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TYPE-II SMOOTHING IN MEAN CURVATURE FLOW

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Velázquez (1994) constructed a smooth $O(4)\times O(4)$ invariant mean curvature flow that forms a type-II singularity at the origin in spacetime. Stolarski (2023) showed that the mean curvature on this solution is uniformly bounded. In a work in preparation, Angenent, Ilmanen, and Velázquez (Angenent et al.) also provided formal asymptotic expansions for a possible smooth continuation of the solution after the singularity.

Here we prove short-time existence of Velázquez' formal continuation, and we verify that the mean curvature is also uniformly bounded on the continuation. Combined with the earlier results of Velázquez and Stolarski we therefore show that there exists a solution $\{M_t^7\subset\mathbb{R}^8 \mid -t_0<t<t_0\}$ that has an isolated singularity at the origin $0\in\mathbb{R}^8$ and at $t=0$; moreover, the mean curvature is uniformly bounded on this solution, even though the second fundamental form is unbounded near the singularity.

1. Introduction

We say that a family of hypersurfaces $\{M_t\}_{t\in[0,T)}\subset\mathbb{R}^{n+1}$ moves by the mean curvature flow if

$$\frac{\partial \vec{F}}{\partial t}=\vec{H}, \tag{MCF}$$

where $\vec{H}(\,\cdot\,,t)$ is the mean curvature vector of the hypersurface M_t, and $\vec{F}(\,\cdot\,,t):M\to M_t\subset\mathbb{R}^{n+1}$ is a smooth family of parametrizations of the moving hypersurface. In the case of closed hypersurfaces, Huisken [1984] showed the norm of the second fundamental form blows up at finite time $T<\infty$; that is

$$\limsup_{t\to T}\max_{M_t}|A|(\,\cdot\,,t)=\infty.$$

Very often, even in a complete, noncompact setting, mean curvature flow (MCF) develops a singularity at a finite time $T<\infty$. It is very natural to ask whether the mean curvature also needs to blow up at a finite time singularity, or equivalently, whether a uniform bound on $|\vec{H}|$ for all $t\in[0,T)$ guarantees the existence of smooth solution past time T.

For mean convex flows it is well known [Huisken 1984] that the mean curvature bounds the second fundamental form A, i.e., $|A|/|\vec{H}|$ attains its maximum at $t=0$ and therefore is uniformly bounded. This implies that for mean convex flows the mean curvature is never bounded near a singularity. Dropping the assumption of mean convexity, it was shown by Le, Lin, and Sesum [Le and Sesum 2011a; 2011b; Lin and Sesum 2016] and by Xu, Ye, and Zhao [Xu et al. 2011] that for mean curvature flow of closed

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hypersurfaces the mean curvature needs to blow up at the first singular time, given some extra assumptions, such as having only type-I singularities or being close to a sphere in the L^2 sense. More recently, Li and Wang [2019] showed, using a quite involved argument that in the case of closed surfaces in $\mathbb{R}^3$ the mean curvature always blows up at the first singular time. The question of boundedness of the mean curvature on a singular mean curvature flow is therefore completely settled in the case of compact surfaces in $\mathbb{R}^3$, and for hypersurfaces in higher dimensions under a variety of extra assumptions.

For $n \geq 4$, Velázquez [1994] constructed $O(n) \times O(n)$ symmetric solutions of dimension $N = 2n - 1$ that converge to the Simons cone at parabolic scales around the singularity, and converge to a smooth minimal surface desingularizing Simons cone at the scale at which the norm of the second fundamental form blows up at the origin. Using formal asymptotic expansions, Angenent, Ilmanen, and Velázquez [Angenent et al.] also suggested a way in which the solution $\{M_t\}$ might be continued smoothly after the singularity, i.e., for $t > 0$.

These complete noncompact solutions were expected to provide examples of higher-dimensional mean curvature flow with the property that the mean curvature stays bounded at the first singular time. Stolarski [2023] used precise asymptotics of these solutions together with sophisticated blow up arguments to rigorously prove that this is indeed the case for $t < 0$; i.e., he showed that before the singularity forms the mean curvature on some of Velázquez' solutions is uniformly bounded. (To be precise, he requires the parameter k that appears in Velázquez' solutions to be even and not less than 4.)

Here we consider the case $n = 4$, i.e., the case of 7-dimensional hypersurfaces in $\mathbb{R}^8$. We first prove existence and regularity of Velázquez' formal extension of the Velázquez–Stolarski solutions, and we thereby obtain a solution $\{M_t \subset \mathbb{R}^8 \mid -t_0 < t < t_0\}$ of MCF that is smooth everywhere except at the origin $(0, 0) \in \mathbb{R}^8 \times (-t_0, t_0)$ in spacetime, and whose *mean curvature is uniformly bounded* even though its *second fundamental form blows up near* $(0, 0)$. In particular, we show that the singular hypersurface $M_0 = \lim_{t \nearrow 0} M_t$ that remains after the Velázquez–Stolarski solution forms its singularity can be used as initial data for MCF, and that at least one of the ensuing solutions has uniformly bounded mean curvature.

Stolarski [2023] indicates he expects his result to be true for closed mean curvature flow that can be obtained by compactifying Velázquez examples, but it still remains open. Another question that remains completely open is what happens in dimensions $3 \leq N \leq 6$ where neither an example of a singular solution with bounded mean curvature nor a theorem proving the impossibility of such an example are known.

1.1. *Outline.* In this paper we consider an $O(4) \times O(4)$ symmetric hypersurface M_0 defined by the profile function

$$u = u_0(x),$$

where $u_0 : (0, \infty) \to \mathbb{R}$ is a smooth function that near the origin satisfies

$$u_0(x) = x + K_0 x^{2(k-1)} + o(x^{2(k-1)}), \quad x \searrow 0, \tag{1.1.1}$$

for some integer

$$k \geq 4$$

and some constant $K_0 > 0$. We will also assume that for all $x > 0$ one has

$$0 \le u_0'(x) \le C_0, \quad |u_0''(x)| \le C_0, \quad |u_0''(x)| \le C_0 x^{2k-4} \tag{1.1.2}$$

for some constant $C_0 > 0$. The last assumption implies, after integration, that for all $x > 0$ one has

$$|u_0'(x) - 1| \le C x^{2k-3} \tag{1.1.3}$$

for some constant $C > 0$, depending on C_0. This implies that for x small enough we have $u_0'(x) \ge \frac{1}{2}$. By rescaling we may assume that

$$u_0'(x) \ge c > 0 \quad \text{for } x \in [0, 1]. \tag{1.1.4}$$

It turns out that such a function $u_0(x)$ is the final-time profile near a singularity $(0, 0)$ of the $O(4)\times O(4)$ MCF solution M_t, $-t_1 < t < 0$, for some small $t_1 < 0$, which was constructed by Velázquez [1994]. Under the assumption that k is even, Stolarski [2023] showed that the Velázquez solution has bounded mean curvature at the singularity, i.e., the mean curvature of M_t remains bounded as $t \to 0^-$ near $(0, 0)$.

Our goal in this paper is to show that the MCF starting at M_0 can be continued for $0 < t < t_0$, for some $t_0 > 0$ small, with a smooth solution M_t, $t \in (0, t_0)$, which is $O(4) \times O(4)$ symmetric. Furthermore, the mean curvature of M_t as $t \to 0^+$ will *remain uniformly bounded* despite the fact that M_0 is singular at $x = 0$.

The solution M_t will be defined by a profile function $u : (0, \infty) \times (0, t_0) \to (0, \infty)$ that satisfies the initial value problem

$$u_t = \frac{u_{xx}}{1+u_x^2} + \frac{3}{x}u_x - \frac{3}{u}, \tag{1.1.5a}$$

$$\lim_{x\to 0} u_x(x, t) = 0, \tag{1.1.5b}$$

$$\lim_{t\to 0} u(x, t) = u_0(x). \tag{1.1.5c}$$

Note the condition $\lim_{x\to 0} u_x(x, t) = 0$ ensures that $u(x, t)$ defines an $O(4) \times O(4)$ hypersurface M_t that is smooth at the origin and hence everywhere.

We will prove the following:

Main Theorem. *Assume that M_0 is an $O(4) \times O(4)$ symmetric hypersurface defined by the profile function $u_0 : [0, \infty) \to \mathbb{R}$ which is smooth for $x > 0$ and at $x = 0$ satisfies condition (1.1.1) for some $k > 3$. Then, there exists $t_0 > 0$ and a C^∞-smooth $O(4) \times O(4)$ symmetric MCF solution M_t, $0 < t \le t_0$, defined by a profile function $u : (0, \infty) \times (0, t_0] \to (0, \infty)$ which satisfies the initial value problem (1.1.5a)–(1.1.5c). Furthermore the mean curvature $H(x, t)$ of the hypersurface M_t satisfies*

$$\sup_{(x,t)\in[0,\infty)\times(0,a]} |H(x, t)| < +\infty$$

for some $0 < a \le t_0$. In particular, $H(x, t)$ is uniformly bounded near the origin as $t \to 0^+$ despite the fact that the mean curvature of M_0 is undefined at the origin.

As a corollary of the main theorem and the results in [Stolarski 2023] we have the following result.

Corollary 1.1.1. *There exists an $O(4)\times O(4)$ symmetric complete noncompact mean curvature flow solution $\{M_t\}_{t\in(-t_0,t_0)}$ such that M_t is smooth for all $t\in(-t_0,t_0)\backslash\{0\}$, has a type-II singularity at the origin and at time $t=0$, and has uniformly bounded mean curvature away from $t=0$. More precisely, there exists a uniform constant C, so that*

$$\sup_{(x,t)\in[0,\infty)\times(-t_0,t_0)\backslash\{0\}} |H(x,t)| \le C.$$

The short-time existence of a smooth MCF solution starting at M_0 follows by standard quasilinear parabolic PDE theory. The challenge here is to establish the *uniform bound* on $H(\,\cdot\,,t)$ near the singularity $(0,0)$. For this purpose we will construct sharp upper and lower barriers which will capture the exact behavior of the profile function $u(x,t)$ of our solution M_t as $(x,t)\to(0,0)$. This will be done in Section 3. In Section 4 we will then construct the profile function $u(x,t)$, namely a solution of the initial boundary value problem (1.1.5a)–(1.1.5c). The boundary condition $u_x(0,t)=0$ and the fact that $u>0$ will guarantee that $u(x,t)$ defines a smooth MCF solution M_t which is $O(4)\times O(4)$ symmetric. In Section 5 we will show that $H(x,t)$ remains bounded as $t\to0$. The barrier construction in Section 3 is based on the formal asymptotic expansion of the profile solution $u(x,t)$ as $(x,t)\to(0,0)$. For the convenience of the reader we will start by giving this expansion in the next section.

2. Formal asymptotic expansion of $u(x,t)$

We start with the construction by Angenent, Ilmanen, and Velázquez [Angenent et al.] of a formal asymptotic expansion of the profile solution $u(x,t)$ for small $t>0$. This construction motivates our choice of barriers in different regions later in order to rigorously prove the existence of a mean curvature flow past the singular time with the following properties. Our solution before the singularity at $t=0$ coincides with the solution constructed by Velázquez [1994], it continues as a smooth solution for $t\in(0,t_1)$, for some $t_1>0$, and has uniformly bounded mean curvature for all times $t<0$, for which it exists, and all $t\in(0,t_1)$.

2.1. ***Outer variables.*** We can approximate any smooth solution for small $t>0$ by using the Taylor expansion $u(x,t)=u(x,0)+tu_t(x,0)+o(t)$. In view of the PDE (1.1.5a) this implies that any solution $u(x,t)$ must satisfy

$$u(x,t)=u_0(x)+t\left\{\frac{u_0''(x)}{1+u_0'(x)^2}+\frac{3}{x}u_0'(x)-\frac{3}{u_0(x)}\right\}+o(t),\qquad t\to0. \tag{2.1.1}$$

We will see that under our assumptions (1.1.1)–(1.1.3) on the initial data, the expansion (2.1.1) holds if $x^2\gg t$. To describe possible solutions for $x^2\sim t$ we introduce a new set of coordinates, the intermediate variables.

2.2. ***Intermediate variables.*** Consider the function $v(y,\tau)$ defined by

$$u(x,t)=\sqrt{t}\,v\left(\frac{x}{\sqrt{t}},\log t\right). \tag{2.2.1}$$

It satisfies

$$v_\tau = \frac{v_{yy}}{1+v_y^2} + \left(\frac{3}{y} + \frac{y}{2}\right)v_y - \frac{v}{2} - \frac{3}{v}. \tag{2.2.2}$$

Assuming that $v(y,\tau)$ is close to the cone, we set

$$v(y,\tau) = y + f(y,\tau)$$

and compute the equation for f,

$$f_\tau = \mathcal{L}f + \mathcal{N}[f], \tag{2.2.3}$$

where $\mathcal{L}$ is the linear differential operator

$$\mathcal{L}f := \frac{1}{2}f_{yy} + \left(\frac{3}{y} + \frac{y}{2}\right)f_y + \left(\frac{3}{y^2} - \frac{1}{2}\right)f, \tag{2.2.4}$$

and where

$$\mathcal{N}[f] := -3\frac{f^2}{y^2(y+f)} - \frac{2+f_y}{1+(1+f_y)^2}f_y f_{yy} \tag{2.2.5}$$

collects the nonlinear terms in the equation for f.

If we assume that the nonlinear terms are much smaller than the linear terms, then f should be approximated by a solution of the linear equation $f_\tau = \mathcal{L}f$. The outer approximation $u(x,t) = u_0(x) + \mathcal{O}(t)$ together with the assumption that the initial function satisfies $u(x,0) = x + K_0x^{2(k-1)} + \cdots$ leads to

$$v(y,\tau) = y + K_0e^{(k-3/2)\tau}y^{2(k-1)} + \cdots \tag{2.2.6}$$

for $y \gg e^{-\tau/2}$. This prompts us to look for approximate solutions of the form

$$v(y,\tau) = y + K_1e^{(k-3/2)\tau}\varphi_k(y), \tag{2.2.7}$$

where φ_k is a solution of the differential equation

$$\mathcal{L}\varphi_k = \left(k - \tfrac{3}{2}\right)\varphi_k.$$

It turns out that there are positive and convex solutions of this equation that are defined for all $y > 0$. Their asymptotic behavior for small and large values of y is given by

$$\varphi_k(y) = \frac{1+o(1)}{y^2}, \quad y \to 0, \qquad \varphi_k(y) = \frac{1+o(1)}{(2k+1)!!}y^{2k-2}, \quad y \to \infty.$$

In Section A.1 we present some more details regarding the eigenfunctions φ_k.

This implies that our intermediate solution $v(y,\tau)$ from (2.2.7) is given by

$$v(y,\tau) = y + K_1e^{(k-3/2)\tau}\frac{y^{2(k-1)}}{(2k+1)!!} + \cdots$$

when y is large.[1] Comparing with (2.2.6) we see that K_0 and K_1 are related by

$$K_1 = K_0(2k+1)!!. \tag{2.2.8}$$

[1]Notation: $(2k+1)!! = 1 \cdot 3 \cdot 5 \cdots (2k-1) \cdot (2k+1)$.

2.3. ***Inner variables.*** One can only expect the intermediate approximation to hold if the nonlinear terms are small compared with the linear terms. Since the linear terms are all of order $\sim f/y^2$ and the nonlinear terms are of order f^2/y^3, we see that the nonlinear terms are dominated by the linear terms if $|f/y| \ll 1$.

When y is small we have $f(y,\tau) \sim e^{(k-3/2)\tau} y^{-2}$, so $|f/y| \ll 1$ holds if

$$e^{(k-3/2)\tau} y^{-3} \ll 1, \quad \text{i.e., } y \gg e^{(k/3-1/2)\tau} = e^{\gamma\tau},$$

where we write

$$\gamma = \frac{k}{3} - \frac{1}{2}.$$

In the original (x,t) coordinates we have $y = e^{\gamma\tau}$ exactly if $x = t^{k/3}$.

This leads us to introduce the new variable

$$z = ye^{-\gamma\tau} = xt^{-k/3}$$

and a new function $w(z,\tau)$ defined by

$$v(y,\tau) = e^{\gamma\tau} w(ye^{-\gamma\tau},\tau). \tag{2.3.1}$$

Equation (2.2.2) is equivalent to

$$\frac{w_{zz}}{1+w_z^2} + \frac{3}{z} w_z - \frac{3}{w} = e^{2\gamma\tau}\left\{ w_\tau + \frac{k}{3}(w - zw_z)\right\}. \tag{2.3.2}$$

For $\tau \to -\infty$ we assume the terms on the right vanish, so it is natural to look for an approximate solution of the form

$$w(z,\tau;K_2) = K_2 W\left(\frac{z}{K_2}\right) + \text{correction terms}, \tag{2.3.3}$$

where $W(z)$ is Alencar's solution[2] of the minimal surface equation

$$\frac{W''(z)}{1+W'(z)^2} + \frac{3}{z} W'(z) - \frac{3}{W(z)} = 0. \tag{2.3.4}$$

By scaling invariance of the minimal surface equation, $KW(z/K)$, with $K > 0$ an arbitrary constant, is always a solution of (2.3.4) if W is one. We choose W so that it is normalized by

$$W(z) = z + \frac{1}{z^2} + o(z^{-2}), \quad z \to \infty. \tag{2.3.5}$$

The matching condition for the inner solution $w(z,\tau) = K_2 W(z/K_2) + \cdots$ with the intermediate solution $v(y,\tau) = y + K_1 e^{(k-3/2)\tau} \varphi_k(y) + \cdots$ is then

$$w(z,\tau) \approx e^{-\gamma\tau} v(e^{\gamma\tau} z, \tau);$$

[2] See Section A.3 for a discussion of this solution. Alencar [1993] analyzed the ODEs that appear when one considers $SO(m) \times SO(m)$ invariant minimal surfaces of this type, although he mostly considered the cases $m = 2, 3$ in that first paper. Velázquez [1994] dealt with the case $m \geq 4$, and later Alencar, Barros, Palmas, Reyes, and Santos gave a complete classification in [Alencar et al. 2005]. Much earlier, Hardt and Simon [1985] proved a general existence theorem for smooth minimal hypersurfaces that accompany strictly minimizing cones, without assuming any kind of symmetry.

i.e.,

$$z+\frac{K_2^3}{z^2}+\cdots=z+K_1\frac{e^{(k-3/2)\tau}e^{-3\gamma\tau}}{z^2}+\cdots=z+\frac{K_1}{z^2}+\cdots.$$

Hence the constants K_1 and K_2 are related by

$$K_2^3=K_1=K_0(2k+1)!!, \tag{2.3.6}$$

and our approximate inner solution is given by

$$w(z,\tau)=K_1^{1/3}W(K_1^{-1/3}z).$$

3. Barriers

3.1. ***The three regions.*** Our goal in this section is to construct upper and lower barriers for

$$u_t=\frac{u_{xx}}{1+u_x^2}+\frac{3}{x}u_x-\frac{3}{u} \tag{1.1.5a}$$

that are valid for all $x\in(0,+\infty)$ and $0<t\le t_0$, for some small enough $t_0>0$.

To do this we modify the approximate solutions from Section 2 in each of the three regions and glue the resulting locally defined barriers into one set of globally defined upper and lower barriers.

First we define *the three regions*. In what follows we regard the three regions as subsets of spacetime and use the different sets of coordinates (x,t), (y,τ), and (z,τ) on spacetime to describe them.

- For any given $M>0$ we define the *outer region* to be

  $$\mathcal{O}_M=\{(x,t)\mid x\ge M\sqrt{t},\ 0<t<M^{-2}\}.$$

  We will assume that $M>1$.

- For any $R>0$ and $\tau_*\in\mathbb{R}$ we define the *intermediate region* to be

  $$\mathcal{M}_{R,\tau_*}=\{(y,\tau)\mid Re^{\gamma\tau}\le y\le e^{-\tau/2},\ \tau\le\tau_*\}.$$

  Since $y=x/\sqrt{t}=xe^{-\tau/2}$, the intermediate region is defined up to $x=1$; hence the intermediate and outer regions clearly overlap.

- Finally, we declare the *inner region* to be

  $$\mathcal{I}_{Z,\tau_*}=\{(z,\tau)\mid 0\le z\le Z,\ \tau\le\tau_*\}.$$

  Since $z=e^{-\gamma\tau}y$, we see that the intermediate and inner regions overlap if $Z>R$.

In Section 4 we will construct a nested sequence of barriers

$$U^-_{\delta_{n-1}}<U^-_{\delta_n}<U^+_{\delta_n}<U^+_{\delta_{n-1}},$$

where $\delta_n=2^{-n}\delta_0$ for some $\delta_0>0$. These barriers will be defined for all $\tau\le\tau_{\delta_n}$, where $\tau_{\delta_n}\to-\infty$ as $\delta_n\to 0$. As a result we will see that we need to take $Z=Z_{\delta_n}$ and $\tau^*=\tau_{\delta_n}$ in the definitions of the intermediate and inner regions above. In addition we will see that $Z_{\delta_n}\to+\infty$ as $\delta_n\to 0$.

3.2. *Fixing the parameters.* From here on we fix the parameters $k > 3$ and $K_0 > 0$, and we let K_1, K_2 be defined by (2.3.6). In all our estimates c and C will be generic constants that can depend *only* on k, K_0, K_1, and K_2. We use C in upper bounds, and c in lower bounds.

3.3. *Barriers in the outer region.*

Lemma 3.3.1. *For sufficiently large $M > 0$ the functions*

$$u^{\pm}(x,t) = u_0(x) \pm Mt \min\{1, x^{2k-4}\} \tag{3.3.1}$$

are super- or subsolutions in the outer region $\mathcal{O}_M$.

Proof. We only consider the upper barrier u^+. Similar arguments apply to the lower barrier.

When $x > 1$ we have $u^+(x,t) = u_0(x) + Mt$, so that for $t \in (0, M^{-2})$ one has $u^+(x,t) \geq \inf_{x \geq 1} u_0(x) =: c$. This implies

$$\left| \frac{u^+_{xx}}{1 + (u^+_x)^2} + \frac{3}{x} u^+_x - \frac{3}{u^+} \right| \leq C$$

for all $x \geq 1$ and $t \leq M^{-2}$. Here C does not depend on M. On the other hand $u^+_t = M$, so for large enough M we get

$$u^+_t \geq \frac{u^+_{xx}}{1 + (u^+_x)^2} + \frac{3}{x} u^+_x - \frac{3}{u^+},$$

i.e., u^+ is an upper barrier for $x \geq 1$.

If $x \geq M\sqrt{t}$ and $x \leq 1$, we have $u^+(x,t) = u_0(x) + Mtx^{2k-4}$, so that

$$|u^+_{xx}| \leq |u_{0,xx}| + CMtx^{2k-6} \leq Cx^{2k-4} + CMtx^{2k-6} \leq Cx^{2k-4}.$$

Similar estimates hold for $u^+_x - 1$ and $u^+(x,t) - x$, namely,

$$x^2 |u^+_{xx}| + x|u^+_x - 1| + |u^+ - x| \leq Cx^{2k-2}.$$

Hence

$$\frac{|u^+_{xx}|}{1 + (u^+_x)^2} \leq Cx^{2k-4},$$

and also

$$\left| \frac{3}{x} u^+_x - \frac{3}{u^+} \right| \leq \frac{3}{x} |u^+_x - 1| + 3 \frac{|u^+ - x|}{xu^+} \leq Cx^{2k-4}.$$

Together we get

$$\left| \frac{u^+_{xx}}{1 + (u^+_x)^2} + \frac{3}{x} u^+_x - \frac{3}{u^+} \right| \leq Cx^{2k-4},$$

where C does not depend on M. On the other hand, $u^+_t = Mx^{2k-4}$. Hence, it now follows that $u_0(x) + Mtx^{2k-4}$ is an upper barrier if M is large enough.

Finally we observe that at the point $x = 1$ the function $u^+(x,t)$ has a concave corner, so that $u^+(x,t) = u_0(x) + Mt \min\{1, x^{2k-4}\}$ is indeed an upper barrier for all $x \geq M\sqrt{t}$, $t < M^{-2}$.

Similar arguments show that

$$u^-(x,t) = u_0(x) - Mt\min\{1, x^{2k-4}\}$$

is a lower barrier in the same region. The only difference is that one now uses for $x > 1,\ t \in (0, M^{-2})$ the lower bound

$$u^-(x,t) \geq \inf_{x\geq 1} u_0(x) - Mt \geq \tfrac{1}{2}c,$$

for M sufficiently large, where $c := \inf_{x\geq 1} u_0(x)$. □

3.4. ***Barriers in the intermediate region.*** We model the upper and lower barriers in the intermediate region on the approximate solution $v(y,\tau) = y + f(y,\tau)$ from Section 2.2, where f is assumed to be a small function that satisfies (2.2.3), i.e., $f_\tau = \mathcal{L}f + \mathcal{N}[f]$. A function f defines an upper barrier for (2.2.3) in $\mathcal{M}_{R,\tau_*}$ if

$$f_\tau - \mathcal{L}f \geq \mathcal{N}[f] \tag{3.4.1}$$

holds throughout $\mathcal{M}_{R,\tau_*}$. For a lower barrier the reverse inequality must hold.

It turns out that the approximate solution $f_0(y,\tau) = Ke^{3\gamma\tau}\varphi_k(y)$ is neither a sub- nor supersolution for any choice of the constant K. To obtain barriers we therefore add a small correction term $f_1(y,\tau)$. While the resulting function $f_0(y,\tau) + f_1(y,\tau)$ does provide a barrier, it does not match the barrier we construct later in the inner region. To remedy this we add a second correction term $f_2(y,\tau)$. The resulting barriers $f_0 + f_1 + f_2$ will contain a small parameter $\delta > 0$. By choosing $\delta > 0$ smaller we get more accurate barriers, but we also have to reduce the time interval $-\infty < \tau \leq \tau_\delta$ on which they are defined. In the end this will allow us to prove convergence as $\tau \to -\infty$ of the actual solution that we construct using our barriers.

Our construction uses an auxiliary function $g : (0,\infty) \to \mathbb{R}$, which is the solution of the boundary value problem

$$\begin{cases} 6\gamma g(y) - \mathcal{L}g(y) = y^{-7} + y^{4k-7}, & 0 < y < \infty, \\ g(y) = -\frac{1}{3}y^{-5} + o(y^{-5}), & y \to 0, \\ g(y) = y^{4k-7} + o(y^{4k-7}), & y \to \infty. \end{cases} \tag{3.4.2}$$

The choice of forcing term in the equation for g above will become apparent in what follows. In Section A.2 we prove:

Lemma 3.4.1. *Equations (3.4.2) have a solution $g : (0,\infty) \to \mathbb{R}$.*

In fact, the proof in Section A.2 shows that there is a one-parameter family of solutions g. We choose one of these, for example the one for which the constant B from the proof in Section A.2 vanishes.

Assuming that Lemma 3.4.1 holds, we look for barriers in the family of functions

$$v_\delta^\pm(y,\tau) = y + f_\delta^\pm(y,\tau), \tag{3.4.3}$$

where

$$f_\delta^\pm(y,\tau) = f_0^\pm(y,\tau,\delta) \pm \{f_1(y,\tau) + f_2(y,\tau)\} \tag{3.4.4}$$

and

$$\begin{aligned} f_0^{\pm}(y,\tau,\delta) &= (K_1 \pm \delta)e^{3\gamma\tau}\varphi_k(y), \\ f_1(y,\tau) &= BK_1^2 e^{6\gamma\tau} g(y), \\ f_2(y,\tau) &= e^{(p+1)\gamma\tau} y^{-p}. \end{aligned} \tag{3.4.5}$$

Here, as in Section 3.2, we have $K_1 = (2k+1)!!\, K_0$, while $B, \delta > 0$ and $p \in (2,3)$ are parameters.

Proposition 3.4.2. *There exist B_*, R_*, and τ_* that only depend on k, K_0 such that for all $\delta \in \big(0, \frac{1}{2}K_1\big)$, $p \in (2,3)$, the functions $f_\delta^{\pm}$ defined in (3.4.4)–(3.4.5) are upper and lower barriers for (2.2.3) in the intermediate region $\mathcal{M}_{R_*,\tau_*}$. It follows that the functions $v_\delta^{\pm}$ defined in (3.4.3) are upper and lower barriers for (2.2.2) in $\mathcal{M}_{R_*,\tau_*}$.*

We begin with two lemmas that will simplify the proof of Proposition 3.4.2.

Lemma 3.4.3. *Wherever $f(y,\tau) \ge 0$ holds, one has*

$$|\mathcal{N}[f]| \le \frac{3}{y^3}[f]_2^2,$$

where, by definition, for any function $F(y,\tau)$ we define

$$[F]_2(y,\tau) := |F(y,\tau)| + |yF_y(y,\tau)| + |y^2 F_{yy}(y,\tau)|. \tag{3.4.6}$$

Proof. Using $2|1+x| \le 1 + (1+x)^2$ one finds for all $x \in \mathbb{R}$

$$\left|\frac{2+x}{1+(1+x)^2}\right| \le \frac{1}{1+(1+x)^2} + \frac{|1+x|}{1+(1+x)^2} \le \frac{3}{2}.$$

Using $f(y,\tau) \ge 0$ this implies

$$\begin{aligned} |\mathcal{N}[f]| &= \left|\frac{-3f^2}{y^2(y+f)} - \frac{2+f_y}{1+(1+f_y)^2} f_y f_{yy}\right| \\ &\le 3\frac{f^2}{y^3} + \frac{3}{2}|f_y f_{yy}| \le \frac{3}{y^3}\{f^2 + |yf_y||y^2 f_{yy}|\} \le \frac{3}{y^3}[f]_2^2. \end{aligned}$$

□

Lemma 3.4.4. *For any $B > 0$ there exist $R(B) > 0$ and $\tau(B) \in \mathbb{R}$ such that if $0 < \delta < \frac{1}{2}K_1$, then $f_\delta^{\pm}$ as defined in (3.4.4)–(3.4.5) satisfies*

$$f_\delta^{\pm}(y,\tau) > 0$$

and

$$|\mathcal{N}[f_\delta^{\pm}]| \le C_* e^{6\gamma\tau}(y^{-7} + y^{4k-7})$$

in the intermediate region $R(B)e^{\gamma\tau} \le y \le e^{-\tau/2}$, $\tau \le \tau(B)$.

As promised in Section 3.2, the constant C_* only depends on the constants k, K_0 but not on B.

Proof. Recall the notation from (3.4.6). The explicit expression (A.1.2) for φ_k implies

$$[\varphi_k]_2 \le Cy^{-2}(1 + y^{2k}),$$

and the construction of the auxiliary function g implies

$$[g]_2 \le Cy^{-5}(1+y^{4k-2}).$$

We also have for all $y > 0$

$$[y^{-p}]_2 = y^{-p} + py^{-p} + p(p+1)y^{-p} = (p+1)^2 y^{-p} < 16y^{-p}$$

because $2 < p < 3$. Hence the three terms f_j in (3.4.5) that add up to $f_\delta^\pm$ satisfy

$$\begin{aligned} [f_0]_2 &\le Ce^{3\gamma\tau} y^{-2}(1+y^{2k}), \\ [f_1]_2 &\le CBe^{6\gamma\tau} y^{-5}(1+y^{4k-2}), \\ [f_2]_2 &\le Ce^{(p+1)\gamma\tau} y^{-p}, \end{aligned}$$

assuming that $0 < \delta \le \frac{1}{2}K_1$.

If $Re^{\gamma\tau} \le y \le e^{-\tau/2}$, then we can estimate $f_\delta^\pm$ as

$$\begin{aligned} [f_\delta^\pm]_2 &\le C\frac{e^{3\gamma\tau}}{y^2}(1+y^{2k}) + CB\frac{e^{6\gamma\tau}}{y^5}(1+y^{4k-2}) + C\frac{e^{(p+1)\gamma\tau}}{y^p} \\ &\le C\frac{e^{3\gamma\tau}}{y^2}(1+y^{2k})\left\{1 + B\frac{e^{3\gamma\tau}}{y^3} + Be^{3\gamma\tau}y^{2k-5} + \frac{e^{(p-2)\gamma\tau}}{y^{p-2}}\right\} \\ &\le C\frac{e^{3\gamma\tau}}{y^2}(1+y^{2k})\{1 + BR^{-3} + Be^{\tau} + R^{-(p-2)}\}, \end{aligned}$$

where in estimating the third term in the bracket we used $3\gamma = k - \frac{3}{2}$. Thus, if we require

$$R \ge \max\{1, B^{1/3}\} \quad \text{and} \quad \tau \le \tau(B) := -\log B, \tag{3.4.7}$$

then $1 + BR^{-3} + Be^{\tau} + R^{-(p-2)} \le 4$ and so

$$[f_\delta^\pm]_2 \le Ce^{3\gamma\tau} y^{-2}(1+y^{2k}).$$

Combined with Lemma 3.4.3 this yields

$$|\mathcal{N}[f_\delta^\pm]| \le \frac{3}{y^3} Ce^{6\gamma\tau} y^{-4}(1+y^{2k})^2 \le \widetilde{C}e^{6\gamma\tau} y^{-7}(1+y^{4k})$$

in the intermediate region, provided that we verify $f_\delta^\pm \ge 0$ when $Re^{\gamma\tau} \le y \le e^{-\tau/2}$.

To prove $f_\delta^\pm \ge 0$ in the intermediate region we recall the assumption $\delta < \frac{1}{2}K_1$, which implies

$$f_\delta^\pm(y,\tau) \ge \tfrac{1}{2}K_1 e^{3\gamma\tau}\varphi_k(y) - \{BK_1^2 e^{6\gamma\tau}|g(y)| + e^{(p+1)\gamma\tau}y^{-p}\}.$$

Use the lower bound $\varphi_k(y) \ge cy^{-2}(1+y^{2k})$ and the upper bound $|g(y)| \le Cy^{-5}(1+y^{4k-2})$ to arrive at

$$f_\delta^\pm(y,\tau) \ge c\frac{e^{3\gamma\tau}}{y^2}(1+y^{2k}) - \left\{CB\frac{e^{6\gamma\tau}}{y^5}(1+y^{4k-2}) + \frac{e^{(p+1)\gamma\tau}}{y^p}\right\},$$

which, because $(1+xy)/(1+x) \le 1+y$ for all $x, y \ge 0$, implies

$$\frac{y^2 e^{-3\gamma\tau}}{c(1+y^{2k})} f_\delta^\pm(y,\tau) \ge 1 - CB\frac{e^{3\gamma\tau}}{y^3}(1+y^{2k-2}) - \frac{1}{c(1+y^{2k})}\frac{e^{(p-2)\gamma\tau}}{y^{p-2}}.$$

In the region $Re^{\gamma\tau} \le y \le e^{-\tau/2}$ we get

$$\frac{y^2 e^{-3\gamma\tau}}{c(1+y^{2k})} f_\delta^\pm(y,\tau) \ge 1 - \frac{CB}{R^3} - CBe^\tau - \frac{1}{cR^{p-2}}.$$

We adjust our choice of $R(B),\ \tau(B)$ in (3.4.7) to

$$R(B) = \widetilde{C}\max\{1, B^{1/3}\}, \quad \tau(B) = -\log(\widetilde{C}B) \tag{3.4.8}$$

for large enough $\widetilde{C} \ge 1$. Then, for $y \ge R(B)$ and $\tau \le \tau(B)$, we have

$$\frac{y^2 e^{-3\gamma\tau}}{c(1+y^{2k})} f_\delta^\pm(y,\tau) \ge \frac{1}{2} > 0,$$

and thus $f_\delta^\pm(y,\tau) > 0$. □

Proof of Proposition 3.4.2. We consider the case of upper barriers, where we have

$$(\partial_\tau - \mathcal{L}) f_\delta^+ = (\partial_\tau - \mathcal{L}) f_0^+ + (\partial_\tau - \mathcal{L}) f_1 + (\partial_\tau - \mathcal{L}) f_2. \tag{3.4.9}$$

The first term vanishes because $f_0^\pm$ is a solution of the linear equation $f_\tau = \mathcal{L}f$. For the last term in (3.4.9) we note that for any $r \in \mathbb{R}$ one has

$$\mathcal{L}[y^r] = \tfrac{1}{2}(r+2)(r+3)y^{r-2} + \tfrac{1}{2}(r-1)y^r.$$

Hence, if $p \in (2,3)$ then $\mathcal{L}[y^{-p}] < 0$ for all $y > 0$. It follows that

$$(\partial_\tau - \mathcal{L}) f_2 > \partial_\tau f_2 = (p+1)\gamma f_2 > 0.$$

The middle term in (3.4.9) satisfies

$$(\partial_\tau - \mathcal{L}) f_1 = BK_1^2 e^{6\gamma\tau}(6\gamma g - \mathcal{L}g) = BK_1^2 e^{6\gamma\tau}(y^{-7} + y^{4k-7}).$$

If we choose $B_* = C_* K_1^{-2}$ where C_* is the constant from Lemma 3.4.4, and if we set $R_* = R(B_*)$ and $\tau_* = \tau(B_*)$ according to (3.4.8), then we clearly have $(\partial_\tau - \mathcal{L}) f_\delta^+ > \mathcal{N}[f_\delta^+]$ in the intermediate region $\mathcal{M}_{R_*,\tau_*}$.

We conclude that f_δ^+ is an upper barrier, i.e., (3.4.1) holds. With minor modifications this argument also shows that f_δ^- is a lower barrier. □

We next show that the barriers $f_\delta^\pm$ form a nested sequence, in the sense of the lemma below. The nesting of barriers will allow us to construct a solution that is bounded by all barriers at once and will enable us to prove the convergence of our solution in the inner region to the Alencar minimal surface, as $\tau \to -\infty$.

Lemma 3.4.5. *The constant R_* from Proposition 3.4.2 can be chosen so that*

$$f_\delta^-(y,\tau) < f_{\delta/2}^-(y,\tau) < f_{\delta/2}^+(y,\tau) < f_\delta^+(y,\tau) \tag{3.4.10}$$

for all (y,τ) with $R_* e^{\gamma\tau} \le y$.

Proof. We can write the barrier functions $f_\delta^\pm$ as

$$f_\delta^\pm(y,\tau) = K_1 e^{3\gamma\tau}\varphi_k(y) \pm \{\delta e^{3\gamma\tau}\varphi_k(y) + B_* K_1^2 e^{6\gamma\tau} g(y) + e^{(p+1)\gamma\tau} y^{-p}\}.$$

Since $\varphi_k(y) > 0$ for all $y > 0$, it is immediately clear that

$$f_\delta^-(y,\tau) < f_{\delta/2}^-(y,\tau) \quad \text{and} \quad f_{\delta/2}^+(y,\tau) < f_\delta^+(y,\tau)$$

for all y,τ.

To prove the middle inequality we note that $f_{\delta/2}^-(y,\tau) < f_{\delta/2}^+(y,\tau)$ holds if and only if

$$\frac{\delta}{2} e^{3\gamma\tau}\varphi_k(y) + B_* K_1^2 e^{6\gamma\tau} g(y) + e^{(p+1)\gamma\tau} y^{-p} > 0,$$

which, in view of $\varphi_k(y) > 0$, will certainly hold if

$$B_* K_1^2 e^{6\gamma\tau} g(y) + e^{(p+1)\gamma\tau} y^{-p} > 0. \tag{3.4.11}$$

Since $g(y) > 0$ for large $y > 0$, there is a constant $C_g > 0$ such that $g(y) \ge -C_g y^{-5}$ for all $y > 0$. Hence (3.4.11) follows from

$$e^{(p+1)\gamma\tau} y^{-p} - C_g B_* K_1^2 e^{6\gamma\tau} y^{-5} > 0, \quad \text{i.e., } y e^{-\gamma\tau} > (C_g B_* K_1^2)^{1/(5-p)}.$$ □

3.5. *Barriers in the inner region.* In this section we present a family of sub- and supersolutions to (2.3.2) for $w(z,\tau)$ in the inner region $0 \le z \le Z$.

We recall our notation from Section 2.3 where $W(z)$ denotes the unique Alencar solution to (2.3.4), normalized so that

$$W(z) = z + \frac{1}{z^2} + \frac{\Gamma}{z^3} + \mathcal{O}(z^{-5}), \quad z \to \infty, \tag{3.5.1}$$

holds for a certain constant $\Gamma \in \mathbb{R}$.

Lemma 3.5.1. *For all $z > 0$ one has $W_K(z) > z W_K'(z)$.*

Proof. The inequality is invariant under rescaling, so we may assume $K = 1$. The asymptotics (3.5.1) show that $W(z) - zW_z(z) \to 0$ as $z \to \infty$. On the other hand, convexity of W implies $(W - zW_z)_z = -zW_{zz} < 0$ for all $z > 0$. Hence $W(z) - zW_z(z) > \lim_{Z\to\infty} W(Z) - ZW_z(Z) = 0$ for all $z \ge 0$. □

Lemma 3.5.2. *For any $K > 0$ the function $w^+(z,\tau) = W_K(z)$ is a supersolution of (2.3.2) on $[0,\infty) \times \mathbb{R}$.*

Proof. The function w^+ satisfies $w_\tau^+ = 0$ and

$$\frac{w_{zz}^+}{1 + (w_z^+)^2} + \frac{3}{z} w_z^+ - \frac{3}{w^+} = 0.$$

From Lemma 3.5.1 we have $w^+ - zw^+_z > 0$, and thus

$$e^{2\gamma\tau}\left(w^+_\tau + \frac{k}{3}(w^+ - zw^+_z)\right) > \frac{w^+_{zz}}{1+(w^+_z)^2} + \frac{3}{z}w^+_z - \frac{3}{w^+},$$

as claimed. □

Lemma 3.5.3. *There exist $D_* > 0$, $\zeta > 0$ such that for all $K \in \left(\frac{1}{2}K_2, 2K_2\right)$ and $D \geq D_*$ there is a $\tau_*(D)$ such that*

$$w^-(z,\tau) := W_K(z) + De^{2\gamma\tau}$$

is a subsolution of (2.3.2) for $0 \leq z \leq \zeta e^{-\gamma\tau}$, $\tau \leq \tau_*(D)$.

Proof. Choose

$$\tau_*(D) \leq \frac{1}{2\gamma}\log\frac{W_K(0)}{D}.$$

Then $\tau \leq \tau_*(D)$ and $z \geq 0$ implies

$$De^{2\gamma\tau} \leq W_K(0) \leq W_K(z),$$

so that

$$W_K(z) \leq w^-(z,\tau) \leq 2W_K(z).$$

If we substitute $w = w^-$ in (2.3.2) and use $2\gamma + \frac{1}{3}k = k-1$, then on one hand

$$e^{2\gamma\tau}\left(w^-_\tau + \frac{k}{3}(w^- - zw^-_z)\right) = e^{2\gamma\tau}\left((k-1)De^{2\gamma\tau} + \frac{k}{3}(W_K - zW'_K)\right)$$

and on the other hand

$$\frac{w^-_{zz}}{1+(w^-_z)2} + \frac{3}{z}w^-_z - \frac{3}{w^-} = \frac{W''_K}{1+(W'_K)^2} + \frac{3}{z}W'_K - \frac{3}{w^-} = \frac{3}{W_K} - \frac{3}{w^-} = \frac{3De^{2\gamma\tau}}{W_K w^-}.$$

Hence w^- is a subsolution if

$$\frac{3D}{W_K(z)w^-(z,\tau)} > (k-1)De^{2\gamma\tau} + \frac{k}{3}(W_K(z) - zW'_K(z)). \tag{3.5.2}$$

Since $W_K \leq w^- \leq 2W_K \leq C(1+z)$, there is a constant C_1 such that the terms on the left are bounded from below by

$$\frac{3D}{W_K(z)w^-(z,\tau)} \geq \frac{C_1 D}{(1+z)^2}.$$

The terms on the right in (3.5.2) satisfy

$$(k-1)e^{2\gamma\tau} \leq C_2\frac{\zeta^2}{(1+z)^2}$$

in the region $1+z \leq \zeta e^{-\gamma\tau}$ and, due to the asymptotic expansion of $W_K(z)$ as $z \to \infty$ (which follows from (3.5.1)), they also satisfy

$$W_K(z) - zW'_K(z) \leq \frac{C_3}{(1+z)^2} \quad \text{for all } z \geq 0.$$

Hence

$$(k-1)De^{2\gamma\tau} + \frac{k}{3}(W_K(z) - zW_K'(z)) \leq \frac{C_2\zeta^2 D + C_3}{(1+z)^2}.$$

Choose $\zeta < \sqrt{C_1/(2C_2)}$, and choose D large enough that $C_3 < \frac{1}{2}C_1 D$. Then we have

$$(k-1)De^{2\gamma\tau} + \frac{k}{3}(W_K(z) - zW_K'(z)) < \frac{C_1 D}{(1+z)^2} \leq \frac{3D}{W_K(z)w(z,\tau)},$$

which implies (3.5.2) and thus that w^- is a lower barrier in the region $1+z \leq \zeta e^{-\gamma\tau}$. Choose τ_* so that $\zeta e^{-\gamma\tau_*} \geq 2$. Then $1+z \leq \zeta e^{-\gamma\tau}$ holds for all $z \leq 1$ and $\tau \leq \tau_*$, while for $z \geq 1$ it follows from $2z \leq \zeta e^{-\gamma\tau}$ that $1+z \leq \zeta e^{-\gamma\tau}$.

Thus w^- is a lower barrier in the region $z \leq \frac{1}{2}\zeta e^{-\gamma\tau}$, $\tau \leq \tau_*$. □

3.6. ***Matching outer and intermediate barriers.*** We show that upper and lower barriers constructed in the inner, the intermediate, and the outer regions match in the overlapping region. We begin here with the overlap of the outer and intermediate regions.

We start with an $M > 0$ large enough that the functions

$$u^{\pm}(x,t) = u_0(x) \pm Mt \min\{1, x^{2k-4}\}$$

are sub- and supersolutions of (1.1.5a) in the outer region $\mathcal{O}_M$ (see Lemma 3.3.1). In order to match the outer barriers with the barriers in the intermediate region, we express the outer barriers $u = u^{\pm}(x,t)$ in the intermediate variables (v, y, τ):

$$v_{\text{out}}^{\pm}(y,\tau) := e^{-\tau/2}u^{\pm}(e^{\tau/2}y, e^{\tau}).$$

In (3.3.1) we defined $u^{\pm}(x,t) = u_0(x) \pm Mtx^{2k-4}$ for $0 < x \leq 1$. If we write the assumption (1.1.1) on the initial data in the form

$$u_0(x) = x + (K_0 + \epsilon_0(x))x^{2k-2}, \tag{3.6.1}$$

where $\epsilon_0 : (0,\infty) \to \mathbb{R}$ satisfies $\lim_{x\to 0}\epsilon_0(x) = 0$, then we get the following expression for the outer barriers in the intermediate variables:

$$v_{\text{out}}^{\pm}(y,\tau) = y + (K_0 + \epsilon_0(ye^{\tau/2}))e^{3\gamma\tau}y^{2k-2} \pm Me^{3\gamma\tau}y^{2k-4}. \tag{3.6.2}$$

The outer barriers only contain the parameter M and thus do not depend on other parameters such as δ and B that appeared in the barriers we constructed for the intermediate and inner regions.

We now consider the intermediate barriers, continuing to use the conventions from Section 3.2 which relate the constants K_0, K_1, etc.

In Proposition 3.4.2 we found B_*, R_*, and τ_*, such that for any $\delta \in \left(0, \frac{1}{2}K_1\right)$ and $p \in (2,3)$ the functions

$$v_\delta^{\pm}(y,\tau) = y + (K_1 \pm \delta)e^{3\gamma\tau}\varphi_k(y) \pm \{e^{(p+1)\gamma\tau}y^{-p} + B_*K_1^2 e^{6\gamma\tau}g(y)\}$$

are upper and lower barriers in the intermediate region $\mathcal{M}_{R_*,\tau_*} = \{R_*e^{\gamma\tau} \leq y \leq e^{-\tau/2},\ \tau \leq \tau_*\}$.

To compare $v_{\rm out}^{\pm}$ and $v_\delta^{\pm}$ we rewrite them as

$$e^{-3\gamma\tau}(v_{\rm out}^{\pm}(y,\tau)-y) = (K_0+\epsilon_0(ye^{\tau/2}))y^{2k-2} \pm My^{2k-4},$$
$$e^{-3\gamma\tau}(v_\delta^{\pm}(y,\tau)-y) = (K_1\pm\delta)\varphi_k(y) \pm e^{(p-2)\gamma\tau}y^{-p} \pm B_* K_1^2 e^{3\gamma\tau} g(y).$$

We now let $\tau\to-\infty$ and conclude that

$$\begin{cases} e^{-3\gamma\tau}(v_{\rm out}^{\pm}(y,\tau)-y) \to K_0 y^{2k-2} \pm My^{2k-4}, \\ e^{-3\gamma\tau}(v_\delta^{\pm}(y,\tau)-y) \to (K_1\pm\delta)\varphi_k(y), \end{cases} \tag{3.6.3}$$

uniformly for bounded y.

The explicit expression (A.1.2) for φ_k implies

$$\varphi_k(y) = \frac{y^{2k-2}}{(2k+1)!!} + c(y)y^{2k-4},$$

where

$$c(y) = c_0 + \frac{c_1}{y^2} + \cdots + \frac{c_{k-1}}{y^{2k-2}}, \qquad c_j = \frac{\binom{k}{j+1}}{(2(k-j)-1)!!}.$$

Substitute this expression for φ_k in (3.6.3) and keep in mind that $K_1 = (2k+1)!!\,K_0$. Then

$$e^{-3\gamma\tau}(v_{\rm out}^{\pm}(y,\tau) - v_\delta^{\pm}(y,\tau)) \to \pm y^{2k-4}\left\{-\frac{\delta y^2}{(2k+1)!!} + M - c(y)\right\}.$$

The function $c(y)$ is clearly bounded for $y\geq 1$, so if M is sufficiently large, one can neglect $c(y)$ and conclude that $v_{\rm out}^{\pm}(y,\tau) - v_\delta^{\pm}(y,\tau)$ changes sign when

$$\frac{\delta y^2}{(2k+1)!!} = M - c(y) \approx M.$$

To make this more precise we introduce

$$Y_\delta := 2\sqrt{(2k+1)!!\,M/\delta} \tag{3.6.4}$$

and compare the barriers $v_{\rm out}^{\pm}(y,\tau)$ and $v_\delta^{\pm}(y,\tau)$ at the endpoints $y_\delta(\tau)\in\left(\frac14 Y_\delta, Y_\delta\right)$.

Lemma 3.6.1. *For any $\delta>0$ there is a $\tau_\delta\in\mathbb{R}$ such that for all $\tau\leq\tau_\delta$ one has*

$$v_{\rm out}^{+}\left(\tfrac14 Y_\delta,\tau\right) > v_\delta^{+}\left(\tfrac14 Y_\delta,\tau\right) \quad \textit{and} \quad v_{\rm out}^{-}\left(\tfrac14 Y_\delta,\tau\right) < v_\delta^{-}\left(\tfrac14 Y_\delta,\tau\right).$$

Moreover, we also have

$$v_{\rm out}^{+}(Y_\delta,\tau) < v_\delta^{+}(Y_\delta,\tau) \quad \textit{and} \quad v_{\rm out}^{-}(Y_\delta,\tau) > v_\delta^{-}(Y_\delta,\tau)$$

for all $\tau\leq\tau_\delta$.

Proof. We only consider the upper barriers, the other case being nearly identical.

We have found that as $\tau\to-\infty$

$$e^{-3\gamma\tau}\left(v_{\rm out}^{+}\left(\tfrac14 Y_\delta,\tau\right) - v_\delta^{+}\left(\tfrac14 Y_\delta,\tau\right)\right) \to \left(\tfrac14 Y_\delta\right)^{2k-4}\left\{-\tfrac14 M + M - c\left(\tfrac14 Y_\delta\right)\right\}.$$

Since $c(y)$ is bounded for $y\geq 1$, given any large M we will still have

$$\tfrac34 M - c\left(\tfrac14 Y_\delta\right) > 0.$$

Hence

$$\lim_{\tau\to-\infty} e^{-3\gamma\tau}\big(v_{\text{out}}^{+}\big(\tfrac14 Y_\delta,\tau\big)-v_\delta^{+}\big(\tfrac14 Y_\delta,\tau\big)\big)>0,$$

which implies that for $-\tau$ sufficiently large one has $v_{\text{out}}^{+}\big(\tfrac14 Y_\delta,\tau\big)>v_\delta^{+}\big(\tfrac14 Y_\delta,\tau\big)$, as claimed.

If on the other hand we compare v_{out}^{+} and v_δ^{+} at $y=Y_\delta$, then we find that for $\tau\to-\infty$

$$e^{-3\gamma\tau}(v_{\text{out}}^{+}(Y_\delta,\tau)-v_\delta^{+}(Y_\delta,\tau))\to Y_\delta^{2k-4}\{-4M+M-c(Y_\delta)\}=-Y_\delta^{2k-4}\{3M+c(Y_\delta)\}.$$

Since $c(y)$ is bounded for $y\ge 1$, it follows that for M large enough we indeed have $v_{\text{out}}^{+}(Y_\delta,\tau)<v_\delta^{+}(Y_\delta,\tau)$, as $\tau\to-\infty$. □

3.7. ***Matching intermediate and inner barriers.*** For any $\delta\in\big(0,\tfrac12 K_1\big)$, $p\in(2,3)$ and $B=B_*$ the barriers $v_\delta^{\pm}(y,\tau)=y+f_\delta^{\pm}(y,\tau)$ constructed above are defined in the intermediate region

$$\mathcal{M}_{R_*,\tau_*}=\{R_* e^{2\gamma\tau}\le y\le e^{-\tau/2},\ \tau\le\tau_*\}.$$

If we assume that $Z>2R_*$, then it follows that $v_\delta^{\pm}(y,\tau)$ are defined in parts of the inner region $\mathcal{I}_{Z,\tau_*}=\{(z,\tau)\mid 0\le z\le Z,\ \tau\le\tau_*\}$. Define

$$w_{\text{md}}^{\pm}(z,\tau):=e^{-\gamma\tau}v_\delta^{\pm}(e^{\gamma\tau}z,\tau).$$

Then

$$w_{\text{md}}^{\pm}(z,\tau)=z+\frac{K_1\pm\delta}{z^2}(1+\epsilon_1(z,\tau))\pm\frac{1}{z^p}\pm\frac{B_*K_1^2}{z^5}(1+\epsilon_2(z,\tau)),$$

where $\epsilon_i(z,\tau)$ are generic functions for which $\epsilon_i(z,\tau)\to 0$ as $\tau\to-\infty$, uniformly for $0\le z\le Z$. In particular, for all $z\in[0,Z]$ we have

$$\lim_{\tau\to-\infty} w_{\text{md}}^{\pm}(z,\tau)=z+\frac{K_1}{z^2}\pm\left\{\frac{\delta}{z^2}+\frac{1}{z^p}+\frac{B_*K_1^2}{z^5}\right\}. \tag{3.7.1}$$

We will now use Lemmas 3.5.2 and 3.5.3 to match $w_{\text{md}}^{\pm}(z,\tau)$ with appropriately chosen barriers $w_\delta^{\pm}(z,\tau)$ in the inner region $0\le z\le Z$. For suitable δ-dependent constants $K_2^{\pm}\in\big(\tfrac12 K_2,2K_2\big)$, with $(K_2)^3=K_1$, we consider

$$w_\delta^{+}(z,\tau):=W_{K_2^+}(z),\qquad w_\delta^{-}(z,\tau):=W_{K_2^-}(z)+De^{2\gamma\tau},$$

where D depends on K_2^- and Z as described in Lemma 3.5.3.

It follows from Lemmas 3.5.2 and 3.5.3 that, for each $K_2^+>0$ and $K_2^->0$, w_δ^+ and w_δ^- are the upper barrier and the lower barrier, respectively, for (2.3.2) in the inner region. Furthermore the asymptotics at infinity of the Alencar solution in (3.5.1) imply that

$$\lim_{\tau\to-\infty} w_\delta^{\pm}(z,\tau)=z+\frac{(K_2^{\pm})^3}{z^2}+\frac{\Gamma(K_2^{\pm})^4}{z^3}+\mathcal{O}(z^{-5}),\qquad z\gg 1.$$

Comparing the asymptotic expansions of $w_{\text{md}}^{\pm}$ and $w_\delta^{\pm}$ we see that they match when $(K_2^{\pm})^3=K_1\pm\delta$. However with this choice the barriers $w_{\text{md}}^{\pm}$ and $w_\delta^{\pm}$ may not intersect. For this reason we choose the constants $K_2^{\pm}$ such that

$$(K_2^{\pm})^3=K_1\pm 2\delta.$$

With this choice we then have

$$\lim_{\tau\to-\infty} w_\delta^\pm(z,\tau) = z + \frac{K_1 \pm 2\delta}{z^2} + \frac{\Gamma(K_1 \pm 2\delta)^{4/3}}{z^3} + \mathcal{O}(z^{-5}), \quad z \gg 1. \tag{3.7.2}$$

Lemma 3.7.1. *Let $p \in (2,3)$ be given, and let $B = B_k$ as in Proposition 3.4.2. Then there exist $\bar\delta > 0$ and $R = R(B)$ so that for any $\delta \in (0,\bar\delta)$ and $\tau \le \tau_\delta$ the barriers $w_\delta^\pm$ and $w_{\mathrm{md}}^\pm$ cross in the interval $\left(\tfrac12 Z_\delta, Z_\delta\right)$, where $Z_\delta := \tfrac43 \delta^{-1/(p-2)}$, in the sense that*

$$w_{\mathrm{md}}^+\left(\tfrac12 Z_\delta,\tau\right) > w_\delta^+\left(\tfrac12 Z_\delta,\tau\right) \quad \text{and} \quad w_{\mathrm{md}}^-\left(\tfrac12 Z_\delta,\tau\right) < w_\delta^-\left(\tfrac12 Z_\delta,\tau\right),$$

and

$$w_{\mathrm{md}}^+(Z_\delta,\tau) < w_\delta^+(Z_\delta,\tau) \quad \text{and} \quad w_{\mathrm{md}}^-(Z_\delta,\tau) > w_\delta^-(Z_\delta,\tau).$$

Proof. We only consider the upper barriers, the other case being nearly identical. Proposition 3.4.2 asserts that for $\delta < \tfrac12 K_1$ the function $w_{\mathrm{md}}^+(z,\tau)$ is an upper barrier in the intermediate region $R_* \le z \le e^{-(k/3)\tau}$ and it satisfies (3.7.1) with this choice of constants; that is

$$\lim_{\tau\to-\infty} w_{\mathrm{md}}^+(z,\tau) = z + \frac{K_1+\delta}{z^2} + \frac{1}{z^p} + \mathcal{O}(z^{-5}), \quad z \to \infty,$$

where the $\mathcal{O}(z^{-5})$ term is uniform in $\delta \in \left(0, \tfrac12 K_1\right)$. We have also seen that

$$\lim_{\tau\to-\infty} w_\delta^+(z,\tau) = z + \frac{K_1+2\delta}{z^2} + \mathcal{O}(z^{-3}), \quad z \to \infty,$$

where $\mathcal{O}(z^{-3})$ is again uniform in δ. Therefore

$$\lim_{\tau\to-\infty} w_\delta^+(z,\tau) - w_{\mathrm{md}}^+(z,\tau) = \frac{\delta}{z^2} - \frac{1}{z^p} + O(z^{-3}), \quad z \to \infty.$$

Consider $Z_\delta := \tfrac43\delta^{-1/(p-2)}$. For small enough $\delta > 0$ one has $Z_\delta \ge 2R_*$, so that $w_\delta^\pm(z,\tau)$ and $w_{\mathrm{md}}^\pm(z,\tau)$ are defined for all $z \ge \tfrac12 Z_\delta$ and all $\tau \le \tau_*$. We evaluate these differences at $z = Z_\delta$ and $z = \tfrac12 Z_\delta$. Eliminating δ by using $\delta = \left(\tfrac34 Z_\delta\right)^{-(p-2)}$ we find

$$\lim_{\tau\to-\infty} w_\delta^+(Z_\delta,\tau) - w_{\mathrm{md}}^+(Z_\delta,\tau) = \left(\left(\tfrac43\right)^{p-2} - 1\right) Z_\delta^{-p} + \mathcal{O}(Z_\delta^{-3}).$$

For small enough $\delta > 0$, Z_δ is large, and thus the first term dominates the second. This implies that for small $\delta > 0$ there is a $\tau_\delta < 0$ such that

$$w_\delta^+(Z_\delta,\tau) - w_{\mathrm{md}}^+(Z_\delta,\tau) > 0$$

for all $\tau \le \tau_\delta$. Similarly, we have

$$\lim_{\tau\to-\infty} w_\delta^+\left(\tfrac12 Z_\delta,\tau\right) - w_{\mathrm{md}}^+\left(\tfrac12 Z_\delta,\tau\right) = \left(\left(\tfrac23\right)^{p-2} - 1\right) 2^p Z_\delta^{-p} + \mathcal{O}(Z_\delta^{-3}).$$

This implies that if $\delta > 0$ is small then there is a $\tau_\delta < 0$ such that

$$w_\delta^+\left(\tfrac12 Z_\delta,\tau\right) - w_{\mathrm{md}}^+\left(\tfrac12 Z_\delta,\tau\right) < 0$$

for all $\tau \le \tau_\delta$. □

3.8. ***A summary of our construction so far.*** The initial data u_0 determines two constants $k \geq 4$ and K_0. Throughout the paper we let $K_1 = (2k+1)!!\, K_0$ and $K_2 = K_1^{1/3}$.

In Section 3.3 we chose a constant $M > 0$ so that Lemma 3.3.1 holds and constructed upper and lower barriers $u^{\pm}(x,t)$ in the outer region $\mathcal{O}_M$.

For any small enough $\delta > 0$ we then constructed a family of barriers $v_\delta^{\pm}$ in the intermediate region defined by $R_* e^{\gamma\tau} \leq y \leq e^{-\tau/2}$, $\tau \leq \tau_\delta$. Here Propositions 3.4.2 and 3.4.5 specify R_*, while τ_δ is determined when we match the intermediate and inner barriers in Lemma 3.6.1.

For small $\delta > 0$ we then considered the inner region

$$\mathcal{I}_{Z_\delta,\tau_\delta} = \{(z,\tau) \mid 0 \leq z \leq Z_\delta,\ \tau \leq \tau_\delta\}$$

with $Z_\delta := \frac{4}{3}\delta^{-1/(p-2)}$ and where τ_δ is as above. Since $\delta > 0$ is small and R_* does not depend on δ, we have $\delta < \left(\frac{3}{2}R_*\right)^{2-p}$, which implies $Z_\delta > 2R_*$. Hence the intermediate and inner regions overlap at least on $\frac{1}{2}Z_\delta \leq z \leq Z_\delta$.

Lemma 3.5.2 with K_2^+ satisfying $(K_2^+)^3 = K_1 + 2\delta$ defines the upper barrier w_δ^+ in the inner region $\mathcal{I}_{Z_\delta,\tau_\delta}$, and Lemma 3.5.3 with K_2^- satisfying $(K_2^-)^3 = K_1 - 2\delta$ defines the constant $D = D(K_2^-)$ and the lower barrier w_δ^- in $\mathcal{I}_{Z_\delta,\tau_\delta}$.

3.9. ***The upper and lower barriers $U_\delta^+(x,t)$ and $U_\delta^-(x,t)$.*** In the previous subsections, we constructed upper barriers $u^+(x,t)$, $v_\delta^+(y,\tau)$, $w_\delta^+(z,\tau)$ and lower barriers $u^-(x,t)$, $v_\delta^-(y,\tau)$, $w_\delta^-(z,\tau)$ in the outer, intermediate, and inner regions, respectively, and showed that they are correctly ordered in the overlaps between the three regions. These barriers exist for all $0 < t \leq t_\delta$ or equivalently $-\infty < \tau \leq \tau_\delta$. We now define the global barrier $U_\delta^+(x,t)$ by taking the minimum of the upper barriers when all are expressed in the unrescaled (x,t) variables. More precisely, we define

$$U_\delta^+(x,t) = \begin{cases} u^+(x,t), & y \geq Y_\delta, \\ \min\{u^+(x,t), t^{1/2} v_\delta^+(y,\log t)\}, & \frac{1}{4}Y_\delta \leq y \leq Y_\delta, \\ t^{1/2} v_\delta^+(y,\log t), & Z_\delta \leq z \text{ and } y \leq \frac{1}{4}Y_\delta, \\ \min\{t^{1/2} v_\delta^+(y,\log t), t^{k/3} w_\delta^+(z,\log t)\}, & \frac{1}{2}Z_\delta \leq z \leq Z_\delta, \\ t^{k/3} w_\delta^+(z,\log t), & 0 \leq z \leq \frac{1}{2}Z_\delta, \end{cases} \tag{3.9.1}$$

where, as before, $y = xt^{-1/2}$ and $z = xt^{-k/3}$. Lemmas 3.7.1 and 3.6.1 imply that U_δ^+ is a weak supersolution of (1.1.5a) and, similarly,

$$U_\delta^-(x,t) = \begin{cases} u^-(x,t), & y \geq Y_\delta, \\ \max\{u^-(x,t), t^{1/2} v_\delta^-(y,\log t)\}, & \frac{1}{4}Y_\delta \leq y \leq Y_\delta, \\ t^{1/2} v_\delta^-(y,\log t), & Z_\delta \leq z \text{ and } y \leq \frac{1}{4}Y_\delta, \\ \max\{t^{1/2} v_\delta^-(y,\log t), t^{k/3} w_\delta^-(z,\log t)\}, & \frac{1}{2}Z_\delta \leq z \leq Z_\delta, \\ t^{k/3} w_\delta^-(z,\log t), & 0 \leq z \leq \frac{1}{2}Z_\delta, \end{cases} \tag{3.9.2}$$

is a weak subsolution of (1.1.5a). This is summarized in the following proposition.

Proposition 3.9.1. *There exist a number $\delta_0 > 0$ and a sequence of times $t_n \searrow 0$ such that the functions $U^{\pm}_{\delta_n}(x,t)$ given in (3.9.1) and (3.9.2), with $\delta_n = 2^{-n}\delta_0$, define weak super- and subsolutions of (1.1.5a) for all $0 < t \le t_n$.*

Moreover, one has

$$U^-_{\delta_n}(x,t) \le U^-_{\delta_{n+1}}(x,t) < U^+_{\delta_{n+1}}(x,t) \le U^+_{\delta_n}(x,t) \tag{3.9.3}$$

for all $x > 0$ and $0 < t \le t_{n+1}$.

Proof. The fact that $U^{\pm}_{\delta_n}(x,t)$, $0 < t \le t_n$, define weak super- and subsolutions of (1.1.5a) follows from Lemma 3.3.1, Proposition 3.4.2, Lemmas 3.5.2 and 3.5.3, and the matching of our barriers in Sections 3.6 and 3.7.

For (3.9.3), we recall that our barriers $u^{\pm}(x,t)$ in the outer region do not depend on δ; hence they are ordered in their common domain and furthermore it is clear that $u^-(x,t) < u^+(x,t)$. In Proposition 3.4.2 we proved (3.4.10), which implies that (3.9.3) holds in the intermediate region for $0 < t \le t_{n+1}$. To finish the proof of (3.9.3) it is sufficient to show that for any $\delta \le \delta_0$ the inequalities

$$w^-_{\delta}(z,\tau) < w^-_{\delta/2}(z,\tau) < w^+_{\delta/2}(z,\tau) < w^+_{\delta}(z,\tau) \tag{3.9.4}$$

hold for all $0 \le z \le Z_\delta$, $\tau \le \tau_\delta$. This follows from the definition of $w^{\pm}_{\delta}(z,\tau)$ in Section 3.7 by observing that the rescaled Alencar solutions $W_K(z) := K\,W(z/K)$ are ordered for $K > 0$; that is,

$$\kappa < \bar\kappa \implies W_\kappa(z) < W_{\bar\kappa}(z) \quad \text{for all } z \in [0,+\infty). \tag{3.9.5}$$

To see this, recall the inequality $W - zW_z > 0$, $z \ge 0$, which is a consequence of the convexity of W and was shown in Lemma 3.5.1. This inequality implies that

$$\frac{d}{d\kappa} W_\kappa(z) = \frac{d}{d\kappa}\Big(\kappa W\Big(\frac{z}{\kappa}\Big)\Big) = W\Big(\frac{z}{\kappa}\Big) - \frac{z}{\kappa} W'\Big(\frac{z}{\kappa}\Big) > 0; \tag{3.9.6}$$

i.e., $\kappa \to W_\kappa(z)$ is monotone increasing in κ. We conclude that (3.9.4), holds which finishes the proof of (3.9.3) and the proof of the proposition. ☐

4. Existence of a smooth solution

4.1. ***Outline of the existence proof.*** In this section we return to the $O(4)\times O(4)$ symmetric hypersurface M_0 with profile function $u_0 : [0,\infty) \to \mathbb{R}$. Recall that u_0 is smooth for $x > 0$ and satisfies conditions (1.1.1) and (1.1.2) for some fixed $k > 3$ and some constant $C_0 > 0$. In Proposition 3.9.1 we constructed sequences of nested upper and lower barriers for (1.1.5a). We will show in this section how to use them to prove the existence of a smooth solution $u(x,t)$ to the initial value problem (1.1.5a)–(1.1.5c) defined for all $0 < t \le t_0$, for some $t_0 > 0$. Our main result in this section is as follows.

Theorem 4.1.1 (existence of a smooth solution). *Assume that M_0 is an $O(4)\times O(4)$ symmetric hypersurface defined by a profile function $u_0 : [0,\infty) \to \mathbb{R}$ which is smooth for $x > 0$ and satisfies conditions (1.1.1)–(1.1.2). Then there exists $t_0 > 0$ and a C^∞-smooth $O(4)\times O(4)$ symmetric MCF solution M_t,*

$0 < t \le t_0$, *defined by a profile function* $u : (0, \infty) \times (0, t_0] \to (0, \infty)$ *which satisfies the initial value problem* (1.1.5a)–(1.1.5c). *Furthermore,* $u(x, t)$ *satisfies*

$$U^-_{\delta_n}(x, t) \le u(x, t) \le U^+_{\delta_n}(x, t), \quad (x, t) \in [0, \infty) \times (0, t_n), \tag{4.1.1}$$

where $\delta_n = 2^{-n}\delta_0$ *and* $U^{\pm}_{\delta_n}(x, t)$, *for* $t \in (0, t_n)$, *are the upper and lower barriers constructed in Proposition 3.9.1.*

It follows from (4.1.1) *that*

$$\lim_{t \searrow 0} t^{-k/3} u(t^{k/3} z, t) = W_{K_2}(z), \tag{4.1.2}$$

uniformly for bounded $z \ge 0$.

Since equation (1.1.5a) is singular at $u = 0$, we cannot directly apply one of the standard short-time existence results to obtain our solution $u(x, t)$. Instead, we will construct it as the limit of a sequence of approximating solutions $u_n(x, t)$, each of which is defined on some time interval starting at a carefully chosen initial time s_n, where $s_n \searrow 0$. We will define the approximating solutions u_n by choosing their initial times s_n and values $u_n(x, s_n)$ in such a way that they satisfy

$$U^-_{\delta_n}(x, s_n) \le u_n(x, s_n) \le U^+_{\delta_n}(x, s_n) \quad \text{for all } x \ge 0, \tag{4.1.3}$$

where $\delta_n := 2^{-n}\delta_0$ and where $U^{\pm}_{\delta_n}(\,\cdot\,, t)$ are the barriers constructed in Proposition 3.9.1.

The barrier $U^-_{\delta_n}$ is bounded away from $u = 0$, and this allows us to invoke a classical short-time existence theorem for the quasilinear parabolic initial value problem (1.1.5a)–(1.1.5b). The short-time existence theorem guarantees that our solution exists for $s_n \le t < \bar{t}_n$, i.e., until some time $\bar{t}_n > s_n$. This time may exceed the life time t_n of the barriers $U^{\pm}_{\delta_n}$. In fact, by finding *a priori* estimates for the solutions $u_n(x, t)$ we will show that there is an n_0 such that for all $n \ge n_0$ we have $\bar{t}_n > t_{n_0}$, and that we can extract a convergent subsequence $u_{n_j}(x, t)$ whose limit $u(x, t)$ is a solution of the full initial value problem (1.1.5a)–(1.1.5c), and which is defined for $x \ge 0$ and $0 \le t \le t_{n_0}$.

The first a priori estimate we derive for the u_n follows directly from the maximum principle applied to the barriers $U^{\pm}_{\delta_n}$. Since the barriers are ordered by (3.9.3), the a priori bound (4.1.3) implies that for all n_0, $n \ge n_0$ and $x \ge 0$ one has

$$U^-_{\delta_{n_0}}(x, s_n) \le U^-_{\delta_n}(x, s_n) \le u_n(x, s_n) \le U^+_{\delta_n}(x, s_n) \le U^+_{\delta_{n_0}}(x, s_n). \tag{4.1.4}$$

The maximum principle tells us that for all $n \ge n_0$ and $x \ge 0$ one has

$$U^-_{\delta_{n_0}}(x, t) \le u_n(x, t) \le U^+_{\delta_{n_0}}(x, t) \tag{4.1.5}$$

for all $t \ge s_n$ at which $U^{\pm}_{\delta_{n_0}}(x, t)$ and $u_n(x, t)$ are defined, i.e., for $s_n \le t < \min\{\bar{t}_n, t_{n_0}\}$.

Thereafter we establish a priori estimates for the higher-order derivatives of the u_n. We conclude this work in the next Section 5 by showing that the mean curvatures $H_n(x, t)$ of the evolving surfaces corresponding to the approximating solutions $u_n(x, t)$ are uniformly bounded for all x, n, t, and hence that the mean curvature of the limit solution $u(x, t)$ also is uniformly bounded.

The simplest choice for the initial value for u_n would be to simply set $u_n(x, s_n) = U^-_{\delta_n}(x, s_n)$, but this function is not necessarily smooth in the overlaps between inner, intermediate, and outer regions, and this complicates the estimation of the higher derivatives of u_n. Furthermore, to prove that the mean curvatures $H_n(x, t)$ are uniformly bounded, it will be important to have $H_n(x, s_n) = 0$ on $0 \le x \le \epsilon s_n^{1/2}$ for some small fixed $\epsilon > 0$. For these reasons we will construct $u_n(x, s_n)$ by smoothly gluing the lower barrier $U^-_{\delta_n}(x, s_n)$ to an Alencar surface in the inner region $x \le \epsilon s_n^{1/2}$. Let us now turn to the details of this construction.

4.2. ***Short time existence and the comparison principle.*** Equation (1.1.5a) for $u(x, t)$ has a singular term at $x = 0$ which is there because we consider radially symmetric solutions only. To derive short-time existence from existing results, it is more convenient to consider the more general case of hypersurfaces that are only partially symmetric, i.e., with $\{1\} \times O(4)$ rather than $O(4) \times O(4)$ symmetry. For any positive function $r : \mathbb{R}^4 \times [0, t_0) \to \mathbb{R}$ we consider the family of hypersurfaces parametrized by

$$F : \mathbb{R}^4 \times S^3 \times [0, t_0) \to \mathbb{R}^8,$$

where

$$F(x, \Omega, t) = (x, r(x, t)\Omega).$$

A direct computation shows that F evolves by MCF if and only if r satisfies

$$r_t = g^{ij}(Dr) r_{x_i x_j} - \frac{3}{r}, \tag{4.2.1}$$

in which

$$g_{ij}(p) = \delta_{ij} + p_i p_j, \quad g^{ij}(p) = \delta_{ij} - \frac{p_i p_j}{1 + |p|^2}.$$

As long as Dr is uniformly bounded, (4.2.1) is a uniformly parabolic quasilinear equation. The solutions that interest us are not bounded, so we choose a reference function $R : \mathbb{R}^4 \to \mathbb{R}$ that is uniformly bounded from below, has uniformly bounded derivatives up to third order, and for which $R(x) - u_0(\|x\|)$ is uniformly bounded.

All initial data we prescribe in the following sections are bounded perturbations of $R(x)$. We therefore consider solutions of the form $r(x, t) = R(x) + a(x, t)$ and derive the equation for a:

$$a_t = g^{ij}(DR + Da) a_{x_i x_j} + g^{ij}(DR + Da) R_{x_i x_j} - \frac{3}{R + a}. \tag{4.2.2}$$

Since we assume that DR and $D^2 R$ are uniformly bounded, this equation is uniformly parabolic, as long as Da is bounded. By assumption $D^m R$ with $m \le 3$ are all uniformly bounded, so (4.2.2) is of the form

$$a_t = A_{ij}(x, Da) a_{x_i x_j} + B(x, a, Da),$$

where A_{ij} are uniformly parabolic, and where the functions A_{ij}, B are C^1 in $x \in \mathbb{R}^4$ and real analytic in (a, Da).

This implies the existence of a short-time solution $a(x, t)$ for any initial $a(x, 0)$ with $a(\cdot, 0) \in C^{1,\alpha}(\mathbb{R}^4)$ and for which $\inf_x R(x) + a(x, 0) > 0$. The classical theory for quasilinear parabolic equations [Ladyženskaja et al. 1968, §VI.1] implies that as long as $\sup_x |a(x, t)|$ and $\sup_x |Da(x, t)|$ are bounded, and as long as $\inf_x R(x) + a(x,t)$ has a positive lower bound, one can show that $Da(\,\cdot\,, t)$ is uniformly Hölder continuous. This in turn implies higher derivative bounds, and hence that the solution can be extended to a larger time interval.

For such solutions the standard comparison principle also holds: if $a_\pm : \mathbb{R}^4 \times [0, t_0) \to \mathbb{R}$ are two solutions with $Da_\pm$ bounded, for which $a_-(x,0) \le a_+(x,0)$ holds for all $x \in \mathbb{R}^4$, then $a_-(x,t) \le a_+(x,t)$ for all $x \in \mathbb{R}^4$ and $t < t_0$.

4.3. ***The approximating sequence of solutions u_n with $n \ge n_0$.*** For a fixed small $\epsilon > 0$ (independent of n) we choose functions Ψ, ψ_n with

$$\psi_n(x) = \Psi\Big(\frac{x}{\epsilon\sqrt{s_n}}\Big), \qquad \Psi \in C^\infty(\mathbb{R}), \qquad \Psi(\xi) = \begin{cases} 1, & 0 \le \xi \le 1, \\ 0, & \xi \ge 2. \end{cases}$$

We define

$$u_{0n}(x) := \psi_n(x) s_n^{k/3} W_{K_2}(x s_n^{-k/3}) + (1 - \psi_n(x)) U^-_{\delta_n}(x, s_n) \tag{4.3.1}$$

and let $u_n : (0,\infty) \times [s_n, \bar{t}_n) \to (0,\infty)$ be the solution to the initial value problem (1.1.5a)–(1.1.5c) with initial data $u_n(\,\cdot\,, s_n) = u_{0n}(x)$ instead of $u_0(x)$.

We will only consider the initial data for sufficiently large n; i.e., we choose an $n_0 \in \mathbb{N}$ and only consider those solutions u_n with $n \ge n_0$. Throughout this section "for all n" will mean "for all $n \ge n_0$," and in each lemma we assume that n_0 has been chosen large enough for the statement to hold.

In Corollary 4.8.2 we verify that our chosen initial data are caught between the barriers, as in (4.1.1). Before doing that we establish some derivative bounds for $u_{0n}(x)$.

Lemma 4.3.1 (monotonicity and derivative bounds). *For large enough n_0 and any $n \ge n_0$ there is an $s_n \in (0, t_n)$ such that the sequence $\{s_n : n \ge n_0\}$ is decreasing and such that $u_n(x, s_n)$ satisfies the following estimates for all n:*

(i) *The function $x \mapsto u_n(x, s_n)$ is locally Lipschitz and*

$$0 \le (u_n)_x(x, s_n) \le C_1 \tag{4.3.2}$$

for almost all $x > 0$, for some $C_1 > 0$

(ii) *The function $x \mapsto u_n(x, s_n)$ is C^3 on the interval $0 \le x \le M s_n^{1/2}$, where for $j = 2, 3$, and all n, one has*

$$(1 + s_n^{-k/3} x)^{j+2} |\partial_x^j u_n(x, s_n)| \le C s_n^{-(j-1)k/3}. \tag{4.3.3}$$

We present the proof in the following Sections 4.4–4.7. Along the way we finally choose the initial times $s_n \searrow 0$, and we use generic constants C that only depend on the various parameters defining the barriers, and the fixed small parameter ϵ, but not on n.

4.4. ***Proof of the first derivative bound (4.3.2).*** We have

$$(u_n)_x(x, s_n) = \psi_n' s_n^{k/3} W_{K_2}(x s_n^{k/3}) - \psi_n' U^-_{\delta_n} + \psi_n W'_{K_2}(x s_n^{-k/3}) + (1 - \psi_n)(U^-_{\delta_n})'. \tag{4.4.1}$$

We estimate these terms one by one.

The terms in (4.4.1) involving ψ_n' vanish outside the interval $\epsilon s_n^{1/2} \le x \le 2\epsilon s_n^{1/2}$. Thus we have

$$|\psi_n'(x) s_n^{k/3} W_{K_2}(x s_n^{-k/3})| \le \max_{x\ge 0} |\psi_n'(x)| \cdot \max_{x \le 2\epsilon s_n^{1/2}} |s_n^{k/3} W_{K_2}(x s_n^{-k/3})| \le C s_n^{-1/2} \cdot C s_n^{1/2} \le C,$$

where we have estimated $W_{K_2}(z) \le C(1+z)$ for all $z \ge 0$.

To estimate the other term involving $\psi_n'(x)$ we recall that in the region $z \ge Z_{\delta_n},\ y \le \frac{1}{4} Y_{\delta_n}$ the definition (3.9.2) implies that U_δ^- is given by

$$U_\delta^-(x,t) = t^{1/2} v_\delta^-(x t^{-1/2}, \log t).$$

Hence, if we choose $s_n > 0$ so small that $\epsilon s_n^{-\gamma} > Z_{\delta_n} = \frac{4}{3}\delta_n^{-1/(p-2)}$ then in the region $\epsilon s_n^{1/2} \le x \le 2\epsilon s_n^{1/2}$ we have

$$U_{\delta_n}^-(x, s_n) = s_n^{1/2} v_{\delta_n}(x s_n^{-1/2}, \log s_n),$$

and thus also

$$|\psi_n'||U_{\delta_n}^-| \le C s_n^{-1/2} |s_n^{1/2} v_{\delta_n}(x s_n^{-1/2}, \log s_n)| \le C v_{\delta_n}(y, \log s_n),$$

where $y = x/\sqrt{s_n}$ lies in the interval $[\epsilon, 2\epsilon]$. This implies that $\psi_n'(x) U_{\delta_n}^-(x, s_n)$ is uniformly bounded.

To estimate the third term we recall that $0 \le W_{K_2}'(z) \le 1$, which implies

$$|\psi_n(x) W_{K_2}'(x s_n^{-k/3})| \le \psi_n(x) \le 1.$$

Finally, the term $(1-\psi_n)(U_{\delta_n}^-)'$ vanishes for $x \le \epsilon\sqrt{s_n}$. For $x \ge \epsilon\sqrt{s_n}$ we have

$$U_{\delta_n}^-(x, s_n) = \begin{cases} \sqrt{s_n}\, v_{\delta_n}^-\big(\frac{x}{\sqrt{s_n}}, \log s_n\big), & x \le \frac{1}{4} Y_{\delta_n}\sqrt{s_n}, \\ \max\big\{\sqrt{s_n}\, v_{\delta_n}^-\big(\frac{x}{\sqrt{s_n}}, \log s_n\big),\ u^-(x, s_n)\big\}, & \frac{1}{4} Y_{\delta_n}\sqrt{s_n} \le x \le Y_{\delta_n}\sqrt{s_n}, \\ u^-(x, s_n), & x \ge Y_{\delta_n}\sqrt{s_n}, \end{cases}$$

with $Y_{\delta_n} = 2\sqrt{(2k+1)!!\, M/\delta_n}$ as in Lemma 3.6.1.

It follows that $x \mapsto U_{\delta_n}^-(x, s_n)$ is a Lipschitz continuous function whose derivative is almost everywhere given by $(v_{\delta_n}^-)_y$ or $u_x^-(x, s_n)$. If $y = x/\sqrt{s_n} \in [\epsilon, Y_{\delta_n}]$ then

$$(v_{\delta_n})_y(y, \log s_n) = 1 + (K_1^- - \delta_n) s_n^{3\gamma} \varphi_k'(y) - B K_1^2 s_n^{6\gamma} g'(y) + p\frac{s_n^{(p+1)\gamma}}{y^{p+1}} \le C,$$

for a uniform constant C, independent of n and for $n \ge n_0$, sufficiently big.

On the other hand, $u^-(x, s_n) = u_0(x) - M s_n \min\{1, x^{2k-4}\}$. For $x \ge 1$ we have $u_x^-(x, s_n) = u_0'(x)$, which is uniformly bounded by the assumption (1.1.2), while for $x < 1$ we have $u_x^-(x, s_n) = u_0'(x) - (2k-4) M s_n x^{2k-5}$, which is also uniformly bounded because we assume $k \ge 4$.

Combining all these estimates together with (4.4.1) yields the uniform Lipschitz bound on u_n.

4.5. *Proof of the second derivative estimate (4.3.3).* We will show

$$|(u_n)_{xx}(x, s_n)| \le C s_n^{-k/3} (1 + x s_n^{-k/3})^{-4} \tag{4.5.1}$$

for all $x \in [0, M\sqrt{s_n}]$.

Writing $z = xs_n^{-k/3}$, we estimate the terms on the right-hand side of

$$(u_n)_{xx} = \psi_n'' s_n^{k/3} W_{K_2}(z) + 2\psi_n' W_{K_2}'(z) + \psi_n W_{K_2}''(z) s_n^{-k/3} + (1-\psi_n)(U_{\delta_n}^-)_{xx} - 2\psi_n'(U_{\delta_n}^-)_x - \psi_n'' U_{\delta_n}^-. \quad (4.5.2)$$

For $0 \le x \le \epsilon s_n^{1/2}$ we have

$$(u_n)_{xx}(x, s_n) = s_n^{-k/3} W_{K_2}''(z).$$

The asymptotic expansion (3.5.1) for W implies that for all $z \ge 0$

$$0 \le W_{K_2}''(z) \le C(1+z)^{-4}.$$

Hence (4.5.1) holds for $x \le \epsilon s_n^{1/2}$.

If $2\epsilon s_n^{1/2} \le x \le M s_n^{1/2}$, i.e., if $2\epsilon \le y \le M$, then $u_n(x, s_n) = s_n^{-1/2} v_{\delta_n}^-(y, \log s_n)$ and thus, using the definition (3.4.3) for $v_{\delta_n}^-$, we find for $2\epsilon \le y \le M$

$$(u_n)_{xx}(x, s_n) = s_n^{-1/2}(v_{\delta_n}^-)_{yy}(y, \log s_n) \le C s_n^{-1/2} \frac{s_n^{3\gamma}}{y^4} \le \frac{C s_n^{-k/3}}{(1 + x s_n^{-k/3})^4}.$$

Finally, if $\epsilon s_n^{1/2} \le x \le 2\epsilon s_n^{1/2}$, then similarly to the previous two cases we get

$$|\psi_n W_{K_2}''(z) s_n^{-k/3} + (1-\psi_n)(U_{\delta_n}^-)_{xx}| \le C s_n^{-k/3}(1 + x s_n^{-k/3})^{-4}.$$

To bound the remaining terms in (4.5.2) it is enough to estimate

$$2|\psi_n'||W_{K_2}'(z) - (U_{\delta_n})_x| + |\psi_n''||s_n^{k/3} W_{K_2}(z) - U_{\delta_n}^-|.$$

Both ψ_n' and ψ_n'' vanish unless $\epsilon s_n^{1/2} \le x \le 2\epsilon s_n^{1/2}$. In this region one has $x s_n^{-k/3} \ge 1$, and thus our desired upper bound satisfies

$$\frac{1}{C} s_n^{k-2} \le s_n^{-k/3}(1 + x s_n^{-k/3})^{-4} \le C s_n^{k-2}.$$

By the asymptotic expansion (3.5.1) of the Alencar solution W for large z, we have $W_{K_2}(z) = z + \mathcal{O}(z^{-2})$ and $W_{K_2}'(z) = 1 + \mathcal{O}(z^{-3})$. When $\epsilon s_n^{1/2} \le x \le 2\epsilon s_n^{1/2}$ this implies

$$\begin{aligned} s_n^{k/3} W_{K_2}(x s_n^{-k/3}) - x &= \mathcal{O}(s_n^k x^{-2}) = \mathcal{O}(s_n^{k-1}), \\ W_{K_2}'(x s_n^{-k/3}) - 1 &= \mathcal{O}(s_n^k x^{-3}) = \mathcal{O}(s_n^{k-3/2}). \end{aligned} \quad (4.5.3)$$

In the region $\epsilon s_n^{1/2} \le x \le 2\epsilon s_n^{1/2}$ we have, by definition, and by the asymptotic expansions of the terms f_0^-, f_1, f_2 in (3.4.5),

$$\begin{aligned} U_{\delta_n}^-(x, s_n) &= s_n^{1/2} v_{\delta_n}^-(y, \log s_n) \qquad \text{where } y = x s_n^{-1/2} \\ &= s_n^{1/2} y + s_n^{1/2} \mathcal{O}(s_n^{k-3/2} y^{-2}) i \\ &= x + \mathcal{O}(s_n^k x^{-2}). \end{aligned} \quad (4.5.4)$$

This expansion may be differentiated with respect to x, resulting in

$$|(U_{\delta_n}^-)_x - 1| \le C s_n^k x^{-3} \le C s_n^{k-3/2}. \quad (4.5.5)$$

The bounds $|\psi_n'| = \mathcal{O}(s_n^{-1/2})$ and $|\psi_n''| = \mathcal{O}(s_n^{-1})$ now lead to

$$|\psi_n''||s_n^{k/3} W_{K_2}(xs_n^{-k/3}) - U_{\delta_n}^-| \le C s_n^{-1} s_n^{k-1} = C s_n^{k-2} \le \frac{C s_n^{-k/3}}{(1 + xs_n^{-k/3})^4}$$

and also

$$|\psi_n'||W_{K_2}'(xs_n^{-k/3}) - (U_{\delta_n}^-)_x| \le C s_n^{-1/2} s_n^{k-3/2} \le \frac{\bar{C} s_n^{-k/3}}{(1 + xs_n^{-k/3})^4}.$$

This concludes the proof of the stated weighted C^2 estimate for u_n at time $t = s_n$.

4.6. ***Proof of the third-order derivative bound (4.3.3).*** We outline the arguments, which are similar to those for the second derivative estimate.

For $0 \le x \le \epsilon s_n^{1/2}$ the definition (4.3.1) of $u_{0n}(x) = u_n(x, s_n)$ directly implies

$$|(u_n)_{xxx}(x, s_n)| = |W_{K_2}'''(z)| s_n^{-2k/3}, \quad \text{where again } z = xs_n^{-k/3}.$$

Using the asymptotic expansion for $W(z)$ as $z \to \infty$ one then verifies the third derivative estimate for $x \le \epsilon s_n^{1/2}$.

If $2\epsilon s_n^{1/2} \le x \le M s_n^{1/2}$, i.e., if $2\epsilon \le y \le M$, then

$$(u_n)_{xxx}(x, s_n) = (U_{\delta_n}^-)_{xxx}(x, s_n) = s_n^{-1} (v_{\delta_n}^-)_{yyy}(y, \log s_n),$$

and the estimate follows from the explicit expression (3.4.3) for $v_{\delta_n}^-(y, \tau)$.

If $\epsilon s_n^{1/2} \le x \le 2\epsilon s_n^{1/2}$, then u_n is given by

$$u_n(x, s_n) = s_n^{k/3} W_{K_2}(z) + \psi_n(x)\{s_n^{k/3} W_{K_2}(z) - U_{\delta_n}^-(x, s_n)\}, \quad z = xs_n^{-k/3}.$$

The third derivative of the first term can be estimated exactly as in the region $x \le \epsilon s_n^{1/2}$. After differentiating the second term three times one ends up with terms of the form

$$\psi_n^{(3-\ell)}(x) \Big(\frac{\partial}{\partial x}\Big)^{\ell} \{s_n^{k/3} W_{K_2}(z) - U_{\delta_n}^-(x, s_n)\}, \quad 0 \le \ell \le 3.$$

Using the asymptotic descriptions we have for W and $U_{\delta_n}^-$, and taking care to cancel the leading terms in these descriptions when $\ell \in \{0, 1\}$, we get the third derivative bounds in (4.3.3). The estimates are similar to the first and second-order estimates.

4.7. ***Proof that $x \mapsto u_n(x, s_n)$ is nondecreasing.*** We consider four regions: the region $0 \le x \le \epsilon s_n^{1/2}$, the region $\epsilon s_n^{1/2} \le x \le 2\epsilon s_n^{1/2}$ where we glue the inner and intermediate barriers, the intermediate region $2\epsilon s_n^{1/2} \le x \le 1$, and finally the region $x \ge 1$.

In the region $0 < x \le \epsilon s_n^{1/2}$ we have $u_n(x, s_n) = s_n^{k/3} W_{K_2}(xs_n^{-k/3})$, which is an increasing function of x because W is increasing.

In the region $\epsilon s_n^{1/2} \le x \le 2\epsilon s_n^{1/2}$, we have

$$(u_n)_x(x, s_n) = \psi_n'(x)(s_n^{k/3} W_{K_2}(xs_n^{-k/3}) - U_{\delta_n}^-(x, s_n)) + \psi_n(x) W_{K_2}'(xs_n^{-k/3}) + (1 - \psi_n(x))(U_{\delta_n}^-)_x(x, s_n).$$

Using (4.5.3), (4.5.4), as well as $|\psi_n'(x)| \le C s_n^{-1/2}$, we estimate the first term above by

$$|\psi_n'(x)|\,|s_n^{k/3} W_{K_2}(x s_n^{-k/3}) - U_{\delta_n}^-(x, s_n)| \le C|\psi_n'(x)| s_n^{k-1} \le C s_n^{k-3/2}.$$

Furthermore, (4.5.3) and (4.5.5) imply

$$|W_{K_2}'(x s_n^{-k/3}) - 1| + |(U_{\delta_n}^-)_x(x, s_n) - 1| \le C s_n^{k-3/2}.$$

It follows that

$$|(u_n)_x(x, s_n) - 1| \le C s_n^{k-3/2}$$

throughout the region $\epsilon s_n^{1/2} \le x \le 2\epsilon s_n^{1/2}$. Since $s_n \to 0$ and $k \ge 4$, so $k - \frac{3}{2} > 0$, we see that for large enough n the function $x \mapsto u_n(x, s_n)$ is strictly increasing when $\epsilon s_n^{1/2} \le x \le 2\epsilon s_n^{1/2}$.

Next, in the region $2\epsilon\sqrt{s_n} \le x \le 1$ we have

$$u_n(x, s_n) = U_{\delta_n}^-(x, s_n) = \max\{s_n^{1/2} v_{\delta_n}^-(x s_n^{-1/2}, \log s_n),\ u^-(x, s_n)\}$$

if $x s_n^{-1/2} \le Y_{\delta_n}$, and $u_n(x, s_n) = u^-(x, s_n)$ otherwise. It is easy to see that $x \mapsto u^-(x, s_n)$ is an increasing function. Concerning $v_{\delta_n}^-(y, \log s_n)$ we recall definition (3.4.3), i.e.,

$$v_{\delta_n}^-(y, \log s_n) = y + (K_1 - \delta) s_n^{3\gamma} \varphi_k(y) - B K_1^2 s_n^{6\gamma} g(y) - s_n^{(p+1)\gamma} y^{-p}.$$

If we choose s_n small enough, then the last three terms will be uniformly small in C^1 on the fixed interval $2\epsilon \le y \le Y_{\delta_n}$ compared to the leading term y, so that $y \mapsto v_{\delta_n}(y, \log s_n)$ is also increasing on the interval $2\epsilon \le y \le Y_{\delta_n}$. It follows that $x \mapsto u_n(x, s_n)$ is increasing on $2\epsilon s_n^{1/2} \le x \le 1$.

The very last situation we must consider is where $x \ge 1$. In this case (1.1.2) implies

$$(U_{\delta_n}^-)_x(x, s_n) = u_0'(x) \ge 0.$$

Since we have covered all cases, the proof of monotonicity of $x \mapsto u_n(x, s_n)$ is complete.

4.8. *Proof of* (4.1.3). We turn to the proof that the initial data $u_n(x, s_n)$ is sandwiched between the two barriers $U_{\delta_n}^{\pm}$, as in (4.1.3).

Lemma 4.8.1. *If n_0 is large enough then, for each $n \ge n_0$, we can choose $s_n \in (0, t_n)$ small enough that*

$$U_{\delta_n}^-(x, s_n) \le s_n^{k/3} W_{K_2}(x s_n^{-k/3}) \le U_{\delta_n}^+(x, s_n) \tag{4.8.1}$$

for $0 \le x \le 2\epsilon s_n^{1/2}$.

Proof. In this proof we write $y = x s_n^{-1/2}$ and $z = x s_n^{-k/3}$.

In the region $0 \le y \le 2\epsilon$ the barriers $U_{\delta_n}^{\pm}$ as defined in (3.9.1), (3.9.2) are given by

$$U_{\delta_n}^+(x, s_n) = \min\{s_n^{1/2} v_{\delta_n}^+(y, \log s_n),\ s_n^{k/3} W_{K_2^+(n)}(z)\},$$
$$U_{\delta_n}^-(x, s_n) = \max\{s_n^{1/2} v_{\delta_n}^-(y, \log s_n),\ s_n^{k/3} W_{K_2^-(n)}(z) + D s_n^{k-1}\},$$

where $K_2^{\pm}(n) = (K_2^3 \pm 2\delta_n)^{1/3}$ (see Section 3.8).

In Lemma 3.7.1 we defined $Z_n := Z_{\delta_n} = \frac{4}{3}\delta_n^{-1/(p-2)}$ and showed that the functions whose max/min define $U^{\pm}_{\delta_n}$ cross in the interval $\frac{1}{2}Z_n \le z \le Z_n$. To prove (4.8.1) we therefore must show

$$s_n^{k/3} W_{K_2^-(n)}(z) + D s_n^{k-1} \le s_n^{k/3} W_{K_2}(z) \le s_n^{k/3} W_{K_2^+(n)}(z) \tag{4.8.2}$$

if $0 \le z \le Z_n$, and

$$s_n^{1/2} v^-_{\delta_n}(y, \log s_n) \le s_n^{k/3} W_{K_2}(z) \le s_n^{1/2} v^+_{\delta_n}(y, \log s_n) \tag{4.8.3}$$

if $z \ge \frac{1}{2}Z_n$ and $y \le 2\epsilon$.

Since $\kappa \mapsto W_\kappa(z) = \kappa W(z/\kappa)$ is strictly increasing (see (3.9.5)), from $K^+_{2,n} = (K_2^3 + 2\delta_n)^{1/3} > K_2$ it follows that $W_{K_2}(z) \le W_{K_2^+(n)}(z)$ holds for all $z \ge 0$. Thus the second inequality in (4.8.2) holds.

The first inequality in (4.8.2) is equivalent to

$$W_{K_2}(z) - W_{K_2^-(n)}(z) \ge D s_n^{2k/3-1} \quad \text{for all } z \le Z_n.$$

By integrating

$$\frac{\partial}{\partial \kappa}\frac{\partial}{\partial z} W_\kappa(z) = -\frac{z}{\kappa^2} W''\Big(\frac{z}{\kappa}\Big) < 0$$

from $\kappa = K_2^-(n)$ to K_2 we see that $W_{K_2}(z) - W_{K_2^-(n)}(z)$ is a decreasing function of z. We therefore must guarantee

$$W_{K_2}(Z_n) - W_{K_2^-(n)}(Z_n) \ge D s_n^{2k/3-1}.$$

This holds for each n provided we choose $s_n \in (0, t_n)$ small enough.

We now consider (4.8.3), which is equivalent to

$$s_n^{-\gamma} v^-_{\delta_n}(s_n^\gamma z, \log s_n) \le W_{K_2}(z) \le s_n^{-\gamma} v^+_{\delta_n}(s_n^\gamma z, \log s_n), \tag{4.8.4}$$

and we must establish these inequalities for $\frac{1}{2}Z_n \le z \le 2\epsilon s_n^{-\gamma}$. Both inequalities can be proved in the same way, and we focus on the one involving $v^-_{\delta_n}$.

Keeping in mind that $K_2 = K_1^3$, the asymptotics (3.5.1) for the Alencar function W imply that there is a constant C such that

$$z + K_1 z^{-2} - C z^{-3} \le W_{K_2}(z) \le z + K_1 z^{-2} + C z^{-3} \tag{4.8.5}$$

for $z \ge 1$. On the other hand, the definition (3.4.3) of v^-_δ implies

$$\begin{aligned} s_n^{-\gamma} v^-_{\delta_n}(s_n^\gamma z, \log s_n) &= z + (K_1 - \delta_n) s_n^{2\gamma} \varphi_k(s_n^\gamma z) - z^{-(p-1)} - B K_1^2 s_n^{5\gamma} g(s_n^{-\gamma} z) \\ &= z + K_1 s_n^{2\gamma} \varphi_k(s_n^\gamma z) - \{\delta_n s_n^{2\gamma} \varphi_k(s_n^\gamma z) + z^{-(p-1)}\} - B K_1^2 s_n^{5\gamma} g(s_n^{-\gamma} z). \end{aligned}$$

For $y \le 2\epsilon$ we have

$$|\varphi_k(y) - y^{-2}| \le C \quad \text{and} \quad |g(y)| \le C y^{-5}.$$

Hence

$$s_n^{-\gamma} v^-_{\delta_n}(s_n^\gamma z, \log s_n) \le z + K_1 z^{-2} - \{\delta_n z^{-2} + z^{-(p-1)}\} + C(s_n^{2\gamma} + z^{-5}), \tag{4.8.6}$$

where C is the same for all sufficiently large $n \in \mathbb{N}$ and for $1 \le z \le 2\epsilon s_n^{-\gamma}$.

If $z \geq 1$ then $z^{-5} \leq z^{-3}$, so (4.8.5) and (4.8.6) together lead to

$$W_{K_2}(z) - s_n^{-\gamma} v_{\delta_n}^-(s_n^\gamma z, \log s_n) \geq \delta_n z^{-2} - C s_n^{2\gamma} + z^{-(p-1)} - C z^{-3}. \tag{4.8.7}$$

Now choose s_n small enough that $s_n < (\delta_n Z_n / C)^{1/2\gamma}$. Then for all $z \geq Z_n$ one has

$$\delta_n z^{-2} - C s_n^{2\gamma} \geq \delta_n Z_n^{-2} - C s_n^{2\gamma} > 0.$$

If we also require n to be large enough that $Z_n > C^{1/(4-p)}$, then we have for all $z \geq Z_n$

$$z^{-(p-1)} - C z^{-3} \geq (z^{4-p} - C) z^{-3} \geq (Z_n^{4-p} - C) z^{-3} > 0.$$

Applying the last two inequalities to (4.8.7) we conclude that the first inequality in (4.8.4) holds. A slight modification of these arguments also proves the second inequality in (4.8.4). □

Corollary 4.8.2. *If for each $n \geq n_0$ we choose $s_n \in (0, t_n)$ as in Lemma 4.8.1, then (4.1.3) holds, i.e., $U_{\delta_n}^-(x, s_n) \leq u_n(x, s_n) \leq U_{\delta_n}^+(x, s_n)$ for all $x \geq 0$.*

Proof. If $x \geq 2\epsilon s_n^{1/2}$ then $u_n(x, s_n) = U_{\delta_n}^-(x, s_n)$ and there is nothing to prove.

If $0 \leq x \leq 2\epsilon s_n^{1/2}$, then $u_n(x, s_n)$ is a convex combination of $U_{\delta_n}^-(x, s_n)$ and $s_n^{k/3} W_{K_2}(s_n^{-k/3} x)$. We have just shown that this second function lies between the barriers so the convex combination u_n also lies between the barriers $U_{\delta_n}^{\pm}$. □

4.9. *Monotonicity and uniform C^1 bound for $u_n(x, t)$.* In the following lemma we show that the initial uniform C^1 bound $\|u_n(\,\cdot\,, s_n)\|_{C^1} \leq C$ persists for as long as each $u_n(x, t)$ exists, provided that n is sufficiently large.

Lemma 4.9.1. *If C_1 is the upper bound for $(u_n)_x(x, s_n)$ from Lemma 4.3.1 then for sufficiently large n we have $0 \leq (u_n)_x(x, t) \leq C_1$ for all $(x, t) \in [0, \infty) \times [s_n, \bar{t}_n)$.*

In order to prove this lemma we will apply the maximum principle to the evolution equation of $(u_n)_x$. For this we first need the following observation.

Lemma 4.9.2. *Let M be the same constant as in Lemma 3.3.1. There is an $\alpha > 0$ such that for all sufficiently large n one has $U_{\delta_n}^-(x, t) \geq x$ for all $x \in [0, \alpha]$ and all $t \in (0, t_n)$.*

Proof. In the part of the outer region where $M\sqrt{t} \leq x \leq 1$ we have $t \leq M^{-2} x^2$, so that

$$\begin{aligned} U_{\delta_n}^-(x, t) &= u_0(x) - M t x^{2(k-2)} \\ &= x + (K_1 + o(1)) x^{2(k-1)} - M t x^{2(k-2)} \quad (x \to 0) \\ &\geq x + (K_1 - M^{-1} + o(1)) x^{2(k-1)} \quad (x \to 0). \end{aligned}$$

If we choose $M > 2/K_1$, then there is an $\alpha > 0$ such that $K_1 - M^{-1} + o(1) > 0$ and hence such that $U_{\delta_n}^-(x, t) > x$ holds when $M\sqrt{t} \leq x \leq \alpha$.

In the intermediate region the lower barrier is given by $t^{1/2} v_{\delta_n}^-(t^{-1/2} x, \log t)$, where in the rescaled variables (y, τ) we have $v_{\delta_n}^-(y, \tau) = y + f_{\delta_n}^-(y, \tau)$. Lemma 3.4.4 tells us that $f_{\delta_n}^-(y, \tau) \geq 0$, so in the intermediate region we have $v_{\delta_n}^-(y, \tau) \geq y$ and hence $U_{\delta_n}^-(x, t) \geq x$.

Finally, in the inner region we have

$$U_{\delta_n}^-(x,t) = t^{k/3} w_n^-(t^{-k/3}x, \log t)$$

and, according to the definition in Lemma 3.5.3,

$$w_n^-(z,\tau) = W_{K_2^-}(z) + De^{2\gamma\tau} > W_{K_2^-}(z) > z$$

because $W_\kappa(z) > z$ for all $z \geq 0$. This implies $U_{\delta_n}^-(x,t) \geq x$ in the inner region as well. □

Proof of Lemma 4.9.1. If u_n is one of the approximating solutions of (1.1.5a), then by differentiating in x we find that $\eta := (u_n)_x$ satisfies

$$\eta_t = \mathcal{M}_n[\eta] - Q_n(x,t)\eta, \tag{4.9.1}$$

where

$$\mathcal{M}_n[\eta] := \frac{\eta_{xx}}{1+(u_n)_x^2} + \frac{3}{x}\eta_x \quad \text{and} \quad Q_n(x,t) := \frac{2(u_n)_{xx}^2}{(1+(u_n)_x^2)^2} - \frac{3}{u_n^2} + \frac{3}{x^2}.$$

Lemma 4.9.2 says that $u_n(x,t) \geq U_{\delta_n}^-(x,t) \geq x$, so $Q_n(x,t) \geq 0$.

If the domain of η were bounded we could directly apply the maximum principle and conclude that η is bounded by its initial values. Since the domain is not bounded, we consider $\Omega(x,t) := x^{-1} + \kappa e^t x^2$ in the domain $x > 0,\ 0 \leq t \leq 1$. (Without loss of generality we assume that $\bar{t}_n \leq 1$ for all n.) In this region Ω satisfies

$$\begin{aligned}
\Omega_t - \mathcal{M}_n[\Omega] + Q_n(x,t)\Omega &\geq \kappa e^t x^2 - \frac{2x^{-3}}{1+(u_n)_x^2} + 3x^{-3} - \frac{2\kappa e^t}{1+(u_n)_x^2} - 6\kappa e^t \\
&\geq \kappa e^t x^2 - 2x^{-3} + 3x^{-3} - 2\kappa e^t - 6\kappa e^t \\
&\geq \kappa e^t x^2 + x^{-3} - 8\kappa e^t \\
&\geq \kappa(x^2 - 8e) + x^{-3}.
\end{aligned}$$

If we choose $\kappa > 0$ sufficiently small, then the left-hand side is positive for all $x > 0$ and $t \in [0,1]$.

For any $\epsilon > 0$ we therefore have

$$\Big(\frac{\partial}{\partial t} - \mathcal{M}_n + Q_n\Big)(\eta + \epsilon\Omega) > 0 \quad \text{in } (0,\infty) \times [s_n, \bar{t}_n).$$

Furthermore $\eta + \epsilon\Omega \to \infty$ as $x \to \{0,\infty\}$, so the maximum principle implies that $\eta + \epsilon\Omega$ attains its minimum at the initial time $t = s_n$. Since $0 \leq u_{n,x}(x,s_n) \leq C_1$ (by Lemma 4.3.1), we find that $\eta(x,t) + \epsilon\Omega(x,t) \geq 0$ for all $\epsilon > 0$, which implies that $u_{n,x}(x,t) = \eta(x,t) \geq 0$ for all $x > 0$ and $t \in [s_n, \bar{t}_n)$.

