3.3. *The pigeonhole argument.* The above assumptions on initial data are sufficient for global existence in the slab but are not sufficient to iterate. For this we shall need strengthened assumptions.

Proposition 6.3.3. *Under the assumptions of Proposition 6.3.2, there exists a constant $C > 0$, implicit in the inequalities $\lesssim$ below, a parameter $\alpha_0 \gg 1$ and, for all $\alpha \geq \alpha_0$, a parameter $\hat{\varepsilon}_{\text{slab}}(\alpha)$ such that, for all $0 < \hat{\varepsilon}_0 \leq \hat{\varepsilon}_{\text{slab}}(\alpha)$ the following holds*:

Let us assume in addition to (6-76) that we have

$$\overset{(1)}{\underset{k}{\mathfrak{E}}}(\tau_0) \leq \hat{\varepsilon}_0, \quad \overset{(2-\delta)}{\underset{k-2}{\mathfrak{E}}}(\tau_0) \leq \hat{\varepsilon}_0, \quad \overset{(0)}{\underset{k-2}{\mathfrak{E}}}(\tau_0) \leq \hat{\varepsilon}_0\alpha L^{-1}, \quad \overset{(1)}{\underset{k-4}{\mathfrak{E}}}(\tau_0) \leq \hat{\varepsilon}_0\alpha L^{-1+\delta}, \quad \overset{(0)}{\underset{k-6}{\mathfrak{E}}}(\tau_0) \leq \hat{\varepsilon}_0\alpha^2 L^{-2+\delta}. \tag{6-85}$$

Then the solution ψ of (4-1) on $\mathcal{R}(\tau_0, \tau_0 + L)$ given by Proposition 6.2.3 satisfies the additional estimates

$$\sup_{\tau_0 \leq \tau \leq \tau_0 + L} \overset{(1)}{\underset{k}{\mathcal{E}}}(\tau) \leq \alpha\hat{\varepsilon}_0, \tag{6-86}$$

$$\sup_{\tau_0 \leq \tau \leq \tau_0 + L} \overset{(1)}{\underset{k}{\mathfrak{E}}}(\tau) \leq \hat{\varepsilon}_0(1 + \alpha L^{-\frac{1}{4}}), \tag{6-87}$$

$$\sup_{\tau_0 \leq \tau \leq \tau_0 + L} \overset{(2-\delta)}{\underset{k-2}{\mathfrak{E}}}(\tau) \leq \hat{\varepsilon}_0(1 + \alpha L^{-\frac{1}{4}}), \tag{6-88}$$

$${}^{\rho}\overset{(1)}{\underset{k}{\mathcal{X}}} + {}^{\chi}\overset{(1)}{\underset{k-1}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0, \quad {}^{\rho}\overset{(2-\delta)}{\underset{k-2}{\mathcal{X}}} + {}^{\chi}\overset{(2-\delta)}{\underset{k-3}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0, \quad {}^{\rho}\overset{(0)}{\underset{k-2}{\mathcal{X}}} + {}^{\chi}\overset{(0)}{\underset{k-3}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0\alpha L^{-1}, \quad {}^{\rho}\overset{(1)}{\underset{k-4}{\mathcal{X}}} + {}^{\chi}\overset{(1)}{\underset{k-5}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0\alpha L^{-1+\delta}, \tag{6-89}$$

$${}^{\rho}\overset{(0)}{\underset{k-6}{\mathcal{X}}} + {}^{\chi}\overset{(0)}{\underset{k-7}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0\alpha^2 L^{-2+\delta}. \tag{6-90}$$

Moreover, for all times τ' with $L \geq \tau' - \tau_0 \geq \frac{1}{2}L$, we have that

$$\overset{(0)}{\underset{k-2}{\mathfrak{E}}}(\tau') \leq \tfrac{1}{2}\hat{\varepsilon}_0\alpha L^{-1}, \tag{6-91}$$

$$\overset{(1)}{\underset{k-4}{\mathfrak{E}}}(\tau') \leq \tfrac{1}{2}\hat{\varepsilon}_0\alpha L^{-1+\delta}, \tag{6-92}$$

$$\overset{(0)}{\underset{k-6}{\mathfrak{E}}}(\tau') \leq \tfrac{1}{4}\hat{\varepsilon}_0\alpha^2 L^{-2+\delta}. \tag{6-93}$$

Proof. The proof follows closely that of Proposition 6.2.5.

Note again that, by the statement of the proposition, we are in particular also assuming a priori inequalities (6-76) for *some* $\varepsilon_0 \leq \varepsilon_{\text{slab}}$ since this is included in the assumptions of Proposition 6.3.2. In view now of Corollary 4.5.3, for α_0 sufficiently large, if say $\hat{\varepsilon}_{\text{slab}}(\alpha) \ll \alpha^{-3}\varepsilon_{\text{slab}}$, it follows from the additional assumptions (6-85) that the estimates (6-76) in fact hold with the specific constant $\varepsilon_0 := \hat{\varepsilon}_0\alpha^3$ for all $\alpha \geq \alpha_0$.

We now revisit (6-70)–(6-75) and add again pairwise. We now obtain that

$${}^{\rho}\overset{(1)}{\underset{k}{\mathcal{X}}} + {}^{\chi}\overset{(1)}{\underset{k-1}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0, \quad {}^{\rho}\overset{(2-\delta)}{\underset{k-2}{\mathcal{X}}} + {}^{\chi}\overset{(2-\delta)}{\underset{k-3}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0, \quad {}^{\rho}\overset{(1)}{\underset{k-4}{\mathcal{X}}} + {}^{\chi}\overset{(1)}{\underset{k-5}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0\alpha L^{-1+\delta}, \tag{6-94}$$

$${}^{\rho}\overset{(0)}{\underset{k-2}{\mathcal{X}}} + {}^{\chi}\overset{(0)}{\underset{k-3}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0\alpha L^{-1} + \Big(\big({}^{\rho}\overset{(0)}{\underset{k-2}{\mathcal{X}}}\big)^{\frac{1-\delta}{1+\delta}}\big({}^{\rho}\overset{(1)}{\underset{k-2}{\mathcal{X}}}\big)^{\frac{2\delta}{1+\delta}}\Big)\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}} + \big(\overset{(0)}{\underset{\ll k}{\mathcal{X}}}\big)^{\frac{1-\delta}{1+\delta}}\big(\overset{(1)}{\underset{\ll k}{\mathcal{X}}}\big)^{\frac{2\delta}{1+\delta}}}, \tag{6-95}$$

$${}^{\rho}\overset{(0)}{\underset{k-6}{\mathcal{X}}} + {}^{\chi}\overset{(0)}{\underset{k-7}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0\alpha^2 L^{-2+\delta} + \hat{\varepsilon}_0\varepsilon_0^{\frac{1}{2}} L^{-\frac{3}{2}\frac{(1-\delta)(\gamma+2\delta)}{1+\delta}}. \tag{6-96}$$

The inequalities (6-94) give the first, second, and fourth inequality of (6-89).

Note that $-2+\delta < -\frac{3}{2}+\delta$. We may however improve (6-96) iteratively as follows: If

$$^{\rho}\overset{(0)}{\underset{k-6}{\mathcal{X}}} + {}^{\chi}\overset{(0)}{\underset{k-7}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0 \alpha^2 L^{-\gamma}$$

for $\gamma \leq 2-\delta$, then

$$\overset{(0)}{\underset{\ll k}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0 \alpha^2 L^{-\gamma} \lesssim \varepsilon_0 L^{-\gamma};$$

hence, plugging this again into (6-81), we obtain

$$^{\rho}\overset{(0)}{\underset{k-6}{\mathcal{X}}} + {}^{\chi}\overset{(0)}{\underset{k-7}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0 \alpha^2 L^{-2+\delta} + \hat{\varepsilon}_0 \varepsilon_0^{\frac{1}{2}} L^{-\frac{3}{2}\frac{(1-\delta)(\gamma+2\delta)}{1+\delta}}. \tag{6-97}$$

Setting $\gamma_0 = \frac{3}{2}\frac{(1-\delta)(1+2\delta)}{1+\delta}$ and defining inductively $\gamma_i = \frac{3}{2}\frac{(1-\delta)(\gamma_{i-1}+2\delta)}{1+\delta}$, we have that there exists a first $i \geq 1$ such that

$$\frac{3}{2}\frac{(1-\delta)(\gamma_i+2\delta)}{1+\delta} > 2-\delta.$$

It follows that (6-97) holds for $\gamma = \gamma_i$; hence

$$^{\rho}\overset{(0)}{\underset{k-6}{\mathcal{X}}} + {}^{\chi}\overset{(0)}{\underset{k-7}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0 \alpha^2 L^{-2+\delta}.$$

This yields (6-90). Note that this implies

$$\overset{(0)}{\underset{\ll k}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0 \alpha^2 L^{(-2+2\delta)z} \lesssim \varepsilon_0 L^{-2+\delta}. \tag{6-98}$$

On the other hand, the fourth inequality of (6-89) implies

$$\overset{(1)}{\underset{\ll k}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0 \alpha L^{-1+\delta} \lesssim \varepsilon_0 L^{-1+\delta}. \tag{6-99}$$

We may now infer the third inequality of (6-89) by plugging the derived bounds into (6-95). This concludes the proof of (6-89).

To show (6-91)–(6-93), we first apply the pigeonhole principle as in [Dafermos and Rodnianski 2010b] to the inequality

$$\int_{\tau_0}^{\tau_0+L} \big(\overset{(0)}{\underset{k-2}{\mathcal{E}'}}(\tau') + \overset{(1-\delta)}{\underset{k-4}{\mathcal{E}'}}(\tau') + \alpha^{-1} L^{1-\delta} \overset{(0)}{\underset{k-6}{\mathcal{E}'}}(\tau') \big)\, d\tau' \lesssim \hat{\varepsilon}_0,$$

which, upon addition, follows from the inequalities of the estimate (6-89) already shown. Recalling from (3-93) that we have

$$\overset{(0)}{\underset{k-2}{\mathcal{E}'}} \gtrsim \overset{(0)}{\underset{k-2}{\mathcal{E}}}, \quad \overset{(1-\delta)}{\underset{k-4}{\mathcal{E}'}} \gtrsim \overset{(1-\delta)}{\underset{k-4}{\mathcal{E}}}, \quad \overset{(0)}{\underset{k-6}{\mathcal{E}'}} \gtrsim \overset{(0)}{\underset{k-6}{\mathcal{E}}},$$

we obtain that there exists $\tau'' \in \big[\tau_0, \tau_0 + \frac{1}{2}L\big]$, whose precise value depends on the solution, such that

$$\overset{(0)}{\underset{k-2}{\mathcal{E}}}(\tau'') \lesssim \hat{\varepsilon}_0 \cdot L^{-1}, \tag{6-100}$$

$$\overset{(1-\delta)}{\underset{k-4}{\mathcal{E}}}(\tau'') \lesssim \hat{\varepsilon}_0 \cdot L^{-1}, \tag{6-101}$$

$$\overset{(0)}{\underset{k-6}{\mathcal{E}}}(\tau'') \lesssim \hat{\varepsilon}_0 \alpha L^{-1+\delta} \cdot L^{-1}. \tag{6-102}$$

Now in view of the interpolation estimate (3-113) of Proposition 3.6.12, we have

$$\overset{(1)}{\underset{k-4}{\mathcal{E}}}(\tau'') \lesssim \big(\overset{(1-\delta)}{\underset{k-4}{\mathcal{E}}}(\tau'')\big)^{1-\delta} \big(\overset{(2-\delta)}{\underset{k-4}{\mathcal{E}}}(\tau'')\big)^{\delta} \leq \big(\overset{(1-\delta)}{\underset{k-4}{\mathcal{E}}}(\tau'')\big)^{1-\delta} \big(\sup_{\tau_0 \leq \tau \leq \tau_0+L} \overset{(2-\delta)}{\underset{k-4}{\mathcal{E}}}(\tau)\big)^{\delta} \tag{6-103}$$

and thus

$$\underset{k-4}{\overset{(1)}{\mathcal{E}}}(\tau'') \lesssim \hat{\varepsilon}_0 L^{-1+\delta}, \tag{6-104}$$

where we have used (6-101) and the estimate for the second factor on the right-hand side of (6-103) contained in the second inequality of (6-89).

Now we apply (6-80) and (6-81) again, with τ'' in place of τ_0, using (6-100)–(6-102) to bound the initial data, to obtain that, for all $\tau_0 + L \geq \tau' \geq \tau_0 + \frac{1}{2}L$, we have

$$\underset{k-2}{\overset{(1)}{\mathfrak{E}}}(\tau') \sim \underset{k-2}{\overset{(1)}{\mathcal{E}}}(\tau) \lesssim \hat{\varepsilon}_0 L^{-1}, \tag{6-105}$$

$$\underset{k-4}{\overset{(1)}{\mathfrak{E}}}(\tau') \sim \underset{k-4}{\overset{(1)}{\mathcal{E}}}(\tau) \lesssim \hat{\varepsilon}_0 L^{-1+\delta}, \tag{6-106}$$

$$\underset{k-6}{\overset{(0)}{\mathfrak{E}}}(\tau') \sim \underset{k-6}{\overset{(0)}{\mathcal{E}}}(\tau) \lesssim \hat{\varepsilon}_0 \alpha L^{-2+\delta}. \tag{6-107}$$

Thus, in view of the requirement $\alpha \geq \alpha_0 \gg 1$, for sufficiently large α_0 we may absorb the constants implicit in $\lesssim$ by explicit constants of our choice by adding extra positive α powers to the right-hand side of (6-106) and (6-107). In this way, we obtain the specific estimates (6-91), (6-92) and (6-93). In the same way, we also obtain the specific constant of the estimate of (6-86) which will be convenient in our scheme.

We finally turn to (6-87) and (6-88). Let us first note that revisiting (6-75), with a little bit of averaging, we have the bounds

$$\begin{aligned}
{}^{\chi}\underset{k-1}{\overset{(0)}{\mathcal{X}}}\big(\tau_0 + \tfrac{1}{2}, \tau_0 + L\big) &\lesssim \int_{\tau_0}^{\tau_0+1/2} \underset{k-1}{\overset{(0)}{\mathcal{E}}}(\tau)\, d\tau + \Big({}^{\rho}\underset{k-1}{\overset{(0)}{\mathcal{X}}} + \big({}^{\rho}\underset{k-1}{\overset{(0)}{\mathcal{X}}}\big)^{\frac{1-\delta}{1+\delta}} \big({}^{\rho}\underset{k-1}{\overset{(1)}{\mathcal{X}}}\big)^{\frac{2\delta}{1+\delta}}\Big) \sqrt{\underset{\ll k}{\overset{(0)}{\mathcal{X}}} + \big(\underset{\ll k}{\overset{(0)}{\mathcal{X}}}\big)^{\frac{1-\delta}{1+\delta}} \big(\underset{\ll k}{\overset{(1)}{\mathcal{X}}}\big)^{\frac{2\delta}{1+\delta}}} + {}^{\rho}\underset{k}{\overset{(0)}{\mathcal{X}}} \sqrt{\underset{\ll k}{\overset{(0)}{\mathcal{X}}}} \sqrt{L} \\
&\lesssim \sqrt{\int_{\tau_0}^{\tau_0+1/2} \overset{(0)}{\mathcal{E}}(\tau)\, d\tau} \sqrt{\int_{\tau_0}^{\tau_0+1/2} \underset{k-2}{\overset{(0)}{\mathcal{E}}}(\tau)\, d\tau} + \hat{\varepsilon}_0 \varepsilon^{\frac{1}{2}} L^{-\frac{1}{4}} \\
&\lesssim \hat{\varepsilon}_0 \alpha^{\frac{1}{2}} L^{-\frac{1}{2}} + \hat{\varepsilon}_0 \varepsilon^{\frac{1}{2}} L^{-\frac{1}{4}} \\
&\lesssim \hat{\varepsilon}_0 \alpha^{\frac{1}{2}} L^{-\frac{1}{4}},
\end{aligned}$$

where we have used also the interpolation inequality (3-115) $\big($and that $L \geq \frac{1}{2}\big)$.

We now notice that the above bound clearly holds also for

$${}^{\chi}\underset{k-1}{\overset{(0)}{\mathcal{X}}}{}' := {}^{\chi}\underset{k-1}{\overset{(0)}{\mathcal{X}}}\big(\tau_0 + \tfrac{1}{2}, \tau_0 + L\big) + \int_{\tau_0}^{\tau_0+1/2} \underset{k-1}{\overset{(0)}{\mathcal{E}}}(\tau)\, d\tau.$$

For (6-87), we revisit (6-72), noting that we may replace ${}^{\chi}\underset{k-1}{\overset{(0)}{\mathcal{X}}}$ by ${}^{\chi}\underset{k-1}{\overset{(0)}{\mathcal{X}}}{}'$ on the right-hand side, and thus we have the bound

$$\begin{aligned}
\sup_{\tau \in [\tau_0, \tau_0 + L]} \underset{k}{\overset{(1)}{\mathfrak{E}}}(\tau) &\leq \underset{k}{\overset{(1)}{\mathfrak{E}}}(\tau_0) + AC\, {}^{\chi}\underset{k-1}{\overset{(0)}{\mathcal{X}}}{}' + C\, {}^{\rho}\underset{k}{\overset{(1)}{\mathcal{X}}} \sqrt{\underset{\ll k}{\overset{(1)}{\mathcal{X}}}} + C\, {}^{\rho}\underset{k}{\overset{(0)}{\mathcal{X}}} \sqrt{\underset{\ll k}{\overset{(0)}{\mathcal{X}}}} \sqrt{L} \\
&\leq \hat{\varepsilon}_0 + AC(\hat{\varepsilon}_0 \alpha^{\frac{1}{2}} L^{-\frac{1}{4}}) + C \hat{\varepsilon}_0 \varepsilon_0^{\frac{1}{2}} L^{\frac{-1+\delta}{2}} + C \hat{\varepsilon}_0 \varepsilon_0^{\frac{1}{2}} L^{-\frac{3}{2}+\delta} \\
&\leq \hat{\varepsilon}_0 + \alpha \hat{\varepsilon}_0 L^{-\frac{1}{4}},
\end{aligned}$$

and we have used a higher power of α to absorb constants.

Finally, for (6-88), we revisit (6-75) with $k-2$ replacing $k-1$ to obtain

$$ {}^{\chi}\underset{k-2}{\overset{(0)}{\mathcal{X}}} \lesssim \underset{k-2}{\overset{(0)}{\mathcal{E}}}(\tau_0) + \Big({}^{\rho}\underset{k-2}{\overset{(0)}{\mathcal{X}}} + \big({}^{\rho}\underset{k-2}{\overset{(0)}{\mathcal{X}}}\big)^{\frac{1-\delta}{1+\delta}}\big({}^{\rho}\underset{k-2}{\overset{(1)}{\mathcal{X}}}\big)^{\frac{2\delta}{1+\delta}}\Big)\sqrt{\underset{\ll k}{\overset{(0)}{\mathcal{X}}} + \big(\underset{\ll k}{\overset{(0)}{\mathcal{X}}}\big)^{\frac{1-\delta}{1+\delta}}\big(\underset{\ll k}{\overset{(1)}{\mathcal{X}}}\big)^{\frac{2\delta}{1+\delta}}} + {}^{\rho}\underset{k-1}{\overset{(0)}{\mathcal{X}}}\sqrt{\underset{\ll k}{\overset{(0)}{\mathcal{X}}}}\sqrt{L} \lesssim \hat{\varepsilon}_0 \alpha L^{-1}, $$

whence we have

$$ \begin{aligned} \sup_{\tau\in[\tau_0,\tau_0+L]} \underset{k-1}{\overset{(2-\delta)}{\mathfrak{E}}}(\tau) &\le \underset{k-1}{\overset{(2-\delta)}{\mathfrak{E}}}(\tau_0) + A\, {}^{\chi}\underset{k-2}{\overset{(0)}{\mathcal{X}}} + C\Big({}^{\rho}\underset{k-1}{\overset{(2-\delta)}{\mathcal{X}}}\sqrt{\underset{\ll k}{\overset{(0)}{\mathcal{X}}}} + \sqrt{{}^{\rho}\underset{k-1}{\overset{(2-\delta)}{\mathcal{X}}}}\sqrt{{}^{\rho}\underset{k-1}{\overset{(0)}{\mathcal{X}}}}\sqrt{\underset{\ll k}{\overset{(2-\delta)}{\mathcal{X}}}}\Big) + C\,{}^{\rho}\underset{k-1}{\overset{(0)}{\mathcal{X}}}\sqrt{\underset{\ll k}{\overset{(0)}{\mathcal{X}}}}\sqrt{L} \\ &\le \hat{\varepsilon}_0 + A\hat{\varepsilon}_0\alpha^{\frac{1}{2}}L^{-1} + C\hat{\varepsilon}_0\varepsilon_0^{\frac{1}{2}}L^{\frac{-2+\delta}{2}} + C\hat{\varepsilon}_0\varepsilon_0^{\frac{1}{2}}L^{\frac{-2+\delta}{2}}L^{\frac{1}{2}} \\ &\le \hat{\varepsilon}_0 + \alpha\hat{\varepsilon}_0 L^{-\frac{1}{4}}, \end{aligned} $$

where we have again used a higher power of α to absorb all other constants. □

6.3.4. *The iteration: proof of Theorem 5.1 in case* (iii). We may now prove Theorem 5.1 in case (iii).

As in case (ii), we shall proceed iteratively. The proof is a simple modification of that of Section 6.2.4. We define

$$ \tau_0 = 1, \quad L_0 = 1, \quad L_i = 2^i, \quad \tau_{i+1} = \tau_i + L_i $$

and fix $\alpha \ge \alpha_0$ so that the statement of Proposition 6.3.3 holds; for instance, set $\alpha := \alpha_0$. Define the parameter

$$ d := \prod_{i=1}^{\infty}(1 + 2^{-\frac{i}{4}}\alpha) < \infty. \tag{6-108} $$

(Note that we shall no longer note the dependence of constants and parameters on α since it is now considered fixed. Thus implicit constants depending on the choice of α will from now on be incorporated in the notations $\sim$ and $\lesssim$.)

