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AN $L^{3/2}$ SL_2 KAKEYA MAXIMAL INEQUALITY

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We prove a special case of the Kakeya maximal function conjecture in $\mathbb{R}^3$, with $C_\epsilon \delta^{-\epsilon}$ loss, when the centre lines of the tubes are SL_2 lines and the tubes satisfy a 2-dimensional ball condition (equivalent to the Wolff axioms in this case). We show that the exponent $p = \frac{3}{2}$ is sharp and that some loss (such as $C_\epsilon \delta^{-\epsilon}$) is necessary, even in the SL_2 case where the δ-tubes have δ-separated directions and the cardinality of the tube family is maximal ($\sim \delta^{-2}$).

The SL_2 Kakeya maximal inequality is deduced from an $L^{3/2}$ inequality for restricted families of projections onto planes. A related $L^{3/2-\epsilon}$ inequality is also derived for restricted projections onto lines, and an application is given to generic intersections of sets in $\mathbb{R}^3$ with "light rays" and "light planes".

1. Introduction

The aim of this work is to prove the following special case of the Kakeya maximal function conjecture:

Theorem 1.1. *Let $\mathbb{T}$ be a collection of SL_2 tubes in $\mathbb{R}^3$ of length 1 and radius δ, satisfying the condition*

$$|\{T \in \mathbb{T} : T \subseteq S\}| \leq (r/\delta)^2 \tag{1-1}$$

for any tube S of length 2 and radius r, and any $r \in [\delta, 1]$. Then for any $\epsilon > 0$,

$$\left\| \sum_{T \in \mathbb{T}} \chi_T \right\|^{3/2}_{L^{3/2}(B_3(0,1))} \leq C_\epsilon \delta^{-\epsilon} |\mathbb{T}| \delta^2. \tag{1-2}$$

In Theorem 1.1, a tube is an SL_2 tube if its centre line is an SL_2 line. A line $\ell \subseteq \mathbb{R}^3$ is called an SL_2 line if there exist $a, b, c, d \in \mathbb{R}$ with $ad - bc = 1$ such that

$$\ell = \{(a, b, 0) + \lambda(c, d, 1) : \lambda \in \mathbb{R}\},$$

or if there exists $t_0 \in \mathbb{R}$ and $(c, d) \in \mathbb{R}^2 \setminus \{0\}$ such that

$$\ell = \{(0, 0, t_0) + \lambda(c, d, 0) : \lambda \in \mathbb{R}\}.$$

In the SL_2 case on $B_3(0, 1)$, the Wolff axioms (see [Katz and Zahl 2019, Definition 1.1]) are equivalent [Katz et al. 2023, Remark 2] to the condition (1-1), which is why Theorem 1.1 is a special case of the Kakeya maximal function conjecture in $\mathbb{R}^3$ for families of tubes satisfying the Wolff axioms. We note that the Wolff axioms (and (1-1)) hold whenever the tubes in $\mathbb{T}$ have $\sim \delta$-separated directions.

Theorem 1.1 will be deduced from an $L^{3/2}$ inequality for restricted families of projections onto planes stated below, and the point-line duality principle from [Liu 2022; Fässler and Orponen 2023b].

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Theorem 1.2. *If $\alpha > 2$, then for any Borel measure μ on $B_3(0,1)$ with $c_\alpha(\mu) < \infty$,*

$$\int_0^{2\pi} \|\pi_{\theta\sharp}\mu\|_{L^{3/2}}^{3/2}\, d\theta \le C_\alpha \mu(\mathbb{R}^3) c_\alpha(\mu)^{1/2}.$$

Here π_θ is the orthogonal projection onto the plane $\frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, 1)^\perp$, and

$$c_\alpha(\mu) = \sup_{r>0,\, x\in\mathbb{R}^3} \frac{\mu(B(x,r))}{r^\alpha}.$$

As a counterpart to Theorem 1.2, the following version for projections ρ_θ onto the line defined as the span of $\frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, 1)$ will be proved with a similar method, but using a maximal estimate from [Pramanik et al. 2022] as a crucial input.

Theorem 1.3. *If $\alpha > 1$ and $1 \le p < \frac{3}{2}$, then for any Borel measure μ on $B_3(0,1)$ with $c_\alpha(\mu) < \infty$,*

$$\int_0^{2\pi} \|\rho_{\theta\sharp}\mu\|_{L^p}^{p}\, d\theta \le C_\alpha \mu(\mathbb{R}^3) c_\alpha(\mu)^{p-1}.$$

The proofs of Theorems 1.2 and 1.3, as well as Theorem 1.4 below, can easily be generalised to include C^2 curves $\gamma : [0,1] \to S^2$ with $\det(\gamma, \gamma', \gamma'')$ nonvanishing (i.e., nondegenerate curves), with π_θ and ρ_θ redefined analogously, but for simplicity they are only written here for the model curve $\gamma(\theta) = \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, 1)$.

In Section 2 it is shown via an explicit construction that there exist closed SL_2 Besicovitch sets of measure zero, using the same point-line duality as that used in [Liu 2022]. This is then used to show the corollary that a uniform L^p bound (i.e., without some loss such as $|\log\delta|$) is not possible in the SL_2 Kakeya maximal inequality for any nontrivial p. In Section 3, we show that the exponent $\frac{3}{2}$ in Theorems 1.1, 1.2, and 1.3 is sharp. This yields a counterexample to [Katz et al. 2023, Remark 1], where it was suggested that $p = 2$ might be possible in the SL_2 case of Theorem 1.1. The version of the SL_2 Kakeya maximal inequality in [Katz et al. 2023] appears slightly different to that used here, but the equivalence of the various forms is discussed in Section 1.2.

By a recent result of Mattila [2023a, Theorem 1.2], a consequence of Theorems 1.2 and 1.3 is the following result about generic intersections of sets in $\mathbb{R}^3$ with "light rays" and "light planes".

Theorem 1.4. *Let $A \subseteq \mathbb{R}^3$ be $\mathcal{H}^s$-measurable with $0 < \mathcal{H}^s(A) < \infty$. If $s > 1$ then for $(\mathcal{H}^s \times \mathcal{H}^1)$-a.e. $(x,\theta) \in A \times [0,2\pi)$,*

$$\dim\big(A \cap \rho_\theta^{-1}(\rho_\theta(x))\big) = s - 1,$$

and for a.e. $\theta \in [0,2\pi)$,

$$\mathcal{H}^1\big\{w \in \mathrm{span}(\gamma(\theta)) : \dim(A \cap \rho_\theta^{-1}(w)) = s-1\big\} > 0.$$

If $s > 2$ then for $(\mathcal{H}^s \times \mathcal{H}^1)$-a.e. $(x,\theta) \in A \times [0,2\pi)$,

$$\dim\big(A \cap \pi_\theta^{-1}(\pi_\theta(x))\big) = s - 2,$$

and for a.e. $\theta \in [0,2\pi)$,

$$\mathcal{H}^2\big\{w \in \gamma(\theta)^\perp : \dim(A \cap \pi_\theta^{-1}(w)) = s-2\big\} > 0.$$

Here and throughout the paper, dim denotes the Hausdorff dimension, and $\mathcal{H}^s$ refers to the s-dimensional Hausdorff measure in Euclidean space. Theorem 1.4 is a refinement of the Marstrand–Mattila slicing theorem in $\mathbb{R}^3$ (see Theorems 10.10 and 10.11 in [Mattila 1995]), which has the same statement as the above, but with lines and planes varying over all lines and planes through a point, rather than just "light rays" and "light planes". By Fubini's theorem and a simple scaling argument, Theorem 1.4 implies the standard Marstrand–Mattila slicing theorem in $\mathbb{R}^3$. Although much work has been done on restricted projection families since they were introduced in [Järvenpää et al. 2008; Fässler and Orponen 2014], the only "restricted intersection" type result in Euclidean space of which we are aware is from [Mattila 2023b], which is different from Theorem 1.4. There also exist results on the dimension of exceptional sets in intersection theorems (see, e.g., [Mattila 2023b] for a survey), but Theorem 1.4 seems to be the first refinement of Marstrand's intersection theorem for (curved) families of subspaces of dimension less than the full Grassmannian.

Note that it is necessary to work with nondegenerate curves γ. Indeed, consider an example of a set $A = [0,1]^2 \times C \subset \mathbb{R}^2 \times \mathbb{R}$, where C is a set with $0 < \mathcal{H}^\epsilon(C) < \infty$. Then $0 < \mathcal{H}^s(A) < \infty$, where $s = 2+\epsilon$. However, if one considers the curve $\gamma(\theta) = (\cos(\theta), \sin(\theta), 0)$, then for $\mathcal{H}^s$-a.e. $x \in A$, $A \cap \pi_\theta^{-1}(\pi_\theta(x))$ contains a line segment, hence has dimension $1 \neq s - 2 = \epsilon$.

Theorem 1.4 also implies that if $A \subseteq \mathbb{R}^3$ is Borel with $\dim A > 1$, then $\mathcal{H}^1(\rho_\theta(A)) > 0$ for a.e. $\theta \in [0, 2\pi)$, which was first proved in [Harris 2024]. This implication follows by applying a theorem of Davies [1952] to reduce to the case where A has positive finite s-dimensional $\mathcal{H}^s$-measure for some $s > 1$, and then observing that

$$\mathcal{H}^1(\rho_\theta(A)) = \mathcal{H}^1\big\{w \in \operatorname{span}(\gamma(\theta)) : A \cap \rho_\theta^{-1}(w) \neq \varnothing\big\},$$

and that clearly the nonempty condition above is weaker than the dimension $s-1$ condition in Theorem 1.4. Similarly, Theorem 1.4 implies that if $A \subseteq \mathbb{R}^3$ is Borel with $\dim A > 2$, then $\mathcal{H}^2(\pi_\theta(A)) > 0$ for a.e. $\theta \in [0, 2\pi)$, which was first proved in [Gan et al. 2022]. These length and area statements in turn imply the Hausdorff dimension versions $\dim(\rho_\theta(A)) = \min\{1, \dim A\}$ and $\dim(\pi_\theta(A)) = \min\{2, \dim A\}$ for a.e. $\theta \in [0, 2\pi)$; Fässler and Orponen proved that for general families of Euclidean projections, the dimension exponents in projection theorems can always be lowered by randomly adding points to lower dimensional sets to obtain higher dimensional sets (see, e.g., [Fässler and Orponen 2023b] for this type of argument), though this method probably loses information about the exceptional set. The a.e. equality $\dim(\rho_\theta(A)) = \min\{1, \dim A\}$ was first obtained by Käenmäki, Orponen and Venieri [Käenmäki et al. 2025], and then for C^2 curves by Pramanik, Yang, and Zahl [Pramanik et al. 2022] and Gan, Guth, and Maldague [Gan et al. 2024] (and all of [Käenmäki et al. 2025; Pramanik et al. 2022; Gan et al. 2024] contain better information about the exceptional set than what follows from Theorem 1.4). The Hausdorff dimension theorem for the projections π_θ was obtained in [Gan et al. 2022] (and by a different method in [Katz et al. 2023]). The a.e. equalities $\dim(\rho_\theta(A)) = \min\{1, \dim A\}$ and $\dim(\pi_\theta(A)) = \min\{2, \dim A\}$ can also be obtained as corollaries of Theorem 1.3 and 1.2 directly (by Hölder's inequality and the definition of Hausdorff dimension).

L^p estimates for projected measures have recently received quite some attention and have proven to be useful in the study of many important problems in geometric measure theory. For instance, an L^p

estimate for radial projections in $\mathbb{R}^d$ was obtained by Orponen [2019], which was used as an essential tool in the proof of the $\frac{5}{4}$ threshold for the two dimensional Falconer distance set problem in [Guth et al. 2020]. In the case of orthogonal projections, an L^p estimate for the orthogonal projected measures over the full Grassmannian was obtained by Dąbrowski, Orponen, and Villa [Dąbrowski et al. 2022], where they considered the case of general orthogonal projections from $\mathbb{R}^d$ to n dimensional subspaces, $1 \le n < d$. Their result is known to be sharp when the dimension of the measure is at least $d-1$, and implies improved Furstenberg set estimates and discretized sum-product estimates. (Very recently, a similar L^p estimate with more general measures on the Grassmannian was derived by Li and Liu [2026].) Our L^p estimates extend this line of investigation of regularity of projected measures and seem to be the first of their kind concerning curved families of subspaces of dimension less than the full Grassmannian.

The SL_2 example related to the Kakeya problem first appeared in Katz and Zahl's $\frac{5}{2}+\epsilon$ lower bound for the Hausdorff dimension of Besicovitch sets in $\mathbb{R}^3$ [2019], where the proof needed to negotiate a hypothetical SL_2 "almost-counterexample". Later, Wang and Zahl [2026] conjectured that SL_2 Besicovitch sets in $\mathbb{R}^3$ have Hausdorff dimension 3 (as a special case of the Kakeya conjecture in $\mathbb{R}^3$). This was proved by Fässler and Orponen [2023a], and also by Katz, Wu, and Zahl [Katz et al. 2023]. By a standard argument (see, e.g., [Wolff 2003, Proposition 10.2]), Theorem 1.1 also implies that SL_2 Besicovitch sets have Hausdorff dimension 3. A type of SL_2 Kakeya maximal inequality first appeared in [Katz et al. 2023]. The inequality there implies that SL_2 Besicovitch sets have Hausdorff dimension 3, but does not imply the SL_2 maximal inequality in (1-2) for any $p > 1$ $\left(\text{in place of } \frac{3}{2}\right)$. By applying Heisenberg dilations $(x, y, z) \mapsto (\lambda x, \lambda y, \lambda^2 z)$, which preserve SL_2 lines, Theorem 1.1 still holds if $B_3(0,1)$ is replaced by $B_3(0,R)$ for any $R \ge 1$ in the left-hand side of (1-2), if the constant in the right-hand side of (1-2) is allowed to depend on R. By then running the argument from [Orponen 2017] with the SL_2 Kakeya maximal function (see Section 1.2) instead of the general Kakeya maximal function, Theorem 1.1 also implies that SL_2 Besicovitch sets in $\mathbb{R}^3$ have *conformal* dimension 3. Conformal dimension is a notion of dimension which is less than or equal to the Hausdorff dimension, and for general sets may be smaller; see [Orponen 2017] for the precise definition. Unlike the Hausdorff dimension case, the argument from [Orponen 2017] relies crucially on the value of p in the Kakeya maximal inequality, and a value of $p < \frac{3}{2}$ would give a nonsharp result for the conformal dimension.

Finally, see [Mattila 2019, Sections 6.7 and 6.8] for a general discussion of connections between restricted projection problems and Kakeya-type problems. We also mention that a related maximal function to the SL_2 Kakeya maximal function was considered in [Fässler et al. 2025], where the optimal $L^{3/2}$ inequality was obtained. This maximal function was a function of the angle $\theta \in [0, \pi)$ of the tube, rather than the direction. The tubes considered were also Heisenberg tubes rather than Euclidean tubes, and the problem was related to the projections ρ_θ rather than π_θ.

1.1. ***History and novelty of the method.*** This subsection outlines a brief history of the "good-bad" decomposition of a measure in the context of this problem, as this may give some motivation for the proofs of Theorems 1.2 and 1.3. The decomposition of a measure into "good" and "bad" parts was first developed and applied to the planar distance set problem in [Guth et al. 2020]. This decomposition was modified in [Harris 2022] to work for restricted projections onto planes in $\mathbb{R}^3$. The argument from

[Harris 2022] contained significant loss in the refined decoupling step, arising from defining the bad part of the measure with tubes rather than planks, which are the natural objects to consider in wave packet decompositions adapted to the cone. The reason for using tubes in [Harris 2022] was partly that the argument used a lemma from Orponen and Venieri's previous work [2020] on the restricted projection problem as a black box to handle the "bad" part, and this lemma was formulated in terms of tubes. In [Harris 2023b], a bootstrapping argument for handling the "bad" part was developed, which removed the dependence on the lemma from [Orponen and Venieri 2020], but it was not yet realised in [Harris 2023b] that "tubes" could be replaced with "planks". This improvement was made in the second part of [Gan et al. 2022], and the possibility of the improvement was suggested by the first part of [Gan et al. 2022], which developed a different and independent approach to this problem via the "high-low method". In the second part of [Gan et al. 2022], it was shown that pushforwards under π_θ of measures of dimension greater than 2 are almost surely in L^1, and the method used to control the L^1 norm of the "bad" part of the measure was an incidence bound proved via the "high-low method". In this work, for projections onto planes we use an entirely recursive approach to bound the $L^{3/2}$ norm of the "bad" part, which allows us to make efficient use of L^2 bounds on the "good" part. The proof is mostly self-contained (in particular, the proof does not use the "high-low method"), and the only nontrivial result from [Gan et al. 2022] that we use is the refined decoupling theorem. Note however that we need only L^4 decoupling for the cone to prove Theorem 1.2, which has a much shorter[1] proof than L^6 decoupling.

The novelty in the proof of Theorem 1.2 is a "self-improving" argument for the exponents, where in practice we need to prove Theorem 1.2 for an increasing sequence of exponents $\{p_n\}_{n=0}^{\infty}$, where $p_0 = 1$, $p_1 = \frac{4}{3}$ and $p_\infty = \frac{3}{2}$. For each fixed exponent, we use a separate bootstrapping argument to prove a weaker version of Theorem 1.2 over a union of "bad" rectangles in each plane, where the meaning of "bad" is gradually weakened until we integrate over the whole planes. At each exponent, a crucial input is a bound on the measure of the unions of "bad" rectangles, which we get by applying the projection inequality for the previous exponent p_{n-1}. Because this projection inequality is better for each n than the previous n, the best exponent we can obtain in Theorem 1.2 improves at each step and we calculate that it converges to $\frac{3}{2}$.

The "good-bad" decomposition was modified to work for projections onto lines in [Harris 2024], where it was shown that the projections of measures of dimensions greater than 1 are almost surely in L^1. By modifying the recursive argument from [Harris 2024] to get an L^p bound as in the planar case, the best possible exponent via a purely recursive argument seems to be $\frac{4}{3}$, which is not the sharp exponent in Theorem 1.3. To get the exponent $\frac{3}{2} - \epsilon$, we need to use an $L^{3/2}$ Kakeya bound for curves satisfying a nonconcentration condition of Pramanik, Yang, and Zahl [Pramanik et al. 2022], as a black box, and therefore our only substantial contribution to Theorem 1.3 is to combine this with L^6 decoupling and the $\alpha > 1$ assumption in Theorem 1.3 to get a uniform bound as in Theorem 1.3.

We remark that this work supersedes the results from [Harris 2023a]. For completeness, and since the work from [Harris 2023a] will not be submitted for publication, some of the work from [Harris 2023a] is

[1] We also note that the refined decoupling theorem for the standard light cone has a simpler proof than in [Gan et al. 2022]; because Lorentz rescalings preserve the light cone exactly, the technicalities introduced in [Gan et al. 2022] to track how the curvature of the cone varies under rescaling are not necessary.

reproduced here (especially in Sections 2 and 3). We also want to point out that, even though the L^p bounds in analogues of Theorems 1.2 and 1.3 in [Harris 2023a] are weaker (at only $p = \frac{4}{3}$ and $p = \frac{6}{5}$, respectively), they are sufficient to derive the same restricted intersection results (with light rays and light planes) as in Theorem 1.4.

1.2. ***Different forms of the*** $\mathbf{SL_2}$ ***Kakeya maximal inequality.*** Given a Borel function f on $\mathbb{R}^3$, let $f^*_{\delta,\mathrm{SL}_2} : S^2 \setminus \{(0,0,\pm 1)\} \to \mathbb{R}$ be the SL_2 Kakeya maximal function at scale δ, which is defined in the same way as for the standard Kakeya maximal function except that the centre lines $\ell(T)$ of the δ-tubes T are required to be SL_2 lines:

$$f^*_{\delta,\mathrm{SL}_2}(e) = \sup_{\substack{\mathrm{dir}(T)=e \\ T=\delta\times 1\text{-tube} \\ \ell(T)\in \mathrm{SL}_2}} \frac{1}{m(T)} \int_T |f|,$$

where $m(T)$ is the Lebesgue measure of T. The terms "SL_2" and "horizontal" will be used interchangeably, though strictly speaking they are slightly different.

If $p^* \in (1,\infty)$ is fixed and p^* is the Hölder conjugate to p, then the inequality

$$\|f^*_{\delta,\mathrm{SL}_2}\|_{L^{p^*}(S^2)} \le C_\epsilon \delta^{-\epsilon} \|f\|_{p^*} \tag{1-3}$$

(for any $\epsilon > 0$) is equivalent (by the standard duality argument; see, e.g., [Wolff 2003, Lemma 10.4]) to the inequality

$$\left\| \sum_{T\in\mathbb{T}} \chi_T \right\|_p^p \le C_\epsilon \delta^{-\epsilon} |\mathbb{T}| \delta^2 \tag{1-4}$$

whenever $\mathbb{T}$ is a family of SL_2 δ-tubes of length 1 with δ-separated directions. It is known that (1-4) is equivalent to

$$\mathcal{H}^3\left(\bigcup_{T\in\mathbb{T}} Y(T) \right) \ge C_\epsilon^{-1} \delta^\epsilon \lambda^{p^*} \delta^2 |\mathbb{T}| \tag{1-5}$$

whenever $\mathbb{T}$ is a family of SL_2 δ-tubes of length 1 with δ-separated directions, $\delta \le \lambda \le 1$, and Y is a λ-shading of $\mathbb{T}$ (i.e., a union of δ-cubes such that at least a fraction λ of each tube in $\mathbb{T}$ is covered by the cubes in Y, meaning that $\mathcal{H}^3(Y(T)) \ge \lambda \mathcal{H}^3(T)$ for any $T \in \mathbb{T}$, where $Y(T) = Y \cap T$). The inequality from [Katz et al. 2023] states that (1-4) holds with $p^* = M$ depending on ϵ (and possibly tending to ∞ as $\epsilon \to 0$). The proof of the equivalence of (1-4) and (1-5) is similar to that for the standard Kakeya maximal function, but will be summarised here for convenience. That (1-4) implies (1-5) follows from Hölder's inequality. The reverse implication follows by pigeonholing a set Y of δ-cubes on which the value of $\sum_{T\in\mathbb{T}} \chi_T$ is approximately constant in order to reverse Hölder's inequality, then by pigeonholing again to a subset $\mathbb{T}' \subseteq \mathbb{T}$ (of potentially smaller cardinality) of tubes containing an approximately equal fraction λ of the cubes from Y in each tube, and then by applying (1-5) to this subset.