By considering $\eta - \epsilon\Omega$ for arbitrary $\epsilon > 0$ we similarly conclude that η is bounded by its largest initial value, i.e., $(u_n)_x(x,t) = \eta(x,t) \leq C_1$ for all $x > 0$ and $t \in [s_n, \bar{t}_n)$. This finishes the proof of Lemma 4.9.1. □

Corollary 4.9.3. *Let $u_n(x,t)$ be a solution to the initial value problem (1.1.5a)–(1.1.5c) with initial data $u_n(x,s_n)$ as above, and let $n \geq n_0$, where n_0 is large enough that all previous results hold. Then the solution $u_n(x,t)$, which exists for all $t \in [s_n, \bar{t}_n)$, satisfies $U_{\delta_{n_0}}^-(x,t) \leq u_n(x,t) \leq U_{\delta_{n_0}}^+(x,t)$ and $0 \leq (u_n)_x \leq C_1$ for all $x \geq 0$ and all $t \in [s_n, \min\{\bar{t}_n, t_{n_0}\})$, where C_1 is as in Lemma 4.3.1.*

Proof. We have shown that $(u_n)_x$ is uniformly bounded, that $u_n \geq U^-_{\delta_n}$ has a positive lower bound, and that $u_n(x,t) - u_0(x)$ is uniformly bounded (because $U^\pm_{\delta_n} - u_0$ is bounded). The discussion in Section 4.2 and (4.1.4) then shows that the maximum principle can be applied to conclude that the solution u_n remains between the barriers $U^\pm_{\delta_{n_0}}$ for as long as both u_n and $U^\pm_{\delta_{n_0}}$ are defined. □

4.10. *Uniform lower bound for $\bar{t}_n$.* Each of the approximating solutions u_n exists at least until time $\bar{t}_n$. We now argue that if n_0 is large enough, then $\bar{t}_n > t_{n_0}$ for all $n \geq n_0$.

We have already verified for all $x \geq 0$ and $t \in [s_n, \min\{\bar{t}_n, t_{n_0}\}]$ that the solution $u_n(x,t)$ remains between the barriers $U^\pm_{\delta_{n_0}}(x,t)$ and that its derivative $(u_n)_x(x,t)$ is uniformly bounded. Standard estimates for quasilinear parabolic equations applied to (4.2.1) or (4.2.2) then imply that higher derivatives of u_n also are uniformly bounded. If we had $\bar{t}_n \leq t_{n_0}$, then $\lim_{t\nearrow\bar{t}_n} u(x,t)$ would exist, and we could extend the solution to a larger time interval. Therefore $\bar{t}_n$ would not be the maximal time of existence for the solution u_n after all.

4.11. *Proof of the main existence Theorem 4.1.1.* We have constructed the sequence of solutions u_n and have established a priori bounds for its derivatives, which imply that there is a subsequence u_{n_j} that converges locally uniformly to a function $u : [0,\infty) \times (0, t_{n_0}] \to \mathbb{R}$. The derivative bounds for the approximating solutions u_n imply that u_n, $u_{n,x}$, $u_{n,xx}$, and $u_{n,t}$ also converge locally uniformly, and that the limit u is a solution of (1.1.5a).

We now verify that u also satisfies the initial and boundary conditions (1.1.5b), (1.1.5c), as well as the asymptotic description (4.1.2) of the inner region.

4.11.1. *The initial condition.* Let n_0 be large enough that all previous results in this section hold. Then all solutions u_{n_j} are caught between the barriers $U^\pm_{n_0}$, so the limit also lies between $U^\pm_{n_0}$. In the outer region, defined by $x \geq M\sqrt{t}$, the lower (upper) barriers are defined in (3.3.1) to be the maximum (minimum) of $u^\pm(x,t) = u_0(x) \pm Mt\min\{1, x^{2k-4}\}$ and the barriers defined in the intermediate region. This implies that for $x \geq M\sqrt{t}$ we have

$$u_0(x,t) - Mt\max\{1, x^{2k-4}\} \leq u(x,t) \leq u_0(x,t) + Mt\max\{1, x^{2k-4}\}.$$

Therefore $\lim_{t\searrow 0} u(x,t) = u_0(x)$ uniformly for all $x > 0$.

4.11.2. *Boundary condition.* The solutions $u_n(x,t)$ all satisfy $u_{n,x}(0,t) = 0$. They converge in C^1 to $u(x,t)$, so we have $u_x(0,t) = 0$ for all $t \in (0, t_{n_0}]$.

4.11.3. *Asymptotics in the inner region.* To finish the proof of the theorem, we will show that

$$\lim_{\tau\to-\infty} w(z,\tau) = W_{K_2}(z),$$

uniformly on compact sets in z. This follows almost immediately from (4.1.1) and the definition of our barriers $\tilde{u}^\pm_n(x,t)$ in the inner region. Using the definitions $w^-_n(z,\tau) = W_{K_2^-(n)}(z) + De^{\gamma\tau}$ and $w^-_n(z,\tau) = W_{K_2^+(n)}(z)$ from Section 3.5, (4.1.1) implies $w^-_n(z,\tau) \leq w(z,\tau) \leq w^+_n(z,\tau)$, and hence

$$W_{K_2^-(n)}(z) + De^{\gamma\tau_n} \leq w(z,\tau) \leq W_{K_2^+(n)}(z) \tag{4.11.1}$$

for all $z \in [0, Z_{\delta_n}]$, and $\tau \leq \tau_n := \log t_n$.

Since $Z_{\delta_n} := \frac{4}{3}\delta_n^{-1/(p-2)} \to +\infty$ and $K_2^{\pm}(n) = (K_2^3 \pm 2\delta_n)^{1/3} \to K_2$ as $n \to +\infty$, (4.11.1) holds on $[0, Z] \times (-\infty, \tau_n)$ for any $Z > 0$, provided n is sufficiently large. The rescaled Alencar solution $W_K(z) = K\,W(z/K)$ depends continuously on K, so after taking the limit $n \to \infty$ in (4.11.1) we conclude that $\lim_{\tau\to 0} w(z, \tau) = W_{K_2}(z)$, uniformly on any bounded interval $0 \le z \le Z$, as claimed in Theorem 4.1.1.

5. Uniform L^∞ bound on the mean curvature

5.1. ***Bounding H.*** In Theorem 4.1.1 we showed the short-time existence of an $O(4) \times O(4)$ symmetric MCF solution $\mathcal{M}_t$, $0 < t \le t_0$, which is smooth for $t > 0$ and defined by a profile function

$$u : [0, +\infty) \times (0, t_0] \to \mathbb{R}$$

which satisfies the initial value problem (1.1.5a)–(1.1.5c) for the given initial data $u_0(x)$. In this section we will show that the mean curvature of $\mathcal{M}_t$ is uniformly bounded on $[0, +\infty) \times (0, t_0]$ despite the fact that the initial data u_0 is singular at the origin. The life time of the solution is $t_0 = t_{n_0}$ for some large enough n_0.

Theorem 5.1.1. *Let $\mathcal{M}_t$, $0 < t \le t_0$, be the $O(4) \times O(4)$ symmetric MCF solution constructed in Theorem 4.1.1. Then*

$$\sup_{0<t\le t_0} \sup_{\mathcal{M}_t} |H| < \infty. \tag{5.1.1}$$

To prove this theorem we will first show, using a direct argument, that $H(x, t)$ is uniformly bounded in the outer region $x \ge M\sqrt{t}$, $0 < t \le t_0$. Then, using an argument by contradiction that is strongly inspired by the approach of Stolarski [2023], we will show that $H(x, t)$ is uniformly bounded in the remaining region $x \le M\sqrt{t}$, $0 < t \le t_0$.

5.2. ***Bounding $H(x, t)$ in the outer region.*** Assuming that t_0 is sufficiently small we show in this section that (5.1.1) holds in the part of the outer region where $x \gg \sqrt{t}$ and $0 < t \le t_0$.

Lemma 5.2.1. *Let Y_{δ_0} be as in (3.6.4). There exist $t_0 > 0$ and a uniform constant $C > 0$ so that for all (x, t) with $t \in (0, t_0)$ and $x \ge Y_{\delta_0}\sqrt{t}$ one has*

$$|H(x, t)| \le C. \tag{5.2.1}$$

Proof. We fix (x_1, t_1) with $0 < t_1 < t_0$ and $x_1 \ge Y_{\delta_0}\sqrt{t_1}$. We first deal with the case when $x_1 \in \left(0, \frac{1}{2}\right)$.

By (4.1.1) the solution u lies between our upper and lower barriers $U_{\delta_0}^{\pm}$ constructed in Proposition 3.9.1. In the region $x \ge \frac{1}{4}Y_{\delta_0}\sqrt{t}$ we have $U_{\delta_0}^{\pm}(x, t) = u^{\pm}(x, t)$, and hence

$$u^-(x, t) \le u(x, t) \le u^+(x, t), \quad \tfrac{1}{4}Y_{\delta_0}\sqrt{t} \le x \le 1, \tag{5.2.2}$$

and, by definition (3.3.1) of $u^{\pm}$,

$$|u(x, t) - u_0(x)| \le M t x^{2k-4}, \quad \tfrac{1}{4}Y_{\delta_0}\sqrt{t} \le x \le 1.$$

Rescaling the solution $u(x,t)$, we consider

$$U(\xi,s)=x_1^{-1}u(x_1\xi,t_1+x_1^2s)\quad \text{for } (\xi,s)\in \mathcal{Q}:=\left(\frac{1}{4},2\right)\times\left(-\frac{t_1}{x_1^2},0\right),$$

which satisfies

$$U_\tau=\frac{U_{\xi\xi}}{1+U_\xi^2}+\frac{3}{\xi}U_\xi-\frac{3}{U}. \tag{5.2.3}$$

If $(\xi,s)\in\mathcal{Q}$, then $x=x_1\xi$ and $t=t_1+x_1^2s$ satisfy $\frac{1}{4}Y_{\delta_0}\sqrt{t}\le x\le 1$, so that (5.2.2) applies, and so that

$$\begin{aligned}|U(\xi,s)-x_1^{-1}u_0(x_1\xi)|\le M(t_1+x_1^2s)x_1^{2k-5}\xi^{2k-4}&\le CMt_1x_1^{2k-5}\\ &\le\frac{CM}{Y_{\delta_0}^2}x_1^{2k-3}\qquad \text{since } x_1\ge Y_{\delta_0}\sqrt{t_1}.\end{aligned}$$

The initial profile u_0 satisfies $x\le u_0(x)\le x+Cx^{2k-2}$ for $0<x<2$, and thus

$$|x_1^{-1}u_0(x_1\xi)-\xi|\le Cx_1^{2k-3}\quad \text{for } \xi\in\left(\tfrac{1}{4},2\right).$$

The last two inequalities together imply that

$$|U(\xi,s)-\xi|\le Cx_1^{2k-3} \tag{5.2.4}$$

holds on $\mathcal{Q}$. Therefore the function

$$F(\xi,s):=\frac{U(\xi,s)-\xi}{x_1^{2k-3}},$$

which satisfies the equation

$$F_s=\frac{F_{\xi\xi}}{1+U_\xi^2}+\frac{3}{\xi}F_\xi+\frac{3}{\xi U(\xi,s)}F, \tag{5.2.5}$$

is bounded on $\mathcal{Q}$ by $|F(\xi,s)|\le C$ for some constant C that does not depend on (x_1,t_1).

Claim 5.2.2. *U and $1+U_\xi^2$ are Hölder continuous on*

$$\mathcal{Q}'=\left(\frac{1}{2},\frac{3}{2}\right)\times\left(-\frac{t_1}{x_1^2},0\right],$$

uniformly in (x_1,t_1).

Proof of Claim 5.2.2. By (5.2.4) we have $\|U\|_{C^0(\mathcal{Q})}\le C$ for a uniform constant C, independent of (x_1,t_1), where $x_1\in\left(0,\frac{1}{2}\right)$. Furthermore, in $\mathcal{Q}$ we also have

$$|U_\xi(\xi,s)|=|u_x(x_1\xi,t_1+x_1^2s)|\le C, \tag{5.2.6}$$

where C is a uniform constant, independent of (x_1,t_1). This follows by Lemma 4.9.1 and the fact that $u_n(x,t)$ smoothly converges as $n\to\infty$ to $u(x,t)$ for all $x>0$ and $t\in(0,t_1]$. Since $U(\xi,s)$ satisfies the uniformly parabolic equation (5.2.3), standard regularity theory applied to (5.2.3) implies that there exists a uniform constant C, independent of (x_1,t_1), so that $|U_{\xi\xi}(\xi,s)|\le C$ in $\mathcal{Q}'$. This implies U and $1+U_\xi^2$ are uniformly Hölder continuous functions on $\mathcal{Q}'$ as claimed. □

Interior parabolic regularity for (5.2.5) then implies that F, F_ξ, and $F_{\xi\xi}$ are uniformly bounded (and even Hölder continuous) on $\mathcal{Q}'$. We conclude that for some constant C that does not depend on (x_1, t_1) we have

$$|F_s(1,0)| \le C.$$

In terms of the original solution $u(x,t)$ this then implies

$$|u_t(x_1,t_1)| \le C x_1^{2k-4} \le C,$$

where we have used $k \ge 4$ and $x_1 \le \frac{1}{2}$ in the last step. We conclude that $|H(x_1,t_1)| \le |u_t(x_1,t_1)|$ is uniformly bounded whenever $Y_{\delta_0}\sqrt{t_1} \le x_1 \le \frac{1}{2}$ and $0 < t_1 < t_0$.

To deal with the case where $x_1 \ge \frac{1}{2}$ we recall that the single variable PDE (1.1.5a) for $u_n(x,t)$ can be interpreted as a parabolic equation in more variables (4.2.1) or (4.2.2). As discussed in Section 4.2, the uniform lower bound $u_n(x,t) \ge U^-_{\delta_0}(x,t)$ combined with the estimate $0 \le (u_n)_x(x,t) \le C_1$ allows one to invoke classical parabolic estimates which imply that $(u_n)_{xx}(x,t)$ is uniformly bounded for all $x \ge \frac{1}{2}$, sufficiently large n, and all $t \in (s_n, \bar{t}_n)$. This then implies that the mean curvature on the limiting solution is bounded when $x_1 \ge \frac{1}{2}$.

Combining the two cases $x_1 \in \left(0, \frac{1}{2}\right)$ and $x_1 \ge \frac{1}{2}$ leads to (5.2.1), which finishes the proof of the proposition. □

5.3. ***Second-order derivative bounds for $x \le M\sqrt{t}$.*** Before we bound $H(x,t)$ in the intermediate and inner regions, we will establish the following weighted C^2 bound, which is crucial for our purposes, for our approximating sequence of solutions $u_n(x,t)$ defined in Section 4.

Lemma 5.3.1. *There exists n_0 sufficiently large and a constant C independent of n so that for all $n \ge n_0$ the bound*

$$|(u_n)_{xx}(x,t)| \le C t^{-k/3}(1 + t^{-k/3}x)^{-4} \tag{5.3.1}$$

holds for all $0 \le x \le M\sqrt{t}$, $t \in [s_n, t_0]$.

Proof. The proof follows from scaling and standard regularity theory for linear and quasilinear parabolic equations. We repeatedly use the first-order derivative bound $0 \le (u_n)_x(x,t) \le C_1$ from Lemma 4.9.1, as well as the initial derivative bounds (4.3.3)

$$|\partial^j u_n(x, s_n)| \le C s_n^{-(j-1)k/3}(1 + s_n^{-k/3}x)^{-(j+2)}, \quad j = 2, 3,$$

which were shown in Lemma 4.3.1.

Since our solutions $u_n(x,t)$ scale differently in the intermediate and inner regions, we need to treat the cases $x \in [2Rt^{k/3}, Mt^{1/2}]$ and $x \in [0, 2Rt^{k/3}]$ separately. We will choose R in the proof of Case 1 below to be a sufficiently large constant which is independent of n. Then for this choice of R we will show that Case 2 holds. In both cases we will assume that $n \ge n_0$ and $s_n \le t \le t_0$, and n_0 will be chosen to be sufficiently large and t_0 will be chosen to be sufficiently small, uniformly in n.

We start by fixing $n \ge n_0$ and a point (x_1, t_1), where $0 \le x_1 \le M\sqrt{t_1}$, $t_1 \in [s_n, t_0]$.

Case 1: Assume $x_1 \in [2Rt_1^{k/3}, Mt_1^{1/2}]$, where R is a sufficiently large constant. Similarly to the proof of Lemma 5.2.1, we consider the rescaling

$$U_n(\xi, s) = x_1^{-1} u_n(x_1\xi, t_1 + x_1^2 s)$$

which satisfies the equation

$$(U_n)_s = \frac{(U_n)_{\xi\xi}}{1 + U_{n\xi}^2} + \frac{3}{\xi}(U_n)_\xi - \frac{3}{U_n} \tag{5.3.2}$$

in the region

$$\mathcal{Q}_n = \left\{(\xi, s) : \frac{1}{2} < \xi < \frac{3}{2}, -\frac{t_1 - s_n}{x_1^2} < s \leq \frac{t_0 - t_1}{x_1^2}\right\}.$$

We subdivide into the *two cases*

$$\frac{t_1 - s_n}{x_1^2} > \frac{1}{2M^2} \quad \text{and} \quad \frac{t_1 - s_n}{x_1^2} \leq \frac{1}{2M^2}.$$

Case 1a: If $(t_1 - s_n)/x_1^2 > 1/(2M^2)$, then the parabolic square

$$\mathcal{Q}'_M = \left\{(\xi, s) : \frac{1}{2} < \xi < \frac{3}{2}, -\frac{1}{2M^2} < s \leq 0\right\}$$

has fixed size (independent of (x_1, t_1) and n) and satisfies $\mathcal{Q}'_M \subset \mathcal{Q}_n$. We will restrict to $\mathcal{Q}'_M$.

For any $(\xi, s) \in \mathcal{Q}'_M$ we have $x := x_1\xi \in [Rt_1^{k/3}, 2Mt_1^{1/2}]$ and $t := t_1 + x_1^2 s \in \left[\frac{1}{2}t_1, t_1\right]$. In particular, we have $y := xt^{-1/2} \in [Rt_1^\gamma, 2\sqrt{2}M]$, i.e., (x, t) lies in the intermediate region, a fact that will be used momentarily.

To obtain the desired bound on $u_{xx}(x_1, t_1)$, we will bound $U_{\xi\xi}(1, 0)$ by applying interior parabolic regularity estimates to the function $U_n(\xi, s) - \xi$ defined in $\mathcal{Q}'_M$.

We first estimate the L^∞ norm of this function on $\mathcal{Q}'_M$ by bounding $|u_n(x, t) - x|$ for $x = x_1\xi$ and $t = t_1 + x_1^2 s$, where $(\xi, s) \in \mathcal{Q}'_M$.

By (4.1.1) the solution u_n lies between our upper and lower barriers constructed in Proposition 3.9.1. Hence

$$|u_n(x, t) - x| \leq \max\{|U_{\delta_0}^+(x, t) - x|, |U_{\delta_0}^-(x, t) - x|\} \tag{5.3.3}$$

for all $n \geq n_0$ sufficiently large. Using the definition of our barriers $U_{\delta_0}^\pm(x, t)$ (see (3.9.1) and (3.9.2)) the difference $|U_{\delta_0}^\pm(x, t) - x|$ for $n \geq n_0$ is bounded by $t^{1/2}|f_{\delta_{n_0}}^\pm(xt^{-1/2}, t)|$; $f_{\delta_{n_0}}^\pm$ was defined in (3.4.4). The latter can be bounded by $2K_1 t^{k-1}\varphi_k(xt^{-1/2})$, provided that t_0 is sufficiently small. This follows from the definition of $f_{\delta_{n_0}}^\pm$ and our estimates in Section 3.4, after expressing these estimates in the (x, t) variables using (2.2.1). Since $\varphi_k(y) \leq C_k(y^{2k-2} + y^{-2})$ with $y := xt^{-1/2} \in [Rt_1^\gamma, 2\sqrt{2}M]$ and $t \in \left[\frac{1}{2}t_1, t_1\right]$, we get

$$\max\{|U_{\delta_0}^+(x, t) - x|, |U_{\delta_0}^-(x, t) - x|\} \leq Ct^{k-1}(xt^{-1/2})^{-2} \leq Cx_1^{-2}t^k \tag{5.3.4}$$

for some constant C (depending only on k and M) which is uniform in (x_1, t_1) and n. Combining (5.3.3) and (5.3.4) while using $t = t_1 + x_1^2 s \leq t_1$ yields

$$|U_n(\xi, s) - \xi| \leq Cx_1^{-3}t_1^k \quad \text{in } \mathcal{Q}'_M. \tag{5.3.5}$$

It follows that the function

$$F_n(\xi,s) := x_1^3 t_1^{-k}(U_n(\xi,s)-\xi)$$

which satisfies the equation

$$(F_n)_s = \frac{(F_n)_{\xi\xi}}{1+U_{n\xi}^2} + \frac{3}{\xi}(F_n)_\xi + \frac{3}{\xi U_n(\xi,s)} F_n \tag{5.3.6}$$

is uniformly bounded in the parabolic cube $\mathcal{Q}'_M$, namely $\|F_n\|_{C^0(\mathcal{Q}'_M)} \le C$, where the constant C is independent of (x_1,t_1) and n.

Claim 5.3.2. *U_n and $1+U_{n\xi}^2$ are Hölder continuous on the parabolic cube*

$$\mathcal{Q}''_M = \left\{(\xi,s) : \frac{1}{4} < \xi < \frac{5}{4},\ -\frac{1}{4M^2} < s \le 0\right\} \subset \mathcal{Q}'_M,$$

uniformly in (x_1,t_1) and n. Furthermore, $\frac{1}{4} \le U_n(\xi,s) \le 2$ for all $(\xi,s) \in \mathcal{Q}'_M$.

Proof. Since $x_1 \ge R t_1^{k/3}$, by (5.3.5) we have that $|U_n(\xi,s)-\xi| \le C R^{-3}$, and since the constant C doesn't depend on R, we may choose R large enough that $\frac{1}{4} \le U_n(\xi,s) \le 2$ for all $(\xi,s) \in \mathcal{Q}'_M$. In addition Lemma 4.9.1 implies that $|U_{n\xi}(\xi,s)| = |(u_n)_x(x_1\xi, t_1 + x_1^2 s)| \le C_1$ in $\mathcal{Q}'_M$. It follows that $U_n(\xi,s)$ satisfies in $\mathcal{Q}'_M$ a uniformly parabolic equation (5.3.2) with bounded coefficients, and therefore standard interior (in spacetime) regularity theory applied to the quasilinear equation (5.3.2) implies the existence of a uniform constant C, independent of (x_1,t_1) and n, so that $|U_{n\xi\xi}(\xi,s)| \le C$ in $\mathcal{Q}''_M \subset \mathcal{Q}'_M$. All the above give us that U_n and $1+U_{n\xi}^2$ are uniformly Hölder continuous functions on $\mathcal{Q}''_M$ as claimed. □

Claim 5.3.2 implies that equation (5.3.6) is uniformly parabolic in $\mathcal{Q}''_M$ and its coefficients are Hölder continuous (uniformly in (x_1,t_1) and n). Interior (in spacetime) Schauder theory applied to (5.3.6) in $\mathcal{Q}''_M$ bounds $|(F_n)_{\xi\xi}(1,0)|$ in terms of $\|F_n\|_{C^0(\mathcal{Q}''_M)}$, concluding that $|(F_n)_{\xi\xi}(1,0)| \le C$ for a uniform constant C. Equivalently, $|(U_n)_{\xi\xi}(1,0)| \le C x_1^{-3} t_1^k$, and converting back to the original solution gives the bound $|(u_n)_{xx}(x_1,t_1)| \le C x_1^{-4} t_1^k$. In the considered region we have $x_1 t^{-k/3} \ge R$, therefore $t_1^k x_1^{-4} = t_1^{-k/3}(t_1^{-k/3}x_1)^{-4} \le C t_1^{-k/3}(1 + x_1 t_1^{-k/3})^{-4}$ (where C depends on R). We conclude that the desired bound (5.3.1) holds when $x_1 \in [2Rt_1^{k/3}, Mt_1^{1/2}]$ and $(t_1 - s_n)/x_1^2 > 1/(2M^2)$.

Case 1b: If $(t_1-s_n)/x_1^2 \le 1/(2M^2)$, then $x_1 \le M t_1^{1/2}$ implies that $t_1 - s_n \le x_1^2/(2M^2) \le \frac{1}{2}t_1$, and hence in this case $t_1 \in [s_n, 2s_n]$. This in turn gives $x_1 \le M\sqrt{2s_n}$, implying in particular that

$$\frac{t_0 - t_1}{x_1^2} \ge \frac{t_0 - 2s_n}{2M^2 s_n} \ge 1,$$

provided that $n \ge n_0$ with n_0 sufficiently large. Hence the cube

$$\mathcal{Q}'_n = \left\{(\xi,s) : \frac{1}{2} < \xi < \frac{3}{2},\ -\frac{t_1 - s_n}{x_1^2} < s \le -\frac{t_1 - s_n}{x_1^2} + 1\right\}$$

has fixed size and satisfies $\mathcal{Q}'_n \subset \mathcal{Q}_n$. The difference between this and the previous case is that the cube $\mathcal{Q}'_n$ starts at $s = -(t_1 - s_n)/x_1^2$ corresponding to the initial time $t = s_n$ for the solution $u_n(x,t)$. This means that our estimates need to include bounds on the initial data $u_n(x,s_n)$.

As in the previous case, we will begin by bounding $|U_n(\xi,s)-\xi|$ in $\mathcal{Q}'_n$. For any $(\xi,s)\in\mathcal{Q}'_n$ we have $x:=x_1\xi\in[Rt_1^{k/3},2M\sqrt{t_1}]\subset[Rs_n^{k/3},2M\sqrt{2s_n}]$ (using $t_1\in[s_n,2s_n]$) and $t:=t_1+x_1^2 s\in[s_n,(2M^2+2)s_n]$ (using $x_1\le Mt_1^{1/2}$). Hence $y:=xt^{-1/2}\in[R/(\sqrt{2}M)s_n^{\gamma},2\sqrt{2}M]$, which shows that the point (x,t) belongs to the intermediate region. Now similar arguments as in Case 1a imply that bounds (5.3.3) and (5.3.4) hold (with s_n instead of t_1). We conclude that $|u_n(x,t)-x|\le Cx_1^{-2}s_n^{3\gamma+3/2}$ holds at $x=x_1\xi,\ t:=t_1+s\xi_1^2$ for any $(\xi,s)\in\mathcal{Q}'_n$, where C is independent of (x_1,t_1) and n. In terms of $U_n(\xi,s)$ we obtain

$$|U_n(\xi,s)-\xi|\le Cx_1^{-3}s_n^{3\gamma+3/2}\le Cx_1^{-3}t_1^{k}\quad\text{in } \mathcal{Q}'_n. \tag{5.3.7}$$

Claim 5.3.3. *U_n and $1+U_{n\xi}^2$ are Hölder continuous on the parabolic cube*

$$\mathcal{Q}''_n:=\left\{(\xi,s):\frac{3}{4}<\xi<\frac{5}{4},\ -\frac{t_1-s_n}{x_1^2}<s\le-\frac{t_1-s_n}{x_1^2}+1\right\}\subset\mathcal{Q}'_n,$$

uniformly in (x_1,t_1) and n. Furthermore, $\frac{1}{4}\le U_n(\xi,s)\le 2$ for all $(\xi,s)\in\mathcal{Q}'_n$.

Proof. Similarly to Claim 5.3.2, the bounds (5.3.7) and Lemma 4.9.1 imply that on $\mathcal{Q}'_n$ we have $\frac{1}{4}\le U_n\le 2$ and $0\le U_{n\xi}\le C_1$. In addition, for $j=2,3$ we have

$$\sup_{1/2\le\xi\le3/2}\left|\partial_\xi^j U_n\left(\xi,-\frac{t_1-s_n}{x_1^2}\right)\right|\le x_1^{j-1}\sup_{x_1/2\le x\le3x_1/2}|\partial_x^j u_n(x,s_n)|\le Cx_1^{-3}s_n^k\le C, \tag{5.3.8}$$

where we used (4.3.3) and our assumption $x_1\ge 2Rt_1^{k/3}$ combined with $t_1\in[s_n,2s_n]$. In all the above bounds C is a uniform constant, independent of (x_1,t_1) and n. Since $U_n(\xi,s)$ satisfies a uniformly parabolic equation (5.3.2) in $\mathcal{Q}'_n$, standard interior (in space) theory for quasilinear equations applied to (5.3.2) yields the C^2 bound $\|U_{n\xi\xi}\|_{C^2(\mathcal{Q}''_n)}\le C$ (and even a $C^{2,1}$ bound), where C is a constant that depends only on

$$\|U_n\|_{C^0(\mathcal{Q}'_n)}\quad\text{and}\quad\left\|U_n\left(\cdot,-\frac{t_1-s_n}{x_1^2}\right)\right\|_{C^3([\xi/2,3\xi/2])},$$

therefore C is uniform in (x_1,t_1) and n, since these bounds are as well. We conclude that U_n and $1+U_{n\xi}^2$ are uniformly Hölder continuous functions on $\mathcal{Q}''_n$, finishing the proof of the claim. □

Consider the function $F_n(\xi,s):=x_1^3t_1^{-k}(U_n(\xi,s)-\xi)$ on $\mathcal{Q}''_n$ which satisfies (5.3.6) and the uniform bound $\|F_n\|_{C^0(\mathcal{Q}''_n)}\le C$, where C is independent of (x_1,t_1) and n. Claim 5.3.3 implies that $F_n(\xi,s)$ satisfies a uniformly parabolic equation (5.3.6) on $\mathcal{Q}''_n$ with coefficients which are uniformly Hölder continuous. Therefore, standard interior (in space) Schauder estimates applied to (5.3.2) on the cube $\mathcal{Q}''_n$ imply that $|(F_n)_{\xi\xi}(1,0)|$ can be bounded in terms of

$$\|F_n\|_{C^0(\mathcal{Q}''_n)}\quad\text{and}\quad\left\|F_n\left(\cdot,-\frac{t_1-s_n}{x_1^2}\right)\right\|_{C^{2,1}([3/4,5/4])}.$$

We have just seen that $\|F_n\|_{C^0(\mathcal{Q}''_n)}\le C$. We will next show the bound

$$\left\|F_n\left(\cdot,-\frac{t_1-s_n}{x_1^2}\right)\right\|_{C^3([3/4,5/4])}\le C.$$

First, (5.3.8) and the definition of F_n give

$$\left|\partial_\xi^j F_n\left(\xi, -\frac{t_1-s_n}{x_1^2}\right)\right| = x_1^3 t_1^{-k} |\partial_\xi^j U_n(\xi, s)| \le C t_1^{-k} s_n^k \le C$$

for $j = 2, 3$ and all $\xi \in \left[\frac{3}{4}, \frac{5}{4}\right]$. The bound for $j = 1$ follows similarly from $0 \le (u_n)_x(x, s_n) \le C$. In all the above bounds C is independent of (x_1, t_1) and n.

We conclude that $|(F_n)_{\xi\xi}(1, 0)| \le C$, where C is independent of (x_1, t_1) and n, and, similarly to Case 1a, the desired bound (5.3.1) holds for $x_1 \in [2Rt_1^{k/3}, Mt_1^{1/2}]$ and $(t_1 - s_n)/x_1^2 \le 1/(2M^2)$. This completes the argument in Case 1b.

Case 2: Suppose next that $x_1 \in [0, Rt_1^{k/3}]$, that is (x_1, t_1) belongs to the tip region. Here R is a large fixed constant, chosen as in Case 1. In this case we will not scale around x_1 but around the origin, and we will show

$$\sup_{x \in [0, Rt_1^{k/3}]} |(u_n)_{xx}(x, t_1)| \le C t_1^{-k/3}, \quad 0 < t_1 \le t_0, \tag{5.3.9}$$

for a uniform constant C independent of n and t_1 (C may depend on R). This estimate is equivalent to (5.3.1) because in the considered region one has $x_1 t_1^{-k/3} \le R$.

To this end we set $\alpha := \frac{1}{3}k \ge 1$ for simplicity and introduce the rescaled function

$$U_n(\xi, s) = t_1^{-\alpha} u_n(t_1^\alpha \xi, t_1 + t_1^{2\alpha} s), \tag{5.3.10}$$

which satisfies (5.3.2) in the region

$$\mathcal{Q}_n = \left\{(\xi, s) : 0 \le \xi \le 2R, \ -\frac{t_1 - s_n}{t_1^{2\alpha}} < s \le \frac{t_0 - t_1}{t_1^{2\alpha}}\right\}.$$

Bound (5.3.9) is equivalent to

$$\sup_{\xi \in [0, R]} |(U_n)_{\xi\xi}(\xi, 0)| \le C \tag{5.3.11}$$

and will follow by applying standard regularity theory to (5.3.2) in an appropriate cube $\mathcal{Q}_n' \subset \mathcal{Q}_n$.

First, one needs to bound U_n on $\mathcal{Q}_n'$ from above and below away from zero. To this end, observe that (4.1.5), (3.9.1)–(3.9.2) and the definition of the inner region barriers in Section 3.5 give

$$t^\alpha W_{K_2^-(n_0)}(xt^{-\alpha}) + De^{2\gamma \log t} \le u_n(x, t) \le t^\alpha W_{K_2^-(n_0)}(xt^{-\alpha}) \tag{5.3.12}$$

for all $n \ge n_0$ sufficiently large and all $x \in [0, Zt^\alpha]$ (for any $Z > 0$) and $t \le t_0$. Here $D > 0$, and thus we can drop the small term $De^{2\gamma \log t}$. The above estimate when expressed in terms of $U_n(\xi, s)$ gives

$$\vartheta_n(s) W_{K_2^-(n_0)}\left(\frac{\xi}{\vartheta_n(s)}\right) \le U_n(\xi, s) \le \vartheta(s) W_{K_2^+(n_0)}\left(\frac{\xi}{\vartheta_n(s)}\right), \tag{5.3.13}$$

where $\vartheta_n(s) := t^\alpha t_1^{-\alpha} = (1 + t_1^{2\alpha-1}s)^\alpha$. Note that in order to obtain (5.3.13) from (5.3.12) we need to have $\xi/\vartheta_n(s) \le Z$ for all $(\xi, s) \in \mathcal{Q}_n'$ and for some $Z > 0$ which is independent of $(\xi, s) \in \mathcal{Q}_n'$. This will be checked below. We need to consider two cases, $(t_1 - s_n)t_1^{-2\alpha} > 1$ and $(t_1 - s_n)t_1^{-2\alpha} \le 1$, and choose $\mathcal{Q}_n'$ appropriately.

Case 2a: If $(t_1 - s_n)t_1^{-2\alpha} > 1$, then we restrict to the parabolic cube of fixed size

$$\mathcal{Q}' = \{(\xi, s) : 0 \le \xi \le 2R,\ -1 < s \le 0\}$$

(independent of t_1 and n), which obviously satisfies $\mathcal{Q}' \subset \mathcal{Q}_n$. We will restrict to $\mathcal{Q}'$, where $s \in (-1, 0]$ implies the bounds $\vartheta_n(s) \ge (1 - t_1^{2\alpha-1})^\alpha \ge \frac{1}{2}$ and $\vartheta_n(s) \le 1$ (for the former use $t_1 \le t_0$, where t_0 can be chosen sufficiently small).

Using $\xi\vartheta_n^{-1} \le 4R$ and $\frac{1}{2} \le \vartheta_n \le 1$, we conclude from (5.3.13) that there exists a uniform in n and t_1 constant $C > 0$ (depending on $\inf_{z\in[0,4R]} W_{K_2^-(n)}(z)$ and $\sup_{z\in[0,4R]} W_{K_2^+(n)}(z)$) such that

$$0 < C^{-1} \le U_n(\xi, s) \le C \quad \text{for all } (\xi, s) \in \mathcal{Q}'. \tag{5.3.14}$$

Furthermore, arguing as in (5.2.6), we note that the definition (5.3.10) of U_n implies

$$U_{n\xi}(\xi, s) = u_{nx}(t_1^\alpha, t_1 + t_1^{2\alpha} s),$$

so that $\|U_{n\xi}\|_{C^0(\mathcal{Q}')} \le \sup |u_{nx}| \le C$, where C does not depend on n or t_1. Standard interior (in spacetime) regularity theory applied to (5.3.2) implies that there exists a uniform constant C, independent of n and t_1, so that $\sup_{\xi\in[0,R]} |(U_n)_{\xi\xi}(\xi, 0)| \le C$; that is (5.3.11) holds. In terms of the original solution $u_n(x, t)$ this implies the desired bound (5.3.9) in the case $(t_1 - s_n)t_1^{-2\alpha} > 1$, with $\alpha = \frac{1}{3}k$.

Case 2b: Finally, if $(t_1 - s_n)t_1^{-2\alpha} \le 1$, then since $t_1 \le t_0$ is small and $\alpha \ge 1$, we have $t_1 \le s_n + t_1^{2\alpha} \le s_n + \frac{1}{2}t_1$; that is $t_1 \in [s_n, 2s_n]$. In this case we restrict to the parabolic cube of fixed size

$$\mathcal{Q}_n' = \left\{(\xi, s) : 0 \le \xi \le 2R,\ -\frac{t_1 - s_n}{t_1^{2\alpha}} < s \le -\frac{t_1 - s_n}{t_1^{2\alpha}} + 1\right\},$$

which contains the point $(1, 0)$ and satisfies $\mathcal{Q}_n' \subset \mathcal{Q}_n$. Since $0 < (t_1 - s_n)/t_1^{2\alpha} \le 1$, for any $(\xi, s) \in \mathcal{Q}_n'$ we have $s \in [-1, 1]$; thus $\vartheta_n := (1 + t_1^{2\alpha-1}s)^\alpha$ satisfies the bounds $\frac{1}{2} \le \vartheta_n(s) \le \frac{3}{2}$ for all $t_1 \le t_0$ with t_0 sufficiently small.

Claim 5.3.4. *The bounds* $0 < C^{-1} \le U_n(\xi, s) \le C$ *and* $|(U_n)_\xi(\xi, s)| \le C$ *hold on* $\mathcal{Q}_n'$. *Furthermore,*

$$\left\| U_n\left(\,\cdot\,, -\frac{t_1 - s_n}{t_1^{2\alpha}}\right)\right\|_{C^3([0,2R])} \le C.$$

In all these bounds C *is a uniform constant independent of* n *and* t_1.

Proof. Since $\frac{1}{2} \le \vartheta_n(s) \le \frac{3}{2}$, similarly to Case 2a we can apply (5.3.13) to obtain $0 < C^{-1} \le U_n(\xi, s) \le C$ in $\mathcal{Q}_n'$. Also, similarly to the previous cases, $0 \le (U_n)_\xi(\xi, s) \le C_1$ in $\mathcal{Q}_n'$ follows from Lemma 4.9.1. For the third bound it is sufficient to just estimate second- and third-order derivatives. To this end we use (4.3.3) which implies that $|\partial_x^j u_n(x, s_n)| \le C s_n^{-(j-1)k/3}$ for $j = 2, 3$ and for all $x \in [0, 2Rt_1^{k/3}]$ (recall that $t_1 \sim s_n$).

In terms of U_n we get $|\partial_\xi^j U_n(\xi, -(t_1 - s_n)/t_1^{2\alpha})| \le C$ for $j = 2, 3$ and for all $\xi \in [0, 2R]$. The above bounds imply that $\|U_n(\,\cdot\,, -(t_1 - s_n)/t_1^{2\alpha})\|_{C^3([0,2R])} \le C$. In all these bounds the constant C is uniform, independent of n and t_1. □

The previous claim and standard interior (in space) regularity theory applied to (5.3.2) on the cube $\mathcal{Q}'_n$ implies that $\sup_{0\le\xi\le R}|(U_n)_{\xi\xi}(\xi,0)|$ (even $\|U_n(\,\cdot\,,0)\|_{C^{2,1}([0,R])}$) can be bounded in terms of $\|U_n\|_{C^0(\mathcal{Q}'_n)}$ and $\|U_n(\,\cdot\,,-(t_1-s_n)/t_1^{2\alpha})\|_{C^3([0,2R])}$, and thus both are bounded by a constant C which is uniform in t_1 and n. We conclude that (5.3.11) holds, which expressed in terms of $u_n(x,t)$ gives that (5.3.9) holds in the last case where $(t_1-s_n)t_1^{-2\alpha}>1$, with $\alpha=\frac{k}{3}$.

Combining Cases 1a–1b and Cases 2a–2b concludes the proof that the desired bound (5.3.1) holds for all (x,t) satisfying $0\le x\le M\sqrt{t},\ t\in[s_n,t_0]$ and all $n\ge n_0$, provided n_0 is sufficiently large and $t_0>0$ is sufficiently small. □

5.4. ***Bounding H in the intermediate and inner regions.*** We will now show that $H(x,t)$ is bounded in the region $x\le M\sqrt{t},\ 0<t\le t_0$. Instead of showing that H is bounded, we will prove that

$$h(x,t):=u_t=H\sqrt{1+u_x^2}$$

is bounded. Since u_x is uniformly bounded (Lemma 4.9.1), the bounds for h and H are equivalent. Arguments in this section have been inspired by arguments from [Stolarski 2023].

The PDE for u implies that $h=u_t$ satisfies

$$h_t=\frac{\partial}{\partial x}\left(\frac{h_x}{1+u_x^2}\right)+\frac{3}{x}h_x+\frac{3}{u^2}h.$$

For $n\ge n_0$, define $h_n(x,t):=\partial_t u_n(x,t)$, where $u_n:[0,\infty)\times[s_n,t_0]\to\mathbb{R}$ is our approximating sequence of solutions from the proof of Theorem 4.1.1 in Section 4. We choose a fixed $m\in(2,3)$ and set

$$\Lambda_n=\max\{(1+t^{-k/3}x)^m|h_n(x,t)|:0\le x\le M\sqrt{t},\ t\in[s_n,t_0]\}.$$

We claim the following holds.

Lemma 5.4.1. *We have* $\sup_n\Lambda_n<\infty$.

This lemma implies that $|h_n(x,t)|$ is uniformly bounded and hence that $H_n=h_n/\sqrt{1+u_x^2}$ is also uniformly bounded. Since the bound is uniform in n, by passing to the limit as $n\to+\infty$ we will then obtain that the mean curvature $H(x,t)$ of our solution is bounded for $0\le x\le M\sqrt{t},\ 0\le t\le t_0$.

5.5. ***Choice of the blow-up sequences.*** For the proof of Lemma 5.4.1 we argue by contradiction and assume that $\sup_n\Lambda_n=\infty$. Then we can pass to a subsequence so that we may assume without loss of generality that

$$\lim_{n\to\infty}\Lambda_n=+\infty. \tag{5.5.1}$$

Our goal in this section is *to contradict* (5.5.1).

The bound (5.3.1) for u_n implies the same bound for h_n; namely, we have

$$|h_n(x,t)|\lesssim t^{-k/3}(1+t^{-k/3}x)^{-4},\quad x\le M\sqrt{t},\quad t\in[s_n,t_0]. \tag{5.5.2}$$

The quantity $(1+t^{-k/3}x)^m|h_n(x,t)|$ attains its maximum in the region $\{(x,t) \mid 0 \le x \le M\sqrt{t},\ s_n \le t \le t_0\}$, so we can choose $T_n \in [s_n, t_0]$ and $a_n \in [0, M\sqrt{T_n}]$ such that

$$|h(a_n, T_n)| = \Lambda_n(1+T_n^{-k/3}a_n)^{-m}. \tag{5.5.3}$$

The inequality (5.5.2) implies

$$T_n^{k/3}(1+T_n^{-k/3}a_n)^{4-m} \lesssim \Lambda_n^{-1},$$

and thus

$$\max\{T_n^{k/3}, T_n^{(m-3)k/3}a_n^{4-m}\} \lesssim \Lambda_n^{-1}.$$

Since $\Lambda_n \to \infty$, we find that $T_n \to 0$ and also

$$a_n \ll T_n^{\frac{3-m}{4-m}\frac{k}{3}}.$$

At this point we use our assumption that $k > 3$ and choose m close enough to $m = 2$ that the exponent of T_n satisfies $\frac{3-m}{4-m}\frac{k}{3} > \frac{1}{2}$, which then implies

$$a_n \ll T_n^{1/2}. \tag{5.5.4}$$

To complete the proof we distinguish between two cases $a_n \lesssim T_n^{k/3}$ and $T_n^{k/3} \ll a_n \ll T_n^{1/2}$, depending on where the maximum a_n is attained.

5.6. *Case 1*: $a_n \lesssim T_n^{k/3}$. We choose the scale $\alpha_n = T_n^{k/3}$ and form the blow-up sequences

$$\bar{u}_n(\xi, s) = \alpha_n^{-1}u_n(\xi\alpha_n, T_n + s\alpha_n^2), \tag{5.6.1}$$

$$\bar{h}_n(\xi, s) = \Lambda_n^{-1}h_n(\xi\alpha_n, T_n + s\alpha_n^2). \tag{5.6.2}$$

These functions are defined for

$$\xi > 0 \quad \text{and} \quad -S_n \le s \le 0, \qquad \text{where } S_n = \frac{T_n - s_n}{\alpha_n^2},$$

and they satisfy the equations

$$\frac{\partial \bar{u}_n}{\partial s} = \frac{\bar{u}_{n\xi\xi}}{1+\bar{u}_{n\xi}^2} + \frac{3}{\xi}\bar{u}_{n\xi} - \frac{3}{\bar{u}_n}, \tag{5.6.3}$$

$$\frac{\partial \bar{h}_n}{\partial s} = \frac{\partial}{\partial \xi}\left(\frac{\bar{h}_{n\xi}}{1+\bar{u}_{n\xi}^2}\right) + \frac{3}{\xi}\bar{h}_{n\xi} + \frac{3}{\bar{u}_n^2}\bar{h}_n. \tag{5.6.4}$$

We use (5.6.1) with $\alpha_n = T_n^{k/3}$ and the definition of the inner region rescaling $w_n(z,\tau)$ of $u_n(x,t)$, i.e.,

$$u_n(x,t) = t^{k/3}w_n(t^{-k/3}x, \log t),$$

with $t = T_n + T_n^{2k/3}s$ to express $\bar{u}_n(\xi, s)$ in terms of $w_n(z,\tau)$. We get

$$\bar{u}_n(\xi, s) = \vartheta_n(s)w_n\left(\frac{\xi}{\vartheta_n(s)}, \log t\right),$$

where

$$\vartheta_n(s) := t^{k/3}T_n^{-k/3} = (T_n + T_n^{2k/3}s)^{k/3}T_n^{-k/3} = (1+T_n^{2k/3-1}s)^{k/3}.$$

Since $T_n \to 0$, we have $\vartheta_n(s) \to 1$ uniformly for bounded s, and thus

$$\log t = \log T_n + \frac{3}{k} \log \vartheta_n(s) \to -\infty,$$

uniformly for bounded s. Similarly to the last statement of Theorem 4.1.1 we claim the following.

Claim 5.6.1. *$\bar{u}_n(\xi, s) \to W_{K_2}(\xi)$ in C^∞_{loc}.*

Proof. For every fixed $\xi > 0$ there exists an n_0 such that for all $n \geq n_0$ we have

$$\vartheta_n w_n^- \left(\frac{\xi}{\vartheta_n(s)}, \log t \right) \leq \bar{u}_n(\xi, s) \leq \vartheta_n w_n^+ \left(\frac{\xi}{\vartheta_n(s)}, \log t \right),$$

where $\log t = \log T_n + \frac{3}{k} \log \vartheta_n(s)$ and w_n^- and w_n^+ are the lower and the upper barriers in the inner region, respectively. See Lemmas 3.5.2 and 3.5.3. This implies

$$\vartheta_n W_{K_2^-(n)} \left(\frac{\xi}{\vartheta_n} \right) + D(T_n \vartheta_n^{3/k})^{2\gamma} \leq \bar{u}_n(\xi, s) \leq \vartheta_n W_{K_2^+(n)} \left(\frac{\xi}{\vartheta_n} \right),$$

where we recall that $(K_2^\pm(n))^3 = K_2^3 \pm \delta_n$. Since $\lim_{n\to\infty} T_n = 0$, $\lim_{n\to\infty} \vartheta_n = 1$, and $\lim_{n\to\infty} K_2^\pm(n) = K_2$, we conclude that $\bar{u}_n(\xi, s) \to W_{K_2}(\xi)$ uniformly for bounded $\xi \geq 0$ and bounded s.

Furthermore, since $(\bar{u}_n)_{\xi\xi}(\xi, s) = \vartheta_n(s)^{-1}(w_n)_{zz}(z, \tau)$ is uniformly bounded for bounded ξ and s, it follows that $\bar{u}_{n\xi}$ also converges locally uniformly. After bootstrapping the nondegenerate parabolic equation (5.6.3) for $\bar{u}_n$ we find that $\bar{u}_n(\xi, s) \to W_{K_2}(\xi)$ in C^∞_{loc}. □

Recall next that by the definition of Λ_n we have

$$|\bar{h}_n(\xi, s)| \leq (1 + T_n^{k/3} \xi (T_n + T_n^{2k/3} s)^{-k/3})^{-m} = (1 + \xi(1 + T_n^{2k/3-1} s)^{-k/3})^{-m}.$$

For $s \leq 0$ and $\xi > 0$ this implies

$$|\bar{h}_n(\xi, s)| \leq \frac{1}{(1+\xi)^m}.$$

Lemma 5.6.2. *Let $\Phi(\xi) = W(\xi) - \xi W'(\xi)$, where $W(\xi)$ is a solution to (2.3.4). Then for any $S_* > 0$ there is a $\kappa_* > 0$ such that $e^{\kappa_* s}\Phi(\xi)$ is a supersolution for (5.6.4) in the region $-\min\{S_n, S_*\} \leq s \leq 0$, $0 < \xi < \frac{1}{2} M T_n^{1/2-k/3}$, where $S_n = (T_n - s_n)/\alpha_n^2$.*

Proof. Expanding the derivative in (5.6.4) leads to

$$\bar{h}_{ns} = \mathcal{M}_n(\bar{h}_n) := \frac{\bar{h}_{n\xi\xi}}{1 + \bar{u}_{n\xi}^2} + \left\{ \frac{3}{\xi} - \frac{2\bar{u}_{n\xi}\bar{u}_{n\xi\xi}}{(1 + \bar{u}_{n\xi}^2)^2} \right\} \bar{h}_{n\xi} + \frac{3}{\bar{u}_n^2} \bar{h}_n. \tag{5.6.5}$$

In order to estimate $\mathcal{M}_n[\Phi]$ we write the right-hand side as

$$\mathcal{M}_n[\bar{h}_n] = \mathcal{M}_\infty[\bar{h}_n] + \mathcal{R}_n[\bar{h}_n]$$

with

$$\mathcal{M}_\infty[\eta] := \frac{\eta_{\xi\xi}}{1 + W'(\xi)^2} + \left\{ \frac{3}{\xi} - \frac{2W'(\xi)W''(\xi)}{(1 + W'(\xi)^2)^2} \right\} \eta_\xi + \frac{3}{W(\xi)^2} \eta \tag{5.6.6}$$

and where the remainder is given by

$$\mathcal{R}_n[\eta] = a_n(\xi, s)\eta_{\xi\xi} + b_n(\xi, s)\eta_\xi + c_n(\xi, s)\eta$$

for certain coefficients a_n, b_n, c_n which one obtains by subtracting (5.6.6) and (5.6.5).

We now argue that a_n, b_n, c_n are uniformly bounded if s remains bounded. The estimate (5.3.1) says $|u_{nxx}| \lesssim t^{-k/3}(1 + t^{-k/3}x)^{-4}$ for small $t > 0$ and for $0 < x < M\sqrt{t}$. By definition (5.6.1) of $\bar{u}_n$ this implies that for $\xi < MT_n^{1/2-k/3}\sqrt{1 + T_n^{2k/3-1}s}$ one has

$$\begin{aligned}|\bar{u}_{n\xi\xi}(\xi, s)| &= T_n^{k/3}|u_{nxx}(T_n^{k/3}\xi, T_n + T_n^{2k/3}s)| \\ &\lesssim (1 + T_n^{2k/3-1}s)^{-k/3}(1 + (1 + T_n^{2k/3-1}s)^{-k/3}\xi)^{-4}.\end{aligned}$$

Under the assumption that $-\min\{S_n, S_*\} \le s \le 0$, and because $T_n \to 0$, we may conclude

$$|\bar{u}_{n\xi\xi}| \lesssim (1 + \xi)^{-4} \quad \text{for } -\min\{S_n, S_*\} \le s \le 0, \quad 0 < \xi < \tfrac{1}{2}MT_n^{1/2-k/3}.$$

This implies that $\bar{u}_{n\xi\xi}$ is uniformly bounded for $0 < \xi < \frac{1}{2}MT_n^{1/2-k/3}$.

Since $\bar{u}_{n\xi}(\xi, s) = u_{nx}(T_n^{k/3}\xi, T_n + T_n^{2k/3}s)$, we have $0 \le \bar{u}_{n\xi} \le C_1$ (by Lemma 4.9.1), so that $\bar{u}_{n\xi}$ is uniformly bounded.

Finally, by comparing $\bar{u}_n$ with the lower barrier $U^-_{\delta_0}$ from (3.9.2) we have

$$\begin{aligned}\bar{u}_n(\xi, s) \ge \bar{u}_n(0, s) &= T_n^{-k/3}u_n(0, T_n + T_n^{2k/3}s) \\ &\ge T_n^{-k/3}U^-_{\delta_0}(0, T_n + T_n^{2k/3}s) && \text{use (3.9.2)} \\ &= T_n^{-k/3}(T_n + T_n^{2k/3}s)^{k/3}w^-_{\delta_0}(0, \log(T_n + T_n^{2k/3}s)) \\ &\ge (1 + T_n^{2k/3-1}s)^{k/3}W_{K_2/2}(0).\end{aligned}$$

The assumption $-\min\{S_n, S_*\} \le s \le 0$ together with $T_n \to 0$ and $W_K(0) = KW(0) = K$ then implies

$$\bar{u}_n(\xi, s) \ge \tfrac{1}{4}K_2 \quad \text{for all } \xi > 0 \text{ and } -\min\{S_n, S_*\} \le s \le 0.$$

This implies that $\bar{u}_n(\xi, s)^{-2}$ is uniformly bounded, and hence that the coefficients a_n, b_n, c_n are uniformly bounded.

Therefore there is a $\kappa > 0$ such that

$$|\mathcal{R}_n[\eta]| \le \kappa(|\eta_{\xi\xi}| + |\eta_\xi| + |\eta|)$$

whenever $-\min\{S_n, S_*\} \le s \le 0$ and $0 \le \xi \le \frac{1}{2}MT_n^{1/2-k/3}$. Since

$$\mathcal{M}_\infty[\Phi] = 0 \quad \text{and} \quad |\Phi''(x)| + |\Phi'(x)| \lesssim \Phi(x),$$

we find that

$$\mathcal{M}_n[\Phi] \le C\kappa\Phi.$$

If we define $\kappa_* = C\kappa$, then we have found that $e^{\kappa s}\Phi(x)$ is an upper barrier for $\bar{h}_{ns} = \mathcal{M}_n[\bar{h}_n]$. □

Lemma 5.6.3. $S_n \to \infty$.

Proof. We argue by contradiction. Assume that there is a subsequence of S_n along which the limit is finite. Without loss of generality we can take this to be S_n itself, that is assume that

$$S_n = \frac{T_n - s_n}{\alpha_n^2} \le \bar{S} < +\infty \quad \text{for all } n.$$

This implies that $T_n \le s_n + \bar{S}\alpha_n^2 = s_n + \bar{S}T_n^{2k/3}$. Since $T_n \to 0$ and $k > 3$, we then conclude that $T_n \le 2s_n$ for $n \gg 1$.

We will now apply the maximum principle to $\bar{h}_n$ in the region

$$-S_n \le s \le 0, \quad 0 \le \xi \le \epsilon T_n^{-(k/3-1/2)}.$$

Observe first that the definition (4.3.1) of our initial data $u_n(x, s_n)$ is such that the surface coincides with an Alencar surface in the region $y = o(1)$, i.e., for $x \le \epsilon\sqrt{s_n}$ with ϵ as in Section 4.3. This implies that $h_n(x, s_n) = 0$ for $x \le \epsilon\sqrt{s_n}$. Using $T_n \le 2s_n$ for $n \gg 1$, we conclude that by taking $n \gg 1$ and ϵ sufficiently small we can guarantee that $\bar{h}_n(\xi, -S_n) = \Lambda_n^{-1} h_n(\alpha_n \xi, s_n) = 0$ for $\xi \le \epsilon\alpha_n^{-1} T_n^{1/2} = \epsilon T_n^{-(k/3-1/2)}$. At the end of this region, where $\xi = \epsilon T_n^{1/2-k/3}$, we have

$$|\bar{h}_n(\xi, s)| \le (1+\xi)^{-m} = (1+\xi)^{-2}(1+\xi)^{-(m-2)} \lesssim T_n^{(m-2)(k/3-1/2)}\Phi(\xi).$$

Here we have used the expansion (A.3.1) combined with the fact that $\Phi(\xi) = W(\xi) - \xi W'(\xi) > 0$ (Lemma A.3.1) to conclude that $\Phi(\xi) \gtrsim (1+\xi)^{-2}$. Choosing κ_* as in Lemma 5.6.2, we see that for suitably large $\widetilde{C}$ the function

$$\widetilde{C} T_n^{(m-2)(k/3-1/2)} e^{\kappa_* s}\Phi(\xi)$$

is an upper bound for both $\bar{h}_n(\xi, s)$ and $-\bar{h}_n(\xi, s)$ in the region $-S_n \le s \le 0$, $\xi \le \epsilon T_n^{1/2-k/3}$ and for all n.

Finally, at $s = 0$ this implies

$$|\bar{h}_n(\xi, 0)| \lesssim T_n^{(m-2)(k/3-1/2)} \to 0 \quad \text{as } n \to \infty.$$

This cannot be, because $\max_\xi |\bar{h}_n(\xi, 0)| = 1$, thus showing that $S_n \to \infty$. □

We can now complete the blow up argument, at least in the case where $a_n \lesssim T_n^{k/3}$. Since $S_n \to \infty$, we can pass to another subsequence along which $\bar{h}_n$ converges in C^∞_{loc} to an ancient solution $\bar{h}$ of

$$\bar{h}_s = \frac{\partial}{\partial\xi}\left(\frac{\bar{h}_\xi}{1+W'(\xi)^2}\right) + \frac{3}{\xi}\bar{h}_\xi + \frac{3}{W(\xi)^2}\bar{h}. \tag{5.6.7}$$

The ancient solution $\bar{h}$ satisfies the bound

$$|\bar{h}(\xi, s)| \le (1+\xi)^{-m}, \quad \xi \ge 0, \quad s \le 0.$$

By the definition of a_n (see (5.5.3)) the function $(1+\xi)^m|\bar{h}_n(\xi, s)|$ attains its maximum at $\xi_n = a_n T_n^{-k/3}$. We assumed here that $a_n \lesssim T_n^{k/3}$, so we may assume also that $\xi_n \to \bar{\xi}$ for some finite $\bar{\xi} \ge 0$. Thus we have

$$\bar{h}(\bar{\xi}, 0) = (1+\bar{\xi})^{-m}. \tag{5.6.8}$$

To complete the proof we compare this ancient solution with the stationary solution $\Phi(\xi)=W(\xi)-\xi W'(\xi)$. By the asymptotic expansion of the Alencar solution we have

$$\Phi(\xi)=(\Gamma_1+o(1))\xi^{-2}, \quad \xi\to\infty,$$

for some constant $\Gamma_1>0$.

Choose a large number $\ell>0$ and consider the function

$$\Psi(\xi)=\Phi(\xi)-\tfrac{1}{2}\Phi(\ell).$$

Since $\Phi(\xi)$ is a decreasing function of ξ, we have

$$\tfrac{1}{2}\Phi(\xi)\le\Psi(\xi)\le\Phi(\xi) \quad \text{for all } \xi\in[0,\ell].$$

Furthermore, it follows from $\mathcal{M}_\infty[\Phi]=0$ that

$$\mathcal{M}_\infty[\Psi](\xi)=-\frac{3\Phi(\ell)}{2W(\xi)^2}.$$

Since $W(\xi)=\xi+o(1)$ and $\Phi(\xi)\sim\xi^{-2}$ for large ξ, there is a $c>0$ such that $W(\xi)^{-2}\ge c\Phi(\xi)\ge c\Psi(\xi)$. There is also a constant $c>0$ with $\Phi(\ell)\ge c\ell^{-2}$. Therefore we get

$$\mathcal{M}_\infty[\Psi]\le -c\ell^{-2}\Psi(\xi) \quad \text{for } \xi\in[0,\ell].$$

It follows that for any s_0

$$\hat{h}(\xi,s)=e^{-c\ell^{-2}(s+s_0)}\Psi(\xi)$$

satisfies $\hat{h}_s\ge\mathcal{M}[\hat{h}]$ for $\xi\in[0,\ell]$.

We will next compare $\bar{h}$ with $\hat{h}$ in the domain $\{0<\xi<\ell,\ -s_0<s<0\}$ which will lead to a contradiction. At $\xi=\ell$ we have

$$\frac{|\bar{h}(\ell,s)|}{\hat{h}(\ell,s)}\le\frac{(1+\ell)^{-m}}{\Psi(\ell)}e^{c\ell^{-2}(s+s_0)}.$$

Using

$$\Psi(\ell)\ge\frac{1}{2}\Phi(\ell)\ge\frac{1}{C}(1+\ell)^{-2}$$

we therefore find for $-s_0\le s\le 0$

$$\frac{|\bar{h}(\ell,s)|}{\hat{h}(\ell,s)}\le C(1+\ell)^{-(m-2)}e^{c\ell^{-2}(s+s_0)}\le C(1+\ell)^{-(m-2)}e^{c\ell^{-2}s_0}.$$

Since $\Psi(\xi)\ge c(1+\xi)^{-2}$ for a uniform c, at time $-s_0$ we have

$$\frac{|\bar{h}(\xi,-s_0)|}{\hat{h}(\xi,-s_0)}\le\frac{(1+\xi)^{-m}}{\Psi(\xi)}\le C(1+\xi)^{-(m-2)}\le C.$$

To conclude our argument, for any given $\ell>0$ we choose $s_0>0$ large enough that

$$C(1+\ell)^{-(m-2)}e^{c\ell^{-2}s_0}>1.$$

Applying the maximum principle to the linear equation $h_s = \mathcal{M}_\infty[h]$ on the domain $\{0<\xi<\ell, -s_0<s<0\}$, we have

$$\frac{|\bar h(\xi,s)|}{\hat h(\xi,s)} \le C(1+\ell)^{-(m-2)}e^{c\ell^{-2}s_0} \quad \text{for } 0\le\xi\le\ell, \quad -s_0\le s\le 0.$$

In particular,

$$\frac{|\bar h(\xi,0)|}{\hat h(\xi,0)} \le C(1+\ell)^{-(m-2)}e^{c\ell^{-2}s_0} \quad \text{for } 0\le\xi\le\ell,$$

and hence, using the definition of $\hat h$,

$$|\bar h(\xi,0)| \le C(1+\ell)^{-m}\Psi(\xi) \quad \text{for } 0\le\xi\le\ell.$$

The constant C does not depend on ℓ, so by choosing ℓ large enough we reach a contradiction if $\bar h(\xi,0)\ne 0$ for some $\xi\ge 0$ since (5.6.8) needs to hold at the same time as well.

This completes the proof of Lemma 5.4.1 in the case $a_n \lesssim T_n^{-k/3}$.

5.7. *Case 2*: $a_n \gg T_n^{-k/3}$. If we are not in Case 1, i.e., if it is not true that $a_n \lesssim T_n^{-k/3}$, then there is a subsequence along which $a_n T_n^{k/3}\to\infty$. In this case we choose our scale to be $\alpha_n = a_n$, and we define the blow-ups

$$\bar u_n(\xi,s) = a_n^{-1}u_n(a_n\xi, T_n + a_n^2 s), \quad \bar h_n(\xi,s) = \frac{h_n(a_n\xi, T_n+a_n^2 s)}{h_n(a_n,T_n)}. \tag{5.7.1}$$

These blow-ups are defined for all $\xi\ge 0$ and for

$$-S_n\le s\le 0, \quad \text{with } S_n = \frac{T_n - s_n}{a_n^2}.$$

By our intermediate region asymptotics for u_n^- and u_n^+, since $e^{(\gamma+1/2)\tau} \ll a_n \ll T_n^{1/2}$ (see (5.5.4)) and $u_n^-(x,s)\le u_n(x,s)\le u_n^+(x,s)$, we have

$$\bar u_n(\xi,s)\to \bar u_\infty(\xi) = \xi$$

uniformly for bounded $\xi\ge 0$ and s and in C^∞_{loc} for $\xi>0$ and $s\le 0$.

Lemma 5.7.1. *For $\bar h_n(\xi,s)$ we have the pointwise bound*

$$|\bar h_n(\xi,s)| \le \left(1+\frac{T_n^{k/3}}{a_n}\right)\left(1+\frac{a_n^2 s}{T_n}\right)^{km/3}\xi^{-m} \tag{5.7.2}$$

for all ξ with $0<a_n\xi\le M\sqrt{T_n+a_n^2 s}$. In particular, for large enough n we also have

$$|\bar h_n(\xi,s)|\le 2\xi^{-m} \tag{5.7.3}$$

for all ξ with $0<a_n\xi\le M\sqrt{T_n+a_n^2 s}$ and for bounded s.

Proof. By definition of Λ_n, a_n, and T_n we have for all $x\le M\sqrt{t}$ and $t\in[s_n,t_0]$

$$|h_n(x,t)|\le \Lambda_n(1+t^{-k/3}x)^{-m}, \quad |h_n(a_n,T_n)| = \Lambda_n(1+T_n^{-k/3}a_n)^{-m}.$$

Hence

$$\left|\frac{h_n(a_n\xi, T_n + a_n^2 s)}{h_n(a_n, T_n)}\right| \le \left\{\frac{1 + T_n^{-k/3} a_n}{1 + (T_n + a_n^2 s)^{-k/3} a_n \xi}\right\}^m.$$

Discarding the "+1" in the denominator and multiplying numerator and denominator with $T_n^{k/3} a_n^{-1}$ we find

$$\left|\frac{h_n(a_n\xi, T_n + a_n^2 s)}{h_n(a_n, T_n)}\right| \le \left(\frac{T_n^{k/3}}{a_n} + 1\right)^m \left(1 + \frac{a_n^2 s}{T_n}\right)^{mk/3} \xi^{-m}.$$

This proves (5.7.2). Since $T_n^{k/3} \ll a_n \ll T_n^{1/2}$ (recall that (5.5.4) implies $a_n \ll T_n^{1/2}$), we have

$$\left(\frac{T_n^{k/3}}{a_n} + 1\right)^m \left(1 + \frac{a_n^2 s}{T_n}\right)^{mk/3} \to 1$$

uniformly for bounded s which implies (5.7.3). □

This lemma tells us we have a sequence of solutions $\bar{h}_n$ of the linear equation

$$\frac{\partial \bar{h}_n}{\partial t} = \frac{\partial}{\partial \xi}\left\{\frac{\bar{h}_{n\xi}}{1 + \bar{u}_{n\xi}^2}\right\} + \frac{3}{\xi}\frac{\partial \bar{h}_n}{\partial \xi} + \frac{3}{\bar{u}_n^2}\bar{h}_n = \frac{\bar{h}_{n\xi\xi}}{1 + \bar{u}_{n\xi}^2} + \left\{\frac{3}{\xi} - \frac{2\bar{u}_{n\xi} u_{n\xi\xi}}{(1 + \bar{u}_{n\xi}^2)^2}\right\}\frac{\partial \bar{h}_n}{\partial \xi} + \frac{3}{\bar{u}_n^2}\bar{h}_n \tag{5.7.4}$$

which satisfies the uniform bound (5.7.3) for all $n \ge n_0 \gg 1$. As before we have:

Lemma 5.7.2. $S_n \to \infty$.

Proof. Assume that S_n is bounded and, after passing to a subsequence, that we have $S_n \to S_\infty$.

The function $\bar{u}_n$ converges in C^∞_{loc} to $\bar{u}_\infty(\xi, s) = \xi$, so interior estimates for the divergence form (5.7.4) imply that $\bar{h}_n$ is locally uniformly Hölder continuous for $\xi > 0$ and $-S_n \le s \le 0$. Moreover, the construction of $u_n(\cdot, s_n)$ guarantees that $\bar{h}_n$ starts out with $\bar{h}_n(\xi, -S_n) = 0$ for all $a_n\xi \ll T_n^{1/2}$. We may therefore assume that there is a convergent subsequence $\bar{h}_n(\xi, s) \to \bar{h}(\xi, s)$, where

$$|\bar{h}(\xi, s)| \le \xi^{-m}$$

for all $\xi > 0$ and $s \in [-S_\infty, 0]$ and where $\bar{h}$ is a solution of

$$\bar{h}_s = \frac{1}{2}\bar{h}_{\xi\xi} + \frac{3}{\xi}\bar{h}_\xi + \frac{3}{\xi^2}\bar{h} := \mathcal{M}_0[\bar{h}],$$

with $\bar{h}(1, 0) = \pm 1$ and $\bar{h}(\xi, -S_\infty) = 0$ for all $\xi > 0$. The limiting function $\bar{h}$ is smooth for $\xi > 0$, $-S_\infty \le s \le 0$. We note that $\hat{h}(\xi) = \xi^{-2} + \xi^{-3}$ is a stationary solution of $\hat{h}_s = \mathcal{M}_0[\hat{h}]$, so that for any $\eta > 0$ the functions $\pm\eta\hat{h}$ provide upper and lower barriers for $\bar{h}$, provided we can show that $-\eta\hat{h} < \bar{h} < \eta\hat{h}$ as $\xi \to 0$ or $\xi \to \infty$. This boundary condition is fulfilled because $|\bar{h}(\xi, s)| \le \xi^{-m}$ with $2 < m < 3$. The maximum principle therefore implies that $|\bar{h}| \le \eta\hat{h}$ for all $\eta > 0$. Letting $\eta \to 0$ this yields $\bar{h}(\xi, s) = 0$ for all $\xi > 0$ and all $s \in [-S_\infty, 0]$. This contradicts $\bar{h}(1, 0) = \pm 1$ and shows that the sequence S_n is indeed unbounded. □

Let $\bar{h}(\xi, s)$ be a limit of the $\bar{h}_n(\xi, s)$ along some subsequence $n = n_k \nearrow \infty$. We now show that $\bar{h}(1, 0) = 0$, which contradicts the fact that $\bar{h}(1, 0) = \pm 1$ and therefore *completes the proof of Lemma 5.4.1*.

Lemma 5.7.3. $\bar{h}(1,0)=0$.

Proof. Choose a small $\epsilon>0$ and consider the function

$$k(\xi,s)=\bar{h}(\xi,s)-\epsilon\xi^{-2}-\epsilon\xi^{-3}.$$

This function is a solution of the linear equation $k_s=\mathcal{M}_0[k]$. In view of the bound $\bar{h}(\xi,s)\le\xi^{-m}$, which holds for all $\xi>0$ and $s\le 0$, we have

$$k(\xi,s)\le\xi^{-m}-\epsilon\xi^{-2}-\epsilon\xi^{-3}.$$

Since $2<m<3$, this implies that $k(\xi,s)<0$ if $\xi\le\epsilon^{1/(3-m)}$ or $\xi\ge\epsilon^{-1/(m-2)}$.

The differential operator $\mathcal{M}_0$ is a standard Sturm–Liouville operator with smooth coefficients on the interval $I_\epsilon=[\epsilon^{1/(3-m)},\epsilon^{-1/(m-2)}]$. Since ξ^{-2} is a strictly positive solution of $\mathcal{M}_0[\varphi]=0$, the principal eigenvalue λ_0 of

$$\mathcal{M}_0[\Omega(\xi)]=-\lambda_0\Omega(\xi),\quad \Omega(\epsilon^{1/(3-m)})=\Omega(\epsilon^{-1/(m-2)})=0$$

is positive, and the corresponding eigenfunction $\Omega(\xi)$ is also positive for all ξ in the interior of the interval I_ϵ. Choose $C_\epsilon>0$ so that

$$\xi^{-m}-\epsilon\xi^{-2}-\epsilon\xi^{-3}\le C_\epsilon\Omega(\xi)$$

for all $\xi\in I_\epsilon$.

For any given $s_0>0$ we then have

$$k(\xi,-s_0)\le C_\epsilon\Omega(\xi)\quad\text{for all }\xi\in I_\epsilon.$$

Moreover, $\hat{k}(\xi,s)=C_\epsilon e^{-\lambda_0(s+s_0)}\Omega(\xi)$ is a solution of $\hat{k}_s=\mathcal{M}[\hat{k}]$, so the maximum principle applied on the domain $I_\epsilon\times[-s_0,0]$ implies that at time $s=0$ we have

$$k(\xi,0)\le\hat{k}(\xi,0)=C_\epsilon e^{-\lambda_0 s_0}\Omega(\xi).$$

Since this is true for all $s_0>0$, we conclude $k(\xi,0)\le 0$. By definition of $k(\xi,s)$ this implies that $\bar{h}(\xi,0)\le\epsilon\xi^{1/(3-m)}+\epsilon\xi^{-1/(m-2)}$ for all $\xi\in I_\epsilon$. In particular, this holds for $\xi=1$ where it implies $\bar{h}(1,0)\le 2\epsilon$. This argument goes through for all $\epsilon>0$, so we find $\bar{h}(1,0)\le 0$.

Applying the whole argument once more to $\tilde{k}(\xi,s)=-\bar{h}(\xi,s)-\epsilon\xi^{1/(3-m)}-\epsilon\xi^{-1/(2-m)}$ instead, we find $-\bar{h}(1,0)\le 0$. Hence $\bar{h}(1,0)=0$, as claimed. □

The proof of Lemma 5.4.1 is now complete. We can now conclude the proof of Theorem 5.1.1.

Proof of Theorem 5.1.1. Lemma 5.4.1 implies $\sup_n\Lambda_n<\infty$. Using the definition of Λ_n this implies that $|H_n|=|h_n|/\sqrt{1+u_{nx}^2}$ is also uniformly bounded. The derivative bounds from sections Section 4.4–4.6 imply that $u_n(x,t)$ converges uniformly smoothly to $u(x,t)$ for $t\in(0,t_0]$ as $n\to\infty$. Therefore we get $|H(x,t)|=\lim_{n\to\infty}|H_n|\le C$ for all $0\le x\le M\sqrt{t}$ and $t\in(0,t_0]$. On the other hand, Lemma 5.2.1 shows that $H(x,t)$ is bounded when $x\ge Y_{\delta_0}\sqrt{t}$. By definition (3.6.4) we have $Y_{\delta_0}:=2\sqrt{(2k+1)!!\,M/\delta_0}$; if we choose M large enough then $Y_{\delta_0}<M$ so the two regions $0\le x\le M\sqrt{t}$ and $x\ge Y_{\delta_0}\sqrt{t}$ overlap. The mean curvature H is therefore uniformly bounded on the whole solution, which concludes the proof of Theorem 5.1.1. □

Appendix

A.1. ***The linear equation in the intermediate region.*** The eigenvalue equation $\mathcal{L}\varphi = \left(k - \frac{3}{2}\right)\varphi$ is

$$\frac{1}{2}\varphi_{yy} + \left(\frac{3}{y} + \frac{y}{2}\right)\varphi_y + \left(\frac{3}{y^2} - \frac{1}{2}\right)\varphi = \left(k - \frac{3}{2}\right)\varphi;$$

i.e.,

$$\varphi_{yy} + \left(\frac{6}{y} + y\right)\varphi_y + \frac{6}{y^2}\varphi = 2(k-1)\varphi.$$

Let $\varphi(y) = y^{-2}\chi_k(y)$. Then χ_k satisfies the equation

$$\chi_k'' + \left(\frac{2}{y} + y\right)\chi_k' = 2k\chi_k.$$

For every *real* $k > 0$ there is a unique solution with $\chi_k(0) = 1,\ \chi_k'(0) = 0$. This solution is monotone increasing and for large y has the expansion

$$\chi_k(y) = C_k y^{2k} + o(y^{2k}), \quad y \to \infty.$$

For $k \in \mathbb{N}$ it is given by the series expansion

$$\chi_k(y) = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!\,(2n+1)!!} y^{2n}, \tag{A.1.1}$$

where $(2n+1)!! := 1\cdot 3\cdot 5\cdot 7\cdots(2n+1)$. This defines φ_k for all real k. We will only need these functions for integer values of k, in which case χ_k is a polynomial, and $\varphi_k(y) = y^{-2}\chi_k(y)$ is given by

$$\varphi_k(y) = y^{-2}\sum_{n=0}^{k}\binom{k}{n}\frac{y^{2n}}{(2n+1)!!}. \tag{A.1.2}$$

There is a second solution $\hat{\chi}_k$ that satisfies

$$\hat{\chi}_k(y) = e^{-y^2/2 + o(y^2)}, \quad y \to \infty.$$

At $y = 0$ this solution is singular:

$$\hat{\chi}_k(y) = \frac{C}{y} + \mathcal{O}(y), \quad y \to 0.$$

A.2. ***Proof of Lemma 3.4.1.*** The homogeneous equation $6\gamma\varphi - \mathcal{L}\varphi = 0$ has solutions of the form

$$\varphi = C\varphi_k^1(y) + B\psi_k^1(y), \quad C, B \in \mathbb{R},$$

where $\varphi_k^1(y)$ and ψ_k^1 are solutions with

$$\varphi_k^1(y) = \begin{cases} y^{-2}, & y \to 0, \\ \mathcal{O}(y^{4k-5}), & y \to \infty, \end{cases}$$

and

$$\psi_k^1(y) = \begin{cases} y^{-3}, & y \to 0, \\ \mathcal{O}(e^{-y^2/2 + o(y^2)}), & y \to \infty. \end{cases}$$

Since $y=0$ is a regular singular point for the differential equation $6\gamma g-\mathcal{L}g=G(y)=y^{-7}+y^{4k-7}$, we look for the solution in the form of a power series. From

$$(6\gamma-\mathcal{L})[y^r]=-\tfrac{1}{2}(r+2)(r+3)y^{r-2}+\tfrac{1}{2}(4k-7-r)y^r \tag{A.2.1}$$

it follows that (3.4.2) has a particular solution of the form

$$g_{0p}(y)=C_0y^{-5}P_0(y^2)+C_1y^{-3}\log(y)P_1(y^2),$$

where $P_j(y^2)$ are power series in y^2 with $P_j(0)=1$. The logarithmic term appears because $r=-3$ is one of the characteristic exponents. The coefficient C_0 is obtained by substitution in the equation. One finds $C_0=-\frac{1}{3}$.

Every solution φ of the homogeneous equation satisfies $\varphi=\mathcal{O}(y^{-3})=o(g_{0p})$ as $y\to 0$, and therefore every solution g of the inhomogeneous equation satisfies

$$g=g_{0p}+\mathcal{O}(y^{-3})=-\tfrac{1}{3}y^{-5}+\mathcal{O}(y^{-3}\log y),\quad \text{as } y\to 0. \tag{A.2.2}$$

The differential equation $6\gamma g-\mathcal{L}g=G$ has an irregular singular point at $y=\infty$, so we cannot use the power series method. Instead, we obtain a solution using sub- and supersolutions. For any $m\in\mathbb{R}$ the functions $g_\pm(y)=y^{4k-7}\pm my^{4k-9}$ satisfy

$$(6\gamma-\mathcal{L})g_\pm=\big(-\tfrac{1}{2}(4k-5)(4k-4)\pm m\big)y^{4k-9}+\mathcal{O}(y^{4k-11}),\quad y\to\infty.$$

For $m>\frac{1}{2}(4k-5)(4k-4)$ it follows that $g_-<g_+$ are sub- and supersolutions for $6\gamma g-\mathcal{L}g=G$ on the interval $[y_0,\infty)$ if y_0 is large enough. Hence there is a particular solution $g_{\infty p}$ satisfying

$$g_{\infty p}(y)=y^{4k-7}+\mathcal{O}(y^{4k-9}),\quad y\to\infty.$$

At $y=0$ all solutions satisfy (A.2.2), so $g_{\infty p}$ also satisfies $g_{\infty p}(y)=-\frac{1}{3}y^{-5}+\mathcal{O}(y^{-3}\log y)$. The general solution of the nonhomogeneous equation (3.4.2) is then of the form $g:=g_{\infty p}+C\varphi_k^1+B\psi_k^1$ for $C,B\in\mathbb{R}$. However, the boundary condition $g(y)=y^{4k-5}+o(y^{4k-5})$ as $y\to\infty$ requires $C=0$. One concludes that $g_B:=g_{\infty p}+B\psi_k^1$, $B\in\mathbb{R}$, is a one-parameter set of solutions to (3.4.2) which satisfies the conditions of our lemma, thus finishing the proof.

A.3. *The Alencar solution.*

Lemma A.3.1. *Let $W:[0,\infty)\to\mathbb{R}$ be the solution of*

$$\frac{W_{zz}}{1+W_z^2}+\frac{3}{z}W_z-\frac{3}{W}=0,\quad W(0)=1,\quad W'(0)=0.$$

Then $W_{zz}>0$ and $0<W-zW_z\le 1$ for all $z\ge 0$.

For large z the solution $W(z)$ has the expansion

$$W=z+\frac{\Gamma_2}{z^2}+\frac{\Gamma_3}{z^3}+\frac{\Gamma_5}{z^5}+\cdots \tag{A.3.1}$$

for certain coefficients $\Gamma_i\in\mathbb{R}$.

Proof. The differential equation for W has been thoroughly studied. In particular, $W_{zz} > 0$ and $W > zW_z$ were shown by Velázquez [1994, Proposition 2.2], ($B''(u) > 0$ and $G_a(r) < 0$ in his notation). Here we prove that $W(z)$ has the stated asymptotic expansion. Let

$$P = W_z \quad \text{and} \quad Q = \frac{z}{W}.$$

Then (P, Q) as a function of $\log z$ satisfy an autonomous system of differential equations,

$$\begin{cases} zP_z = 3(1+P^2)(Q-P), \\ zQ_z = P - P^2 Q. \end{cases} \tag{A.3.2}$$

This system has two fixed points with $Q \geq 0$, namely, the origin $(0, 0)$ and the point $(1, 1)$.

The origin corresponds to the boundary condition $W_z = 0,\ z = 0$, while the fixed point corresponds to the Simons cone on which $W = z$ and $W_z = 1$.

The matrix of the linearization at $(0, 0)$ is $\left(\begin{smallmatrix} 1 & 0 \\ 3 & -3 \end{smallmatrix}\right)$. Its eigenvalues are $\lambda_1 = +1$ and $\lambda_2 = -3$. The eigenvector corresponding to the unstable eigenvalue is $\left(\begin{smallmatrix} 4 \\ 3 \end{smallmatrix}\right)$. The unique orbit in the unstable manifold of the origin is the Alencar solution. It approaches the fixed point $(1, 1)$ as $z \to \infty$. The matrix of the linearization at $(1, 1)$ is $\left(\begin{smallmatrix} -1 & -1 \\ 6 & -6 \end{smallmatrix}\right)$ with eigenvalues/vectors

$$\lambda_1 = -3, \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \lambda_2 = -4, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

The eigenvalues are both negative and they satisfy the "no resonance" condition, i.e., neither eigenvalue is an integer multiple of the other. This implies that there is a real analytic conjugacy of the nonlinear system (A.3.2) near the fixed point $(1, 1)$ with the linearization (see the chapter on normal forms and Poincaré's theorem in [Arnold 1983]). The general solution of the linear system is

$$C_1 z^{-3} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 z^{-4} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} C_1 z^{-3} + C_2 z^{-4} \\ 2C_1 z^{-3} + 3C_2 z^{-4} \end{pmatrix}.$$

This in turn implies that all solutions of (A.3.2) that converge to $(1, 1)$ are convergent power series in z^{-3} and z^{-4}. In particular, $1/Q = W/z$ has an expansion of the form

$$\frac{W}{z} = 1 + C_3 z^{-3} + C_4 z^{-4} + C_6 z^{-6} + C_7 z^{-7} + \cdots = 1 + \sum_{l,m \geq 1} C_{l,m} z^{-3l-4m}.$$

Therefore $W(z)$ satisfies

$$W = z + C_3 z^{-2} + C_4 z^{-3} + C_6 z^{-5} + C_7 z^{-6} + \cdots = z + \sum_{l,m \geq 1} C_{l,m} z^{1-3l-4m}.$$

So if we set $\Gamma_m = C_{m+1}$ we have proved the expansion (A.3.1) □

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QUASILINEAR WAVE EQUATIONS ON ASYMPTOTICALLY FLAT SPACETIMES WITH APPLICATIONS TO KERR BLACK HOLES

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We prove global existence and decay for small-data solutions to a class of quasilinear wave equations on a wide variety of asymptotically flat spacetime backgrounds, allowing in particular for the presence of horizons, ergoregions and trapped null geodesics, and including as a special case the Schwarzschild and very slowly rotating $|a| \ll M$ Kerr family of black holes in general relativity. There are two distinguishing aspects of our approach. The first aspect is its dyadically localised nature: The nontrivial part of the analysis is reduced entirely to time-translation-invariant r^p-weighted estimates, in the spirit of Dafermos and Rodnianski (2010b), to be applied on dyadic time-slabs which for large r are outgoing. Global existence and decay then both immediately follow by elementary iteration on consecutive such time-slabs, without further global bootstrap. The second, and more fundamental, aspect is our direct use of a "black box" linear inhomogeneous energy estimate on exactly stationary metrics, together with a novel but elementary physical-space top-order identity that need not capture the structure of trapping and is robust to perturbation. In the specific example of Kerr black holes, the required linear inhomogeneous estimate can then be quoted directly from the literature (Dafermos et al. (2016)), while the additional top-order physical-space identity can be shown easily in many cases (we include in the Appendix a proof for the Kerr case $|a| \ll M$, which can in fact be understood in this context simply as a perturbation of Schwarzschild). In particular, the approach circumvents the need either for producing a purely physical-space identity capturing trapping or for a careful analysis of the commutation properties of frequency projections with the wave operator of time-dependent metrics.

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1. Introduction

We consider here quasilinear equations of the form

$$\Box_{g(\psi,x)}\psi = N(\partial\psi, \psi, x) \tag{1-1}$$

on a 4-dimensional manifold $\mathcal{M}$, where $g(\psi, x)$ and $N(\partial\psi, \psi, x)$ are appropriate nonlinear terms and where $g(0, x) = g_0(x)$ defines a stationary asymptotically flat Lorentzian metric on $\mathcal{M}$ foliated by a suitable family of hypersurfaces $\Sigma(\tau)$, for $\tau \geq 0$, which for large r are outgoing null. In this paper, we prove the following:

Theorem. *For equations (1-1), under appropriate assumptions on the background spacetime $(\mathcal{M}, g_0)$ and on the nonlinearities $g(\psi, x)$ and $N(\partial\psi, \psi, x)$ to be described below, we have the following*:

- <u>*Global existence of small data solutions*</u>: *solutions ψ arising from smooth initial data on Σ_0, sufficiently small as measured in a suitable weighted Sobolev energy, exist globally on $\mathcal{M}$.*
- <u>*Orbital stability*</u>: *under the above assumptions, the above weighted energy flux through $\Sigma(\tau)$ is uniformly bounded by a constant times its initial value.*
- <u>*Asymptotic stability*</u>: *under the above assumptions, a suitable lower-order unweighted energy flux through $\Sigma(\tau)$ decays inverse polynomially in τ (implying also pointwise inverse polynomial decay for ψ).*

One possible motivation for studying (1-1) is as an illustrative model problem for issues relating to the nonlinear stability of the spacetimes $(\mathcal{M}, g_0)$, when these themselves are solutions to the celebrated Einstein equations of general relativity. An important example of such spacetimes $(\mathcal{M}, g_0)$ allowed by our theorem is provided by the Schwarzschild family of metrics for $M > 0$, and the more general Kerr family (parametrised by a and M) in the very slowly rotating regime $|a| \ll M$. For a discussion of these spacetimes and the stability problem in general relativity, see [Dafermos et al. 2021].

Among other features, the Schwarzschild and Kerr spacetimes exhibit *trapped null geodesics*, along which energy can concentrate over large timescales. Moreover, these are *asymptotically flat* spacetimes, meaning that linear waves ψ decay like $\sim r^{-1}$ along outgoing null cones. This then constrains the decay of ψ in the near region to also be at best only inverse polynomial. (In the above black hole examples, this slow decay in the near region is already a linear effect due to scattering off of far-away curvature associated to the nontrivial spacetime mass at infinity; in Minkowski space, this slow decay in the near region arises due to purely nonlinear scattering effects, even for compactly supported initial data.) It is the combination of these two specific analytical difficulties — trapped null geodesics and the relatively slow decay in the near region necessitated by asymptotic flatness — of the black hole stability problem (and related problems) which we wish to capture with our assumptions in the present paper. It is for this reason that we include both the nonlinear term implicit in the expression $\Box_{g(\psi,x)}\psi$ on the left-hand side of (1-1) (the "quasilinearity"), as this is the most dangerous term in the vicinity of trapping, as well as the nonlinear term $N(\partial\psi, \psi, x)$ on the right-hand side (the "semilinearity"), as this term models the true null structure at infinity of the Einstein equations, when the latter are written in appropriate geometric gauges. To avoid inessential complications, we will in fact assume that $g(\psi, x) = g_0$ for large r and that $N(\partial\psi, \psi, x)$ satisfies a generalised version of the null condition [Klainerman 1986].

In the case where $(\mathcal{M}, g_0)$ is Minkowski space, a version of the above theorem follows from classical work of [Klainerman 1986; Christodoulou 1986], and there have been many further amplifications over the years, particularly in the context of the obstacle problem; see, e.g., [Metcalfe and Sogge 2005]. Concerning specifically the Schwarzschild and very slowly rotating $|a| \ll M$ Kerr setting, versions of the above theorem have been shown previously in various special cases; see already [Luk 2013] in the semilinear and [Lindblad and Tohaneanu 2018; 2020] in the quasilinear case, as well as the recent [Lindblad and Tohaneanu 2024] (the latter three works concerning slightly different classes of equations, but satisfying the weak null condition [Lindblad and Rodnianski 2003]). For axisymmetric solutions to certain semilinear equations on Kerr in the full subextremal range $|a| < M$, see [Stogin 2017]. See also [Pasqualotto 2019] for a related physically motivated quasilinear problem and [Hintz and Vasy 2016; Mavrogiannis 2024] for the study of (1-1) on the nonasymptotically flat Schwarzschild–de Sitter and Kerr–de Sitter black hole backgrounds, where the cosmological constant Λ is positive and decay of ψ is in fact exponential. For work specifically connected to the related problem of stability of these spacetimes themselves under the Einstein equations, see already Section 1.4.4.

In comparison to previous related work, there are two distinguishing features of the present approach.

The first distinguishing feature is our purely dyadic framework: Rather than relying on a global bootstrap based on time-weighted norms, the argument is entirely reduced to r^p-weighted but *time-translation-invariant* estimates to be applied in spacetime slabs of time length L which are outgoing null for large r. Global existence (and decay) is then inferred by proceeding iteratively in time in consecutive slabs of length $L = 2^i$, in the spirit of [Dafermos and Rodnianski 2010b]. No bootstrap is necessary for the iteration itself, and to estimate the i-th slab, no information is necessary to remember about the past other than information on the data of the slab itself. In this sense, the argument is truly dyadically localised. This yields a more streamlined proof even restricted to the semilinear case. (In fact, for both the semilinear and quasilinear cases, if $(\mathcal{M}, g_0)$ is a suitably small perturbation of Minkowski space, the time-translation-invariant estimate can be applied directly without iteration, and the approach proves global existence without explicit reference to time decay, cf. the recent [Facci and Metcalfe 2022]. In our formalism we shall always assume g_0 to be stationary, though we note that results can also be obtained on nonstationary perturbations of Minkowski space [Yang 2013].)

The second, and more fundamental, feature is our direct use of a "black box" linear inhomogeneous energy estimate (which in applications captures both complicated trapping phenomena as well as low-frequency obstructions) on the exactly stationary background, together with a physical-space top-order identity which may be applied directly to the quasilinear equation (1-1) and is in fact completely insensitive to trapping (and, when it holds, robust to perturbation of the metric g_0). In the example of Kerr, in fact for the full subextremal case $|a| < M$, our black box assumption follows directly from [Dafermos et al. 2016], while we show explicitly how to retrieve the additional top-order physical-space estimate in the case of $|a| \ll M$ (which in this context, can in fact be thought of simply as a perturbation of Schwarzschild) in Appendix A. The correct formulation of the companion estimate to be used in connection with [Dafermos et al. 2016] for the full subextremal case $|a| < M$ will appear elsewhere. (We emphasise that it is the phenomenon of *superradiance* which is the primary difficulty in producing this identity, not the complicated structure of trapping per se; for instance, the necessary top-order physical-space identity can be shown for general stationary spacetimes without horizons or ergospheres, irrespective of the structure of trapping.) The approach thus does not depend on whether or not there exists a purely physical-space based identity capturing trapping (something very fragile!) nor does it require a careful analysis of the commutation properties of frequency projections with the wave operator $\Box_{g(\psi,x)}$ of the time-dependent metric $g(\psi, x)$ corresponding to the actual solution ψ (something quite technical, which was successfully done, however, for instance in [Lindblad and Tohaneanu 2020]).

We emphasise that the role of the companion physical-space estimate is purely in order to deal with quasilinear terms. When our method is restricted to the semilinear case, i.e., when $g(\psi, x) = g_0$ identically in (1-1), we have that the black box linear inhomogeneous energy estimate can be applied on its own. In particular, in view of [Dafermos et al. 2016], our main theorem immediately applies to semilinear equations on Kerr in the full subextremal range $|a| < M$.

We note that the direct appeal to a linear "black box" statement is similar in spirit to [Metcalfe and Sogge 2005] for instance, where results on quasilinear equations outside obstacles in Minkowski space were studied under an exponential decay assumption concerning the linear problem.

The remainder of this introduction is structured as follows: In Section 1.1 below, we will discuss in more detail the assumptions we make on the background $(\mathcal{M}, g_0)$, introducing both the black box linear inhomogeneous estimate and the additional physical-space identity, followed by the assumptions on the nonlinearity in Section 1.2. We will then sketch our proof of the above theorem in Section 1.3, introducing our purely dyadic approach. Finally, we shall end in Section 1.4 with a general discussion, giving various extensions of the method, describing in more detail the relation with the nonlinear stability problem for black holes and comparing in particular with the case of backgrounds modelled on Schwarzschild–de Sitter and Kerr–de Sitter spacetimes and with the recent work [Mavrogiannis 2024].

1.1. *Assumptions on $(\mathcal{M}, g_0)$: "black box" estimates and physical-space identities.* We will make assumptions directly on the geometry of $(\mathcal{M}, g_0)$ and on properties of the linear inhomogeneous wave equation

$$\Box_{g_0}\psi = F \tag{1-2}$$

on the fixed background g_0.

1.1.1. *Geometric assumptions on $(\mathcal{M}, g_0)$.* The purely geometric assumptions on $(\mathcal{M}, g_0)$ will include assumptions concerning a suitable notion of asymptotic flatness, that of stationarity (existence of a Killing field T which is timelike near infinity), a smooth T-invariant strictly positive function $r \geq r_0$, which behaves like the Euclidean area-radius as $r \to \infty$, and the existence of a T-translation-invariant foliation of $\mathcal{M}$ by $\Sigma(\tau)$ for $\tau \geq 0$ hypersurfaces which are spacelike for $r \leq R$ and "outgoing" null for $r \geq R$. (In the case of Minkowski space, which will be the most basic example, we note that the function r will only coincide with the usual radial coordinate for large values.)

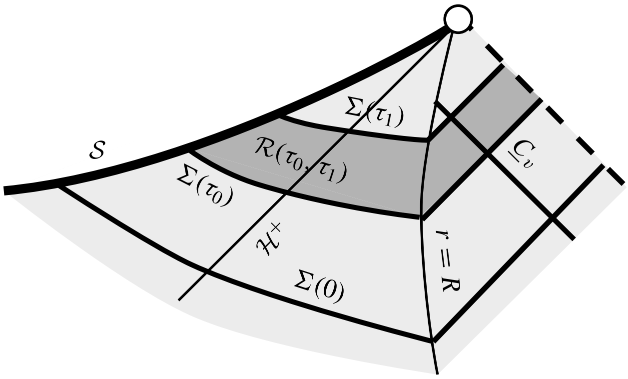

Figure 1. The underlying spacetime $(\mathcal{M}, g_0)$ in the case $\mathcal{S} \neq \varnothing$ with hypersurfaces and regions depicted.

To encompass the black hole cases of interest, we will allow for a possibly empty *space-like* future boundary $\mathcal{S} = \{r = r_0\}$ of $\mathcal{M}$, in which case we will also assume the presence of a Killing horizon $\mathcal{H}^+$ at $r = r_{\text{Killing}}$ for an $r_0 < r_{\text{Killing}}$, whose surface gravity will be required to be positive and whose Killing generator Z need not coincide with T (in which case however Z will be required to lie in the span of T and an additional assumed Killing vector field Ω_1). We will furthermore require that, for $r > r_{\text{Killing}}$, T (or more generally the span of T and Ω_1, if the latter is assumed Killing) is timelike. This will incorporate Schwarzschild and Kerr black holes in the full subextremal range $|a| < M$.

With minor modifications, we could also allow $\mathcal{S}$ to be a time-like boundary along which suitable boundary conditions are imposed, and this would allow one to consider waves outside of obstacles, cf. [Metcalfe and Sogge 2005].

We will denote by $\mathcal{R}(\tau_0, \tau_1)$ spacetime "slabs" $\mathcal{R}(\tau_0, \tau_1) = \bigcup_{\tau_0 \le \tau \le \tau_1} \Sigma(\tau) = J^-(\Sigma(\tau_1)) \cap J^+(\Sigma(\tau_0))$. The region $r \ge R$ will also be foliated by T-translation-invariant "ingoing" null hypersurfaces $\underline{C}_v$.

Refer to Figure 1 and already to Section 2 for a detailed discussion.

1.1.2. *Assumptions on* $\Box_{g_0}\psi = F$. In this section, we discuss the assumptions which we make at the level of equation (1-2).

Basic degenerate integrated local energy estimate. Our fundamental "black box" assumption at the level of solutions of the inhomogeneous linear equation (1-2) will be the validity of an integrated local energy estimate

$$\mathcal{F}(v), \quad \mathcal{E}(\tau) + c\int_{\tau_0}^{\tau_1} \overset{(-1-\delta)}{\chi\mathcal{E}'}(\tau')\,d\tau' + c\int_{\tau_0}^{\tau_1} \underset{-1}{\mathcal{E}'}(\tau')\,d\tau' \le \lambda\mathcal{E}(\tau_0) + C\int_{\mathcal{R}(\tau_0,\tau_1)} |(V_0^\mu \partial_\mu\psi + w_0\psi)F| + C\int_{\mathcal{R}(\tau_0,\tau_1)} F^2, \tag{1-3}$$

where $\mathcal{E}(\tau)$ denotes an energy flux through the hypersurface $\Sigma(\tau)$, $\mathcal{F}(v)$ denotes an energy flux through $\underline{C}_v \cap \mathcal{R}(\tau_0, \tau_1)$, $\lambda \ge 1$, $C > 0$, $c > 0$ are constants, and V_0 and w are a fixed vector field and function, respectively, and $\tau_0 \le \tau \le \tau_1$ are arbitrary. In Minkowski space, such estimates go back to Morawetz [1968].

The energy flux $\mathcal{E}(\tau)$ will control all first-order derivatives of ψ on the space-like part of $\Sigma(\tau)$ while it will only control all tangential derivatives on the null parts of $\Sigma(\tau)$; similarly, $\mathcal{F}(v)$ will control all tangential derivatives on $\underline{C}_v$. The fluxes will also control a zeroth-order term with weight r^{-2}. The two additional terms on the left-hand side are as follows: The quantity ${}^{\chi}\overset{(-1-\delta)}{\mathcal{E}'}(\tau')$ denotes an integral over $\Sigma(\tau')$ controlling all first-order derivatives of ψ, whose density is multiplied by $\chi r^{-1-\delta}$, where $0 \le \chi \le 1$ denotes a function which is allowed to degenerate but must satisfy $\chi = 1$ in a region to be discussed below, and where $\delta > 0$ can be taken to be an arbitrarily small constant which will be fixed throughout. The quantity $\underset{-1}{\mathcal{E}'}(\tau')$ denotes simply the zeroth-order quantity

$$\underset{-1}{\mathcal{E}'}(\tau) := \int_{\Sigma(\tau)} r^{-3-\delta}\psi^2.$$

The subscript -1 indicates that it is of order 1 less in differentiability than the other energies considered. We recall that, in our setup, what we define to be r satisfies $r \ge r_0 > 0$, and thus r weights are only relevant as $r \to \infty$.

In the case where $\mathcal{M}$ has no space-like boundary, i.e., $\mathcal{S} = \varnothing$, and T is globally timelike, our only assumption on χ will be that $\chi = 1$ for large r. In the case where $\mathcal{S} \ne \varnothing$, we will require that $\chi = 1$ for $r \le r_2$ for some $r_2 > r_{\text{Killing}}$. See already Section 3.2 for the detailed assumptions.

Let us note that estimate (1-3) with χ satisfying the above properties has indeed been shown in [Dafermos et al. 2016] in the Kerr case for the full subextremal range of parameters $|a| < M$. (See already Theorem D.1.) One essential feature in deriving (1-3) in Kerr is the fact that the trapped null geodesics of g_0, corresponding to photons in bound orbits around the black hole, are unstable in a suitable sense. We note that one can show in general that estimate (1-3) in the above form *cannot* hold in the presence of stable trapping. (Thus, (1-3) in particular constrains properties of geodesic flow which can be expressed purely geometrically in terms of g_0.) Similarly, the possibility of an estimate (1-3) with $\chi = 1$ in a neighbourhood of $\mathcal{H}^+$ is related to the nondegenerate property of the horizon of subextremal Kerr and its celebrated local *red-shift effect*. See [Dafermos and Rodnianski 2013].

Physical-space energy identities. One way to try to prove (1-3) is through a physical-space energy *identity* arising from integrating the divergence identity

$$\nabla^\mu J^{V,w,q,\varpi}_\mu[\psi] = K^{V,w,q}[\psi] + (V^\mu \partial_\mu \psi)F + w\psi F \tag{1-4}$$

for well-chosen currents $J^{V,w,q,\varpi}[\psi, g_0]$ (associated to a vector field V, a scalar function w, a 1-form q and a 2-form ϖ) with (degenerate) coercivity properties for the arising bulk and boundary terms. In this case, if one defines

$$\mathfrak{E}(\tau) = \int_{\Sigma(\tau)} J^{V,w,q,\varpi}_\mu[\psi] n^\mu_{\Sigma(\tau)}, \quad \mathfrak{F}(v) = \int_{\underline{C}_v \cap \mathcal{R}(\tau_0,\tau_1)} J^{V,w,q,\varpi}_\mu[\psi] n^\mu_{\underline{C}_v},$$

then one has $\mathfrak{E}(\tau) \sim \mathcal{E}(\tau)$, $\mathfrak{F}(v) \sim \mathcal{F}(v)$, and one can reexpress (1-3) with $\mathfrak{E}(\tau)$ and $\mathfrak{E}(\tau_0)$ replacing $\mathcal{E}(\tau)$ and $\mathcal{E}(\tau_0)$, respectively, but with λ in (1-3) now equal to 1. This is precisely the situation in the Schwarzschild case, where appropriate coercive purely physical-space identities of the form (1-4) can be deduced from [Dafermos and Rodnianski 2007a; Marzuola et al. 2010].

If (1-3) indeed holds as a consequence of the coercivity properties of a divergence identity (1-4), then this situation appears very advantageous for nonlinear applications. The advantage of such coercive identities (1-4) is that they are robust and easily amenable to *direct* application to the quasilinear (1-1). This is because the latter can be viewed as an inhomogeneous wave equation *with respect to the wave operator* $\Box_{g(\psi,x)}$ *corresponding to the solution itself*, and both the existence of energy identities like (1-4) and their coercivity properties are stable to passing from g_0 to $g(\psi, x)$, i.e., still hold for $J^{V,w,q,\varpi}[g(\psi, x), \psi]$.

If one only has estimate (1-3) as a "black box", i.e., not proven via a physical-space identity (1-4), one may still of course apply the estimate to (1-1) by rewriting (1-1) as an inhomogeneous equation with respect to $\Box_{g_0}$:

$$\Box_{g_0}\psi = (\Box_{g_0} - \Box_{g(\psi,x)})\psi + N(\partial\psi, \psi, x). \tag{1-5}$$

Applied in this manner, however, the resulting estimate *loses derivatives* in view of the quasilinear second-order terms on the right-hand side of (1-5). (Note in contrast that in the purely semilinear case, where $g(\psi, x) = g_0$, there is no such loss in estimating (1-5), and we may base our argument entirely on (1-3). See already Section 1.4.1.) At first glance, this loss would appear to present a fundamental difficulty.

An auxiliary physical-space estimate. Here we come to the key observation of the present approach: to avoid the loss of derivatives described above, one does *not* need that (1-3) be proven via a physical-space identity (1-4); one only needs a much weaker (from the coercivity point of view) estimate arising from (1-4) which need not imply (1-3) but can be used *in conjunction* with (1-3).

Specifically, our main assumption is that, in addition to (1-3), one has a physical-space identity (1-4) with (a) coercive boundary terms and with (b) bulk terms which are only assumed to be nonnegative *at highest order*, i.e., up to lower-order "error" terms. Importantly, however, these allowed lower-order error terms must moreover be supported entirely in the region where the degeneration function χ of (1-3) satisfies $\chi = 1$, i.e., where the estimate of (1-3) is in fact nondegenerate. The identity (1-4) applied to (1-2), on its own, gives rise thus to an estimate of the form

$$\mathfrak{F}(v), \quad \mathfrak{E}(\tau) + c\int_{\tau_0}^{\tau_1} {}^{\rho}\overset{(-1-\delta)}{\mathcal{E}'}(\tau')\,d\tau' + c\int_{\tau_0}^{\tau_1} {}^{\rho}\underset{-1}{\mathcal{E}'}(\tau')\,d\tau' \\ \leq \mathfrak{E}(\tau_0) + A\int_{\tau_0}^{\tau_1} \overset{\xi}{\underset{-1}{\mathcal{E}'}}(\tau')\,d\tau' + \int_{\mathcal{R}(\tau_0,\tau_1)} |(V^{\mu}\partial_{\mu}\psi + w\psi)F| \tag{1-6}$$

for all $\tau \in [\tau_0, \tau_1]$ and sufficiently large v, where ${}^{\rho}\overset{(-1-\delta)}{\mathcal{E}'}$ is defined similarly to ${}^{\chi}\overset{(-1-\delta)}{\mathcal{E}'}$ but with degeneration function ρ which in general vanishes on a bigger set than χ. The zeroth-order term ${}^{\rho}\underset{-1}{\mathcal{E}'}$ on the left-hand side also degenerates, unlike the analogous term in (1-3). The presence of the term multiplying the new constant A on the right-hand side is necessary in view of the fact that nonnegativity is only assumed for the highest-order terms. Here, $\overset{\xi}{\underset{-1}{\mathcal{E}'}}$ is a zeroth-order energy whose density is supported only in the support of a function $\xi(r)$:

$$\overset{\xi}{\underset{-1}{\mathcal{E}'}}(\tau) := \int_{\Sigma(\tau)} \xi(r)\psi^2.$$

Importantly, in view of our comments above, we will require that ξ vanishes identically where χ of (1-3) degenerates, i.e., $\xi = 0$ where $\chi \neq 1$. We will also require ξ to vanish for large r and, in the case where $\mathcal{S} \neq \varnothing$, in the region $r \leq r_2$.

In addition, we will make on ρ the same asymptotic nonvanishing assumption that we made previously on χ; namely that $\rho = 1$ for large r and, in the presence of a nonempty space-like boundary $\mathcal{S} \neq \varnothing$, that $\rho = 1$ in $r \leq r_2$. See already Section 3.4.3 for the detailed assumptions and Figure 2 therein for a depiction of the supports of χ, ρ and ξ.

We note that, in the case where $\mathcal{S} = \varnothing$ and T is globally timelike, one can in fact easily construct a current satisfying (a) and (b) and leading to (1-6) *irrespective of the structure of trapping and the validity of an estimate of the form* (1-3). We will show in Appendix A that, for the very slowly rotating Kerr case $|a| \ll M$, although there is no globally time-like T, there is a similar elementary construction, based essentially only on the fact that superradiance is effectively governed by a small parameter and the ergoregion is confined to a part of spacetime which can be understood using only the red-shift effect (cf. the original treatment of the boundedness problem on Kerr [Dafermos and Rodnianski 2011]). In general, however, the main difficulty in proving (1-6) is ensuring the boundary coercivity (a).

Though on its own estimate (1-6) is clearly weaker than (1-3), its robustness to direct application to (1-1) allows one to avoid the above loss of derivatives, when (1-6) is used in conjunction with (1-3) applied to (1-5). Before turning to describe how this is done in the context of the proof, we will in fact have to slightly strengthen our assumed estimates (1-3) and (1-6) as follows.

Extending to r^p-weighted estimates. First of all, we will need to extend (1-3) and (1-6) to r^p-weighted estimates, for $0 \leq p < 2$, of the type first introduced in [Dafermos and Rodnianski 2010b]:

$$\overset{(p)}{\mathcal{E}}(\tau) + \overset{(p)}{\mathcal{F}}(v) + \int_{\tau_0}^{\tau_1} {}^{\chi}\overset{(p-1)}{\mathcal{E}'}(\tau')\, d\tau' + \int_{\tau_0}^{\tau_1} \underset{-1}{\overset{(p-1)}{\mathcal{E}'}}(\tau')\, d\tau' \lesssim \overset{(p)}{\mathcal{E}}(\tau_0) + \text{inhomogeneous terms}, \tag{1-7}$$

$$\overset{(p)}{\mathfrak{F}}(v), \quad \overset{(p)}{\mathfrak{E}}(\tau) + c\int_{\tau_0}^{\tau_1} {}^{\rho}\overset{(p-1)}{\mathcal{E}'}(\tau')\, d\tau' + c\int_{\tau_0}^{\tau_1} {}^{\rho}\underset{-1}{\mathcal{E}'}(\tau')\, d\tau'$$
$$\leq \overset{(p)}{\mathfrak{E}}(\tau_0) + A\int_{\tau_0}^{\tau_1} \underset{-1}{\overset{\xi}{\mathcal{E}'}}(\tau')\, d\tau' + \text{inhomogeneous terms}. \tag{1-8}$$

In the above, $\overset{(p)}{\mathcal{E}}$ (resp. ${}^{\chi}\overset{(p-1)}{\mathcal{E}'}$, etc.) are r^p (resp. r^{p-1}, etc.) weighted versions of $\mathcal{E}$ (resp. ${}^{\chi}\overset{(-1-\delta)}{\mathcal{E}'}$, etc.), while $\overset{(p)}{\mathfrak{E}}$ and $\overset{(p)}{\mathfrak{F}}$ satisfy $\overset{(p)}{\mathfrak{E}} \sim \overset{(p)}{\mathcal{E}}$ and $\overset{(p)}{\mathfrak{F}} \sim \overset{(p)}{\mathcal{F}}$ but can moreover be represented as the flux term of a current,

$$\overset{(p)}{\mathfrak{E}}(\tau) = \int_{\Sigma(\tau)} \overset{(p)}{J}{}^{V,w,q,\varpi}_{\mu}[\psi] n^{\mu}, \qquad \overset{(p)}{\mathfrak{F}}(v) = \int_{\underline{C}_v} \overset{(p)}{J}{}^{V,w,q,\varpi}_{\mu}[\psi] n^{\mu},$$

and we in fact assume that (1-8), just as (1-6), is the result of a pointwise coercive energy identity (1-4) associated to the current $\overset{(p)}{J}$.

