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Considering an integer $d > 0$, we show the existence of convex-cocompact representations of surface groups into SO(4, 1) admitting an embedded minimal map with small second fundamental form, and whose associated hyperbolic 4-manifolds are disc bundles of degree d over the surface, provided its genus is large enough. We also show that we can realize these representations as complex variation of Hodge structures. This gives examples of quasicircles in $\mathbb{S}^3$ bounding superminimal discs in $\mathbb{H}^4$ of arbitrarily small second fundamental form. Those are examples of generalized almost-fuchsian representations which are not deformations of fuchsian representations by discrete and faithful representations.

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1. Introduction

1.1. *Context.*

1.1.1. *Minimal surfaces and convex-cocompact representations.* In this paper, we consider representations of a closed surface group into $\mathrm{SO}(4,1) \approx \mathrm{Isom}(\mathbb{H}^4)$ and minimal surfaces in the hyperbolic 4-space equivariant under the group action. We prove the existence of "exotic" minimal surfaces of small curvature, that is, minimal surfaces with arbitrarily small second fundamental form while the quotient of the hyperbolic space by the group action is a nontrivial disc bundle over the surface considered.

Minimal surfaces are surfaces which are locally critical points of the area functional. Considering an immersion from a Riemann surface Σ to $\mathbb{H}^4$, the image of an immersion is a minimal surface if the immersion is conformal and harmonic. The study of minimal surfaces in hyperbolic manifolds is a rich and diverse topic. Schoen and Yau [1979] and Sacks and Uhlenbeck [1982] showed that in any isotopy class of an incompressible surface in a compact manifold, there is a weakly branched immersion whose image is minimal. See [Meeks and Yau 1982] for the case when the target manifold has mean convex boundary. In the case of a hyperbolic three-dimensional manifold with convex boundary, there is an immersion, which will be an embedding if we're in the isotopy class of an incompressible surface.

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Considering a faithful and discrete representation ρ in $\mathrm{Isom}(\mathbb{H}^n)$, it is then natural to ask whether the quotient manifold $\rho\backslash\mathbb{H}^n$ contains a nonempty compact convex set. When that is the case, we say that ρ is *convex-cocompact*. In particular, for a convex-cocompact representation ρ from a closed surface group in $\mathrm{Isom}(\mathbb{H}^n)$, there is always an equivariant weak immersion $\mathbb{D}^2 \to \mathbb{H}^4$ whose image is minimal (but has branch points in general). While branch points can be ruled out for least area minimal surfaces in hyperbolic 3-manifolds with convex boundary, it is not true from dimension 4 on. In this regard, our theorem gives examples of minimal maps into a disc bundle over a surface which are embeddings, so without branch points and self-intersections.

1.1.2. *Hyperbolic structures on disc bundles.* Among the family of hyperbolic 4-manifolds whose fundamental group is a surface group, one family is of specific interest to us: disc bundles over a Riemann surface. Considering a fuchsian representation of a surface group into $\mathrm{SO}(4, 1)$, it preserves a totally geodesic disc in $\mathbb{H}^4$ and the quotient manifold identifies with the normal bundle to the quotient of the disc. Topologically the quotient manifold is the product of a disc with the Riemann surface. However, these are not the only examples of hyperbolic disc bundles. Gromov, Lawson and Thurston [Gromov et al. 1988] were the first to give examples of hyperbolic structures on nontrivial disc bundles over a surface, considering equivariant piecewise-linear embeddings of the disc into $\mathbb{H}^4$. Kuiper [1988] also gave examples of those, using tessellations of the hyperbolic 4-space.

Topologically, disc bundles over a Riemann surface are classified by the Euler characteristic of the surface and the Euler class of the bundle. The precise list of disc bundles over a closed surface admitting a hyperbolic structure is still an open problem. Gromov, Lawson and Thurston conjectured that a necessary condition to admit a hyperbolic structure would be that the degree d is smaller in absolute value than the Euler characteristic $2g - 2$. Kapovich [1989] showed that for $0 \leq d \leq (g - 1)/11$, there is indeed a hyperbolic structure on the corresponding disc bundle.

Our result gives a new method of finding some of those hyperbolic structures, with more geometric properties, although we don't have an explicit condition on how small the degree has to be with regard to g. Check [Ho 2014] for the latest results concerning the Gromov–Lawson–Thurston conjecture.

1.1.3. *Almost-fuchsian representations.* Another important question regarding minimal surfaces is the potential uniqueness in a fixed isotopy class. Already in dimension 3, there are examples of hyperbolic 3-manifolds containing arbitrarily many isotopic minimal surfaces [Huang and Wang 2015]. However, we may get uniqueness if we add some conditions on the manifold: Uhlenbeck [1983] introduced the notion of *almost-fuchsian representations* which are faithful and discrete representations of a surface group into $\mathrm{PSL}(2, \mathbb{C})$ admitting an equivariant immersion $\mathbb{D}^2 \to \mathbb{H}^3$ minimal and with principal curvatures in $(-1, 1)$.

Note that asking the principal curvatures to be zero is equivalent to asking that our immersion is totally geodesic. Here as we relaxed this closed condition into asking the principal curvatures to be in $(-1, 1)$, the notion is open in the character variety. This differential geometric assumption has several consequences: the representation is then convex-cocompact, the image of f is only ρ-invariant minimal disc, and f is an embedding.

Since then, the notion of almost-fuchsian representation has been extended to almost-fuchsian immersions [Krasnov and Schlenker 2007]. Seppi [2016] studied almost-fuchsian discs in $\mathbb{H}^3$, Donaldson [2003], Hodge [2005] and Trautwein [2019] considered the hyperkähler structure on the moduli space of almost-fuchsian representations extending the hyperkähler structure on Teichmüller space. With Smith [Bronstein and Smith 2023], we have exhibited a parametrization of the set of almost-fuchsian discs in $\mathbb{H}^3$ by a convex set of bounded holomorphic quadratic differentials on the disc. There are important questions related to foliations by surfaces of almost-fuchsian manifolds [Guo et al. 2010; Choudhury et al. 2024].

Jiang [2025] considered almost-fuchsian discs in $\mathbb{H}^n$. For faithful and discrete representations into $\mathrm{SO}(n, 1)$, the existence of an equivariant minimally immersed disc whose scalar second fundamental forms have curvatures in $(-1, 1)$ actually implies the uniqueness of the minimal surface in its isotopy class, its embeddedness, and the convex-cocompactness of the representation. Recently, Davalo [2025] extended the notion of almost-fuchsian immersion into nearly geodesic immersion into a higher rank symmetric space of noncompact type. The only known examples of nearly geodesic immersions are deformations of totally geodesic immersions. While in $\mathrm{PSL}(2, \mathbb{C})$ those are indeed the only examples, we bring here examples of such immersions which cannot be deformed to totally geodesic ones. In particular, the limit set of our representations is a quasicircle, and there is a quasiconformal homeomorphism of $\mathbb{S}^3$ sending our limit set to a round curve, but this homeomorphism cannot be made equivariant.

1.1.4. *Superminimal maps and complex variations of Hodge structures.* The construction we will exhibit will naturally give examples of complex variations of Hodge structures in $\mathrm{SO}(4, 1)$. The notion of complex variation of Hodge structures for representations of surface groups come from the nonabelian Hodge correspondence; see for instance the survey [Wentworth 2016]. The nonabelian Hodge correspondence is a one-to-one correspondence between the moduli space of representations of surface groups with the moduli space of an object called the Higgs bundle, on which there is an action of $\mathbb{C}^*$. The fixed points of this action are called complex variations of Hodge structures. Loftin and McIntosh [2019] give an extensive description of those representations in their article about equivariant minimal surfaces in $\mathbb{H}^4$. For a representation ρ with an equivariant minimal immersion f, asking that ρ is a complex variation of Hodge structure is equivalent to asking that f is superminimal, i.e., that it has circular ellipse of curvature (see Definition 2.5).

The term superminimal was coined in [Kommerell 1897]. In particular, it implies the minimality of the image of the map. Regarding the study of superminimal surfaces in the sphere $\mathbb{S}^4$, there are some impressive results: see for instance [Bryant 1982], which exhibited a duality between superminimal surfaces in $\mathbb{S}^4$ and holomorphic legendrian curves in $\mathbb{C}P^3$. Check [Forstnerič 2021] for the analogous statement in $\mathbb{H}^4$. More precisely, Loftin and McIntosh showed that for any pair of integers d and $g \geq 2$ such that $|d| < 2g-2$, there are representations from $\pi_1 \Sigma_g$ into $\mathrm{SO}(4, 1)$ with an equivariant superminimal immersion whose normal bundle is of degree d over Σ_g. Our result states that for d very small compared to g, there are such pairs (ρ, f) where f is actually embedded and ρ is convex-cocompact. We also have, as the normal bundle to f will identify to the quotient manifold $\rho\backslash\mathbb{H}^4$, hyperbolic structures on degree d disc bundles over Σ_g.

1.1.5. *Solving "mixed" Toda systems.* Finally, the method used to build our examples is to solve a nonlinear PDE system which we call a mixed Toda system. Toda systems, and Toda systems with opposite signs, are in [Guest and Lin 2014] and designate PDE systems where the unknown is (w_i) and satisfy an equation of the type

$$\Delta w_i = e^{w_i - w_{i-1}} - e^{w_i - w_{i+1}}. \tag{1-1}$$

While our system definitely cannot be written in this form, it belongs to a class of equations arising frequently when dealing with flat curvature equations.

This kind of system appears quite easily in the theory of flat bundles. Li and Mochizuki [2025] encounter Toda systems (of the opposite type) when dealing with cyclic Higgs bundles associated to representations in rank 2 Lie groups. Malchiodi and Ruiz [2013] and Battaglia, Jevnikar, Malchiodi and Ruiz [Battaglia et al. 2015] made use of the Moser–Trudinger inequality to solve some Toda systems motivated by the nonabelian Chern–Simons theory. Here in rank 1 with the complex Hodge variation structure assumption, we get something alike, but with tweaked signs:

$$\begin{cases} \Delta u = -1 + e^{2u} + e^{-2u} e^{2v} |\alpha|^2, \\ \Delta v = \dfrac{1}{2g-2} - e^{-2u} e^{2v} |\alpha|^2. \end{cases} \tag{1-2}$$

We think the method used in this article to solve this system could be applied to a broader class of PDE systems, yet to be described.

1.2. ***Statement of the results.*** First, we state our result in dimension 4 concerning degree 1-disc bundles over Riemann surfaces.

Theorem 1.1. *Fix $r \in (0, 1)$. There is $g_0 \geq 2$ such that, for any $g \geq g_0$ and Σ a closed Riemann surface of genus g, there is a representation $\rho : \pi_1 \Sigma \to \mathrm{Isom}(\mathbb{H}^4)$ and an immersion $f : \mathbb{D}^2 \to \mathbb{H}^4$ satisfying*

(a) *ρ is faithful, discrete, convex-cocompact, and almost-fuchsian;*

(b) *f is a ρ-equivariant embedding, superminimal;*

(c) *the second fundamental form of f satisfies $|\mathbb{I}_f| < r$ everywhere;*

(d) *the quotient hyperbolic 4-manifold $\rho\backslash\mathbb{H}^4$ is a degree 1 disc bundle over Σ.*

As a corollary, we have the existence of almost-fuchsian representations uniformizing degree d disc bundles over Riemann surfaces of high genus.

Corollary 1.2. *Let $d > 0$. For any $g_0 > 0$, there is $g \geq g_0$, Σ a surface of genus g and $\rho : \pi_1 \Sigma \to \mathrm{Isom}(\mathbb{H}^4)$ and $f : \mathbb{D}^2 \to \mathbb{H}^4$ satisfying*

(a) *ρ is faithful, discrete, convex-cocompact, almost-fuchsian;*

(b) *f is ρ-equivariant embedding, superminimal;*

(c) *the second fundamental form of f satisfies $|\mathbb{I}_f| < r$ everywhere;*

(d) *the quotient manifold $\rho\backslash\mathbb{H}^4$ is a degree d disc bundle over Σ.*

1.3. *Scheme of proof.*

1.3.1. *Writing the flat curvature equations.* In order to build a hyperbolic structure on a degree d disc bundle, we first show the existence of a hyperbolic structure on a degree 1 disc bundle over a surface with the desired properties. Once this example is built, we can take a cover of degree d of this bundle in order to obtain a hyperbolic structure on a degree d disc bundle over a surface (of genus large enough).

Consider a Riemann surface Σ of genus g. Our proof will rely on solving a PDE system of the following kind:

$$\begin{cases} \Delta u = -1 + e^{2u} + e^{-2u}e^{2v}|\alpha|^2, \\ \Delta v = \dfrac{1}{2g-2} - e^{2v}e^{-2u}|\alpha|^2, \end{cases} \tag{1-3}$$

where α has to be a holomorphic section of a degree $4g-3$ line bundle over Σ. The Laplacian is always taken with respect to the Poincaré volume form on the surface, and every line bundle will be considered endowed with a metric whose curvature form is proportional to the Poincaré volume form, hence defining the norm $|\alpha|$.

