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FLOW OF THE ZEROS OF POLYNOMIALS UNDER ITERATED DIFFERENTIATION

ANDREI MARTÍNEZ-FINKELSHTEIN AND EVGUENII A. RAKHMANOV

For a monic polynomial Q_n of degree n, let $Q_{n,k}$ be its k-th derivative normalized to be monic. Under the only assumption that the sequence $\{Q_n\}$ has a weak-* limiting zero distribution (an empirical distribution of zeros) represented by a probability measure μ_0 with compact support in the complex plane, we show that, as $n, k \to \infty$ such that $k/n \to t \in (0, 1)$, the Cauchy transform of the normalized zero-counting measure of the polynomials $Q_{n,k}$ converges in a neighborhood of infinity to an analytic function, uniquely determined by μ_0 and t, that can be written as the Cauchy transform of a measure μ_t, not necessarily uniquely determined unless μ_0 is supported on the real line.

The family of these Cauchy transforms and, when well defined, the corresponding measures μ_t, $t \in (0, 1)$, whose dependence on the parameter t can be interpreted as a flow of the zeros under iterated differentiation, has several interesting connections with the inviscid Burgers equation, the fractional free convolution of μ_0, or a nonlocal diffusion equation governing the density of μ_t on $\mathbb{R}$.

We provide an elementary and unified approach that not only recovers but also explains various phenomena observed in prior works — from Burgers-type PDEs to free probability limits.

1. Introduction

In recent years, the problem of repeated differentiation of polynomials has been intensively studied from different points of view and in different contexts. In this introduction, we start by formulating our main result. After that, we make a few remarks about the classical roots of the problem and briefly comment on some recent results.

Hereafter, the asymptotic zero distribution of a sequence of polynomials (also known as its empirical distribution of zeros) is understood in terms of the weak-* limits of their zero-counting measures. We define the zero-counting measure of a polynomial P as

$$\chi(P) := \sum_{P(x)=0} \delta_x, \tag{1-1}$$

where the sum is taken over all zeros of P, accounting for multiplicities, and δ_x is the Dirac measure (unit mass) supported at x. This way, $\chi(P)$ is independent of normalization of P, and

$$\chi(P)(\mathbb{C}) = \deg(P).$$

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We are interested in the following problem: given a sequence of (monic) polynomials Q_n, $\deg Q_n = n$, $n \in \mathbb{N}$, and the triangular table of their iterated derivatives

$$Q_{n,k}(x) := \frac{(n-k)!}{n!}\frac{d^k}{dx^k}Q_n(x) = x^{n-k} + \text{lower degree terms}, \quad k = 0, 1, \dots, n-1, \tag{1-2}$$

what are the possible limit distributions of the normalized zero-counting measures

$$\sigma_{n,k} := \frac{1}{n}\chi(Q_{n,k}) \tag{1-3}$$

as $n \to \infty$ in such a way that $k/n \to t \in [0, 1)$, if we only know that the $\sigma_{n,0}$ have a weak-* limit, compactly supported on $\mathbb{C}$?

A weaker form of this question deals with the convergence of the logarithmic derivatives

$$v_{n,k}(z) := \frac{Q'_{n,k}(z)}{nQ_{n,k}(z)}. \tag{1-4}$$

The main result of this paper is the following:

Theorem 1.1. *Let Q_n be a sequence of polynomials whose zeros belong to a compact convex set S. Assume also that the limit*

$$\lim_n \frac{Q'_n(z)}{nQ_n(z)} = u(z)$$

exists uniformly on compact subsets of $\Omega = \mathbb{C} \setminus S$.

Then for any $t \in [0, 1)$ and any sequence k_n of natural numbers with $k_n/n \to t$ we have that

$$\lim_n v_{n,k}(z) = \lim_n \frac{Q'_{n,k}(z)}{nQ_{n,k}(z)} = u(z, t) \tag{1-5}$$

uniformly on compact subsets of Ω, where $u(z, t)$ is uniquely determined by the equation

$$\frac{\partial u}{\partial t}(z, t) = \frac{1}{u(z, t)}\frac{\partial u}{\partial z}(z, t) \tag{1-6}$$

with the initial condition $u(z, 0) = u(z)$.

Equation (1-6) is a special case of a nonlinear wave equation, known as the inviscid Burgers or Hopf equation.[1] It is traditionally considered for the real values of the variables z and t, but it has also been studied on the complex plane, both in its viscous and inviscid form; see, for example, [del Monaco and Schleißinger 2016; Hall et al. 2025; 2026; Hotta and Katori 2018; Hotta and Schleißinger 2021; VandenHeuvel et al. 2023].

The Hopf equation is known for a remarkable phenomenon: its solutions can develop singularities (or shocks) even when posed on the real line. However, this behavior does not occur when the initial condition $u(z, 0)$ is given by the Cauchy transform of a unit *positive* measure supported on a bounded convex set S. In this case, the solution $u(z, t)$ to equation (1-6) remains analytic in the domain $\Omega = \mathbb{C} \setminus S$.

[1] It goes by various other names such as dispersionless Burgers equation, dispersionless KdV (Korteweg-de Vries) equation, inviscid KdV; see, e.g., [Arendt and Urban 2023, Section 2.1.3], [Evans 1998, Section 3.4] or [Markowich 2007, Chapter 9].

Moreover, for any fixed $t \in (0, 1)$, the solution $u(z, t)$ can itself be represented as the Cauchy transform of a positive measure supported on S — although this measure is not necessarily unique. This apparently new observation is implicitly contained in the statement of Theorem 1.1. Indeed, the initial measure can be approximated by the zero-counting measures of a sequence of polynomials Q_n whose zeros are all contained in S. The analyticity of $u(z, t)$ in Ω then follows from the convergence in (1-5). The representation of u as a Cauchy transform follows from the weak compactness of the associated sequence of zero-counting measures.

It is important to note that the analyticity statement fails if the assumption of positivity of μ_0 is dropped; see Remark 2.2 in Section 2.

The fact that, in the situation of complex zeros, not much more than (1-5) can be said is illustrated by the paradigmatic example of two sequences of polynomials, $Q_n(z) = z^n$ and $Q_n(z) = z^n - 1$. In both cases, the sequence of normalized zero-counting measures σ_{n,k_n}, defined in (1-3), converges as $k_n/n \to t$ to the same measure $(1-t)\delta_0$, although the μ_0 are clearly different: δ_0 in the former case, and the unit Lebesgue measure on the unit circle in the latter. In the notation of Theorem 1.1, in the domain $0 < |z| < 1$,

$$u(z) = 0 \ \text{ for } z^n \quad \text{and} \quad u(z) = 1/z \ \text{ for } z^n - 1,$$

which shows that $u(z)$ does not determine $u(z, t)$ there.

The situation is different if the Q_n have only real zeros or, more generally, lie on a segment of a straight line. It will be shown that $u(z, t)$ is the Cauchy transform of a unique positive measure compactly supported on the same segment. In this context, we denote this measure by μ_t. It is completely determined by the values of $u(z, t)$ in a neighborhood of infinity, and measures $\sigma_{n,k}$ weakly converge to μ_t. In more general cases, when uniqueness is not guaranteed, we use μ_t to denote one of the measures whose Cauchy transform coincides with $u(z, t)$ in a neighborhood of infinity; see for example the statement of (iii) in Theorem 2.1.

Let us comment on the classical roots of the problem of zeros of derivatives of polynomials.

A standard example of a flow generated by repeated differentiation is provided by the sequence of polynomials

$$Q_{2n,k}(z) = \frac{(2n-k)!}{(2n)!}\frac{d^k}{dx^k}(x^2-1)^n, \quad n = 0, 1, 2, \ldots, \quad k = [2tn], \tag{1-7}$$

for some $t \in [0, 1)$ (we denote by $[\,\cdot\,]$ the integer part of the argument). The notion of "flow" is justified by considering the parameter t as "time". The corresponding normalized sequence of zero-counting measures $\frac{1}{2n}\chi(Q_{2n,[2tn]})$ converges in the weak-* sense[2] to a measure μ_t with total mass $1-t$, which has an explicit representation

$$\mu_t = \max\left\{\tfrac{1}{2} - t, 0\right\}(\delta_{-1} + \delta_1) + \mu_t^{\mathrm{ac}}, \tag{1-8}$$

[2]The weak-* convergence in this context is a classical result and may be regarded as part of the folklore in the analytic theory of polynomials. For explicit formulas in this setting, see, for example, [Bøgvad et al. 2024; Hoskins and Kabluchko 2023] or the derivation provided in Section 5.

where μ_t^{ac} is an absolutely continuous measure supported on the interval $\big[-2\sqrt{t(1-t)}, 2\sqrt{t(1-t)}\big]$, with

$$d\mu_t^{\mathrm{ac}}(x) = \frac{1}{2\pi}\frac{\sqrt{4t(1-t)-x^2}}{1-x^2}\,dx. \tag{1-9}$$

As t progresses, μ_t represents the flow of zeros of iterated derivatives, starting from its initial distribution $\mu_0 = \frac{1}{2}(\delta(-1)+\delta(1))$. The measure μ_t is well known: it is related to zeros of Gegenbauer polynomials and hence to the angular part of the hyperspherical harmonics; see, e.g., [Saff and Totik 2024, Examples IV.1.18 and IV.6.2]. In fact, for $k = n$, (1-7) is just the Rodrigues formula for the Legendre polynomials, orthogonal in $[-1, 1]$ with respect to the weight $w(x) = 1$. This also explains why μ_t, for $t = \frac{1}{2}$, is one-half of the equilibrium measure (or the Robin distribution) of the interval $[-1, 1]$. Notice also the curious symmetry of μ_t^{ac} with respect to the time reversal ($t \mapsto 1-t$). Since every classical family of orthogonal polynomials satisfies a Rodrigues formula, each of these families originates its own flow of measures with interesting properties.