An a priori weaker inequality than (1-4) is

$$\left\| \sum_{T\in\mathbb{T}} \chi_T \right\|_{L^p(B_3(0,1))}^p \le C_\epsilon \delta^{-\epsilon} \tag{1-6}$$

whenever $\mathbb{T}$ is a family of SL_2 δ-tubes of length 1 with δ-separated directions. Clearly (1-4) implies (1-6), and the implication can be reversed for the standard Kakeya maximal function by rotational averaging (see, e.g., [Mattila 2015, Proposition 22.7]), but rotations in $O(3)$ do not necessarily preserve SL_2 lines, so it does not seem obvious that (1-6) implies (1-4) in the SL_2 case. In Section 3 it is shown that (1-6) cannot hold in the SL_2 case for all $\epsilon > 0$ unless $p \le \frac{3}{2}$ (and therefore neither can (1-5), (1-4) or (1-3)).

2. SL_2 Besicovitch sets of measure zero

The construction of a measure zero SL_2 Besicovitch set in this section is based on Kahane's construction of a measure zero Besicovitch set in the plane [1969]; for an exposition see [Stein and Shakarchi 2005, Chapter 7] or [Mattila 2015, Chapter 11].

Given $(x_1, y_1, t_1), (x_2, y_2, t_2) \in \mathbb{R}^3$, let

$$(x_1, y_1, t_1) * (x_2, y_2, t_2) = \big(x_1 + x_2, y_1 + y_2, t_1 + t_2 + \tfrac{1}{2}(x_1 y_2 - x_2 y_1)\big)$$

be the Heisenberg product. A line $\ell \subseteq \mathbb{R}^3$ is called horizontal if there exists $\theta \in [0, \pi)$ and $p \in \mathbb{R}^3$ such that

$$\ell = p * \mathbb{V}_\theta,$$

where $\mathbb{V}_\theta = \{(\lambda e^{i\theta}, 0) : \lambda \in \mathbb{R}\}$, and we identify $\lambda e^{i\theta} \in \mathbb{C}$ with $(\lambda \cos\theta, \lambda \sin\theta) \in \mathbb{R}^2$. Horizontal line segments and SL_2 line segments are line segments which are subsets of horizontal and SL_2 lines, respectively. A set $K \subseteq \mathbb{R}^3$ will be called a horizontal Besicovitch set if K contains a horizontal unit line segment in every direction in $S^2 \setminus \{(0, 0, \pm 1)\}$. Similarly, K will be called an SL_2 Besicovitch set if K contains an SL_2 unit line segment in every direction in $S^2 \setminus \{(0, 0, \pm 1)\}$. If $F(x, y, z) = (x, y, 2z)$, then F sends horizontal lines to SL_2 lines, and $\ell \mapsto F(\ell)$ is a bijection between horizontal and SL_2 lines with inverse $\ell \mapsto F^{-1}(\ell)$, where $F^{-1}(x, y, z) = (x, y, z/2)$. For any horizontal line segment I, the length of $F(I)$ is comparable to the length of I, so SL_2 lines and horizontal lines are more or less the same.

Every horizontal line can be written in the form $p * \mathbb{V}_\theta$ with (p_1, p_2) in the span of $ie^{i\theta}$, where $p = (p_1, p_2, p_3)$. If $R > 0$ and if such a horizontal line intersects $B(0, R)$, then $|(p_1, p_2)| \le R$ and the direction of $p * \mathbb{V}_\theta$ has distance at least $2/\sqrt{R^2 + 4}$ from $(0, 0, \pm 1)$. It follows that a bounded set cannot contain a horizontal unit line segment in every direction in $S^2 \setminus \{0, 0, \pm 1\}$ (and similarly for SL_2 line segments). For this reason, the compact sets of horizontal and SL_2 line segments in Theorem 2.2 below contain only unit line segments in every direction in $S^2 \setminus \mathcal{N}_\epsilon(\{(0, 0, \pm 1)\})$ for a given $\epsilon > 0$, and to get all directions in $S^2 \setminus \{(0, 0, \pm 1)\}$ it is necessary to replace "compact" with "closed".

The following (straightforward) lemma is used to make the proof of Theorem 2.2 below constructive, though it is possible to give a nonconstructive proof of Theorem 2.2 below by substituting the application of Lemma 2.1 in the proof of Theorem 2.2 with an averaging/Fubini/scaling argument.

Lemma 2.1. *If $E \subseteq (-\pi/2, \pi/2)$ is Lebesgue measurable with $\mathcal{H}^1(E) = 0$ and if F is the union of great circles defined by*

$$F = \{(\lambda e^{i\theta}, \mu) \in S^2 : (\lambda, \mu) \in S^1,\ \theta \in E\},$$

then $\mathcal{H}^1(F \cap C) = 0$ for any circle $C \subseteq S^2$ with the property that either $(0, 0, 1) \notin C$ or $(0, 0, -1) \notin C$.

Proof. Let E, C be given. By stereographic projection on the Riemann sphere (which maps circles to circles or lines) it suffices to show that if $\widetilde{C}$ is either a circle in $\mathbb{R}^2$ or a line in $\mathbb{R}^2$ which does not pass through the origin, then $\mathcal{H}^1(\widetilde{F} \cap \widetilde{C}) = 0$, where

$$\widetilde{F} = \{\lambda e^{i\theta} : \lambda \in \mathbb{R},\ \theta \in E\}.$$

To show this, it suffices to show that for any $(x_0, y_0) \in \widetilde{C} \setminus \{0\}$, there exists an $\epsilon > 0$ such that $\mathcal{H}^1(\widetilde{F} \cap \widetilde{C} \cap B((x_0, y_0), \epsilon)) = 0$. By scaling and rotation it may be assumed that $(x_0, y_0) = (1, 0)$. If $\widetilde{C}$ is a circle, then let (a, b) and $r > 0$ be such that $\widetilde{C}$ is parametrised by $\gamma(t) = (\gamma_1(t), \gamma_2(t)) = (a + r\cos t, b + r\sin t)$, where $\gamma : [0, 2\pi) \to \widetilde{C}$ satisfies $\gamma(t_0) = (1, 0)$ for some $t_0 \in [0, 2\pi)$. Otherwise let $\gamma(t) = (1, 0) + t(v_1, v_2)$ be the parametrisation of $\widetilde{C}$, where $v_2 \neq 0$ and $t \in \mathbb{R}$, and define $t_0 = 0$. It suffices to show that for $\delta > 0$ sufficiently small,

$$\mathcal{H}^1\big\{t \in (t_0 - \delta, t_0 + \delta) : \arctan(\gamma_2(t)/\gamma_1(t)) \in E\big\} = 0.$$

Equivalently, if $G : (t_0 - \delta, t_0 + \delta) \to (-\pi/2, \pi/2)$ is defined by $G(t) = \arctan(\gamma_2(t)/\gamma_1(t))$, then $(G_\sharp \mathcal{H}^1)(E) = 0$ for $\delta > 0$ sufficiently small. Since $\mathcal{H}^1(E) = 0$, it is enough to show that $G_\sharp \mathcal{H}^1 \ll \mathcal{H}^1$ for $\delta > 0$ sufficiently small. If $\widetilde{C}$ is a circle and $(a, b) = 0$, then $G'(t) = \frac{d}{dt} \arctan(\gamma_2(t)/\gamma_1(t)) = 1$ for all t in a neighbourhood of $t = t_0 = 0$, and $G_\sharp \mathcal{H}^1 \ll \mathcal{H}^1$ follows. If $\widetilde{C}$ is a line, then

$$G'(t) = \frac{v_2}{|\gamma(t)|^2} \neq 0$$

for all t in a neighbourhood of $t_0 = 0$, and again $G_\sharp \mathcal{H}^1 \ll \mathcal{H}^1$ follows. The remaining case is where $\widetilde{C}$ is a circle and $(a, b) \neq 0$. Let $H(t) = \det(\gamma(t), \gamma'(t))$. Then

$$G'(t) = \frac{H(t)}{|\gamma(t)|^2}$$

for all t in a neighbourhood of 0, so if $H(t)$ is nonvanishing at $t = t_0$, it follows that $G_\sharp \mathcal{H}^1 \ll \mathcal{H}^1$ for $\delta > 0$ sufficiently small. If $H(t_0) = 0$, then $H(t) \neq 0$ for all $t \in (t_0 - \delta, t_0 + \delta) \setminus \{t_0\}$ for some $\delta > 0$ sufficiently small, which follows from the identity $|H'(t)|^2 + |H''(t)|^2 = r^2(a^2 + b^2)$. Therefore $G_\sharp \mathcal{H}^1 \ll \mathcal{H}^1$ for $\delta > 0$ sufficiently small. This finishes the proof. □

Theorem 2.2. (1) *For any $\epsilon > 0$, there exists a compact set of measure zero containing a horizontal unit line segment in every direction in $S^2 \setminus \mathcal{N}_\epsilon(\{(0, 0, \pm 1)\})$.*

(2) *There exists a closed set of measure zero containing a horizontal unit line segment in every direction in $S^2 \setminus \{(0, 0, \pm 1)\}$.*

(3) *There exists a closed set of measure zero containing a horizontal line in every direction in*

$$S^2 \setminus \{(0, 0, \pm 1)\}.$$

The above claims all hold if "a horizontal" is replaced by "an SL_2*".*

Proof. Let A be any compact subset of $\mathbb{R}^3$ such that $0 < \mathcal{H}^2(A) < \infty$, such that the projection of A down to the (x, y)-plane contains a nonempty open disc, and such that for any nongreat circle $C \subseteq S^2$,

$\mathcal{H}^2(\pi_{v^\perp}(A)) = 0$ for $\mathcal{H}^1$-a.e. $v \in C$, where $\pi_{v^\perp}$ denotes projection to the orthogonal complement of v. Such a set can be constructed as follows. Let $\mathcal{C}$ be the Cantor set in $\mathbb{R}$ of dimension $\frac{1}{2}$ obtained by starting with the unit interval $[0, 1]$ and removing the open middle half interval at each step. It is a standard result that

$$\{2x + y : x, y \in \mathcal{C}\} = [0, 3],$$

as can be seen, e.g., by characterising $\mathcal{C}$ as those elements of $[0, 1]$ of the form $\sum_{j=1}^{\infty} \varepsilon_j/4^j$ with $\varepsilon_j \in \{0, 3\}$ for all j. It follows that if $B = \mathcal{C} \times \mathcal{C} \times [0, 1]$, then the projection of B onto the plane spanned by $\frac{1}{\sqrt{5}}(2, 1, 0)$ and $(0, 0, 1)$ contains a nonempty open disc. Moreover, if $v \in S^2$ is of the form $(\lambda_1 e^{i\theta}, \lambda_2)$, then

$$\pi_{v^\perp}(B) \subseteq \pi_{(ie^{i\theta},0)}(\mathcal{C} \times \mathcal{C} \times \{0\}) + \pi_{(\lambda_2 e^{i\theta}, -\lambda_1)}(\mathbb{R}^3).$$

Hence, by Fubini's theorem,

$$\big\{v \in S^2 : \mathcal{H}^2(\pi_{v^\perp}(B)) > 0\big\} \subseteq F := \big\{(\lambda_1 e^{i\theta}, \lambda_2) : \theta \in E,\ (\lambda_1, \lambda_2) \in S^1\big\},$$

where

$$E = \big\{\theta \in [0, \pi) : \mathcal{H}^1(\pi_{ie^{i\theta}}(\mathcal{C} \times \mathcal{C})) > 0\big\}.$$

But $\mathcal{H}^1(E) = 0$ (see, e.g., [Mattila 2015, Theorem 10.1]), and by Lemma 2.1 this implies that $\mathcal{H}^1(F \cap C) = 0$ for any nongreat circle C. It follows that for any nongreat circle C, $\mathcal{H}^2(\pi_{v^\perp}(B)) = 0$ for $\mathcal{H}^1$-a.e. $v \in C$. Since this property is rotation-invariant, replacing B by a rotation of B which sends the plane spanned by $\frac{1}{\sqrt{5}}(2, 1, 0)$ and $(0, 0, 1)$ to the (x, y)-plane in $\mathbb{R}^3$ yields the required set A.

For the first claim in the theorem statement, let

$$K = K(A) = \bigcup_{(a,b,c)\in A} \left\{\big(as + b, s, c + \tfrac{1}{2}bs\big) : |s| \le \frac{1}{\sqrt{4 + 4a^2 + b^2}}\right\}.$$

Then, by Fubini's theorem,

$$\mathcal{H}^3(K) \le \int_{-1}^{1} \mathcal{H}^2\big\{(x, y) \in \mathbb{R}^2 : (x, y) = \big(as + b, c + \tfrac{1}{2}bs\big) : (a, b, c) \in A\big\}\, ds.$$

For each $s \in (-1, 1)$, the set

$$\big\{(x, y) \in \mathbb{R}^2 : (x, y) = \big(as + b, c + \tfrac{1}{2}bs\big) : (a, b, c) \in A\big\}$$

can be written as

$$\big\{(\langle p, (s, 1, 0)\rangle, \langle p, (0, s/2, 1)\rangle) \in \mathbb{R}^2 : p \in A\big\}.$$

The vectors $(s, 1, 0)$ and $(0, s/2, 1)$ are both orthogonal to $(1, -s, s^2/2)$. The vectors $(s, 1, 0)$ and $(0, s/2, 1)$ are not necessarily orthogonal to each other, but by following the Gram–Schmidt process,[2] the above can be written as

$$A_s\left\{\left(\langle p, (s, 1, 0)\rangle, \left\langle p, (0, s/2, 1) - \frac{s}{2(1 + s^2)}(s, 1, 0)\right\rangle\right) \in \mathbb{R}^2 : p \in A\right\},$$

[2]The same observation was made in [Fässler and Orponen 2023a].

where $A_s : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear map

$$A_s(x, y) = \left(x, y + \frac{sx}{2(1+s^2)}\right).$$

Since $\det A_s = 1$,

$$\mathcal{H}^2\left\{\left(\langle p, (s, 1, 0)\rangle, \langle p, (0, s/2, 1)\rangle\right) \in \mathbb{R}^2 : p \in A\right\}$$
$$= \mathcal{H}^2\left\{\left(\langle p, (s, 1, 0)\rangle, \left\langle p, (0, s/2, 1) - \frac{s}{2(1+s^2)}(s, 1, 0)\right\rangle\right) \in \mathbb{R}^2 : p \in A\right\} \sim \mathcal{H}^2(\pi_{(1,-s,s^2/2)^\perp}(A)).$$

The curve $(1, -s, s^2/2)$ is a parabola inside the cone $\eta_2^2 = 2\eta_1\eta_3$, and this cone is a clockwise rotation of the cone $\xi_3^2 = \xi_1^2 + \xi_2^2$ by $\pi/4$ in the (ξ_1, ξ_3)-plane. It follows that the normalised curve $(1/\sqrt{1+s^2+s^4/4})(1, -s, s^2/2)$ lies in the intersection of the cone $\eta_2^2 = 2\eta_1\eta_3$ with the sphere S^2, and this intersection is a nongreat circle in S^2. Hence $\mathcal{H}^2(\pi_{(1,-s,s^2/2)^\perp}(A)) = 0$ for a.e. $s \in \mathbb{R}$, which yields $\mathcal{H}^3(K) = 0$. Since the projection of A down to the (x, y)-plane contains an open disc $U \subseteq \mathbb{R}^2$, for all $(a, b) \in U$, the set K contains a horizontal unit line segment with direction $1/(a^2+1+(b/2)^2)(a, 1, b/2)$.

Any translate $A_{(x_0,y_0)} := A + (x_0, y_0, 0)$ of A with $(x_0, y_0) \in \mathbb{R}^2$ has the property that the projection of $A_{(x_0,y_0)}$ to $\mathbb{R}^2 \times \{0\}$ contains the open disc $U_{(x_0,y_0)} := U + (x_0, y_0)$, and also has the property that for any nongreat circle C, the projection of $A_{(x_0,y_0)}$ onto $v^\perp$ has $\mathcal{H}^2$-measure zero for $\mathcal{H}^1$-a.e. $v \in C$. Fix $(a_0, b_0) \in U$. For any $v \in S^2 \setminus \mathbb{R} \times \{0\} \times \mathbb{R}$, choose $(a_v, b_v) \in \mathbb{R}^2$ such that $(a_v + a_0, 1, (b_v + b_0)/2)$ is parallel to v. Then $K(A_{(a_v,b_v)})$ contains a horizontal unit line segment with direction $1/(a^2+1+(b/2)^2)(a, 1, b/2)$ for all $(a, b) \in U + (a_v, b_v)$. This shows that for any $v \in S^2 \setminus \mathbb{R} \times \{0\} \times \mathbb{R}$, there is an open neighbourhood $U_v \subseteq S^2$ of v and a measure zero compact set $K(A_{(a_v,b_v)})$ containing a horizontal unit line segment with every direction in U_v. Since the collection of horizontal lines is invariant under rotations of the (x, y)-plane (which fix the third coordinate), this implies that for any $v \in S^2 \setminus \{(0, 0, \pm 1\}$, there is an open neighbourhood U_v of v and a measure zero compact set containing a horizontal unit line segment with every direction in U_v. This implies that for any $\epsilon > 0$, there exists a measure zero compact set containing a horizontal unit line segment in every direction in $S^2 \setminus \mathcal{N}_\epsilon(\{(0, 0, \pm 1)\})$. This verifies the first claim.

For the second claim, take a countable sequence of translates $A_1, A_2, \ldots$ of A by vectors in $\mathbb{R}^2 \times \{0\}$ such that $d(A_j, 0) \to \infty$, and such that the projections of the A_j's onto $\mathbb{R}^2 \times \{0\}$ cover $\mathbb{R}^2$. Let

$$K_1 = (\mathbb{R} \times \{0\} \times \mathbb{R}) \cup \bigcup_{(a,b,c) \in \bigcup_j A_j} K(A_j)$$
$$= (\mathbb{R} \times \{0\} \times \mathbb{R}) \cup \bigcup_{(a,b,c) \in \bigcup_j A_j} \left\{\left(as + b, s, c + \tfrac{1}{2}bs\right) : |s| \le \frac{1}{\sqrt{4+4a^2+b^2}}\right\},$$

and let K_2 be a rotation of K_1 by $\pi/2$ in the (x, y)-plane. It has already been shown that $K = K_1 \cup K_2$ has measure zero and contains a horizontal unit line segment in every direction in $S^2 \setminus \{(0, 0, \pm 1)\}$. By the condition $d(0, A_j) \to \infty$, together with the compactness of A and the definition of K, it is straightforward to check that K is closed. This verifies the second claim.

The third claim follows from a similar argument with the same A_j's; by taking

$$K_1 = (\mathbb{R}\times\{0\}\times\mathbb{R}) \cup \bigcup_{(a,b,c)\in\bigcup_j A_j} \{(as+b, s, c+\tfrac{1}{2}bs) : s\in\mathbb{R}\}.$$

Then $K = K_1 \cup K_2$ has measure zero by a similar argument to the above, and is a closed set of measure zero containing a horizontal line in every direction in $S^2 \setminus \{(0,0,\pm 1)\}$.

Finally, the proof for SL_2 lines is similar. □

Corollary 2.3. *If $p\in(1,\infty]$ and*

$$\left\| \sum_{T\in\mathbb{T}} \chi_T \right\|_{L^p(B(0,1))} \le C_\delta \tag{2-1}$$

for any set $\mathbb{T}$ of $\delta\times 1$ SL_2 tubes with δ-separated directions in S^2, then $\lim_{\delta\to 0} C_\delta = \infty$.

Let U be any Borel subset of $S^2\setminus\{(0,0,\pm1)\}$ with $\mathcal{H}^2(U)>0$. If, for some fixed $p\in[1,\infty)$,

$$\|f^*_{\delta,\mathrm{SL}_2}\|_{L^p(U)} \le C_{\delta,U}\|f\|_p$$

for all $\delta>0$ and for any nonnegative Borel function f on $\mathbb{R}^3$, then $\lim_{\delta\to 0} C_{\delta,U}=\infty$.

Proof. For the first part, using Theorem 2.2 let B be a compact set of measure zero containing a unit line segment in every direction in $U = S^2\setminus\mathcal{N}_{0.01}\{(0,0,\pm1)\}$. Let $\widetilde{U}=\{v_1,\dots,v_N\}$ be a maximal δ-separated subset of U, where $N\sim\delta^{-2}$. For each $v_k\in\widetilde{U}$, let T_k be a $\delta\times\delta\times 1$ SL_2 tube with direction v_k such that $T_k\subseteq\mathcal{N}_\delta(B)$. Then (for a sufficiently large absolute constant C)

$$1 \lesssim \int_{B(0,C)} \sum_{k=1}^N \chi_{T_k} \le \mathcal{H}^3(\mathcal{N}_\delta(B))^{1-1/p} \left\| \sum_{k=1}^N \chi_{T_k} \right\|_{L^p(B(0,C))}.$$

Hence, by (2-1),

$$1 \lesssim \mathcal{H}^3(\mathcal{N}_\delta(B))^{1-1/p} C_\delta.$$

Letting $\delta\to 0$ proves the first part.

For the second part, let $V\subseteq U$ be a compact subset of U such that $\mathcal{H}^2(V)>0$. Let B be a compact set of measure zero containing an SL_2 unit line segment in every direction in V, which exists by Theorem 2.2. Then

$$1 \lesssim \|(\chi_{\mathcal{N}_\delta}(B))^*_{\delta,\mathrm{SL}_2}\|_{L^p(V)} \le C_{\delta,U}\mathcal{H}^3(\mathcal{N}_\delta(B))^{1/p}.$$

Letting $\delta\to 0$ proves the second part. □

3. Train tracks of planks

Given a Borel measure μ on $\mathbb{R}^3$ and $\alpha\ge 0$, recall that

$$c_\alpha(\mu) = \sup_{x\in\mathbb{R}^3,\, r>0} \frac{\mu(B(x,r))}{r^\alpha}.$$

Let

$$I_\alpha(\mu) = \iint \frac{1}{|x-y|^\alpha}\, d\mu(x)\, d\mu(y).$$

Recall that $\gamma : [0, 2\pi) \to S^2$ is defined by $\gamma(\theta) = \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, 1)$, and recall that for each $\theta \in [0, 2\pi)$, π_θ is the orthogonal projection onto the plane $\gamma(\theta)^\perp$. Oberlin and Oberlin [2015] used Erdoğan's L^2 bound for the decay of conical averages of Fourier transforms of fractal measures from [Erdoğan 2004] to prove a bound on the average L^2 norms of pushforwards of fractal measures under π_θ. When $\alpha > \frac{5}{2}$, the uncertainty principle suggests that there is no loss in the approach from [Oberlin and Oberlin 2015] (up to the endpoint, provided only L^2 norms are considered), and this suggests that the Knapp examples from [Erdoğan 2004] used to prove sharpness of the L^2 conical decay rates can also be used to prove sharpness of the bound from [Oberlin and Oberlin 2015]. The following proposition verifies this intuition.