The second section will be devoted to showing how solving this system yields a minimal disc in $\mathbb{H}^4$ equivariant under a representation ρ. With the good pointwise control on $e^{2v}e^{-2u}|\alpha|^2$, the representation ρ will be almost-fuchsian and the quotient manifold $\rho\backslash\mathbb{H}^4$ will be a degree 1 disc bundle over the Riemann surface.

The associated minimal map will have a holomorphic second fundamental form (Definition 2.2) $\alpha \oplus 0$, hence be superminimal.

1.3.2. *Controlling the geometric data.* The third section will be devoted to finding families (Σ_n) of surfaces with genus going to infinity, with data (α_n) associated, while controlling the behavior of solutions of PDEs on (Σ_n) depending on (α_n). To do so, we need to control the systole and spectral gap of (Σ_n), and the ratio $\left(\fint |\alpha_n|^2\right)/|\alpha_n|^2_\infty$.

Controlling the systole is the easiest part, as the systole is nondecreasing when taking the Riemannian cover of a surface. Controlling the spectral gap is trickier, but can be done by taking (Σ_n) a random cover of a genus 2 surface thanks to a result of Magee, Naud and Petri [Magee et al. 2022]: considering a cover $\Sigma' \to \Sigma$ of closed Riemann surfaces, the relative spectrum of Σ' is defined to be the part of the spectrum of Σ' which is not in the spectrum of Σ. Magee, Naud and Petri states that for a random cover Σ of large degree, the probability that the relative spectral gap is lower than $\frac{3}{16} - \varepsilon$ goes to zero when the degree goes to infinity. Using this result, we are able to find families of coverings of a Riemann surface whose spectral gap is uniformly lower bounded.

Finally, we need to build α_n such that the ratios $|\alpha_n|^2_\infty / \fint |\alpha_n|^2$ are uniformly bounded. This property will be called *balanced* family of sections. We will show that multiplying a cover of a section with a section of a line bundle of degree 1 yields a balanced family of sections of the desired type.

1.3.3. *Building the maps Φ and Ψ.* Once this is done, we need to solve our PDE system. For this, we will consider each line of the system and build two maps. Fixing v and solving the first line in u, we will

build a map $\Phi : C^{0,\alpha}(\Sigma) \to C^{2,\alpha}(\Sigma)$ which will be well-defined on the set $\mathcal{K}_v$, where

$$\mathcal{K}_v := \left\{ v \in C^{0,\alpha}(\Sigma) : e^{2v}|\alpha|^2 \leq \tfrac{1}{2} \right\}. \tag{1-4}$$

And it happens that the image of $\mathcal{K}_v$ is in the set $\mathcal{K}_u$, where

$$\mathcal{K}_u := \left\{ u \in C^{0,\alpha}(\Sigma) : -\tfrac{1}{2}\ln 2 \leq u \leq 0 \right\}. \tag{1-5}$$

We would like to do the same work for the second equation, fixing v, solving in u to get a map Ψ. However, the work to build Ψ is considerably harder. The third section will be devoted to the study of Ψ. The method used to solve it is heavily inspired from the analysis of the prescribed curvature equation in the projective plane $\mathbb{R}P^2$: for an even function on the sphere which is positive on an open set, it is the curvature of a metric conformal to the round metric on $\mathbb{S}^2$ (and even). See [Chang and Yang 2003] for a precise explanation of this. With the same tools, we are able to solve the second line and to get the following estimate: there is a constant $C > 0$ such that the solution v satisfies

$$e^{2v}e^{-2u}|\alpha|^2 \leq \frac{C}{2g-2}. \tag{1-6}$$

The constant C will depend on three parameters: the systole δ of the surface, its spectral gap Λ and the ratio between the L^∞-norm of α and its L^2-average. Hence, thanks to the third section it can be bounded on surfaces of sufficiently large genus.

1.3.4. *Piecing everything together.* Considering a family of covers (Σ_n) with a balanced family of sections (α_n) and controlled systole and spectral gap, we are able, provided n is large enough, to build a continuous map $\Psi : \mathcal{K}_u \to \mathcal{K}_v$.

With Ψ constructed and well-defined, we will show that the composed map $\Phi \circ \Psi$ defined on $\mathcal{K}_u$ has image in a compact convex subset of $\mathcal{K}_u$. Hence, by Schauder's fixed-point theorem, $C^{0,\alpha}(\Sigma)$ being a Banach space, $\Phi \circ \Psi$ has a fixed point $u \in \mathcal{K}_u$. Write $v = \Psi(u)$.

It is clear that (u, v) is a solution of the PDE system considered, and so it corresponds to an almost-fuchsian representation ρ whose quotient manifold is a degree 1 disc bundle over the Riemann surface.

2. The curvature equations

2.1. ***Minimal equivariant immersions in*** $\mathbb{H}^4$**.** Let Σ be a closed oriented surface of genus $g \geq 2$.

Consider a representation

$$\rho : \pi_1\Sigma \to \mathrm{SO}_0(4,1) \approx \mathrm{Isom}_+(\mathbb{H}^4).$$

We will always assume that ρ is faithful and discrete. Let $f : \mathbb{D}^2 = \widetilde{\Sigma} \to \mathbb{H}^4$ be an equivariant immersion with minimal image.

Endowing Σ with the induced metric by f, we get a conformal and harmonic map.

Associated to f, there is a flat $\mathbb{R}^{4,1}$ vector bundle E, which decomposes as follows:

Let $\mathbb{I}$ be the second fundamental form of f, and B its shape operator. The vector bundle (E, ∇_E) splits as $E = O_{\mathbb{R}} \oplus T_{\mathbb{R}} \oplus N_{\mathbb{R}}$, where $O_{\mathbb{R}}$ is the trivial vector bundle of rank 1, $T_{\mathbb{R}}$ the tangent bundle of rank 2,

and $N_{\mathbb{R}}$ is the normal bundle to f, again of rank 2. In this splitting, the connection has the expression

$$\nabla_E = \begin{pmatrix} \nabla & a^\dagger & 0 \\ a & \nabla_T & B \\ 0 & \mathbb{I} & \nabla_N \end{pmatrix}. \tag{2-1}$$

Denote by q the quadratic form in $\mathbb{R}^{4,1}$ given by

$$q(x_1, \dots, x_5) := x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_5^2. \tag{2-2}$$

As the map f is valued in the quadric $q = -1$ in $\mathbb{R}^{4,1}$, the image of f gives a reduction of E into an $S(O(1) \times O(4))$ bundle. The differential of f being valued, by definition, in the tangent bundle, we denote by $a = df$ the nonvanishing section of $T^*\Sigma \otimes \mathrm{Hom}(O_{\mathbb{R}}, T)$. The definition of $\mathbb{I}$ and of B gives the desired decomposition of ∇_E.

Remark. This splitting is orthogonal with respect to b the signature $(4, 1)$ bilinear form, with $O_{\mathbb{R}}$ being a negative line. As the surface is oriented and our representation is into orientation-preserving isometries, $\mathrm{SO}_0(4, 1) \approx \mathrm{Isom}_+(\mathbb{H}^4)$, and $T_{\mathbb{R}}$ and $N_{\mathbb{R}}$ are endowed with an SO(2) structure corresponding to the induced metrics on the tangent and normal bundle to f. The nondegenerate metric b also implies that $\mathbb{I}$ and B are antiadjoint with respect to b:

$$b(B(w)u, v) + b(\mathbb{I}(u, v), w) = 0 \quad \text{for all } (u, v) \in T_{\mathbb{R}},\ w \in N_{\mathbb{R}}. \tag{2-3}$$

We let $B = -\mathbb{I}^\dagger$, where $\dagger$ denotes the transposition, identifying $\mathrm{Hom}(E, F)$ with $\mathrm{Hom}(F^*, E^*)$.

Complexify now the bundles $T_{\mathbb{R}}$ and $N_{\mathbb{R}}$. As each of these are equipped with an SO(2)-structure, their complexification splits into the sum of two line bundles, the restriction of B giving a pairing between those. Now for each line subbundle of E coming into play, we consider ∇_L the induced connection of ∇_E and we take its $(0, 1)$-part to get $\bar\partial_L$ a holomorphic structure on L. Remark that the $(0, 1)$-part is well-defined because the induced metric endows the surface with a Riemann surface structure.

The diagonalization of the SO(2) structure with regard to the metrics yields:

Proposition 2.1. *Viewing* $(\mathbb{I}_{\mathbb{C}})$ *as a* 1*-form valued in* $\mathrm{Hom}(K^{-1}, N)$, *the complexification of the second fundamental form can be written*

$$(T_{\mathbb{R}})_{\mathbb{C}} = \overline{K} \oplus K = K^{-1} \oplus K, \tag{2-4}$$

$$(N_{\mathbb{R}})_{\mathbb{C}} = N \oplus \overline{N} = N \oplus N^{-1}, \tag{2-5}$$

$$(\mathbb{I})_{\mathbb{C}} = \begin{pmatrix} \alpha & \beta^* \\ \beta & \alpha^* \end{pmatrix}, \tag{2-6}$$

with α *and* β $(1, 0)$*-forms.*

Proof. The splitting of the complexification of an SO(2)-bundle is given by the eigenspaces of J the fiberwise rotation of angle $\frac{\pi}{2}$. Remark that this splitting diagonalizes ∇_T and ∇_N. In this splitting, write down the second fundamental form

$$(\mathbb{I})_{\mathbb{C}} = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}. \tag{2-7}$$

As $\mathbb{I}$ is real-valued, we have

$$\gamma = \overline{\beta} = \beta^*, \tag{2-8}$$

$$\delta = \overline{\alpha} = \alpha^*. \tag{2-9}$$

Finally, as f is minimal, $\mathbb{I}$ is traceless, and so the $(1,1)$-part of the second fundamental form is zero. This implies that both $\alpha \in \Gamma(K \otimes \mathrm{Hom}(K^{-1}, N))$ and $\beta \in \Gamma(K \otimes \mathrm{Hom}(K^{-1}, N^{-1}))$ are $(1,0)$-forms. □

Remark. In the process, we endowed the bundle $E \otimes \mathbb{C}$ with the following holomorphic structure:

$$E \otimes \mathbb{C} = \mathcal{O} \oplus K^{-1} \oplus K \oplus N \oplus N^{-1}, \tag{2-10}$$

$$\bar{\partial}_E = \bar{\partial}_{\mathcal{O}} \oplus \bar{\partial}_{K^{-1}} \oplus \bar{\partial}_K \oplus \bar{\partial}_N \oplus \bar{\partial}_{N^{-1}}. \tag{2-11}$$

It is clear that the sections α, β characterize wholly the second fundamental form. Hence we define the *holomorphic second fundamental form*:

Definition 2.2. Let $f : \mathbb{D}^2 \hookrightarrow \mathbb{H}^4$ be a minimal immersion. The *holomorphic second fundamental form* of f is the $(2,0)$-part of the complexification of the second fundamental form

$$\mathbb{I}^{2,0} = \alpha \oplus \beta \in \Gamma(K^2(N \oplus N^{-1})). \tag{2-12}$$

2.2. ***Writing the flat curvature equations as a scalar PDE system.*** In order to write this system as a scalar PDE system, we need to write down each metric as a conformal change of a preferred one. For the metric on K, we denote by $e^{2u}\sigma$ the induced metric, where σ is the unique complete hyperbolic metric in the conformal class.

Let ω be the Poincaré $(1,1)$-form associated to σ. Also, we denote by Δ the Laplace–Beltrami operator on (Σ, σ). Note that Δ is defined to be locally $\partial_x^2 + \partial_y^2$, so it is a nonpositive operator satisfying

$$\int_\Sigma (\Delta f) g\, \omega + \int_\Sigma \nabla f \cdot \nabla g\, \omega = 0 \quad \text{for all } (f, g) \in C(\Sigma). \tag{2-13}$$

Generally, on a holomorphic line bundle L of degree d, there is a hermitian metric h_L whose curvature form is

$$iF_{L,h_L} = \frac{d}{2g-2}\omega. \tag{2-14}$$

The metric h_L is uniquely defined up to multiplicative constant. From now on, every line bundle will be endowed with a metric of this kind, and a given fiberwise norm $|\cdot|_L$.