To obtain (1-7) and (1-8), given (1-3) and (1-6), it suffices to assume the existence of currents defined only in the region $r \geq \widetilde{R}$ with suitable far-away coercivity properties. For maximal generality, we will here simply directly postulate the existence of such currents as an additional assumption. See already

Section 3.5 for the precise formulation. This assumption can be shown to follow from suitable pointwise asymptotically flat assumptions on the metric g_0 and holds of course in all our examples of interest, where the currents can easily be explicitly constructed. See Appendix B, and also [Moschidis 2016], for an even more general setting.

Extending to higher-order estimates. Secondly, we will need, through suitable commutations, to extend (1-3), (1-6), (1-7) and (1-8) to higher-order statements. We will distinguish the two energies

$$\overset{(p)}{\underset{k}{\mathcal{E}}} \quad \text{and} \quad \overset{(p)}{\underset{k}{\mathfrak{E}}}$$

at k-th order.

The energy $\overset{(p)}{\underset{k}{\mathcal{E}}}$ is the sum of the usual energy, with r^p weights, $0 \leq p < 2$, and with commutation vector fields $\widetilde{\mathfrak{D}}^k$ up to order k, *spanning the entire tangent space*. The set will include the vector fields Ω_i, $i = 1, \dots, 3$, which for large r correspond to the usual rotational vector fields. (Note that the latter will be the only commutation vector fields which are r-weighted.) This is a fundamental energy with respect to which initial data can be measured and with respect to which suitable Sobolev inequalities hold.

The energy $\overset{(p)}{\underset{k}{\mathfrak{E}}}$, on the other hand, denotes the precise energy flux of the current leading to (1-4) applied moreover with a new set $\mathfrak{D}^k$ of commutation vector fields, which will include T (and Ω_1, if this is assumed to be Killing), but for which *all non-Killing vectors are cut off to vanish in a suitable region of finite r*. (See already Sections 3.6.1–3.6.4 for the definition of these commutation vector fields and energies.) Nonetheless, by our geometric assumptions and elliptic estimates, we will have that the energies

$$\overset{(p)}{\underset{k}{\mathfrak{E}}} \sim \overset{(p)}{\underset{k}{\mathcal{E}}} \tag{1-9}$$

are equivalent for solutions of $\Box_{g_0}\psi = 0$.

To extend our estimates to higher order, we will need some properties of the commutation errors arising from $[\Box_{g_0}, \mathfrak{D}^k]$. Let us note that, in the case where $\mathcal{S} = \varnothing$ and T is globally timelike, these errors are only supported in the asymptotic region of large r and can be controlled if the metric is suitably asymptotically flat. For maximum generality, we will formulate the relevant asymptotic assumption directly in terms of pointwise decay bounds for these commutators. This will include all examples of interest. See already Section 3.6.2.

In the case where $\mathcal{S} \neq \varnothing$, there will be additional commutation errors $[\Box_{g_0}, \mathfrak{D}^k]$ arising in a neighbourhood of the boundary $\mathcal{S}$ including the Killing horizon $\mathcal{H}^+$. Key to their control is the assumption of positive surface gravity of $\mathcal{H}^+$, discussed in Section 1.1.1, and the inclusion among the collection $\mathfrak{D}$ of a well-chosen vector field Y which is null and transversal to $\mathcal{H}^+$. As first shown in [Dafermos and Rodnianski 2013], commutation by Y generates a term with a good sign at $\mathcal{H}^+$ (see already Proposition 3.6.1 of Section 3.6.3), and this fact, together with elliptic estimates in $r > r_{\text{Killing}}$, an enhanced red-shift estimate near $\mathcal{H}^+$ and enhanced positivity in the black hole interior up to $\mathcal{S}$ (see already Propositions 3.4.1 and 3.4.2 of Section 3.4.5), can be used to absorb the errors arising from commutation and to extend the estimates to higher order. We note that the above-mentioned good sign is yet another manifestation of the local red-shift at the horizon $\mathcal{H}^+$ associated to the positivity of the surface gravity. To understand how all commutation errors can indeed be absorbed, see already Section 3.6.6.

The final extended estimates on (1-2) resulting from (1-3) and (1-6) take the form

$$\overset{(p)}{\underset{k}{\mathcal{E}}}(\tau)+\overset{(p)}{\underset{k}{\mathcal{F}}}+\int_{\tau_0}^{\tau_1}{}^{\chi}\overset{(p-1)}{\underset{k}{\mathcal{E}}}{}'(\tau')\,d\tau'+\int_{\tau_0}^{\tau_1}\overset{(p-1)}{\underset{k-1}{\mathcal{E}}}{}'(\tau')\,d\tau'\lesssim\overset{(p)}{\mathcal{E}}(\tau_0)+\text{inhomogeneous terms,}\tag{1-10}$$

$$\begin{aligned}\overset{(p)}{\underset{k}{\mathfrak{F}}}(v),\quad \overset{(p)}{\underset{k}{\mathfrak{E}}}(\tau)+c\int_{\tau_0}^{\tau_1}{}^{\rho}\overset{(p-1)}{\underset{k}{\mathcal{E}}}{}'(\tau')\,d\tau'+c\int_{\tau_0}^{\tau_1}{}^{\rho}\overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}}}{}'(\tau')\,d\tau'&\\ \le\overset{(p)}{\underset{k}{\mathfrak{E}}}(\tau_0)+A\int_{\tau_0}^{\tau_1}\overset{\xi}{\underset{k-1}{\mathcal{E}}}{}'(\tau')\,d\tau'+\text{inhomogeneous terms,}&\end{aligned}\tag{1-11}$$

where again (1-11) derives from the pointwise coercivity properties of the energy identity (1-4) associated to a current $\overset{(p)}{\underset{k}{J}}$, whose flux terms are precisely given by $\overset{(p)}{\underset{k}{\mathfrak{E}}}$ and $\overset{(p)}{\underset{k}{\mathfrak{F}}}$. See already Section 3.6.8 for the precise form of the estimates and inhomogeneous terms.

1.2. ***Assumptions on the nonlinearities $g(\psi, x)$ and $N(\partial\psi, \psi, x)$.*** We now turn to the nonlinear equation (1-1). In formulating assumptions on the allowed nonlinearities, we are here largely motivated by geometric formulations of the Einstein equations, as in [Dafermos et al. 2021], which produce "almost decoupled" equations similar to (1-1), which satisfy an appropriate form of the null condition. We emphasise, however, that the resulting equations in such a formulation do not constitute a pure system of wave equations in the form (1-1), and the quasilinear structure is mitigated through coupling with transport equations. Thus, in this context, one should really only think of (1-1) as an indicative model equation. For more specific discussion of the Einstein equations, see already Section 1.4.4.

In addressing the combined difficulties of the interaction of quasilinear terms, trapped null geodesics, and merely polynomial decay at the level of model scalar equations of the form (1-1), the most natural simplest setting is to require that the *quasilinear* term $g(\psi, x) - g_0(x)$ be supported entirely in the region $r \le R$, and to impose on the *semilinear* term $N(\partial\psi, \psi, x)$ a generalised version of the null condition [Klainerman 1986]. This also allows us to use the exact null hypersurfaces of the background $(\mathcal{M}, g_0)$ without additional complications, making the null hierarchical structure of the estimates clearer. (In the case of the Einstein equations, in our framework, these would be replaced by null hypersurfaces of the actual dynamic spacetime $(\mathcal{M}, g)$, for instance associated to a double-null gauge.) We note, however, that use of exact null hypersurfaces is in no way fundamental; one can always replace these with suitable hyperboloidal hypersurfaces, for instance, using the setup of [Moschidis 2016].

For maximum generality, rather than attempt a general algebraic definition of the null condition for the semilinear terms $N(\partial\psi, \psi, x)$ of (1-1), we will formalise a "null condition assumption" in the form of a time-translation-invariant r^p-weighted estimate for the inhomogeneous terms of (1-10)–(1-11) when applied to (1-1), restricted entirely to the asymptotic region $r \ge R$.

To describe the condition, it will be convenient to introduce "master" energies

$${}^{\rho}\overset{(p)}{\underset{k}{\mathcal{X}}}(\tau_0,\tau_1):=\sup_v\overset{(0)}{\underset{k}{\mathcal{F}}}(v)+\sup_{\tau_0\le\tau\le\tau_1}\overset{(p)}{\underset{k}{\mathcal{E}}}(\tau)+\int_{\tau_0}^{\tau_1}\Big({}^{\rho}\overset{(p-1)}{\underset{k}{\mathcal{E}}}{}'(\tau)+{}^{\rho}\overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}}}{}'(\tau)\Big)\,d\tau\tag{1-12}$$

for all $\delta \le p \le 2-\delta$. In the case $p = 0$, which is anomalous, we will define

$$\begin{aligned}{}^{\rho}\overset{(0)}{\underset{k}{\mathcal{X}}}(\tau_0,\tau_1)&:=\sup_v\overset{(0)}{\underset{k}{\mathcal{F}}}(v)+\sup_{\tau_0\le\tau\le\tau_1}\overset{(0)}{\mathcal{E}}(\tau)+\int_{\tau_0}^{\tau_1}\Big({}^{\rho}\overset{(-1-\delta)}{\underset{k}{\mathcal{E}}}{}'(\tau)+{}^{\rho}\overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}}}{}'(\tau)\Big)\,d\tau,\\ {}^{\rho}\overset{(0+)}{\underset{k}{\mathcal{X}}}(\tau_0,\tau_1)&:=\sup_v\overset{(0)}{\underset{k}{\mathcal{F}}}(v)+\sup_{\tau_0\le\tau\le\tau_1}\overset{(0)}{\mathcal{E}}(\tau)+\int_{\tau_0}^{\tau_1}\Big({}^{\rho}\overset{(\delta-1)}{\underset{k}{\mathcal{E}}}{}'(\tau')+{}^{\rho}\overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}}}{}'(\tau')\Big)\,d\tau.\end{aligned}\tag{1-13}$$

We may define similar master energies ${}^{\chi}\overset{(p)}{\underset{k}{\mathcal{X}}}$, etc., with χ replacing ρ, and where extra nondegenerate lower-order terms $\overset{(p-1)}{\underset{k}{\mathcal{E}'}}$ and $\overset{(-1-\delta)}{\underset{k-1}{\mathcal{E}'}}$ are added to the integrands on the right-hand side of (1-12) and (1-13), respectively, and finally $\overset{(p)}{\underset{k}{\mathcal{X}}}$, where ρ is removed. Let us note that

$$ {}^{\rho}\overset{(p)}{\underset{k}{\mathcal{X}}} \lesssim {}^{\chi}\overset{(p)}{\underset{k}{\mathcal{X}}} \lesssim \overset{(p)}{\underset{k}{\mathcal{X}}}, \qquad \overset{(p)}{\underset{k-1}{\mathcal{X}}} \lesssim {}^{\chi}\overset{(p)}{\underset{k}{\mathcal{X}}}. $$

Specifically, our null condition assumption is that in any spacetime slab $\mathcal{R}(\tau_0, \tau_1)$, the far-away (i.e., $r \geq R$) contribution of the inhomogeneous terms arising from $N(\partial\psi, \psi, x)$ in the estimates (1-10) and (1-11) can be bounded by the expressions

$$ {}^{\rho}\overset{(p)}{\underset{k}{\mathcal{X}}}(\tau_0, \tau_1)\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}(\tau_0, \tau_1)} + \sqrt{{}^{\rho}\overset{(p)}{\underset{k}{\mathcal{X}}}(\tau_0, \tau_1)}\sqrt{{}^{\rho}\overset{(0)}{\underset{k}{\mathcal{X}}}(\tau_0, \tau_1)}\sqrt{\overset{(p)}{\underset{\ll k}{\mathcal{X}}}(\tau_0, \tau_1)} \quad \text{and} \quad {}^{\rho}\overset{(0+)}{\underset{k}{\mathcal{X}}}(\tau_0, \tau_1)\sqrt{\overset{(0+)}{\underset{\ll k}{\mathcal{X}}}(\tau_0, \tau_1)} \tag{1-14} $$

for $\delta \leq p \leq 2 - \delta$ and $p = 0$, respectively, and where all energies may in fact be restricted to the region $r \geq \frac{8}{9}R$, where $\rho = 1$. (Note how the $p = 0$-weighted estimate is anomalous, in that the nonlinear terms require the boundedness of a bulk term associated to the energy identity of a higher ($p > 0$) weight so as to be estimated.) See already Section 4.7 for the precise formulation of the assumption.

The assumption that one can bound the relevant inhomogeneous terms arising from $N(\partial\psi, \psi, x)$ by (1-14) isolates precisely that aspect of the usual null condition which is relevant for global existence in our method. We will then show in Appendix C that our assumption indeed holds in particular for the usual nonlinearities allowed by the classical null condition [Klainerman 1986] and includes also the class of semilinear terms $N(\partial\psi, \psi, x)$ on Kerr studied previously in [Luk 2013].

1.3. ***The dyadically localised approach to global existence*: *sketch of the proof of the main theorem.*** We now turn to the proof of the main theorem and the other novel aspect of the present work, namely the replacement of a global bootstrap built on global decay-in-time estimates with dyadic iteration in consecutive spacetime slabs based entirely on dyadically localised, r^p-weighted — but time-translation-invariant — estimates.

We sketch the argument here. For details, see already Section 6. In our discussion below, we will focus directly on our main case of interest, referred to later in the paper as case (iii), where the black box inhomogeneous estimate (1-3) has nontrivial degeneration and indeed does not arise from a physical-space identity, and thus must be used in conjunction with the auxiliary identity (1-6). We emphasise, however, that the dyadic approach described here is already novel in the case, referred to later in the paper as case (ii), where the black box estimate (1-3) again has degeneration but is actually a consequence of a physical-space identity (as in Schwarzschild; see the discussion following (1-4)). Finally, in the case where the black box estimate (1-3) derives from a physical-space identity and is moreover nondegenerate (i.e., $\chi = 1$ identically), referred to in the paper as case (i), then dyadic iteration is in fact unnecessary, and the approach reduces to a completely elementary time-translation-invariant estimate. (This latter case applies for instance when g_0 is a small stationary perturbation of Minkowski space.) *In the paper*, *we will provide self-contained treatments of all three cases so the reader can compare with the more traditional approach.*

1.3.1. *The r^p-weighted estimate hierarchy on a spacetime slab of time length L.* In the dyadic approach, the key element is an appropriate time-translation-invariant r^p-weighted hierarchy on a slab of time length L. The hierarchy will be formed by combining the r^p and higher k-order estimates associated to (1-3) and (1-6) with various choices of p weights and differentiability-order k. The estimates will depend only on a time-translation-invariant bootstrap assumption, to be retrieved within the slab, concerning only lower-order energy quantities.

Specifically, from the assumptions of Sections 1.1 and 1.2, on any spacetime slab $\mathcal{R}(\tau_0, \tau_0 + L)$ of time length $L \geq 1$, we obtain, for $p = 2 - \delta$ and $p = 1$, and for $0 < \delta < \frac{1}{10}$, the following hierarchy of estimates:

$$\overset{(p)}{\underset{k}{\mathfrak{F}}}(v), \quad \overset{(p)}{\underset{k}{\mathfrak{E}}}(\tau), \quad {}^{\rho}\overset{(p)}{\underset{k}{\mathcal{X}}}(\tau_0, \tau) \leq \overset{(p)}{\underset{k}{\mathfrak{E}}}(\tau_0) + \boxed{A \int_{\tau_0}^{\tau} \overset{\xi}{\underset{k-1}{\mathcal{E}'}}} + C\, {}^{\rho}\overset{(p)}{\underset{k}{\mathcal{X}}}(\tau_0, \tau) \sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}(\tau_0, \tau)}$$
$$+ C \sqrt{{}^{\rho}\overset{(p)}{\underset{k}{\mathcal{X}}}(\tau_0, \tau)} \sqrt{{}^{\rho}\overset{(0)}{\underset{k}{\mathcal{X}}}(\tau_0, \tau)} \sqrt{\overset{(p)}{\underset{\ll k}{\mathcal{X}}}(\tau_0, \tau)}$$
$$+ \boxed{C \sup_{\tau_0 \leq \tau' \leq \tau} \overset{(0)}{\mathcal{E}}(\tau') \sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}(\tau_0, \tau)} \sqrt{L}}, \tag{1-15}$$

$${}^{\chi}\overset{(p)}{\underset{k-1}{\mathcal{X}}}(\tau_0, \tau) \lesssim \overset{(p)}{\underset{k-1}{\mathcal{E}}}(\tau_0) + {}^{\rho}\overset{(p)}{\underset{k-1}{\mathcal{X}}}(\tau_0, \tau) \sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}(\tau_0, \tau)} + \sqrt{{}^{\rho}\overset{(p)}{\underset{k-1}{\mathcal{X}}}(\tau_0, \tau)} \sqrt{{}^{\rho}\overset{(0)}{\underset{k-1}{\mathcal{X}}}(\tau_0, \tau)} \sqrt{\overset{(p)}{\underset{\ll k}{\mathcal{X}}}(\tau_0, \tau)}$$
$$+ \boxed{\sup_{\tau_0 \leq \tau' \leq \tau} \overset{(0)}{\mathcal{E}}(\tau')} \sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}(\tau_0, \tau)} \sqrt{L}. \tag{1-16}$$

The estimates are in fact contingent on an appropriate time-translation-invariant bootstrap assumption on the lower-order energy

$$\overset{(p)}{\underset{\ll k}{\mathcal{X}}}(\tau_0, \tau) \lesssim \varepsilon \tag{1-17}$$

for $p = 0$. The hierarchy (1-15)–(1-16) descends also to $p = 0$, but where

$${}^{\rho}\overset{(0+)}{\underset{k}{\mathcal{X}}}(\tau_0, \tau) \sqrt{\overset{(0+)}{\underset{\ll k}{\mathcal{X}}}(\tau_0, \tau)} \quad \text{and} \quad {}^{\rho}\overset{(0+)}{\underset{k-1}{\mathcal{X}}}(\tau_0, \tau) \sqrt{\overset{(0+)}{\underset{\ll k}{\mathcal{X}}}(\tau_0, \tau)}$$

replace the sum of the third-to-last and second-to-last terms of (1-15) and (1-16), respectively. (This anomaly, referred to already immediately after (1-14), is a fundamental aspect of the estimates.) See already Section 6.3.1.

Estimate (1-15) arises from applying the physical-space identity associated to the current $\overset{(p)}{\underset{k}{J}}$, which in the linear case led to (1-11), *directly* to the k-times commuted equation (1-1), i.e., it arises from the divergence identity (1-4) associated to the current $\overset{(p)}{\underset{k}{J}}[g(\psi, x), \psi]$, where $g(\psi, x)$ replaces g_0 covariantly in the definition of $\overset{(p)}{\underset{k}{J}}$. (The energies $\overset{(p)}{\underset{k}{\mathfrak{E}}}$ and $\overset{(p)}{\underset{k}{\mathfrak{F}}}$ in (1-15) now in fact denote the exact fluxes arising from these currents; in view of the bootstrap assumption (1-17), one can again show the equivalence (1-9) for solutions of (1-1).) As discussed already in Section 1.1.2, it follows that (1-15) does not "lose derivatives", as is clear from examining the order of differentiability of the terms on the right-hand side. We remark that the first boxed term in (1-15) reflects the analogous term in (1-11). On the right-hand side of (1-15), we recognise the bound for the far-away contribution of the nonlinear term (1-14) from our assumption capturing the null condition, while the second boxed term in (1-15) is necessary to control the nonlinear terms on the set where ρ degenerates, for there, this boxed term cannot be absorbed in the spacetime bulk term on the left-hand side. (The bad explicit dependence on the length L arises because one must take the supremum over τ for the energy of the highest-order terms there.)

Estimate (1-16), on the other hand, arises from applying the "black box" estimate (1-10) to the nonlinear equation written in the form (1-5), i.e., now thought of simply as an inhomogeneous equation with respect to the background g_0. The boxed term in (1-16), which is k-th order, reflects the loss of derivatives due to the quasilinearity discussed already in Section 1.1.2.

The estimates can in principle close because, in view of our restrictions on the support of ξ, we have the fundamental relation

$$\underset{k-1}{\overset{\xi}{\mathcal{E}}}{}' \lesssim {}^{\chi}\underset{k-1}{\overset{(-1-\delta)}{\mathcal{E}}}{}' + \underset{k-2}{\overset{(-1-\delta)}{\mathcal{E}}}{}', \tag{1-18}$$

and thus we may bound the term

$$A\int_{\tau_0}^{\tau} \underset{k-1}{\overset{\xi}{\mathcal{E}}}{}'(\tau')\,d\tau' \lesssim {}^{\chi}\underset{k-1}{\overset{(0)}{\mathcal{X}}}(\tau_0,\tau),$$

which is in turn bounded by the term on the left-hand side of (1-16) for any p.

1.3.2. *Global existence in a slab.* The first task is to show global existence in a given slab $\mathcal{R}(\tau_0, \tau_0 + L)$ for arbitrary $L \geq 1$, given suitable (L-dependent) smallness assumptions at $\Sigma(\tau_0)$.

We first note that the estimate (1-15) alone can be used to easily show *local* existence, by which we mean existence in an entire smaller slab of the form $\mathcal{R}(\tau_0, \tau_0 + \epsilon)$, provided that some $\underset{k}{\overset{(p)}{\mathcal{E}}} \lesssim \varepsilon_0$ for sufficiently high k. This is in fact true also for $p = 0$. (Note that this "local" result already uses nontrivially the null condition assumption of Section 1.2. This is because our foliation $\Sigma(\tau)$ is outgoing null for $r \geq R$. Equations (1-1) not satisfying some version of the null condition can fail to yield solutions in $\mathcal{R}(\tau_0, \tau_0 + \epsilon)$ for any $\epsilon > 0$, no matter how small the data are.) Moreover, we show that smallness of a suitably high $\underset{k}{\overset{(p)}{\mathcal{E}}}$ norm defines a continuation criterion.

In view of the above, for global existence, it suffices to show that, under suitable assumptions at $\tau = \tau_0$, the quantity $\underset{k}{\overset{(p)}{\mathcal{E}}}$, for some $p \in \{0\} \cup [\delta, 2 - \delta]$ and suitably high k, remains globally small in the slab. Examining the hierarchy, for global existence in a slab $\mathcal{R}(\tau_0, \tau_0 + L)$ of length L, it seems necessary to have at least

$$\underset{\ll k}{\overset{(0)}{\mathcal{X}}}(\tau_0,\tau) \lesssim \varepsilon L^{-1}. \tag{1-19}$$

Indeed, in view of the nonlinear terms, the L^{-1} factor in (1-19) represents a natural threshold, as $s = 1$ represents the minimum value of s for which $\sqrt{L^{-s}}\sqrt{L} \lesssim 1$, allowing the quantities in a fixed slab to be easily bounded by their initial values.

Assuming

$$\underset{k}{\overset{(1)}{\mathcal{E}}}(\tau_0) \lesssim \varepsilon_0, \quad \underset{k-2}{\overset{(0)}{\mathcal{E}}}(\tau_0) \lesssim L^{-1}\varepsilon_0, \tag{1-20}$$

which of course imply, for $\varepsilon_0 \ll \varepsilon$, both (1-17) (for $p = 1$) and (1-19) at $\tau = \tau_0$, an easy continuity argument, with (1-17) and (1-19) taken as bootstrap assumptions, shows that if estimates (1-20) hold for $\tau = \tau_0$, then the solution indeed exists globally in the entire slab $\mathcal{R}(\tau_0, \tau_0 + L)$.

See already Proposition 6.3.2 for the details of the proof. Note that our above appeal to (1-17) and (1-19) is in fact the only instance that a bootstrap assumption must be used in the proof of our main theorem.

1.3.3. *Improved estimates and the pigeonhole principle.* Once global existence within a slab has been established, one can improve a posteriori the estimates given stronger assumptions on initial data. Anticipating dyadic iteration, one seeks a set of initial estimates at $\tau = \tau_0$ with the property that they are *exactly recoverable* at the top of the slab $\tau_0 + L$, with suitable redefinition of ε_0 and with L replaced by $2L$.

Re-examining the hierarchy (1-15)–(1-16), we show that, for an appropriate large constant α and any $L \geq 1$, $\tau_0 \geq 1$, for $\hat{\epsilon}_0$ sufficiently small and given for example the stronger estimates

$$\underset{k}{\overset{(1)}{\mathfrak{E}}}(\tau_0) \leq \hat{\varepsilon}_0, \quad \underset{k-2}{\overset{(2-\delta)}{\mathfrak{E}}}(\tau_0) \leq \hat{\varepsilon}_0, \quad \underset{k-2}{\overset{(0)}{\mathfrak{E}}}(\tau_0) \leq \hat{\varepsilon}_0 \alpha L^{-1}, \quad \underset{k-4}{\overset{(1)}{\mathfrak{E}}}(\tau_0) \leq \hat{\varepsilon}_0 \alpha L^{-1+\delta}, \quad \underset{k-6}{\overset{(0)}{\mathfrak{E}}}(\tau_0) \leq \hat{\varepsilon}_0 \alpha^2 L^{-2+\delta}, \tag{1-21}$$

we have that (1-21) holds also at the final time $\tau_0 + L$, where τ_0, L and $\hat{\varepsilon}_0$ are now however replaced by

$$\tau_0 + L, \quad 2L \quad \text{and} \quad \hat{\varepsilon}_0(1 + \alpha L^{-\frac{1}{4}}), \tag{1-22}$$

respectively. Note that the statement that the last three inequalities of (1-21) hold at $\tau_0 + L$ with $2L$ in place of L is a *stronger* smallness assumption than that assumed for these quantities at τ_0, signifying that these quantities have in fact decayed, and relies on a pigeonhole argument as in [Dafermos and Rodnianski 2010b], based on the fundamental relation that, for $p - 1 \geq 0$, the *bulk* integrand $\underset{k-2}{\overset{(p-1)}{\mathcal{E}'}}$ of the p-weighted identity (1-16) at order $k-1$ (when integrated, included in the $\chi \underset{k-2}{\overset{(p-1)}{\mathcal{X}}}$ master energy) controls the *boundary* term of the $(p-1)$-weighted identity at arbitrary order k, denoted without the prime:

$$\underset{k}{\overset{(p-1)}{\mathcal{E}'}} \gtrsim \underset{k}{\overset{(p-1)}{\mathcal{E}}}. \tag{1-23}$$

Briefly, the pigeonhole principle is applied as follows: The integral of the left-hand side of (1-23) for $p = 1$ and k replaced by $k-2$ can be shown to be bounded $\lesssim \hat{\varepsilon}_0$, whence we obtain the existence of a $\tau = \tau''$ slice for which the right-hand side $\underset{k-2}{\overset{(0)}{\mathcal{E}}}(\tau'')$ is bounded $\lesssim \hat{\varepsilon}_0 L^{-1}$, where the L^{-1} factor arises from the length of the interval. We can then propagate this to $\tau_0 + L$ and use that $\alpha \gg 1$ to arrange that we have the precise new version of the third inequality of (1-21) at $\tau_0 + L$, i.e., for $\underset{k-2}{\overset{(0)}{\mathfrak{E}}}(\tau_0 + L)$. Similarly, the integral of the left-hand side of (1-23) for $p = 2 - \delta$ and k replaced by $k-4$ can be shown to be bounded $\lesssim \hat{\varepsilon}_0$, whence we obtain the existence of a $\tau = \tau''$ slice for which the right-hand side $\underset{k-4}{\overset{(1-\delta)}{\mathcal{E}}}(\tau'')$ is bounded $\lesssim \hat{\varepsilon}_0 L^{-1}$, the L^{-1} factor again arising from the length of the interval. Interpolating with the boundedness statement $\underset{k-4}{\overset{(2-\delta)}{\mathcal{E}}}(\tau'') \lesssim \hat{\varepsilon}_0$, we obtain that on this slice

$$\underset{k-4}{\overset{(1)}{\mathcal{E}}}(\tau'') \lesssim \hat{\varepsilon}_0 L^{-1+\delta}.$$

Again, we can then propagate this to $\tau_0 + L$ and use that $\alpha \gg 1$ to arrange that we have the precise new version of the fourth inequality of (1-21) at $\tau_0 + L$, i.e., for $\underset{k-4}{\overset{(1)}{\mathfrak{E}}}(\tau_0 + L)$. Finally, a similar argument, applying (1-23) for $p = 1$ and $k-6$, yields the new version of the last inequality of (1-21).

Note that, to obtain the new versions of the first two inequalities of (1-21) at $\tau_0 + L$, which refer to quantities that do not decay with respect to their initial values, it is important that the constant appearing in the first term on the right-hand side of (1-15) is indeed 1.

1.3.4. *Iteration over consecutive spacetime slabs of dyadic time length $L_i = 2^i$.* Let us assume that our initial data on $\tau_0 := 1$ satisfy

$$\overset{(1)}{\underset{k}{\mathcal{E}}}(\tau_0) + \overset{(2-\delta)}{\underset{k-2}{\mathcal{E}}}(\tau_0) \leq \varepsilon_0$$

for sufficiently small ε_0. It follows that (1-21) is satisfied for $\tau_0 = 1,\ L := L_0 = 1$ and appropriate $\hat{\varepsilon}_0$.

Examining closely (1-21) and (1-22), we see that we have now obtained a series of estimates which, in addition to being sufficient to obtain existence in $\mathcal{R}(\tau_0, \tau_0 + L)$, are moreover preserved under dyadic iteration. We simply iterate the above statement on consecutive spacetime slabs of dyadic time length $L_i = 2^i$, defining $\tau_{i+1} = \tau_i + L_i = 2^i$ and $\hat{\varepsilon}_{i+1} = \hat{\varepsilon}_i(1 + \alpha L_i^{-1/4})$. Note that $\hat{\varepsilon}_{i+1} \lesssim \hat{\varepsilon}_0 \lesssim \varepsilon_0$ for all i.

This immediately yields global existence in $\mathcal{R}(\tau_0, \infty) = \bigcup_{i\geq 0} \mathcal{R}(\tau_i, \tau_{i+1})$, the top-order boundedness statement

$$\overset{(1)}{\underset{k}{\mathcal{E}}}(\tau) + \overset{(2-\delta)}{\underset{k-2}{\mathcal{E}}}(\tau) \lesssim \varepsilon_0 \tag{1-24}$$

and the decay estimate

$$\overset{(0)}{\underset{k-6}{\mathcal{E}}}(\tau_i) \lesssim \varepsilon_0 \tau^{-2+\delta}.$$

Thus, our result is a true orbital stability statement at highest order in addition to yielding asymptotic stability.

1.4. ***Discussion.*** We end with some additional discussion.

1.4.1. *The semilinear case.* We have already remarked that in the semilinear case, i.e., the case where $g(\psi, x) = g_0$ identically and the equation takes the form

$$\Box_{g_0} \psi = N(\partial\psi, \psi, x), \tag{1-25}$$

there is no loss of derivatives when estimating (1-25) as a linear inhomogeneous equation. Thus, one can base the entire argument directly on the black box estimate (1-3), without the need for the auxiliary physical-space based estimate (1-6), provided of course that the appropriate asymptotic flatness and commutation assumptions are made so that (1-3) can be extended to the higher-order weighted (1-10). See already Remark 5.2 for the modified statement appropriate in this case and a guide to its proof. In particular, our theorem applies directly to the purely semilinear version of (1-1) on Kerr in the full subextremal range $|a| < M$.

1.4.2. *Additional examples and extensions.* We collect here some additional examples to which our results apply, as well as various natural future extensions.

Other spacetimes $(\mathcal{M}, g_0)$ *satisfying our assumptions.* We have already specifically mentioned the examples where $(\mathcal{M}, g_0)$ is Minkowski space, Schwarzschild and Kerr, but let us remark that our black box assumption (1-3) holds in fact for the full subextremal Kerr–Newman family [Civin 2015]. We also note that combining the use of an energy current construction similar to Appendix A with [Wunsch and Zworski 2011], one can show that (1-3) holds for *arbitrary* stationary perturbations $(\mathcal{M}, g_0)$ of Schwarzschild, provided that they are close to Schwarzschild in a suitably regular norm. Since we have already remarked that the physical-space based estimate (1-6) holds also for such spacetimes, in fact under much weaker regularity assumptions, this means that our main theorem indeed applies in this case. Thus, for very slowly rotating $|a| \ll M$ Kerr spacetimes, one sees that the special additional structure of Kerr (like the various manifestations of Carter separability, the Killing tensor, etc.) has no significance for the problem, and the spacetimes can just be understood as a very small stationary perturbation of Schwarzschild with a Killing horizon. *It is only in the case $|a| \sim M$ where true Kerr properties are of any relevance.* A treatment of this case, reducing directly to the black box estimate proven in [Dafermos et al. 2016], will appear elsewhere.

The axisymmetric case. In the case where we assume that the metric g_0 admits an additional Killing field Ω_1, one may also restrict to nonlinear equations of the form (1-1) which moreover preserve this symmetry, i.e., with the property that if the data are preserved by Ω_1, then so is the solution. Let us call such equations and solutions axisymmetric. Under the geometric assumptions of this paper, for axisymmetric equations (1-1) and restricted to axisymmetric solutions, one may in fact easily derive the existence of the necessary auxiliary physical-space identity yielding (1-6), just as in Schwarzschild. This in particular applies to Kerr–Newman in the full subextremal case. Thus, in view of [Dafermos et al. 2016; Civin 2015] which obtain (1-3), the main theorem of the paper applies also in these settings. We leave the details as an exercise for the reader. Note that in this case, if one prefers, one may quote the original [Dafermos and Rodnianski 2010a] for (1-3) for axisymmetric solutions in place of the more general (1-3) obtained for all solutions in [Dafermos et al. 2016].

Potentials. Our method applies also if a suitably decaying T-independent potential $\mathcal{V}(x)$ is added to (1-1), i.e., for

$$\Box_{g(\psi,x)}\psi - \mathcal{V}(x)\psi = N(\partial\psi, \psi, x),$$

provided that both the black box assumption (1-3) and the physical-space based (1-6) hold for solutions of the inhomogeneous version of the new linearised equation $\Box_{g_0}\psi - \mathcal{V}(x)\psi = F$ in place of (1-2). We note that, in the case where T is globally timelike and $\mathcal{V}$ decays suitably and is nonnegative, one can always obtain also the analogue of (1-6). The same statement applies for any such potential in the slowly rotating Kerr case $|a| \ll M$.

Systems. For notational convenience, we have restricted here to scalar equations. The considerations generalise readily to systems, for which a generalised null condition can again be formulated in terms of bounds of the nonlinear terms by (1-14); this now includes many examples. See also the discussion of the weak null condition in Section 1.4.5 below.

The obstacle problem. We have already remarked that, with a mild adaptation of the setup, one can also consider the case where the boundary $\mathcal{S}$ is in fact timelike, imposing now also suitable boundary conditions on $\mathcal{S}$. This is for instance the setting for the classical obstacle problem. The analogue of (1-3) has indeed been proven in the nontrapping case (with $\chi = 1$ identically, i.e., without degeneration) but also, with suitable degeneration functions χ, for many examples with nontrivial hyperbolic trapping; see for example [Lafontaine 2022]. We note again that the additional physical-space estimate (1-6) can always be retrieved in this case. Thus, this situation can also be incorporated in our framework.

Extremal black holes. Extremal black holes do not satisfy our assumptions on $(\mathcal{M}, g_0)$, already because the stationary vector field T will not be tangential to the boundary $\mathcal{S}$, but more seriously because χ must degenerate at the black hole horizon [Sbierski 2015], where T will not be timelike. Nonetheless, a version of (1-3) has been proven in the extremal Reissner–Nordström case [Aretakis 2011] (and in the extremal Kerr case, under the assumption of axisymmetry [Aretakis 2012]), with an additional hierarchy of degeneration associated to the horizon, and nonlinear stability for semilinear equations has been proven [Angelopoulos et al. 2020], where, however, an additional null structural condition is required at the horizon, in full analogy with the situation at null infinity. It would be interesting to reformulate this latter work in the dyadic setup used here. This is related to the nonlinear stability problem of extremal black holes. See the discussion in [Dafermos et al. 2021] and Conjecture IV.2 therein.

The asymptotically AdS case. Finally, although our assumptions are modelled on the asymptotically flat setting, one could also try to reformulate things in the asymptotically anti-de Sitter case, appropriate for solutions of the Einstein equations with negative cosmological constant $\Lambda < 0$ (for a discussion of the $\Lambda > 0$ case see already Section 1.4.6 below). Here, one must also impose boundary conditions *at infinity*, since infinity itself can be thought of as an asymptotic *time-like* boundary. We note that, under reflecting boundary conditions, there exist periodic (and thus nondecaying) solutions of the wave equation on pure AdS, while general solutions on the Schwarzschild–anti de Sitter and Kerr–anti de Sitter case [Holzegel and Smulevici 2013; 2014], again under such boundary conditions, decay only inverse logarithmically. (This is due to stable photon orbits repeatedly reflecting off of infinity only to return later having been repelled centrifugally by the black hole.) Based on this lack of decay, pure AdS has in fact been proven to be nonlinearly *unstable* under reflecting boundary conditions as a solution to various Einstein-matter systems, where the problem can be studied already under spherical symmetry [Moschidis 2023]. Thus, if one hopes to show nonlinear stability for an equation of the type (1-1) on asymptotically AdS backgrounds, it is more natural to consider *dissipative* boundary conditions like those considered in [Holzegel et al. 2020]. Note that on Kerr–AdS, one expects to indeed satisfy the analogue of (1-6) with the help of the Hawking–Reall Killing vector field, provided that the parameters satisfy the Hawking–Reall bound. Thus, given a version of (1-3), nonlinear stability for a suitable class of equations (1-1) should be tractable for all such Kerr–AdS parameters.

1.4.3. *The dyadic approach vs. the traditional approach.* The dyadic approach followed here is of course closely related to the more traditional approach. Our L-weighted smallness translates easily into t-decay assumptions, and if one wishes one can rewrite the argument using a bootstrap with t-decaying norms. We believe, however, that the dyadic localisation of the argument both makes the proof more modular and may serve to better identify possible future refinements with respect to regularity and decay. One sees clearly, for instance, that the L^{-1} smallness assumptions necessary for global existence *within a slab* are manifestly weaker than the $L^{-2+\delta}$ smallness necessary to improve and iterate (in fact one may weaken this to $L^{-1-\epsilon}$). Since bootstrap is only used to show existence *within a slab*, this already simplifies the argument considerably. We also note that the $L^{-1-\epsilon}$ threshold corresponds to pointwise decay $t^{-1/2-\epsilon/2}$, considerably weaker than the $t^{-1-\epsilon}$ "improved decay" which is often invoked for global existence and stability for (1-1). Proving sharper decay is of course an extremely important problem in itself (see [Angelopoulos et al. 2018; 2023; Hintz 2022; Kehrberger 2022]) but is not necessary (or even particularly helpful) for nonlinear stability.

Let us also point out that for problems with gauge invariance, like the Einstein equations, dyadic localisation also provides a convenient opportunity to refresh the gauge. This suggests an alternative approach to the global teleological normalisations of gauge done in [Dafermos et al. 2019b; 2021].

1.4.4. *Applications to the Teukolsky equation and derived equations and to the nonlinear stability of black holes.* As discussed in Section 1.2, one motivation for the precise assumptions on the nonlinearities of (1-1) made here is viewing these as a model for understanding the Einstein equations, when the latter are considered under suitable geometrically defined gauges. Such gauges were first exploited analytically in the monumental proof of the nonlinear stability of Minkowski space [Christodoulou and Klainerman 1993].

The geometric-gauge approach to black hole stability was taken up in [Dafermos et al. 2019b], where the problem was studied in double-null gauge and the full linear stability of Schwarzschild was first proven. (See [Christodoulou 2009] for an introduction to double-null gauge, including a discussion of the characteristic initial value problem and a derivation of all equations.) The setup is of course intimately connected to the formalism of Newman and Penrose [1962]. In the system resulting from linearising the reduced Einstein equations in double-null gauge around Schwarzschild, the gauge-invariant quantities are determined by the extremal curvature components α and $\underline{\alpha}$, which each satisfy the Teukolsky equation, first derived in this setting by [Bardeen and Press 1973] (and generalised to Kerr in [Teukolsky 1973]). The transport equations satisfied by the residual gauge-dependent quantities, on the other hand, are always already *linearly* coupled to the gauge-invariant ones. To analyse first the gauge-invariant quantities, [Dafermos et al. 2019b] introduced a pair of novel physical-space quantities P and $\underline{P}$, related to α and $\underline{\alpha}$ by second-order weighted null differential operators, but satisfying the more tractable Regge–Wheeler equation (an equation first derived in a different context in [Regge and Wheeler 1957]), which, unlike the Teukolsky equation, could be understood by the exact same methods as the linear wave equation. In particular, an analogue of (1-3) was proven, via a physical-space identity, for P and $\underline{P}$, and this led to boundedness and decay through the hierarchical structure of the system, first for the quantities P and $\underline{P}$ themselves, then for the original gauge-independent quantities α and $\underline{\alpha}$, and then, after teleological normalisation of the double-null gauge, for all residual gauge-dependent quantities as well. (For a complete scattering theory for this system, see [Masaood 2022; 2024]. For other approaches to the gauge in Schwarzschild under linear theory, see [Benomio 2024] and the references discussed in Section 1.4.5 below.) The origin of this relation between Teukolsky and Regge–Wheeler goes back to the fixed frequency transformation theory of [Chandrasekhar 1975]. On the basis of this linear theory, the full nonlinear stability of the Schwarzschild family, without symmetry, was proven in our [Dafermos et al. 2021]. (For previous nonlinear results for Schwarzschild under symmetry assumptions, see [Christodoulou 1987; Dafermos and Rodnianski 2005] in the case of the Einstein-scalar field system under spherical symmetry, and [Holzegel 2010; Klainerman and Szeftel 2020] for vacuum, the former in the higher-dimensional case under biaxial Bianchi symmetry, and the latter under polarised axisymmetry, reducing to 2+1 dimensions, the first such result beyond 1+1-dimensional reductions.) Note that since Schwarzschild is a codimension-3 subfamily of Kerr, nonlinear asymptotic stability of Schwarzschild refers to constructing the full (teleologically defined) codimension-3 set of initial data which asymptote to Schwarzschild in the future.

A generalisation of the quantities P and $\underline{P}$ of [Dafermos et al. 2019b] to the slowly rotating Kerr case $|a| \ll M$ was given independently by [Dafermos et al. 2019a; Ma 2020]. The equations satisfied by these quantities are now however (weakly) coupled to the quantities satisfying the Teukolsky equation. The works [Dafermos et al. 2019a; Ma 2020] both show an analogue of (1-3) for this system, not via a physical-space identity however but based on the framework for frequency-localised estimates on Kerr introduced in [Dafermos and Rodnianski 2010a]. These results were followed by full linear stability statements for the gauge-dependent quantities in various gauges [Andersson et al. 2025; Häfner et al. 2021] (the work [Häfner et al. 2021] considers in fact harmonic gauge; cf. the discussion in Section 1.4.5). The estimates of [Dafermos et al. 2019a; Ma 2020] have recently been reformulated by [Giorgi et al. 2024] in the language of the higher-order physical-space commutation by the Carter tensor first introduced in [Andersson and Blue 2015]. The work [Giorgi et al. 2024] then uses this, among other ingredients, in the context of the formalism of [Giorgi et al. 2020] to obtain nonlinear stability results for the very slowly rotating Kerr case $|a| \ll M$, completing an impressive series of preprints. For generalisations to the Einstein–Maxwell system, see [Giorgi 2020; 2021; Apetroaie 2023].

The full subextremal Kerr case $|a| < M$ is more subtle, as it cannot be understood as a small perturbation of Schwarzschild. Remarkably, however, an analogue of (1-3) for this system has been obtained in the full subextremal case [Shlapentokh-Rothman and da Costa 2020; 2023] using the already highly nontrivial mode stability results [Whiting 1989; Shlapentokh-Rothman 2015; Teixeira da Costa 2020] (see also [Andersson et al. 2017]) but, also, nontrivial new high-frequency structure with no apparent precise analogue in the context of the wave equation. To exploit frequency analysis, the proof of [Shlapentokh-Rothman and da Costa 2020] uses a version of the frequency localisation framework of [Dafermos et al. 2016]. (In connection with the Teukolsky equation, we also note [Ma and Zhang 2023; Millet 2023] for precise asymptotics in the special case of smooth compactly supported data. For a discussion of what are the appropriate initial assumptions on data for the Teukolsky equation in various physical settings, see [Gajic and Kehrberger 2022].)

In view of the method introduced in the present paper, it should be clear that there is absolutely nothing to fear in these types of frequency localisations for nonlinear applications, provided that they are indeed used to prove a spacetime-localised statement which can be expressed in the form (1-3). The results of [Dafermos et al. 2019a; Ma 2020] and [Shlapentokh-Rothman and da Costa 2020] can thus in principle be used *directly* for the nonlinear problem, in fact, as "black box" results. Thus, in our view, given the breakthrough of [Shlapentokh-Rothman and da Costa 2020], the technical complications in extending the nonlinear Schwarzschild analysis of [Dafermos et al. 2021] to the full subextremal case $|a| < M$ of Kerr may not be as severe as one might naively have thought.

1.4.5. *Harmonic gauge and the weak null condition.* In our discussion in Section 1.4.4 we have focussed above primarily on "geometric" gauges, but, as is well known, another way to relate (1-1) to the Einstein equations is via the harmonic gauge condition (see for instance [Lindblad and Rodnianski 2010], where the nonlinear stability of Minkowski space is proven in this gauge). The resulting reduced equations (for $\psi^{\mu\nu} = g^{\mu\nu} - g_0^{\mu\nu}$) produce additional complicated linear terms and moreover fail to satisfy the null condition, although they do satisfy the so-called *weak null condition* introduced in [Lindblad and Rodnianski 2003]. Note that a linear stability result has been given for this system in the Schwarzschild case [Johnson 2019] (see also [Hung 2018]) and, as mentioned earlier, also on very slowly rotating Kerr $|a| \ll M$ in [Häfner et al. 2021].

In order to model the Einstein equations in harmonic gauge, classes of equations (1-1) satisfying the weak null condition have been studied in the recent [Lindblad and Tohaneanu 2018; 2020; 2024]. Though we do not here implement our method in this latter setting, nonetheless, we emphasise that our analysis can in principle be extended to equations or systems satisfying the weak null condition, following [Keir 2018], under an appropriate black box assumption for the linearisation.

1.4.6. *Comparison with the* $\Lambda > 0$ *case.* Finally, it is interesting to compare with the $\Lambda > 0$ case. Here, the analogue of the Schwarzschild and Kerr black holes are the Schwarzschild–de Sitter and Kerr–de Sitter family. See [Dafermos and Rodnianski 2013] for a discussion of their geometry. The study of decay for semilinear and quasilinear equations of the type (1-1) on Kerr–de Sitter was pioneered by Hintz and Vasy [2016], eventually leading to their groundbreaking proof of the nonlinear stability of very slowly rotating ($|a| \ll M,\ \Lambda$) Kerr–de Sitter in harmonic gauge. The proof appeals to extensive machinery from microlocal analysis, with an elaborate compactification of spacetime, and with a Nash–Moser iteration. For a more recent approach replacing Nash–Moser iteration with a global bootstrap, see [Fang 2021].

Returning to the model equation (1-1), an elementary new method was recently introduced by Mavrogiannis [Mavrogiannis 2024; 2022] to treat nonlinear stability on Schwarzschild–de Sitter and Kerr–de Sitter backgrounds. In [Mavrogiannis 2024], the entire analysis is reduced to a "black box" estimate for the linear problem on time-slabs of some *fixed* length L. The required black box linear estimate, however, is not just the analogue of the (degenerate) integrated local energy estimate (1-3) (first proven in this context in [Dafermos and Rodnianski 2007b]), but a higher-order refinement of (1-3) which is *relatively nondegenerate*, i.e., where the bulk and boundary term have identical degeneration function and thus again are comparable. This is possible through a commutation with a globally defined well-chosen vector field orthogonal to the photon sphere and vanishing at the horizons. This is an analogue of an energy originally constructed in the Schwarzschild case in [Holzegel and Kauffman 2020]. (In the Kerr–de Sitter case, both the degenerate integrated local energy decay statement and the accompanying relatively nondegenerate statement are now proven using frequency localisation in a framework similar to [Dafermos et al. 2016].) Exponential decay is then a derived statement which follows from iterating a suitable estimate on consecutive slabs, each now of fixed length L.

In comparison to the asymptotically flat case studied in the present paper, it is noteworthy that in the Kerr–de Sitter case one does *not* require an additional nontrivial physical-space-based ingredient, analogous to (1-6), other than this new, relatively nondegenerate version of the black box linear estimate (1-3). From the time-slab-localised point of view of the two works, the reason for the difference is clear: since estimates in [Mavrogiannis 2024] have been reduced to a fixed time scale L, the role of (1-6) is essentially provided by *Cauchy stability*, a completely soft statement.

From this point of view, the difference between the Schwarzschild and Kerr case on the one hand and their de Sitter analogues on the other is more than simply one between polynomial and exponential decay, but is one between *dyadically localised* and *truly local*. In this approach, it is this fundamentally local nature of the analysis in the de Sitter case that renders the *nonlinear* aspects of stability problems on such de Sitter backgrounds to be essentially soft.

1.5. *Outline of the paper.* We end with an outline of the remainder of the paper.

In Section 2 we shall describe the geometric assumptions on the background spacetime, followed in Section 3 by the assumptions on properties of the linear inhomogeneous equation (1-2). We shall then introduce in Section 4 the class of nonlinear equations (1-1) that we shall consider, stating in particular a local well-posedness theorem and continuation criterion. We shall give the precise statement of the main theorem in Section 5. The proof will then be carried out in Section 6.

The paper also contains four appendices. In Appendix A, we show how to obtain (1-6) from a physical-space identity in the very slowly rotating Kerr case. In Appendix B, we will show how to define the far-away currents encoding the r^p method necessary to extend the estimates (1-3) and (1-6) to (1-7) and (1-8), respectively. In Appendix C, we show that our null condition assumption encoded by the bounds (1-14) indeed includes the classical null condition [Klainerman 1986] and also the more general class of semilinear terms considered on Kerr in [Luk 2013]. In Appendix D, we show explicitly how to obtain the inhomogeneous estimate (1-3) in the Kerr case from the homogeneous estimates of [Dafermos et al. 2016]. Together, these appendices show that our main theorem holds in particular for a wide class of equations of the form (1-1) on very slowly rotating $|a| \ll M$ Kerr backgrounds (and, for the semilinear case $g = g_0$, in the full subextremal range $|a| < M$).

2. Geometric assumptions on the background spacetime

We consider a manifold $\mathcal{M}$ with a background metric g_0 satisfying certain assumptions. In this section, we collect the assumptions which concern directly the geometry of $(\mathcal{M}, g_0)$. The assumptions here will not be the most general possible but are sufficient to include the main examples of interest. They can be easily further generalised in various directions if desired. Some of the assumptions are in principle redundant, but we have not attempted to derive them from the most minimal considerations. We note already that the assumptions of this section are modelled on (and are satisfied by — see Section 2.7!) the basic cases of Minkowski space and the (extended) exterior regions of Schwarzschild and subextremal $|a| < M$ Kerr spacetimes.

2.1. *Underlying differential structure and the positive function r.* For definiteness, we consider the underlying differential structure of $\mathcal{M}$ to be given by $\mathbb{R}^4_{(x^0,x^1,x^2,x^3)}$ or alternatively by the manifold with boundary $\mathbb{R}^4 \setminus \{|x^1|^2 + |x^2|^2 + |x^3|^2 < r_0^2\}$. (The black hole examples in fact correspond to the latter case.) In this latter case, let us define

$$r := \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}. \tag{2-1}$$

In the former case where the underlying manifold is $\mathbb{R}^4$, let us pick an arbitrary $r_0 > 0$ and define

$$r := f(\sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}), \tag{2-2}$$

where $f:[0,\infty)\to[r_0,\infty)$ is a smooth function with $f'(v)>0$ such that $f(z)=\sqrt{r_0^2+z^2}$ near $z=0$ and $f(z)=z$ for $z\geq 2r_0$. Thus, in all cases r is a smooth positive function on $\mathcal{M}$ satisfying $r\geq r_0>0$. Let us also fix a large $R\geq 20r_0$.

We may also define ambient spherical coordinates in the usual way by the relation

$$(x^1,\,x^2,\,x^3)=(r\cos\varphi\sin\vartheta,\,r\sin\varphi\sin\vartheta,\,r\cos\vartheta). \tag{2-3}$$

In the case of nonempty boundary, we will denote by $\Omega_1=\partial_\varphi$, Ω_2, and Ω_3 the standard rotation vector fields associated to the ambient spherical coordinates $(x^0,r,\vartheta,\varphi)$. These are globally regular vector fields.

In the case where the underlying structure is $\mathbb{R}^4$, the above vector fields would only be regular for $r>r_0$. In this case then, we will use the notation $\Omega_i,\ i=1,\ldots,3$, for the above standard vector fields multiplied by $\omega(r)$, for a smooth cut-off function satisfying $\omega(r)=1$ for $r\geq\frac{1}{2}R$ and $\omega(r)=0$ in a neighbourhood of r_0. These are again globally regular vector fields.

2.2. ***The Lorentzian metric g_0, time orientation, and the causality of the boundary.*** We assume that g_0 is a time-oriented Lorentzian metric on $\mathcal{M}_0$ such that the coordinate vector field ∂_{x^0} is future-directed timelike for $r\geq\frac{1}{6}R$.

In the case where $\mathcal{M}$ is a manifold with boundary $\mathcal{S}=\{r=r_0\}$, we assume that the boundary $\mathcal{S}$ is spacelike and the future-directed normal points out of the spacetime.

2.3. ***The stationary Killing field T.*** Denoting now by T the coordinate vector field

$$T=\partial_{x^0}$$

with respect to the ambient coordinates (x^0,x^1,x^2,x^3), we assume that T is Killing with respect to the metric g_0. It follows by the previous assumptions that the ambient function r above is T-invariant and that T is future-directed timelike in the region $r\geq\frac{1}{6}R$. Let us further assume that $g(T,T)\to -1$ as $r\to\infty$.

We will denote by ϕ_τ the 1-parameter group of diffeomorphisms generated by T.

In the case where $\mathcal{S}=\varnothing$, it is natural to already assume that T is globally timelike, in view of the Friedman instability [Moschidis 2018] and its evanescent analogue [Keir 2020] which applies in the marginal case, both of which would be incompatible with assumptions we will make later on concerning the behaviour of waves.

In the case where $\mathcal{S}\neq\varnothing$, we notice that T cannot be globally timelike as T is tangential to and thus spacelike on $\mathcal{S}=\{r=r_0\}$. In this case we will need to assume the existence of a Killing horizon.

2.4. ***The Killing horizon $\mathcal{H}^+$.*** If $\mathcal{S}\neq\varnothing$, we will assume the following further properties:

- We assume that there exists an r_{Killing} such that the hypersurface $\mathcal{H}^+:=\{r=r_{\text{Killing}}\}$ is a Killing horizon with future-directed null generator $Z=T$ or, more generally, with future-directed null generator Z in the span of T and Ω_1, in which case we will assume that Ω_1 is globally Killing. We will denote by Z the globally defined Killing field given by this combination of T and Ω_1.

- We will assume furthermore that $\mathcal{H}^+$ has strictly positive surface gravity, i.e., $\nabla_Z Z = \kappa(\vartheta, \varphi) Z$ for some $\kappa > 0$, and we will assume that T, or more generally the span of T and Ω_1, is timelike for $r > r_{\text{Killing}}$.
- Finally, in the case $\mathcal{S} \neq \varnothing$, we will assume that the vector field ∇r is future-directed timelike in the region $r < r_{\text{Killing}}$. Note that ∇r on $\mathcal{H}^+ = \{r = r_{\text{Killing}}\}$ will be null and in the direction of Z.

2.5. ***Foliations, subregions, and volume forms.***

2.5.1. *The foliation* $\Sigma(\tau)$. We will assume $\mathcal{M}$ admits a hypersurface Σ_0 with 2-dimensional corner at $\Sigma_0 \cap \{r = R\}$ such that $\Sigma_0 \cap \{r \leq R\}$ is strictly spacelike, $\Sigma_0 \cap \{r \geq R\}$ is null, and Σ_0 is transversal to T.

We assume that, for $r' \geq \frac{1}{2}R$, Σ_0 is transversal to the hypersurface $\{r = r'\}$ and that Ω_1, Ω_2, Ω_3 are tangent to $\Sigma_0 \cap \left\{r \geq \frac{1}{2}R\right\}$. In particular, the 2-dimensional space $T_p(\Sigma_0 \cap \{r \leq R\}) \cap T_p(\Sigma_0 \cap \{r \geq R\})$ should coincide with the span of these vectors.

If $\mathcal{S} \neq \varnothing$, we assume that Σ_0 is transversal to the hypersurface $\mathcal{S}$ and $\Sigma_0 \cap \mathcal{S}$ is diffeomorphic to the 2-sphere. We assume finally that the vector field Z of Section 2.4 is orthogonal to $\Sigma_0 \cap \mathcal{H}^+$, while Ω_1, Ω_2, Ω_3 are tangent to $\Sigma_0 \cap \mathcal{H}^+$.

We assume that Σ_0 separates $\mathcal{M}$ into two connected components and that Σ_0 is a past Cauchy hypersurface for $J^+(\Sigma_0)$.

Writing $\Sigma(\tau) := (\phi_\tau)_*(\Sigma_0)$, we assume

$$J^+(\Sigma_0) = \bigcup_{\tau \geq 0} \Sigma(\tau),$$

where J^+ denotes causal future in $\mathcal{M}$ with respect to the metric g_0.

Clearly $\{\Sigma(\tau)\}_{\tau \in \mathbb{R}}$ defines a foliation of $\mathcal{M}$ and thus defines globally on $\mathcal{M}$ a Lipschitz function τ, which is smooth separately on $r \leq R$ and $r \geq R$.

Note that by our assumptions above, for $r \geq \frac{1}{2}R$, we have that $\tau = \tau(x^0, r)$ and the triple (r, ϑ, φ) represent a smooth coordinate system on $\Sigma(\tau) \cap \{r \geq R\}$ (modulo the spherical degeneration).

We will denote by L a smooth choice of the future-directed null generator of $\Sigma_0 \cap \{r \geq R\}$ normalised to satisfy the constraint $g(L, T) \sim -1$. By translation invariance, this extends to a smooth vector field on all of $\{r \geq R\}$ in the direction of the null generator of $\Sigma(\tau)$. (Note that we also use L to denote a general length parameter; in practice these two notations will not interfere with one another.)

2.5.2. *The regions* $\mathcal{R}(\tau_0, \tau_1)$. Let us define

$$\mathcal{R}(\tau_0, \tau_1) := \bigcup_{\tau_0 \tau \leq \tau_1} \Sigma(\tau).$$

We shall refer to such regions as spacetime slabs. We will also use the notation

$$\mathcal{R}(\tau_0, \infty) := \bigcup_{\tau \geq \tau_0} \Sigma(\tau).$$

Note that $\mathcal{R}(\tau_0, \infty) = J^+(\Sigma_0)$.

We shall define

$$\mathcal{S}(\tau_0, \tau_1) := \mathcal{S} \cap \mathcal{R}(\tau_0, \tau_1).$$

2.5.3. *The ingoing cones $\underline{C}_v$ and truncated regions.* We also assume that the region $r \geq R$ is foliated by translation-invariant "ingoing" null cones $\underline{C}_v$ parametrised by a smooth function v defined on $r \geq R$, increasing towards the future, which may moreover be expressed as $v(x^0, r)$. In particular, the vector fields Ω_i are tangent to $\underline{C}_v$. We again define a smooth future-directed null generator $\underline{L}$ of $\underline{C}_v$, normalised by the constraint $g(\underline{L}, T) \sim -1$ and translation-invariant; this defines a smooth vector field on $r \geq R$.

Let us assume moreover that $g(\underline{L}, L) \sim -1$ globally in $r \geq R$.

Note that, under the above assumptions, r is constant on $\underline{C}_v \cap \Sigma(\tau)$ and the future boundary of $\underline{C}_v$ is the set $\{r = R\} \cap \Sigma(\tau(v))$, where this relation defined $\tau(v)$.

If $\tau_0 \leq \tau_1 \leq \tau(v)$, we shall define

$$\mathcal{R}(\tau_0, \tau_1, v) := \mathcal{R}(\tau_0, \tau_1) \setminus \bigcup_{\tilde{v} > v} \underline{C}_{\tilde{v}}$$

and

$$\Sigma(\tau, v) := \Sigma(\tau) \setminus \bigcup_{\tilde{v} > v} \underline{C}_{\tilde{v}}.$$

The spacetime region $\mathcal{R}(\tau_0, \tau_1, v)$ is a compact subset of spacetime with past boundary $\Sigma(\tau_0, v)$ and future boundary $\mathcal{S}(\tau_0, \tau_1) \cup \Sigma(\tau_1, v) \cup \underline{C}_v$.

2.5.4. *Comparison of volume forms.* We will assume that, in the region $r \geq R$, writing the volume form for $(\mathcal{M}, g)$ as

$$dV_{\mathcal{M}} = \upsilon(r, \vartheta, \varphi)\, d\tau r^2\, dr \sin\vartheta\, d\vartheta\, d\varphi,$$

for $\Sigma(\tau) \cap \{r \geq R\}$, with the choice L for the normal, as

$$dV_{\Sigma(\tau)\cap\{r\geq R\}} := \tilde{\upsilon}(r, \vartheta, \varphi) r^2\, dr \sin\vartheta\, d\vartheta\, d\varphi,$$

and for $\underline{C}_v$, with the choice $\underline{L}$ for the normal, as

$$dV_{\underline{C}_v} := \tilde{\tilde{\upsilon}}(r, \vartheta, \varphi) r^2\, dr \sin\vartheta\, d\vartheta\, d\varphi,$$

we then have

$$\upsilon \sim \tilde{\upsilon} \sim \tilde{\tilde{\upsilon}} \sim 1.$$

With this assumption, it follows by the coarea formula and compactness that, globally, the volume form of $(\mathcal{M}, g_0)$ is related to the volume form of $\Sigma(\tau)$,

$$dV_{\mathcal{M}} \sim d\tau\, dV_{\Sigma(\tau)},$$

where $\sim$ is interpreted for 4-forms in the obvious sense.

Note that when volume forms are omitted from integrals, the above induced volume forms from the metric g_0 will be understood, unless otherwise noted.

2.6. ***Other vector fields.*** It will be useful to extend the vector fields defined above to a global set of vector fields which span the tangent space $T_x\mathcal{M}$ for all $x \in \mathcal{M}$.

2.6.1. *The global extensions of the vector fields L, $\underline{L}$, and Ω_i.* For notational convenience, in the case where $\mathcal{S} = \varnothing$, let us define $\Omega_4 = (1-\omega(r))\partial_{x^1}$, $\Omega_5 = (1-\omega(r))\partial_{x^2}$, $\Omega_6 = (1-\omega(r))\partial_{x^3}$, where ω is as in Section 2.1, and let us define L and $\underline{L}$ to be translation-invariant extensions of the vector fields defined already with the property that L, $\underline{L}$ and $\Omega_1, \dots, \Omega_6$ span the tangent space globally. Note that this is easy to satisfy in general. We are moreover not requiring that L and $\underline{L}$ be null, and the Ω_i, $i = 1, \dots, 6$, are not required to be tangential to the ambient spheres for $r \le \frac{1}{2}R$. (See already Section 2.7.1.)

In the case where $\mathcal{S} \neq \varnothing$, we will define L and $\underline{L}$ to be smooth ϕ_τ-invariant extensions of L and $\underline{L}$ to $r_0 \le r \le R$, with the property that L, $\underline{L}$ everywhere span the same plane spanned by the coordinate vector fields ∂_{x^0} and ∂_r of ambient spherical coordinates $(x^0, r, \vartheta, \varphi)$. This again can easily seen to be possible. Note again that we are not requiring these vector fields to be globally null.

2.6.2. *The notation $|\nabla\!\!\!/ \psi|^2$.* We define the notation

$$|\nabla\!\!\!/ \psi|^2 := \sum_i r^{-2} |\Omega_i \psi|^2.$$

We note that, under our assumptions, the above expression is comparable to the induced gradient squared on the space-like spheres $\Sigma(\tau) \cap \{r = r'\}$ in the region $r \ge \frac{1}{2}R$.

In view, however, of our spanning assumptions, in both the case $\mathcal{S} = \varnothing$ and $\mathcal{S} \neq \varnothing$, the expression

$$|L\psi|^2 + |\underline{L}\psi|^2 + |\nabla\!\!\!/ \psi|^2$$

will always be a translation-invariant coercive expression on first derivatives of ψ, similar (by compactness) to any other such coercive expression in $r \le R$, while in the region $r \ge R$ it will have the right weights in r so as to be a suitable reference point for natural energy expressions.

2.6.3. *The 1-forms ϱ^L, $\varrho^{\underline{L}}$, ϱ^1, ϱ^2, and ϱ^3.* We will need to introduce a set of weighted spanning 1-forms on the sphere so as to properly measure weighted boundedness of forms. Define 1-forms ϱ^L, $\varrho^{\underline{L}}$, and ϱ^1, ϱ^2, ϱ^3 on the far region $\mathcal{M} \cap \{r \ge R\}$ as follows.

The region $\mathcal{M} \cap \{r \ge R\}$ can be written as the product manifold $\mathcal{M} \cap \{r \ge R\} = \mathbb{R} \times [R, \infty) \times S^2$. Let $\pi : \mathcal{M} \cap \{r \ge R\} \to \mathbb{R} \times [R, \infty)$ denote the canonical projection. Let ϱ^L, $\varrho^{\underline{L}}$ be defined by $\varrho^L = \pi^* \tilde{\varrho}^L$ and $\varrho^{\underline{L}} = \pi^* \tilde{\varrho}^{\underline{L}}$, where $\tilde{\varrho}^L$ and $\tilde{\varrho}^{\underline{L}}$ are the dual coframes of $\pi_* L$ and $\pi_* \underline{L}$, respectively, on $\mathbb{R} \times [R, \infty)$. It follows that

$$\varrho^L(L) = \varrho^{\underline{L}}(\underline{L}) = 1, \quad \varrho^L(\underline{L}) = \varrho^{\underline{L}}(L) = \varrho^L(\Omega_i) = \varrho^{\underline{L}}(\Omega_i) = 0, \quad i = 1, 2, 3.$$

For all smooth functions ψ, one can then write

$$d\psi = L\psi\, \varrho^L + \underline{L}\psi\, \varrho^{\underline{L}} + d\!\!\!/ \psi,$$

where $d\!\!\!/ \psi(L) = d\!\!\!/ \psi(\underline{L}) = 0$.

For each $i = 1, 2, 3$, there is a polar coordinate system (ϑ^i, φ^i) on S^2, associated to the corresponding rotation vector field Ω_i, with the property that, in the $(x^0, r, \vartheta^i, \varphi^i)$ coordinate system for $\mathcal{M} \cap \{r \ge R\}$, one has $\partial_{\varphi^i} = \Omega_i$. Such a (ϑ^i, φ^i) coordinate system on S^2 is unique up to a choice of meridian. Define then, for $i = 1, 2, 3$,

$$\varrho^1 = r \sin\vartheta^1\, d\varphi^1, \quad \varrho^2 = r \sin\vartheta^2\, d\varphi^2, \quad \varrho^3 = r \sin\vartheta^3\, d\varphi^3.$$

There exist (nonunique, but universal) bounded functions $a^i{}_j(\vartheta, \varphi)$ for $i, j = 1, 2, 3$ such that, for any smooth function $\psi : \mathcal{M} \cap \{r \geq R\} \to \mathbb{R}$,

$$d\psi = L\psi \varrho^L + \underline{L}\psi \varrho^{\underline{L}} + r^{-1} a^i{}_j(\vartheta, \varphi)\Omega_i \psi \varrho^j. \tag{2-4}$$

2.6.4. *The vector field Y.* In the case $\mathcal{S} \neq \varnothing$, we may define Y to be a ϕ_τ-invariant vector field such that: Y is future-directed null on $\mathcal{H}^+$, transversal to $\mathcal{H}^+$, and orthogonal to $\mathcal{H}^+ \cap \Sigma_0$; Y is supported in $r \leq r_1 + \frac{1}{2}(r_2 - r_1)$ for some $r_2 > r_1 > r_{\text{Killing}}$; and Z, Y, and Ω_i span the tangent space in $r \leq r_1 + \frac{1}{4}(r_2 - r_1)$.

The existence of such a vector field in a neighbourhood of a Killing horizon follows from [Dafermos and Rodnianski 2013].

In the case where $\mathcal{S} = \varnothing$, since we are assuming that T is globally timelike, we may simply set $Y = 0$.

2.7. *Examples*: *Minkowski, Schwarzschild and Kerr.* We note that Schwarzschild and Kerr in the full subextremal black hole range of parameters $|a| < M$ satisfy the assumptions of this section with an appropriate definition of the underlying differential structure.

2.7.1. *Minkowski.* For the Minkowski case, we consider the underlying manifold to be $\mathbb{R}^4$, i.e., without boundary, and we define the metric to be the familiar expression

$$g_0 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2.$$

We define $r_0 := 1$, which will determine the function r. Let us distinguish this function from $\tilde{r} := \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$ which we may think of as a function $\tilde{r}(r)$. Fixing any $R \geq 20$, we note that $u = t - \tilde{r}$, $v = t + \tilde{r}$ are null coordinates. We may define $L = \partial_u$, $\underline{L} = \partial_v$, with respect to coordinates $(u, v, \vartheta, \varphi)$, in the region $r \geq 2$.

We may take

$$\Sigma_0 = (\{t = 0\} \cap \{r \leq R\}) \cup (\{v = \tilde{r}(R)\} \cap \{r \geq R\})$$

and $\underline{C}_v$ defined by the level sets of v.

Note that $r/\tilde{r} \sim 1$ for $r \geq r_0$ and that the spacetime volume form is $\tilde{r}^2 \sin\vartheta \, d\tilde{r} \, d\tau \, d\vartheta \, d\varphi$, while the volume form on $\Sigma_0 \cap \{r \geq R\}$ and $\underline{C}_v$ is $\tilde{r}^2 \sin\vartheta \, d\tilde{r} \, d\vartheta \, d\varphi$ with our choice of L and $\underline{L}$.

Note also, in this case, we define $\Omega_4 = \eta(r)\partial_{x^1}$, $\Omega_5 = \eta(r)\partial_{x^2}$, and $\Omega_6 = \eta(r)\partial_{x^3}$, where η is a cutoff vanishing with $\eta = 1$ for $r \leq \frac{1}{4}$ and $\eta = 0$ for $r \geq \frac{1}{2}$, and we can extend L and $\underline{L}$ simply by $L = (1 - \eta)\partial_u + \eta\partial_v$ and $\underline{L} = (1 - \eta)\partial_v$. (We may then define the cutoff ω of Section 2.1, so that $\omega = 1$ for $r \geq \frac{1}{4}$ and $\omega = 0$ for $r \leq \frac{1}{8}$.) We have of course $Y = 0$ in this case.

2.7.2. *Schwarzschild.* For the Schwarzschild case, given a real parameter $M > 0$, we may define $r_0 := (2 - \delta_1)M$ for sufficiently small $0 < \delta_1 \ll M$, denote x^0 by t^*, let r be defined by (1-1) and standard spherical coordinates (ϑ, φ) on $\mathbb{R}^3$ by (2-3), and define the metric g_M to be

$$-\left(1 - \frac{2M}{r}\right)(dt^*)^2 + \frac{4M}{r}\, dr \, dt^* + \left(1 + \frac{2M}{r}\right) dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta \, d\varphi^2).$$

We define the vector fields Ω_i to be the standard Killing vector fields associated to spherical symmetry, and we define $T = \partial_{t^*}$ to be the coordinate vector field with respect to the above coordinates.

Note that $r = 2M$ is a Killing horizon with null generator T and positive surface gravity, and the hypersurfaces $r = r'$ for $r' \in [r_0, 2M)$ are indeed spacelike.

We may define the function t in the region $r > 2M$ by

$$t = t^* - 2M \log(r - 2M).$$

We note that, in the coordinates (t, r, θ, ϕ), the metric takes the familiar form

$$-\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2).$$

We may define $r^* = r + 2M \log(r - 2M)$, and we may define coordinates u and v in $r > 2M$ by $u = t - r^*$ and $v = t + r^*$.

We may fix $R \geq 20r_0$ and define now $L = \partial_v$ and $\underline{L} = \partial_u$ to be the coordinate vector fields with respect to (u, v, θ, ϕ) coordinates in $r \geq R$. We may extend these to be globally defined linearly independent translation-invariant vector fields in the span of the coordinate vector fields ∂_{t*} and ∂_r.

We then may define

$$\Sigma_0 = (\{t^* = 0\} \cap \{r \leq R\}) \cup (\{v = R\} \cap \{r \geq R\}),$$

which satisfies all the required transversality properties, etc.

Finally, we may define for instance Y to be a translation-invariant vector field in the span of ∂_{t*} and ∂_r, which is null and future-directed and satisfies $g(\partial_{t*}, Y) = -2$ at $r = 2M$, such that Y is supported entirely in $r \leq r_1 + \frac{1}{2}(r_2 - r_1)$, with $r_1 := (2 + \delta_1)M$.

2.7.3. *Kerr.* To put Kerr in our preferred form, one uses a combination of Kerr star coordinates (based on Boyer–Lindquist coordinates) and double-null coordinates. We leave the details to the reader but note the following.

Given subextremal parameters $|a| < M$ and defining

$$r_+ = M + \sqrt{M^2 - a^2},$$

we set $r_0 = r_+ - \delta_1$ for a small δ_1.

One may define the ambient differential structure such that the r of (2-1) will coincide with the Boyer–Lindquist r of Appendix A in the region $r \leq \frac{1}{3}R$, while for $r \geq \frac{1}{2}R$ it will coincide with the coordinate r_* of Section C.2. The coordinate v coincides with the double-null v of Section C.2, and $\Sigma(\tau) \cap \{r \geq R\}$ will be a hypersurface of constant u, where again u is as defined in Section C.2.

Further, one may set things up so that $\Omega_1 = \partial_\varphi$ is the axisymmetric Killing field.

Let us note finally that extremal Kerr (corresponding to $|a| = M$) cannot in fact be put into our preferred form already because of our requirement that T be tangential to $\mathcal{S}$ and thus spacelike. Note also that the Killing horizon of extremal Kerr has zero surface gravity, which would also contradict our assumptions of Section 2.4.

2.8. ***Table of r-parameters.*** We collect finally a list of important r-values in increasing order in Table 1. Some will only be introduced later in the paper. The parameters r_{Killing}, r_1, r_2 only occur if $\mathcal{S} \neq \varnothing$.

r_0	$r \geq r_0$; moreover, if $\mathcal{S} \neq \varnothing$, then $\mathcal{S} = \{r = r_0\}$
r_{Killing}	$\mathcal{H}^+ = \{r = r_{\text{Killing}}\}$ a Killing horizon with positive surface gravity; span of T and Ω_1 is timelike for $r > r_{\text{Killing}}$
r_1	parameter related to the vector field Y if $\mathcal{S} \neq \varnothing$
$r_1 + \frac{1}{4}(r_2 - r_1)$	$Z,\ Y,\ \Omega_i$ span the tangent space for $r_0 \leq r_1 + \frac{1}{4}(r_2 - r_1)$
$r_1 + \frac{1}{2}(r_2 - r_1)$	commutation vector fields $\mathfrak{D}$ all Killing for $r_1 + \frac{1}{2}(r_2 - r_1) \leq r \leq \frac{1}{2}R$
r_2	$\rho = 1,\ \chi = 1$ for $r \leq r_2$
$\frac{1}{6}R$	T timelike for $r \geq \frac{1}{6}R$
$\frac{1}{4}R$	$\rho = 1,\ \chi = 1$ for $r \geq \frac{1}{4}R$
$\frac{1}{2}R$	commutation vector fields $\mathfrak{D}$ all Killing for $r_1 + \frac{1}{2}(r_2 - r_1) \leq r \leq \frac{1}{2}R$; $g = g_0$ for $r \geq \frac{1}{2}R$
$\frac{8}{9}R$	the generalised null condition assumption concerns $r \geq \frac{8}{9}R$
R	$\Sigma(\tau) \cap \{r \geq R\}$ is null, $\underline{C}_v \subset \{r \geq R\}$
$\widetilde{R}$	parameter related to positivity properties of far-away currents

Table 1. Important r-values in this paper.

3. Assumed identities and estimates for $\Box_{g_0}\psi = F$

Our fundamental assumptions in this paper are connected with the behaviour of solutions of the linear inhomogeneous equation (1-2) on the exactly stationary background g_0.

3.1. ***Constants and parameters.*** Before stating assumptions, we make a remark concerning constants and parameters.

Given a spacetime $(\mathcal{M}, g_0)$ satisfying the assumptions of Section 2, we will consider the parameters of Section 2.8 as fixed. Let us also fix once and for all a

$$0 < \delta < \tfrac{1}{10}. \tag{3-1}$$

We will use k to denote integers ≥ 0 which will parametrise number of derivatives.

In inequalities, we will denote by C and c generic positive constants depending only on (a) $(\mathcal{M}, g_0)$ (with the choice of r-parameters), (b) the above choice of δ, and (c) if there is k-dependence in the relevant statement, also on k. (We use C for large constants and c for small constants.)

For nonnegative quantities A and B, the notation

$$A \lesssim B$$

means $A \leq CB$, while

$$A \sim B$$

means $cB \leq A \leq CB$.

The reader should be prepared to distinguish between $\leq$ and $\lesssim$, as both will appear!

For a discussion of additional smallness parameters depending also on the nonlinearity, see already Section 4.2.

3.2. ***Basic degenerate integrated local energy estimate.*** As discussed in the introduction, our basic assumption will be that of a (degenerate) spacetime-localised integrated local energy decay statement for the inhomogeneous equation (1-2).

The statement is that a certain energy flux on $\Sigma(\tau)$ plus a nonnegative bulk are controlled by an initial energy flux on $\Sigma(\tau_0)$ and a spacetime integral over the slab $\mathcal{R}(\tau_0, \tau)$ relating to the inhomogeneous term F. The controlled bulk integral is allowed to degenerate where a certain degeneration function $\chi = \chi(r)$ vanishes, but it is assumed to control a zeroth-order term without degeneration (with decaying weight at infinity).

Let us thus define

$$\chi : [r_0, \infty) \to [0, 1] \tag{3-2}$$

to be a function such that $\chi = 1$ for $r \geq \frac{1}{4}R$ and $r \leq r_2$.

The assumed estimate is

$$\sup_{v:\tau\leq\tau(v)} \overset{(0)}{\mathcal{F}}(v, \tau_0, \tau), \quad \overset{(0)}{\mathcal{E}}(\tau) + c\overset{(0)}{\mathcal{E}}_{\mathcal{S}}(\tau) + c\int_{\tau_0}^{\tau} {}^{\chi}\overset{(-1-\delta)}{\mathcal{E}'}(\tau')\,d\tau' + c\int_{\tau_0}^{\tau} \underset{-1}{\mathcal{E}'}(\tau')\,d\tau'$$
$$\leq \lambda\overset{(0)}{\mathcal{E}}(\tau_0) + C\int_{\mathcal{R}(\tau_0,\tau)} |(V_0^{\mu}\partial_{\mu}\psi + w_0\psi)F| + C\int_{\mathcal{R}(\tau_0,\tau)} F^2 \tag{3-3}$$

for some constants $\lambda \geq 1,\ C \geq 1,\ 0 < c < 1$, and where V_0 is a fixed vector field and w_0 is a fixed function satisfying

$$|g_0(V_0, L)| \lesssim 1, \quad |g_0(V_0, \underline{L})| \lesssim 1, \quad \sum |g_0(V_0, \Omega_i)|^2 \lesssim 1, \quad |w_0| \lesssim r^{-1}. \tag{3-4}$$

Here, the unprimed energies are defined by

$$\overset{(0)}{\mathcal{E}}(\tau) := \int_{\Sigma(\tau)} |L\psi|^2 + |\nabla\!\!\!/\psi|^2 + \iota_{r\leq R}|\underline{L}\psi|^2 + r^{-2}\psi^2, \tag{3-5}$$

$$\overset{(0)}{\mathcal{E}}_{\mathcal{S}}(\tau) := \int_{\mathcal{S}(\tau_0,\tau)} |L\psi|^2 + |\nabla\!\!\!/\psi|^2 + |\underline{L}\psi|^2 + \psi^2, \tag{3-6}$$

$$\overset{(0)}{\mathcal{F}}(\tau_0, \tau, v) := \int_{\underline{C}_v\cap\mathcal{R}(\tau_0,\tau)} |\underline{L}\psi|^2 + |\nabla\!\!\!/\psi|^2 + r^{-2}\psi^2, \tag{3-7}$$

while the primed energies are defined by

$$\begin{aligned} {}^{\chi}\overset{(-1-\delta)}{\mathcal{E}'}(\tau) &:= \int_{\Sigma(\tau)} r^{-1-\delta}\chi(r)(|L\psi|^2 + |\underline{L}\psi|^2 + |\nabla\!\!\!/\psi|^2), \\ \underset{-1}{\mathcal{E}'}(\tau) &:= \int_{\Sigma(\tau)} r^{-3-\delta}\psi^2. \end{aligned} \tag{3-8}$$

(The prime $'$ notation will be used in general to denote energies that naturally appear in bulk terms.) The estimate (3-3) is to hold for all smooth ψ such that the right-hand side of (3-3) is finite. In the above, we already see the δ fixed in (3-1). For future reference let us also define the quantity

$$\overset{(-1-\delta)}{\mathcal{E}'}(\tau) := \int_{\Sigma(\tau)} r^{-1-\delta}(|L\psi|^2 + |\underline{L}\psi|^2 + |\nabla\!\!\!/\psi|^2) + r^{-3-\delta}\psi^2. \tag{3-9}$$

Note that if $\chi = 1$ identically, then ${}^{\chi}\overset{(-1-\delta)}{\mathcal{E}'} + \underset{-1}{\mathcal{E}'} = \overset{(-1-\delta)}{\mathcal{E}'}$. We note that all expressions defined above are T-invariant.

We distinguish the constants C and λ in (3-3) to highlight the significance of the case $\lambda = 1$, when (3-3) is derived from a suitable energy identity and the energies are replaced by exact fluxes. See already Section 3.4.1.

Note that in cases (i) and (ii), we shall see that (3-3) holds, and in fact one may drop the $\int F^2$ term on the right-hand side.

In fact, in all of the above cases more precise estimates than (3-3) are available with respect to what can actually be controlled by the right-hand side; we shall not need to make use of this here.

Again, the assumptions are motivated by our model cases of Minkowski space, Schwarzschild, and subextremal Kerr black hole exteriors. The estimate (3-3) indeed holds in the case of Kerr in the full subextremal range $|a| < M$ (this is the statement of Theorem D.1 of Appendix D). We note, however, the following remark.

Remark 3.2.1. In our arguments in subsequent sections, we will in fact only use the statement that follows from (3-3) by partitioning the middle term on the right-hand side into the regions $r \le R$ and $r \ge R$, and applying Cauchy–Schwarz to the former, replacing thus the full middle term with the expression

$$\sqrt{\int_{\mathcal{R}(\tau_0,\tau_1)\cap\{r\le R\}} |L\psi|^2 + |\underline{L}\psi|^2 + |\nabla\!\!\!/\psi|^2 + r^{-2}|\psi|^2}\sqrt{\int_{\mathcal{R}(\tau_0,\tau_1)\cap\{r\le R\}} F^2} + \int_{\mathcal{R}(\tau_0,\tau)\cap\{r\ge R\}} |(V_0^{\mu}\partial_\mu\psi + w_0\psi)F|. \tag{3-10}$$

We could thus alternatively consider this weaker statement, i.e., (3-3) but with (3-10) replacing the middle term on the right-hand side, as our main assumption. We prefer, however, to keep our general assumption in the form of (3-3) because it is more compact and we do not know of a physical example where this weaker assumption holds but our original assumption does not. On the other hand, it is often easier to prove the weaker assumption directly than to prove (3-3). This is indeed the case for Kerr in the full subextremal range $|a| < M$, where in Appendix D we will in fact only give a proof of this weaker version.

3.3. ***Physical-space identities on a general Lorentzian metric.*** As explained in the introduction, for nonlinear applications to (1-1), it is most convenient when (3-3) is the result of a *physical-space energy* identity (1-4) for solutions ψ of (1-2). Thus, we will recall some general properties of such identities here.

3.3.1. *Definition of energy currents.* Physical-space energy identities can be associated to a quadruple (V, w, q, ϖ), where V^{μ} is a vector field on $\mathcal{M}$, w is a scalar function, q_{μ} is a 1-form, and $\varpi_{\mu\nu}$ is a 2-form. Given such a quadruple, a general Lorentzian metric g, and a suitably regular function ψ, we define

$$J_\mu^{V,w,q,\varpi}[g,\psi] := T_{\mu\nu}[g,\psi]V^\nu + w\psi\partial_\mu\psi + q_\mu\psi^2 + *d(\psi^2\varpi)_\mu, \tag{3-11}$$

$$K^{V,w,q}[g,\psi] := \pi^V_{\mu\nu}[g]T^{\mu\nu}[g,\psi] + \nabla^\mu w\psi\partial_\mu\psi + w\nabla^\mu\psi\partial_\mu\psi + \nabla^\mu q_\mu\psi^2 + 2\psi q_\mu g^{\mu\nu}\partial_\nu\psi, \tag{3-12}$$

$$H^{V,w}[\psi] := V^\mu\partial_\mu\psi + w\psi, \tag{3-13}$$

where

$$T_{\mu\nu}[g,\psi] = \partial_\mu\psi\partial_\nu\psi - \tfrac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\psi\partial_\beta\psi, \quad \pi^X_{\mu\nu}[g] = \tfrac{1}{2}(\mathcal{L}_X g)_{\mu\nu} = \tfrac{1}{2}(\nabla_\mu X_\nu + \nabla_\nu X_\mu),$$

and where $*\colon \Lambda^3\mathcal{M} \to \Lambda^1\mathcal{M}$ denotes the Hodge star operator.

Let us note already that it is often more natural to parametrise choices of currents in a slightly different way as twisted currents [Holzegel and Warnick 2014]; for instance, given a scalar function, one may define a twisted energy momentum tensor $\widetilde{T}_{\mu\nu}[g,\psi]$ according to (A-8) and then consider for instance currents of the form $\widetilde{J}^V_\mu[g,\psi] := \widetilde{T}_{\mu\nu}[g,\psi]V^\nu$, etc. Such a current can always be rewritten as (3-11) for some w, q, ϖ.

We note that the additional ϖ component in (3-11) arises naturally for twisted currents and is useful in generating positive zeroth-order flux terms on the boundary.

3.3.2. *The divergence identity.* With the above definitions, one can compute the following identity for a general function ψ:

$$\nabla^\mu_g J_\mu^{V,w,q,\varpi}[g,\psi] = K^{V,w,q}[g,\psi] + H^{V,w}[\psi]\Box_g\psi. \tag{3-14}$$

In particular, for solutions of the covariant wave equation $\Box_g\psi = 0$, one obtains a divergence relation between the currents J and K which both depend only on the 1-jet of ψ. See [Christodoulou 2000] for a discussion of the classification of currents with this property. Note that ϖ does not contribute to the bulk current K and neither ϖ nor q contribute to H, which moreover is independent of the metric g.

The significance of identity (3-14) is that it can be integrated in a spacetime region bounded by homologous hypersurfaces to obtain a relation between boundary fluxes of J and a bulk integral of K. We give the form of this relation below in the special case of the region $\mathcal{R}(\tau_0,\tau_1,v) \subset \mathcal{M}$.

3.3.3. *The integrated identity on $\mathcal{R}(\tau_0,\tau_1,v)$.* For a solution of (1-2), identity (3-14) upon integration in $\mathcal{R}(\tau_0,\tau_1,v)$ yields

$$\int_{\Sigma(\tau_1,v)} J[\psi]\cdot n + \int_{\mathcal{S}(\tau_0,\tau_1)} J[\psi]\cdot n + \int_{\underline{C}_v(\tau_0,\tau_1)} J[\psi]\cdot n + \int_{\mathcal{R}(\tau_0,\tau_1,v)} K[\psi] = \int_{\Sigma(\tau_0,v)} J[\psi]\cdot n - \int_{\mathcal{R}(\tau_0,\tau_1,v)} H[\psi]F, \tag{3-15}$$

where the normals and volume forms are with respect to the metric g_0 according to the convention of Section 2.5.4.

Identity (3-15) will be useful when it satisfies suitable *positivity properties for its bulk and boundary terms*.

3.4. ***Assumed unweighted first-order physical-space identities: cases* (i)–(iii).** As discussed already in the introduction, we shall *not* in general require that (3-3) is the result of an integrated divergence identity (3-15) associated to currents with a pointwise coercivity property. We shall, however, require, *in addition* to assuming (3-3), the existence of currents generating an identity (3-15) with much weaker nonnegativity properties, properties which in particular are insensitive to the presence and structure of possible trapping.

It is indeed useful to see first, however, how the existence of a physical-space proof of (3-3) via an identity of type (3-15) simplifies the considerations. Thus, we shall distinguish two simpler cases, to be called case (i) and (ii), to be discussed in Sections 3.4.1 and 3.4.2 below, where indeed (3-3) is proven via an identity (3-15), (with case (i) corresponding to the even simpler setting where there is no degeneration at all in the coercivity).

The most general case, case (iii), which represents the main goal of this paper, will be discussed in Section 3.4.3. We will introduce some helpful common notation in Section 3.4.4.

Finally, in Section 3.4.5, for the case $\mathcal{S}\neq\varnothing$, we will provide a family of currents with enhanced red-shift control at $\mathcal{H}^+$ and in the black hole interior, parametrised by a parameter ς, which will be useful for obtaining higher-order estimates in Section 3.6.

3.4.1. *Case* (i). The simplest case to consider is when (3-3) indeed follows from a physical-space energy identity (3-15), and when moreover there is in fact no degeneration in the estimate (3-3), i.e., the function χ of (3-2) satisfies $\chi=1$ identically.

That is to say, we assume that there exists a T-invariant quadruple (V,w,q,ϖ), where V is a vector field, w is a scalar function, q is a 1-form, and ϖ is a 2-form, satisfying

$$\begin{gathered}
|g_0(V,L)|\lesssim 1,\quad |g_0(V,\underline{L})|\lesssim 1,\quad \sum|g_0(V,\Omega_i)|^2\lesssim 1,\\
|w|\lesssim r^{-1},\quad |q_\mu L^\mu|\lesssim r^{-2},\quad |q_\mu\underline{L}^\mu|\lesssim r^{-2},\\
|(*\,d\varpi)_\mu L^\mu|\lesssim r^{-2},\quad |(*\,d\varpi)_\mu\underline{L}^\mu|\lesssim r^{-2},\\
|*(\varrho^L\wedge\varpi)_\mu L^\mu|\lesssim r^{-1},\quad |*(\varrho^L\wedge\varpi)_\mu\underline{L}^\mu|\lesssim r^{-1},\\
|*(\varrho^{\underline{L}}\wedge\varpi)_\mu L^\mu|\lesssim r^{-1},\quad |*(\varrho^{\underline{L}}\wedge\varpi)_\mu\underline{L}^\mu|\lesssim r^{-1},\\
|*(\varrho^i\wedge\varpi)_\mu L^\mu|\lesssim r^{-1},\quad |*(\varrho^i\wedge\varpi)_\mu\underline{L}^\mu|\lesssim r^{-1},\qquad i=1,2,3,
\end{gathered}\tag{3-16}$$

such that, defining $J^{V,w,q,\varpi}[\psi]$ and $K^{V,w,q}$ by (3-11)–(3-12), the energy identity (3-15) corresponding to these currents has the following properties: (a) the boundary terms of (3-15) on $\Sigma(\tau)$ are coercive, (b) the remaining boundary terms are nonnegative, and (c) the bulk term K is nonnegative and coercive, with no degeneration, except "at infinity", where standard derivatives are in general only controlled with weight $r^{-1-\delta}$.

More precisely, we assume the pointwise bulk coercivity relation

$$K^{V,w,q}[\psi]\gtrsim r^{-1-\delta}((L\psi)^2+(\underline{L}\psi)^2+|\nabla\!\!\!/\psi|^2)+r^{-3-\delta}\psi^2\tag{3-17}$$

and pointwise boundary coercivity relations

$$\begin{aligned} J^{V,w,q,\varpi}_\mu[\psi]n^\mu_{\Sigma(\tau)} &\gtrsim (L\psi)^2 + |\nabla\!\!\!/\psi|^2 + \iota_{r\le R}(\underline{L}\psi)^2 + r^{-2}\psi^2, \\ J^{V,w,q,\varpi}_\mu[\psi]n^\mu_{\underline{C}_v} &\gtrsim (\underline{L}\psi)^2 + |\nabla\!\!\!/\psi|^2 + r^{-2}\psi^2, \\ J^{V,w,q,\varpi}_\mu[\psi]n^\mu_{\mathcal{S}} &\gtrsim ((L\psi)^2 + (\underline{L}\psi)^2 + |\nabla\!\!\!/\psi|^2 + \psi^2). \end{aligned} \tag{3-18}$$

Here the normals are of course taken with respect to the metric g_0.

Let us note that the boundary coercivity statement on $\Sigma(\tau)$ can only possibly hold if V is timelike on $\Sigma(\tau)$.

Defining

$$\begin{gathered} \overset{(0)}{\mathfrak{E}}(\tau) := \int_{\Sigma(\tau)} J^{V,w,q,\varpi}_\mu[\psi]n^\mu_{\Sigma(\tau)}, \quad \overset{(0)}{\mathfrak{E}}_{\mathcal{S}}(\tau) := \int_{\mathcal{S}} J^{V,w,q,\varpi}_\mu[\psi]n^\mu_{\mathcal{S}}, \\ \overset{(0)}{\mathfrak{F}}(v,\tau_0,\tau_1) := \int_{\underline{C}_v\cap\mathcal{R}(\tau_0,\tau_1)} J^{V,w,q,\varpi}_\mu[\psi]n^\mu_{\underline{C}_v}, \end{gathered} \tag{3-19}$$

it follows from (3-18) that

$$\overset{(0)}{\mathcal{E}}(\tau) \lesssim \overset{(0)}{\mathfrak{E}}(\tau), \quad \overset{(0)}{\mathcal{E}}_{\mathcal{S}}(\tau) \lesssim \overset{(0)}{\mathfrak{E}}_{\mathcal{S}}(\tau), \quad \overset{(0)}{\mathcal{F}}(v,\tau_0,\tau_1) \lesssim \overset{(0)}{\mathfrak{F}}(v,\tau_0,\tau_1). \tag{3-20}$$

From (3-16), it follows that, in addition to the coercivity (3-18), we have the corresponding boundedness

$$\begin{aligned} J^{V,w,q,\varpi}_\mu[\psi]n^\mu_{\Sigma(\tau)} &\lesssim (L\psi)^2 + |\nabla\!\!\!/\psi|^2 + \iota_{r\le R}(\underline{L}\psi)^2 + r^{-2}\psi^2, \\ J^{V,w,q,\varpi}_\mu[\psi]n^\mu_{\underline{C}_v} &\lesssim (\underline{L}\psi)^2 + |\nabla\!\!\!/\psi|^2 + r^{-2}\psi^2, \\ J^{V,w,q,\varpi}_\mu[\psi]n^\mu_{\mathcal{S}} &\lesssim (L\psi)^2 + (\underline{L}\psi)^2 + |\nabla\!\!\!/\psi|^2 + \psi^2, \end{aligned} \tag{3-21}$$

and thus

$$\overset{(0)}{\mathcal{E}}(\tau) \sim \overset{(0)}{\mathfrak{E}}_r(\tau), \quad \overset{(0)}{\mathcal{E}}_{\mathcal{S}}(\tau) \sim \overset{(0)}{\mathfrak{E}}_{\mathcal{S}}(\tau), \quad \overset{(0)}{\mathcal{F}}(v,\tau_0,\tau_1) \sim \overset{(0)}{\mathfrak{F}}(v,\tau_0,\tau_1). \tag{3-22}$$

Note that, to estimate the boundary terms arising from the 2-form ϖ, we have used (2-4) and the fact that

$$*d(\psi^2\varpi) = 2\psi*(d\psi\wedge\varpi) + \psi^2*d\varpi.$$

Under the above assumptions, in the notation of Section 3.2 (recall now (3-9)), the identity (3-15) gives rise to the estimate

$$\sup_{v:\tau\le\tau(v)} \overset{(0)}{\mathfrak{F}}(v,\tau_0,\tau), \quad \overset{(0)}{\mathfrak{E}}(\tau) + \overset{(0)}{\mathfrak{E}}_{\mathcal{S}}(\tau) + c\int_{\tau_0}^{\tau_1} \overset{(-1-\delta)}{\mathcal{E}'}(\tau')\,d\tau' \le \overset{(0)}{\mathfrak{E}}(\tau_0) + \int_{\mathcal{R}(\tau_0,\tau_1)} |H[\psi]F|. \tag{3-23}$$

We note that, in view of (3-20), (3-22), and the fact that we may express

$$H[\psi] = V^\mu\partial_\mu\psi + w\psi,$$

this gives (3-3) for some $\lambda \ge 1$ and without degeneration, i.e., with $\chi = 1$, and with $V_0 = V$ and $w_0 = w$, and where including the final $\int F^2$ term in (3-3) is here unnecessary. The point of expressing the estimate in terms of the fraktur energies (3-19) is that (3-23) is a sharper statement than (3-3), corresponding to $\lambda = 1$, which will be useful for us.

Let us note immediately that Minkowski space itself, but also suitably small stationary perturbations of the Minkowski metric, satisfy the assumptions of this section (see Appendix B). More generally, given a metric g_0 as in Section 2 and a T-invariant quadruple (V, w, q, ϖ) satisfying (3-16), whose energy currents $J^{V,w,q,\varpi}[g_0, \psi]$, $K^{V,w,q}[g_0, \psi]$ satisfy the above coercivity properties (3-17), (3-18), and (3-21), it is clear that $J^{V,w,q,\varpi}[g_\epsilon, \psi]$, $K^{V,w,q}[g_\epsilon, \psi]$ retain the coercivity properties (3-17), (3-18), and (3-21) for any stationary perturbation g_ϵ of g_0 satisfying the assumptions of Section 2, sufficiently close to g, such that $g = g_\epsilon$ in $r \geq R$. Thus, we see that when, as in the present section, estimate (3-3) is proven via (3-23) and there is no degeneration, i.e., $\chi = 1$, then the estimate (3-3) can immediately be inferred to be stable to suitably small perturbations of the metric g_0.

Unfortunately, however, for most of our examples of spacetimes $(\mathcal{M}, g_0)$ of interest, it turns out that the assumptions of the present section cannot in fact hold. Specifically, if $(\mathcal{M}, g_0)$ contains trapped null geodesics, then one can show that (3-3) cannot hold with $\chi = 1$ identically, and thus no quadruple (V, w, q, ϖ) as above can exist satisfying (3-17); see [Sbierski 2015]. In particular, the assumptions of this section do not encompass the black hole cases of interest like Schwarzschild or Kerr.

3.4.2. *Case* (ii). The next simplest case is when estimate (3-3) again follows from the coercivity properties of a suitable physical-space energy identity (3-15), but where the degeneration function χ of (3-2) is now nontrivial, potentially vanishing on some set.

More precisely, we assume that there exists a T-invariant quadruple (V, w, q, ϖ), again bounded in the sense of (3-16), but now satisfying the following relaxed coercivity properties: defining currents $J^{V,w,q,\varpi}$, $K^{V,w,q}$ by (3-11)–(3-12), we again assume the boundary coercivity properties (3-18) on $J^{V,w,q,\varpi}$, but we weaken the bulk coercivity assumption on $K^{V,w,q}$ to

$$K^{V,w,q}[\psi] \gtrsim \chi(r) r^{-1-\delta}((L\psi)^2 + (\underline{L}\psi)^2 + |\nabla\!\!\!/\psi|^2) + r^{-3-\delta}\psi^2, \tag{3-24}$$

where χ is the function (3-2) in Section 3.2.

We define the fraktur energies again by (3-19), and we note that again we have (3-20) and, in view of (3-16), also (3-21) and thus (3-22). Under the above assumptions, identity (3-15) gives rise to the estimate

$$\sup_{v:\tau\leq\tau(v)} \overset{(0)}{\mathfrak{F}}(v, \tau_0, \tau), \quad \overset{(0)}{\mathfrak{E}}(\tau) + \overset{(0)}{\mathfrak{E}}_{\mathcal{S}}(\tau) + c\int_{\tau_0}^{\tau} \chi \overset{(-1-\delta)}{\mathcal{E}'}(\tau')\, d\tau' + c\int_{\tau_0}^{\tau} \mathcal{E}'_{-1}(\tau')\, d\tau' \leq \overset{(0)}{\mathfrak{E}}(\tau_0) + \int_{\mathcal{R}(\tau_0,\tau)} |H[\psi]F|. \tag{3-25}$$

We note again that, in view of (3-20) and (3-22), estimate (3-25) indeed implies (3-3), for some $\lambda \geq 1$, with $V_0 = V$ and $w_0 = w$ and where including the final $\int F^2$ term in (3-3) is here unnecessary. As with (3-23) of case (i), the point of expressing the estimate in terms of the fraktur energies (3-19) is that (3-23) is a sharper statement than (3-3), corresponding to $\lambda = 1$, which will be useful for us.

We note that a current-defining quadruple (V, w, q, ϖ) satisfying the properties of this section indeed exists for the Schwarzschild metric and can be constructed from the considerations in [Dafermos and Rodnianski 2007a; Marzuola et al. 2010]. Note that a prerequisite for even the degenerate bulk coercivity property (3-24) is that any trapped null geodesics be "unstable" in a suitable sense (cf. the

Schwarzschild–AdS case with reflective boundary conditions at infinity, where there exist stably trapped null geodesics [Holzegel and Smulevici 2014]).

In the Kerr case for all $|a| \neq 0$, even though all trapped null geodesics are again unstable, and even though estimate (3-3) is true (by [Dafermos et al. 2016]), one can show that no quadruple (V, w, q, ϖ) can give rise to currents satisfying the coercivity properties of this section; see [Alinhac 2009]. (For a higher-order current defined using second-order operators which gives an analogue of the coercivity properties here in the $|a| \ll M$ case, see however [Andersson and Blue 2015].)

One of the main motivations of this paper is to show that, from a suitable point of view, not only is a purely physical-space proof of (3-3) unnecessary for nonlinear applications, but such a proof would only result in a minor and inessential simplification of the argument. We now turn to our main case of interest, case (iii).

3.4.3. *Case* (iii). The most general case we wish to consider in this paper is where the estimate (3-3) is assumed as a "black box", i.e., it is *not* necessarily the consequence of the coercivity properties of some more fundamental physical-space identity (3-15). We note for instance that (3-3) indeed holds when $(\mathcal{M}, g_0)$ is Kerr, in fact for the full subextremal range $|a| < M$ of parameters [Dafermos et al. 2016], but as discussed above, it does not arise (see [Alinhac 2009]) from a current as in case (ii).

In this most general case, however, in addition to assuming (3-3), we will still make a further assumption on the existence of an auxiliary pair of currents $J^{V,w,q,\varpi}$, $K^{V,w,q}$ whose energy identity will *not* in general imply (3-3) but rather will be used *in combination* with (3-3). This auxiliary current will have the following properties: The bulk K current will be nonnegative in a neighbourhood of the set $\{\chi \neq 1\}$, where χ is the function (3-2) of Section 3.2 appearing in (3-3). In nontrivial applications, K will often vanish identically in this region, making it completely insensitive to the possible presence and nature of trapping. Where $\chi = 1$, on the other hand, the bulk K current will be assumed nonnegative only modulo lower-order terms, provided these lower-order terms are supported in the region $r_2 \leq r \leq \frac{1}{2}R$. Finally, for $r \geq R$, the bulk current K will control the terms familiar from cases (i) and (ii).

We now lay out the assumptions in detail.

We will define functions

$$\rho : [r_0, \infty) \to [0, 1], \qquad \xi : [r_0, \infty) \to [0, 1] \tag{3-26}$$

such that $\rho = 1$ for $r \geq \frac{1}{4}R$ and $r \leq r_2$ and such that

$$\xi = 0 \text{ in } \{\chi \neq 1\} \cup \{r \leq r_2\} \cup \left\{r \geq \tfrac{1}{4}R\right\}, \tag{3-27}$$

where χ is the function (3-2) in Section 3.2 appearing in the assumed estimate (3-3). (In nontrivial applications, ρ will vanish identically in a neighbourhood containing the set where $\chi \neq 1$.) The supports of the various functions are indicated in Figure 2. We note finally that we lose no generality in the present paper in always taking ρ, χ, ξ to be indicator functions of appropriate sets, but the sharpest estimates one could prove would often involve degeneration on sets of positive codimension.

We assume the existence of a T-invariant quadruple (V, w, q, ϖ) satisfying the boundedness properties (3-16) such that the associated current $J^{V,w,q,\varpi}[\psi]$ still satisfies (3-18) but where the bulk coercivity

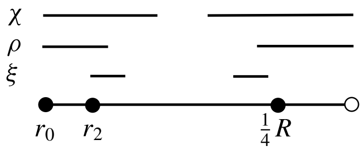

Figure 2. The supports of χ, ρ and ξ.

assumption (3-24) on $K^{V,w,q}[\psi]$ is now further relaxed to

$$K^{V,w,q}[\psi]+\tilde{A}\xi(r)\psi^2 \gtrsim \rho(r)r^{-1-\delta}((L\psi)^2+(\underline{L}\psi)^2+|\nabla\!\!\!/\psi|^2)+\rho(r)r^{-3-\delta}\psi^2, \tag{3-28}$$

where $\tilde{A}\ge 0$ is a possibly large constant.

With this current, we define the fraktur energies again by (3-19), and we again have (3-20) and, in view of (3-16), also (3-21) and thus (3-22).

In view of the relaxed coercivity properties, the identity (3-15) gives rise to

$$\sup_{v:\tau\le\tau(v)} \overset{(0)}{\mathfrak{F}}(v,\tau_0,\tau),\quad \overset{(0)}{\mathfrak{E}}(\tau)+\overset{(0)}{\mathfrak{E}}_{\mathcal{S}}(\tau)+c\int_{\tau_0}^{\tau} {}^{\rho}\overset{(-1-\delta)}{\mathcal{E}'}(\tau')+{}^{\rho}\underset{-1}{\mathcal{E}'}(\tau')\,d\tau' \le \overset{(0)}{\mathfrak{E}}(\tau_0)+A\int_{\tau_0}^{\tau'} \overset{\xi}{\underset{-1}{\mathcal{E}'}}(\tau')\,d\tau'+\int_{\mathcal{R}(\tau_0,\tau)}|H[\psi]F| \tag{3-29}$$

for an $A\ge 0$, where ${}^{\rho}\overset{(-1-\delta)}{\mathcal{E}'}(\tau)$ is defined analogously to ${}^{\chi}\overset{(-1-\delta)}{\mathcal{E}'}(\tau)$; i.e.,

$${}^{\rho}\overset{(-1-\delta)}{\mathcal{E}'}(\tau):=\int_{\Sigma(\tau)} r^{-1-\delta}\rho(r)(|L\psi|^2+|\underline{L}\psi|^2+|\nabla\!\!\!/\psi|^2),\qquad {}^{\rho}\underset{-1}{\mathcal{E}'}(\tau):=\int_{\Sigma(\tau)} r^{-3-\delta}\rho(r)\psi^2,$$

and

$$\overset{\xi}{\underset{-1}{\mathcal{E}'}}(\tau):=\int_{\Sigma(\tau)}\xi(r)\psi^2. \tag{3-30}$$

Again, we note that it is expressing things with respect to the fraktur energies which allows the constant to be exactly 1 in the above estimate (3-29), and this fact will be useful for us.

The existence of currents $J^{V,w,q,\varpi}$, $K^{V,w,q}$ satisfying the above assumptions can indeed be shown for Kerr in the range $|a|\ll M$ (and in fact for general stationary suitably small perturbations of Schwarzschild satisfying the assumptions of Section 2 and appropriate assumptions at infinity). See Appendix A. Note that estimate (3-29) is manifestly weaker than (3-3). The point, as discussed in the introduction, is that, being derived from the (relaxed) coercivity properties of (3-28) applied to (3-14), estimate (3-29), or more properly the identity (3-14) itself, can be applied directly to (1-1). See already Section 4.3.1.

3.4.4. *Summary of the unweighted assumptions for cases* (i), (ii), *and* (iii). So as to not have to always refer to separate formulas in the distinct cases (i), (ii), and (iii), we summarise the assumptions in a way which can be subsequently interpreted for all cases simultaneously.

In case (i), we set $\tilde{\rho}=\rho=\chi=1$ and $A=\tilde{A}=0$.

In case (ii), we set χ to be the function (3-2) appearing in both (3-3) and (3-24), and we set $\rho=\chi$, $\tilde{\rho}=1$, and $A=\tilde{A}=0$.