The formula (1-7) can be extended in several directions. For instance, the type-II Hermite–Padé (or multiple orthogonal) polynomials Q_{2n} of degree $2n$, defined up to normalization by the orthogonality conditions

$$\int_{-1}^{0} Q_{2n}(x)x^k\,dx = \int_{0}^{1} Q_{2n}(x)x^k\,dx = 0, \quad k = 0, 1, \ldots, n-1, \tag{1-10}$$

have the representation

$$Q_{2n}(x) = \frac{(2n)!}{(3n)!}\frac{d^n}{dx^n}(x^3-x)^n, \tag{1-11}$$

obtained originally by Kalyagin [1979], who used it to study the asymptotic properties of Q_{2n}. As before, we can consider

$$Q_{2n,k}(z) = \frac{(3n-k)!}{(3n)!}\frac{d^k}{dx^k}(x^3-x)^n, \quad n = 0, 1, 2, \ldots, \quad k = [3tn],$$

and the corresponding family of measures μ_t of their limiting zero distributions. In this case, the explicit formulas for μ_t are not as simple as before. Similar Rodrigues-type formulas exist for other families of semiclassical and multiple orthogonal polynomials [Aptekarev et al. 1997; Sorokin 2022; 2023]. Recently, the asymptotics of polynomials given by (1-7), where x^2-1 on the right-hand side is replaced by an arbitrary polynomial P, possibly with complex zeros, has been considered in [Bøgvad et al. 2024]; see Section 5 for more details and comments on some results of this work.

The interest in this subject has recently been revitalized by a series of papers by Steinerberger [O'Rourke and Steinerberger 2021; Steinerberger 2019; 2023], and several other works have followed in rapid succession, establishing interesting connections with free probability and the inviscid Burgers (or Hopf) partial differential equation.

First, Steinerberger [2019] wrote a nonlocal diffusion equation that should describe the evolution of the absolutely continuous part μ_t^{ac} of μ_t. Observing the similarities with an equation in [Shlyakhtenko and Tao 2022], he conjectured in [Steinerberger 2023] that μ_t can be written in terms of the fractional free additive convolution of the initial measure μ_0. Hoskins and Kabluchko [2023] rigorously proved this; later,

it was also established in [Arizmendi et al. 2023], this time using elegant arguments involving only finite free convolution of polynomials, a notion developed in a series of works of Markus and collaborators; see [Marcus et al. 2022]. In the case of complex random zeros, the free probability interpretation of repeated differentiation has been obtained in [Campbell et al. 2024].

The paper [Marcus et al. 2022] also contains another interesting motivation for studying our main problem: the average characteristic polynomial of all $k \times k$ minors of an $n \times n$ matrix A can be expressed in terms of the $(n-k)$-th derivative of the characteristic polynomial of A; see [Marcus et al. 2022, Lemma 1.17] for a precise formulation.

Other works addressing several aspects of repeated differentiation (many of them in the context of random polynomials) are [Angst et al. 2024; Byun et al. 2022; Campbell et al. 2024; Galligo 2022; Hoskins and Steinerberger 2022; Kabluchko 2026; Kiselev and Tan 2022]. A related differential operator (the heat-flow operator), which is degree-preserving and contains derivatives of all orders, was studied in [Hall et al. 2025]. The authors present a transport map T_t, defined in terms of the Cauchy (or Stieltjes) transform of μ_0, such that $T_t(\mu_0) = \mu_t$. Moreover, they show that the Cauchy transform of μ_t satisfies the inviscid Burgers equation (written in a slightly different form than the one appearing in the following); see [Hall et al. 2025, Equation (5.4)]. As they point out, the complex Burgers equation with initial value given by a Cauchy transform has recently been used in the description of the growth of multiple SLE; see [del Monaco and Schleißinger 2016; Hotta and Katori 2018; Hotta and Schleißinger 2021]. It also appears in [Aptekarev and Rykov 2006] in the description of the evolution of the end-points of the support of the equilibrium measure in a specific external field depending on t; see also [Martínez-Finkelshtein et al. 2015]. Also, equation (1.17) in [Shlyakhtenko and Tao 2022] can be reduced to (1-6). Another equivalent formulation of (1-6) appears again in [Hall et al. 2026]; the authors even address the behavior of individual zeros and conjecture (proving it in the rotationally invariant case) that the Cauchy transform of the measure, evaluated along the path of a single root, must remain constant, and the roots should move along the characteristic curves of a certain nonlinear PDE.

In this paper, we address the problem of the macroscopic description of the zeros of polynomials under repeated differentiation, with the most general assumptions stated in Theorem 1.1. However, our goal is not only to prove (and partially reprove) the results but also to show why they are true. The main feature of our approach is that it is totally elementary and uses only basic facts from complex analysis.

Our main findings can be summarized as follows. The complex Hopf equation (1-6) plays the main role. The connection of the problem on the flow of derivatives under differentiation with this equation lies in the elementary fact that the Cauchy transform of the zero-counting measure of a polynomial can be expressed in terms of the logarithmic derivative of this polynomial. As a first step, we show that if its initial value is given by the Cauchy transform of a compactly supported probability measure μ_0, then there is a universal domain, depending only on the support of μ_0, where not only the solutions of the equation remain analytic, but they can be written as the Cauchy transform of a measure for all values of the parameter $t \in (0, 1)$.

We show that the solution of the Hopf equation by the method of characteristics is precisely the expression of the fact that μ_t can be written as the fractional free convolution of μ_0.

Finally, we show that the nonlocal diffusion from [Steinerberger 2019] is merely a projection of the Hopf equation onto the real line via the Sohotski–Plemelj identities (or the Stieltjes–Perron inversion formula).

2. Main results

Let us start by introducing the following notation: for $0 < R < +\infty$, define $\mathbb{D}_R = \{z \in \mathbb{C} : |z| < R\}$, $\overline{\mathbb{D}}_R = \{z \in \mathbb{C} : |z| \le R\}$, and $\Omega_R = \{z \in \mathbb{C} : R < |z| < +\infty\} = \mathbb{C} \setminus \overline{\mathbb{D}}_R$. Notice that the open domain $\Omega_R \subset \mathbb{C}$ does not include infinity, while its closure in $\overline{\mathbb{C}}$ — that is, $\overline{\Omega}_R = \{z \in \mathbb{C} : R \le |z| \le +\infty\}$ — does. We also denote by ∂_z the differentiation operator with respect to the real or complex parameter z (each time, it will be clear from the context).

The *inviscid Burgers* or *Hopf equation*, considered in this paper, is written as

$$\partial_t u = \frac{\partial_z u}{u}, \quad u = u(z; t). \tag{2-1}$$

A more familiar form $\partial_t u + u\,\partial_z u = 0$ can be obtained, for instance, by replacing u by $-1/u$. We require that the parameter t takes real values in $[0, 1]$, while z is in the complex plane; in our case, z is in a neighborhood of infinity, that is, a domain of the form Ω_R.

It is well known that the solutions of (2-1) can exhibit shocks in a finite time t; we will be interested in a very particular type of initial value problem for (2-1), when $u(z, 0)$ is the Cauchy transform of a probability measure.

Given a finite Borel (in general, signed) measure μ on $\mathbb{C}$, we define its *Cauchy* (also known as Stieltjes or Markov) *transform* as

$$\mathcal{C}^\mu(z) := \int \frac{d\mu(y)}{z - y}, \tag{2-2}$$

which is a holomorphic function in $\mathbb{C} \setminus \operatorname{supp}(\mu)$. Moreover, if μ is compactly supported, then

$$\mathcal{C}^\mu(z) = \sum_{k=0}^{\infty} \frac{m_k}{z^{k+1}} \tag{2-3}$$

converges in a neighborhood of infinity; the coefficients

$$m_j = \int z^j \, d\mu(z), \quad j = 0, 1, 2, \ldots, \tag{2-4}$$

are the *moments* of the measure μ.

Let μ_0 be a compactly supported Borel probability measure on $\mathbb{C}$, and let $0 < r < +\infty$ be such that $\operatorname{supp} \mu_0 \subset \mathbb{D}_r$. One of the central results of this paper is the fact that the initial value problem for equation (2-1) with initial datum

$$u(z, 0) = \mathcal{C}^{\mu_0}(z) \tag{2-5}$$

has an analytic solution $u(z, t)$ for all $t \in [0, T]$, $0 < T < 1$, and z in a neighborhood of infinity that depends only on r and T but not on the measure μ_0 itself:

Theorem 2.1. *Let* $0 < T < 1$ *and* $0 < r < +\infty$ *be fixed. Then*:

(i) *There exists an* $R = R(T) > 2r$ *such that, for any probability measure* μ_0 *with* $\operatorname{supp}(\mu_0) \subset \mathbb{D}_r$ *and for* $(z, t) \in \Omega_R \times [0, T]$, *there exists a unique solution* $u(z, t)$ *of* (2-1) *satisfying the initial condition*

$$u(z, 0) = \mathcal{C}^{\mu_0}(z).$$

(ii) *For* $(z, t) \in \Omega_R \times [0, T]$, $u(z, t)$ *is given by*

$$u(z, t) = u(s, 0), \tag{2-6}$$

where the unique $s = s(z)$ *is implicitly defined by the equation* $F(z, s) = 0$, *with*

$$F(z, s) := z - s + \frac{t}{u(s, 0)}. \tag{2-7}$$

(iii) *For each* $t \in [0, 1)$, *there is a measure* μ_t, *with* $\operatorname{supp}(\mu_t) \subset \mathbb{D}_r$, *such that*

$$u(z, t) = \mathcal{C}^{\mu_t}(z), \quad z \in \Omega_r. \tag{2-8}$$

The moments

$$m_j(t) := \int z^j \, d\mu_t(z), \quad j = 0, 1, 2, \ldots, \tag{2-9}$$

of μ_t *are determined uniquely by* μ_0,[3] *with* $m_0(t) = 1 - t$. *Moreover, for* $k \in \mathbb{N}$, $m_k(t)$ *is a polynomial in* t *of degree at most* k *such that*

$$\lim_{t \to 1} \frac{m_k(t)}{1 - t} = m_1(0)^k, \quad k \geq 1. \tag{2-10}$$

In essence, this theorem guarantees that, for any compactly supported initial positive measure μ_0, the flow governed by the Hopf equation preserves analyticity and admits a representation as a Cauchy transform throughout its evolution.