Proposition 3.1. *Let $\alpha \in [2, 3]$. If*

$$\int_0^{2\pi} \|\pi_{\theta\sharp}\mu\|^2_{L^2(\gamma(\theta)^\perp)}\, d\theta \le C_\alpha \mu(\mathbb{R}^3) c_\alpha(\mu) \tag{3-1}$$

for some nonzero finite C_α depending only on α, for all Borel measures μ on $B_3(0, 1)$, then $\alpha \ge \frac{5}{2}$. The same is true if $\mu(\mathbb{R}^3)c_\alpha(\mu)$ in (3-1) is replaced by $I_\alpha(\mu)$.

Remark 3.2. Oberlin and Oberlin [2015] proved that (3-1) does hold if $\alpha > \frac{5}{2}$, with either $\mu(\mathbb{R}^3)c_\alpha(\mu)$ or $I_\alpha(\mu)$ on the right-hand side.

Proof of Proposition 3.1. Let $\alpha \in [2, 3]$ and suppose that (3-1) holds. Let $\delta > 0$ be small. Fix any $\theta_0 \in [0, 2\pi)$. Let ψ be a smooth nonnegative bump function supported in $B(0, 1)$, with $\psi \sim 1$ on $B\big(0, \frac{1}{2}\big)$, such that $\int \psi = 1$. Let μ be the measure with Radon–Nikodym derivative equal to

$$\mu(x) = \delta^{-3/2}\psi\big(\langle x, \gamma(\theta_0)\rangle, \langle \delta^{-1/2}x, \sqrt{2}\gamma'(\theta_0)\rangle, \langle \delta^{-1}x, \sqrt{2}(\gamma \times \gamma')(\theta_0)\rangle\big),$$

i.e., μ is a Schwartz function of L^1 norm 1 supported on a plank of dimensions $\sim 1 \times \delta^{1/2} \times \delta$ centred at the origin, such that the longest direction is parallel to $\gamma(\theta_0)$, the medium direction is parallel to $\gamma'(\theta_0)$ and the shortest direction is parallel to $(\gamma \times \gamma')(\theta_0)$. If $|y_1| \le \frac{1}{100}\delta^{1/2}$, $|y_2| \le \frac{1}{100}\delta$, and $|y_3| \le \frac{1}{100}$, then

$$\mu\big(y_1\gamma'(\theta_0) + y_2(\gamma \times \gamma')(\theta_0) + y_3\gamma(\theta)\big) \gtrsim \delta^{-3/2}.$$

By second-order Taylor approximation, it follows that if $|\theta - \theta_0| < c\delta^{1/2}$ for $c > 0$ a sufficiently small absolute constant, then for all $|x_1| \le 10^{-3}\delta^{1/2}$ and $|x_2| \le 10^{-3}\delta$,

$$(\pi_{\theta\sharp}\mu)\big(x_1\gamma'(\theta) + x_2(\gamma \times \gamma')(\theta)\big) \ge \int_{-10^{-3}}^{10^{-3}} \mu\big(x_1\gamma'(\theta) + x_2(\gamma \times \gamma')(\theta) + t\gamma(\theta)\big)\, dt \gtrsim \delta^{-3/2}.$$

Therefore

$$\begin{aligned}\int_0^{2\pi} &\|\pi_{\theta\sharp}\mu\|^2_{L^2(\gamma(\theta)^\perp)}\, d\theta \\ &\ge \int_{|\theta-\theta_0|\le c\delta^{1/2}} \int_{[-10^{-3}\delta^{1/2}, 10^{-3}\delta^{1/2}]} \int_{[-10^{-3}\delta, 10^{-3}\delta]} \big|(\pi_{\theta\sharp}\mu)\big(x_1\gamma'(\theta) + x_2(\gamma \times \gamma')(\theta)\big)\big|^2\, dx_1\, dx_2\, d\theta \\ &\gtrsim \delta^{-1}.\end{aligned}$$

But $\mu(\mathbb{R}^3) = 1$, and it is straightforward to check that

$$c_\alpha(\mu) \sim \delta^{3/2-\alpha}.$$

Similarly $I_\alpha(\mu) \sim \delta^{3/2-\alpha}$ (e.g., via the Plancherel formula for the energy). Hence, by the assumed inequality (3-1),

$$\delta^{-1} \lesssim \delta^{3/2-\alpha}.$$

Letting $\delta \to 0$ gives $-1 \geq \frac{3}{2} - \alpha$ or $\alpha \geq \frac{5}{2}$. □

In the previous example, $c_\alpha(\mu)$ is much larger than $\mu(\mathbb{R}^3)$. But if ν is the sum of N copies of μ translated in the $\gamma'(\theta_0)$ and $(\gamma \times \gamma')(\theta_0)$ directions, such that the projections of these translated copies under π_θ are pairwise disjoint for $|\theta - \theta_0| \lesssim \delta^{1/2}$, then the lower bound for the left-hand side will be multiplied by N. If the translations are chosen sufficiently sparse, to ensure that $c_\alpha(\nu)$ is not larger than $c_\alpha(\mu)$, then the right-hand side will also be scaled by N, since $\mu(\mathbb{R}^3)$ will be scaled by N. If N could be chosen large enough to make $\nu(\mathbb{R}^3) \sim \delta^{3/2-\alpha} \sim c_\alpha(\nu) \sim c_\alpha(\mu)$, then this would show that no inequality is possible even with the larger right-hand side of $c_\alpha(\nu)^2$ instead of $\nu(\mathbb{R}^3)c_\alpha(\nu)$ in (3-1). The proposition below shows that this is possible using "parallel train tracks of planks". The spacing is based on the example of "parallel train tracks" from [Guth et al. 2020, Proposition 6.1], where the long spaces are a $\delta^{-1/2}$-multiple of the short spaces (the R in [Guth et al. 2020] corresponds to δ^{-1} here).

Proposition 3.3. *Let $\alpha \in [2, 3]$. If*

$$\int_0^{2\pi} \|\pi_{\theta\sharp}\nu\|^2_{L^2(\gamma(\theta)^\perp)}\, d\theta \leq C_\alpha c_\alpha(\nu)^2 \tag{3-2}$$

for some nonzero finite C_α depending only on α, for all Borel measures ν on $B_3(0, 1)$, then $\alpha \geq \frac{5}{2}$.

Proof. Let $\alpha \in [2, 3]$ and suppose that (3-2) holds. Let $\delta > 0$ be small, and fix any $\theta_0 \in [0, 2\pi)$. Let $\mu = \mu_{\delta,\theta_0}$ be the measure from the proof of Proposition 3.1, given by

$$\mu(x) = \delta^{-3/2}\psi\big(\langle x, \gamma(\theta_0)\rangle, \langle \delta^{-1/2}x, \sqrt{2}\gamma'(\theta_0)\rangle, \langle \delta^{-1}x, \sqrt{2}(\gamma \times \gamma')(\theta_0)\rangle\big),$$

where ψ is a smooth nonnegative bump function supported in $B(0, 1)$, with $\psi \sim 1$ on $B\big(0, \frac{1}{2}\big)$, such that $\int \psi = 1$. Let ν be the sum of translated copies of μ; spacing $\delta^{\alpha/2-1/2}$ in the short direction $(\gamma \times \gamma')(\theta_0)$, and spacing $\delta^{\alpha/2-1}$ in the medium direction $\gamma'(\theta_0)$ given by

$$\nu(x) = \sum_{|m|\leq 10^{-3}\delta^{1/2-\alpha/2}} \sum_{|n|\leq 10^{-3}\delta^{1-\alpha/2}} \mu\big(x - \delta^{\alpha/2-1/2}m(\gamma \times \gamma')(\theta_0) - n\delta^{\alpha/2-1}\gamma'(\theta_0)\big).$$

There are $\sim \delta^{3/2-\alpha}$ such copies, and the supports of the projections of these translated copies under π_θ are pairwise disjoint for $|\theta - \theta_0| \leq c\delta^{1/2}$, for a sufficiently small absolute constant c (using $\alpha \leq 3$ and second-order Taylor approximation), so the lower bound of δ^{-1} from the proof of Proposition 3.1 is multiplied by $\delta^{3/2-\alpha}$ to get

$$\int_0^{2\pi} \|\pi_{\theta\sharp}\nu\|^2_{L^2(\gamma(\theta)^\perp)}\, d\theta \gtrsim \delta^{1/2-\alpha}.$$

Moreover, $\nu(\mathbb{R}^3) \sim \delta^{3/2-\alpha}$, and

$$c_\alpha(\nu) \sim \delta^{3/2-\alpha}. \tag{3-3}$$

The lower bound $c_\alpha(\nu) \gtrsim \delta^{3/2-\alpha}$ in (3-3) follows from $c_\alpha(\nu) \gtrsim \nu(\mathbb{R}^3)$. The upper bound can be shown by considering different ranges of r separately, as follows. The maximum of $\nu(B(x,r))/r^\alpha$ over the range $0 < r \le \delta$ occurs near $r = \delta$ and is $\lesssim \delta^{3/2-\alpha}$. The maximum for the range $\delta \le r \le \delta^{\alpha/2-1/2}$ also occurs near $r = \delta$ (since the ball cannot intersect multiple "slats"), and is $\lesssim \delta^{3/2-\alpha}$. The maximum in the range $\delta^{\alpha/2-1/2} \le r \le \delta^{1/2}$ occurs near $r = \delta^{1/2}$, and is $\lesssim \delta^{3/2-\alpha}$. The maximum for the range $\delta^{1/2} \le r \le \delta^{\alpha/2-1}$ also occurs near $r = \delta^{1/2}$ (since the ball cannot intersect multiple "tracks"), and is $\lesssim \delta^{3/2-\alpha}$. The maximum for the range $\delta^{\alpha/2-1} \le r \le 1$ occurs near $r = 1$, and is $\lesssim \delta^{3/2-\alpha}$. Applying the assumed inequality (3-2) gives

$$\delta^{1/2-\alpha} \lesssim \delta^{3-2\alpha}.$$

Letting $\delta \to 0$ gives $\frac{1}{2} - \alpha \ge 3 - 2\alpha$ or $\alpha \ge \frac{5}{2}$. □

We will see that it is possible to weaken the requirement $\alpha \ge \frac{5}{2}$ by replacing the L^2 norm on the left in the previous examples by an L^p norm with $1 < p < 2$. Proposition 3.4 below shows that at the critical exponent $\alpha = 2$, the average L^p norms of δ-discretised measures cannot be bounded by $C_\epsilon \delta^{-\epsilon} c_2(\mu)^p$ unless $p \le \frac{3}{2}$. The example is similar to the "train tracks of planks" used in Proposition 3.3, except that as α approaches 2, the spaces between the parallel "train tracks" tends to a distance ~ 1, and the example reduces to a single "train track of planks", which is a plank version of the original "train track" example from [Wolff 1999, p. 563; Katz and Tao 2001, p. 151]. When $\alpha = 2$, a uniform bound (without the $C_\epsilon \delta^{-\epsilon}$ factor) is known to not be possible, by considering a purely 2-unrectifiable set in $\mathbb{R}^3$ and applying the Besicovitch–Federer projection theorem. Finally, the counterexample for projections onto lines below is based on the Knapp example from [Erdoğan 2004] used to prove sharpness of the L^2 conical decay rates of 1-dimensional fractal measures.

Proposition 3.4. *Let $p \in [1, \infty)$ and let $\epsilon > 0$.*

(1) *If, for any $\delta > 0$,*

$$\int_0^{2\pi} \|\pi_{\theta\sharp}\mu\|_{L^p(\gamma(\theta)^\perp)}^p \, d\theta \le C_\epsilon \delta^{-\epsilon} c_2(\mu)^p \tag{3-4}$$

for all Borel measures μ on $B_3(0,1)$ of the form

$$\mu = \frac{1}{\delta^3 \mathcal{H}^3(B(0,1))} \sum_{B \in \mathcal{B}} a_B \chi_B,$$

where $a_B > 0$ for all $B \in \mathcal{B}$ and where $\mathcal{B}$ is a disjoint family of δ-balls, then $p \le \frac{3}{2} + \epsilon$.

(2) *If, for any $\delta > 0$,*

$$\int_0^{2\pi} \|\rho_{\theta\sharp}\mu\|_{L^p(\mathrm{span}(\gamma(\theta)))}^p \, d\theta \le C_\epsilon \delta^{-\epsilon} c_1(\mu)^p \tag{3-5}$$

for all Borel measures μ on $B_3(0,1)$ of the form

$$\mu = \frac{1}{\delta^3 \mathcal{H}^3(B(0,1))} \sum_{B \in \mathcal{B}} a_B \chi_B,$$

where $a_B > 0$ for all $B \in \mathcal{B}$ and where $\mathcal{B}$ is a disjoint family of δ-balls, then $p \le \frac{3}{2} + \epsilon$.

Proof. Let $\delta > 0$ be small, fix $\theta_0 \in [0, 2\pi)$ and assume that (3-4) holds. Let $\mu = \mu_{\delta,\theta_0}$ be the measure from the proof of Proposition 3.1 given by

$$\mu(x) = \delta^{-3/2}\psi\big(\langle x, \gamma(\theta_0)\rangle, \langle \delta^{-1/2}x, \sqrt{2}\gamma'(\theta_0)\rangle, \langle \delta^{-1}x, \sqrt{2}(\gamma \times \gamma')(\theta_0)\rangle\big),$$

where ψ is a smooth nonnegative bump function supported in $B(0, 1)$, with $\psi \sim 1$ on $B\big(0, \frac{1}{2}\big)$, such that $\int \psi = 1$. Similarly to the proof of Proposition 3.1,

$$\int_0^{2\pi} \|\pi_{\theta\sharp}\mu\|^p_{L^p(\gamma(\theta)^\perp)}\, d\theta \geq \int_{|\theta-\theta_0|\leq c\delta^{1/2}} \int_{-10^{-3}\delta^{1/2}}^{10^{-3}\delta^{1/2}} \int_{-10^{-3}\delta}^{10^{-3}\delta} \delta^{-3p/2}\, dx_1\, dx_2\, d\theta \gtrsim \delta^{2-3p/2}. \tag{3-6}$$

But $\mu(\mathbb{R}^3) = 1$, and

$$c_2(\mu) \sim \delta^{-1/2}.$$

Moreover, μ is essentially a sum of indicator functions over a boundedly overlapping family of δ-balls. Hence, if (3-4) holds with the smaller $\mu(\mathbb{R}^3)c_2(\mu)^{p-1}$ on the right-hand side instead of $c_2(\mu)^p$, then

$$\delta^{2-3p/2} \lesssim \delta^{1/2-p/2-\epsilon}.$$

Letting $\delta \to 0$ gives $2 - \frac{3p}{2} \geq \frac{1}{2} - \frac{p}{2} - \epsilon$ or $p \leq \frac{3}{2} + \epsilon$.

In order to get the same restriction on p if the larger right-hand side $c_2(\mu)^p$ is assumed, let ν be the sum of translated copies of μ spacing $\delta^{1/2}$ in the short direction $(\gamma \times \gamma')(\theta_0)$, given by

$$\nu(x) = \sum_{|m|\leq 10^{-3}\delta^{-1/2}} \mu\big(x - \delta^{1/2}m(\gamma \times \gamma')(\theta_0)\big).$$

There are $\sim \delta^{-1/2}$ such copies, and the supports of the projections of these translated copies under π_θ are pairwise disjoint for $|\theta - \theta_0| \leq c\delta^{1/2}$, for a sufficiently small absolute constant c, so the lower bound of $\delta^{2-3p/2}$ from (3-6) is multiplied by $\delta^{-1/2}$ to get

$$\int_0^{2\pi} \|\pi_{\theta\sharp}\nu\|^p_{L^p(\gamma(\theta)^\perp)}\, d\theta \gtrsim \delta^{(3-3p)/2}. \tag{3-7}$$

But $\nu(\mathbb{R}^3) \sim \delta^{-1/2} \sim c_2(\nu)$ (as shown in the proof of Proposition 3.3), and ν is essentially a sum of indicator functions over a boundedly overlapping family of δ-balls. Hence, by the assumed (3-4),

$$\delta^{(3-3p)/2} \lesssim \delta^{-p/2-\epsilon}.$$

Letting $\delta \to 0$ gives $\frac{1}{2}(3-3p) \geq -\frac{1}{2}p - \epsilon$ or $p \leq \frac{3}{2} + \epsilon$. This proves the restriction on p for projections onto planes.

It remains to consider the restriction on p for projections onto lines. Let $\mu = \mu_{\delta,\theta_0}$ be the measure similar to the one from the proof of Proposition 3.1, given by

$$\mu(x) = \delta^{-3/2}\psi\big(\langle \delta^{-1}x, \gamma(\theta_0)\rangle, \langle \delta^{-1/2}x, \sqrt{2}\gamma'(\theta_0)\rangle, \langle x, \sqrt{2}(\gamma \times \gamma')(\theta_0)\rangle\big),$$

where ψ is a smooth nonnegative bump function supported in $B(0, 1)$, with $\psi \sim 1$ on $B\big(0, \frac{1}{2}\big)$, such that $\int \psi = 1$. Then $\mu(\mathbb{R}^3) \sim c_1(\mu) \sim 1$. For any $|\theta - \theta_0| \lesssim \delta^{1/2}$, μ is roughly $\delta^{-3/2}$ times the indicator

function of a $\sim \delta \times \delta^{1/2} \times 1$ plank in directions $\gamma(\theta)$, $\gamma'(\theta)$, and $(\gamma \times \gamma')(\theta)$, respectively, and so $\rho_{\theta\sharp}\mu$ is roughly a δ^{-1} multiple of the indicator function of an interval of length $\sim \delta$. Hence

$$\int_{[\theta_0-\delta^{1/2},\theta_0+\delta^{1/2}]} \int |\rho_{\theta\sharp}\mu|^p \, d\theta \gtrsim \delta^{3/2-p}.$$

Hence, by the assumed (3-5),

$$\delta^{3/2-p} \lesssim \delta^{-\epsilon}.$$

Letting $\delta \to 0$ gives $p \le \frac{3}{2} + \epsilon$. This proves the restriction on p for projections onto lines. □

For $(x, y, t) \in \mathbb{R}^3$, define $\ell^*(x, y, t)$ to be the line

$$\ell^*(x, y, t) = \big(0, x, t - \tfrac{1}{2}xy\big) + L_y,$$

where L_y is the "light ray" in the light cone

$$\widetilde{\Gamma} := \{\eta \in \mathbb{R}^3 : \eta_2^2 = 2\eta_1\eta_3\}$$

given by

$$L_y = \big\{\lambda\big(1, -y, \tfrac{1}{2}y^2\big) : \lambda \in \mathbb{R}\big\}.$$

The cone $\widetilde{\Gamma}$ is the image of the light cone

$$\Gamma := \{\xi \in \mathbb{R}^3 : \xi_3^2 = \xi_1^2 + \xi_2^2\}$$

in $\mathbb{R}^3$, under the orthogonal transformation $U(\xi_1, \xi_2, \xi_3) = (\eta_1, \eta_2, \eta_3)$ given by

$$\eta_1 = \frac{\xi_1 + \xi_3}{\sqrt{2}}, \quad \eta_3 = \frac{-\xi_1 + \xi_3}{\sqrt{2}}, \quad \eta_2 = \xi_2.$$

The restriction of ℓ^* to $|y| \le \sqrt{2}$, followed by U^* (an anticlockwise rotation by $\pi/4$ in the (ξ_1, ξ_3)-plane), parametrises the family of light rays parallel to some $\gamma(\theta)$ with $|\theta| \le \pi/2$.

Given $(a, b, c) \in \mathbb{R}^3$, define $\ell(a, b, c)$ to be the horizontal line

$$\ell(a, b, c) = \{(b, 0, c) + s(a, 1, b/2) : s \in \mathbb{R}\}. \tag{3-8}$$

The following is the point-line duality principle.

Lemma 3.5 [Fässler and Orponen 2023b, Lemma 4.11]. *Let $p \in \mathbb{R}^3$ and $p^* \in \mathbb{H}$. Then*

$$p \in \ell^*(p^*) \quad \textit{if and only if} \quad p^* \in \ell(p).$$

The symmetry of the above lemma means that many of the ideas from [Fässler and Orponen 2023b] for vertical projections in the Heisenberg group can be reversed as in the lemma below.

Lemma 3.6. *The measure $\mathfrak{m}$ given by*

$$\mathfrak{m}(F) = \int_{-\pi/2}^{\pi/2} \mathcal{H}^2\big\{w \in \gamma(\theta)^\perp : \pi_\theta^{-1}(w) \in U^*F\big\} \, d\theta,$$

for a Borel set F of light rays parallel to lines in $\widetilde{\Gamma}$, is comparable to the pushforward of the Lebesgue measure on $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : |x_2| \le \sqrt{2}\}$ under ℓ^*, meaning that

$$(\ell^*_\sharp \mathcal{H}^3 \chi_{|x_2|\le\sqrt{2}})(F) \sim \mathfrak{m}(F) \tag{3-9}$$

for any Borel set F of light rays parallel to lines in $\widetilde{\Gamma}$. As a consequence, for any nonnegative Borel function f on the set of light rays parallel to lines in $\widetilde{\Gamma}$,

$$\int_{-\pi/2}^{\pi/2} \int_{\gamma(\theta)^\perp} f(U\pi_\theta^{-1}(w))\, d\mathcal{H}^2(w)\, d\theta \sim \int f\, d(\ell^*_\sharp \mathcal{H}^3 \chi_{|x_2|\le\sqrt{2}}). \tag{3-10}$$

Proof. Let F be given. Let m be the Lebesgue measure on $\mathbb{R}^3 \cap \{|x_2| \le \sqrt{2}\}$. Then

$$\begin{aligned}
(\ell^*_\sharp m)(F) &= m\big\{(x, y, t) \in \mathbb{R}^3 : |y| \le \sqrt{2},\ \big(0, x, t - \tfrac{1}{2}xy\big) + L_y \in F\big\} \\
&= \int_{-\sqrt{2}}^{\sqrt{2}} \mathcal{H}^2\big\{(x, t) \in \mathbb{R}^2 : \big(0, x, t - \tfrac{1}{2}xy\big) + \operatorname{span}(1, -y, y^2/2) \in F\big\}\, dy \\
&\sim \int_{-\sqrt{2}}^{\sqrt{2}} \mathcal{H}^2\big\{w \in (1, -y, y^2/2)^\perp : \pi^{-1}_{(1,-y,y^2/2)^\perp}(w) \in F\big\}\, dy \\
&= \int_{-\sqrt{2}}^{\sqrt{2}} \mathcal{H}^2\big\{w \in (U^*(1, -y, y^2/2))^\perp : \pi^{-1}_{(U^*(1,-y,y^2/2))^\perp}(w) \in U^*F\big\}\, dy \\
&\sim \int_{-\pi/2}^{\pi/2} \mathcal{H}^2\big\{w \in (\cos\theta, \sin\theta, 1)^\perp : \pi^{-1}_{(\cos\theta,\sin\theta,1)^\perp}(w) \in U^*F\big\}\, d\theta \\
&= \mathfrak{m}(F).
\end{aligned}$$

This proves (3-9) and yields (3-10) whenever $f = \chi_F$ for a Borel set F of light rays. The equivalence (3-10) for general Borel functions follows by approximating f with simple functions and then applying the monotone convergence theorem. □

Proposition 3.7. *If, for any $\epsilon > 0$ and $\delta > 0$,*

$$\Big\| \sum_{T\in\mathbb{T}} \chi_T \Big\|^p_{L^p(B(0,1))} \le C_\epsilon \delta^{-\epsilon} \tag{3-11}$$

for any family $\mathbb{T}$ of SL_2 δ-tubes of length 1 with δ-separated directions, then $p \le \frac{3}{2}$.