Let's write down the flat curvature condition as a scalar PDE system:

Theorem 2.3. *Let $f : \Sigma \hookrightarrow M$ be a conformal harmonic immersion into a complete hyperbolic 4-manifold. Consider $\alpha \oplus \beta$ its holomorphic second fundamental form. Let $e^{2u}\sigma$ be the induced metric on Σ and $e^{2v}h_N$ the induced metric on the normal bundle. Then the flat curvature equations are equivalent to the system*

$$\begin{cases} \bar{\partial}\alpha = 0, \\ \bar{\partial}\beta = 0, \\ \Delta u = e^{2u} - 1 + e^{-2u}(e^{2v}|\alpha|^2 + e^{-2v}|\beta|^2), \\ \Delta v = c - e^{-2u}(e^{2v}|\alpha|^2 - e^{-2v}|\beta|^2). \end{cases} \tag{2-15}$$

Proof. First, as $B = -\mathbb{I}^\dagger$, we have the explicit description

$$B = \begin{pmatrix} -\alpha^{\dagger,*} & -\beta^{\dagger,*} \\ -\beta^\dagger & -\alpha^\dagger \end{pmatrix}. \tag{2-16}$$

Now the flat curvature equation is $F_{\nabla_E} = 0$, so coordinatewise,

$$\begin{cases} \nabla\alpha = 0, \\ \nabla\beta = 0, \\ F_{K^{-1}} + aa^* - \alpha^{*,\dagger}\alpha - \beta^{*,\dagger}\beta = 0, \\ F_N - \alpha\alpha^{*,\dagger} - \beta^*\beta^\dagger = 0, \end{cases} \tag{2-17}$$

where we wrote $\alpha\alpha^*$ for the wedge product $\alpha \wedge \alpha^*$. Also, the notation $\nabla\alpha = 0$ means that α is parallel for the connection on $\mathrm{Hom}(K^{-1}, N)$, namely $\nabla_N \wedge \alpha + \alpha \wedge \nabla_{K^{-1}} = 0$.

The standard formula for the curvature under a conformal change of metric is

$$\begin{cases} iF_{K^{-1}} = e^{-2u}(-\Delta u - 1)\omega, \\ iF_N = e^{-2u}(-\Delta v + c)\omega. \end{cases} \tag{2-18}$$

Now recall that as α is a $(1,0)$-form, and locally $dz \wedge d\bar{z} = -2i\,dx \wedge dy$, we have

$$\begin{cases} iaa^* = \omega, \\ i\alpha\alpha^* = e^{-4u}e^{2v}|\alpha|^2\omega, \\ i\beta\beta^* = e^{-4u}e^{-2v}|\beta|^2\omega. \end{cases} \tag{2-19}$$

So the flat curvature equation is equivalent to

$$\begin{cases} e^{-2u}(\Delta u + 1) = 1 + e^{-4u}(e^{2v}|\alpha|^2 + e^{-2v}|\beta|^2), \\ e^{-2u}(\Delta v - c) = e^{-4u}(-e^{2v}|\alpha|^2 + e^{-2v}|\beta|^2). \end{cases} \tag{2-20}$$

Also, as α is of type $(1,0)$, necessarily $\partial\alpha = 0$. Hence, $\nabla\alpha = 0$ rewrites as $\bar{\partial}\alpha = 0$, and the same applies to β. Finally, rearranging the right- and left-hand sides, we get the scalar PDE system

$$\begin{cases} \bar{\partial}\alpha = 0, \\ \bar{\partial}\beta = 0, \\ \Delta u = -1 + e^{2u} + e^{-2u}(e^{2v}|\alpha|^2 + e^{-2v}|\beta|^2), \\ \Delta v = c - e^{-2u}(e^{2v}|\alpha|^2 - e^{-2v}|\beta|^2), \end{cases} \tag{2-21}$$

as desired. □

Remark. The Higgs bundle associated to f can be read with our notation. It will be

$$\mathcal{E} = \mathcal{O} \oplus K^{-1} \oplus K \oplus N \oplus N^{-1},$$

and the holomorphic structure with the Higgs field has the explicit description

$$\bar\partial_{\mathcal{E}} = \begin{pmatrix} \bar\partial & 0 & 0 & 0 & 0 \\ 0 & \bar\partial & 0 & -\alpha^* & -\beta^* \\ 0 & 0 & \bar\partial & 0 & 0 \\ 0 & 0 & \beta^* & \bar\partial & 0 \\ 0 & 0 & \alpha^* & 0 & \bar\partial \end{pmatrix}, \quad \Phi_{\mathcal{E}} = \begin{pmatrix} 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ a^* & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2-22}$$

Recall that we chose N so that $\deg N \geq 0$. Observing that $\mathcal{O} \oplus K^{-1} \oplus N$ is an invariant subbundle, the stability condition implies

$$0 \leq \deg N \leq 2g - 2. \tag{2-23}$$

Proposition 2.4. *Let f be a minimal map with $\alpha = 0$. Then $\deg N = 0$ and f is totally geodesic.*

Proof. As $\alpha = 0$, N is an invariant subbundle of the corresponding Higgs bundle to f. Then, by semistability, $\deg N = 0$. Now if $\beta \neq 0$ then v is solution of

$$\Delta v = e^{-2u} e^{-2v} |\beta|^2. \tag{2-24}$$

Integrating this identity implies

$$\int_\Sigma e^{-2u} e^{-2v} |\beta|^2 = 0. \tag{2-25}$$

Hence β is the zero section and f is totally geodesic, as asserted. □

We will be interested in the specific case when the second fundamental form, viewed as a map $T_{\mathbb{R}} \to (T_{\mathbb{R}})^*$, has *circular ellipse of curvature*, i.e., the image of the unit circle in $T_{\mathbb{R}}$ by the map $X \mapsto \mathbb{I}(X, X)$ is a circle. This corresponds to asking the $(2, 0)$ and $(0, 2)$ parts of $\mathbb{I}_{\mathbb{C}}$ to be orthogonal.

Definition 2.5 [Loftin and McIntosh 2019]. We say that f is "superminimal" if f is harmonic, conformal and its holomorphic second fundamental form $\alpha \oplus \beta$ satisfies

$$\alpha \cdot \beta = 0 \in H^0(K^4). \tag{2-26}$$

Remark. The notion of superminimal maps goes back to [Kommerell 1897], which studied superminimal immersions in $\mathbb{S}^4$. See [Forstnerič 2021] for a recent review of the topic. Another interpretation of the superminimality is that it is equivalent to asking that the normal Gauss map, valued in the Grassmannian of geodesic discs in $\mathbb{H}^4$, is conformal.

Depending on the degree of N, superminimal maps behave differently:

Proposition 2.6. *Let f be a superminimal map with $\deg N = 0$, equivariant under a faithful and discrete representation $\rho : \pi_1\Sigma \to \mathrm{SO}(4, 1)$, Σ being a closed surface. Then f is totally geodesic and ρ is a fuchsian representation.*

Proof. While this proposition can be found in [Loftin and McIntosh 2019], we reproduce it here as it can be seen as a consequence of our PDE system: As the normal bundle is trivial, we have $c = 0$ and the

superminimality implies $\alpha = 0$ or $\beta = 0$. For convenience's sake, assume $\beta = 0$. Then (u, v) must be a solution of the system

$$\begin{cases} \Delta u = -1 + e^{2u} + e^{-2u} e^{2v} |\alpha|^2, \\ \Delta v = -e^{2u} e^{2v} |\alpha|^2. \end{cases} \tag{2-27}$$

Fix u, and observe that, on a closed Riemann surface, we must have $\int \Delta v = 0$. Hence, $\alpha = 0$, the immersion f is totally geodesic and ρ is fuchsian. □

2.3. ***Almost-fuchsian discs in*** $\mathbb{H}^4$**.** First introduced in [Uhlenbeck 1983] for immersions into $\mathbb{H}^3$, the notion of almost-fuchsian immersion naturally generalizes to immersions into $\mathbb{H}^4$.

Definition 2.7. Let $f : \mathbb{D} \hookrightarrow \mathbb{H}^4$ be a proper immersion. Then f is said to be *almost-fuchsian* if it is minimal and there is $\delta > 0$ such that the second fundamental form satisfies

$$|\mathbb{I}(u, u)| \leq (1 - \delta)|u|^2 \quad \text{for all } u \in T_f. \tag{2-28}$$

Proposition 2.8. *Let $f : \mathbb{D} \to \mathbb{H}^4$ be a proper almost-fuchsian immersion. Then the exponential map $N_f \to \mathbb{H}^4$ is a diffeomorphism.*

As this constitutes, with the following result, [Jiang 2025, Lemma 4], we won't reproduce a proof here. Note that Jiang's convention for the second fundamental form differs from ours by a factor one half, hence the difference in the condition. More specifically, almost-fuchsian discs in $\mathbb{H}^4$ are embeddings, bound by a quasicircle at infinity, and if they are ρ-equivariant for some $\rho : \Gamma \to \mathrm{SO}(4, 1)$ discrete and faithful, ρ is convex-cocompact.

Proposition 2.9. *Let Σ be a closed hyperbolic surface, and $\rho : \pi_1 \Sigma \to \mathrm{SO}(4, 1)$ a faithful and discrete representation such that there exists $f : \mathbb{D} \hookrightarrow \mathbb{H}^4$ a ρ-equivariant proper almost-fuchsian immersion. Then ρ is convex-cocompact, f is an embedding and the hyperbolic 4-manifold $M = \rho \backslash \mathbb{H}^4$ is diffeomorphic via $\exp_f$ to a disc bundle over Σ. Also, f is the unique ρ-equivariant minimal immersion.*

Definition 2.10. A representation $\rho : \Gamma \to \mathrm{SO}(4, 1)$ is said to be *almost-fuchsian* if it is discrete, faithful and it admits a proper almost-fuchsian ρ-equivariant immersion $f : \widetilde{\Sigma} \to \mathbb{H}^4$.

It is not known which disc bundles can be uniformized by almost-fuchsian representations. Fuchsian representations uniformize the trivial bundle, and we will show in this paper that for a surface of large enough genus, there are almost-fuchsian representations uniformizing vector bundles with positive degree over the surface.

Proposition 2.11. *Let Σ be a closed hyperbolic surface of genus g. Denote its volume form by w. Consider N a line bundle of degree $d \geq 0$, endowed with a metric h of curvature form*

$$c\omega = \frac{d}{2g-2}\, \omega.$$

Let $\alpha \in H^0(K^2N)$ be such that there exist smooth functions u, v on Σ satisfying

$$\Delta u = -1 + e^{2u} + e^{-2u}e^{2v}|\alpha|^2, \tag{2-29}$$

$$\Delta v = c - e^{-2u}e^{2v}|\alpha|^2, \tag{2-30}$$

$$\sup e^{-4u}e^{2v}|\alpha|^2 < 1. \tag{2-31}$$

Then there is a convex-cocompact representation $\rho : \pi_1\Sigma \to \mathrm{SO}(4,1)$ and a minimal, superminimal, ρ-equivariant and almost-fuchsian immersion $f : \mathbb{D} \to \mathbb{H}^4$ such that

(1) *the induced metric by f is $e^{2u}w$,*

(2) *the hyperbolic manifold $\rho\backslash\mathbb{H}^4$ identifies with the normal bundle to f and has degree d,*

(3) *the metric induced on the normal bundle is $e^{2v}h$,*

(4) *the holomorphic second fundamental form of f is α.*

Proof. With the given data, consider the bundle $\mathcal{E} = \mathcal{O} \oplus K^{-1} \oplus K \oplus N \oplus N^{-1}$, with the metric achieved by product of the metrics on each line. Adding α and its h-dual α^*, the two equations are exactly the flat curvature equation, corresponding to a representation ρ and a map $f : \mathbb{D} \to \mathbb{H}^4$ whose holomorphic second fundamental form satisfies

$$(\mathbb{I}_f)_{\mathbb{C}} = \alpha \oplus 0 \in \Gamma(K^2(N \oplus N^{-1})). \tag{2-32}$$

It is clear that such an f is minimal and superminimal. The last hypothesis

$$\sup e^{-4u}e^{2v}|\alpha|^2 < 1 \tag{2-33}$$

ensures that f and ρ are almost-fuchsian, as asserted. □

3. Balanced holomorphic sections of positive line bundles

In this section, we show that there are families of holomorphic sections of line bundles of degree $4g-3$ which behave roughly like covers in the following sense:

Definition 3.1 (balanced family). Let (Σ_g) be a family of genus g Riemann surfaces, equipped with their hyperbolic metric. Let N_g be a family of line bundles of degree $n_g \geq 0$. Let (α_g) be a family of holomorphic sections of N_g.

On each Σ_g, denote by ω_g the Poincaré volume form. On each N_g, denote by h_g the metric on N_g with curvature form

$$\frac{n_g}{2g-2}\omega_g.$$

We say that (α_g, N_g) is *balanced* if there is $C > 0$ such that

$$\sup_g \frac{\sup_{x\in X_g} |\alpha_g|^2_{\omega_g,h_g}}{⨍_{X_g} |\alpha_g|^2_{h_g} d\omega_g} \leq C. \tag{3-1}$$

The main idea behind this control is that when solving elliptic PDEs depending on α_g, as α_g is coarsely periodic, the solutions should be too, and we will show that the solutions satisfy estimates which are independent of the genus g.

We will make extensive use of the following notation:

Notation 3.2. In this section, Σ denotes a closed hyperbolic surface of genus g. We denote by δ and Λ the systole and spectral gap of Σ, respectively. Any N positive line bundle over Σ is endowed with a metric of curvature form proportional to the Poincaré volume form ω, i.e., whose curvature form satisfies

$$iF(N, h_N) = \frac{\deg(N)}{2g-2}\omega. \tag{3-2}$$

Remark. It is easy to exhibit examples of balanced sections for n_g proportional to $g-1$. Indeed, if $n_g = a(g-1)$, consider N_2 a degree a line bundle over Σ_2, and α a holomorphic section of N_2. and consider covers $\Sigma_g \to \Sigma_2$. Then the family of lifts of N_2 and α yields a balanced family.