For $0 < \varepsilon_0 \le \varepsilon_{\text{global}}$ and $\varepsilon_{\text{global}}$ sufficiently small, since

$$ \underset{k}{\overset{(1)}{\mathcal{E}}}(\tau_0) \le \varepsilon_0, \quad \underset{k-2}{\overset{(2-\delta)}{\mathcal{E}}}(\tau_0) \le \varepsilon_0, \tag{6-109} $$

in view of Corollary 4.5.3, we have

$$ \underset{k}{\overset{(1)}{\mathfrak{E}}}(\tau_0) \lesssim \varepsilon_0, \quad \underset{k-2}{\overset{(2-\delta)}{\mathfrak{E}}}(\tau_0) \lesssim \varepsilon_0. $$

Thus, for sufficiently small $\varepsilon_{\text{global}} \ll \hat{\varepsilon}_{\text{slab}}$ and all $0 < \varepsilon_0 \le \varepsilon_{\text{global}}$, if (6-109) is satisfied, it follows that there exists a $\hat{\varepsilon}_0(\varepsilon_0) \sim \varepsilon_0$ satisfying

$$ \alpha\hat{\varepsilon}_0 \le \frac{1}{d}\varepsilon_{\text{slab}}, \quad \hat{\varepsilon}_0 \le \frac{1}{d}\hat{\varepsilon}_{\text{slab}} \tag{6-110} $$

and

$$ \underset{k}{\overset{(1)}{\mathfrak{E}}}(\tau_0) \le \hat{\varepsilon}_0, \quad \underset{k-2}{\overset{(2-\delta)}{\mathfrak{E}}}(\tau_0) \le \hat{\varepsilon}_0, \quad \underset{k-2}{\overset{(0)}{\mathfrak{E}}}(\tau_0) \le \hat{\varepsilon}_0\alpha L_0^{-1}, \quad \underset{k-4}{\overset{(1)}{\mathfrak{E}}}(\tau_0) \le \hat{\varepsilon}_0\alpha L_0^{-1+\delta}, \quad \underset{k-6}{\overset{(0)}{\mathfrak{E}}}(\tau_0) \le \hat{\varepsilon}_0\alpha^2 L_0^{-2+\delta}. $$

Finally, with our restriction on the definition of ε_0, we may also write

$$ \underset{k}{\overset{(1)}{\mathcal{E}}}(\tau_0) \le \alpha\hat{\varepsilon}_0. $$

In general, given $\tau_i \geq 1$ defined above, $\hat{\varepsilon}_i \leq \hat{\varepsilon}_{\text{slab}}$, and a solution ψ of (4-1) on $\mathcal{R}(\tau_0, \tau_i)$ such that

$$\overset{(1)}{\underset{k}{\mathcal{E}}}(\tau_i) \leq \alpha \hat{\varepsilon}_i, \tag{6-111}$$

and

$$\overset{(1)}{\underset{k}{\mathfrak{E}}}(\tau_i) \leq \hat{\varepsilon}_i, \quad \overset{(2-\delta)}{\underset{k-2}{\mathfrak{E}}}(\tau_i) \leq \hat{\varepsilon}_i, \quad \overset{(0)}{\underset{k-2}{\mathfrak{E}}}(\tau_i) \leq \hat{\varepsilon}_i \alpha L_i^{-1}, \quad \overset{(1)}{\underset{k-4}{\mathfrak{E}}}(\tau_i) \leq \hat{\varepsilon}_i \alpha L_i^{-1+\delta}, \quad \overset{(0)}{\underset{k-6}{\mathfrak{E}}}(\tau_i) \leq \hat{\varepsilon}_i \alpha^2 L_i^{-2+\delta}, \tag{6-112}$$

we note that, by our restrictions on $\hat{\varepsilon}_{\text{slab}}$, the assumptions of Proposition 6.2.3 hold with $\alpha^2 \hat{\varepsilon}_i$ in place of ε_0 (here we are using (6-111) to invoke Corollary 4.5.3 to rewrite (6-60) in terms of the calligraphic energies; cf. the first lines of the proof of Proposition 6.3.3) and the assumptions of Proposition 6.3.3 then apply with $\hat{\varepsilon}_i$ in place of $\hat{\varepsilon}_0$, where both propositions are understood now with τ_i, τ_{i+1} in place of τ_0, τ_1. It follows that the solution ψ extends to a solution defined also in $\mathcal{R}_i := \mathcal{R}(\tau_i, \tau_i + L_i)$ satisfying the estimates (6-86)–(6-89), while for $\tau' = \tau_{i+1} = \tau_i + L_i$ we have in addition (6-92)–(6-93). We have thus

$$\begin{aligned}
&\overset{(1)}{\underset{k}{\mathcal{E}}}(\tau_{i+1}) \leq \hat{\varepsilon}_i \alpha \leq \hat{\varepsilon}_{i+1} \alpha, \\
&\overset{(1)}{\underset{k}{\mathfrak{E}}}(\tau_{i+1}) \leq \hat{\varepsilon}_i (1 + \alpha L_i^{-\frac{1}{4}}) \leq \hat{\varepsilon}_{i+1}, \\
&\overset{(2-\delta)}{\underset{k-2}{\mathfrak{E}}}(\tau_{i+1}) \leq \hat{\varepsilon}_i (1 + \alpha L_i^{-\frac{1}{4}}) \leq \hat{\varepsilon}_{i+1}, \\
&\overset{(1)}{\underset{k-2}{\mathfrak{E}}}(\tau_{i+1}) \leq \tfrac{1}{2} \hat{\varepsilon}_i \alpha L_i^{-1} \leq \hat{\varepsilon}_{i+1} \alpha L_{i+1}^{-1}, \\
&\overset{(1)}{\underset{k-4}{\mathfrak{E}}}(\tau_{i+1}) \leq \tfrac{1}{2} \hat{\varepsilon}_i \alpha L_i^{-1+\delta} \leq \hat{\varepsilon}_{i+1} \alpha L_{i+1}^{-1+\delta}, \\
&\overset{(0)}{\underset{k-6}{\mathfrak{E}}}(\tau_{i+1}) \leq \tfrac{1}{4} \hat{\varepsilon}_i \alpha^2 L_i^{-2+\delta} \leq \hat{\varepsilon}_{i+1} \alpha^2 L_{i+1}^{-2+\delta}
\end{aligned}$$

as long as

$$\hat{\varepsilon}_{i+1} := \hat{\varepsilon}_i (1 + \alpha L_i^{-\frac{1}{4}}). \tag{6-113}$$

By our requirement (6-110) and the definition (6-108) of the parameter d and then defining $\hat{\varepsilon}_{i+1}$ inductively by (6-113), it follows that $\hat{\varepsilon}_{i+1} \leq \hat{\varepsilon}_{\text{slab}}$ and $\alpha \hat{\varepsilon}_{i+1} \leq \varepsilon_{\text{slab}}$.

It follows that a solution exists in $\mathcal{R}(\tau_0, \infty) = \cup \mathcal{R}(\tau_i, \tau_i + L_i)$ and in each interval the estimates (6-89)–(6-90) hold, with $L = L_i$.

We obtain finally that for all $\tau \geq 1$ we have (among other estimates)

$$\overset{(1)}{\underset{k}{\mathcal{E}}}(\tau) \lesssim \varepsilon_0, \tag{6-114}$$

$$\int_{\tau_0}^{\tau} {}^{\rho}\overset{(0)}{\mathcal{E}'} + {}^{\chi}\overset{(0)}{\underset{k-1}{\mathcal{E}'}} + \overset{(0)}{\underset{k-2}{\mathcal{E}'}} \lesssim \varepsilon_0 \log(\tau + 1), \tag{6-115}$$

$$\overset{(2-\delta)}{\underset{k-2}{\mathcal{E}}}(\tau) \lesssim \varepsilon_0, \tag{6-116}$$

$$\overset{(1)}{\underset{k-4}{\mathcal{E}}}(\tau) \lesssim \varepsilon_0 \tau^{-1+\delta}, \quad \overset{(0)}{\underset{k-4}{\mathcal{F}}}(v, \tau) \lesssim \varepsilon_0 \tau^{-1+\delta}, \tag{6-117}$$

$$\int_{\tau}^{\infty} {}^{\rho}\overset{(0)}{\underset{k-4}{\mathcal{E}'}} + {}^{\chi}\overset{(0)}{\underset{k-5}{\mathcal{E}'}} + \overset{(0)}{\underset{k-6}{\mathcal{E}'}} \lesssim \varepsilon_0 \tau^{-1+\delta}, \tag{6-118}$$

$$\overset{(0)}{\underset{k-6}{\mathcal{E}}}(\tau) \lesssim \varepsilon_0 \tau^{-2+\delta}, \quad \overset{(0)}{\underset{k-6}{\mathcal{F}}}(v, \tau) \lesssim \varepsilon_0 \tau^{-2+\delta}, \tag{6-119}$$

$$\int_{\tau}^{\infty} {}^{\rho}\overset{(-1-\delta)}{\underset{k-6}{\mathcal{E}'}} + {}^{\chi}\overset{(-1-\delta)}{\underset{k-7}{\mathcal{E}'}} + \overset{(-1-\delta)}{\underset{k-8}{\mathcal{E}'}} \lesssim \varepsilon_0 \tau^{-2+\delta}. \tag{6-120}$$

Finally, let us note that we may improve (6-115) to

$$\int_{\tau_0}^{\infty} {}^{\rho}\overset{(0)}{\mathcal{E}}{}' + {}^{\chi}\overset{(0)}{\underset{k-1}{\mathcal{E}}}{}' + \overset{(0)}{\underset{k-2}{\mathcal{E}}}{}' \lesssim \varepsilon_0. \tag{6-121}$$

This follows, for all τ, by applying again both estimates of Proposition 4.8.1 appropriate to case (iii) globally in $\mathcal{R}(\tau_0, \tau)$. One may now re-estimate all nonlinear spacetime integrals arising in the estimates on dyadic intervals and sum.

Appendix A: The physical-space identity on very slowly rotating Kerr

Here we show that very slowly rotating Kerr metrics indeed satisfy the assumptions of Section 3.4.3.

Theorem A.1. *Consider the manifold $(\mathcal{M}, g_{a,M})$ of Section 2.7.3, where $g_{a,M}$ denotes the Kerr metric and $|a| \ll M$. Then there exist currents $J^{V,w,q,\varpi}$, $K^{V,w,q}$ associated to the wave operator $\Box_{g_{a,M}}$ satisfying the assumptions of Section 3.4.3.*

Theorem A.1 actually holds for general suitably small stationary perturbations of Schwarzschild satisfying the assumptions of Section 2 and appropriate decay at infinity. So as not to complicate matters more, here we will simply do explicitly the computation for slowly rotating Kerr, $|a| \ll M$.

The nontrivial computations necessary to produce $J^{V,w,q,\varpi}$ and $K^{V,w,q}$ all involve the exterior region. As a result, we may work entirely in Boyer–Lindquist coordinates. Thus, *in this appendix*, r *will denote the Boyer–Lindquist coordinate and* ***not*** *the function* (2-1) *of Section 2.1*. We will show explicitly the positivity properties in the region $r > r_+$, which can be covered globally by such Boyer–Lindquist coordinates. The extension of the coercivity properties to the manifold of Section 2.7.3 follows softly, without computation. (See already the end of Section A.4).

Our currents will be explicit, but rather than give formulae for the final (V, w, q, ϖ), we will build the current from its natural constituent pieces. In particular, so as to directly generate positive zeroth-order terms in the boundary currents, we will make use of the twisted energy momentum tensor as defined in [Holzegel and Warnick 2014].

This appendix is organised as follows. We first recall the Kerr metric in Boyer–Lindquist coordinates in Section A.1. We shall then review in Section A.2 the twisted energy momentum tensor and its basic properties. In the next two sections we shall build our current as a combination of a "Morawetz current", constructed in Section A.3 (vanishing, however, identically in an open set containing trapped null geodesics), and the twisted stationary current and red-shift currents, constructed in Section A.4. Section A.5 contains some auxiliary calculations.

A.1. ***The Kerr metric in Boyer–Lindquist coordinates.*** We define

$$\Delta = r^2 - 2Mr + a^2, \quad \Sigma = r^2 + a^2\cos^2\theta. \tag{A-1}$$

We recall the (t, r, θ, ϕ) Boyer–Lindquist coordinates and the associated $(t, r^\star, \theta, \phi)$ coordinates (which we will also refer to as Boyer–Lindquist coordinates) with the familiar relation

$$\frac{dr^\star}{dr} = \frac{r^2+a^2}{\Delta}.$$

We shall consider these in the domain $\mathbb{R} \times (r_+, \infty) \times \mathbb{S}^2$, where (θ, ϕ) are identified with the usual spherical coordinates of $\mathbb{S}^2$. The Kerr metric then has the form

$$g = g_{a,M} = g_{tt}\, dt \otimes dt + g_{t\phi}(dt \otimes d\phi + d\phi \otimes dt) + g_{r^\star r^\star}\, dr^\star \otimes dr^\star + g_{\theta\theta}\, d\theta \otimes d\theta + g_{\phi\phi}\, d\phi \otimes d\phi,$$

with

$$\begin{aligned} g_{tt} &= -\left(1 - \frac{2Mr}{\Sigma}\right), \quad g_{r^\star r^\star} = \frac{\Sigma\Delta}{(r^2+a^2)^2}, \quad g_{\theta\theta} = \Sigma, \\ g_{t\phi} &= -\frac{2Mr}{\Sigma} a \sin^2\theta, \quad g_{\phi\phi} = \left(r^2 + a^2 + \frac{2Mra^2}{\Sigma}\sin^2\theta\right)\sin^2\theta. \end{aligned} \tag{A-2}$$

We compute the inverse metric components as

$$\begin{aligned} g^{tt} &= -\frac{1}{\Delta}\left(r^2 + a^2 + \frac{2Mra^2}{\Sigma}\sin^2\theta\right), \quad g^{r^\star r^\star} = \frac{(r^2+a^2)^2}{\Sigma\Delta}, \quad g^{\theta\theta} = \frac{1}{\Sigma}, \\ g^{t\phi} &= -\frac{2Mr}{\Sigma\Delta} a, \quad g^{\phi\phi} = \frac{\Delta - a^2\sin^2\theta}{\Sigma\Delta\sin^2\theta}. \end{aligned} \tag{A-3}$$

For the determinant in these coordinates, we note

$$\sqrt{|g|} = \frac{\Delta}{r^2+a^2}\Sigma\sin\theta = g_{r^\star r^\star}(r^2+a^2)\sin\theta. \tag{A-4}$$

A.2. ***The twisted energy momentum tensor.*** We consider the covariant wave equation

$$\Box_g \psi = g^{\alpha\beta}\nabla_\alpha\nabla_\beta\psi = 0$$

for $g = g_{a,M}$ the Kerr metric. Note that this equation can be rewritten as follows: with the function $\beta = (\sqrt{r^2+a^2})^{-1}$, we define as in [Holzegel and Warnick 2014] the twisted operators

$$\begin{aligned} \widetilde{\nabla}_\mu(\,\cdot\,) &= \beta\nabla_\mu(\beta^{-1}\,\cdot\,), \\ \widetilde{\nabla}^\dagger_\mu(\,\cdot\,) &= -\beta^{-1}\nabla_\mu(\beta\,\cdot\,) \end{aligned} \tag{A-5}$$

and rewrite the wave equation as

$$\begin{aligned} 0 = \Box_g\psi &= -\widetilde{\nabla}^\dagger_\mu\widetilde{\nabla}^\mu\psi - \mathcal{V}\psi = 0 \\ &= \sqrt{r^2+a^2}\,\nabla_\mu((r^2+a^2)^{-1}\nabla^\mu(\psi\sqrt{r^2+a^2})) - \mathcal{V}\psi, \end{aligned} \tag{A-6}$$

where

$$\begin{aligned} \mathcal{V} &= -\frac{\Box_g\beta}{\beta} = -\frac{1}{g_{r^\star r^\star}(r^2+a^2)\beta}\partial_{r^\star}((r^2+a^2)\partial_{r^\star}\beta) \\ &= \frac{1}{\Sigma}\cdot\frac{2Mr^3 + a^2r(r-4M) + a^4}{(r^2+a^2)^2} =: \frac{1}{\Sigma}\mathcal{V}_0. \end{aligned} \tag{A-7}$$

We now define the twisted energy momentum tensor

$$\begin{aligned} \widetilde{T}_{\mu\nu}[\psi] &= \widetilde{\nabla}_\mu\psi\,\widetilde{\nabla}_\nu\psi - \tfrac{1}{2}g_{\mu\nu}(\widetilde{\nabla}^\alpha\psi\,\widetilde{\nabla}_\alpha\psi + \mathcal{V}\psi^2) \\ &= \beta^2\big[\nabla_\mu(\beta^{-1}\psi)\nabla_\nu(\beta^{-1}\psi) - \tfrac{1}{2}g_{\mu\nu}(\nabla^\alpha(\beta^{-1}\psi)\nabla_\alpha(\beta^{-1}\psi) + \mathcal{V}\beta^{-2}\psi^2)\big]. \end{aligned} \tag{A-8}$$

We note that, for all $|a| < M$, $\mathcal{V}$ is strictly positive in the exterior; in fact

$$\mathcal{V} \geq \frac{2Mr(r-M)(r+M)}{\Sigma(r^2+a^2)^2} \gtrsim r^{-3} \tag{A-9}$$

since $r \geq r_+ > M$, and $\widetilde{T}_{\mu\nu}[\psi]$ satisfies the dominant energy condition, i.e., $\widetilde{T}_{\mu\nu}[\psi]\xi^\mu\xi^\nu$ is nonnegative if ξ is timelike and in fact controls coercively first derivatives of ψ as well as ψ itself (the latter with the weight r^{-3}). (Indeed, this direct control of the zeroth-order term is our motivation for considering (A-8) in place of the usual $T_{\mu\nu}$.).

From Proposition 3 of [Holzegel and Warnick 2014] we infer the following.

Proposition A.2.1. *For ψ a C^2 solution of $\Box_g\psi = 0$ and X a smooth spacetime vector field, we have the identity*

$$\nabla^\mu \widetilde{J}^X_\mu[\psi] = \widetilde{K}^X[\psi], \tag{A-10}$$

where

$$\widetilde{J}^X_\mu[\psi] = \widetilde{T}_{\mu\nu}[\psi]X^\nu, \tag{A-11}$$

$$\widetilde{K}^X[\psi] = {}^{(X)}\pi^{\mu\nu}\widetilde{T}_{\mu\nu}[\psi] + X^\nu\widetilde{S}_\nu[\psi], \tag{A-12}$$

and

$$\widetilde{S}_\mu[\psi] = \frac{\widetilde{\nabla}^\dagger_\mu(\beta\mathcal{V})}{2\beta}\psi^2 + \frac{\widetilde{\nabla}^\dagger_\mu\beta}{2\beta}\widetilde{\nabla}^\nu\psi\widetilde{\nabla}_\nu\psi. \tag{A-13}$$

We also have the Lagrangian identity for an arbitrary spacetime function w:

Proposition A.2.2. *For ψ a C^2 solution of $\Box_g\psi = 0$ and w a smooth spacetime function, we have the identity*

$$\nabla^\mu \widetilde{J}^{\mathrm{aux},w}_\mu = \widetilde{K}^{\mathrm{aux},w}[\psi], \tag{A-14}$$

with

$$\widetilde{K}^{\mathrm{aux},w}[\psi] := w\beta^2\nabla^\alpha(\beta^{-1}\psi)\nabla_\alpha(\beta^{-1}\psi) + (\beta^{-1}\psi)^2\big(-\tfrac{1}{2}\nabla^\mu(\beta^2\nabla_\mu w) + \mathcal{V}w\beta^2\big),$$
$$\widetilde{J}^{\mathrm{aux},w}_\mu := w\beta^2(\beta^{-1}\psi\nabla_\mu(\beta^{-1}\psi)) - \tfrac{1}{2}\psi^2\nabla_\mu w.$$

Proof. This follows from

$$\begin{aligned}\nabla^\mu(w(\psi\beta^{-1}\beta^2\nabla_\mu(\beta^{-1}\psi)))\\ = w\beta^2\nabla^\mu(\beta^{-1}\psi)\nabla^\mu(\beta^{-1}\psi) + w\psi(-\widetilde{\nabla}^\dagger_\mu\widetilde{\nabla}^\mu\psi) + (\nabla^\mu w)\beta^2\tfrac{1}{2}\nabla_\mu(\beta^{-1}\psi)^2\end{aligned} \tag{A-15}$$

after rearranging and inserting (A-6). □

Remark A.2.3. Note that when w is a function of r only (as will always be the case in the applications below) we have

$$\nabla^\mu(\beta^2\nabla_\mu w) = \frac{1}{\sqrt{g}}\partial_{r^\star}(\sqrt{g}g^{r^\star r^\star}\beta^2\partial_{r^\star}w) = \frac{r^2+a^2}{\Delta\Sigma}\partial^2_{r^\star}w,$$

a formula which is useful in the computations below.

A.3. ***A Morawetz current vanishing identically in a neighbourhood of trapping.*** In this section, we shall produce a current giving the desired bulk positivity properties modulo suitable zeroth-order terms but with additional degeneration at the horizon and without the boundary positivity properties. (The horizon degeneration and the boundary positivity properties will be dealt with in Section A.4.)

The point about this current is that it will completely vanish in the set $\frac{11}{4}M \le r \le \frac{7}{2}M$, which contains all trapped null geodesics for $|a| \ll M$. Thus, *this current is insensitive to the precise nature of the dynamics near trapping*. The requirement of vanishing on such a set will necessarily generate lower-order terms, however, with an unfavourable sign. These too will be supported away from trapping.

Our current will be defined by combining those of Propositions A.2.1 and A.2.2, and the vector field component of the current will be in the direction of $\partial_{r^\star}$. We begin with a computation; note that the prime $'$ below will denote $\frac{d}{dr^*}$.