Proof. It is first shown that if, for any $\epsilon > 0$, (3-11) holds for any family $\mathbb{T}$ of SL_2 δ-tubes of length 1 satisfying the *weaker* property that

$$|\{T \in \mathbb{T} : T \subseteq S\}| \le (r/\delta)^2,$$

for any tube S of length 2 and radius r, and any $r \in [\delta, 1]$, then $p \le \frac{3}{2}$.

Let $\delta > 0$ be small and choose $\theta_0 = \pi/2$. Let $\mu = \mu_{\delta,\theta_0}$ be a $\delta^{1/2}$-multiple of the (slightly modified) measure from the proof of Proposition 3.1, given by

$$\mu(x) = \delta^{-1}\psi\big(\langle x, \gamma(\theta_0)\rangle, \langle \delta^{-1/2}x, \sqrt{2}\gamma'(\theta_0)\rangle, \langle \delta^{-1}x, \sqrt{2}(\gamma \times \gamma')(\theta_0)\rangle\big),$$

where ψ is a smooth nonnegative bump function supported in $B(0, c)$, with $\psi \sim 1$ on $B(0, c/2)$, such that $\int \psi = 1$ (here c is a small absolute constant to be chosen; this is the only difference from the proof of Proposition 3.1 where $c = 1$). As in the proof of Proposition 3.4, let ν be the sum of translated copies of μ spacing $\delta^{1/2}$ in the short direction $(\gamma \times \gamma')(\theta_0)$, given by

$$\nu(x) = \sum_{|m| \leq c\delta^{-1/2}} \mu\big(x - \delta^{1/2} m(\gamma \times \gamma')(\theta_0)\big).$$

Then ν is supported in a ball around the origin of radius $\sim c$, and ν is very similar to a $\delta^{1/2}$-multiple of the measure from the proof of Proposition 3.4, so by a similar argument to (3-7),

$$\delta^{3/2-p} \lesssim \int_{-\pi/2}^{\pi/2} \|\pi_{\theta\sharp}\nu\|^p_{L^p(\gamma(\theta)^\perp)}\, d\theta.$$

Let $\mathcal{B}$ be a family of δ-balls, such that the centres of the balls in $\mathcal{B}$ form a maximal δ-separated subset of the support of ν. Then

$$\nu \leq \sum_{B \in \mathcal{B}} \delta^{-1} \chi_B.$$

Hence, by Lemma 3.6,

$$\begin{aligned}
\int_{-\pi/2}^{\pi/2} \|\pi_{\theta\sharp}\nu\|^p_{L^p(\gamma(\theta)^\perp)}\, d\theta &= \int_{-\pi/2}^{\pi/2} \int_{\gamma(\theta)^\perp} \left| \int_{\pi_\theta^{-1}(y)} \nu \, d\mathcal{H}^1 \right|^p d\mathcal{H}^2(y)\, d\theta \\
&= \int_{-\pi/2}^{\pi/2} \int_{\gamma(\theta)^\perp} \left| \int_{U\pi_\theta^{-1}(y)} (U_\sharp \nu)\, d\mathcal{H}^1 \right|^p d\mathcal{H}^2(y)\, d\theta \\
&\sim \int \left| \int_L U_\sharp \nu \, d\mathcal{H}^1 \right|^p d(\ell^*_\sharp \mathcal{H}^3 \chi_{|x_2| \leq \sqrt{2}})(L) \\
&\lesssim \int |\{B \in \mathcal{B} : UB \cap L \neq \varnothing\}|^p \, d(\ell^*_\sharp \mathcal{H}^3 \chi_{|x_2| \leq \sqrt{2}})(L) \\
&= \int_{|p_2^*| \leq \sqrt{2}} |\{B \in \mathcal{B} : UB \cap \ell^*(p^*) \neq \varnothing\}|^p \, d\mathcal{H}^3(p^*). \qquad (3\text{-}12)
\end{aligned}$$

If $|p_2^*| \leq \sqrt{2}$ and $\ell^*(p^*)$ intersects $B(0, c)$, then $|p^*| \leq 2$ (provided c is now chosen sufficiently small; this follows easily from the definition of ℓ^*). Hence

$$(3\text{-}12) \leq \int_{|p^*| \leq 2} |\{B \in \mathcal{B} : UB \cap \ell^*(p^*) \neq \varnothing\}|^p \, d\mathcal{H}^3(p^*). \qquad (3\text{-}13)$$

By Lemma 3.5 (point-line duality),

$$(3\text{-}13) \leq \int_{|p^*| \leq 2} \left| \sum_{B \in \mathcal{B}} \chi_{\ell(UB)}(p^*) \right|^p d\mathcal{H}^3(p^*).$$

The measure ν satisfies $c_2(\nu) \lesssim 1$, and similarly so does the measure $\sum_{B \in \mathcal{B}} \delta^{-1} \chi_B$ (which is roughly the same as ν), which means that for any $\delta \leq r \leq 1$, the number of δ-balls from $\mathcal{B}$ intersecting any r-ball is

$\lesssim (r/\delta)^2$. By the formula for ℓ, this implies that for any $r \in [\delta, 1]$ and any tube r-tube T of length 2,

$$|\{\ell(UB) : B \in \mathcal{B}, \ell(UB) \subseteq T\}| \lesssim (r/\delta)^2.$$

Therefore, if (3-11) holds for any family of SL_2 tubes satisfying the 2-dimensional ball condition above, then

$$\delta^{3/2-p} \lesssim \int_{|p^*|\le 2} \left| \sum_{B\in\mathcal{B}} \chi_{\ell(UB)}(p^*) \right|^p d\mathcal{H}^3(p^*) \le C_\epsilon \delta^{-\epsilon}.$$

This yields $p \le \frac{3}{2} + \epsilon$.

To get the same restriction on p when (3-11) holds only for those sets $\mathbb{T}$ of SL_2 δ-tubes of length 1 with δ-separated directions, it suffices to show that the collection of tubes $\ell(UB)$ have $\sim \delta$-separated directions, for any subfamily of balls B in $\mathcal{B}$ separated by 100δ. Let $B_1, B_2 \in \mathcal{B}$ be 100δ-separated balls in $\mathcal{B}$. Let (a_j, b_j, c_j) be the centre of UB_j, where $j \in \{1, 2\}$. The direction of $\ell(UB_j)$ is parallel to $(a_j, 1, b_j/2)$, so it is required to show that

$$|a_1 - a_2| + |b_1 - b_2| \gtrsim \delta.$$

Write

$$B_2 = B_1 + \lambda_1\gamma(\pi/2) + \lambda_2\gamma'(\pi/2) + \lambda_3(\gamma \times \gamma')(\pi/2),$$

where $|\lambda_3| \le 1$, $|\lambda_2| \le c\delta^{1/2}$ and $|\lambda_1| \le 1$, and $|(\lambda_1, \lambda_2, \lambda_3)| \ge 100\delta$. This can be written as

$$B_2 = B_1 + \frac{\lambda_1}{\sqrt{2}}(0, 1, 1) + \frac{\lambda_2}{\sqrt{2}}(-1, 0, 0) + \frac{\lambda_3}{2}(0, -1, 1).$$

Hence

$$UB_2 = UB_1 + \frac{\lambda_1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, 1, \frac{1}{\sqrt{2}}\right) + \frac{\lambda_2}{\sqrt{2}}\left(\frac{-1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) + \frac{\lambda_3}{2}\left(\frac{1}{\sqrt{2}}, -1, \frac{1}{\sqrt{2}}\right).$$

It follows that

$$(a_2, b_2) = (a_1, b_1) + \left(\frac{\lambda_1}{2} - \frac{\lambda_2}{2} + \frac{\lambda_3}{2\sqrt{2}}, \frac{\lambda_1}{\sqrt{2}} - \frac{\lambda_3}{2}\right).$$

By the triangle inequality,

$$\left|\frac{\lambda_1}{2} - \frac{\lambda_2}{2} + \frac{\lambda_3}{2\sqrt{2}}\right| \ge \left|\frac{\lambda_3}{\sqrt{2}} - \frac{\lambda_2}{2}\right| - \frac{1}{\sqrt{2}}\left|\frac{\lambda_1}{\sqrt{2}} - \frac{\lambda_3}{2}\right|.$$

Hence

$$|a_1 - a_2| + \frac{|b_1 - b_2|}{\sqrt{2}} \ge \left|\frac{\lambda_3}{\sqrt{2}} - \frac{\lambda_2}{2}\right|.$$

If B_1 and B_2 lie in different planks, then $|\lambda_3| \ge \delta^{1/2}$, and since $|\lambda_2| \le c\delta^{1/2}$, this yields

$$|a_1 - a_2| + |b_1 - b_2| \gtrsim \delta^{1/2}.$$

It remains to consider the case where B_1 and B_2 lie in the same plank. In this case, $|\lambda_3| \le \delta$. If $|\lambda_1| \ge 10\delta$, then

$$|b_1 - b_2| \sim \left|\frac{\lambda_1}{\sqrt{2}} - \frac{\lambda_3}{2}\right| \gtrsim \delta.$$

Otherwise $|\lambda_1| < 10\delta$ and $|\lambda_3| \le \delta$ imply that $|\lambda_2| \ge 50\delta$, and this gives

$$|a_1 - a_2| = \left|\frac{\lambda_1}{2} - \frac{\lambda_2}{2} + \frac{\lambda_3}{2\sqrt{2}}\right| \gtrsim \delta.$$

This shows that the directions of 100δ-separated balls in $\mathcal{B}$ are $\sim \delta$-separated. It follows that if (3-11) holds for any family $\mathbb{T}$ of SL_2 δ-tubes of length 1 with δ-separated directions, then it can be applied to the family $\{\ell(UB) : B \in \mathcal{B}\}$, and therefore

$$\delta^{3/2-p} \lesssim \int_{|p^*|\le 2} \left|\sum_{B\in\mathcal{B}} \chi_{\ell(UB)}(p^*)\right|^p d\mathcal{H}^3(p^*) \le C_\epsilon \delta^{-\epsilon}.$$

This yields $p \le \frac{3}{2} + \epsilon$. Letting $\epsilon \to 0$ gives $p \le \frac{3}{2}$. □

4. An $L^{3/2}$ inequality for projections onto planes

To begin this section, we will outline a decomposition of the outer part of the light cone

$$\{(\xi_1, \xi_2, \xi_3) : |\xi_3| = |(\xi_1, \xi_2)|\},$$

which will be important in the proof of Theorem 1.2. The "outer part" means the set

$$\{(\xi_1, \xi_2, \xi_3) : |(\xi_1, \xi_2)| > |\xi_3|\}.$$

For each $\theta \in [0, 2\pi)$ and $0 \le k < j$, let

$$\tau(\theta, j, k, +) = \big\{\lambda_1(\gamma \times \gamma')(\theta) + \lambda_2\gamma'(\theta) + \lambda_3\gamma(\theta) : |\lambda_1| \sim 2^j,\ |\lambda_2| \lesssim 2^{j-k/2},\ |\lambda_3| \sim 2^{j-k},\ \lambda_1 > 0,\ \lambda_3 < 0\big\},$$

and for $k = j$ let

$$\tau(\theta, j, j, +) = \big\{\lambda_1(\gamma \times \gamma')(\theta) + \lambda_2\gamma'(\theta) + \lambda_3\gamma(\theta) : |\lambda_1| \sim 2^j,\ |\lambda_2| \lesssim 2^{j/2},\ |\lambda_3| \lesssim 1,\ \lambda_1 > 0\big\}.$$

To be precise, $|\lambda_1| \sim 2^j$ can be taken to mean $2^j/100 \le |\lambda_1| \le 100 \cdot 2^j$, and similarly for the other implied constants here; the exact constants we choose are not important. Similarly, we define

$$\tau(\theta, j, k, -) = \big\{\lambda_1(\gamma \times \gamma')(\theta) + \lambda_2\gamma'(\theta) + \lambda_3\gamma(\theta) : |\lambda_1| \sim 2^j,\ |\lambda_2| \lesssim 2^{j-k/2},\ |\lambda_3| \sim 2^{j-k},\ \lambda_1 < 0,\ \lambda_3 > 0\big\},$$

and for $k = j$,

$$\tau(\theta, j, j, -) = \big\{\lambda_1(\gamma \times \gamma')(\theta) + \lambda_2\gamma'(\theta) + \lambda_3\gamma(\theta) : |\lambda_1| \sim 2^j,\ |\lambda_2| \lesssim 2^{j/2},\ |\lambda_3| \lesssim 1,\ \lambda_1 < 0\big\}.$$

For each such τ, we write $\theta = \theta_\tau$. For each k, choose a maximal $c2^{-k/2}$-separated subset Θ_k of $[0, 2\pi)$, let $\Lambda_{j,k} = \{\tau(\theta, j, k, \pm) : \theta \in \Theta_k\}$, where c is a small constant such that the set $\Lambda = \bigcup_{j\ge1}\bigcup_{0\le k\le j} \Lambda_{j,k}$ forms a boundedly overlapping cover of the set $\big\{\xi \in \mathbb{R}^3 : \frac{1}{3} \le |\xi_3| \le |(\xi_1, \xi_2)| \le 3|\xi_3|\big\}$.

The covering property can be seen as follows. If ξ is such that $\frac{1}{3} \le |\xi_3| \le |(\xi_1, \xi_2)| \le 3|\xi_3|$, we can choose $j \ge 1$ such that $2^j/2 \le |\xi| \le 2 \cdot 2^j$, and let $\omega = 2^{-j}\xi$. Then ω lies outside the light cone and

has a distance ~ 1 to the origin, and distance $\lesssim 1$ to the light cone. Therefore, we can find $0 \leq k \leq j$ such that the distance from ω to the light cone is either $\lesssim 2^{-j}$ or $\sim 2^{-k}$. We can then express ω as $\lambda_1(\gamma \times \gamma')(\theta_0) + \lambda_3\gamma(\theta_0)$ for some $\theta_0 \in [0, 2\pi)$, $|\lambda_1| \sim 1$ and either $|\lambda_3| \lesssim 2^{-j}$ or $\lambda_3 \sim 2^{-k}$ (with appropriate signs depending on whether ω is closer to the forward or backward light cone). By writing $\theta_0 = \theta + O(2^{-k/2})$ for some $\theta \in \Theta_k$ and applying a Taylor expansion, and then scaling up $\xi = 2^j\omega$, we get that ξ is in one of the sets $\tau(\theta, j, k, \pm)$, which verifies the covering property.

Let $\{\psi_\tau\}_{\tau\in\Lambda}$ be a smooth partition of unity such that each ψ_τ is supported in τ and such that the ψ_τ's sum to 1 on the set

$$\left\{\xi \in \mathbb{R}^3 : \tfrac{1}{2} \leq |\xi_3| \leq |(\xi_1, \xi_2)| \leq 2|\xi_3|\right\}.$$

The partition of unity can be constructed so that the derivatives of each ψ_τ satisfy natural bounds; meaning that for any $m \geq 1$, the size of the m-th derivative of ψ_τ in the direction $(\gamma \times \gamma')(\theta_\tau)$ is $\lesssim_m 2^{-jm}$, the size of the m-th derivative of ψ_τ in the direction $\gamma'(\theta_\tau)$ is $\lesssim_m 2^{-(j-k/2)m}$, and the size of the m-th derivative of ψ_τ in the direction $\gamma(\theta_\tau)$ is $\lesssim_m 2^{-(j-k)m}$ (see [Harris 2023b, Lemma 3.3] for a more precise statement).

Given $\delta > 0$ (which may be thought of as vanishingly small) and $0 \leq k \leq j$, for each $\tau \in \Lambda_{j,k}$ cover $\mathbb{R}^3$ by a boundedly overlapping collection of planks $T \in \mathbb{T}_\tau$ of dimensions

$$2^{k\delta-(j-k)} \times 2^{k(-1/2+\delta)-(j-k)} \times 2^{k(-1+\delta)-(j-k)}$$

dual to τ (meaning that the long direction of T is the short direction of τ, and vice-versa). Given a measure μ on $B_3(0, 1)$ and a plank T, define

$$M_T\mu = \eta_T(\mu * \check{\psi}_\tau),$$

where $\sum_{T\in\mathbb{T}_\tau} \eta_T = 1$, and such that for each T, η_T is a bump function supported on T, with Fourier transform essentially supported in a dual box to T.

Lemma 4.2 below states that the projections of measures of dimension greater than 2 have small L^p norms on small unions of rectangles, on average, for exponents $1 \leq p \leq \frac{3}{2}$. The main use of Lemma 4.2 will be in proving that pushforwards of measures of dimension greater than 2 are almost surely in $L^{3/2}$; by splitting μ into "good" and "bad" parts, and bounding the $L^{3/2}$ norm of the projections of the "bad" part of a measure by the $L^{3/2}$ norm of the projections of the original measure on small unions of rectangles. Before we state the lemma, we give a definition which makes "small unions of rectangles" more precise. Let ϕ be a smooth bump function equal to 1 on $B_3(0, 1)$ and vanishing outside $B_3(0, 2)$. For each $R > 0$ let $\phi_R(x) = R^3\phi(Rx)$.

Definition 4.1. If $\alpha \in [0, 3]$, $\beta > 0$, $R \geq 1$, and μ is a Borel measure on $B_3(0, 1)$ with $c_\alpha(\mu) < \infty$, then a triple $(\mu, R, \{\mathbb{D}_\theta\})$ is called β-admissible if $\{\mathbb{D}_\theta\}$ is a family of sets, where each $\mathbb{D}_\theta$ is a set of rectangles of dimensions $R^{-1/2} \times R^{-1}$ in $\gamma(\theta)^\perp$ with long direction parallel to $\gamma'(\theta)$ and short direction parallel to $(\gamma \times \gamma')(\theta)$, with $(\pi_{\theta\sharp}(\mu * \phi_R))(D) \geq c_\alpha(\mu)R^{\beta-3/2}$ for each $D \in \mathbb{D}_\theta$, such that the set

$$\left\{(\theta, x) \in [0, 2\pi) \times \mathbb{R}^3 : x \in \bigcup_{D\in\mathbb{D}_\theta} D\right\}$$

is Borel measurable.

Lemma 4.2. *Let $\alpha > 2$ and $1 \le p \le \frac{3}{2}$. Then for any $\beta > 0$, there exists a $\kappa > 0$ (which may depend on α, β, and p) such that*

$$\int_0^{2\pi} \int_{\bigcup_{D\in\mathbb{D}_\theta}} |\pi_{\theta\sharp}(\mu * \phi_R)|^p \, d\mathcal{H}^2 \, d\theta \le C_{\alpha,\beta,p} R^{-\kappa} \mu(\mathbb{R}^3) c_\alpha(\mu)^{p-1} \tag{4-1}$$

for all β-admissible datum $(\mu, R, \{\mathbb{D}_\theta\})$.

Remark 4.3. Having $\mu * \phi_R$ instead of μ in (4-4) is similar to assuming that μ is a linear combination of characteristic functions over a boundedly overlapping collection of R^{-1}-balls.

Proof. Apparently, the result follows from the endpoint cases $p = 1$ and $p = \frac{3}{2}$ and interpolation. Instead of proving the two endpoint cases directly, we will in fact show that the claimed result holds true for a sequence of increasing exponents $\{p_n\}$. Briefly speaking, the proof is based on a self-improving argument: if (4-1) holds for the exponent p_n, then it holds for p_{n+1}.

Here are the details. Define the sequence $\{p_n\}_{n\in\mathbb{N}\cup\{0,\infty\}}$ by

$$\frac{1}{p_n} = \frac{2}{3} + \frac{1}{3\cdot 4^n}, \quad n \in \mathbb{N} \cup \{0, \infty\}. \tag{4-2}$$

We claim that the following holds true: if either

(1) $n \in \{0, 1\}$, or if

(2) $2 \le n \le \infty$, and for all $m \in \mathbb{N} \setminus \{0\}$ with $m < n$, for any $\beta > 0$, there exists a $\kappa > 0$ (depending on α, β, m) such that

$$\int_0^{2\pi} \int_{\bigcup_{D\in\mathbb{D}_\theta}} |\pi_{\theta\sharp}(\mu * \phi_R)|^{p_m} \, d\mathcal{H}^2 \, d\theta \le C_{\alpha,\beta,m} R^{-\kappa} \mu(\mathbb{R}^3) c_\alpha(\mu)^{p_m - 1} \tag{4-3}$$

for all β-admissible datum $(\mu, R, \{\mathbb{D}_\theta\})$,

then for any $\beta > 0$, there exists a $\kappa > 0$ (which may depend on α, β, and n) such that

$$\int_0^{2\pi} \int_{\bigcup_{D\in\mathbb{D}_\theta}} |\pi_{\theta\sharp}(\mu * \phi_R)|^{p_n} \, d\mathcal{H}^2 \, d\theta \le C_{\alpha,\beta,n} R^{-\kappa} \mu(\mathbb{R}^3) c_\alpha(\mu)^{p_n - 1} \tag{4-4}$$

for all β-admissible datum $(\mu, R, \{\mathbb{D}_\theta\})$.

It is easy to see that the claim would imply the desired result. Indeed, the endpoint case $p = 1$ follows directly from setting $n = 0$ (for which (4-4) holds unconditionally). Induction then shows that (4-4) holds for all finite n. Lastly, applying the claim again to $n = \infty$ proves the endpoint case $p = \frac{3}{2}$.