Here we will prove the existence of balanced families for line bundles of degree $4g-3$:

Theorem 3.3. *Let Σ be a closed hyperbolic surface of genus $g \geq 2$. Consider (Σ_n) a family of covers of Σ of degree n. Then there is a balanced family (α_n, N_n), with N_n a line bundle of degree $(4g-4)n+1$ over Σ_n.*

The main idea will be to consider covers to get a balanced section of a degree $4n(g-1)$ line bundle, and then to tensorize with a well-chosen degree 1 line bundle to get the desired family of sections. This requires a description of the oscillatory behavior of a degree 1 bundle section, which we provide here.

Before all, we will make use of the continuous embedding $W^{2,2} \hookrightarrow L^\infty$.

Lemma 3.4. *Let Σ be a hyperbolic surface with systole δ. Let M be the constant of continuity of the embedding $W^{2,2}(\mathbb{H}^2) \hookrightarrow L^\infty(\mathbb{H}^2)$. Then, for any $u \in W^{2,2}(\Sigma)$,*

$$|u|_\infty \leq \frac{M}{\delta}|u|_{2,2}. \tag{3-3}$$

Proof. As smooth maps are dense in $W^{2,2}(\Sigma)$, consider $u \in W^{2,2}(\Sigma)$ smooth. Let $z_0 \in \Sigma$. By definition of the systole, there is an isometric embedding from a hyperbolic disc of radius δ around z_0. Fix χ a smooth radial map on $D(z_0, \delta)$, $1/\delta$ Lipschitz, between 0 and 1, such that $\chi(z_0) = 1$ and $\chi(\partial D(z_0, \delta)) = 0$. Choose for χ a rescaling of a cutoff map on $D(0,1) \subset \mathbb{H}^2$, so that

$$\sup D\chi = \sup D^2\chi = \frac{1}{\delta}. \tag{3-4}$$

Then we can extend by zero to consider $u\chi \in W^{2,2}(\mathbb{H}^2)$. Using our cutoff function, we get

$$|u(z_0)| = |u\chi(z_0)| \leq |u\chi|_\infty \leq M|u\chi|_{2,2} \leq \frac{M}{\delta}|u|_{2,2}, \tag{3-5}$$

as asserted. □

Proposition 3.5. *Let $z_0 \in \Sigma$. Denote by $D = D(z_0, \delta)$ the disc of radius δ around z_0. Denote by a the constant $a = 2\pi/\mathrm{Vol}(D)$. Consider $k : \Sigma \to \mathbb{R}$ the C^1 function satisfying, in the weak sense,*

$$\begin{cases} \displaystyle\int_\Sigma k = 0, \\ \Delta k = \dfrac{1}{2g-2} - a\mathbf{1}_D. \end{cases} \tag{3-6}$$

Then there is a bound $C(\delta, \Lambda)$ such that

$$|\sup k - \inf k| \le C(\delta, \Lambda). \tag{3-7}$$

Proof. As $a = 2\pi/\mathrm{Vol}\, D$,

$$\int_\Sigma \frac{1}{2g-2} - a\mathbf{1}_D = 0. \tag{3-8}$$

It is clear that $1/(2g-2) - a\mathbf{1}_D \in L^2(\Sigma)$. Hence there exists a unique $k \in W^{2,2}(\Sigma)$ satisfying the stated conditions. From the Sobolev embedding $W^{2,2} \hookrightarrow C^{0,\alpha}(\Sigma)$, we get the k is Hölder-continuous, and hence in particular it belongs to $L^p(\Sigma)$ for any $p > 0$. From the L^p-interior estimates (see [Gilbarg and Trudinger 1977, Theorem 9.11]) it follows that k belongs to $W^{2,p}(\Sigma)$ for any $p > 1$. From the Sobolev embedding $W^{2,p} \hookrightarrow C^{1,\alpha}$, provided $\alpha < (p-2)/p$, we deduce that k is of regularity $C^{1,\alpha}$ for any $\alpha < 1$.

We now compute an estimate of the sup norm of k. By construction $k \in W^{2,2}(\Sigma)$, so it admits a Hessian in the distributional sense. First, we have $\int k\Delta k = -|\nabla k|_2^2$, so

$$|\nabla k|_2^2 = a\int_D k \le 2\pi \sup g. \tag{3-9}$$

Recall Bochner's identity

$$\int |\nabla^2 k|^2 = \int (\Delta k)^2 + \int |\nabla k|^2, \tag{3-10}$$

which translates to give the estimate

$$|\nabla^2 k|_2^2 \le \frac{4\pi^2}{\mathrm{Vol}(D)} - \frac{2\pi}{2g-2} + 2\pi \sup k. \tag{3-11}$$

As $\int k = 0$, by definition of the spectral gap we have

$$|k|_2^2 \le \Lambda |\nabla k|_2^2 \le 2\pi\Lambda \sup k. \tag{3-12}$$

Combining all this, we bound the $W^{2,2}$-norm

$$|k|_{2,2}^2 = |\nabla^2 k|_2^2 + |\nabla k|_2^2 + |k|_2^2 \le \frac{4\pi^2}{\mathrm{Vol}(D)} - \frac{2\pi}{2g-2} + 2\pi(2+\Lambda)\sup k. \tag{3-13}$$

By Lemma 3.4, this implies

$$|\sup k - \inf k| \le \frac{2M}{\delta}\left(\frac{4\pi^2}{\mathrm{Vol}(D)} - \frac{2\pi}{2g-2} + 2\pi(2+\Lambda)\sup k\right)^{1/2}. \tag{3-14}$$

Because $\int k = 0$, we know that $\inf k \leq 0$ and $\sup k \geq 0$. Thus,

$$0 \leq (\sup k)^2 \leq \frac{4M^2}{\delta^2}\left(\frac{4\pi^2}{\mathrm{Vol}(D)} - \frac{\pi}{g-1} + 2\pi(2+\Lambda)\sup k\right). \tag{3-15}$$

Going to the squares, this is a control of the type $X^2 \leq AX + B$, with $A, B > 0$. Explicitly, A and B are given by

$$\begin{cases} A = \dfrac{8\pi(2+\Lambda)M^2}{\delta^2}, \\ B = \dfrac{4M^2}{\delta^1}\left(\dfrac{4\pi^2}{\mathrm{Vol}(D)} - \dfrac{\pi}{g-1}\right), \end{cases} \tag{3-16}$$

so it implies the existence of $C(\delta, \Lambda) = \frac{1}{2}\left(A + \sqrt{A+4B}\right)$ such that

$$\sup k \leq C(\delta, \Lambda)s. \tag{3-17}$$

Plugging this back into the inequality (3-14), we get the existence of $\widetilde{C}(\delta, \Lambda)$ satisfying

$$|\sup k - \inf k| \leq \widetilde{C}(\delta, \Lambda), \tag{3-18}$$

as asserted. □

It happens that the oscillations of k away from $D(z_0, r)$ are actually the same as those of the log of the section s associated to the line bundle of divisor $\{z_0\}$.

Proposition 3.6. *Let N be a degree 1 line bundle over Σ and $s \in H^0(N)$ a section with a unique zero z_0. Denote by $D = D(z_0, \delta)$ the disc of radius δ around z_0. Set $a = 2\pi/\mathrm{Vol}(D)$. Consider $k : \Sigma \to \mathbb{R}$ the function associated to z_0 defined in Proposition 3.5. Denote by f the function $f = -\frac{1}{2}\log|s|^2$. Then $f - k$ is bounded on Σ and constant on $\Sigma - D(z_0, \delta)$.*

Remark. As s is well-defined up to a multiplicative constant, we can choose the value of $f - k$ on $\Sigma - D(z_0, \delta)$ at will.

Proof. Let r be the radial geodesic coordinate on $D(z_0, \delta)$. With the notation above, define the map $h : D \to \mathbb{R}$ by

$$h(r) = -2a \ln\cosh\left(\frac{r}{2}\right) + b + \ln\left(\tanh\frac{r}{2}\right) \tag{3-19}$$

for some b to be specified later in the proof. As h is radial, we can explicitly compute the Laplacian of h:

$$\Delta h = \partial_r^2 h + \frac{1}{\tanh r}\partial_r h = -a. \tag{3-20}$$

We also have

$$|z - z_0| = \tanh\left(\frac{r}{2}\right). \tag{3-21}$$

Because z_0 is a simple zero of s, locally

$$|s(z)|^2 \approx K|z - z_0|^2, \quad f(z) = -\ln\tanh\frac{r}{2} + \mathcal{O}(1). \tag{3-22}$$

And so $f + h$ is bounded.

Also, we compute the radial derivative at the boundary of the disc of h:

$$\partial_r h(\delta) = \left(\frac{1}{2} - a\right) \tanh \frac{\delta}{2} + \frac{1}{2\tanh(\delta/2)}. \tag{3-23}$$

But by definition of a,

$$a = \frac{2\pi}{\mathrm{Vol}(D)} = \frac{2\pi}{4\pi \sinh^2(r/2)} = \frac{1}{2\sinh^2(r/2)}, \tag{3-24}$$

which implies

$$\partial_r h(\delta) = 0. \tag{3-25}$$

Hence, with the appropriate choice of b, we can extend h by zero to get a C^1 function on Σ such that $f + h$ is bounded and satisfies weakly

$$\Delta(f + h) = \Delta f - a\mathbf{1}_D. \tag{3-26}$$

It remains to make Δf explicit. Since s is a holomorphic section, f can be expressed from the curvature form of the metric h_N on N. Denoting $H = h(s, s)$ and using local holomorphic coordinates on S, one has the formula (cf. [Huybrechts 2005, Chapter 4, Proposition 4.2.14 and Example 4.2.15]) for the Chern connection which is locally $d + \bar{\partial} \ln H$. From [Huybrechts 2005, Lemma 4.3.4], we then compute the curvature of that connection:

$$F(h) = d\partial \ln(H) + \partial \ln(H) \wedge \partial \ln(H) = \tfrac{1}{4}\Delta^e \ln |s|^2 \, d\bar{z} \wedge dz, \tag{3-27}$$

where Δ^e is the Laplacian with regard to the metric $dz\, d\bar{z}$. Note that the second term is zero since it is a $(2, 0)$-form on a Riemann surface. Keeping in mind that $d\bar{z} \wedge dz = 2i\, dx \wedge dy$, we get the formula

$$F(h_N) = \tfrac{1}{2}\Delta^e \ln |s|^2 \cdot i\, dx \wedge dy. \tag{3-28}$$

Finally, with regard to the hyperbolic metric on the Riemann surface which is locally $e^{\varphi}\, dz\, d\bar{z}$, the hyperbolic Laplacian is $\Delta = e^{-\varphi}\Delta^e$ while the volume form is $\omega = e^{\varphi}\, dx \wedge dy$. All in all, this shows that

$$F(h_N) = \tfrac{1}{2}\Delta \ln |s|^2 \cdot i\omega. \tag{3-29}$$

But we chose the Hermitian metric h_N so that its curvature form satisfies

$$iF(h_N) = \frac{\deg(N)}{2g - 2}\omega. \tag{3-30}$$

Since $f = -\frac{1}{2}\ln |s|^2$, we deduce that $\Delta f = 1/(2g - 2)$, and $f + h$ satisfies

$$\Delta(f + h) = \frac{1}{2g-2} - a\mathbf{1}_D. \tag{3-31}$$

Satisfied weakly, it is in particular true in $L^2(\Sigma)$. This implies that $f + h = k + A$ for some constant A. As h is constant out of D, this means that $f - k$ is constant on $\Sigma - D$, as asserted. □

Corollary 3.7. *Let Σ be a hyperbolic surface with systole δ and spectral gap Λ. Let N be a degree 1 line bundle over Σ and $\alpha \in H^0(N)$ section, with a unique zero z_0. Let $D = D(z_0, \delta)$ be the disc of radius δ around z_0. Then there is some $C(\delta, \Lambda)$ and $\lambda > 0$ such that*

$$\frac{1}{C(\delta, \Lambda)} \leq \lambda |\alpha|^2 \leq C(\delta, \Lambda) \quad \textit{for all } z \in \Sigma - D(z_0, \delta). \tag{3-32}$$

Proof. With the g defined in Proposition 3.5, we know

$$|\sup g - \inf g| \leq C(\delta, \Lambda). \tag{3-33}$$

Let $f = -\frac{1}{2} \ln |\alpha|^2$. By Proposition 3.6, $f - g$ is constant on $\Sigma - D$. Thus

$$\left| \sup_{\Sigma - D} f - \inf_{\Sigma - D} f \right| = \left| \sup_{\Sigma - D} g - \inf_{\Sigma - D} g \right| \leq C(\delta, \Lambda), \tag{3-34}$$

which directly gives, on $\Sigma - D$,

$$\frac{\sup |\alpha|^2}{\inf |\alpha|^2} \leq e^{2C(\delta, \Lambda)}. \tag{3-35}$$