Finally, in case (iii), we set χ to be the function (3-2) of Section 3.2 appearing in (3-3), we let ρ and ξ be the functions (3-26) and the constants A and $\tilde{A}$ be as in Section 3.4.3, and we set $\tilde{\rho}=\rho$.

Our assumptions, applicable for all cases, are (a) that (3-3) holds and (b) that there exist T-invariant (V, w, q, ϖ) satisfying the bounds (3-16) such that, defining the currents $J^{V,w,q,\varpi}[\psi]$, $K^{V,w,q}[\psi]$, and $H^{V,w}[\psi]$ by (3-11)–(3-13), we have the bulk coercivity property

$$K^{V,w,q}[\psi]+\tilde{A}\xi(r)\psi^2 \gtrsim \rho(r)r^{-1-\delta}((L\psi)^2+(\underline{L}\psi)^2+|\nabla\!\!\!/\psi|^2+r^{-3}\psi^2)+\tilde{\rho}(r)r^{-3-\delta}\psi^2 \tag{3-31}$$

and the boundary coercivity properties

$$\begin{aligned} J^{V,w,q,\varpi}_\mu[\psi]n^\mu_{\Sigma(\tau)} &\gtrsim (L\psi)^2+|\nabla\!\!\!/\psi|^2+\iota_{r\le R}(\underline{L}\psi)^2+r^{-2}\psi^2,\\ J^{V,w,q,\varpi}_\mu[\psi]n^\mu_{\underline{C}_v} &\gtrsim (\underline{L}\psi)^2+|\nabla\!\!\!/\psi|^2+r^{-2}\psi^2,\\ J^{V,w,q,\varpi}_\mu[\psi]n^\mu_{\mathcal{S}} &\gtrsim (L\psi)^2+(\underline{L}\psi)^2+|\nabla\!\!\!/\psi|^2+\psi^2. \end{aligned} \tag{3-32}$$

With these currents, we define the fraktur energies again by (3-19), and we again have (3-20) and, in view of (3-16), also (3-21) and thus (3-22).

The energy identity (3-15) gives rise then to

$$\sup_{v:\tau\le\tau(v)} \overset{(0)}{\mathfrak{F}}(v,\tau_0,\tau),\quad \overset{(0)}{\mathfrak{E}}(\tau)+\overset{(0)}{\mathfrak{E}}_{\mathcal{S}}(\tau)+c\int_{\tau_0}^{\tau} {}^{\rho}\overset{(-1-\delta)}{\mathcal{E}'}(\tau')\,d\tau'+c\int_{\tau_0}^{\tau} {}^{\tilde{\rho}}\underset{-1}{\mathcal{E}'}(\tau')\,d\tau' \\ \le \overset{(0)}{\mathfrak{E}}(\tau_0)+A\int_{\tau_0}^{\tau} \underset{-1}{\overset{\xi}{\mathcal{E}'}}(\tau')\,d\tau'+\int_{\mathcal{R}(\tau_0,\tau)}|H[\psi]F|, \tag{3-33}$$

where we define

$${}^{\tilde{\rho}}\underset{-1}{\mathcal{E}'}(\tau):=\int_{\Sigma(\tau)}\tilde{\rho}(r)r^{-3-\delta}\psi^2.$$

We emphasise again that in cases (i) and (ii) the statement that (3-3) holds need not be taken as an independent assumption, as (3-3) in fact follows from (3-33) with the above definitions in these two cases.

3.4.5. *An enhanced red-shift current and enhanced positivity in the black hole interior.* In the case $\mathcal{S}\neq\varnothing$, for the purpose of higher-order estimates to be considered in Section 3.6, we will need to enhance the positivity near $\mathcal{H}^+$.

We first state the following proposition which allows us to introduce an arbitrary largeness factor ς in front of some of the terms of our coercivity estimate.

Proposition 3.4.1. *Under the assumptions of Section 3.4.4, there exists a constant $c>0$ such that the following holds*:

Given arbitrary $\varsigma\ge1$, there exist parameters $r_0\le r_0'(\varsigma)<r_{\text{Killing}}<r_1(\varsigma)\le r_1$, a translation-invariant vector field V'_ς, and a 1-form q'_ς such that, defining $J^{V'_\varsigma,w,q'_\varsigma,\varpi}[\psi]$, $K^{V'_\varsigma,w,q'_\varsigma}[\psi]$, $H^{V'_\varsigma,w}[\psi]$ as in Section 3.4.4, we have

$$K^{V'_\varsigma,w,q'_\varsigma}[\psi]\ge c(Y\psi)^2+c\varsigma\left(\sum_{i=1}^{3}(\Omega_i\psi)^2+(T\psi)^2+\psi^2\right) \tag{3-34}$$

in $r_0'(\varsigma)\le r\le r_1(\varsigma)$ and all properties of Section 3.4.4 hold for these currents with implicit constants in (3-16), (3-31), (3-32), (3-21), which may be taken independently of ς. Finally, $V'_\varsigma=V$, $q'_\varsigma=q$ for $r\ge r_1$.

Proof. This is clear by examining the proof of Theorem 7.1 of [Dafermos and Rodnianski 2013]. □

Using the above and the time-like character of ∇r in the "black hole interior" region $r < r_{\mathrm{Killing}}$, we may further modify our current to obtain the following enhanced positivity in $r_0 \le r \le r_1(\varsigma)$, in particular, all the way up to $\mathcal{S}$, at the expense of a lower-order term. The resulting modified current will be used for higher-order estimates.

Proposition 3.4.2. *Under the assumptions of Section 3.4.4, there exist constants $C > 0$ and $c > 0$ such that the following holds:*

Given arbitrary $\varsigma \ge 1$, let V'_ς and q'_ς be as given in Proposition 3.4.1. Then there exists a translation-invariant quadruple $(V_\varsigma,\ w_\varsigma,\ q_\varsigma,\ \varpi_\varsigma)$, with $V_\varsigma = V'_\varsigma$, $w_\varsigma = w$, $q_\varsigma = q'$, $\varpi_\varsigma = \varpi$ for $r \ge r_{\mathrm{Killing}}$, and a positive function $\lambda_\varsigma(r)$ such that, defining $J^{V_\varsigma,w_\varsigma,q_\varsigma,\varpi_\varsigma}[\psi]$, $K^{V_\varsigma,w_\varsigma,q_\varsigma}[\psi]$, $H^{V_\varsigma,w_\varsigma}[\psi]$ as in Section 3.4.4, the enhanced positivity (modulo lower-order terms)

$$K^{V_\varsigma,w_\varsigma,q_\varsigma}[\psi] \ge c\varsigma\lambda_\varsigma(r)\left(\sum_{i=1}^{3}(\Omega_i\psi)^2+(T\psi)^2+|r_{\mathrm{Killing}}-r|(Y\psi)^2\right)+c\lambda_\varsigma(r)(Y\psi)^2-C\varsigma\lambda_\varsigma(r)\psi^2 \quad (3\text{-}35)$$

holds in $r_0 \le r \le r_{\mathrm{Killing}}$.

Moreover, all boundedness and coercivity properties of the currents of Section 3.4.4 hold for these currents as well with constants independent of ς in the region $r \ge r_{\mathrm{Killing}}$, while in the region $r < r_{\mathrm{Killing}}$ the properties of Section 3.4.4 again hold for these currents but with (3-31) replaced by (3-35), and with coercivity bounds that now however depend in general on ς. In particular, we have

$$|H^{V_\varsigma,w_\varsigma}[\psi]| \le C\lambda_\varsigma(r)\left(|Y\psi|+\sum_{i=1}^{3}|\Omega_i\psi|+|T\psi|\right)+C\lambda_\varsigma(r)\psi^2. \quad (3\text{-}36)$$

Proof. Given ς, let $r'_0(\varsigma)$ be as in Proposition 3.4.1. Let us define $\lambda_\varsigma(r)$ to be a smooth positive function such that $\lambda_\varsigma(r) = 1$ for $r \ge r_{\mathrm{Killing}}$,

$$\lambda_\varsigma(r) = e^{-\varsigma(r_{\mathrm{Killing}}-r)} \quad (3\text{-}37)$$

for $\{r_0 \le r \le r_{\mathrm{Killing}}\} \cap (\{\varsigma \ge (r_{\mathrm{Killing}}-r)^{-1}\} \cup \{r \le r'_0(\varsigma)\})$, and

$$0 \le \frac{d\lambda_\varsigma(r)}{dr} \le 2\varsigma\lambda_\varsigma(r).$$

We define now

$$V_\varsigma := \lambda_\varsigma(r)V'_\varsigma, \quad w_\varsigma := \lambda_\varsigma(r)w, \quad q_\varsigma := \lambda_\varsigma(r)q - *(d\lambda_\varsigma \wedge \varpi), \quad \varpi_\varsigma = \lambda_\varsigma(r)\varpi.$$

We note that, under these definitions, $J^{V_\varsigma,w_\varsigma,q_\varsigma,\varpi_\varsigma}[\psi] = \lambda_\varsigma(r)J^{V'_\varsigma,w,q'_\varsigma,\varpi}[\psi]$, and thus the positivity properties of the boundary currents are preserved but with constants that now depend on ς.

We have that

$$\nabla^\mu V_\varsigma^\nu = \nabla^\mu(\lambda_\varsigma(r))V_\varsigma'^{\nu} + \lambda_\varsigma(r)\nabla^\mu V_\varsigma'^{\nu} = \frac{d\lambda_\varsigma(r)}{dr}\nabla^\mu r V_\varsigma'^{\nu} + \lambda_\varsigma(r)\nabla^\mu V_\varsigma'^{\nu},$$

and thus

$$\begin{aligned}
K^{V_\varsigma^\nu}[\psi]+w_\varsigma\nabla^\mu\psi\partial_\mu\psi &= T_{\mu\nu}\tfrac{1}{2}(\nabla^\mu V_\varsigma^\nu+\nabla^\nu V_\varsigma^\mu)+w_\varsigma\nabla^\mu\psi\partial_\mu\psi \\
&= \frac{d\lambda_\varsigma(r)}{dr}T_{\mu\nu}\nabla^\mu r V_\varsigma'^{\nu}+\lambda_\varsigma(r)\big(T_{\mu\nu}\tfrac{1}{2}(\nabla^\mu V_\varsigma'^{\nu}+\nabla^\nu V_\varsigma'^{\mu})+w\nabla^\mu\psi\partial_\mu\psi\big) \\
&\geq c\frac{d\lambda_\varsigma(r)}{dr}\bigg(\sum_{i=1}^{3}(\Omega_i\psi)^2+(T\psi)^2+|r_{\text{Killing}}-r|(Y\psi)^2\bigg) \\
&\qquad\qquad +c\lambda_\varsigma(r)\bigg((Y\psi)^2+\sum_{i=1}^{3}(\Omega_i\psi)^2+(T\psi)^2\bigg) \\
&\geq c\varsigma\lambda_\varsigma(r)\bigg(\sum_{i=1}^{3}(\Omega_i\psi)^2+(T\psi)^2+|r_{\text{Killing}}-r|(Y\psi)^2\bigg)+c\lambda_\varsigma(r)(Y\psi)^2.
\end{aligned}$$

For the first inequality, we are using the fact that ∇r is a smooth vector field which is future-directed null on $\mathcal{H}^+$ and future-directed timelike in $r<r_{\text{Killing}}$ by our assumptions from Section 2.4, as well as the fact that

$$T_{\mu\nu}[\psi]\tfrac{1}{2}(\nabla^\mu V_\varsigma'^{\nu}+\nabla^\nu V_\varsigma'^{\mu})+w\nabla^\mu\psi\partial_\mu\psi\geq c\bigg((Y\psi)^2+\sum_{i=1}^{3}(\Omega_i\psi)^2+(T\psi)^2\bigg), \tag{3-38}$$

which follows because coercivity of the current $K^{V_\varsigma',w,q_\varsigma'}$ asserted in (3-34) requires in particular coercivity of its highest-order terms. Note that the second inequality above clearly follows from the first wherever (3-37) holds. On the other hand, wherever (3-37) does not hold, we again obtain the second inequality in view of the definition of $\lambda_\varsigma(r)$ since, in this region, the enhanced positivity (3-34) applies, allowing us to put extra ς factors on the second two terms of (3-38) as well as the bound $\varsigma(r_{\text{Killing}}-r)\leq 1$.

On the other hand,

$$\begin{aligned}
&|K^{w_\varsigma,q_\varsigma}[\psi]-w_\varsigma\nabla^\mu\psi\partial_\mu\psi| \\
&\quad=|\nabla^\mu w_\varsigma\psi\partial_\mu\psi+\nabla^\mu(q_\varsigma)_\mu\psi^2+2\psi(q_\varsigma)_\mu g^{\mu\nu}\partial_\nu\psi| \\
&\quad\leq C\bigg(\frac{d\lambda_\varsigma(r)}{dr}+\lambda_\varsigma(r)\bigg)|\psi|\bigg(|Y\psi|+\sum_{i=1}^{3}|\Omega_i\psi|+|T\psi|\bigg) \\
&\qquad\qquad+C\bigg(\frac{d\lambda_\varsigma(r)}{dr}+\lambda_\varsigma(r)\bigg)\psi^2+C\bigg(\frac{d\lambda_\varsigma(r)}{dr}+C\lambda_\varsigma(r)\bigg)|\psi|\bigg(|Y\psi|+\sum_{i=1}^{3}|\Omega_i\psi|+|T\psi|\bigg),
\end{aligned}$$

whence we deduce

$$|K^{w_\varsigma,q_\varsigma}[\psi-w_\varsigma\nabla^\mu\psi\partial_\mu\psi]|\leq\tfrac{1}{2}(K^{V_\varsigma^\nu}[\psi]+w_\varsigma\nabla^\mu\psi\partial_\mu\psi)+C\varsigma\lambda_\varsigma(r)\psi^2.$$

Further,

$$|H^{V_\varsigma,w_\varsigma}[\psi]|=|V_\varsigma^\nu\partial_\nu\psi|=|\lambda_\varsigma(r)V_\varsigma'^{\nu}\psi+w_\varsigma(r)\psi|\leq C\lambda_\varsigma(r)\bigg(|Y\psi|+\sum_{i=1}^{3}|\Omega_i\psi|+|T\psi|\bigg)+C\lambda_\varsigma(r)|\psi|,$$

giving (3-36). The statement (3-35) now follows since we have

$$K^{V_\varsigma,w_\varsigma,q_\varsigma}=K^{V_\varsigma}+w_\varsigma\nabla^\mu\psi\partial_\mu\psi+K^{w_\varsigma,q_\varsigma}-w_\varsigma\nabla^\mu\psi\partial_\mu\psi$$

and the above bounds. □

3.5. ***The r^p hierarchy.*** For nonlinear applications, we will need to extend our estimates to suitable weighted estimates satisfying the r^p hierarchy [Dafermos and Rodnianski 2010b].

In a suitable framework, very general assumptions on stationary metrics g_0 allowing one to apply the r^p hierarchy are contained in [Moschidis 2016]. Here, let us note that this class was shown in particular to contain Minkowski space, Schwarzschild, and Kerr in the full range of parameters.

So as not to translate to the setup of [Moschidis 2016], however, and rather than formulate the most general asymptotically flat assumptions which we will allow in terms of g_0, it will be convenient to make the assumptions *directly* in terms of coercivity properties of appropriate currents defined in a region $r \geq \widetilde{R} \geq R$ for some large $\widetilde{R}$.

To give our precise assumption, let us first define $\overset{(p)}{V}{}^{\circ}_{\text{far}} := r^p L$ and let $\overset{(p)}{w}{}^{\circ}_{\text{far}}$, $\overset{(p)}{q}{}^{\circ}_{\text{far}}$, $\overset{(p)}{\varpi}{}^{\circ}_{\text{far}}$ be as defined in Section B.2. We assume then that, for any $\delta \leq p \leq 2-\delta$, there exists a T-invariant quadruple $(\overset{(p)}{V}_{\text{far}}, \overset{(p)}{w}_{\text{far}}, \overset{(p)}{q}_{\text{far}}, \overset{(p)}{\varpi}_{\text{far}})$, with $\overset{(p)}{V}_{\text{far}} = \overset{(p)}{V}{}^{\circ}_{\text{far}} + \widetilde{V}_{\text{far}}$ a vector field, $\overset{(p)}{w}_{\text{far}} = \overset{(p)}{w}{}^{\circ}_{\text{far}} + \tilde{w}_{\text{far}}$ a scalar function, $\overset{(p)}{q}_{\text{far}} = \overset{(p)}{q}{}^{\circ}_{\text{far}} + \tilde{q}_{\text{far}}$ a 1-form, and $\overset{(p)}{\varpi}_{\text{far}} = \overset{(p)}{\varpi}{}^{\circ}_{\text{far}} + \tilde{\varpi}_{\text{far}}$ a 2-form, defined on $r \geq \widetilde{R}$ and satisfying

$$\begin{gathered} |g(\widetilde{V}_{\text{far}}, L)| \lesssim 1, \quad |g(\widetilde{V}_{\text{far}}, \underline{L})| \lesssim 1, \quad \sum |g(\widetilde{V}_{\text{far}}, \Omega_i)|^2 \lesssim 1, \\ |\tilde{w}_{\text{far}}| \lesssim r^{-1}, \quad |L^\mu \tilde{q}_{\text{far}\mu}| \lesssim r^{-2}, \quad |\underline{L}^\mu \tilde{q}_{\text{far}\mu}| \lesssim r^{-2}, \\ |(* d\tilde{\varpi}_{\text{far}})_\mu L^\mu| \lesssim r^{-2}, \quad |(* d\tilde{\varpi}_{\text{far}})_\mu \underline{L}^\mu| \lesssim r^{-2}, \\ |* (\varrho^L \wedge \tilde{\varpi}_{\text{far}})_\mu L^\mu| \lesssim r^{-1}, \quad |* (\varrho^L \wedge \tilde{\varpi}_{\text{far}})_\mu \underline{L}^\mu| \lesssim r^{-1}, \\ |* (\varrho^{\underline{L}} \wedge \tilde{\varpi}_{\text{far}})_\mu L^\mu| \lesssim r^{-1}, \quad |* (\varrho^{\underline{L}} \wedge \tilde{\varpi}_{\text{far}})_\mu \underline{L}^\mu| \lesssim r^{-1}, \\ |* (\varrho^i \wedge \tilde{\varpi}_{\text{far}})_\mu L^\mu| \lesssim r^{-1}, \quad |* (\varrho^i \wedge \tilde{\varpi}_{\text{far}})_\mu \underline{L}^\mu| \lesssim r^{-1}, \qquad i = 1, 2, 3, \end{gathered} \tag{3-39}$$

such that, defining the associated currents $\overset{(p)}{J}_{\text{far}}$, $\overset{(p)}{K}_{\text{far}}$ by (3-11), (3-12), these satisfy the weighted bulk coercivity property

$$\overset{(p)}{K}_{\text{far}}[\psi] \gtrsim r^{p-1}((r^{-1}L(r\psi))^2 + (L\psi)^2 + |\nabla\!\!\!/ \psi|^2) + r^{-1-\delta}(\underline{L}\psi)^2 + r^{p-3}\psi^2 \tag{3-40}$$

and the weighted boundary coercivity properties

$$\begin{aligned} \overset{(p)}{J}_{\text{far}\mu}[\psi] n^\mu_{\Sigma(\tau)} &\gtrsim r^p (r^{-1}L(r\psi))^2 + r^{\frac{p}{2}}(L\psi)^2 + |\nabla\!\!\!/ \psi|^2 + r^{\frac{p}{2}-2}\psi^2, \\ \overset{(p)}{J}_{\text{far}\mu}[\psi] n^\mu_{\underline{C}_v} &\gtrsim (\underline{L}\psi)^2 + r^p |\nabla\!\!\!/ \psi|^2 + r^{p-2}\psi^2. \end{aligned} \tag{3-41}$$

We note that the $r^{p-1}(r^{-1}L(r\psi))^2$ term is redundant in (3-40) as it can be estimated pointwise from $r^{p-1}(L\psi)^2$ and $r^{p-3}\psi^2$. We retain it to compare with the boundary term (3-41) where it is necessary to retain explicitly the $(r^{-1}L(r\psi))^2$ term, as it is not controlled by the terms $r^{p/2}(L\psi)^2$ and $r^{p/2-2}\psi^2$.

See Appendix B for the construction of such a current on Minkowski space and a broad class of spacetimes with suitable asymptotic flatness assumptions at infinity, including Schwarzschild and Kerr in the full subextremal range $|a| < M$.

Let us note immediately that, given such currents satisfying the far-away coercivity assumptions (3-40) and (3-41) and given (V, w, q, ϖ) as in Section 3.4.4, by defining a suitable cut-off function $\zeta(r)$ with $\zeta = 0$ for $r \leq \widetilde{R}$ and $\zeta(r) = 1$ for $r \geq \widetilde{R} + 1$, introducing a small fixed parameter $e > 0$, and defining the

T-invariant vector field, functions and forms

$$\overset{(p)}{V} = V + e\zeta(r)\overset{(p)}{V}_{\mathrm{far}}, \quad \overset{(p)}{w} = w + e\zeta\overset{(p)}{w}_{\mathrm{far}}, \quad \overset{(p)}{q} = q + e\zeta\overset{(p)}{q}_{\mathrm{far}}, \quad \overset{(p)}{\varpi} = \varpi + e\zeta\overset{(p)}{\varpi}_{\mathrm{far}}, \tag{3-42}$$

one sees immediately that the associated currents $\overset{(p)}{J}$, $\overset{(p)}{K}$ satisfy the global (relaxed) weighted bulk coercivity assumptions

$$\overset{(p)}{K}[\psi]+\tilde{A}\xi(r)\psi^2 \gtrsim r^{p-1}\rho(r)((r^{-1}L(r\psi))^2+(L\psi)^2+|\nabla\!\!\!/\psi|^2)+r^{-1-\delta}\rho(r)(\underline{L}\psi)^2+r^{p-3}\tilde{\rho}(r)\psi^2 \tag{3-43}$$

and the global weighted boundary coercivity assumptions

$$\begin{aligned} \overset{(p)}{J}_\mu[\psi]n^\mu_{\Sigma(\tau)} &\gtrsim r^p(r^{-1}L(r\psi))^2 + r^{\frac{p}{2}}(L\psi)^2 + |\nabla\!\!\!/\psi|^2 + \iota_{r\le R}(\underline{L}\psi)^2 + r^{\frac{p}{2}-2}\psi^2, \\ \overset{(p)}{J}_\mu[\psi]n^\mu_{\underline{C}_v} &\gtrsim (\underline{L}\psi)^2 + r^p|\nabla\!\!\!/\psi|^2 + r^{p-2}\psi^2, \\ \overset{(p)}{J}_\mu[\psi]n^\mu_{\mathcal{S}} &\gtrsim (L\psi)^2 + (\underline{L}\psi)^2 + |\nabla\!\!\!/\psi|^2 + \psi^2, \end{aligned} \tag{3-44}$$

as one can absorb the error terms in the region $\widetilde{R} \le r \le \widetilde{R}+1$ arising from the cutoff by terms controlled by the coercivity relations (3-31) and (3-32), provided e is suitably chosen. We define also the associated $\overset{(p)}{H} = \overset{(p)}{V}{}^\mu\partial_\mu\psi + \overset{(p)}{w}$, so that the divergence identity (3-14) holds with $\overset{(p)}{J}$, $\overset{(p)}{K}$, and $\overset{(p)}{H}$.

From the bounds (3-39) and (3-44), we see that we have in fact

$$\begin{aligned} \overset{(p)}{J}_\mu[\psi]n^\mu_{\Sigma(\tau)} &\sim r^p(r^{-1}L(r\psi))^2 + r^{\frac{p}{2}}(L\psi)^2 + |\nabla\!\!\!/\psi|^2 + \iota_{r\le R}(\underline{L}\psi)^2 + r^{\frac{p}{2}-2}\psi^2, \\ \overset{(p)}{J}_\mu[\psi]n^\mu_{\underline{C}_v} &\sim (\underline{L}\psi)^2 + r^p|\nabla\!\!\!/\psi|^2 + r^{p-2}\psi^2, \\ \overset{(p)}{J}_\mu[\psi]n^\mu_{\mathcal{S}} &\sim (L\psi)^2 + (\underline{L}\psi)^2 + |\nabla\!\!\!/\psi|^2 + \psi^2. \end{aligned} \tag{3-45}$$

To state the resulting weighted versions of estimate (3-3) let us introduce some notation. For $\delta \le p \le 2-\delta$ we define

$$\overset{(p)}{\mathcal{E}}(\tau) := \overset{(0)}{\mathcal{E}}(\tau) + \int_{\Sigma(\tau)\cap\{r\ge R\}} r^p(r^{-1}L(r\psi))^2 + r^{\frac{p}{2}}(L\psi)^2 + r^{\frac{p}{2}-2}\psi^2, \tag{3-46}$$

$$\overset{(p)}{\mathcal{F}}(v,\tau_0,\tau) := \overset{(0)}{\mathcal{F}}(v,\tau_0,\tau) + \int_{\underline{C}_v\cap\mathcal{R}(\tau_0,\tau)} r^p|\nabla\!\!\!/\psi|^2 + r^{p-2}\psi^2, \tag{3-47}$$

$$\overset{(p-1)}{\mathcal{E}'}(\tau) := \overset{(-1-\delta)}{\mathcal{E}'}(\tau) + \int_{\Sigma(\tau)\cap\{r\ge R\}} r^{p-1}((r^{-1}L(r\psi))^2 + (L\psi)^2 + |\nabla\!\!\!/\psi|^2) + r^{p-3}\psi^2. \tag{3-48}$$

We also define versions of (3-48) where the first term is replaced by ${}^{\chi}\overset{(-1-\delta)}{\mathcal{E}'}(\tau)$ or ${}^{\rho}\overset{(-1-\delta)}{\mathcal{E}'}(\tau)$, respectively. We will call these ${}^{\chi}\overset{(p-1)}{\mathcal{E}'}(\tau)$ and ${}^{\rho}\overset{(p-1)}{\mathcal{E}'}(\tau)$. Note that these latter two expressions do not control a zeroth-order term in the region $r < R$.

Note the following properties. For $2-\delta \ge p \ge \delta$, we have

$$\overset{(p)}{\mathcal{E}} \gtrsim \overset{(p')}{\mathcal{E}},\ \overset{(p)}{\mathcal{F}} \gtrsim \overset{(p')}{\mathcal{F}} \text{ for } p \ge p' \ge \delta \text{ or } p'=0, \quad \overset{(p-1)}{\mathcal{E}'} \gtrsim \overset{(p-1)}{\mathcal{E}} \text{ for } p \ge 1+\delta, \quad \overset{(p-1)}{\mathcal{E}'} \gtrsim \overset{(0)}{\mathcal{E}} \text{ for } p \ge 1. \tag{3-49}$$

Let us also define the fluxes

$$\overset{(p)}{\mathfrak{E}}(\tau) := \int_{\Sigma(\tau)} \overset{(p)}{J}_\mu[\psi]n^\mu_{\Sigma(\tau)}, \quad \overset{(p)}{\mathfrak{E}}_{\mathcal{S}}(\tau) := \int_{\mathcal{S}(\tau_0,\tau)} \overset{(p)}{J}_\mu[\psi]n^\mu_{\mathcal{S}}, \quad \overset{(p)}{\mathfrak{F}}(v,\tau_0,\tau) := \int_{\underline{C}_v\cap\mathcal{R}(\tau_0,\tau)} \overset{(p)}{J}_\mu[\psi]n^\mu_{\underline{C}_v}. \tag{3-50}$$

We note that, by (3-45), it follows that

$$\overset{(p)}{\mathfrak{E}}(\tau) \sim \overset{(p)}{\mathcal{E}}(\tau), \quad \overset{(p)}{\mathfrak{E}}_{\mathcal{S}}(\tau_0, \tau) = \overset{(0)}{\mathfrak{E}}_{\mathcal{S}}(\tau_0, \tau) \sim \overset{(0)}{\mathcal{E}}_{\mathcal{S}}(\tau_0, \tau), \tag{3-51}$$

$$\overset{(p)}{\mathfrak{F}}(v, \tau_0, \tau) \sim \overset{(p)}{\mathcal{F}}(v, \tau_0, \tau). \tag{3-52}$$

We may now state the main result of this section.

Proposition 3.5.1. *Let $(\mathcal{M}, g_0)$ satisfy the assumptions of Sections 2 and 3.4.4, and let (V, w, q, ϖ) be as in Section 3.4.4. (In particular, the estimates (3-3) and (3-33) are assumed to hold.) Assume for all $0 < \delta \le p \le 2 - \delta$ the existence of a quadruple $(\overset{(p)}{V}_{\mathrm{far}}, \overset{(p)}{w}_{\mathrm{far}}, \overset{(p)}{q}_{\mathrm{far}}, \overset{(p)}{\varpi}_{\mathrm{far}})$ as above satisfying (3-39) and the far-away coercivity properties (3-40) and (3-41).*

Then, for all $0 < \delta \le p \le 2 - \delta$, the estimate

$$\sup_{v:\tau \le \tau(v)} \overset{(p)}{\mathfrak{F}}(v, \tau_0, \tau), \quad \overset{(p)}{\mathfrak{E}}(\tau) + \overset{(p)}{\mathfrak{E}}_{\mathcal{S}}(\tau_0, \tau) + c\int_{\tau_0}^{\tau} {}^{\rho}\overset{(p-1)}{\mathcal{E}'}(\tau')\,d\tau' + c\int_{\tau_0}^{\tau} {}^{\tilde{\rho}}\underset{-1}{\mathcal{E}'}(\tau')\,d\tau'$$
$$\le \overset{(p)}{\mathfrak{E}}(\tau_0) + A\int_{\tau_0}^{\tau} \overset{\xi}{\underset{-1}{\mathcal{E}'}}(\tau') + \int_{\mathcal{R}(\tau_0,\tau)} |\overset{(p)}{H}[\psi]F| + C\int_{\mathcal{R}(\tau_0,\tau)} F^2 \tag{3-53}$$

as well as the estimate

$$\sup_{v:\tau \le \tau(v)} \overset{(p)}{\mathcal{F}}(v, \tau_0, \tau) + \overset{(p)}{\mathcal{E}}(\tau) + \overset{(0)}{\mathcal{E}}_{\mathcal{S}}(\tau_0, \tau) + \int_{\tau_0}^{\tau} {}^{\chi}\overset{(p-1)}{\mathcal{E}'}(\tau')\,d\tau' + \int_{\tau_0}^{\tau} \underset{-1}{\mathcal{E}'}(\tau')\,d\tau'$$
$$\lesssim \overset{(p)}{\mathcal{E}}(\tau_0) + \int_{\mathcal{R}(\tau_0,\tau)} (|r^p r^{-1} L(r\psi)| + |\widetilde{V}_p^{\mu}\partial_\mu\psi| + |\tilde{w}_p\psi|)|F| + \int_{\mathcal{R}(\tau_0,\tau)} F^2 \tag{3-54}$$

hold for all $\tau_0 \le \tau$, where $\widetilde{V}_p$ is a fixed vector field and $\tilde{w}_p$ is a fixed function satisfying

$$|g(\widetilde{V}_p, L)| \lesssim 1, \quad |g(\widetilde{V}_p, \underline{L})| \lesssim 1, \quad \sum |g(\widetilde{V}_p, \Omega_i)|^2 \lesssim 1, \quad |\tilde{w}_p| \lesssim r^{-1}. \tag{3-55}$$

Remark 3.5.2. In the case where we replace the middle term of (3-3) with (3-10), we should add the first term of (3-10) to the right-hand side of (3-54).

Proof. The estimate (3-53) follows immediately from the energy identity (3-15) corresponding to the current $\overset{(p)}{J}$ in view of the properties assumed.

Note that in cases (i) or (ii), estimate (3-53) already implies (3-54) in view also of (3-39). In general, to obtain (3-54), we add a large multiple of estimate (3-3) to (3-53). This allows us to absorb the term multiplying A on the right-hand side of (3-53) in view of the trivial relation

$$\overset{\xi}{\underset{-1}{\mathcal{E}'}} \lesssim \underset{-1}{\mathcal{E}'}. \qquad \square$$

For consistency, we will in what follows often denote the quadruple (V, w, q, ϖ) of Section 3.4.4 as $\overset{(p)}{V}, \overset{(p)}{w}, \overset{(p)}{q}, \overset{(p)}{\varpi}$, and the currents as $\overset{(0)}{J}, \overset{(0)}{K}, \overset{(0)}{H}$.

Finally, in view of Proposition 3.4.2, given $\varsigma \ge 1$, we may define versions of the above currents where $V_\varsigma, w_\varsigma, q_\varsigma, \varpi_\varsigma$ replace V, w, q, ϖ, respectively, in definition (3-42). We will denote these currents as

$$\overset{(p)}{J}_{\varsigma}, \quad \overset{(p)}{K}_{\varsigma}, \quad \overset{(p)}{H}_{\varsigma}.$$

All boundedness and coercivity inequalities will continue to hold with constants independent of ς in the region $r \geq r_{\text{Killing}}$, while in the region $r < r_{\text{Killing}}$, the resulting $\overset{(p)}{K}_{\varsigma}$ will satisfy the enhanced positivity property (3-35), at the expense of lower-order terms. (In the region $r < r_{\text{Killing}}$, however, we emphasise the dependence of the coercivity constants on ς.)

3.6. *Higher-order estimates.* As is well known, in applications to nonlinear problems, one must be able to prove higher-order estimates in order for the estimates to close.

3.6.1. *The commutation vector fields $\mathfrak{D}$ and the auxiliary $\widetilde{\mathfrak{D}}$.* Let $\zeta(r)$ denote a smooth cut-off function such that $\zeta = 1$ for $r \geq \frac{3}{4}R$ and $\zeta = 0$ for $r \leq \frac{1}{2}R$. If $\mathcal{S} = \varnothing$, let $\hat{\zeta} = 0$. Otherwise, let $\hat{\zeta}(r)$ denote a smooth cut-off function such that $\hat{\zeta} = 1$ for $r \leq r_1 + \frac{1}{4}(r_2 - r_1)$ and $\hat{\zeta} = 0$ for $r \geq r_1 + \frac{1}{2}(r_2 - r_1)$. If Ω_1 is Killing, we define $\upsilon = 1$, otherwise we set $\upsilon = 0$. (More generally, we may take $\upsilon = 0$ if T coincides with the Killing generator and is timelike in $r > r_{\text{Killing}}$.)

We define

$$\mathfrak{D}_1 = T, \quad \mathfrak{D}_2 = \upsilon\Omega_1, \quad \mathfrak{D}_3 = Y, \quad \mathfrak{D}_4 = \zeta L, \quad \mathfrak{D}_5 = \zeta \underline{L}, \quad \mathfrak{D}_{5+i} = (\zeta + \hat{\zeta})\Omega_i, \quad i = 1, \dots, 3. \tag{3-56}$$

We will write $\boldsymbol{k} = (k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8)$ and $|\boldsymbol{k}| = \sum_{i=1}^{8} k_i$, and we will denote by $\mathfrak{D}^{\boldsymbol{k}}\psi$ the expression

$$\mathfrak{D}^{\boldsymbol{k}}\psi = \mathfrak{D}_1^{k_1}\mathfrak{D}_2^{k_2}\cdots\mathfrak{D}_8^{k_8}\psi. \tag{3-57}$$

Note that the vector fields (3-56) span the tangent space for $r \geq \frac{3}{4}R$ and, if $\mathcal{S} \neq \varnothing$, in $r \leq r_1 + \frac{1}{4}(r_2 - r_1)$, but not in general for $r_1 + \frac{1}{4}(r_2 - r_1) \leq r \leq \frac{3}{4}R$. It is also useful to have a collection of operators that do. We thus define $\widetilde{\mathfrak{D}}^{\boldsymbol{k}}$ to denote commutation strings of operators from the collection

$$\widetilde{\mathfrak{D}}_1 = L, \quad \widetilde{\mathfrak{D}}_2 = \underline{L}, \quad \widetilde{\mathfrak{D}}_{2+i} = \Omega_i, \quad i = 1, \dots, 6; \tag{3-58}$$

i.e., we define

$$\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi = \widetilde{\mathfrak{D}}_1^{k_1}\widetilde{\mathfrak{D}}_2^{k_2}\cdots\widetilde{\mathfrak{D}}_8^{k_8}\psi. \tag{3-59}$$

(Recall that the additional Ω_i, $i = 4, 5, 6$, were introduced in the case $\mathcal{S} = \varnothing$ to ensure the existence of a convenient globally defined spanning set of vectors. In the case $\mathcal{S} \neq \varnothing$, we may understand the above formula with $\Omega_4 = \Omega_5 = \Omega_6 = 0$.)

3.6.2. *Assumption on commutation errors in $r \geq R_k$.* As is to be expected, in order to obtain higher-order estimates on (3-3), we will need to strengthen our asymptotic flatness assumption to a higher-order statement on g_0. Again, instead of formulating sufficient conditions in terms of the decay properties of g_0 for large r, it will be convenient here to directly assume exactly the statement we shall need in terms of decay properties of the coefficients appearing in $[\mathfrak{D}^{\boldsymbol{k}}, \Box_{g_0}]\psi$.

Our assumption is thus simply the following: for all $k \geq 1$, there exists an R_k such that the following pointwise bound for $p = 0$ and for $\delta \leq p \leq 2 - \delta$ holds in $r \geq R_k$:

$$\left|\sum_{|\boldsymbol{k}|=k} \overset{(p)}{H}[\mathfrak{D}^{\boldsymbol{k}}\psi][\mathfrak{D}^{\boldsymbol{k}}, \Box_{g_0}]\psi\right| \leq \frac{1}{12}\sum_{|\boldsymbol{k}|=k} \overset{(p)}{K}[\mathfrak{D}^{\boldsymbol{k}}\psi] + C\sum_{|\boldsymbol{k}|\leq k-1} \overset{(p)}{K}[\mathfrak{D}^{\boldsymbol{k}}\psi] \quad \text{for all smooth functions } \psi. \tag{3-60}$$

Again, we emphasise that, by our conventions of Section 3.1, the constant C on the right-hand side of (3-60) in general depends on k. (In actuality, we need only assume (3-60) for all $k \le k_{\text{asympt}}$ for some sufficiently large k_{asympt}, but the statement of our theorem will then be restricted to such k.)

Assumption (3-60) is easily seen to be satisfied in all examples of Section 2.7. We note that the k-dependence of R_k is in general necessary, even for Minkowski space, as we must take $R_k \to \infty$ as $k \to \infty$.

3.6.3. *The red-shift commutation.* We recall the following:

Proposition 3.6.1 [Dafermos and Rodnianski 2013]. *Let Y be the vector field of Section 2.6.4, and let $\mathcal{H}^+$ be a Killing horizon with positive surface gravity as assumed in Section 2.4 such that the generator Z lies in the span of T and Ω_1. Then, along $\mathcal{H}^+$, we have*

$$[Y^k, \Box_{g_0}]\psi = \kappa_k(\vartheta, \varphi)(Y^{k+1}\psi) + Y^k \Box_{g_0}\psi + \sum_{\substack{|\boldsymbol{k}| \le k+1 \\ k_3 < k+1}} \alpha_{\boldsymbol{k}}(\vartheta, \phi)(\mathfrak{D}^{\boldsymbol{k}}\psi)$$

for a $\kappa_k \ge c > 0$.

We have the following:

Corollary 3.6.2. *Let $\overset{(p)}{H}$, $\overset{(p)}{H}_\varsigma$ be as defined in Section 3.5. Then there exist constants $c > 0$, $C > 0$ (independent of ς) such that, along $\mathcal{H}^+$, we have*

$$\overset{(p)}{H}[Y^k\psi][Y^k, \Box_{g_0}]\psi \ge c(Y^{k+1}\psi)^2 - C \sum_{\substack{|\boldsymbol{k}| \le k+1 \\ k_3 \ne k+1}} (\mathfrak{D}^{\boldsymbol{k}}\psi)^2. \tag{3-61}$$

Again, we emphasise that, by our conventions of Section 3.1, the constants c and C on the right-hand side of (3-61) in general depend on k.

3.6.4. *Divergence identity for the higher-order master currents.* We define the positive signature function for $1 \le |\boldsymbol{k}| \le k$,

$$\sigma(\boldsymbol{k}, k) = \begin{cases} \sigma_{12}(k) & \text{if } k_1 + k_2 = |\boldsymbol{k}|, \\ 1 & \text{otherwise}, \end{cases} \tag{3-62}$$

where σ_{12} will be chosen later such that moreover $\sigma_{12} \ge 1$.

We define an additional signature function $\varsigma(k)$, for $k \ge 1$ also to be determined later, which will be used to select the parameter of Proposition 3.4.2 to be used in the currents for higher-order estimates.

Let us finally fix a positive function

$$\varkappa(\boldsymbol{k}, k) = \varkappa(|\boldsymbol{k}|, k) = (\varkappa_0(k))^{1-|\boldsymbol{k}|} \tag{3-63}$$

for a $\varkappa_0(k) \ge 1$ to be determined later.

Given the above commutation vector fields (3-56) and the notation (3-57), we may now define currents

$$\begin{aligned} \underset{k}{\overset{(p)}{J}}[\psi] &:= \varkappa(0, k)\overset{(p)}{J}[\psi] + \sum_{1 \le |\boldsymbol{k}| \le k} \varkappa(\boldsymbol{k}, k)\sigma(\boldsymbol{k}, k)\overset{(p)}{J}_{\varsigma(k)}[\mathfrak{D}^{\boldsymbol{k}}\psi], \\ \underset{k}{\overset{(p)}{K}}[\psi] &:= \varkappa(0, k)\overset{(p)}{K}[\psi] + \sum_{1 \le |\boldsymbol{k}| \le k} \varkappa(\boldsymbol{k}, k)\sigma(\boldsymbol{k}, k)\overset{(p)}{K}_{\varsigma(k)}[\mathfrak{D}^{\boldsymbol{k}}\psi] \end{aligned} \tag{3-64}$$

and, given a collection $G_k = \{G_{\boldsymbol{k}}\}_{|\boldsymbol{k}|\le k}$ of scalar functions, the current

$$\underset{k}{\overset{(p)}{H}}[\psi]\cdot G_k := \varkappa_0(k)\overset{(p)}{H}[\psi]G_0 + \sum_{1\le|\boldsymbol{k}|\le k}\varkappa(\boldsymbol{k},k)\sigma(\boldsymbol{k},k)\overset{(p)}{H}_{\varsigma(k)}[\mathfrak{D}^{\boldsymbol{k}}\psi]G_{\boldsymbol{k}}, \tag{3-65}$$

where $\overset{(p)}{J}$, $\overset{(p)}{K}$ and $\overset{(p)}{H}$, as well as $\overset{(p)}{J}_\varsigma$, $\overset{(p)}{K}_\varsigma$ and $\overset{(p)}{H}_\varsigma$, are as defined in Section 3.5.

If ψ satisfies the inhomogeneous equation (3-3), then $\mathfrak{D}^{\boldsymbol{k}}\psi$ satisfies the equation

$$\Box_{g_0}(\mathfrak{D}^{\boldsymbol{k}}\psi) = [\mathfrak{D}^{\boldsymbol{k}}, \Box_{g_0}]\psi + \mathfrak{D}^{\boldsymbol{k}}F,$$

and the currents satisfy

$$\nabla^\mu \underset{k}{\overset{(p)}{J}}_\mu[\psi] = \underset{k}{\overset{(p)}{K}}[\psi] + \underset{k}{\overset{(p)}{H}}[\psi]\cdot\{[\mathfrak{D}^{\boldsymbol{k}}, \Box_{g_0}]\psi\} + \underset{k}{\overset{(p)}{H}}[\psi]\cdot\{\mathfrak{D}^{\boldsymbol{k}}F\}, \tag{3-66}$$

which can again be integrated in the region $\mathcal{R}(\tau_0,\tau_1,v)$ to yield

$$\begin{aligned}\int_{\Sigma(\tau_1,v)}\underset{k}{\overset{(p)}{J}}[\psi]\cdot n + \int_{\mathcal{S}(\tau_0,\tau_1)}\underset{k}{\overset{(p)}{J}}[\psi]\cdot n + \int_{\underline{C}_v(\tau_0,\tau_1)}\underset{k}{\overset{(p)}{J}}[\psi]\cdot n + \int_{\mathcal{R}(\tau_0,\tau_1,v)}\underset{k}{\overset{(p)}{K}}[\psi]\\ = \int_{\Sigma(\tau_0,v)}\underset{k}{\overset{(p)}{J}}[\psi]\cdot n - \int_{\mathcal{R}(\tau_0,\tau_1,v)}\underset{k}{\overset{(p)}{H}}[\psi]\cdot\{[\mathfrak{D}^{\boldsymbol{k}}, \Box_{g_0}]\psi\} - \int_{\mathcal{R}(\tau_0,\tau_1,v)}\underset{k}{\overset{(p)}{H}}[\psi]\cdot\{\mathfrak{D}^{\boldsymbol{k}}F\}.\end{aligned} \tag{3-67}$$

Note that

$$\underset{k}{\overset{(p)}{H}}[\psi]\cdot\{[\mathfrak{D}^{\boldsymbol{k}}, \Box_{g_0}]\psi\} = 0 \tag{3-68}$$

in $r_1 + \frac{1}{2}(r_2 - r_1) \le r \le \frac{1}{2}R$.

Finally, if

$$\varkappa_0(k) \gg \varsigma(k) \tag{3-69}$$

is sufficiently large, then we notice that $\underset{k}{\overset{(p)}{K}}[\psi] \ge 0$ even for $r \le r_{\text{Killing}}$, and in fact, in $r_0 \le r \le r_1(\varsigma)$, we have

$$\begin{aligned}\underset{k}{\overset{(p)}{K}}[\psi] \ge c\varkappa_0(k)\overset{(p)}{K}[\psi] + c\sum_{1\le|\boldsymbol{k}|\le k}\varkappa(\boldsymbol{k},k)\varsigma\lambda_\varsigma(r)\biggl(\sum_{i=1}^{3}(\Omega_i\mathfrak{D}^{\boldsymbol{k}}\psi)^2 + (T\mathfrak{D}^{\boldsymbol{k}}\psi)^2 + |r_{\text{Killing}} - r|(Y\mathfrak{D}^{\boldsymbol{k}}\psi)^2\biggr)\\ + \varkappa(\boldsymbol{k},k)\lambda_\varsigma(r)(Y\mathfrak{D}^{\boldsymbol{k}}\psi)^2,\end{aligned} \tag{3-70}$$

where we have absorbed the lower-order term in (3-35) with the wrong sign by the largeness of

$$\varkappa_0(k) = \varkappa(|\boldsymbol{k}|,k)^{-1}\varkappa(|\boldsymbol{k}|-1,k)$$

and the positivity of $\overset{(p)}{K}[\psi]$, and we have dropped the factor $\sigma(\boldsymbol{k},k)$ using that $\sigma \ge 1$. We will always assume (3-69) in what follows so that (3-70) holds.

Defining

$$\begin{gathered}\underset{k}{\overset{(p)}{\mathfrak{E}}}(\tau) := \int_{\Sigma(\tau)}\underset{k}{\overset{(p)}{J}}_\mu[\psi]n^\mu_{\Sigma(\tau)}, \quad \underset{k}{\overset{(p)}{\mathfrak{E}}}_{\mathcal{S}}(\tau_0,\tau) := \int_{\mathcal{S}(\tau_0,\tau)}\underset{k}{\overset{(p)}{J}}_\mu[\psi]n^\mu_{\mathcal{S}},\\ \underset{k}{\overset{(p)}{\mathfrak{F}}}(v,\tau_0,\tau) := \int_{\underline{C}_v\cap\mathcal{R}(\tau_0,\tau)}\underset{k}{\overset{(p)}{J}}_\mu[\psi]n^\mu_{\underline{C}_v},\end{gathered} \tag{3-71}$$

note that by construction

$$\overset{(p)}{\underset{k}{\mathfrak{E}}}(\tau) \geq 0, \quad \overset{(p)}{\underset{k}{\mathfrak{E}}}_{\mathcal{S}}(\tau_0, \tau) \geq 0, \quad \overset{(p)}{\underset{k}{\mathfrak{F}}}(v, \tau_0, \tau) \geq 0,$$

and thus, from (3-67), we have in particular

$$\begin{aligned}\sup_{v:\tau_1 \leq \tau(v)} \overset{(p)}{\underset{k}{\mathfrak{F}}}(v, \tau_0, \tau_1) + \overset{(p)}{\underset{k}{\mathfrak{E}}}(\tau) + \overset{(p)}{\underset{k}{\mathfrak{E}}}_{\mathcal{S}}(\tau_0, \tau) + \int_{\mathcal{R}(\tau_0,\tau_1)} \overset{(p)}{\underset{k}{K}}[\psi] \\ \leq \overset{(p)}{\underset{k}{\mathfrak{E}}}(\tau_0) - \int_{\mathcal{R}(\tau_0,\tau_1)\cap\{r\leq r_2\}} \overset{(p)}{\underset{k}{H}}[\psi] \cdot \{[\mathfrak{D}^{\boldsymbol{k}}, \Box_{g_0}]\psi\} + \int_{\mathcal{R}(\tau_0,\tau_1)\cap\{r\geq R/2\}} |\overset{(p)}{\underset{k}{H}}[\psi] \cdot \{[\mathfrak{D}^{\boldsymbol{k}}, \Box_{g_0}]\psi\}| \\ + \int_{\mathcal{R}(\tau_0,\tau_1)} |\overset{(p)}{\underset{k}{H}}[\psi] \cdot \{\mathfrak{D}^{\boldsymbol{k}} F\}|, \end{aligned} \tag{3-72}$$

provided the terms on the right-hand side of the above inequalities are suitably integrable.

3.6.5. *Elliptic estimates for $\Box_{g_0}\psi = F$.* Since in the region $r \leq \frac{3}{4}R$ our commutation operators $\mathfrak{D}^{\boldsymbol{k}}$ do not necessarily span the tangent space, we will need to also invoke elliptic estimates. These will allow one to estimate, for solutions of $\Box_{g_0}\psi = F$, all highest-order derivatives $\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi$ from only derivatives $\mathfrak{D}^{\boldsymbol{k}}\psi$ with respect to our commutation operators together with appropriate terms involving F. These estimates require integration over hypersurfaces $\Sigma(\tau)$ or spacetime regions $\mathcal{R}(\tau_0, \tau_1)$ and rely on the fact that the commutation operators $\mathfrak{D}_i$ span an appropriate time-like direction.

For estimates at order k, we will in fact more generally need spacetime elliptic estimates in $r \leq \frac{9}{8}R_k$, despite the fact that the $\mathfrak{D}^{\boldsymbol{k}}$ span the tangent space as long as $r \geq \frac{3}{4}R$, in order to absorb commutation error terms where the formula (3-60) does not yet apply.

We have the following proposition.

Proposition 3.6.3. *Let $(\mathcal{M}, g_0)$ satisfy the assumptions of Section 2. Let ψ be a solution of the inhomogeneous equation (4-4) in $\mathcal{R}(\tau_0, \tau_1)$, and let $\tau_0 \leq \tau' \leq \tau_1$. Then, for all $k \geq 1$ and for all $r_{\text{Killing}} < r'_- < r' < r'' < r''_+ \leq R$,*

$$\begin{aligned}\int_{\Sigma(\tau')\cap\{r'\leq r\leq r''\}} \sum_{|\boldsymbol{k}|\leq k+1} (\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2 \\ \lesssim \int_{\Sigma(\tau')\cap\{r'_-\leq r\leq r''_+\}} \sum_{\substack{1\leq|\boldsymbol{k}|\leq k+1 \\ k_1+k_2\geq|\boldsymbol{k}|-1}} (\mathfrak{D}^{\boldsymbol{k}}\psi)^2 + \sum_{|\boldsymbol{k}|\leq 1} (\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2 + \sum_{|\boldsymbol{k}|\leq k-1} (\widetilde{\mathfrak{D}}^{\boldsymbol{k}} F)^2, \end{aligned} \tag{3-73}$$

where here $\lesssim$ depends on the choice of $r'_- < r' < r'' < r''_+$. We also have the estimate, for all $r' \leq r_1$,

$$\int_{\Sigma(\tau')\cap\{r\geq r'\}} \sum_{|\boldsymbol{k}|\leq k+1} (\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2 \lesssim \int_{\Sigma(\tau')\cap\{r\geq r'\}} \sum_{|\boldsymbol{k}|\leq k+1} (\mathfrak{D}^{\boldsymbol{k}}\psi)^2 + \sum_{|\boldsymbol{k}|\leq k-1} (\widetilde{\mathfrak{D}}^{\boldsymbol{k}} F)^2. \tag{3-74}$$

Note that the analogous statements to (3-73), (3-74) with integration on $\mathcal{R}(\tau_0, \tau_1) \cap \{r' \leq r \leq r''\}$, etc., follow immediately in view of the coarea formula.

In fact, even without assuming $r''_+ \leq R$, we still have the spacetime elliptic estimate

$$\int_{\mathcal{R}(\tau_0,\tau_1)\cap\{r'\leq r\leq r''\}} \sum_{|\boldsymbol{k}|\leq k+1} (\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2 \lesssim \int_{\mathcal{R}(\tau_0,\tau_1)\cap\{r'_-\leq r\leq r''_+\}} \sum_{\substack{1\leq|\boldsymbol{k}|\leq k+1\\ k_1+k_2\geq|\boldsymbol{k}|-1}} (\mathfrak{D}^{\boldsymbol{k}}\psi)^2 + \sum_{|\boldsymbol{k}|\leq 1} (\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2 + \sum_{|\boldsymbol{k}|\leq k-1} (\widetilde{\mathfrak{D}}^{\boldsymbol{k}}F)^2, \quad (3\text{-}75)$$

where again $\lesssim$ depends on the choice of $r'_- < r' < r'' < r''_+$.

Proof. Estimate (3-73) is a standard elliptic estimate, using the fact that the span of T, Ω_1 always contains a time-like direction in the region $r \geq r'_- > r_{\text{Killing}}$. Estimate (3-74) then follows from the previous in view of the fact that the $\mathfrak{D}^k$ span the tangent space for $r \leq r_1$ and $r \geq \frac{1}{2}R$.

As we have remarked, in the case $r''_+ \leq R$, estimate (3-75) follows from (3-73) by the coarea formula. It can be obtained more generally, even if $R \leq r''$ or $R \leq r''_+$, by a suitable integration by parts argument. We sketch this here for the case $k = 1$. Let us consider the most difficult case where $R < r'' < r''_+$. One first does the usual elliptic estimates on $\Sigma(\tau)\cap\{r \leq R\}$, then integrates over τ to estimate the left-hand side of (3-75) integrated over $\mathcal{R}(\tau_0, \tau_1)\cap\{r \leq R\}$ from the right-hand side, but with an additional boundary term on $r = R$. Squaring the inhomogeneous wave equation in $r \geq R$ and integrating by parts along the null cone $\Sigma(\tau)\cap\{r \geq R\}$, one may again bound the left-hand side of (3-75), integrated over $\mathcal{R}(\tau_0, \tau_1)\cap\{R \leq r \leq r''\}$, from the right-hand side and an additional boundary term on $r = R$. These boundary terms can be arranged to cancel modulo a term of the form

$$\int_{\mathcal{R}(\tau_0,\tau_1)\cap\{r=R\}} T\psi \,\not\Delta\psi,$$

where $\not\Delta$ denotes the Laplacian on $\Sigma(\tau)\cap\{r = R\}$, and this term can in turn be related to a bulk integral which can again be controlled by the right-hand side of (3-75). □

3.6.6. *Global control of the commutation errors.* We may now give the final statement allowing for the global control of the commutation error terms in the identity (3-66).

Proposition 3.6.4. *Under the assumptions of Section 3.6.2, we may choose our weight functions $\sigma_{12}(k)$ in the definition (3-62), $\varkappa_0(k)$ in the definition (3-63), and $\varsigma(k)$ such that the following holds:*

For all $k \geq 1$, there exists $r_{\text{Killing}} < r_1(k) \leq r_1$ such that, for all ψ satisfying the inhomogeneous equation (4-4) in $\mathcal{R}(\tau_0, \tau_1)$, the following estimates hold:

$$-\overset{(p)}{\underset{k}{H}}[\psi]\cdot\{[\mathfrak{D}^{\boldsymbol{k}}, \Box_{g_0}]\psi\} \leq \tfrac{1}{3}\overset{(p)}{\underset{k}{K}}[\psi], \quad r_0 \leq r \leq r_1(k), \quad (3\text{-}76)$$

$$\int_{\mathcal{R}(\tau_0,\tau_1)\cap\{r_1(k)\leq r_2\}} |\overset{(p)}{\underset{k}{H}}[\psi]\cdot\{[\mathfrak{D}^{\boldsymbol{k}}, \Box_{g_0}]\psi\}| \leq \int_{\mathcal{R}(\tau_0,\tau_1)\cap\{r_{\text{Killing}}\leq r\leq r_2\}} \tfrac{1}{3}\overset{(p)}{\underset{k}{K}}[\psi] + C\sum_{|\boldsymbol{k}|\leq k-1} |\widetilde{\mathfrak{D}}^{\boldsymbol{k}}F|^2, \quad (3\text{-}77)$$

$$\overset{(p)}{\underset{k}{H}}[\psi]\cdot\{[\mathfrak{D}^{\boldsymbol{k}}, \Box_{g_0}]\psi\} = 0, \quad r_1 + \tfrac{1}{2}(r_2 - r_1) \leq r \leq \tfrac{1}{2}R, \quad (3\text{-}78)$$

$$\int_{\mathcal{R}(\tau_0,\tau_1)\cap\{R/2\leq r\leq R_k\}} |\overset{(p)}{\underset{k}{H}}[\psi]\cdot\{[\mathfrak{D}^{\boldsymbol{k}}, \Box_{g_0}]\psi\}| \leq \int_{\mathcal{R}(\tau_0,\tau_1)\cap\{R/4\leq r\leq 9R_k/8\}} \tfrac{1}{3}\overset{(p)}{\underset{k}{K}}[\psi] + C\sum_{|\boldsymbol{k}|\leq k-1} |\widetilde{\mathfrak{D}}^{\boldsymbol{k}}F|^2, \quad (3\text{-}79)$$

$$|\overset{(p)}{\underset{k}{H}}[\psi]\cdot\{[\mathfrak{D}^{\boldsymbol{k}}, \Box_{g_0}]\psi\}| \leq \tfrac{1}{10}\overset{(p)}{\underset{k}{K}}[\psi], \quad r \geq R_k. \quad (3\text{-}80)$$

Proof. We first prove (3-80). Here we will use (3-60). We have that the left-hand side of (3-80) is bounded in $r \geq R_k$ by

$$
\begin{aligned}
&\Big|\overset{(p)}{\underset{k}{H}}[\psi]\cdot\{[\mathfrak{D}^{\boldsymbol{k}},\Box_{g_0}]\psi\}_{|\boldsymbol{k}|\leq k}\Big| \\
&\quad=\Big|\sum_{|\boldsymbol{k}|\leq k}\varkappa(\boldsymbol{k},k)\overset{(p)}{H}[\mathfrak{D}^{\boldsymbol{k}}\psi][\mathfrak{D}^{\boldsymbol{k}},\Box_{g_0}]\Big| \\
&\quad\leq\frac{1}{12}\sum_{|\boldsymbol{k}|\leq k}\sigma(\boldsymbol{k},k)\varkappa(\boldsymbol{k},k)\overset{(p)}{K}[\mathfrak{D}^{\boldsymbol{k}}\psi]+C\sum_{|\boldsymbol{k}|\leq k-1}\varkappa(|\boldsymbol{k}|+1,k)\sigma(\boldsymbol{k},k)\overset{(p)}{K}[\mathfrak{D}^{\boldsymbol{k}}\psi] \\
&\quad\leq\frac{1}{12}\sum_{|\boldsymbol{k}|\leq k}\sigma(\boldsymbol{k},k)\varkappa(\boldsymbol{k},k)\overset{(p)}{K}[\mathfrak{D}^{\boldsymbol{k}}\psi]+C\sum_{|\boldsymbol{k}|\leq k-1}(\varkappa(|\boldsymbol{k}|+1,k)\varkappa^{-1}(|\boldsymbol{k}|,k))\varkappa(|\boldsymbol{k}|,k)\sigma(\boldsymbol{k},k)\overset{(p)}{K}[\mathfrak{D}^{\boldsymbol{k}}\psi] \\
&\quad\leq\tfrac{1}{12}\overset{(p)}{\underset{k}{K}}[\psi]+\tfrac{1}{100}\overset{(p)}{\underset{k}{K}}[\psi]\leq\tfrac{1}{10}\overset{(p)}{\underset{k}{K}}[\psi]
\end{aligned}
$$

provided that $\varkappa(\boldsymbol{k},k)$ is defined so that

$$\varkappa_0^{-1}(k)\ll 1. \tag{3-81}$$

To prove (3-76), we note that, in the region $r_0\leq r\leq r_1$, we may estimate

$$
\begin{aligned}
&-\overset{(p)}{\underset{k}{H}}[\psi]\cdot\{[\mathfrak{D}^{\boldsymbol{k}},\Box_{g_0}]\psi\} \\
&\quad=-\sum_{\substack{1\leq|\boldsymbol{k}|\leq k\\ k_3<k}}\varkappa(|\boldsymbol{k}|,k)\overset{(p)}{H}_{\varsigma(k)}[\mathfrak{D}^{\boldsymbol{k}}\psi][\mathfrak{D}^{\boldsymbol{k}},\Box_{g_0}]\psi-\varkappa(k,k)\overset{(p)}{H}_{\varsigma(k)}[Y^k\psi][Y^k,\Box_{g_0}]\psi \\
&\quad\leq C\varsigma(k)^{\frac12}\lambda_{\varsigma(k)}(r)\sum_{1\leq|\boldsymbol{k}|\leq k}\varkappa(|\boldsymbol{k}|,k)\bigg((T\mathfrak{D}^{\boldsymbol{k}}\psi)^2+\sum_{i=1}^{3}(\Omega_i\mathfrak{D}^{\boldsymbol{k}}\psi)^2+(\mathfrak{D}^{\boldsymbol{k}}\psi)^2\bigg) \\
&\qquad\quad+C\varsigma(k)^{-\frac12}\lambda_{\varsigma(k)}(r)\sum_{1\leq|\boldsymbol{k}|\leq k+1}\varkappa(|\boldsymbol{k}|-1,k)(\mathfrak{D}^{\boldsymbol{k}}\psi)^2-c\lambda_{\varsigma(k)}(r)\varkappa(k,k)(Y^{k+1}\psi)^2 \\
&\qquad\qquad+C\lambda_{\varsigma(k)}(r)|r-r_{\text{Killing}}|\varkappa(k,k)\sum_{1\leq|\boldsymbol{k}|\leq k+1}(\mathfrak{D}^{\boldsymbol{k}}\psi)^2.
\end{aligned}
\tag{3-82}
$$

Here, we have applied (3-61) from Corollary 3.6.2 and the bounds on $\overset{(p)}{H}_{\varsigma(k)}$ following from (3-36) of Proposition 3.4.2. Note the dependence on $\varsigma(k)$ through $\lambda_{\varsigma(k)}(r)$, and note that the terms in the commutation identity not present on $\mathcal{H}^+$ appear with an extra $(r-r_{\text{Killing}})$ factor. (We emphasise again the conventions from Section 3.1 that constants C, c also depends on k!)

We now have that, given $\varsigma=\varsigma(k)$, from (3-70), it follows that in the region $r_0\leq r\leq r_1(\varsigma)$, provided that (3-69) holds, we have

$$
\begin{aligned}
\varsigma(k)^{\frac12}\lambda_{\varsigma(k)}(r)\sum_{1\leq|\boldsymbol{k}|\leq k}\varkappa(|\boldsymbol{k}|,k)\bigg((T\mathfrak{D}^{\boldsymbol{k}}\psi)^2+\sum_{i=1}^{3}(\Omega_i\mathfrak{D}^{\boldsymbol{k}}\psi)^2+(\mathfrak{D}^{\boldsymbol{k}}\psi)^2\bigg)&\leq C\varsigma^{-\frac12}(k)\overset{(p)}{\underset{k}{K}}[\psi]. \\
\varsigma^{-\frac12}(k)\lambda_{\varsigma(k)}(r)\sum_{1\leq|\boldsymbol{k}|\leq k+1}\varkappa(|\boldsymbol{k}|-1)(\mathfrak{D}^{\boldsymbol{k}}\psi)^2&\leq C\varsigma^{-\frac12}(k)\overset{(p)}{\underset{k}{K}}[\psi].
\end{aligned}
\tag{3-83}
$$

Finally, again provided (3-69) holds and we further restrict $r_1(\varsigma)$ such that $0 < r_1(\varsigma) - r_{\text{Killing}} \le \varsigma^{-1}$, we have

$$\lambda_{\varsigma(k)}(r)|r - r_{\text{Killing}}|\varkappa(k,k) \sum_{|\boldsymbol{k}|\le k+1} (\mathfrak{D}^{\boldsymbol{k}}\psi)^2 \le C\varsigma^{-1}(k)\overset{(p)}{\underset{k}{K}}[\psi].$$

It follows that, for $r_0 \le r \le r_1(\varsigma)$, we have

$$-\overset{(p)}{\underset{k}{H}}[\psi]\cdot\{[\mathfrak{D}^{\boldsymbol{k}}, \Box_{g_0}]\psi\} \le C\varsigma^{-\frac{1}{2}}(k)\overset{(p)}{\underset{k}{K}}[\psi]. \tag{3-84}$$

Now we fix $\varsigma = \varsigma(k)$ to be sufficiently large such that $C\varsigma^{-1/2}(k) \le \frac{1}{3}$. This yields (3-76).

Since $\varsigma(k)$ is now fixed, we also have $r_1(k) := r_1(\varsigma(k))$, and, according to our conventions, the dependence on these parameters may now be absorbed into the constants C, c, etc.

To prove (3-77), we estimate

$$\begin{aligned}
&\sum_{|\boldsymbol{k}|\le k}\int_{\mathcal{R}(\tau_0,\tau_1)\cap\{r_1(k)\le r\le r_2\}} |\overset{(p)}{\underset{k}{H}}[\psi]\cdot\{[\mathfrak{D}^{\boldsymbol{k}}, \Box_{g_0}]\psi\}| \\
&\qquad \le C\int_{\mathcal{R}(\tau_0,\tau_1)\cap\{r_1(k)\le r\le r_2\}} \sum_{|\boldsymbol{k}|\le k+1} \varkappa(|\boldsymbol{k}|-1,k)(\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2 \\
&\qquad \le C\int_{\mathcal{R}(\tau_0,\tau_1)\cap\{r_{\text{Killing}}\le r\le r_2\}} \sum_{\substack{1\le|\boldsymbol{k}|\le k+1\\ k_1+k_2\ge|\boldsymbol{k}|-1}} \varkappa(|\boldsymbol{k}|-1,k)(\mathfrak{D}^{\boldsymbol{k}}\psi)^2 + \sum_{|\boldsymbol{k}|\le 1}(\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2 + C\sum_{|\boldsymbol{k}|\le k-1}(\widetilde{\mathfrak{D}}^{\boldsymbol{k}}F)^2 \\
&\qquad \le C\int_{\mathcal{R}(\tau_0,\tau_1)\cap\{r_{\text{Killing}}\le r\le r_2\}} \sigma_{12}^{-1}(k)\overset{(p)}{\underset{k}{K}}[\psi] + C\varkappa^{-1}(0,k)\overset{(p)}{\underset{k}{K}}[\psi] + C\sum_{|\boldsymbol{k}|\le k-1}(\widetilde{\mathfrak{D}}^{\boldsymbol{k}}F)^2.
\end{aligned}$$

(We emphasise again the conventions from Section 3.1 that the constants C also depends on k.) We have used the spacetime version of estimate (3-73) of Proposition 3.6.3 in the above (i.e., (3-75)), with

$$r'_- := r_{\text{Killing}} + \tfrac{1}{2}(r_1(k) - r_{\text{Killing}}), \quad r''_+ := r_1 + \tfrac{3}{4}(r_2 - r_1), \quad r' = r_1(k), \quad r'' = r_1 + \tfrac{1}{2}(r_2 - r_1),$$

where we also use that the integrand on the left-hand side is nonzero only in $r' \le r \le r''$.

Requiring now $\sigma_{12}(k) \gg 1$ to be sufficiently large and $\varkappa(0,k) = \varkappa_0(k) \gg 1$ to be sufficiently large, we obtain (3-77).

The proof of (3-79) is analogous to (3-77) and also constrains $\sigma_{12}(k)$, $\varkappa_0(k)$ to be sufficiently large. This finally fixes $\sigma_{12}(k)$ and $\varkappa_0(k)$. Note here we use (3-75) with $r'' = R_k$ and $r''_+ = \frac{9}{8}R_k$ and in general we may have $\frac{9}{8}R_k \ge R$.

Identity (3-78) follows immediately from the definition of the commutation vector fields $\mathfrak{D}$. □

In what follows, we shall consider ς, σ, and $\varkappa$ fixed so as to satisfy the above Proposition. Thus, from now on, dependence of constants on ς, σ, and $\varkappa$ will be absorbed in the C and $\lesssim$ notation. We emphasise again that, according to our conventions from Section 3.1, all our constants c, C will in general depend on k.

We have the following immediate corollary of Proposition 3.6.4.

Corollary 3.6.5. *Let $k \geq 1$, and let ψ be a solution of the inhomogeneous equation (4-4) in $\mathcal{R}(\tau_0, \tau_1)$. Then we have the global (one-sided) bound*

$$\int_{\mathcal{R}(\tau_0,\tau_1)\cap\{r\leq r_2\}} -\overset{(p)}{\underset{k}{H}}[\psi]\cdot\{[\mathfrak{D}^{\boldsymbol{k}}, \Box_{g_0}]\psi\} + \int_{\mathcal{R}(\tau_0,\tau_1)\cap\{r\geq r_2\}} |\overset{(p)}{\underset{k}{H}}[\psi]\cdot\{[\mathfrak{D}^{\boldsymbol{k}}, \Box_{g_0}]\psi\}| \\ \leq \int_{\mathcal{R}(\tau_0,\tau_1)\cap\{r\leq r_2\}\cap\{r\geq R/4\}} \tfrac{1}{2}\overset{(p)}{\underset{k}{K}}[\psi] + C\sum_{|\boldsymbol{k}|\leq k-1} |\widetilde{\mathfrak{D}}^{\boldsymbol{k}} F|^2. \tag{3-85}$$

Proof. Note that the presence of $\frac{1}{2}$ instead of $\frac{1}{3}$ is due to the fact that both estimates (3-79) and (3-80) borrow from the bulk of the region $r \geq R_k$. Note that restricted to the region $r \geq R$, if desired, we may of course replace the $\widetilde{\mathfrak{D}}$ commutation on the last term on the right-hand side by $\mathfrak{D}$. □

3.6.7. *The higher-order energy notation.* We are now ready to derive the higher-order weighted estimates which will allow us in particular to address nonlinear problems. We will turn to the estimates themselves in Section 3.6.8. In the meantime, let us introduce some notation below.

We proceed to summarise the definitions: given $k \geq 1$, $1 \leq \tau_0 \leq \tau$, v, and a spacetime function ψ, we define the following energies.

In general, our fundamental energies without degeneration functions χ or ρ will be defined with $\widetilde{\mathfrak{D}}^{\boldsymbol{k}}$ as commutation vector fields, i.e., they will contain all derivatives at the relevant order. Specifically, we define first the unweighted energies:

$$\overset{(0)}{\mathcal{E}}(\tau) := \sum_{|\boldsymbol{k}|\leq k} \int_{\Sigma(\tau)} (L\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2 + |\nabla\!\!\!/\,\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi|^2 + \iota_{r\leq R}(\underline{L}\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2 + r^{-2}(\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2, \tag{3-86}$$

$$\overset{(0)}{\mathcal{E}}_{\mathcal{S}}(\tau_0, \tau) := \sum_{|\boldsymbol{k}|\leq k} \int_{\mathcal{S}(\tau_0,\tau)} (L\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2 + |\nabla\!\!\!/\,\psi|^2 + (\underline{L}\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2 + (\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2, \tag{3-87}$$

$$\overset{(0)}{\underset{k}{\mathcal{F}}}(v, \tau_0, \tau) := \sum_{|\boldsymbol{k}|\leq k} \int_{\underline{C}_v\cap\mathcal{R}(\tau_0,\tau)} (\underline{L}\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2 + (\nabla\!\!\!/\,\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2 + r^{-2}(\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2, \tag{3-88}$$

$$\overset{(-1-\delta)}{\underset{k}{\mathcal{E}'}}(\tau) := \sum_{|\boldsymbol{k}|\leq k} \int_{\Sigma(\tau)} r^{-1-\delta}((L\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2 + (\underline{L}\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2 + |\nabla\!\!\!/\,\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi|^2) + r^{-3-\delta}(\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2. \tag{3-89}$$

We now define (in analogy with (3-46)–(3-48)) the higher-order p-weighted energies for $\delta \leq p \leq 2-\delta$:

$$\overset{(p)}{\underset{k}{\mathcal{E}}}(\tau) := \overset{(0)}{\mathcal{E}}(\tau) + \sum_{|\boldsymbol{k}|\leq k} \int_{\Sigma(\tau)\cap\{r\geq R\}} r^p(r^{-1}L(r\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi))^2 + r^{\frac{p}{2}}(L\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2 + r^{\frac{p}{2}-2}(\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2, \tag{3-90}$$

$$\overset{(p)}{\underset{k}{\mathcal{F}}}(v, \tau_0, \tau) := \overset{(0)}{\underset{k}{\mathcal{F}}}(v) + \sum_{|\boldsymbol{k}|\leq k} \int_{\underline{C}_v\cap\mathcal{R}(\tau_0,\tau)} r^p|\nabla\!\!\!/\,\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi|^2 + r^{p-2}(\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2, \tag{3-91}$$

$$\overset{(p-1)}{\underset{k}{\mathcal{E}'}}(\tau) := \overset{(-1-\delta)}{\underset{k}{\mathcal{E}'}}(\tau) + \sum_{|\boldsymbol{k}|\leq k} \int_{\Sigma(\tau)} r^{p-1}((r^{-1}L(r\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi))^2 + (L\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2 + |\nabla\!\!\!/\,\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi|^2) + r^{p-3}(\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2. \tag{3-92}$$

Note the following important relation:

$$\overset{(p)}{\underset{k}{\mathcal{E}}} \gtrsim \overset{(p')}{\underset{k}{\mathcal{E}}},\ \overset{(p)}{\underset{k}{\mathcal{F}}} \gtrsim \overset{(p')}{\underset{k}{\mathcal{F}}} \text{ for } p \geq p' \geq \delta \text{ or } p' = 0, \quad \overset{(p-1)}{\underset{k}{\mathcal{E}'}} \gtrsim \overset{(p-1)}{\underset{k}{\mathcal{E}}} \text{ for } p \geq 1+\delta, \quad \overset{(p-1)}{\underset{k}{\mathcal{E}'}} \gtrsim \overset{(0)}{\mathcal{E}} \text{ for } p \geq 1, \tag{3-93}$$

representing the higher-order analogue of (3-49)

In contrast, we define the higher-order energies carrying the degeneration functions χ, ρ and $\tilde{\rho}$ in terms of the $\mathfrak{D}$ commutators:

$$\overset{\chi(-1-\delta)}{\underset{k}{\mathcal{E}'}}(\tau) := \sum_{|\boldsymbol{k}|\leq k} \int_{\Sigma(\tau)} r^{-1-\delta}\chi(r)((L\mathfrak{D}^k\psi)^2 + (\underline{L}\mathfrak{D}^k\psi)^2 + |\nabla\!\!\!/\mathfrak{D}^k\psi|^2), \tag{3-94}$$

$$\overset{\rho(-1-\delta)}{\underset{k}{\mathcal{E}'}}(\tau) := \sum_{|\boldsymbol{k}|\leq k} \int_{\Sigma(\tau)} r^{-1-\delta}\rho(r)((L\mathfrak{D}^k\psi)^2 + (\underline{L}\mathfrak{D}^k\psi)^2 + |\nabla\!\!\!/\mathfrak{D}^k\psi|^2), \tag{3-95}$$

$$\overset{\tilde{\rho}(-3-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau) := \sum_{|\boldsymbol{k}|\leq k} \int_{\Sigma(\tau)} \tilde{\rho}(r) r^{-3-\delta}(\mathfrak{D}^k\psi)^2, \tag{3-96}$$

and then

$$\overset{\chi(p-1)}{\underset{k}{\mathcal{E}'}}(\tau) := \overset{\chi(-1-\delta)}{\underset{k}{\mathcal{E}'}}(\tau) + \sum_{|\boldsymbol{k}|\leq k} \int_{\Sigma(\tau)\cap\{r\geq R\}} r^{p-1}((r^{-1}L(r\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi))^2 + (L\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2 + |\nabla\!\!\!/\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi|^2) + r^{p-3}(\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2, \tag{3-97}$$

$$\overset{\rho(p-1)}{\underset{k}{\mathcal{E}'}}(\tau) := \overset{\rho(-1-\delta)}{\underset{k}{\mathcal{E}'}}(\tau) + \sum_{|\boldsymbol{k}|\leq k} \int_{\Sigma(\tau)\cap\{r\geq R\}} r^{p-1}((r^{-1}L(r\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi))^2 + (L\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2 + |\nabla\!\!\!/\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi|^2) + r^{p-3}(\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2. \tag{3-98}$$

(Note that in the integrals over $r \geq R$, it does not matter whether we use $\widetilde{\mathfrak{D}}$ or $\mathfrak{D}$ as these would here define directly comparable energies.)

In analogy with (3-30), we define

$$\overset{\xi}{\underset{k-1}{\mathcal{E}'}}(\tau) := \sum_{|\boldsymbol{k}|\leq k} \int_{\Sigma(\tau)} \xi(r)(\mathfrak{D}^k\psi)^2. \tag{3-99}$$

We note the fundamental relation

$$\overset{\xi}{\underset{k-1}{\mathcal{E}'}} \lesssim \overset{\chi(-1-\delta)}{\underset{k-1}{\mathcal{E}'}} + \overset{(-1-\delta)}{\underset{k-2}{\mathcal{E}'}} \tag{3-100}$$

which follows from our constraints on the support of ξ and the degeneration of χ.

Proposition 3.6.6. *For a general smooth function ψ, we have the following relations between energies*:

$$\overset{(p)}{\underset{k}{\mathfrak{E}}}(\tau) \lesssim \overset{(p)}{\underset{k}{\mathcal{E}}}(\tau), \tag{3-101}$$

$$\overset{(p)}{\underset{k}{\mathfrak{E}}}_{\mathcal{S}}(\tau) = \overset{(0)}{\underset{k}{\mathfrak{E}}}_{\mathcal{S}}(\tau) \sim \overset{(p)}{\underset{k}{\mathcal{E}}}_{\mathcal{S}}(\tau), \quad \overset{(p)}{\underset{k}{\mathfrak{F}}}(v, \tau_0, \tau) \sim \overset{(p)}{\underset{k}{\mathcal{F}}}(v, \tau_0, \tau), \tag{3-102}$$

$$\begin{aligned} \int_{\mathcal{R}(\tau_0,\tau_1)} \overset{(p)}{\underset{k}{K}}[\psi] + \sum_{|\boldsymbol{k}|\leq k} \tilde{A}\xi(r)(\mathfrak{D}^k\psi)^2 &\gtrsim \int_{\tau_0}^{\tau_1} \overset{\rho(p-1)}{\underset{k}{\mathcal{E}'}}(\tau')\,d\tau' + \int_{\tau_0}^{\tau_1} \overset{\tilde{\rho}(-3-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau')\,d\tau', \quad 2-\delta \geq p \geq \delta, \\ \int_{\mathcal{R}(\tau_0,\tau_1)} \overset{(p)}{\underset{k}{K}}[\psi] + \sum_{|\boldsymbol{k}|\leq k} \tilde{A}\xi(r)(\mathfrak{D}^k\psi)^2 &\gtrsim \int_{\tau_0}^{\tau_1} \overset{\rho(-1-\delta)}{\underset{k}{\mathcal{E}'}}(\tau')\,d\tau' + \int_{\tau_0}^{\tau_1} \overset{\tilde{\rho}(-3-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau')\,d\tau', \quad p = 0. \end{aligned} \tag{3-103}$$

For ψ a solution of the inhomogeneous equation $\Box_{g_0}\psi = F$, we have

$$\overset{(p)}{\underset{k}{\mathcal{E}}}(\tau) \lesssim \overset{(p)}{\underset{k}{\mathfrak{E}}}(\tau) + \int_{\Sigma(\tau)\cap\{r\leq R\}} \sum_{|\boldsymbol{k}|\leq k-1} (\widetilde{\mathfrak{D}}^{\boldsymbol{k}} F)^2. \tag{3-104}$$

If $\chi = 1$ and $\tilde{\rho} = 1$ identically (*as in case* (i)), *then*

$$\overset{(p-1)}{\underset{k}{\mathcal{E}'}}(\tau) \lesssim {}^{\chi}\overset{(p-1)}{\underset{k}{\mathcal{E}'}}(\tau) + {}^{\tilde{\rho}}\overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau) + \int_{\Sigma(\tau)\cap\{r\leq R\}} \sum_{|\boldsymbol{k}|\leq k-1} (\widetilde{\mathfrak{D}}^{\boldsymbol{k}} F)^2, \quad 2-\delta \geq p \geq \delta, \tag{3-105}$$

$$\overset{(-1-\delta)}{\underset{k}{\mathcal{E}'}}(\tau) \lesssim {}^{\chi}\overset{(-1-\delta)}{\underset{k}{\mathcal{E}'}}(\tau) + {}^{\tilde{\rho}}\overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau) + \int_{\Sigma(\tau)\cap\{r\leq R\}} \sum_{|\boldsymbol{k}|\leq k-1} (\widetilde{\mathfrak{D}}^{\boldsymbol{k}} F)^2, \tag{3-106}$$

and if $\tilde{\rho} = 1$ identically (*i.e., as in cases* (i) *and* (ii)), *then*

$$\overset{(p-1)}{\underset{k-1}{\mathcal{E}'}}(\tau) \lesssim {}^{\rho}\overset{(p-1)}{\underset{k}{\mathcal{E}'}}(\tau) + {}^{\tilde{\rho}}\overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau) + \int_{\Sigma(\tau)\cap\{r\leq R\}} \sum_{|\boldsymbol{k}|\leq k-2} (\widetilde{\mathfrak{D}}^{\boldsymbol{k}} F)^2, \quad 2-\delta \geq p \geq \delta, \tag{3-107}$$

$$\overset{(-1-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau) \lesssim {}^{\rho}\overset{(-1-\delta)}{\underset{k}{\mathcal{E}'}}(\tau) + {}^{\tilde{\rho}}\overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau) + \int_{\Sigma(\tau)\cap\{r\leq R\}} \sum_{|\boldsymbol{k}|\leq k-2} (\widetilde{\mathfrak{D}}^{\boldsymbol{k}} F)^2. \tag{3-108}$$

Proof. The relations (3-101)–(3-103) follow immediately from our coercivity and boundedness assumptions on the currents, while the inequalities (3-104)–(3-108) follow easily from the elliptic estimate (3-74) of Proposition 3.6.3. □

We note the following immediate corollary of the above proposition:

Corollary 3.6.7. *Let ψ be a solution of*

$$\Box_{g_0}\psi = 0 \quad \text{in } \mathcal{R}(\tau_0, \tau_1).$$

Then calligraphic and fraktur energies on $\Sigma(\tau)$ are equivalent:

$$\overset{(p)}{\underset{k}{\mathcal{E}}}(\tau) \sim \overset{(p)}{\underset{k}{\mathfrak{E}}}(\tau).$$

If $\chi = 1$ and $\tilde{\rho} = 1$ identically (*as in case* (i)), *then*

$$\overset{(p-1)}{\underset{k}{\mathcal{E}'}}(\tau) \sim {}^{\chi}\overset{(p-1)}{\underset{k}{\mathcal{E}'}}(\tau) + {}^{\tilde{\rho}}\overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau),$$
$$\overset{(-1-\delta)}{\underset{k}{\mathcal{E}'}}(\tau) \sim {}^{\chi}\overset{(-1-\delta)}{\underset{k}{\mathcal{E}'}}(\tau) + {}^{\tilde{\rho}}\overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau).$$

If $\tilde{\rho} = 1$ identically (*i.e., as in cases* (i) *and* (ii)), *then*

$$\overset{(p-1)}{\underset{k-1}{\mathcal{E}'}}(\tau) \lesssim {}^{\rho}\overset{(p-1)}{\underset{k}{\mathcal{E}'}}(\tau) + {}^{\tilde{\rho}}\overset{(-1-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau),$$
$$\overset{(-1-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau) \lesssim {}^{\rho}\overset{(-1-\delta)}{\underset{k}{\mathcal{E}'}}(\tau) + {}^{\tilde{\rho}}\overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau).$$

3.6.8. *The final higher-order estimates.* We conclude with our higher-order version of Proposition 3.5.1.

Proposition 3.6.8. *Under the assumptions of Proposition 3.5.1, let us assume the additional assumptions of Section 3.6.2.*

Fix $k \geq 1$. Then, for all $0 < \delta \leq p \leq 2-\delta$ and for all $\tau_0 \leq \tau \leq \tau_1$, we have the following statement: Let ψ be a solution of the inhomogeneous equation (4-4) in $\mathcal{R}(\tau_0, \tau_1)$. Then we have

$$
\begin{aligned}
\sup_{v:\tau\leq\tau(v)} \underset{k}{\overset{(p)}{\mathfrak{F}}}(v,\tau_0,\tau), \quad & \underset{k}{\overset{(p)}{\mathfrak{E}}}(\tau) + \underset{k\,\mathcal{S}}{\overset{(0)}{\mathfrak{E}}}(\tau_0,\tau) + c\int_{\tau_0}^{\tau} {}^{\rho}\underset{k}{\overset{(p-1)}{\mathcal{E}'}}(\tau')\,d\tau' + c\int_{\tau_0}^{\tau} {}^{\tilde{\rho}}\underset{k-1}{\overset{(-3-\delta)}{\mathcal{E}'}}(\tau')\,d\tau' \\
&\leq \underset{k}{\overset{(p)}{\mathfrak{E}}}(\tau_0) + A\int_{\tau_0}^{\tau} \underset{k-1}{\overset{\xi}{\mathcal{E}'}}(\tau') + \int_{\mathcal{R}(\tau_0,\tau)} |\underset{k}{\overset{(p)}{H}}[\psi]\cdot\{\mathfrak{D}^{\boldsymbol{k}}F\}| + C\int_{\mathcal{R}(\tau_0,\tau)} \sum_{|\boldsymbol{k}|\leq k} (\mathfrak{D}^{\boldsymbol{k}}F)^2 \\
&\qquad + C\int_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq R\}} \sum_{|\boldsymbol{k}|\leq k-1} (\widetilde{\mathfrak{D}}^{\boldsymbol{k}}F)^2 \qquad (3\text{-}109)
\end{aligned}
$$

as well as the estimate

$$
\begin{aligned}
\sup_{v:\tau\leq\tau(v)} \underset{k}{\overset{(p)}{\mathcal{F}}}(v,\tau_0,\tau) &+ \underset{k}{\overset{(p)}{\mathcal{E}}}(\tau) + \overset{(0)}{\mathcal{E}}_{\mathcal{S}}(\tau_0,\tau) + \int_{\tau_0}^{\tau} {}^{\chi}\underset{k}{\overset{(p-1)}{\mathcal{E}'}}(\tau')\,d\tau' + \int_{\tau_0}^{\tau} \underset{k-1}{\overset{(p-1)}{\mathcal{E}'}}(\tau')\,d\tau' \\
&\lesssim \underset{k}{\overset{(p)}{\mathcal{E}}}(\tau_0) + \int_{\mathcal{R}(\tau_0,\tau)} \sum_{|\boldsymbol{k}|\leq k} (|V_p^{\mu}(\mathfrak{D}^{\boldsymbol{k}}\psi)_{\mu}| + |w_p\mathfrak{D}^{\boldsymbol{k}}\psi|)|\mathfrak{D}^{\boldsymbol{k}}F| + \int_{\mathcal{R}(\tau_0,\tau)} \sum_{|\boldsymbol{k}|\leq k} (\mathfrak{D}^{\boldsymbol{k}}F)^2 \\
&\qquad + C\int_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq R\}} \sum_{|\boldsymbol{k}|\leq k-1} (\widetilde{\mathfrak{D}}^{\boldsymbol{k}}F)^2 + \int_{\Sigma(\tau)\cap\{r\leq R\}} \sum_{|\boldsymbol{k}|\leq k-1} (\widetilde{\mathfrak{D}}^{\boldsymbol{k}}F)^2. \qquad (3\text{-}110)
\end{aligned}
$$

For $p = 0$, identical statements hold with $p-1$ replaced by $-1-\delta$.

Remark 3.6.9. In the case where we replace the middle term of (3-3) with (3-10), we should add

$$
\sum_{|\boldsymbol{k}|\leq k} \sqrt{\int_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq R\}} |L\mathfrak{D}^{\boldsymbol{k}}\psi|^2 + |\underline{L}\mathfrak{D}^{\boldsymbol{k}}\psi|^2 + |\nabla\!\!\!/\,\mathfrak{D}^{\boldsymbol{k}}\psi|^2 + r^{-2}|\mathfrak{D}^{\boldsymbol{k}}\psi|^2} \sqrt{\int_{\mathcal{R}(\tau_0,\tau_1)\cap\{r\leq R\}} (\mathfrak{D}^{\boldsymbol{k}}F)^2}
$$

to the right-hand side of (3-110).

Proof. We first prove (3-109). This follows from (3-72) and applying Proposition 3.6.4 (in the form of Corollary 3.6.5) and the properties of Section 3.6.7.

To prove (3-110), let us first commute the equation by $T^{\tilde{k}}$ (and $\Omega_1^{\tilde{k}}$, if assumed Killing) and apply the black box estimate (3-54) to $T^{\tilde{k}}\psi$ (and $\Omega_1^{\tilde{k}}\psi$) for $\tilde{k} \leq k$. This gives

$$
\begin{aligned}
&\sup_{v} \overset{(p)}{\mathcal{F}}[T^{\tilde{k}}\psi](v,\tau_0,\tau) + \overset{(p)}{\mathcal{E}}[T^{\tilde{k}}\psi](\tau) + \overset{(p)}{\mathcal{E}}_{\mathcal{S}}[T^{\tilde{k}}\psi](\tau) + \int_{\tau_0}^{\tau} {}^{\chi}\overset{(p-1)}{\mathcal{E}'}[T^{\tilde{k}}\psi](\tau')\,d\tau + \int_{\tau_0}^{\tau} \underset{-1}{\mathcal{E}'}[T^{\tilde{k}}\psi](\tau')\,d\tau' \\
&\lesssim \overset{(p)}{\mathcal{E}}[T^{\tilde{k}}\psi](\tau_0) + \int_{\mathcal{R}(\tau_0,\tau)} (|r^p r^{-1}L(rT^{\tilde{k}}\psi)| + |\widetilde{V}_p^{\mu}\partial_{\mu}T^{\tilde{k}}\psi| + |\tilde{w}_p\psi|)|T^{\tilde{k}}F| + \int_{\mathcal{R}(\tau_0,\tau)} (T^{\tilde{k}}F)^2, \qquad (3\text{-}111)
\end{aligned}
$$

and similarly with $\Omega_1^{\tilde{k}}\psi$.

Now note that, in the support of ξ, all $\mathfrak{D}$ vanish except for T (and Ω_1), whence we manifestly have

$$A\int_{\tau_0}^{\tau} \overset{\xi}{\underset{k-1}{\mathcal{E}'}}(\tau')\,d\tau' \lesssim \sum_{\tilde{k}\le k}\int_{\tau_0}^{\tau} \underset{-1}{\mathcal{E}'}[T^{\tilde{k}}\psi](\tau') + \underset{-1}{\mathcal{E}'}[\upsilon\Omega_1^{\tilde{k}}\psi](\tau'),$$

where we recall that $\upsilon = 1$ only if Ω_1 is Killing. It follows that the left-hand side of (3-109) is bounded by the right-hand side of (3-110).

Now apply the black box estimate (3-54) to $\mathfrak{D}^{\boldsymbol{k}}\psi$ and note that we can bound the terms arising from $[\Box_{g_0}, \mathfrak{D}^{\boldsymbol{k}}]\psi$ by $c\int_{\tau_0}^{\tau} {}^{\rho}\overset{(p-1)}{\underset{k}{\mathcal{E}'}}(\tau') + \overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau')\,d\tau'$ and the spacetime term involving $\widetilde{\mathcal{D}}^{\boldsymbol{k}}F$ on the right-hand side of (3-110).

We thus have that (3-111) holds with $\mathfrak{D}^{\boldsymbol{k}}\psi$ in place of $T^k\psi$.

We apply our elliptic estimate of Proposition 3.6.6 to bound $\overset{(p)}{\underset{k}{\mathcal{E}}}(\tau)$ from $\overset{(p)}{\underset{k}{\mathfrak{E}}}(\tau)$, generating the last term on the right-hand side of (3-110).

Finally, by (3-74) of Proposition 3.6.3, we may estimate

$$\int_{\tau_0}^{\tau} \overset{(p-1)}{\underset{k-1}{\mathcal{E}'}}(\tau')\,d\tau' \lesssim \int_{\tau_0}^{\tau} \Big({}^{\rho}\overset{(p-1)}{\underset{k-1}{\mathcal{E}'}}(\tau') + \overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau')\Big)\,d\tau' + \int_{\mathcal{R}(\tau_0,\tau)\cap\{r\le R\}} \sum_{|\boldsymbol{k}|\le k-1} |\widetilde{\mathfrak{D}}^{\boldsymbol{k}}F|^2. \qquad \square$$

Remark 3.6.10. Note that for convenience we have used (3-109) to obtain (3-110). This was because our precise commutation assumptions were stated with respect to the global currents. If one does not assume estimate (3-33), and thus one does not have (3-109), one may still obtain (3-110) under the assumption of asymptotic flatness of Sections 3.5 and 3.6.2, where we insert simply the far-away currents $\overset{(p)}{J}_{\text{far}}$, $\overset{(p)}{K}_{\text{far}}$ in (3-60). (We also use that, from the assumptions of Section 2, we can still define currents as in Section 3.4.5 giving enhanced positivity near $\mathcal{H}^+$ and in the black hole interior.) Thus, in particular, estimate (3-110) holds in the Kerr case in the full subextremal range $|a| < M$.

3.6.9. *Sobolev inequalities and interpolation of p-weighted energies.* We end this section recording two easy statements about the energies we have defined.

In anticipation of studying nonlinear equations, one will need to estimate lower-order pointwise quantities from higher-order energies by Sobolev inequalities. We have the following:

Proposition 3.6.11. *Let ψ be a smooth function on a neighbourhood of $\Sigma(\tau)$. Then we have*

$$\sum_{|\boldsymbol{k}|\le k-3} \sup_{x\in\Sigma(\tau)\cap\{r\le R\}} (\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi(x))^2 \lesssim \min\Big\{\overset{(-1-\delta)}{\underset{k}{\mathcal{E}'}}(\tau), \overset{(0)}{\underset{k}{\mathcal{E}}}(\tau)\Big\}. \tag{3-112}$$

Proof. We note that the left-hand side of (3-112) is in fact bounded by the energy restricted to $\Sigma\cap\{r\le R\}$, which is why we may choose either of the two quantities on the right-hand side (3-112). $\square$

Weighted Sobolev inequalities will also hold globally on $\Sigma(\tau)$, and in practice these are used to estimate nonlinearities in the region near infinity. Because this use is incorporated in our assumption on the null condition (see already Section 4.7), we shall not need to state such inequalities, although, in practice, they will appear in the context of verifying the assumptions of Section 4.7. See already Appendix C.

Finally, we have the following easy interpolation result.

Proposition 3.6.12. *For δ as fixed in (3-1), one has the following interpolation inequalities*:

$$\overset{(1)}{\underset{k}{\mathcal{E}}}(\tau) \lesssim \big(\overset{(1-\delta)}{\underset{k}{\mathcal{E}}}(\tau)\big)^{1-\delta}\big(\overset{(2-\delta)}{\underset{k}{\mathcal{E}}}(\tau)\big)^{\delta}, \tag{3-113}$$

$$\overset{(\delta-1)}{\underset{k}{\mathcal{E}'}}(\tau) \lesssim \big(\overset{(-1-\delta)}{\underset{k}{\mathcal{E}'}}(\tau)\big)^{\frac{1-\delta}{1+\delta}}\big(\overset{(0)}{\mathcal{E}'}(\tau)\big)^{\frac{2\delta}{1+\delta}}, \tag{3-114}$$

$$\int_{\tau_0}^{\tau_0+1/2}\overset{(0)}{\underset{k-1}{\mathcal{E}}}(\tau) \lesssim \sqrt{\int_{\tau_0}^{\tau_0+1/2}\overset{(0)}{\mathcal{E}}(\tau)}\sqrt{\int_{\tau_0}^{\tau_0+1/2}\overset{(0)}{\underset{k-2}{\mathcal{E}}}(\tau)}. \tag{3-115}$$

Proof. The proof of these inequalities is standard and is left to the reader. ☐

4. Quasilinear equations: preliminaries, the null condition, and local existence

We assume throughout that $(\mathcal{M}, g_0)$ satisfies the assumptions of Sections 2 and 3 (for cases (i), (ii) or (iii)). We will introduce in this section the class of quasilinear equations to be considered in this paper, and derive some preliminary results which will be used in the proof of our main theorem.

4.1. ***The class of equations.*** We will consider solutions ψ to quasilinear equations of the form

$$\Box_{g(\psi,x)}\psi = N^{\mu\nu}(\psi, x)\partial_\mu\psi\,\partial_\nu\psi, \tag{4-1}$$

where

$$g : \mathbb{R}\times\mathcal{M} \to T^*\mathcal{M}\otimes T^*\mathcal{M} \quad \text{and} \quad N : \mathbb{R}\times\mathcal{M} \to T\mathcal{M}\otimes T\mathcal{M} \tag{4-2}$$

are such that $\pi\circ g(\psi, x) = x$ and $\pi\circ N(\psi, x) = x$, where

$$\pi : T\mathcal{M}\otimes T\mathcal{M} \to \mathcal{M} \quad \text{and} \quad \pi : T^*\mathcal{M}\otimes T^*\mathcal{M} \to \mathcal{M}$$

denote the canonical projections, and such that $g(0, x) = g_0(x)$ for all x, while $g(\cdot\,, x) = g_0(x)$ for $r(x) \geq \frac{1}{2}R$, and N and g are smooth maps.

We will assume moreover that, for each k,

$$\partial_{\boldsymbol{x}}^{\boldsymbol{k}}\partial_\xi^s g^{\alpha\beta}(\xi, x) \quad \text{and} \quad \partial_{\boldsymbol{x}}^{\boldsymbol{k}}\partial_\xi^s N^{\alpha\beta}(\xi, x)$$

are uniformly bounded for all $|\boldsymbol{k}| + s \leq k$, all $r \leq R$, and $|\xi| \leq 1$. Here $\partial_{\boldsymbol{x}}^{\boldsymbol{k}}$ denote multi-indices with respect to the ambient Cartesian coordinates of Section 2.1.

For $N^{\mu\nu}(\psi, x)$, we will eventually need in addition to assume some version of the null condition. We will only introduce this in Section 4.7 (see already Assumption 4.7.1). Let us note that our assumption on the support of $g - g_0$ is merely so as not to deal with formulations of the null condition for the quasilinear part.

We will sometimes view the equation in (4-1) as an inhomogeneous equation on a fixed g_0 background, i.e., we will write it as

$$\Box_{g_0}\psi = N^{\mu\nu}(\psi, x)\partial_\mu\psi\,\partial_\nu\psi + (\Box_{g_0} - \Box_{g(\psi,x)})\psi, \tag{4-3}$$

and similarly for its commuted versions. We may thus apply estimates to the inhomogeneous equation

$$\Box_{g_0}\psi = F. \tag{4-4}$$

4.2. ***Smallness parameters.*** Starting in this section, we shall introduce smallness parameters (i.e., parameters related to making smallness assumptions on solutions), denoted using the symbol ε and various subscripts, e.g., $\varepsilon_{\text{prelim}}$, $\varepsilon_{\text{local}}$. Unless otherwise noted, these will depend only on $(\mathcal{M}, g_0)$, on the nonlinearities of (4-1) defined by (4-2) and, in general, on k, if there is k dependence in the statement.

When these smallness parameters depend on an additional quantity, this will be indicated in parenthesis; e.g., $\hat{\varepsilon}_{\text{slab}}(\alpha)$.

Note finally that our convention on constants denoted C, c, etc. remains the same as set in Section 3.1, i.e., these will *not* depend on (4-2).

4.3. ***Energy currents for $\Box_{g(\psi,x)}$ and the stability of coercivity properties.*** Let ψ denote a solution of (4-1) on a domain $\mathcal{R}(\tau_0, \tau_1)$. Because we shall use energy identities connected with $\Box_{g(\psi,x)}$, we shall require certain basic smallness assumptions on ψ which ensure that the causal nature of relevant hypersurfaces is retained and that induced volume forms of g and g_0 are comparable.

In the sections to follow, the main smallness parameter we will consider will be $\varepsilon_{\text{prelim}} > 0$. We emphasise that, according to our conventions of Section 4.2, $\varepsilon_{\text{prelim}} > 0$ will in general depend on the nonlinearity (4-2) (and also on k). The parameter $\varepsilon_{\text{prelim}} > 0$ can be taken as fixed everywhere in the paper, but note that its smallness is constrained in multiple propositions whose preambles refer to its existence.

As we shall see, the propositions in this section will always refer to solutions satisfying

$$\sum_{|\boldsymbol{k}|\le 1} (\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)^2 \le \sqrt{\varepsilon} \tag{4-5}$$

in $r \le R$ for $0 < \varepsilon \le \varepsilon_{\text{prelim}}$. (We recall here that, for $r(x) \ge R$, we have $g(\psi, x) = g_0$.) Eventually, (4-5) will be the consequence of a stronger estimate (see already (4-23)).

4.3.1. *Currents for $\Box_{g(x,\psi)}\psi = F$ and the stability of coercivity properties.* Let us for the moment consider more generally the equation

$$\Box_{g(\psi,x)}\psi = F \tag{4-6}$$

for arbitrary F, where ψ satisfies (4-5) for sufficiently small ε.

We may define now the currents

$$\overset{(p)}{\underset{k}{J}}[g(\psi,x),\psi], \qquad \overset{(p)}{\underset{k}{K}}[g(\psi,x),\psi] \tag{4-7}$$

again by the expressions (3-64), where the constituent (3-11), (3-12) are defined with the same quadruples (3-42) as before, but now with $g = g(\psi, x)$ replacing g_0. These currents satisfy (3-15), with respect now to the normals and volume forms of the metric g, and where $\overset{(p)}{\underset{k}{H}}[\psi]$ is defined by (3-13). (Notice that the definition of $\overset{(p)}{\underset{k}{H}}[\psi]$ does not depend on the metric.) In the region $r \ge R$ of course, all currents coincide with their g_0 versions since $g = g_0$ in this region.

We have the following proposition.

Proposition 4.3.1. *There exists an* $\varepsilon_{\text{prelim}} > 0$ *such that the following statement holds*:

Let ψ *be a solution of* (4-6) *in* $\mathcal{R}(\tau_0, \tau_1)$ *satisfying* (4-5) *for* $0 < \varepsilon \le \varepsilon_{\text{prelim}}$. *Then* $x \mapsto g(\psi(x), x)$ *defines a Lorentzian metric on* $\mathcal{R}(\tau_0, \tau_1)$, *and the identity* (3-15) *holds in* $\mathcal{R}(\tau_0, \tau_1, v)$ *for all* v *such that* $\tau_1 \le \tau(v)$, *where the coefficients, normals and volume forms are* $\varepsilon^{1/4}$ *close to those of the currents* (4-7) *corresponding to* $\Box_{g_0}\phi = F$.

In particular, there exist constants C, c *such that, for* $p = 0$ *or* $\delta \le p \le 2-\delta$, *the corresponding coercivity properties* (3-45) *are retained for the currents* (4-7), *while* (3-43) *is replaced by*

$$\overset{(p)}{K}[g(\psi, x), \psi] + \tilde{A}\xi(r)\psi^2 \ge cr^{p-1}\rho(r)((L\psi)^2 + |\nabla\!\!\!/\psi|^2) + cr^{-1-\delta}\rho(r)(\underline{L}\psi)^2 + cr^{-3}\tilde{\rho}(r)\psi^2 \\ - C\varepsilon^{\frac14}\,\iota_{\{r\le R\}\cap\{\rho\le C\varepsilon^{1/4}<1/2\}}((L\psi)^2 + (\underline{L}\psi)^2 + |\nabla\!\!\!/\psi|^2 + \psi^2). \quad (4\text{-}8)$$

In particular, we still have (3-43) *in* $r \ge R$ *and also in* $\{r \le R\} \cap \{\rho \ge C\varepsilon^{1/4}\}$.

Given $\varsigma = \varsigma(k)$ *as fixed previously, we have that the above applies also for the currents* $\overset{(p)}{J}_{\varsigma}[g(\psi, x), \psi]$, $\overset{(p)}{K}_{\varsigma}[g(\psi, x), \psi]$ *in the region* $r \ge r_{\text{Killing}}$, *while in the region* $r \le r_1(\varsigma)$ *the coercivity properties of Propositions* 3.4.1 *and* 3.4.2 *hold for these currents.*

Proof. This is clear from the properties of tensor identities. Note how the smallness constraint on $\varepsilon_{\text{prelim}}$ provided by this proposition depends of course on the map g of (4-2) and is needed even simply to ensure that g is Lorentzian and that the inverse metric to be well defined. $\square$

Note that, since $\rho = \tilde{\rho} = 1$ in case (i), the bound (4-8) implies that the full coercivity applies in that case. (Note that if ρ is a step function valued in $\{0, 1\}$, then $\{\rho \ge C\varepsilon^{1/4}\} = \left\{\rho \ge \frac12\right\} = \{\rho = 1\}$ and $\{\rho \le C\varepsilon^{1/4}\} = \{\rho = 0\}$.)

Corollary 4.3.2. *Under the assumption of Proposition 4.3.1, we have the coercivity statement*

$$\overset{(p)}{K}_{k}[g(\psi, x), \psi] \ge \tfrac{8}{9}\overset{(p)}{K}_{k}[g_0, \psi]$$

in $\{r \le r_2\} \cup \left\{r \ge \frac14 R\right\}$ *and an analogue of the coercivity statement of* (3-85) *holds in the form*

$$\int_{\mathcal{R}(\tau_0,\tau_1)\cap\{r\le r_2\}} -\overset{(p)}{H}_{k}[\psi]\cdot\{[\mathfrak{D}^{\boldsymbol{k}}, \Box_{g_0}]\psi\} + \int_{\mathcal{R}(\tau_0,\tau_1)\cap\{r\ge r_2\}} |\overset{(p)}{H}_{k}[\psi]\cdot\{[\mathfrak{D}^{\boldsymbol{k}}, \Box_{g_0}]\psi\}| \\ \le \int_{\mathcal{R}(\tau_0,\tau_1)\cap\{r\le r_2\}\cap\{r\ge R/4\}} \tfrac23\overset{(p)}{K}_{k}[g(\psi, x), \psi] + C\sum_{|\boldsymbol{k}|\le k-1}(\widetilde{\mathfrak{D}}^{\boldsymbol{k}}(F + (\Box_{g_0} - \Box_{g(\psi,x)})\psi))^2. \quad (4\text{-}9)$$

We have moreover the following version of (3-103):

$$\int_{\mathcal{R}(\tau_0,\tau_1)} \overset{(p)}{K}_{k}[g(\psi, x), \psi] + \sum_{|\boldsymbol{k}|\le k}\tilde{A}\xi(r)(\mathfrak{D}^{k}\psi)^2 \gtrsim \int_{\tau_0}^{\tau_1} {}^{\rho}\overset{(p-1)}{\mathcal{E}'_{k}}(\tau')\,d\tau' + \int_{\tau_0}^{\tau_1} {}^{\tilde{\rho}}\overset{(-3-\delta)}{\mathcal{E}'_{k-1}}(\tau')\,d\tau', \quad 2-\delta \ge p \ge \delta, \\ \int_{\mathcal{R}(\tau_0,\tau_1)} \overset{(p)}{K}_{k}[g(\psi, x), \psi] + \sum_{|\boldsymbol{k}|\le k}\tilde{A}\xi(r)(\mathfrak{D}^{k}\psi)^2 \gtrsim \int_{\tau_0}^{\tau_1} {}^{\rho}\overset{(-1-\delta)}{\mathcal{E}'_{k}}(\tau')\,d\tau' + \int_{\tau_0}^{\tau_1} {}^{\tilde{\rho}}\overset{(-3-\delta)}{\mathcal{E}'_{k-1}}(\tau')\,d\tau', \quad p = 0. \quad (4\text{-}10)$$

Proof. Note that we have retained g_0 on the left-hand side of (4-9), so this is simply a statement about the stability of coercivity properties of the first term on the right-hand side. $\square$

4.3.2. *Relations of energy fluxes.* In view of the above we will define a new set of energy quantities, $\overset{(0)}{\underset{k}{\mathfrak{F}}}(g)$, $\overset{(p)}{\underset{k}{\mathfrak{E}}}(g)$, etc., defined in parallel with those of $\overset{(0)}{\underset{k}{\mathfrak{F}}}$, $\overset{(p)}{\underset{k}{\mathfrak{E}}}$ of Section 3.6.7 (to be denoted below as $\overset{(0)}{\underset{k}{\mathfrak{F}}}(g_0)$, $\overset{(p)}{\underset{k}{\mathfrak{E}}}(g_0)$), but where the flux is defined with respect to the divergence identity with respect to $g = g(\psi, x)$, i.e., we define

$$\begin{aligned}\overset{(p)}{\underset{k}{\mathfrak{E}}}(g,\tau) &:= \int_{\Sigma(\tau)} \overset{(p)}{\underset{k}{J}}{}_{\mu}[g,\psi] n(g)^{\mu}_{\Sigma(\tau)}, \qquad \overset{(p)}{\underset{k}{\mathfrak{E}}}{}_{\mathcal{S}}(g,\tau) := \int_{\mathcal{S}} \overset{(p)}{\underset{k}{J}}{}_{\mu}[g,\psi] n(g)^{\mu}_{\mathcal{S}}, \\ \overset{(p)}{\underset{k}{\mathfrak{F}}}(g,v,\tau_0,\tau) &:= \int_{\underline{C}_v \cap \mathcal{R}(\tau_0,\tau)} \overset{(p)}{\underset{k}{J}}{}_{\mu}[g,\psi] n(g)^{\mu}_{\underline{C}_v}, \qquad \tau_0 \le \tau \le \tau(v),\end{aligned} \tag{4-11}$$

where the omitted volume forms above are here understood with respect to $g = g(\psi, x)$.