Remark 2.2. (a) Using the change of variables $w = u(s, 0)$ we can rewrite (2-6)–(2-7) as

$$u^{-1}(w, t) = u^{-1}(w, 0) - \frac{t}{w}, \tag{2-11}$$

where u^{-1} is the functional inverse of u, or equivalently as

$$w = u\Big(z + \frac{t}{w}, 0\Big), \tag{2-12}$$

which is an equation on $w = u(z, t)$.

(b) The assertion in (i) of Theorem 2.1 about R depending only on r and T is in general false for a unit ($m_0 = 1$) but signed measure. A simple example is the case

$$\mu = (a + 1)\delta_0 - a\delta_1, \quad a > 0,$$

for which

$$u(z, 0) = \frac{a+1}{z} - \frac{a}{z-1} \tag{2-13}$$

[3]Recall that μ_t itself is not necessarily uniquely defined by its moments.

is holomorphic in Ω_1, but

$$u(z,t)=\frac{(1-2t)z-(a+1-t)+\sqrt{z^2-2(t-1+a(1-2t))z+(a+1-t)^2}}{2(z-1)z}$$

has branch points at $\zeta_+(t)=\overline{\zeta_-(t)}$, with $|\zeta_\pm|=a+1-t$. This shows that in this case there is no bounded R that works for all values of the parameter $a>0$.

(c) We can formally define the derivative with respect to t of the measures μ_t:

$$\tau_t:=-\lim_{\varepsilon\to 0+}\frac{1}{\varepsilon}(\mu_{t+\varepsilon}-\mu_t).$$

By (iii) of Theorem 2.1, $\tau_t(\mathbb{C})=1$ for all $t\in(0,1)$. In the terminology of [Kerov 1998], Theorem 2.1 states that τ_t is the *inverse Markov transform* of the measure μ_t.

(d) Formula (2-10) can be interpreted as follows: μ_t disappears, as $t\to 1-$, as a mass point at $m_1(0)$, i.e., at the center of mass of the initial distribution μ_0. This is the case, at least, when $\operatorname{supp}\mu_0\subset\mathbb{R}$.

Recall that the Hilbert transform of a measure μ on $\mathbb{R}$ is defined by

$$\mathcal{H}_\mu(x):=\frac{1}{\pi}\,\text{p.v.}\int\frac{d\mu(y)}{x-y}. \tag{2-14}$$

Theorem 2.1 has the following consequence when the initial data is supported on the real line: the flow remains uniquely determined and real, and satisfies a nonlinear nonlocal transport equation.

Corollary 2.3. *Let μ_0 be a compactly supported probability measure on $\mathbb{R}$, and let μ_t, $t\in[0,1)$, be as described in* (iii) *of Theorem 2.1. Then, for each $t\in[0,1)$, the measure μ_t can also be taken as supported on $\mathbb{R}$, and under this condition it is unique. If μ_t is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ then the density*

$$f(x,t):=\frac{d\mu_t(x)}{dx}$$

satisfies

$$\partial_t f(x,t)=-\frac{1}{\pi}\partial_x\arctan\left(\frac{\mathcal{H}_{\mu_t}(x)}{f(x,t)}\right), \tag{2-15}$$

where H_{μ_t} is its Hilbert transform.

The "nonlocal transport equation" (2-15) coincides with the one found in [Steinerberger 2019].

The initial value problem for the Hopf equation, whose existence and analyticity have been established in Theorem 2.1, is directly related to the central problem of this paper, on the flow of zeros of a sequence of polynomials under iterated differentiation, as explained in the Introduction.

Namely, recall that for a sequence of monic polynomials Q_n, with $\deg Q_n=n$, $n\in\mathbb{N}$, and the associated polynomials (1-2), we defined the normalized zero-counting measures (1-3) such that

$$\sigma_{n,k}(\mathbb{C})=\frac{n-k}{n}.$$

The only assumption on $\{Q_n\}$ is that there exists $0<r<+\infty$ such that, for each sufficiently large $n\in\mathbb{N}$, all zeros of Q_n belong to $\mathbb{D}_r$, and by the Gauss–Lucas theorem the same property is shared by all

polynomials $Q_{n,k}$, defined in (1-2). Equivalently, we can say that, for all sufficiently large $n \in \mathbb{N}$, the measures $\sigma_{n,k},\ k \in \{0, 1, \dots, n-1\}$, are supported in $\mathbb{D}_r$.

A key consequence of Theorem 2.1 is the existence of a universal value $R = R(T) > 2r$ such that Hopf's equation (2-1) has a holomorphic and uni-valued solution in Ω_R for any $0 < t < T$ and for *any* initial condition given by the Cauchy transform of *any* probability measure supported in $\overline{\mathbb{D}}_r$. In particular, if for each $n \in \mathbb{N}$ we denote by $u_n(z, t)$ the solution of (2-1) with the initial condition

$$u_n(z, 0) := \mathcal{C}^{\sigma_{n,0}}(z) = \frac{Q_n'(z)}{n Q_n(z)}, \tag{2-16}$$

then all $u_n(z, t)$ are guaranteed to exist and to be analytic in $\Omega_R \times [0, T]$.

As in (1-4), define for $k \in \{0, 1, \dots, n-1\},\ n \in \mathbb{N}$,

$$v_{n,k}(z) := \mathcal{C}^{\sigma_{n,k}}(z) = \frac{Q_{n,k}'(z)}{n Q_{n,k}(z)} = \frac{n-k}{n} \frac{Q_{n,k+1}(z)}{Q_{n,k}(z)}. \tag{2-17}$$

For all such k and n, $v_{n,k}$ is holomorphic in $\overline{\Omega}_r$, and

$$v_{n,k}(z) = \frac{n-k}{n} \frac{1}{z} + \mathcal{O}(z^{-2}), \quad z \to \infty. \tag{2-18}$$

Taking the logarithmic derivative on both sides of (2-17) we obtain that

$$v_{n,k+1}(z) - v_{n,k}(z) = \frac{1}{n} \frac{\partial_z v_{n,k}(z)}{v_{n,k}(z)}. \tag{2-19}$$

This identity can be regarded as a discrete version of Hopf's equation (2-1), and thus it is natural to expect that, under our assumptions, functions $v_{n,k}(z)$ and $u_n(z, k/n)$ are close for sufficiently large n.

If additionally

$$\sigma_{n,0} \xrightarrow{*} \mu_0, \quad n \to \infty, \tag{2-20}$$

where $\xrightarrow{*}$ denotes the weak-* convergence and μ_0 is a probability measure on $\mathbb{C}$ supported on $\overline{\mathbb{D}}_r$, then the initial conditions (2-16) converge to the initial condition (2-5), at least in the fixed neighborhood of infinity, and we can expect that the solutions $u_n(z, t)$ of equation (2-1) with initial conditions (2-16) will converge there to $u(z, t)$, the solution of the same equation (2-1) but with initial condition (2-5). These heuristic arguments can be made rigorous, although the trivial-looking assertion about the proximity of the solutions to the "discrete" and "continuous" Hopf equations requires some careful analysis due to the nonlinear character of these equations. For convenience of the reader, we formulate Theorem 1.1 here in a more precise form:

Theorem 2.4. *Let Q_n be a sequence of polynomials with uniformly bounded zeros such that* (2-20) *holds. Then, for any $t \in [0, 1)$, there exists a positive Borel measure μ_t with* $\operatorname{supp} \mu_t \subset \operatorname{conv}(\operatorname{supp} \mu_0)$ *and $\mu_t(\mathbb{C}) = 1 - t$ such that in a neighborhood of infinity*

$$\mathcal{C}^{\mu_t}(z) = u(z, t), \tag{2-21}$$

where u is the solution of (2-1) *with the initial condition* (2-5).

Furthermore, for any sequence k_n of natural numbers with $k_n/n \to t$ we have that

$$\lim_n \mathcal{C}^{\sigma_{n,k_n}}(z) = \mathcal{C}^{\mu_t}(z)$$

locally uniformly in $\mathbb{C} \setminus (\mathrm{conv}(\mathrm{supp}\,\mu_0))$.

Finally, if all zeros of Q_n are real, then μ_t is uniquely determined from u, and

$$\sigma_{n,k_n} \xrightarrow{*} \mu_t, \quad n \to \infty.$$

Remark 2.5. In the statement above, $\mathrm{conv}(K)$ denotes the convex hull of the set K.

With the assumptions of the theorem, let $T = \frac{1}{2}(1+t)$, and let $R = R(T)$ be as in Theorem 2.1. Then to establish Theorem 2.4, it is sufficient to prove that

$$\lim_n \mathcal{C}^{\sigma_{n,k_n}}(z) = u(z,t)$$

locally uniformly in $\Omega_R = \{z \in \mathbb{C} : R < |z| < +\infty\} = \mathbb{C} \setminus \overline{\mathbb{D}}_R$. Indeed, by the weak-* compactness of the sequence σ_{n,k_n}, there exists a measure μ_t, with the properties indicated above, such that $\sigma_{n,k_n} \xrightarrow{*} \mu_t$ along a subsequence of $\mathbb{N}$, and thus (2-21) holds in Ω_R. The rest of the assertion follows by elementary analytic continuation arguments.