We now prove the claim. For a given n, the proof is a bootstrapping argument which gradually lowers the value of β, which for simplicity will be written as a proof by contradiction. Define $p_m = p_0 = 1$ for $m < 0$. Let $n \in \mathbb{N} \cup \{0, \infty\}$ be given and let $\beta_{\inf}$ be the infimum over all positive β for which the conclusion of the lemma is true for $p = p_n$. Suppose for a contradiction that $\beta_{\inf} > 0$. Clearly $\beta_{\inf} \le \frac{3}{2}$ (in particular it is finite); this follows from the inequality $\mu(\mathbb{R}^3) \le c_\alpha(\mu)$. Choose a constant $c \in (0, 1)$ sufficiently close to 1 to ensure that $1 - c < \frac{1}{1000}(\alpha - 2)$, then let $\beta_0 \in (\beta_{\inf}, 2\beta_{\inf})$ be sufficiently close to $\beta_{\inf}$ so that $c\beta_0 < \beta_{\inf}$. To obtain a contradiction, it will suffice to show that the lemma holds for all

$\beta > c\beta_0$. Fix such a $\beta > c\beta_0$. Let $\delta > 0$ be such that $\delta \ll \kappa(\beta_0)$ and $\delta \ll \beta$. The measure $\mu * \phi_R$ will be relabelled as μ to simplify the notation (so it may be assumed that $\widehat{\mu}$ is negligible outside $B(0, R^{1+\delta})$).

We first derive an estimate of the measure of the integration domain. Let

$$F = \left\{(\theta, x) \in [0, 2\pi) \times \mathbb{R}^3 : x \in \bigcup_{D \in \mathbb{D}_\theta} D\right\}.$$

By the assumption that $(\pi_{\theta\sharp}\mu)(D) \geq R^{\beta-3/2} c_\alpha(\mu)$ and the Vitali covering lemma (to handle the possibly nondisjoint case),

$$(\mathcal{H}^1 \times \mathcal{H}^2)(F) \lesssim c_\alpha(\mu)^{-1} R^{-\beta} \int_0^{2\pi} \int_{\bigcup_{D \in \mathbb{D}_\theta} D} |\pi_{\theta\sharp}\mu| \, d\mathcal{H}^2 \, d\theta.$$

If $n < \infty$, then by Hölder's inequality applied to χ_F and $|\pi_{\theta\sharp}\mu|$,

$$(\mathcal{H}^1 \times \mathcal{H}^2)(F)^{1/p_{n-1}} \lesssim c_\alpha(\mu)^{-1} R^{-\beta} \left(\int_0^{2\pi} \int_{\bigcup_{D \in \mathbb{D}_\theta} D} |\pi_{\theta\sharp}\mu|^{p_{n-1}} \, d\mathcal{H}^2 \, d\theta \right)^{1/p_{n-1}}.$$

Therefore

$$(\mathcal{H}^1 \times \mathcal{H}^2)(F) \lesssim c_\alpha(\mu)^{-p_{n-1}} R^{-\beta p_{n-1}} \int_0^{2\pi} \int_{\bigcup_{D \in \mathbb{D}_\theta} D} |\pi_{\theta\sharp}\mu|^{p_{n-1}} \, d\mathcal{H}^2 \, d\theta.$$

By the assumed (4-3) (which also holds trivially without the $R^{-\kappa}$ decay if $m \in \{-1, 0\}$) this gives

$$(\mathcal{H}^1 \times \mathcal{H}^2)(F) \lesssim \mu(\mathbb{R}^3) c_\alpha(\mu)^{-1} R^{-\beta p_{n-1}}.$$

To include the case $n = \infty$, one can replace the above by a slightly weaker estimate

$$(\mathcal{H}^1 \times \mathcal{H}^2)(F) \lesssim \mu(\mathbb{R}^3) c_\alpha(\mu)^{-1} R^{\delta - \beta p_{n-1}}, \tag{4-5}$$

which, when $n = \infty$, follows by rerunning the above argument with p_{n-1} replaced by p_m for some sufficiently large finite m $\left(\text{more precisely, taking } m < \infty \text{ such that } p_m > \frac{3}{2} - \frac{\delta}{\beta}\right)$. Here, $n - 1 := \infty$ if $n = \infty$. Since the measure $\mathcal{H}^2$ on $\mathbb{R}^3$ is not σ-finite, to avoid issues of applying Fubini's theorem above and below we identify the measure $\mathcal{H}^1 \times \mathcal{H}^2$ on $[0, 2\pi) \times \mathbb{R}^3$ with that obtained by integrating first in $x \in \gamma(\theta)^\perp$ then in $\theta \in [0, 2\pi)$ (rather than in the other order; we could alternatively rewrite the measure as a measure on $[0, 2\pi) \times \mathbb{R}^2$ by using coordinates).

Note that the above is the only place in the proof that the assumption (4-3) is used. From now on, estimate (4-5) of the integration domain is all we need.

Let J be such that $2^J \sim R^\epsilon$, where $\epsilon > 0$ is chosen so that $\delta \ll \epsilon \ll \min\{\beta, \alpha - 2\}$. Given (j, k) and $\tau \in \Lambda_{j,k}$, let

$$\mathbb{T}_{\tau,b} = \left\{T \in \mathbb{T}_\tau : \mu(4T) \geq 2^{-k(3/2-\beta_0)} 2^{-(j-k)\alpha + 1000 j\epsilon} c_\alpha(\mu)\right\}, \quad \mathbb{T}_{\tau,g} = \mathbb{T}_\tau \setminus \mathbb{T}_{\tau,b}, \tag{4-6}$$

and define the bad and good part of the measure as

$$\mu_b = \sum_{j \geq J} \sum_{k \in [j\epsilon, j]} \sum_{\tau \in \Lambda_{j,k}} \sum_{T \in \mathbb{T}_{\tau,b}} M_T \mu, \quad \mu_g = \mu - \mu_b. \tag{4-7}$$

Note that by the trivial upper bound on each $\mu(4T)$ implied by the Frostman condition of μ, $\mathbb{T}_{\tau,b}$ is empty whenever $\tau \in \Lambda_{j,k}$ and $0 \le k \le j\epsilon$, so the range of k in the definition of μ_b in (4-7) could be replaced by $k \in [0, j]$. This will be used later.

By the triangle inequality, followed by Hölder's inequality (using (4-5)),

$$\begin{aligned}\int_0^{2\pi} \int_{\bigcup_{D\in\mathbb{D}_\theta}} |\pi_{\theta\sharp}|^{p_n}\, d\mathcal{H}^2\, d\theta \\ \lesssim \int_0^{2\pi}\int |\pi_{\theta\sharp}\mu_b|^{p_n}\, d\mathcal{H}^2\, d\theta + \left(\frac{\mu(\mathbb{R}^3)}{c_\alpha(\mu)}\right)^{1-p_n/2} R^{\delta-\beta p_{n-1}(1-p_n/2)} \left(\int_0^{2\pi}\int |\pi_{\theta\sharp}\mu_g|^2\, d\mathcal{H}^2\, d\theta\right)^{p_n/2}. \end{aligned} \tag{4-8}$$

In what follows, negligible tail terms of the form $R^{-N}\mu(\mathbb{R}^3)$ (for arbitrarily large N) will be left out, since whenever such terms dominate, the desired inequalities follow trivially. In particular, this means that it suffices to consider scales $J \le j \lesssim \log R$.

We first estimate the bad part, which doesn't rely on (4-5). By the triangle inequality, followed by Hölder's inequality again (noting that $j \lesssim \log R$),

$$\int_0^{2\pi}\int |\pi_{\theta\sharp}\mu_b|^{p_n}\, d\mathcal{H}^2\, d\theta \lesssim (\log R)^{O(1)} \sum_{j\ge J}\sum_{k\in[j\epsilon,\,j]} \int_0^{2\pi}\int \left|\sum_{\tau\in\Lambda_{j,k}}\sum_{T\in\mathbb{T}_{\tau,b}} \pi_{\theta\sharp}M_T\mu\right|^{p_n} d\mathcal{H}^2\, d\theta. \tag{4-9}$$

Fix a pair (j, k) occurring in the above sum. We will prove the following:

Claim.

$$\begin{aligned}\int_0^{2\pi}\int \left|\sum_{\tau\in\Lambda_{j,k}}\sum_{T\in\mathbb{T}_{\tau,b}} \pi_{\theta\sharp}M_T\mu\right|^{p_n} d\mathcal{H}^2\, d\theta \\ \lesssim 2^{kO(\delta)} \int_0^{2\pi} \sum_{\substack{\tau\in\Lambda_{j,k}:\\ |\theta_\tau-\theta|\le 2^{k(\delta-1/2)}}} \sum_{T\in\mathbb{T}_{\tau,b}} \mu(2T)^{p_n} (m(T)2^{j-k})^{-(p_n-1)}\, d\theta, \end{aligned} \tag{4-10}$$

where θ_τ is defined by $\tau = \tau(\theta_\tau, j, k)$, and $m(T)$ is the Lebesgue measure of T.

Proof of the Claim. In order to prove (4-10), we will sometimes need to make use of the intermediate inequality, for any $\theta \in [0, 2\pi]$ with $|\theta_\tau - \theta| \le 2^{k(\delta-1/2)}$,

$$\|\pi_{\theta\sharp}M_T\mu\|_{L^{p_n}(\mathcal{H}^2)}^{p_n} \lesssim 2^{kO(\delta)}\mu(2T)^{p_n}(m(T)2^{j-k})^{-(p_n-1)}. \tag{4-11}$$

The inequality (4-11) holds up to a rapidly decaying tail, and is a consequence of the uncertainty principle (see Bernstein's inequality in [Wolff 2003, Proposition 5.3]). The idea is that, since each $M_T\mu$ is a wave packet that is concentrated on T and locally constant, up to a rapidly decaying tail, one has $\|\pi_{\theta\sharp}M_T\mu\|_{L^{p_n}}^{L^{p_n}} \sim \|\pi_{\theta\sharp}M_T\mu\|_{L^1}^{p_n} m(\pi_\theta(T))^{1-p_n}$. Since $\|\pi_{\theta\sharp}M_T\mu\|_{L^1} \le \|M_T\mu\|_{L^1}$, one can then use Lemma 4 of [Gan et al. 2022] to get $\|\pi_{\theta\sharp}M_T\mu\|_{L^1} \lesssim 2^{kO(\delta)}\mu(2T)$. The condition $|\theta_\tau - \theta| \le 2^{k(\delta-1/2)}$ implies that $\pi_\theta(T)$ is contained in a rectangle of area $\lesssim 2^{kO(\delta)}2^{j-k}m(T)$, and putting this all together yields (4-11).

An elementary way to obtain (4-11) which avoids using Lemma 4 of [Gan et al. 2022] or Bernstein's inequality, is to use the definition $M_T\mu = \eta_T(\mu * \check{\psi}_\tau)$ and the definition of π_θ to get

$$\|\pi_{\theta\sharp} M_T\mu\|_{L^\infty} \lesssim 2^{k-j+kO(\delta)} \|M_T\mu\|_{L^\infty} \lesssim 2^{k-j+kO(\delta)} \|\mu * \check{\psi}_\tau\|_{L^\infty(T)}.$$

If $x \in T$, then apart from a negligible error term, since $\check{\psi}_\tau$ is negligible outside the plank parallel to T centred at the origin,

$$|\mu * \check{\psi}_\tau(x)| \le \int |\check{\psi}_\tau(x-y)|\, d\mu(y) \lesssim m(\tau)\mu(2T) \lesssim 2^{kO(\delta)} m(T)^{-1}\mu(2T).$$

Hence

$$\|\pi_{\theta\sharp} M_T\mu\|_{L^\infty} \lesssim 2^{k-j+kO(\delta)} m(T)^{-1}\mu(2T). \tag{4-12}$$

Again the condition $|\theta_\tau - \theta| \le 2^{k(\delta-1/2)}$ implies that $\pi_\theta(T)$ is contained in a rectangle with

$$\text{area} \lesssim 2^{kO(\delta)} 2^{j-k} m(T),$$

so substituting (4-12) into the left-hand side of (4-11) yields

$$\|\pi_{\theta\sharp} M_T\mu\|_{L^{p_n}(\mathcal{H}^2)}^{p_n}\, d\theta \lesssim 2^{kO(\delta)} 2^{j-k} m(T) (2^{k-j} m(T)^{-1}\mu(2T))^{p_n} \lesssim 2^{kO(\delta)} \mu(2T)^{p_n} (2^{k-j} m(T)^{-1})^{p_n - 1},$$

which matches (4-11).

We now prove (4-10). First, note that the left-hand side of (4-10) is

$$\lesssim \int_0^{2\pi} \int \Bigg| \sum_{\substack{\tau\in\Lambda_{j,k}:\\ |\theta_\tau-\theta|\le 2^{k(\delta-1/2)}}} \sum_{T\in\mathbb{T}_{\tau,b}} \pi_{\theta\sharp} M_T\mu \Bigg|^{p_n} d\mathcal{H}^2\, d\theta + \int_0^{2\pi} \int \Bigg| \sum_{\substack{\tau\in\Lambda_{j,k}:\\ |\theta_\tau-\theta|> 2^{k(\delta-1/2)}}} \sum_{T\in\mathbb{T}_{\tau,b}} \pi_{\theta\sharp} M_T\mu \Bigg|^{p_n} d\mathcal{H}^2\, d\theta. \tag{4-13}$$

The second term is negligible by the trivial L^∞ bound on each $\pi_{\theta\sharp} M_T\mu$, followed by the same stationary phase bound used in the L^1 case (see Lemma 5 of [Gan et al. 2022]). So it suffices to only consider τ such that θ_τ is close to θ.

Assume first that $k = j$. By the $\lesssim 2^{kO(\delta)}$-overlapping property of the $\pi_\theta(T)$'s (this property uses $j = k$), one gets the bound

$$\int_0^{2\pi} \int \Bigg| \sum_{\substack{\tau\in\Lambda_{j,k}:\\ |\theta_\tau-\theta|\le 2^{k(\delta-1/2)}}} \sum_{T\in\mathbb{T}_{\tau,b}} \pi_{\theta\sharp} M_T\mu \Bigg|^{p_n} d\mathcal{H}^2\, d\theta \lesssim 2^{kO(\delta)} \int_0^{2\pi} \sum_{\substack{\tau\in\Lambda_{j,k}:\\ |\theta_\tau-\theta|\le 2^{k(\delta-1/2)}}} \sum_{T\in\mathbb{T}_{\tau,b}} \|\pi_{\theta\sharp} M_T\mu\|_{L^{p_n}(\mathcal{H}^2)}^{p_n}\, d\theta,$$

and then (4-10) follows by applying (4-11) to the above.

Now consider the case $k < j$. If p_n were replaced by $p = 2$ then (4-10) would follow by orthogonality. More precisely, by applying Plancherel in two dimensions, and then changing variables to an integral over $\mathbb{R}^3$ (fixing (j,k) ensures that the Jacobian of the change of variables is the constant $2^{-j+k/2}$; we

use (4-18) below and the straightforward observation that if $\eta_1\gamma'(\theta)+\eta_2(\gamma\times\gamma')(\theta)$ intersects $\tau\in\Lambda_{j,k}$, then $|\eta_1|\sim 2^{j-k/2}$, whose justification can be found in [Gan et al. 2022, equation 106]),

$$\int_0^{2\pi}\int\left|\sum_{\tau\in\Lambda_{j,k}}\sum_{T\in\mathbb{T}_{\tau,b}}\pi_{\theta\sharp}M_T\mu\right|^2 d\mathcal{H}^2\,d\theta\lesssim 2^{-j+k/2}\int_{\mathbb{R}^3}\left|\sum_{\tau\in\Lambda_{j,k}}\sum_{T\in\mathbb{T}_{\tau,b}}M_T\mu(x)\right|^2 dx. \tag{4-14}$$

We then apply Plancherel's theorem in $\mathbb{R}^3$ to decouple the L^2 norms (as in [Gan et al. 2022, equation 113]), to bound the above by

$$2^{-j+k/2}\sum_{\tau\in\Lambda_{j,k}}\sum_{T\in\mathbb{T}_{\tau,b}}\int_{\mathbb{R}^3}|M_T\mu(x)|^2\,dx.$$

Each $M_T\mu$ is just a wave packet, so it is easy to bound the above by

$$2^{-j+k/2+kO(\delta)}\sum_{\tau\in\Lambda_{j,k}}\sum_{T\in\mathbb{T}_{\tau,b}}\mu(2T)^2m(T)^{-1},$$

and by Fubini, the above is comparable to the right-hand side of (4-10) if p_n is replaced by $p=2$.

If p_n were replaced by $p=1$, then (4-10) would follow by applying the triangle inequality and then Lemma 4 of [Gan et al. 2022] to the first term in (4-13), and again the stationary phase bound from Lemma 5 of [Gan et al. 2022] to the second term in (4-13). This proves (4-10) if $p=1$.

The arguments used above to prove (4-10) when p_n is replaced by $p\in\{1,2\}$ apply similarly if the collections of T's in the left-hand side is replaced by any subcollection, so (4-10) holds more generally for any subcollection of T's in the left-hand side, if p_n is replaced by $p\in\{1,2\}$.

The inequality (4-10) for the exponent $p=p_n$ will be shown via the following application of the interpolation argument from [Demeter 2020, Exercise 9.21]. By pigeonholing, there exists a subcollection $\mathbb{T}$ of T's in the left-hand side of (4-10), such that $\mu(2T)$ is constant over $T\in\mathbb{T}$ up to a factor of 2, and such that

$$\int_0^{2\pi}\int\left|\sum_{\tau\in\Lambda_{j,k}}\sum_{T\in\mathbb{T}_{\tau,b}}\pi_{\theta\sharp}M_T\mu\right|^{p_n} d\mathcal{H}^2\,d\theta\lesssim 2^{kO(\delta)}\int_0^{2\pi}\int\left|\sum_{T\in\mathbb{T}}\pi_{\theta\sharp}M_T\mu\right|^{p_n} d\mathcal{H}^2\,d\theta.$$

If we write $p_n=(1-\phi)\cdot 1+\phi\cdot 2$ and apply Hölder's inequality with $q=1/(1-\phi)$ and $q'=1/\phi$, this bounds the above by

$$2^{kO(\delta)}\left(\int_0^{2\pi}\int\left|\sum_{T\in\mathbb{T}}\pi_{\theta\sharp}M_T\mu\right| d\mathcal{H}^2\,d\theta\right)^{1-\phi}\left(\int_0^{2\pi}\int\left|\sum_{T\in\mathbb{T}}\pi_{\theta\sharp}M_T\mu\right|^2 d\mathcal{H}^2\,d\theta\right)^{\phi}.$$

If we apply (4-10) in the cases $p=1$ and $p=2$ (with the sum in left-hand side replaced by a sum over $\mathbb{T}$), the above is bounded by

$$2^{kO(\delta)}\left(\int_0^{2\pi}\sum_{\substack{\tau\in\Lambda_{j,k}:\\ |\theta_\tau-\theta|\le 2^{k(\delta-1/2)}}}\sum_{T\in\mathbb{T}_{\tau,b}\cap\mathbb{T}}\mu(2T)\,d\theta\right)^{1-\phi}\times\left(\int_0^{2\pi}\sum_{\substack{\tau\in\Lambda_{j,k}:\\ |\theta_\tau-\theta|\le 2^{k(\delta-1/2)}}}\sum_{T\in\mathbb{T}_{\tau,b}\cap\mathbb{T}}\mu(2T)^2(m(T)2^{j-k})^{-1}\,d\theta\right)^{\phi}.$$

By Fubini's theorem and the essentially constant property of $\mu(2T)$, the above is bounded by

$$2^{kO(\delta)}(|\mathbb{T}|\mu(2T)2^{-k/2})^{1-\phi}\big(|\mathbb{T}|\mu(2T)^2(m(T)2^{j-k})^{-1}2^{-k/2}\big)^{\phi}.$$

This simplifies to

$$2^{kO(\delta)}|\mathbb{T}|\mu(2T)^{p_n}2^{-k/2}(m(T)2^{j-k})^{-(p_n-1)},$$

which, by Fubini's theorem again, is bounded by

$$2^{kO(\delta)}\int_0^{2\pi}\sum_{\substack{\tau\in\Lambda_{j,k}:\\ |\theta_\tau-\theta|\le 2^{k(\delta-1/2)}}}\sum_{T\in\mathbb{T}_{\tau,b}}\mu(2T)^{p_n}(m(T)2^{j-k})^{-(p_n-1)}\,d\theta.$$

This matches the right-hand side of (4-10), so this finishes the proof of (4-10). □

We continue with the proof of the theorem, and the bound on the "bad" part. The right-hand side of (4-10) is bounded by

$$2^{kO(\delta)}\int_0^{2\pi}\sum_m\sum_{\substack{\tau\in\Lambda_{j,k}:\\ |\theta_\tau-\theta|\le 2^{k(\delta-1/2)}}}\sum_{\substack{T\in\mathbb{T}_{\tau,b}:\\ T\cap B_m\neq\varnothing}}\mu(2T)^{p_n}(m(T)2^{j-k})^{-(p_n-1)}\,d\theta, \tag{4-15}$$

where, for each pair (j,k), $\{B_m\}_m$ is a boundedly overlapping cover of $\mathbb{R}^3$ by balls of radius $2^{-(j-k)+k\delta}$. The size of the balls B_m is chosen so that, for any fixed B_m, after a rescaling later, the badness of planks T (intersecting B_m) implies the β_0-admissibility, i.e., the rescaled version of the measure μ restricted on B_m and the rescaled planks satisfy the hypotheses for (4-4).