Lemma A.3.1. *For*

$$\beta^{-1} = \sqrt{r^2+a^2}, \quad h = \Big(1-\frac{3M}{r}\Big)\frac{\Delta}{(r^2+a^2)^2}, \quad \textit{and} \quad X = f(r^*)\partial_{r^\star}$$

(*with f arbitrary*), *we have*, *for any $\delta \in \mathbb{R}$*, *the divergence identity*

$$\nabla^\mu(\widetilde{J}^X_\mu[\psi] + \widetilde{J}^{\mathrm{aux},\frac12 f'}_\mu[\psi] - \delta\cdot\widetilde{J}^{\mathrm{aux},f\cdot h}_\mu[\psi]) = \widetilde{K}^X[\psi] + \widetilde{K}^{\mathrm{aux},\frac12 f'}[\psi] - \delta\cdot\widetilde{K}^{\mathrm{aux},f\cdot h}_\mu[\psi], \tag{A-16}$$

where

$$\begin{aligned}&\widetilde{J}^X_\mu[\psi] + \widetilde{J}^{\mathrm{aux},\frac12 f'}_\mu[\psi] - \delta\widetilde{J}^{\mathrm{aux},f\cdot h}_\mu[\psi]\\ &= \widetilde{T}_{\mu\nu}[\psi]X^\nu + f'\beta^2(\beta^{-1}\psi\nabla_\mu(\beta^{-1}\psi)) - \tfrac12\psi^2\nabla_\mu f' - \delta\big[f\cdot h\beta^2(\beta^{-1}\psi\nabla_\mu(\beta^{-1}\psi)) - \tfrac12\psi^2\nabla_\mu f\cdot h\big]\end{aligned} \tag{A-17}$$

and

$$\begin{aligned}&\widetilde{K}^{f\partial_{r^\star}}[\psi] + \widetilde{K}^{\mathrm{aux},\frac12 f'}[\psi] - \delta\widetilde{K}^{\mathrm{aux},f\cdot h}_\mu[\psi]\\ &\quad= \beta^2(g^{r^\star r^\star}\partial_{r^\star} f - \delta f h)(\partial_{r^\star}(\beta^{-1}\psi))^2 + \tfrac12 f\beta^2 \sum_{\mu,\nu\neq r^\star}(\mathcal{A}^{\mu\nu} - 2\delta h g^{\mu\nu})\partial_\mu(\beta^{-1}\psi)\partial_\nu(\beta^{-1}\psi)\\ &\qquad + \frac{r^2+a^2}{\Delta\Sigma}\bigg(-\tfrac14 f''' - \tfrac12 f\partial_{r^\star}\Big(\frac{\Delta}{(r^2+a^2)^2}\mathcal{V}_0\Big) + \delta\Big[\tfrac12(f\cdot h)'' - \mathcal{V}_0 f\cdot h\frac{\Delta}{(r^2+a^2)^2}\Big]\bigg)(\beta^{-1}\psi)^2\end{aligned} \tag{A-18}$$

with

$$\mathcal{A}^{\mu\nu} = -\partial_{r^\star}(g^{\mu\nu}) + g^{\mu\nu}g_{r^\star r^\star}\partial_{r^\star}(g^{r^\star r^\star}). \tag{A-19}$$

Proof. We compute (see Section A.5)

$$\begin{aligned}\widetilde{K}^{f\partial_{r^\star}}[\psi] &= \beta^2 g^{r^\star r^\star}\partial_{r^\star} f(\partial_{r^\star}(\beta^{-1}\psi))^2 - \tfrac12 f\beta^2\sum_{\mu,\nu\neq r^\star}[\partial_{r^\star}(g^{\mu\nu}) - g^{\mu\nu}g_{r^\star r^\star}\partial_{r^\star}(g^{r^\star r^\star})]\partial_\mu(\beta^{-1}\psi)\partial_\nu(\beta^{-1}\psi)\\ &\quad - \tfrac12\beta^2(\partial_{r^\star} f)g^{\alpha\beta}\partial_\alpha(q^{-1}\psi)\partial_\beta(\beta^{-1}\psi) - \tfrac12 f\partial_{r^\star}(\beta^2\mathcal{V})(\beta^{-1}\psi)^2\\ &\quad - \frac12\Big(\partial_{r^\star} f + f\frac{1}{\sqrt{g}}\partial_{r^\star}\sqrt{g}\Big)\mathcal{V}\psi^2.\end{aligned} \tag{A-20}$$

Adding the Lagrangian identity of Proposition A.2.2 with $w = \frac{1}{2}f'$ (and using Remark A.2.3), we deduce the result for $\delta = 0$. For arbitrary δ, we simply add the Lagrangian identity of Proposition A.2.2 with

$$w = \Big(1 - \frac{3M}{r}\Big)\frac{\Delta}{(r^2+a^2)} f = f \cdot h$$

and group terms. □

We now exploit the divergence identity of Lemma A.3.1. For this we first define the following (disjoint) decomposition of the range of the r-variable:

$$\begin{aligned}[r_+, \infty) &= R_1 \cup R_2 \cup R_3 \cup R_4 \cup R_5 \\ &= \big[r_+, \tfrac{5}{2}M\big) \cup \big[\tfrac{5}{2}M, \tfrac{11}{4}M\big) \cup \big[\tfrac{11}{4}M, \tfrac{7}{2}M\big) \cup \big[\tfrac{7}{2}M, 4M\big) \cup \big[4M, \infty\big).\end{aligned}$$

Note that $\mathcal{M} \cap \{r \in R_3\}$ includes the region containing all trapped null geodesics if $\frac{a}{M}$ is suitably small.

Proposition A.3.2. *There exists an* (*explicit*) *function f such that*, *for all $|a|/M$ sufficiently small*, *we can choose $\delta > 0$ sufficiently small* (*depending only on M*) *such that the estimate*

$$\begin{aligned}&\widetilde{K}^{f\partial_{r^\star}}[\psi] + \widetilde{K}^{\mathrm{aux}, \frac{1}{2}f'}[\psi] - \delta \cdot \widetilde{K}_\mu^{\mathrm{aux}, \frac{f\Delta}{(r^2+a^2)^2}}[\psi] \\ &\geq c\mathbb{1}_{R_1\cup R_5}\bigg(\frac{(\partial_t(\beta^{-1}\psi))^2 + (\partial_{r^\star}(\beta^{-1}\psi))^2 + (\beta^{-1}\psi)^2}{r^4} + \frac{1}{r^5}|\mathring{\nabla}(\beta^{-1}\psi)|^2\bigg) - C\mathbb{1}_{R_2\cup R_4}(\beta^{-1}\psi)^2 \qquad \text{(A-21)}\end{aligned}$$

holds for constants c and C depending only on M. Here we have defined the shorthand

$$|\mathring{\nabla}(\beta^{-1}\psi)|^2 := (\partial_\theta(\beta^{-1}\psi))^2 + \frac{1}{\sin^2\theta}(\partial_\phi(\beta^{-1}\psi))^2.$$

Proof. In the proof, we let $\tilde{c}$ and $\widetilde{C}$ be constants depending only on M (but which might change from line to line). Starting from (A-18) we define

$$\mathcal{B}^{\mu\nu} = \mathcal{A}^{\mu\nu} - 2\delta\Big(1 - \frac{3M}{r}\Big)\frac{\Delta}{(r^2+a^2)^2} g^{\mu\nu}$$

and compute

$$\begin{aligned}\mathcal{B}^{tt} &= 2a^2 \cdot \frac{r^3 - 3Mr^2 + a^2r + a^2M}{\Sigma(r^2+a^2)^2}\sin^2\theta + 2\delta\Big(1 - \frac{3M}{r}\Big)\frac{r^2 + a^2 + \frac{2Mra^2}{\Sigma}\sin^2\theta}{(r^2+a^2)^2}, \\ \mathcal{B}^{t\phi} &= \frac{2aM(a^2 - 3r^2)}{\Sigma(r^2+a^2)^2} + 2\delta \cdot a\Big(1 - \frac{3M}{r}\Big)\frac{2Mr}{\Sigma(r^2+a^2)^2}, \\ \mathcal{B}^{\phi\phi} &= 2\frac{r^2(r-3M) + a^2r\cos[2\theta] + a^2M}{\Sigma(r^2+a^2)^2\sin^2\theta} - 2\delta\Big(1 - \frac{3M}{r}\Big)\frac{\Delta - a^2\sin^2\theta}{\Sigma(r^2+a^2)^2\sin^2\theta}, \\ \mathcal{B}^{\theta\theta} &= 2\frac{r^2(r-3M) + a^2r + a^2M}{\Sigma(r^2+a^2)^2\sin^2\theta} - 2\delta\Big(1 - \frac{3M}{r}\Big)\frac{\Delta}{\Sigma(r^2+a^2)^2}.\end{aligned} \qquad \text{(A-22)}$$

From this one sees that, for $|a|/M$ sufficiently small, we can choose δ sufficiently small (depending only on M) such that in $\mathcal{M} \cap \{r \notin R_3\}$ $\big($where $\big|1 - \frac{3M}{r}\big| \geq \frac{1}{12}\big)$ the estimates

$$\frac{\mathcal{B}^{tt}}{1 - \frac{3M}{r}} \geq \frac{\tilde{c}}{r^2}, \qquad \frac{\mathcal{B}^{\phi\phi}}{1 - \frac{3M}{r}} \geq \frac{\tilde{c}}{r^3\sin^2\theta}, \qquad \frac{\mathcal{B}^{\theta\theta}}{1 - \frac{3M}{r}} \geq \frac{\tilde{c}}{r^3} \qquad \text{(A-23)}$$

hold for a constant $\tilde{c}$ depending only on M. Moreover, we have

$$\left|\frac{\mathcal{B}^{t\phi}}{1-\frac{3M}{r}}\right| \le \frac{|a|}{M}\frac{\widetilde{C}}{r^4}. \tag{A-24}$$

We next choose $f:[r_+,\infty)\to\mathbb{R}$ to be bounded, monotonically increasing as follows:

$$f=\begin{cases}-\frac{M}{r} & \text{if } r\in R_1,\\ C^3 \text{ interpolate} & \text{if } r\in R_2,\\ 0 & \text{if } r\in R_3,\\ C^3 \text{ interpolate} & \text{if } r\in R_4,\\ 1-\frac{M}{r} & \text{if } r\in R_5.\end{cases}$$

Specifically, we do the C^3 interpolation such that we have f monotonically increasing and

$$f'\ge\tilde{c}(-f) \text{ in } R_2 \quad \text{and} \quad f'\ge\tilde{c}f \text{ in } R_4 \tag{A-25}$$

for a fixed constant $\tilde{c}>0$ depending only on M. In particular (potentially making δ slightly smaller), we can achieve that

$$g^{r^\star r^\star}\partial_{r^\star}f-\delta f\Big(1-\frac{3M}{r}\Big)\frac{\Delta}{(r^2+a^2)^2}\ge f\Big(1-\frac{3M}{r}\Big)\frac{\tilde{c}}{r^2}$$

holds in all of $\mathcal{M}\cap\{r\notin R_3\}$ for a $\tilde{c}$ depending only on M. Since $f\big(1-\frac{3M}{r}\big)$ is globally nonnegative and in fact bounded uniformly below by a $\tilde{c}$ in $R_1\cup R_5$, the estimate (A-21) now follows except for the zeroth-order term on the right-hand side. (For this the only thing to notice is that in

$$\tfrac{1}{2}f\beta^2\Big(1-\frac{3M}{r}\Big)\Bigg[\sum_{\mu,\nu\neq r^\star}\frac{\mathcal{B}^{\mu\nu}}{1-\frac{3M}{r}}\partial_\mu(\beta^{-1}\psi)\partial_\nu(\beta^{-1}\psi)\Bigg]$$

the quadratic form in the square bracket is positive definite by the estimates (A-23) and (A-24).)

For the zeroth-order term, clearly we only need to establish a lower bound in $R_1\cup R_5$. We start with R_5, where we have for $f=1-M/r$ that

$$-\tfrac{1}{2}f'''-f\bigg(\frac{\Delta}{(r^2+a^2)^2}V_0\bigg)'=\frac{3M}{r^4}+\mathcal{O}(r^{-5})$$

and

$$\frac{r^2+a^2}{r^2-2Mr+a^2}\bigg(-\tfrac{1}{2}f'''-f\bigg(\frac{\Delta}{(r^2+a^2)^2}\mathcal{V}_0\bigg)'\bigg)=\frac{M}{r^4}\bigg(3-\frac{2M}{r}-14\frac{M^2}{r^2}\bigg) \quad \text{if } a=0.$$

Since the left-hand side of the second identity is continuous in a and the bracket on the right-hand side is uniformly bounded below by $\frac{1}{3}$ for $r\ge 3M$, the estimate claimed for the zeroth-order term follows for sufficiently small $|a|/M$ after potentially making δ smaller; note that

$$\delta\frac{r^2+a^2}{r^2-2Mr+a^2}\bigg[\tfrac{1}{2}(f\cdot h)''-\mathcal{V}_0 f\cdot h\frac{\Delta}{(r^2+a^2)^2}\bigg]\le\frac{\widetilde{C}\delta}{r^4}.$$

Similarly, in R_1 we have for $f = -M/r$ the identity

$$\frac{r^2+a^2}{r^2-2Mr+a^2}\left(-\tfrac{1}{2}f''' - f(w\mathcal{V}_0)'\right) = -\frac{M}{r^6}(14M^2 - 14Mr + 3r^2) \quad \text{if } a = 0.$$

The expression on the right-hand side is easily shown to be uniformly bounded below by $M^3/(4r^6)$ for $r \in \left[\frac{9}{5}M, \frac{11}{4}M\right]$. Since the expression on the left is in particular continuous in a on $\left[r_+, \frac{11}{4}M\right]$, the estimate claimed for the zeroth-order term also follows in R_1. $\square$

A.4. ***Adding the red-shift and stationary currents and completing the proof.*** Let T denote the stationary Killing field of Kerr (which in Boyer–Lindquist coordinates is $T = \partial_t$) and N the time-like (including on $\mathcal{H}^+$) red-shift vector field constructed in Theorem 7.1 of [Dafermos and Rodnianski 2013]. We have that $\widetilde{K}^T = 0$, while

$$\widetilde{K}^N[\psi] \geq \tilde{c}\Bigg((\partial_t(\beta^{-1}\psi))^2 + (\partial_{r^\star}(\beta^{-1}\psi))^2 + |\mathring{\nabla}(\beta^{-1}\psi)|^2 + (\beta^{-1}\psi)^2 + \left[\frac{r^2+a^2}{\Delta}\left(\partial_t - \partial_{r^\star} + \frac{a}{r^2+a^2}\partial_\phi\right)(\beta^{-1}\psi)\right]^2\Bigg) \quad \text{in } r \leq \tfrac{9}{4}M$$

provided a is sufficiently small. We also have $N = T$ in the complement of R_1.

We shall now complete the proof of Theorem A.1 by adding

$$\Upsilon \cdot \widetilde{J}^T_\mu[\psi] + \eta \widetilde{J}^N_\mu[\psi]$$

to the current of Proposition A.3.2 for constants $\Upsilon > 0$ (large) and $\eta > 0$ (small).

We consider the divergence identity of Lemma A.3.1 with δ and f now chosen as in Proposition A.3.2. We expand the identity as follows for constants $\Upsilon > 0$ (large) and $\eta > 0$ (small) to be chosen below:

$$\nabla^\mu\big(\widetilde{J}^X_\mu[\psi] + \widetilde{J}^{\mathrm{aux},\frac{1}{2}f'}_\mu[\psi] - \delta\cdot\widetilde{J}^{\mathrm{aux},f\cdot h}_\mu[\psi] + \Upsilon\cdot\widetilde{J}^T_\mu[\psi] + \eta\cdot\widetilde{J}^N_\mu[\psi]\big) = \widetilde{K}^X[\psi] + \widetilde{K}^{\mathrm{aux},\frac{1}{2}f'}[\psi] - \delta\cdot\widetilde{K}^{\mathrm{aux},f\cdot h}_\mu[\psi] + \eta\cdot\widetilde{K}^N[\psi]. \quad \text{(A-26)}$$

It is now clear that we can choose η sufficiently small (depending only on M) such that for $r \geq \frac{9}{4}M$ we can absorb the term $\eta\widetilde{K}^N[\psi]$, which is supported only in R_1, by the positivity of

$$\widetilde{K}^X[\psi] + \widetilde{K}^{\mathrm{aux},\frac{1}{2}f'}[\psi] - \delta\cdot\widetilde{K}^{\mathrm{aux},f\cdot h}_\mu[\psi]$$

established in Proposition A.3.2 to deduce

$$\widetilde{K}^{f\partial_{r^\star}}[\psi] + \widetilde{K}^{\mathrm{aux},\frac{1}{2}f'}[\psi] - \delta\widetilde{K}^{\mathrm{aux},\frac{f\Delta}{(r^2+a^2)^2}}_\mu[\psi] + \eta\widetilde{K}^N[\psi] \geq c\mathbb{1}_{R_1\cup R_5}\left(\frac{(\partial_t(\beta^{-1}\psi))^2 + (\partial_{r^\star}(\beta^{-1}\psi))^2 + (\beta^{-1}\psi)^2}{r^4} + \frac{1}{r^5}|\mathring{\nabla}(\beta^{-1}\psi)|^2\right) + c\mathbb{1}_{R_1}\left[\frac{r^2+a^2}{\Delta}\left(\partial_t - \partial_{r^\star} + \frac{a}{r^2+a^2}\partial_\phi\right)(\beta^{-1}\psi)\right]^2 - C\mathbb{1}_{R_2\cup R_4}|(\beta^{-1}\psi)|^2, \quad \text{(A-27)}$$

where c may be smaller than in (A-21) but still depends only on M. We finally claim that we can choose Υ sufficiently large in (A-26) such that

$$\begin{aligned}\big(\widetilde{J}^X_\mu[\psi] + \widetilde{J}^{\mathrm{aux},\frac{1}{2}f'}_\mu[\psi] - \delta \widetilde{J}^{\mathrm{aux},f\cdot h}_\mu[\psi] + \Upsilon \widetilde{J}^T_\mu[\psi] + \eta \widetilde{J}^N_\mu[\psi]\big) n^\mu_{\Sigma_\tau} \\ = \boxed{\widetilde{T}_{\mu\nu}[\psi](X^\nu + \Upsilon T^\nu + \eta N^\nu) n^\mu_{\Sigma_\tau}} + f'\beta^2(\beta^{-1}\psi\nabla_\mu(\beta^{-1}\psi)) n^\mu_{\Sigma_\tau} \\ - \tfrac{1}{2}\psi^2\nabla_\mu f' n^\mu_{\Sigma_\tau} - \delta\big[f\cdot h\beta^2(\beta^{-1}\psi\nabla_\mu(\beta^{-1}\psi)) - \tfrac{1}{2}\psi^2\nabla_\mu f\cdot h\big] n^\mu_{\Sigma_\tau} \\ \geq \beta^2\left(|L(\beta^{-1}\psi)|^2 + \iota_{r\leq R}|\underline{L}(\beta^{-1}\psi)|^2 + \frac{1}{r^2}|\mathring{\nabla}(\beta^{-1}\psi)|^2 + \frac{1}{r^3}(\beta^{-1}\psi)^2\right).\end{aligned} \tag{A-28}$$

For the boxed term, the estimate is an immediate consequence of the twisted energy momentum tensor (A-8) satisfying the dominant energy condition, as discussed in Section A.2, the positivity (A-9), and the fact that the vector field $\Upsilon T + \eta N + X$ is timelike for sufficiently large Υ, provided that we then restrict to $|a|$ sufficiently small. Moreover, for the boxed term the zeroth-order term in the estimate scales with Υ; i.e., the larger we choose Υ the larger the zeroth-order term becomes. This can be used to absorb the remaining terms using the Cauchy–Schwarz inequality for $|a|$ sufficiently small depending on this final choice of Υ.

The relations (A-27) and (A-28) give the coercivity property (3-31) and the first property of (3-32), restricted to the exterior region, where we define $\rho = \mathbb{1}_{R_1\cup R_5}$ and $\xi = \mathbb{1}_{R_2\cup R_4}$, except that the r decay for the bulk current is not optimised to the $r^{-1-\delta}$ and $r^{-3-\delta}$ weights for the first and zeroth-order terms, respectively, and the r decay for the boundary current is not optimised to the r^{-2} weight for the zeroth-order term. (Note that the Boyer–Lindquist r and the r of (2-1) are comparable for large r values, so one can compare directly the r-decay as if the coordinates were the same.) To remedy this, it suffices to add $\epsilon J^{\hat{\chi}V,\hat{\chi}w,\hat{\chi}q,\hat{\chi}\varpi}$ and $\epsilon K^{\hat{\chi}V,\hat{\chi}w,\hat{\chi}q}$, respectively, to the currents, where (V, w, q, ϖ) here denotes the current of Section B.1, $\hat{\chi}$ is a cutoff supported far away, and ϵ is sufficiently small. The resulting currents now indeed satisfy (3-31) and the first inequality of (3-32), restricted to the exterior, with weights as stated. We note finally that, for small $|a|$, we may take the χ in the estimate (3-3) proven in [Dafermos and Rodnianski 2010a] or [Dafermos et al. 2016] to be identically 1 outside of a small neighbourhood of $r = 3M$. Thus, our ξ indeed satisfies (3-27).

The currents trivially may be extended to the slightly larger domain of Section 2.7.3. Note finally that the additional positivity statements of (3-32) are easily shown to hold. Rewriting the current in terms of a single quadruple (V, w, q, ϖ), one easily sees that the boundedness statements (3-16) hold as well. This completes the proof of Theorem A.1.

A.5. ***Computation of the X-deformation tensor.*** We collect here some computations which were used in the proof of Theorem A.1.