As a consequence, the Cauchy transform of the measure μ_t is completely determined in the complement of $\mathrm{conv}(\mathrm{supp}\,\mu_0)$. However, in its maximum generality, this does not determine measure μ_t uniquely, which makes the statement of Theorem 2.4 apparently less straightforward.

Finally, we recast Theorem 2.4 in the terminology of free probability; see e.g., [Arizmendi et al. 2023; Marcus et al. 2022; Mingo and Speicher 2017; Nica and Speicher 2006] for a background. Along with the Cauchy transform $\mathcal{C}^{\mu}$ of a compactly supported probability measure μ, as in (2-2), we can define its *R-transform* implicitly by

$$\mathcal{R}_\mu(\mathcal{C}^\mu(z)) = z - \frac{1}{\mathcal{C}^\mu(z)}; \tag{2-22}$$

in particular, $\mathcal{R}_\mu$ is an analytic function in a neighborhood of the origin. For $s > 0$, the *fractional free additive convolution* $\mu^{\boxplus s}$ of μ is defined via the identity (see [Shlyakhtenko and Tao 2022])

$$\mathcal{R}_{\mu^{\boxplus s}}(w) = s\mathcal{R}_\mu(w). \tag{2-23}$$

Corollary 2.6. *Let Q_n be a sequence of polynomials as in Theorem 2.4 such that* (2-20) *holds, where μ_0 is a compactly supported probability measure on $\mathbb{R}$. For $t \in [0,1)$, define*

$$\widehat{Q}_{n,k}(x) := Q_{n,k}((1-t)x),$$

where we use the notation (1-2).

Then, for $k/n \to t$,

$$\frac{1}{n-k}\chi(\widehat{Q}_{n,k}) \xrightarrow{*} \mu_0^{\boxplus 1/(1-t)}. \tag{2-24}$$

As mentioned in the Introduction, (2-24) — established heuristically in [Steinerberger 2023] — was rigorously proven in [Hoskins and Kabluchko 2023]. A short proof was presented in [Arizmendi et al. 2023],

based entirely on arguments from free probability. We will show that this result is an immediate consequence of solving Hopf's equation by the method of characteristics.

3. Proof of Theorem 2.1 and Corollary 2.3

We start by addressing the evolution in a neighborhood of infinity of the Cauchy transform (2-2) of a finite Borel *signed* measure μ, compactly supported on $\mathbb{C}$, according to the Hopf equation (2-1). Recall that $\mathcal{C}^\mu$ is a holomorphic function in $\mathbb{C}\setminus \operatorname{supp}(\mu)$, and that the series in (2-3) converges locally uniformly in any $\Omega_r = \{z\in\mathbb{C} : r < |z| < +\infty\}$, disjoint from $\operatorname{supp}(\mu)$. We assume additionally that

$$m_0 = \mu(\mathbb{C}) = 1 \tag{3-1}$$

and denote by $|\mu|$ the total variation of μ; since μ is signed, in general, $|\mu|(\mathbb{C}) \geq m_0$.

As a first step, we need some uniform estimates on $\mathcal{C}^\mu$:

Proposition 3.1. *Let μ be a finite Borel and compactly supported signed measure on $\mathbb{C}$ such that* (3-1) *holds, and let $0 < r < +\infty$ be such that* $\operatorname{supp}(\mu)\subset \mathbb{D}_r$. *Then, for any $R > (1+|\mu|(\mathbb{C}))r$, $\mathcal{C}^\mu$ is analytic in $\overline{\Omega}_R$ and does not vanish for $z\in\Omega_R$.*

If in addition μ is a positive (*probability*) *measure, then for $|z|\geq R > 2r$*

$$\frac{2}{9} < \frac{1-r/R}{(1+r/R)^2} \leq |z\mathcal{C}^\mu(z)| \leq \frac{R}{R-r}, \tag{3-2}$$

$$\frac{1-2r/R}{(1+r/R)^4} \leq \left|z^2\frac{d}{dz}\mathcal{C}^\mu(z)\right| \leq \frac{1}{(1-r/R)^2} < 4. \tag{3-3}$$

Moreover, if $R > (\sqrt{2}+1)r$, then C^μ is univalent (*injective*) *in Ω_R, and*

$$\overline{\mathbb{D}}_{1/(4R)} \subset C^\mu(\overline{\Omega}_R) \subset \overline{\mathbb{D}}_{1/(R-r)}. \tag{3-4}$$

Remark 3.2. As formula (2-13) in Remark 2.2(b) shows, for a signed measure μ, R in the first assertion of Proposition 3.1, in general, cannot be made independent of $|\mu|$.

Proof. For an arbitrary $R > r$, the function $z\mathcal{C}^\mu(z)$ is analytic in $\overline{\Omega}_R$ and can be written as

$$z\mathcal{C}^\mu(z) = 1 + \int \frac{y}{z-y}\, d\mu(t).$$

Since for $|z| = R$

$$\left|\int \frac{y}{z-y}\, d\mu(t)\right| \leq \frac{r}{R-r}|\mu(\mathbb{C})|,$$

if $R > r$ is such that

$$\frac{r}{R-r}|\mu|(\mathbb{C}) < 1, \quad \text{or equivalently } \ R > r(1+|\mu|(\mathbb{C})),$$

then by Rouché's theorem, $\mathcal{C}^\mu(z)\neq 0$ in $\overline{\Omega}_R$. In particular, if μ is a probability measure, this is true for any $R > 2r$.

Assuming that μ is a probability measure with $\operatorname{supp}(\mu) \subset \mathbb{D}_r$, we have $|\mu|(\mathbb{C}) = 1$, and the bound obtained above gives us that, for $|z| = R > 2r$,

$$|z\mathcal{C}^\mu(z)| = \left|1 + \int \frac{y}{z-y}\, d\mu(t)\right| \le 1 + \frac{r}{R-r} = \frac{R}{R-r}.$$

On the other hand,

$$|z\mathcal{C}^\mu(z)| \ge \operatorname{Re}(z\mathcal{C}^\mu(z)) = \int \operatorname{Re}\left(\frac{1}{1-y/z}\right) d\mu(y),$$

which gives us that, for $|z| = R > 2r$ and $|y| \le r < R$,

$$\operatorname{Re}\left(\frac{1}{1-y/z}\right) = \frac{1-\operatorname{Re}(y/z)}{|1-y/z|^2} \ge \frac{1-r/R}{(1+r/R)^2} > \frac{2}{9},$$

and the bounds in (3-2) follow.

In a similar fashion,

$$\left|z^2 \frac{d}{dz}\mathcal{C}^\mu(z)\right| = \left|\int \frac{1}{(1-y/z)^2}\, d\mu(t)\right| \le \frac{1}{(1-r/R)^2} < 4$$

and

$$\left|z^2 \frac{d}{dz}\mathcal{C}^\mu(z)\right| \ge \operatorname{Re}\left(-z^2 \frac{d}{dz}\mathcal{C}^\mu(z)\right) = \int \operatorname{Re}\frac{1}{(1-y/z)^2}\, d\mu(y).$$

For $|z| \ge R > 2r$ and $y \in \mathbb{D}_r$,

$$\begin{aligned}
\operatorname{Re}(1-y/z)^2 &= 1 - 2\operatorname{Re}\left(\frac{y}{z}\right) + \operatorname{Re}\left(\frac{y}{z}\right)^2 = 1 - 2\operatorname{Re}\left(\frac{y}{z}\right) + \left(\operatorname{Re}\frac{y}{z}\right)^2 - \left(\operatorname{Im}\frac{y}{z}\right)^2 \\
&= \left(1 - \operatorname{Re}\frac{y}{z}\right)^2 - \left(\operatorname{Im}\frac{y}{z}\right)^2 \ge \left(1 - \operatorname{Re}\frac{y}{z}\right)^2 - \left|\frac{y}{z}\right|^2 \ge \left(1 - \frac{r}{R}\right)^2 - \left(\frac{r}{R}\right)^2 \\
&= 1 - \frac{2r}{R} > 0,
\end{aligned}$$

so that

$$\operatorname{Re}\frac{1}{(1-y/z)^2} = \frac{\operatorname{Re}(1-y/z)^2}{|1-y/z|^4} \ge \frac{1-2r/R}{(1+r/R)^4},$$

and hence

$$\left|z^2 \frac{d}{dz}\mathcal{C}^\mu(z)\right| \ge \frac{1-2r/R}{(1+r/R)^4}, \quad |z| \ge R > 2r.$$

This concludes the proof of (3-3).

Furthermore, for $R > r$, consider the function

$$f_R(z) := R\mathcal{C}^\mu\left(\frac{R}{z}\right) = z \int \frac{1}{1-zy/R}\, d\mu(y) = z + \mathcal{O}(z^2), \quad z \to 0, \tag{3-5}$$

defined in the unit disk $\mathbb{D}_1 = \{z \in \mathbb{C} : |z| < 1\}$. We have that

$$\operatorname{Re}\left(\frac{f_R(z)}{z}\right) = \int \operatorname{Re}\left(\frac{1}{1-zy/R}\right) d\mu(y) = \int \frac{1-\operatorname{Re}(zy)/R}{|1-zy/R|^2}\, d\mu(y).$$

Since

$$1-\frac{\operatorname{Re}(zy)}{R}\geq 1-\frac{|zy|}{R}\geq 1-\frac{r}{R}>0,$$

we conclude that $\operatorname{Re}(f_R(z)/z)>0$ in $|z|<1$, and by [MacGregor 1962, Theorem 3], f_R is univalent in $|z|<\sqrt{2}-1$ (or, equivalently, $\mathcal{C}^\mu$ is univalent in $\Omega_{(\sqrt{2}+1)R}$).