Corollary 4.3.3. *We have that, under the assumptions of Proposition 4.3.1, for all $\tau_0 \le \tau \le \tau_1$ and v such that $\tau_1 \le \tau(v)$,*

$$\overset{(p)}{\underset{k}{\mathfrak{E}}}(g,\tau) \sim \overset{(p)}{\underset{k}{\mathfrak{E}}}(g_0,\tau) \lesssim \overset{(p)}{\underset{k}{\mathcal{E}}}(\tau), \quad \overset{(0)}{\mathfrak{E}}_{\mathcal{S}}(g,\tau) = \overset{(p)}{\mathfrak{E}}_{\mathcal{S}}(g,\tau) \sim \overset{(p)}{\underset{k}{\mathfrak{E}}}{}_{\mathcal{S}}(g_0,\tau) = \overset{(0)}{\underset{k}{\mathfrak{E}}}{}_{\mathcal{S}}(g_0,\tau) \sim \overset{(0)}{\mathcal{E}}_{\mathcal{S}}(\tau),$$
$$\overset{(p)}{\underset{k}{\mathfrak{F}}}(g,v,\tau_0,\tau) = \overset{(p)}{\underset{k}{\mathfrak{F}}}(g_0,v,\tau_0,\tau) \sim \overset{(p)}{\underset{k}{\mathcal{F}}}(v,\tau_0,\tau),$$

and in fact

$$\begin{aligned}\overset{(0)}{\underset{k}{\mathfrak{E}}}(g) \le (1 + C\varepsilon^{\frac14}) \overset{(0)}{\underset{k}{\mathfrak{E}}}(g_0), &\qquad \overset{(0)}{\underset{k}{\mathfrak{E}}}(g_0) \le (1 + C\varepsilon^{\frac14}) \overset{(0)}{\underset{k}{\mathfrak{E}}}(g), \\ \overset{(p)}{\underset{k}{\mathcal{E}}}(\tau) \lesssim \overset{(p)}{\underset{k}{\mathfrak{E}}}[g](\tau) + \int_{\Sigma(\tau)\cap\{r_1 \le r \le R\}} &\sum_{|\boldsymbol{k}| \le k-1} (\widetilde{\mathfrak{D}}^{\boldsymbol{k}}(F + (\Box_{g_0} - \Box_{g(\psi,x)})\psi))^2.\end{aligned} \tag{4-12}$$

As always, the significance of expressing the above with respect to the energies $\overset{(p)}{\underset{k}{\mathfrak{E}}}(g)$, etc., is that the constants in our nonlinear inequalities will be exactly 1.

Note that, in all integrals below with volume form omitted, we will continue to use the volume form induced by g_0 unless otherwise noted. (We shall only use the volume form induced by $g(\psi, x)$ when applying the divergence identity involving (4-11). The corresponding volume forms are of course equivalent under the assumptions of Proposition 4.3.1, but again one must distinguish where appropriate so as to obtain the exact constant.)

4.4. ***Higher-order estimates for the quasilinear equation.*** Using the above, we can finally obtain the following:

Proposition 4.4.1. *For all $k \ge 1$, there exists an $\varepsilon_{\text{prelim}} > 0$ such that the following statement holds:*

Let $0 < \delta \le p \le 2 - \delta$, $\tau_0 \le \tau_1$, and let ψ be a solution of (4-1) in $\mathcal{R}(\tau_0, \tau_1)$ satisfying (4-5) for $0 < \varepsilon \le \varepsilon_{\text{prelim}}$. Then, for all $\tau \in [\tau_0, \tau_1]$, we have

$$\begin{aligned}&\sup_{v:\tau\le\tau(v)} \overset{(p)}{\underset{k}{\mathfrak{F}}}(v,\tau_0,\tau), \quad \overset{(p)}{\underset{k}{\mathfrak{E}}}(\tau) + \overset{(p)}{\underset{k}{\mathfrak{E}}}{}_{\mathcal{S}}(\tau_0,\tau) + c\int_{\tau_0}^{\tau} {}^{\rho}\overset{(p-1)}{\underset{k}{\mathcal{E}'}}(\tau')\,d\tau' + c\int_{\tau_0}^{\tau} {}^{\tilde\rho}\overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau')\,d\tau' \\ &\quad \le \overset{(p)}{\underset{k}{\mathfrak{E}}}(\tau_0) + A\int_{\tau_0}^{\tau} \overset{\xi}{\underset{k-1}{\mathcal{E}'}}(\tau')\,d\tau' \\ &\qquad + \int_{\mathcal{R}(\tau_0,\tau)} \big| \overset{(p)}{\underset{k}{H}}[\psi] \cdot \{\mathfrak{D}^{\boldsymbol{k}}(N^{\alpha\beta}(\psi,x)\partial_\alpha\psi\partial_\beta\psi)\}\big|\end{aligned} \tag{4-13}$$

$$+\int_{\mathcal{R}(\tau_0,\tau)\cap\{r\le R\}}\big|\overset{(p)}{\underset{k}{H}}[\psi]\cdot\{[\Box_{g(\psi,x)}-\Box_{g_0},\mathfrak{D}^{\boldsymbol{k}}]\psi\}\big| \tag{4-14}$$

$$+C\int_{\mathcal{R}(\tau_0,\tau)}\sum_{|\boldsymbol{k}|\le k}(\mathfrak{D}^{\boldsymbol{k}}(N^{\alpha\beta}(\psi,x)\partial_\alpha\psi\partial_\beta\psi))^2+C\int_{\mathcal{R}(\tau_0,\tau)\cap\{r\le R\}}\sum_{|\boldsymbol{k}|\le k}([\Box_{g(\psi,x)}-\Box_{g_0},\mathfrak{D}^{\boldsymbol{k}}]\psi)^2$$

$$+C\int_{\mathcal{R}(\tau_0,\tau)\cap\{r\le R\}}\sum_{|\boldsymbol{k}|\le k-1}(\widetilde{\mathfrak{D}}^{\boldsymbol{k}}(N^{\alpha\beta}(\psi,x)\partial_\alpha\psi\,\partial_\beta\psi+(\Box_{g_0}-\Box_{g(\psi,x)})\psi))^2 \tag{4-15}$$

$$+C\boxed{\varepsilon^{\frac14}}\int_{\mathcal{R}(\tau_0,\tau)\cap\{r\le R\}\cap\{\rho\le C\sqrt{\varepsilon}\}}\sum_{|\boldsymbol{k}|\le k}((L(\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi))^2+(\underline{L}(\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi))^2+|\nabla\!\!\!/(\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\psi)|^2)+\psi^2, \tag{4-16}$$

as well as the estimate

$$\sup_{v:\tau\le\tau(v)}\overset{(p)}{\underset{k-1}{\mathcal{F}}}(v,\tau_0,\tau)+\overset{(p)}{\underset{k-1}{\mathcal{E}}}(\tau)+\overset{(0)}{\underset{k-1}{\mathcal{E}}}{}_{\mathcal{S}}(\tau_0,\tau)+\int_{\tau_0}^{\tau}\chi\overset{(p-1)}{\underset{k-1}{\mathcal{E}'}}(\tau')\,d\tau'+\int_{\tau_0}^{\tau}\overset{(p-1)}{\underset{k-2}{\mathcal{E}'}}(\tau')\,d\tau' \tag{4-17}$$

$$\lesssim\overset{(p)}{\underset{k-1}{\mathcal{E}}}(\tau_0)$$

$$+\int_{\mathcal{R}(\tau_0,\tau)}\sum_{|\boldsymbol{k}|\le k-1}(|V_p^\mu\partial_\mu(\mathfrak{D}^{\boldsymbol{k}}\psi)|+|w_p\mathfrak{D}^{\boldsymbol{k}}\psi|)|\mathfrak{D}^{\boldsymbol{k}}(N^{\alpha\beta}(\psi,x)\partial_\alpha\psi\,\partial_\beta\psi+(\Box_{g_0}-\Box_{g(\psi,x)})\psi)| \tag{4-18}$$

$$+\int_{\mathcal{R}(\tau_0,\tau)}\sum_{|\boldsymbol{k}|\le k-1}(\mathfrak{D}^{\boldsymbol{k}}(N^{\alpha\beta}(\psi,x)\partial_\alpha\psi\,\partial_\beta\psi+(\Box_{g_0}-\Box_{g(\psi,x)})\psi))^2 \tag{4-19}$$

$$+\int_{\mathcal{R}(\tau_0,\tau)\cap\{r\le R\}}\sum_{|\boldsymbol{k}|\le k-2}(\widetilde{\mathfrak{D}}^{\boldsymbol{k}}(N^{\alpha\beta}(\psi,x)\partial_\alpha\psi\,\partial_\beta\psi+(\Box_{g_0}-\Box_{g(\psi,x)})\psi))^2 \tag{4-20}$$

$$+\int_{\Sigma(\tau)\cap\{r\le R\}}\sum_{|\boldsymbol{k}|\le k-2}(\widetilde{\mathfrak{D}}^{\boldsymbol{k}}(N^{\alpha\beta}(\psi,x)\partial_\alpha\psi\,\partial_\beta\psi+(\Box_{g_0}-\Box_{g(\psi,x)})\psi))^2. \tag{4-21}$$

Moreover, for $p=0$, *identical statements hold with* $-1-\delta$ *replacing* $p-1$.

Remark 4.4.2. In the case where we replace the middle term of (3-3) with (3-10), we should add

$$\sum_{|\boldsymbol{k}|\le k-1}\sqrt{\int_{\mathcal{R}(\tau_0,\tau)\cap\{r\le R\}}|L\mathfrak{D}^{\boldsymbol{k}}\psi|^2+|\underline{L}\mathfrak{D}^{\boldsymbol{k}}\psi|^2+|\nabla\!\!\!/\mathfrak{D}^{\boldsymbol{k}}\psi|^2+r^{-2}|\mathfrak{D}^{\boldsymbol{k}}\psi|^2}$$
$$\times\sqrt{\int_{\mathcal{R}(\tau_0,\tau_1)\cap\{r\le R\}}\mathfrak{D}^{\boldsymbol{k}}(N^{\alpha\beta}(\psi,x)\partial_\alpha\psi\,\partial_\beta\psi+(\Box_{g_0}-\Box_{g(\psi,x)})\psi)^2}$$

to the right-hand side of (4-18); cf. Remark 3.6.9.

Proof. Note the term on line (4-14) arose by expanding the term appearing in the actual identity as follows. Write

$$\int_{\mathcal{R}(\tau_0,\tau)\cap\{r\le R\}}\overset{(p)}{\underset{k}{H}}[\psi]\cdot\{[\Box_{g(\psi,x)},\mathfrak{D}^{\boldsymbol{k}}]\psi\}$$
$$=\int_{\mathcal{R}(\tau_0,\tau)\cap\{r\le R\}}\overset{(p)}{\underset{k}{H}}[\psi]\cdot\{[\Box_{g(\psi,x)}-\Box_{g_0},\mathfrak{D}^{\boldsymbol{k}}]\psi\}+\int_{\mathcal{R}(\tau_0,\tau)\cap\{r\le R\}}\overset{(p)}{\underset{k}{H}}[\psi]\cdot\{[\Box_{g_0},\mathfrak{D}^{\boldsymbol{k}}]\psi\},$$

then bring the second term to the left-hand side to use Corollary 4.3.2 and absorb it in the bulk. (Recall that, in the region $r \geq R$, we have $g = g_0$.) The term on line (4-15) arose from our application of elliptic estimates to ψ, considering it as a solution of (4-3). The term on line (4-16) arose from the term on the last line of (4-8). We note that, restricted to $r \geq R$, we may replace $\widetilde{\mathfrak{D}}$ commutation with $\mathfrak{D}$ commutation.

The inequality (4-17)–(4-21), on the other hand, simply arises from applying the estimate (3-110) to the nonlinear equation written in the form (4-3). Note that we have applied it at one order less, i.e., with $k-1$ in place of k, in view of the fact that the right-hand side is of order k. □

4.5. ***Summed norm notation and the lower-order smallness assumption.*** In addition to the primitive assumption (4-5), in the context of our proof, we will need to introduce stronger a priori energy smallness assumptions on ψ in our region $\mathcal{R}(\tau_0, \tau_1)$ under consideration. In the context of the proof of the main theorem, this will appear as a bootstrap assumption. This will ensure that higher-order nonlinear terms are indeed absorbable.

We first explain some additional notation. We define the following summed quantities for $\delta \leq p \leq 2-\delta$:

$$\overset{(p)}{\underset{k}{\mathcal{X}}}(\tau_0,\tau_1) := \sup_{\tau'\in[\tau_0,\tau_1]} \overset{(p)}{\underset{k}{\mathcal{E}}}(\tau') + \sup_{v:\tau_1\leq v(\tau)} \overset{(p)}{\underset{k}{\mathcal{F}}}(v,\tau_0,\tau_1) + \int_{\tau_0}^{\tau_1} \overset{(p-1)}{\underset{k}{\mathcal{E}'}}(\tau')\,d\tau',$$

$${}^{\rho}\overset{(p)}{\underset{k}{\mathcal{X}}}(\tau_0,\tau_1) := \sup_{\tau'\in[\tau_0,\tau_1]} \overset{(p)}{\underset{k}{\mathcal{E}}}(\tau') + \sup_{v:\tau_1\leq \tau(v)} \overset{(p)}{\underset{k}{\mathcal{F}}}(v,\tau_0,\tau_1) + \int_{\tau_0}^{\tau_1} \Big({}^{\rho}\overset{(p-1)}{\underset{k}{\mathcal{E}'}}(\tau') + {}^{\tilde{\rho}}\overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau')\Big)\,d\tau',$$

$${}^{\chi}\overset{(p)}{\underset{k}{\mathcal{X}}}(\tau_0,\tau_1) := \sup_{\tau'\in[\tau_0,\tau_1]} \overset{(p)}{\underset{k}{\mathcal{E}}}(\tau') + \sup_{v:\tau_1\leq \tau(v)} \overset{(p)}{\underset{k}{\mathcal{F}}}(v,\tau_0,\tau_1) + \int_{\tau_0}^{\tau_1} \Big({}^{\chi}\overset{(p-1)}{\underset{k}{\mathcal{E}'}}(\tau') + \overset{(p-1)}{\underset{k-1}{\mathcal{E}'}}(\tau')\Big)\,d\tau'.$$

For $p=0$, we first define the analogous quantities, where $p-1$ is replaced by $-1-\delta$:

$$\overset{(0)}{\underset{k}{\mathcal{X}}}(\tau_0,\tau_1) := \sup_{\tau'\in[\tau_0,\tau_1]} \overset{(0)}{\underset{k}{\mathcal{E}}}(\tau') + \sup_{v:\tau_1\leq \tau(v)} \overset{(0)}{\underset{k}{\mathcal{F}}}(v,\tau_0,\tau_1) + \int_{\tau_0}^{\tau_1} \overset{(-1-\delta)}{\underset{k}{\mathcal{E}'}}(\tau')\,d\tau',$$

$${}^{\rho}\overset{(0)}{\underset{k}{\mathcal{X}}}(\tau_0,\tau_1) := \sup_{\tau'\in[\tau_0,\tau_1]} \overset{(0)}{\underset{k}{\mathcal{E}}}(\tau') + \sup_{v:\tau_1\leq \tau(v)} \overset{(0)}{\underset{k}{\mathcal{F}}}(v,\tau_0,\tau_1) + \int_{\tau_0}^{\tau_1} \Big({}^{\rho}\overset{(-1-\delta)}{\underset{k}{\mathcal{E}'}}(\tau') + {}^{\tilde{\rho}}\overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau')\Big)\,d\tau',$$

$${}^{\chi}\overset{(0)}{\underset{k}{\mathcal{X}}}(\tau_0,\tau_1) := \sup_{\tau'\in[\tau_0,\tau_1]} \overset{(0)}{\underset{k}{\mathcal{E}}}(\tau') + \sup_{v:\tau_1\leq \tau(v)} \overset{(0)}{\underset{k}{\mathcal{F}}}(v,\tau_0,\tau_1) + \int_{\tau_0}^{\tau_1} \Big({}^{\chi}\overset{(-1-\delta)}{\underset{k}{\mathcal{E}'}}(\tau') + \overset{(-1-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau')\Big)\,d\tau'.$$

Because $p=0$ is anomalous, however, we will need in addition the following stronger energies which will appear *on the right-hand side* of $p=0$ estimates:

$$\overset{(0+)}{\underset{k}{\mathcal{X}}}(\tau_0,\tau_1) := \sup_{\tau'\in[\tau_0,\tau_1]} \overset{(0)}{\underset{k}{\mathcal{E}}}(\tau') + \sup_{v:\tau_1\leq \tau(v)} \overset{(0)}{\underset{k}{\mathcal{F}}}(v,\tau_0,\tau_1) + \int_{\tau_0}^{\tau_1} \overset{(\delta-1)}{\underset{k}{\mathcal{E}'}}(\tau')\,d\tau',$$

$${}^{\rho}\overset{(0+)}{\underset{k}{\mathcal{X}}}(\tau_0,\tau_1) := \sup_{\tau'\in[\tau_0,\tau_1]} \overset{(0)}{\underset{k}{\mathcal{E}}}(\tau') + \sup_{v:\tau_1\leq \tau(v)} \overset{(0)}{\underset{k}{\mathcal{F}}}(v,\tau_0,\tau_1) + \int_{\tau_0}^{\tau_1} \Big({}^{\rho}\overset{(\delta-1)}{\underset{k}{\mathcal{E}'}}(\tau') + {}^{\tilde{\rho}}\overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau')\Big)\,d\tau',$$

$${}^{\chi}\overset{(0+)}{\underset{k}{\mathcal{X}}}(\tau_0,\tau_1) := \sup_{\tau'\in[\tau_0,\tau_1]} \overset{(0)}{\underset{k}{\mathcal{E}}}(\tau') + \sup_{v:\tau_1\leq \tau(v)} \overset{(0)}{\underset{k}{\mathcal{F}}}(v,\tau_0,\tau_1) + \int_{\tau_0}^{\tau_1} \Big({}^{\chi}\overset{(\delta-1)}{\underset{k}{\mathcal{E}'}}(\tau') + \overset{(\delta-1)}{\underset{k-1}{\mathcal{E}'}}(\tau')\Big)\,d\tau'.$$

Note the general properties that $p' \geq p$, $k' \geq k$ implies $\overset{(p')}{\underset{k'}{\mathcal{X}}} \gtrsim \overset{(p)}{\underset{k}{\mathcal{X}}}$, ${}^{\chi}\overset{(p')}{\underset{k'}{\mathcal{X}}} \gtrsim {}^{\chi}\overset{(p)}{\underset{k}{\mathcal{X}}}$, ${}^{\rho}\overset{(p')}{\underset{k'}{\mathcal{X}}} \gtrsim {}^{\rho}\overset{(p)}{\underset{k}{\mathcal{X}}}$, while

$$\overset{(p)}{\underset{k-1}{\mathcal{X}}} \lesssim {}^{\chi}\overset{(p)}{\underset{k}{\mathcal{X}}}. \tag{4-22}$$

We will use the notation $\ll k$ to denote some particular positive integer, depending on k, which may vary across different instances of our use of the notation, such that $\ll k \leq k$ and, in fact, $\ll k$ is "much less than k", provided k is sufficiently large. In particular, for all positive integers n, we assume there exists a $k(n)$ for which $k \geq k(n)$ implies $\ll k \leq k - n$.

We have the following:

Proposition 4.5.1. *Let $k \geq 4$ be sufficiently large. There exists an $\varepsilon_{\text{prelim}} > 0$ such that, for all $0 < \varepsilon \leq \varepsilon_{\text{prelim}}$, the following holds:*

Let ψ satisfy

$$\underset{\ll k}{\overset{(p)}{\mathcal{X}}} \leq \varepsilon \tag{4-23}$$

in $\mathcal{R}(\tau_0, \tau_1)$ for $p = 0$ (or some $\delta \leq p \leq 2 - \delta$), where $\ll k \geq 4$. Then the following improved version of (4-5) holds:

$$\sup_{r \leq R} \sum_{|\boldsymbol{k}| \leq 1} (\widetilde{\mathfrak{D}}^{\boldsymbol{k}} \psi)^2 \lesssim \varepsilon \leq \sqrt{\varepsilon}. \tag{4-24}$$

Proof. This follows of course immediately from the Sobolev inequality (3-112). □

In the context of the proof of the main theorem, inequality (4-23) will be introduced as a bootstrap assumption, and Proposition 4.5.1 will be applied to retrieve the assumption (4-5), which is necessary for the results of Sections 4.3–4.4. Note that, with the assumption (4-23), we may replace the boxed $\varepsilon^{1/4}$ in (4-16) with $\sqrt{\varepsilon}$.

We note that, in what follows, restrictions on k sufficiently large will always include the condition $\ll k \geq 4$.

4.5.1. *Comparability of $\mathfrak{E}$ and $\mathcal{E}$ energies.* We note first the following:

Proposition 4.5.2. *Let $k \geq 4$ be sufficiently large. There exists an $\varepsilon_{\text{prelim}} > 0$ such that the following holds:*

Let ψ be a solution of (4-1) in $\mathcal{R}(\tau_0, \tau_1)$ satisfying (4-23) for $p = 0$ and some $0 < \varepsilon \leq \varepsilon_{\text{prelim}}$. Then, setting $F = N^{\mu\nu}(\psi, x)\partial_\mu \psi\, \partial_\nu \psi + (\Box_{g_0} - \Box_{g(\psi,x)})\psi$, we have

$$\int_{\Sigma(\tau') \cap \{r \leq R\}} \sum_{|\boldsymbol{k}| \leq k-1} (\widetilde{\mathfrak{D}}^{\boldsymbol{k}} F)^2 \lesssim \min\Big\{ \underset{\ll k}{\overset{(0)}{\mathcal{E}}}(\tau'), \underset{\ll k}{\overset{(-1-\delta)}{\mathcal{E}'}}(\tau') \Big\} \overset{(0)}{\mathcal{E}}(\tau') \lesssim \varepsilon\, \overset{(0)}{\mathcal{E}}(\tau'). \tag{4-25}$$

Proof. This is standard in view of our assumptions on N and $g(\psi, x)$ from Section 4.1 and can be proven using the Sobolev inequality (3-112) on $\Sigma(\tau') \cap \{r \leq R\}$. □

We may now obtain the following result, which can be viewed as a corollary of Proposition 3.6.6.

Corollary 4.5.3. *Let $k \geq 4$ be sufficiently large. Then there exists an $\varepsilon_{\text{prelim}} > 0$ such that the following statement holds:*

Let ψ be a solution of (4-1) in $\mathcal{R}(\tau_0, \tau_1)$ satisfying (4-23) for $p = 0$ and some $0 < \varepsilon \leq \varepsilon_{\text{prelim}}$. We have the analogue of Corollary 3.6.7: The calligraphic and fraktur (when applicable) energies are comparable, i.e.,

$$\underset{k}{\overset{(p)}{\mathcal{E}}}(\tau) \sim \underset{k}{\overset{(p)}{\mathfrak{E}}}(\tau)(g),$$

for all $\tau_0 \le \tau \le \tau_1$. *Moreover, if* $\chi = 1$ *and* $\tilde\rho = 1$ *identically* (*as in case* (i)), *then*

$$\overset{(p-1)}{\underset{k}{\mathcal{E}'}}(\tau) \sim {}^{\chi}\overset{(p-1)}{\underset{k}{\mathcal{E}'}}(\tau) + {}^{\tilde\rho}\overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau), \qquad \overset{(-1-\delta)}{\underset{k}{\mathcal{E}'}}(\tau) \sim {}^{\chi}\overset{(-1-\delta)}{\underset{k}{\mathcal{E}'}}(\tau) + {}^{\tilde\rho}\overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau)$$

and thus

$${}^{\chi}\overset{(p)}{\underset{k}{\mathcal{X}}}(\tau_0,\tau_1) \sim {}^{\rho}\overset{(p)}{\underset{k}{\mathcal{X}}}(\tau_0,\tau_1) \sim \overset{(p)}{\underset{k}{\mathcal{X}}}(\tau_0,\tau_1).$$

If $\tilde\rho = 1$ *identically and* $\rho = \chi$, (*i.e., as in cases* (i) *and* (ii)), *then*

$${}^{\chi}\overset{(p-1)}{\underset{k}{\mathcal{E}'}}(\tau) + \overset{(p-1)}{\underset{k-1}{\mathcal{E}'}}(\tau) \sim {}^{\rho}\overset{(p-1)}{\underset{k}{\mathcal{E}'}}(\tau) + {}^{\tilde\rho}\overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau), \qquad {}^{\chi}\overset{(-1-\delta)}{\underset{k}{\mathcal{E}'}}(\tau) + \overset{(-1-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau) \sim {}^{\rho}\overset{(-1-\delta)}{\underset{k}{\mathcal{E}'}}(\tau) + {}^{\tilde\rho}\overset{(-3-\delta)}{\underset{k-1}{\mathcal{E}'}}(\tau)$$

and thus

$${}^{\chi}\overset{(p)}{\underset{k}{\mathcal{X}}}(\tau_0,\tau_1) \sim {}^{\rho}\overset{(p)}{\underset{k}{\mathcal{X}}}(\tau_0,\tau_1).$$

Proof. This follows from (3-104)–(3-108), with F defined as in Proposition 4.5.2, using the estimate (4-25) to absorb the inhomogeneous term, for sufficiently small $\varepsilon_{\text{prelim}}$. □

4.6. *Estimates on the nonlinear terms in the near region.*

Proposition 4.6.1. *Let* $k \ge 4$ *be sufficiently large. There exists an* $\varepsilon_{\text{prelim}} > 0$ *such that the following holds*:

Let ψ *be a solution of* (4-1) *in* $\mathcal{R}(\tau_0,\tau_1)$ *satisfying* (4-23) *for* $p = 0$ *and some* $0 < \varepsilon \le \varepsilon_{\text{prelim}}$. *Then all integrals on lines* (4-13)–(4-16), ***restricted to*** $\boldsymbol{\mathcal{R}(\tau_0,\tau_1) \cap \{r \le R\}}$, *may be estimated by*

$$\cdots \lesssim \int_{\tau_0}^{\tau} \overset{(0)}{\mathcal{E}}(\tau') \sqrt{\overset{(-1-\delta)}{\underset{\ll k}{\mathcal{E}'}}(\tau')}\, d\tau', \tag{4-26}$$

while the integrals on lines (4-18)–(4-19), *again* ***restricted to*** $\boldsymbol{\mathcal{R}(\tau_0,\tau_1) \cap \{r \le R\}}$, *with or without the extra term of Remark 4.4.2, may be similarly estimated by*

$$\cdots \lesssim \int_{\tau_0}^{\tau} \overset{(0)}{\mathcal{E}}(\tau') \sqrt{\overset{(-1-\delta)}{\underset{\ll k}{\mathcal{E}'}}(\tau')}\, d\tau'. \tag{4-27}$$

Proof. This is standard given the assumptions in Section 4.1 and can be proven using the Sobolev inequality (3-112) on $\Sigma(\tau) \cap \{r \le R\}$; cf. the proof of (4-25) which in fact already bounds several of the terms. (Note that the boxed $\varepsilon^{1/4}$ in (4-16) may clearly now be replaced by the quantity $\sqrt{\overset{(-1-\delta)}{\underset{\ll k}{\mathcal{E}'}}(\tau')}$ within the integral.) □

We remark again that, as opposed to (4-26), the control (4-27) loses a derivative, as the energy on the left-hand side of (4-18) is of $(k-1)$-th order while the right-hand side of (4-27) is of k-th order.

4.7. ***Assumptions for the nonlinearity in the far region*: *the "null" condition.*** We may now state our specific assumption capturing the null condition for the semilinear terms.

Rather than formulate algebraically the null condition in terms of the form of N, for maximum generality we will make our basic assumption directly at the level of an estimate for terms on the right-hand side of the inequalities of Proposition 4.4.1.

Our assumptions will only depend on the region $r \ge \frac{8}{9}R$, so we will need some notation to denote the restriction of energies, etc., to this region. Given $r_* \ge r_0$ and v, by

$$\overset{(p)}{\underset{k}{\mathcal{E}}}{}_{r_*,v}, \qquad \overset{(p)}{\underset{k}{\mathcal{X}}}{}_{r_*,v}, \qquad {}^{\rho}\overset{(p)}{\underset{k}{\mathcal{X}}}{}_{r_*,v}, \qquad \ldots$$

we shall mean the expressions defined as before, but where all domains of integration are restricted to the region $\mathcal{R}(\tau_0, \tau_1, v) \cap \{r \geq r_*\}$. We may thus define these expressions for functions

$$\psi : \mathcal{R}(\tau_0, \tau_1, v) \cap \{r \geq r_*\} \to \mathbb{R}.$$

Similarly, the above expressions without the v subscript, e.g., $\overset{(p)}{\underset{k}{\mathcal{E}}}{}_{r_*}$, will be defined where all domains of integration are restricted to $\mathcal{R}(\tau_0, \tau_1) \cap \{r \geq r_*\}$. These can then be defined for functions

$$\psi : \mathcal{R}(\tau_0, \tau_1) \cap \{r \geq r_*\} \to \mathbb{R}.$$

Let us note that

$$r_* \geq \tfrac{8}{9} R \quad \Longrightarrow \quad \overset{(p)}{\underset{k}{\mathcal{X}}}{}_{r_*, v} = {}^{\chi}\overset{(p)}{\underset{k}{\mathcal{X}}}{}_{r_*, v} = {}^{\rho}\overset{(p)}{\underset{k}{\mathcal{X}}}{}_{r_*, v}, \tag{4-28}$$

and similarly for the quantities without the v subscript.

Assumption 4.7.1 (null condition for semilinear terms). *There exists a* k_{null} *and*, *for all* $k \geq k_{\text{null}}$, *an* $\varepsilon_{\text{null}} > 0$ *such that the following holds for all* $\tau_0 \leq \tau_1$, v.

Let ψ *be a smooth function defined on* $\mathcal{R}(\tau_0, \tau_1, v) \cap \left\{r \geq \frac{8}{9} R\right\}$ *satisfying the bound*

$$\overset{(0)}{\underset{\ll k}{\mathcal{X}}}{}_{8R/9, v} \leq \varepsilon \tag{4-29}$$

for some $0 < \varepsilon \leq \varepsilon_{\text{null}}$. *Then*, *for all* $\delta \leq p \leq 2 - \delta$,

$$\begin{aligned}\int_{\mathcal{R}(\tau_0,\tau_1,v)\cap\{r\geq R\}} \sum_{|\boldsymbol{k}|\leq k} &(|r^p r^{-1} L(r\mathfrak{D}^{\boldsymbol{k}}\psi)| + |L\mathfrak{D}^{\boldsymbol{k}}\psi| + |\underline{L}\mathfrak{D}^{\boldsymbol{k}}\psi| + |r^{-1}\nabla\!\!\!/\,\mathfrak{D}^{\boldsymbol{k}}\psi| + |r^{-1}\mathfrak{D}^{\boldsymbol{k}}\psi|) \\ &\qquad \times |\mathfrak{D}^{\boldsymbol{k}}(N^{\alpha\beta}(\psi, x)\partial_\alpha\psi\,\partial_\beta\psi)| + (\mathfrak{D}^{\boldsymbol{k}}(N^{\alpha\beta}(\psi, x)\partial_\alpha\psi\,\partial_\beta\psi))^2 \\ &\lesssim \overset{(p)}{\underset{k}{\mathcal{X}}}{}_{8R/9,v}(\tau_0, \tau_1)\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}{}_{8R/9,v}(\tau_0, \tau_1)} + \sqrt{\overset{(p)}{\underset{k}{\mathcal{X}}}{}_{8R/9,v}(\tau_0, \tau_1)}\sqrt{\overset{(0)}{\underset{k}{\mathcal{X}}}{}_{8R/9,v}(\tau_0, \tau_1)}\sqrt{\overset{(p)}{\underset{\ll k}{\mathcal{X}}}{}_{8R/9,v}(\tau_0, \tau_1)},\end{aligned} \tag{4-30}$$

while, *corresponding to* $p = 0$, *we have*

$$\begin{aligned}\int_{\mathcal{R}(\tau_0,\tau_1,v)\cap\{r\geq R\}} \sum_{|\boldsymbol{k}|\leq k} &(|r^{-1} L(r\mathfrak{D}^{\boldsymbol{k}}\psi)| + |L\mathfrak{D}^{\boldsymbol{k}}\psi| + |\underline{L}\mathfrak{D}^{\boldsymbol{k}}\psi| + |r^{-1}\nabla\!\!\!/\,\mathfrak{D}^{\boldsymbol{k}}\psi| + |r^{-1}\mathfrak{D}^{\boldsymbol{k}}\psi|) \\ &\qquad \times |\mathfrak{D}^{\boldsymbol{k}}(N^{\alpha\beta}(\psi, x)\partial_\alpha\psi\,\partial_\beta\psi)| + (\mathfrak{D}^{\boldsymbol{k}}(N^{\alpha\beta}(\psi, x)\partial_\alpha\psi\,\partial_\beta\psi))^2 \\ &\lesssim \overset{(0+)}{\underset{k}{\mathcal{X}}}{}_{8R/9,v}(\tau_0, \tau_1)\sqrt{\overset{(0+)}{\underset{\ll k}{\mathcal{X}}}{}_{8R/9,v}(\tau_0, \tau_1)}.\end{aligned} \tag{4-31}$$

We note the following.

Proposition 4.7.2. *Assumption 4.7.1 holds for equations (4-1) when* g_0 *is Minkowski and the semilinear terms* N *satisfy the classical null condition of Klainerman [1986]*, *and more generally when* g_0 *is the Kerr metric and the semilinear terms* N *belong to the class considered by Luk [2013].*

Proof. See Appendix C. □

Remark 4.7.3. Let us note that, for ψ defined on the slab $\mathcal{R}(\tau_1, \tau_2) \cap \left\{r \geq \frac{8}{9} R\right\}$, from the trivial bound $\overset{(p)}{\underset{k}{\mathcal{X}}}{}_{r_*,v} \leq \overset{(p)}{\underset{k}{\mathcal{X}}}{}_{r_*}$, we may drop the v subscripts. For ψ defined globally on the slab $\mathcal{R}(\tau_1, \tau_2)$, in view of (4-28) and the trivial bounds ${}^{\rho}\overset{(p)}{\underset{k}{\mathcal{X}}}{}_{r_*} \leq {}^{\rho}\overset{(p)}{\underset{k}{\mathcal{X}}}$, etc., we may replace the $\frac{8}{9}M$, v subscripts in all the $\overset{(p)}{\underset{k}{\mathcal{X}}}$, etc., energies on the right-hand side with the ρ superscript, i.e., replace $\overset{(p)}{\underset{k}{\mathcal{X}}}{}_{8M/9,v}$, etc., with ${}^{\rho}\overset{(p)}{\underset{k}{\mathcal{X}}}$, etc. Also, for such ψ, we may clearly replace (4-29) with assumption (4-23).

4.8. ***The final estimates.*** Putting everything together we have the following:

Proposition 4.8.1. *Consider* $(\mathcal{M}, g_0)$ *satisfying the assumptions of Sections 2 and 3* (*for cases* (i), (ii), *or* (iii)) *and equation (4-1) satisfying the assumptions of Sections 4.1 and 4.7.*

Then, for all $k \geq 4$ *sufficiently large, there exist constants* $C > 0$ (*also implicit in the* $\lesssim$ *below*), $c > 0$, *and an* $\varepsilon_{\text{prelim}} > 0$, *such that the following is true*:

Let $\tau_0 \leq \tau_1$, *and let* ψ *be a solution of (4-1) in* $\mathcal{R}(\tau_0, \tau_1)$ *satisfying (4-23) with* $p = 0$ *for some* $0 < \varepsilon \leq \varepsilon_{\text{prelim}}$. *Then, for* $\delta \leq p \leq 2 - \delta$, *one has the estimates*

$$\begin{aligned}
&\sup_{v:\tau\leq\tau(v)} \overset{(p)}{\mathfrak{F}}(v, \tau_0, \tau), \quad c^{\rho}\overset{(p)}{\underset{k}{\mathcal{X}}}(\tau_0, \tau_1), \quad \overset{(p)}{\underset{k}{\mathfrak{E}}}(\tau_1) \\
&\qquad \leq \overset{(p)}{\underset{k}{\mathfrak{E}}}(\tau_0) + A\int_{\tau_0}^{\tau_1} \overset{\xi}{\underset{k-1}{\mathcal{E}'}}(\tau')\,d\tau' + C\overset{(p)}{\underset{k}{\mathcal{X}}}{}_{8R/9}(\tau_0, \tau_1)\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}{}_{8R/9}(\tau_0, \tau_1)} \\
&\qquad\quad + C\sqrt{\overset{(p)}{\underset{k}{\mathcal{X}}}{}_{8R/9}(\tau_0, \tau_1)}\sqrt{\overset{(0)}{\underset{k}{\mathcal{X}}}{}_{8R/9}(\tau_0, \tau_1)}\sqrt{\overset{(p)}{\underset{\ll k}{\mathcal{X}}}{}_{8R/9}(\tau_0, \tau_1)} + C\int_{\tau_0}^{\tau_1} \overset{(0)}{\mathcal{E}}(\tau')\sqrt{\overset{(-1-\delta)}{\underset{\ll k}{\mathcal{E}'}}(\tau')}\,d\tau',
\end{aligned} \tag{4-32}$$

$$\begin{aligned}
\chi\overset{(p)}{\underset{k-1}{\mathcal{X}}}(\tau_0, \tau_1) \lesssim{}& \overset{(p)}{\underset{k-1}{\mathcal{E}}}(\tau_0) + \overset{(p)}{\underset{k-1}{\mathcal{X}}}{}_{8R/9}(\tau_0, \tau_1)\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}{}_{8R/9}(\tau_0, \tau_1)} + \sqrt{\overset{(p)}{\underset{k-1}{\mathcal{X}}}{}_{8R/9}(\tau_0, \tau_1)}\sqrt{\overset{(0)}{\underset{k-1}{\mathcal{X}}}{}_{8R/9}(\tau_0, \tau_1)}\sqrt{\overset{(p)}{\underset{\ll k}{\mathcal{X}}}{}_{8R/9}(\tau_0, \tau_1)} \\
&+ \int_{\tau_0}^{\tau_1} \overset{(0)}{\mathcal{E}}(\tau')\sqrt{\overset{(-1-\delta)}{\underset{\ll k}{\mathcal{E}'}}(\tau')}\,d\tau',
\end{aligned} \tag{4-33}$$

while, for $p = 0$, *one has the estimates*

$$\begin{aligned}
&\sup_{v:\tau\leq\tau(v)} \overset{(0)}{\mathfrak{F}}(v, \tau_0, \tau), \quad c^{\rho}\overset{(0)}{\underset{k}{\mathcal{X}}}(\tau_0, \tau_1), \quad \overset{(0)}{\underset{k}{\mathfrak{E}}}(\tau_1) \\
&\quad \leq \overset{(0)}{\underset{k}{\mathfrak{E}}}(\tau_0) + A\int_{\tau_0}^{\tau_1} \overset{\xi}{\underset{k-1}{\mathcal{E}'}}(\tau')\,d\tau' + C\overset{(0+)}{\underset{k}{\mathcal{X}}}{}_{8R/9}(\tau_0, \tau_1)\sqrt{\overset{(0+)}{\underset{\ll k}{\mathcal{X}}}{}_{8R/9}(\tau_0, \tau_1)} + C\int_{\tau_0}^{\tau_1} \overset{(0)}{\mathcal{E}}(\tau')\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{E}}}(\tau')}\,d\tau',
\end{aligned} \tag{4-34}$$

$$\chi\overset{(0)}{\underset{k-1}{\mathcal{X}}}(\tau_0, \tau_1) \lesssim \overset{(0)}{\underset{k-1}{\mathcal{E}}}(\tau_0) + \overset{(0+)}{\underset{k}{\mathcal{X}}}{}_{8R/9}(\tau_0, \tau_1)\sqrt{\overset{(0+)}{\underset{\ll k}{\mathcal{X}}}{}_{8R/9}(\tau_0, \tau_1)} + \int_{\tau_0}^{\tau_1} \overset{(0)}{\mathcal{E}}(\tau')\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{E}}}(\tau')}\,d\tau'. \tag{4-35}$$

Finally, in both the case $p = 0$ *and the case* $\delta \leq p \leq 2 - \delta$, *one has the alternative estimate*

$$\overset{(p)}{\underset{k}{\mathcal{X}}}(\tau_0, \tau_1) \lesssim \overset{(p)}{\underset{k}{\mathcal{E}}}(\tau_0) + (1 + \tau_1 - \tau_0)\overset{(p)}{\underset{k}{\mathcal{X}}}(\tau_0, \tau_1)\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}(\tau_0, \tau_1)}, \tag{4-36}$$

depending on $\tau_1 - \tau_0$.

Proof. This follows from Proposition 4.4.1, Corollary 4.5.3, Proposition 4.6.1 and Assumption 4.7.1. □

Remark 4.8.2. Note how the case $p = 0$ is anomalous in that one sees "0+" on the right-hand side of (4-34)–(4-35). It is this fact which will require us to use $p > 0$ weights to close the global estimates (4-32)–(4-33), even in the case (i). On the other hand, (4-36) is sufficient to show local existence even using only the $p = 0$ weight.

Remark 4.8.3. In using the above we shall always simply replace terms like $\overset{(p)}{\underset{k}{\mathcal{X}}}{}_{8R/9}(\tau_0, \tau_1)$, etc., by $\rho\overset{(p)}{\underset{k}{\mathcal{X}}}(\tau_0, \tau_1)$, etc., according to Remark 4.7.3. We have kept the terms in the above form simply to highlight their origin in the estimate of Assumption 4.7.1.

Remark 4.8.4. In the case where we only assume the black box inequality (3-3) and the assumptions of asymptotic flatness of Sections 3.5 and 3.6.2 as interpreted in Remark 3.6.10 (necessary to obtain (3-110)) and, moreover, the equation (4-1) is semilinear, i.e., $g(x,\psi)=g_0$ for all $x\in\mathcal{M}$, we then obtain the statement of Proposition 4.8.1 without inequalities (4-32) and (4-34) and where (4-33) and (4-35) are replaced by

$$\chi\,\overset{(p)}{\underset{k}{\mathcal{X}}}(\tau_0,\tau_1)\lesssim \overset{(p)}{\underset{k}{\mathcal{E}}}(\tau_0)+\overset{(p)}{\underset{k}{\mathcal{X}}}{}_{8R/9}(\tau_0,\tau_1)\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}{}_{8R/9}(\tau_0,\tau_1)}+\sqrt{\overset{(p)}{\underset{k}{\mathcal{X}}}{}_{8R/9}(\tau_0,\tau_1)}\sqrt{\overset{(0)}{\underset{k}{\mathcal{X}}}{}_{8R/9}(\tau_0,\tau_1)}\sqrt{\overset{(p)}{\underset{\ll k}{\mathcal{X}}}{}_{8R/9}(\tau_0,\tau_1)}$$
$$+\int_{\tau_0}^{\tau_1}\overset{(0)}{\mathcal{E}}(\tau')\sqrt{\overset{(-1-\delta)}{\underset{\ll k}{\mathcal{E}'}}(\tau')}\,d\tau', \tag{4-37}$$

$$\chi\,\overset{(0)}{\underset{k}{\mathcal{X}}}(\tau_0,\tau_1)\lesssim \overset{(0)}{\mathcal{E}}(\tau_0)+\overset{(0+)}{\underset{k}{\mathcal{X}}}{}_{8R/9}(\tau_0,\tau_1)\sqrt{\overset{(0+)}{\underset{\ll k}{\mathcal{X}}}{}_{8R/9}(\tau_0,\tau_1)}+\int_{\tau_0}^{\tau_1}\overset{(0)}{\mathcal{E}}(\tau')\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{E}}}(\tau')}\,d\tau', \tag{4-38}$$

respectively. (In particular, (4-36) also still holds as stated.)

4.9. *Local well-posedness.*

Finally, we give a local well-posedness statement and an extension criterion.

Since our initial data hypersurfaces $\Sigma(\tau_0)$ are in part null, it will be convenient to assume that our initial data are smooth. We define a smooth initial data set on $\Sigma(\tau_0)$ to be a pair $(\uppsi,\uppsi')$ where $\uppsi$ is a function on $\Sigma(\tau_0)$ smooth on $\Sigma(\tau_0)\cap\{r\le R\}$ and smooth on $\Sigma(\tau_0)\cap\{r\ge R\}$ and $\uppsi'$ is a smooth function on $\Sigma(\tau_0)\cap\{r\le R\}$, such that moreover there exists a smooth function Ψ on $\mathcal{R}(\tau_0,\tau_1)$ for some τ_1 such that

$$\Psi|_{\Sigma(\tau_0)}=\uppsi,\quad n\Psi|_{\Sigma(\tau_0)\cap\{r\le R\}}=\uppsi',$$

where n denotes the normal to $\Sigma(\tau_0)\cap\{r\le R\}$.

Note that given such initial data on $\Sigma(\tau_0)$ one may define

$$\overset{(p)}{\underset{k}{\mathcal{E}}}[\uppsi,\uppsi'] \tag{4-39}$$

as follows:

Using (4-1) we may compute along $\Sigma(\tau_0)$ the $(k+1)$-jet of any smooth solution ψ of (4-1) such that

$$\psi|_{\Sigma(\tau_0)}=\uppsi,\quad n\psi|_{\Sigma(\tau_0)\cap\{r\le R\}}=\uppsi'. \tag{4-40}$$

We may define $\overset{(p)}{\underset{k}{\mathcal{E}}}[\uppsi,\uppsi']$ to equal the usual $\overset{(p)}{\underset{k}{\mathcal{E}}}(\tau_0)$ where the derivatives are interpreted in terms of this formal computation. Note of course that the expression (4-39) may well be infinite.

Alternatively, we may express this as follows. Using the above computation, we may in fact define a smooth Ψ as in the previous paragraph such that Ψ moreover satisfies (4-1) along $\Sigma(\tau_0)$ and, for all $|\boldsymbol{k}|\le k$, $\widetilde{\mathfrak{D}}^{\boldsymbol{k}}\Psi$ satisfies along $\Sigma(\tau_0)$ the equation one obtains by commuting (4-1) by $\widetilde{\mathfrak{D}}^{\boldsymbol{k}}$. We may then simply define (4-39) by substituting this Ψ for ψ in the usual expression for $\overset{(p)}{\underset{k}{\mathcal{E}}}(\tau_0)$.

With this, we can now state our local well-posedness:

Proposition 4.9.1 (local well-posedness). *Consider $(\mathcal{M},g_0)$ satisfying the assumptions of Sections 2 and 3 (for cases* (i), (ii), *or* (iii)) *and equation (4-1) satisfying the assumptions of Sections 4.1 and 4.7. Fix either $p=0$ or $\delta\le p\le 2-\delta$.*

There exists a positive integer $k_{\mathrm{loc}} \geq 4$ *such that the following holds: Let* $k \geq k_{\mathrm{loc}}$. *Then there exists a positive real constant* $C > 0$ *sufficiently large, a positive real parameter* $\varepsilon_{\mathrm{loc}} > 0$ *sufficiently small, and a decreasing positive function* $\tau_{\mathrm{exist}} : (0, \varepsilon_{\mathrm{loc}}) \to \mathbb{R}$ *such that, for all smooth initial data* (ψ, ψ') *on* $\Sigma(\tau_0)$ *such that*

$$\overset{(p)}{\underset{k}{\mathcal{E}}}[\psi, \psi'] \leq \varepsilon_0 \leq \varepsilon_{\mathrm{loc}},$$

there exists a smooth solution ψ *of* (4-1) *in* $\mathcal{R}(\tau_0, \tau_1)$ *for* $\tau_1 := \tau_0 + \tau_{\mathrm{exist}}$ *which satisfies*

$$\overset{(p)}{\underset{k}{\mathcal{X}}}(\tau_0, \tau_1) \leq C\varepsilon_0.$$

Moreover, for all $\tau_0 \leq \tau \leq \tau_1$, *any other smooth* $\tilde{\psi}$ *defined on* $\mathcal{R}(\tau_0, \tau)$ *satisfying* (4-1) *in* $\mathcal{R}(\tau_0, \tau)$ *and attaining the initial data, i.e., satisfying* (4-40) *(with* $\tilde{\psi}$ *replacing* ψ*), coincides with the restriction of* ψ; *i.e.,* $\tilde{\psi} = \psi|_{\mathcal{R}(\tau_0, \tau)}$.

We have in addition the following propagation of higher-order regularity and/or higher weighted estimates. Given p, $k \geq k_{\mathrm{loc}}$, $\varepsilon_0 \leq \varepsilon_{\mathrm{loc}}$, τ_0, τ_1 *as above,* $2-\delta \geq p' \geq p$ *if* $p \geq \delta$ *and either* $2-\delta \geq p' \geq \delta$ *or* $p' = 0$ *if* $p = 0$, *and* $k' \geq k$. *Then there exists a constant* $C(k', \tau_1 - \tau_0)$ *such that, given* ψ *as above,*

$$\overset{(p')}{\underset{k'}{\mathcal{X}}}(\tau_0, \tau_1) \leq C(k', \tau_1 - \tau_0) \overset{(p')}{\underset{k'}{\mathcal{E}}}(\tau_0).$$

Proof. This can be easily proven using estimate (4-36). We leave the details to the reader. □

Note also the following easy corollary, which we will use as an extension criterion.

Corollary 4.9.2 (continuation criterion). *Fix* $p = 0$ *or* $\delta \leq p \leq 2-\delta$, *and let* $k \geq k_{\mathrm{loc}}$ *and* $\varepsilon_{\mathrm{loc}}$ *be as in Proposition 4.9.1. There exists a constant* $C > 0$ *and an* $\epsilon > 0$ *such that the following is true:*

Let $\tau_f > \tilde{\tau}_0$, *and suppose that there exists a smooth solution* ψ *of* (4-1) *on* $\bigcup_{\tilde{\tau}_0 < \tau < \tau_f} \mathcal{R}(\tilde{\tau}_0, \tau)$ *such that*

$$\overset{(p)}{\underset{k}{\mathcal{E}}}(\tau) \leq \varepsilon_{\mathrm{loc}}$$

for $k \geq k_{\mathrm{loc}}$ *and all* $\tilde{\tau}_0 \leq \tau < \tau_f$. *Then, defining* $\tau_1 := \tau_f + \epsilon$, ψ *extends uniquely to a smooth function on* $\mathcal{R}(\tilde{\tau}_0, \tau_1)$ *satisfying* (4-1) *on this set and, setting* $\tau_0 := \max\{\tau_f - \epsilon, \tilde{\tau}_0\}$, *satisfying the estimate*

$$\overset{(p)}{\underset{k}{\mathcal{X}}}(\tau_0, \tau_1) \leq C\varepsilon_{\mathrm{loc}}. \tag{4-41}$$

Moreover, for all $k' \geq k$ *and all* $2-\delta \geq p' \geq p$, *there exist constants* $C(k')$ *such that*

$$\overset{(p')}{\underset{k'}{\mathcal{X}}}(\tau_0, \tau_1) \leq C(k') \overset{(p')}{\underset{k'}{\mathcal{E}}}(\tau_0). \tag{4-42}$$

Remark 4.9.3. Recall that by our conventions the constants C, k_{loc}, $\varepsilon_{\mathrm{loc}}$ depend in addition on k. We may assume that k_{loc} is large enough that $k \geq k_{\mathrm{loc}}$ satisfies the largeness constraint of all previous propositions in this section and that $\varepsilon_{\mathrm{loc}} \leq \varepsilon_{\mathrm{prelim}}$.

Remark 4.9.4. In view of Remark 4.8.4 and the fact that one needs only (4-36), all the statements of this section also hold in the semilinear case under the relaxed assumptions described there.

5. The main theorem: global existence and stability

We may now state the main result of this paper.

Theorem 5.1 (global existence and stability). *Consider* $(\mathcal{M}, g_0)$ *satisfying the assumptions of Sections 2 and 3 (for cases* (i), (ii), *or* (iii)) *and equation (4-1) satisfying the assumptions of Section 4.1 and Assumption 4.7.1 of Section 4.7.*

There exists a positive integer $k_{\mathrm{global}} \geq k_{\mathrm{local}}$ *large enough that, given* $k \geq k_{\mathrm{global}}$, *there exists a positive* $0 < \varepsilon_{\mathrm{global}} < \varepsilon_{\mathrm{loc}}$ *sufficiently small and a positive constant* $C > 0$ *sufficiently large, so that the following holds*:

Fix $\tau_0 = 1$ *and consider as in Proposition 4.9.1 initial data* (ψ, ψ') *on* $\Sigma(\tau_0)$ *for (4-1) satisfying*

$$\overset{(p)}{\underset{k}{\mathcal{E}}}[\psi, \psi'] \leq \varepsilon_0 \leq \varepsilon_{\mathrm{global}},$$

$$\overset{(1)}{\underset{k}{\mathcal{E}}}[\psi, \psi'] + \overset{(2-\delta)}{\underset{k-1}{\mathcal{E}}}[\psi, \psi'] \leq \varepsilon_0 \leq \varepsilon_{\mathrm{global}},$$

$$\overset{(1)}{\underset{k}{\mathcal{E}}}[\psi, \psi'] + \overset{(2-\delta)}{\underset{k-2}{\mathcal{E}}}[\psi, \psi'] \leq \varepsilon_0 \leq \varepsilon_{\mathrm{global}},$$

according to case (i), (ii), *and* (iii), *respectively, where in the former case* $\delta \leq p \leq 2 - \delta$. *Then the* ψ *given by Proposition 4.9.1 extends to a unique globally defined solution of (4-1) in* $\mathcal{R}(\tau_0, \infty)$ *satisfying the estimates*

$$^{\rho}\overset{(p)}{\underset{k}{\mathcal{X}}}(\tau_0, \tau) \leq C\varepsilon_0, \quad ^{\rho}\overset{(1)}{\underset{k}{\mathcal{X}}}(\tau_0, \tau) + {^{\rho}\overset{(2-\delta)}{\underset{k-1}{\mathcal{X}}}}(\tau_0, \tau) \leq C\varepsilon_0, \quad ^{\rho}\overset{(1)}{\underset{k}{\mathcal{X}}}(\tau_0, \tau) + {^{\rho}\overset{(2-\delta)}{\underset{k-2}{\mathcal{X}}}}(\tau_0, \tau) \leq C\varepsilon_0, \tag{5-1}$$

according to cases (i), (ii), (iii), *respectively, for all* $\tau \geq \tau_0$; *in particular*

$$\overset{(p)}{\underset{k}{\mathcal{E}}}(\tau) \leq C\varepsilon_0, \quad \overset{(1)}{\underset{k}{\mathcal{E}}}(\tau) + \overset{(2-\delta)}{\underset{k-1}{\mathcal{E}}}(\tau) \leq C\varepsilon_0, \quad \overset{(1)}{\underset{k}{\mathcal{E}}}(\tau) + \overset{(2-\delta)}{\underset{k-2}{\mathcal{E}}}(\tau) \leq C\varepsilon_0,$$

respectively.

The solution will satisfy moreover estimates (6-8) in case (i), *estimates (6-62) and (6-64)–(6-68) in case* (ii), *and estimates (6-116)–(6-121) in case* (iii).

We shall give the proof of this theorem in Section 6.

Remark 5.2. Our theorem also applies under the relaxed assumptions of Remark 4.8.4, where (4-1) is however required to be semilinear. Here we can again distinguish case (i) where $\chi = 1$ and case (ii) where χ is allowed to degenerate. In case (i), the theorem holds as stated except for (6-8), which no longer applies. Similarly, in case (ii), the theorem holds as stated except for (6-62), which no longer applies. The necessary modifications will be collected in a series of remarks in Sections 6.1 and 6.2. (See already Section 6.2.5 for the completion of case (ii).)

We reiterate (cf. Propositions 4.7.2, Theorem A.1 and Theorem D.1) that the above theorem in particular holds for equations (4-1) on the Kerr spacetime $(\mathcal{M}, g_{a,M})$ for $|a| \ll M$, with quasilinear terms as described in Section 4.1 and the general semilinear terms considered in [Luk 2013], and, by the above remark, restricted to the purely semilinear case, for the full range of parameters $|a| < M$.

6. The estimate hierarchies and global existence and decay

We proceed with the proof of Theorem 5.1 in cases (i), (ii), and (iii). These will be treated in Sections 6.1, 6.2, and 6.3 below, respectively.

Note that we will now suppress some of the cumbersome notation with the following conventions:

In this section, the fraktur energies $\mathfrak{E}$, $\mathfrak{F}$ will everywhere denote $\mathfrak{E}(g)$, $\mathfrak{F}(g)$, so we will drop explicit reference to g here. When the region $\mathcal{R}(\tau_0, \tau_1)$ is assumed, we will omit the τ_0 or (τ_0, τ_1) arguments in various energies, etc., and denote $\mathcal{X}(\tau_0, \tau_1)$ by $\mathcal{X}$.

The smallness parameter $\varepsilon_{\text{global}}$ will be determined in the context of the proof. We will in fact introduce other parameters on the way; these will be in the relation

$$\varepsilon_{\text{global}} \le \hat{\varepsilon}_{\text{slab}} \le \varepsilon_{\text{slab}} \le \varepsilon_{\text{boot}} \le \varepsilon_{\text{loc}}, \tag{6-1}$$

where parameters $\hat{\varepsilon}_{\text{slab}}$, $\varepsilon_{\text{slab}}$ will only appear for cases (ii) and (iii). The parameter $\varepsilon_{\text{boot}}$ will be used in the context of a continuity or bootstrap argument. Let us remark already that if the basic bootstrap estimate (4-23) holds for an $\varepsilon \le \varepsilon_{\text{boot}}$, then by Remark 4.9.3 ε will satisfy the smallness requirements of all propositions of Section 4 for the regions under consideration.

6.1. ***Case* (i).** Case (i) is the most elementary case, where moreover only the minimal nontrivial r^p weights for $2-\delta \ge p \ge \delta$ are required on data. (For instance, one may fix $p = \delta$.) We present first the fundamental estimate in Section 6.1.1 that holds under the bootstrap assumption (4-23) and then carry out the global existence proof in Section 6.1.2.

The proof will also hold for the semilinear case with the relaxed assumptions of Remark 4.8.4 if we are in the analogue of case (i), i.e., if $\chi = 1$. The modifications necessary to treat this case are described in a series of remarks (see already Remarks 6.1.2 and 6.1.3).

6.1.1. *The fundamental estimate.* The fundamental estimate is given simply by the following:

Proposition 6.1.1. *Let* $\delta \le p \le 2-\delta$, *let k be sufficiently large, and let us assume the case* (i) *assumptions. There exists an* $\varepsilon_{\text{boot}} > 0$ *small enough that the following is true*:

Let ψ *solve* (4-1) *on* $\mathcal{R}(\tau_0, \tau_1)$ *and satisfy moreover* (4-23) *with our chosen p and with* $0 < \varepsilon \le \varepsilon_{\text{boot}}$. *Then we have*

$$\overset{(p)}{\underset{k}{\mathcal{X}}} \lesssim \overset{(p)}{\underset{k}{\mathcal{E}}}(\tau_0), \qquad \sup_{\tau_0 \le \tau \le \tau_1} \overset{(p)}{\underset{k}{\mathfrak{E}}}(\tau) \le \overset{(p)}{\underset{k}{\mathfrak{E}}}(\tau_0)(1 + C\varepsilon^{\frac{1}{2}}). \tag{6-2}$$

Proof. Let $\varepsilon_{\text{boot}} \le \varepsilon_{\text{loc}}$. The assumption of Proposition 4.8.1 is satisfied in view of our discussion following (6-1). From estimate (4-32) of Proposition 4.8.1, since in case (i) we have $\rho = 1$, $A = 0$, $\tilde{A} = 0$, it follows that

$$\overset{(p)}{\underset{k}{\mathcal{X}}} \lesssim \overset{(p)}{\underset{k}{\mathcal{E}}}(\tau_0) + \overset{(p)}{\underset{k}{\mathcal{X}}} \sqrt{\overset{(p)}{\underset{\ll k}{\mathcal{X}}}}, \qquad \sup_{\tau_0 \le \tau \le \tau_1} \overset{(p)}{\underset{k}{\mathfrak{E}}}(\tau) \le \overset{(p)}{\underset{k}{\mathfrak{E}}}(\tau_0) + C \overset{(p)}{\underset{k}{\mathcal{X}}} \sqrt{\overset{(p)}{\underset{\ll k}{\mathcal{X}}}}. \tag{6-3}$$

By our bootstrap assumption (4-23), we have

$$\overset{(p)}{\underset{\ll k}{\mathcal{X}}} \le \varepsilon. \tag{6-4}$$

Thus, possibly requiring $\varepsilon_{\text{boot}}$ to be even smaller, we may absorb the nonlinear term in the first inequality of (6-3) to obtain the first inequality of (6-2).

To obtain the second inequality of (6-2), we remark that Corollary 4.5.3 applies to yield in particular

$$\overset{(p)}{\underset{k}{\mathcal{E}}}(\tau_0) \sim \overset{(p)}{\underset{k}{\mathfrak{E}}}(\tau_0). \tag{6-5}$$

The second inequality of (6-2) now immediately follows from the second inequality of (6-3), plugging in the first inequality of (6-2) just established, (6-5) and (6-4). □

Remark 6.1.2. Under the relaxed assumptions of Remark 4.8.4, where (4-1) is however required to be semilinear and we are in the analogue of case (i), i.e., $\chi = 1$, from (4-38) we obtain the first inequality of (6-3). Thus, the above proposition holds as stated for the first inequality of (6-2).

6.1.2. *Proof of Theorem 5.1 in case* (i). We now carry out the proof of Theorem 5.1 proper in case (i).

Let $\delta \le p \le 2-\delta$ be as in the statement, and recall the assumption

$$\overset{(p)}{\underset{k}{\mathcal{E}}}(\tau_0) \le \varepsilon_0 \le \varepsilon_{\text{global}}$$

on initial data for a sufficiently small $\varepsilon_{\text{global}}$ to be determined.

Consider the set $\mathfrak{B}$ consisting of all $\tau_f > \tau_0$ such that a solution ψ of (4-1) achieving the data exists on $\mathcal{R}(\tau_0, \tau_f)$ and such that moreover the energy bootstrap assumption (4-23) holds on $\mathcal{R}(\tau_0, \tau_1 := \tau_f)$ with our chosen p and where $0 < \varepsilon \le \varepsilon_{\text{boot}}$ is chosen to satisfy

$$1 \gg \varepsilon \gg \varepsilon_{\text{global}}. \tag{6-6}$$

(The above relation in particular constrains $\varepsilon_{\text{global}}$ to be small.) By the local well-posedness statement Proposition 4.9.1, it follows that, since $k \ge k_{\text{loc}}$ and $\varepsilon_0 \le \varepsilon_{\text{global}} \le \varepsilon_{\text{loc}}$, we have $\tau_0 + \tau_{\text{exist}} \in \mathfrak{B}$, and thus $\mathfrak{B} \ne \varnothing$. Also note that, a fortiori, if $\tau_f \in \mathfrak{B}$, then $(\tau_0, \tau_f] \subset \mathfrak{B}$ and thus $\mathfrak{B}$ is manifestly a connected subset of (τ_0, ∞).

For any $\tau_1 \in \mathfrak{B}$, Proposition 6.1.1 applies in $\mathcal{R}(\tau_0, \tau_1)$. The first estimate of (6-2) then yields

$$\overset{(p)}{\underset{k}{\mathcal{X}}} \lesssim \overset{(p)}{\underset{k}{\mathcal{E}}}(\tau_0) \lesssim \varepsilon_0. \tag{6-7}$$

If follows that, for sufficiently small $\varepsilon_{\text{glob}}$ and all $\varepsilon_0 \le \varepsilon_{\text{glob}}$, the continuation criterion Corollary 4.9.2 applies to obtain that ψ extends as a solution to (4-1) on some $\mathcal{R}(\tau_0, \tau_f + \epsilon)$, where ϵ is independent of τ_f, and that, considering now $\tau_1 := \tau_f + \epsilon$, inequality (6-7) holds also in $\mathcal{R}(\tau_0, \tau_1 := \tau_f + \epsilon)$ (with a slightly different implicit constant than that of (6-7)). (To infer this from (4-41), which was an estimate on $\overset{(p)}{\underset{k}{\mathcal{X}}}(\tau_f, \tau_f + \epsilon)$, note that

$$\overset{(p)}{\underset{k}{\mathcal{X}}}(\tau_0, \tau_1) \lesssim \overset{(p)}{\underset{k}{\mathcal{X}}}(\tau_0, \tau_f) + \overset{(p)}{\underset{k}{\mathcal{X}}}(\tau_f, \tau_f + \epsilon).)$$

Finally we note that, for ε satisfying (6-6), inequality (6-7) in $\mathcal{R}(\tau_0, \tau_1 := \tau_f + \epsilon)$ shows in particular that inequalities (4-23) hold in this set. It follows by the definition of $\mathfrak{B}$ that $\tau_f + \epsilon \in \mathfrak{B}$ and thus, given also its connectedness, the set $\mathfrak{B}$ is open. Since ϵ does not depend on τ_f, it follows that $\mathfrak{B}$ is also closed, as any limit point τ_{limit} of $\mathfrak{B}$ in (τ_0, ∞) satisfies $\tau_{\text{limit}} \le \tau_f + \epsilon$ for some $\tau_f \in \mathfrak{B}$. We have shown that $\mathfrak{B}$ is a nonempty open and closed subset of (τ_0, ∞), and thus $\mathfrak{B} = (\tau_0, \infty)$. Hence the solution ψ exists globally in $\mathcal{R}(\tau_0, \infty)$ and satisfies (6-7) in $\mathcal{R}(\tau_0, \tau_1)$ where τ_1 is now any $\tau_1 > \tau_0$. This gives (5-1).

Revisiting (6-2) we see finally that we have the more precise global estimate

$$\overset{(p)}{\underset{k}{\mathfrak{E}}}(\tau) \leq \overset{(p)}{\underset{k}{\mathfrak{E}}}(\tau_0)(1 + C\varepsilon_0^{\frac{1}{2}}). \tag{6-8}$$

This completes the proof.

Remark 6.1.3. Note that, in view of Remark 6.1.2, the above proof goes through, except for the last paragraph involving (6-8), also under the relaxed assumptions of Remark 4.8.4, where (4-1) is however required to be semilinear and we are in the analogue of case (i), i.e., $\chi = 1$. Thus, in this case, one indeed obtains the statement as given in Remark 5.2.

6.2. *Case* (ii). Case (ii) is the second simplified case which we shall consider. Unlike case (i), we shall require proving actual τ-decay above a certain threshold in order to close, and this in turn will require raising p to $p = 2 - \delta$ in our initial energy assumptions.

The logic of the proof is a simplified version of the scheme we shall use for the general case and which has already been summarised in the introduction:

(1) Rather than bootstrap directly t-weighted estimates, we formulate the fundamental estimates, depending only on the bootstrap assumption (4-23), as a hierarchy of estimates on a spacetime slab of τ length at most L. This is the content of Section 6.2.1.

(2) We shall then show by a bootstrap argument, *restricted to such a slab*, how global existence and estimates on the slab can be proven, with suitable assumptions on the initial data of the slab, which now involve L. This is the content of Section 6.2.2.

(3) Still restricted to a given slab of length L, we will show that one can in fact a posteriori improve the above estimates under suitable additional assumptions on the initial data and, using a pigeonhole argument, show moreover an improved estimate for any $\Sigma(\tau')$ slice near the top of the slab. This is the content of Section 6.2.3.

(4) Finally, global existence and τ decay now follow by iterating the above estimates on a consecutive sequence of spacetime slabs of dyadic time length $L_i = 2^i$. This is the content of Section 6.2.4.

We shall in addition give an alternative iteration proof in Section 6.2.5 which does not use the exact boundedness statement of the fraktur energies. The advantage of this alternative proof is that it allows us to treat the semilinear case with the relaxed assumptions of Remark 4.8.4. The modifications necessary to treat this case are described in a series of remarks (see already Remarks 6.2.2, 6.2.4, 6.2.7 and 6.2.8).

6.2.1. *The hierarchy of inequalities.*

Proposition 6.2.1. *Let k be sufficiently large, and let us assume the case* (ii) *assumptions. There exist constants $C > 0$, $c > 0$ and an $\varepsilon_{\mathrm{boot}} > 0$ small enough that the following is true*:

Consider a region $\mathcal{R}(\tau_0, \tau_1)$ and a ψ solving (4-1) on $\mathcal{R}(\tau_0, \tau_1)$, satisfying moreover (4-23) with $p = 0$ and with $0 < \varepsilon \leq \varepsilon_{\mathrm{boot}}$. Let us assume moreover that

$$\tau_1 \leq \tau_0 + L$$

for some arbitrary $L > 0$. *We have the following hierarchy of inequalities on* $\mathcal{R}(\tau_0, \tau_1)$:

$$\underset{k}{\overset{(2-\delta)}{\mathfrak{F}}}(v, \tau_1),\ \ \underset{k}{\overset{(2-\delta)}{\mathfrak{E}}}(\tau_1),\ \ c\,{}^{\chi}\underset{k}{\overset{(2-\delta)}{\mathcal{X}}} \le \underset{k}{\overset{(2-\delta)}{\mathfrak{E}}}(\tau_0) + C\left({}^{\chi}\underset{k}{\overset{(2-\delta)}{\mathcal{X}}}\sqrt{\underset{\ll k}{\overset{(0)}{\mathcal{X}}}} + \sqrt{{}^{\chi}\underset{k}{\overset{(2-\delta)}{\mathcal{X}}}}\sqrt{{}^{\chi}\underset{k}{\overset{(0)}{\mathcal{X}}}}\sqrt{\underset{\ll k}{\overset{(2-\delta)}{\mathcal{X}}}}\right) + C\,{}^{\chi}\underset{k}{\overset{(0)}{\mathcal{X}}}\sqrt{\underset{\ll k}{\overset{(0)}{\mathcal{X}}}}\sqrt{L}, \tag{6-9}$$

$$\underset{k}{\overset{(1)}{\mathfrak{F}}}(v, \tau_1),\ \ \underset{k}{\overset{(1)}{\mathfrak{E}}}(\tau_1),\ \ c\,{}^{\chi}\underset{k}{\overset{(1)}{\mathcal{X}}} \le \underset{k}{\overset{(1)}{\mathfrak{E}}}(\tau_0) + C\,{}^{\chi}\underset{k}{\overset{(1)}{\mathcal{X}}}\sqrt{\underset{\ll k}{\overset{(1)}{\mathcal{X}}}} + C\,{}^{\chi}\underset{k}{\overset{(0)}{\mathcal{X}}}\sqrt{\underset{\ll k}{\overset{(0)}{\mathcal{X}}}}\sqrt{L}, \tag{6-10}$$

$$\begin{aligned}\underset{k}{\overset{(0)}{\mathfrak{F}}}(v, \tau_1),\ \ \underset{k}{\overset{(0)}{\mathfrak{E}}}(\tau_1),\ \ c\,{}^{\chi}\underset{k}{\overset{(0)}{\mathcal{X}}} \le \underset{k}{\overset{(0)}{\mathfrak{E}}}(\tau_0) + C\Big({}^{\chi}\underset{k}{\overset{(0)}{\mathcal{X}}} + \big({}^{\chi}\underset{k}{\overset{(0)}{\mathcal{X}}}\big)^{\frac{1-\delta}{1+\delta}}\big({}^{\chi}\underset{k}{\overset{(1)}{\mathcal{X}}}\big)^{\frac{2\delta}{1+\delta}}\Big)\sqrt{\underset{\ll k}{\overset{(0)}{\mathcal{X}}} + \big(\underset{\ll k}{\overset{(0)}{\mathcal{X}}}\big)^{\frac{1-\delta}{1+\delta}}\big(\underset{\ll k}{\overset{(1)}{\mathcal{X}}}\big)^{\frac{2\delta}{1+\delta}}}\\ + C\,{}^{\chi}\underset{k}{\overset{(0)}{\mathcal{X}}}\sqrt{\underset{\ll k}{\overset{(0)}{\mathcal{X}}}}\sqrt{L}.\end{aligned} \tag{6-11}$$

Proof. Again we recall that if $\varepsilon_{\text{boot}} \le \varepsilon_{\text{loc}}$, then the assumption of Proposition 4.8.1 holds. The first two inequalities follow from estimate (4-32) of Proposition 4.8.1 applied to $p = 2-\delta$, $p = 1$ in view of the fact that $\rho = \chi$, where we have used the relations (4-22) and (4-28) to replace the far-away supported nonlinear terms with those displayed above, together with the estimate

$$\int_{\tau_0}^{\tau_1} \overset{(0)}{\mathcal{E}}(\tau')\sqrt{\underset{\ll k}{\overset{(-1-\delta)}{\mathcal{E}'}}(\tau')}\,d\tau' \lesssim \sup_{\tau_0\le\tau\le\tau_1} \overset{(0)}{\mathcal{E}}(\tau)\sqrt{\int_{\tau_0}^{\tau_1}\underset{\ll k}{\overset{(-1-\delta)}{\mathcal{E}'}}(\tau')\,d\tau'}\cdot\sqrt{L} \lesssim {}^{\chi}\underset{k}{\overset{(0)}{\mathcal{X}}}\sqrt{\underset{\ll k}{\overset{(0)}{\mathcal{X}}}}\sqrt{L}.$$

Note that for the $p = 1$ estimate we retain only the less precise expression ${}^{\chi}\underset{k}{\overset{(1)}{\mathcal{X}}}\sqrt{\underset{\ll k}{\overset{(1)}{\mathcal{X}}}}$, which will be sufficient for our purposes.

The third inequality follows similarly using estimate (4-34), where we use also the interpolation inequality (3-114) to obtain

$$\underset{k}{\overset{(0+)}{\mathcal{X}}}_{8R/9}(\tau_0, \tau_1)\sqrt{\underset{\ll k}{\overset{(0+)}{\mathcal{X}}}_{8R/9}(\tau_0, \tau_1)} \lesssim \Big({}^{\chi}\underset{k}{\overset{(0)}{\mathcal{X}}} + \big({}^{\chi}\underset{k}{\overset{(0)}{\mathcal{X}}}\big)^{\frac{1-\delta}{1+\delta}}\big({}^{\chi}\underset{k}{\overset{(1)}{\mathcal{X}}}\big)^{\frac{2\delta}{1+\delta}}\Big)\sqrt{\underset{\ll k}{\overset{(0)}{\mathcal{X}}} + \big(\underset{\ll k}{\overset{(0)}{\mathcal{X}}}\big)^{\frac{1-\delta}{1+\delta}}\big(\underset{\ll k}{\overset{(1)}{\mathcal{X}}}\big)^{\frac{2\delta}{1+\delta}}}. \qquad \square$$

Remark 6.2.2. We note that inequalities (6-9)–(6-11) imply of course

$${}^{\chi}\underset{k}{\overset{(2-\delta)}{\mathcal{X}}} \lesssim \underset{k}{\overset{(2-\delta)}{\mathcal{E}}}(\tau_0) + \left({}^{\chi}\underset{k}{\overset{(2-\delta)}{\mathcal{X}}}\sqrt{\underset{\ll k}{\overset{(0)}{\mathcal{X}}}} + \sqrt{{}^{\chi}\underset{k}{\overset{(2-\delta)}{\mathcal{X}}}}\sqrt{{}^{\chi}\underset{k}{\overset{(0)}{\mathcal{X}}}}\sqrt{\underset{\ll k}{\overset{(2-\delta)}{\mathcal{X}}}}\right) + {}^{\chi}\underset{k}{\overset{(0)}{\mathcal{X}}}\sqrt{\underset{\ll k}{\overset{(0)}{\mathcal{X}}}}\sqrt{L}, \tag{6-12}$$

$${}^{\chi}\underset{k}{\overset{(1)}{\mathcal{X}}} \lesssim \underset{k}{\overset{(1)}{\mathcal{E}}}(\tau_0) + {}^{\chi}\underset{k}{\overset{(1)}{\mathcal{X}}}\sqrt{\underset{\ll k}{\overset{(1)}{\mathcal{X}}}} + {}^{\chi}\underset{k}{\overset{(0)}{\mathcal{X}}}\sqrt{\underset{\ll k}{\overset{(0)}{\mathcal{X}}}}\sqrt{L}, \tag{6-13}$$

$${}^{\chi}\underset{k}{\overset{(0)}{\mathcal{X}}} \lesssim \underset{k}{\overset{(0)}{\mathcal{E}}}(\tau_0) + \Big({}^{\chi}\underset{k}{\overset{(0)}{\mathcal{X}}} + \big({}^{\chi}\underset{k}{\overset{(0)}{\mathcal{X}}}\big)^{\frac{1-\delta}{1+\delta}}\big({}^{\chi}\underset{k}{\overset{(1)}{\mathcal{X}}}\big)^{\frac{2\delta}{1+\delta}}\Big)\sqrt{\underset{\ll k}{\overset{(0)}{\mathcal{X}}} + \big(\underset{\ll k}{\overset{(0)}{\mathcal{X}}}\big)^{\frac{1-\delta}{1+\delta}}\big(\underset{\ll k}{\overset{(1)}{\mathcal{X}}}\big)^{\frac{2\delta}{1+\delta}}} + {}^{\chi}\underset{k}{\overset{(0)}{\mathcal{X}}}\sqrt{\underset{\ll k}{\overset{(0)}{\mathcal{X}}}}\sqrt{L}. \tag{6-14}$$

Following the above proof but now using equations (4-37) and (4-38), we may in fact directly deduce inequalities (6-12)–(6-14) under the relaxed assumptions of Remark 4.8.4, where (4-1) is however required to be semilinear and we are in the analogue of case (ii). Thus, the analogue of Proposition 6.2.1 holds in that case where (6-9)–(6-11) are replaced by (6-12)–(6-14).

6.2.2. *Global existence in L-slabs.* The presence of positive powers of L in (6-9)–(6-11) means that our smallness assumptions must involve negative powers of L in order for the estimates to close.

Proposition 6.2.3. *Let $k - 1 \ge k_{\text{loc}}$ be sufficiently large, and let us assume the case* (ii) *assumptions. Then there exists an $0 < \varepsilon_{\text{slab}} \le \varepsilon_{\text{loc}}$ and a constant $C > 0$ implicit in the sign $\lesssim$ below such that the following is true*:

Given arbitrary $L \geq 1$, $\tau_0 \geq 0$, $0 < \varepsilon_0 \leq \varepsilon_{\text{slab}}$ and initial data (ψ, ψ') on $\Sigma(\tau_0)$ as in Proposition 4.9.1 satisfying moreover

$$\underset{k}{\overset{(1)}{\mathcal{E}}}(\tau_0) \leq \varepsilon_0, \quad \underset{k-1}{\overset{(0)}{\mathcal{E}}}(\tau_0) \leq \varepsilon_0 L^{-1}, \tag{6-15}$$

we have that the unique solution of Proposition 4.9.1 achieving the data can be extended to a ψ defined on the entire spacetime slab $\mathcal{R}(\tau_0, \tau_0 + L)$ satisfying (4-1) and the estimates

$${}^{\chi}\underset{k}{\overset{(1)}{\mathcal{X}}} \lesssim \varepsilon_0, \quad {}^{\chi}\underset{k-1}{\overset{(0)}{\mathcal{X}}} \lesssim \varepsilon_0 L^{-1}. \tag{6-16}$$

Proof. Consider the set $\mathfrak{B} \subset (\tau_0, \tau_0 + L]$ consisting of all $\tau_0 + L \geq \tau_f \geq \tau_0$ such that a solution ψ of (4-1) achieving the data exists on $\mathcal{R}(\tau_0, \tau_f)$ and such that the bootstrap assumption (4-23) with $p = 1$ and also the additional bootstrap assumption

$$\underset{\ll k}{\overset{(0)}{\mathcal{X}}} \leq \varepsilon L^{-1} \tag{6-17}$$

hold in $\mathcal{R}(\tau_0, \tau_1 := \tau_f)$, where $0 < \varepsilon \leq \varepsilon_{\text{boot}}$ is a small constant satisfying

$$1 \gg \varepsilon \gg \varepsilon_{\text{slab}}. \tag{6-18}$$

(The above relation in particular already constrains $\varepsilon_{\text{slab}}$ to be sufficiently small.)

By the local well-posedness statement Proposition 4.9.1, it follows that, since $k - 1 \geq k_{\text{loc}}$ and $\varepsilon_0 \leq \varepsilon_{\text{slab}} \leq \varepsilon_{\text{loc}}$, we have $\tau_0 + \tau_{\text{exist}} \in \mathfrak{B}$ and thus $\mathfrak{B} \neq \varnothing$, provided that ε satisfies (6-18). Also note that, a fortiori, if $\tau_f \in \mathfrak{B}$, then $(\tau_0, \tau_f] \in \mathfrak{B}$ and thus $\mathfrak{B}$ is manifestly a connected subset of (τ_0, ∞).

For $\tau_1 := \tau_f \in \mathcal{B}$, Proposition 6.2.1 applies in $\mathcal{R}(\tau_0, \tau_1)$. From (6-10) and (6-15) we obtain

$${}^{\chi}\underset{k}{\overset{(1)}{\mathcal{X}}} \lesssim \varepsilon_0 + \varepsilon^{\frac{1}{2}}\, {}^{\chi}\underset{k}{\overset{(1)}{\mathcal{X}}}, \tag{6-19}$$

where we have used the bootstrap assumptions (4-23) and (6-17).

From (6-11) for $k-1$ and (6-15) we obtain

$${}^{\chi}\underset{k-1}{\overset{(0)}{\mathcal{X}}} \lesssim \varepsilon_0 L^{-1} + \varepsilon^{\frac{1}{2}} \Big({}^{\chi}\underset{k-1}{\overset{(0)}{\mathcal{X}}}\Big)^{\frac{1-\delta}{1+\delta}} \Big({}^{\chi}\underset{k-1}{\overset{(1)}{\mathcal{X}}}\Big)^{\frac{2\delta}{1+\delta}} L^{-\frac{1}{2}\frac{1-\delta}{1+\delta}} + \varepsilon^{\frac{1}{2}}\, {}^{\chi}\underset{k-1}{\overset{(0)}{\mathcal{X}}}. \tag{6-20}$$

It follows that, for ε satisfying (6-18), we obtain

$${}^{\chi}\underset{k}{\overset{(2-\delta)}{\mathcal{X}}} \lesssim \varepsilon_0, \quad {}^{\chi}\underset{k-1}{\overset{(0)}{\mathcal{X}}} \lesssim \varepsilon_0 L^{-\frac{1}{2}\frac{1-\delta}{1+\delta}}, \tag{6-21}$$

and plugging the second inequality of (6-21) into (6-20) we obtain

$${}^{\chi}\underset{k-1}{\overset{(0)}{\mathcal{X}}} \lesssim \varepsilon_0 L^{-1} + \varepsilon^{\frac{1}{2}} \varepsilon_0 L^{-\frac{1}{2}\left(\frac{1-\delta}{1+\delta}\right)^2} L^{-\frac{1}{2}\frac{1-\delta}{1+\delta}}. \tag{6-22}$$

Now defining

$$\gamma_0 := \frac{1}{2}\left(\frac{1-\delta}{1+\delta}\right)^2 + \frac{1}{2}\frac{1-\delta}{1+\delta},$$

we have

$${}^{\chi}\underset{k-1}{\overset{(0)}{\mathcal{X}}} \lesssim \varepsilon_0 L^{-\gamma_0}, \tag{6-23}$$

whence, plugging (6-23) into (6-20), we improve to

$${}^{\chi}\underset{k-1}{\overset{(0)}{\mathcal{X}}} \lesssim \varepsilon_0 L^{-1} + \varepsilon^{\frac{1}{2}} \varepsilon_0 L^{-\gamma_0\left(\frac{1-\delta}{1+\delta}\right)^2} L^{-\frac{1}{2}\frac{1-\delta}{1+\delta}}. \tag{6-24}$$

Defining γ_i iteratively by

$$\gamma_i = \gamma_{i-1}\left(\frac{1-\delta}{1+\delta}\right)^2 + \frac{1}{2}\frac{1-\delta}{1+\delta}$$

and in view of the restriction (3-1), there exists a first i such that $\gamma_i \geq 1$, whence we obtain

$$ {}^{\chi}\underset{k-1}{\overset{(0)}{\mathcal{X}}} \lesssim \varepsilon_0 L^{-1}. \tag{6-25}$$

We may now already apply our continuation criterion Corollary 4.9.2, applied with $p = 0$ and $k-1$, to assert the existence of an ϵ, independent of τ_1, such that, now defining $\tau_1 := \min\{\tau_f + \epsilon, \tau_0 + L\}$, the solution ψ extends to a smooth solution of (4-1) on $\mathcal{R}(\tau_0, \tau_1)$ and, moreover, from (4-41), that

$$ {}^{\chi}\underset{k-1}{\overset{(0)}{\mathcal{X}}} \lesssim \varepsilon_0 L^{-1} \tag{6-26}$$

holds on $\mathcal{R}(\tau_0, \tau_1)$ and, from (4-42), that

$$ {}^{\chi}\underset{k}{\overset{(1)}{\mathcal{X}}} \lesssim \varepsilon_0 \tag{6-27}$$

holds on $\mathcal{R}(\tau_0, \tau_1)$. It follows that (6-17) and (4-23) (with $p = 1$) hold on $\mathcal{R}(\tau_0, \tau_1)$, and thus we have $\tau_1 = \min\{\tau_f + \epsilon, \tau_0 + L\} \in \mathfrak{B}$.

Since ϵ is independent of τ_f and in view also of the connectivity of $\mathfrak{B}$, it follows that $\mathfrak{B}$ is a nonempty open and closed subset of $(\tau_0, \tau_0 + L]$ and thus $\mathfrak{B} = (\tau_0, \tau_0 + L]$, and hence the solution exists in the entire spacetime slab $\mathcal{R}(\tau_0, \tau_0 + L)$.

The estimates (6-21) and (6-25) thus hold in the entire spacetime slab. This gives (6-16). □

Remark 6.2.4. Examining the proof, it is clear that we have only used inequalities (6-12)–(6-14) and not the full (6-9)–(6-11) of Proposition 6.2.1. Thus, in view of the comments of Remark 6.2.2, the proof holds also for the semilinear case under the relaxed assumptions described in Remark 4.8.4. In general, examining the proof, we may in fact relax the assumption of the first inequality of (6-15) to the assumption

$$\underset{k}{\overset{(1)}{\mathcal{E}}}(\tau_0) \leq \varepsilon_0 L^{\beta} \tag{6-28}$$

for a sufficiently small β, in which case the first inequality of (6-16) is replaced by

$$ {}^{\chi}\underset{k}{\overset{(1)}{\mathcal{X}}}(\tau_0) \leq \varepsilon_0 L^{\beta}. \tag{6-29}$$

This will again be useful for the semilinear case.

6.2.3. *The pigeonhole argument.* The above assumptions on initial data are sufficient for global existence in the slab but are not sufficient to iterate. For this we shall need strengthened assumptions.

Proposition 6.2.5. *Under the assumptions of Proposition 6.2.3, there exists a constant $C > 0$, implicit in the inequalities $\lesssim$ below, a parameter $\alpha_0 \gg 1$ and, for all $\alpha \geq \alpha_0$, a parameter $\hat{\varepsilon}_{\mathrm{slab}}(\alpha)$ such that, for all $0 < \hat{\varepsilon}_0 \leq \hat{\varepsilon}_{\mathrm{slab}}(\alpha)$, the following holds*:

Let us assume that in addition to (6-15) we have

$$\underset{k}{\overset{(1)}{\mathfrak{E}}}(\tau_0) \leq \hat{\varepsilon}_0, \quad \underset{k-1}{\overset{(2-\delta)}{\mathfrak{E}}}(\tau_0) \leq \hat{\varepsilon}_0, \quad \underset{k-1}{\overset{(0)}{\mathfrak{E}}}(\tau_0) \leq \hat{\varepsilon}_0 \alpha L^{-1}, \quad \underset{k-2}{\overset{(1)}{\mathfrak{E}}}(\tau_0) \leq \hat{\varepsilon}_0 \alpha L^{-1+\delta}, \quad \underset{k-3}{\overset{(0)}{\mathfrak{E}}}(\tau_0) \leq \hat{\varepsilon}_0 \alpha^2 L^{-2+\delta}. \tag{6-30}$$

Then the solution ψ of (4-1) on $\mathcal{R}(\tau_0, \tau_0+L)$ given by Proposition 6.2.3 satisfies the additional estimates

$$\sup_{\tau_0\le\tau\le\tau_0+L} \overset{(1)}{\underset{k}{\mathcal{E}}}(\tau) \le \alpha\hat{\varepsilon}_0, \tag{6-31}$$

$$\sup_{\tau_0\le\tau\le\tau_0+L} \overset{(1)}{\underset{k}{\mathfrak{E}}}(\tau) \le \hat{\varepsilon}_0(1+\hat{\varepsilon}_0^{\frac{1}{4}}L^{\frac{-1+\delta}{2}}), \tag{6-32}$$

$$\sup_{\tau_0\le\tau\le\tau_0+L} \overset{(2-\delta)}{\underset{k-1}{\mathfrak{E}}}(\tau) \le \hat{\varepsilon}_0(1+\hat{\varepsilon}_0^{\frac{1}{4}}L^{-\frac{1}{2}}), \tag{6-33}$$

$$\chi\overset{(1)}{\underset{k}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0, \quad \chi\overset{(2-\delta)}{\underset{k-1}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0, \quad \chi\overset{(0)}{\underset{k-1}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0\alpha L^{-1}, \quad \chi\overset{(1)}{\underset{k-2}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0\alpha L^{-1+\delta}, \quad \chi\overset{(0)}{\underset{k-3}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0\alpha^2 L^{-2+\delta}. \tag{6-34}$$

Moreover, for all times τ' with $L \ge \tau'-\tau_0 \ge \frac{1}{2}L$, we have

$$\overset{(0)}{\underset{k-1}{\mathfrak{E}}}(\tau') \le \tfrac{1}{2}\hat{\varepsilon}_0\alpha L^{-1}, \tag{6-35}$$

$$\overset{(1)}{\underset{k-1}{\mathfrak{E}}}(\tau') \le \tfrac{1}{2}\hat{\varepsilon}_0\alpha L^{-1+\delta}, \tag{6-36}$$

$$\overset{(0)}{\underset{k-2}{\mathfrak{E}}}(\tau') \le \tfrac{1}{4}\hat{\varepsilon}_0\alpha^2 L^{-2+\delta}. \tag{6-37}$$

Proof. Note that, by the statement of the proposition, we are in particular also assuming a priori (6-15) for *some* $\varepsilon_0 \le \varepsilon_{\text{slab}}$ since this is included in the assumptions of Proposition 6.2.3. In view now of Corollary 4.5.3, however, for α_0 sufficiently large, if say $\hat{\varepsilon}_{\text{slab}}(\alpha) \ll \alpha^{-3}\varepsilon_{\text{slab}}$, it follows from the additional assumptions (6-30) that the estimates (6-15) in fact hold with the specific constant $\varepsilon_0 := \hat{\varepsilon}_0\alpha^3$ for all $\alpha \ge \alpha_0$.

To obtain (6-34) we argue as follows. We note from the proof of Proposition 6.2.3 that we have the following system of inequalities:

$$\chi\overset{(1)}{\underset{k}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0 + \varepsilon_0^{\frac{1}{2}}\chi\overset{(1)}{\underset{k}{\mathcal{X}}}, \tag{6-38}$$

$$\chi\overset{(2-\delta)}{\underset{k-1}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0 + \varepsilon_0^{\frac{1}{2}}\chi\overset{(2-\delta)}{\underset{k-1}{\mathcal{X}}}, \tag{6-39}$$

$$\chi\overset{(0)}{\underset{k-1}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0\alpha L^{-1} + \big(\chi\overset{(0)}{\underset{k-1}{\mathcal{X}}}\big)^{\frac{1-\delta}{1+\delta}}\big(\chi\overset{(1)}{\underset{k-1}{\mathcal{X}}}\big)^{\frac{2\delta}{1+\delta}}\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}} + \big(\overset{(0)}{\underset{\ll k}{\mathcal{X}}}\big)^{\frac{1-\delta}{1+\delta}}\big(\overset{(1)}{\underset{\ll k}{\mathcal{X}}}\big)^{\frac{2\delta}{1+\delta}}} + \varepsilon_0^{\frac{1}{2}}\chi\overset{(0)}{\underset{k-1}{\mathcal{X}}}, \tag{6-40}$$

$$\chi\overset{(1)}{\underset{k-2}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0\alpha L^{-1+\delta} + \varepsilon_0^{\frac{1}{2}}\chi\overset{(1)}{\underset{k-2}{\mathcal{X}}}, \tag{6-41}$$

$$\chi\overset{(0)}{\underset{k-3}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0\alpha^2 L^{-2+\delta} + \big(\chi\overset{(0)}{\underset{k-3}{\mathcal{X}}}\big)^{\frac{1-\delta}{1+\delta}}\big(\chi\overset{(1)}{\underset{k-3}{\mathcal{X}}}\big)^{\frac{2\delta}{1+\delta}}\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}} + \big(\overset{(0)}{\underset{\ll k}{\mathcal{X}}}\big)^{\frac{1-\delta}{1+\delta}}\big(\overset{(1)}{\underset{\ll k}{\mathcal{X}}}\big)^{\frac{2\delta}{1+\delta}}} + \varepsilon_0^{\frac{1}{2}}\chi\overset{(0)}{\underset{k-3}{\mathcal{X}}}, \tag{6-42}$$

where we have used also the initial data assumptions (6-30).

For $\hat{\varepsilon}_{\text{slab}}(\alpha)$ sufficiently small, we have $\varepsilon_0 \ll 1$ and thus we obtain immediately that

$$\begin{gathered}
\chi\overset{(1)}{\underset{k}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0, \quad \chi\overset{(2-\delta)}{\underset{k-1}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0, \quad \chi\overset{(1)}{\underset{k-2}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0\alpha L^{-1+\delta}, \\
\chi\overset{(0)}{\underset{k-1}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0\alpha L^{-1} + \big(\chi\overset{(0)}{\underset{k-1}{\mathcal{X}}}\big)^{\frac{1-\delta}{1+\delta}}\big(\chi\overset{(1)}{\underset{k-1}{\mathcal{X}}}\big)^{\frac{2\delta}{1+\delta}}\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}} + \big(\overset{(0)}{\underset{\ll k}{\mathcal{X}}}\big)^{\frac{1-\delta}{1+\delta}}\big(\overset{(1)}{\underset{\ll k}{\mathcal{X}}}\big)^{\frac{2\delta}{1+\delta}}}, \\
\chi\overset{(0)}{\underset{k-3}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0\alpha^2 L^{-2+\delta} + \big(\chi\overset{(0)}{\underset{k-3}{\mathcal{X}}}\big)^{\frac{1-\delta}{1+\delta}}\big(\chi\overset{(1)}{\underset{k-3}{\mathcal{X}}}\big)^{\frac{2\delta}{1+\delta}}\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}} + \big(\overset{(0)}{\underset{\ll k}{\mathcal{X}}}\big)^{\frac{1-\delta}{1+\delta}}\big(\overset{(1)}{\underset{\ll k}{\mathcal{X}}}\big)^{\frac{2\delta}{1+\delta}}}.
\end{gathered} \tag{6-43}$$

The first two inequalities of (6-43) yield the first two inequalities of (6-34), while the third inequality of (6-43) yields the fourth inequality of (6-34). On the other hand, from the latter inequality of (6-43), we obtain

$$ {}^{\chi}\overset{(0)}{\mathcal{X}}_{k-3} \lesssim \hat{\varepsilon}_0 \alpha^2 L^{-2+\delta} + \hat{\varepsilon}_0 \varepsilon_0^{\frac{1}{2}} L^{-\frac{3}{2}\frac{(1-\delta)(1+2\delta)}{1+\delta}}. \tag{6-44} $$

Note that $-\frac{3}{2}\frac{(1-\delta)(1+2\delta)}{1+\delta} > -2+\delta$. We may however improve (6-44) iteratively as follows: If

$$ {}^{\chi}\overset{(0)}{\mathcal{X}}_{k-3} \lesssim \hat{\varepsilon}_0 \alpha^2 L^{-\gamma}, $$

for $\gamma < 2-\delta$, then

$$ \overset{(0)}{\mathcal{X}}_{\ll k} \lesssim \hat{\varepsilon}_0 \alpha^2 L^{-\gamma} \lesssim \varepsilon_0 L^{-\gamma}; $$

hence, plugging this again into the final inequality of (6-43) to estimate the nonlinear term, we obtain

$$ {}^{\chi}\overset{(0)}{\mathcal{X}}_{k-3} \lesssim \hat{\varepsilon}_0 \alpha^2 L^{-2+\delta} + \hat{\varepsilon}_0 \varepsilon_0^{\frac{1}{2}} L^{-\frac{3}{2}\frac{(1-\delta)(\gamma+2\delta)}{1+\delta}}. \tag{6-45} $$

Setting $\gamma_0 = \frac{3}{2}\frac{(1-\delta)(1+2\delta)}{1+\delta}$ and defining inductively $\gamma_i = \frac{3}{2}\frac{(1-\delta)(\gamma_{i-1}+2\delta)}{1+\delta}$, we have that there exists a first $i \geq 1$ such that

$$ \frac{3}{2}\frac{(1-\delta)(\gamma_i+2\delta)}{1+\delta} > 2-\delta. $$

It follows that (6-45) holds for $\gamma = \gamma_i$; hence

$$ {}^{\chi}\overset{(0)}{\mathcal{X}}_{k-3} \lesssim \hat{\varepsilon}_0 \alpha^2 L^{-2+\delta}. $$

This yields the fifth inequality of (6-34). Note that this implies

$$ \overset{(0)}{\mathcal{X}}_{\ll k} \lesssim \hat{\varepsilon}_0 \alpha^2 L^{-2+\delta} \lesssim \varepsilon_0 L^{-2+\delta}. \tag{6-46} $$

Note also that the third inequality of (6-43) implies

$$ \overset{(1)}{\mathcal{X}}_{\ll k} \lesssim \hat{\varepsilon}_0 \alpha L^{-1+\delta} \lesssim \varepsilon_0 L^{-1+\delta}. \tag{6-47} $$

Finally, plugging these bounds into the fourth inequality of (6-43) yields the third inequality of (6-34), completing the proof of (6-34).

To obtain (6-32), we recall (6-10) from Proposition 6.2.1. This gives, for sufficiently small $\hat{\varepsilon}_{\mathrm{slab}}$,

$$ \begin{aligned} \sup_{\tau_0 \leq \tau \leq \tau_0+L} \overset{(1)}{\mathfrak{E}}_k(\tau) &\leq \overset{(1)}{\mathfrak{E}}_k(\tau_0) + C\, {}^{\chi}\overset{(1)}{\mathcal{X}}_k \sqrt{\overset{(1)}{\mathcal{X}}_{\ll k}} + C\, {}^{\chi}\overset{(0)}{\mathcal{X}}_k \sqrt{\overset{(0)}{\mathcal{X}}_{\ll k}} \sqrt{L}, \\ &\leq \hat{\varepsilon}_0 + C\hat{\varepsilon}_0 \varepsilon_0^{\frac{1}{2}} L^{\frac{-1+\delta}{2}} + C\hat{\varepsilon}_0 \varepsilon_0^{\frac{1}{2}} L^{\frac{-2+\delta}{2}} L^{\frac{1}{2}} \leq \hat{\varepsilon}_0 (1 + \hat{\varepsilon}_0^{\frac{1}{4}} L^{\frac{-1+\delta}{2}}), \end{aligned} $$

where we have used the first inequality of (6-30), the first inequality of (6-43), the estimate (6-47) and the estimate (6-46), and we have replaced $\varepsilon_0^{1/2}$ by $\hat{\varepsilon}_0^{1/4}$ in the penultimate line, sacrificing a quarter power to absorb the resulting α term and the constant C.

To obtain (6-33), we now recall (6-9) from Proposition 6.2.1. This gives, for sufficiently small $\hat{\varepsilon}_{\mathrm{slab}}$,

$$ \begin{aligned} \sup_{\tau_0 \leq \tau \leq \tau_0+L} \overset{(2-\delta)}{\mathfrak{E}}_{k-1}(\tau) &\leq \overset{(2-\delta)}{\mathfrak{E}}_{k-1}(\tau_0) + C\Big({}^{\chi}\overset{(2-\delta)}{\mathcal{X}}_{k-1} \sqrt{\overset{(0)}{\mathcal{X}}_{\ll k}} + \sqrt{{}^{\chi}\overset{(2-\delta)}{\mathcal{X}}_{k-1}} \sqrt{{}^{\chi}\overset{(0)}{\mathcal{X}}_{k-1}} \sqrt{\overset{(2-\delta)}{\mathcal{X}}_{\ll k}} \Big) + C\, {}^{\chi}\overset{(0)}{\mathcal{X}}_{k-1} \sqrt{\overset{(0)}{\mathcal{X}}_{\ll k}} \sqrt{L} \\ &\leq \hat{\varepsilon}_0 + C\hat{\varepsilon}_0 \varepsilon_0^{\frac{1}{2}} L^{\frac{-2+\delta}{2}} + C\hat{\varepsilon}_0 \alpha^{\frac{1}{2}} \varepsilon_0^{\frac{1}{2}} L^{-\frac{1}{2}} + C\hat{\varepsilon}_0 \alpha \varepsilon_0^{\frac{1}{2}} L^{\frac{-2+\delta}{2}} \\ &\leq \hat{\varepsilon}_0 (1 + \hat{\varepsilon}_0^{\frac{1}{4}} L^{-\frac{1}{2}}), \end{aligned} $$

where we have used the second inequality of (6-30), the second and third inequalities of (6-34), the estimate (6-46) and the first inequality of (6-44), and we have replaced $\varepsilon_0^{1/2}$ by $\hat{\varepsilon}_0^{1/4}$ in the penultimate line, sacrificing a quarter power to absorb the resulting α term and the constant C.

To show (6-35)–(6-37), we first apply the pigeonhole principle as in [Dafermos and Rodnianski 2010b] to the inequality

$$\int_{\tau_0}^{\tau_0+L}\big(\overset{(0)}{\mathcal{E}}{}'_{k-1}(\tau')+\overset{(1-\delta)}{\mathcal{E}}{}'_{k-2}(\tau')+\alpha^{-1}L^{1-\delta}\overset{(0)}{\mathcal{E}}{}'_{k-3}(\tau')\big)\,d\tau' \lesssim \hat{\varepsilon}_0,$$

which, upon addition, follows from the first, second and fourth inequalities of the estimate (6-34) already shown. Recalling from (3-93) that, for both $p=2-\delta$ and $p=1$, we have

$$\overset{(p-1)}{\mathcal{E}}{}'_{k-2} \gtrsim \overset{(p-1)}{\mathcal{E}}_{k-2}, \quad \overset{(p-1)}{\mathcal{E}}{}'_{k-3} \gtrsim \overset{(p-1)}{\mathcal{E}}_{k-3},$$

we obtain that there exists $\tau'' \in \big[\tau_0, \tau_0+\tfrac{1}{2}L\big]$, whose precise value depends on the solution, such that

$$\overset{(0)}{\mathcal{E}}_{k-1}(\tau'') \lesssim \hat{\varepsilon}_0 \cdot L^{-1}, \tag{6-48}$$

$$\overset{(1-\delta)}{\mathcal{E}}_{k-2}(\tau'') \lesssim \hat{\varepsilon}_0 \cdot L^{-1}, \tag{6-49}$$

$$\overset{(0)}{\mathcal{E}}_{k-3}(\tau'') \lesssim \hat{\varepsilon}_0\alpha L^{-1+\delta} \cdot L^{-1}. \tag{6-50}$$

Now in view of the interpolation estimate (3-113) of Proposition 3.6.12, we have

$$\begin{aligned}\overset{(1)}{\mathcal{E}}_{k-2}(\tau'') &\lesssim \big(\overset{(1-\delta)}{\mathcal{E}}_{k-2}(\tau'')\big)^{1-\delta}\big(\overset{(2-\delta)}{\mathcal{E}}_{k-2}(\tau'')\big)^{\delta}\\ &\le \big(\overset{(1-\delta)}{\mathcal{E}}_{k-2}(\tau'')\big)^{1-\delta}\Big(\sup_{\tau\in[\tau_0,\tau_0+L]}\overset{(2-\delta)}{\mathcal{E}}_{k-2}(\tau)\Big)^{\delta}\end{aligned}$$

and thus

$$\overset{(1)}{\mathcal{E}}_{k-2}(\tau'') \lesssim \hat{\varepsilon}_0 L^{-1+\delta}, \tag{6-51}$$

where we have used (6-49) and the estimate for $\overset{(2-\delta)}{\mathcal{E}}_{k-2}$ contained in (6-34).

Now we apply (6-10) and (6-11) again, with τ'' in place of τ_0, using (6-48), (6-51) and (6-50) to bound the initial data, to obtain that, for all $\tau_0+L \ge \tau' \ge \tau_0+\frac{1}{2}L$, we have

$$\overset{(0)}{\mathfrak{E}}_{k-1}(\tau') \sim \overset{(0)}{\mathcal{E}}_{k-1}(\tau') \lesssim \hat{\varepsilon}_0 L^{-1}, \tag{6-52}$$

$$\overset{(1)}{\mathfrak{E}}_{k-2}(\tau') \sim \overset{(1)}{\mathcal{E}}_{k-2}(\tau') \lesssim \hat{\varepsilon}_0 L^{-1+\delta}, \tag{6-53}$$

$$\overset{(0)}{\mathfrak{E}}_{k-3}(\tau') \sim \overset{(0)}{\mathcal{E}}_{k-3}(\tau') \lesssim \hat{\varepsilon}_0 \alpha L^{-2+\delta}. \tag{6-54}$$

Thus, in view of the requirement $\alpha \ge \alpha_0 \gg 1$, for sufficiently large α_0 we may absorb the constants implicit in $\lesssim$ by explicit constants of our choice by adding extra positive α powers to the right-hand side of (6-53) and (6-54). In this way, we obtain the specific estimates (6-36) and (6-37) for all $\tau' \ge \tau''$ and thus in particular for all $L \ge \tau' - \tau_0 \ge \frac{1}{2}L$. In the same way, we also obtain the specific constant of the estimate of (6-31) which will be convenient in our scheme. □

Let us state an alternative "relaxed" version of the above proposition.

Proposition 6.2.6. *Under the relaxed assumptions for Proposition 6.2.3 described in Remark 6.2.4, there exist a constant $C > 0$, implicit in the inequalities $\lesssim$ below, a parameter $\beta > 0$ sufficiently small, a parameter $\alpha_0 \gg 1$ and, for all $\alpha \geq \alpha_0$, a parameter $\hat{\varepsilon}_{\text{slab}}(\alpha)$ such that the following holds*:

Given $0 < \hat{\varepsilon}_0 \leq \hat{\varepsilon}_{\text{slab}}$, let us assume in addition that

$$\underset{k}{\overset{(1)}{\mathcal{E}}}(\tau_0) \leq \hat{\varepsilon}_0 L^{\beta}, \quad \underset{k-1}{\overset{(2-\delta)}{\mathcal{E}}}(\tau_0) \leq \hat{\varepsilon}_0 L^{\beta}, \quad \underset{k-1}{\overset{(0)}{\mathcal{E}}}(\tau_0) \leq \hat{\varepsilon}_0 \alpha^{\frac{1}{2}} L^{-1+\beta}, \\ \underset{k-2}{\overset{(1)}{\mathcal{E}}}(\tau_0) \leq \hat{\varepsilon}_0 \alpha L^{-1+\delta+\beta}, \quad \underset{k-3}{\overset{(0)}{\mathcal{E}}}(\tau_0) \leq \hat{\varepsilon}_0 \alpha^2 L^{-2+\delta+\beta}. \tag{6-55}$$

Then the solution ψ of (4-1) on $\mathcal{R}(\tau_0, \tau_0 + L)$ given by Proposition 6.2.3 satisfies the additional estimates

$$\sup_{\tau \leq \tau \leq \tau_0 + L} \underset{k}{\overset{(1)}{\mathcal{E}}}(\tau) \leq \alpha^{\beta} \hat{\varepsilon}_0 L^{\beta}, \qquad \sup_{\tau_0 \leq \tau \leq \tau_0 + L} \underset{k-1}{\overset{(2-\delta)}{\mathcal{E}}}(\tau) \leq \alpha^{\beta} \hat{\varepsilon}_0 L^{\beta}, \tag{6-56}$$

$$\chi \underset{k}{\overset{(1)}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0 L^{\beta}, \quad \chi \underset{k-1}{\overset{(2-\delta)}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0 L^{\beta}, \quad \chi \underset{k-1}{\overset{(0)}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0 \alpha L^{-1+\beta}, \quad \chi \underset{k-2}{\overset{(1)}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0 \alpha L^{-1+\delta+\beta}, \quad \chi \underset{k-3}{\overset{(0)}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0 \alpha^2 L^{-2+\delta+\beta}. \tag{6-57}$$

Moreover, for all times τ' with $L \geq \tau' - \tau_0 \geq \frac{1}{2} L$, we have that

$$\underset{k-1}{\overset{(0)}{\mathcal{E}}}(\tau') \leq \hat{\varepsilon}_0 \alpha^{2\beta} L^{-1+\beta}, \quad \underset{k-1}{\overset{(1)}{\mathcal{E}}}(\tau') \leq \hat{\varepsilon}_0 \alpha^{2\beta} L^{-1+\delta+\beta}, \quad \underset{k-2}{\overset{(0)}{\mathcal{E}}}(\tau') \leq \hat{\varepsilon}_0 \alpha^{1+2\beta} L^{-2+\delta+\beta}. \tag{6-58}$$

Proof. The proof is similar to that of Proposition 6.2.5 and is left to the reader. □

Remark 6.2.7. The above proposition can now immediately be seen to also hold in the semilinear case, with the relaxed assumptions described in Remark 4.8.4.