We have

$$-2^{(X)}\pi^{\mu\nu} = X^\alpha\partial_\alpha(g^{\mu\nu}) - \partial_\alpha X^\mu g^{\alpha\nu} - \partial_\alpha X^\nu g^{\alpha\mu}; \tag{A-29}$$

hence for $X = f(r)\partial_{r^\star}$

$$-2^{(X)}\pi^{\mu\nu} = f\partial_{r^\star}(g^{\mu\nu}) - \partial_\alpha X^\mu g^{\alpha\nu} - \partial_\alpha X^\nu g^{\alpha\mu}, \tag{A-30}$$

which we can write (using $g^{r^\star r^\star} g_{r^\star r^\star} = 1$ for Kerr in Boyer–Lindquist) as

$$\begin{aligned} -2{}^{(X)}\pi^{\mu\nu} &= f\partial_{r^\star}(g^{\mu\nu}) + fg^{\mu\nu} g_{r^\star r^\star}\partial_{r^\star}(g^{r^\star r^\star}) - fg^{\mu\nu} g_{r^\star r^\star}\partial_{r^\star}(g^{r^\star r^\star}) \quad \text{unless } \mu=\nu=r^\star, \\ -2{}^{(X)}\pi^{r^\star r^\star} &= fg^{r^\star r^\star} g_{r^\star r^\star}\partial_{r^\star}(g^{r^\star r^\star}) - 2g^{r^\star r^\star}\partial_{r^\star} f, \end{aligned} \tag{A-31}$$

so

$$-2{}^{(X)}\pi^{\mu\nu} = g^{\mu\nu}\cdot fg_{r^\star r^\star}\partial_{r^\star}(g^{r^\star r^\star}) + \begin{cases} -2g^{r^\star r^\star}\partial_{r^\star} f & \text{if } \mu=\nu=r^\star, \\ f\partial_{r^\star}(g^{\mu\nu}) - fg^{\mu\nu} g_{r^\star r^\star}\partial_{r^\star}(g^{r^\star r^\star}) & \text{otherwise.} \end{cases}$$

We note also

$$\begin{aligned} \operatorname{tr}{}^{(X)}\pi &= g_{\mu\nu}{}^{(X)}\pi^{\mu\nu} \\ &= -\tfrac{1}{2} g_{\mu\nu} f\partial_{r^\star}(g^{\mu\nu}) + \partial_{r^\star} f \\ &= f\frac{1}{\sqrt{g}}\partial_{r^\star}\sqrt{g} + \partial_{r^\star} f. \end{aligned} \tag{A-32}$$

Therefore

$$\begin{aligned} &{}^{(X)}\pi^{\mu\nu}\widetilde{T}_{\mu\nu} \\ &= {}^{(X)}\pi^{\mu\nu}\widetilde{\nabla}_\mu\psi\widetilde{\nabla}_\nu\psi - \tfrac{1}{2}(\operatorname{tr}{}^{(X)}\pi) g^{\mu\nu}\widetilde{\nabla}_\mu\psi\widetilde{\nabla}_\nu\psi - \tfrac{1}{2}(\operatorname{tr}{}^{(X)}\pi)\mathcal{V}\psi^2 \\ &= \beta^2 g^{r^\star r^\star}\partial_{r^\star} f(\partial_{r^\star}(\beta^{-1}\psi))^2 - \tfrac{1}{2}\beta^2 f\sum_{\mu,\nu\neq r^\star}[\partial_{r^\star}(g^{\mu\nu}) - g^{\mu\nu} g_{r^\star r^\star}\partial_{r^\star}(g^{r^\star r^\star})]\partial_\mu(\beta^{-1}\psi)\partial_\nu(\beta^{-1}\psi) \\ &\quad -\frac{1}{2}\left(\partial_{r^\star} f + f\frac{1}{\sqrt{g}}\partial_{r^\star}\sqrt{g} + fg_{r^\star r^\star}\partial_{r^\star}(g_{r^\star r^\star})^{-1}\right) g^{\mu\nu}\widetilde{\nabla}_\mu\psi\widetilde{\nabla}_\nu\psi - \frac{1}{2}\left(\partial_{r^\star} f + f\frac{1}{\sqrt{g}}\partial_{r^\star}\sqrt{g}\right)\mathcal{V}\psi^2. \end{aligned} \tag{A-33}$$

Noting that the determinant is given by

$$\sqrt{g} = \frac{\Delta}{r^2+a^2}\Sigma\sin\theta = g_{r^\star r^\star}(r^2+a^2)\sin\theta$$

for Kerr in Boyer–Lindquist $(t, r^\star, \theta, \phi)$,

$$\begin{aligned} {}^{(X)}\pi^{\mu\nu}\widetilde{T}_{\mu\nu} &= \beta^2 g^{r^\star r^\star}\partial_{r^\star} f(\partial_{r^\star}(\beta^{-1}\psi))^2 - \tfrac{1}{2}\beta^2 f\sum_{\mu,\nu\neq r^\star}[\partial_{r^\star}(g^{\mu\nu}) - g^{\mu\nu} g_{r^\star r^\star}\partial_{r^\star}(g^{r^\star r^\star})]\partial_\mu(\beta^{-1}\psi)\partial_\nu(\beta^{-1}\psi) \\ &\quad -\frac{1}{2}\left(\partial_{r^\star} f + f\frac{2r\Delta}{(r^2+a^2)^2}\right) g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi - \frac{1}{2}\left(\partial_{r^\star} f + f\frac{1}{\sqrt{g}}\partial_{r^\star}\sqrt{g}\right)\mathcal{V}\psi^2. \end{aligned}$$

We now recall from Proposition A.2.1 the definition $\widetilde{K}^X[\psi] = {}^{(X)}\pi^{\mu\nu}\widetilde{T}_{\mu\nu}[\psi] + X^\nu\widetilde{S}_\nu[\psi]$ and compute

$$\begin{aligned} \widetilde{K}^X[\psi] &= \beta^2 g^{r^\star r^\star}\partial_{r^\star} f(\partial_{r^\star}(\beta^{-1}\psi))^2 - \tfrac{1}{2} f\beta^2\sum_{\mu,\nu\neq r^\star}[\partial_{r^\star}(g^{\mu\nu}) - g^{\mu\nu} g_{r^\star r^\star}\partial_{r^\star}(g^{r^\star r^\star})]\partial_\mu(\beta^{-1}\psi)\partial_\nu(\beta^{-1}\psi) \\ &\quad - \tfrac{1}{2}\beta^2\left(\partial_{r^\star} f + f\frac{2r\Delta}{(r^2+a^2)^2} + f\frac{\partial_{r^\star}(\beta^2)}{\beta^2}\right) g^{\mu\nu}\partial_\mu(\beta^{-1}\psi)\partial_\nu(\beta^{-1}\psi) - \tfrac{1}{2} f\partial_{r^\star}(\beta^2\mathcal{V})(\beta^{-1}\psi)^2 \\ &\quad - \frac{1}{2}\left(\partial_{r^\star} f + f\frac{1}{\sqrt{g}}\partial_{r^\star}\sqrt{g}\right)\mathcal{V}\psi^2, \end{aligned} \tag{A-34}$$

which indeed simplifies to (A-20).

Appendix B: Energy currents in Minkowski space

In this section the energy currents used to obtain the assumptions of Sections 3.4.1 and 3.5 are described in Minkowski space. In Section B.1 a quadruple (V, w, q, ϖ) satisfying the assumptions of Section 3.4.1 is introduced, and in Section B.2 a quadruple $(\widetilde{V}_{\mathrm{far}}, \tilde{w}_{\mathrm{far}}, \tilde{q}_{\mathrm{far}}, \tilde{\varpi}_{\mathrm{far}})$ satisfying the assumptions of Section 3.5 is introduced.

Recall that, for a given spacetime $(\mathcal{M}, g)$ and suitably regular function $\psi : \mathcal{M} \to \mathbb{R}$, for a given vector field V, a function w, a 1-form q, and a 2-form ϖ, the energy current $J^{V,w,q,\varpi}$ takes the form

$$J^{V,w,q,\varpi}_{\mu}[g, \psi] := T_{\mu\nu}[g, \psi]V^{\nu} + w\psi\partial_{\mu}\psi + \psi^2 q_{\mu} + *d(\psi^2\varpi)_{\mu}.$$

Here, for $0 \le k \le 4$, $*\colon \Lambda^k\mathcal{M} \to \Lambda^{4-k}\mathcal{M}$ denotes the Hodge star operator which satisfies, for all $\alpha, \beta \in \Lambda^k\mathcal{M}$,

$$\alpha \wedge *\beta = g(\alpha, \beta)d\,\mathrm{Vol}_{\mathcal{M}}\,.$$

The divergence of $J^{V,w,q,\varpi}$ takes the form

$$\nabla^{\mu} J^{V,w,q,\varpi}_{\mu}[g, \psi] = K^{V,w,q}[g, \psi] + H^{V,w}[\psi]\Box_g\psi,$$

where

$$K^{V,w,q}[g, \psi] := \pi^{V}_{\mu\nu}[g]T^{\mu\nu}[g, \psi] + \psi\nabla^{\mu}w\nabla_{\mu}\psi + w\nabla^{\mu}\psi\nabla_{\mu}\psi + \psi^2\nabla^{\mu}q_{\mu} + 2\psi g^{\mu\nu}q_{\mu}\partial_{\nu}\psi,$$
$$H^{V,w}[\psi] := V^{\mu}\partial_{\mu}\psi + w\psi.$$

The divergence of a 1-form $\xi \in \Lambda^1\mathcal{M}$ can be expressed in terms of d and $*$ by

$$\mathrm{Div}\,\xi = *d*\xi.$$

In particular it follows that, for any 2-form $\varpi \in \Lambda^2\mathcal{M}$ and any function $\psi \in C^{\infty}(\mathcal{M})$,

$$\mathrm{Div} * d(\psi^2\varpi) = -*dd(\psi^2\varpi) = 0.$$

Thus the choice of ϖ in the current $J^{V,w,q,\varpi}_{\mu}[g, \psi]$ never contributes to the associated $K^{V,w,q}[g, \psi]$ or $H^{V,w}[g, \psi]$. Moreover q does not contribute to $H^{V,w}[g, \psi]$.

Throughout this section, $(\mathcal{M}, g_0)$ denotes Minkowski space (see Section 2.7.1) and

$$r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} \tag{B-1}$$

denotes the standard radial coordinate and *not* the r of (2-1). (Note that (B-1) was denoted $\tilde{r}$ in Section 2.7.1. Since (2-1) and (B-1) are comparable for large r, the associated r-weighted coercivity properties will be equivalent.) Recall the $(t, r, \vartheta, \varphi)$ and $(u, v, \vartheta, \varphi)$ coordinate systems. The Minkowski metric takes the form

$$g_0 = -dt^2 + dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2) = -du\,dv + r^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2).$$

The following basic properties of Minkowski space are used. The spacetime volume form of Minkowski space can be written

$$d\,\mathrm{Vol}_{\mathcal{M}} = r^2 \sin\vartheta\, dt \wedge dr \wedge d\vartheta \wedge d\varphi = \tfrac{1}{2}r^2 \sin\vartheta\, du \wedge dv \wedge d\vartheta \wedge d\varphi,$$

and so the Hodge star operator $*$ in particular satisfies

$$*(r^2 \sin\vartheta\, dr \wedge d\vartheta \wedge d\varphi) = -dt, \quad *(r^2 \sin\vartheta\, dt \wedge d\vartheta \wedge d\varphi) = -dr, \tag{B-2}$$

$$*(r^2 \sin\vartheta\, du \wedge d\vartheta \wedge d\varphi) = du, \quad *(r^2 \sin\vartheta\, dv \wedge d\vartheta \wedge d\varphi) = -dv. \tag{B-3}$$

The normals to the hypersurfaces $\Sigma(\tau)$ and $\underline{C}_v$ take the form

$$n_{\Sigma_\tau} = \partial_v + \iota_{r\le R}\partial_u, \quad n_{\underline{C}_v} = \partial_u. \tag{B-4}$$

B.1. ***The*** $J^{V,w,q,\varpi}$ ***current.*** In the case that $(\mathcal{M}, g_0)$ is Minkowski space, the tuple (V, w, q, ϖ) of Section 3.4.1 can be defined as follows. Consider

$$V_1 = T = \partial_u + \partial_v, \quad w_1 = 0, \quad q_1 = 0, \quad \varpi_1 = 0,$$

$$V_2 = \delta_1 T = \delta_1(\partial_u + \partial_v), \quad w_2 = 0, \quad q_2 = 0, \quad \varpi_2 = -\frac{\delta_1}{2} r^{-1} r^2 \sin\vartheta\, d\vartheta \wedge d\varphi,$$

$$V_3 = \frac{\delta_2}{2}\left(1 - \frac{\delta_3}{(1+r)^\delta}\right)(\partial_v - \partial_u), \quad w_3 = \frac{\delta_2}{r}\left(1 - \frac{\delta_3}{(1+r)^\delta}\right), \quad (q_3)_\mu = -\frac{\partial_{x^\mu} w_3}{2}, \quad \varpi_3 = 0,$$

for appropriate

$$0 < \delta_3 \ll \delta_2 \ll \delta_1 \ll 1, \tag{B-5}$$

and define

$$V = \sum_{i=1}^{3} V_i, \quad w = \sum_{i=1}^{3} w_i, \quad q = \sum_{i=1}^{3} q_i, \quad \varpi = \sum_{i=1}^{3} \varpi_i, \tag{B-6}$$

and then, for a given ψ, define currents $J_\mu^{V_i,w_i,q_i,\varpi_i}[\psi]$ and $J_\mu^{V,w,q,\varpi}[\psi]$ by (3-11). These satisfy the boundedness properties (3-16). (Note that the current $J_\mu^{V_2,w_2,q_2,\alpha_2}[g_0, \psi]$ can be viewed as arising from contracting the twisted energy momentum tensor, defined in (A-8), with the Killing vector field $\delta_1 T$.)

Proposition B.1.1 (the $J^{V,w,q,\varpi}$ current in Minkowski space). *With (V, w, q, ϖ) defined as above, if δ_1, δ_2 and δ_3 are chosen according to (B-5), the current $J_\mu^{V,w,q,\varpi}[\psi]$ satisfies the coercivity relations (3-18), and the corresponding $K^{V,w,q}[\psi]$ satisfies the coercivity relation (3-17). More precisely,*

$$J_\mu^{V,w,q,\varpi}[\psi] n^\mu_{\Sigma_\tau} \gtrsim (\partial_v\psi)^2 + |\nabla\!\!\!/ \psi|^2 + r^{-2}|\partial_v(r\psi)|^2 + \iota_{r\le R}((\partial_u\psi)^2 + r^{-2}|\partial_u(r\psi)|^2) + (1+r)^{-2}\psi^2, \tag{B-7}$$

$$J_\mu^{V,w,q,\varpi}[\psi] n^\mu_{\underline{C}_v} \gtrsim (\partial_u\psi)^2 + r^{-2}|\partial_u(r\psi)|^2 + |\nabla\!\!\!/ \psi|^2 + (1+r)^{-2}\psi^2, \tag{B-8}$$

$$K^{V,w,q}[\psi] \gtrsim r^{-1}|\nabla\!\!\!/ \psi|^2 + (1+r)^{-1-\delta}((\partial_u\psi)^2 + (\partial_v\psi)^2) + (1+r)^{-3-\delta}\psi^2. \tag{B-9}$$

Moreover, the corresponding $H^{V,w}[\psi]$ satisfies

$$|H^{V,w}[\psi]| \lesssim |\partial_u\psi| + |\partial_v\psi| + \frac{1}{r}|\psi|. \tag{B-10}$$

Proof. Recalling (B-2) and (B-3), note first that

$$*d(\psi^2\varpi_2) = -\delta_1\left(\frac{\psi^2}{r^2}+\frac{2}{r}\psi\partial_r\psi\right)dt - \delta_1\frac{2}{r^2}\psi\partial_t\psi x^i\,dx^i$$
$$= \delta_1\left(-\frac{\psi^2}{2r^2}+\frac{2}{r}\psi\partial_u\psi\right)du - \delta_1\left(\frac{\psi^2}{2r^2}+\frac{2}{r}\psi\partial_v\psi\right)dv.$$

Recall the expressions (B-4) for the normals to the hypersurfaces Σ_τ and $\underline{C}_v$. The fluxes corresponding to $J^{V_i,w_i,q_i,\varpi_i}[\psi]$ satisfy

$$J^{V_1,w_1,q_1,\varpi_1}_\mu[\psi]n^\mu_{\Sigma_\tau} = (\partial_v\psi)^2+\tfrac14|\nabla\!\!\!/\psi|^2+\iota_{r\le R}(\partial_u\psi)^2,$$
$$J^{V_1,w_1,q_1,\varpi_1}_\mu[\psi]n^\mu_{\underline{C}_v} = (\partial_u\psi)^2+\tfrac14|\nabla\!\!\!/\psi|^2,$$
$$J^{V_2,w_2,q_2,\varpi_2}_\mu[\psi]n^\mu_{\Sigma_\tau} = \frac{\delta_1}{r^2}(\partial_v(r\psi))^2+\frac{\delta_1}{4r^2}|\nabla\!\!\!/(r\psi)|^2+\frac{\delta_1}{r^2}\iota_{r\le R}(\partial_u(r\psi))^2$$
$$= \delta_1 J^{V_1,w_1,q_1,\varpi_1}_\mu[\psi]n^\mu_{\Sigma_\tau}+\delta_1\left(\frac{1}{4r^2}\psi^2+\frac1r\psi\partial_v\psi+\frac{1}{4r^2}\psi^2\iota_{r\le R}-\frac1r\psi\partial_u\psi\iota_{r\le R}\right),$$
$$J^{V_2,w_2,q_2,\varpi_2}_\mu[\psi]n^\mu_{\underline{C}_v} = \frac{\delta_1}{4r^2}(\partial_u(r\psi))^2+\frac{\delta_1}{8r^2}|\nabla\!\!\!/(r\psi)|^2$$
$$= \delta_1 J^{V_1,w_1,q_1,\varpi_1}_\mu[\psi]n^\mu_{\underline{C}_v}+\delta_1\left(\frac{1}{4r^2}\psi^2-\frac1r\psi\partial_u\psi\right),$$
$$|J^{V_3,w_3,q_3,\varpi_3}_\mu[\psi]n^\mu_{\Sigma_\tau}| \le \delta_2 C((\partial_v\psi)^2+|\nabla\!\!\!/\psi|^2+(1+r)^{-2}|\psi|^2+\iota_{r\le R}(\partial_u\psi)^2),$$
$$|J^{V_3,w_3,q_3,\varpi_3}_\mu[\psi]n^\mu_{\underline{C}_v}| \le \delta_2 C((\partial_u\psi)^2+|\nabla\!\!\!/\psi|^2+(1+r)^{-2}|\psi|^2),$$

and the bulk terms satisfy

$$K^{V_1,w_1,q_1}[\psi]=0,$$
$$H^{V_1,w_1}[\psi]=(\partial_u\psi+\partial_v\psi),$$
$$K^{V_2,w_2,q_2}[\psi]=0,$$
$$H^{V_2,w_2}[\psi]=\delta_1(\partial_u\psi+\partial_v\psi),$$
$$K^{V_3,w_3,q_3}[\psi]=\delta_2\left(\frac1r-\delta_3\left(\frac{1}{r(1+r)^\delta}+\frac{\delta}{2(1+r)^{1+\delta}}\right)\right)|\nabla\!\!\!/\psi|^2$$
$$+\frac{\delta_2\delta_3\delta}{2(1+r)^{1+\delta}}((\partial_t\psi)^2+(\partial_r\psi)^2)+\frac{\delta_2\delta_3\delta(1+\delta)}{2r(1+r)^{2+\delta}}\psi^2,$$
$$H^{V_3,w_3}[\psi]=\frac{\delta_2}{2}\left(1-\frac{\delta_3}{(1+r)^\delta}\right)(\partial_v\psi-\partial_u\psi)+\frac{\delta_2}{r}\left(1-\frac{\delta_3}{(1+r)^\delta}\right)\psi.$$

It thus follows that the coercivity relations (B-7)–(B-9), along with the property (B-10), hold if δ_1, δ_2 and δ_3 are chosen according to (B-5). □

B.2. ***The J_{far} current.*** The quadruples $(\overset{(p)}{V}{}^{\circ}_{\mathrm{far}}, \overset{(p)}{w}{}^{\circ}_{\mathrm{far}}, \overset{(p)}{q}{}^{\circ}_{\mathrm{far}}, \overset{(p)}{\varpi}{}^{\circ}_{\mathrm{far}})$ and $(\widetilde{V}_{\mathrm{far}}, \tilde{w}_{\mathrm{far}}, \tilde{q}_{\mathrm{far}}, \tilde{\varpi}_{\mathrm{far}})$ of Section 3.5 are defined as follows. Consider some

$$0 < \delta_6 \ll \delta_5 \ll \delta_4 \ll \delta_3 \ll \delta_2 \ll \delta_1 \ll 1. \tag{B-11}$$

Noting that $-\frac{1}{2}\partial_{x^\mu} u\, dx^\mu = -\frac{1}{2}du = (\partial_v)^\flat$, define

$$\overset{(p)}{V}{}^{\circ}_{\mathrm{far}} = r^p \partial_v, \quad \overset{(p)}{w}{}^{\circ}_{\mathrm{far}} = \frac{r^{p-1}}{2}, \quad (\overset{(p)}{q}{}^{\circ}_{\mathrm{far}})_\mu = -\frac{\partial_{x^\mu} \overset{(p)}{w}{}^{\circ}_{\mathrm{far}}}{2} - \left(\frac{p}{4} - \frac{\delta_4}{2}\right) r^{p-2} \partial_{x^\mu} u,$$

$$\overset{(p)}{\varpi}{}^{\circ}_{\mathrm{far}} = \left(-\frac{r^{p-1}}{4} + 2\delta_5 r^{\frac{p}{2}-1}\right) r^2 \sin\vartheta\, d\vartheta \wedge d\varphi.$$

Define then

$$\widetilde{V}_{\mathrm{far}} = \frac{1}{\delta_6} V, \quad \tilde{w}_{\mathrm{far}} = \frac{1}{\delta_6} w, \quad \tilde{q}_{\mathrm{far}} = \frac{1}{\delta_6} q, \quad \tilde{\varpi}_{\mathrm{far}} = \frac{1}{\delta_6} \varpi,$$

where V, w, q, ϖ are defined by (B-6), so that

$$\overset{(p)}{V}_{\mathrm{far}} = \overset{(p)}{V}{}^{\circ}_{\mathrm{far}} + \frac{1}{\delta_6} V, \quad \overset{(p)}{w}_{\mathrm{far}} = \overset{(p)}{w}{}^{\circ}_{\mathrm{far}} + \frac{1}{\delta_6} w, \quad \overset{(p)}{q}_{\mathrm{far}} = \overset{(p)}{q}{}^{\circ}_{\mathrm{far}} + \frac{1}{\delta_6} q, \quad \overset{(p)}{\varpi}_{\mathrm{far}} = \overset{(p)}{\varpi}{}^{\circ}_{\mathrm{far}} + \frac{1}{\delta_6} \varpi.$$

Proposition B.2.1 (the J_{far} current in Minkowski space). *With $(\overset{(p)}{V}_{\mathrm{far}}, \overset{(p)}{w}_{\mathrm{far}}, \overset{(p)}{q}_{\mathrm{far}}, \overset{(p)}{\varpi}_{\mathrm{far}})$ defined as above, if $\delta_1, \dots, \delta_6$ are chosen according to (B-11), the associated currents $\overset{(p)}{J}_{\mathrm{far}}$, $\overset{(p)}{K}_{\mathrm{far}}$, defined by (3-11), (3-12), satisfy the weighted bulk coercivity property (3-40) and the weighted boundary coercivity properties (3-41). More precisely, for $\delta \le p \le 2-\delta$ and $r \ge R$,*

$$\overset{(p)}{J}_{\mathrm{far}\,\mu}[\psi] n^\mu_{\Sigma_\tau} \gtrsim r^{p-2} |\partial_v(r\psi)|^2 + r^{\frac{p}{2}} (\partial_v \psi)^2 + |\nabla\!\!\!/ \psi|^2 + r^{\frac{p}{2}-2} \psi^2, \tag{B-12}$$

$$\overset{(p)}{J}_{\mathrm{far}\,\mu}[\psi] n^\mu_{\underline{C}_v} \gtrsim (\partial_u \psi)^2 + r^p |\nabla\!\!\!/ \psi|^2 + r^{p-2} \psi^2, \tag{B-13}$$

$$\overset{(p)}{K}_{\mathrm{far}}[\psi] \gtrsim r^{p-3} |\partial_v(r\psi)|^2 + r^{p-1} |\partial_v \psi|^2 + r^{p-1} |\nabla\!\!\!/ \psi|^2 + r^{-1-\delta} |\partial_u \psi|^2 + r^{p-3} \psi^2. \tag{B-14}$$