Moreover, with $R>(\sqrt{2}+1)r$, by the Koebe $1/4$ theorem (see, e.g., [Duren 1983, §2.2]), the image $f_R(\mathbb{D}_1)$ of the unit disk by the function f_R contains the disk $\overline{\mathbb{D}}_{1/4}$, or, in other words, the image of $\overline{\Omega}_R$ by $\mathcal{C}^\mu$ contains the disk $\overline{\mathbb{D}}_{1/(4R)}$. □

It is well known that the Hopf equation (2-1) can be solved by the method of characteristics. In preparation for the analysis of this solution, we prove the following auxiliary result:

Proposition 3.3. *Let μ be a finite Borel signed measure μ on $\mathbb{C}$ such that* (3-1) *holds. Assume $0<r<+\infty$ is such that* $\operatorname{supp}\mu\subset\mathbb{D}_r$, *and let $0<T<1$ be fixed. Then there exists an $R>2r$, depending only on the values of T, $|m_1|$, and $|\mu|(\mathbb{C})$, such that*

$$f_1(s)=\mathcal{C}^\mu\Big(\frac{1}{s}\Big)=\int\frac{s}{1-sy}\,d\mu(y)$$

is analytic in $\overline{\mathbb{D}}_{1/R}$, does not vanish for $0<|s|\leq 1/R$, and, for each $z\in\mathbb{D}_{1/(2R)}$ and each $t\in[0,T]$, the equation

$$z-s-\frac{zst}{f_1(s)}=0 \tag{3-6}$$

has a unique solution $s\in\mathbb{D}_{1/R}$.

Proof. By Proposition 3.1, if $R>r(1+|\mu|(\mathbb{C}))$ then $f_1(z)$ is analytic and nonvanishing for $0<|z|<1/R$. The assertion about the unisolvence of (3-6) is obviously true for $z=0$, so we assume that $z\neq 0$.

Write

$$f_1(s)=s+s^2g(s), \tag{3-7}$$

with

$$g(s):=\int\frac{y}{1-sy}\,d\mu(y), \tag{3-8}$$

so that

$$g(0)=\int y\,d\mu(y)=m_1,$$

the first moment of μ; see (2-4). Notice that

$$|g(s)-g(0)|=\left|\int\frac{y}{1-sy}\,d\mu(y)-\int y\,d\mu(y)\right|=|s|\left|\int\frac{y^2}{1-sy}\,d\mu(y)\right|,$$

and for $|s|=1/R$

$$|g(s)-g(0)|=\frac{1}{R}\left|\int\frac{y^2}{1-sy}\,d\mu(y)\right|\leq\frac{r^2}{R-r}|\mu|(\mathbb{C}).$$

If we take $R>r(1+|\mu|(\mathbb{C}))$ such that

$$\frac{r^2}{R-r}|\mu|(\mathbb{C})<1,\quad\text{or equivalently}\quad R>r(1+r|\mu|(\mathbb{C})),$$

then

$$|g(s)| \leq 1 + |m_1|, \quad s \in \overline{\mathbb{D}}_{1/R}. \tag{3-9}$$

Now, given $T \in (0, 1)$, let $R > \max\{r(1 + |\mu|(\mathbb{C})), r(1 + r|\mu|(\mathbb{C}))\}$ be such that

$$R > (1 + |m_1|)\frac{1+T}{1-T}, \quad \text{or equivalently } 0 < \frac{T}{1 - (|m_1| + 1)/R} < \frac{1+T}{2} < 1.$$

Then, for $s \in \mathbb{D}_{1/R}$,

$$0 \leq \frac{|g(s)|}{R} \leq \frac{1 + |m_1|}{R} < \frac{1-T}{1+T} < 1,$$

so that

$$\left|\frac{t}{1 + sg(s)}\right| \leq \frac{T}{|1 - |g(s)|/R|} \leq \frac{T}{1 - |g(s)|/R} < \frac{1+T}{2} < 1, \tag{3-10}$$

and as a consequence, for $z \neq 0$ and $|s| = 1/R$,

$$\left|\frac{zst}{f_1(s)}\right| = \left|\frac{zt}{1 + sg(s)}\right| < |z|. \tag{3-11}$$

Thus, since for $0 < |z| < 1/(2R)$ and $|s| = 1/R$ we have

$$|z - s| > \frac{1}{2R} > |z|,$$

we can apply Rouché's theorem to claim that, for every $z \in \mathbb{D}_{1/(2R)} \setminus \{0\}$, equation (3-6) has a unique solution.

All assertions have been proved for any

$$R > \max\left\{r(1 + |\mu|(\mathbb{C})),\ r(1 + r|\mu|(\mathbb{C})),\ (1 + |m_1|)\frac{1+T}{1-T}\right\} \geq 2r,$$

which concludes the proof. □

Remark 3.4. Under the assumption of Proposition 3.3, for a signed measure μ, both $|\mu|$ and $|m_1|$ can be arbitrarily large even though $m_0 = \mu(\mathbb{C}) = 1$.

However, if μ is a probability measure, then $|\mu|(\mathbb{C}) = m_0 = 1$ and

$$\operatorname{supp}(\mu) \subset \mathbb{D}_r \quad \Longrightarrow \quad |m_1| = \left|\int t\, d\mu(t)\right| \leq r,$$

showing that we can take

$$R = R(T) > \max\left\{2r,\ r(1 + r),\ (1 + r)\frac{1+T}{1-T}\right\}, \tag{3-12}$$

which depends only on the values of T and r.

Now we are ready to prove the first two statements of Theorem 2.1.

Proof of statements (i)–(ii) *of Theorem 2.1.* Let $0 < T < 1$ and $0 < r < +\infty$ be fixed and take $R = R(T) > 2r$, satisfying (3-12). Consider equation (2-1) with initial condition (2-5), where μ_0 is a probability measure supported by $\mathbb{D}_r$. We solve (2-1) using the standard method of characteristics; see, e.g., [Salih 2018; Whitham 1974].

If $z = z(t)$ is differentiable, then

$$\frac{d}{dt}u(z(t), t) = \frac{\partial u}{\partial t} + z'(t)\frac{\partial u}{\partial z}.$$

Comparing it with (2-1) we see that if $z(t)$ satisfies

$$z'(t) = -\frac{1}{u(z(t), t)}, \tag{3-13}$$

then for a solution u of (2-1) we should have

$$\frac{d}{dt}u(z(t), t) = 0. \tag{3-14}$$

Differentiating (3-13) with respect to t again and using (3-14) we conclude that $z''(t) \equiv 0$, so that $z(t)$ must be linear in t: $z(t) = Lt + s$.

To find the slope L, we use the initial values,

$$L = z'(0) = -\frac{1}{u(z(0), 0)} = -\frac{1}{u(s, 0)},$$

so that $z(t) = s - t/u(s, 0)$. This is the characteristics line, along which $u(z, t)$ is constant, with the constant given by $u(z(0), 0) = u(s, 0)$. Thus, for (z, t), the solution of (2-1) is $u(z, t) = u(s, 0)$, where $s = s(z)$ is the solution of

$$z - s + \frac{t}{u(s, 0)} = 0,$$

that is, of $F(z, s) = 0$. Making the change of variables $z \mapsto 1/z$ and $s \mapsto 1/s$, this is equivalent to writing

$$u\Big(\frac{1}{z}, t\Big) = u\Big(\frac{1}{s}, 0\Big), \tag{3-15}$$

where $s = s(z)$ is defined implicitly by

$$z - s - \frac{zst}{u(1/s, 0)} = 0. \tag{3-16}$$

Thus, it remains to apply Proposition 3.3 with $f_1(z) = u(1/z, 0)$ to conclude the proof. □

Now we turn to the proof of statement (iii) of Theorem 2.1. The fact that there exists a measure μ_t such that (2-8) holds will follow directly from Theorem 2.4. Hence we only need to establish the second part of the statement, on the properties of the moments $m_k(t)$. Notice that the fact that they are uniquely determined by the initial data $\{m_k(0)\}_{k\geq 0}$ is also a consequence of the uniqueness of the solution $u(z, t)$, already proved before.

Proof of statement (iii) *of Theorem 2.1.* Assume that μ_0 is a probability measure compactly supported in the disk $\mathbb{D}_r$, $0 < T < 1$, and take $R = R(T) > 2r$ satisfying (3-12). The unique solution $u(z, t)$ of

equation (2-1) for $t \in [0, T]$, satisfying the initial condition (2-5), has the Laurent expansion, convergent in Ω_R, given by

$$u(z,t) = \sum_{k=0}^{\infty} \frac{m_k(t)}{z^{k+1}}, \quad |z| > R, \tag{3-17}$$

with $m_0(0) = 1$. Replacing this expansion in (2-1), rewritten as

$$\partial_z u = u\, \partial_t u, \tag{3-18}$$

we get an infinite system of equations

$$\sum_{j=0}^{k} m_{k-j}(t) m_j'(t) = -(k+1) m_k(t), \quad k = 0, 1, \ldots, \tag{3-19}$$

where we have written

$$m_k'(t) = \frac{d}{dt} m_k(t).$$

For $k = 0$, we immediately get $m_0'(t) = -1$, so that $m_0(t) = 1 - t$. For $k = 1$, we obtain $m_1(t) = m_1(0)(1-t)$, so that (2-10) is valid for $k = 1$.