6.2.4. *The iteration*: *proof of Theorem 5.1 in case* (ii). We may now prove Theorem 5.1 in case (ii).

We shall proceed iteratively.

We define $\tau_0 = 1$, $L_0 = 1$, $L_i = 2^i$, $\tau_{i+1} = \tau_i + L_i$ and fix $\alpha \geq \alpha_0$ so that the statement of Proposition 6.2.5 holds. (Note that we shall no longer track the dependence of constants and parameters on α since it is now fixed; one could simply take $\alpha = \alpha_0$.)

For $0 < \varepsilon_0 \leq \varepsilon_{\text{global}}$ and $\varepsilon_{\text{global}}$ sufficiently small, since

$$\underset{k}{\overset{(1)}{\mathcal{E}}}(\tau_0) + \underset{k-1}{\overset{(2-\delta)}{\mathcal{E}}}(\tau_0) \leq \varepsilon_0,$$

in view of Corollary 4.5.3, we have

$$\underset{k}{\overset{(1)}{\mathfrak{E}}}(\tau_0) \lesssim \varepsilon_0, \quad \underset{k-1}{\overset{(2-\delta)}{\mathfrak{E}}}(\tau_0) \lesssim \varepsilon_0.$$

Thus, for sufficiently small $\varepsilon_{\text{global}} \ll \hat{\varepsilon}_{\text{slab}}$, it follows that there exists an $\hat{\varepsilon}_0(\varepsilon_0) \sim \varepsilon_0$, satisfying moreover $2\alpha\hat{\varepsilon}_0 \leq \varepsilon_{\text{slab}}$, such that

$$\hat{\varepsilon}_0 \leq \tfrac{1}{2} \hat{\varepsilon}_{\text{slab}}$$

and

$$\underset{k}{\overset{(1)}{\mathfrak{E}}}(\tau_0) \leq \hat{\varepsilon}_0, \quad \underset{k-1}{\overset{(2-\delta)}{\mathfrak{E}}}(\tau_0) \leq \hat{\varepsilon}_0, \quad \underset{k-1}{\overset{(0)}{\mathfrak{E}}}(\tau_0) \leq \hat{\varepsilon}_0 \alpha L^{-1}, \quad \underset{k-2}{\overset{(1)}{\mathfrak{E}}}(\tau_0) \leq \hat{\varepsilon}_0 \alpha L^{-1+\delta}, \quad \underset{k-3}{\overset{(0)}{\mathfrak{E}}}(\tau_0) \leq \hat{\varepsilon}_0 \alpha^2 L^{-2+\delta}.$$

Finally, with our restriction on the definition of ε_0, we may also write

$$\underset{k}{\overset{(1)}{\mathcal{E}}}(\tau_0) \leq \alpha \hat{\varepsilon}_0.$$

In general, given $\tau_i \geq 1$ defined above, $\hat{\varepsilon}_i \leq \hat{\varepsilon}_{\mathrm{slab}}$, and a solution ψ of (4-1) on $\mathcal{R}(\tau_0, \tau_i)$ such that

$$\overset{(1)}{\underset{k}{\mathcal{E}}}(\tau_i) \leq \alpha \hat{\varepsilon}_i \tag{6-59}$$

and

$$\overset{(1)}{\underset{k}{\mathfrak{E}}}(\tau_i) \leq \hat{\varepsilon}_i, \quad \overset{(2-\delta)}{\underset{k-1}{\mathfrak{E}}}(\tau_i) \leq \hat{\varepsilon}_i, \quad \overset{(0)}{\underset{k-1}{\mathfrak{E}}}(\tau_i) \leq \hat{\varepsilon}_i \alpha L^{-1}, \quad \overset{(1)}{\underset{k-1}{\mathfrak{E}}}(\tau_i) \leq \hat{\varepsilon}_i \alpha L_i^{-1+\delta}, \quad \overset{(0)}{\underset{k-2}{\mathfrak{E}}}(\tau_i) \leq \hat{\varepsilon}_i \alpha^2 L_i^{-2+\delta}, \tag{6-60}$$

we note that, by our restrictions on $\hat{\varepsilon}_{\mathrm{slab}}$, the assumptions of Proposition 6.2.3 hold with $\alpha^2 \hat{\varepsilon}_i$ in place of ε_0 (here we are using (6-59) to invoke Corollary 4.5.3 to rewrite (6-60) in terms of the calligraphic energies; cf. the first lines of the proof of Proposition 6.2.5) and the assumptions of Proposition 6.2.5 then apply with $\hat{\varepsilon}_i$ in place of $\hat{\varepsilon}_0$, where both propositions are understood now with τ_i, τ_{i+1} in place of τ_0, τ_1. It follows that the solution ψ extends to a solution defined also in $\mathcal{R}_i := \mathcal{R}(\tau_i, \tau_i + L_i)$ and satisfying the estimates (6-31)–(6-34), while for $\tau' = \tau_{i+1} = \tau_i + L_i$ we have in addition (6-36)–(6-37). We have thus

$$\begin{aligned}
&\overset{(1)}{\underset{k}{\mathcal{E}}}(\tau_{i+1}) \leq \alpha \hat{\varepsilon}_i \leq \alpha \hat{\varepsilon}_{i+1},\\
&\overset{(1)}{\underset{k}{\mathfrak{E}}}(\tau_{i+1}) \leq \hat{\varepsilon}_i (1 + \hat{\varepsilon}_i^{\frac{1}{4}} L_i^{-\frac{1+\delta}{2}}) \leq \hat{\varepsilon}_{i+1},\\
&\overset{(2-\delta)}{\underset{k-1}{\mathfrak{E}}}(\tau_{i+1}) \leq \hat{\varepsilon}_i (1 + \hat{\varepsilon}_i^{\frac{1}{4}} L_i^{-\frac{1}{2}}) \leq \hat{\varepsilon}_{i+1},\\
&\overset{(1)}{\underset{k-1}{\mathfrak{E}}}(\tau_{i+1}) \leq \tfrac{1}{2} \hat{\varepsilon}_i \alpha L_i^{-1} \leq \hat{\varepsilon}_{i+1} \alpha L_{i+1}^{-1},\\
&\overset{(1)}{\underset{k-2}{\mathfrak{E}}}(\tau_{i+1}) \leq \tfrac{1}{2} \hat{\varepsilon}_i \alpha L_i^{-1+\delta} \leq \hat{\varepsilon}_{i+1} \alpha L_{i+1}^{-1+\delta},\\
&\overset{(0)}{\underset{k-3}{\mathfrak{E}}}(\tau_{i+1}) \leq \tfrac{1}{4} \hat{\varepsilon}_i \alpha^2 L_i^{-2+\delta} \leq \hat{\varepsilon}_{i+1} \alpha^2 L_{i+1}^{-2+\delta}
\end{aligned}$$

as long as

$$\hat{\varepsilon}_{i+1} := \hat{\varepsilon}_i (1 + \hat{\varepsilon}_i^{\frac{1}{4}} L_i^{-\frac{1+\delta}{2}}). \tag{6-61}$$

Now note that, for $\varepsilon_{\mathrm{global}}$ sufficiently small, the above inductive definition (6-61) of $\hat{\varepsilon}_i$ ensures that $\hat{\varepsilon}_i \leq 2\hat{\varepsilon}_0$ for all i.

It follows that a solution exists in $\mathcal{R}(\tau_0, \infty) = \bigcup \mathcal{R}(\tau_i, \tau_i + L_i)$ and in each interval the estimates (6-34) hold.

We obtain finally that for all $\tau \geq 1$ we have

$$\overset{(1)}{\underset{k}{\mathfrak{E}}}(\tau) \leq \hat{\varepsilon}_0 + C \hat{\varepsilon}_0^{\frac{3}{2}}, \quad \overset{(2-\delta)}{\underset{k-1}{\mathfrak{E}}}(\tau) \leq \hat{\varepsilon}_0 + C \hat{\varepsilon}_0^{\frac{3}{2}}, \tag{6-62}$$

$$\int_{\tau_0}^{\tau} \chi \overset{(1-\delta)}{\underset{k-1}{\mathcal{E}'}} + \overset{(1-\delta)}{\underset{k-2}{\mathcal{E}'}} \lesssim \hat{\varepsilon}_0 \log(\tau + 1), \tag{6-63}$$

$$\overset{(1)}{\underset{k-2}{\mathcal{E}}}(\tau) \lesssim \hat{\varepsilon}_0 \tau^{-1+\delta}, \quad \overset{(1)}{\underset{k-2}{\mathcal{F}}}(v, \tau) \lesssim \hat{\varepsilon}_0 \tau^{-1+\delta}, \tag{6-64}$$

$$\int_{\tau}^{\infty} \chi \overset{(0)}{\underset{k-2}{\mathcal{E}'}} + \overset{(0)}{\underset{k-3}{\mathcal{E}'}} \lesssim \hat{\varepsilon}_0 \tau^{-1+\delta}, \tag{6-65}$$

$$\overset{(0)}{\underset{k-3}{\mathcal{E}}}(\tau) \lesssim \hat{\varepsilon}_0 \tau^{-2+\delta}, \quad \overset{(0)}{\underset{k-3}{\mathcal{F}}}(v, \tau) \lesssim \hat{\varepsilon}_0 \tau^{-2+\delta}, \tag{6-66}$$

$$\int_{\tau}^{\infty} \chi \overset{(-1-\delta)}{\underset{k-3}{\mathcal{E}'}} + \overset{(-1-\delta)}{\underset{k-4}{\mathcal{E}'}} \lesssim \hat{\varepsilon}_0 \tau^{-2+\delta}. \tag{6-67}$$

Here, by $\mathcal{F}(v, \tau)$ we denote the restriction of the flux on $\underline{C}_v$ to $J^+(\Sigma(\tau))$.

Let us note that we may improve, a posteriori, the inequality (6-63) to

$$\int_{\tau_0}^{\infty} \chi \overset{(1-\delta)}{\underset{k-1}{\mathcal{E}'}} + \overset{(1-\delta)}{\underset{k-2}{\mathcal{E}'}} \lesssim \hat{\varepsilon}_0. \tag{6-68}$$

To show (6-68), for all $\tau \geq \tau_0$, we reapply the estimates of Proposition 4.8.1 globally on $\mathcal{R}(\tau_0, \tau)$. To control the nonlinear error bulk integrals, we redecompose the domain in dyadic time slabs, apply the estimates proven, and then sum. In view of the estimates, all these error bulk integrals can be controlled by $C\hat{\varepsilon}_0^{3/2}$.

6.2.5. *Alternative proof using Proposition 6.2.6 and the semilinear case.* We note that we may prove the theorem alternatively using Proposition 6.2.6. We first fix $\alpha \geq \alpha_0$ and now define $L_i := \alpha^i$. Here our iterative assumption is

$$\overset{(1)}{\underset{k}{\mathcal{E}}}(\tau_0) \leq \hat{\varepsilon}_0 L_i^{\beta}, \quad \overset{(2-\delta)}{\underset{k-1}{\mathcal{E}}}(\tau_0) \leq \hat{\varepsilon}_0 L_i^{\beta}, \quad \overset{(0)}{\underset{k-1}{\mathcal{E}}}(\tau_0) \leq \hat{\varepsilon}_0 \alpha L_i^{-1+\beta}, \quad \overset{(1)}{\underset{k-2}{\mathcal{E}}}(\tau_0) \leq \hat{\varepsilon}_0 \alpha L_i^{-1+\delta+\beta}, \quad \overset{(0)}{\underset{k-3}{\mathcal{E}}}(\tau_0) \leq \hat{\varepsilon}_0 \alpha^2 L_i^{-2+\delta+\beta}.$$

In view of (6-56) and (6-58) and the definition of L_i, we see that this is closed under iteration. We obtain thus existence in $\mathcal{R}(\tau_0, \infty)$ and at first instance the bounds

$$\overset{(1)}{\underset{k}{\mathcal{E}}}(\tau) \lesssim \hat{\varepsilon}_0 \tau^{\beta}, \quad \overset{(2-\delta)}{\underset{k-1}{\mathcal{E}}}(\tau) \lesssim \hat{\varepsilon}_0 \tau^{\beta},$$
$$\overset{(0)}{\underset{k-1}{\mathcal{E}}}(\tau) \lesssim \hat{\varepsilon}_0 \tau^{-1+\beta}, \quad \overset{(1)}{\underset{k-2}{\mathcal{E}}}(\tau) \lesssim \hat{\varepsilon}_0 \tau^{-1+\delta+\beta}, \quad \overset{(0)}{\underset{k-3}{\mathcal{E}}}(\tau_0) \lesssim \hat{\varepsilon}_0 \tau^{-2+\delta+\beta}.$$

These bounds are of course weaker than those of (6-62)–(6-67). The τ^{β} factor terms may be, however, removed a posteriori, as in the last paragraph of Section 6.2.4, by revisiting the estimates globally on $\mathcal{R}(\tau_0, \tau)$, controlling the nonlinear error bulk integrals by redecomposing into dyadic time slabs, applying the estimates already proven, and summing.

Thus this proof in the end yields finally the same estimates as before, but where (6-62) is replaced by

$$\overset{(0)}{\mathcal{E}} \lesssim \hat{\varepsilon}_0, \quad \overset{(1)}{\underset{k-2}{\mathcal{E}}}(\tau) \lesssim \hat{\varepsilon}_0. \tag{6-69}$$

Remark 6.2.8. In view of Remark 6.2.7, this proof in particular applies to the semilinear case with the relaxed assumptions of Remark 4.8.4, giving the result for case (ii) as stated in Remark 5.2.

Remark 6.2.9. The disadvantage of this alternative proof over the proof given in Section 6.2.4 is that, to obtain the fundamental top-order orbital stability statement (6-69), one must revisit the estimates globally. We thus believe that the proof in Section 6.2.4 better reflects the dyadically localised philosophy of our method.

6.3. ***Case* (iii).** We now turn to the most general case we consider, case (iii), where we do not assume that (3-3) follows from a physical-space identity (3-15) but only assume (3-3) as a black box, together with the weaker physical-space identity described in Section 3.4.3.

This case is slightly more involved because we must combine the estimate originating from Section 3.4.3 with the estimate originating in Section 3.2. This alters a bit the numerology of the number of derivatives we must take, but the basic scheme is the same as case (ii).

6.3.1. *The hierarchy of inequalities.*

Proposition 6.3.1. *Let k be sufficiently large, and let us assume the case* (iii) *assumptions. There exist constants $C>0$, $c>0$ and an $\varepsilon_{\mathrm{boot}}>0$ small enough that the following is true*:

Consider a region $\mathcal{R}(\tau_0,\tau_1)$ and a ψ solving (4-1) on $\mathcal{R}(\tau_0,\tau_1)$ and satisfying moreover (4-23) for $p=0$ and $0<\varepsilon\le\varepsilon_{\mathrm{boot}}$. Let us assume moreover that

$$\tau_1\le\tau_0+L$$

for some arbitrary $L>0$. We have the following hierarchy of inequalities:

$$\overset{(2-\delta)}{\underset{k}{\mathfrak{F}}}(v,\tau_1),\quad \overset{(2-\delta)}{\underset{k}{\mathfrak{E}}}(\tau_1),\quad c\,{}^{\rho}\overset{(2-\delta)}{\underset{k}{\mathcal{X}}}\le \overset{(2-\delta)}{\underset{k}{\mathfrak{E}}}(\tau_0)+AC\,{}^{\chi}\overset{(0)}{\underset{k-1}{\mathcal{X}}}+C\Big({}^{\rho}\overset{(2-\delta)}{\underset{k}{\mathcal{X}}}\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}}+\sqrt{{}^{\rho}\overset{(2-\delta)}{\underset{k}{\mathcal{X}}}}\sqrt{{}^{\rho}\overset{(0)}{\underset{k}{\mathcal{X}}}}\sqrt{\overset{(2-\delta)}{\underset{\ll k}{\mathcal{X}}}}\Big)+C\,{}^{\rho}\overset{(0)}{\underset{k}{\mathcal{X}}}\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}}\sqrt{L},\tag{6-70}$$

$${}^{\chi}\overset{(2-\delta)}{\underset{k-1}{\mathcal{X}}}\lesssim \overset{(2-\delta)}{\underset{k-1}{\mathcal{E}}}(\tau_0)+{}^{\rho}\overset{(2-\delta)}{\underset{k-1}{\mathcal{X}}}\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}}+\sqrt{{}^{\rho}\overset{(2-\delta)}{\underset{k-1}{\mathcal{X}}}}\sqrt{{}^{\rho}\overset{(0)}{\underset{k-1}{\mathcal{X}}}}\sqrt{\overset{(2-\delta)}{\underset{\ll k}{\mathcal{X}}}}+{}^{\rho}\overset{(0)}{\underset{k}{\mathcal{X}}}\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}}\sqrt{L},\tag{6-71}$$

$$\overset{(1)}{\underset{k}{\mathfrak{F}}}(v,\tau_1),\quad \overset{(1)}{\underset{k}{\mathfrak{E}}}(\tau_1),\quad c\,{}^{\rho}\overset{(1)}{\underset{k}{\mathcal{X}}}\le \overset{(1)}{\underset{k}{\mathfrak{E}}}(\tau_0)+AC\,{}^{\chi}\overset{(0)}{\underset{k-1}{\mathcal{X}}}+C\,{}^{\rho}\overset{(1)}{\underset{k}{\mathcal{X}}}\sqrt{\overset{(1)}{\underset{\ll k}{\mathcal{X}}}}+C\,{}^{\rho}\overset{(0)}{\underset{k}{\mathcal{X}}}\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}}\sqrt{L},\tag{6-72}$$

$${}^{\chi}\overset{(1)}{\underset{k-1}{\mathcal{X}}}\lesssim \overset{(1)}{\underset{k-1}{\mathcal{E}}}(\tau_0)+C\,{}^{\rho}\overset{(1)}{\underset{k-1}{\mathcal{X}}}\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}}+C\,{}^{\rho}\overset{(0)}{\underset{k}{\mathcal{X}}}\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}}\sqrt{L},\tag{6-73}$$

$$\begin{aligned}\overset{(0)}{\underset{k}{\mathfrak{F}}}(v,\tau_1),\quad \overset{(0)}{\underset{k}{\mathfrak{E}}}(\tau_1),\quad c\,{}^{\rho}\overset{(0)}{\underset{k}{\mathcal{X}}}\le \overset{(0)}{\underset{k}{\mathfrak{E}}}(\tau_0)+AC\,{}^{\chi}\overset{(0)}{\underset{k-1}{\mathcal{X}}}+C\Big({}^{\rho}\overset{(0)}{\underset{k}{\mathcal{X}}}+\big({}^{\rho}\overset{(0)}{\underset{k}{\mathcal{X}}}\big)^{\frac{1-\delta}{1+\delta}}\big({}^{\rho}\overset{(1)}{\underset{k}{\mathcal{X}}}\big)^{\frac{2\delta}{1+\delta}}\Big)\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}+\big(\overset{(0)}{\underset{\ll k}{\mathcal{X}}}\big)^{\frac{1-\delta}{1+\delta}}\big(\overset{(1)}{\underset{\ll k}{\mathcal{X}}}\big)^{\frac{2\delta}{1+\delta}}}\\ +C\,{}^{\rho}\overset{(0)}{\underset{k}{\mathcal{X}}}\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}}\sqrt{L},\end{aligned}\tag{6-74}$$

$$\begin{aligned}{}^{\chi}\overset{(0)}{\underset{k-1}{\mathcal{X}}}\lesssim \overset{(0)}{\underset{k-1}{\mathcal{E}}}(\tau_0)+\Big({}^{\rho}\overset{(0)}{\underset{k-1}{\mathcal{X}}}+\big({}^{\rho}\overset{(0)}{\underset{k-1}{\mathcal{X}}}\big)^{\frac{1-\delta}{1+\delta}}\big({}^{\rho}\overset{(1)}{\underset{k-1}{\mathcal{X}}}\big)^{\frac{2\delta}{1+\delta}}\Big)\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}+\big(\overset{(0)}{\underset{\ll k}{\mathcal{X}}}\big)^{\frac{1-\delta}{1+\delta}}\big(\overset{(1)}{\underset{\ll k}{\mathcal{X}}}\big)^{\frac{2\delta}{1+\delta}}}\\ +{}^{\rho}\overset{(0)}{\underset{k}{\mathcal{X}}}\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}}\sqrt{L}.\end{aligned}\tag{6-75}$$

Proof. Again we recall that if $\varepsilon_{\mathrm{boot}}\le\varepsilon_{\mathrm{loc}}$, then the assumption of Proposition 4.8.1 holds. The proposition then follows from Proposition 4.8.1 as in the proof of Proposition 6.2.1, where we have used also the crucial estimate

$$A\int_{\tau_0}^{\tau_1}\overset{\xi}{\underset{k-1}{\mathcal{E}}}{}'(\tau')\,d\tau'\lesssim AC\,{}^{\chi}\overset{(0)}{\underset{k-1}{\mathcal{X}}},$$

which follows from the fundamental relation (3-100). □

6.3.2. *Global existence in L-slabs.* To show global existence in L-slabs, we require a minor modification of the assumptions of Proposition 6.2.3.

Proposition 6.3.2. *Let $k-2\ge k_{\mathrm{loc}}$ be sufficiently large, and let us assume the case* (iii) *assumptions. Then there exists a positive constant $\varepsilon_{\mathrm{slab}}\le\varepsilon_{\mathrm{loc}}$ and a constant $C>0$ implicit in the sign $\lesssim$ below such that the following is true*:

Given arbitrary $L\ge 1$, $\tau_0\ge 0$, $0<\varepsilon_0\le\varepsilon_{\mathrm{slab}}$ and initial data (ψ,ψ') on $\Sigma(\tau_0)$ as in Proposition 4.9.1 satisfying moreover

$$\overset{(1)}{\underset{k}{\mathcal{E}}}(\tau_0)\le\varepsilon_0,\qquad \overset{(0)}{\underset{k-2}{\mathcal{E}}}(\tau_0)\le\varepsilon_0L^{-1},\tag{6-76}$$

we have that the unique solution of Proposition 4.9.1 achieving the data can be extended to a ψ defined on the entire spacetime slab $\mathcal{R}(\tau_0, \tau_0 + L)$ satisfying (4-1) and the estimates

$$ {}^{\rho}\overset{(1)}{\underset{k}{\mathcal{X}}} + {}^{\chi}\overset{(1)}{\underset{k-1}{\mathcal{X}}} \lesssim \varepsilon_0, \quad {}^{\rho}\overset{(0)}{\underset{k-2}{\mathcal{X}}} + {}^{\chi}\overset{(0)}{\underset{k-3}{\mathcal{X}}} \lesssim \varepsilon_0 L^{-1}. \tag{6-77} $$

Proof. Consider the set $\mathfrak{B} \subset (\tau_0, \tau_0 + L]$ consisting of all $\tau_0 + L \geq \tau_f \geq \tau_0$ such that a solution ψ of (4-1) achieving the data exists on $\mathcal{R}(\tau_0, \tau_f)$ and such that the bootstrap assumption (4-23) (with $p = 1$) and also the additional bootstrap assumption

$$ \overset{(0)}{\underset{\ll k}{\mathcal{X}}} \leq \varepsilon L^{-1} \tag{6-78} $$

hold in $\mathcal{R}(\tau_0, \tau_1 := \tau_f)$, where $0 < \varepsilon \leq \varepsilon_{\text{boot}}$ is a small constant satisfying

$$ 1 \gg \varepsilon \gg \varepsilon_{\text{slab}}. \tag{6-79} $$

(The above relation in particular already constrains $\varepsilon_{\text{slab}}$ to be sufficiently small.)

By the local well-posedness statement Proposition 4.9.1, it follows that, since $k - 2 \geq k_{\text{loc}}$ and $\varepsilon_0 \leq \varepsilon_{\text{slab}} \leq \varepsilon_{\text{loc}}$, we have $\tau_0 + \tau_{\text{exist}} \subset \mathfrak{B}$ and thus $\mathfrak{B} \neq \varnothing$, provided that ε satisfies (6-79). Also note that, a fortiori, if $\tau_f \in \mathfrak{B}$, then $(\tau_0, \tau_f] \in \mathfrak{B}$ and thus $\mathfrak{B}$ is manifestly a connected subset of (τ_0, ∞).

Proposition 6.3.1 holds for $\mathcal{R}(\tau_0, \tau_1)$ with $\tau_1 := \tau_f$ for any $\tau_f \in \mathfrak{B}$. Adding the equations (6-72)–(6-75) pairwise with a suitable constant so as to absorb the term multiplying A, with $k-2$ replacing k for the latter pair, we obtain the system

$$ {}^{\rho}\overset{(1)}{\underset{k}{\mathcal{X}}} + {}^{\chi}\overset{(1)}{\underset{k-1}{\mathcal{X}}} \lesssim \overset{(1)}{\underset{k}{\mathfrak{E}}}(\tau_0) + {}^{\rho}\overset{(1)}{\underset{k}{\mathcal{X}}}\sqrt{\overset{(1)}{\underset{\ll k}{\mathcal{X}}}} + {}^{\rho}\overset{(0)}{\underset{k}{\mathcal{X}}}\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}}\sqrt{L}, \tag{6-80} $$

$$ {}^{\rho}\overset{(0)}{\underset{k-2}{\mathcal{X}}} + {}^{\chi}\overset{(0)}{\underset{k-3}{\mathcal{X}}} \lesssim \overset{(0)}{\underset{k-2}{\mathfrak{E}}}(\tau_0) + \Big({}^{\rho}\overset{(0)}{\underset{k-2}{\mathcal{X}}} + \big({}^{\rho}\overset{(0)}{\underset{k-2}{\mathcal{X}}}\big)^{\frac{1-\delta}{1+\delta}}\big({}^{\rho}\overset{(1)}{\underset{k-2}{\mathcal{X}}}\big)^{\frac{2\delta}{1+\delta}}\Big)\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}} + \big(\overset{(0)}{\underset{\ll k}{\mathcal{X}}}\big)^{\frac{1-\delta}{1+\delta}}\big(\overset{(1)}{\underset{\ll k}{\mathcal{X}}}\big)^{\frac{2\delta}{1+\delta}}} + {}^{\rho}\overset{(0)}{\underset{k-2}{\mathcal{X}}}\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}}\sqrt{L}. \tag{6-81} $$

We now obtain

$$ {}^{\rho}\overset{(1)}{\underset{k}{\mathcal{X}}} + {}^{\chi}\overset{(1)}{\underset{k-1}{\mathcal{X}}} \lesssim \varepsilon_0, \quad {}^{\rho}\overset{(0)}{\underset{k-2}{\mathcal{X}}} + {}^{\chi}\overset{(0)}{\underset{k-3}{\mathcal{X}}} \lesssim \varepsilon_0 L^{-1}. \tag{6-82} $$

We may now already apply our continuation criterion Corollary 4.9.2, applied with $p = 0$ and $k - 2$, to assert the existence of an ϵ, independent of τ_1, such that, now defining $\tau_1 := \min\{\tau_f + \epsilon, \tau_0 + L\}$, the solution ψ extends to a smooth solution of (4-1) on $\mathcal{R}(\tau_0, \tau_1)$ and, moreover, from (4-41), that

$$ {}^{\chi}\overset{(0)}{\underset{k-3}{\mathcal{X}}} + {}^{\rho}\overset{(0)}{\underset{k-2}{\mathcal{X}}} \lesssim \varepsilon_0 L^{-1} \tag{6-83} $$

holds on $\mathcal{R}(\tau_0, \tau_1)$ and, from (4-42), that

$$ {}^{\chi}\overset{(1)}{\underset{k}{\mathcal{X}}} \lesssim \varepsilon_0 \tag{6-84} $$

holds on $\mathcal{R}(\tau_0, \tau_1)$. It follows that (6-78) and (4-23) (with $p = 1$) hold on $\mathcal{R}(\tau_0, \tau_1)$, and we thus have $\tau_1 = \min\{\tau_f + \epsilon, \tau_0 + L\} \in \mathfrak{B}$.

Since ϵ is independent of τ_f, and in view also of the connectivity of $\mathfrak{B}$, it follows that $\mathfrak{B}$ is a nonempty open and closed subset of $(\tau_0, \tau_0 + L]$ and thus $\mathfrak{B} = (\tau_0, \tau_0 + L]$, and hence the solution exists in the entire spacetime slab $\mathcal{R}(\tau_0, \tau_0 + L)$.

The estimates (6-82) thus hold in the entire spacetime slab. This gives (6-77). □

6.3.3. *The pigeonhole argument.* The above assumptions on initial data are sufficient for global existence in the slab but are not sufficient to iterate. For this we shall need strengthened assumptions.

Proposition 6.3.3. *Under the assumptions of Proposition 6.3.2, there exists a constant $C > 0$, implicit in the inequalities $\lesssim$ below, a parameter $\alpha_0 \gg 1$ and, for all $\alpha \geq \alpha_0$, a parameter $\hat{\varepsilon}_{\text{slab}}(\alpha)$ such that, for all $0 < \hat{\varepsilon}_0 \leq \hat{\varepsilon}_{\text{slab}}(\alpha)$ the following holds:*

Let us assume in addition to (6-76) that we have

$$\overset{(1)}{\underset{k}{\mathfrak{E}}}(\tau_0) \leq \hat{\varepsilon}_0, \quad \overset{(2-\delta)}{\underset{k-2}{\mathfrak{E}}}(\tau_0) \leq \hat{\varepsilon}_0, \quad \overset{(0)}{\underset{k-2}{\mathfrak{E}}}(\tau_0) \leq \hat{\varepsilon}_0 \alpha L^{-1}, \quad \overset{(1)}{\underset{k-4}{\mathfrak{E}}}(\tau_0) \leq \hat{\varepsilon}_0 \alpha L^{-1+\delta}, \quad \overset{(0)}{\underset{k-6}{\mathfrak{E}}}(\tau_0) \leq \hat{\varepsilon}_0 \alpha^2 L^{-2+\delta}. \tag{6-85}$$

Then the solution ψ of (4-1) on $\mathcal{R}(\tau_0, \tau_0 + L)$ given by Proposition 6.2.3 satisfies the additional estimates

$$\sup_{\tau_0 \leq \tau \leq \tau_0 + L} \overset{(1)}{\underset{k}{\mathcal{E}}}(\tau) \leq \alpha \hat{\varepsilon}_0, \tag{6-86}$$

$$\sup_{\tau_0 \leq \tau \leq \tau_0 + L} \overset{(1)}{\underset{k}{\mathfrak{E}}}(\tau) \leq \hat{\varepsilon}_0 (1 + \alpha L^{-\frac{1}{4}}), \tag{6-87}$$

$$\sup_{\tau_0 \leq \tau \leq \tau_0 + L} \overset{(2-\delta)}{\underset{k-2}{\mathfrak{E}}}(\tau) \leq \hat{\varepsilon}_0 (1 + \alpha L^{-\frac{1}{4}}), \tag{6-88}$$

$${}^{\rho}\overset{(1)}{\underset{k}{\mathcal{X}}} + {}^{\chi}\overset{(1)}{\underset{k-1}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0, \quad {}^{\rho}\overset{(2-\delta)}{\underset{k-2}{\mathcal{X}}} + {}^{\chi}\overset{(2-\delta)}{\underset{k-3}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0, \quad {}^{\rho}\overset{(0)}{\underset{k-2}{\mathcal{X}}} + {}^{\chi}\overset{(0)}{\underset{k-3}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0 \alpha L^{-1}, \quad {}^{\rho}\overset{(1)}{\underset{k-4}{\mathcal{X}}} + {}^{\chi}\overset{(1)}{\underset{k-5}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0 \alpha L^{-1+\delta}, \tag{6-89}$$

$${}^{\rho}\overset{(0)}{\underset{k-6}{\mathcal{X}}} + {}^{\chi}\overset{(0)}{\underset{k-7}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0 \alpha^2 L^{-2+\delta}. \tag{6-90}$$

Moreover, for all times τ' with $L \geq \tau' - \tau_0 \geq \frac{1}{2} L$, we have that

$$\overset{(0)}{\underset{k-2}{\mathfrak{E}}}(\tau') \leq \tfrac{1}{2} \hat{\varepsilon}_0 \alpha L^{-1}, \tag{6-91}$$

$$\overset{(1)}{\underset{k-4}{\mathfrak{E}}}(\tau') \leq \tfrac{1}{2} \hat{\varepsilon}_0 \alpha L^{-1+\delta}, \tag{6-92}$$

$$\overset{(0)}{\underset{k-6}{\mathfrak{E}}}(\tau') \leq \tfrac{1}{4} \hat{\varepsilon}_0 \alpha^2 L^{-2+\delta}. \tag{6-93}$$

Proof. The proof follows closely that of Proposition 6.2.5.

Note again that, by the statement of the proposition, we are in particular also assuming a priori inequalities (6-76) for *some* $\varepsilon_0 \leq \varepsilon_{\text{slab}}$ since this is included in the assumptions of Proposition 6.3.2. In view now of Corollary 4.5.3, for α_0 sufficiently large, if say $\hat{\varepsilon}_{\text{slab}}(\alpha) \ll \alpha^{-3} \varepsilon_{\text{slab}}$, it follows from the additional assumptions (6-85) that the estimates (6-76) in fact hold with the specific constant $\varepsilon_0 := \hat{\varepsilon}_0 \alpha^3$ for all $\alpha \geq \alpha_0$.

We now revisit (6-70)–(6-75) and add again pairwise. We now obtain that

$${}^{\rho}\overset{(1)}{\underset{k}{\mathcal{X}}} + {}^{\chi}\overset{(1)}{\underset{k-1}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0, \quad {}^{\rho}\overset{(2-\delta)}{\underset{k-2}{\mathcal{X}}} + {}^{\chi}\overset{(2-\delta)}{\underset{k-3}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0, \quad {}^{\rho}\overset{(1)}{\underset{k-4}{\mathcal{X}}} + {}^{\chi}\overset{(1)}{\underset{k-5}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0 \alpha L^{-1+\delta}, \tag{6-94}$$

$${}^{\rho}\overset{(0)}{\underset{k-2}{\mathcal{X}}} + {}^{\chi}\overset{(0)}{\underset{k-3}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0 \alpha L^{-1} + \Big(\big({}^{\rho}\overset{(0)}{\underset{k-2}{\mathcal{X}}}\big)^{\frac{1-\delta}{1+\delta}} \big({}^{\rho}\overset{(1)}{\underset{k-2}{\mathcal{X}}}\big)^{\frac{2\delta}{1+\delta}}\Big) \sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}} + \big(\overset{(0)}{\underset{\ll k}{\mathcal{X}}}\big)^{\frac{1-\delta}{1+\delta}} \big(\overset{(1)}{\underset{\ll k}{\mathcal{X}}}\big)^{\frac{2\delta}{1+\delta}}}, \tag{6-95}$$

$${}^{\rho}\overset{(0)}{\underset{k-6}{\mathcal{X}}} + {}^{\chi}\overset{(0)}{\underset{k-7}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0 \alpha^2 L^{-2+\delta} + \hat{\varepsilon}_0 \varepsilon_0^{\frac{1}{2}} L^{-\frac{3}{2} \frac{(1-\delta)(\gamma+2\delta)}{1+\delta}}. \tag{6-96}$$

The inequalities (6-94) give the first, second, and fourth inequality of (6-89).

Note that $-2+\delta < -\frac{3}{2}+\delta$. We may however improve (6-96) iteratively as follows: If

$$ {}^{\rho}\underset{k-6}{\overset{(0)}{\mathcal{X}}} + {}^{\chi}\underset{k-7}{\overset{(0)}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0\alpha^2 L^{-\gamma} $$

for $\gamma \le 2-\delta$, then

$$ \underset{\ll k}{\overset{(0)}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0\alpha^2 L^{-\gamma} \lesssim \varepsilon_0 L^{-\gamma}; $$

hence, plugging this again into (6-81), we obtain

$$ {}^{\rho}\underset{k-6}{\overset{(0)}{\mathcal{X}}} + {}^{\chi}\underset{k-7}{\overset{(0)}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0\alpha^2 L^{-2+\delta} + \hat{\varepsilon}_0\varepsilon_0^{\frac{1}{2}} L^{-\frac{3}{2}\frac{(1-\delta)(\gamma+2\delta)}{1+\delta}}. \tag{6-97} $$

Setting $\gamma_0 = \frac{3}{2}\frac{(1-\delta)(1+2\delta)}{1+\delta}$ and defining inductively $\gamma_i = \frac{3}{2}\frac{(1-\delta)(\gamma_{i-1}+2\delta)}{1+\delta}$, we have that there exists a first $i \ge 1$ such that

$$ \frac{3}{2}\frac{(1-\delta)(\gamma_i+2\delta)}{1+\delta} > 2-\delta. $$

It follows that (6-97) holds for $\gamma = \gamma_i$; hence

$$ {}^{\rho}\underset{k-6}{\overset{(0)}{\mathcal{X}}} + {}^{\chi}\underset{k-7}{\overset{(0)}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0\alpha^2 L^{-2+\delta}. $$

This yields (6-90). Note that this implies

$$ \underset{\ll k}{\overset{(0)}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0\alpha^2 L^{(-2+2\delta)z} \lesssim \varepsilon_0 L^{-2+\delta}. \tag{6-98} $$

On the other hand, the fourth inequality of (6-89) implies

$$ \underset{\ll k}{\overset{(1)}{\mathcal{X}}} \lesssim \hat{\varepsilon}_0\alpha L^{-1+\delta} \lesssim \varepsilon_0 L^{-1+\delta}. \tag{6-99} $$

We may now infer the third inequality of (6-89) by plugging the derived bounds into (6-95). This concludes the proof of (6-89).

To show (6-91)–(6-93), we first apply the pigeonhole principle as in [Dafermos and Rodnianski 2010b] to the inequality

$$ \int_{\tau_0}^{\tau_0+L} \Big(\underset{k-2}{\overset{(0)}{\mathcal{E}'}}(\tau') + \underset{k-4}{\overset{(1-\delta)}{\mathcal{E}'}}(\tau') + \alpha^{-1}L^{1-\delta}\underset{k-6}{\overset{(0)}{\mathcal{E}'}}(\tau') \Big)\, d\tau' \lesssim \hat{\varepsilon}_0, $$

which, upon addition, follows from the inequalities of the estimate (6-89) already shown. Recalling from (3-93) that we have

$$ \underset{k-2}{\overset{(0)}{\mathcal{E}'}} \gtrsim \underset{k-2}{\overset{(0)}{\mathcal{E}}}, \quad \underset{k-4}{\overset{(1-\delta)}{\mathcal{E}'}} \gtrsim \underset{k-4}{\overset{(1-\delta)}{\mathcal{E}}}, \quad \underset{k-6}{\overset{(0)}{\mathcal{E}'}} \gtrsim \underset{k-6}{\overset{(0)}{\mathcal{E}}}, $$

we obtain that there exists $\tau'' \in \big[\tau_0, \tau_0 + \frac{1}{2}L\big]$, whose precise value depends on the solution, such that

$$ \underset{k-2}{\overset{(0)}{\mathcal{E}}}(\tau'') \lesssim \hat{\varepsilon}_0 \cdot L^{-1}, \tag{6-100} $$

$$ \underset{k-4}{\overset{(1-\delta)}{\mathcal{E}}}(\tau'') \lesssim \hat{\varepsilon}_0 \cdot L^{-1}, \tag{6-101} $$

$$ \underset{k-6}{\overset{(0)}{\mathcal{E}}}(\tau'') \lesssim \hat{\varepsilon}_0\alpha L^{-1+\delta} \cdot L^{-1}. \tag{6-102} $$

Now in view of the interpolation estimate (3-113) of Proposition 3.6.12, we have

$$ \underset{k-4}{\overset{(1)}{\mathcal{E}}}(\tau'') \lesssim \big(\underset{k-4}{\overset{(1-\delta)}{\mathcal{E}}}(\tau'')\big)^{1-\delta} \big(\underset{k-4}{\overset{(2-\delta)}{\mathcal{E}}}(\tau'')\big)^{\delta} \le \big(\underset{k-4}{\overset{(1-\delta)}{\mathcal{E}}}(\tau'')\big)^{1-\delta} \Big(\sup_{\tau_0 \le \tau \le \tau_0+L} \underset{k-4}{\overset{(2-\delta)}{\mathcal{E}}}(\tau)\Big)^{\delta} \tag{6-103} $$

and thus

$$\overset{(1)}{\underset{k-4}{\mathcal{E}}}(\tau'') \lesssim \hat{\varepsilon}_0 L^{-1+\delta}, \tag{6-104}$$

where we have used (6-101) and the estimate for the second factor on the right-hand side of (6-103) contained in the second inequality of (6-89).

Now we apply (6-80) and (6-81) again, with τ'' in place of τ_0, using (6-100)–(6-102) to bound the initial data, to obtain that, for all $\tau_0 + L \geq \tau' \geq \tau_0 + \frac{1}{2}L$, we have

$$\overset{(1)}{\underset{k-2}{\mathfrak{E}}}(\tau') \sim \overset{(1)}{\underset{k-2}{\mathcal{E}}}(\tau) \lesssim \hat{\varepsilon}_0 L^{-1}, \tag{6-105}$$

$$\overset{(1)}{\underset{k-4}{\mathfrak{E}}}(\tau') \sim \overset{(1)}{\underset{k-4}{\mathcal{E}}}(\tau) \lesssim \hat{\varepsilon}_0 L^{-1+\delta}, \tag{6-106}$$

$$\overset{(0)}{\underset{k-6}{\mathfrak{E}}}(\tau') \sim \overset{(0)}{\underset{k-6}{\mathcal{E}}}(\tau) \lesssim \hat{\varepsilon}_0 \alpha L^{-2+\delta}. \tag{6-107}$$

Thus, in view of the requirement $\alpha \geq \alpha_0 \gg 1$, for sufficiently large α_0 we may absorb the constants implicit in $\lesssim$ by explicit constants of our choice by adding extra positive α powers to the right-hand side of (6-106) and (6-107). In this way, we obtain the specific estimates (6-91), (6-92) and (6-93). In the same way, we also obtain the specific constant of the estimate of (6-86) which will be convenient in our scheme.

We finally turn to (6-87) and (6-88). Let us first note that revisiting (6-75), with a little bit of averaging, we have the bounds

$$\begin{aligned}
{}^{\chi}\overset{(0)}{\underset{k-1}{\mathcal{X}}}\big(\tau_0+\tfrac{1}{2},\, \tau_0+L\big) &\lesssim \int_{\tau_0}^{\tau_0+1/2} \overset{(0)}{\underset{k-1}{\mathcal{E}}}(\tau)\,d\tau + \Big({}^{\rho}\overset{(0)}{\underset{k-1}{\mathcal{X}}} + \big({}^{\rho}\overset{(0)}{\underset{k-1}{\mathcal{X}}}\big)^{\frac{1-\delta}{1+\delta}} \big({}^{\rho}\overset{(1)}{\underset{k-1}{\mathcal{X}}}\big)^{\frac{2\delta}{1+\delta}}\Big) \sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}} + \big(\overset{(0)}{\underset{\ll k}{\mathcal{X}}}\big)^{\frac{1-\delta}{1+\delta}} \big(\overset{(1)}{\underset{\ll k}{\mathcal{X}}}\big)^{\frac{2\delta}{1+\delta}}} + {}^{\rho}\overset{(0)}{\underset{k}{\mathcal{X}}} \sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}} \sqrt{L} \\
&\lesssim \sqrt{\int_{\tau_0}^{\tau_0+1/2} \overset{(0)}{\mathcal{E}}(\tau)\,d\tau} \sqrt{\int_{\tau_0}^{\tau_0+1/2} \overset{(0)}{\underset{k-2}{\mathcal{E}}}(\tau)\,d\tau} + \hat{\varepsilon}_0 \varepsilon^{\frac{1}{2}} L^{-\frac{1}{4}} \\
&\lesssim \hat{\varepsilon}_0 \alpha^{\frac{1}{2}} L^{-\frac{1}{2}} + \hat{\varepsilon}_0 \varepsilon^{\frac{1}{2}} L^{-\frac{1}{4}} \\
&\lesssim \hat{\varepsilon}_0 \alpha^{\frac{1}{2}} L^{-\frac{1}{4}},
\end{aligned}$$

where we have used also the interpolation inequality (3-115) $\big($and that $L \geq \frac{1}{2}\big)$.

We now notice that the above bound clearly holds also for

$${}^{\chi}\overset{(0)}{\underset{k-1}{\mathcal{X}}}{}' := {}^{\chi}\overset{(0)}{\underset{k-1}{\mathcal{X}}}\big(\tau_0+\tfrac{1}{2},\, \tau_0+L\big) + \int_{\tau_0}^{\tau_0+1/2} \overset{(0)}{\underset{k-1}{\mathcal{E}}}(\tau)\,d\tau.$$

For (6-87), we revisit (6-72), noting that we may replace ${}^{\chi}\overset{(0)}{\underset{k-1}{\mathcal{X}}}$ by ${}^{\chi}\overset{(0)}{\underset{k-1}{\mathcal{X}}}{}'$ on the right-hand side, and thus we have the bound

$$\begin{aligned}
\sup_{\tau\in[\tau_0,\tau_0+L]} \overset{(1)}{\underset{k}{\mathfrak{E}}}(\tau) &\leq \overset{(1)}{\underset{k}{\mathfrak{E}}}(\tau_0) + AC\, {}^{\chi}\overset{(0)}{\underset{k-1}{\mathcal{X}}}{}' + C\, {}^{\rho}\overset{(1)}{\underset{k}{\mathcal{X}}} \sqrt{\overset{(1)}{\underset{\ll k}{\mathcal{X}}}} + C\, {}^{\rho}\overset{(0)}{\underset{k}{\mathcal{X}}} \sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}} \sqrt{L} \\
&\leq \hat{\varepsilon}_0 + AC(\hat{\varepsilon}_0 \alpha^{\frac{1}{2}} L^{-\frac{1}{4}}) + C\hat{\varepsilon}_0 \varepsilon_0^{\frac{1}{2}} L^{\frac{-1+\delta}{2}} + C\hat{\varepsilon}_0 \varepsilon_0^{\frac{1}{2}} L^{-\frac{3}{2}+\delta} \\
&\leq \hat{\varepsilon}_0 + \alpha \hat{\varepsilon}_0 L^{-\frac{1}{4}},
\end{aligned}$$

and we have used a higher power of α to absorb constants.

Finally, for (6-88), we revisit (6-75) with $k-2$ replacing $k-1$ to obtain

$$
{}^{\chi}\overset{(0)}{\underset{k-2}{\mathcal{X}}} \lesssim \overset{(0)}{\underset{k-2}{\mathcal{E}}}(\tau_0) + \big({}^{\rho}\overset{(0)}{\underset{k-2}{\mathcal{X}}} + \big({}^{\rho}\overset{(0)}{\underset{k-2}{\mathcal{X}}}\big)^{\frac{1-\delta}{1+\delta}}\big({}^{\rho}\overset{(1)}{\underset{k-2}{\mathcal{X}}}\big)^{\frac{2\delta}{1+\delta}}\big)\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}} + \big(\overset{(0)}{\underset{\ll k}{\mathcal{X}}}\big)^{\frac{1-\delta}{1+\delta}}\big(\overset{(1)}{\underset{\ll k}{\mathcal{X}}}\big)^{\frac{2\delta}{1+\delta}}} + {}^{\rho}\overset{(0)}{\underset{k-1}{\mathcal{X}}}\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}}\sqrt{L} \lesssim \hat{\varepsilon}_0 \alpha L^{-1},
$$

whence we have

$$
\begin{aligned}
\sup_{\tau\in[\tau_0,\tau_0+L]} \overset{(2-\delta)}{\underset{k-1}{\mathfrak{E}}}(\tau) &\le \overset{(2-\delta)}{\underset{k-1}{\mathfrak{E}}}(\tau_0) + A\,{}^{\chi}\overset{(0)}{\underset{k-2}{\mathcal{X}}} + C\Big({}^{\rho}\overset{(2-\delta)}{\underset{k-1}{\mathcal{X}}}\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}} + \sqrt{{}^{\rho}\overset{(2-\delta)}{\underset{k-1}{\mathcal{X}}}}\sqrt{{}^{\rho}\overset{(0)}{\underset{k-1}{\mathcal{X}}}}\sqrt{\overset{(2-\delta)}{\underset{\ll k}{\mathcal{X}}}}\Big) + C\,{}^{\rho}\overset{(0)}{\underset{k-1}{\mathcal{X}}}\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{X}}}}\sqrt{L} \\
&\le \hat{\varepsilon}_0 + A\hat{\varepsilon}_0\alpha^{\frac{1}{2}}L^{-1} + C\hat{\varepsilon}_0\varepsilon_0^{\frac{1}{2}}L^{\frac{-2+\delta}{2}} + C\hat{\varepsilon}_0\varepsilon_0^{\frac{1}{2}}L^{\frac{-2+\delta}{2}}L^{\frac{1}{2}} \\
&\le \hat{\varepsilon}_0 + \alpha\hat{\varepsilon}_0 L^{-\frac{1}{4}},
\end{aligned}
$$

where we have again used a higher power of α to absorb all other constants. □

6.3.4. *The iteration*: *proof of Theorem 5.1 in case* (iii). We may now prove Theorem 5.1 in case (iii).

As in case (ii), we shall proceed iteratively. The proof is a simple modification of that of Section 6.2.4. We define

$$
\tau_0 = 1, \quad L_0 = 1, \quad L_i = 2^i, \quad \tau_{i+1} = \tau_i + L_i
$$

and fix $\alpha \ge \alpha_0$ so that the statement of Proposition 6.3.3 holds; for instance, set $\alpha := \alpha_0$. Define the parameter

$$
d := \prod_{i=1}^{\infty}(1 + 2^{-\frac{i}{4}}\alpha) < \infty. \tag{6-108}
$$

(Note that we shall no longer note the dependence of constants and parameters on α since it is now considered fixed. Thus implicit constants depending on the choice of α will from now on be incorporated in the notations $\sim$ and $\lesssim$.)

For $0 < \varepsilon_0 \le \varepsilon_{\text{global}}$ and $\varepsilon_{\text{global}}$ sufficiently small, since

$$
\overset{(1)}{\underset{k}{\mathcal{E}}}(\tau_0) \le \varepsilon_0, \quad \overset{(2-\delta)}{\underset{k-2}{\mathcal{E}}}(\tau_0) \le \varepsilon_0, \tag{6-109}
$$

in view of Corollary 4.5.3, we have

$$
\overset{(1)}{\underset{k}{\mathfrak{E}}}(\tau_0) \lesssim \varepsilon_0, \quad \overset{(2-\delta)}{\underset{k-2}{\mathfrak{E}}}(\tau_0) \lesssim \varepsilon_0.
$$

Thus, for sufficiently small $\varepsilon_{\text{global}} \ll \hat{\varepsilon}_{\text{slab}}$ and all $0 < \varepsilon_0 \le \varepsilon_{\text{global}}$, if (6-109) is satisfied, it follows that there exists a $\hat{\varepsilon}_0(\varepsilon_0) \sim \varepsilon_0$ satisfying

$$
\alpha\hat{\varepsilon}_0 \le \frac{1}{d}\varepsilon_{\text{slab}}, \quad \hat{\varepsilon}_0 \le \frac{1}{d}\hat{\varepsilon}_{\text{slab}} \tag{6-110}
$$

and

$$
\overset{(1)}{\underset{k}{\mathfrak{E}}}(\tau_0) \le \hat{\varepsilon}_0, \quad \overset{(2-\delta)}{\underset{k-2}{\mathfrak{E}}}(\tau_0) \le \hat{\varepsilon}_0, \quad \overset{(0)}{\underset{k-2}{\mathfrak{E}}}(\tau_0) \le \hat{\varepsilon}_0\alpha L_0^{-1}, \quad \overset{(1)}{\underset{k-4}{\mathfrak{E}}}(\tau_0) \le \hat{\varepsilon}_0\alpha L_0^{-1+\delta}, \quad \overset{(0)}{\underset{k-6}{\mathfrak{E}}}(\tau_0) \le \hat{\varepsilon}_0\alpha^2 L_0^{-2+\delta}.
$$

Finally, with our restriction on the definition of ε_0, we may also write

$$
\overset{(1)}{\underset{k}{\mathcal{E}}}(\tau_0) \le \alpha\hat{\varepsilon}_0.
$$

In general, given $\tau_i \geq 1$ defined above, $\hat{\varepsilon}_i \leq \hat{\varepsilon}_{\text{slab}}$, and a solution ψ of (4-1) on $\mathcal{R}(\tau_0, \tau_i)$ such that

$$\underset{k}{\overset{(1)}{\mathcal{E}}}(\tau_i) \leq \alpha \hat{\varepsilon}_i, \tag{6-111}$$

and

$$\underset{k}{\overset{(1)}{\mathfrak{E}}}(\tau_i) \leq \hat{\varepsilon}_i, \quad \underset{k-2}{\overset{(2-\delta)}{\mathfrak{E}}}(\tau_i) \leq \hat{\varepsilon}_i, \quad \underset{k-2}{\overset{(0)}{\mathfrak{E}}}(\tau_i) \leq \hat{\varepsilon}_i \alpha L_i^{-1}, \quad \underset{k-4}{\overset{(1)}{\mathfrak{E}}}(\tau_i) \leq \hat{\varepsilon}_i \alpha L_i^{-1+\delta}, \quad \underset{k-6}{\overset{(0)}{\mathfrak{E}}}(\tau_i) \leq \hat{\varepsilon}_i \alpha^2 L_i^{-2+\delta}, \tag{6-112}$$

we note that, by our restrictions on $\hat{\varepsilon}_{\text{slab}}$, the assumptions of Proposition 6.2.3 hold with $\alpha^2 \hat{\varepsilon}_i$ in place of ε_0 (here we are using (6-111) to invoke Corollary 4.5.3 to rewrite (6-60) in terms of the calligraphic energies; cf. the first lines of the proof of Proposition 6.3.3) and the assumptions of Proposition 6.3.3 then apply with $\hat{\varepsilon}_i$ in place of $\hat{\varepsilon}_0$, where both propositions are understood now with τ_i, τ_{i+1} in place of τ_0, τ_1. It follows that the solution ψ extends to a solution defined also in $\mathcal{R}_i := \mathcal{R}(\tau_i, \tau_i + L_i)$ satisfying the estimates (6-86)–(6-89), while for $\tau' = \tau_{i+1} = \tau_i + L_i$ we have in addition (6-92)–(6-93). We have thus

$$\begin{aligned}
&\underset{k}{\overset{(1)}{\mathcal{E}}}(\tau_{i+1}) \leq \hat{\varepsilon}_i \alpha \leq \hat{\varepsilon}_{i+1} \alpha, \\
&\underset{k}{\overset{(1)}{\mathfrak{E}}}(\tau_{i+1}) \leq \hat{\varepsilon}_i (1 + \alpha L_i^{-\frac{1}{4}}) \leq \hat{\varepsilon}_{i+1}, \\
&\underset{k-2}{\overset{(2-\delta)}{\mathfrak{E}}}(\tau_{i+1}) \leq \hat{\varepsilon}_i (1 + \alpha L_i^{-\frac{1}{4}}) \leq \hat{\varepsilon}_{i+1}, \\
&\underset{k-2}{\overset{(1)}{\mathfrak{E}}}(\tau_{i+1}) \leq \tfrac{1}{2} \hat{\varepsilon}_i \alpha L_i^{-1} \leq \hat{\varepsilon}_{i+1} \alpha L_{i+1}^{-1}, \\
&\underset{k-4}{\overset{(1)}{\mathfrak{E}}}(\tau_{i+1}) \leq \tfrac{1}{2} \hat{\varepsilon}_i \alpha L_i^{-1+\delta} \leq \hat{\varepsilon}_{i+1} \alpha L_{i+1}^{-1+\delta}, \\
&\underset{k-6}{\overset{(0)}{\mathfrak{E}}}(\tau_{i+1}) \leq \tfrac{1}{4} \hat{\varepsilon}_i \alpha^2 L_i^{-2+\delta} \leq \hat{\varepsilon}_{i+1} \alpha^2 L_{i+1}^{-2+\delta}
\end{aligned}$$

as long as

$$\hat{\varepsilon}_{i+1} := \hat{\varepsilon}_i (1 + \alpha L_i^{-\frac{1}{4}}). \tag{6-113}$$

By our requirement (6-110) and the definition (6-108) of the parameter d and then defining $\hat{\varepsilon}_{i+1}$ inductively by (6-113), it follows that $\hat{\varepsilon}_{i+1} \leq \hat{\varepsilon}_{\text{slab}}$ and $\alpha \hat{\varepsilon}_{i+1} \leq \varepsilon_{\text{slab}}$.

It follows that a solution exists in $\mathcal{R}(\tau_0, \infty) = \cup \mathcal{R}(\tau_i, \tau_i + L_i)$ and in each interval the estimates (6-89)–(6-90) hold, with $L = L_i$.

We obtain finally that for all $\tau \geq 1$ we have (among other estimates)

$$\underset{k}{\overset{(1)}{\mathcal{E}}}(\tau) \lesssim \varepsilon_0, \tag{6-114}$$

$$\int_{\tau_0}^{\tau} {}^{\rho}\overset{(0)}{\mathcal{E}}{}' + {}^{\chi}\underset{k-1}{\overset{(0)}{\mathcal{E}}}{}' + \underset{k-2}{\overset{(0)}{\mathcal{E}}}{}' \lesssim \varepsilon_0 \log(\tau + 1), \tag{6-115}$$

$$\underset{k-2}{\overset{(2-\delta)}{\mathcal{E}}}(\tau) \lesssim \varepsilon_0, \tag{6-116}$$

$$\underset{k-4}{\overset{(1)}{\mathcal{E}}}(\tau) \lesssim \varepsilon_0 \tau^{-1+\delta}, \quad \underset{k-4}{\overset{(0)}{\mathcal{F}}}(v, \tau) \lesssim \varepsilon_0 \tau^{-1+\delta}, \tag{6-117}$$

$$\int_{\tau}^{\infty} {}^{\rho}\underset{k-4}{\overset{(0)}{\mathcal{E}}}{}' + {}^{\chi}\underset{k-5}{\overset{(0)}{\mathcal{E}}}{}' + \underset{k-6}{\overset{(0)}{\mathcal{E}}}{}' \lesssim \varepsilon_0 \tau^{-1+\delta}, \tag{6-118}$$

$$\underset{k-6}{\overset{(0)}{\mathcal{E}}}(\tau) \lesssim \varepsilon_0 \tau^{-2+\delta}, \quad \underset{k-6}{\overset{(0)}{\mathcal{F}}}(v, \tau) \lesssim \varepsilon_0 \tau^{-2+\delta}, \tag{6-119}$$

$$\int_{\tau}^{\infty} {}^{\rho}\underset{k-6}{\overset{(-1-\delta)}{\mathcal{E}}}{}' + {}^{\chi}\underset{k-7}{\overset{(-1-\delta)}{\mathcal{E}}}{}' + \underset{k-8}{\overset{(-1-\delta)}{\mathcal{E}}}{}' \lesssim \varepsilon_0 \tau^{-2+\delta}. \tag{6-120}$$

Finally, let us note that we may improve (6-115) to

$$\int_{\tau_0}^{\infty} {}^{\rho}\overset{(0)}{\mathcal{E}'} + {}^{\chi}\underset{k-1}{\overset{(0)}{\mathcal{E}'}} + \underset{k-2}{\overset{(0)}{\mathcal{E}'}} \lesssim \varepsilon_0. \tag{6-121}$$

This follows, for all τ, by applying again both estimates of Proposition 4.8.1 appropriate to case (iii) globally in $\mathcal{R}(\tau_0, \tau)$. One may now re-estimate all nonlinear spacetime integrals arising in the estimates on dyadic intervals and sum.

Appendix A: The physical-space identity on very slowly rotating Kerr

Here we show that very slowly rotating Kerr metrics indeed satisfy the assumptions of Section 3.4.3.

Theorem A.1. *Consider the manifold $(\mathcal{M}, g_{a,M})$ of Section 2.7.3, where $g_{a,M}$ denotes the Kerr metric and $|a| \ll M$. Then there exist currents $J^{V,w,q,\varpi}$, $K^{V,w,q}$ associated to the wave operator $\Box_{g_{a,M}}$ satisfying the assumptions of Section 3.4.3.*

Theorem A.1 actually holds for general suitably small stationary perturbations of Schwarzschild satisfying the assumptions of Section 2 and appropriate decay at infinity. So as not to complicate matters more, here we will simply do explicitly the computation for slowly rotating Kerr, $|a| \ll M$.

The nontrivial computations necessary to produce $J^{V,w,q,\varpi}$ and $K^{V,w,q}$ all involve the exterior region. As a result, we may work entirely in Boyer–Lindquist coordinates. Thus, *in this appendix*, r *will denote the Boyer–Lindquist coordinate and* ***not*** *the function* (2-1) *of Section 2.1*. We will show explicitly the positivity properties in the region $r > r_+$, which can be covered globally by such Boyer–Lindquist coordinates. The extension of the coercivity properties to the manifold of Section 2.7.3 follows softly, without computation. (See already the end of Section A.4).

Our currents will be explicit, but rather than give formulae for the final (V, w, q, ϖ), we will build the current from its natural constituent pieces. In particular, so as to directly generate positive zeroth-order terms in the boundary currents, we will make use of the twisted energy momentum tensor as defined in [Holzegel and Warnick 2014].

This appendix is organised as follows. We first recall the Kerr metric in Boyer–Lindquist coordinates in Section A.1. We shall then review in Section A.2 the twisted energy momentum tensor and its basic properties. In the next two sections we shall build our current as a combination of a "Morawetz current", constructed in Section A.3 (vanishing, however, identically in an open set containing trapped null geodesics), and the twisted stationary current and red-shift currents, constructed in Section A.4. Section A.5 contains some auxiliary calculations.

A.1. ***The Kerr metric in Boyer–Lindquist coordinates.*** We define

$$\Delta = r^2 - 2Mr + a^2, \quad \Sigma = r^2 + a^2 \cos^2\theta. \tag{A-1}$$

We recall the (t, r, θ, ϕ) Boyer–Lindquist coordinates and the associated $(t, r^\star, \theta, \phi)$ coordinates (which we will also refer to as Boyer–Lindquist coordinates) with the familiar relation

$$\frac{dr^\star}{dr} = \frac{r^2+a^2}{\Delta}.$$

We shall consider these in the domain $\mathbb{R}\times(r_+,\infty)\times\mathbb{S}^2$, where (θ,ϕ) are identified with the usual spherical coordinates of $\mathbb{S}^2$. The Kerr metric then has the form

$$g=g_{a,M}=g_{tt}\,dt\otimes dt+g_{t\phi}(dt\otimes d\phi+d\phi\otimes dt)+g_{r^\star r^\star}\,dr^\star\otimes dr^\star+g_{\theta\theta}\,d\theta\otimes d\theta+g_{\phi\phi}\,d\phi\otimes d\phi,$$

with

$$\begin{aligned} g_{tt}&=-\left(1-\frac{2Mr}{\Sigma}\right),\quad g_{r^\star r^\star}=\frac{\Sigma\Delta}{(r^2+a^2)^2},\quad g_{\theta\theta}=\Sigma,\\ g_{t\phi}&=-\frac{2Mr}{\Sigma}a\sin^2\theta,\quad g_{\phi\phi}=\left(r^2+a^2+\frac{2Mra^2}{\Sigma}\sin^2\theta\right)\sin^2\theta.\end{aligned}\tag{A-2}$$

We compute the inverse metric components as

$$\begin{aligned} g^{tt}&=-\frac{1}{\Delta}\left(r^2+a^2+\frac{2Mra^2}{\Sigma}\sin^2\theta\right),\quad g^{r^\star r^\star}=\frac{(r^2+a^2)^2}{\Sigma\Delta},\quad g^{\theta\theta}=\frac{1}{\Sigma},\\ g^{t\phi}&=-\frac{2Mr}{\Sigma\Delta}a,\quad g^{\phi\phi}=\frac{\Delta-a^2\sin^2\theta}{\Sigma\Delta\sin^2\theta}.\end{aligned}\tag{A-3}$$

For the determinant in these coordinates, we note

$$\sqrt{|g|}=\frac{\Delta}{r^2+a^2}\Sigma\sin\theta=g_{r^\star r^\star}(r^2+a^2)\sin\theta.\tag{A-4}$$

A.2. ***The twisted energy momentum tensor.*** We consider the covariant wave equation

$$\Box_g\psi=g^{\alpha\beta}\nabla_\alpha\nabla_\beta\psi=0$$

for $g=g_{a,M}$ the Kerr metric. Note that this equation can be rewritten as follows: with the function $\beta=(\sqrt{r^2+a^2})^{-1}$, we define as in [Holzegel and Warnick 2014] the twisted operators

$$\begin{aligned}\widetilde{\nabla}_\mu(\,\cdot\,)&=\beta\nabla_\mu(\beta^{-1}\,\cdot\,),\\ \widetilde{\nabla}^\dagger_\mu(\,\cdot\,)&=-\beta^{-1}\nabla_\mu(\beta\,\cdot\,)\end{aligned}\tag{A-5}$$

and rewrite the wave equation as

$$\begin{aligned}0=\Box_g\psi&=-\widetilde{\nabla}^\dagger_\mu\widetilde{\nabla}^\mu\psi-\mathcal{V}\psi=0\\ &=\sqrt{r^2+a^2}\nabla_\mu((r^2+a^2)^{-1}\nabla^\mu(\psi\sqrt{r^2+a^2}))-\mathcal{V}\psi,\end{aligned}\tag{A-6}$$

where

$$\begin{aligned}\mathcal{V}&=-\frac{\Box_g\beta}{\beta}=-\frac{1}{g_{r^\star r^\star}(r^2+a^2)\beta}\partial_{r^\star}((r^2+a^2)\partial_{r^\star}\beta)\\ &=\frac{1}{\Sigma}\cdot\frac{2Mr^3+a^2r(r-4M)+a^4}{(r^2+a^2)^2}=:\frac{1}{\Sigma}\mathcal{V}_0.\end{aligned}\tag{A-7}$$

We now define the twisted energy momentum tensor

$$\begin{aligned}\widetilde{T}_{\mu\nu}[\psi]&=\widetilde{\nabla}_\mu\psi\widetilde{\nabla}_\nu\psi-\tfrac{1}{2}g_{\mu\nu}(\widetilde{\nabla}^\alpha\psi\widetilde{\nabla}_\alpha\psi+\mathcal{V}\psi^2)\\ &=\beta^2\big[\nabla_\mu(\beta^{-1}\psi)\nabla_\nu(\beta^{-1}\psi)-\tfrac{1}{2}g_{\mu\nu}(\nabla^\alpha(\beta^{-1}\psi)\nabla_\alpha(\beta^{-1}\psi)+\mathcal{V}\beta^{-2}\psi^2)\big].\end{aligned}\tag{A-8}$$

We note that, for all $|a| < M$, $\mathcal{V}$ is strictly positive in the exterior; in fact

$$\mathcal{V} \geq \frac{2Mr(r-M)(r+M)}{\Sigma(r^2+a^2)^2} \gtrsim r^{-3} \tag{A-9}$$

since $r \geq r_+ > M$, and $\widetilde{T}_{\mu\nu}[\psi]$ satisfies the dominant energy condition, i.e., $\widetilde{T}_{\mu\nu}[\psi]\xi^\mu\xi^\nu$ is nonnegative if ξ is timelike and in fact controls coercively first derivatives of ψ as well as ψ itself (the latter with the weight r^{-3}). (Indeed, this direct control of the zeroth-order term is our motivation for considering (A-8) in place of the usual $T_{\mu\nu}$.).

From Proposition 3 of [Holzegel and Warnick 2014] we infer the following.

Proposition A.2.1. *For ψ a C^2 solution of $\Box_g\psi = 0$ and X a smooth spacetime vector field, we have the identity*

$$\nabla^\mu \widetilde{J}^X_\mu[\psi] = \widetilde{K}^X[\psi], \tag{A-10}$$

where

$$\widetilde{J}^X_\mu[\psi] = \widetilde{T}_{\mu\nu}[\psi]X^\nu, \tag{A-11}$$

$$\widetilde{K}^X[\psi] = {}^{(X)}\pi^{\mu\nu}\widetilde{T}_{\mu\nu}[\psi] + X^\nu\widetilde{S}_\nu[\psi], \tag{A-12}$$

and

$$\widetilde{S}_\mu[\psi] = \frac{\widetilde{\nabla}^\dagger_\mu(\beta\mathcal{V})}{2\beta}\psi^2 + \frac{\widetilde{\nabla}^\dagger_\mu\beta}{2\beta}\widetilde{\nabla}^\nu\psi\widetilde{\nabla}_\nu\psi. \tag{A-13}$$

We also have the Lagrangian identity for an arbitrary spacetime function w:

Proposition A.2.2. *For ψ a C^2 solution of $\Box_g\psi = 0$ and w a smooth spacetime function, we have the identity*

$$\nabla^\mu \widetilde{J}^{\mathrm{aux},w}_\mu = \widetilde{K}^{\mathrm{aux},w}[\psi], \tag{A-14}$$

with

$$\widetilde{K}^{\mathrm{aux},w}[\psi] := w\beta^2\nabla^\alpha(\beta^{-1}\psi)\nabla_\alpha(\beta^{-1}\psi) + (\beta^{-1}\psi)^2\big(-\tfrac{1}{2}\nabla^\mu(\beta^2\nabla_\mu w) + \mathcal{V}w\beta^2\big),$$
$$\widetilde{J}^{\mathrm{aux},w}_\mu := w\beta^2(\beta^{-1}\psi\nabla_\mu(\beta^{-1}\psi)) - \tfrac{1}{2}\psi^2\nabla_\mu w.$$

Proof. This follows from

$$\nabla^\mu(w(\psi\beta^{-1}\beta^2\nabla_\mu(\beta^{-1}\psi))) = w\beta^2\nabla^\mu(\beta^{-1}\psi)\nabla^\mu(\beta^{-1}\psi) + w\psi(-\widetilde{\nabla}^\dagger_\mu\widetilde{\nabla}^\mu\psi) + (\nabla^\mu w)\beta^2\tfrac{1}{2}\nabla_\mu(\beta^{-1}\psi)^2 \tag{A-15}$$

after rearranging and inserting (A-6). □

Remark A.2.3. Note that when w is a function of r only (as will always be the case in the applications below) we have

$$\nabla^\mu(\beta^2\nabla_\mu w) = \frac{1}{\sqrt{g}}\partial_{r^\star}(\sqrt{g}g^{r^\star r^\star}\beta^2\partial_{r^\star}w) = \frac{r^2+a^2}{\Delta\Sigma}\partial^2_{r^\star}w,$$

a formula which is useful in the computations below.

A.3. ***A Morawetz current vanishing identically in a neighbourhood of trapping.*** In this section, we shall produce a current giving the desired bulk positivity properties modulo suitable zeroth-order terms but with additional degeneration at the horizon and without the boundary positivity properties. (The horizon degeneration and the boundary positivity properties will be dealt with in Section A.4.)

The point about this current is that it will completely vanish in the set $\frac{11}{4}M \leq r \leq \frac{7}{2}M$, which contains all trapped null geodesics for $|a| \ll M$. Thus, *this current is insensitive to the precise nature of the dynamics near trapping*. The requirement of vanishing on such a set will necessarily generate lower-order terms, however, with an unfavourable sign. These too will be supported away from trapping.

Our current will be defined by combining those of Propositions A.2.1 and A.2.2, and the vector field component of the current will be in the direction of $\partial_{r^\star}$. We begin with a computation; note that the prime $'$ below will denote $\frac{d}{dr^*}$.

Lemma A.3.1. *For*

$$\beta^{-1} = \sqrt{r^2+a^2}, \quad h = \Big(1-\frac{3M}{r}\Big)\frac{\Delta}{(r^2+a^2)^2}, \quad \textit{and} \quad X = f(r^*)\partial_{r^\star}$$

(*with f arbitrary*), *we have*, *for any $\delta \in \mathbb{R}$*, *the divergence identity*

$$\nabla^\mu(\widetilde{J}^X_\mu[\psi] + \widetilde{J}^{\text{aux},\frac{1}{2}f'}_\mu[\psi] - \delta \cdot \widetilde{J}^{\text{aux},f\cdot h}_\mu[\psi]) = \widetilde{K}^X[\psi] + \widetilde{K}^{\text{aux},\frac{1}{2}f'}[\psi] - \delta \cdot \widetilde{K}^{\text{aux},f\cdot h}_\mu[\psi], \tag{A-16}$$

where

$$\begin{aligned}&\widetilde{J}^X_\mu[\psi] + \widetilde{J}^{\text{aux},\frac{1}{2}f'}_\mu[\psi] - \delta\widetilde{J}^{\text{aux},f\cdot h}_\mu[\psi]\\ &= \widetilde{T}_{\mu\nu}[\psi]X^\nu + f'\beta^2(\beta^{-1}\psi\nabla_\mu(\beta^{-1}\psi)) - \tfrac{1}{2}\psi^2\nabla_\mu f' - \delta\big[f\cdot h\beta^2(\beta^{-1}\psi\nabla_\mu(\beta^{-1}\psi)) - \tfrac{1}{2}\psi^2\nabla_\mu f\cdot h\big]\end{aligned} \tag{A-17}$$

and

$$\begin{aligned}&\widetilde{K}^{f\partial_{r^\star}}[\psi] + \widetilde{K}^{\text{aux},\frac{1}{2}f'}[\psi] - \delta\widetilde{K}^{\text{aux},f\cdot h}_\mu[\psi]\\ &\quad = \beta^2(g^{r^\star r^\star}\partial_{r^\star} f - \delta f h)(\partial_{r^\star}(\beta^{-1}\psi))^2 + \tfrac{1}{2}f\beta^2\sum_{\mu,\nu\neq r^\star}(\mathcal{A}^{\mu\nu} - 2\delta h g^{\mu\nu})\partial_\mu(\beta^{-1}\psi)\partial_\nu(\beta^{-1}\psi)\\ &\qquad + \frac{r^2+a^2}{\Delta\Sigma}\bigg(-\tfrac{1}{4}f''' - \tfrac{1}{2}f\partial_{r^\star}\Big(\frac{\Delta}{(r^2+a^2)^2}\mathcal{V}_0\Big) + \delta\Big[\tfrac{1}{2}(f\cdot h)'' - \mathcal{V}_0 f\cdot h\frac{\Delta}{(r^2+a^2)^2}\Big]\bigg)(\beta^{-1}\psi)^2\end{aligned} \tag{A-18}$$

with

$$\mathcal{A}^{\mu\nu} = -\partial_{r^\star}(g^{\mu\nu}) + g^{\mu\nu}g_{r^\star r^\star}\partial_{r^\star}(g^{r^\star r^\star}). \tag{A-19}$$

Proof. We compute (see Section A.5)

$$\begin{aligned}\widetilde{K}^{f\partial_{r^\star}}[\psi] = \beta^2 g^{r^\star r^\star}\partial_{r^\star} f(\partial_{r^\star}(\beta^{-1}\psi))^2 - \tfrac{1}{2}f\beta^2\sum_{\mu,\nu\neq r^\star}[\partial_{r^\star}(g^{\mu\nu}) - g^{\mu\nu}g_{r^\star r^\star}\partial_{r^\star}(g^{r^\star r^\star})]\partial_\mu(\beta^{-1}\psi)\partial_\nu(\beta^{-1}\psi)\\ - \tfrac{1}{2}\beta^2(\partial_{r^\star} f)g^{\alpha\beta}\partial_\alpha(q^{-1}\psi)\partial_\beta(\beta^{-1}\psi) - \tfrac{1}{2}f\partial_{r^\star}(\beta^2\mathcal{V})(\beta^{-1}\psi)^2\\ - \frac{1}{2}\Big(\partial_{r^\star} f + f\frac{1}{\sqrt{g}}\partial_{r^\star}\sqrt{g}\Big)\mathcal{V}\psi^2.\end{aligned} \tag{A-20}$$

Adding the Lagrangian identity of Proposition A.2.2 with $w = \frac{1}{2}f'$ (and using Remark A.2.3), we deduce the result for $\delta = 0$. For arbitrary δ, we simply add the Lagrangian identity of Proposition A.2.2 with

$$w = \Big(1 - \frac{3M}{r}\Big)\frac{\Delta}{(r^2+a^2)} f = f \cdot h$$

and group terms. □

We now exploit the divergence identity of Lemma A.3.1. For this we first define the following (disjoint) decomposition of the range of the r-variable:

$$\begin{aligned}[r_+, \infty) &= R_1 \cup R_2 \cup R_3 \cup R_4 \cup R_5 \\ &= \big[r_+, \tfrac{5}{2}M\big) \cup \big[\tfrac{5}{2}M, \tfrac{11}{4}M\big) \cup \big[\tfrac{11}{4}M, \tfrac{7}{2}M\big) \cup \big[\tfrac{7}{2}M, 4M\big) \cup \big[4M, \infty\big).\end{aligned}$$

Note that $\mathcal{M} \cap \{r \in R_3\}$ includes the region containing all trapped null geodesics if $\frac{a}{M}$ is suitably small.

Proposition A.3.2. *There exists an (explicit) function f such that, for all $|a|/M$ sufficiently small, we can choose $\delta > 0$ sufficiently small (depending only on M) such that the estimate*

$$\begin{aligned}&\widetilde{K}^{f\partial_{r^\star}}[\psi] + \widetilde{K}^{\mathrm{aux},\frac{1}{2}f'}[\psi] - \delta \cdot \widetilde{K}_\mu^{\mathrm{aux},\frac{f\Delta}{(r^2+a^2)^2}}[\psi] \\ &\geq c\mathbb{1}_{R_1\cup R_5}\bigg(\frac{(\partial_t(\beta^{-1}\psi))^2 + (\partial_{r^\star}(\beta^{-1}\psi))^2 + (\beta^{-1}\psi)^2}{r^4} + \frac{1}{r^5}|\mathring{\nabla}(\beta^{-1}\psi)|^2\bigg) - C\mathbb{1}_{R_2\cup R_4}(\beta^{-1}\psi)^2 \end{aligned} \tag{A-21}$$

holds for constants c and C depending only on M. Here we have defined the shorthand

$$|\mathring{\nabla}(\beta^{-1}\psi)|^2 := (\partial_\theta(\beta^{-1}\psi))^2 + \frac{1}{\sin^2\theta}(\partial_\phi(\beta^{-1}\psi))^2.$$

Proof. In the proof, we let $\tilde{c}$ and $\widetilde{C}$ be constants depending only on M (but which might change from line to line). Starting from (A-18) we define

$$\mathcal{B}^{\mu\nu} = \mathcal{A}^{\mu\nu} - 2\delta\Big(1 - \frac{3M}{r}\Big)\frac{\Delta}{(r^2+a^2)^2} g^{\mu\nu}$$

and compute

$$\begin{aligned}\mathcal{B}^{tt} &= 2a^2 \cdot \frac{r^3 - 3Mr^2 + a^2 r + a^2 M}{\Sigma(r^2+a^2)^2}\sin^2\theta + 2\delta\Big(1 - \frac{3M}{r}\Big)\frac{r^2 + a^2 + \frac{2Mra^2}{\Sigma}\sin^2\theta}{(r^2+a^2)^2}, \\ \mathcal{B}^{t\phi} &= \frac{2aM(a^2 - 3r^2)}{\Sigma(r^2+a^2)^2} + 2\delta \cdot a\Big(1 - \frac{3M}{r}\Big)\frac{2Mr}{\Sigma(r^2+a^2)^2}, \\ \mathcal{B}^{\phi\phi} &= 2\frac{r^2(r-3M) + a^2 r\cos[2\theta] + a^2 M}{\Sigma(r^2+a^2)^2\sin^2\theta} - 2\delta\Big(1 - \frac{3M}{r}\Big)\frac{\Delta - a^2\sin^2\theta}{\Sigma(r^2+a^2)^2\sin^2\theta}, \\ \mathcal{B}^{\theta\theta} &= 2\frac{r^2(r-3M) + a^2 r + a^2 M}{\Sigma(r^2+a^2)^2\sin^2\theta} - 2\delta\Big(1 - \frac{3M}{r}\Big)\frac{\Delta}{\Sigma(r^2+a^2)^2}.\end{aligned} \tag{A-22}$$

From this one sees that, for $|a|/M$ sufficiently small, we can choose δ sufficiently small (depending only on M) such that in $\mathcal{M} \cap \{r \notin R_3\}$ $\big($where $\big|1 - \frac{3M}{r}\big| \geq \frac{1}{12}\big)$ the estimates

$$\frac{\mathcal{B}^{tt}}{1 - \frac{3M}{r}} \geq \frac{\tilde{c}}{r^2}, \qquad \frac{\mathcal{B}^{\phi\phi}}{1 - \frac{3M}{r}} \geq \frac{\tilde{c}}{r^3\sin^2\theta}, \qquad \frac{\mathcal{B}^{\theta\theta}}{1 - \frac{3M}{r}} \geq \frac{\tilde{c}}{r^3} \tag{A-23}$$

hold for a constant $\tilde{c}$ depending only on M. Moreover, we have

$$\left|\frac{\mathcal{B}^{t\phi}}{1-\frac{3M}{r}}\right| \leq \frac{|a|}{M}\frac{\widetilde{C}}{r^4}. \tag{A-24}$$

We next choose $f : [r_+, \infty) \to \mathbb{R}$ to be bounded, monotonically increasing as follows:

$$f = \begin{cases} -\frac{M}{r} & \text{if } r \in R_1, \\ C^3 \text{ interpolate} & \text{if } r \in R_2, \\ 0 & \text{if } r \in R_3, \\ C^3 \text{ interpolate} & \text{if } r \in R_4, \\ 1-\frac{M}{r} & \text{if } r \in R_5. \end{cases}$$

Specifically, we do the C^3 interpolation such that we have f monotonically increasing and

$$f' \geq \tilde{c}(-f) \ \text{ in } R_2 \quad \text{and} \quad f' \geq \tilde{c}f \ \text{ in } R_4 \tag{A-25}$$

for a fixed constant $\tilde{c} > 0$ depending only on M. In particular (potentially making δ slightly smaller), we can achieve that

$$g^{r^\star r^\star}\partial_{r^\star} f - \delta f\Big(1-\frac{3M}{r}\Big)\frac{\Delta}{(r^2+a^2)^2} \geq f\Big(1-\frac{3M}{r}\Big)\frac{\tilde{c}}{r^2}$$

holds in all of $\mathcal{M} \cap \{r \notin R_3\}$ for a $\tilde{c}$ depending only on M. Since $f\big(1-\frac{3M}{r}\big)$ is globally nonnegative and in fact bounded uniformly below by a $\tilde{c}$ in $R_1 \cup R_5$, the estimate (A-21) now follows except for the zeroth-order term on the right-hand side. (For this the only thing to notice is that in

$$\tfrac{1}{2} f\beta^2\Big(1-\frac{3M}{r}\Big)\Bigg[\sum_{\mu,\nu\neq r^\star}\frac{\mathcal{B}^{\mu\nu}}{1-\frac{3M}{r}}\partial_\mu(\beta^{-1}\psi)\partial_\nu(\beta^{-1}\psi)\Bigg]$$

the quadratic form in the square bracket is positive definite by the estimates (A-23) and (A-24).)

For the zeroth-order term, clearly we only need to establish a lower bound in $R_1 \cup R_5$. We start with R_5, where we have for $f = 1 - M/r$ that

$$-\tfrac{1}{2}f''' - f\bigg(\frac{\Delta}{(r^2+a^2)^2}V_0\bigg)' = \frac{3M}{r^4} + \mathcal{O}(r^{-5})$$

and

$$\frac{r^2+a^2}{r^2-2Mr+a^2}\bigg(-\tfrac{1}{2}f''' - f\bigg(\frac{\Delta}{(r^2+a^2)^2}\mathcal{V}_0\bigg)'\bigg) = \frac{M}{r^4}\bigg(3-\frac{2M}{r}-14\frac{M^2}{r^2}\bigg) \quad \text{if } a=0.$$

Since the left-hand side of the second identity is continuous in a and the bracket on the right-hand side is uniformly bounded below by $\frac{1}{3}$ for $r \geq 3M$, the estimate claimed for the zeroth-order term follows for sufficiently small $|a|/M$ after potentially making δ smaller; note that

$$\delta\frac{r^2+a^2}{r^2-2Mr+a^2}\bigg[\tfrac{1}{2}(f\cdot h)'' - \mathcal{V}_0 f\cdot h\frac{\Delta}{(r^2+a^2)^2}\bigg] \leq \frac{\widetilde{C}\delta}{r^4}.$$

Similarly, in R_1 we have for $f = -M/r$ the identity

$$\frac{r^2+a^2}{r^2-2Mr+a^2}\left(-\tfrac{1}{2}f''' - f(w\mathcal{V}_0)'\right) = -\frac{M}{r^6}(14M^2-14Mr+3r^2) \quad \text{if } a=0.$$

The expression on the right-hand side is easily shown to be uniformly bounded below by $M^3/(4r^6)$ for $r \in \left[\frac{9}{5}M, \frac{11}{4}M\right]$. Since the expression on the left is in particular continuous in a on $\left[r_+, \frac{11}{4}M\right]$, the estimate claimed for the zeroth-order term also follows in R_1. □

A.4. ***Adding the red-shift and stationary currents and completing the proof.*** Let T denote the stationary Killing field of Kerr (which in Boyer–Lindquist coordinates is $T = \partial_t$) and N the time-like (including on $\mathcal{H}^+$) red-shift vector field constructed in Theorem 7.1 of [Dafermos and Rodnianski 2013]. We have that $\widetilde{K}^T = 0$, while

$$\widetilde{K}^N[\psi] \geq \tilde{c}\Bigg((\partial_t(\beta^{-1}\psi))^2 + (\partial_{r^\star}(\beta^{-1}\psi))^2 + |\mathring{\nabla}(\beta^{-1}\psi)|^2 + (\beta^{-1}\psi)^2 \\ + \left[\frac{r^2+a^2}{\Delta}\left(\partial_t - \partial_{r^\star} + \frac{a}{r^2+a^2}\partial_\phi\right)(\beta^{-1}\psi)\right]^2\Bigg) \quad \text{in } r \leq \tfrac{9}{4}M$$

provided a is sufficiently small. We also have $N = T$ in the complement of R_1.

We shall now complete the proof of Theorem A.1 by adding

$$\Upsilon \cdot \widetilde{J}^T_\mu[\psi] + \eta \widetilde{J}^N_\mu[\psi]$$

to the current of Proposition A.3.2 for constants $\Upsilon > 0$ (large) and $\eta > 0$ (small).

We consider the divergence identity of Lemma A.3.1 with δ and f now chosen as in Proposition A.3.2. We expand the identity as follows for constants $\Upsilon > 0$ (large) and $\eta > 0$ (small) to be chosen below:

$$\nabla^\mu\left(\widetilde{J}^X_\mu[\psi] + \widetilde{J}^{\mathrm{aux},\frac{1}{2}f'}_\mu[\psi] - \delta\cdot \widetilde{J}^{\mathrm{aux},f\cdot h}_\mu[\psi] + \Upsilon\cdot\widetilde{J}^T_\mu[\psi] + \eta\cdot\widetilde{J}^N_\mu[\psi]\right) \\ = \widetilde{K}^X[\psi] + \widetilde{K}^{\mathrm{aux},\frac{1}{2}f'}[\psi] - \delta\cdot\widetilde{K}^{\mathrm{aux},f\cdot h}_\mu[\psi] + \eta\cdot\widetilde{K}^N[\psi]. \quad \text{(A-26)}$$

It is now clear that we can choose η sufficiently small (depending only on M) such that for $r \geq \frac{9}{4}M$ we can absorb the term $\eta\widetilde{K}^N[\psi]$, which is supported only in R_1, by the positivity of

$$\widetilde{K}^X[\psi] + \widetilde{K}^{\mathrm{aux},\frac{1}{2}f'}[\psi] - \delta\cdot\widetilde{K}^{\mathrm{aux},f\cdot h}_\mu[\psi]$$

established in Proposition A.3.2 to deduce

$$\widetilde{K}^{f\partial_{r^\star}}[\psi] + \widetilde{K}^{\mathrm{aux},\frac{1}{2}f'}[\psi] - \delta\widetilde{K}^{\mathrm{aux},\frac{f\Delta}{(r^2+a^2)^2}}_\mu[\psi] + \eta\widetilde{K}^N[\psi] \\ \geq c\mathbb{1}_{R_1\cup R_5}\left(\frac{(\partial_t(\beta^{-1}\psi))^2 + (\partial_{r^\star}(\beta^{-1}\psi))^2 + (\beta^{-1}\psi)^2}{r^4} + \frac{1}{r^5}|\mathring{\nabla}(\beta^{-1}\psi)|^2\right) \\ + c\mathbb{1}_{R_1}\left[\frac{r^2+a^2}{\Delta}\left(\partial_t - \partial_{r^\star} + \frac{a}{r^2+a^2}\partial_\phi\right)(\beta^{-1}\psi)\right]^2 - C\mathbb{1}_{R_2\cup R_4}|(\beta^{-1}\psi)|^2, \quad \text{(A-27)}$$

where c may be smaller than in (A-21) but still depends only on M. We finally claim that we can choose Υ sufficiently large in (A-26) such that

$$\begin{aligned}&\big(\widetilde{J}^X_\mu[\psi]+\widetilde{J}^{\mathrm{aux},\frac{1}{2}f'}_\mu[\psi]-\delta\widetilde{J}^{\mathrm{aux},f\cdot h}_\mu[\psi]+\Upsilon\widetilde{J}^T_\mu[\psi]+\eta\widetilde{J}^N_\mu[\psi]\big)n^\mu_{\Sigma_\tau}\\ &\quad=\boxed{\widetilde{T}_{\mu\nu}[\psi](X^\nu+\Upsilon T^\nu+\eta N^\nu)n^\mu_{\Sigma_\tau}}+f'\beta^2(\beta^{-1}\psi\nabla_\mu(\beta^{-1}\psi))n^\mu_{\Sigma_\tau}\\ &\qquad\qquad-\tfrac{1}{2}\psi^2\nabla_\mu f' n^\mu_{\Sigma_\tau}-\delta\big[f\cdot h\beta^2(\beta^{-1}\psi\nabla_\mu(\beta^{-1}\psi))-\tfrac{1}{2}\psi^2\nabla_\mu f\cdot h\big]n^\mu_{\Sigma_\tau}\\ &\quad\geq\beta^2\bigg(|L(\beta^{-1}\psi)|^2+\iota_{r\leq R}|\underline{L}(\beta^{-1}\psi)|^2+\frac{1}{r^2}|\mathring{\nabla}(\beta^{-1}\psi)|^2+\frac{1}{r^3}(\beta^{-1}\psi)^2\bigg).\end{aligned}\tag{A-28}$$

For the boxed term, the estimate is an immediate consequence of the twisted energy momentum tensor (A-8) satisfying the dominant energy condition, as discussed in Section A.2, the positivity (A-9), and the fact that the vector field $\Upsilon T+\eta N+X$ is timelike for sufficiently large Υ, provided that we then restrict to $|a|$ sufficiently small. Moreover, for the boxed term the zeroth-order term in the estimate scales with Υ; i.e., the larger we choose Υ the larger the zeroth-order term becomes. This can be used to absorb the remaining terms using the Cauchy–Schwarz inequality for $|a|$ sufficiently small depending on this final choice of Υ.

The relations (A-27) and (A-28) give the coercivity property (3-31) and the first property of (3-32), restricted to the exterior region, where we define $\rho=\mathbb{1}_{R_1\cup R_5}$ and $\xi=\mathbb{1}_{R_2\cup R_4}$, except that the r decay for the bulk current is not optimised to the $r^{-1-\delta}$ and $r^{-3-\delta}$ weights for the first and zeroth-order terms, respectively, and the r decay for the boundary current is not optimised to the r^{-2} weight for the zeroth-order term. (Note that the Boyer–Lindquist r and the r of (2-1) are comparable for large r values, so one can compare directly the r-decay as if the coordinates were the same.) To remedy this, it suffices to add $\epsilon J^{\hat{\chi}V,\hat{\chi}w,\hat{\chi}q,\hat{\chi}\varpi}$ and $\epsilon K^{\hat{\chi}V,\hat{\chi}w,\hat{\chi}q}$, respectively, to the currents, where (V,w,q,ϖ) here denotes the current of Section B.1, $\hat{\chi}$ is a cutoff supported far away, and ϵ is sufficiently small. The resulting currents now indeed satisfy (3-31) and the first inequality of (3-32), restricted to the exterior, with weights as stated. We note finally that, for small $|a|$, we may take the χ in the estimate (3-3) proven in [Dafermos and Rodnianski 2010a] or [Dafermos et al. 2016] to be identically 1 outside of a small neighbourhood of $r=3M$. Thus, our ξ indeed satisfies (3-27).

The currents trivially may be extended to the slightly larger domain of Section 2.7.3. Note finally that the additional positivity statements of (3-32) are easily shown to hold. Rewriting the current in terms of a single quadruple (V,w,q,ϖ), one easily sees that the boundedness statements (3-16) hold as well. This completes the proof of Theorem A.1.

A.5. ***Computation of the X-deformation tensor.*** We collect here some computations which were used in the proof of Theorem A.1.

We have

$$-2^{(X)}\pi^{\mu\nu}=X^\alpha\partial_\alpha(g^{\mu\nu})-\partial_\alpha X^\mu g^{\alpha\nu}-\partial_\alpha X^\nu g^{\alpha\mu};\tag{A-29}$$

hence for $X=f(r)\partial_{r^\star}$

$$-2^{(X)}\pi^{\mu\nu}=f\partial_{r^\star}(g^{\mu\nu})-\partial_\alpha X^\mu g^{\alpha\nu}-\partial_\alpha X^\nu g^{\alpha\mu},\tag{A-30}$$

which we can write (using $g^{r^\star r^\star} g_{r^\star r^\star} = 1$ for Kerr in Boyer–Lindquist) as

$$\begin{aligned} -2^{(X)}\pi^{\mu\nu} &= f\partial_{r^\star}(g^{\mu\nu}) + fg^{\mu\nu} g_{r^\star r^\star}\partial_{r^\star}(g^{r^\star r^\star}) - fg^{\mu\nu} g_{r^\star r^\star}\partial_{r^\star}(g^{r^\star r^\star}) \quad \text{unless } \mu=\nu=r^\star, \\ -2^{(X)}\pi^{r^\star r^\star} &= fg^{r^\star r^\star} g_{r^\star r^\star}\partial_{r^\star}(g^{r^\star r^\star}) - 2g^{r^\star r^\star}\partial_{r^\star} f, \end{aligned} \tag{A-31}$$

so

$$-2^{(X)}\pi^{\mu\nu} = g^{\mu\nu}\cdot fg_{r^\star r^\star}\partial_{r^\star}(g^{r^\star r^\star}) + \begin{cases} -2g^{r^\star r^\star}\partial_{r^\star} f & \text{if } \mu=\nu=r^\star, \\ f\partial_{r^\star}(g^{\mu\nu}) - fg^{\mu\nu} g_{r^\star r^\star}\partial_{r^\star}(g^{r^\star r^\star}) & \text{otherwise.} \end{cases}$$

We note also

$$\begin{aligned} \operatorname{tr}{}^{(X)}\pi &= g_{\mu\nu}{}^{(X)}\pi^{\mu\nu} \\ &= -\tfrac{1}{2} g_{\mu\nu} f\partial_{r^\star}(g^{\mu\nu}) + \partial_{r^\star} f \\ &= f\frac{1}{\sqrt{g}}\partial_{r^\star}\sqrt{g} + \partial_{r^\star} f. \end{aligned} \tag{A-32}$$

Therefore

$$\begin{aligned} &{}^{(X)}\pi^{\mu\nu}\widetilde{T}_{\mu\nu} \\ &= {}^{(X)}\pi^{\mu\nu}\widetilde{\nabla}_\mu\psi\widetilde{\nabla}_\nu\psi - \tfrac{1}{2}(\operatorname{tr}{}^{(X)}\pi)g^{\mu\nu}\widetilde{\nabla}_\mu\psi\widetilde{\nabla}_\nu\psi - \tfrac{1}{2}(\operatorname{tr}{}^{(X)}\pi)\mathcal{V}\psi^2 \\ &= \beta^2 g^{r^\star r^\star}\partial_{r^\star} f(\partial_{r^\star}(\beta^{-1}\psi))^2 - \tfrac{1}{2}\beta^2 f\sum_{\mu,\nu\neq r^\star}[\partial_{r^\star}(g^{\mu\nu}) - g^{\mu\nu} g_{r^\star r^\star}\partial_{r^\star}(g^{r^\star r^\star})]\partial_\mu(\beta^{-1}\psi)\partial_\nu(\beta^{-1}\psi) \\ &\quad -\frac{1}{2}\left(\partial_{r^\star} f + f\frac{1}{\sqrt{g}}\partial_{r^\star}\sqrt{g} + fg_{r^\star r^\star}\partial_{r^\star}(g_{r^\star r^\star})^{-1}\right)g^{\mu\nu}\widetilde{\nabla}_\mu\psi\widetilde{\nabla}_\nu\psi - \frac{1}{2}\left(\partial_{r^\star} f + f\frac{1}{\sqrt{g}}\partial_{r^\star}\sqrt{g}\right)\mathcal{V}\psi^2. \end{aligned} \tag{A-33}$$

Noting that the determinant is given by

$$\sqrt{g} = \frac{\Delta}{r^2+a^2}\Sigma\sin\theta = g_{r^\star r^\star}(r^2+a^2)\sin\theta$$

for Kerr in Boyer–Lindquist $(t, r^\star, \theta, \phi)$,

$$\begin{aligned} {}^{(X)}\pi^{\mu\nu}\widetilde{T}_{\mu\nu} &= \beta^2 g^{r^\star r^\star}\partial_{r^\star} f(\partial_{r^\star}(\beta^{-1}\psi))^2 - \tfrac{1}{2}\beta^2 f\sum_{\mu,\nu\neq r^\star}[\partial_{r^\star}(g^{\mu\nu}) - g^{\mu\nu} g_{r^\star r^\star}\partial_{r^\star}(g^{r^\star r^\star})]\partial_\mu(\beta^{-1}\psi)\partial_\nu(\beta^{-1}\psi) \\ &\quad -\frac{1}{2}\left(\partial_{r^\star} f + f\frac{2r\Delta}{(r^2+a^2)^2}\right)g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi - \frac{1}{2}\left(\partial_{r^\star} f + f\frac{1}{\sqrt{g}}\partial_{r^\star}\sqrt{g}\right)\mathcal{V}\psi^2. \end{aligned}$$

We now recall from Proposition A.2.1 the definition $\widetilde{K}^X[\psi] = {}^{(X)}\pi^{\mu\nu}\widetilde{T}_{\mu\nu}[\psi] + X^\nu\widetilde{S}_\nu[\psi]$ and compute

$$\begin{aligned} \widetilde{K}^X[\psi] &= \beta^2 g^{r^\star r^\star}\partial_{r^\star} f(\partial_{r^\star}(\beta^{-1}\psi))^2 - \tfrac{1}{2} f\beta^2\sum_{\mu,\nu\neq r^\star}[\partial_{r^\star}(g^{\mu\nu}) - g^{\mu\nu} g_{r^\star r^\star}\partial_{r^\star}(g^{r^\star r^\star})]\partial_\mu(\beta^{-1}\psi)\partial_\nu(\beta^{-1}\psi) \\ &\quad -\tfrac{1}{2}\beta^2\left(\partial_{r^\star} f + f\frac{2r\Delta}{(r^2+a^2)^2} + f\frac{\partial_{r^\star}(\beta^2)}{\beta^2}\right)g^{\mu\nu}\partial_\mu(\beta^{-1}\psi)\partial_\nu(\beta^{-1}\psi) - \tfrac{1}{2} f\partial_{r^\star}(\beta^2\mathcal{V})(\beta^{-1}\psi)^2 \\ &\quad -\frac{1}{2}\left(\partial_{r^\star} f + f\frac{1}{\sqrt{g}}\partial_{r^\star}\sqrt{g}\right)\mathcal{V}\psi^2, \end{aligned} \tag{A-34}$$

which indeed simplifies to (A-20).

Appendix B: Energy currents in Minkowski space

In this section the energy currents used to obtain the assumptions of Sections 3.4.1 and 3.5 are described in Minkowski space. In Section B.1 a quadruple (V, w, q, ϖ) satisfying the assumptions of Section 3.4.1 is introduced, and in Section B.2 a quadruple $(\widetilde{V}_{\text{far}}, \tilde{w}_{\text{far}}, \tilde{q}_{\text{far}}, \tilde{\varpi}_{\text{far}})$ satisfying the assumptions of Section 3.5 is introduced.

Recall that, for a given spacetime $(\mathcal{M}, g)$ and suitably regular function $\psi : \mathcal{M} \to \mathbb{R}$, for a given vector field V, a function w, a 1-form q, and a 2-form ϖ, the energy current $J^{V,w,q,\varpi}$ takes the form

$$J^{V,w,q,\varpi}_{\mu}[g, \psi] := T_{\mu\nu}[g, \psi]V^{\nu} + w\psi\partial_{\mu}\psi + \psi^2 q_{\mu} + *d(\psi^2\varpi)_{\mu}.$$

Here, for $0 \le k \le 4$, $* \colon \Lambda^k\mathcal{M} \to \Lambda^{4-k}\mathcal{M}$ denotes the Hodge star operator which satisfies, for all $\alpha, \beta \in \Lambda^k\mathcal{M}$,

$$\alpha \wedge *\beta = g(\alpha, \beta)d\,\mathrm{Vol}_{\mathcal{M}}\,.$$

The divergence of $J^{V,w,q,\varpi}$ takes the form

$$\nabla^{\mu}J^{V,w,q,\varpi}_{\mu}[g, \psi] = K^{V,w,q}[g, \psi] + H^{V,w}[\psi]\Box_g\psi,$$

where

$$K^{V,w,q}[g, \psi] := \pi^{V}_{\mu\nu}[g]T^{\mu\nu}[g, \psi] + \psi\nabla^{\mu}w\nabla_{\mu}\psi + w\nabla^{\mu}\psi\nabla_{\mu}\psi + \psi^2\nabla^{\mu}q_{\mu} + 2\psi g^{\mu\nu}q_{\mu}\partial_{\nu}\psi,$$
$$H^{V,w}[\psi] := V^{\mu}\partial_{\mu}\psi + w\psi.$$

The divergence of a 1-form $\xi \in \Lambda^1\mathcal{M}$ can be expressed in terms of d and $*$ by

$$\mathrm{Div}\,\xi = *d*\xi.$$

In particular it follows that, for any 2-form $\varpi \in \Lambda^2\mathcal{M}$ and any function $\psi \in C^{\infty}(\mathcal{M})$,

$$\mathrm{Div}*d(\psi^2\varpi) = -*dd(\psi^2\varpi) = 0.$$

Thus the choice of ϖ in the current $J^{V,w,q,\varpi}_{\mu}[g, \psi]$ never contributes to the associated $K^{V,w,q}[g, \psi]$ or $H^{V,w}[g, \psi]$. Moreover q does not contribute to $H^{V,w}[g, \psi]$.

Throughout this section, $(\mathcal{M}, g_0)$ denotes Minkowski space (see Section 2.7.1) and

$$r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} \tag{B-1}$$

denotes the standard radial coordinate and *not* the r of (2-1). (Note that (B-1) was denoted $\tilde{r}$ in Section 2.7.1. Since (2-1) and (B-1) are comparable for large r, the associated r-weighted coercivity properties will be equivalent.) Recall the $(t, r, \vartheta, \varphi)$ and $(u, v, \vartheta, \varphi)$ coordinate systems. The Minkowski metric takes the form

$$g_0 = -dt^2 + dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2) = -du\,dv + r^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2).$$

The following basic properties of Minkowski space are used. The spacetime volume form of Minkowski space can be written

$$d\,\mathrm{Vol}_{\mathcal{M}} = r^2\sin\vartheta\, dt \wedge dr \wedge d\vartheta \wedge d\varphi = \tfrac{1}{2}r^2\sin\vartheta\, du \wedge dv \wedge d\vartheta \wedge d\varphi,$$

and so the Hodge star operator $*$ in particular satisfies

$$*(r^2 \sin\vartheta\, dr \wedge d\vartheta \wedge d\varphi) = -dt, \quad *(r^2 \sin\vartheta\, dt \wedge d\vartheta \wedge d\varphi) = -dr, \tag{B-2}$$

$$*(r^2 \sin\vartheta\, du \wedge d\vartheta \wedge d\varphi) = du, \quad *(r^2 \sin\vartheta\, dv \wedge d\vartheta \wedge d\varphi) = -dv. \tag{B-3}$$

The normals to the hypersurfaces $\Sigma(\tau)$ and $\underline{C}_v$ take the form

$$n_{\Sigma_\tau} = \partial_v + \iota_{r\le R}\partial_u, \quad n_{\underline{C}_v} = \partial_u. \tag{B-4}$$

B.1. ***The $J^{V,w,q,\varpi}$ current.*** In the case that $(\mathcal{M}, g_0)$ is Minkowski space, the tuple (V, w, q, ϖ) of Section 3.4.1 can be defined as follows. Consider

$$V_1 = T = \partial_u + \partial_v, \quad w_1 = 0, \quad q_1 = 0, \quad \varpi_1 = 0,$$

$$V_2 = \delta_1 T = \delta_1(\partial_u + \partial_v), \quad w_2 = 0, \quad q_2 = 0, \quad \varpi_2 = -\frac{\delta_1}{2} r^{-1} r^2 \sin\vartheta\, d\vartheta \wedge d\varphi,$$

$$V_3 = \frac{\delta_2}{2}\Big(1 - \frac{\delta_3}{(1+r)^\delta}\Big)(\partial_v - \partial_u), \quad w_3 = \frac{\delta_2}{r}\Big(1 - \frac{\delta_3}{(1+r)^\delta}\Big), \quad (q_3)_\mu = -\frac{\partial_{x^\mu} w_3}{2}, \quad \varpi_3 = 0,$$

for appropriate

$$0 < \delta_3 \ll \delta_2 \ll \delta_1 \ll 1, \tag{B-5}$$

and define

$$V = \sum_{i=1}^{3} V_i, \quad w = \sum_{i=1}^{3} w_i, \quad q = \sum_{i=1}^{3} q_i, \quad \varpi = \sum_{i=1}^{3} \varpi_i, \tag{B-6}$$

and then, for a given ψ, define currents $J_\mu^{V_i,w_i,q_i,\varpi_i}[\psi]$ and $J_\mu^{V,w,q,\varpi}[\psi]$ by (3-11). These satisfy the boundedness properties (3-16). (Note that the current $J_\mu^{V_2,w_2,q_2,\alpha_2}[g_0, \psi]$ can be viewed as arising from contracting the twisted energy momentum tensor, defined in (A-8), with the Killing vector field $\delta_1 T$.)