Moreover, the corresponding $\overset{(p)}{H}_{\mathrm{far}}[\psi]$ satisfies

$$|\overset{(p)}{H}_{\mathrm{far}}[\psi]| \lesssim r^{p-1} |\partial_v(r\psi)| + |\partial_u \psi| + |\partial_v \psi| + \frac{1}{r} |\psi|. \tag{B-15}$$

Proof. Recall again (B-3) and note that

$$\begin{aligned} * d(\psi^2 \overset{(p)}{\varpi}{}^{\circ}_{\mathrm{far}}) = {} & \left(\left(\frac{p+1}{8} r^{p-2} - \delta_5 \frac{p+2}{2} r^{\frac{p}{2}-2} \right) \psi^2 - \left(\frac{r^{p-1}}{2} - 4\delta_5 r^{\frac{p}{2}-1} \right) \psi \partial_u \psi \right) du \\ & + \left(\left(\frac{p+1}{8} r^{p-2} - \delta_5 \frac{p+2}{2} r^{\frac{p}{2}-2} \right) \psi^2 + \left(\frac{r^{p-1}}{2} - 4\delta_5 r^{\frac{p}{2}-1} \right) \psi \partial_v \psi \right) dv. \end{aligned}$$

Note moreover that

$$(\pi^{\overset{(p)}{V}{}^{\circ}_{\mathrm{far}}})^{\sharp\sharp} = p r^{p-1} \partial_v \otimes \partial_v - \frac{p r^{p-1}}{2} (\partial_u \otimes \partial_v + \partial_v \otimes \partial_u) + \frac{r^{p-3}}{2} (\partial_\vartheta \otimes \partial_\vartheta + \sin^{-2}\vartheta\, \partial_\varphi \otimes \partial_\varphi)$$

and

$$\nabla^\mu (\overset{(p)}{q}{}^{\circ}_{\mathrm{far}})_\mu = \frac{p r^{p-3}}{4} - \frac{\delta_4 p r^{p-3}}{2}.$$

The fluxes corresponding to $J^{\overset{(p)}{\mathring{V}}_{\text{far}},\overset{(p)}{\mathring{w}}_{\text{far}},\overset{(p)}{\mathring{q}}_{\text{far}},\overset{(p)}{\mathring{\varpi}}_{\text{far}}}[\psi]$ satisfy, for $r \geq R$,

$$J_\mu^{\overset{(p)}{\mathring{V}}_{\text{far}},\overset{(p)}{\mathring{w}}_{\text{far}},\overset{(p)}{\mathring{q}}_{\text{far}},\overset{(p)}{\mathring{\varpi}}_{\text{far}}}[\psi]n^\mu_{\Sigma_\tau} = r^{p-2}|\partial_v(r\psi)|^2 - \delta_5\left(\frac{p+2}{2}r^{\frac{p}{2}-2}\psi^2 + 4r^{\frac{p}{2}-1}\psi\partial_v\psi\right)$$

$$= r^{p-2}|\partial_v(r\psi)|^2 + \delta_5\left(\frac{(2-p)}{2}r^{\frac{p}{2}-2}\psi^2 - 4r^{\frac{p}{2}-2}\psi\partial_v(r\psi)\right), \tag{B-16}$$

$$J_\mu^{\overset{(p)}{\mathring{V}}_{\text{far}},\overset{(p)}{\mathring{w}}_{\text{far}},\overset{(p)}{\mathring{q}}_{\text{far}},\overset{(p)}{\mathring{\varpi}}_{\text{far}}}[\psi]n^\mu_{\underline{C}_v} = \frac{r^p}{4}|\nabla\!\!\!/\psi|^2 + \frac{\delta_4}{2}r^{p-2}\psi^2 - \delta_5\left(\frac{p+2}{2}r^{\frac{p}{2}-2}\psi^2 - 4r^{\frac{p}{2}-1}\psi\partial_u\psi\right), \tag{B-17}$$

and the bulk terms satisfy

$$K^{\overset{(p)}{\mathring{V}}_{\text{far}},\overset{(p)}{\mathring{w}}_{\text{far}},\overset{(p)}{\mathring{q}}_{\text{far}}}[\psi] = pr^{p-3}|\partial_v(r\psi)|^2 + \frac{(2-p)r^{p-1}}{4}|\nabla\!\!\!/\psi|^2 + \frac{\delta_4(2-p)}{2}r^{p-3}\psi^2 - 2\delta_4 r^{p-3}\psi\partial_v(r\psi), \tag{B-18}$$

$$H^{\overset{(p)}{\mathring{V}}_{\text{far}},\overset{(p)}{\mathring{w}}_{\text{far}}}[\psi] = r^{p-1}\partial_v(r\psi). \tag{B-19}$$

In particular, if δ_4 and δ_5 are chosen according to (B-11) (depending on p), then

$$J_\mu^{\overset{(p)}{\mathring{V}}_{\text{far}},\overset{(p)}{\mathring{w}}_{\text{far}},\overset{(p)}{\mathring{q}}_{\text{far}},\overset{(p)}{\mathring{\varpi}}_{\text{far}}}[\psi]n^\mu_{\Sigma_\tau} \gtrsim r^{p-2}|\partial_v(r\psi)|^2 + r^{\frac{p}{2}}(\partial_v\psi)^2 + r^{\frac{p}{2}-2}\psi^2, \tag{B-20}$$

$$J_\mu^{\overset{(p)}{\mathring{V}}_{\text{far}},\overset{(p)}{\mathring{w}}_{\text{far}},\overset{(p)}{\mathring{q}}_{\text{far}},\overset{(p)}{\mathring{\varpi}}_{\text{far}}}[\psi]n^\mu_{\underline{C}_v} + (\partial_u\psi)^2 \gtrsim r^p|\nabla\!\!\!/\psi|^2 + r^{p-2}\psi^2, \tag{B-21}$$

and

$$K^{\overset{(p)}{\mathring{V}}_{\text{far}},\overset{(p)}{\mathring{w}}_{\text{far}},\overset{(p)}{\mathring{q}}_{\text{far}}}[\psi] \gtrsim pr^{p-3}|\partial_v(r\psi)|^2 + (2-p)r^{p-1}|\nabla\!\!\!/\psi|^2 + p(2-p)r^{p-1}|\partial_v\psi|^2 + (2-p)r^{p-3}\psi^2. \tag{B-22}$$

It follows from Proposition B.1.1 that the currents $\overset{(p)}{J}_{\text{far}}$, $\overset{(p)}{K}_{\text{far}}$ satisfy the weighted bulk coercivity properties (B-12)–(B-14), provided $\delta_1, \dots, \delta_6$ are chosen according to (B-11), and that $\overset{(p)}{H}_{\text{far}}$ satisfies (B-15). □

The inequalities (B-10) and (B-15) in particular mean that the $r^{-1}|\nabla\!\!\!/\psi|$ term on the left-hand side of the assumed inequality (4-30) is superfluous in the case that $(\mathcal{M}, g_0)$ is Minkowski space. This term is estimated in the proof of Proposition 4.7.2 nonetheless in order to illustrate how it is estimated in the case of Kerr.

Appendix C: Verifying the null condition assumption

In this section the proof of Proposition 4.7.2 is given. Proposition 4.7.2 can be more precisely stated as follows.

Proposition C.1. *Assumption 4.7.1 holds for the classical null condition of Klainerman [1986] on Minkowski space and more generally the class of equations on Kerr considered in Luk [2013].*

More precisely, if $(\mathcal{M}, g_0)$ is either Minkowski space or a member of the Kerr family and if

$$F = N^{\mu\nu}(\psi, x)\partial_\mu\psi\partial_\nu\psi$$

satisfies the assumption (C-22), then there exists $k_{\text{null}} > 0$ and, for all $k \geq k_{\text{null}}$, there exists $\varepsilon_{\text{null}} > 0$ such that we have the following.

Let ψ be a smooth function in $\mathcal{R}(\tau_0, \tau_1, v)$ satisfying (4-29) for $0 < \varepsilon \le \varepsilon_{\text{null}}$. Then, for all $\delta \le p \le 2-\delta$,

$$\sum_{|\boldsymbol{k}|\le k} \int_{\mathcal{R}(\tau_0,\tau',v)\cap\{r\ge R\}} \Big(|\underline{L}\mathfrak{D}^{\boldsymbol{k}}\psi| + |L\mathfrak{D}^{\boldsymbol{k}}\psi| + \frac{1}{r}|\not\nabla\mathfrak{D}^{\boldsymbol{k}}\psi| + \frac{1}{r}|\mathfrak{D}^{\boldsymbol{k}}\psi|\Big)|\mathfrak{D}^{\boldsymbol{k}}F| + \int_{\mathcal{R}(\tau_0,\tau',v)\cap\{r\ge R\}} (\mathfrak{D}^{\boldsymbol{k}}F)^2 \lesssim \sqrt{\overset{(0+)}{\mathcal{X}}_{\ll k\, 8R/9}(\tau_0,\tau_1)\overset{(0+)}{\mathcal{X}}_{k\, R}(\tau_0,\tau_1)}$$

and, moreover,

$$\sum_{|\boldsymbol{k}|\le k} \int_{\mathcal{R}(\tau_0,\tau',v)\cap\{r\ge R\}} \Big(r^{p-1}|L(r\mathfrak{D}^{\boldsymbol{k}}\psi)| + |\underline{L}\mathfrak{D}^{\boldsymbol{k}}\psi| + |L\mathfrak{D}^{\boldsymbol{k}}\psi| + \frac{1}{r}|\not\nabla\mathfrak{D}^{\boldsymbol{k}}\psi| + \frac{1}{r}|\mathfrak{D}^{\boldsymbol{k}}\psi|\Big)|\mathfrak{D}^{\boldsymbol{k}}F|$$
$$+ \int_{\mathcal{R}(\tau_0,\tau',v)\cap\{r\ge R\}} (\mathfrak{D}^{\boldsymbol{k}}F)^2 \lesssim \sqrt{\overset{(0)}{\mathcal{X}}_{\ll k\, 8R/9}(\tau_0,\tau_1)\overset{(p)}{\mathcal{X}}_{k\, R}(\tau_0,\tau_1)} + \sqrt{\overset{(p)}{\mathcal{X}}_{\ll k\, R}(\tau_0,\tau_1)}\sqrt{\overset{(p)}{\mathcal{X}}_{k\, R}(\tau_0,\tau_1)}\sqrt{\overset{(0)}{\mathcal{X}}_{k\, R}(\tau_0,\tau_1)}.$$

The estimate of Proposition C.1 is only nontrivial for large r and, thus, the main content appears already around Minkowski space. Accordingly, the proof will be given in detail for the case that $(\mathcal{M}, g_0)$ is Minkowski space for the two model nonlinearities

$$N^{\mu\nu}(\psi, x)\partial_\mu\psi\partial_\nu\psi = \partial_u\psi\partial_v\psi,$$
$$N^{\mu\nu}(\psi, x)\partial_\mu\psi\partial_\nu\psi = \not g^{AB}\not\nabla_A\psi\not\nabla_B\psi,$$

where $\not g = r^2\gamma$ is the induced round metric and $\not\nabla$ its associated connection. See Section C.1. The more general nonlinearities and the Kerr case are discussed in Section C.2.

C.1. ***Two model nonlinearities on Minkowski space.*** Throughout this section, we will write $(\mathcal{M}, g_0)$ to denote Minkowski space (see Section 2.7.1). Note that, in the region $r \ge \frac{8}{9}R$ which will be considered here, the r of (2-1) coincides with the standard radial coordinate (B-1) used in Appendix B and thus coincides with what was denoted as $\tilde{r}$ in Section 2.7.1.

Recall the (u, v, θ, ϕ) coordinate system. In the region $r \ge R$ the null vector fields take the form

$$L = \partial_v, \quad \underline{L} = \partial_u.$$

The spheres $\Sigma(\tau) \cap \underline{C}_v$ are round, $\not g = r^2\gamma$ denotes the induced round metric and $\not\nabla$ its associated connection. Define

$$\overset{(0)}{\mathcal{F}}{}^*_k := \sup_{\tau_0\le\tau\le\tau_1} \overset{(0)}{\mathcal{F}}_k(\tau),$$
$$\overset{(p)}{\mathcal{E}}{}^*_k := \sup_{\tau_0\le\tau\le\tau_1} \overset{(p)}{\mathcal{E}}_k(\tau),$$
$$\int^* \overset{(p-1)}{\mathcal{E}'_k} := \int_{\tau_0}^{\tau_1} \overset{(p-1)}{\mathcal{E}'_k}(\tau)\,d\tau,$$

and similarly for $\ll k$ replacing k, and with R subscripts added, etc. This notation will be used throughout this section.

The main result of this section is the following.

Proposition C.1.1 (model nonlinearities on Minkowski space). *Fix $\delta \le p \le 2-\delta$ and $k \ge 7$, and let ψ be a smooth function in $\mathcal{R}(\tau_0, \tau_1, v) \cap \{r \ge \frac{8}{9}R\}$ satisfying (4-29).*

Then the nonlinearities

$$F = \partial_u\psi\,\partial_v\psi \quad \textit{and} \quad F = \not{g}^{AB}\not\nabla_A\psi\not\nabla_B\psi$$

both satisfy

$$\sum_{|\boldsymbol{k}|\le k}\int_{\mathcal{R}(\tau_0,\tau',v)\cap\{r\ge R\}}\Big(|\partial_u\mathfrak{D}^{\boldsymbol{k}}\psi| + |\partial_v\mathfrak{D}^{\boldsymbol{k}}\psi| + \frac{1}{r}|\not\nabla\mathfrak{D}^{\boldsymbol{k}}\psi| + \frac{1}{r}|\mathfrak{D}^{\boldsymbol{k}}\psi|\Big)|\mathfrak{D}^{\boldsymbol{k}}F| + \int_{\mathcal{R}(\tau_0,\tau',v)\cap\{r\ge R\}}(\mathfrak{D}^{\boldsymbol{k}}F)^2$$
$$\lesssim \sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{F}}}{}^*_R + \overset{(0)}{\underset{\ll k}{\mathcal{E}}}{}^*_{8R/9} + \int^*\overset{(-1-\delta)}{\underset{\ll k}{\mathcal{E}'}}_R}\Big(\overset{(0)}{\underset{k}{\mathcal{F}}}{}^*_R + \overset{(0)}{\underset{k}{\mathcal{E}}}{}^*_R + \int^*\overset{(\delta-1)}{\underset{k}{\mathcal{E}'}}_R\Big) + \sqrt{\int^*\overset{(\delta-1)}{\underset{\ll k}{\mathcal{E}'}}_R}\Big(\overset{(0)}{\underset{k}{\mathcal{F}}}{}^*_R + \int^*\overset{(-1-\delta)}{\underset{k}{\mathcal{E}}}_R\Big)$$

and

$$\sum_{|\boldsymbol{k}|\le k}\int_{\mathcal{R}(\tau_0,\tau',v)\cap\{r\ge R\}}\Big(r^{p-1}|\partial_v(r\mathfrak{D}^{\boldsymbol{k}}\psi)| + |\partial_u\mathfrak{D}^{\boldsymbol{k}}\psi| + |\partial_v\mathfrak{D}^{\boldsymbol{k}}\psi| + \frac{1}{r}|\not\nabla\mathfrak{D}^{\boldsymbol{k}}\psi| + \frac{1}{r}|\mathfrak{D}^{\boldsymbol{k}}\psi|\Big)|\mathfrak{D}^{\boldsymbol{k}}F|$$
$$+ \int_{\mathcal{R}(\tau_0,\tau',v)\cap\{r\ge R\}}(\mathfrak{D}^{\boldsymbol{k}}F)^2$$
$$\lesssim \sqrt{\int^*\overset{(p-1)}{\underset{\ll k}{\mathcal{E}'}}_R}\sqrt{\overset{(0)}{\underset{k}{\mathcal{F}}}{}^*_R}\sqrt{\int^*\overset{(p-1)}{\underset{k}{\mathcal{E}'}}_R} + \sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{F}}}{}^*_R + \overset{(0)}{\underset{\ll k}{\mathcal{E}}}{}^*_{8R/9} + \int^*\overset{(-1-\delta)}{\underset{\ll k}{\mathcal{E}'}}_R}\Big(\overset{(0)}{\underset{k}{\mathcal{F}}}{}^*_R + \overset{(p)}{\underset{k}{\mathcal{E}}}{}^*_R + \int^*\overset{(p-1)}{\underset{k}{\mathcal{E}'}}_R\Big).$$

The following properties of Minkowski space will be used in the proof of Proposition C.1.1. First, for any $\boldsymbol{k}$ and any function ψ, for $r \ge R$,

$$|\mathfrak{D}^{\boldsymbol{k}}L\psi| \lesssim \sum_{|\tilde{\boldsymbol{k}}|\le|\boldsymbol{k}|}|L\mathfrak{D}^{\tilde{\boldsymbol{k}}}\psi|, \qquad |\mathfrak{D}^{\boldsymbol{k}}\underline{L}\psi| \lesssim \sum_{|\tilde{\boldsymbol{k}}|\le|\boldsymbol{k}|}|\underline{L}\mathfrak{D}^{\tilde{\boldsymbol{k}}}\psi|. \tag{C-1}$$

Indeed, (C-1) follows from $L = \partial_v$, $\underline{L} = \partial_u$, $T = N = \partial_u + \partial_v$, and $[\partial_u, \Omega_i] = [\partial_v, \Omega_i] = 0$ for $i = 1, 2, 3$. Thus ∂_u and ∂_v commute with all components of $\mathfrak{D}$. Second, the fact that, for any $\boldsymbol{k}$ and any function ψ, for $r \ge R$,

$$|\mathfrak{D}^{\boldsymbol{k}}(\not{g}^{AB}\not\nabla_A\psi\not\nabla_B\psi)| \lesssim \sum_{|\boldsymbol{k}_1|+|\boldsymbol{k}_2|\le|\boldsymbol{k}|}|\not\nabla\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\not\nabla\mathfrak{D}^{\boldsymbol{k}_2}\psi| \tag{C-2}$$

will also be used.

In the above and what follows, note that when the volume form is omitted, the usual spacetime volume form is to be understood, and this contains in particular an omitted $r^2\sin\vartheta$ factor.

The proof of Proposition C.1.1 relies on weighted Sobolev inequalities, which are discussed in Section C.1.1. In Section C.1.2 the proof of Proposition C.1.1 for the case of the nonlinearity $F = \partial_u\psi\,\partial_v\psi$ is given, followed by the proof for the nonlinearity $F = \not{g}^{AB}\not\nabla_A\psi\not\nabla_B\psi$ in Section C.1.3.

C.1.1. *Weighted Sobolev inequalities.* The proof of Proposition C.1 relies on certain weighted Sobolev inequalities.

Let $d\omega$ denote the unit volume form on S^2:

$$d\omega = \sin\vartheta\, d\vartheta\, d\varphi.$$

Recall (see Section 2.7.1) that $r(u, v) = \frac{1}{2}(v-u)$. Define $v(R) = v(R, u) = 2R + u$, so that we have $r(u, v(R, u)) = R$, and similarly define $u(R, v) = v - 2R$, so that $r(u(R, v), v) = R$.

Recall the region $\mathcal{R}(\tau_0, \tau_1, v)$. In order to avoid the blowup of constants in Sobolev inequalities, the region near the corner $\{u = \tau_0\} \cap \{r = R\}$ will typically be considered separately from its complement. Accordingly, define

$$\mathcal{R}_R(\tau_0, \tau_1, v) := \mathcal{R}(\tau_0, \tau_1, v) \cap \{r \geq R\}, \quad \check{\mathcal{R}}_R(\tau_0, \tau_1, v) := \mathcal{R}_R(\tau_0, \tau_1, v) \setminus \{v' \leq v(R, \tau_0 + R)\}, \tag{C-3}$$

and

$$\mathcal{R}_{\text{Corner}}(\tau_0, \tau_1, v) := \mathcal{R}(\tau_0, \tau_1, v) \cap \{r \geq R\} \cap \{v' \leq v(R, \tau_0 + R)\}. \tag{C-4}$$

Note that

$$r \leq \tfrac{1}{2} 3R \quad \text{in } \mathcal{R}_{\text{Corner}}(\tau_0, \tau_1, v). \tag{C-5}$$

Define also

$$T_1 = T_1(v) = \min\{\tau_1, u(R, v)\}, \quad T_1' = T_1(v'), \tag{C-6}$$

and

$$\Sigma_R(\tau, v) = \Sigma(\tau, v) \cap \{r \geq R\}.$$

Proposition C.1.2 (Sobolev inequality on incoming cones $\underline{C}_v$). *For any $\tau_0 \leq u \leq \tau_1$, $v(R, \tau_0 + R) \leq v' \leq v$, $(\vartheta, \varphi) \in S^2$ (i.e., for any $(u, v', \vartheta, \varphi) \in \check{\mathcal{R}}_R(\tau_0, \tau_1, v)$), and any function $f : \mathcal{R}(\tau_0, \tau_1, v) \to \mathbb{R}$,*

$$r|f(u, v', \vartheta, \varphi)| \lesssim (1 + (\tau_1 - \tau_0)^{-\frac{1}{2}}) \sum_{k \leq 1} \left(\int_{\tau_0}^{T_1'} \int_{S^2} \left(|\underline{L}^k f|^2 + \sum_{i=1}^{3} |\Omega_i \underline{L}^k f|^2 + \sum_{i,j=1}^{3} |\Omega_i \Omega_j \underline{L}^k f|^2 \right)(u', v', \vartheta, \varphi) r^2 \, d\omega \, du' \right)^{\frac{1}{2}}.$$

Proof. For simplicity, consider first the case that $v' \geq v(R, \tau_1)$, so that $T_1' = \tau_1$. Suppose first that $u \geq \frac{1}{2}(\tau_1 + \tau_2)$. Let ϕ be a smooth cut-off function such that $\phi(\tau_1) = 0$, $\phi(\tau) = 1$ for $\tau \geq \frac{1}{2}(\tau_1 + \tau_2)$, and

$$|\phi'(\tau)| \leq \frac{4}{\tau_2 - \tau_1} \quad \text{for } \tau_1 \leq \tau \leq \frac{\tau_1 + \tau_2}{2}.$$

Now

$$\partial_u(\phi(u)|f(u, v', \vartheta, \varphi)|^2) = \phi'(u)|f(u, v', \vartheta, \varphi)|^2 + 2\phi(u) f(u, v', \vartheta, \varphi) \partial_u f(u, v', \vartheta, \varphi).$$

Integrating from τ_0 to u gives

$$r^2 |f(u, v', \vartheta, \varphi)|^2 \lesssim (1 + (\tau_1 - \tau_0)^{-1}) \int_{\tau_0}^{\tau_1} |f(u', v', \vartheta, \varphi)|^2 r^2 \, du' + \int_{\tau_0}^{\tau_1} |\partial_u f(u', v', \vartheta, \varphi)|^2 r^2 \, du'.$$

The result then follows from the standard Sobolev inequality on S^2,

$$\sup_{(\vartheta, \varphi) \in S^2} |f(u', v', \vartheta, \varphi)| \lesssim r^{-1} \left(\int_{S^2} \left(|f|^2 + \sum_{i=1}^{3} |\Omega_i f|^2 + \sum_{i,j=1}^{3} |\Omega_i \Omega_j f|^2 \right)(u', v', \vartheta, \varphi) r^2 \sin\vartheta \, d\vartheta \, d\varphi \right)^{\frac{1}{2}}.$$

Similarly for $u \leq \frac{1}{2}(\tau_1 + \tau_2)$, and similarly for $v(R, \tau_0 + R) \leq v' \leq v(R, \tau_1)$. $\square$

Proposition C.1.2 in particular implies that

$$\sup_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)} \sum_{|\boldsymbol{k}|\le k-3} (r|\partial_u \mathfrak{D}^{\boldsymbol{k}}\psi| + r|\nabla\!\!\!/(\mathfrak{D}^{\boldsymbol{k}}\psi)|) \lesssim (1+(\tau_1-\tau_0)^{-\frac{1}{2}})\sqrt{\overset{(0)}{\underset{k}{\mathcal{F}}}{}^*} \tag{C-7}$$

and, for any $s \ge 0$ and any $\tau_0 \le u \le \tau_1,\ v(R,\tau_0+R) \le v' \le v,\ (\vartheta,\varphi) \in S^2$,

$$r^s|\partial_v \mathfrak{D}^{\boldsymbol{k}}\psi(u,v',\vartheta,\varphi)| \lesssim (1+(\tau_1-\tau_0)^{-\frac{1}{2}}) \sum_{|\tilde{\boldsymbol{k}}|\le|\boldsymbol{k}|+3} \left(\int_{\tau_0}^{T_1'}\int_{S^2} r^{s-2}|\partial_v \mathfrak{D}^{\boldsymbol{k}}\psi(u',v',\vartheta,\varphi)| r^2\, d\omega\, du'\right)^{\frac{1}{2}}. \tag{C-8}$$

It is in the forms (C-7) and (C-8) that Proposition C.1.2 will be used below.