Multiply by λ such that $(\sup \lambda |\alpha|^2)(\inf \lambda |\alpha|^2) = 1$ to get

$$e^{-C(\delta, \Lambda)} \leq \lambda |\alpha|^2(z) \leq e^{C(\delta, \Lambda)} \quad \text{for all } z \in \Sigma - D, \tag{3-36}$$

as desired. □

It remains to check that we can also have an upper bound on D. We can do this by some version of the Schwarz lemma adapted to sections of positive line bundles:

Proposition 3.8. *Let Σ be a hyperbolic surface with systole δ and spectral gap Λ. Let N be a degree 1 line bundle over Σ and $\alpha \in H^0(N)$ a section with a unique zero z_0. Let $D = D(z_0, \delta)$, the disc of radius δ around z_0. Then there is $C(\delta) > 0$ satisfying*

$$|\alpha|^2(z) \leq C(\delta) |z|^2 \sup_{\partial D} |\alpha|^2 \quad \textit{for all } z \in D. \tag{3-37}$$

Proof. Let $f = -\frac{1}{2} \ln |\alpha|^2$. Then f satisfies

$$\Delta f = \frac{1}{2g - 2}. \tag{3-38}$$

Let r be the radial geodesic coordinate on D, and consider the map

$$h = f + \ln \tanh \frac{r}{2} - \frac{2}{2g - 2} \ln \cosh \frac{r}{2}. \tag{3-39}$$

Then h is bounded on D and satisfies

$$\Delta h = 0. \tag{3-40}$$

By the minimum principle,

$$\inf_D h = \inf_{\partial D} h = \inf_{\partial D} f + \ln \tanh \frac{\delta}{2} - \frac{1}{g - 1} \ln \cosh \frac{\delta}{2}. \tag{3-41}$$

Exponentiating, we get the upper bound on $|\alpha|$:

$$\frac{|\alpha|^2}{|z|^2}\cosh\left(\frac{r}{2}\right)^{2/(g-1)} \leq \sup_{\partial D}|\alpha|^2\frac{(\cosh(\delta/2))^{2/(g-1)}}{(\tanh(\delta/2))^2} \quad \text{for all } z\in D. \tag{3-42}$$

Because $g>1$ and $\cosh(t)\geq 1$, this implies

$$|\alpha|^2(z)\leq |z|^2\sup_{\partial D}|\alpha|^2\frac{(\cosh(\delta/2))^2}{(\tanh(\delta/2))^2}, \tag{3-43}$$

as required. □

Finally, we combine these results to show our estimate of oscillations of sections of degree 1 line bundles:

Proposition 3.9. *Let Σ be a hyperbolic surface. Denote by δ its systole, and by Λ its spectral gap. Consider $r<\delta/2$. There are constants $C_i(\delta,\Lambda,r)$ such that, for any holomorphic section s of a degree 1 line bundle with a unique zero $z_0\in\Sigma$, there is a $\lambda>0$ such that*

$$|\lambda s|^2\leq C_2 \qquad \text{on } B(z_0,r), \tag{3-44}$$

$$\frac{1}{C_1}\leq|\lambda s|^2\leq C_1 \quad \text{on } \Sigma - B(z_0,r). \tag{3-45}$$

Remark. Up to renormalizing our section, one can always assume the first condition to be satisfied on the ball of radius r, and then the result states that the oscillations away from the zero of the section are controlled, essentially by the spectral gap and the systole of the surface.

Proof. Let Σ be a hyperbolic surface with systole δ and spectral gap Λ. Let N be a degree 1 line bundle over Σ, with $\alpha\in H^0(N)$ a holomorphic section having one unique zero z_0. Consider $D=D(z_0,\delta)$. Thanks to Corollary 3.7, up to renormalizing α, there is $C(\delta,\Lambda)>0$ such that

$$\frac{1}{C(\delta,\Lambda)}\leq|\alpha|^2(z)\leq C(\delta,\Lambda) \quad \text{for all } z\in\Sigma-D. \tag{3-46}$$

Now because of Proposition 3.8, there is $C(\delta)$ such that

$$|\alpha|^2\leq C(\delta)C(\delta,\Lambda) \quad \text{for all } z\in D, \tag{3-47}$$

as asserted. □

Multiplying those sections with covers of a section of a degree $4(g-1)$ line bundle will result in a balanced section of a bundle of the desired degree $4n(g-1)+1$.

Proof of Theorem 3.3. Let Σ be a closed hyperbolic surface of genus $g\geq 2$. Consider Σ_n a family of degree n covers of Σ such that the spectral gap of Σ_n remains lower bounded by $\Lambda>0$. This kind of cover exists; for instance we can use the result of [Magee et al. 2022] which states that for a random cover of Σ, the probability that the relative spectral gap (i.e., the part of the spectral gap which is not in Σ) of the cover is bigger than $\frac{3}{16}-\varepsilon$ goes to 1 when the degree is large. Hence, for any $\eta<\min\left\{\Lambda,\frac{3}{16}\right\}$, we can choose a family of covers whose spectral gap is larger than η.

Of course, the systole is nondecreasing by taking a finite cover, and hence we can consider δ the systole of Σ.

Consider s a holomorphic section of K^2 over Σ which is of degree $4g-4$. Denote by s_n the lift of s to Σ_n, which is a section of a degree $4n(g-1)$ line bundle. Obviously, we have

$$\int |s_n|^2 = n \int |s|^2, \quad |s_n|_\infty = |s|_\infty. \tag{3-48}$$

Also, consider (z_n) a sequence of points $z_n \in \Sigma_n$. Define $L_n = \mathcal{O}(z_n)$ the degree 1 line bundle associated to the divisor $\{z_n\}$.

Thanks to Proposition 3.9, there are sections τ_n of L_n, and constants $C_i(\delta, \Lambda)$ such that

$$\begin{cases} \dfrac{1}{C_1(\delta,\Lambda)} \le |\tau_n|^2 \le C_1(\delta,\Lambda) & \text{out of } D(z_n,\delta), \\ |\tau_n|^2 \le C_2(\delta,\Lambda) & \text{on } D(z_n,\delta). \end{cases} \tag{3-49}$$

Consider now $N_n = K^2 L_n$ and $\alpha_n = s_n \tau_n$ a section of N_n line bundle of degree $4n(g-1)+1$. By construction, we have the estimates

$$\frac{1}{C_1(\delta,\Lambda)} |s_n|_\infty^2 \le |\alpha_n|_\infty^2 \le C_2(\delta,\Lambda) |s_n|_\infty^2, \tag{3-50}$$

$$\frac{1}{C_1(\delta,\Lambda)} \int_{\Sigma_n - D(z_n,\delta)} |s_n|^2 \le \int |\alpha_n|^2. \tag{3-51}$$

Considering that

$$|s_n|_\infty = |s|_\infty, \tag{3-52}$$

$$\int_{\Sigma_n - D(z_n,\delta)} |s_n|^2 \ge (n-1) \int_\Sigma |s|^2, \tag{3-53}$$

we eventually obtain

$$\frac{|\alpha_n|_\infty^2}{⨍ |\alpha_n|^2} \le \frac{n}{n-1} \frac{|s|_\infty^2}{\int |s|^2} C_1(\delta,\Lambda) C_2(\delta,\Lambda) \operatorname{Vol}(\Sigma), \tag{3-54}$$

which is bounded in n, so the family (N_n, α_n) is balanced. □

Remark. In a broader setting, given a section s of a degree d line bundle, we expect the ratio between the L^2 average and the L^∞ norm to be a quantitative indicator of how much s looks like a lift of a section on a smaller surface. With this in mind, we expect this ratio to be controlled by the lowest distance between the zeroes of s. This would have the advantage of being an open property, and that when it is actually possible, the minimal ratio is achieved by a lift.

Remark. The same proof can be adapted to show that for any $a, d \ge 0$, we can build a balanced family $(\Sigma_g, N_g, \alpha_g)$ with N_g of degree $a(g-1)+d$.

4. Estimates on the solutions to the flat curvature equations

Throughout this section, Σ will be a closed hyperbolic surface of genus $g \geq 2$. Unless otherwise stated, N will be a degree 1 line bundle over Σ and α will be a holomorphic 2-form valued in N, i.e., $\alpha \in H^0(K^2N)$. Fix $c = 1/(2g-2)$, and consider the PDE system

$$\begin{cases} \Delta u = e^{2u} - 1 + e^{-2u}e^{2v}|\alpha|^2, \\ \Delta v = c - e^{-2u}e^{2v}|\alpha|^2. \end{cases} \tag{4-1}$$

In this part, we consider each equation separately, considering the first as giving u depending on v, and vice versa for the second.

4.1. ***Dealing with Gauss's equation.*** Fixing a map u, the first equation is the famous Gauss equation, and is well-behaved. As the data v and α are fixed, we set $f = e^{2v}|\alpha|^2$, and our problem boils down to solving

$$\Delta u = -1 + e^{2u} + e^{-2u} f. \tag{4-2}$$

Denote by $C^{0,\alpha}(\Sigma)$ the set of continuous α-Hölder functions on Σ. Looking at the results of [Bronstein and Smith 2023], we can apply the same sub-supersolution method to get:

Theorem 4.1. *Let $\eta \in (0, 1]$. Consider the set*

$$\mathcal{C} := \left\{ f \in C^{0,\alpha}(\Sigma) : 0 \leq f \leq \frac{\eta}{(1+\eta)^2} \right\}. \tag{4-3}$$

Then for any $f \in \mathcal{C}$, there is a unique $u \in C^{2,\alpha}(\Sigma)$ satisfying

$$-\tfrac{1}{2}\ln 2 \leq u \leq 0, \tag{4-4}$$

$$\Delta u = e^{2u} - 1 + e^{-2u} f. \tag{4-5}$$

Moreover, u satisfies the upper bound

$$e^{-2u} f \leq \eta. \tag{4-6}$$

Proof. We proceed with the sub-supersolution method; see, for instance, [Evans 2010, Section 9.3]. A subsolution to the equation $\Delta u = F(u)$ is a function v_- satisfying $\Delta v_- \geq F(v_-)$. Analogously, a supersolution is a function v_+ satisfying $\Delta v_+ \leq F(v_+)$.

The sub-supersolution principle states that the existence of a subsolution v_- and a supersolution v_+ satisfying $v_- \leq v_+$ ensures the existence of a solution v satisfying $v_- \leq v \leq v_+$.

Freely, 0 is a supersolution to the equation (4-5). As $f \in \mathcal{C}$, the constant map $-\frac{1}{2}\ln(1+\eta)$ is a subsolution. Hence, by the sub-supersolution principle, there is a solution u satisfying

$$-\tfrac{1}{2}\ln 2 \leq u \leq 0. \tag{4-7}$$

To prove uniqueness, assume u_1 and u_2 are both solutions in $\left[-\frac{1}{2}\ln 2, 0\right]$ with

$$\Delta u_i = e^{2u_i} - 1 + e^{-2u_i} f. \tag{4-8}$$

Then the difference $w = u_1 - u_2$ satisfies

$$\Delta w = (e^{2u_2} - e^{-2u_2} f)(e^{2w} - 1) + f e^{-2u_2}(e^{2w} + e^{-2w} - 2). \tag{4-9}$$

We observe that the second term is nonnegative, and thus

$$\Delta w \geq (e^{2u_2} - e^{-2u_2} f)(e^{2w} - 1). \tag{4-10}$$

But as $f \in \mathcal{C}$ and $u_2 \geq -\frac{1}{2}\ln 2$, we get

$$e^{-4u_2} f \leq 1, \tag{4-11}$$

which implies

$$e^{2u_2} - e^{-2u_2} f \geq 0. \tag{4-12}$$

Hence, by the maximum principle, at a maximum of w,

$$(e^{2u_2} - e^{-2u_2} f)(e^{2w} - 1) \leq 0. \tag{4-13}$$

Hence

$$w \leq 0. \tag{4-14}$$

Exchanging the roles of u_1 and u_2 shows

$$u_1 = u_2. \tag{4-15}$$

So there is a unique solution in $\left[-\frac{1}{2}\ln 2, 0\right]$.

Finally, because $-\frac{1}{2}\ln(1+\eta) \leq u \leq 0$, we deduce

$$e^{-2u} f \leq \frac{\eta}{1+\eta} \leq \eta, \tag{4-16}$$

as asserted. □

Proposition 4.2. *The map $\Phi : C^{0,\alpha}(\Sigma) \to C^{2,\alpha}(\Sigma)$ which to f associates the unique solution u of*

$$\Delta u = e^{2u} - 1 + e^{-2u} f \tag{4-17}$$

satisfying $-\frac{1}{2}\ln 2 \leq u \leq 0$ is continuous on the set $\left\{f \in C^{0,\alpha}(\Sigma) : 0 \leq f < \frac{1}{2}\right\}$.