Proposition B.1.1 (the $J^{V,w,q,\varpi}$ current in Minkowski space). *With (V, w, q, ϖ) defined as above, if δ_1, δ_2 and δ_3 are chosen according to (B-5), the current $J_\mu^{V,w,q,\varpi}[\psi]$ satisfies the coercivity relations (3-18), and the corresponding $K^{V,w,q}[\psi]$ satisfies the coercivity relation (3-17). More precisely,*

$$J_\mu^{V,w,q,\varpi}[\psi] n^\mu_{\Sigma_\tau} \gtrsim (\partial_v\psi)^2 + |\nabla\!\!\!/\psi|^2 + r^{-2}|\partial_v(r\psi)|^2 + \iota_{r\le R}((\partial_u\psi)^2 + r^{-2}|\partial_u(r\psi)|^2) + (1+r)^{-2}\psi^2, \tag{B-7}$$

$$J_\mu^{V,w,q,\varpi}[\psi] n^\mu_{\underline{C}_v} \gtrsim (\partial_u\psi)^2 + r^{-2}|\partial_u(r\psi)|^2 + |\nabla\!\!\!/\psi|^2 + (1+r)^{-2}\psi^2, \tag{B-8}$$

$$K^{V,w,q}[\psi] \gtrsim r^{-1}|\nabla\!\!\!/\psi|^2 + (1+r)^{-1-\delta}((\partial_u\psi)^2 + (\partial_v\psi)^2) + (1+r)^{-3-\delta}\psi^2. \tag{B-9}$$

Moreover, the corresponding $H^{V,w}[\psi]$ satisfies

$$|H^{V,w}[\psi]| \lesssim |\partial_u\psi| + |\partial_v\psi| + \frac{1}{r}|\psi|. \tag{B-10}$$

Proof. Recalling (B-2) and (B-3), note first that

$$
\begin{aligned}
*d(\psi^2\varpi_2) &= -\delta_1\left(\frac{\psi^2}{r^2}+\frac{2}{r}\psi\,\partial_r\psi\right)dt-\delta_1\frac{2}{r^2}\psi\,\partial_t\psi\, x^i\,dx^i\\
&=\delta_1\left(-\frac{\psi^2}{2r^2}+\frac{2}{r}\psi\,\partial_u\psi\right)du-\delta_1\left(\frac{\psi^2}{2r^2}+\frac{2}{r}\psi\,\partial_v\psi\right)dv.
\end{aligned}
$$

Recall the expressions (B-4) for the normals to the hypersurfaces Σ_τ and $\underline{C}_v$. The fluxes corresponding to $J^{V_i,w_i,q_i,\varpi_i}[\psi]$ satisfy

$$
\begin{aligned}
J_\mu^{V_1,w_1,q_1,\varpi_1}[\psi]n^\mu_{\Sigma_\tau} &= (\partial_v\psi)^2+\tfrac14|\nabla\!\!\!/\psi|^2+\iota_{r\le R}(\partial_u\psi)^2,\\
J_\mu^{V_1,w_1,q_1,\varpi_1}[\psi]n^\mu_{\underline{C}_v} &= (\partial_u\psi)^2+\tfrac14|\nabla\!\!\!/\psi|^2,\\
J_\mu^{V_2,w_2,q_2,\varpi_2}[\psi]n^\mu_{\Sigma_\tau} &= \frac{\delta_1}{r^2}(\partial_v(r\psi))^2+\frac{\delta_1}{4r^2}|\nabla\!\!\!/(r\psi)|^2+\frac{\delta_1}{r^2}\iota_{r\le R}(\partial_u(r\psi))^2\\
&=\delta_1 J_\mu^{V_1,w_1,q_1,\varpi_1}[\psi]n^\mu_{\Sigma_\tau}+\delta_1\left(\frac{1}{4r^2}\psi^2+\frac1r\psi\,\partial_v\psi+\frac{1}{4r^2}\psi^2\iota_{r\le R}-\frac1r\psi\,\partial_u\psi\,\iota_{r\le R}\right),\\
J_\mu^{V_2,w_2,q_2,\varpi_2}[\psi]n^\mu_{\underline{C}_v} &= \frac{\delta_1}{4r^2}(\partial_u(r\psi))^2+\frac{\delta_1}{8r^2}|\nabla\!\!\!/(r\psi)|^2\\
&=\delta_1 J_\mu^{V_1,w_1,q_1,\varpi_1}[\psi]n^\mu_{\underline{C}_v}+\delta_1\left(\frac{1}{4r^2}\psi^2-\frac1r\psi\,\partial_u\psi\right),\\
|J_\mu^{V_3,w_3,q_3,\varpi_3}[\psi]n^\mu_{\Sigma_\tau}| &\le \delta_2 C((\partial_v\psi)^2+|\nabla\!\!\!/\psi|^2+(1+r)^{-2}|\psi|^2+\iota_{r\le R}(\partial_u\psi)^2),\\
|J_\mu^{V_3,w_3,q_3,\varpi_3}[\psi]n^\mu_{\underline{C}_v}| &\le \delta_2 C((\partial_u\psi)^2+|\nabla\!\!\!/\psi|^2+(1+r)^{-2}|\psi|^2),
\end{aligned}
$$

and the bulk terms satisfy

$$
\begin{aligned}
K^{V_1,w_1,q_1}[\psi]&=0,\\
H^{V_1,w_1}[\psi]&=(\partial_u\psi+\partial_v\psi),\\
K^{V_2,w_2,q_2}[\psi]&=0,\\
H^{V_2,w_2}[\psi]&=\delta_1(\partial_u\psi+\partial_v\psi),\\
K^{V_3,w_3,q_3}[\psi]&=\delta_2\left(\frac1r-\delta_3\left(\frac{1}{r(1+r)^\delta}+\frac{\delta}{2(1+r)^{1+\delta}}\right)\right)|\nabla\!\!\!/\psi|^2\\
&\qquad+\frac{\delta_2\delta_3\delta}{2(1+r)^{1+\delta}}((\partial_t\psi)^2+(\partial_r\psi)^2)+\frac{\delta_2\delta_3\delta(1+\delta)}{2r(1+r)^{2+\delta}}\psi^2,\\
H^{V_3,w_3}[\psi]&=\frac{\delta_2}{2}\left(1-\frac{\delta_3}{(1+r)^\delta}\right)(\partial_v\psi-\partial_u\psi)+\frac{\delta_2}{r}\left(1-\frac{\delta_3}{(1+r)^\delta}\right)\psi.
\end{aligned}
$$

It thus follows that the coercivity relations (B-7)–(B-9), along with the property (B-10), hold if δ_1, δ_2 and δ_3 are chosen according to (B-5). □

B.2. ***The J_{far} current.*** The quadruples $(\overset{(p)}{V}{}^{\circ}_{\mathrm{far}}, \overset{(p)}{w}{}^{\circ}_{\mathrm{far}}, \overset{(p)}{q}{}^{\circ}_{\mathrm{far}}, \overset{(p)}{\varpi}{}^{\circ}_{\mathrm{far}})$ and $(\widetilde{V}_{\mathrm{far}}, \tilde{w}_{\mathrm{far}}, \tilde{q}_{\mathrm{far}}, \tilde{\varpi}_{\mathrm{far}})$ of Section 3.5 are defined as follows. Consider some

$$0 < \delta_6 \ll \delta_5 \ll \delta_4 \ll \delta_3 \ll \delta_2 \ll \delta_1 \ll 1. \tag{B-11}$$

Noting that $-\frac{1}{2}\partial_{x^\mu} u \, dx^\mu = -\frac{1}{2} du = (\partial_v)^\flat$, define

$$\overset{(p)}{V}{}^{\circ}_{\mathrm{far}} = r^p \partial_v, \quad \overset{(p)}{w}{}^{\circ}_{\mathrm{far}} = \frac{r^{p-1}}{2}, \quad (\overset{(p)}{q}{}^{\circ}_{\mathrm{far}})_\mu = -\frac{\partial_{x^\mu} \overset{(p)}{w}{}^{\circ}_{\mathrm{far}}}{2} - \left(\frac{p}{4} - \frac{\delta_4}{2}\right) r^{p-2} \partial_{x^\mu} u,$$

$$\overset{(p)}{\varpi}{}^{\circ}_{\mathrm{far}} = \left(-\frac{r^{p-1}}{4} + 2\delta_5 r^{\frac{p}{2}-1}\right) r^2 \sin\vartheta \, d\vartheta \wedge d\varphi.$$

Define then

$$\widetilde{V}_{\mathrm{far}} = \frac{1}{\delta_6} V, \quad \tilde{w}_{\mathrm{far}} = \frac{1}{\delta_6} w, \quad \tilde{q}_{\mathrm{far}} = \frac{1}{\delta_6} q, \quad \tilde{\varpi}_{\mathrm{far}} = \frac{1}{\delta_6} \varpi,$$

where V, w, q, ϖ are defined by (B-6), so that

$$\overset{(p)}{V}_{\mathrm{far}} = \overset{(p)}{V}{}^{\circ}_{\mathrm{far}} + \frac{1}{\delta_6} V, \quad \overset{(p)}{w}_{\mathrm{far}} = \overset{(p)}{w}{}^{\circ}_{\mathrm{far}} + \frac{1}{\delta_6} w, \quad \overset{(p)}{q}_{\mathrm{far}} = \overset{(p)}{q}{}^{\circ}_{\mathrm{far}} + \frac{1}{\delta_6} q, \quad \overset{(p)}{\varpi}_{\mathrm{far}} = \overset{(p)}{\varpi}{}^{\circ}_{\mathrm{far}} + \frac{1}{\delta_6} \varpi.$$

Proposition B.2.1 (the J_{far} current in Minkowski space). *With $(\overset{(p)}{V}_{\mathrm{far}}, \overset{(p)}{w}_{\mathrm{far}}, \overset{(p)}{q}_{\mathrm{far}}, \overset{(p)}{\varpi}_{\mathrm{far}})$ defined as above, if $\delta_1, \dots, \delta_6$ are chosen according to* (B-11), *the associated currents $\overset{(p)}{J}_{\mathrm{far}}$, $\overset{(p)}{K}_{\mathrm{far}}$, defined by* (3-11), (3-12), *satisfy the weighted bulk coercivity property* (3-40) *and the weighted boundary coercivity properties* (3-41). *More precisely, for $\delta \le p \le 2 - \delta$ and $r \ge R$,*

$$\overset{(p)}{J}_{\mathrm{far}\,\mu}[\psi] n^\mu_{\Sigma_\tau} \gtrsim r^{p-2} |\partial_v (r\psi)|^2 + r^{\frac{p}{2}} (\partial_v \psi)^2 + |\nabla\!\!\!/ \psi|^2 + r^{\frac{p}{2}-2} \psi^2, \tag{B-12}$$

$$\overset{(p)}{J}_{\mathrm{far}\,\mu}[\psi] n^\mu_{\underline{C}_v} \gtrsim (\partial_u \psi)^2 + r^p |\nabla\!\!\!/ \psi|^2 + r^{p-2} \psi^2, \tag{B-13}$$

$$\overset{(p)}{K}_{\mathrm{far}}[\psi] \gtrsim r^{p-3} |\partial_v (r\psi)|^2 + r^{p-1} |\partial_v \psi|^2 + r^{p-1} |\nabla\!\!\!/ \psi|^2 + r^{-1-\delta} |\partial_u \psi|^2 + r^{p-3} \psi^2. \tag{B-14}$$

Moreover, the corresponding $\overset{(p)}{H}_{\mathrm{far}}[\psi]$ satisfies

$$|\overset{(p)}{H}_{\mathrm{far}}[\psi]| \lesssim r^{p-1} |\partial_v (r\psi)| + |\partial_u \psi| + |\partial_v \psi| + \frac{1}{r} |\psi|. \tag{B-15}$$

Proof. Recall again (B-3) and note that

$$\begin{aligned} * d(\psi^2 \overset{(p)}{\varpi}{}^{\circ}_{\mathrm{far}}) = &\left(\left(\frac{p+1}{8} r^{p-2} - \delta_5 \frac{p+2}{2} r^{\frac{p}{2}-2}\right) \psi^2 - \left(\frac{r^{p-1}}{2} - 4\delta_5 r^{\frac{p}{2}-1}\right) \psi \partial_u \psi\right) du \\ &+ \left(\left(\frac{p+1}{8} r^{p-2} - \delta_5 \frac{p+2}{2} r^{\frac{p}{2}-2}\right) \psi^2 + \left(\frac{r^{p-1}}{2} - 4\delta_5 r^{\frac{p}{2}-1}\right) \psi \partial_v \psi\right) dv. \end{aligned}$$

Note moreover that

$$(\pi^{\overset{(p)}{V}{}^{\circ}_{\mathrm{far}}})^{\sharp\sharp} = p r^{p-1} \partial_v \otimes \partial_v - \frac{p r^{p-1}}{2} (\partial_u \otimes \partial_v + \partial_v \otimes \partial_u) + \frac{r^{p-3}}{2} (\partial_\vartheta \otimes \partial_\vartheta + \sin^{-2}\vartheta \, \partial_\varphi \otimes \partial_\varphi)$$

and

$$\nabla^\mu (\overset{(p)}{q}{}^{\circ}_{\mathrm{far}})_\mu = \frac{p r^{p-3}}{4} - \frac{\delta_4 p r^{p-3}}{2}.$$

The fluxes corresponding to $J^{\overset{(p)}{\mathring{V}}_{\text{far}},\overset{(p)}{\mathring{w}}_{\text{far}},\overset{(p)}{\mathring{q}}_{\text{far}},\overset{(p)}{\mathring{\varpi}}_{\text{far}}}[\psi]$ satisfy, for $r \geq R$,

$$J_\mu^{\overset{(p)}{\mathring{V}}_{\text{far}},\overset{(p)}{\mathring{w}}_{\text{far}},\overset{(p)}{\mathring{q}}_{\text{far}},\overset{(p)}{\mathring{\varpi}}_{\text{far}}}[\psi] n^\mu_{\Sigma_\tau} = r^{p-2}|\partial_v(r\psi)|^2 - \delta_5\left(\frac{p+2}{2} r^{\frac{p}{2}-2}\psi^2 + 4r^{\frac{p}{2}-1}\psi\partial_v\psi\right)$$

$$= r^{p-2}|\partial_v(r\psi)|^2 + \delta_5\left(\frac{(2-p)}{2} r^{\frac{p}{2}-2}\psi^2 - 4r^{\frac{p}{2}-2}\psi\partial_v(r\psi)\right), \tag{B-16}$$

$$J_\mu^{\overset{(p)}{\mathring{V}}_{\text{far}},\overset{(p)}{\mathring{w}}_{\text{far}},\overset{(p)}{\mathring{q}}_{\text{far}},\overset{(p)}{\mathring{\varpi}}_{\text{far}}}[\psi] n^\mu_{\underline{C}_v} = \frac{r^p}{4}|\nabla\!\!\!/\psi|^2 + \frac{\delta_4}{2} r^{p-2}\psi^2 - \delta_5\left(\frac{p+2}{2} r^{\frac{p}{2}-2}\psi^2 - 4r^{\frac{p}{2}-1}\psi\partial_u\psi\right), \tag{B-17}$$

and the bulk terms satisfy

$$K^{\overset{(p)}{\mathring{V}}_{\text{far}},\overset{(p)}{\mathring{w}}_{\text{far}},\overset{(p)}{\mathring{q}}_{\text{far}}}[\psi] = pr^{p-3}|\partial_v(r\psi)|^2 + \frac{(2-p)r^{p-1}}{4}|\nabla\!\!\!/\psi|^2 + \frac{\delta_4(2-p)}{2} r^{p-3}\psi^2 - 2\delta_4 r^{p-3}\psi\partial_v(r\psi), \tag{B-18}$$

$$H^{\overset{(p)}{\mathring{V}}_{\text{far}},\overset{(p)}{\mathring{w}}_{\text{far}}}[\psi] = r^{p-1}\partial_v(r\psi). \tag{B-19}$$

In particular, if δ_4 and δ_5 are chosen according to (B-11) (depending on p), then

$$J_\mu^{\overset{(p)}{\mathring{V}}_{\text{far}},\overset{(p)}{\mathring{w}}_{\text{far}},\overset{(p)}{\mathring{q}}_{\text{far}},\overset{(p)}{\mathring{\varpi}}_{\text{far}}}[\psi] n^\mu_{\Sigma_\tau} \gtrsim r^{p-2}|\partial_v(r\psi)|^2 + r^{\frac{p}{2}}(\partial_v\psi)^2 + r^{\frac{p}{2}-2}\psi^2, \tag{B-20}$$

$$J_\mu^{\overset{(p)}{\mathring{V}}_{\text{far}},\overset{(p)}{\mathring{w}}_{\text{far}},\overset{(p)}{\mathring{q}}_{\text{far}},\overset{(p)}{\mathring{\varpi}}_{\text{far}}}[\psi] n^\mu_{\underline{C}_v} + (\partial_u\psi)^2 \gtrsim r^p|\nabla\!\!\!/\psi|^2 + r^{p-2}\psi^2, \tag{B-21}$$

and

$$K^{\overset{(p)}{\mathring{V}}_{\text{far}},\overset{(p)}{\mathring{w}}_{\text{far}},\overset{(p)}{\mathring{q}}_{\text{far}}}[\psi] \gtrsim pr^{p-3}|\partial_v(r\psi)|^2 + (2-p)r^{p-1}|\nabla\!\!\!/\psi|^2 + p(2-p)r^{p-1}|\partial_v\psi|^2 + (2-p)r^{p-3}\psi^2. \tag{B-22}$$

It follows from Proposition B.1.1 that the currents $\overset{(p)}{J}_{\text{far}}$, $\overset{(p)}{K}_{\text{far}}$ satisfy the weighted bulk coercivity properties (B-12)–(B-14), provided $\delta_1, \dots, \delta_6$ are chosen according to (B-11), and that $\overset{(p)}{H}_{\text{far}}$ satisfies (B-15). □

The inequalities (B-10) and (B-15) in particular mean that the $r^{-1}|\nabla\!\!\!/\psi|$ term on the left-hand side of the assumed inequality (4-30) is superfluous in the case that $(\mathcal{M}, g_0)$ is Minkowski space. This term is estimated in the proof of Proposition 4.7.2 nonetheless in order to illustrate how it is estimated in the case of Kerr.

Appendix C: Verifying the null condition assumption

In this section the proof of Proposition 4.7.2 is given. Proposition 4.7.2 can be more precisely stated as follows.

Proposition C.1. *Assumption 4.7.1 holds for the classical null condition of Klainerman [1986] on Minkowski space and more generally the class of equations on Kerr considered in Luk [2013].*

More precisely, if $(\mathcal{M}, g_0)$ is either Minkowski space or a member of the Kerr family and if

$$F = N^{\mu\nu}(\psi, x)\partial_\mu\psi\partial_\nu\psi$$

satisfies the assumption (C-22), then there exists $k_{\text{null}} > 0$ and, for all $k \geq k_{\text{null}}$, there exists $\varepsilon_{\text{null}} > 0$ such that we have the following.

Let ψ be a smooth function in $\mathcal{R}(\tau_0, \tau_1, v)$ satisfying (4-29) for $0 < \varepsilon \leq \varepsilon_{\mathrm{null}}$. Then, for all $\delta \leq p \leq 2-\delta$,

$$\sum_{|\boldsymbol{k}|\leq k} \int_{\mathcal{R}(\tau_0,\tau',v)\cap\{r\geq R\}} \Big(|\underline{L}\mathfrak{D}^{\boldsymbol{k}}\psi| + |L\mathfrak{D}^{\boldsymbol{k}}\psi| + \frac{1}{r}|\nabla\!\!\!/\,\mathfrak{D}^{\boldsymbol{k}}\psi| + \frac{1}{r}|\mathfrak{D}^{\boldsymbol{k}}\psi|\Big)|\mathfrak{D}^{\boldsymbol{k}}F| + \int_{\mathcal{R}(\tau_0,\tau',v)\cap\{r\geq R\}} (\mathfrak{D}^{\boldsymbol{k}}F)^2 \lesssim \sqrt{\overset{(0+)}{\mathcal{X}}_{\ll k\, 8R/9}(\tau_0,\tau_1)\overset{(0+)}{\mathcal{X}}_{k\, R}(\tau_0,\tau_1)}$$

and, moreover,

$$\sum_{|\boldsymbol{k}|\leq k} \int_{\mathcal{R}(\tau_0,\tau',v)\cap\{r\geq R\}} \Big(r^{p-1}|L(r\mathfrak{D}^{\boldsymbol{k}}\psi)| + |\underline{L}\mathfrak{D}^{\boldsymbol{k}}\psi| + |L\mathfrak{D}^{\boldsymbol{k}}\psi| + \frac{1}{r}|\nabla\!\!\!/\,\mathfrak{D}^{\boldsymbol{k}}\psi| + \frac{1}{r}|\mathfrak{D}^{\boldsymbol{k}}\psi|\Big)|\mathfrak{D}^{\boldsymbol{k}}F|$$
$$+ \int_{\mathcal{R}(\tau_0,\tau',v)\cap\{r\geq R\}} (\mathfrak{D}^{\boldsymbol{k}}F)^2 \lesssim \sqrt{\overset{(0)}{\mathcal{X}}_{\ll k\, 8R/9}(\tau_0,\tau_1)\overset{(p)}{\mathcal{X}}_{k\, R}(\tau_0,\tau_1)} + \sqrt{\overset{(p)}{\mathcal{X}}_{\ll k\, R}(\tau_0,\tau_1)}\sqrt{\overset{(p)}{\mathcal{X}}_{k\, R}(\tau_0,\tau_1)}\sqrt{\overset{(0)}{\mathcal{X}}_{k\, R}(\tau_0,\tau_1)}.$$

The estimate of Proposition C.1 is only nontrivial for large r and, thus, the main content appears already around Minkowski space. Accordingly, the proof will be given in detail for the case that $(\mathcal{M}, g_0)$ is Minkowski space for the two model nonlinearities

$$N^{\mu\nu}(\psi, x)\partial_\mu\psi\partial_\nu\psi = \partial_u\psi\partial_v\psi,$$
$$N^{\mu\nu}(\psi, x)\partial_\mu\psi\partial_\nu\psi = g\!\!\!/^{AB}\nabla\!\!\!/_A\psi\nabla\!\!\!/_B\psi,$$

where $g\!\!\!/ = r^2\gamma$ is the induced round metric and $\nabla\!\!\!/$ its associated connection. See Section C.1. The more general nonlinearities and the Kerr case are discussed in Section C.2.

C.1. ***Two model nonlinearities on Minkowski space.*** Throughout this section, we will write $(\mathcal{M}, g_0)$ to denote Minkowski space (see Section 2.7.1). Note that, in the region $r \geq \frac{8}{9}R$ which will be considered here, the r of (2-1) coincides with the standard radial coordinate (B-1) used in Appendix B and thus coincides with what was denoted as $\tilde{r}$ in Section 2.7.1.

Recall the (u, v, θ, ϕ) coordinate system. In the region $r \geq R$ the null vector fields take the form

$$L = \partial_v, \quad \underline{L} = \partial_u.$$

The spheres $\Sigma(\tau) \cap \underline{C}_v$ are round, $g\!\!\!/ = r^2\gamma$ denotes the induced round metric and $\nabla\!\!\!/$ its associated connection. Define

$$\overset{(0)}{\mathcal{F}}{}^*_k := \sup_{\tau_0\leq\tau\leq\tau_1} \overset{(0)}{\mathcal{F}}_k(\tau),$$
$$\overset{(p)}{\mathcal{E}}{}^*_k := \sup_{\tau_0\leq\tau\leq\tau_1} \overset{(p)}{\mathcal{E}}_k(\tau),$$
$$\int^* \overset{(p-1)}{\mathcal{E}'_k} := \int_{\tau_0}^{\tau_1} \overset{(p-1)}{\mathcal{E}'_k}(\tau)\, d\tau,$$

and similarly for $\ll k$ replacing k, and with R subscripts added, etc. This notation will be used throughout this section.

The main result of this section is the following.

Proposition C.1.1 (model nonlinearities on Minkowski space). *Fix $\delta \le p \le 2-\delta$ and $k \ge 7$, and let ψ be a smooth function in $\mathcal{R}(\tau_0, \tau_1, v) \cap \{r \ge \frac{8}{9}R\}$ satisfying (4-29).*

Then the nonlinearities

$$F = \partial_u\psi\,\partial_v\psi \quad \text{and} \quad F = \not{g}^{AB}\not\nabla_A\psi\not\nabla_B\psi$$

both satisfy

$$\sum_{|\boldsymbol{k}|\le k}\int_{\mathcal{R}(\tau_0,\tau',v)\cap\{r\ge R\}}\Big(|\partial_u\mathfrak{D}^{\boldsymbol{k}}\psi| + |\partial_v\mathfrak{D}^{\boldsymbol{k}}\psi| + \frac{1}{r}|\not\nabla\mathfrak{D}^{\boldsymbol{k}}\psi| + \frac{1}{r}|\mathfrak{D}^{\boldsymbol{k}}\psi|\Big)|\mathfrak{D}^{\boldsymbol{k}}F| + \int_{\mathcal{R}(\tau_0,\tau',v)\cap\{r\ge R\}}(\mathfrak{D}^{\boldsymbol{k}}F)^2$$
$$\lesssim \sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{F}}}{}^{*}_{R} + \overset{(0)}{\underset{\ll k}{\mathcal{E}}}{}^{*}_{8R/9} + \int^{*}\overset{(-1-\delta)}{\underset{\ll k}{\mathcal{E}'}}_{R}}\Big(\overset{(0)}{\underset{k}{\mathcal{F}}}{}^{*}_{R} + \overset{(0)}{\underset{k}{\mathcal{E}}}{}^{*}_{R} + \int^{*}\overset{(\delta-1)}{\underset{k}{\mathcal{E}'}}_{R}\Big) + \sqrt{\int^{*}\overset{(\delta-1)}{\underset{\ll k}{\mathcal{E}'}}_{R}}\Big(\overset{(0)}{\underset{k}{\mathcal{F}}}{}^{*}_{R} + \int^{*}\overset{(-1-\delta)}{\underset{k}{\mathcal{E}}}_{R}\Big)$$

and

$$\sum_{|\boldsymbol{k}|\le k}\int_{\mathcal{R}(\tau_0,\tau',v)\cap\{r\ge R\}}\Big(r^{p-1}|\partial_v(r\mathfrak{D}^{\boldsymbol{k}}\psi)| + |\partial_u\mathfrak{D}^{\boldsymbol{k}}\psi| + |\partial_v\mathfrak{D}^{\boldsymbol{k}}\psi| + \frac{1}{r}|\not\nabla\mathfrak{D}^{\boldsymbol{k}}\psi| + \frac{1}{r}|\mathfrak{D}^{\boldsymbol{k}}\psi|\Big)|\mathfrak{D}^{\boldsymbol{k}}F|$$
$$+ \int_{\mathcal{R}(\tau_0,\tau',v)\cap\{r\ge R\}}(\mathfrak{D}^{\boldsymbol{k}}F)^2$$
$$\lesssim \sqrt{\int^{*}\overset{(p-1)}{\underset{\ll k}{\mathcal{E}'}}_{R}}\sqrt{\overset{(0)}{\underset{k}{\mathcal{F}}}{}^{*}_{R}}\sqrt{\int^{*}\overset{(p-1)}{\underset{k}{\mathcal{E}'}}_{R}} + \sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{F}}}{}^{*}_{R} + \overset{(0)}{\underset{\ll k}{\mathcal{E}}}{}^{*}_{8R/9} + \int^{*}\overset{(-1-\delta)}{\underset{\ll k}{\mathcal{E}'}}_{R}}\Big(\overset{(0)}{\underset{k}{\mathcal{F}}}{}^{*}_{R} + \overset{(p)}{\underset{k}{\mathcal{E}}}{}^{*}_{R} + \int^{*}\overset{(p-1)}{\underset{k}{\mathcal{E}'}}_{R}\Big).$$

The following properties of Minkowski space will be used in the proof of Proposition C.1.1. First, for any $\boldsymbol{k}$ and any function ψ, for $r \ge R$,

$$|\mathfrak{D}^{\boldsymbol{k}}L\psi| \lesssim \sum_{|\tilde{\boldsymbol{k}}|\le|\boldsymbol{k}|}|L\mathfrak{D}^{\tilde{\boldsymbol{k}}}\psi|, \qquad |\mathfrak{D}^{\boldsymbol{k}}\underline{L}\psi| \lesssim \sum_{|\tilde{\boldsymbol{k}}|\le|\boldsymbol{k}|}|\underline{L}\mathfrak{D}^{\tilde{\boldsymbol{k}}}\psi|. \tag{C-1}$$

Indeed, (C-1) follows from $L = \partial_v$, $\underline{L} = \partial_u$, $T = N = \partial_u + \partial_v$, and $[\partial_u, \Omega_i] = [\partial_v, \Omega_i] = 0$ for $i = 1, 2, 3$. Thus ∂_u and ∂_v commute with all components of $\mathfrak{D}$. Second, the fact that, for any $\boldsymbol{k}$ and any function ψ, for $r \ge R$,

$$|\mathfrak{D}^{\boldsymbol{k}}(\not{g}^{AB}\not\nabla_A\psi\not\nabla_B\psi)| \lesssim \sum_{|\boldsymbol{k}_1|+|\boldsymbol{k}_2|\le|\boldsymbol{k}|}|\not\nabla\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\not\nabla\mathfrak{D}^{\boldsymbol{k}_2}\psi| \tag{C-2}$$

will also be used.

In the above and what follows, note that when the volume form is omitted, the usual spacetime volume form is to be understood, and this contains in particular an omitted $r^2\sin\vartheta$ factor.

The proof of Proposition C.1.1 relies on weighted Sobolev inequalities, which are discussed in Section C.1.1. In Section C.1.2 the proof of Proposition C.1.1 for the case of the nonlinearity $F = \partial_u\psi\,\partial_v\psi$ is given, followed by the proof for the nonlinearity $F = \not{g}^{AB}\not\nabla_A\psi\not\nabla_B\psi$ in Section C.1.3.

C.1.1. *Weighted Sobolev inequalities.* The proof of Proposition C.1 relies on certain weighted Sobolev inequalities.

Let $d\omega$ denote the unit volume form on S^2:

$$d\omega = \sin\vartheta\, d\vartheta\, d\varphi.$$

Recall (see Section 2.7.1) that $r(u,v)=\frac{1}{2}(v-u)$. Define $v(R)=v(R,u)=2R+u$, so that we have $r(u,v(R,u))=R$, and similarly define $u(R,v)=v-2R$, so that $r(u(R,v),v)=R$.

Recall the region $\mathcal{R}(\tau_0,\tau_1,v)$. In order to avoid the blowup of constants in Sobolev inequalities, the region near the corner $\{u=\tau_0\}\cap\{r=R\}$ will typically be considered separately from its complement. Accordingly, define

$$\mathcal{R}_R(\tau_0,\tau_1,v):=\mathcal{R}(\tau_0,\tau_1,v)\cap\{r\geq R\},\quad \check{\mathcal{R}}_R(\tau_0,\tau_1,v):=\mathcal{R}_R(\tau_0,\tau_1,v)\setminus\{v'\leq v(R,\tau_0+R)\},\tag{C-3}$$

and

$$\mathcal{R}_{\text{Corner}}(\tau_0,\tau_1,v):=\mathcal{R}(\tau_0,\tau_1,v)\cap\{r\geq R\}\cap\{v'\leq v(R,\tau_0+R)\}.\tag{C-4}$$

Note that

$$r\leq\tfrac{1}{2}3R\quad\text{in }\mathcal{R}_{\text{Corner}}(\tau_0,\tau_1,v).\tag{C-5}$$

Define also

$$T_1=T_1(v)=\min\{\tau_1,u(R,v)\},\quad T_1'=T_1(v'),\tag{C-6}$$

and

$$\Sigma_R(\tau,v)=\Sigma(\tau,v)\cap\{r\geq R\}.$$

Proposition C.1.2 (Sobolev inequality on incoming cones $\underline{C}_v$). *For any $\tau_0\leq u\leq\tau_1,\ v(R,\tau_0+R)\leq v'\leq v$, $(\vartheta,\varphi)\in S^2$ (i.e., for any $(u,v',\vartheta,\varphi)\in\check{\mathcal{R}}_R(\tau_0,\tau_1,v)$), and any function $f:\mathcal{R}(\tau_0,\tau_1,v)\to\mathbb{R}$,*

$$r|f(u,v',\vartheta,\varphi)|$$
$$\lesssim(1+(\tau_1-\tau_0)^{-\frac{1}{2}})\sum_{k\leq 1}\biggl(\int_{\tau_0}^{T_1'}\int_{S^2}\Bigl(|\underline{L}^kf|^2+\sum_{i=1}^{3}|\Omega_i\underline{L}^kf|^2+\sum_{i,j=1}^{3}|\Omega_i\Omega_j\underline{L}^kf|^2\Bigr)(u',v',\vartheta,\varphi)r^2\,d\omega\,du'\biggr)^{\frac{1}{2}}.$$

Proof. For simplicity, consider first the case that $v'\geq v(R,\tau_1)$, so that $T_1'=\tau_1$. Suppose first that $u\geq\frac{1}{2}(\tau_1+\tau_2)$. Let ϕ be a smooth cut-off function such that $\phi(\tau_1)=0$, $\phi(\tau)=1$ for $\tau\geq\frac{1}{2}(\tau_1+\tau_2)$, and

$$|\phi'(\tau)|\leq\frac{4}{\tau_2-\tau_1}\quad\text{for }\tau_1\leq\tau\leq\frac{\tau_1+\tau_2}{2}.$$

Now

$$\partial_u(\phi(u)|f(u,v',\vartheta,\varphi)|^2)=\phi'(u)|f(u,v',\vartheta,\varphi)|^2+2\phi(u)f(u,v',\vartheta,\varphi)\partial_uf(u,v',\vartheta,\varphi).$$

Integrating from τ_0 to u gives

$$r^2|f(u,v',\vartheta,\varphi)|^2\lesssim(1+(\tau_1-\tau_0)^{-1})\int_{\tau_0}^{\tau_1}|f(u',v',\vartheta,\varphi)|^2r^2\,du'+\int_{\tau_0}^{\tau_1}|\partial_uf(u',v',\vartheta,\varphi)|^2r^2\,du'.$$

The result then follows from the standard Sobolev inequality on S^2,

$$\sup_{(\vartheta,\varphi)\in S^2}|f(u',v',\vartheta,\varphi)|\lesssim r^{-1}\biggl(\int_{S^2}\Bigl(|f|^2+\sum_{i=1}^{3}|\Omega_if|^2+\sum_{i,j=1}^{3}|\Omega_i\Omega_jf|^2\Bigr)(u',v',\vartheta,\varphi)r^2\sin\vartheta\,d\vartheta\,d\varphi\biggr)^{\frac{1}{2}}.$$

Similarly for $u\leq\frac{1}{2}(\tau_1+\tau_2)$, and similarly for $v(R,\tau_0+R)\leq v'\leq v(R,\tau_1)$. □

Proposition C.1.2 in particular implies that

$$\sup_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)} \sum_{|\boldsymbol{k}|\leq k-3} (r|\partial_u \mathfrak{D}^{\boldsymbol{k}}\psi| + r|\nabla\!\!\!/(\mathfrak{D}^{\boldsymbol{k}}\psi)|) \lesssim (1+(\tau_1-\tau_0)^{-\frac{1}{2}})\sqrt{\overset{(0)}{\underset{k}{\mathcal{F}}}{}^*} \tag{C-7}$$

and, for any $s \geq 0$ and any $\tau_0 \leq u \leq \tau_1$, $v(R, \tau_0+R) \leq v' \leq v$, $(\vartheta,\varphi) \in S^2$,

$$r^s|\partial_v \mathfrak{D}^{\boldsymbol{k}}\psi(u,v',\vartheta,\varphi)| \lesssim (1+(\tau_1-\tau_0)^{-\frac{1}{2}}) \sum_{|\tilde{\boldsymbol{k}}|\leq|\boldsymbol{k}|+3} \left(\int_{\tau_0}^{T_1'}\int_{S^2} r^{s-2}|\partial_v \mathfrak{D}^{\boldsymbol{k}}\psi(u',v',\vartheta,\varphi)|r^2\,d\omega\,du'\right)^{\frac{1}{2}}. \tag{C-8}$$

It is in the forms (C-7) and (C-8) that Proposition C.1.2 will be used below.

Proposition C.1.3 (Sobolev inequality on $\Sigma(\tau)$). *For any $\tau_0 \leq \tau \leq \tau_1$, $v(R,u) \leq v' \leq v$, $(\vartheta,\varphi) \in S^2$, and any function $f : \mathcal{R}(\tau_0,\tau_1,v) \to \mathbb{R}$,*

$$|f(\tau,v',\vartheta,\varphi)| \lesssim \sum_{k=0}^{1}\left(\int_{\Sigma_R(\tau,v)} r^{-2}\left(|L^k f|^2 + \sum_{i=1}^{3}|\Omega_i L^k f|^2 + \sum_{i,j=1}^{3}|\Omega_i\Omega_j L^k f|^2\right) r^2\,d\omega\,dv'\right)^{\frac{1}{2}} + \left(\int_{\Sigma_{8R/9}(\tau,v)} r^{-2}|f|^2 r^2\,d\omega\,dv'\right)^{\frac{1}{2}}.$$

Proof. The proof follows from the fundamental theorem of calculus and the standard Sobolev inequality on S^2, as in the proof of Proposition C.1.2. □

Proposition C.1.3 in particular implies that

$$\sup_{\mathcal{R}_{\text{Corner}}(\tau_0,\tau_1,v)} \sum_{|\boldsymbol{k}|\leq k-4} |\mathfrak{D}^{\boldsymbol{k}}\psi| \lesssim \sqrt{\overset{(0)}{\underset{k}{\mathcal{E}}}{}^*_{8R/9}}, \tag{C-9}$$

where the region $\mathcal{R}_{\text{Corner}}$ is defined in (C-4). It is in the form (C-9) that Proposition C.1.3 will typically be used.

C.1.2. *The proof of Proposition C.1.1 for the nonlinearity $F = \partial_u\psi\partial_v\psi$.* Proposition C.1.1 for the nonlinearity

$$F = \partial_u\psi\partial_v\psi$$

follows from the following proposition, whose proof is the subject of the present section.

Proposition C.1.4 (nonlinear error estimates for $F = \partial_u\psi\partial_v\psi$ on Minkowski space). *Under the assumptions of Proposition C.1.1, for each $|\boldsymbol{k}| \leq k$,*

$$\sum_{|\boldsymbol{k}_1|+|\boldsymbol{k}_2|\leq|\boldsymbol{k}|}\Bigg(\int_{\mathcal{R}(\tau_0,\tau_1,v)\cap\{r\geq R\}}\left(|\partial_u\mathfrak{D}^{\boldsymbol{k}}\psi| + |\partial_v\mathfrak{D}^{\boldsymbol{k}}\psi| + \frac{1}{r}|\nabla\!\!\!/\mathfrak{D}^{\boldsymbol{k}}\psi| + \frac{1}{r}|\mathfrak{D}^{\boldsymbol{k}}\psi|\right)\cdot|\mathfrak{D}^{\boldsymbol{k}_1}\partial_u\psi|\cdot|\mathfrak{D}^{\boldsymbol{k}_2}\partial_v\psi| + \int_{\mathcal{R}(\tau_0,\tau_1,v)\cap\{r\geq R\}}|\mathfrak{D}^{\boldsymbol{k}_1}\partial_u\psi|^2\cdot|\mathfrak{D}^{\boldsymbol{k}_2}\partial_v\psi|^2\Bigg)$$

$$\lesssim \sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{F}}}{}^*_R + \int^*\overset{(-1-\delta)}{\underset{\ll k}{\mathcal{E}'}}_R}\left(\overset{(0)}{\underset{k}{\mathcal{F}}}{}^*_R + \overset{(0)}{\underset{k}{\mathcal{E}}}{}^*_R + \int^*\overset{(\delta-1)}{\underset{k}{\mathcal{E}'}}_R\right) + \sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{E}}}{}^*_{8R/9} + \int^*\overset{(\delta-1)}{\underset{\ll k}{\mathcal{E}'}}_R}\left(\overset{(0)}{\underset{k}{\mathcal{F}}}{}^*_R + \int^*\overset{(-1-\delta)}{\underset{k}{\mathcal{E}}}_R\right)$$

and

$$\sum_{|\boldsymbol{k}_1|+|\boldsymbol{k}_2|\le|\boldsymbol{k}|}\int_{\mathcal{R}(\tau_0,\tau_1,v)\cap\{r\ge R\}} r^{p-1}|\partial_v(r\mathfrak{D}^{\boldsymbol{k}}\psi)|\cdot|\mathfrak{D}^{\boldsymbol{k}_1}\partial_u\psi|\cdot|\mathfrak{D}^{\boldsymbol{k}_2}\partial_v\psi| \lesssim \sqrt{\int^* \overset{(p-1)}{\mathcal{E}}{}'_{\ll k\,R}}\sqrt{\overset{(0)}{\mathcal{F}}{}^*_{k\,R}}\sqrt{\int^*\overset{(p-1)}{\mathcal{E}}{}'_{k\,R}} + \sqrt{\overset{(0)}{\mathcal{F}}{}^*_{\ll k\,R}+\overset{(0)}{\mathcal{E}}{}^*_{\ll k\,8R/9}+\int^*\overset{(-1-\delta)}{\mathcal{E}}{}'_{\ll k\,R}}\left(\overset{(0)}{\mathcal{F}}{}^*_{k\,R}+\overset{(p)}{\mathcal{E}}{}^*_{k\,R}+\int^*\overset{(p-1)}{\mathcal{E}}{}'_{k\,R}\right).$$

Proof. The separate terms are estimated individually. We will use (4-29) to replace quartic bounds with cubic ones without further comment. In view of the fact (C-1), the proof follows from the estimates (C-10)–(C-16) below.

The notation of Section C.1.1 will be used throughout. Recall, in particular, the regions $\check{\mathcal{R}}_R$ and $\mathcal{R}_{\text{Corner}}$ defined in (C-3) and (C-4), respectively, the boundedness property (C-5) of r in $\mathcal{R}_{\text{Corner}}$, and the T_1 and T_1' notation defined in (C-6).

The constant in the Sobolev inequality (C-7) blows up as $\tau_1-\tau_0\to 0$. Accordingly, the cases

$$\tau_1-\tau_0\ge 1 \quad\text{and}\quad \tau_1-\tau_0<1$$

are typically considered separately. Similarly, the region $\mathcal{R}_{\text{Corner}}$, near the corner $\{u=\tau_0\}\cap\{r=R\}$, is treated separately.

Estimate in the corner region $\{v'\le v(R,\tau_0+R)\}$: For any $|\tilde{\boldsymbol{k}}|\le k+1$,

$$\sum_{|\boldsymbol{k}_1|+|\boldsymbol{k}_2|\le|\tilde{\boldsymbol{k}}|-1}\int_{\mathcal{R}_{\text{Corner}}(\tau_0,\tau_1,v)}|\mathfrak{D}^{\tilde{\boldsymbol{k}}}\psi|\cdot|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi| \lesssim \sqrt{\overset{(0)}{\mathcal{E}}{}^*_{\ll k\,8R/9}}\int^*\overset{(-1-\delta)}{\mathcal{E}}{}'_{k\,R}. \tag{C-10}$$

Indeed, consider some $|\boldsymbol{k}_1|+|\boldsymbol{k}_2|\le|\tilde{\boldsymbol{k}}|-1$. Since $k\ge 7$, it follows that either $|\boldsymbol{k}_1|\le k-4$ or $|\boldsymbol{k}_2|\le k-4$. In the former case,

$$\begin{aligned}\int_{\mathcal{R}_{\text{Corner}}(\tau_0,\tau_1,v)}&|\mathfrak{D}^{\tilde{\boldsymbol{k}}}\psi|\cdot|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|\\ &\lesssim \sup_{\mathcal{R}_{\text{Corner}}(\tau_0,\tau_1,v)}|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|\left(\int_{\mathcal{R}_{\text{Corner}}(\tau_0,\tau_1,v)}|\mathfrak{D}^{\tilde{\boldsymbol{k}}}\psi|\right)^{\frac{1}{2}}\left(\int_{\mathcal{R}_{\text{Corner}}(\tau_0,\tau_1,v)}|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|\right)^{\frac{1}{2}}\\ &\lesssim\sqrt{\overset{(0)}{\mathcal{E}}{}^*_{\ll k\,8R/9}}\int^*\overset{(-1-\delta)}{\mathcal{E}}{}'_{k\,R}\end{aligned}$$

by the Sobolev inequality (C-9). Similarly when $|\boldsymbol{k}_2|\le k-4$.

It follows, in view of the boundedness property (C-5), that the estimates of the proposition are trivial in the corner region. The remainder of the proof thus concerns the region $\check{\mathcal{R}}_R(\tau_0,\tau_1,v)$.

Estimate for the $\partial_u\psi$ *term*: First note that, for any $|\boldsymbol{k}|\le k$,

$$\sum_{|\boldsymbol{k}_1|+|\boldsymbol{k}_2|\le|\boldsymbol{k}|}\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)}|\partial_u\mathfrak{D}^{\boldsymbol{k}}\psi|\cdot|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi| \lesssim \sqrt{\overset{(0)}{\mathcal{F}}{}^*_{\ll k}}\left(\overset{(0)}{\mathcal{E}}{}^*_{k}+\int^*\overset{(\delta-1)}{\mathcal{E}}{}'_{k}\right)+\sqrt{\overset{(0)}{\mathcal{E}}{}^*_{\ll k}+\int^*\overset{(\delta-1)}{\mathcal{E}}{}'_{\ll k}}\left(\overset{(0)}{\mathcal{F}}{}^*_{k}+\int^*\overset{(-1-\delta)}{\mathcal{E}}_{k}\right). \tag{C-11}$$

Indeed, suppose $|\boldsymbol{k}_1|+|\boldsymbol{k}_2| \le |\boldsymbol{k}|$. Since $k \ge 7$, it follows that either $|\boldsymbol{k}_1| \le k-3$ or $|\boldsymbol{k}_2| \le k-3$. If $|\boldsymbol{k}_1| \le k-3$, then, for $\tau_1-\tau_0 \ge 1$,

$$\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)} |\partial_u \mathfrak{D}^{\boldsymbol{k}}\psi|\cdot|\partial_u \mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\partial_v \mathfrak{D}^{\boldsymbol{k}_2}\psi|$$

$$\lesssim \int_{v(R,\tau_0+R)}^{v} \sup_{u,\theta} r|\partial_u \mathfrak{D}^{\boldsymbol{k}_1}\psi| \int_{\tau_0}^{T_1'}\int_{S^2} (r^{-1-\delta}|\partial_u \mathfrak{D}^{\boldsymbol{k}}\psi|^2 + r^{-1+\delta}|\partial_v \mathfrak{D}^{\boldsymbol{k}_2}\psi|^2) r^2\, d\omega\, du\, dv' \lesssim \sqrt{\overset{(0)}{\mathcal{F}}{}^*_{\ll k}} \int^* \overset{(\delta-1)}{\mathcal{E}}{}'_k$$

by the Sobolev inequality on the incoming cones (C-7). For $\tau_1-\tau_0<1$,

$$\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)} |\partial_u \mathfrak{D}^{\boldsymbol{k}}\psi|\cdot|\partial_u \mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\partial_v \mathfrak{D}^{\boldsymbol{k}_2}\psi|$$

$$\lesssim \int_{v(R,\tau_0)}^{v} \sup_{\substack{\tau_0\le u\le T_1'\\ \theta\in S^2}} |r\partial_u \mathfrak{D}^{\boldsymbol{k}_1}\psi| \left(\int_{\tau_0}^{T_1'}\int_{S^2} r^{-1-\delta}|\partial_u \mathfrak{D}^{\boldsymbol{k}}\psi| r^2\, d\omega\, du\right)^{\frac12} \left(\int_{\tau_0}^{T_1'}\int_{S^2} r^{-1+\delta}|\partial_v \mathfrak{D}^{\boldsymbol{k}_2}\psi| r^2\, d\omega\, du\right)^{\frac12} dv'$$

$$\lesssim \frac{\sqrt{\overset{(0)}{\mathcal{F}}{}^*_{\ll k}}}{(\tau_1-\tau_0)^{1/2}} \left(\int_{v(R,\tau_0)}^{v}\int_{\tau_0}^{T_1'}\int_{S^2} r^{-1-\delta}|\partial_u \mathfrak{D}^{\boldsymbol{k}}\psi| r^2\, d\omega\, du\, dv'\right)^{\frac12}$$
$$\times \left(\int_{\tau_0}^{T_1}\int_{v(R,u)}^{v}\int_{S^2} r^{-1+\delta}|\partial_v \mathfrak{D}^{\boldsymbol{k}_2}\psi| r^2\, d\omega\, dv'\, du\right)^{\frac12}$$

$$\lesssim \frac{\sqrt{\overset{(0)}{\mathcal{F}}{}^*_{\ll k}}}{(\tau_1-\tau_0)^{1/2}} \sqrt{\int^* \overset{(-1-\delta)}{\mathcal{E}}_k} \sqrt{\overset{(0)}{\mathcal{E}}{}^*_k} \left(\int_{\tau_0}^{T_1} du\right)^{\frac12} \lesssim \sqrt{\overset{(0)}{\mathcal{E}}{}^*_k} \sqrt{\int^* \overset{(-1-\delta)}{\mathcal{E}}_k} \sqrt{\overset{(0)}{\mathcal{F}}{}^*_{\ll k}}$$

by the Sobolev inequality on the incoming cones (C-7).

If $|\boldsymbol{k}_2| \le k-2$, then

$$\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)} |\partial_u \mathfrak{D}^{\boldsymbol{k}}\psi|\cdot|\partial_u \mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\partial_v \mathfrak{D}^{\boldsymbol{k}_2}\psi|$$

$$\lesssim \int_{v(R,\tau_0+R)}^{v} \sup_{u,\theta} |r^{\frac{1+\delta}{2}}\partial_v \mathfrak{D}^{\boldsymbol{k}_2}\psi| \int_{\tau_0}^{T_1'}\int_{S^2} r^{-\frac{1+\delta}{2}}|\partial_u \mathfrak{D}^{\boldsymbol{k}}\psi|\cdot|\partial_u \mathfrak{D}^{\boldsymbol{k}_1}\psi| r^2\, d\omega\, du\, dv'$$

$$\lesssim (1+(\tau_1-\tau_0)^{-\frac12}) \sum_{|\tilde{\boldsymbol{k}}|\le|\boldsymbol{k}_2|+3} \int_{v(R)}^{v} \left(\int_{\tau_0}^{T_1'}\int_{S^2} r^{-1+\delta}|\partial_v \mathfrak{D}^{\tilde{\boldsymbol{k}}}\psi|^2 r^2\, d\omega\, du\right)^{\frac12}$$
$$\times \left(\int_{\tau_0}^{T_1'}\int_{S^2} r^{-1-\delta}|\partial_u \mathfrak{D}^{\boldsymbol{k}}\psi|^2 r^2\, d\omega\, du\right)^{\frac12} \left(\int_{\tau_0}^{T_1'}\int_{S^2} |\partial_u \mathfrak{D}^{\boldsymbol{k}_1}\psi|^2 r^2\, d\omega\, du\right)^{\frac12} dv'$$

$$\lesssim (1+(\tau_1-\tau_0)^{-\frac12}) \sqrt{\overset{(0)}{\mathcal{F}}{}^*_k} \sqrt{\int^* \overset{(-1-\delta)}{\mathcal{E}}_k} \sum_{|\tilde{\boldsymbol{k}}|\le|\boldsymbol{k}_2|+3} \left(\int_{\tau_0}^{\tau_1}\int_{v(R,u)}^{v}\int_{S^2} r^{-1+\delta}|\partial_v \mathfrak{D}^{\tilde{\boldsymbol{k}}}\psi|^2 r^2\, d\omega\, du\, dv'\right)^{\frac12}$$

by the Sobolev inequality on the incoming cones (C-8). Hence, if $\tau_1-\tau_0\ge 1$,

$$\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)} |\partial_u \mathfrak{D}^{\boldsymbol{k}}\psi|\cdot|\partial_u \mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\partial_v \mathfrak{D}^{\boldsymbol{k}_2}\psi| \lesssim \sqrt{\overset{(0)}{\mathcal{F}}{}^*_k} \sqrt{\int^* \overset{(-1-\delta)}{\mathcal{E}}_k} \sqrt{\int^* \overset{(\delta-1)}{\mathcal{E}}{}'_{\ll k}}$$

and, if $\tau_1-\tau_0<1$,

$$\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)} |\partial_u \mathfrak{D}^{\boldsymbol{k}}\psi|\cdot|\partial_u \mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\partial_v \mathfrak{D}^{\boldsymbol{k}_2}\psi| \lesssim \sqrt{\overset{(0)}{\mathcal{F}}{}^*_k} \sqrt{\int^* \overset{(-1-\delta)}{\mathcal{E}}_k} \sqrt{\overset{(0)}{\mathcal{E}}{}^*_{\ll k}}.$$

Estimate for the $\partial_v\psi$ term: Next, for any $|\boldsymbol{k}| \le k$,

$$\sum_{|\boldsymbol{k}_1|+|\boldsymbol{k}_2|\le|\boldsymbol{k}|} \int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)} |\partial_v \mathfrak{D}^{\boldsymbol{k}}\psi| \cdot |\partial_u \mathfrak{D}^{\boldsymbol{k}_1}\psi| \cdot |\partial_v \mathfrak{D}^{\boldsymbol{k}_2}\psi| \lesssim \sqrt{\underset{\ll k}{\overset{(0)}{\mathcal{F}}}{}^* + \int^* \underset{\ll k}{\overset{(-1-\delta)}{\mathcal{E}'}}} \Big(\underset{k}{\overset{(0)}{\mathcal{F}}}{}^* + \underset{k}{\overset{(0)}{\mathcal{E}}}{}^* + \int^* \underset{k}{\overset{(\delta-1)}{\mathcal{E}'}} \Big). \tag{C-12}$$

Indeed, if $|\boldsymbol{k}_1| \le k-2$, then

$$\begin{aligned} &\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)} |\partial_v \mathfrak{D}^{\boldsymbol{k}}\psi| \cdot |\partial_u \mathfrak{D}^{\boldsymbol{k}_1}\psi| \cdot |\partial_v \mathfrak{D}^{\boldsymbol{k}_2}\psi| \\ &\qquad \lesssim \sup_{u,v,\theta} |r\partial_u \mathfrak{D}^{\boldsymbol{k}_1}\psi| \Big(\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)} r^{-1} |\partial_v \mathfrak{D}^{\boldsymbol{k}}\psi|^2 \Big)^{\frac12} \Big(\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)} r^{-1} |\partial_v \mathfrak{D}^{\boldsymbol{k}_2}\psi|^2 \Big)^{\frac12} \\ &\qquad \lesssim (1+(\tau_1-\tau_0)^{-\frac12}) \sqrt{\underset{\ll k}{\overset{(0)}{\mathcal{F}}}{}^*} \sqrt{\int^* \underset{k}{\overset{(\delta-1)}{\mathcal{E}'}}} \Big(\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)} r^{-1} |\partial_v \mathfrak{D}^{\boldsymbol{k}}\psi|^2 \Big)^{\frac12} \end{aligned}$$

by the Sobolev inequality on the incoming cones (C-7). Hence, if $\tau_1 - \tau_0 \ge 1$,

$$\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)} |\partial_v \mathfrak{D}^{\boldsymbol{k}}\psi| \cdot |\partial_u \mathfrak{D}^{\boldsymbol{k}_1}\psi| \cdot |\partial_v \mathfrak{D}^{\boldsymbol{k}_2}\psi| \lesssim \sqrt{\underset{\ll k}{\overset{(0)}{\mathcal{F}}}{}^*} \int^* \underset{k}{\overset{(\delta-1)}{\mathcal{E}'}}$$

and, if $\tau_1 - \tau_0 < 1$,

$$\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)} |\partial_v \mathfrak{D}^{\boldsymbol{k}}\psi| \cdot |\partial_u \mathfrak{D}^{\boldsymbol{k}_1}\psi| \cdot |\partial_v \mathfrak{D}^{\boldsymbol{k}_2}\psi| \lesssim \sqrt{\underset{\ll k}{\overset{(0)}{\mathcal{F}}}{}^*} \sqrt{\underset{k}{\overset{(0)}{\mathcal{E}}}{}^*} \sqrt{\int^* \underset{k}{\overset{(\delta-1)}{\mathcal{E}'}}}.$$

If $|\boldsymbol{k}_2| \le |\boldsymbol{k}| - 2$, then one similarly estimates

$$\begin{aligned} &\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)} |\partial_v \mathfrak{D}^{\boldsymbol{k}}\psi| \cdot |\partial_u \mathfrak{D}^{\boldsymbol{k}_1}\psi| \cdot |\partial_v \mathfrak{D}^{\boldsymbol{k}_2}\psi| \\ &\lesssim \int_{v(R,\tau_0+R)}^{v} \sup_{\substack{\tau_0\le u\le T_1' \\ \theta\in S^2}} |r^{\frac12}\partial_v \mathfrak{D}^{\boldsymbol{k}_2}\psi| \Big(\int_{\tau_0}^{T_1'} \int_{S^2} r^{-1} |\partial_v \mathfrak{D}^{\boldsymbol{k}}\psi| r^2 \, d\omega \, du \Big)^{\frac12} \Big(\int_{\tau_0}^{T_1'} \int_{S^2} |\partial_u \mathfrak{D}^{\boldsymbol{k}_1}\psi| r^2 \, d\omega \, du \Big)^{\frac12} dv' \\ &\lesssim (1+(\tau_1-\tau_0)^{-\frac12}) \sqrt{\underset{k}{\overset{(0)}{\mathcal{F}}}{}^*} \\ &\qquad \times \sum_{|\tilde{\boldsymbol{k}}|\le|\boldsymbol{k}_2|+3} \int_{v(R,\tau_0+R)}^{v} \Big(\int_{\tau_0}^{T_1'} \int_{S^2} r^{-1-\delta} |\partial_v \mathfrak{D}^{\tilde{\boldsymbol{k}}}\psi| r^2 \, d\omega \, du \Big)^{\frac12} \Big(\int_{\tau_0}^{T_1'} \int_{S^2} r^{-1+\delta} |\partial_v \mathfrak{D}^{\boldsymbol{k}}\psi| r^2 \, d\omega \, du \Big)^{\frac12} dv' \\ &\lesssim (1+(\tau_1-\tau_0)^{-\frac12}) \sqrt{\underset{k}{\overset{(0)}{\mathcal{F}}}{}^*} \sqrt{\int^* \underset{\ll k}{\overset{(-1-\delta)}{\mathcal{E}'}}} \Big(\int_{v(R,\tau_0+R)}^{v} \int_{\tau_0}^{T_1'} \int_{S^2} r^{-1+\delta} |\partial_v \mathfrak{D}^{\boldsymbol{k}}\psi| r^2 \, d\omega \, du \, dv' \Big)^{\frac12} \end{aligned}$$

by the Sobolev inequality on the incoming cones (C-8). Hence, if $\tau_1 - \tau_0 \ge 1$,

$$\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)} |\partial_v \mathfrak{D}^{\boldsymbol{k}}\psi| \cdot |\partial_u \mathfrak{D}^{\boldsymbol{k}_1}\psi| \cdot |\partial_v \mathfrak{D}^{\boldsymbol{k}_2}\psi| \lesssim \sqrt{\underset{k}{\overset{(0)}{\mathcal{F}}}{}^*} \sqrt{\int^* \underset{\ll k}{\overset{(-1-\delta)}{\mathcal{E}'}}} \sqrt{\int^* \underset{k}{\overset{(\delta-1)}{\mathcal{E}'}}}$$

and, if $\tau_1 - \tau_0 < 1$,

$$\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)} |\partial_v \mathfrak{D}^{\boldsymbol{k}}\psi| \cdot |\partial_u \mathfrak{D}^{\boldsymbol{k}_1}\psi| \cdot |\partial_v \mathfrak{D}^{\boldsymbol{k}_2}\psi| \lesssim \sqrt{\underset{k}{\overset{(0)}{\mathcal{F}}}{}^*} \sqrt{\int^* \underset{\ll k}{\overset{(-1-\delta)}{\mathcal{E}'}}} \sqrt{\underset{k}{\overset{(0)}{\mathcal{E}}}{}^*}.$$

Estimate for the $r^{-1}\not\nabla\psi$ and $r^{-1}\psi$ terms: For any $|\boldsymbol{k}|\le k$,

$$\sum_{|\boldsymbol{k}_1|+|\boldsymbol{k}_2|\le|\boldsymbol{k}|}\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)} r^{-1}|\not\nabla\mathfrak{D}^{\boldsymbol{k}}\psi|\cdot|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi| \lesssim \sqrt{\underset{\ll k}{\overset{(0)}{\mathcal{F}}}{}^* + \int^* \underset{\ll k}{\overset{(-1-\delta)}{\mathcal{E}'}}}\left(\underset{k}{\overset{(0)}{\mathcal{F}}}{}^* + \underset{k}{\overset{(0)}{\mathcal{E}}}{}^* + \int^* \underset{k}{\overset{(\delta-1)}{\mathcal{E}'}}\right) \tag{C-13}$$

and

$$\sum_{|\boldsymbol{k}_1|+|\boldsymbol{k}_2|\le|\boldsymbol{k}|}\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)} r^{-1}|\mathfrak{D}^{\boldsymbol{k}}\psi|\cdot|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\partial_u\mathfrak{D}^{\boldsymbol{k}_2}\psi| \lesssim \sqrt{\underset{\ll k}{\overset{(0)}{\mathcal{F}}}{}^* + \int^* \underset{\ll k}{\overset{(-1-\delta)}{\mathcal{E}'}}}\left(\underset{k}{\overset{(0)}{\mathcal{F}}}{}^* + \underset{k}{\overset{(0)}{\mathcal{E}}}{}^* + \int^* \underset{k}{\overset{(\delta-1)}{\mathcal{E}'}}\right). \tag{C-14}$$

Indeed, (C-13) and (C-14) follow as in the proof of (C-12), using now the fact that

$$\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)} r^{-1-\delta}|r^{-1}\not\nabla\psi|^2 + r^{-1-\delta}|r^{-1}\psi|^2 \le \int^* \underset{k}{\overset{(-1-\delta)}{\mathcal{E}'}}$$

and

$$\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)} r^{-1+\delta}|r^{-1}\not\nabla\psi|^2 + r^{-1+\delta}|r^{-1}\psi|^2 \le \int^* \underset{k}{\overset{(\delta-1)}{\mathcal{E}'}},$$

$$\int_{v(R,\tau)}^{v}\int_{S^2}(|r^{-1}\not\nabla\psi|^2 + |r^{-1}\psi|^2)r^2\,d\omega\,dv' \le \underset{k}{\overset{(0)}{\mathcal{E}}}{}^*.$$

Estimate for the quartic term: For any $|\boldsymbol{k}|\le k$,

$$\sum_{|\boldsymbol{k}_1|+|\boldsymbol{k}_2|\le|\boldsymbol{k}|}\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)} |\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|^2\cdot|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|^2 \lesssim \left(\underset{k}{\overset{(0)}{\mathcal{E}}}{}^* + \int^* \underset{k}{\overset{(-1-\delta)}{\mathcal{E}'}}\right)\underset{\ll k}{\overset{(0)}{\mathcal{F}}}{}^* + \left(\underset{\ll k}{\overset{(0)}{\mathcal{E}}}{}^* + \int^* \underset{\ll k}{\overset{(-1-\delta)}{\mathcal{E}'}}\right)\underset{k}{\overset{(0)}{\mathcal{F}}}{}^*. \tag{C-15}$$

Indeed, supposing first that $|\boldsymbol{k}_1|\le k-3$,

$$\begin{aligned}\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)} &|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|^2\cdot|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|^2\\ &\lesssim \sup_{u,\theta}(r|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|)^2\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)} r^{-2}|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|^2 \lesssim \underset{\ll k}{\overset{(0)}{\mathcal{F}}}{}^*(1+(\tau_1-\tau_0)^{-\frac{1}{2}})\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)} r^{-2}|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|^2\end{aligned}$$

by the Sobolev inequality on the incoming cones (C-7). Hence, if $\tau_1-\tau_0\ge 1$,

$$\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)} |\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|^2\cdot|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|^2 \lesssim \underset{\ll k}{\overset{(0)}{\mathcal{F}}}{}^*\int^* \underset{k}{\overset{(-1-\delta)}{\mathcal{E}'}}$$

and, if $\tau_1-\tau_0<1$,

$$\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)} |\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|^2\cdot|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|^2 \lesssim \underset{\ll k}{\overset{(0)}{\mathcal{F}}}{}^*\underset{k}{\overset{(0)}{\mathcal{E}}}{}^*.$$

Similarly, if $|\boldsymbol{k}_2|\le k-3$,

$$\begin{aligned}\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)} |\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|^2\cdot|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|^2 &\lesssim \int_{v(R,\tau_0+R)}^{v}\sup_{u,\theta}|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|^2\int_{\tau_0}^{\tau_1}\int_{S^2}|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|^2 r^2\,d\omega\,du\,dv'\\ &\lesssim \underset{k}{\overset{(0)}{\mathcal{F}}}{}^*(1+(\tau_1-\tau_0)^{-\frac{1}{2}})\sum_{|\tilde{\mathbf{k}}|\le|\mathbf{k}_2|+3}\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)} r^{-2}|\partial_v\mathfrak{D}^{\tilde{\mathbf{k}}}\psi|^2,\end{aligned}$$

and so one similarly has

$$\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)} |\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|^2\cdot|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|^2 \lesssim \left(\underset{\ll k}{\overset{(0)}{\mathcal{E}}}{}^* + \int^* \underset{\ll k}{\overset{(-1-\delta)}{\mathcal{E}'}}\right)\underset{k}{\overset{(0)}{\mathcal{F}}}{}^*.$$

Estimate for the $r^{p-1}\partial_v(r\psi)$ term: Finally, for any $|\boldsymbol{k}| \le k$,

$$\sum_{|\boldsymbol{k}_1|+|\boldsymbol{k}_2|\le|\boldsymbol{k}|}\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)} r^{p-1}|\partial_v(r\mathfrak{D}^{\boldsymbol{k}}\psi)|\cdot|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|$$
$$\lesssim \Big(\overset{(0)}{\mathcal{F}}{}^*_{k} + \overset{(p)}{\mathcal{E}}{}^*_{k} + \int^* \overset{(p-1)}{\mathcal{E}}{}'_{k}\Big)\sqrt{\overset{(0)}{\mathcal{F}}{}^*_{\ll k} + \overset{(0)}{\mathcal{E}}{}^*_{\ll k}} + \sqrt{\overset{(0)}{\mathcal{F}}{}^*_{k}}\sqrt{\int^* \overset{(p-1)}{\mathcal{E}}{}'_{k}}\sqrt{\int^* \overset{(p-1)}{\mathcal{E}}{}'_{\ll k}}. \quad \text{(C-16)}$$

Indeed, consider first the case $\tau_1 - \tau_0 \ge 1$. Suppose first that $|\boldsymbol{k}_1| \le k-3$. Then, if $\tau_1-\tau_0 \ge 1$,

$$\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)} r^{p-1}|\partial_v(r\mathfrak{D}^{\boldsymbol{k}}\psi)|\cdot|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|$$
$$\lesssim \sup_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)}(r|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|)\Big(\int_{\mathcal{R}_R(\tau_0,\tau_1,v)} r^{p-3}|\partial_v(r\mathfrak{D}^{\boldsymbol{k}}\psi)|^2\Big)^{\frac12}\Big(\int_{\mathcal{R}_R(\tau_0,\tau_1,v)} r^{p-1}|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|^2\Big)^{\frac12} \lesssim \sqrt{\overset{(0)}{\mathcal{F}}{}^*_{\ll k}}\int^* \overset{(p-1)}{\mathcal{E}}{}'_{k}$$

by the Sobolev inequality on the incoming cones (C-7). Similarly, if $\tau_1-\tau_0<1$,

$$\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)} r^{p-1}|\partial_v(r\mathfrak{D}^{\boldsymbol{k}}\psi)|\cdot|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|$$
$$\lesssim (\tau_1-\tau_0)^{\frac12}\sqrt{\overset{(0)}{\mathcal{F}}{}^*_{\ll k}}\Big(\int_{\mathcal{R}_R(\tau_0,\tau_1,v)} r^{p-2}|\partial_v(r\mathfrak{D}^{\boldsymbol{k}}\psi)|^2\Big)^{\frac12}\Big(\int_{\mathcal{R}_R(\tau_0,\tau_1,v)} r^{p-2}|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|^2\Big)^{\frac12} \lesssim \sqrt{\overset{(0)}{\mathcal{F}}{}^*_{\ll k}}\sqrt{\overset{(p)}{\mathcal{E}}{}^*_{k}}\sqrt{\overset{(0)}{\mathcal{E}}{}^*_{k}}.$$

Suppose now $|\boldsymbol{k}_2| \le k-3$. Then, if $\tau_1-\tau_0\ge 1$,

$$\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)} r^{p-1}|\partial_v(r\mathfrak{D}^{\boldsymbol{k}}\psi)|\cdot|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|$$
$$\lesssim \sqrt{\overset{(0)}{\mathcal{F}}{}^*_{k}}\int_{v(R,\tau_0)}^{v}\sup_{u,\theta}(r^{\frac{p+1}{2}}|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|)\Big(\int_{\tau_0}^{\tau_1}\int_{S^2} r^{p-3}|\partial_v(r\mathfrak{D}^{\boldsymbol{k}}\psi)|^2r^2\,d\omega\,du\Big)^{\frac12}dv$$
$$\lesssim \sqrt{\overset{(0)}{\mathcal{F}}{}^*_{k}}\Big(\int_{\mathcal{R}_R(\tau_0,\tau_1,v)} r^{p-3}|\partial_v(r\mathfrak{D}^{\boldsymbol{k}}\psi)|^2r^2\,d\omega\,du\,dv\Big)^{\frac12}\sum_{|\tilde{\mathbf{k}}|\le|\boldsymbol{k}_2|+3}\Big(\int_{\mathcal{R}_R(\tau_0,\tau_1,v)} r^{p-1}|\partial_v\mathfrak{D}^{\tilde{\boldsymbol{k}}}\psi|^2r^2\,d\omega\,du\,dv\Big)^{\frac12}$$
$$\lesssim \sqrt{\overset{(0)}{\mathcal{F}}{}^*_{k}}\sqrt{\int^* \overset{(p-1)}{\mathcal{E}}{}'_{k}}\sqrt{\int^* \overset{(p-1)}{\mathcal{E}}{}'_{\ll k}}$$

using again the Sobolev inequality on the incoming cones (C-8). Similarly, if $\tau_1-\tau_0<1$,

$$\int_{\check{\mathcal{R}}_R(\tau_0,\tau_1,v)} r^{p-1}|\partial_v(r\mathfrak{D}^{\boldsymbol{k}}\psi)|\cdot|\partial_u\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\partial_v\mathfrak{D}^{\boldsymbol{k}_2}\psi|$$
$$\lesssim (\tau_1-\tau_0)^{-\frac12}\sqrt{\overset{(0)}{\mathcal{F}}{}^*_{k}}\Big(\int_{\mathcal{R}_R(\tau_0,\tau_1,v)} r^{p-2}|\partial_v(r\mathfrak{D}^{\boldsymbol{k}}\psi)|^2\Big)^{\frac12}\sum_{|\tilde{\mathbf{k}}|\le|\boldsymbol{k}_2|+3}\Big(\int_{\mathcal{R}_R(\tau_0,\tau_1,v)} r^{p-2}|\partial_v\mathfrak{D}^{\tilde{\boldsymbol{k}}}\psi|^2\Big)^{\frac12}$$
$$\lesssim \sqrt{\overset{(0)}{\mathcal{F}}{}^*_{k}}\sqrt{\overset{(p)}{\mathcal{E}}{}^*_{k}}\sqrt{\overset{(0)}{\mathcal{E}}{}^*_{\ll k}}. \qquad \square$$

C.1.3. *The proof of Proposition C.1.1 for the nonlinearity $F = \not{g}^{AB}\not\nabla_A\psi\not\nabla_B\psi$.* We now consider the nonlinearity

$$F = \not{g}^{AB}\not\nabla_A\psi\not\nabla_B\psi.$$

In view of the property (C-2), the proof of Proposition C.1.1 follows from the next proposition.

Proposition C.1.5 (nonlinear error estimates for $F = \slashed{g}^{AB}\slashed{\nabla}_A\psi\slashed{\nabla}_B\psi$ on Minkowski space). *Under the assumptions of Proposition C.1.1, for each* $|\boldsymbol{k}| \le k$,

$$\sum_{|\boldsymbol{k}_1|+|\boldsymbol{k}_2|\le|\boldsymbol{k}|}\Bigg(\int_{\mathcal{R}(\tau_0,\tau_1,v)\cap\{r\ge R\}}\Big(|\partial_u\mathfrak{D}^{\boldsymbol{k}}\psi|+|\partial_v\mathfrak{D}^{\boldsymbol{k}}\psi|+\frac{1}{r}|\slashed{\nabla}\mathfrak{D}^{\boldsymbol{k}}\psi|+\frac{1}{r}|\mathfrak{D}^{\boldsymbol{k}}\psi|\Big)\cdot|\slashed{\nabla}\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\slashed{\nabla}\mathfrak{D}^{\boldsymbol{k}_2}\psi| + \int_{\mathcal{R}(\tau_0,\tau_1,v)\cap\{r\ge R\}}|\slashed{\nabla}\mathfrak{D}^{\boldsymbol{k}_1}\psi|^2\cdot|\slashed{\nabla}\mathfrak{D}^{\boldsymbol{k}_2}\psi|^2\Bigg)$$
$$\lesssim\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{F}}}{}^*_R+\overset{(0)}{\underset{\ll k}{\mathcal{E}}}{}^*_{8R/9}}\Big(\overset{(0)}{\underset{k}{\mathcal{E}}}{}^*_R+\int^*\overset{(\delta-1)}{\underset{k}{\mathcal{E}}}{}'_R\Big)$$

and

$$\sum_{|\boldsymbol{k}_1|+|\boldsymbol{k}_2|\le|\boldsymbol{k}|}\int_{\mathcal{R}(\tau_0,\tau_1,v)\cap\{r\ge R\}}r^{p-1}|\partial_v(r\mathfrak{D}^{\boldsymbol{k}}\psi)|\cdot|\slashed{\nabla}\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\slashed{\nabla}\mathfrak{D}^{\boldsymbol{k}_2}\psi|\lesssim\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{F}}}{}^*_R+\overset{(0)}{\underset{\ll k}{\mathcal{E}}}{}^*_{8R/9}}\Big(\overset{(p)}{\underset{k}{\mathcal{E}}}{}^*_R+\int^*\overset{(p-1)}{\underset{k}{\mathcal{E}}}{}'_R\Big).$$

Proof. The separate terms are estimated individually, and the proof follows from estimates (C-17)–(C-21) below.

The notation of Section C.1.1 will again be used throughout. As in the proof of Proposition C.1.4, the cases $\tau_1-\tau_0\ge 1$ and $\tau_1-\tau_0<1$ are typically considered separately. Similarly, the region near the corner $\{u=\tau_0\}\cap\{r=R\}$ is treated separately.

Estimate in the corner region $\{v'\le v(R,\tau_0+R)\}$: For any $|\tilde{\boldsymbol{k}}|\le k+1$,

$$\sum_{|\boldsymbol{k}_1|+|\boldsymbol{k}_2|\le|\tilde{\boldsymbol{k}}|-1}\int_{\mathcal{R}_{\mathrm{Corner}}(\tau_0,\tau_1,v)}|\mathfrak{D}^{\tilde{\boldsymbol{k}}}\psi|\cdot|\slashed{\nabla}\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\slashed{\nabla}\mathfrak{D}^{\boldsymbol{k}_2}\psi|\lesssim\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{E}}}{}^*_{8R/9}}\int^*\overset{(-1-\delta)}{\underset{k}{\mathcal{E}}}{}'_R. \tag{C-17}$$

Indeed, (C-17) follows exactly as in the proof of Proposition C.1.4 using the Sobolev inequality (C-9).

The remainder of the proof concerns the region $\check{\mathcal{R}}_R(\tau_0,\tau_1,v)$.

Estimate for the $\partial_u\psi$ *term*: First note that, for any $|\boldsymbol{k}|\le k$,

$$\sum_{|\boldsymbol{k}_1|+|\boldsymbol{k}_2|\le|\boldsymbol{k}|}\int_{\check{\mathcal{R}}(\tau_0,\tau_1,v)}|\partial_u\mathfrak{D}^{\boldsymbol{k}}\psi|\cdot|\slashed{\nabla}\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\slashed{\nabla}\mathfrak{D}^{\boldsymbol{k}_2}\psi|\lesssim\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{F}}}{}^*}\int^*\overset{(\delta-1)}{\underset{k}{\mathcal{E}}}{}'. \tag{C-18}$$

Indeed, for $\tau_1-\tau_0\ge 1$, assuming without loss of generality that $|\boldsymbol{k}_1|\le k-3$,

$$\begin{aligned}\int_{\check{\mathcal{R}}(\tau_0,\tau_1,v)}&|\partial_u\mathfrak{D}^{\boldsymbol{k}}\psi|\cdot|\slashed{\nabla}\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\slashed{\nabla}\mathfrak{D}^{\boldsymbol{k}_2}\psi|\\ &\lesssim\sup_{u,v,\theta}r|\slashed{\nabla}\mathfrak{D}^{\boldsymbol{k}_1}\psi|\Big(\int_{\check{\mathcal{R}}(\tau_0,\tau_1,v)}r^{-1-\delta}|\partial_u\mathfrak{D}^{\boldsymbol{k}}\psi|^2\Big)^{\frac12}\Big(\int_{\check{\mathcal{R}}(\tau_0,\tau_1,v)}r^{-1+\delta}|\slashed{\nabla}\mathfrak{D}^{\boldsymbol{k}_2}\psi|^2\Big)^{\frac12}\\ &\lesssim\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{F}}}{}^*}\int^*\overset{(\delta-1)}{\underset{k}{\mathcal{E}}}{}'\end{aligned}$$

by the Sobolev inequality on the incoming cones (C-7) since $p\ge\delta$. For $\tau_1-\tau_0<1$,

$$\int_{\check{\mathcal{R}}(\tau_0,\tau_1,v)}|\partial_u\mathfrak{D}^{\boldsymbol{k}}\psi|\cdot|\slashed{\nabla}\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\slashed{\nabla}\mathfrak{D}^{\boldsymbol{k}_2}\psi|\lesssim\sqrt{\overset{(0)}{\underset{\ll k}{\mathcal{F}}}{}^*}\sqrt{\overset{(0)}{\underset{k}{\mathcal{E}}}{}^*}\sqrt{\overset{(0)}{\underset{k}{\mathcal{F}}}{}^*}.$$

Estimates for the $\partial_v\psi$, $r^{-1}\nabla\!\!\!/\psi$, and $r^{-1}\psi$ terms: Next, for any $|\boldsymbol{k}|\le k$,

$$\sum_{|\boldsymbol{k}_1|+|\boldsymbol{k}_2|\le|\boldsymbol{k}|}\int_{\check{\mathcal{R}}(\tau_0,\tau_1,v)}(|\partial_v\mathfrak{D}^{\boldsymbol{k}}\psi|+r^{-1}|\nabla\!\!\!/\mathfrak{D}^{\boldsymbol{k}}\psi|+r^{-1}|\mathfrak{D}^{\boldsymbol{k}}\psi|)\cdot|\nabla\!\!\!/\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\nabla\!\!\!/\mathfrak{D}^{\boldsymbol{k}_2}\psi| \lesssim\sqrt{\underset{\ll k}{\overset{(0)}{\mathcal{F}}}{}^*}\biggl(\underset{k}{\overset{(0)}{\mathcal{E}}}{}^*+\int^*\underset{k}{\overset{(\delta-1)}{\mathcal{E}'}}\biggr). \tag{C-19}$$

Indeed, for $\tau_1-\tau_0\ge 1$, assuming again without loss of generality that $|\boldsymbol{k}_1|\le k-3$,

$$\int_{\check{\mathcal{R}}(\tau_0,\tau_1,v)}|\partial_v\mathfrak{D}^{\boldsymbol{k}}\psi|\cdot|\nabla\!\!\!/\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\nabla\!\!\!/\mathfrak{D}^{\boldsymbol{k}_2}\psi| \lesssim\sup_{u,v,\theta} r|\nabla\!\!\!/\mathfrak{D}^{\boldsymbol{k}_1}\psi|\biggl(\int_{\check{\mathcal{R}}(\tau_0,\tau_1,v)}r^{-1}|\partial_v\mathfrak{D}^{\boldsymbol{k}}\psi|^2\biggr)^{\frac12}\biggl(\int_{\check{\mathcal{R}}(\tau_0,\tau_1,v)}r^{-1}|\nabla\!\!\!/\mathfrak{D}^{\boldsymbol{k}_2}\psi|^2\biggr)^{\frac12}\lesssim\sqrt{\underset{\ll k}{\overset{(0)}{\mathcal{F}}}{}^*}\int^*\underset{k}{\overset{(\delta-1)}{\mathcal{E}'}},$$

and similarly, for $\tau_1-\tau_0<1$,

$$\int_{\check{\mathcal{R}}(\tau_0,\tau_1,v)}|\partial_v\mathfrak{D}^{\boldsymbol{k}}\psi|\cdot|\nabla\!\!\!/\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\nabla\!\!\!/\mathfrak{D}^{\boldsymbol{k}_2}\psi|\lesssim\sqrt{\underset{\ll k}{\overset{(0)}{\mathcal{F}}}{}^*}\underset{k}{\overset{(0)}{\mathcal{E}}}{}^*$$

using again the Sobolev inequality on the incoming cones (C-7). Similarly for the $r^{-1}\psi$ and $r^{-1}\nabla\!\!\!/\psi$ terms, we use the fact that

$$\int_{\check{\mathcal{R}}(\tau_0,\tau_1,v)}(r^{-1}|\nabla\!\!\!/\mathfrak{D}^{\boldsymbol{k}}\psi|^2+r^{-3}|\mathfrak{D}^{\boldsymbol{k}}\psi|^2)\le\int^*\underset{k}{\overset{(\delta-1)}{\mathcal{E}'}},\quad \int_{v(R,u)}^{v}\int_{S^2}(|r^{-1}\nabla\!\!\!/\psi|^2+|r^{-1}\psi|^2)r^2\,d\omega\,dv'\le\underset{k}{\overset{(0)}{\mathcal{E}}}{}^*.$$

Estimate for the quartic term: For any $|\boldsymbol{k}|\le k$,

$$\sum_{|\boldsymbol{k}_1|+|\boldsymbol{k}_2|\le|\boldsymbol{k}|}\int_{\check{\mathcal{R}}(\tau_0,\tau_1,v)}|\nabla\!\!\!/\mathfrak{D}^{\boldsymbol{k}_1}\psi|^2\cdot|\nabla\!\!\!/\mathfrak{D}^{\boldsymbol{k}_2}\psi|^2\lesssim\underset{\ll k}{\overset{(0)}{\mathcal{F}}}{}^*\biggl(\underset{k}{\overset{(0)}{\mathcal{E}}}{}^*+\int^*\underset{k}{\overset{(-1-\delta)}{\mathcal{E}}}\biggr), \tag{C-20}$$

which follows as an easy consequence of the Sobolev inequality (C-7).

Estimate for the $r^{p-1}\partial_v(r\psi)$ term: Finally, for any $|\boldsymbol{k}|\le k$,

$$\sum_{|\boldsymbol{k}_1|+|\boldsymbol{k}_2|\le|\boldsymbol{k}|}\int_{\check{\mathcal{R}}(\tau_0,\tau_1,v)}r^{p-1}|\partial_v(r\mathfrak{D}^{\boldsymbol{k}}\psi)|\cdot|\nabla\!\!\!/\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\nabla\!\!\!/\mathfrak{D}^{\boldsymbol{k}_2}\psi|\lesssim\sqrt{\underset{\ll k}{\overset{(0)}{\mathcal{F}}}{}^*}\biggl(\underset{k}{\overset{(p)}{\mathcal{E}}}{}^*+\int^*\underset{k}{\overset{(p-1)}{\mathcal{E}'}}\biggr). \tag{C-21}$$

Assume again, without loss of generality, that $|\boldsymbol{k}_1|\le k-3$. Then, using (C-7), for $\tau_1-\tau_0\ge 1$,

$$\int_{\check{\mathcal{R}}(\tau_0,\tau_1,v)}r^{p-1}|\partial_v(r\mathfrak{D}^{\boldsymbol{k}}\psi)|\cdot|\nabla\!\!\!/\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\nabla\!\!\!/\mathfrak{D}^{\boldsymbol{k}_2}\psi| \lesssim\sup_{\check{\mathcal{R}}(\tau_0,\tau_1,v)}r|\nabla\!\!\!/\mathfrak{D}^{\boldsymbol{k}_1}\psi|\biggl(\int_{\check{\mathcal{R}}(\tau_0,\tau_1,v)}r^{p-3}|\partial_v(r\mathfrak{D}^{\boldsymbol{k}}\psi)|^2\biggr)^{\frac12}\biggl(\int_{\check{\mathcal{R}}(\tau_0,\tau_1,v)}r^{p-1}|\nabla\!\!\!/\mathfrak{D}^{\boldsymbol{k}_2}\psi|^2\biggr)^{\frac12}\lesssim\sqrt{\underset{\ll k}{\overset{(0)}{\mathcal{F}}}{}^*}\int^*\underset{k}{\overset{(p-1)}{\mathcal{E}'}}.$$

Similarly, for $\tau_1-\tau_0<1$,

$$\int_{\check{\mathcal{R}}(\tau_0,\tau_1,v)}r^{p-1}|\partial_v(r\mathfrak{D}^{\boldsymbol{k}}\psi)|\cdot|\nabla\!\!\!/\mathfrak{D}^{\boldsymbol{k}_1}\psi|\cdot|\nabla\!\!\!/\mathfrak{D}^{\boldsymbol{k}_2}\psi| \lesssim(\tau_1-\tau_0)^{-\frac12}\sqrt{\underset{\ll k}{\overset{(0)}{\mathcal{F}}}{}^*}\biggl(\int_{\check{\mathcal{R}}(\tau_0,\tau_1,v)}r^{p-2}|\partial_v(r\mathfrak{D}^{\boldsymbol{k}}\psi)|^2\biggr)^{\frac12}\biggl(\int_{\check{\mathcal{R}}(\tau_0,\tau_1,v)}r^{p-2}|\nabla\!\!\!/\mathfrak{D}^{\boldsymbol{k}_2}\psi|^2\biggr)^{\frac12}\lesssim\sqrt{\underset{\ll k}{\overset{(0)}{\mathcal{F}}}{}^*}\sqrt{\underset{k}{\overset{(p)}{\mathcal{E}}}{}^*}\sqrt{\underset{k}{\overset{(0)}{\mathcal{E}}}{}^*}. \qquad\square$$

C.2. ***More general nonlinearities and Kerr.*** For more general nonlinearities on Minkowski space of the form

$$F = N^{\mu\nu}(\psi, x)\partial_\mu\psi\,\partial_\nu\psi,$$

with N satisfying a null condition of the form

$$\sup_{|\xi|\le 1, r\ge R} \sum_{|\mathbf{k}|+s\le k} \sum_{A,B=1,2} \mathfrak{D}^{\mathbf{k}}\partial_\xi^s(rN^{uu}+N^{uv}+N^{vv}+rN^{Au}+rN^{Av}+r^2N^{AB})(\xi, x) \lesssim D_k, \tag{C-22}$$

where the A, B indices refer to coordinates ϑ, φ (recall the $(u, v, \vartheta, \varphi)$ coordinate system of Section 2.7.1) and D_k are arbitrary constants, the proof of Proposition C.1 follows exactly as above, where the bound (4-29) is now also used to obtain pointwise bounds on ψ through an easy weighted Sobolev estimate which allow as to invoke (C-22).

Remark C.2.1. Note that, besides the nonlinearities of the classical null condition [Klainerman 1986], our class (C-22) includes for instance also nonlinearities which for large r take the form $F = (\sin r)\partial_u\psi\,\partial_v\psi$. It does not, however, include even more general examples like $F = (\sin x)\partial_u\psi\,\partial_v\psi$ considered recently in [Anderson and Zbarsky 2024] due to the presence of the weighted vector fields Ω_i in our set of commutation vector fields $\mathfrak{D}^{\mathbf{k}}$.

More generally, if g_0 is a metric with asymptotics suitably close to those of Minkowski space, with extra terms suitably small, then the above proof again applies.

We will consider explicitly the case that g_0 is the Kerr metric. Define double-null $(u, v, \theta^1, \theta^2)$ coordinates on Kerr, when $0 < |a| < M$, in terms of the Boyer–Lindquist coordinates (t, r, ϑ, ϕ) of Section A.1, by

$$u = t - r_*, \qquad v = t + r_*,$$

with θ^1 defined implicitly by the relation

$$F(\theta^1, r, \vartheta) = 0,$$

$$\text{where } F(\theta^1, r, \vartheta) = \int_{\theta^1}^{\vartheta} \frac{1}{a\sqrt{\sin^2\theta^1 - \sin^2\theta'}}\,d\theta' + \int_r^\infty \frac{1}{\sqrt{((r')^2+a^2)^2 - a^2\sin^2\theta^1\Delta'}}\,dr', \tag{C-23}$$

and θ^2 defined by

$$\theta^2 = \phi + h(r_*, \theta^1), \qquad \text{where } h \text{ satisfies } \quad \partial_{r_*}h(r_*, \theta^1) = \frac{2Mar}{\Sigma R^2}, \qquad \lim_{r_*\to\infty} h(r_*, \theta^1) = 0.$$

See for instance [Pretorius and Israel 1998; Dafermos and Luk 2025]. Here

$$r_*(r, \vartheta) = \int \frac{r^2+a^2}{\Delta}\,dr + \int_r^\infty \frac{(r')^2+a^2-\sqrt{((r')^2+a^2)^2 - a^2\sin^2\theta^1\Delta'}}{\Delta'}\,dr' + \int_{\theta^1}^{\vartheta} a\sqrt{\sin^2\theta^1 - \sin^2\theta'}\,d\theta' \tag{C-24}$$

(note that the expression (C-24) is independent of θ^1 in view of (C-23)), where

$$\int \frac{r^2+a^2}{\Delta}\,dr$$

is a function satisfying

$$\partial_r \int \frac{r^2+a^2}{\Delta}\, dr = \frac{r^2+a^2}{\Delta}$$

and

$$\Delta = r^2+a^2-2Mr, \quad \Delta' = (r')^2+a^2-2Mr', \quad \Sigma = r^2+a^2\cos^2\vartheta,$$

$$\mathrm{R}^2 = r^2+a^2+\frac{2Ma^2 r\sin^2\vartheta}{\Sigma}.$$

Note in particular that r_* is distinct from $r^\star$ defined in Section A.1.

In these double-null coordinates, the Kerr metric takes the form

$$g_{a,M} = -\Omega^2\, du\, dv + \not{g}_{AB}(d\theta^A - b^A\, dv)(d\theta^B - b^B\, dv),$$

where

$$\Omega^2 = \frac{\Delta}{\mathrm{R}^2}, \quad b^1 = 0, \quad b^2 = \frac{4Mar}{\Sigma \mathrm{R}^2},$$

and

$$\not{g}_{11} = \frac{a^2(\partial_{\theta^1}F)^2(\sin^2\theta^1-\sin^2\vartheta)((r^2+a^2)^2-a^2\sin^2\theta^1\Delta)}{\mathrm{R}^2} + (\partial_{\theta^1}h)^2\mathrm{R}^2\sin^2\vartheta,$$

$$\not{g}_{22} = \mathrm{R}^2\sin^2\vartheta, \quad \not{g}_{12} = \not{g}_{21} = -\mathrm{R}^2\sin^2\vartheta\,\partial_{\theta^1}h.$$

In particular, one has

$$L = \partial_v + b^A\partial_{\theta^A}, \quad \underline{L} = \partial_u.$$

Moreover,

$$T = \partial_u + \partial_v \quad \text{and} \quad \Omega_i = \Omega_i(\theta^1,\theta^2)^A\partial_{\theta^A} \ \text{ for } i = 1, 2, 3,$$

for functions $\Omega_i(\theta^1,\theta^2)^A$, so that, in particular,

$$[\partial_u, T] = [\partial_u, \Omega_i] = [\partial_v, T] = [\partial_v, \Omega_i] = 0.$$

The function $b^2 = b^2(u, v, \theta^1, \theta^2)$ satisfies, for any $k \geq 0$,

$$\sum_{k_1+k_2+|\mathbf{k}_3|\leq k} |\partial_u^{k_1}(r\partial_v)^{k_2}\Omega^{\mathbf{k}_3}b^2| \leq \frac{C_k}{r^3}$$

for large r, and hence the commutation relations

$$|\mathfrak{D}^{\boldsymbol{k}}L\psi| \lesssim \sum_{|\tilde{\boldsymbol{k}}|\leq|\boldsymbol{k}|} |L\mathfrak{D}^{\tilde{\boldsymbol{k}}}\psi| + \frac{1}{r^3}\sum_{i=1}^{3}\sum_{|\tilde{\boldsymbol{k}}|\leq|\boldsymbol{k}|-1} |\Omega_i\mathfrak{D}^{\tilde{\boldsymbol{k}}}\psi|, \tag{C-25}$$

$$|\mathfrak{D}^{\boldsymbol{k}}\underline{L}\psi| \lesssim \sum_{|\tilde{\boldsymbol{k}}|\leq|\boldsymbol{k}|} |\underline{L}\mathfrak{D}^{\tilde{\boldsymbol{k}}}\psi| + \frac{1}{r^3}\sum_{i=1}^{3}\sum_{|\tilde{\boldsymbol{k}}|\leq|\boldsymbol{k}|-1} |\Omega_i\mathfrak{D}^{\tilde{\boldsymbol{k}}}\psi| \tag{C-26}$$

hold for large r. Moreover (C-2) remains true. The proof of Proposition C.1 in this case then follows just as in the case where g_0 was the Minkowski metric, using now (C-25) and (C-26) in place of (C-1).

We note finally that the condition (C-22) applied to Kerr indeed includes in particular the nonlinearities considered in [Luk 2013].

Appendix D: The inhomogeneous estimate (3-3) on Kerr in the full subextremal case $|a| < M$

The main theorem of [Dafermos et al. 2016] only states (3-3) for the homogeneous case, i.e., the case $F = 0$. In this section, we explicitly address the issue of the inhomogeneous estimate (3-3). We have the following:

Theorem D.1. *The inhomogeneous estimate (3-3) holds on Kerr in the full subextremal case $|a| < M$.*

Though the inhomogeneous estimate (3-3) for general F, precisely as stated, indeed can be shown to hold in the full subextremal case $|a| < M$, it is a little bit delicate to produce the physical-space expression corresponding to the middle term on the right-hand side of (3-3), except in the case $|a| \ll M$. (This difficulty is entirely due to the presence of the term $\overset{(0)}{\mathcal{E}}(\tau)$ on the left-hand side.) The weaker version of the estimate in Section 3.2, on the other hand, which is all that is actually used here, can essentially be immediately read off from the proof of the main result of [loc. cit.]. Since we have no real use for the stronger statement, we prefer here to give the details of how to directly obtain the weaker statement (though in the case $|a| \ll M$, we note that our argument indeed recovers the stronger statement).

Proof (of the weaker version of Remark 3.2.1). We will obtain the estimate in three steps. The reader should refer to [Dafermos et al. 2016] to follow along.

Step 1. We first obtain (3-3) (in its original form) for general F, but without the future boundary terms on the left-hand side, and where the regions are all restricted to $r \geq r_+$.

For this, note that one can assume without loss of generality that F is compactly supported in spacetime, and thus it is clearly sufficient to prove Proposition 9.1.1 of [loc. cit.] now for solutions of (1-2), with the extra inhomogeneous terms of (3-3) now on the right-hand side.

We note that the cutoffs used in the proof of Proposition 9.1.1 of [loc. cit.] already gave rise to inhomogeneous terms

$$\mathcal{T} := \int_{-\infty}^{\infty} \sum_{m\ell} \left(\int_{-\infty}^{\infty} H \cdot (f, h, y, \chi) \cdot (u, u') \right) d\omega\, dr^*; \tag{D-1}$$

see, e.g., (165) of [loc. cit.]. An inhomogeneity F on the right-hand side of (1-2) contributes an additional term of the form (D-1) in the proof of this proposition, where H is replaced by the Carter separated coefficients of F according to formula (43) of [loc. cit.]. Let us denote this term as $\mathcal{T}_F$. One easily sees, however, that, in view of the inclusion of our inhomogeneity F, the problem can be reduced to the case where one has trivial initial data, in which case one may work directly with ψ without applying a cutoff. Thus we may assume now that we *only* have $\mathcal{T}_F$ on the right-hand side, and H below will denote the Carter separated coefficients of F. We must now essentially repeat the steps of the proof of the proposition in order to produce the desired right-hand side.

For this, we first immediately partition $\mathcal{T}_F$ as $\mathcal{T}_F^1 + \mathcal{T}_F^2 + \mathcal{T}_F^3$, where the summands correspond to the integral of (D-1) over the region $R_- \leq r \leq R_+$ (i.e., $R_-^* \leq r^* \leq R_+^*$, etc.), $\{r \leq R_-\} \cup \{R_+ \leq r \leq R\}$ and $r \geq R$, respectively, where $R_\pm$, R_∞ are as in [loc. cit.] and we require $R \geq R_\infty$.

Considering $\mathcal{T}_F^1$, we further partition $\mathcal{T}_F^1$ as $\mathcal{T}_{F,\omega}^1+\mathcal{T}_{F,\not\omega}^1$, where $\mathcal{T}_{F,\omega}^1$ is the contribution of the currents in $H\cdot(f,h,y,\chi)\cdot(u,u')$ that multiply ω and $\mathcal{T}_{F,\not\omega}^1$ denotes the rest. We note (see formula (102) of [Dafermos et al. 2016]) that the coefficients of the currents in the sum defining $\mathcal{T}_{F,\omega}^1$ are then frequency-independent for all "trapped" frequencies $(\omega,m,\ell)\in\mathcal{G}_\natural$ (the expression is simply $E\omega\,\mathrm{Im}(H\bar{u})$). Thus, by adding a suitable compensating term $\mathcal{T}_{\mathrm{comp}}^1$ supported only in the untrapped frequencies, we have that $\mathcal{T}_{F,\omega}^1+\mathcal{T}_{\mathrm{comp}}^1$ is a sum and integral over the precise expression $E\omega\,\mathrm{Im}(H\bar{u})$, and thus this integral and sum may be rewritten via the Plancherel formulae on page 820 of [loc. cit.] as a spacetime integral precisely like the middle term of (3-3), restricted to $r\le R$, with $V_0=\partial_t$.

For the remaining terms $\mathcal{T}_{F,\not\omega}^1-\mathcal{T}_{\mathrm{comp}}^1$, note that these are supported entirely in the *nontrapped* frequencies, i.e., in the complement frequency set $\mathcal{G}_\natural^c$. Recall now that in estimate (165) of [loc. cit.], we in fact manifestly control a stronger left-hand side than what is written; namely, we may substitute ζ with $(1-r_{\mathrm{trap}}/r)^2$ (which was the original form of the expression before (163) of [loc. cit.] was invoked). In particular, since $r_{\mathrm{trap}}=0$ for all nontrapped frequencies, we control all derivatives without degeneration for frequencies in $\mathcal{G}_\natural^c$. One may apply Cauchy–Schwarz to the expression

$$H\cdot(f,h,y,\chi)\cdot(u,u')$$

and to the integrand summands of $\mathcal{T}_{\mathrm{comp}}^1$, absorbing the resulting terms proportional to $|u|^2$, together with any frequency coefficients ω^2 and Λ, and proportional to $|u'|^2$, into the bulk controlled in view of our nondegenerate bulk, at the expense of an additional term $|H|^2$ on the right-hand side. Upon application of the Plancherel formulae of page 820 of [loc. cit.], this produces a bulk term $\int F^2$ of the form already present on the right-hand side of (3-3). The term $\mathcal{T}_F^2$ may be treated exactly in the same way, as here the controlled bulk term is nondegenerate for all frequencies since the integral is supported away from trapping.

This leaves the term $\mathcal{T}_F^3$. One notes that, for $r\ge R$, the coefficients of the currents appearing in (D-1) are frequency-independent *up to multipliers which decay exponentially in* r. Again, by adding and subtracting a compensating term as above, one may thus estimate the additional term here by the middle term on the right-hand side of (3-10) (now restricted to $r\ge R$), again with a suitable vector field V_0, after applying Plancherel, up to an additional exponentially decaying term, which may be estimated by Cauchy–Schwarz, absorbing the term containing derivatives of ψ into the bulk on the left-hand side just as above, and producing an extra bulk term $\int F^2$ of the form already present on the right-hand side of (3-3).

Finally, we note that to complete the proof of Proposition 9.1.1 in [loc. cit.] in the inhomogeneous case, it remains to apply the analogue of Proposition 9.7.1 in [loc. cit.] (this is where the quantitative version of mode stability [Shlapentokh-Rothman 2015] is appealed to), which also however produces an additional inhomogeneous term when applied to (1-2). Examining the original [Shlapentokh-Rothman 2015], in particular the proof of Lemma 3.3 of that paper, one sees that this extra term may be bounded by the $\int F^2$ term on the right-hand side of (3-3). This completes the proof of Step 1.

Step 2. We now apply a sufficiently small multiple of the red-shift multiplier and of a cutoff version of the J_μ^T current to obtain the terms

$$\sup_{v:\tau\le\tau(v)}\overset{(0)}{\mathcal{F}}(v,\tau_0,\tau)+\overset{(0)}{\mathcal{E}}_{\mathcal{S}}(\tau)$$

on the left-hand side of (3-3) and the part of the spacetime integral supported in $r_0 \le r \le r_+$, exploiting that one may always absorb the resulting error terms in the bulk already controlled in Step 1. We add these identities to the estimate obtained above. As these are physical-space identities, the inhomogeneity F manifestly generates a term of the form of the middle term in (3-3) which may be combined with the previous such terms to define a new vector field V_0. One obtains in this way also a bound on the restriction of the integral defining $\overset{(0)}{\mathcal{E}}(\tau)$ to $\{r \le R_-\} \cup \{r \ge R_+\}$.

Note that in the case of very slow rotation $|a| \ll M$, one may choose the support of the gradient of the cutoff applied to J^T_μ in the identity above to in fact be in the region $r \le R_-$, whereas J^T_μ is moreover coercive in the region $r \ge R_-$. Thus, the above argument in fact yields control of the full $\overset{(0)}{\mathcal{E}}(\tau)$. This would then complete the proof. In this case, note that we thus obtain the estimate (3-3) in its original from.

Step 3. In the general subextremal case, however, J^T_μ is not everywhere coercive in the region $R_- \le r \le R_+$, and it remains to control the contribution of the finite region $R_- \le r \le R_+$ to the energy integral of the missing term $\overset{(0)}{\mathcal{E}}(\tau)$ from the left-hand side of (3-3).

To obtain the missing part of the energy flux we must thus repeat the proof of Proposition 13.1 of [Dafermos et al. 2016], allowing now an inhomogeneous term F. Recall that in this proof one considered a cutoff version $\tilde{\psi}$, decomposed orthogonally using Carter's separation as $\tilde{\psi} = \tilde{\psi}_1 + \cdots + \tilde{\psi}_N$ for some large N and applied a distinct current $J^{V_i}_\mu$ to each summand $\tilde{\psi}_i$.

Due to the presence of cutoffs in the proof, one can again read off the additional terms arising from the new inhomogeneity. Since, outside the region $R_- \le r \le R_+$, all errors can be absorbed as in the previous steps, while the cutoffs vanish inside the region $R_- \le r \le R_+$, all inhomogeneous terms in $R_- \le r \le R_+$ now arise from F, and we may use the notation $\widetilde{F}_i$ of the proof of Proposition 13.1 of [loc. cit.] to denote precisely these new terms. The resulting inhomogeneous terms that must be controlled are then

$$\sum_i \int_{R_- \le r \le R_+} V_i \tilde{\psi}_i \cdot \widetilde{F}_i.$$

By orthogonality, we may indeed manifestly bound this term by the expression in the first term of (3-10).

Since all terms of the form of the middle term on the right-hand side of (3-3) supported in $r \le R$ (which we generated in the previous two steps) can of course manifestly be bounded by the first term of (3-10), we obtain finally that (3-3) indeed holds with (3-10) replacing the middle term on the right-hand side. □

We note that obtaining (3-3) in the $|a| < M$ case with its middle term in its original form requires a slightly more delicate decomposition of $\tilde{\psi}$ than the one used in Proposition 13.1 of [loc. cit.]. Again, since this is not used, we spare the reader the details of this argument.

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MEAN CURVATURE FLOW WITH MULTIPLICITY 2 CONVERGENCE IN $\mathbb{R}^3$

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We construct a new example of an immortal mean curvature flow of smooth embedded connected surfaces in $\mathbb{R}^3$, which converges to a plane with multiplicity 2 as time approaches infinity.

1. Introduction

Higher multiplicity convergence is a significant phenomenon in differential geometry and geometric measure theory. Roughly speaking, it means that several different sheets of a surface collapse to the same sheet in the limit. One example is the dilation of the minimal surface catenoid, which ultimately converges to a plane with multiplicity 2.

When higher multiplicity convergence occurs, the geometry and topology may change dramatically. In the case of the dilating catenoid converging to a plane with multiplicity 2, the nontrivial topology and high-curvature region of the catenoids vanish in the limit plane. Therefore, higher multiplicity convergence is a central topic in the study of geometry and topology using geometric measure theory.

In this paper, we construct a new example of the mean curvature flow of smooth embedded connected surfaces in $\mathbb{R}^3$, which converges to a plane with multiplicity 2 as time approaches ∞. A smooth one-parameter family of hypersurfaces $M(t) \subset \mathbb{R}^{n+1}$, $t \in (0, T)$ evolves by its mean curvature if

$$\partial_t x = \vec{H}(x),$$

where $\vec{H}(x)$ is the mean curvature vector of the surface at point x. A smooth mean curvature flow that exists for all future time is called *immortal*.

Theorem 1.1. *There exists an immortal mean curvature flow of connected embedded rotationally symmetric surfaces* $(M(t))_{t\in[0,\infty)}$ *in* $\mathbb{R}^3$, *which is smooth for* $t > 0$, *whose limit as* $t \to \infty$ *is a plane with multiplicity* 2.

Our example contrasts sharply with the picture of the singular behavior of mean curvature flow in $\mathbb{R}^3$. Recall that Huisken [1990] introduced the rescaled mean curvature flow to blow up the singularities of mean curvature flow. If (y, T) is a singularity of a mean curvature flow $\{M(t)\}_{t\in[0,T)}$, the rescaled mean curvature flow is defined to be $\widetilde{M}(\tau) := e^{\tau/2}(M(T - e^{-\tau}) - y)$, and it satisfies the equation

$$\partial_\tau x = \vec{H}(x) + x^\perp/2,$$

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where $x^{\perp}$ denotes the projection of the position vector to the normal direction. Using Huisken's monotonicity formula, Ilmanen [Ilmanen 1995] and White [White 1997] showed that the (subsequential) limit of a rescaled mean curvature flow is a self-shrinker, namely a (possibly singular) hypersurface satisfying the equation $\vec{H}(x) + x^{\perp}/2 = 0$. It is worth noting that the plane is one of the self-shrinkers. The limit self-shrinker of a rescaled mean curvature flow may have higher multiplicity. Ilmanen [1995] made the following conjecture:

Conjecture 1.2 (multiplicity one conjecture of mean curvature flow). *For a mean curvature flow of closed embedded surfaces in* $\mathbb{R}^3$, *the rescaled mean curvature flow blowing up a singularity must converge to a self-shrinker with multiplicity* 1.

This conjecture has been verified in many special cases; for example, when the initial surface is mean convex or the blow-up rate of the mean curvature is at most type I. We refer the readers to [Huisken and Sinestrari 1999a; 1999b; White 2000; 2003; Haslhofer and Kleiner 2017; Li and Wang 2019; 2022].

While we were preparing this paper, Bamler and Kleiner [2023] posted a proof confirming the multiplicity one conjecture. Consequently, a smooth rescaled mean curvature flow cannot converge to the plane with higher multiplicity as time approaches ∞, which contrasts sharply with our example. One reason for this lies in the stability of the plane as a static point in mean curvature flow, meanwhile, the plane is unstable as a static point in the rescaled mean curvature flow, as observed by Colding and Minicozzi [2012].

To elaborate, imagine over a large region of the plane, the flow converging to it with multiplicity 2 can be represented as double graphs. Let the difference between these two graphs be denoted by f. This function satisfies a nonlinear equation, with its linear part given by $\partial_t f = Lf$, where L is the linearized operator of the plane. For mean curvature flow, $L = \Delta$ is the Laplacian, and for rescaled mean curvature flow, $L = \Delta - \frac{1}{2}\langle x, \nabla \cdot \rangle + \frac{1}{2}$. Because $f > 0$, its evolution is dominated by the first eigenfunction of L, and the first eigenvalue determines its growth rate. For $L = \Delta$, the first eigenvalue is 0, allowing the double graphs to tend to each other due to the nonlinear factor, with a subexponential speed. However, for $L = \Delta - \frac{1}{2}\langle x, \nabla \cdot \rangle + \frac{1}{2}$, the first eigenvalue is 1, implying that the double graphs will move away from each other with an exponential speed.

We would like to compare our example with another higher multiplicity example in the min-max theory, a significant tool for constructing minimal hypersurfaces in Riemannian manifolds. Recent study by Marques and Neves [2016; 2017; 2021] and Xin Zhou [2020] provided a comprehensive description of the min-max minimal hypersurfaces in Riemannian manifolds with dimensions between 3 and 7. These minimal hypersurfaces are either unstable with multiplicity 1, or stable with potentially higher multiplicity.

Zhichao Wang and Xin Zhou [2025] constructed a closed Riemannian manifold to show the existence of higher multiplicity min-max minimal surfaces. While the min-max theory involves a different mechanism, the reason that higher multiplicity is a feature of stable minimal surfaces is similar: when a minimal surface is unstable, the linearized operator has a positive eigenfunction with negative eigenvalue, which pushes the nearby graphs away from each other, preventing higher multiplicity convergence.

Recall that, after taking a quotient of the rotation, the Euclidean space becomes a half-plane. A rotationally symmetric surface can then be represented by a curve in the half-plane, known as the profile curve. When considering the evolution of the profile curve, our example can be interpreted as a free

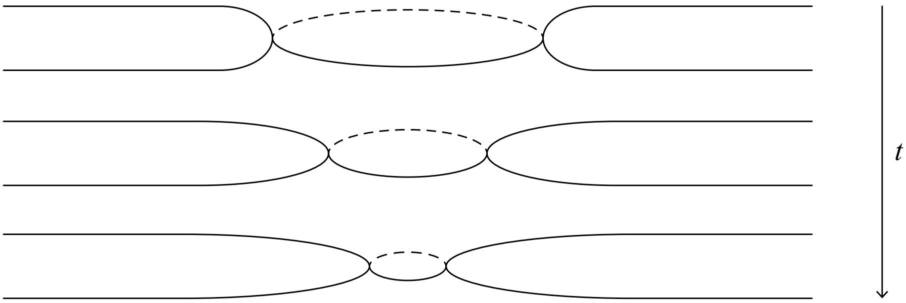

Figure 1. The shape of the immortal flow as time goes by. The lower figure shows the flow at a larger time.

boundary curve shortening flow in a surface with a boundary, up to a conformal factor in the speed. In our case, the long-time behavior of our flow is "collapsing", which is very different from the free boundary curve shortening flow in a surface with convex boundary as studied in [Langford and Zhu 2023; Ko 2023].

We remark that there was an example by Evans and Ilmanen in [Ilmanen 1994, Appendix E], showing that a Brakke flow can have a higher density limit as $t \to \infty$ even if initially the Brakke flow is supported on a smooth embedded surface with density 1. A Brakke flow is a geometric measure theoretic weak flow of mean curvature flow, and the density is the ratio of the measure compared to the Hausdorff measure. To our knowledge, the example in [Ilmanen 1994, Appendix E] is not found in the existing literature, and Ilmanen did not explain the method to construct such an example. While our example is inspired by their description, it should be noted that from the perspective of Brakke flows, the example suggested by Evans and Ilmanen might not be smoothly connected.

1.1. ***Shape of the immortal flow.*** The flow constructed in Theorem 1.1 looks like two parallel planes connected by a neck. As time goes by, the size of the neck shrinks and shrinks, and as time tends to infinity, the size of the neck tends to 0 (See the Figure 1).

Although this may seem intuitive, showing that such a flow converges to a plane with multiplicity 2 as time tends to infinity is not straightforward. One challenge is to prove that the neck does not pinch at a finite time, and the flow does not expand to infinity. One novelty of this paper is to turn intuition into mathematically rigorous arguments.

There are several tricky facts about this immortal flow. Firstly, our flow appears as two sheets, with each sheet asymptotically approaching a plane. As detailed in the Appendix, the asymptotically planar surface will be asymptotic to the same plane under mean curvature flow. Consequently, the immortal flow does not uniformly converge to the plane with multiplicity 2, but instead converges uniformly in any compact region.

Secondly, this flow seems highly unstable. In fact, our construction reveals that even a minor perturbation can lead to a finite-time neck pinch singularity or cause the flow to escape to infinity. Therefore, we anticipate that such higher multiplicity convergence is nongeneric.