For fixed (j,k), m, τ, and a T occurring in the sum above, recalling that the shortest sidelength of the plank T is $2^{k\delta-j}$, so for any $\theta\in[0,2\pi)$,

$$\mu(2T)^{p_n}(m(T)2^{j-k})^{-(p_n-1)}\lesssim\left(\int_{\pi_\theta(2T)}\big|\pi_{\theta\sharp}(\mu_{m,j,k}*\phi_{2^{j-k\delta}})\big|\,d\mathcal{H}^2\right)^{p_n}(m(T)2^{j-k})^{-(p_n-1)},$$

where $\mu_{m,j,k}$ is the restriction of μ to $100B_m$. By Hölder's inequality applied to the above,

$$\mu(2T)^{p_n}(m(T)2^{j-k})^{-(p_n-1)}\lesssim 2^{kO(\delta)}\int_{\pi_\theta(2T)}|\pi_{\theta\sharp}(\mu_{m,j,k}*\phi_{2^{j-k\delta}})|^{p_n}\,d\mathcal{H}^2,\quad\text{when } |\theta-\theta_\tau|\le 2^{k(\delta-1/2)}.$$

If we substitute this inequality into (4-15), and then (4-15) into (4-10), and then sum (4-10) over j and k in (4-9), we get

$$\begin{aligned}&\int_0^{2\pi}\int|\pi_{\theta\sharp}\mu_b|^{p_n}\,d\mathcal{H}^2\,d\theta\\ &\quad\lesssim(\log R)^{O(1)}\sum_{j\ge J}\sum_{k\in[j\epsilon,j]}2^{kO(\delta)}\sum_m\int_0^{2\pi}\sum_{\substack{\tau\in\Lambda_{j,k}:\\ |\theta_\tau-\theta|\le 2^{k(\delta-1/2)}}}\sum_{\substack{T\in\mathbb{T}_{\tau,b}:\\ T\cap B_m\neq\varnothing}}\int_{\pi_\theta(2T)}|\pi_{\theta\sharp}(\mu_{m,j,k}*\phi_{2^{j-k\delta}})|^{p_n}\,d\mathcal{H}^2\,d\theta.\end{aligned}$$

The rectangles $\pi_\theta(2T)$ in the sum above are $\lesssim 2^{kO(\delta)}$-overlapping over the two innermost sums, so the above implies that

$$\int_0^{2\pi}\int |\pi_{\theta\sharp}\mu_b|^{p_n}\,d\mathcal{H}^2\,d\theta \lesssim (\log R)^{O(1)}\sum_{j\geq J}\sum_{k\in[j\epsilon,j]} 2^{kO(\delta)}\sum_m \int_0^{2\pi}\int_{E_{m,j,k}(\theta)} |\pi_{\theta\sharp}(\mu_{m,j,k}*\phi_{2^{j-k\delta}})|^{p_n}\,d\mathcal{H}^2\,d\theta,$$

where

$$E_{m,j,k}(\theta) := \bigcup_{\substack{\tau\in\Lambda_{j,k}:\\ |\theta_\tau-\theta|\leq 2^{k(\delta-1/2)}}}\ \bigcup_{\substack{T\in\mathbb{T}_{\tau,b}:\\ T\cap B_m\neq\varnothing}} \pi_\theta(2T).$$

We will abbreviate $\nu_{m,j,k}=\mu_{m,j,k}*\phi_{2^{j-k\delta}}$ to simplify the notation. If

$$F_{m,j,k}(\theta) := 2^{j-k-k\delta}E_{m,j,k}(\theta) = \bigcup_{\substack{\tau\in\Lambda_{j,k}:\\ |\theta_\tau-\theta|\leq 2^{k(\delta-1/2)}}}\ \bigcup_{\substack{T\in\mathbb{T}_{\tau,b}:\\ T\cap B_m\neq\varnothing}} \pi_\theta(2^{j-k-k\delta}2T),$$

then by rescaling each ball B_m to a ball of radius 1, using that orthogonal projections commute with rescalings,

$$\begin{aligned}\int_0^{2\pi}\int |\pi_{\theta\sharp}\mu_b|^{p_n}\,d\mathcal{H}^2\,d\theta &\\ \lesssim (\log R)^{O(1)}\sum_{j\geq J}\sum_{k\in[j\epsilon,j]} 2^{kO(\delta)}\sum_m 2^{2(j-k)(p_n-1)}&\int_0^{2\pi}\int_{F_{m,j,k}(\theta)} |\pi_{\theta\sharp}2^{j-k-k\delta}_\sharp\nu_{m,j,k}|^{p_n}\,d\mathcal{H}^2\,d\theta.\end{aligned}$$

Each rescaled measure $2^{j-k-k\delta}_\sharp\nu_{m,j,k}$ and collection of rescaled planks $2^{j-k-k\delta}2T$ satisfy the hypotheses for (4-4); by the definition of β_0 and of the "bad" planks. Moreover,

$$c_\alpha(2^{j-k-k\delta}_\sharp\nu_{m,j,k}) \leq 2^{-\alpha(j-k)+kO(\delta)}c_\alpha(\mu).$$

Hence, by the conclusion of (4-4) for β_0 and the assumption that $\alpha>2$,

$$\begin{aligned}\int_0^{2\pi}\int |\pi_{\theta\sharp}\mu_b|^{p_n}\,d\mathcal{H}^2\,d\theta &\lesssim (\log R)^{O(1)}\sum_{j\geq J}\sum_{k\in[j\epsilon,j]}\sum_m \mu(100B_m)2^{kO(\delta)}c_\alpha(\mu)^{p_n-1}2^{-k\kappa}\\ &\lesssim 2^{-J\kappa/2}\mu(\mathbb{R}^3)c_\alpha(\mu)^{p_n-1} \lesssim R^{-\epsilon\kappa/100}\mu(\mathbb{R}^3)c_\alpha(\mu)^{p_n-1}.\end{aligned}$$

It remains to bound the "good" part. The L^2 method we use is similar to the second part of [Gan et al. 2022], but we include all details for completeness. By (4-8), it suffices to show that

$$\int_0^{2\pi}\int |\pi_{\theta\sharp}\mu_g|^2\,d\mathcal{H}^2\,d\theta \lesssim \mu(\mathbb{R}^3)c_\alpha(\mu)R^{-100\delta+p_{n-1}\beta(2/p_n-1)}. \tag{4-16}$$

By Plancherel's theorem in two dimensions and the projection-slice theorem,

$$\int_0^{2\pi}\int |\pi_{\theta\sharp}\mu_g|^2\,d\mathcal{H}^2\,d\theta = 2\int_0^{2\pi}\int \left|\widehat{\mu}_g(\eta_1\gamma'(\theta)+\eta_2(\gamma\times\gamma')(\theta))\right|^2 d\eta_1\,d\eta_2\,d\theta.$$

Therefore, to prove (4-16), it suffices to show that

$$\int_0^{2\pi}\int_{\{|\eta|\geq 2^{2J}\}} \left|\widehat{\mu}_g(\eta_1\gamma'(\theta)+\eta_2(\gamma\times\gamma')(\theta))\right|^2 d\eta_1\, d\eta_2\, d\theta \lesssim \mu(\mathbb{R}^3)c_\alpha(\mu)R^{-100\delta+p_{n-1}\beta(2/p_n-1)};$$

since the part near the origin can be handled by a trivial bound on $\|\widehat{\mu}_g\|_{L^\infty(B(0,2^{2J})}$ by

$$\|\widehat{\mu}\|_{L^\infty}+\|\widehat{\mu}_b\|_{L^\infty(B(0,2^{2J}))}\lesssim 2^{O(1)J}\lesssim R^{O(1)\epsilon}$$

(this is where we use that $\epsilon\ll\beta$). We break the left-hand side above into two integrals:

$$\int_0^{2\pi}\int_{\{|\eta_1|\geq|\eta|^{1-\epsilon}\}\cap\{|\eta|\geq 2^{2J}\}} \left|\widehat{\mu}_g(\eta_1\gamma'(\theta)+\eta_2(\gamma\times\gamma')(\theta))\right|^2 d\eta_1\, d\eta_2\, d\theta + \int_0^{2\pi}\int_{\{|\eta_1|\leq|\eta|^{1-\epsilon}\}\cap\{|\eta|\geq 2^{2J}\}} \left|\widehat{\mu}_g(\eta_1\gamma'(\theta)+\eta_2(\gamma\times\gamma')(\theta))\right|^2 d\eta_1\, d\eta_2\, d\theta.$$

For the first integral, $\widehat{\mu}_b$ is negligible on the domain of integration, since $\mathbb{T}_{\tau,b}$ is empty whenever $\tau\in\Lambda_{j,k}$ and $k\leq 10j\epsilon$ (as mentioned previously, this follows from the definition of $\mathbb{T}_{\tau,b}$ and the fractal property of μ). Therefore, to bound the first integral, it suffices to prove that

$$\int_0^{2\pi}\int_{\{|\eta_1|\geq|\eta|^{1-\epsilon}\}\cap\{|\eta|\geq 2^{2J}\}} \left|\widehat{\mu}(\eta_1\gamma'(\theta)+\eta_2(\gamma\times\gamma')(\theta))\right|^2 d\eta_1\, d\eta_2\, d\theta \lesssim \mu(\mathbb{R}^3)c_\alpha(\mu). \tag{4-17}$$

The change of variables

$$(\xi_1,\xi_2,\xi_3)=\eta_1\gamma'(\theta)+\eta_2(\gamma\times\gamma')(\theta)$$

has Jacobian

$$\left|\frac{\partial(\eta_1,\eta_2,\theta)}{\partial(\xi_1,\xi_2,\xi_3)}\right|\sim|\eta_1|^{-1}, \tag{4-18}$$

so the integral in (4-17) is bounded by

$$\int|\xi|^{\epsilon-1}|\widehat{\mu}(\xi)|^2\,d\xi.$$

By the Fourier formula for the energy (see, e.g., [Mattila 2015, Theorem 3.10]), the above is comparable to $I_{2+\epsilon}(\mu)$, which is bounded by $\mu(\mathbb{R}^3)c_\alpha(\mu)$ since $\alpha>2+\epsilon$ (see for instance the inequality at the end of page 19 in [Mattila 2015]).

Therefore, it remains to prove that

$$\int_0^{2\pi}\int_{\{|\eta_1|\leq|\eta|^{1-\epsilon}\}\cap\{|\eta|\geq 2^{2J}\}} \left|\widehat{\mu}_g(\eta_1\gamma'(\theta)+\eta_2(\gamma\times\gamma')(\theta))\right|^2 d\eta_1\, d\eta_2\, d\theta \lesssim \mu(\mathbb{R}^3)c_\alpha(\mu)R^{-100\delta+p_{n-1}\beta(2/p_n-1)}. \tag{4-19}$$

As remarked above, that $\mathbb{T}_{\tau,b}$ is empty whenever $k\leq 10j\epsilon$ implies that the definition of μ_b in (4-7) can be written as

$$\mu_b=\sum_{j\geq J}\sum_{k\in[0,j]}\sum_{\tau\in\Lambda_{j,k}}\sum_{T\in\mathbb{T}_{\tau,b}} M_T\mu,\quad \mu_g=\mu-\mu_b.$$

It follows that on the domain of integration in (4-19), μ_g can be replaced by the sum over the good planks (ignoring rapidly decaying error terms), and it suffices to prove that

$$\int_0^{2\pi}\int_{\{|\eta_1|\le|\eta|^{1-\epsilon}\}\cap\{|\eta|\ge 2^{2J}\}}\left|\sum_{j\ge J}\sum_{k\in[j\epsilon/1000,\,j]}\sum_{\tau\in\Lambda_{j,k}}\sum_{T\in\mathbb{T}_{\tau,g}}\widehat{M_T\mu}(\eta_1\gamma'(\theta)+\eta_2(\gamma\times\gamma')(\theta))\right|^2 d\eta_1\,d\eta_2\,d\theta \lesssim \mu(\mathbb{R}^3)c_\alpha(\mu)R^{-100\delta+p_{n-1}\beta(2/p_n-1)}.$$

By disjointness, it is enough to show that

$$\sum_{j\ge J}\sum_{k\in[j\epsilon/1000,\,j]}\sum_{\tau\in\Lambda_{j,k}}\int_0^{2\pi}\int_{\{|\eta_1|\le|\eta|^{1-\epsilon}\}\cap\{|\eta|\ge 2^{2J}\}}\left|\sum_{T\in\mathbb{T}_{\tau,g}}\widehat{M_T\mu}(\eta_1\gamma'(\theta)+\eta_2(\gamma\times\gamma')(\theta))\right|^2 d\eta_1\,d\eta_2\,d\theta \lesssim \mu(\mathbb{R}^3)c_\alpha(\mu)R^{-100\delta+p_{n-1}\beta(2/p_n-1)}.$$

If $k<j$ and $\tau\in\Lambda_{j,k}$, then by a change of variables and the formula (4-18),

$$\int_0^{2\pi}\int_{\{|\eta_1|\le|\eta|^{1-\epsilon}\}\cap\{|\eta|\ge 2^{2J}\}}\left|\sum_{T\in\mathbb{T}_{\tau,g}}\widehat{M_T\mu}(\eta_1\gamma'(\theta)+\eta_2(\gamma\times\gamma')(\theta))\right|^2 d\eta_1\,d\eta_2\,d\theta \lesssim 2^{-j+k/2}\int_{\mathbb{R}^3}\left|\sum_{T\in\mathbb{T}_{\tau,g}}\widehat{M_T\mu}(\xi)\right|^2 d\xi,$$

where we used that if $\eta_1\gamma'(\theta)+\eta_2(\gamma\times\gamma')(\theta)$ intersects $\tau\in\Lambda_{j,k}$, then $|\eta_1|\sim 2^{j-k/2}$. We then use essential orthogonality of wave packets to get

$$\int_0^{2\pi}\int_{\{|\eta_1|\le|\eta|^{1-\epsilon}\}\cap\{|\eta|\ge 2^{2J}\}}\left|\sum_{T\in\mathbb{T}_{\tau,g}}\widehat{M_T\mu}(\eta_1\gamma'(\theta)+\eta_2(\gamma\times\gamma')(\theta))\right|^2 d\eta_1\,d\eta_2\,d\theta \lesssim 2^{-j+k/2+kO(\delta)}\sum_{T\in\mathbb{T}_{\tau,g}}\int_{\mathbb{R}^3}|\widehat{M_T\mu}(\xi)|^2\,d\xi. \quad (4\text{-}20)$$

The above inequality also holds if $k=j$ by the following argument. Use Plancherel's theorem in two dimensions to change the integral in the left-hand side of (4-20) back to an integral of $\sum_{T\in\mathbb{T}_{\tau,g}}\pi_{\theta\sharp}M_T\mu$. The only significant θ in the integration over θ satisfies $|\theta-\theta_\tau|\lesssim 2^{k(-1/2+\delta)}$, so the sets $\pi_\theta(T)$ are $\lesssim 2^{kO(\delta)}$-overlapping. This means that the L^2 norms can be decoupled, and then (4-20) follows by a calculation using that each $M_T\mu$ is a wave packet.

Therefore, it suffices to prove that

$$\sum_{j\ge J}\sum_{k\in[\epsilon j/1000,\,j]}2^{-j+k/2+kO(\delta)}\sum_{\tau\in\Lambda_{j,k}}\sum_{T\in\mathbb{T}_{\tau,g}}\|M_T\mu\|_2^2\lesssim\mu(\mathbb{R}^3)c_\alpha(\mu)R^{-100\delta+p_{n-1}\beta(2/p_n-1)}. \quad (4\text{-}21)$$

This will be shown via

$$\sum_{j\ge J}\sum_{k\in[\epsilon j/1000,\,j]}2^{-j+k/2+kO(\delta)}\sum_{m}\sum_{\tau\in\Lambda_{j,k}}\sum_{T\in\mathbb{T}_{\tau,m,g}}\|M_T\mu\|_2^2\lesssim\mu(\mathbb{R}^3)c_\alpha(\mu)R^{-100\delta+p_{n-1}\beta(2/p_n-1)},$$

where, for each (j,k), $\{B_m\}$ is a boundedly overlapping covering of $\mathbb{R}^3$ by balls of radius $2^{-(j-k)+k\delta}$, and where $\mathbb{T}_{\tau,g,m}=\{T\in\mathbb{T}_{\tau,g}:T\cap B_m\ne\varnothing\}$. Fix j, k, and m. By writing out the L^2 norm as an inner product

and expanding out the definition of one of the $M_T\mu$'s, the innermost double sum can be written as

$$\sum_{\tau\in\Lambda_{j,k}}\sum_{T\in\mathbb{T}_{\tau,g,m}}\|M_T\mu\|_2^2=\int\sum_{\tau\in\Lambda_{j,k}}\sum_{T\in\mathbb{T}_{\tau,g,m}}(\eta_T M_T\mu)*\check{\psi}_\tau\,d\mu. \tag{4-22}$$

By Cauchy–Schwarz,

$$(4\text{-}22)\le\mu(100B_m)^{1/2}\bigg(\int\bigg|\sum_{\tau\in\Lambda_{j,k}}\sum_{T\in\mathbb{T}_{\tau,g,m}}(\eta_T M_T\mu)*\check{\psi}_\tau\bigg|^2 d(\mu*\phi_{2^j})\bigg)^{1/2}, \tag{4-23}$$

where we used the Fourier support of the integrand in (4-22) to replace μ by its smoothed out version $\mu*\phi_{2^j}$ at the scale 2^{-j}. Let $p_{\mathrm{dec}}\in(2,6]$ be an exponent to be chosen. We abbreviate $f_T=(\eta_T M_T\mu)*\check{\psi}_\tau$. By dyadic pigeonholing, we can find a subset $\mathbb{W}\subseteq\bigcup_{\tau\in\Lambda_{j,k}}\mathbb{T}_{\tau,g,m}$ such that $\|f_T\|_{p_{\mathrm{dec}}}$ is constant over $T\in\mathbb{W}$ up to a factor of 2, such that

$$(4\text{-}23)\le O(j)\mu(100B_m)^{1/2}\bigg(\int\bigg|\sum_{T\in\mathbb{W}}f_T\bigg|^2 d(\mu*\phi_{2^j})\bigg)^{1/2}. \tag{4-24}$$

By partitioning the domain of integration into boundedly overlapping balls of radius 2^{-j}, we can find a dyadic number M and a union Y of balls of radius 2^{-j} such that each ball $B\subseteq Y$ intersects $\sim M$ planks $2T$ with $T\in\mathbb{W}$, and such that

$$(4\text{-}24)\lesssim O(j)\mu(100B_m)^{1/2}\bigg(\int_Y\bigg|\sum_{T\in\mathbb{W}}f_T\bigg|^2 d(\mu*\phi_{2^j})\bigg)^{1/2}. \tag{4-25}$$

By Hölder's inequality with respect to Lebesgue measure,

$$\bigg(\int_Y\bigg|\sum_{T\in\mathbb{W}}f_T\bigg|^2 d(\mu*\phi_{2^j})\bigg)^{1/2}\le\bigg(\int_Y\bigg|\sum_{T\in\mathbb{W}}f_T\bigg|^{p_{\mathrm{dec}}}\bigg)^{1/p_{\mathrm{dec}}}\bigg(\int_Y(\mu*\phi_{2^j})^{p_{\mathrm{dec}}/p_{\mathrm{dec}}-2}\bigg)^{1/2-1/p_{\mathrm{dec}}}, \tag{4-26}$$

By a rescaling, followed by an application of the refined decoupling inequality (Theorem A.1), the first factor in the right-hand side of (4-26) satisfies

$$\bigg(\int_Y\bigg|\sum_{T\in\mathbb{W}}f_T\bigg|^{p_{\mathrm{dec}}}\bigg)^{1/p_{\mathrm{dec}}}\lesssim 2^{k\epsilon}\bigg(\frac{M}{|\mathbb{W}|}\bigg)^{1/2-1/p_{\mathrm{dec}}}\bigg(\sum_{T\in\mathbb{W}}\|f_T\|_{p_{\mathrm{dec}}}^2\bigg)^{1/2}. \tag{4-27}$$

By the definition of M, and the definition of "bad" planks in (4-6), the second factor in the right-hand side of (4-26) satisfies

$$\begin{aligned}\bigg(\int_Y(\mu*\phi_{2^j})^{p_{\mathrm{dec}}/p_{\mathrm{dec}}-2}\bigg)^{1/2-1/p_{\mathrm{dec}}}&\lesssim\bigg(\int_Y d(\mu*\phi_{2^j})\bigg)^{1/2-1/p_{\mathrm{dec}}}2^{j(3-\alpha)/p_{\mathrm{dec}}}c_\alpha(\mu)^{1/p_{\mathrm{dec}}}\\&\lesssim\bigg(\frac{1}{M}\sum_{T\in\mathbb{W}}\int_{Y\cap T}d(\mu*\phi_{2^j})\bigg)^{1/2-1/p_{\mathrm{dec}}}2^{j(3-\alpha)/p_{\mathrm{dec}}}c_\alpha(\mu)^{1/p_{\mathrm{dec}}}\end{aligned}\qquad\text{(cont.)}$$

$$\lesssim \left(\frac{|\mathbb{W}|}{M} 2^{-\alpha(j-k)-k(3/2-\beta_0)+1000j\epsilon} c_\alpha(\mu)\right)^{1/2-1/p_{\mathrm{dec}}} 2^{j(3-\alpha)/p_{\mathrm{dec}}} c_\alpha(\mu)^{1/p_{\mathrm{dec}}}$$

$$= \left(\frac{|\mathbb{W}|}{M}\right)^{1/2-1/p_{\mathrm{dec}}} 2^{(1/2-1/p_{\mathrm{dec}})\left(-\alpha(j-k)-k(3/2-\beta_0)+1000j\epsilon\right)+j(3-\alpha)/p_{\mathrm{dec}}} c_\alpha(\mu)^{1/2}. \quad (4\text{-}28)$$

Substituting (4-27) and (4-28) into (4-26) yields

$$\left(\int_Y \left|\sum_{T\in\mathbb{W}} f_T\right|^2 d\mu\right)^{1/2} \lesssim 2^{k\epsilon+(1/2-1/p_{\mathrm{dec}})\left(-\alpha(j-k)-k(3/2-\beta_0)+1000j\epsilon\right)+j(3-\alpha)/p_{\mathrm{dec}}} c_\alpha(\mu)^{1/2} \left(\sum_{T\in\mathbb{W}} \|f_T\|_{p_{\mathrm{dec}}}^2\right)^{1/2}. \quad (4\text{-}29)$$

By recalling that $f_T = (\eta_T M_T \mu) * \check{\psi}_\tau$ and applying the Hausdorff–Young inequality, followed by Hölder's inequality,

$$\|f_T\|_{p_{\mathrm{dec}}} \lesssim \|M_T\mu\|_2 2^{(3j-3k/2)(1/2-1/p_{\mathrm{dec}})}.$$

Therefore, (4-29) becomes

$$\left(\int_Y \left|\sum_{T\in\mathbb{W}} f_T\right|^2 d\mu\right)^{1/2} \lesssim 2^{k\epsilon+(1/2-1/p_{\mathrm{dec}})\left(-\alpha(j-k)-k(3/2-\beta_0)+1000j\epsilon\right)+j(3-\alpha)/p_{\mathrm{dec}}+(3j-3k/2)(1/2-1/p_{\mathrm{dec}})} \times c_\alpha(\mu)^{1/2} \left(\sum_{\tau\in\Lambda_{j,k}} \sum_{T\in\mathbb{T}_{\tau,g,m}} \|M_T\mu\|_2^2\right)^{1/2}, \quad (4\text{-}30)$$

where we now sum over all "good" planks in $\mathbb{T}_{\tau,g,m}$ rather than just $\mathbb{W}$, since the property of $\mathbb{W}$ will no longer be needed. By substituting (4-30) into (4-25), then into (4-24), then into (4-23), and then finally into (4-22), we get

$$\sum_{\tau\in\Lambda_{j,k}} \sum_{T\in\mathbb{T}_{\tau,g,m}} \|M_T\mu\|_2^2 \lesssim 2^{jO(\epsilon)+(1/2-1/p_{\mathrm{dec}})\left(-\alpha(j-k)-k(3/2-\beta_0)+1000j\epsilon\right)+j(3-\alpha)/p_{\mathrm{dec}}+(3j-3k/2)(1/2-1/p_{\mathrm{dec}})} \times \mu(100B_m)^{1/2} c_\alpha(\mu)^{1/2} \left(\sum_{\tau\in\Lambda_{j,k}} \sum_{T\in\mathbb{T}_{\tau,g,m}} \|M_T\mu\|_2^2\right)^{1/2}.$$

By cancelling the common factor in both sides, this simplifies to

$$\sum_{\tau\in\Lambda_{j,k}} \sum_{T\in\mathbb{T}_{\tau,g,m}} \|M_T\mu\|_2^2 \lesssim 2^{jO(\epsilon)+j(3-\alpha)-2k(1/2-1/p_{\mathrm{dec}})(3-\alpha-\beta_0)} \mu(100B_m) c_\alpha(\mu).$$

Summing over m gives

$$\sum_{\tau\in\Lambda_{j,k}} \sum_{T\in\mathbb{T}_{\tau,g}} \|M_T\mu\|_2^2 \lesssim 2^{jO(\epsilon)+j(3-\alpha)-2k(1/2-1/p_{\mathrm{dec}})(3-\alpha-\beta_0)} \mu(\mathbb{R}^3) c_\alpha(\mu).$$

If we substitute this bound into the left-hand side of (4-21), we get

$$\sum_{j\ge J} \sum_{k\in[\epsilon j/1000,\, j]} \sum_{\tau\in\Lambda_{j,k}} \sum_{T\in\mathbb{T}_{\tau,g}} 2^{-j+k/2+kO(\delta)} \|M_T\mu\|_2^2 \lesssim \mu(\mathbb{R}^3) c_\alpha(\mu) \sum_{J\le j\le(1+\delta)\log_2(R)} \sum_{k\in[\epsilon j/1000,\, j]} 2^{-j+k/2+jO(\epsilon)} 2^{j(3-\alpha)-2k(1/2-1/p_{\mathrm{dec}})(3-\alpha-\beta_0)}. \quad (4\text{-}31)$$

By taking $p_{\mathrm{dec}} = 4$ and summing the two geometric series, (4-31) simplifies to

$$(4\text{-}31) \lesssim \mu(\mathbb{R}^3) c_\alpha(\mu) R^{O(\epsilon)} \max\{1, R^{\beta_0/2-(\alpha-2)/2}\}. \tag{4-32}$$

Using that $\delta \ll \epsilon \ll \min\{\beta, \alpha - 2\}$, to verify that (4-32) implies (4-21), it suffices to check that

$$\tfrac{1}{2}\beta_0 \le p_{n-1}\beta\Big(\frac{2}{p_n} - 1\Big) + (\alpha - 2)/100.$$

Since $\beta > c\beta_0$, where $1 - c \ll \alpha - 2$, it suffices to check that

$$\tfrac{1}{2} \le p_{n-1}\Big(\frac{2}{p_n} - 1\Big). \tag{4-33}$$

But by (4-2), equality[3] holds in (4-33) for any $n \in \mathbb{N} \cup \{\infty\}$ (using that $p_{-1} = 1$ when $n = 0$). We have verified (4-16), so this finishes the proof. □

We are now ready to prove the following main result of the section, which includes Theorem 1.2 as a special case.