Proof. Thanks to Theorem 4.1, Φ is well-defined. For f given, consider the linearized equation

$$\Delta \dot{u} = 2(e^{2u} - e^{-2u}) f \dot{u}. \tag{4-18}$$

The assumptions on u imply

$$e^{2u} - e^{-2u} f > 0. \tag{4-19}$$

Denote by L the operator $Lu = -\Delta u + 2(e^{2u} - e^{-2u} f)u$, which is a bounded linear operator $C^{2,\alpha} \to C^{0,\alpha}$. We claim that L is an isomorphism. First, L is surjective. This is a consequence of the positivity of $e^{2u} - e^{-2u}$ and the ellipticity of L; see [Gilbarg and Trudinger 1977, Section 6.3]. Also, L is injective: if $Lv = 0$, then at a maximum of v, we have

$$2(e^{2u} - e^{-2u})v = \Delta v \leq 0. \tag{4-20}$$

Hence we deduce that $v \leq 0$. But since $L(-v) = 0$ too, we get $v = 0$ and L is an isomorphism. By a local inversion principle, we get that the map Φ is continuous. □

4.2. ***Dealing with Ricci's equation.*** Considering the second equation, we will fix v and express a solution v as the maximizer of the following functional:

Notation 4.3. For $u \in C^{1,\alpha}(\Sigma)$, consider $W_0^{1,2}(\Sigma)$ the Sobolev space of zero average functions that are square integrable with square integrable gradient. Define the functional $J_u : W_0^{1,2}(\Sigma) \to \mathbb{R}$ as

$$J_u(w) := \log\left(⨍ e^{-2u}|\alpha|^2 e^{2w}\right) - \frac{1}{c \operatorname{Vol}\Sigma} \int |\nabla w|^2 \tag{4-21}$$

with $c = 1/(2g-2)$. The notation $⨍$ is used for the average of the considered function.

We write $|w|_2^2$ for the squared L^2-norm:

$$|w|_2^2 = \int_\Sigma |w|^2. \tag{4-22}$$

When no confusion is possible, we write J for J_u. We prove that J attains its maximum at a function w which is, up to an additive constant, the solution v that we are looking for. This proof is heavily inspired from the method used to solve the prescribed curvature equation on the sphere when the prescribed data is even. Chang and Yang [2003] used this method in this precise case. Keep in mind that our equation is not a prescribed curvature equation on the surface, but rather a prescribed curvature equation on a disc bundle on it.

Lemma 4.4. *J is well-defined and weakly upper semicontinuous on $W_0^{1,2}(\Sigma)$.*

Proof. We rely on the Moser–Trudinger inequality proved in [Li 2001] on any compact Riemannian surface. There is a constant $C(\Sigma)$ such that, for any $w \in W^{1,2}(\Sigma)$ with zero average,

$$\int_\Sigma \exp\left(\frac{4\pi w^2}{\|\nabla w\|_2^2}\right) \le C(\Sigma). \tag{4-23}$$

To prove that J is well-defined, we dominate e^{2w} in the following way:

$$e^{-2u}|\alpha|^2 e^{2w} \le \max(e^{-2u}|\alpha|^2) \exp\left(\frac{4\pi w^2}{\|\nabla w\|_2^2}\right) \exp\left(\frac{\|\nabla w\|_2^2}{4\pi}\right). \tag{4-24}$$

We deduce that $e^{-2u}|\alpha|^2 e^{2w}$ is integrable, and J_u is well-defined. It remains to prove the weak semicontinuity of J_u.

First, since closed convex sets are weakly closed, the L^2-norm squared of the gradient, being a continuous convex functional on $W^{1,2}(\Sigma)$, is weakly lower semicontinuous, and hence its opposite is upper semicontinuous. We finish this proof by proving that the functional

$$H(w) = \int e^{-2u}|\alpha|^2 e^{2w} \tag{4-25}$$

is weakly continuous. Let $w \in W^{1,2}(\Sigma)$, and $w_n \rightharpoonup w$. Since it is weakly converging, it is bounded, and there is a uniform upper bound $A > 0$ on the sequence $\|\nabla(w_n - w)\|_2^2$. For $t > 0$, $e^{t|w|}$ is integrable since we can control

$$t|w| \le \frac{4\pi w^2}{\|\nabla w\|_2^2} + \frac{t\|\nabla w\|_2^2}{4\pi}, \tag{4-26}$$

and apply the Moser–Trudinger inequality to obtain the upper bound

$$\int_\Sigma e^{t|w|} \leq C(\Sigma) \exp \frac{t\|\nabla w\|_2^2}{4\pi}. \tag{4-27}$$

This yields a constant $C(t)$ such that

$$\int_\Sigma e^{t|w_n - w|} \leq C(t). \tag{4-28}$$

To deduce the weak continuity, we use the bound

$$|e^x - 1| \leq |x| e^{|x|} \tag{4-29}$$

to control the convergence:

$$\int_\Sigma (e^{-2u}|\alpha|^2(e^{2w_n} - e^{2w})) \leq \max e^{-2u}|\alpha|^2 \int_\Sigma e^{2w}|e^{2(w_n - w)} - 1| \tag{4-30}$$

$$\leq \max e^{-2u}|\alpha|^2 \int_\Sigma e^{2w} e^{2|w_n - w|}|w_n - w| \tag{4-31}$$

$$\leq \max e^{-2u}|\alpha|^2 \left(\int e^{4w} e^{4|w_n - w|} \right)^{1/2} \|w_n - w\|_2 \tag{4-32}$$

$$\leq \max e^{-2u}|\alpha|^2 \left(\int e^{8w} \right)^{1/4} C(8)^{1/4} \|w_n - w\|_2. \tag{4-33}$$

In this last inequality all terms are bounded in n apart from $\|w_n - w\|_2$, which goes to zero since weak convergence in $W^{1,2}$ implies strong convergence in $L^2(\Sigma)$. This shows that H is weakly continuous. Hence, the functional J_u is weakly upper semicontinuous, as desired. □

In order to study explicitly the behavior of J, we will need a more precise control on the Sobolev constants of the functions involved, namely, we will use a precise Moser–Trudinger inequality obtained in [Bronstein 2023]. Recall that the space $W_0^{1,2}$ is the space of functions in $W^{1,2}$ whose average is zero.

Theorem 4.5 (Moser–Trudinger, Bronstein). *Denote by δ and Λ the systole and spectral gap of Σ, respectively. There is a constant $C(\delta, \Lambda)$ independent of the genus, depending only on δ, Λ and continuously in those parameters such that*

$$\int |\nabla u|^2 \leq 1 \;\Rightarrow\; \int e^{4\pi u^2} \leq \mathrm{Vol}(\Sigma) + C(\delta, \Lambda) \quad \textit{for all } u \in W_0^{1,2}(\Sigma). \tag{4-34}$$

Corollary 4.6. *For any $u \in W^{1,2}(\Sigma)$, $k \geq 1$ and $t > 0$, $e^{tu} \in L^k(\Sigma)$.*

Proof. Since $(e^{tu})^k = e^{tku}$ and $tku \in W^{1,2}(\Sigma)$, it is enough to prove it for $t = k = 1$. Also, since $e^{x+u} = e^x e^u$, it is sufficient to prove that for any $u \in W_0^{1,2}(\Sigma)$, e^u is integrable. Fixing now $u \in W_0^{1,2}(\Sigma)$ we know from the Moser–Trudinger inequality that for some $r > 0$, e^{ru^2} is integrable. Finally, since $0 < e^u \leq e^{ru^2} e^{1/4r}$, we deduce that e^u is integrable, as claimed. □

Corollary 4.7. *The functional J is upper bounded on $W_0^{1,2}(\Sigma)$, and we have the estimate*

$$\log\left(⨍ e^{-2u}|\alpha|^2\right) \le \sup J \le \log\max e^{-2u}|\alpha|^2 + \log\left(1+\frac{C(\delta,\Lambda)}{\mathrm{Vol}(\Sigma)}\right). \tag{4-35}$$

Also, J is proper on $W_0^{1,2}(\Sigma)$, since

$$J(w) + \frac{1}{4\pi}|\nabla w|_2^2 \le \log\max e^{-2u}|\alpha|^2 + \log\left(1+\frac{C(\delta,\Lambda)}{\mathrm{Vol}(\Sigma)}\right) \quad \textit{for all } w \in W_0^{1,2}(\Sigma). \tag{4-36}$$

Remark. The fact that the functions considered here have zero average is crucial, so that as a consequence of the Poincaré inequality the energy is proper on that space.

Proof. Let $w \in W_0^{1,2}(\Sigma)$. Consider that

$$2w \le 4\pi\frac{w^2}{|\nabla w|_2^2} + \frac{|\nabla w|_2^2}{4\pi}. \tag{4-37}$$

Exponentiating this inequality, we get

$$e^{2w} \le e^{|\nabla w|_2^2/4\pi} e^{4\pi(w/|\nabla w|_2)^2}. \tag{4-38}$$

Using Theorem 4.5 to bound the integral of the right-hand side,

$$\int e^{2w} \le (C(\delta,\Lambda)+\mathrm{Vol}(\Sigma))e^{|\nabla w|_2^2/4\pi}. \tag{4-39}$$

We deduce

$$J(w) \le \log\max e^{-2u}|\alpha|^2 + \left(\frac{1}{4\pi}-\frac{1}{c\,\mathrm{Vol}\,\Sigma}\right)|\nabla w|_2^2 + \log\left(1+\frac{C(\delta,\Lambda)}{\mathrm{Vol}\,\Sigma}\right). \tag{4-40}$$

As $\mathrm{Vol}(\Sigma) = 4\pi(g-1)$ and $c = 1/(2g-2)$, it follows that

$$\frac{1}{4\pi} - \frac{1}{c\,\mathrm{Vol}(\Sigma)} = -\frac{1}{4\pi}. \tag{4-41}$$

Inserting it on the left-hand side, we get

$$J(w) + \frac{1}{4\pi}|\nabla w|_2^2 \le \log\max e^{-2u}|\alpha|^2 + \log\left(1+\frac{C(\delta,\Lambda)}{\mathrm{Vol}(\Sigma)}\right). \tag{4-42}$$

So J is indeed proper and upper bounded with the desired explicit upper bound.

The lower estimate for $\sup J$ is achieved by explicitly calculating $J(0) \le \sup J$. □

Proposition 4.8. *The functional J attains its maximum at $w \in W_0^{1,2}(\Sigma)$ satisfying*

$$\frac{1}{4\pi}|\nabla w|_2^2 \le \log\frac{\max e^{-2u}|\alpha|^2}{⨍ e^{-2u}|\alpha|^2} + \log\left(1+\frac{C(\delta,\Lambda)}{\mathrm{Vol}\,\Sigma}\right). \tag{4-43}$$

Proof. Thanks to Corollary 4.7, we know that J is upper bounded. Consider a maximizing sequence (w_n) of J in $W_0^{1,2}(\Sigma)$. We have the estimate

$$J(w_n) + \frac{1}{4\pi}|\nabla w_n|_2^2 \le \log\max e^{-2u}|\alpha|^2 + \log\left(1+\frac{C(\delta,\Lambda)}{\mathrm{Vol}\,\Sigma}\right). \tag{4-44}$$

As $(J(w_n))$ is upper bounded by $\sup J$, so is $(|\nabla w_n|_2^2)$. We deduce that $(|\nabla w_n|_2^2)$ is bounded. The explicit bound, assuming $J(w_n) \geq J(0)$, is

$$\frac{1}{4\pi}|\nabla w_n|_2^2 \leq \log \frac{\max e^{-2u}|\alpha|^2}{⨍ e^{-2u}|\alpha|^2} + \log\Big(1 + \frac{C(\delta, \Lambda)}{\operatorname{Vol}\Sigma}\Big). \tag{4-45}$$

As (w_n) is bounded in $W_0^{1,2}(\Sigma)$, extracting a subsequence we can assume it converges weakly towards some $w \in W_0^{1,2}(\Sigma)$. By Lemma 4.4, J is weakly continuous, so w is a maximizer of J. As $|\nabla w|_2^2 \leq \limsup |\nabla w_n|_2^2$, we get the desired bound on w. □

Theorem 4.9. *Let N be of degree 1, $\alpha \in H^0(K^2 N)$, $c = 1/(2g-2)$ and $u \in C^{0,\alpha}\big(\Sigma, \big[-\frac{1}{2}\ln 2, 0\big]\big)$. Then there is a solution v to*

$$\Delta v = c - e^{2v} e^{-2u}|\alpha|^2 \tag{4-46}$$

satisfying

$$\frac{1}{4\pi}|\nabla v|_2^2 \leq \log \frac{\max e^{-2u}|\alpha|^2}{⨍ e^{-2u}|\alpha|^2} + \log\Big(1 + \frac{C(\delta, \Lambda)}{\operatorname{Vol}\Sigma}\Big). \tag{4-47}$$

Proof. Thanks to Proposition 4.8, there exists $w \in W_0^{1,2}(\Sigma)$ a maximizer of the functional J. For any $s \in W_0^{1,2}$, we have $D_w J \cdot s = 0$ and so

$$\frac{2}{\operatorname{Vol}(\Sigma)} \frac{\int e^{-2u}|\alpha|^2 e^{2w} s}{⨍ e^{-2u}|\alpha|^2 e^{2w}} - \frac{2}{c\operatorname{Vol}(\Sigma)} \int \nabla w \cdot \nabla s = 0. \tag{4-48}$$

Weakly, this means

$$\frac{1}{c}\Delta w + \frac{e^{-2u}|\alpha|^2 e^{2w}}{⨍ e^{-2u}|\alpha|^2 e^{2w}} \in (L_0^2)^{\perp}. \tag{4-49}$$

As the orthogonal of the zero average functions are the constant maps, there is a constant A such that

$$\Delta w = A - c\frac{e^{-2u}|\alpha|^2 e^{2w}}{⨍ e^{-2u}|\alpha|^2 e^{2w}}. \tag{4-50}$$

Note that since $w \in W_0^{1,2}(\Sigma)$, applying Corollary 4.6, $e^{2w} \in L^2(\Sigma)$. Hence, by [Gilbarg and Trudinger 1977, Theorem 8.8], it follows that $w \in W^{2,2}(\Sigma)$ and the equation (4-50) is satisfied strongly. Via the Morrey embedding $W^{2,2} \subset C^{0,\alpha}$ for any $\alpha < 1$, it follows from (4-50) that Δw is Hölder continuous. By elliptic regularity, we deduce that $w \in C^{2,\alpha}(\Sigma)$, and the equality (4-50) is satisfied in $C^{0,\alpha}(\Sigma)$.