Proposition C.1.3 (Sobolev inequality on $\Sigma(\tau)$). *For any $\tau_0 \le \tau \le \tau_1,\ v(R,u) \le v' \le v,\ (\vartheta,\varphi) \in S^2$, and any function $f : \mathcal{R}(\tau_0,\tau_1,v) \to \mathbb{R}$,*

$$|f(\tau,v',\vartheta,\varphi)| \lesssim \sum_{k=0}^{1}\left(\int_{\Sigma_R(\tau,v)} r^{-2}\left(|L^k f|^2 + \sum_{i=1}^{3}|\Omega_i L^k f|^2 + \sum_{i,j=1}^{3}|\Omega_i\Omega_j L^k f|^2\right) r^2\, d\omega\, dv'\right)^{\frac{1}{2}} + \left(\int_{\Sigma_{8R/9}(\tau,v)} r^{-2}|f|^2 r^2\, d\omega\, dv'\right)^{\frac{1}{2}}.$$

Proof. The proof follows from the fundamental theorem of calculus and the standard Sobolev inequality on S^2, as in the proof of Proposition C.1.2. □

Proposition C.1.3 in particular implies that

$$\sup_{\mathcal{R}_{\text{Corner}}(\tau_0,\tau_1,v)} \sum_{|\boldsymbol{k}|\le k-4} |\mathfrak{D}^{\boldsymbol{k}}\psi| \lesssim \sqrt{\overset{(0)}{\underset{k}{\mathcal{E}}}{}^*_{8R/9}}, \tag{C-9}$$

where the region $\mathcal{R}_{\text{Corner}}$ is defined in (C-4). It is in the form (C-9) that Proposition C.1.3 will typically be used.

C.1.2. *The proof of Proposition C.1.1 for the nonlinearity $F = \partial_u\psi\partial_v\psi$.* Proposition C.1.1 for the nonlinearity

$$F = \partial_u\psi\partial_v\psi$$

follows from the following proposition, whose proof is the subject of the present section.

Proposition C.1.4 (nonlinear error estimates for $F = \partial_u\psi\partial_v\psi$ on Minkowski space). *Under the assumptions of Proposition C.1.1, for each $|\boldsymbol{k}| \le k$,*

$$\sum_{|\boldsymbol{k}_1|+|\boldsymbol{k}_2|\le|\boldsymbol{k}|}\left(\int_{\mathcal{R}(\tau_0,\tau_1,v)\cap\{r\ge R\}}\left(|\partial_u\mathfrak{D}^{\boldsymbol{k}}\psi| + |\partial_v\mathfrak{D}^{\boldsymbol{k}}\psi| + \frac{1}{r}|\nabla\!\!\!/\mathfrak{D}^{\boldsymbol{k}}\psi| + \frac{1}{r}|\mathfrak{D}^{\boldsymbol{k}}\psi|\right)\cdot|\mathfrak{D}^{\boldsymbol{k}_1}\partial_u\psi|\cdot|\mathfrak{D}^{\boldsymbol{k}_2}\partial_v\psi| + \int_{\mathcal{R}(\tau_0,\tau_1,v)\cap\{r\ge R\}}|\mathfrak{D}^{\boldsymbol{k}_1}\partial_u\psi|^2\cdot|\mathfrak{D}^{\boldsymbol{k}_2}\partial_v\psi|^2\right)$$

$$\lesssim \sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{F}}}{}^*_R + \int^*\overset{(-1-\delta)}{\underset{\ll k}{\mathcal{E}'}}_R}\left(\overset{(0)}{\underset{k}{\mathcal{F}}}{}^*_R + \overset{(0)}{\underset{k}{\mathcal{E}}}{}^*_R + \int^*\overset{(\delta-1)}{\underset{k}{\mathcal{E}'}}_R\right) + \sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{E}}}{}^*_{8R/9} + \int^*\overset{(\delta-1)}{\underset{\ll k}{\mathcal{E}'}}_R}\left(\overset{(0)}{\underset{k}{\mathcal{F}}}{}^*_R + \int^*\overset{(-1-\delta)}{\underset{k}{\mathcal{E}}}_R\right)$$

and

$$\sum_{|\boldsymbol{k}_1|+|\boldsymbol{k}_2|\le|\boldsymbol{k}|}\int_{\mathcal{R}(\tau_0,\tau_1,v)\cap\{r\ge R\}} r^{p-1}|\partial_v(r\mathfrak{D}^{\boldsymbol{k}}\psi)|\cdot|\mathfrak{D}^{\boldsymbol{k}_1}\partial_u\psi|\cdot|\mathfrak{D}^{\boldsymbol{k}_2}\partial_v\psi|$$
$$\lesssim \sqrt{\int^{*}\overset{(p-1)}{\mathcal{E}'}_{\ll k\,R}}\sqrt{\overset{(0)}{\mathcal{F}}{}^{*}_{k\,R}}\sqrt{\int^{*}\overset{(p-1)}{\mathcal{E}'}_{k\,R}}+\sqrt{\overset{(0)}{\mathcal{F}}{}^{*}_{\ll k\,R}+\overset{(0)}{\mathcal{E}}{}^{*}_{\ll k\,8R/9}+\int^{*}\overset{(-1-\delta)}{\mathcal{E}'}_{\ll k\,R}}\Bigl(\overset{(0)}{\mathcal{F}}{}^{*}_{k\,R}+\overset{(p)}{\mathcal{E}}{}^{*}_{k\,R}+\int^{*}\overset{(p-1)}{\mathcal{E}'}_{k\,R}\Bigr).$$

Proof. The separate terms are estimated individually. We will use (4-29) to replace quartic bounds with cubic ones without further comment. In view of the fact (C-1), the proof follows from the estimates (C-10)–(C-16) below.

The notation of Section C.1.1 will be used throughout. Recall, in particular, the regions $\check{\mathcal{R}}_R$ and $\mathcal{R}_{\mathrm{Corner}}$ defined in (C-3) and (C-4), respectively, the boundedness property (C-5) of r in $\mathcal{R}_{\mathrm{Corner}}$, and the T_1 and T_1' notation defined in (C-6).

The constant in the Sobolev inequality (C-7) blows up as $\tau_1-\tau_0\to 0$. Accordingly, the cases

$$\tau_1-\tau_0\ge 1 \quad\text{and}\quad \tau_1-\tau_0<1$$

are typically considered separately. Similarly, the region $\mathcal{R}_{\mathrm{Corner}}$, near the corner $\{u=\tau_0\}\cap\{r=R\}$, is treated separately.

Estimate in the corner region $\{v'\le v(R,\tau_0+R)\}$: For any $|\tilde{\boldsymbol{k}}|\le k+1$,

$$\sum_{|\boldsymbol{k}_1|+|\boldsymbol{k}_2|\le|\tilde{\boldsymbol{k}}|-1}\int_{\mathcal{R}_{\mathrm{Corner}}(\tau_0,\tau_1,v)}|\mathfrak{D}^{\tilde{\boldsymbol{k}}}\psi|\cdot|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|\lesssim\sqrt{\overset{(0)}{\mathcal{E}}{}^{*}_{\ll k\,8R/9}}\int^{*}\overset{(-1-\delta)}{\mathcal{E}'}_{k\,R}. \tag{C-10}$$

Indeed, consider some $|\boldsymbol{k}_1|+|\boldsymbol{k}_2|\le|\tilde{\boldsymbol{k}}|-1$. Since $k\ge 7$, it follows that either $|\boldsymbol{k}_1|\le k-4$ or $|\boldsymbol{k}_2|\le k-4$. In the former case,

$$\int_{\mathcal{R}_{\mathrm{Corner}}(\tau_0,\tau_1,v)}|\mathfrak{D}^{\tilde{\boldsymbol{k}}}\psi|\cdot|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|$$
$$\lesssim\sup_{\mathcal{R}_{\mathrm{Corner}(\tau_0,\tau_1,v)}}|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|\Bigl(\int_{\mathcal{R}_{\mathrm{Corner}}(\tau_0,\tau_1,v)}|\mathfrak{D}^{\tilde{\boldsymbol{k}}}\psi|\Bigr)^{\frac12}\Bigl(\int_{\mathcal{R}_{\mathrm{Corner}}(\tau_0,\tau_1,v)}|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|\Bigr)^{\frac12}$$
$$\lesssim\sqrt{\overset{(0)}{\mathcal{E}}{}^{*}_{\ll k\,8R/9}}\int^{*}\overset{(-1-\delta)}{\mathcal{E}'}_{k\,R}$$

by the Sobolev inequality (C-9). Similarly when $|\boldsymbol{k}_2|\le k-4$.

It follows, in view of the boundedness property (C-5), that the estimates of the proposition are trivial in the corner region. The remainder of the proof thus concerns the region $\check{\mathcal{R}}_R(\tau_0,\tau_1,v)$.

Estimate for the $\partial_u\psi$ *term*: First note that, for any $|\boldsymbol{k}|\le k$,

$$\sum_{|\boldsymbol{k}_1|+|\boldsymbol{k}_2|\le|\boldsymbol{k}|}\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)}|\partial_u\mathfrak{D}^{\boldsymbol{k}}\psi|\cdot|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|$$
$$\lesssim\sqrt{\overset{(0)}{\mathcal{F}}{}^{*}_{\ll k}}\Bigl(\overset{(0)}{\mathcal{E}}{}^{*}_{k}+\int^{*}\overset{(\delta-1)}{\mathcal{E}'}_{k}\Bigr)+\sqrt{\overset{(0)}{\mathcal{E}}{}^{*}_{\ll k}+\int^{*}\overset{(\delta-1)}{\mathcal{E}'}_{\ll k}}\Bigl(\overset{(0)}{\mathcal{F}}{}^{*}_{k}+\int^{*}\overset{(-1-\delta)}{\mathcal{E}}_{k}\Bigr). \tag{C-11}$$

Indeed, suppose $|\boldsymbol{k}_1|+|\boldsymbol{k}_2|\leq|\boldsymbol{k}|$. Since $k\geq 7$, it follows that either $|\boldsymbol{k}_1|\leq k-3$ or $|\boldsymbol{k}_2|\leq k-3$. If $|\boldsymbol{k}_1|\leq k-3$, then, for $\tau_1-\tau_0\geq 1$,

$$\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)}|\partial_u\mathfrak{D}^{\boldsymbol{k}}\psi|\cdot|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|$$
$$\lesssim\int_{v(R,\tau_0+R)}^{v}\sup_{u,\theta}r|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|\int_{\tau_0}^{T_1'}\int_{S^2}(r^{-1-\delta}|\partial_u\mathfrak{D}^{\boldsymbol{k}}\psi|^2+r^{-1+\delta}|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|^2)r^2\,d\omega\,du\,dv'\lesssim\sqrt{\overset{(0)}{\mathcal{F}}{}^*_{\ll k}}\int^*\overset{(\delta-1)}{\mathcal{E}}{}'_k$$

by the Sobolev inequality on the incoming cones (C-7). For $\tau_1-\tau_0<1$,

$$\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)}|\partial_u\mathfrak{D}^{\boldsymbol{k}}\psi|\cdot|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|$$
$$\lesssim\int_{v(R,\tau_0)}^{v}\sup_{\substack{\tau_0\leq u\leq T_1'\\ \theta\in S^2}}|r\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|\left(\int_{\tau_0}^{T_1'}\int_{S^2}r^{-1-\delta}|\partial_u\mathfrak{D}^{\boldsymbol{k}}\psi|r^2\,d\omega\,du\right)^{\frac12}\left(\int_{\tau_0}^{T_1'}\int_{S^2}r^{-1+\delta}|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|r^2\,d\omega\,du\right)^{\frac12}dv'$$
$$\lesssim\frac{\sqrt{\overset{(0)}{\mathcal{F}}{}^*_{\ll k}}}{(\tau_1-\tau_0)^{1/2}}\left(\int_{v(R,\tau_0)}^{v}\int_{\tau_0}^{T_1'}\int_{S^2}r^{-1-\delta}|\partial_u\mathfrak{D}^{\boldsymbol{k}}\psi|r^2\,d\omega\,du\,dv'\right)^{\frac12}$$
$$\times\left(\int_{\tau_0}^{T_1}\int_{v(R,u)}^{v}\int_{S^2}r^{-1+\delta}|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|r^2\,d\omega\,dv'\,du\right)^{\frac12}$$
$$\lesssim\frac{\sqrt{\overset{(0)}{\mathcal{F}}{}^*_{\ll k}}}{(\tau_1-\tau_0)^{1/2}}\sqrt{\int^*\overset{(-1-\delta)}{\mathcal{E}}_k}\sqrt{\overset{(0)}{\mathcal{E}}{}^*_k}\left(\int_{\tau_0}^{T_1}du\right)^{\frac12}\lesssim\sqrt{\overset{(0)}{\mathcal{E}}{}^*_k}\sqrt{\int^*\overset{(-1-\delta)}{\mathcal{E}}_k}\sqrt{\overset{(0)}{\mathcal{F}}{}^*_{\ll k}}$$

by the Sobolev inequality on the incoming cones (C-7).

If $|\boldsymbol{k}_2|\leq k-2$, then

$$\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)}|\partial_u\mathfrak{D}^{\boldsymbol{k}}\psi|\cdot|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|$$
$$\lesssim\int_{v(R,\tau_0+R)}^{v}\sup_{u,\theta}|r^{\frac{1+\delta}{2}}\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|\int_{\tau_0}^{T_1'}\int_{S^2}r^{-\frac{1+\delta}{2}}|\partial_u\mathfrak{D}^{\boldsymbol{k}}\psi|\cdot|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|r^2\,d\omega\,du\,dv'$$
$$\lesssim(1+(\tau_1-\tau_0)^{-\frac12})\sum_{|\tilde{\boldsymbol{k}}|\leq|\boldsymbol{k}_2|+3}\int_{v(R)}^{v}\left(\int_{\tau_0}^{T_1'}\int_{S^2}r^{-1+\delta}|\partial_v\mathfrak{D}^{\tilde{\boldsymbol{k}}}\psi|^2r^2\,d\omega\,du\right)^{\frac12}$$
$$\times\left(\int_{\tau_0}^{T_1'}\int_{S^2}r^{-1-\delta}|\partial_u\mathfrak{D}^{\boldsymbol{k}}\psi|^2r^2\,d\omega\,du\right)^{\frac12}\left(\int_{\tau_0}^{T_1'}\int_{S^2}|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|^2r^2\,d\omega\,du\right)^{\frac12}dv'$$
$$\lesssim(1+(\tau_1-\tau_0)^{-\frac12})\sqrt{\overset{(0)}{\mathcal{F}}{}^*_k}\sqrt{\int^*\overset{(-1-\delta)}{\mathcal{E}}_k}\sum_{|\tilde{\boldsymbol{k}}|\leq|\boldsymbol{k}_2|+3}\left(\int_{\tau_0}^{\tau_1}\int_{v(R,u)}^{v}\int_{S^2}r^{-1+\delta}|\partial_v\mathfrak{D}^{\tilde{\boldsymbol{k}}}\psi|^2r^2\,d\omega\,du\,dv'\right)^{\frac12}$$

by the Sobolev inequality on the incoming cones (C-8). Hence, if $\tau_1-\tau_0\geq 1$,

$$\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)}|\partial_u\mathfrak{D}^{\boldsymbol{k}}\psi|\cdot|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|\lesssim\sqrt{\overset{(0)}{\mathcal{F}}{}^*_k}\sqrt{\int^*\overset{(-1-\delta)}{\mathcal{E}}_k}\sqrt{\int^*\overset{(\delta-1)}{\mathcal{E}}{}'_{\ll k}}$$

and, if $\tau_1-\tau_0<1$,

$$\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)}|\partial_u\mathfrak{D}^{\boldsymbol{k}}\psi|\cdot|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|\lesssim\sqrt{\overset{(0)}{\mathcal{F}}{}^*_k}\sqrt{\int^*\overset{(-1-\delta)}{\mathcal{E}}_k}\sqrt{\overset{(0)}{\mathcal{E}}{}^*_{\ll k}}.$$

Estimate for the $\partial_v\psi$ term: Next, for any $|\boldsymbol{k}|\le k$,

$$\sum_{|\boldsymbol{k}_1|+|\boldsymbol{k}_2|\le|\boldsymbol{k}|}\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)}|\partial_v\mathfrak{D}^{\boldsymbol{k}}\psi|\cdot|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|\lesssim\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{F}}}{}^*+\int^*\overset{(-1-\delta)}{\underset{\ll k}{\mathcal{E}'}}}\Big(\overset{(0)}{\underset{k}{\mathcal{F}}}{}^*+\overset{(0)}{\underset{k}{\mathcal{E}}}{}^*+\int^*\overset{(\delta-1)}{\underset{k}{\mathcal{E}'}}\Big).\tag{C-12}$$

Indeed, if $|\boldsymbol{k}_1|\le k-2$, then

$$\begin{aligned}\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)}&|\partial_v\mathfrak{D}^{\boldsymbol{k}}\psi|\cdot|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|\\&\lesssim\sup_{u,v,\theta}|r\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|\Big(\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)}r^{-1}|\partial_v\mathfrak{D}^{\boldsymbol{k}}\psi|^2\Big)^{\frac12}\Big(\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)}r^{-1}|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|^2\Big)^{\frac12}\\&\lesssim(1+(\tau_1-\tau_0)^{-\frac12})\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{F}}}{}^*}\sqrt{\int^*\overset{(\delta-1)}{\underset{k}{\mathcal{E}'}}}\Big(\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)}r^{-1}|\partial_v\mathfrak{D}^{\boldsymbol{k}}\psi|^2\Big)^{\frac12}\end{aligned}$$

by the Sobolev inequality on the incoming cones (C-7). Hence, if $\tau_1-\tau_0\ge1$,

$$\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)}|\partial_v\mathfrak{D}^{\boldsymbol{k}}\psi|\cdot|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|\lesssim\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{F}}}{}^*}\int^*\overset{(\delta-1)}{\underset{k}{\mathcal{E}'}}$$

and, if $\tau_1-\tau_0<1$,

$$\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)}|\partial_v\mathfrak{D}^{\boldsymbol{k}}\psi|\cdot|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|\lesssim\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{F}}}{}^*}\sqrt{\overset{(0)}{\underset{k}{\mathcal{E}}}{}^*}\sqrt{\int^*\overset{(\delta-1)}{\underset{k}{\mathcal{E}'}}}.$$

If $|\boldsymbol{k}_2|\le|\boldsymbol{k}|-2$, then one similarly estimates

$$\begin{aligned}&\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)}|\partial_v\mathfrak{D}^{\boldsymbol{k}}\psi|\cdot|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|\\&\lesssim\int_{v(R,\tau_0+R)}^{v}\sup_{\substack{\tau_0\le u\le T_1'\\\theta\in S^2}}|r^{\frac12}\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|\Big(\int_{\tau_0}^{T_1'}\int_{S^2}r^{-1}|\partial_v\mathfrak{D}^{\boldsymbol{k}}\psi|r^2\,d\omega\,du\Big)^{\frac12}\Big(\int_{\tau_0}^{T_1'}\int_{S^2}|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|r^2\,d\omega\,du\Big)^{\frac12}dv'\\&\lesssim(1+(\tau_1-\tau_0)^{-\frac12})\sqrt{\overset{(0)}{\underset{k}{\mathcal{F}}}{}^*}\\&\qquad\times\sum_{|\tilde{\boldsymbol{k}}|\le|\boldsymbol{k}_2|+3}\int_{v(R,\tau_0+R)}^{v}\Big(\int_{\tau_0}^{T_1'}\int_{S^2}r^{-1-\delta}|\partial_v\mathfrak{D}^{\tilde{\boldsymbol{k}}}\psi|r^2\,d\omega\,du\Big)^{\frac12}\Big(\int_{\tau_0}^{T_1'}\int_{S^2}r^{-1+\delta}|\partial_v\mathfrak{D}^{\boldsymbol{k}}\psi|r^2\,d\omega\,du\Big)^{\frac12}dv'\\&\lesssim(1+(\tau_1-\tau_0)^{-\frac12})\sqrt{\overset{(0)}{\underset{k}{\mathcal{F}}}{}^*}\sqrt{\int^*\overset{(-1-\delta)}{\underset{\ll k}{\mathcal{E}'}}}\Big(\int_{v(R,\tau_0+R)}^{v}\int_{\tau_0}^{T_1'}\int_{S^2}r^{-1+\delta}|\partial_v\mathfrak{D}^{\boldsymbol{k}}\psi|r^2\,d\omega\,du\,dv'\Big)^{\frac12}\end{aligned}$$

by the Sobolev inequality on the incoming cones (C-8). Hence, if $\tau_1-\tau_0\ge1$,

$$\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)}|\partial_v\mathfrak{D}^{\boldsymbol{k}}\psi|\cdot|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|\lesssim\sqrt{\overset{(0)}{\underset{k}{\mathcal{F}}}{}^*}\sqrt{\int^*\overset{(-1-\delta)}{\underset{\ll k}{\mathcal{E}'}}}\sqrt{\int^*\overset{(\delta-1)}{\underset{k}{\mathcal{E}'}}}$$

and, if $\tau_1-\tau_0<1$,

$$\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)}|\partial_v\mathfrak{D}^{\boldsymbol{k}}\psi|\cdot|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|\lesssim\sqrt{\overset{(0)}{\underset{k}{\mathcal{F}}}{}^*}\sqrt{\int^*\overset{(-1-\delta)}{\underset{\ll k}{\mathcal{E}'}}}\sqrt{\overset{(0)}{\underset{k}{\mathcal{E}}}{}^*}.$$