We conclude the proof by induction, assuming that the assertion is established for $m_1(t), \ldots, m_{k-1}(t)$, with $k \geq 2$. We need to show that $m_k(t)$ is a polynomial in t of degree at most k, whose coefficients depend only on the initial moments $m_j(0)$, $0 \leq j \leq k$, and such that

$$m_k(1) = 0, \quad -m_k'(1) = \lim_{t \to 1} \frac{m_k(t)}{1-t} = m_1(0)^k.$$

Let us rewrite (3-19) as

$$(1-t) m_k'(t) + k m_k(t) = -\sum_{j=1}^{k-1} m_{k-j}(t) m_j'(t) = -f_k'(t), \tag{3-20}$$

where

$$f_k(t) := \frac{1}{2} \sum_{j=1}^{k-1} m_{k-j}(t) m_j(t).$$

Multiplying both sides of (3-20) by the integrating factor $(1-t)^{-k-1}$ we get

$$\left(\frac{m_k(t)}{(1-t)^k} \right)' = -\frac{f_k'(t)}{(1-t)^{k+1}},$$

so that, taking into account the initial conditions,

$$m_k(t) = (1-t)^k m_k(0) - (1-t)^k \int_0^t \frac{f_k'(u)}{(1-u)^{k+1}}\, du. \tag{3-21}$$

Notice that

$$f_2(t) = \tfrac{1}{2} m_1(t)^2 = \tfrac{1}{2} m_1(0)^2 (1-t)^2, \quad f_2''(1) = m_1(0)^2,$$

and by the induction hypothesis, f_k is a polynomial of degree at most k in t, with

$$\begin{aligned} &f_k(1) = f_k'(1) = 0, \\ &f_k''(1) = \lim_{t\to 1} \frac{2f_k(t)}{(1-t)^2} = -\lim_{t\to 1} \frac{f_k'(t)}{1-t} = \sum_{j=1}^{k-1} m_{k-j}'(1) m_j'(1) = (k-1)m_1(0)^k. \end{aligned} \tag{3-22}$$

Integrating in (3-21) by parts k times and using that $f^{(k+1)} \equiv 0$, we get

$$\int_0^t \frac{f_k'(u)}{(1-u)^{k+1}}\,du = \left(\sum_{j=1}^{k} \frac{(-1)^{j-1}(k-j)!}{k!} \frac{f_k^{(j)}(u)}{(1-u)^{k+1-j}}\right)\Bigg|_0^t,$$

so that

$$-(1-t)^k\left[\int_0^t \frac{f_k'(u)}{(1-u)^{k+1}}\,du\right] = (1-t)^k d + \sum_{j=1}^{k} \frac{(-1)^j (k-j)!}{k!} f_k^{(j)}(t)(1-t)^{j-1},$$

where d is a constant:

$$d = \sum_{j=1}^{k} \frac{(-1)^{j-1}(k-j)!}{k!} f_k^{(j)}(0).$$

Combining all in (3-21), we conclude that

$$\begin{aligned} m_k(t) &= m_k(0)(1-t)^k + \sum_{j=1}^{k} (-1)^j \frac{(k-j)!}{k!}\big[f_k^{(j)}(t) - f_k^{(j)}(0)(1-t)^{k-j+1}\big](1-t)^{j-1} \\ &= -\frac{1}{k} f_k'(t) + \frac{1}{k(k-1)} f_k''(t)(1-t) + \mathcal{O}((1-t)^2), \quad t \to 1. \end{aligned}$$

By (3-22), $f_k'(1) = 0$, so that $m_k(1) = 0$, and

$$-m_k'(1) = \lim_{t\to 1} \frac{m_k(t)}{1-t} = \lim_{t\to 1}\left(-\frac{1}{k}\frac{f_k'(t)}{1-t} + \frac{1}{k(k-1)} f_k''(t)\right) = \frac{f_k''(1)}{k-1} = m_1(0)^k,$$

which proves the assertion. □

Proof of Corollary 2.3. Assume that μ_t is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ with density

$$f(x,t) := \frac{d\mu_t(x)}{dx}.$$

The Sokhotski–Plemelj formulas

$$(\mathcal{C}^\mu)_\pm(x) = \lim_{y\to 0+} \mathcal{C}^\mu(x \pm iy) = -\pi(\mathcal{H}_\mu(x) \pm i\mu'(x)), \quad x \in \mathbb{R}, \tag{3-23}$$

where $\mathcal{H}_\mu$ is the Hilbert transform defined in (2-14), imply that

$$u_\pm(x,t) = -\pi(1-t)(\mathcal{H}_{\mu_t}(x) \pm if).$$

Using this in (2-1) and subtracting both boundary values, we get the identity

$$2\pi i\,\partial_t[(1-t)f(x,t)] = 2i\,\partial_x\left(\frac{f(x,t)}{\mathcal{H}_{\mu_t}(x)}\right)\frac{1}{1+\left(\frac{f(x,t)}{\mathcal{H}_{\mu_t}(x)}\right)^2},$$

or

$$\partial_t[(1-t)f(x,t)] = \frac{1}{\pi}\partial_x \arctan\left(\frac{f(x,t)}{\mathcal{H}_{\mu_t}(x)}\right) = -\frac{1}{\pi}\partial_x \arctan\left(\frac{\mathcal{H}_{\mu_t}(x)}{f(x,t)}\right),$$

which proves (2-15). □

4. Proof of Theorem 2.4 and Corollary 2.6

The triangular table of monic polynomials $Q_{n,k}$, $\deg Q_{n,k} = n-k$, $n \in \mathbb{N}$,

$$Q_{n,k}(z) = \frac{(n-k)!}{n!}\frac{d^k}{dz^k}Q_n(z), \quad k \in \{0, 1, \ldots, n-1\},$$

was introduced in (1-2) along with their zero-counting measures $\sigma_{n,k}$ in (1-3).

The heuristic arguments supporting Theorem 2.4 were presented in Section 2, where we also introduced the functions $u_n(z,t)$, the solution of Hopf's equation (2-1) with the initial condition (2-16). As explained, our goal is to show that

$$u_{n,k}(z) := u(z, k/n)$$

and $v_{n,k}$, defined in (2-17), are asymptotically close. The solution of the Hopf equation is characterized by the identity (2-11), which motivates us to study the behavior of their functional inverses.

Since $v_{n,k}$ is the Cauchy transform of a measure of size $1-k/n$, by Proposition 3.1, the inverse function $v_{n,k}^{-1}(\zeta)$ is guaranteed to exist for all $(\zeta, k/n) \in \mathbb{D}_\rho \times [0,T]$, $0 < T < 1$, where $\rho = (1-T)/(4R)$ and $R > (\sqrt{2}+1)r$; moreover, $v_{n,k}^{-1}(\mathbb{D}_\rho) \subset \Omega_R$.

On the other hand, recall that

$$u_n(z,t) = u_n(s,0) = \mathcal{C}^{\sigma_{n,0}}(s) = \frac{Q_n'(s)}{nQ_n(s)},$$

and by Proposition 3.3, the inverse function $u_{n,k}^{-1}(\zeta)$ is guaranteed to exist for all $(\zeta, k/n) \in \mathbb{D}_{1/(4R)} \times [0,T]$, $0 < T < 1$, where $R > (\sqrt{2}+1)r$; also, $u_{n,k}^{-1}(\mathbb{D}_\rho) \subset \Omega_R$.

In summary, given $R > (\sqrt{2}+1)r$ and $0 < T < 1$, there exists a $0 < \rho < (1-T)/(4R)$ such that we can consider the inverse functions $u_{n,k}^{-1}(\zeta)$ and $v_{n,k}^{-1}(\zeta)$, conformal mappings of $\mathbb{D}_\rho$ into Ω_R for every $k/n \in [0,T]$. We fix these values of R and ρ for the rest of the proof.

Proposition 4.1. *For the inverse functions $v_{n,k}^{-1}(\zeta)$ and $v_{n,k+1}^{-1}(\zeta)$, with $\zeta \in \mathbb{D}_\rho$, we have*

$$v_{n,k+1}^{-1}(\zeta) - v_{n,k}^{-1}(\zeta) = -\frac{1}{n\zeta} + \Delta_{n,k}(\zeta). \tag{4-1}$$

There is $\rho' < \rho$ and $M > 0$, depending only on r, such that the $\Delta_{n,k}$ are analytic in $\mathbb{D}_{\rho'}$ and

$$|\Delta_{n,k}(\zeta)| \le \frac{M}{n^2}, \quad \zeta \in \mathbb{D}_{\rho'}. \tag{4-2}$$

Proof. Notice that, by (2-18),

$$v_{n,k+1}^{-1}(\zeta) - v_{n,k}^{-1}(\zeta) + \frac{1}{n\zeta}$$

is analytic at $\zeta = 0$, so it is enough to establish the bound (4-2) on a circle $|\zeta| = \rho'$.

To simplify the notation, define $\varphi_{n,k}(\zeta) := v_{n,k}^{-1}(\zeta)$ and let $'$ stand for the derivative with respect to the corresponding variable that should be clear in each context: $v'_{n,k}(z) = \partial_z v_{n,k}(z)$, $\varphi'_{n,k}(\zeta) = \partial_\zeta v_{n,k}^{-1}(\zeta)$, etc. We also define $z_k := \varphi_{n,k}(\zeta)$ and $z_{k+1} := \varphi_{n,k+1}(\zeta)$.