Theorem 4.4. *Let $\alpha > 2$ and $1 \le p \le \frac{3}{2}$. Then*

$$\int_0^{2\pi} \int |\pi_{\theta\sharp}\mu|^p \, d\mathcal{H}^2 \, d\theta \le C_{\alpha,p} \mu(\mathbb{R}^3) c_\alpha(\mu)^{p-1} \tag{4-34}$$

for any Borel measure μ on $B_3(0, 1)$ with $c_\alpha(\mu) < \infty$.

Proof. Since (4-34) is trivial if the projections are a.e. absolutely continuous, it suffices to prove (4-34) for $p > 1$. By interpolation (or just the log convexity of L^p norms, or Hölder's inequality), it suffices to prove the endpoint case $p = \frac{3}{2}$. Since $p = \frac{3}{2} > 1$, by a standard limiting argument (which uses that the continuous compactly supported functions are dense in the dual of L^p), it may be assumed that μ is a smooth function.

Take $\beta > 0$ such that $\beta \ll \alpha - 2$, and let $\delta, \epsilon > 0$ be such that $\delta \ll \epsilon \ll \beta$ and $\delta \ll \kappa = \kappa(\beta)$, the parameter in Lemma 4.2. Write $\mu = \mu_b + \mu_g$, where the "bad" part is defined as in (4-7), except that we take $J = 1$ and the parameter β is used instead of β_0 in (4-6). By the triangle inequality, and then by following the same argument as in the proof of Lemma 4.2,

$$\begin{aligned}\Big(\int_0^{2\pi} \int |\pi_{\theta\sharp}\mu_b|^p \, d\mathcal{H}^2 \, d\theta\Big)^{1/p} \\ \lesssim \sum_{j\ge 1} \sum_{k\in[j\epsilon, j]} \Big(O(j) 2^{2(j-k)(p-1)+kO(\delta)} \sum_m \int_0^{2\pi} \int_{F_{m,j,k}(\theta)} \big|\pi_{\theta\sharp} 2^{j-k-k\delta}_\sharp (\mu_{m,j,k} * \phi_{2^{j-k\delta}})\big|^p \, d\mathcal{H}^2 \, d\theta \Big)^{1/p},\end{aligned}$$

where

$$F_{m,j,k}(\theta) := \bigcup_{\substack{\tau\in\Lambda_{j,k}:\\ |\theta_\tau - \theta| \le 2^{k(\delta-1/2)}}} \bigcup_{\substack{T\in\mathbb{T}_{\tau,b}:\\ T\cap B_m \ne \varnothing}} \pi_\theta(2^{j-k-k\delta} 2T).$$

[3]This step could only be improved by using $p_{\mathrm{dec}} > 4$ if we could assume that β is small, but for β close to 1 there is no gain from taking $p_{\mathrm{dec}} > 4$ in (4-31). From the train-track example in Proposition 3.4, we expect the $\beta = 1$ case to be critical. Equality at $p_{n-1} = p_n = \frac{3}{2}$ explains why the exponent $\frac{3}{2}$ is important and cannot be increased.

By the $p = \frac{3}{2}$ case of Lemma 4.2,

$$\int_0^{2\pi}\int |\pi_{\theta\sharp}\mu_b|^p \, d\mathcal{H}^2 \, d\theta \lesssim c_\alpha(\mu)^{p-1}\mu(\mathbb{R}^3)\bigg(\sum_{j\geq 1} 2^{-j\epsilon\kappa/1000}\bigg)^p \lesssim c_\alpha(\mu)^{p-1}\mu(\mathbb{R}^3). \tag{4-35}$$

This bounds the bad part, so it remains to bound the good part.

If we follow the same argument as in Lemma 4.2 to bound the good part, the proof works without substantial changes, and yields

$$\int_0^{2\pi}\int |\pi_{\theta\sharp}\mu_g|^2 \, d\mathcal{H}^2 \, d\theta \lesssim I_{2+100\epsilon}(\mu) + \mu(\mathbb{R}^3)c_\alpha(\mu)\sum_{j\geq 1}\sum_{k\in[\epsilon j/1000, j]} 2^{-j+k/2+jO(\epsilon)}2^{j(3-\alpha)-2k(1/2-1/p_{\mathrm{dec}})(3-\alpha-\beta)}.$$

By taking $p_{\mathrm{dec}} = 4$ and using $\alpha > 2$ and $\epsilon \ll \beta \ll \alpha - 2$, the second term is a decaying geometric series, which yields

$$\int_0^{2\pi}\int |\pi_{\theta\sharp}\mu_g|^2 \, d\mathcal{H}^2 \, d\theta \lesssim \mu(\mathbb{R}^3)c_\alpha(\mu).$$

Note that the argument above for the bad part in (4-35) also works if one sets $p = 1$ and applies the corresponding conclusion in Lemma 4.2. Hence, the above shows that $\pi_{\theta\sharp}\mu_b \in L^1$ for a.e. $\theta \in [0, 2\pi)$, and that

$$\int_0^{2\pi}\int |\pi_{\theta\sharp}\mu_b| \, d\mathcal{H}^2 \, d\theta \lesssim \mu(\mathbb{R}^3).$$

By the trivial L^1 estimate of $\pi_{\theta\sharp}\mu$ (using that μ is a smooth function) and the triangle inequality, this gives

$$\int_0^{2\pi}\int |\pi_{\theta\sharp}\mu_g| \, d\mathcal{H}^2 \, d\theta \lesssim \mu(\mathbb{R}^3).$$

By writing $\frac{3}{2} = p = (1-\phi)1 + \phi\cdot 2$, where $\phi = p - 1$, and then applying Hölder's inequality to the function

$$|\pi_{\theta\sharp}\mu_g|^p = |\pi_{\theta\sharp}\mu_g|^{(1-\phi)\cdot 1}\cdot|\pi_{\theta\sharp}\mu_g|^{\phi\cdot 2},$$

with $q = \frac{1}{1-\phi}$ and $q' = \frac{1}{\phi}$, this yields

$$\int_0^{2\pi}\int |\pi_{\theta\sharp}\mu_g|^p \, d\mathcal{H}^2 \, d\theta \lesssim c_\alpha(\mu)^{p-1}\mu(\mathbb{R}^3). \tag{4-36}$$

Combining (4-35) and (4-36) gives, by the triangle inequality,

$$\int_0^{2\pi}\int |\pi_{\theta\sharp}\mu|^p \, d\mathcal{H}^2 \, d\theta \lesssim c_\alpha(\mu)^{p-1}\mu(\mathbb{R}^3). \qquad \square$$

Proof of Theorem 1.1. The class of horizontal (or SL_2) lines is invariant under rotations of the plane $\mathbb{R}^2 \times \{0\}$, so by the triangle inequality it may be assumed that all tubes $T \in \mathbb{T}$ have $\ell(T) = p * \mathbb{V}_\theta$ with

$|\theta| \geq \pi/4$, where we use the convention $-\pi/2 < \theta \leq \pi/2$. Recall that $U(\xi_1, \xi_2, \xi_3) = (\eta_1, \eta_2, \eta_3)$ is the orthogonal transformation given by

$$\eta_1 = \frac{\xi_1 + \xi_3}{\sqrt{2}}, \quad \eta_3 = \frac{-\xi_1 + \xi_3}{\sqrt{2}}, \quad \eta_2 = \xi_2.$$

For each $T \in \mathbb{T}$, let w_T be such that $\ell(Uw_T)$ is the centre line of T. By the assumption that $|\theta| \geq \pi/4$ for the angle of the tubes T, and by the definition of ℓ in (3-8), the property that T intersects $B_3(0, 1)$ implies that $|w_T| \lesssim 1$. This ensures that the measure $\mu := \delta^{-1} \sum_{T \in \mathbb{T}} \chi_{B(w_T, C\delta)}$, where C is a large constant to be chosen, is contained in a ball around the origin of radius $\lesssim 1$. If we run the point-line duality argument from (3-12) in reverse, we get

$$\begin{aligned}
\int_{-\pi/2}^{\pi/2} \|\pi_{\theta\sharp}\mu\|_{L^{3/2}(\gamma(\theta)^\perp)}^{3/2} \, d\theta &= \int_{-\pi/2}^{\pi/2} \int_{\gamma(\theta)^\perp} \left| \int_{\pi_\theta^{-1}(y)} \mu \, d\mathcal{H}^1 \right|^{3/2} d\mathcal{H}^2(y) \, d\theta \\
&= \int_{-\pi/2}^{\pi/2} \int_{\gamma(\theta)^\perp} \left| \int_{U\pi_\theta^{-1}(y)} (U_\sharp \mu) \, d\mathcal{H}^1 \right|^{3/2} d\mathcal{H}^2(y) \, d\theta \\
&\sim \int \left| \int_L U_\sharp \mu \, d\mathcal{H}^1 \right|^{3/2} d(\ell_\sharp^* \mathcal{H}^3 \chi_{|x_2| \leq \sqrt{2}})(L) \qquad \text{(by Lemma 3.6)} \\
&\gtrsim \int \left| \{T \in \mathbb{T} : UB(w_T, 100\delta) \cap L \neq \varnothing\} \right|^{3/2} d(\ell_\sharp^* \mathcal{H}^3 \chi_{|x_2| \leq \sqrt{2}})(L) \qquad (4\text{-}37) \\
&= \int_{|p_2^*| \leq \sqrt{2}} \left| \{T \in \mathbb{T} : UB(w_T, 100\delta) \cap \ell^*(p^*) \neq \varnothing\} \right|^{3/2} d\mathcal{H}^3(p^*) \\
&\geq \int_{|p^*| \leq 1} \left| \{T \in \mathbb{T} : UB(w_T, 100\delta) \cap \ell^*(p^*) \neq \varnothing\} \right|^{3/2} d\mathcal{H}^3(p^*) \\
&= \int_{|p^*| \leq 1} \left| \{T \in \mathbb{T} : p^* \in \ell(UB(w_T, 100\delta))\} \right|^{3/2} d\mathcal{H}^3(p^*) \qquad (4\text{-}38) \\
&= \int_{|p^*| \leq 1} \left| \sum_{T \in \mathbb{T}} \chi_{\ell(UB(w_T, 100\delta))}(p^*) \right|^{3/2} d\mathcal{H}^3(p^*) \\
&\geq \int_{B_3(0,1)} \left| \sum_{T \in \mathbb{T}} \chi_T \right|^{3/2}, \qquad (4\text{-}39)
\end{aligned}$$

where we now choose C large enough to ensure that (4-37) holds. In more detail, if C is large enough then for any SL_2 line L,

$$\int_L U_\sharp \mu \, d\mathcal{H}^1 \gtrsim \left| \{T \in \mathbb{T} : UB(w_T, 100\delta) \cap L \neq \varnothing\} \right|.$$

By the formula defining μ, to verify the above it suffices to check that for any $T \in \mathbb{T}$ such that $UB(w_T, 100\delta) \cap L \neq \varnothing$,

$$\mathcal{H}^1(L \cap UB(w_T, C\delta)) \gtrsim \delta,$$

which is clearly true if $C > 1000$. The inequality (4-38) used Lemma 3.5. The last line of (4-39) used

$$\ell(UB(w_T, 100\delta)) \cap B_3(0, 1) \supseteq \mathcal{N}_\delta(\ell(Uw_T)) \cap B_3(0, 1) \supseteq T \cap B_3(0, 1),$$

where the first part follows from the definition of ℓ in (3-8) (it is locally Lipschitz), and the second part follows from the assumption that $\ell(Uw_T)$ is the centre line of T.

If we apply Theorem 1.2 to the left-hand side of (4-39) with $\alpha = 2+\epsilon$, we get

$$\left\| \sum_{T\in\mathbb{T}} \chi_T \right\|^{3/2}_{L^{3/2}(B(0,1))} \lesssim \int \|\pi_{\theta\sharp}\mu\|^{3/2}_{3/2}\, d\theta \le C_\epsilon \delta^{-\epsilon/2} \mu(\mathbb{R}^3) c_2(\mu)^{1/2}. \tag{4-40}$$

The above inequality used that $c_{2+\epsilon}(\mu) \lesssim \delta^{-\epsilon} c_2(\mu)$ whenever μ is a linear combination of indicator functions over a boundedly overlapping collection of δ-balls. But $\mu(\mathbb{R}^3) \lesssim \delta^2|\mathbb{T}|$, and $c_2(\mu) \lesssim 1$ (this is where we use the 2-dimensional ball condition (1-1) on the family of tubes $\mathbb{T}$), so (4-40) implies (1-2) and proves Theorem 1.1. □

5. An L^p inequality for projections onto lines

Lemma 5.2 below is the analogue of Lemma 4.2 for lines instead of planes. To state it, as in the case of projections to planes, we first need a definition for a "small collection of bad intervals".

Definition 5.1. Given $R > 1$ and $\beta > 0$, and a Borel measure μ on $B_3(0,1)$ with $c_1(\mu) < \infty$, we call the triple $(\mu, R, \{\mathbb{D}_\theta\})$ "β-admissible" if $\{\mathbb{D}_\theta\}$ is a family of sets, where each $\mathbb{D}_\theta$ is a set of intervals of radius R^{-1} in $\mathrm{span}(\gamma(\theta))$, with $(\rho_{\theta\sharp}(\mu * \phi_R))(D) \ge c_1(\mu) R^{\beta-1}$ for each $D \in \mathbb{D}_\theta$, and such that the set $\left\{(\theta, x) \in [0, 2\pi) \times \mathbb{R}^3 : x \in \bigcup_{D\in\mathbb{D}_\theta} D\right\}$ is Borel measurable.

Lemma 5.2. *Let p be such that $1 \le p < \frac{3}{2}$. For any $\beta > 0$, there exists an $\kappa > 0$ (which may depend on β and p), such that*

$$\int_0^{2\pi} \int_{\bigcup_{D\in\mathbb{D}_\theta}} |\rho_{\theta\sharp}(\mu * \phi_R)|^p \, d\mathcal{H}^1 \, d\theta \le C_{\beta,p} R^{-\kappa} \mu(\mathbb{R}^3) c_1(\mu)^{p-1} \tag{5-1}$$

for any β-admissible triple $(\mu, R, \{\mathbb{D}_\theta\})$.

Proof. To prove (5-1), by pigeonholing we can replace μ by a sum of indicator functions over a disjoint family of R^{-1}-balls, and replace $\mu * \phi_R$ with μ in (5-1), i.e.,

$$\mu = \frac{R^3 \mu(\mathbb{R}^3)}{|\mathcal{B}|(4\pi/3)} \sum_{B\in\mathcal{B}} \chi_B. \tag{5-2}$$

By scaling we can additionally assume that $c_1(\mu) = 1$. Then (5-1) is equivalent to

$$\left(\frac{R^3\mu(\mathbb{R}^3)}{|\mathcal{B}|}\right)^p \int_0^{2\pi} \int_{\bigcup_{D\in\mathbb{D}_\theta}} \left| \rho_{\theta\sharp}\left(\sum_{B\in\mathcal{B}} \chi_B\right)\right|^p d\mathcal{H}^1 \, d\theta \le C_{\beta,p} R^{-\kappa} \mu(\mathbb{R}^3).$$

By integrating out the projections ρ_θ, it is enough to show that

$$\left(\frac{R\mu(\mathbb{R}^3)}{|\mathcal{B}|}\right)^p \int_0^{2\pi} \int_{\bigcup_{D\in\mathbb{D}_\theta}} \left| \sum_{B\in\mathcal{B}} \chi_{\rho_\theta(B)}\right|^p d\mathcal{H}^1 \, d\theta \le C_{\beta,p} R^{-\kappa} \mu(\mathbb{R}^3). \tag{5-3}$$

For each $z \in \mathbb{R}^3$, let $\Gamma_z := \{(\theta, \langle z, \gamma(\theta)\rangle) : \theta \in [0, 2\pi)\}$, which is a planar curve. Let $\delta = R^{-1}$, and let Γ_z^δ be the δ-neighbourhood of the planar curve Γ_z. Let Z'_δ be the set of centres of balls in $\mathcal{B}$. It is

straightforward to verify that

$$\sum_{B\in\mathcal{B}} \chi_{\rho_\theta(B)}(y) \le \sum_{z\in Z'_\delta} \chi_{\Gamma^\delta_z}(\theta, \langle y, \gamma(\theta)\rangle),$$

which was observed in [Pramanik et al. 2022, equation 2.12], and also in [Käenmäki et al. 2025]. To show (5-3) it is therefore enough to check that

$$\left(\frac{R\mu(\mathbb{R}^3)}{|\mathcal{B}|}\right)^p \int_0^{2\pi}\int_{\bigcup_{D\in\mathbb{D}_\theta}} \left|\sum_{z\in Z'_\delta} \chi_{\Gamma^\delta_z}(\theta, \langle y, \gamma(\theta)\rangle)\right|^p d\mathcal{H}^1(y)\, d\theta \le C_{\beta,p} R^{-\kappa}\mu(\mathbb{R}^3).$$

By Hölder's inequality applied to the above, it suffices to check that, for any $\epsilon > 0$ (where ϵ is much smaller than κ),

$$\int_0^{2\pi}\int \left|\sum_{z\in Z'_\delta} \chi_{\Gamma^\delta_z}(\theta, \langle y, \gamma(\theta)\rangle)\right|^{3/2} d\mathcal{H}^1(y)\, d\theta \le C_\epsilon R^\epsilon \mu(\mathbb{R}^3)^{-1/2} R^{-3/2} |\mathcal{B}|^{3/2}, \tag{5-4}$$

where we used the Vitali covering lemma to estimate the measure of

$$\left\{(\theta, x) \in [0, 2\pi) \times \mathbb{R}^3 : x \in \bigcup_{D\in\mathbb{D}_\theta} D\right\}.$$

By the Frostman condition on the measure μ, (5-4) follows from the general $L^{3/2}$ maximal function estimate in [Pramanik et al. 2022] (more precisely see Remark 5.10 on page 32 of [Pramanik et al. 2022]) which states (as a special case of [Pramanik et al. 2022, equation 5.63]) with $\alpha = \zeta = 1$)

$$\int_0^{2\pi}\int \left|\sum_{z\in Z'_\delta} \chi_{\Gamma^\delta_z}(\theta, \langle y, \gamma(\theta)\rangle)\right|^{3/2} d\mathcal{H}^1(y)\, d\theta \le C_\epsilon \delta^{-\epsilon} A^{1/2} \delta |Z'_\delta|, \tag{5-5}$$

whenever Z'_δ is a collection of points in $B_3(0, 1)$ such that

$$|\{z \in Z'_\delta \cap B(z_0, r)\}| \le A(r/\delta), \quad r \in [\delta, 1]. \tag{5-6}$$

The formula (5-2) and the assumption that $c_1(\mu) \le 1$ yields that (5-6) holds with $A \sim \delta |Z'_\delta| \mu(\mathbb{R}^3)^{-1}$, so (5-5) implies (5-4), and this finishes the proof. □

Proof of Theorem 1.3. The theorem for $p = 1$ follows from the theorem for any $p > 1$, so it may be assumed that $p > 1$. Therefore, by a limiting argument, it may be assumed that μ is a smooth function. Take $\beta > 0$ such that $\beta \ll \alpha - 1$, and let $\delta, \epsilon > 0$ be such that $\delta \ll \epsilon \ll \beta$, and $\delta \ll \kappa(\beta)$, the parameter in Lemma 5.2. Write $\mu = \mu_b + \mu_g$, where

$$\mu_b := \sum_{j\ge 1} \sum_{\tau\in\Lambda_j} \sum_{T\in\mathbb{T}_{\tau,b}} M_T\mu, \quad \mu_g := \mu - \mu_b, \quad M_T\mu = \eta_T(\mu * \check{\psi}_\tau),$$

and where $\{\psi_\tau\}_{\tau\in\Lambda}$ is a smooth partition of unity on the ~ 1 neighbourhood of the light cone minus the set $B_3(0, 1)$, $\Lambda = \bigcup_{j\ge1} \Lambda_j$, and each Λ_j is a boundedly overlapping cover of the ~ 1 neighbourhood of the light cone intersected with $B(0, 2^j) \setminus B(0, 2^{j-1})$, by boxes of dimensions $\sim 1 \times 2^{j/2} \times 2^j$. For each τ,

the set $\mathbb{T}_\tau$ is a boundedly overlapping cover of $\mathbb{R}^3$ by planks T of dimensions $2^{j\delta} \times 2^{-j(1/2-\delta)} \times 2^{-j(1-\delta)}$ dual to τ, and

$$\mathbb{T}_{\tau,b} = \left\{T \in \mathbb{T}_\tau : \mu(4T) \geq 2^{-j(1-\beta)} c_\alpha(\mu)\right\}, \quad \mathbb{T}_{\tau,g} = \mathbb{T}_\tau \setminus \mathbb{T}_{\tau,b}. \tag{5-7}$$

The functions η_T form a smooth partition of unity subordinate to the cover $\mathbb{T}_\tau$ of $\mathbb{R}^3$, and each $\widehat{\eta_T}$ is rapidly decaying outside the box T_0 around the origin dual to T.