Thirdly, while the immortal flow from our construction appears to be bounded between two parallel planes, we expect that a slight modification could yield an immortal flow converging to a plane with multiplicity 2 whose ends grow, rather than being bounded. It is not clear how much the ends can grow.

Our method suggests that if the ends grow too fast, the immortal flow might simply become a static catenoid (see Remark 4.14).

1.2. *Idea of the construction.* We will be focused on rotationally symmetric mean curvature flow, which is a class of mean curvature flow that has been well-studied by Altschuler, Angenent, and Giga [Altschuler et al. 1995]. Our construction involves two main steps.

In the first step, we construct a smooth free boundary mean curvature flow within a solid cylinder with radius R in $\mathbb{R}^3$. This flow converges to a free boundary disk with multiplicity 2 as time tends to infinity. Given a domain D with boundary ∂D, we say that a family of hypersurfaces with boundary $\{M(t)\}_{t\in[0,T)}$ evolves by its mean curvature with free boundary condition, if $\partial M_t \subset \partial D$ and wherever the flow and the ∂D intersect, they meet orthogonally. From a partial differential equation (PDE) point of view, the free boundary mean curvature flow is a nonlinear PDE subject to Neumann boundary conditions. For the study of free boundary mean curvature flow, we refer the readers to [Stahl 1996a; 1996b; Edelen 2016; Buckland 2005; Edelen et al. 2022], among others.

The construction uses an interpolation argument. We will consider a (singular) foliation of rotationally symmetric surfaces $\{\Gamma_s\}_{s\in I}$ in the cylinder. Each surface consists of two planes with holes connected by a neck. As the size of the hole varies, the behavior of the surface under mean curvature flow changes: when the hole is very small, the surface pinches at the origin; when the hole is very large, the surface contracts to the boundary. Thus, there must be a critical surface Γ_{s_0} where neither pinching nor contracting to the boundary occurs under mean curvature flow. We prove that this critical surface indeed converges to the free boundary disk with multiplicity 2 under the mean curvature flow.

This interpolation argument has been used in the construction of mean curvature flow, and we refer the readers to [White 2002; Chu and Sun 2025] for previous applications of this technique.

In the next step, we let $R \to \infty$ to extract a subsequence of the flows constructed in the first step, obtaining a limit flow. A key aspect of the proof is ensuring that our limit is a nonstatic flow, rather than a static minimal surface such as a catenoid. This requires a careful selection of the (singular) foliation in the first step. Additionally, we derive quantitative estimates of the obtained flow through the use of a barrier argument.

One technique that we frequently use in our proof is the parabolic Sturmian theorem proved by Angenent [1988]. Such a Sturmian theorem was previously used in the study of rotationally symmetric mean curvature flow by Altschuler, Angenent, and Giga [Altschuler et al. 1995]. Roughly speaking, the Sturmian theorem can be used to control the number of intersection points between a mean curvature flow (that we are interested in) and a barrier. This provides a local comparison of the flow with a barrier, which is more powerful than the classical comparison principle of mean curvature flows, as it typically yields a global comparison that may not be sufficiently strong. We believe that exploring the application of the parabolic Sturmian theorem can be useful in other scenarios.

2. Preliminary

2.1. *Rotationally symmetric surface.* We begin with some basic settings and notions. By taking the quotient of an $\mathrm{SO}(n-1)$ action, any rotationally symmetric surface in $\mathbb{R}^n$ can be represented by a curve (known as the profile curve) in the half-plane, denoted as $\{(x, y) \mid x \geq 0\}$. The y-axis serves as the rotational axis. The area of the surface can be described by the length of the profile curve in the metric

$$g_{\text{rot}} := x^{2(n-2)}(dx^2 + dy^2). \tag{2-1}$$

In this paper, we only consider the case $n = 3$.

Throughout this paper, we will study the rotationally symmetric surfaces that are also reflexive symmetric. Given any curve γ in the region $\{(x, y) \mid x, y \geq 0\}$ that touches the x-axis, let $S(\gamma)$ be the rotationally symmetric hypersurface in $\mathbb{R}^3$ obtained by first reflecting γ along the x-axis, then rotating γ and its reflection together about the y-axis. For any continuous function $f(x)$, we denote the graph of $f(x)$ in $\{(x, y) \mid x \geq 0\}$ as G_f.

Given $R > 0$, the solid closed cylinder with radius R will be denoted by C_R. After taking the quotient of the $\mathrm{SO}(n-1)$ action, C_R can be described as $C_R = \{(x, y) \mid x \leq R\}$.

Given $0 < p < R$, $\epsilon \geq 0$, let $\mathcal{F}_{R,p,\epsilon}$ denote the set of all continuous functions $f : [p, R] \to [0, \infty)$ with $f(p) = 0$, $f'(R) = 0$, which is smooth in the following sense:

$$\begin{aligned} &f \text{ is smooth on } (p, R], \text{ and f has a local inverse on } [p, p+\epsilon], \text{ such that} \\ &u : [0, f(p+\epsilon)] \to [p, p+\epsilon], \quad u(y) = f^{-1}(y) \quad \text{ is a smooth function.} \end{aligned} \tag{2-2}$$

Clearly, the graph of f can be expressed as the union of two graphs of smooth functions, u and v, where $u = f^{-1}$ as mentioned above, and $v : [p+\epsilon, R] \to [0, \infty)$, $v(x) = f(x)$. We name u as the *vertical graph function*, and its graph as the *vertical graph*, v as the *horizontal graph function*, and its graph as the *horizontal graph*.

We remark that our terminology is different from [Altschuler et al. 1995], because we view the graph in the direction of the rotationally symmetric plane, while they view the graph in the direction of the rotational axis. So our vertical graph is the horizontal graph in [Altschuler et al. 1995], and our horizontal graph is the vertical graph there.

Let $\mathcal{F}_{R,p} = \bigcup_{\epsilon \geq 0} \mathcal{F}_{R,p,\epsilon}$, and for any $f \in \mathcal{F}_{R,p}$, we name the point $(p, 0)$ as the *neck point* of *f*. Let $\mathcal{F}_R = \bigcup_{p \in (0,R)} \mathcal{F}_{R,p}$ and $\mathcal{F} = \bigcup_{R>0} \mathcal{F}_R$.

For any $f \in \mathcal{F}_R$, $f'(R) = 0$ implies the curve G_f and the line $x = R$ intersect orthogonally, which guarantees that $S(G_f)$ is a free boundary surface.

Let $\mathcal{G}_{R,p}$ denote a subset of $\mathcal{F}_{R,p}$, containing all functions f with the following derivative conditions:

$$f'(x) > 0 \quad \text{for all } x \in (p, R). \tag{2-3}$$

Then by the inverse function theorem, condition (2-3) is equivalent with the following derivative conditions of the vertical graph function u and the horizontal graph function v:

$$u'(y) > 0, \quad v'(x) > 0. \tag{2-4}$$

Finally, $\mathcal{G}_R = \bigcup_{p \in (0,R)} \mathcal{G}_{R,p}$. This is the class of functions whose graphs will be the profile curves of the mean curvature flow we will study. See Figure 2.

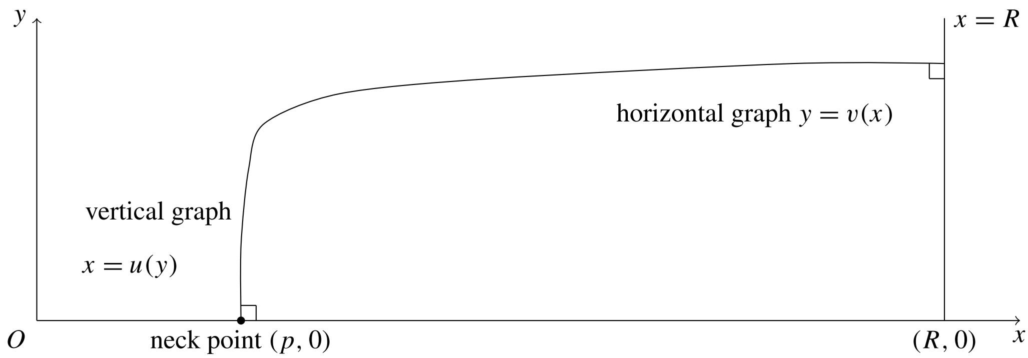

Figure 2. Example of a curve that we study.

2.2. ***Mean curvature flow for rotationally symmetric free boundary surface.*** Suppose $f_0 \in \mathcal{F}_R$, and let $\Gamma = S(G_{f_0})$ be a rotationally symmetric free boundary surface in C_R. Then for $f(\,\cdot\,,t) \in \mathcal{F}_R$, $\Gamma_t = S(G_{f(\,\cdot\,,t)})$ is said to be a mean curvature flow of free boundary surfaces with initial condition f_0, if the corresponding vertical graph function, horizontal graph function (u_0, v_0) of f_0, and $(u(\,\cdot\,,t), v(\,\cdot\,,t))$ of $f(\,\cdot\,,t)$ satisfy the equations (see, e.g., [Altschuler et al. 1995])

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{u_{yy}}{1+(u_y)^2} - \dfrac{1}{u}, \\ \dfrac{\partial v}{\partial t} = \dfrac{v_{xx}}{1+(v_x)^2} + \dfrac{v_x}{x}, \\ u(\,\cdot\,,0) = u_0,\ v(\,\cdot\,,0) = v_0, \\ v_x(R,\,\cdot\,) = 0. \end{cases} \tag{2-5}$$

For simplicity, we will say $f(\,\cdot\,,t)$ is a solution to (2-5) if its corresponding vertical graph function and horizontal graph function solve these equations. We will call the first equation in (2-5) the vertical graph equation, and the second equation in (2-5) the horizontal graph equation.

We have results following from [Stahl 1996b] concerning the mean curvature flow of free boundary surfaces:

Proposition 2.1 [Stahl 1996b]. *Given $f_0 \in \mathcal{F}_R$, there exists a unique solution to (2-5) with a free boundary condition on a maximal time interval $[0, T)$. This solution is smooth for $t > 0$, and in the class $C^{2+\omega,\,1+\omega/2}$ (with arbitrary $0 < \omega < 1$) for $t \geq 0$. Moreover, if $T < \infty$, then the curvature at some point in $G_{f(\,\cdot\,,t)}$ blows up as $t \to T$.*

Following from Proposition 2.1, the first singular time only depends on the initial data f_0, and we will denote it by $T(f_0)$.

Suppose γ_1 and γ_2 are two curves in $\{x, y \geq 0\} \cap C_R$. We say that γ_1 is *on top of* γ_2, if for all pairs (c, y_1, y_2) such that $(c, y_1) \in \gamma_1$, $(c, y_2) \in \gamma_2$, it holds that $y_1 \geq y_2$. If $y_1 > y_2$ for all pairs (c, y_1, y_2), then we say that γ_1 is *strictly on top of* γ_2. We have the following results based on the maximum principle.

Proposition 2.2 (comparison principle). *Suppose $\{S(\gamma_t^1)\}_{t\in[0,T(\gamma_0^1))}$ and $\{S(\gamma_t^2)\}_{t\in[0,T(\gamma_0^2))}$ are families of surfaces in C_R, evolving by mean curvature. If the initial curve γ_0^1 is on top of the initial curve γ_0^2, and the intersection angle between γ_t^1 and the line $\{x = R\}$ is less than or equal to the intersection angle between γ_t^2 and the line $\{x = R\}$ (or the curve γ_t^2 does not intersect the line $\{x = R\}$), then γ_t^1 is on top of γ_t^2 for all time $t \in [0, T)$, where $T = \min\{T(\gamma_0^1), T(\gamma_0^2)\}$, and the function $d(t) = \operatorname{dist}(\gamma_t^1, \gamma_t^2)$ is monotone increasing.*

We refer the readers to [Mantegazza 2011, Section 2.2] for a proof. Note that in [Mantegazza 2011], the comparison principle was proved for mean curvature flow without boundary, however, the proof only uses the parabolic maximum principle, hence the proof also works for mean curvature flows with boundary. The boundary angle condition implies that the shortest distance between γ_t^1 and γ_t^2 is attained either when both points are in the interior of the domain, or when both points are attained on the boundary. This can be obtained by, for example, the first variational formula of the distance function. The first interior case is exactly the same as [Mantegazza 2011, Section 2.2]. The latter case only happens if the intersection angle between γ_t^1 and the line $\{x = R\}$ equals the intersection angle between γ_t^2 and the line $\{x = R\}$, and if we write the flow near the boundary as graphs, the parabolic maximum principle also implies that the distance is increasing.

Remark 2.3. If γ_0^1 is on top of γ_0^2 and they are not the same curve, by strictly maximum principle, γ_t^1 will be strictly on top of γ_t^2 after a short amount of time.

We state below the continuous dependence theorem from [Amann 1988, Theorem 8.1]. While Amann proved this result for general dynamics in fractional Sobolev spaces, we only require the basic case where solutions are smooth functions defined on open subsets of $[0, R] \times (0, +\infty)$. The assumptions of the theorem are satisfied since our equation (2-5) is an autonomous quasilinear parabolic equation whose right-hand side is of class C^1. We refer readers to Sections 7 and 8 of [Amann 1988] for complete details.

Proposition 2.4 (continuous dependence). *Suppose f_t is a solution to (2-5), with initial data $f_0 \in \mathcal{F}$. Write $\mathcal{D} = \{(t, f_0) \in \mathbb{R}_{\geq 0} \times \mathcal{F} \mid 0 \leq t < T(f_0)\}$. Then $\mathcal{D}$ is open in $\mathbb{R}_{\geq 0} \times \mathcal{F}$ and the map*

$$\psi : \mathcal{D} \to \mathcal{F}, \quad \psi(t, f_0) = f_t$$

is a Lipschitz map.

Later we will use rotationally symmetric minimal surfaces as barriers. The minimal surfaces are surfaces with zero mean curvature. They are static solutions to the mean curvature flow. Recall that the catenoids are rotationally symmetric minimal surfaces defined by $S(\gamma_c)$, where γ_c is the curve

$$\gamma_c(s) = (c \cosh(s/c), s), \quad s \in [0, \infty). \tag{2-6}$$

It is well-known that the catenoids and planes are the only rotationally symmetric minimal surfaces in $\mathbb{R}^3$. Furthermore, there is no catenoid with the y-axis as the axis of rotation that satisfies the free boundary condition in C_R. The following lemma is straightforward by the equation of catenoids.

Lemma 2.5. *The only rotationally symmetric free boundary minimal surfaces in C_R are the disks given by the intersection of C_R and the plane $\{y = c\}$ for some constant c.*

2.3. ***Three barriers.*** We will frequently use three known solutions of mean curvature flow as barriers. They are the planes, the catenoids, and the Angenent torus.

The planes are static mean curvature flows. If we write them as a rotational symmetric mean curvature flow, the profile curve is given by the curve $f(\cdot, t) \equiv C$ for a constant C.

The catenoids are also static mean curvature flows. If we write them as a rotational symmetric mean curvature flow, the horizontal graph function u is given by

$$u(y, t) = C \cosh(y/C), \quad y \in [0, \infty),$$

for a positive constant C. In addition, the vertical graph function of the catenoid is given by

$$v(x, t) = C \ln\left(\frac{x}{C} + \sqrt{\frac{x^2 - C^2}{C^2}}\right), \quad x \in [C, \infty).$$

It is worth noting that the catenoids satisfy the following properties. Firstly, for any fixed C, v grows logarithmically, which implies that the catenoids cannot be bounded between two planes. Secondly, as $C \to 0$, the catenoids converge to a plane with multiplicity 2 (see Section 2.5 for the precise meaning).

The Angenent torus was constructed by Angenent [1992]. It is a rotationally symmetric torus in $\mathbb{R}^3$ that is self-shrinking under the mean curvature flow, and it shrinks to a point in finite time. Because the Angenent torus is rotationally symmetric, it is determined by the profile curve. We use $\mathcal{A}$ to denote the Angenent torus that is rotationally symmetric around the y-axis, with its closest point to the y-axis being $\left(\frac{11}{10}, 0\right)$.

Although there is no explicit formula for the Angenent torus, there are some qualitative characterizations. The upper half of the profile curve (in the region $\{x \geq 0,\ y \geq 0\}$) of $\mathcal{A}$ is confined within a rectangle defined by $1 < x < \alpha,\ 0 \leq y < \alpha$, for some real number $\alpha > 2$. We use T' to denote the time at which $\mathcal{A}$ shrinks to a single point under the mean curvature flow.

2.4. ***Sturmian Theorem.*** Briefly speaking, the parabolic Sturmian theorem asserts that the number of zeros of a solution to the parabolic PDE defined on $\mathbb{R}$ or a subinterval of $\mathbb{R}$ can only decrease unless a new intersection point is produced on the boundary. We refer the readers to [Angenent 1988, Theorems B and C] for detailed statements, and [Angenent 1991, Section 2] for the adaption to nonlinear parabolic equations. Here we only state the applications of the parabolic Sturmian theorem to rotationally symmetric mean curvature flows.

Lemma 2.6. *Suppose $f(\cdot, t)$ and $g(\cdot, t)$ are two solutions to the rotationally symmetric mean curvature flow equation (2-5) defined on $C_R \times [0, T]$ (T can be $+\infty$), and let $u(\cdot, t)$ and $v(\cdot, t)$ be the corresponding vertical and horizontal graph functions of $f(\cdot, t)$, and $\tilde{u}(\cdot, t)$ and $\tilde{v}(\cdot, t)$ be the corresponding vertical and horizontal graph functions of $g(\cdot, t)$. The number of intersection points of $f(\cdot, t)$ and $g(\cdot, t)$ is denoted by $z(t)$. Then we have the following properties*:

(1) *$z(t)$ is a finite number for $t > 0$, unless $f(\cdot, t) = g(\cdot, t)$.*

(2) *$z(t)$ is nonincreasing in t, unless $f(\cdot, t)$ and $g(\cdot, t)$ produce a new intersection point on ∂C_R.*

(3) *If $f(\cdot, t_0)$ and $g(\cdot, t_0)$ touch at a point p, in the sense that two curves intersect at p at time t_0 and the unit tangent vector at p coincides, then*

- *either there exists* $\delta > 0$ *such that* $f(\cdot, t)$ *and* $g(\cdot, t)$ *have at least two intersection points for* $t \in (t_0 - \delta, t_0)$ *and at most one intersection point for* $t \in (t_0, t_0 + \delta)$,
- *or there exists* $\delta > 0$ *such that* $f(\cdot, t)$ *and* $g(\cdot, t)$ *have at most one intersection point for* $t \in (t_0 - \delta, t_0)$ *and at least two intersection points for* $t \in (t_0, t_0 + \delta)$.

The proof is an application of Angenent's parabolic Sturmian theorem to u, $\tilde{u}$, and v, $\tilde{v}$.

2.5. ***Multiplicity.*** Multiplicity is a concept in geometric measure theory. Roughly speaking, the multiplicity of a surface being m simply means that we count the surface m times. For example, consider the union of three surfaces $\{y = x^2/n\}$, $\{y = -x^2/n\}$ and $\{y = 0\}$. Let $n \to \infty$, the limit is the plane $\{y = 0\}$ with multiplicity 3. For the detailed definition, we refer the readers to [Simon 1983].

Throughout this paper, the multiplicity 2 convergence is understood as follows. We say a mean curvature flow $\{M(t)\}_{t \in [0,\infty)}$ converges to a plane with multiplicity 2 if for any compact region K and $\epsilon > 0$, there exists $t_\epsilon > 0$, such that for $t > t_\epsilon$, $[M(t) \cap K] \setminus B_\epsilon(0)$ can be written as the union of two graphs of functions $f_1(\cdot, t)$, $f_2(\cdot, t)$ over the plane, and $\|f_1(\cdot, t)\|_{C^1} \to 0$, $\|f_2(\cdot, t)\|_{C^1} \to 0$ as $t \to \infty$.[1] This convergence is stronger than the higher multiplicity convergence in the sense of geometric measure theory.

We remark that the higher multiplicity convergence in the sense of geometric measure theory can be much more complicated than the pattern described above. Nevertheless, in this paper, any multiplicity 2 convergence would be the above pattern.

3. Free boundary cases

In this section, we study the solutions to (2-5). The main goal is to construct an immortal smooth free boundary mean curvature flow within the solid cylinder C_R, converging to the free boundary disk with multiplicity 2 as time approaches ∞. The explicit statements are in Theorems 3.14 and 3.23.

Let us sketch the idea of proof. First, we use the idea of Altschuler, Angenent, and Giga [Altschuler et al. 1995] in the study of rotationally symmetric mean curvature flow to show that the singularities of rotationally symmetric mean curvature flow with free boundary can only occur either on the rotational axis or on the free boundary.

Next, we construct a family of rotational symmetric free boundary surfaces $\{M^s\}_{s \in [0,1]}$, such that under the MCF, $M^0(t)$ has a singularity on the axis while $M^1(t)$ has a singularity on the boundary. Then an interpolation argument shows that there exists $s_0 \in [0, 1]$, such that the MCF starting from M^{s_0} is smooth for all future time.

By the first variational formula,

$$\int_0^\infty \int_{M^{s_0}(t)} |\vec{H}|^2 \, d\mathcal{H}^2 \, dt \leq \text{Area}(M^{s_0}) < +\infty.$$

This implies that $M^{s_0}(t)$ subsequentially converges to a stationary varifold with free boundary. To show that the limit is a plane with multiplicity 2, we need a gradient estimate, as stated in Proposition 3.20. This estimate is inspired by [Altschuler et al. 1995].

[1] Moreover, by Brakke's regularity theorem (see [Brakke 1978; White 2005]), or the Ecker–Huisken estimate [1991], this implies that $\|f_1(\cdot, t)\|_{C^l} \to 0$, $\|f_2(\cdot, t)\|_{C^l} \to 0$ for any $l \in \mathbb{Z}_+$.

3.1. ***Rotationally symmetric MCF with free boundary within a cylinder.*** We will introduce the following notation throughout this and the next section. Suppose $f(\cdot,t)$ is a solution to (2-5); $u(\cdot,t)$ and $v(\cdot,t)$ will denote the vertical graph function and the horizontal graph function of $f(\cdot,t)$.

Proposition 3.1. *Suppose $\Gamma_t = S(G_{f(\cdot,t)})$, where $f(\cdot,t)\in\mathcal{F}_R$ is a solution to (2-5). If the initial condition $f(\cdot,0)\in\mathcal{G}_R$, and $T=T(f(\cdot,0))>0$, then $f(\cdot,t)\in\mathcal{G}_R$ for all $t\in(0,T)$.*

Proof. Recall that $f(\cdot,0)\in\mathcal{G}_R$ means $u(\cdot,0)$ and $v(\cdot,0)$ satisfy the derivative condition (2-4). Since $u(\cdot,t)$ and $v(\cdot,t)$ are solutions to (2-5), we know that u_y and v_x satisfy the equations

$$\begin{aligned} \frac{\partial u_y}{\partial t} &= \frac{(u_y)_{yy}}{1+(u_y)^2} - 2\frac{u_y[(u_y)_y]^2}{(1+(u_y)^2)^2} + \frac{u_y}{u^2}, \\ \frac{\partial v_x}{\partial t} &= \frac{(v_x)_{xx}}{1+(v_x)^2} - 2\frac{v_x[(v_x)_x]^2}{(1+(v_x)^2)^2} + \frac{(v_x)_x}{x} - \frac{v_x}{x^2}. \end{aligned} \tag{3-1}$$

By the strict maximum principle of the quasilinear equation, we claim that for $t>0$,

$$u_y(y,t)>0, \quad v_x(x,t)>0, \tag{3-2}$$

and the proof is complete. □

Remark 3.2. We note that our initial data may not be smooth but only Lipschitz later on. (see Section 3.2). Nevertheless, we can approximate our initial data by smooth monotone increasing functions. Therefore we can still obtain the desired monotonicity for $t>0$.

It follows from Proposition 3.1 that, if $f(\cdot,0)\in\mathcal{G}_R$, then $f(\cdot,t)\in\mathcal{G}_R$. By the derivative condition (2-3), we know that

$$f(R,t)=\max\{y \mid (x,y)\in G_{f(\cdot,t)} \text{ for some } x\in[0,R]\},$$

and $f(R,t)$ can be viewed as the height of the curve $G_{f(\cdot,t)}$; we prove in the next lemma that it is monotone.

Lemma 3.3. *$f(R,t)$ is a decreasing function in t, and it is strictly decreasing if $f(\cdot,0)$ is not a constant function.*

Proof. The horizontal line $y=f(R,t)$ is on top of the curve $G_{f(\cdot,t)}$, and remains static under the mean curvature flow. By Proposition 2.2, for any $t<t'$, $y=f(R,t)$ is on top of the curve $G_{f(\cdot,t')}$; thus $f(R,t')\le f(R,t)$.

Moreover, if $f(\cdot,0)$ is not a constant function, then the strong maximum principle [Stahl 1996a, Theorem 3.1] yields the result. □

Although Lemma 3.3 shows that the height of the profile curve of the flow strictly decreases, we do not have a quantitative estimate of the decreasing rate. To obtain such a bound, we need to construct a new family of barriers.

Given $a\in(0,R)$, Consider a smooth function $l_a\in C^\infty([0,R])$ such that

$$l_a(x)=\begin{cases} 0 & \text{for } 0\le x\le \dfrac{a}{2}, \\ 1 & \text{for } a\le x\le R, \end{cases}$$

and $l_a'(x) > 0$ for $x \in (a/2, a)$. Let Γ^a be the hypersurface given by rotating the graph of l_a along the y-axis, and denote the corresponding solution to the mean curvature flow equation by $\Gamma^a(t)$. Since the initial surface $\Gamma^a(0) = \Gamma^a$ is the graph of a Lipschitz function, it follows from the rotational symmetry and the results in [Ecker and Huisken 1989] that there exists a smooth solution $L_a(x, t)$ to the horizontal graph equation with initial data $L_a(x, 0) = l_a(x)$, and boundary condition $\frac{\partial}{\partial x} L_a(0, t) = \frac{\partial}{\partial x} L_a(R, t) = 0$, such that $\Gamma^a(t)$ is given by rotating the graph of $L_a(x, t)$ along the y-axis.

The following lemma shows that after evolving for a sufficiently long time, the height of the $L_a(x, t)$ will decrease for a definite amount.

Lemma 3.4. *There exist a constant $\beta_{a,R} > 0$ and a time $T_{a,R} > 0$ such that $L_a(R, t) < 1 - \beta_{a,R}$ for all $t > T_{a,R}$.*

Proof. By the maximum principle, we have $0 < L_a(x, t) < 1$ for all $x \in (0, R)$, $t > 0$. In addition, $\frac{\partial}{\partial x} L_a \geq 0$ is bounded. By a similar argument as in the proof of Lemma 3.3, we know that $L_a(R, t)$ is a strictly decreasing function of t. □

Next, we will examine the behavior of the neck point when the singularity emerges during the mean curvature flow.

In the proof of Lemma 3.5, Corollary 3.7, Lemma 3.8, and Proposition 3.9, we use the following setting and notation. Let $f(\cdot, t) \in \mathcal{F}_R$ be a solution to (2-5), with the initial condition $f(\cdot, 0) \in \mathcal{G}_R$, and first singular time $T > 0$. By Proposition 3.1, we know that $f(\cdot, t)$ has a positive derivative with respect to x except at the two endpoints. Then by the inverse function theorem, we can extend the vertical graph equation $u(\cdot, t)$ to be defined on $[0, f(R, t)]$, and $u(\cdot, t)$ is smooth on $[0, f(R, t))$, for $t \in (0, T)$.

We adapt the idea from [Altschuler et al. 1995, Theorem 4.3] to get the following gradient estimate.

Lemma 3.5. *Given $T > 0$, there exists a continuous nonincreasing function $\sigma : (0, R/2] \to \mathbb{R}_+$ that only depends on the neck point of the initial condition (i.e., $u(0, 0)$) and T, such that*

$$0 < u_y(y, t) \leq e^{\sigma(\delta)/t}, \qquad \delta = \min\{u(y, t), R - u(y, t)\},$$

holds for all $0 < t < T$, $0 < y < f(R, t)$.

Proof. Let $a = u(0, 0)$. As shown in (3-1), we know $p(x, t) = \frac{\partial}{\partial x} L_a(x, t)$ satisfies the linear parabolic equation

$$\frac{\partial p}{\partial t} = \frac{1}{1 + p^2} p_{xx} - \left(\frac{2pp_x}{(1 + p^2)^2} - \frac{1}{x} \right) p_x - \frac{1}{x^2} p.$$

Since $p(x, 0)$ is positive on $(a/2, a)$, it follows from equation (4.3) in the proof of Theorem 4.3(b) in [Altschuler et al. 1995] that for every $\delta > 0$, there exists a constant $A_\delta < \infty$ such that

$$p(x, t) \geq e^{-A_\delta/t} \tag{3-3}$$

for all $\delta \leq x \leq R - \delta$ and all $0 < t < T$. We can choose the constant A_δ so that it is nonincreasing in δ.

Next, we translate $\Gamma^a(t)$ along the y-axis by ξ to get a new mean curvature flow $\Gamma^a_\xi(t)$. Its profile curve is given by

$$y = L_a(x, t) + \xi.$$

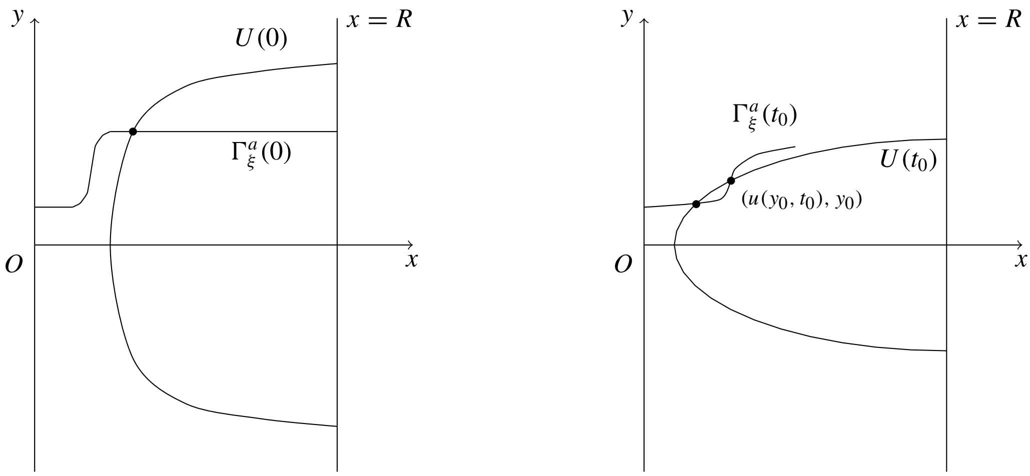

Figure 3. Proof of Lemma 3.5. Left: initial conditions of these flows. Right: by the Sturmian theorem, the phenomenon in this figure cannot happen.

Since $\frac{\partial}{\partial x} L_a(x,t) > 0$ for all $t > 0$, for each $t > 0$, denote the inverse function of $L_a(x,t)$ by $w(y,t)$. Therefore we can also represent Γ^a_ξ by the curve $x = w(y-\xi, t)$. Now we consider $0 < t_0 < T$ and $0 < y_0 < f(R, t_0)$. There is a unique $\xi \in \mathbb{R}$ with

$$w(y_0 - \xi, t_0) = u(y_0, t_0). \tag{3-4}$$

Let $U(t)$ be the union of the graph of $u(y,t)$ and its reflection with respect to the x-axis (the graph of $u(-y,t)$). By the Sturmian theorem, $U(t)$ and the graph of $w(y-\xi,t)$ cannot have fewer intersections when $t < t_0$ than what they have when $t = t_0$.

As $t \downarrow 0$, the graph of $w(y-\xi,t)$ (i.e., $\Gamma^a_\xi(t)$) converges to $\Gamma^a_\xi(0)$. This hypersurface intersects $U(0)$ exactly once (since the neck point of $U(0)$ is $(a,0)$). See Figure 3, left. Therefore, the graph of $w(y-\xi,t_0)$ and $U(t_0)$ only intersect once.

We prove $u_y(y_0,t_0) \le w_y(y_0-\xi,t_0)$ by contradiction. If $u_y(y_0,t_0) > w_y(y_0-\xi,t_0)$, then for $\epsilon > 0$ small, $u(y_0-\epsilon,t_0) < w(y_0-\epsilon-\xi,t_0)$. By using the monotonicity of the functions u, w, we know that as the y coordinate of $U(t_0)$ decreases from y_0 to 0, the x coordinate of $U(t_0)$ is at least $u(0,t_0)$, and $u(0,t_0) > 0$ (otherwise a singularity at the origin appears at time t_0). On the other hand, the x coordinate of the graph of $w(y-\xi,t_0)$ decreases to 0. By applying the intermediate value theorem, $U(t_0)$ and the graph of $w(y-\xi,t_0)$ intersect at least twice, which contradicts the fact that they have at most one intersection point; see Figure 3, right.

Let $\delta = \min\{u(y_0,t_0), R - u(y_0,t_0)\}$. By (3-3), (3-4), and the inverse function theorem,

$$u_y(y_0,t_0) \le w_y(y_0-\xi,t_0) \le e^{A_\delta/t_0}. \qquad \square$$

Remark 3.6. From the free boundary condition, we can see that $\sigma(\delta) \to \infty$ as $\delta \to 0$.

By Lemma 3.3, we know the limit of the height $h = \lim_{t\to T} f(R,t)$ exists. The above gradient estimate implies that if the height of the function $f(\cdot\,,t)$ tends to 0, then the neck point must tend to the boundary of the cylinder. In other words, the flow must shrink to a point on the boundary.

Corollary 3.7. *If* $h = 0$, *then* $\lim_{t\to T} u(0,t) = R$.

Proof. We prove this by contradiction. Suppose not, then there exists $0 < \epsilon < \frac{R}{2}$ and an increasing sequence $\{t_i\}$ such that $\lim_{i\to\infty} t_i = T$, $\lim_{i\to\infty} u(0,t_i) < R - \epsilon$.

Then $f(x,t_i)$ is well-defined on $[R-\epsilon, R-\epsilon/2]$. Let $a(t_i) = f(R-\epsilon, t_i)$ and $b(t_i) = f(R-\epsilon/2, t_i)$. Then by Lemma 3.5,

$$\frac{\epsilon}{2} = u(b(t_i),t_i) - u(a(t_i),t_i) = \int_{a(t_i)}^{b(t_i)} u_y(y,t_i)\,dy \le (b(t_i)-a(t_i))e^{\sigma(\epsilon/2)/t_i}.$$

Hence $f(R,t_i) \ge b(t_i) \ge \frac{1}{2}\epsilon e^{-\sigma(\epsilon/2)/t_i} \ge \frac{1}{2}\epsilon e^{-\sigma(\epsilon/2)/T}$ for all i. Moreover $\lim_{t_i\to T} f(R,t_i) \ge \frac{1}{2}\epsilon e^{-\sigma(\epsilon/2)/T} > 0$, which contradicts $\lim_{t\to T} f(R,t) = 0$. □

On the other hand, if the limit of the height h is not zero, we obtain an improved gradient estimate.

Lemma 3.8. *If* $h > 0$, *then for any* $0 < a < h$, *let* $\lambda = \pi/(h-a)$. *There exists a constant* $\varepsilon > 0$ *such that* $u_y(y,t) \ge \varepsilon e^{-\lambda^2 t}\sin(\lambda(y-a))$ *for all* $y \in [a,h]$, $0 \le t < T$. *In addition,* $u(a,t) \le R - \frac{2\varepsilon}{\lambda}e^{-\lambda^2 T}$.

Proof. For $0 \le t < T$, $u(y,t)$ is well-defined on $[0,h]$ and $u_y(y,t) > 0$ for $y \in (0,h]$. We know u satisfies the vertical graph equation

$$u_t = \frac{u_{yy}}{1+(u_y)^2} - \frac{1}{u}.$$

Define $\theta(y,t) = \arctan u_y(y,t)$. Then $\theta(y,t) \in (0,\pi/2)$ for $y \in [a,h]$, and

$$u_t = \theta_y - \frac{1}{u}, \quad \theta_t = \frac{1}{1+(u_y)^2}(u_y)_t = \frac{1}{1+(u_y)^2}(u_t)_y = \frac{1}{1+(u_y)^2}\left(\theta_{yy} + \frac{u_y}{u^2}\right).$$

Hence $\theta_t - \theta_{yy}/(1+(u_y)^2) > 0$ for $y \in [a,h]$. Since $\theta(y,0) > 0$ for all $y \in [a,h]$, let

$$\varepsilon = \min_{y\in[a,h]} \theta(y,0) > 0, \quad \varphi(y,t) = \varepsilon e^{-\lambda^2 t}\sin(\lambda(y-a)).$$

Then $\varphi_t = \varphi_{yy}$, $\varphi_{yy} \le 0$ on $[a,h]$, and thus

$$\varphi_t - \frac{\varphi_{yy}}{1+(u_y)^2} = \varphi_{yy}\frac{(u_y)^2}{1+(u_y)^2} \le 0.$$

As a consequence

$$\varphi_t - \frac{\varphi_{yy}}{1+(u_y)^2} < \theta_t - \frac{\theta_{yy}}{1+(u_y)^2}, \quad \varphi(y,0) \le \varepsilon \le \theta(y,0), \quad \varphi(a,t) = 0 < \theta(a,t), \quad \varphi(h,t) = 0 < \theta(h,t).$$

We apply the classical maximum principle to conclude that $\theta(y,t) \ge \varphi(y,t)$ for all $a \le y \le h$, $0 \le t < T$. Therefore

$$u_y(y,t) = \tan\theta(y,t) \ge \theta(y,t) \ge \varphi(y,t) = \varepsilon e^{-\lambda^2 t}\sin(\lambda(y-a)),$$

$$u(h,t) - u(a,t) = \int_a^h u_y(y,t)\,dy \ge \int_a^h \varepsilon e^{-\lambda^2 t}\sin(\lambda(y-a))\,dy = \frac{2\epsilon}{\lambda}e^{-\lambda^2 t}.$$

This implies $u(a,t) \le R - \frac{2\varepsilon}{\lambda}e^{-\lambda^2 T}$. □

Now we are ready to describe the possible singular behaviors of the rotationally symmetric flows that we are interested in. There is a trichotomy: either the flow remains smooth forever, or it has a neck singularity on the rotational axis, or it shrinks to a singularity on the boundary.

Proposition 3.9. *Suppose $f(\cdot\,,t)$ is a solution to (2-5) with initial condition $f(\cdot\,,0)\in\mathcal{G}_R$. Then the flow first becomes singular at time T if and only if $\lim_{t\to T}u(0,t)=0$ or $\lim_{t\to T}u(0,t)=R$. In addition, if such T doesn't exist, then the mean curvature flow exists for all future time.*

Proof. It is clear that if $\lim_{t\to T}u(0,t)=0$, then a neck pinch singularity appears at the origin, and if $\lim_{t\to T}u(0,t)=R$, a singularity appears at the boundary. Now we assume neither of the above happens, and we want to show that T is not a singular time.

We prove by contradiction and assume that a singularity appears at time T. By Corollary 3.7, $h>0$. We claim that there exists $\epsilon_1>0$ such that $u(0,t)>\epsilon_1$ for all $0\le t<T$. Otherwise there exists a sequence $\{t_i\}$ in $[0,T)$ such that $\lim_{i\to\infty}u(0,t_i)=0$. Up to extracting from a subsequence, we can assume t_i converges to $T'\in[0,T]$. Since $u(0,T)\neq 0$, $T'<T$, and a singularity appears at the origin at time T', which is a contradiction.

By a similar argument, we can also assume that $u(0,t)<R-\epsilon_1$ for all $0\le t<T$.

For any $0<a<h$, by Lemma 3.8, we know $u(a,t)<R-\frac{2\varepsilon}{\lambda}e^{-\lambda^2T}$. Let

$$\epsilon=\min\Big\{\epsilon_1,\ \frac{2\varepsilon}{\lambda}e^{-\lambda^2T}\Big\}.$$

We know $\epsilon<u(0,t)<u(a,t)<R-\epsilon$.

We claim that $u(y,t)$ is smooth at $(y,t)\in[0,a]\times[0,T]$. This follows from $u_y(0,t)=0$, a priori estimate for u_y in Lemma 3.5, and hence (see [Ladyzhenskaya et al. 1968]) for all higher derivatives of u in the interior. Therefore the singularity can only appear on the boundary, i.e., at (R,h).

Now consider the horizontal graph function $v(x,t)$, which is defined for $R-\epsilon_1\le x\le R,\ 0<t<T$, and it is uniformly bounded by the height of the initial condition. This function is a solution of the horizontal graph equation, so the Evans–Spruck estimates [1992, Corollary 5.3] (also see [Altschuler et al. 1995, page 303]) imply that ∇v as well as all higher space derivatives of v are uniformly bounded on the region $\{(x,t)\mid R-\epsilon_1/2\le x\le R,\ T/2<t<T\}$. Hence $v(x,t)\to v(x,T)$ uniformly in $R-\epsilon_1/2\le x\le R$ as $t\nearrow T$. We have also shown that $v_t(r,t)$ is uniformly bounded for $R-\epsilon_1/2\le x\le R,\ T/2<t<T$, hence $(R,h)=(R,v(R,T))$ cannot be a singular point. □

In the following proposition, we show that the appearance of the neck singularity is an open condition.

Proposition 3.10. *Let $\mathscr{L}$ denote the set of functions $f_0\in\mathcal{G}_R$ such that the solution $f(\cdot\,,t)$ to (2-5) with initial condition $f(\cdot\,,0)=f_0$ becomes singular in finite time, and $\lim_{t\to T}u(0,t)=0$ for some $T>0$. Then $\mathscr{L}$ is an open set in $\mathcal{G}_R$ with respect to the C^1 norm.*

Proof. Suppose $f_0\in\mathscr{L}$, and becomes singular as $t\to T$. It suffices to show that there exists $\epsilon>0$, such that for any $\hat{f}_0$ with $\|\hat{f}_0-f_0\|_{C^1}\le\epsilon$, $\hat{f}_0\in\mathscr{L}$ as well.

Because $\lim_{t\to T}u(0,t)=0$, the mean curvature flow $\Gamma_t=S(G_{f(\cdot\,,t)})$ has a singularity at the origin. By [Altschuler et al. 1995], this is a neck singularity, i.e., $(T-t)^{-1/2}\Gamma_t$ converges to the cylinder $S^1(\sqrt{2})\times\mathbb{R}$

smoothly in B_r, for any $r > 0$, as $t \to T$. In particular, there exists $t_0 < T$, such that $(T - t_0)^{-1/2}\Gamma_{t_0}$ has distance at most $\frac{1}{2}$ away from $S^1(\sqrt{2}) \times \mathbb{R}$, inside the ball $B_{10\alpha}$.

Suppose $\hat{f}(\cdot\,, t)$ is the solution to (2-5) with initial condition $\hat{f}(\cdot\,, 0) = \hat{f}_0$. By the continuity of the mean curvature flow with respect to the initial data, for any $\delta > 0$, there exists $\epsilon > 0$, such that whenever $\|\hat{f}_0 - f_0\|_{C^1} \le \epsilon$, $(T-t_0)^{-1/2}\widehat{\Gamma}_{t_0} = (T-t_0)^{-1/2}S(G_{\hat{f}(\cdot\,,t_0)})$ has distance at most $\frac{1}{2}$ away from $(T-t_0)^{-1/2}\Gamma_{t_0}$. In particular, $(T - t_0)^{-1/2}\widehat{\Gamma}_{t_0}$ has distance at most 1 away from $S^1(\sqrt{2}) \times \mathbb{R}$, inside the ball $B_{10\alpha}$.

Using an Angenent torus as a barrier, we see that $\widehat{\Gamma}_t$ has a finite time singularity, which must occur on the rotational axis. This implies that $\hat{f}_0 \in \mathscr{L}$. □

3.2. *Interpolation family of surfaces.* We consider a family of initial data. Let

$$\rho_\delta = \left\{(\delta, y) \mid y \in \left[0, \frac{\alpha}{\delta}\right]\right\} \cup \left\{\left(x, \frac{\alpha}{\delta}\right) \mid x \in [\delta, \infty)\right\}, \quad \delta \in (0, \infty),$$

and let

$$\rho_{\delta,R} = \rho_\delta \cap \{(x, y) \mid x \le R\} \tag{3-5}$$

for $R > 0$. For any $0 < \delta_1 < \delta_2 < R$, $\rho_{\delta_1,R}$ is on top of $\rho_{\delta_2,R}$, and they form a (singular) foliation.

Remark 3.11. For $\delta \in (0, R)$, even though our initial data $\rho_{\delta,R}$ are not contained in the set of graphs of functions in $\mathcal{G}_R$, they are locally Lipschitz. In fact, Stahl [1996b] obtained a local C^1 estimate of the free boundary mean curvature flow; see [Stahl 1996b, Remark 6.14]. Therefore, one can use an approximation argument to show that there exists a family of functions $f(t) \in \mathcal{G}_R$, $t \in (0, \epsilon)$, for small $\epsilon > 0$, such that $S(G_{f(t)})$ evolves by its mean curvature, and converges to $S(\rho_{\delta,R})$ as $t \to 0$. Such an approximation argument has been used by Ecker and Huisken [1991, Theorem 4.2] and we refer the readers to the discussion before that theorem. Therefore we can apply all arguments above to solutions with initial data $\rho_{\delta,R}$.

Let $f_\delta(\cdot\,, t)$ be the family of solutions to (2-5) with initial data $\rho_{\delta,R}$. Denote the vertical graph function and the horizontal graph function of $f_\delta(\cdot\,, t)$ by $u_\delta(\cdot\,, t)$ and $v_\delta(\cdot\,, t)$. By Proposition 2.1, we can write the first singular time of $f_\delta(\cdot\,, t)$ as $T(\rho_{\delta,R})$.

We need the following lemmas to study the flows with a finite time singularity, and the flow that exists for all future time.

Lemma 3.12. *Given $\delta > 1$, $R > 2\alpha$. If $\lim_{t\to T} u_\delta(0, t) = 0$, then $u_\delta(0, t) \le \alpha < R$ for all $t \in [0, T(\rho_{\delta,R}))$.*

Proof. Since $\delta > 1$, by Proposition 2.2, the horizontal line $y = \alpha$ is always on top of $G_{f_\delta(\cdot\,,t)}$.

We prove this lemma by contradiction. Suppose $u_\delta(0, t_1) > \alpha$ for some $t_1 \in [0, T(\rho_{\delta,R}))$. We choose ϵ small so that $0 < \epsilon < \min\left\{u_\delta(0, t_1) - \alpha, \frac{1}{100}\alpha\right\}$. Consider the curve

$$\mathscr{C} : \left\{(x, y) \,\middle|\, x = (\alpha + \epsilon)\cosh\left(\frac{y}{\alpha + \epsilon}\right),\ y \ge 0\right\}$$

(the upper half profile curve of the catenoid with neck point $(\alpha + \epsilon, 0)$). $\mathscr{C}$ intersect the boundary $x = R$ at

$$(R, \zeta_{R,\alpha,\epsilon}), \quad \text{where } \zeta_{R,\alpha,\epsilon} := (\alpha + \epsilon)\ln\left(\frac{R}{\alpha + \epsilon} + \sqrt{\frac{R^2 - (\alpha + \epsilon)^2}{(\alpha + \epsilon)^2}}\right).$$

Since $R > 2\alpha$, and $\epsilon < \frac{1}{100}\alpha$, we have $\zeta_{R,\alpha,\epsilon} > \alpha$. Thus $\mathscr{C}$ has exactly one intersection point with $\rho_{\delta,R}$.

By the Sturmian theorem, any new intersection point between $\mathscr{C}$ and $G_{f_\delta(x,t)}$ can only appear on the boundary. But we know for $t \in [0, T(\rho_{\delta,R}))$, $f_\delta(R,t) \le \alpha < \zeta_{R,\alpha,\epsilon}$; thus there will be no new intersection point appearing on the boundary.

Because $u_\delta(0,t_1) > \alpha+\epsilon$, $f_\delta(R,t_1) < \zeta_{R,\alpha,\epsilon}$, $G_{f_\delta(x,t_1)}$ and $\mathscr{C}$ cannot intersect, otherwise they have at least two intersection points. Then $\mathscr{C}$ is on top of $G_{f_\delta(x,t_1)}$, which implies $u_\delta(0,t) \ge \alpha+\epsilon$ for all $t \ge t_1$. This contradicts $\lim_{t\to T} u_\delta(0,t) = 0$. □

Remark 3.13. In fact, using the parabolic Sturmian theorem to compare the flow with catenoids, one can show that the neck point moves monotonically to the origin after a sufficiently long time. We do not need this fact and hence we omit the proof here.

We are now ready to proceed with the proof for the free boundary version of our main theorem.

Theorem 3.14. *Given $R > 2\alpha$, there exists an immortal rotationally symmetric free boundary mean curvature flow of surfaces in C_R, which is not a static plane.*

Proof. For $\delta \le 1$, we know that the curve $\rho_{\delta,R}$ is on top of the profile curve of the Angenent torus $\mathcal{A}$. Since the neck point of $\mathcal{A}$ tends to the origin under the mean curvature flow at some finite time T', by the comparison principle, $u_\delta(0,t) \to 0$ at some time $t < T'$. By the comparison principle, Proposition 2.2, if $u_\delta(0,t) \to 0$ in finite time, then $u_{\delta'}(0,t) \to 0$ in finite time for all $0 < \delta' < \delta$.

For $\alpha \le \delta < R$, the curve $\rho_{\delta,R}$ lies within the region $[\delta, R] \times [0,1]$. In contrast, the catenoid, described by $x = \cosh y$, remains static under the mean curvature flow and it lies strictly above the line $y = 1$ within the interval $[\delta, R]$ (since $\cosh 1 < 2 < \alpha \le \delta$). Thus this catenoid is on top of $\rho_{\delta,R}$, and the function $y = \ln(x + \sqrt{x^2-1})$ has positive derivative at $x = R$. Thus by Proposition 2.2, this catenoid is on top of $G_{f_\delta(\,\cdot\,,t)}$ for all time t, which implies $u_\delta(0,t) \ge 1$ for all t.

Hence, by Propositions 2.4 and 3.10, there exists a maximal interval $(0, n_R)$ such that for any δ within this interval, $u_\delta(0,t)$ converges to 0 in finite time. As indicated in the preceding argument, $1 < n_R \le \alpha$. By Propositions 2.2 and 3.9, the singular time $T(\rho_{\delta,R})$ is strictly increasing in $\delta \in (0, n_R)$, and its limit as $\delta \to n_R$ has to be ∞, otherwise $u_{n_R}(0,t)$ will converge to 0 in finite time by Proposition 2.4.

By the selection of n_R, $u_{n_R}(0,t)$ never reaches 0 in finite time. Combining this with Lemma 3.12, we conclude that for $\delta \in (1, n_R)$ and $t \in [0, T(\rho_{\delta,R}))$, we have $u_\delta(0,t) \le 2\alpha)$. Utilizing Proposition 2.4 and the fact that $\lim_{\delta\to n_R} T(\rho_{\delta,R}) = \infty$, we can deduce that $u_{n_R}(0,t) \le 2\alpha$ for all t.

As a result, $u_{n_R}(0,t)$ does not converge to either 0 or R within any finite time. By Proposition 3.9, the solutions $u_{n_R}(\,\cdot\,,t)$ and $v_{n_R}(\,\cdot\,,t)$ exist for all time $t \in [0,\infty)$. □

From the construction above, we know that the solution $f_{n_R}(\,\cdot\,,t)$ to (2-5) exists for all time $t \in [0,\infty)$. In the following, we will show the free boundary mean curvature flow induced from $f_{n_R}(\,\cdot\,,t)$ will converge to the plane with multiplicity 2.

Next, we show that the neck point of the function f_{n_R} converges to 0 as $t \to \infty$.

Lemma 3.15.
$$\lim_{t\to\infty} u_{n_R}(0,t) = 0.$$

Proof. We prove this by contradiction. The idea is to compare the flow with the barrier that we constructed in Lemma 3.4. Suppose there exists $a > 0$, and a sequence $\{t_i\}_{i\in\mathbb{Z}_+}$, $t_i \nearrow \infty$ such that $u_{n_R}(0, t_i) > a$. Up to extracting a subsequence, we may assume that $t_{i+1} > t_i + T_{a,R}$, where $T_{a,R}$ is the constant in Lemma 3.4.

Let $\chi(t)$ be the union of the graph of $f_{n_R}(\,\cdot\,, t)$ and its reflection with respect to the x-axis. $\chi(0)$ is bounded between the lines $y = \pm\alpha$. By the monotonicity of f_{n_R} with respect to x, we know that the graph of $l_a(x) + f_{n_R}(R, t_i) - 1$ is on top of $\chi(t_i)$ for all i. Since $L_a(x, t) + f_{n_R}(R, t_i) - 1$ solves the mean curvature flow equation, by comparison principle Proposition 2.2, we know that the graph of $L_a(x, t) + f_{n_R}(R, t_i) - 1$ is on top of $\chi(t + t_i)$.

By Lemma 3.4,

$$f_{n_R}(R, t_{i+1}) \le L_a(x, t_{i+1} - t_i) + f_{n_R}(R, t_i) - 1 < f_{n_R}(R, t_i) - \beta_{a,R}$$

for all i. Then $f_{n_R}(R, t_{i+1}) < f_{n_R}(R, t_1) - i\beta_{a,R}$. Let $i \to \infty$, we get a contradiction with the fact that $f_{n_R}(R, t) \ge 0$ for all t. Hence $\lim_{t\to\infty} u_{n_R}(0, t) = 0$. □

Remark 3.16. Since $1 < n_R \le \alpha$, and the flow $f_{n_R}(x, t)$ exists for all future time, by Lemmas 3.12 and 3.15, we have $u_{n_R}(0, t) \le \alpha$ for all $R > 2\alpha$ and $t \ge 0$.

Remark 3.17. The above argument works for all rotationally symmetric free boundary mean curvature flows $S(G_{f(\,\cdot\,,t)})$ that exist for all future time, which implies that their neck points must converge to 0 as $t \to \infty$.

As a consequence, we can prove the following height lower bound which depends on the location of the neck point for the solution f_{n_R}.

Corollary 3.18. *Given $R > 2\alpha$, if $u_{n_R}(0, t_0) \ge \kappa$ at time t_0, where $\kappa \in (0, R)$, then $f_{n_R}(R, t_0) > \frac{1}{2}\kappa$. Conversely, if $f_{n_R}(R, t_0) \le \kappa$ at time t_0, then $u_{n_R}(0, t_0) < 2\kappa$.*

Proof. Suppose $u_{n_R}(0, t_0) \ge \kappa$, we prove $f_{n_R}(R, t_0) > \frac{1}{2}\kappa$ by contradiction. Suppose by contrary that $f_{n_R}(R, t_0) \le \frac{1}{2}\kappa$. Then by Proposition 3.1, we know the graph of $f_{n_R}(\,\cdot\,, t_0)$ is bounded in the region $\{(x, y) \mid \kappa < x \le R,\ 0 \le y \le \frac{1}{2}\kappa\}$, and intersects the line $x = R$ orthogonally. Then consider the restricted curve $x = \frac{1}{2}\kappa \cosh(2y/\kappa)$, $x \le R$, which is on top of the graph of $f_{n_R}(\,\cdot\,, t_0)$, and has smaller intersection angle with the line $x = R$. By Proposition 2.2, we know that $u_{n_R}(0, t) > \frac{1}{2}\kappa$ for time $t \ge t_0$. However, By Lemma 3.15, we know that $u(0, t) \to 0$ as $t \to \infty$. This is a contradiction. □

We can also use the Angenent torus as a barrier to obtain a height upper bound.

Lemma 3.19. $f_{n_R}(x, t) < \alpha x$ *for all* $t \in [0, \infty)$, $x \in (u_{n_R}(0, t), R/\alpha]$.

Proof. We prove this by contradiction. Suppose not. Then $f_{n_R}(x_0, t) \ge \alpha x_0$ for some $t \in [0, \infty)$, $x_0 \in (u_{n_R}(0, t), R/\alpha]$. Then $G_{f_{n_R}(\,\cdot\,,t)}$ is on top of the Angenent torus $x_0\mathcal{A}$, which implies that f_{n_R} has a finite time singularity. This is a contradiction. □

We are ready to prove the key long-time gradient estimate of f_{n_R}.

Proposition 3.20. *For any $R > 2\alpha$, given $0 < a < b < R$, there exist a constant ω depending on a, R and a constant T depending on a, b, R, such that*

$$\frac{\partial}{\partial x} f_{n_R}(x,t) \leq \tan\left(\frac{\pi}{2}\frac{\ln R - \ln b}{\ln R - \ln a} + \frac{\omega}{\ln t}\right)$$

for all $x \in [b, R]$, $t \geq T$. In addition, $\lim_{t\to\infty} f_{n_R}(R,t) - f_{n_R}(b,t) = 0$.

Proof. Given any $0 < a < b < R$, by Lemma 3.15, we know that there exists $T > 10$ such that for any $t \geq T$, $u_{n_R}(0,t) < \frac{1}{2}a$. Then $f_{n_R}(x,t)$ is a smooth function over $[a, R]$ for all $t \geq T$. For simplicity, we will use $f(x,t)$ to express the function $f_{n_R}(x,t)$ restricted on the interval $[a, R]$, which is smooth for all time $t \geq T$. We know

$$f_t = \frac{f_{xx}}{1+f_x^2} + \frac{f_x}{x}, \quad f_x > 0 \quad \text{for all } x \in [a, R).$$

Let $\phi(x,t) = \arctan(f_x(x,t))$, $x \in [a, R]$, and $t \in [T, \infty)$. Then

$$0 \leq \phi(x,t) < \frac{\pi}{2} \quad \text{and} \quad f_x(x,t) = \tan(\phi(x,t)),$$

so

$$\phi_x(x,t) = \frac{f_{xx}(x,t)}{1+f_x^2(x,t)}, \quad f_t = \phi_x + \frac{f_x}{x}.$$

Moreover,

$$\phi_t = \frac{1}{1+f_x^2}(f_x)_t = \frac{1}{1+f_x^2}(f_t)_x = \frac{1}{1+f_x^2}\left(\phi_{xx} + \frac{f_{xx}}{x} - \frac{f_x}{x^2}\right).$$

Therefore,

$$\phi_t - \frac{\phi_x}{x} - \frac{1}{1+f_x^2}\phi_{xx} = -\frac{f_x}{x^2(1+f_x^2)} < 0. \tag{3-6}$$

Let $\mu = \pi/(2\ln(R/a))$ and ω be a constant to be determined later. Let

$$\varphi(x,t) = \mu \ln\frac{R}{x} + \frac{\omega - x}{\ln t}.$$

Then

$$\varphi_x = -\frac{\mu}{x} - \frac{1}{\ln t} < 0, \quad \varphi_{xx} = \frac{\mu}{x^2} > 0,$$

$$\varphi_t - \frac{\varphi_x}{x} - \frac{\varphi_{xx}}{1+f_x^2} \geq \varphi_t - \frac{\varphi_x}{x} - \varphi_{xx} = \frac{t\ln t - (\omega - x)x}{xt(\ln t)^2} \geq \frac{t\ln t - \omega R}{xt(\ln t)^2}.$$

Let $T' = T + R(R + \pi/2)$, $\omega = R + \frac{\pi}{2}\ln T'$. Then

$$t\ln t - \omega R \geq T'\ln T' - R\left(R + \frac{\pi}{2}\ln T'\right) \geq R^2(\ln T' - 1) > 0.$$

Thus for $x \in [a, R],\ t \geq T'$, we know that

$$\varphi(x, T') \geq \frac{\omega - x}{\ln T'} \geq \frac{\pi}{2} \geq \phi(x, T'),$$
$$\varphi_t - \frac{\varphi_x}{x} - \frac{\varphi_{xx}}{1+f_x^2} \geq 0 > \phi_t - \frac{\phi_x}{x} - \frac{\phi_{xx}}{1+f_x^2},$$
$$\varphi(a, t) \geq \mu \ln \frac{R}{a} = \frac{\pi}{2} \geq \phi(a, t),$$
$$\varphi(R, t) > 0 = \phi(R, t).$$

Now we can apply the parabolic maximum principle to conclude that $\varphi(x, t) \geq \phi(x, t)$ for all $x \in [a, R]$, $t \geq T'$. On the other hand, for any $x \in [b, R]$,

$$\varphi(x, t) \leq \frac{\pi}{2}\frac{\ln R - \ln b}{\ln R - \ln a} + \frac{\omega - b}{\ln t}.$$

Thus there exists $T'' > T'$ such that for all $x \in [b, R]$ and $t \geq T''$, $\varphi(x, t) < \frac{\pi}{2}$. In summary, for $x \in [b, R]$, $t \geq T''$, we have the gradient estimate

$$f_x(x, t) = \tan(\phi(x, t)) \leq \tan(\varphi(x, t)) \leq \tan(\varphi(b, t)) = \tan\left(\frac{\pi}{2}\frac{\ln R - \ln b}{\ln R - \ln a} + \frac{\omega - b}{\ln t}\right). \tag{3-7}$$

Next, integrating this gradient bound yields

$$f(R, t) - f(b, t) = \int_b^R f_x(x, t)dx \leq (R - b)\tan\left(\frac{\pi}{2}\frac{\ln R - \ln b}{\ln R - \ln a} + \frac{\omega - b}{\ln t}\right).$$

Let $t \to \infty$,

$$0 \leq \limsup_{t\to\infty}[f(R, t) - f(b, t)] \leq (R - b)\tan\left(\frac{\pi}{2}\frac{\ln R - \ln b}{\ln R - \ln a}\right).$$

Finally let $a \to 0$. We have $\lim_{t\to\infty}[f(R, t) - f(b, t)] = 0$. □

The gradient estimate yields the following two corollaries, which can be combined to show that $f_{n_R}(x, t)$ will C^1 converge to 0.

Corollary 3.21. *$f_{n_R}(x, t)$ converges to 0 uniformly as $t \to \infty$ for all $x \in (0, R]$.*

Proof. By Lemma 3.3, we know $\lim_{t\to\infty} f_{n_R}(R, t)$ exists. By Lemma 3.19 and Proposition 3.20, for any $0 < b \leq R/\alpha$,

$$0 \leq \lim_{t\to\infty} f_{n_R}(R, t) = \lim_{t\to\infty} f_{n_R}(b, t) \leq \alpha b.$$

Let $b \to 0$, we get $\lim_{t\to\infty} f_{n_R}(R, t) = 0$. Since $f_{n_R}(\,\cdot\,, t)$ is a strictly increasing function, thus $f_{n_R}(x, t)$ converges to 0 as $t \to \infty$ and this convergence is uniform in x. □

Corollary 3.22. *Given $0 < b < R$, $\frac{\partial}{\partial x} f_{n_R}(x, t)$ converges to 0 uniformly as $t \to \infty$ for all $x \in [b, R]$.*

Proof. For any $\epsilon > 0$, there exists $0 < a < b$ such that

$$\tan\left(\frac{\pi}{2}\frac{\ln R - \ln b}{\ln R - \ln a}\right) < \frac{\epsilon}{2}.$$

Let ω, T be as in Proposition 3.20. Then there exists $T' \geq T$ such that for $t \geq T'$,

$$\tan\left(\frac{\pi}{2}\frac{\ln R - \ln b}{\ln R - \ln a} + \frac{\omega}{\ln t}\right) < \epsilon.$$

By Proposition 3.20, for $x \in [b, R]$, $t \geq T'$,

$$\frac{\partial}{\partial x} f_{n_R}(x, t) \leq \tan\left(\frac{\pi}{2}\frac{\ln R - \ln b}{\ln R - \ln a} + \frac{\omega}{\ln t}\right) < \epsilon. \qquad \square$$

Theorem 3.23. *For any $R > 2\alpha$, $S(G_{f_{n_R}(\cdot,t)})$ is a rotationally symmetric free boundary mean curvature flow in C_R, and the flow exists for all future time. In addition, the forward limit of this flow is the free boundary disk $\{y = 0\} \cap C_R$ with multiplicity 2.*

Proof. Following from the proof of Theorem 3.14, we only need to prove that the forward limit of $S(G_{f_{n_R}(\cdot,t)})$ is the free boundary disk $\{y = 0\} \cap C_R$ with multiplicity 2. In other words, for any $\epsilon > 0$, $f_{n_R}(x, t)|_{x\in[\epsilon,R]}$ converges to 0 in C^1.

Given $\epsilon > 0$, by Lemma 3.15, we know that there exists $T > 0$ such that the neck point of $f_{n_R}(\cdot, t)$ is contained in $B_{\epsilon/(2\alpha)}(0)$ for $t \geq T$. So $f_{n_R}(x, t)|_{[\epsilon/(2\alpha),R]}$ is well defined for all $t \geq T$. By Lemma 3.19, $f_{n_R}(\cdot, t)|_{[0,\epsilon/(2\alpha)]}$ is contained in $B_\epsilon(0)$ for $t \geq T$. By Corollaries 3.21 and 3.22, we know $f_{n_R}(\cdot, t)|_{[\epsilon/(2\alpha),R]}$ C^1 converges to 0 as $t \to \infty$. This completes the proof. $\square$

4. Passing to limit in $\mathbb{R}^3$

For simplicity, we slightly modify the notation. Let $f_R(x, t)$ denote the solution in C_R that we obtained in the last section, and at time t, its graph is denoted by $\Gamma_{R,t}$. Let $u_R(x, t)$ be the vertical graph function of $f_R(x, t)$, and therefore the neck point of $f_R(x, t)$ is $u_R(0, t)$.

In this section, we let $R \to \infty$ to get a limit of the solutions that we have obtained in Section 3. There are two steps: first, we show that we can indeed find a limit mean curvature flow as $R \to \infty$. Second, we show that the limit mean curvature flow converges to the plane with multiplicity 2.

We start with some basic relationships between the solutions f_R with different R. Recall that n_R is the neck point value of the initial data of f_R, and the initial curve $\Gamma_{R,0}$ is $\rho_{n_R,R}$.

Lemma 4.1. $$n_r \leq n_R \quad \textit{for any } 2\alpha < r \leq R.$$

Proof. We argue by contradiction. If there exists $2\alpha < r \leq R$ such that $n_r > n_R$, then $\rho_{n_R,R}$ is strictly on top of $\rho_{n_r,r}$. By the derivative estimate (3-2), the intersection angle between $\Gamma_{R,t}$ and the line $x = r$ is less than $\frac{\pi}{2}$.

Consider the curve $\Gamma_{r,t}$, and the curve $\Gamma_{R,t}$ restricted to the region $x \leq r$. By Proposition 2.2, the distance between these two curves under the mean curvature flow is monotonically increasing, therefore their neck points cannot both converge to 0 as $t \to \infty$ $\square$

Using the fact $1 < n_R \leq \alpha$ for all $R > 2\alpha$ and combining with Lemma 4.1, we get the following corollary, which shows that the initial curves $\rho_{n_R,R}$ has a limit as $R \to \infty$.

Corollary 4.2. *There exists* $\eta \in (1, \alpha]$ *such that* $\lim_{R\to\infty} n_R = \eta$. *Therefore the initial curves* $\rho_{n_R,R}$ *converges to the curve* ρ_η *as* $R \to \infty$.

Next, to take a (subsequential) limit as $R \to \infty$, we need some uniform gradient estimates of f_R. In the following proposition, we use catenoids as barriers to derive a uniform gradient estimate of f_R away from the neck point.

Proposition 4.3. *For any* $R > 2\alpha + 2$, $x \in [\alpha + 2, R]$, *and any time* $t \geq 0$, *we have*

$$\frac{\partial}{\partial x} f_R(x, t) \leq \frac{\alpha + 1}{\sqrt{x^2 - (\alpha + 1)^2}}. \tag{4-1}$$

Proof. Consider the catenoid with neck point $(\alpha + 1, \xi)$, namely we move the catenoid with neck point $(\alpha + 1, 0)$ upward with distance ξ. It is static under the mean curvature flow, and its profile curve $\bar{\mathcal{C}}_\xi$ is given by

$$x = (\alpha + 1) \cosh\left(\frac{y - \xi}{\alpha + 1}\right).$$

Let $\mathcal{C}_\xi$ denote the graph of

$$g_\xi(x) = \xi + (\alpha + 1) \ln\left(\frac{x}{\alpha + 1} + \sqrt{\frac{x^2 - (\alpha + 1)^2}{(\alpha + 1)^2}}\right), \quad x \in [\alpha + 1, \infty).$$

Then $\bar{\mathcal{C}}_\xi$ is the union of $\mathcal{C}_\xi$ and its reflection with respect to the line $y = \xi$.

Now we compare the catenoids with the mean curvature flow $\Gamma_{R,t}$. Let $c = g_0(R) > g_0(2\alpha+2) \geq \alpha+1$. The y-coordinate of the intersection point between $\mathcal{C}_\xi$ and the boundary of the cylinder $\{x = R\}$ is $\xi + c$.

Recall that $\Gamma_{R,0} = \rho_{n_R,R}$ with $n_R \leq \alpha$, while the x-coordinates of any point on $\mathcal{C}_\xi$ is at least $\alpha + 1$. From the construction (3-5) of $\rho_{\delta,R}$, we observe that for $\xi \geq f_R(R, 0) - c$, there is at most one intersection point between $\mathcal{C}_\xi$ and $\Gamma_{R,0}$; for $\xi < f_R(R, 0) - c$, there is no intersection point between $\Gamma_{R,0}$ and $\mathcal{C}_\xi$. See Figure 4, left.

We make the following claims:

Claim 1. *For any* t, *there is at most one intersection point between* $\Gamma_{R,t}$ *and* $\mathcal{C}_\xi$.

Proof. By the Sturmian theorem, under the mean curvature flow, the intersection point between $\Gamma_{R,t}$ and $\mathcal{C}_\xi$ could only appear on the boundary $x = R$. As mentioned above, the boundary point of $\mathcal{C}_\xi$ is $(R, \xi + c)$.

If $\xi \geq f_R(R, 0) - c$, then for any time $t > 0$, $f_R(R, t) < f_R(R, 0) \leq \xi + c$, and no extra intersection points will appear on the boundary.

If $\xi < f_R(R, 0) - c$, by Lemma 3.3, the boundary point of $\Gamma_{R,t}$ moves toward $(R, 0)$ monotonically under the flow. Thus there will be at most one extra intersection point. □

Given time $t_1 \geq 0$, let $D(x) = g_0(x) - f_R(x, t_1)$, which is defined on $[\alpha+1, R]$. Then $D'(R) = g_0'(R) > 0$. Since $D'(x)$ is a smooth function, there exists a maximal interval $[p, R]$ such that $D'(x)$ is nonnegative on this interval. Then

$$d := D(R) - D(p) = \int_p^R D'(x)\, dx > 0.$$

Claim 2. *The lower bound of this maximal interval* $p \leq \alpha + 2$.

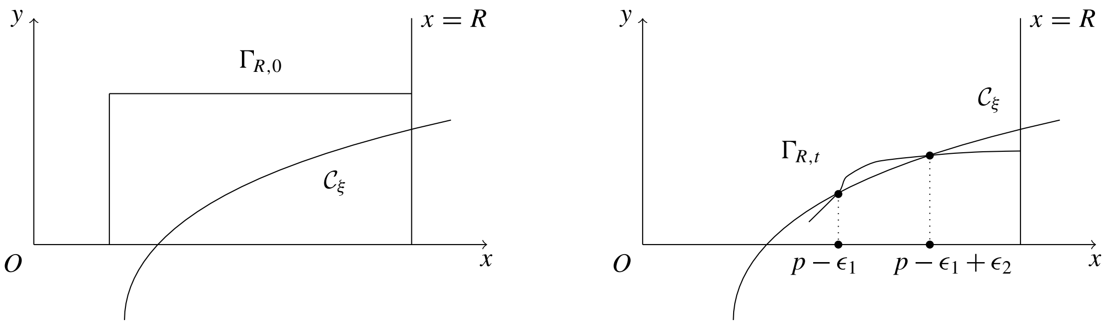

Figure 4. Proof of Proposition 4.3. Left: initial status of the curves. Right: by the Sturmian theorem, the phenomenon in this figure cannot happen.

Proof. We argue by contradiction. If $p > \alpha+2$, by the choice of p, there exists $\epsilon_1 > 0$ small enough such that $D'(p-\epsilon_1) < 0$, and $|D(p) - D(p-\epsilon_1)| < \frac{1}{2}d$, which implies $\frac{\partial}{\partial x} f_R(p-\epsilon_1, t_1) > g_0'(p-\epsilon_1)$ and $D(p-\epsilon_1) < D(R)$.

Let $\xi = -D(p-\epsilon_1)$. Then $\mathcal{C}_\xi$ and Γ_{R,t_1} intersect at $(p-\epsilon_1,\ f_R(p-\epsilon_1, t_1))$. Since $\frac{\partial}{\partial x} f_R(p-\epsilon_1, t_1) > g_0'(p-\epsilon_1)$, for small $\epsilon_2 > 0$, we have $f_R(p-\epsilon_1+\epsilon_2, t_1) > g_i(p-\epsilon_1+\epsilon_2)$. While on the boundary, we have

$$g_i(R) = \xi + c = -D(p-\epsilon_1) + g_0(R) > -D(R) + g_0(R) = f_R(R, t_1).$$

By the intermediate value theorem, there exists $x_1 \in (p-\epsilon_1+\epsilon_2,\ R)$ such that $g_\xi(x_1) = f_R(x_1, t_1)$. See Figure 4, right. Therefore we have two intersection points $(p-\epsilon_1,\ f_R(p-\epsilon_1, t_1)),\ (x_1,\ f_R(x_1, t_1))$ between Γ_{R,t_1} and $\mathcal{C}_\xi$, which contradicts Claim 1. □

By Claim 2, for any $x \in [\alpha+2,\ R]$, $D'(x) \geq 0$. This proves (4-1). □

Next, we prove a uniform gradient estimate of the vertical part of f_R near the neck point. First of all, we show the neck point of $\Gamma_{R,t}$ is nondecreasing in R.

Proposition 4.4. *The distance from the origin to the neck point of* $\Gamma_{R,t}$ *is nondecreasing in* R, *i.e.*, $u_{R_1}(0, t) \leq u_{R_2}(0, t)$ *for all* $2\alpha < R_1 < R_2$, $t \geq 0$.

Proof. Given $R_1 < R_2$, we use $\Gamma'_{R_2,t}$ to denote the curve $\Gamma_{R_2,t}$ restricted on the region $x \leq R_1$. We know $\Gamma_{R_1,0}$ is on top of $\Gamma'_{R_2,0}$. For any $t \geq 0$, $\frac{\partial}{\partial x} f_{R_1}(R_1, t) = 0$ and $\frac{\partial}{\partial x} f_{R_2}(R_1, t) > 0$.

From Lemma 4.1, the neck points of the initial conditions have the comparison $u_{R_1}(0, 0) \leq u_{R_2}(0, 0)$. Now we have two cases: either this inequality is a strict inequality, or $u_{R_1}(0, 0) = u_{R_2}(0, 0)$.

Case 1. If $u_{R_1}(0, 0) < u_{R_2}(0, 0)$, then at $t = 0$, there is no intersection point between $\Gamma_{R_1,0}$ and $\Gamma_{R_2,0}$. By the Sturmian theorem, either $\Gamma_{R_1,t}$ and $\Gamma_{R_2,t}$ have no intersection point, or there is a new intersection point that can only appear on the boundary.

If $f_{R_1}(R_1, t) > f_{R_2}(R_1, t)$ for all time $t \geq 0$, there will be no new intersection point appearing on the boundary. Therefore for any $t \geq 0$, $\Gamma_{R_1,t}$ and $\Gamma'_{R_2,t}$ do not intersect, which implies that $\Gamma_{R_1,t}$ is on top of $\Gamma'_{R_2,t}$, and $u_{R_1}(0, t) < u_{R_2}(0, t)$.

If $f_{R_1}(R_1,t) \le f_{R_2}(R_1,t)$ at some time t, let t_1 be the first time where this inequality holds. Since $\frac{\partial}{\partial x} f_{R_1}(R_1,t) < \frac{\partial}{\partial x} f_{R_2}(R_1,t)$, by the maximum principle, $f_{R_1}(R_1,t) < f_{R_2}(R_1,t)$ for all time $t > t_1$. Hence there will be at most one extra intersection point.

For time $t \le t_1$, we must have $f_{R_1}(R_1,t) \ge f_{R_2}(R_1,t)$, and we get $u_{R_1}(0,t) < u_{R_2}(0,t)$ by an argument similar to above.

For time $t > t_1$, we have $f_{R_1}(R_1,t) < f_{R_2}(R_1,t)$, and $\Gamma_{R_1,t}$ and $\Gamma'_{R_2,t}$ intersect at most at one point. If there is exactly one intersection point, then $u_{R_1}(0,t) < u_{R_2}(0,t)$. If there is no intersection point, $\Gamma'_{R_2,t}$ is on top of $\Gamma_{R_1,t}$, and by a similar argument as in Lemma 4.1, we get a contradiction.

In summary, $u_{R_1}(0,t) \le u_{R_2}(0,t)$ for all $t \ge 0$.

Case 2. If $u_{R_1}(0,0) = u_{R_2}(0,0) = a$ then $f_{R_1}(R_1,\epsilon) < f_{R_2}(R_1,0)$ for any small time $\epsilon > 0$, since the height is strictly decreasing.

If $u_{R_1}(0,\epsilon) \ge u_{R_2}(0,0)$, then $\Gamma'_{R_2,0}$ is on top of $\Gamma_{R_1,t}$. By a similar argument as in Lemma 4.1, we get a contradiction. Hence $u_{R_1}(0,\epsilon) < u_{R_2}(0,0)$, which implies that $\Gamma_{R_1,\epsilon}$ and $\Gamma'_{R_2,0}$ have exactly one intersection point.

Since $f_{R_1}(R_1,\epsilon) < f_{R_2}(R_1,0)$ and $\frac{\partial}{\partial x} f_{R_1}(R_1,t+\epsilon) = 0 < \frac{\partial}{\partial x} f_{R_2}(R_1,t)$, by the Sturmian theorem, $\Gamma_{R_1,t+\epsilon}$ and $\Gamma'_{R_2,t}$ have at most one intersection point. By the maximum principle, $f_{R_1}(R_1,t+\epsilon) < f_{R_2}(R_1,t)$. Therefore if there is exactly one intersection point, we get $u_{R_1}(0,t+\epsilon) < u_{R_2}(0,t)$. If there is no intersection point, $\Gamma'_{R_2,t}$ is on top of $\Gamma_{R_1,t+\epsilon}$. Again, by a similar argument as in Lemma 4.1, we get a contradiction.

Therefore, $u_{R_1}(0,t+\epsilon) < u_{R_2}(0,t)$ is true for all $\epsilon > 0$, by taking the limit as $\epsilon \to 0$, we claim that $u_{R_1}(0,t) \le u_{R_2}(0,t)$. □

From Remark 3.16, we know that $u_R(0,t) \le \alpha$ for all $R > 2\alpha$. Thus by Proposition 4.4, we know that $\eta(t) = \lim_{R\to\infty} u_R(0,t)$ exists.

Let $\tau > 2\alpha + 2$ be a large constant. Then $u_\tau(0,t) \le \eta(t) \le \alpha$ for all $t \ge 0$. Given any $T > 0$, let $\mu(T) = \min_{t\in[0,T]} u_\tau(0,t) > 0$. Then for any $R > \tau$, $t \in [0,T]$, we have $\mu(T) \le u_R(0,t) \le \alpha$.

Proposition 4.5. *Given $R > \tau$, $T > 0$. For any $y \in \left[0, \frac{1}{2}\mu(T)\right]$, $t \in [0,T]$, we have*

$$\frac{\partial}{\partial y} u_R(y,t) \le \frac{2\alpha e^{2y/\mu(T)}}{\mu(T)} \le \frac{2\alpha e}{\mu(T)}. \tag{4-2}$$

Proof. When $t = 0$,

$$\frac{\partial}{\partial y} u_R(y,t) = 0 \le \frac{2\alpha e^{2y/\mu(T)}}{\mu(T)}.$$

Now we focus on $t > 0$.

Let $h_\xi(y) = \frac{1}{2}\mu(T)\cosh(2(y-\xi)/\mu(T))$, defined on $[\xi,\infty)$. Its inverse function is given by

$$h_\xi^{-1}(x) = \xi + \frac{\mu(T)}{2} \ln\left(\frac{2x}{\mu(T)} + \sqrt{\frac{4x^2 - \mu(T)^2}{\mu(T)^2}}\right).$$

Since $u_R(0,t) \ge \mu(T)$ for all $t \in [0,T]$, by Corollary 3.18, $u_R(y,t)$ is well-defined for $y \in \left[0, \frac{1}{2}\mu(T)\right]$, $t \in (0,T]$.

Consider the catenoid with neck point $\big(\frac{1}{2}\mu(T), \xi\big)$. It is static under the mean curvature flow, and its profile curve $\bar{\mathcal{C}}_\xi$ is given by $x = h_\xi(y)$. Let $\mathcal{C}_\xi$ denote the graph of $y = h_\xi^{-1}(x)$, and $\bar{\mathcal{C}}_\xi$ be the union of $\mathcal{C}_\xi$ and its reflection with respect to the line $y = \xi$.

By the comparison principle, Proposition 2.2, for any $t \geq 0$, $\mathcal{C}_\xi$ cannot be on top of $\Gamma_{R,t}$. Therefore there exists $x(t) \in [u_R(0,t), R]$ such that $h_\xi^{-1}(x(t)) < f_R(x(t), t)$.

Let $c = h_0^{-1}(R)$. The y-coordinate of the intersection point between $\mathcal{C}_\xi$ and the boundary of the cylinder $\{x = R\}$ is $(R, \xi + c)$.

$\Gamma_{R,0} = \rho_{n_R,R}$ with $n_R \geq \mu(T)$. From the construction of $\rho_{\delta,R}$ in (3-5), we observe that for $\xi \geq f_R(R,0) - c$, there are at most two intersection points between $\Gamma_{R,0}$ and $\mathcal{C}_\xi$; for $\xi < f_R(R,0) - c$, there is at most one intersection point between $\Gamma_{R,0}$ and $\mathcal{C}_\xi$.

We have the following claims:

Claim 3. *For any $t \in [0,T]$, there are at most two intersection points between $\Gamma_{R,t}$ and $\mathcal{C}_\xi$.*

Proof. By the Sturmian theorem, the new intersection point between $\Gamma_{R,t}$ and $\mathcal{C}_\xi$ can only appear on the boundary $x = R$. As mentioned above, the boundary point of $\mathcal{C}_\xi$ is $(R, \xi + c)$.

If $\xi \geq f_R(R,0) - c$, then for any time $t > 0$, $f_R(R,t) < f_R(R,0) \leq \xi + c$, there will be no extra intersection point appearing on the boundary.

If $\xi < f_R(R,0) - c$, by Lemma 3.3, the boundary point of $\Gamma_{R,t}$ moves toward $(R, 0)$ monotonically under the flow; thus there will be at most one extra intersection point. □

Given time $t_1 \in (0,T]$, let $D(y) = h_0^{-1}(u_R(y,t_1)) - y$, which is well defined on $\big[0, \frac{1}{2}\mu(T)\big]$. Due to the uniqueness and smoothness of the solution $f_R(x,t)$, along with the reflexive symmetry of the profile curve, we have $\frac{\partial}{\partial y} u_R(0,t_1) = 0$. Then

$$D'(0) = (h_0^{-1})'(u_R(0,t_1)) \frac{\partial}{\partial y} u_R(0,t_1) - 1 = -1 < 0.$$

Since $D'(x)$ is a smooth function, there exists a maximal interval $[0,q]$ such that $D'(y)$ is nonpositive on this interval. Then $D(0) - D(q) = -\int_0^q D'(y)\,dy > 0$.

Claim 4. *The upper bound of the maximal interval $q \geq \frac{1}{2}\mu(T)$.*

Proof. We argue by contradiction. If $q < \frac{1}{2}\mu(T)$, by the choice of q, there exists a small $\epsilon_1 > 0$ with $q + \epsilon_1 < \frac{1}{2}\mu(T)$ such that $D'(q+\epsilon_1) > 0$, and $|D(q) - D(q+\epsilon_1)| < \frac{d}{2}$.

Thus $D(0) > D(q+\epsilon_1)$. By the inverse function theorem,

$$(h_0^{-1})'(u_R(q+\epsilon_1, t_1)) > \left(\frac{\partial}{\partial y} u_R(q+\epsilon_1, t_1)\right)^{-1} > \frac{\partial}{\partial x} f_R(u_R(q+\epsilon_1,t_1), t_1).$$

Let $\xi = -D(q+\epsilon_1)$. Then $\mathcal{C}_\xi$ and Γ_{R,t_1} intersect at $(u_R(q+\epsilon_1,t_1), q+\epsilon_1)$.

We know $\xi = q + \epsilon_1 - h_0^{-1}(u_R(q+\epsilon_1), t_1)) \leq q + \epsilon_1 - h_0^{-1}(\mu(T)) < \frac{1}{2}\mu(T) - \frac{3}{5}\mu(T) < 0$.

By the comparison principle, Proposition 2.2, $\mathcal{C}_0$ cannot be on top of Γ_{R,t_1}. Therefore there exists $x_1 \in [u_R(0,t), R]$ such that $h_0^{-1}(x_1) < f_R(x_1, t_1)$. Then

$$x_1 = u_R(f_R(x_1,t_1),t_1) > u_R(h_0^{-1}(x_1),t_1) \geq u_R(h_0^{-1}(u_R(0,t)),t_1) \geq u_R(h_0^{-1}(\mu(T)),t_1) > u_R\big(\tfrac{3}{5}\mu(T), t_1\big).$$

Since $\frac{\partial}{\partial x} f_R(u_R(q+\epsilon_1,t_1),t_1) < (h_0^{-1})'(u_R(q+\epsilon_1,t_1))$, there exists a small $\epsilon_2>0$ with

$$\epsilon_2 < \min\bigl\{u_R(q+\epsilon_1,t_1)-u_R(0,t_1),\ u_R\bigl(\tfrac{3}{5}\mu(T),t_1\bigr)-u_R(q+\epsilon_1,t_1)\bigr\}$$

such that

$$f_R(u_R(q+\epsilon_1,t_1)-\epsilon_2,t_1) > h_i^{-1}(u_R(q+\epsilon_1,t_1)-\epsilon_2),$$
$$f_R(u_R(q+\epsilon_1,t_1)+\epsilon_2,t_1) < h_i^{-1}(u_R(q+\epsilon_1,t_1)+\epsilon_2).$$

Now, for $x=u_R(0,t_1)$ and $x=x_1$, we have

$$f_R(u_R(0,t_1),t_1) = -D(0)+h_0^{-1}(u_R(0,t_1)) < -D(q+\epsilon_1)+h_0^{-1}(u_R(0,t_1)) = h_i^{-1}(u_R(0,t_1)),$$
$$f_R(x_1,t_1) > h_0^{-1}(x_1) > h_i^{-1}(x_1).$$

By the intermediate value theorem, there exist

$$x_2 \in (u_R(0,t_1),u_R(q+\epsilon_1,t_1)-\epsilon_2) \quad\text{and}\quad x_3 \in (u_R(q+\epsilon_1,t_1)+\epsilon_2,x_1)$$

such that $f_R(x_2,t_1)=h_0^{-1}(x_2)$, $f_R(x_3,t_1)=h_0^{-1}(x_3)$. Therefore there are three intersection points $(u_R(q+\epsilon_1,t_1),q+\epsilon_1)$, $(x_2, f_R(x_2,t_1))$, and $(x_3, f_R(x_3,t_1))$ between Γ_{R,t_1} and $\mathcal{C}_\xi$, which contradicts Claim 3. □

Now for any $y\in\bigl[0,\frac{1}{2}\mu(T)\bigr]$ and $t\in(0,T]$, by Claim 4, $D'(y)\le 0$. Thus

$$\frac{\partial}{\partial y}u_R(y,t) \le \frac{1}{(h_0^{-1})'(u_R(y,t))} = \frac{\sqrt{4(u_R(y,t))^2-\mu(T)^2}}{\mu(T)} \le \frac{2u_R(y,t)}{\mu(T)}.$$

Since $u_R(0,t)\le\alpha$, we have $u_R(y,t)\le \alpha e^{2y/\mu(T)}$, and prove (4-2). □

Now we are ready to take the limit of $f_R(x,t)$ as $R\to\infty$.

Proposition 4.6. *There exists a rotationally symmetric mean curvature flow that is the C^∞_{loc} subsequential limit of $S(G_{f_R(\,\cdot\,,t)})$ as $R\to\infty$. This limit flow exists for all future time.*

Proof. Given $T>0$, for any $R>\tau$, $t_1\in(0,T]$, the neck point of $f_R(x,t_1)$ is $(u_R(0,t_1),0)$. Because $f_R(x,t_1)$ is a strictly increasing function, for any $x_1\ge u_R(0,t_1)$, the restricted curve $\{(x, f_R(x,t_1)) \mid x\ge x_1\}$ lies within the region $\{x\ge x_1,\ y\ge f_R(x_1,t_1)\}$. Hence the intersection angle between this restricted curve and the line $\{y=x+f_R(x_1,t_1)-x_1\}$ is not larger than $\frac{\pi}{4}$.

Consider the graph of the function $f_R(x,t_1)$ on the interval $[u_R(0,t_1),\alpha+4]$. We can view it as the graph of a function $g(x)$ over the line $\{y=x\}$. The above argument tells us that $|g'(x)|\le 1$. The two endpoints of this restricted graph are $(u_R(0,t_1),0)$ and $(\alpha+4, f_R(\alpha+4,t_1))$. The x-coordinates of the projection of these two points onto $\{y=x\}$ have the bounds

$$0<\frac{u_R(0,t_1)}{2}\le\frac{\alpha}{2},\qquad \frac{\alpha+4}{2}<\frac{\alpha+4+f_R(\alpha+4,t_1)}{2}<\frac{\alpha+4+\alpha}{2}.$$

Thus this restricted graph can be viewed as a normal graph of a C^1 function with a uniform derivative bound (not dependent on R and t). Together with the uniform gradient estimate given by Proposition 4.3, for any compact region $Q:\{(x,t)\mid 0\le x\le a,\ 0\le t\le b\}$, we can take a subsequential limit of the restricted function $f_R(x,t)|_Q$ as $R\to\infty$ by the Arzelà–Ascoli theorem. By a diagonal sequence argument, we know $\{f_R(x,t), t \in [0,T]\}$ subsequentially converges in R to a function $\tilde{f}(x,t)$. By the uniform gradient estimates in Propositions 4.3 and 4.5, and the interior gradient estimate in [Ecker and Huisken 1991], this convergence is C^∞_{loc} in the spacetime.

From the construction, we can see that $\tilde{f}$ is defined for all $t \in [0,T]$, $x \in [\eta(t), \infty)$. The fact that $\eta(t) \geq u_\tau(0,t) > 0$ with a further diagonal argument guarantees a limit mean curvature flow that exists for all future time. □

Remark 4.7. Since $\tilde{f}$ is a subsequential limit of f_R, all the uniform properties of f_R also hold for $\tilde{f}$. Hence $\tilde{f}$ is a strictly increasing function in x for all $t > 0$, its function value is bounded by α, and its neck point tends to the origin monotonically. The gradient estimate in Proposition 4.3 also holds for $\tilde{f}$.

So far, we have obtained an eternal flow $M(t) := S(G_{\tilde{f}(\cdot,t)})$ defined in the whole $\mathbb{R}^3$. It remains to show that the long time limit of this flow is the plane with multiplicity 2.

We first show that for a sequence of $t_i \to \infty$, $M(t_i)$ converges to the plane with multiplicity 2. As a consequence, this allows us to show that the neck point of $\tilde{f}(\cdot,t)$ tends to 0 as $t \to \infty$.

The proof uses an argument by Ilmanen [1995] saying that if the spacetime L^2-norm of the mean curvature of a sequence of embedded mean curvature flows in $\mathbb{R}^3$ tends to 0, then this sequence of mean curvature flows subsequentially converge to a smooth embedded minimal surface with multiplicity. The following statement can be found in [Bamler and Kleiner 2023, Lemma 2.13], with slight modifications in notation.

Lemma 4.8. *Suppose $\delta > 0$, and $(M^i(t))_{t\in[t_1-\delta,t_2+\delta]}$ is a sequence of smooth mean curvature flows in $\mathbb{R}^3$, with the genus of $M^i(t)$ and the Gaussian density of $M^i(t)$ uniformly bounded for all $i \in \mathbb{Z}_+$ and $t \in [t_1-\delta, t_2+\delta]$. Suppose that for every open subset U of $\mathbb{R}^3$ we have*

$$\int_{t_1}^{t_2}\int_{M^i(t)\cap U} |\vec{H}|^2 \, d\mathcal{H}^2 \, dt \to 0, \quad \text{as } i \to \infty. \tag{4-3}$$

Then there exists a subsequence of $(M^i(t))_{t\in[t_1,t_2]}$ that converges (in the sense of varifold convergence) to a static mean curvature flow, which is a smooth embedded complete minimal surface with multiplicity $k \in \mathbb{Z}_+$.

Given $R > 0$, to obtain the local L^2-bound (4-3) of the limit flow $M(t)$ in C_R, we need to know the behavior of the mean curvature of $M(t)$ on the boundary of C_R.

Lemma 4.9. *For any $x \geq \alpha+2$, $t > 0$, the mean curvature vector $\vec{H}$ of the upper half of $S(G_{\tilde{f}(\cdot,t)})$ points downward at the point $\tilde{f}(x,t)$. More precisely, the projection of $\vec{H}$ to the y-axis is negative.*

Proof. Consider the family of the translations of the catenoids given by the graph of the function

$$U_{\nu,\xi}(x) = \xi + \nu \ln\left(\frac{x}{\nu} + \sqrt{\frac{x^2-\nu^2}{\nu^2}}\right), \quad \text{where } 0 < \nu \leq \alpha+1.$$

Given $x_1 \geq \alpha+2$ and $t_1 > 0$, by Proposition 4.3, we know

$$0 < \frac{\partial}{\partial x}\tilde{f}(x_1,t_1) \leq \frac{\alpha+1}{\sqrt{x^2-(\alpha+1)^2}}.$$

Thus there exist ν and ξ such that

$$U_{\nu,\xi}(x_1) = \tilde{f}(x_1, t_1), \quad U'_{\nu,\xi}(x_1) = \frac{\partial}{\partial x}\tilde{f}(x_1, t_1).$$

If $\frac{\partial^2}{\partial x^2}\tilde{f}(x_1, t_1) > U''_{\nu,\xi}(x_1)$, then the graph of $\tilde{f}(\cdot, t_1)$ is on top of the graph of $U_{\nu,\xi}$ near a neighborhood of x_1. Since $U_{\nu,\xi}$ tends to ∞ as $x \to \infty$ and $\tilde{f}(\cdot, t_1)$ is bounded, these two graphs have another intersection point far away from the origin. The mean curvature at $\tilde{f}(x_1, t_1)$ along its graph is strictly positive by the local comparison with the catenoid, thus at time $t_1 - \epsilon$, for some $\epsilon > 0$ small enough, the graph of $\tilde{f}(\cdot, t_1 - \epsilon)$ and the graph of $U_{\nu,\xi}$ have two intersection points in a neighborhood of $\tilde{f}(x_1, t_1 - \epsilon)$, while there is another intersection point far away from these two intersection points by the boundedness of $\tilde{f}$ and unboundedness of the catenoid.

But at $t = 0$, the graph of $\tilde{f}(\cdot, 0)$ and the graph of $U_{\nu,\xi}$ have at most two intersection points, and for large x, the second graph is always strictly on top of the first graph by the boundedness. By the Sturmian theorem, there are at most two intersection points between the graph of $\tilde{f}(\cdot, t_1 - \epsilon)$ and the graph of $U_{\nu,\xi}$. This yields a contradiction.

Hence $\frac{\partial^2}{\partial x^2}\tilde{f}(x_1, t_1) \le U''_{\nu,\xi}(x_1)$, which implies that locally $\tilde{f}(\cdot, t_1)$ lies below $U_{\nu,\xi}$. Therefore, locally the upper half of $S(G_{\tilde{f}(\cdot, t_1)})$ touches the catenoid from below. This shows that the mean curvature vector of the upper half of $S(G_{\tilde{f}(\cdot, t_1)})$ points downward. □

Proposition 4.10. *For any sequence $t_i \nearrow \infty$, there exists a subsequence t_{i_j} that $M(t_{i_j})$ converges (in the sense of varifold convergence) to the union of planes with possibly higher multiplicity. Moreover, the neck point of $\tilde{f}$ tends to the origin as $t \to \infty$.*

Proof. Without loss of generality, we remove some of the t_i's to assume $t_{i+1} - t_i > 1$. Let $M^i(t) = (M(t + t_i))_{t \in [0,1]}$. Consider the restriction of our family of surfaces $M(t)$ on C_R, denoted by $\overline{M}(t)$. We use $\overline{M}^i(t)$ to denote the flow $(\overline{M}(t + t_i))_{t \in [0,1]}$. By the first variation formula, we have

$$\int_0^T \int_{\overline{M}(t)} |\vec{H}|^2 \, d\mathcal{H}^2 \, dt = \operatorname{Area}(\overline{M}(0)) - \operatorname{Area}(\overline{M}(T)) + \int_0^\infty \int_{\partial \overline{M}(t)} \langle \eta, \vec{H} \rangle \, ds \, dt,$$

where η is the outward unit conormal of $\partial \overline{M}(t)$. By Lemma 4.9, and $\frac{\partial}{\partial x}\tilde{f}(R, t) > 0$, we know $\langle \eta, \vec{H} \rangle \le 0$ along $\partial \overline{M}^t$. Since $\overline{M}(t)$ is a rotationally symmetric surface generated by graphs of a bounded increasing function on a bounded interval, the area of $\overline{M}(t)$ is bounded and independent of t. Hence $\int_0^\infty \int_{\overline{M}(t)} |\vec{H}|^2 \, d\mathcal{H}^2 \, dt$ is bounded, and moreover,

$$\int_0^1 \int_{\overline{M}^i(t)} |\vec{H}|^2 \, d\mathcal{H}^2 \, dt \to 0 \quad \text{as } i \to \infty.$$

By the compactness of weak mean curvature flows (see Section 7 of [Ilmanen 1994]) and Lemma 4.8, we know that M^i subsequentially converges to a static mean curvature flow defined for time [0, 1], which is a smooth embedded minimal surface Σ with possibly higher multiplicity. Such Σ is also rotationally symmetric and lies between two parallel planes as $M(t)$ does. The only smooth embedded minimal surface that satisfies all these properties is the plane or the union of parallel planes.

As a consequence, the neck point of $\tilde{f}$ tends to the origin as $t \to \infty$. □

Remark 4.11. Ilmanen's argument only assures the regularity of the limit minimal surface. It does not ensure the convergence is also regular; for example, it does not ensure the convergence is in the Lipschitz topology.

One reason is that there could be "pimples" in the sequence of flows that vanish in the geometric measure theory limit. Such pimples are described by Simon [1993, Lemma 2.1] in a key lemma that is used by Ilmanen [1995]. On the other hand, if we can show that any subsequential limit in Proposition 4.10 is a fixed plane, we can use Brakke's regularity of mean curvature flow to show that the convergence is multiplicity 2 in our sense. Nevertheless, we decided to show a quantitative gradient estimate directly, because such an estimate gives us a better understanding of this limit eternal flow.

From the gradient estimate (4-1), we know that

$$\frac{\partial}{\partial x}\tilde{f}(x,t) \le \frac{\alpha+1}{\sqrt{x^2-(\alpha+1)^2}}, \tag{4-4}$$

for all $x \ge \alpha+2,\ t \ge 0$. Next, we adapt the same method that we used in Proposition 3.20 to get an improved gradient estimate over any closed interval in (a,∞).

Proposition 4.12. *For any* $0<a<b<c$, *choose* $k > c+\alpha+2$ *large enough that*

$$\frac{(\alpha+1)(\ln k-\ln a)}{\sqrt{k^2-(\alpha+1)^2}} < \frac{\pi}{4}\ln\frac{b}{a}. \tag{4-5}$$

Then there exist a constant ω depending on a, k and a constant T depending on a, b, k, such that

$$\frac{\partial}{\partial x}\tilde{f}(x,t) \le \tan\biggl(\frac{\pi}{2}\frac{\ln k-\ln b}{\ln k-\ln a}+\frac{\omega-b}{\ln t}+\arctan\biggl(\frac{\alpha+1}{\sqrt{k^2-(\alpha+1)^2}}\biggr)\biggr)$$

for all $x\in[b,k],\ t\ge T$. In addition, $\lim_{t\to\infty}[\tilde{f}(c,t)-\tilde{f}(b,t)]=0$.

Proof. By Proposition 4.10, we know that there exists $T>0$ such that for all $t\ge T$, the distance from the neck point of $\tilde{f}(x,t)$ to the origin is less than $\frac{a}{2}$. Hence $\tilde{f}(x,t)$ is a smooth function over $[a,k]$ for all $t\ge T$.

We know

$$\tilde{f}_t = \frac{\tilde{f}_{xx}}{1+\tilde{f}_x^2}+\frac{\tilde{f}_x}{x}, \qquad \tilde{f}_x>0 \qquad \text{for all } x\in[a,k].$$

Let $\phi(x,t)=\arctan(\tilde{f}_x(x,t))$ for $x\in[a,k],\ t\in[T,\infty)$. Then $0\le\phi(x,t)<\frac{\pi}{2}$, and $\tilde{f}_x(x,t)=\tan(\phi(x,t))$. As shown in the proof of Proposition 3.20,

$$\phi_t-\frac{\phi_x}{x}-\frac{1}{1+\tilde{f}_x^2}\phi_{xx} = -\frac{\tilde{f}_x}{x^2(1+\tilde{f}_x^2)}<0. \tag{4-6}$$

Let $\mu=\pi/(2\ln k/a),\ T'=T+k\bigl(k+\frac{\pi}{2}\bigr),\ \omega=k+\frac{\pi}{2}\ln T'$. Let

$$\varphi(x,t)=\mu\ln\frac{k}{x}+\frac{\omega-x}{\ln t}+\arctan\biggl(\frac{\alpha+1}{\sqrt{k^2-(\alpha+1)^2}}\biggr).$$

By the same argument as in the proof of Proposition 3.20, we have

$$\varphi_t - \frac{\varphi_x}{x} - \frac{\varphi_{xx}}{1+\tilde{f}_x^2} > 0.$$

Thus for $x \in [a, k],\ t \geq T'$, we know that

$$\begin{aligned}
\varphi(x, T') &\geq \frac{\omega - x}{\ln T'} \geq \frac{\pi}{2} \geq \phi(x, T'),\\
\varphi_t - \frac{\varphi_x}{x} - \frac{\varphi_{xx}}{1+\tilde{f}_x^2} &> 0 > \phi_t - \frac{\phi_x}{x} - \frac{\phi_{xx}}{1+\tilde{f}_x^2},\\
\varphi(a, t) &\geq \mu \ln \frac{k}{a} = \frac{\pi}{2} \geq \phi(a, t),\\
\varphi(k, t) &> \arctan\left(\frac{\alpha+1}{\sqrt{k^2-(\alpha+1)^2}}\right) \geq \phi(k, t), \quad \text{(by (4-4))}.
\end{aligned}$$

We apply the classical maximum principle to conclude that $\varphi(x, t) \geq \phi(x, t)$ for all $x \in [a, R],\ t \geq T'$. For any $x \in [b, k]$,

$$\phi(x, t) \leq \varphi(x, t) \leq \varphi(b, t) = \frac{\pi}{2}\frac{\ln k - \ln b}{\ln k - \ln a} + \frac{\omega - b}{\ln t} + \arctan\left(\frac{\alpha+1}{\sqrt{k^2-(\alpha+1)^2}}\right).$$

The condition (4-5) guarantees that

$$\arctan\left(\frac{\alpha+1}{\sqrt{k^2-(\alpha+1)^2}}\right) < \frac{\pi}{4}\left(1 - \frac{\ln k - \ln b}{\ln k - \ln a}\right).$$

In addition, there exists $T'' > T'$ such that for all $t \geq T''$, we have

$$\frac{\omega - b}{\ln t} < \frac{\pi}{4}\left(1 - \frac{\ln k - \ln b}{\ln k - \ln a}\right).$$

Therefore for all $x \in [b, k],\ \varphi(x, t) < \frac{\pi}{2}$.

For all $x \in [b, k],\ t \geq T''$ we have

$$\begin{aligned}
\tilde{f}_x(x, t) &= \tan(\phi(x, t)) \leq \tan\left(\frac{\pi}{2}\frac{\ln k - \ln b}{\ln k - \ln a} + \frac{\omega - b}{\ln t} + \arctan\left(\frac{\alpha+1}{\sqrt{k^2-(\alpha+1)^2}}\right)\right),\\
\tilde{f}(c, t) - \tilde{f}(b, t) &= \int_b^c \tilde{f}_x(x, t)\,dx \leq (c-b)\tan\left(\frac{\pi}{2}\frac{\ln k - \ln b}{\ln k - \ln a} + \frac{\omega - b}{\ln t} + \arctan\left(\frac{\alpha+1}{\sqrt{k^2-(\alpha+1)^2}}\right)\right).
\end{aligned}$$

Let $t \to \infty$. We know

$$0 \leq \limsup_{t\to\infty}[\tilde{f}(c, t) - \tilde{f}(b, t)] \leq (c-b)\tan\left(\frac{\pi}{2}\frac{\ln k - \ln b}{\ln k - \ln a} + \arctan\left(\frac{\alpha+1}{\sqrt{k^2-(\alpha+1)^2}}\right)\right).$$

Since the condition (4-5) works for all k large enough and a small enough, we can let $a \to 0$, and $k \to \infty$. Let $a \to 0$. We have

$$0 \leq \limsup_{t\to\infty}[\tilde{f}(c, t) - \tilde{f}(b, t)] \leq \frac{(c-b)(\alpha+1)}{\sqrt{k^2-(\alpha+1)^2}}.$$

Then letting $k \to \infty$, we conclude $\lim_{t\to\infty}[\tilde{f}(c, t) - \tilde{f}(b, t)] = 0$. □

We are now prepared to prove our main theorem.

Theorem 4.13. *There exists a rotationally symmetric surface in $\mathbb{R}^3$ such that the mean curvature flow starting from this surface exists for all future time, and the forward limit is the multiplicity 2 plane $\{y=0\}$.*

Proof. We have already shown the mean curvature flow starting from ρ_η, given by $S(G_{\tilde f(x,t)})$, exists for all future time. Now we show that it converges to the multiplicity 2 plane as $t\to\infty$.

For any subsequential limit of $\tilde f(x,t)$, by Proposition 4.10, we know it is the graph of a function defined on $[0,\infty)$. Lemma 3.19 implies that $\tilde f(x,t)\le \alpha x$, for all $t\in[0,\infty)$. By Proposition 4.12, we know that for any $0<b<c$, we have

$$0\le \limsup_{t\to\infty}\tilde f(c,t)=\limsup_{t\to\infty}\tilde f(b,t)\le \alpha b.$$

Since b can be arbitrarily small, it follows that for all $x>0$ any subsequential limit of $\tilde f(x,t)$ as $t\to\infty$ is 0. Moreover, the proof of Proposition 4.12 also implies the gradient $\frac{\partial}{\partial x}\tilde f(x,t)$ uniformly converges to 0 at $t\to\infty$ for x in a fixed compact subinterval of $(0,\infty)$. Therefore, the forward limit of the immortal flow is the multiplicity 2 plane $\{y=0\}$. □

Remark 4.14. If we replace the family of curves ρ_δ with a family of quarter circles, namely

$$\{(R-x,\sqrt{r^2-x^2})\mid x\in[0,r]\}\quad\text{for } 0\le r\le R,$$

then the interpolation argument as described in Theorem 3.14 (with a possibly different bound on R) remains valid. Therefore, for large R, we can obtain a family of rotationally symmetric free boundary surfaces in C_R, evolving under the mean curvature flow, and converging to a multiplicity 2 disk.

However, if we consider the surface in this family with the neck point at $(1,0)$ for large R, and take the limit as R approaches infinity, these surfaces will converge to the catenoid that is formed by rotating the graph of $x=\cosh y$. This addresses the importance of making a careful choice of the (singular) foliation.

Appendix: Asymptotically planar mean curvature flow

Definition A.1. A surface $M\subset\mathbb{R}^3$ is asymptotic to a plane P if for any $\epsilon>0$ there exists a compact set K_ϵ such that $\overline{M\setminus K_\epsilon}$ is a complete noncompact surface with boundary, $\partial M\subset K_\epsilon$, and $\overline{M\setminus K_\epsilon}\subset P_\epsilon$, where P_ϵ is the ϵ tubular neighborhood of P.

Proposition A.2. *Suppose $\{M(t)\}_{t\in[0,1]}$ is a mean curvature flow, $R>0$, and $N(t)$ is a connected component of $M(t)\setminus\bar B_R$. If $N(0)$ is asymptotic to a plane P, then $N(t)$ is asymptotic to the same plane P for $t\in[0,1]$.*

Proof. Without loss of generality, we assume $P=\{(x,y,0)\mid (x,y)\in\mathbb{R}^2\}$. For any $\epsilon>0$, it suffices to find $R_\epsilon>0$, such that $\overline{N(t)\setminus B(R)}\subset P_\epsilon$. Because $N(0)$ is asymptotic to P, we may assume $N(0)\setminus B_{R'}\subset P_{\epsilon/4}$. Now we choose r large such that $r-\sqrt{r^2-4}\le\epsilon/2$ (e.g., $r^2>8/\epsilon+1$), and we choose $R_\epsilon>r+R'$. Then for any (x,y) with $|(x,y)|\ge R_\epsilon$, $\partial B_r(x,y,\epsilon/2+r)$ has distance at least $\epsilon/2$ away from $P_{\epsilon/4}$, therefore it is disjoint from $N(0)\setminus B_{R'}$. Applying the avoidance principle shows that $N(t)\setminus B_{R_\epsilon}$ is disjoint from $\partial B_{\sqrt{r^2-4t}}(x,y,\epsilon/2+r)$. In particular, this implies that $N(t)\setminus B_{R_\epsilon}\subset P_\epsilon$ for $t\in[0,1]$. □

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