Substituting $z_k = \varphi_{n,k}(\zeta)$ in (2-19) and using that $v'_{n,k}(z_k) = 1/\varphi'_{n,k}(\zeta)$, we obtain

$$v_{n,k+1}(\varphi_{n,k}(\zeta)) = \zeta + \frac{1}{n\zeta}\frac{1}{\varphi'_{n,k}(\zeta)}.$$

Applying now $\varphi_{n,k+1} = v_{n,k+1}^{-1}$ to both sides yields

$$\varphi_{n,k}(\zeta) = \varphi_{n,k+1}\left(\zeta + \frac{1}{n\zeta}\frac{1}{\varphi'_{n,k}(\zeta)}\right),$$

and thus

$$\varphi_{n,k+1}(\zeta) - \varphi_{n,k}(\zeta) = \varphi_{n,k+1}(\zeta) - \varphi_{n,k+1}\left(\zeta + \frac{1}{n\zeta}\frac{1}{\varphi'_{n,k}(\zeta)}\right). \tag{4-3}$$

Using that, for small $s \in \mathbb{C}$,

$$\varphi_{n,k+1}(\zeta + s) = \varphi_{n,k+1}(\zeta) + \varphi'_{n,k+1}(\zeta)s + \tfrac{1}{2}\varphi''_{n,k+1}(\zeta + \theta)s^2, \tag{4-4}$$

we get that

$$\begin{aligned} z_{k+1} - z_k = \varphi_{n,k+1}(\zeta) - \varphi_{n,k}(\zeta) &= -\frac{1}{n\zeta}\frac{\varphi'_{n,k+1}(\zeta)}{\varphi'_{n,k}(\zeta)} - \frac{\delta_{n,k}(\zeta)}{n^2\zeta^2} \\ &= -\frac{1}{n\zeta} - \frac{1}{n\zeta}\frac{\varphi'_{n,k+1}(\zeta) - \varphi'_{n,k}(\zeta)}{\varphi'_{n,k}(\zeta)} - \frac{\delta_{n,k}(\zeta)}{n^2\zeta^2}. \end{aligned} \tag{4-5}$$

With the notation introduced above, let us rewrite

$$\frac{\varphi'_{n,k+1}(\zeta) - \varphi'_{n,k}(\zeta)}{\varphi'_{n,k}(\zeta)} = \frac{v'_{n,k}(z_k) - v'_{n,k+1}(z_{k+1})}{v'_{n,k+1}(z_{k+1})} = \frac{1}{n}\varepsilon_{n,k}^{(1)}(\zeta) - \varepsilon_{n,k}^{(2)}(\zeta)(u - z), \tag{4-6}$$

where, again by (2-19),

$$\begin{aligned} \varepsilon_{n,k}^{(1)}(\zeta) = n\frac{v'_{n,k}(z_{k+1}) - v'_{n,k+1}(z_{k+1})}{v'_{n,k+1}(z_{k+1})} &= -\frac{1}{v'_{n,k+1}(z_{k+1})}\left(\frac{v'_{n,k}(u)}{v_{n,k}(u)}\right)\Big|_{u=z_{k+1}} \\ &= -\frac{1}{z_{k+1}^2 v'_{n,k+1}(z_{k+1})} z_{k+1}^2\left(\frac{v'_{n,k}(u)}{v_{n,k}(u)}\right)'\Big|_{u=z_{k+1}} \end{aligned} \tag{4-7}$$

and

$$\varepsilon_{n,k}^{(2)}(\zeta) = \frac{v'_{n,k}(z_k) - v'_{n,k}(z_{k+1})}{z_k - z_{k+1}}\frac{1}{v'_{n,k+1}(z_{k+1})}.$$

Combining (4-5) and (4-6) we get

$$(z_{k+1}-z_k)\left(1-\frac{\varepsilon_{n,k}^{(2)}(\zeta)}{n\zeta}\right)=-\frac{1}{n\zeta}-\frac{\varepsilon_{n,k}^{(1)}(\zeta)}{n^2\zeta}-\frac{\delta_{n,k}(\zeta)}{n^2\zeta^2}. \tag{4-8}$$

To prove (4-2), it suffices to show that $\varepsilon_{n,k}^{(1)}(\zeta)$, $\varepsilon_{n,k}^{(2)}(\zeta)$, and $\delta_{n,k}(\zeta)$ are uniformly bounded for sufficiently small $|\zeta|$ (or equivalently sufficiently large z_k and z_{k+1}).

By (4-6) and Proposition 3.1, $\varepsilon_{n,k}^{(1)}(\zeta)$ is analytic for $|z_{k+1}|\geq R$, and the functions

$$u\frac{v_{n,k}'(u)}{v_{n,k}(u)} \quad \text{and} \quad u^2v_{n,k+1}'(u)$$

are analytic and nonvanishing in $|u|\geq R$. By (3-2)–(3-3), for $|u|\geq R$,

$$\left|u\frac{v_{n,k}'(u)}{v_{n,k}(u)}\right|=\left|\frac{u^2v_{n,k}'(u)}{uv_{n,k}(u)}\right|\leq 18,$$

and in consequence,

$$\left|\frac{v_{n,k}'(u)}{v_{n,k}(u)}\right|\leq\frac{18}{R}, \quad |u|=R.$$

By the Cauchy integral formula,

$$\left|u^2\left(\frac{v_{n,k}'(u)}{v_{n,k}(u)}\right)'\right|\leq 18(R+1)^2, \quad |u|=R+1. \tag{4-9}$$

On the other hand, by (3-3),

$$\left|\frac{1}{u^2v_{n,k+1}'(u)}\right|\leq\frac{1}{1-(k+1)/n}\frac{(1+r/R)^4}{1-2r/R}\leq\frac{4(3+2\sqrt{2})}{1-(k+1)/n}, \quad |u|\geq R\geq(\sqrt{2}+1)r. \tag{4-10}$$

Using these estimates in (4-7), we conclude that $|\varepsilon_{n,k}^{(1)}(\zeta)|$ is uniformly bounded for $|z_{k+1}|\geq R+1$ and $k/n\leq T<1$.

As for $\varepsilon_{n,k}^{(2)}(\zeta)$, we notice that for $z_k, z_{k+1}\in\Omega_R$

$$\left|\frac{v_{n,k}'(z_k)-v_{n,k}'(z_{k+1})}{z_k-z_{k+1}}\right|\leq\max_{s\in\Omega_R}|v_{n,k}''(s)|.$$

Appealing again to (3-2) and Cauchy's formula, we get the uniform boundedness of the function $\varepsilon_{n,k}^{(2)}(\zeta)$ for $z_k, z_{k+1}\in\Omega_{R+1}$.

Finally, by (4-4), $\delta_{n,k}(\zeta)$ in (4-5) satisfies

$$|\delta_{n,k}(\zeta)|\leq\frac{1}{2}\max_{\zeta\in\mathbb{D}_\rho}\frac{\varphi_{n,k+1}''(\zeta)}{(\varphi_{n,k}'(\zeta))^2}, \quad \zeta\in\mathbb{D}_\rho. \tag{4-11}$$

We have

$$\begin{aligned}\frac{\varphi_{n,k+1}''(\zeta)}{(\varphi_{n,k}'(\zeta))^2}=-\frac{v_{n,k+1}''(z_{k+1})}{(v_{n,k+1}'(z_{k+1}))^3}(v_{n,k}'(z_k))^2&=-\frac{v_{n,k+1}''(z_{k+1})}{v_{n,k+1}'(z_{k+1})}\left(\frac{v_{n,k}'(z_k)}{v_{n,k+1}'(z_{k+1})}\right)^2\\&=-\frac{v_{n,k+1}''(z_{k+1})}{v_{n,k+1}'(z_{k+1})}\left(1+\frac{\varepsilon_{n,k}(\zeta)}{n}\right)^2.\end{aligned}$$

Reasoning as above, by (3-2),

$$|v''_{n,k+1}(u)| \le \frac{4(1-(k+1)/n)}{R}, \quad |u| = R+1,$$

while, by (4-10),

$$\left|\frac{1}{v'_{n,k+1}(u)}\right| \le \frac{4(3+2\sqrt{2})R^2}{1-(k+1)/n}, \quad |u| \ge R \ge (\sqrt{2}+1)r,$$

showing that $\delta_{n,k}$ in (4-5) is uniformly bounded also. □

We are one step away from proving Theorem 2.4. Since $u(z,t)$ is the solution of (2-1) with the initial condition (2-5), it satisfies (2-11), which implies that

$$(u^{-1}_{n,k+1} - u^{-1}_{n,k})(\zeta) = -\frac{1}{n\zeta}.$$

Combining it with (4-1) we get that

$$(v^{-1}_{n,k+1}(\zeta) - u^{-1}_{n,k+1}(\zeta)) - (v^{-1}_{n,k}(\zeta) - u^{-1}_{n,k}(\zeta)) = \Delta_{n,k}(\zeta) = \mathcal{O}\Big(\frac{1}{n^2}\Big), \quad |\zeta| = \rho',$$

uniformly over $n, k \in \mathbb{N}$ such that $k/n < T$. Since $u^{-1}_{n,0} = v^{-1}_{n,0}$, we conclude that

$$u^{-1}_{n,k}(\zeta) - v^{-1}_{n,k}(\zeta) = \mathcal{O}\Big(\frac{1}{n}\Big) \tag{4-12}$$

locally uniformly in $\mathbb{D}_\rho$.