Writing $\int \Delta w = 0$, we deduce

$$A = c. \tag{4-51}$$

Consider the element

$$v = w + \frac{1}{2}\ln\left(\frac{c}{⨍ e^{-2u}|\alpha|^2 e^{2w}}\right) \in W^{1,2}(\Sigma). \tag{4-52}$$

As $\Delta v = \Delta w$, we get

$$\Delta v = c - e^{-2u}|\alpha|^2 e^{2v}. \tag{4-53}$$

As $|\nabla w|_2^2 = |\nabla v|_2^2$, the estimate of Proposition 4.8 ensures the desired bound on $|\nabla v|_2^2$. □

Remark. In contrast with the Gauss equation, we cannot in this case establish the uniqueness of the solution v. We will address this problem by showing stability of the solutions under some conditions after stating some precise estimates on the solution w and v.

4.3. ***Estimating the regularity of the solution.*** In order to apply later a fixed point theorem, we need to understand the behavior of the solutions v. Particularly, we need an L^∞ bound on $e^{2v}|\alpha|^2$. Because of the sign difference with Gauss's equation, applying a maximum principle won't yield bounds on v. We get our upper bound via an estimate of the Hessian of a maximizer w of J_u.

Proposition 4.10. *Assume $u \in C^{0,\alpha}(\Sigma)$. Then there is a constant $C(\delta, \Lambda)$ such that, for any maximizer w of the functional J_u, it satisfies*

$$|\nabla^2 w|_2^2 \le c^2\left((\operatorname{Vol}\Sigma + C(\delta,\Lambda))\left(\frac{\max|\alpha|^2}{⨍|\alpha|^2}\right)^2 - \operatorname{Vol}\Sigma\right) + 4\pi\left(\log\frac{\max e^{-2u}|\alpha|^2}{⨍ e^{-2u}|\alpha|^2} + \log\left(1+\frac{C(\delta,\Lambda)}{\operatorname{Vol}\Sigma}\right)\right). \tag{4-54}$$

Proof. Let w be a maximizer of J. First, we use the Bochner identity to estimate the Hessian of w; see [Jost 2017, Theorem 4.5.1]. We have

$$-\tfrac{1}{2}\Delta(|\nabla w|^2) = -\sigma(\Delta\nabla w, \nabla w) + |\nabla^2 w|^2 + \operatorname{Ric}(\nabla w, \nabla w), \tag{4-55}$$

where σ is the hyperbolic metric and Ric is its Ricci tensor. As the metric is hyperbolic, we have $\operatorname{Ric} = -\sigma$. Integrating this formula yields the identity

$$|\nabla^2 w|_2^2 = \int(\Delta w)^2 + \int|\nabla w|^2. \tag{4-56}$$

Since Δw satisfies

$$\Delta w = c - \frac{c|\alpha|^2 e^{-2u}e^{2w}}{⨍ e^{-2u}|\alpha|^2 e^{2w}}, \tag{4-57}$$

we deduce that

$$\int(\Delta w)^2 = -c^2\operatorname{Vol}\Sigma + c^2(\operatorname{Vol}\Sigma)^2\frac{\int e^{-4u}|\alpha|^4 e^{4w}}{\left(\int e^{-2u}|\alpha|^2 e^{2w}\right)^2}. \tag{4-58}$$

To control the right-hand side, we estimate

$$4w \le \frac{4\pi w^2}{|\nabla w|_2^2} + \frac{|\nabla w|_2^2}{\pi} \tag{4-59}$$

and apply the Moser–Trudinger inequality of Theorem 4.5 to get

$$\int e^{4w} \le (\operatorname{Vol}\Sigma + C(\delta,\Lambda))e^{|\nabla w|_2^2/\pi}. \tag{4-60}$$

This ensures

$$\int|\alpha|^4 e^{-4u}e^{4w} \le (\operatorname{Vol}\Sigma + C(\delta,\Lambda))\max e^{-4u}|\alpha|^4\exp\left(\frac{|\nabla w|_2^2}{\pi}\right). \tag{4-61}$$

It remains to estimate the denominator of the right-hand side of (4-58). As $c\,\mathrm{Vol}\,\Sigma = 2\pi$ and by maximality $J(w) \geq J(0)$, we have

$$\log \fint |\alpha|^2 e^{-2u} \leq \log \fint e^{-2u}|\alpha|^2 e^{2w} - \frac{1}{2\pi}|\nabla w|_2^2. \tag{4-62}$$

Exponentiating this,

$$\int e^{-2u}|\alpha|^2 \exp\Big(\frac{1}{2\pi}|\nabla w|_2^2\Big) \leq \int e^{-2u}|\alpha|^2 e^{2w}. \tag{4-63}$$

Combining this with the upper bound on the numerator, we get

$$\frac{\int e^{-4u}|\alpha|^4 e^{4w}}{\big(\int e^{-2u}|\alpha|^2 e^{2w}\big)^2} \leq \frac{(\mathrm{Vol}\,\Sigma + C(\delta,\Lambda))\max e^{-4u}|\alpha|^4 \exp(|\nabla w|_2^2/\pi)}{\big(\int e^{-2u}|\alpha|^2 \exp(|\nabla w|_2^2/2\pi)\big)^2}. \tag{4-64}$$

Rearranging the right-hand side, we obtain

$$\int (\Delta w)^2 \leq -c^2\,\mathrm{Vol}\,\Sigma + c^2(\mathrm{Vol}\,\Sigma + C(\delta,\Lambda))\frac{(\max e^{-2u}|\alpha|^2)^2}{\big(\fint e^{-2u}|\alpha|^2\big)^2}. \tag{4-65}$$

Combining with the bound on $|\nabla w|_2^2$ from Proposition 4.8, we get the desired result. □

We sum up with the local upper bound on the solution, which will be key to building solutions of the PDE system:

Proposition 4.11. *Let $u : \Sigma \to \mathbb{R}$ be a map in $C^{0,\alpha}(\Sigma)$ such that*

$$1 \leq e^{-2u} \leq 2. \tag{4-66}$$

Then there is a constant $C_1 = C_1\big(\delta, \Lambda, |\alpha|_\infty^2/\fint|\alpha|^2\big)$, and $v \in C^{2,\alpha}(\Sigma)$ a solution of

$$\Delta v = c - e^{-2u}e^{2v}|\alpha|^2 \tag{4-67}$$

satisfying

$$\sup e^{2v}|\alpha|^2 \leq \frac{C_1\big(\delta,\Lambda,|\alpha|_\infty^2/\fint|\alpha|^2\big)}{2g-2}. \tag{4-68}$$

The constant C_1 is uniformly bounded if the parameters δ, Λ are bounded away from zero and $|\alpha|_\infty^2/\fint|\alpha|^2$ is bounded away from infinity.

Proof. Once again, consider $w \in W_0^{1,2}$ a maximizer of J and build the solution v as a translate of w so that

$$w = v - \bar{v}. \tag{4-69}$$

Thanks to Proposition 4.10, there is a bound $C_1\big(\delta,\Lambda,|\alpha|_\infty^2/\fint|\alpha|^2\big)$ such that

$$\|\nabla^2 w\|_2 \leq C_1\bigg(\delta,\Lambda,\frac{|\alpha|_\infty^2}{\fint|\alpha|^2}\bigg). \tag{4-70}$$

But from Proposition 4.8, there is a bound $C_2\big(\delta,\Lambda,|\alpha|_\infty^2/\fint|\alpha|^2\big)$ such that the gradient norm satisfies

$$\|\nabla w\|_2 \leq C_2\bigg(\delta,\Lambda,\frac{|\alpha|_\infty^2}{\fint|\alpha|^2}\bigg). \tag{4-71}$$

Also, since w has zero average, its L^2-norm can be bounded with the help of the Poincaré inequality:

$$\|w\|_2^2 \leq \Lambda C_2\left(\delta, \Lambda, \frac{|\alpha|_\infty^2}{⨍ |\alpha|^2}\right). \tag{4-72}$$

Hence, there is a constant $C_3\big(\delta, \Lambda, |\alpha|_\infty^2/⨍ |\alpha|^2\big)$ such that

$$\|w\|_{2,2} \leq C_3\left(\delta, \Lambda, \frac{|\alpha|_\infty^2}{⨍ |\alpha|^2}\right). \tag{4-73}$$

As the continuous embedding $W^{2,2} \hookrightarrow L^\infty$ depends only on the systole δ, this implies that we have $C_4 > 0$ with the same dependencies such that

$$|w|_\infty \leq C_4\left(\delta, \Lambda, \frac{|\alpha|_\infty^2}{⨍ |\alpha|^2}\right). \tag{4-74}$$

We now compute the average of $e^{2v}|\alpha|^2$:

$$c = ⨍ e^{-2u}e^{2v}|\alpha|^2 = e^{2\bar{v}} ⨍ e^{-2u}e^{2w}|\alpha|^2, \tag{4-75}$$

from which we get the estimate (recalling that $c = 1/(2g-2)$)

$$e^{2\bar{v}}e^{-2C_4} ⨍ |\alpha|^2 \leq \frac{1}{2g-2} \leq 2e^{2\bar{v}}e^{2C_4} ⨍ |\alpha|^2. \tag{4-76}$$

Combining this, we get

$$e^{2v}|\alpha|^2 = e^{2\bar{v}}e^{2w}|\alpha|^2 \leq \frac{e^{4C_4}}{2g-2} \frac{|\alpha|_\infty^2}{⨍ |\alpha|^2}, \tag{4-77}$$

as asserted. As the constant C_1 from Proposition 4.10 is continuous in the parameters, so is the exhibited constant C_4. □

It remains to show the stability of the solutions under suitable perturbations.

Lemma 4.12. *Consider a function $f \geq 0$ in $C^{0,\alpha}(\Sigma)$ such that there is a solution v of*

$$\Delta v = c - e^{2v} f \tag{4-78}$$

satisfying $2e^{2v} f < \Lambda$. Consider the operator $H = \Delta + 2e^{2v} f : W^{2,2}(\Sigma) \to L^2(\Sigma)$. Then H is invertible, and we have the upper bound on its inverse:

$$\|H^{-1}\| \leq \max\{(\Lambda - 2|e^{2v} f|_\infty)^{-1}, c^{-1}\}. \tag{4-79}$$

Proof. H is a self-adjoint operator between Hilbert spaces. Moreover, for λ large enough, we have the estimate

$$|(\lambda I - H)w|_2^2 \, |(\lambda - e^{2v} f)w - \Delta w|_2^2 \tag{4-80}$$

$$\leq \big(\lambda - \tfrac{1}{2}\Lambda\big)^2 \int w^2 + \int (\Delta w)^2 - 2\int (\lambda - e^{2v} f) w \Delta w \tag{4-81}$$

$$\leq C|w|_{2,2}^2. \tag{4-82}$$

Hence, for λ large enough, $(\lambda I - H)$ has compact inverse, so it is a Fredholm operator, and H has discrete spectrum.