Fix $j \geq 1$. It will be shown that

$$\int_0^{2\pi} \int \left| \sum_{\tau \in \Lambda_j} \sum_{T \in \mathbb{T}_{\tau,b}} \rho_{\theta\sharp} M_T \mu \right|^p d\mathcal{H}^1 \, d\theta \lesssim 2^{jO(\delta)} \int_0^{2\pi} \sum_{\substack{\tau \in \Lambda_j : \\ |\theta_\tau - \theta| \leq 2^{j(\delta-1/2)}}} \sum_{T \in \mathbb{T}_{\tau,b}} \mu(2T)^p 2^{j(p-1)} \, d\theta, \tag{5-8}$$

where θ_τ is such that $\gamma(\theta_\tau)$ is parallel to the long axis of τ (the medium axis of τ is parallel to $\gamma'(\theta_\tau)$ and the short axis of τ is parallel to $(\gamma \times \gamma')(\theta_\tau)$). If p were equal to 2 then (5-8) would be a consequence of the following argument. For each θ, apply Plancherel's theorem in one dimension to change the left-hand side of (5-8) to a surface integral over the light cone at distance $\sim 2^j$ from the origin. The uncertainty principle, using that μ is supported in the unit ball, changes this integral to (2^{-j} times) an integral over $\mathbb{R}^3$ (essentially supported in the ~ 1 neighbourhood of the light cone in $\mathbb{R}^3$ at distance $\sim 2^j$ from the origin). Plancherel's theorem in $\mathbb{R}^3$ can then be used to decouple the L^2 norms (this is similar to the argument from equations 2.16 to 2.18 in [Harris 2024]), and then (5-8) follows from the uncertainty principle or direct calculation (and (5-8) continues to hold with the set of planks in the left-hand side of (5-8) replaced by any subcollection). If p were equal to 1 then (5-8) would follow from the triangle inequality and stationary phase (see, e.g., Lemma 2.2 of [Harris 2024]). This implies (5-8) if either $p = 1$ or $p = 2$ (and similarly for any subcollection of T's in the left-hand side). The case $p \in (1, 2)$ of (5-8) then holds by interpolation; by pigeonholing it may be assumed that the sum over the T's in the left-hand side of (5-8) is over a subset of the T's such that $\mu(2T)$ is constant up to a factor of 2, and then (5-8) follows by writing $p = (1-\phi) \cdot 1 + \phi \cdot 2$ and applying Hölder's inequality with $q = 1/(1-\phi)$ and $q' = 1/\phi$. This proves (5-8).

By Hölder's inequality, the integrand in the right-hand side of (5-8) satisfies, for any $\theta \in [0, 2\pi)$,

$$\sum_{\substack{\tau \in \Lambda_j : \\ |\theta_\tau - \theta| \leq 2^{j(\delta-1/2)}}} \sum_{T \in \mathbb{T}_{\tau,b}} \mu(2T)^p 2^{j(p-1)} \lesssim 2^{jO(\delta)} \int_{F_j(\theta)} |\rho_{\theta\sharp}(\mu * \phi_{2^{j(1-\delta)}})|^p \, d\mathcal{H}^1, \tag{5-9}$$

where we used the inequality $\|x\|_p \leq \|x\|_1$ for sequences to deal with the possibility of multiple planks in a slab,[4] and

$$F_j(\theta) := \bigcup_{\substack{\tau \in \Lambda_j : \\ |\theta_\tau - \theta| \leq 2^{j(\delta-1/2)}}} \bigcup_{T \in \mathbb{T}_{\tau,b}} \rho_\theta(100T).$$

[4]This step was not needed in the case of projections onto planes, and was overlooked in [Harris 2023a].

By the triangle inequality, followed by (5-8), and then (5-9),

$$\left(\int_0^{2\pi}\int |\rho_{\theta\sharp}\mu_b|^p\, d\mathcal{H}^1\, d\theta\right)^{1/p} \lesssim \sum_{j\geq 1}\left(2^{jO(\delta)}\int_0^{2\pi}\int_{F_j(\theta)} |\rho_{\theta\sharp}(\mu * \phi_{2^{j(1-\delta)}})|^p\, d\mathcal{H}^1\, d\theta\right)^{1/p}.$$

By the definition of the sets $\mathbb{T}_{\tau,b}$, the collection of intervals formed by $F_j(\theta)$ generate an admissible triple satisfying the assumptions of Lemma 5.2, so applying this yields

$$\int_0^{2\pi}\int |\rho_{\theta\sharp}\mu_b|^p\, d\mathcal{H}^1\, d\theta \lesssim c_\alpha(\mu)^{p-1}\mu(\mathbb{R}^3)\left(\sum_{j\geq 1} 2^{-j\kappa/10^{10}}\right)^p \lesssim c_\alpha(\mu)^{p-1}\mu(\mathbb{R}^3). \tag{5-10}$$

This bounds the "bad" part.

It remains to bound the "good" part. From this point on, the L^2 method we use will be similar to [Harris 2024], but we include the details for completeness. It will be shown that

$$\int_0^{2\pi}\int |\rho_{\theta\sharp}\mu_g|^2\, d\mathcal{H}^1\, d\theta \lesssim \mu(\mathbb{R}^3)c_\alpha(\mu)\sum_{j\geq 1} 2^{j\left((5-2\alpha)/p_{\mathrm{dec}}-1/2+\beta(p_{\mathrm{dec}}-2/p_{\mathrm{dec}})\right)+jO(\delta)}, \tag{5-11}$$

where $2\leq p_{\mathrm{dec}}\leq 6$. Before proving (5-11), we will show that it implies the theorem. By taking $p_{\mathrm{dec}}=6$ and using $\delta\ll\beta\ll\alpha-1$, (5-11) is a decaying geometric series, which yields

$$\int_0^{2\pi}\int |\rho_{\theta\sharp}\mu_g|^2\, d\mathcal{H}^1\, d\theta \lesssim \mu(\mathbb{R}^3)c_\alpha(\mu).$$

By taking $p=1$, the previous argument for the bad part to obtain (5-10) shows that $\rho_{\theta\sharp}\mu_b\in L^1$ for a.e. $\theta\in[0,2\pi)$, and that

$$\int_0^{2\pi}\int |\rho_{\theta\sharp}\mu_b|\, d\mathcal{H}^1\, d\theta \lesssim \mu(\mathbb{R}^3).$$

By the triangle inequality, this implies that

$$\int_0^{2\pi}\int |\rho_{\theta\sharp}\mu_g|\, d\mathcal{H}^1\, d\theta \lesssim \mu(\mathbb{R}^3).$$

For general p with $1\leq p<\frac{3}{2}$, by writing $p=(1-\phi)1+\phi\cdot 2$, where $\phi=p-1$, and by applying Hölder's inequality to the function

$$|\rho_{\theta\sharp}\mu_g|^p = |\rho_{\theta\sharp}\mu_g|^{(1-\phi)\cdot 1}\cdot|\rho_{\theta\sharp}\mu_g|^{\phi\cdot 2},$$

with $q=\frac{1}{1-\phi}$ and $q'=\frac{1}{\phi}$, this yields

$$\int_0^{2\pi}\int |\rho_{\theta\sharp}\mu_g|^p\, d\mathcal{H}^1\, d\theta \lesssim c_\alpha(\mu)^{p-1}\mu(\mathbb{R}^3). \tag{5-12}$$

Combining (5-10) and (5-12) gives

$$\int_0^{2\pi}\int |\rho_{\theta\sharp}\mu|^p\, d\mathcal{H}^1\, d\theta \lesssim c_\alpha(\mu)^{p-1}\mu(\mathbb{R}^3).$$

This proves the theorem, assuming (5-11).

It remains to prove (5-11). By the triangle inequality and by symmetry, to prove (5-11) it suffices to show that for any $j \geq 1$,

$$\int_0^{2\pi} \int_{2^{j-1}}^{2^j} |\widehat{\mu}_g(t\gamma(\theta))|^2 \, dt \, d\theta \lesssim \mu(\mathbb{R}^3) c_\alpha(\mu) 2^{j\left((5-2\alpha)/p_{\mathrm{dec}} - 1/2 + \beta(p_{\mathrm{dec}} - 2/p_{\mathrm{dec}})\right) + jO(\epsilon)}, \tag{5-13}$$

for any $p_{\mathrm{dec}} \in (2, 6]$. By the definition of μ_g, it suffices to show that for any $j \geq 1$,

$$\int_0^{2\pi} \int_{2^{j-10}}^{2^{j+10}} \left| \sum_{\tau \in \Lambda_j} \sum_{T \in \mathbb{T}_{\tau,g}} \widehat{M_T \mu}(t\gamma(\theta)) \right|^2 dt \, d\theta \lesssim \mu(\mathbb{R}^3) c_\alpha(\mu) 2^{j\left((5-2\alpha)/p_{\mathrm{dec}} - 1/2 + \beta(p_{\mathrm{dec}} - 2/p_{\mathrm{dec}})\right) + jO(\epsilon)},$$

where, above and below, we can assume that rapidly decaying tail terms do not dominate (since in this case (5-13) would be trivial). Since μ is supported in the unit ball, the uncertainty principle can be used to change the above surface integral to an integral over $\mathbb{R}^3$. In more detail, we partition the part of the light cone over which we are integrating into balls of radius $2^{-10j\delta}$, and then use the locally constant property of the integrand over each such ball to replace the surface integral $dt \, d\theta$ over the ball intersected with the light cone with the 3-dimensional integral dx over the ball. Because the $dt \, d\theta$ measure above of such a ball intersected with the light cone is $\lesssim 2^{-j}$ (as $t \sim 2^j$ and is varying on an interval of length ~ 1, the angle θ such that $t\gamma(\theta)$ intersects the ball must lie in an interval of size $\sim 2^{-j}$), we gain a factor of 2^{-j} when changing $dt \, d\theta$ to dx. We then replace the integral over the $2^{-10j\delta}$-neighbourhood of the light cone with an integral over all of $\mathbb{R}^3$, and then use Plancherel's theorem in $\mathbb{R}^3$ to decouple the L^2 norms. Due to the 2^{-j} gain, it is enough to show that

$$\sum_{\tau \in \Lambda_j} \sum_{T \in \mathbb{T}_{\tau,g}} \int_{\mathbb{R}^3} |M_T \mu|^2 \, dx \lesssim \mu(\mathbb{R}^3) c_\alpha(\mu) 2^{j\left((5-2\alpha)/p_{\mathrm{dec}} + 1/2 + \beta(p_{\mathrm{dec}} - 2/p_{\mathrm{dec}})\right) + jO(\epsilon)}. \tag{5-14}$$

By writing out the L^2 norm above as an inner product, applying the definition $M_T \mu = \eta_T(\mu * \check{\psi}_\tau)$ to one of the $M_T \mu$'s, and using Fubini's theorem to interchange the μ integral with all the others, the left-hand side of (5-14) is equal to

$$\int \sum_{\tau \in \Lambda_j} \sum_{T \in \mathbb{T}_{\tau,g}} (\eta_T M_T \mu) * \check{\psi}_\tau \, d\mu. \tag{5-15}$$

By the Cauchy–Schwarz inequality (with respect to the measure μ),

$$(5\text{-}15) \lesssim \mu(\mathbb{R}^3)^{1/2} \left(\int \left| \sum_{\tau \in \Lambda_j} \sum_{T \in \mathbb{T}_{\tau,g}} (\eta_T M_T \mu) * \check{\psi}_\tau \right|^2 d\mu \right)^{1/2}. \tag{5-16}$$

For each T above, abbreviate $f_T = (\eta_T M_T \mu) * \check{\psi}_\tau$. By pigeonholing, the sum over T in the left-hand side above can be restricted to a subset $\mathbb{W}$ of $\bigcup_{\tau \in \Lambda_j} \mathbb{T}_{\tau,g}$ on which $\|f_T\|_{p_{\mathrm{dec}}}$ is constant over $T \in \mathbb{W}$ up to a factor of 2. Because the integrand on the left-hand side of (5-16) has frequencies of size $\lesssim 2^j$, the measure μ being integrated against can be replaced by its smoothed out version $\mu * \phi_{2^j}$ at scale 2^{-j}. We then partition the domain into balls B of radius 2^{-j}, and pigeonhole these balls B according to the

number M of planks $T \in \mathbb{W}$ such that $2T$ has nonempty intersection with B. Let Y be the union over these balls B. With this setup, (5-16) will be bounded by

$$\log(2^j)^{O(1)}\mu(\mathbb{R}^3)^{1/2}\bigg(\int_Y \bigg|\sum_{T\in\mathbb{W}} f_T\bigg|^2 d(\mu * \phi_j)\bigg)^{1/2}. \tag{5-17}$$

By Hölder's inequality (with respect to Lebesgue measure),

$$\bigg(\int_Y \bigg|\sum_{T\in\mathbb{W}} f_T\bigg|^2 d(\mu * \phi_j)\bigg)^{1/2} \le \bigg(\int_Y \bigg|\sum_{T\in\mathbb{W}} f_T\bigg|^{p_{\mathrm{dec}}}\bigg)^{1/p_{\mathrm{dec}}} \bigg(\int_Y (\mu * \phi_j)^{p_{\mathrm{dec}}/p_{\mathrm{dec}}-2}\bigg)^{1/2-1/p_{\mathrm{dec}}}. \tag{5-18}$$

By the refined decoupling theorem (see Theorem A.1), the first factor in (5-18) satisfies

$$\bigg(\int_Y \bigg|\sum_{T\in\mathbb{W}} f_T\bigg|^{p_{\mathrm{dec}}}\bigg)^{1/p_{\mathrm{dec}}} \le C_\epsilon 2^{j\epsilon}\bigg(\frac{M}{|\mathbb{W}|}\bigg)^{1/2-1/p_{\mathrm{dec}}} \bigg(\sum_{T\in\mathbb{W}} \|f_T\|_{p_{\mathrm{dec}}}^2\bigg)^{1/2}. \tag{5-19}$$

By using (5-7), the second factor in (5-18) satisfies

$$\begin{aligned}\bigg(\int_Y (\mu * \phi_j)^{p_{\mathrm{dec}}/(p_{\mathrm{dec}}-2)}\bigg)^{1/2-1/p_{\mathrm{dec}}} &\le \bigg(\frac{1}{M}\sum_{T\in\mathbb{W}}\int_{Y\cap 2T} (\mu * \phi_j)^{p_{\mathrm{dec}}/(p_{\mathrm{dec}}-2)}\bigg)^{1/2-1/p_{\mathrm{dec}}}\\ &\lesssim \bigg(\frac{1}{M}\sum_{T\in\mathbb{W}} \mu(4T)\bigg)^{1/2-1/p_{\mathrm{dec}}} c_\alpha(\mu)^{1/p_{\mathrm{dec}}} 2^{j(3-\alpha)/p_{\mathrm{dec}}}\\ &\lesssim c_\alpha(\mu)^{1/2}\bigg(\frac{|\mathbb{W}|}{M}\bigg)^{1/2-1/p_{\mathrm{dec}}} 2^{-j(1-\beta)(1/2-1/p_{\mathrm{dec}})+j(3-\alpha)/p_{\mathrm{dec}}}.\end{aligned} \tag{5-20}$$

Substituting (5-19) and (5-20) into (5-17) and then (5-18) yields

$$\begin{aligned}&\bigg(\int \bigg|\sum_{\tau\in\Lambda_j}\sum_{T\in\mathbb{T}_{\tau,g}} (\eta_T M_T \mu) * \check{\psi}_\tau\bigg|^2 d\mu\bigg)^{1/2}\\ &\qquad\lesssim c_\alpha(\mu)^{1/2} 2^{-j(1-\beta)(1/2-1/p_{\mathrm{dec}})+j(3-\alpha)/p_{\mathrm{dec}}+jO(\epsilon)}\bigg(\sum_{\tau\in\Lambda_j}\sum_{T\in\mathbb{T}_{\tau,b}} \|f_T\|_{p_{\mathrm{dec}}}^2\bigg)^{1/2}.\end{aligned} \tag{5-21}$$

By comparing the left-hand side of the above with (5-15) and then (5-14), and then by using

$$\|f_T\|_{p_{\mathrm{dec}}} \lesssim 2^{jO(\delta)}\|M_T\mu\|_2 2^{3j/2(1/2-1/p_{\mathrm{dec}})}$$

(which follows from the definition of f_T and the Hausdorff–Young inequality, followed by Hölder's inequality), (5-21) implies that

$$\begin{aligned}&\sum_{\tau\in\Lambda_j}\sum_{T\in\mathbb{T}_{\tau,g}} \|M_T\mu\|_2^2\\ &\qquad\lesssim \bigg(\sum_{\tau\in\Lambda_j}\sum_{T\in\mathbb{T}_{\tau,g}} \|M_T\mu\|_2^2\bigg)^{1/2}\mu(\mathbb{R}^3)^{1/2} c_\alpha(\mu)^{1/2} 2^{(3j/2-j(1-\beta))(1/2-1/p_{\mathrm{dec}})+j(3-\alpha)/p_{\mathrm{dec}}+jO(\epsilon)}.\end{aligned}$$

This simplifies to

$$\sum_{\tau\in\Lambda_j}\sum_{T\in\mathbb{T}_{\tau,g}}\|M_T\mu\|_2^2 \lesssim \mu(\mathbb{R}^3)c_\alpha(\mu)2^{j(1+2\beta)(1/2-1/p_{\mathrm{dec}})+2j(3-\alpha)/p_{\mathrm{dec}}+jO(\epsilon)}.$$

This is the same as (5-14), so this finishes the proof. □

6. Some open problems

(1) An open problem that lies in-between the stated intersection and projection theorems, is whether there is a "restricted" Besicovitch–Federer projection theorem in $\mathbb{R}^3$. For example, if $A\subseteq\mathbb{R}^3$ is a Borel set with $0<\mathcal{H}^2(A)<\infty$, such that the projection $\pi_{v^\perp}(A)$ of A onto $v^\perp$ has $\mathcal{H}^2(\pi_{v^\perp}(A))=0$ for a.e. $v\in S^2$, where $\pi_{v^\perp}$ is orthogonal projection onto $v^\perp$, then is $\mathcal{H}^2(\pi_\theta(A))=0$ for a.e. $\theta\in[0,2\pi)$?

(2) We do not know if $p<\frac{3}{2}$ in Theorem 1.3 can be replaced by $p=\frac{3}{2}$. The exponent $p=\frac{3}{2}$ is only sharp in Theorems 1.2 and 1.3 if p is not allowed to depend on α. In Theorem 1.2, if $p=p(\alpha)$ is allowed to depend on α, then up to the endpoint the sharp exponent is likely the one which naturally interpolates between $p=\frac{3}{2}$ at $\alpha=2$ and $p=2$ at $\alpha=\frac{5}{2}$, and similarly for Theorem 1.3.

(3) Another open problem is whether $B_3(0,1)$ in Theorem 1.1 can be replaced by $\mathbb{R}^3$; unlike for the standard Kakeya maximal function, the local and global versions do not seem to be obviously equivalent, since horizontal or SL_2 lines are not preserved by Euclidean translations. For the global problem on $\mathbb{R}^3$, the ball condition (1-1) is no longer equivalent to the Wolff axioms in the SL_2 case. Moreover, the example consisting of $\sim\delta^{-2}$ many $1\times\delta$ tube segments inside a vertical $1\times\delta\times1$ slab parallel to the xz-plane, centred at the point $(\delta^{-1},0,0)$, shows that the global version of (1-2) is *false* if only the ball condition (1-1) is assumed. However, the global version of (1-2) with $B_3(0,1)$ replaced by $\mathbb{R}^3$ may still be true if the full Wolff axioms are assumed instead of just the ball condition (1-1).

Appendix: Refined decoupling

Given $R\geq1$ and a small constant $c>0$, let Θ_R be a $\sim cR^{-1/2}$-separated subset of $[0,2\pi)$. For each $\theta\in\Theta_R$, let $\tau(\theta)$ be a plank of dimensions $1\times R^{-1/2}\times R^{-1}$ of distance ~1 from the origin, tangent to the cone at $\gamma(\theta)$. Given $\delta>0$, for each τ, let $\mathbb{T}_\tau$ be a collection of planks of dimensions $R^\delta\times R^{1/2+\delta}\times R^{1+\delta}$ dual to τ, forming a boundedly overlapping cover of $\mathbb{R}^3$.

Theorem A.1 [Gan et al. 2022, Theorem 9]. *Let $p_{\mathrm{dec}}\in[2,6]$. If $c>0$ is sufficiently small, then for any $\epsilon>0$, there exists $\delta_0>0$ such that the following holds for all $0<\delta<\delta_0$. Let $R>1$, and suppose that $f=\sum_{T\in\mathbb{W}}f_T$, where $\mathbb{W}\subseteq\bigcup_{\theta\in\Theta_R}\mathbb{T}_{\tau(\theta)}$, and each f_T has $\widehat{f_T}$ supported in τ, with f_T essentially supported on T in the sense that $\|f_T\|_{L^\infty(\mathbb{R}^3\setminus T)}\leq AR^{-10000}$, for a large constant A. Suppose that $\|f_T\|_p$ is constant over $T\in\mathbb{W}$ up to a factor of 2, and that Y is a disjoint union of radius 1 balls in $B(0,R)$, such that each radius 1 ball $B\subseteq Y$ intersects at most M planks $2T$ with $T\in\mathbb{W}$. Then*

$$\Big\|\sum_{T\in\mathbb{W}}f_T\Big\|_{L^{p_{\mathrm{dec}}}(Y)}\leq C_{\epsilon,\delta,A,c}R^\epsilon\Big(\frac{M}{|\mathbb{W}|}\Big)^{1/2-1/p_{\mathrm{dec}}}\Big(\sum_{T\in\mathbb{W}}\|f_T\|_{p_{\mathrm{dec}}}^2\Big)^{1/2}.$$

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