Estimate for the $r^{-1}\nabla\!\!\!/\psi$ and $r^{-1}\psi$ terms: For any $|\boldsymbol{k}|\le k$,

$$\sum_{|\boldsymbol{k}_1|+|\boldsymbol{k}_2|\le|\boldsymbol{k}|}\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)} r^{-1}|\nabla\!\!\!/\mathfrak{D}^{\boldsymbol{k}}\psi|\cdot|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi| \lesssim \sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{F}}}{}^*+\int^*\overset{(-1-\delta)}{\underset{\ll k}{\mathcal{E}'}}}\left(\overset{(0)}{\underset{k}{\mathcal{F}}}{}^*+\overset{(0)}{\underset{k}{\mathcal{E}}}{}^*+\int^*\overset{(\delta-1)}{\underset{k}{\mathcal{E}'}}\right) \tag{C-13}$$

and

$$\sum_{|\boldsymbol{k}_1|+|\boldsymbol{k}_2|\le|\boldsymbol{k}|}\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)} r^{-1}|\mathfrak{D}^{\boldsymbol{k}}\psi|\cdot|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\partial_u\mathfrak{D}^{\boldsymbol{k}_2}\psi| \lesssim \sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{F}}}{}^*+\int^*\overset{(-1-\delta)}{\underset{\ll k}{\mathcal{E}'}}}\left(\overset{(0)}{\underset{k}{\mathcal{F}}}{}^*+\overset{(0)}{\underset{k}{\mathcal{E}}}{}^*+\int^*\overset{(\delta-1)}{\underset{k}{\mathcal{E}'}}\right). \tag{C-14}$$

Indeed, (C-13) and (C-14) follow as in the proof of (C-12), using now the fact that

$$\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)} r^{-1-\delta}|r^{-1}\nabla\!\!\!/\psi|^2+r^{-1-\delta}|r^{-1}\psi|^2\le\int^*\overset{(-1-\delta)}{\underset{k}{\mathcal{E}'}}$$

and

$$\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)} r^{-1+\delta}|r^{-1}\nabla\!\!\!/\psi|^2+r^{-1+\delta}|r^{-1}\psi|^2\le\int^*\overset{(\delta-1)}{\underset{k}{\mathcal{E}'}},$$

$$\int_{v(R,\tau)}^{v}\int_{S^2}(|r^{-1}\nabla\!\!\!/\psi|^2+|r^{-1}\psi|^2)r^2\,d\omega\,dv'\le\overset{(0)}{\underset{k}{\mathcal{E}}}{}^*.$$

Estimate for the quartic term: For any $|\boldsymbol{k}|\le k$,

$$\sum_{|\boldsymbol{k}_1|+|\boldsymbol{k}_2|\le|\boldsymbol{k}|}\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)}|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|^2\cdot|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|^2\lesssim\left(\overset{(0)}{\underset{k}{\mathcal{E}}}{}^*+\int^*\overset{(-1-\delta)}{\underset{k}{\mathcal{E}'}}\right)\overset{(0)}{\underset{\ll k}{\mathcal{F}}}{}^*+\left(\overset{(0)}{\underset{\ll k}{\mathcal{E}}}{}^*+\int^*\overset{(-1-\delta)}{\underset{\ll k}{\mathcal{E}'}}\right)\overset{(0)}{\underset{k}{\mathcal{F}}}{}^*. \tag{C-15}$$

Indeed, supposing first that $|\boldsymbol{k}_1|\le k-3$,

$$\begin{aligned}&\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)}|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|^2\cdot|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|^2\\ &\qquad\lesssim\sup_{u,\theta}(r|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|)^2\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)}r^{-2}|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|^2\lesssim\overset{(0)}{\underset{\ll k}{\mathcal{F}}}{}^*(1+(\tau_1-\tau_0)^{-\frac{1}{2}})\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)}r^{-2}|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|^2\end{aligned}$$

by the Sobolev inequality on the incoming cones (C-7). Hence, if $\tau_1-\tau_0\ge1$,

$$\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)}|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|^2\cdot|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|^2\lesssim\overset{(0)}{\underset{\ll k}{\mathcal{F}}}{}^*\int^*\overset{(-1-\delta)}{\underset{k}{\mathcal{E}'}}$$

and, if $\tau_1-\tau_0<1$,

$$\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)}|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|^2\cdot|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|^2\lesssim\overset{(0)}{\underset{\ll k}{\mathcal{F}}}{}^*\overset{(0)}{\underset{k}{\mathcal{E}}}{}^*.$$

Similarly, if $|\boldsymbol{k}_2|\le k-3$,

$$\begin{aligned}\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)}|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|^2\cdot|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|^2&\lesssim\int_{v(R,\tau_0+R)}^{v}\sup_{u,\theta}|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|^2\int_{\tau_0}^{\tau_1}\int_{S^2}|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|^2r^2\,d\omega\,du\,dv'\\ &\lesssim\overset{(0)}{\underset{k}{\mathcal{F}}}{}^*(1+(\tau_1-\tau_0)^{-\frac{1}{2}})\sum_{|\tilde{\mathbf{k}}|\le|\mathbf{k}_2|+3}\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)}r^{-2}|\partial_v\mathfrak{D}^{\tilde{\mathbf{k}}}\psi|^2,\end{aligned}$$

and so one similarly has

$$\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)}|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|^2\cdot|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|^2\lesssim\left(\overset{(0)}{\underset{\ll k}{\mathcal{E}}}{}^*+\int^*\overset{(-1-\delta)}{\underset{\ll k}{\mathcal{E}'}}\right)\overset{(0)}{\underset{k}{\mathcal{F}}}{}^*.$$

Estimate for the $r^{p-1}\partial_v(r\psi)$ term: Finally, for any $|\boldsymbol{k}|\le k$,

$$\sum_{|\boldsymbol{k}_1|+|\boldsymbol{k}_2|\le|\boldsymbol{k}|}\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)} r^{p-1}|\partial_v(r\mathfrak{D}^{\boldsymbol{k}}\psi)|\cdot|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi| \lesssim \Big(\overset{(0)}{\underset{k}{\mathcal{F}}}{}^* + \overset{(p)}{\underset{k}{\mathcal{E}}}{}^* + \int^* \overset{(p-1)}{\underset{k}{\mathcal{E}'}}\Big)\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{F}}}{}^* + \overset{(0)}{\underset{\ll k}{\mathcal{E}}}{}^*} + \sqrt{\overset{(0)}{\underset{k}{\mathcal{F}}}{}^*}\sqrt{\int^*\overset{(p-1)}{\underset{k}{\mathcal{E}'}}}\sqrt{\int^*\overset{(p-1)}{\underset{\ll k}{\mathcal{E}'}}}. \tag{C-16}$$

Indeed, consider first the case $\tau_1-\tau_0\ge 1$. Suppose first that $|\boldsymbol{k}_1|\le k-3$. Then, if $\tau_1-\tau_0\ge 1$,

$$\begin{aligned}&\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)} r^{p-1}|\partial_v(r\mathfrak{D}^{\boldsymbol{k}}\psi)|\cdot|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|\\ &\lesssim \sup_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)}(r|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|)\Big(\int_{\mathcal{R}_R(\tau_0,\tau_1,v)} r^{p-3}|\partial_v(r\mathfrak{D}^{\boldsymbol{k}}\psi)|^2\Big)^{\frac12}\Big(\int_{\mathcal{R}_R(\tau_0,\tau_1,v)} r^{p-1}|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|^2\Big)^{\frac12} \lesssim \sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{F}}}{}^*}\int^*\overset{(p-1)}{\underset{k}{\mathcal{E}'}}\end{aligned}$$

by the Sobolev inequality on the incoming cones (C-7). Similarly, if $\tau_1-\tau_0<1$,

$$\begin{aligned}&\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)} r^{p-1}|\partial_v(r\mathfrak{D}^{\boldsymbol{k}}\psi)|\cdot|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|\\ &\quad\lesssim (\tau_1-\tau_0)^{\frac12}\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{F}}}{}^*}\Big(\int_{\mathcal{R}_R(\tau_0,\tau_1,v)} r^{p-2}|\partial_v(r\mathfrak{D}^{\boldsymbol{k}}\psi)|^2\Big)^{\frac12}\Big(\int_{\mathcal{R}_R(\tau_0,\tau_1,v)} r^{p-2}|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|^2\Big)^{\frac12} \lesssim \sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{F}}}{}^*}\sqrt{\overset{(p)}{\underset{k}{\mathcal{E}}}{}^*}\sqrt{\overset{(0)}{\underset{k}{\mathcal{E}}}{}^*}.\end{aligned}$$

Suppose now $|\boldsymbol{k}_2|\le k-3$. Then, if $\tau_1-\tau_0\ge 1$,

$$\begin{aligned}&\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)} r^{p-1}|\partial_v(r\mathfrak{D}^{\boldsymbol{k}}\psi)|\cdot|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|\\ &\lesssim \sqrt{\overset{(0)}{\underset{k}{\mathcal{F}}}{}^*}\int_{v(R,\tau_0)}^{v}\sup_{u,\theta}(r^{\frac{p+1}{2}}|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|)\Big(\int_{\tau_0}^{\tau_1}\int_{S^2} r^{p-3}|\partial_v(r\mathfrak{D}^{\boldsymbol{k}}\psi)|^2r^2\,d\omega\,du\Big)^{\frac12}dv\\ &\lesssim \sqrt{\overset{(0)}{\underset{k}{\mathcal{F}}}{}^*}\Big(\int_{\mathcal{R}_R(\tau_0,\tau_1,v)} r^{p-3}|\partial_v(r\mathfrak{D}^{\boldsymbol{k}}\psi)|^2r^2\,d\omega\,du\,dv\Big)^{\frac12}\sum_{|\tilde{\mathbf{k}}|\le|\boldsymbol{k}_2|+3}\Big(\int_{\mathcal{R}_R(\tau_0,\tau_1,v)} r^{p-1}|\partial_v\mathfrak{D}^{\tilde{\boldsymbol{k}}}\psi|^2r^2\,d\omega\,du\,dv\Big)^{\frac12}\\ &\lesssim \sqrt{\overset{(0)}{\underset{k}{\mathcal{F}}}{}^*}\sqrt{\int^*\overset{(p-1)}{\underset{k}{\mathcal{E}'}}}\sqrt{\int^*\overset{(p-1)}{\underset{\ll k}{\mathcal{E}'}}}\end{aligned}$$

using again the Sobolev inequality on the incoming cones (C-8). Similarly, if $\tau_1-\tau_0<1$,

$$\begin{aligned}&\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)} r^{p-1}|\partial_v(r\mathfrak{D}^{\boldsymbol{k}}\psi)|\cdot|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|\\ &\qquad\lesssim (\tau_1-\tau_0)^{-\frac12}\sqrt{\overset{(0)}{\underset{k}{\mathcal{F}}}{}^*}\Big(\int_{\mathcal{R}_R(\tau_0,\tau_1,v)} r^{p-2}|\partial_v(r\mathfrak{D}^{\boldsymbol{k}}\psi)|^2\Big)^{\frac12}\sum_{|\tilde{\mathbf{k}}|\le|\boldsymbol{k}_2|+3}\Big(\int_{\mathcal{R}_R(\tau_0,\tau_1,v)} r^{p-2}|\partial_v\mathfrak{D}^{\tilde{\boldsymbol{k}}}\psi|^2\Big)^{\frac12}\\ &\qquad\lesssim \sqrt{\overset{(0)}{\underset{k}{\mathcal{F}}}{}^*}\sqrt{\overset{(p)}{\underset{k}{\mathcal{E}}}{}^*}\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{E}}}{}^*}.\end{aligned}$$

□

C.1.3. *The proof of Proposition C.1.1 for the nonlinearity $F=\not{g}^{AB}\not\nabla_A\psi\not\nabla_B\psi$.* We now consider the nonlinearity

$$F=\not{g}^{AB}\not\nabla_A\psi\not\nabla_B\psi.$$

In view of the property (C-2), the proof of Proposition C.1.1 follows from the next proposition.

Proposition C.1.5 (nonlinear error estimates for $F = \slashed{g}^{AB}\slashed{\nabla}_A\psi\slashed{\nabla}_B\psi$ on Minkowski space). *Under the assumptions of Proposition C.1.1, for each* $|\boldsymbol{k}| \leq k$,

$$\sum_{|\boldsymbol{k}_1|+|\boldsymbol{k}_2|\leq|\boldsymbol{k}|}\Bigg(\int_{\mathcal{R}(\tau_0,\tau_1,v)\cap\{r\geq R\}}\Big(|\partial_u\mathfrak{D}^{\boldsymbol{k}}\psi|+|\partial_v\mathfrak{D}^{\boldsymbol{k}}\psi|+\frac{1}{r}|\slashed{\nabla}\mathfrak{D}^{\boldsymbol{k}}\psi|+\frac{1}{r}|\mathfrak{D}^{\boldsymbol{k}}\psi|\Big)\cdot|\slashed{\nabla}\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\slashed{\nabla}\mathfrak{D}^{\boldsymbol{k}_2}\psi| \\ +\int_{\mathcal{R}(\tau_0,\tau_1,v)\cap\{r\geq R\}}|\slashed{\nabla}\mathfrak{D}^{\boldsymbol{k}_1}\psi|^2\cdot|\slashed{\nabla}\mathfrak{D}^{\boldsymbol{k}_2}\psi|^2\Bigg) \\ \lesssim\sqrt{\overset{(0)}{\mathcal{F}}{}^*_{\ll k\,R}+\overset{(0)}{\mathcal{E}}{}^*_{\ll k\,8R/9}}\Big(\overset{(0)}{\mathcal{E}}{}^*_{k\,R}+\int^*\overset{(\delta-1)}{\mathcal{E}'}_{k\,R}\Big)$$

and

$$\sum_{|\boldsymbol{k}_1|+|\boldsymbol{k}_2|\leq|\boldsymbol{k}|}\int_{\mathcal{R}(\tau_0,\tau_1,v)\cap\{r\geq R\}}r^{p-1}|\partial_v(r\mathfrak{D}^{\boldsymbol{k}}\psi)|\cdot|\slashed{\nabla}\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\slashed{\nabla}\mathfrak{D}^{\boldsymbol{k}_2}\psi|\lesssim\sqrt{\overset{(0)}{\mathcal{F}}{}^*_{\ll k\,R}+\overset{(0)}{\mathcal{E}}{}^*_{\ll k\,8R/9}}\Big(\overset{(p)}{\mathcal{E}}{}^*_{k\,R}+\int^*\overset{(p-1)}{\mathcal{E}'}_{k\,R}\Big).$$

Proof. The separate terms are estimated individually, and the proof follows from estimates (C-17)–(C-21) below.

The notation of Section C.1.1 will again be used throughout. As in the proof of Proposition C.1.4, the cases $\tau_1-\tau_0\geq 1$ and $\tau_1-\tau_0<1$ are typically considered separately. Similarly, the region near the corner $\{u=\tau_0\}\cap\{r=R\}$ is treated separately.

Estimate in the corner region $\{v'\leq v(R,\tau_0+R)\}$: For any $|\tilde{\boldsymbol{k}}|\leq k+1$,

$$\sum_{|\boldsymbol{k}_1|+|\boldsymbol{k}_2|\leq|\tilde{\boldsymbol{k}}|-1}\int_{\mathcal{R}_{\mathrm{Corner}}(\tau_0,\tau_1,v)}|\mathfrak{D}^{\tilde{\boldsymbol{k}}}\psi|\cdot|\slashed{\nabla}\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\slashed{\nabla}\mathfrak{D}^{\boldsymbol{k}_2}\psi|\lesssim\sqrt{\overset{(0)}{\mathcal{E}}{}^*_{\ll k\,8R/9}}\int^*\overset{(-1-\delta)}{\mathcal{E}'}_{k\,R}. \tag{C-17}$$

Indeed, (C-17) follows exactly as in the proof of Proposition C.1.4 using the Sobolev inequality (C-9).

The remainder of the proof concerns the region $\check{\mathcal{R}}_R(\tau_0,\tau_1,v)$.

Estimate for the $\partial_u\psi$ *term*: First note that, for any $|\boldsymbol{k}|\leq k$,

$$\sum_{|\boldsymbol{k}_1|+|\boldsymbol{k}_2|\leq|\boldsymbol{k}|}\int_{\check{\mathcal{R}}(\tau_0,\tau_1,v)}|\partial_u\mathfrak{D}^{\boldsymbol{k}}\psi|\cdot|\slashed{\nabla}\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\slashed{\nabla}\mathfrak{D}^{\boldsymbol{k}_2}\psi|\lesssim\sqrt{\overset{(0)}{\mathcal{F}}{}^*_{\ll k}}\int^*\overset{(\delta-1)}{\mathcal{E}'}_{k}. \tag{C-18}$$

Indeed, for $\tau_1-\tau_0\geq 1$, assuming without loss of generality that $|\boldsymbol{k}_1|\leq k-3$,

$$\int_{\check{\mathcal{R}}(\tau_0,\tau_1,v)}|\partial_u\mathfrak{D}^{\boldsymbol{k}}\psi|\cdot|\slashed{\nabla}\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\slashed{\nabla}\mathfrak{D}^{\boldsymbol{k}_2}\psi| \\ \lesssim\sup_{u,v,\theta}r|\slashed{\nabla}\mathfrak{D}^{\boldsymbol{k}_1}\psi|\Big(\int_{\check{\mathcal{R}}(\tau_0,\tau_1,v)}r^{-1-\delta}|\partial_u\mathfrak{D}^{\boldsymbol{k}}\psi|^2\Big)^{\frac{1}{2}}\Big(\int_{\check{\mathcal{R}}(\tau_0,\tau_1,v)}r^{-1+\delta}|\slashed{\nabla}\mathfrak{D}^{\boldsymbol{k}_2}\psi|^2\Big)^{\frac{1}{2}} \\ \lesssim\sqrt{\overset{(0)}{\mathcal{F}}{}^*_{\ll k}}\int^*\overset{(\delta-1)}{\mathcal{E}'}_{k}$$

by the Sobolev inequality on the incoming cones (C-7) since $p\geq\delta$. For $\tau_1-\tau_0<1$,

$$\int_{\check{\mathcal{R}}(\tau_0,\tau_1,v)}|\partial_u\mathfrak{D}^{\boldsymbol{k}}\psi|\cdot|\slashed{\nabla}\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\slashed{\nabla}\mathfrak{D}^{\boldsymbol{k}_2}\psi|\lesssim\sqrt{\overset{(0)}{\mathcal{F}}{}^*_{\ll k}}\sqrt{\overset{(0)}{\mathcal{E}}{}^*_{k}}\sqrt{\overset{(0)}{\mathcal{F}}{}^*_{k}}.$$

Estimates for the $\partial_v\psi$, $r^{-1}\nabla\!\!\!/\psi$, and $r^{-1}\psi$ terms: Next, for any $|\boldsymbol{k}|\le k$,

$$\sum_{|\boldsymbol{k}_1|+|\boldsymbol{k}_2|\le|\boldsymbol{k}|}\int_{\check{\mathcal{R}}(\tau_0,\tau_1,v)}(|\partial_v\mathfrak{D}^{\boldsymbol{k}}\psi|+r^{-1}|\nabla\!\!\!/\mathfrak{D}^{\boldsymbol{k}}\psi|+r^{-1}|\mathfrak{D}^{\boldsymbol{k}}\psi|)\cdot|\nabla\!\!\!/\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\nabla\!\!\!/\mathfrak{D}^{\boldsymbol{k}_2}\psi| \lesssim \sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{F}^*}}}\bigg(\overset{(0)}{\underset{k}{\mathcal{E}^*}}+\int^*\overset{(\delta-1)}{\underset{k}{\mathcal{E}'}}\bigg). \tag{C-19}$$

Indeed, for $\tau_1-\tau_0\ge 1$, assuming again without loss of generality that $|\boldsymbol{k}_1|\le k-3$,

$$\int_{\check{\mathcal{R}}(\tau_0,\tau_1,v)}|\partial_v\mathfrak{D}^{\boldsymbol{k}}\psi|\cdot|\nabla\!\!\!/\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\nabla\!\!\!/\mathfrak{D}^{\boldsymbol{k}_2}\psi| \lesssim \sup_{u,v,\theta} r|\nabla\!\!\!/\mathfrak{D}^{\boldsymbol{k}_1}\psi|\bigg(\int_{\check{\mathcal{R}}(\tau_0,\tau_1,v)}r^{-1}|\partial_v\mathfrak{D}^{\boldsymbol{k}}\psi|^2\bigg)^{\frac12}\bigg(\int_{\check{\mathcal{R}}(\tau_0,\tau_1,v)}r^{-1}|\nabla\!\!\!/\mathfrak{D}^{\boldsymbol{k}_2}\psi|^2\bigg)^{\frac12}\lesssim\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{F}^*}}}\int^*\overset{(\delta-1)}{\underset{k}{\mathcal{E}'}},$$

and similarly, for $\tau_1-\tau_0<1$,

$$\int_{\check{\mathcal{R}}(\tau_0,\tau_1,v)}|\partial_v\mathfrak{D}^{\boldsymbol{k}}\psi|\cdot|\nabla\!\!\!/\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\nabla\!\!\!/\mathfrak{D}^{\boldsymbol{k}_2}\psi|\lesssim\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{F}^*}}}\,\overset{(0)}{\underset{k}{\mathcal{E}^*}}$$

using again the Sobolev inequality on the incoming cones (C-7). Similarly for the $r^{-1}\psi$ and $r^{-1}\nabla\!\!\!/\psi$ terms, we use the fact that

$$\int_{\check{\mathcal{R}}(\tau_0,\tau_1,v)}(r^{-1}|\nabla\!\!\!/\mathfrak{D}^{\boldsymbol{k}}\psi|^2+r^{-3}|\mathfrak{D}^{\boldsymbol{k}}\psi|^2)\le\int^*\overset{(\delta-1)}{\underset{k}{\mathcal{E}'}},\quad \int_{v(R,u)}^{v}\int_{S^2}(|r^{-1}\nabla\!\!\!/\psi|^2+|r^{-1}\psi|^2)r^2\,d\omega\,dv'\le\overset{(0)}{\underset{k}{\mathcal{E}^*}}.$$

Estimate for the quartic term: For any $|\boldsymbol{k}|\le k$,

$$\sum_{|\boldsymbol{k}_1|+|\boldsymbol{k}_2|\le|\boldsymbol{k}|}\int_{\check{\mathcal{R}}(\tau_0,\tau_1,v)}|\nabla\!\!\!/\mathfrak{D}^{\boldsymbol{k}_1}\psi|^2\cdot|\nabla\!\!\!/\mathfrak{D}^{\boldsymbol{k}_2}\psi|^2\lesssim\overset{(0)}{\underset{\ll k}{\mathcal{F}^*}}\bigg(\overset{(0)}{\underset{k}{\mathcal{E}^*}}+\int^*\overset{(-1-\delta)}{\underset{k}{\mathcal{E}}}\bigg), \tag{C-20}$$

which follows as an easy consequence of the Sobolev inequality (C-7).