To conclude the lemma, it remains to check that H has no eigenvalue in the range $(-\Lambda + |2e^{2v} f|_\infty, 2c)$. Let t be in this range. If $\dot{v}$ is an eigenvector for the value t, we have

$$\Delta \dot{v} + (2e^{2v} f - t)\dot{v} = 0. \tag{4-83}$$

Decompose $\dot{v} = m(\dot{v}) + \check{v}$ with $m(\dot{v})$ being the average of $\dot{v}$. Then it becomes

$$m(\dot{v}) \int 2e^{2v} f - t = -\int (2e^{2v} f - t)\check{v}, \tag{4-84}$$

$$\int |\nabla \check{v}|^2 - (2e^{2v} f - t)\check{v}^2 = m(\dot{v}) \int (2e^{2v} f - t)\check{v}. \tag{4-85}$$

Combining these two equations with $\int e^{2v} f = c$, we get

$$\int |\nabla \check{v}|^2 - (2e^{2v} f - t)\check{v}^2 = -2m(v)^2 (2c - t) \operatorname{Vol} \Sigma. \tag{4-86}$$

As $t < 2c$, the right-hand side is nonpositive, while because $-\Lambda + 2e^{2v} f < t$, the left-hand side is nonnegative. So both must be zero, and $\dot{v} = 0$, as desired. □

A consequence of this lemma is that, for a varying family of functions f, we are locally able to lift branches of solutions to $\Delta v = c - e^{2v} f$. But we will need to be able to lift to a continuous branch for all f obtained from solving the Gauss–Codazzi equation. To this aim, we give a more precise description of the open set on which we are able to do so:

Theorem 4.13. *Let Σ be a closed hyperbolic surface. We denote by (C_i) constants depending on the systole, the spectral gap, and the ratio $|\alpha|_\infty^2 / \fint |\alpha|^2$. Consider v_0 a solution of*

$$\Delta v = c - e^{2v} |\alpha|^2 \tag{4-87}$$

satisfying

$$e^{2v_0} |\alpha|^2 \leq \frac{C_1}{2g - 2} < \Lambda. \tag{4-88}$$

Fix $\varepsilon > 0$. Consider the functional $\mathcal{H} : C^{0,\alpha}(\Sigma) \times W_{2,2}(\Sigma) \to L^2(\Sigma)$ such that

$$\mathcal{H}(f, v) = \Delta v - c + e^{2v} f |\alpha|^2. \tag{4-89}$$

Then there is $C_2 > 0$ such that for $|f - 1|_\infty \leq C_2 / \sqrt{2g - 2}$, there is a unique $v \in W^{2,2}$ satisfying

$$\begin{cases} |v - v_0|_{2,2} \leq \varepsilon, \\ \qquad \Delta v = c - e^{2v} f |\alpha|^2. \end{cases} \tag{4-90}$$

In particular, this v also satisfies

$$e^{2v} |\alpha|^2 \leq \frac{C_3}{2g - 2}. \tag{4-91}$$

Proof. This explicit lift comes from an estimation of the bounds appearing in the implicit function theorem. We will use the version in [Holtzman 1970]. With the notation of the theorem, we need to compute $k_1 = |(D_v\mathcal{H}(f_0, v_0))^{-1}|$. The Lemma 4.12 shows that

$$k_1 \leq \tfrac{1}{2}(2g-2), \tag{4-92}$$

provided that $1/(2(2g-2)) \leq \Lambda$. Now we need to find $\delta > 0$ and g_1 such that, for $s \in [0, \delta]$, $v \in [0, \varepsilon]$,

$$|f-1| \leq \delta,\ |v - v_0|_{2,2} \leq \varepsilon \quad \Rightarrow \quad \|D_v\mathcal{H}(f, v) - D_v\mathcal{H}(1, v_0)\| \leq g_1(|f-1|_\infty, |v-v_0|_{2,2}). \tag{4-93}$$

But, as

$$|D_v\mathcal{H}(f, v)\dot{v} - D_v\mathcal{H}(1, v_0)\dot{v}|_2^2 = \int (2(e^{2v}f - e^{2v_0})|\alpha|^2\dot{v})^2 \leq \frac{4C_1^2}{(2g-2)^2} e^{4|v-v_0|_\infty} |1-f|_\infty |\dot{v}|_2^2, \tag{4-94}$$

we deduce

$$\|D_v\mathcal{H}(f, v) - D_v\mathcal{H}(1, v_0)\| \leq \frac{C_4}{2g-2} e^{2C_5|v-v_0|_{2,2}} |1-f|_\infty. \tag{4-95}$$

Finally, we must find g_2 such that

$$|f-1|_\infty \leq \delta \ \Rightarrow\ \|\mathcal{H}(f, v_0)\| \leq g_2(|f-1|_\infty). \tag{4-96}$$

Explicitly,

$$|\mathcal{H}(f, v_0)|_2^2 = \int (e^{2v_0}|\alpha|^2(f-1)^2)^2 \leq \frac{C_6}{2g-2}, \tag{4-97}$$

from which we get

$$g_2(s) = \frac{C_7 s}{\sqrt{2g-2}}. \tag{4-98}$$

Now we can lift a continuous branch satisfying $\mathcal{H}(f, \Psi(f)) = 0$ provided that δ satisfies

$$k_1 g_1(\delta, \varepsilon) \leq \eta < 1 \quad \text{and} \quad k_1 g_2(\delta) \leq \varepsilon(1-\eta). \tag{4-99}$$

This is verified for δ of the form

$$\delta = \frac{C_8}{\sqrt{2g-2}}, \tag{4-100}$$

and the solutions $v = \Psi(f)$ satisfy

$$|v - v_0|_{2,2} \leq \varepsilon, \tag{4-101}$$

from which we deduce

$$e^{2v}|\alpha|^2 \leq e^{2C_9|v-v_0|_{2,2}} e^{2v_0}|\alpha|^2 \leq \frac{C_{10}}{2g-2}, \tag{4-102}$$

as desired. □

5. Proof of the main theorem

In this section we prove the geometric meaning of all this study:

Theorem 5.1. *Let Σ be a closed hyperbolic surface of genus g. Fix $\eta \in (0, 1)$ and consider a family $(\Sigma_n, N_n, \alpha_n)_{n\geq 2}$ satisfying*

(a) *Σ_n is a degree n cover of Σ, equipped with the lift of the hyperbolic metric on Σ;*

(b) *N_n is a degree $4n(g-1)+1$ line bundle over Σ_n, equipped with a metric of constant sectional curvature*;

(c) *$\alpha_n \in H^0(N_n)$ is a section of N_n*;

(d) *there is $\Lambda_0 > 0$ such that for all n the spectral gap of Σ_n is bigger than Λ_0*;

(e) *the family of sections is balanced, i.e.,*

$$\sup_{n\geq 2} \frac{|\alpha_n|_\infty^2}{\mathcal{f}\,|\alpha_n|^2} < \infty. \tag{5-1}$$

Then, for n large enough, there are $u, v : \Sigma_n \to \mathbb{R}$ such that

$$\sup_{z\in\Sigma_n} e^{-4u}e^{2v}|\alpha|^2(z) < \eta, \tag{5-2}$$

$$\begin{cases} \Delta u = e^{2u} - 1 + e^{-2u}e^{2v}|\alpha_n|^2, \\ \Delta v = \dfrac{1}{n(2g-2)} - e^{-2u}e^{2v}|\alpha_n|^2. \end{cases} \tag{5-3}$$

As a corollary, we get the desired existence theorem for almost-fuchsian representations with nontrivial normal bundle.

Corollary 5.2. *There is a genus g_0 such that for any $g \geq g_0$, there is a representation $\rho : \pi_1\Sigma_g \to \mathrm{SO}(4,1)$ which is almost-fuchsian and a complex variation of Hodge structure, and for which the hyperbolic 4-manifold $\rho\backslash\mathbb{H}^4$ is a degree 1 disc bundle over Σ_g.*

Proof. Assuming Theorem 5.1, let Σ be a closed hyperbolic surface of genus 2. Thanks to the result of [Magee et al. 2022], we know the existence of a family of covers (Σ_n) satisfying condition (d) of Theorem 5.1. Thanks to Theorem 3.3, there is a balanced sequence (α_n, N_n) satisfying conditions (b), (c) and (e). Thus for n large enough, we have (u, v) satisfying

$$\sup_{z\in\Sigma_n} e^{-4u}e^{2v}|\alpha|^2(z) < 1, \tag{5-4}$$

$$\begin{cases} \Delta u = e^{2u} - 1 + e^{-2u}e^{2v}|\alpha_n|^2, \\ \Delta v = \dfrac{1}{n(2g-2)} - e^{-2u}e^{2v}|\alpha_n|^2. \end{cases} \tag{5-5}$$

Thanks to Proposition 2.11, this corresponds to an almost-fuchsian representation and complex variation of Hodge structure, with holomorphic data α, and the hyperbolic 4-manifold is a degree 1 disc bundle over Σ_n. □

The main argument of Theorem 5.1 will be Schauder's fixed point theorem: a continuous map which preserves a compact convex set has a fixed point.

Proof of Theorem 5.1. Denote by $\mathcal{K}_u^n$ the convex set

$$\mathcal{K}_u^n := \left\{u \in C^{0,\alpha}(\Sigma_n) : -\tfrac{1}{2}\ln(1+\eta) \leq u \leq 0\right\}. \tag{5-6}$$

Thanks to Theorem 4.9, for $u = 0$, there is a function v_0 which is a solution of

$$\Delta v_0 = \frac{1}{|\chi(\Sigma_n)|} - e^{2v_0}|\alpha_n|^2. \tag{5-7}$$

By Proposition 4.11, there is $C_0 > 0$ depending continuously on δ, Λ and the ratio $|\alpha_n|^2_\infty / \fint |\alpha_n|^2$ such that

$$e^{2v_0}|\alpha_n|^2 \leq \frac{C_0}{2n(g-1)}. \tag{5-8}$$

The assumptions (a), (d) and (e) ensure that the constant C can be chosen to be independent of n, as it must be for our proof. Recall Λ_0 defined in (d) as a common lower bound to the spectral gaps of (Σ_n). Consider now n large enough that

$$\frac{C_0}{2n(g-1)} < \min\left\{\Lambda_0, \frac{1}{2}\right\}. \tag{5-9}$$

Applying Theorem 4.13 for some fixed $\varepsilon > 0$, there are constants C_1, C_2, C_3 independent of n and a continuous map

$$\Psi : \left\{u \in C^{0,\alpha}(\Sigma_n) : |e^{2u} - 1|_\infty \leq \frac{C_1}{\sqrt{2n(g-1)}}\right\} \to \{v \in C^{2,\beta}(\Sigma_n) : |v - v_0|_\infty \leq C_2\varepsilon\} \tag{5-10}$$

such that $v = \Psi(u)$ is a solution of

$$\Delta v = \frac{1}{|\chi(\Sigma_n)|} - e^{2v}e^{-2u}|\alpha_n|^2, \tag{5-11}$$

and $v = \Psi(u)$ satisfies

$$e^{2v}|\alpha_n|^2 \leq \frac{C_3}{2n(g-1)}. \tag{5-12}$$

Here β is any exponent $0 < \beta < \alpha$. Provided $C_3/(2n(g-1)) \leq \eta/(1+\eta)^2$, Theorem 4.1 gives us a map

$$\Phi : \left\{v \in C^{0,\alpha}(\Sigma_n) : e^{2v}|\alpha_n|^2 \leq \frac{C_3}{2n(g-1)}\right\} \to \left\{u \in C^{2,\beta}(\Sigma_n) : |e^{2u} - 1|_\infty \leq \frac{C_4}{2n(g-1)}\right\} \tag{5-13}$$

for an n-independent constant C_4 such that $u = \Phi(v)$ is a solution of

$$\Delta u = e^{2u} - 1 + e^{-2u}e^{2v}|\alpha_n|^2 \tag{5-14}$$

and u satisfies

$$\sup e^{-4u}e^{2v}|\alpha_n|^2 < \eta. \tag{5-15}$$

Finally, for n sufficiently large,

$$\frac{C_4}{2n(g-1)} \leq \frac{C_1}{\sqrt{2n(g-1)}}, \tag{5-16}$$

and $\Phi \circ \Psi$ is a well-defined self-map of $\mathcal{K} = \{u : |e^{2u} - 1| \leq C_4/(2n(g-1))\} \subset \mathcal{K}^u_n$. We also point out that, by ellipticity, every element in the image of Φ is in $C^{2,\alpha}$ and satisfies

$$\begin{cases} -\frac{1}{2}\ln 2 \leq u \leq 0, \\ \quad -1 \leq \Delta u \leq 1. \end{cases} \tag{5-17}$$

And so the image of Φ is in the set

$$\mathcal{Q} = \left\{u \in C^{2,\alpha}(\Sigma_n) : -1 \leq \Delta u \leq 1 \text{ and } -\tfrac{1}{2}\ln 2 \leq u \leq 0\right\}, \tag{5-18}$$

whose closure in $C^{0,\alpha}(\Sigma_n)$ is convex and compact, since the uniform bounds on u and Δu induce uniform bounds on the L^2-norms of u and ∇u, the Bochner formula and the bound on Δu induce a uniform bound on $\nabla^2 u$, and the embedding $W^{2,2} \hookrightarrow C^{0,\alpha}$ is compact.

Finally, $\Phi \circ \Psi$ is a continuous self-map of $\mathcal{K}$, which is convex, and its image is contained in a compact convex set. Hence, by the Schauder fixed point theorem, it has a fixed point u. Let $v = \Psi(u)$. Then (u, v) is solution of the desired PDE system, as asserted. By construction, it satisfies the estimate

$$e^{-4u}e^{2v}|\alpha_n|^2 \leq \eta, \tag{5-19}$$

completing the proof. □

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