Estimate for the $r^{p-1}\partial_v(r\psi)$ term: Finally, for any $|\boldsymbol{k}|\le k$,

$$\sum_{|\boldsymbol{k}_1|+|\boldsymbol{k}_2|\le|\boldsymbol{k}|}\int_{\check{\mathcal{R}}(\tau_0,\tau_1,v)}r^{p-1}|\partial_v(r\mathfrak{D}^{\boldsymbol{k}}\psi)|\cdot|\nabla\!\!\!/\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\nabla\!\!\!/\mathfrak{D}^{\boldsymbol{k}_2}\psi|\lesssim\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{F}^*}}}\bigg(\overset{(p)}{\underset{k}{\mathcal{E}^*}}+\int^*\overset{(p-1)}{\underset{k}{\mathcal{E}'}}\bigg). \tag{C-21}$$

Assume again, without loss of generality, that $|\boldsymbol{k}_1|\le k-3$. Then, using (C-7), for $\tau_1-\tau_0\ge 1$,

$$\int_{\check{\mathcal{R}}(\tau_0,\tau_1,v)}r^{p-1}|\partial_v(r\mathfrak{D}^{\boldsymbol{k}}\psi)|\cdot|\nabla\!\!\!/\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\nabla\!\!\!/\mathfrak{D}^{\boldsymbol{k}_2}\psi| \lesssim \sup_{\check{\mathcal{R}}(\tau_0,\tau_1,v)} r|\nabla\!\!\!/\mathfrak{D}^{\boldsymbol{k}_1}\psi|\bigg(\int_{\check{\mathcal{R}}(\tau_0,\tau_1,v)}r^{p-3}|\partial_v(r\mathfrak{D}^{\boldsymbol{k}}\psi)|^2\bigg)^{\frac12}\bigg(\int_{\check{\mathcal{R}}(\tau_0,\tau_1,v)}r^{p-1}|\nabla\!\!\!/\mathfrak{D}^{\boldsymbol{k}_2}\psi|^2\bigg)^{\frac12}\lesssim\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{F}^*}}}\int^*\overset{(p-1)}{\underset{k}{\mathcal{E}'}}.$$

Similarly, for $\tau_1-\tau_0<1$,

$$\int_{\check{\mathcal{R}}(\tau_0,\tau_1,v)}r^{p-1}|\partial_v(r\mathfrak{D}^{\boldsymbol{k}}\psi)|\cdot|\nabla\!\!\!/\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\nabla\!\!\!/\mathfrak{D}^{\boldsymbol{k}_2}\psi| \lesssim(\tau_1-\tau_0)^{-\frac12}\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{F}^*}}}\bigg(\int_{\check{\mathcal{R}}(\tau_0,\tau_1,v)}r^{p-2}|\partial_v(r\mathfrak{D}^{\boldsymbol{k}}\psi)|^2\bigg)^{\frac12}\bigg(\int_{\check{\mathcal{R}}(\tau_0,\tau_1,v)}r^{p-2}|\nabla\!\!\!/\mathfrak{D}^{\boldsymbol{k}_2}\psi|^2\bigg)^{\frac12}\lesssim\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{F}^*}}}\sqrt{\overset{(p)}{\underset{k}{\mathcal{E}^*}}}\sqrt{\overset{(0)}{\underset{k}{\mathcal{E}^*}}}.\quad\square$$

C.2. ***More general nonlinearities and Kerr.*** For more general nonlinearities on Minkowski space of the form

$$F = N^{\mu\nu}(\psi, x)\partial_\mu\psi\partial_\nu\psi,$$

with N satisfying a null condition of the form

$$\sup_{|\xi|\leq 1, r\geq R} \sum_{|\mathbf{k}|+s\leq k} \sum_{A,B=1,2} \mathfrak{D}^{\mathbf{k}}\partial_\xi^s(rN^{uu} + N^{uv} + N^{vv} + rN^{Au} + rN^{Av} + r^2N^{AB})(\xi, x) \lesssim D_k, \tag{C-22}$$

where the A, B indices refer to coordinates ϑ, φ (recall the $(u, v, \vartheta, \varphi)$ coordinate system of Section 2.7.1) and D_k are arbitrary constants, the proof of Proposition C.1 follows exactly as above, where the bound (4-29) is now also used to obtain pointwise bounds on ψ through an easy weighted Sobolev estimate which allow as to invoke (C-22).

Remark C.2.1. Note that, besides the nonlinearities of the classical null condition [Klainerman 1986], our class (C-22) includes for instance also nonlinearities which for large r take the form $F = (\sin r)\partial_u\psi\partial_v\psi$. It does not, however, include even more general examples like $F = (\sin x)\partial_u\psi\partial_v\psi$ considered recently in [Anderson and Zbarsky 2024] due to the presence of the weighted vector fields Ω_i in our set of commutation vector fields $\mathfrak{D}^{\mathbf{k}}$.

More generally, if g_0 is a metric with asymptotics suitably close to those of Minkowski space, with extra terms suitably small, then the above proof again applies.

We will consider explicitly the case that g_0 is the Kerr metric. Define double-null $(u, v, \theta^1, \theta^2)$ coordinates on Kerr, when $0 < |a| < M$, in terms of the Boyer–Lindquist coordinates (t, r, ϑ, ϕ) of Section A.1, by

$$u = t - r_*, \qquad v = t + r_*,$$

with θ^1 defined implicitly by the relation

$$F(\theta^1, r, \vartheta) = 0,$$

$$\text{where } F(\theta^1, r, \vartheta) = \int_{\theta^1}^{\vartheta} \frac{1}{a\sqrt{\sin^2\theta^1 - \sin^2\theta'}}\,d\theta' + \int_r^\infty \frac{1}{\sqrt{((r')^2 + a^2)^2 - a^2\sin^2\theta^1\Delta'}}\,dr', \tag{C-23}$$

and θ^2 defined by

$$\theta^2 = \phi + h(r_*, \theta^1), \qquad \text{where } h \text{ satisfies } \quad \partial_{r_*}h(r_*, \theta^1) = \frac{2Mar}{\Sigma R^2}, \qquad \lim_{r_*\to\infty} h(r_*, \theta^1) = 0.$$

See for instance [Pretorius and Israel 1998; Dafermos and Luk 2025]. Here

$$r_*(r, \vartheta) = \int \frac{r^2 + a^2}{\Delta}\,dr + \int_r^\infty \frac{(r')^2 + a^2 - \sqrt{((r')^2 + a^2)^2 - a^2\sin^2\theta^1\Delta'}}{\Delta'}\,dr' + \int_{\theta^1}^{\vartheta} a\sqrt{\sin^2\theta^1 - \sin^2\theta'}\,d\theta' \tag{C-24}$$

(note that the expression (C-24) is independent of θ^1 in view of (C-23)), where

$$\int \frac{r^2 + a^2}{\Delta}\,dr$$

is a function satisfying

$$\partial_r \int \frac{r^2+a^2}{\Delta}\, dr = \frac{r^2+a^2}{\Delta}$$

and

$$\Delta = r^2+a^2-2Mr, \quad \Delta' = (r')^2+a^2-2Mr', \quad \Sigma = r^2+a^2\cos^2\vartheta,$$

$$\mathrm{R}^2 = r^2+a^2+\frac{2Ma^2 r\sin^2\vartheta}{\Sigma}.$$

Note in particular that r_* is distinct from $r^\star$ defined in Section A.1.

In these double-null coordinates, the Kerr metric takes the form

$$g_{a,M} = -\Omega^2\, du\, dv + \not{g}_{AB}(d\theta^A - b^A\, dv)(d\theta^B - b^B\, dv),$$

where

$$\Omega^2 = \frac{\Delta}{\mathrm{R}^2}, \quad b^1 = 0, \quad b^2 = \frac{4Mar}{\Sigma \mathrm{R}^2},$$

and

$$\not{g}_{11} = \frac{a^2(\partial_{\theta^1}F)^2(\sin^2\theta^1 - \sin^2\vartheta)((r^2+a^2)^2 - a^2\sin^2\theta^1\Delta)}{\mathrm{R}^2} + (\partial_{\theta^1}h)^2\mathrm{R}^2\sin^2\vartheta,$$

$$\not{g}_{22} = \mathrm{R}^2\sin^2\vartheta, \quad \not{g}_{12} = \not{g}_{21} = -\mathrm{R}^2\sin^2\vartheta\,\partial_{\theta^1}h.$$

In particular, one has

$$L = \partial_v + b^A\partial_{\theta^A}, \quad \underline{L} = \partial_u.$$

Moreover,

$$T = \partial_u + \partial_v \quad \text{and} \quad \Omega_i = \Omega_i(\theta^1,\theta^2)^A\partial_{\theta^A} \text{ for } i = 1, 2, 3,$$

for functions $\Omega_i(\theta^1,\theta^2)^A$, so that, in particular,

$$[\partial_u, T] = [\partial_u, \Omega_i] = [\partial_v, T] = [\partial_v, \Omega_i] = 0.$$

The function $b^2 = b^2(u, v, \theta^1, \theta^2)$ satisfies, for any $k \geq 0$,

$$\sum_{k_1+k_2+|\mathbf{k}_3|\leq k} |\partial_u^{k_1}(r\partial_v)^{k_2}\Omega^{\mathbf{k}_3}b^2| \leq \frac{C_k}{r^3}$$

for large r, and hence the commutation relations

$$|\mathfrak{D}^{\boldsymbol{k}}L\psi| \lesssim \sum_{|\tilde{\boldsymbol{k}}|\leq|\boldsymbol{k}|} |L\mathfrak{D}^{\tilde{\boldsymbol{k}}}\psi| + \frac{1}{r^3}\sum_{i=1}^{3}\sum_{|\tilde{\boldsymbol{k}}|\leq|\boldsymbol{k}|-1} |\Omega_i\mathfrak{D}^{\tilde{\boldsymbol{k}}}\psi|, \tag{C-25}$$

$$|\mathfrak{D}^{\boldsymbol{k}}\underline{L}\psi| \lesssim \sum_{|\tilde{\boldsymbol{k}}|\leq|\boldsymbol{k}|} |\underline{L}\mathfrak{D}^{\tilde{\boldsymbol{k}}}\psi| + \frac{1}{r^3}\sum_{i=1}^{3}\sum_{|\tilde{\boldsymbol{k}}|\leq|\boldsymbol{k}|-1} |\Omega_i\mathfrak{D}^{\tilde{\boldsymbol{k}}}\psi| \tag{C-26}$$

hold for large r. Moreover (C-2) remains true. The proof of Proposition C.1 in this case then follows just as in the case where g_0 was the Minkowski metric, using now (C-25) and (C-26) in place of (C-1).

We note finally that the condition (C-22) applied to Kerr indeed includes in particular the nonlinearities considered in [Luk 2013].

Appendix D: The inhomogeneous estimate (3-3) on Kerr in the full subextremal case $|a| < M$

The main theorem of [Dafermos et al. 2016] only states (3-3) for the homogeneous case, i.e., the case $F = 0$. In this section, we explicitly address the issue of the inhomogeneous estimate (3-3). We have the following:

Theorem D.1. *The inhomogeneous estimate* (3-3) *holds on Kerr in the full subextremal case* $|a| < M$.

Though the inhomogeneous estimate (3-3) for general F, precisely as stated, indeed can be shown to hold in the full subextremal case $|a| < M$, it is a little bit delicate to produce the physical-space expression corresponding to the middle term on the right-hand side of (3-3), except in the case $|a| \ll M$. (This difficulty is entirely due to the presence of the term $\overset{(0)}{\mathcal{E}}(\tau)$ on the left-hand side.) The weaker version of the estimate in Section 3.2, on the other hand, which is all that is actually used here, can essentially be immediately read off from the proof of the main result of [loc. cit.]. Since we have no real use for the stronger statement, we prefer here to give the details of how to directly obtain the weaker statement (though in the case $|a| \ll M$, we note that our argument indeed recovers the stronger statement).

Proof (*of the weaker version of Remark 3.2.1*). We will obtain the estimate in three steps. The reader should refer to [Dafermos et al. 2016] to follow along.

Step 1. We first obtain (3-3) (in its original form) for general F, but without the future boundary terms on the left-hand side, and where the regions are all restricted to $r \geq r_+$.

For this, note that one can assume without loss of generality that F is compactly supported in spacetime, and thus it is clearly sufficient to prove Proposition 9.1.1 of [loc. cit.] now for solutions of (1-2), with the extra inhomogeneous terms of (3-3) now on the right-hand side.

We note that the cutoffs used in the proof of Proposition 9.1.1 of [loc. cit.] already gave rise to inhomogeneous terms

$$\mathcal{T} := \int_{-\infty}^{\infty} \sum_{m\ell} \left(\int_{-\infty}^{\infty} H \cdot (f, h, y, \chi) \cdot (u, u') \right) d\omega\, dr^*; \tag{D-1}$$

see, e.g., (165) of [loc. cit.]. An inhomogeneity F on the right-hand side of (1-2) contributes an additional term of the form (D-1) in the proof of this proposition, where H is replaced by the Carter separated coefficients of F according to formula (43) of [loc. cit.]. Let us denote this term as $\mathcal{T}_F$. One easily sees, however, that, in view of the inclusion of our inhomogeneity F, the problem can be reduced to the case where one has trivial initial data, in which case one may work directly with ψ without applying a cutoff. Thus we may assume now that we *only* have $\mathcal{T}_F$ on the right-hand side, and H below will denote the Carter separated coefficients of F. We must now essentially repeat the steps of the proof of the proposition in order to produce the desired right-hand side.

For this, we first immediately partition $\mathcal{T}_F$ as $\mathcal{T}_F^1 + \mathcal{T}_F^2 + \mathcal{T}_F^3$, where the summands correspond to the integral of (D-1) over the region $R_- \leq r \leq R_+$ (i.e., $R_-^* \leq r^* \leq R_+^*$, etc.), $\{r \leq R_-\} \cup \{R_+ \leq r \leq R\}$ and $r \geq R$, respectively, where $R_\pm$, R_∞ are as in [loc. cit.] and we require $R \geq R_\infty$.

Considering $\mathcal{T}^1_F$, we further partition $\mathcal{T}^1_F$ as $\mathcal{T}^1_{F,\omega}+\mathcal{T}^1_{F,\phi}$, where $\mathcal{T}^1_{F,\omega}$ is the contribution of the currents in $H\cdot(f, h, y, \chi)\cdot(u, u')$ that multiply ω and $\mathcal{T}^1_{F,\phi}$ denotes the rest. We note (see formula (102) of [Dafermos et al. 2016]) that the coefficients of the currents in the sum defining $\mathcal{T}^1_{F,\omega}$ are then frequency-independent for all "trapped" frequencies $(\omega, m, \ell)\in\mathcal{G}_\natural$ (the expression is simply $E\omega\,\mathrm{Im}(H\bar{u})$). Thus, by adding a suitable compensating term $\mathcal{T}^1_{\mathrm{comp}}$ supported only in the untrapped frequencies, we have that $\mathcal{T}^1_{F,\omega}+\mathcal{T}^1_{\mathrm{comp}}$ is a sum and integral over the precise expression $E\omega\,\mathrm{Im}(H\bar{u})$, and thus this integral and sum may be rewritten via the Plancherel formulae on page 820 of [loc. cit.] as a spacetime integral precisely like the middle term of (3-3), restricted to $r\le R$, with $V_0=\partial_t$.

For the remaining terms $\mathcal{T}^1_{F,\phi}-\mathcal{T}^1_{\mathrm{comp}}$, note that these are supported entirely in the *nontrapped* frequencies, i.e., in the complement frequency set $\mathcal{G}^c_\natural$. Recall now that in estimate (165) of [loc. cit.], we in fact manifestly control a stronger left-hand side than what is written; namely, we may substitute ζ with $(1-r_{\mathrm{trap}}/r)^2$ (which was the original form of the expression before (163) of [loc. cit.] was invoked). In particular, since $r_{\mathrm{trap}}=0$ for all nontrapped frequencies, we control all derivatives without degeneration for frequencies in $\mathcal{G}^c_\natural$. One may apply Cauchy–Schwarz to the expression

$$H\cdot(f, h, y, \chi)\cdot(u, u')$$

and to the integrand summands of $\mathcal{T}^1_{\mathrm{comp}}$, absorbing the resulting terms proportional to $|u|^2$, together with any frequency coefficients ω^2 and Λ, and proportional to $|u'|^2$, into the bulk controlled in view of our nondegenerate bulk, at the expense of an additional term $|H|^2$ on the right-hand side. Upon application of the Plancherel formulae of page 820 of [loc. cit.], this produces a bulk term $\int F^2$ of the form already present on the right-hand side of (3-3). The term $\mathcal{T}^2_F$ may be treated exactly in the same way, as here the controlled bulk term is nondegenerate for all frequencies since the integral is supported away from trapping.

This leaves the term $\mathcal{T}^3_F$. One notes that, for $r\ge R$, the coefficients of the currents appearing in (D-1) are frequency-independent *up to multipliers which decay exponentially in* r. Again, by adding and subtracting a compensating term as above, one may thus estimate the additional term here by the middle term on the right-hand side of (3-10) (now restricted to $r\ge R$), again with a suitable vector field V_0, after applying Plancherel, up to an additional exponentially decaying term, which may be estimated by Cauchy–Schwarz, absorbing the term containing derivatives of ψ into the bulk on the left-hand side just as above, and producing an extra bulk term $\int F^2$ of the form already present on the right-hand side of (3-3).

Finally, we note that to complete the proof of Proposition 9.1.1 in [loc. cit.] in the inhomogeneous case, it remains to apply the analogue of Proposition 9.7.1 in [loc. cit.] (this is where the quantitative version of mode stability [Shlapentokh-Rothman 2015] is appealed to), which also however produces an additional inhomogeneous term when applied to (1-2). Examining the original [Shlapentokh-Rothman 2015], in particular the proof of Lemma 3.3 of that paper, one sees that this extra term may be bounded by the $\int F^2$ term on the right-hand side of (3-3). This completes the proof of Step 1.

Step 2. We now apply a sufficiently small multiple of the red-shift multiplier and of a cutoff version of the J^T_μ current to obtain the terms

$$\sup_{v:\tau\le\tau(v)} \overset{(0)}{\mathcal{F}}(v,\tau_0,\tau)+\overset{(0)}{\mathcal{E}}_{\mathcal{S}}(\tau)$$

on the left-hand side of (3-3) and the part of the spacetime integral supported in $r_0 \le r \le r_+$, exploiting that one may always absorb the resulting error terms in the bulk already controlled in Step 1. We add these identities to the estimate obtained above. As these are physical-space identities, the inhomogeneity F manifestly generates a term of the form of the middle term in (3-3) which may be combined with the previous such terms to define a new vector field V_0. One obtains in this way also a bound on the restriction of the integral defining $\overset{(0)}{\mathcal{E}}(\tau)$ to $\{r \le R_-\} \cup \{r \ge R_+\}$.

Note that in the case of very slow rotation $|a| \ll M$, one may choose the support of the gradient of the cutoff applied to J^T_μ in the identity above to in fact be in the region $r \le R_-$, whereas J^T_μ is moreover coercive in the region $r \ge R_-$. Thus, the above argument in fact yields control of the full $\overset{(0)}{\mathcal{E}}(\tau)$. This would then complete the proof. In this case, note that we thus obtain the estimate (3-3) in its original from.

Step 3. In the general subextremal case, however, J^T_μ is not everywhere coercive in the region $R_- \le r \le R_+$, and it remains to control the contribution of the finite region $R_- \le r \le R_+$ to the energy integral of the missing term $\overset{(0)}{\mathcal{E}}(\tau)$ from the left-hand side of (3-3).

To obtain the missing part of the energy flux we must thus repeat the proof of Proposition 13.1 of [Dafermos et al. 2016], allowing now an inhomogeneous term F. Recall that in this proof one considered a cutoff version $\tilde\psi$, decomposed orthogonally using Carter's separation as $\tilde\psi = \tilde\psi_1 + \cdots + \tilde\psi_N$ for some large N and applied a distinct current $J^{V_i}_\mu$ to each summand $\tilde\psi_i$.

Due to the presence of cutoffs in the proof, one can again read off the additional terms arising from the new inhomogeneity. Since, outside the region $R_- \le r \le R_+$, all errors can be absorbed as in the previous steps, while the cutoffs vanish inside the region $R_- \le r \le R_+$, all inhomogeneous terms in $R_- \le r \le R_+$ now arise from F, and we may use the notation $\widetilde{F}_i$ of the proof of Proposition 13.1 of [loc. cit.] to denote precisely these new terms. The resulting inhomogeneous terms that must be controlled are then

$$\sum_i \int_{R_- \le r \le R_+} V_i \tilde\psi_i \cdot \widetilde{F}_i .$$

By orthogonality, we may indeed manifestly bound this term by the expression in the first term of (3-10).

Since all terms of the form of the middle term on the right-hand side of (3-3) supported in $r \le R$ (which we generated in the previous two steps) can of course manifestly be bounded by the first term of (3-10), we obtain finally that (3-3) indeed holds with (3-10) replacing the middle term on the right-hand side. □

We note that obtaining (3-3) in the $|a| < M$ case with its middle term in its original form requires a slightly more delicate decomposition of $\tilde\psi$ than the one used in Proposition 13.1 of [loc. cit.]. Again, since this is not used, we spare the reader the details of this argument.

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