In order to complete the proof, it remains to show that (4-12) implies proximity of the direct functions; that is,

$$u_{n,k}(z) - v_{n,k}(z) = \mathcal{O}\Big(\frac{1}{n}\Big) \tag{4-13}$$

locally uniformly in Ω_R for some $R > r$. For that, it is more convenient to work with the functions

$$\hat{u}_{n,k}(z) := u_{n,k}\Big(\frac{1}{z}\Big), \quad \hat{v}_{n,k}(z,t) := v_{n,k}\Big(\frac{1}{z}, t\Big),$$

which are holomorphic in $\overline{\mathbb{D}}_{1/R}$ and nonvanishing in $\overline{\mathbb{D}}_{1/R} \setminus \{0\}$. Also,

$$\hat{u}'_{n,k}(0) = \hat{v}'_{n,k}(0) = 1 - \frac{k}{n} \ge 1 - T > 0,$$

and both $\hat{u}_{n,k}$ and $\hat{v}_{n,k}$ are uniformly bounded in $\overline{\mathbb{D}}_{1/R}$: there exists $M > 0$, depending on T and R only, such that

$$|\hat{u}_{n,k}(z)| \le M, \quad |\hat{v}_{n,k}(z)| \le M, \qquad z \in \overline{\mathbb{D}}_{1/R},$$

for every pair $(k,n) \in \mathbb{N} \cup \{0\} \times \mathbb{N}$ with $k/n \in [0,T]$. The assertion (4-13) is a consequence of the general facts gathered in Propositions A.1–A.3 that, for the sake of readability, we defer to the Appendix. This concludes the proof of Theorem 2.4.

Proof of Corollary 2.6. By (2-22)–(2-23),

$$\mathcal{R}_{\mu_0^{\boxplus 1/(1-t)}}(w) = \frac{1}{1-t}\mathcal{R}_{\mu_0}(w) = \frac{1}{1-t}[\mathcal{C}^{\mu_0}]^{-1}(w) - \frac{1}{(1-t)w},$$

where $[\mathcal{C}^{\mu}]^{-1}$ is the functional inverse of $\mathcal{C}^{\mu}$. Taking into account (2-11), we have that

$$\begin{aligned}\mathcal{R}_{\mu_0^{\boxplus 1/(1-t)}}(w) = \frac{1}{1-t}u^{-1}(w,0) - \frac{1}{(1-t)w} &= \frac{1}{1-t}\left(u^{-1}(w,t) + \frac{t}{w}\right) - \frac{1}{(1-t)w} \\ &= \frac{1}{1-t}u^{-1}(w,t) - \frac{1}{w},\end{aligned}$$

so that

$$\left[\mathcal{C}^{\mu_0^{\boxplus 1/(1-t)}}\right]^{-1}(w) = \frac{1}{1-t}u^{-1}(w,t),$$

or equivalently

$$\mathcal{C}^{\mu_0^{\boxplus 1/(1-t)}}(z) = \mathcal{C}^{\mu_t}((1-t)z) = u((1-t)z,t). \tag{4-14}$$

On the other hand, for

$$\hat{\sigma}_{n,k} := \frac{1}{n-k}\chi(\widehat{Q}_{n,k})$$

we have that

$$\mathcal{C}^{\hat{\sigma}_{n,k}}(z) = \frac{1}{n-k}\frac{\partial_z \widehat{Q}_{n,k}(z)}{\widehat{Q}_{n,k}(z)} = (1-t)\frac{1}{n-k}\frac{Q'_{n,k}(z)}{Q_{n,k}(z)}((1-t)z) = (1-t)\frac{n}{n-k}\mathcal{C}^{\sigma_{n,k}}((1-t)z),$$

with $\sigma_{n,k}$ defined in (1-3). Hence, for $k/n \to t$, by Theorem 2.4,

$$\lim_n \mathcal{C}^{\hat{\sigma}_{n,k}}(z) = \mathcal{C}^{\mu_t}((1-t)z),$$

so that, by (4-14),

$$\lim_n \mathcal{C}^{\hat{\sigma}_{n,k}}(z) = \mathcal{C}^{\mu_0^{\boxplus 1/(1-t)}}(z).$$

For measures on the real line, this is equivalent to (2-24). □

5. Some examples

We finish our exposition by presenting a few simple examples, starting with some paradigmatic zero distributions on the real line.

It is well known that under very mild assumptions on the orthogonality measure on the interval $[-1, 1]$ (such as having a strictly positive Radon–Nikodym derivative with respect to the Lebesgue measure almost everywhere on $[-1, 1]$; see, e.g., [Stahl and Totik 1992]), the limit zero distribution of the corresponding orthogonal polynomials is the equilibrium measure (or arcsine distribution) of the interval $[-1, 1]$; that is,

$$d\mu_0(x) = \frac{1}{\pi\sqrt{1-x^2}}\,dx, \quad x \in (-1, 1),$$

for which

$$u(z,0) = \mathcal{C}^{\mu_0}(z) = \frac{1}{\sqrt{z^2-1}} = \frac{1}{z} + \mathcal{O}\left(\frac{1}{z^2}\right), \quad z \in \mathbb{C} \setminus [-1, 1],$$

and consequently,

$$u^{-1}(w,0)=\frac{\sqrt{w^2+1}}{w}.$$

By (2-11), solving the equation

$$z=\frac{\sqrt{w^2+1}-t}{w}$$

we get that

$$w=u(z,t)=\frac{-tz+\sqrt{z^2-(1-t^2)}}{z^2-1}.$$

Since the residues of the right-hand side at ± 1 are 0, we conclude that μ_t is absolutely continuous and

$$d\mu_t(x)=\frac{1}{\pi}\frac{\sqrt{1-t^2-x^2}}{1-x^2}\,dx,\quad x\in\left[-\sqrt{1-t^2},\sqrt{1-t^2}\right], \tag{5-1}$$

where we have used the Stieltjes–Perron inversion formula.

Another notable case is the semicircle distribution on the interval $[-1,1]$,

$$d\mu_0(x)=\frac{2}{\pi}\sqrt{1-x^2}\,dx,\quad x\in(-1,1),$$

corresponding, for instance, to the zero limit distribution of the appropriately rescaled Hermite polynomials. Since

$$u(z,0)=2(z-\sqrt{z^2-1})=\frac{1}{z}+\mathcal{O}\left(\frac{1}{z^2}\right),\quad z\in\mathbb{C}\setminus[-1,1],$$

and

$$u^{-1}(w,0)=\frac{w^2+4}{4w},$$

solving the equation

$$z=\frac{w^2+4-4t}{4w},$$

we get that

$$w=u(z,t)=2(z-\sqrt{z^2-(1-t)}).$$

Again, μ_t is the absolutely continuous probability measure given by

$$d\mu_t(x)=\frac{2}{\pi}\sqrt{1-t-x^2}\,dx,\quad x\in\left[-\sqrt{1-t},\sqrt{1-t}\right], \tag{5-2}$$

the rescaled semicircle distribution for all values of $t\in[0,1)$.

The case of the Marchenko–Pastur distribution on $[0,4]$, given by

$$d\mu_0(x)=\frac{1}{2\pi}\frac{\sqrt{x(4-x)}}{x}\,dx,$$

that describes the asymptotic behavior of singular values of large random matrices (or equivalently, the asymptotic zero distribution of rescaled Laguerre polynomials) is analyzed in a similar way.

On the other hand, a natural case to consider is of the normalized uniform distribution on $[-1, 1]$. It turns out it is remarkably challenging; indeed, in this case

$$u(z, 0) = \frac{1}{2}\log\Big(\frac{z+1}{z-1}\Big), \quad z \in \mathbb{C} \setminus [-1, 1],$$

and

$$u^{-1}(w, 0) = \frac{v+1}{v-1}, \quad v = e^{2w}.$$

However, the equation

$$z = \frac{v+1}{v-1} - \frac{t}{\log v}, \quad v = e^{2w}, \tag{5-3}$$

is highly transcendental and does not have an explicit solution. We can use this representation to find the support of the measure μ_t, which again appears to be a formidable task.

Let us end with the case where μ_0 is discrete and is the normalized zero-counting measure of a given polynomial P of degree m, with real or complex zeros. This is the case, for instance, addressed at the beginning, when the Q_n are generated by a Rodrigues-type formula. As mentioned, this situation has been studied before in [Bøgvad et al. 2024] using different techniques.

Now

$$u(z, 0) = \frac{1}{m}\frac{d}{dz}\log P(z),$$

and (2-12) reduces to[4]

$$mP(z + tv) - vP'(z + tv) = 0, \quad v = \frac{1}{w}.$$

Using that

$$P(z + s) = \sum_{j=0}^{m} \frac{P^{(j)}(z)}{j!} s^j,$$

this yields an algebraic equation for $w = u(z, t)$,

$$\sum_{j=0}^{m} \left(m - \frac{j}{t}\right) \frac{P^{(j)}(z)}{j!} t^j w^{m-j} = 0, \tag{5-4}$$

also derived in [Bøgvad et al. 2024].

For instance, in our first example (1-7), when $P(z) = z^2 - 1$ and $m = 2$,

$$u(z, 0) = \frac{z}{z^2 - 1},$$

and (5-4) yields the quadratic equation for $w = u(z, t)$:

$$(z^2 - 1)w^2 - (1 - 2t)zw - (1 - t)t = 0.$$

Thus,

$$u(z, t) = \frac{(1 - 2t)z - \sqrt{4(t - 1)t + z^2}}{2(z^2 - 1)},$$

and we get the expressions (1-8)–(1-9) for the measure μ_t.

[4]This equation appears in [Bøgvad et al. 2024, Formula (1.6)] in the context that is a particular case of a discrete measure μ_0.

Remark 5.1. Ravichandran [2020] (see also [Totik 2021]) proved that, for $t \geq \frac{1}{2}$, the ratio of the diameters of the zero sets of $Q_{n,k}(x)$, defined in (1-2), and Q_n, with $k = [nt]$, is bounded above by $2\sqrt{t(1-t)}$. Obviously, both (5-1) and (5-2) are in agreement with this bound, and formulas (1-8)–(1-9) show that the assumption that $t \geq \frac{1}{2}$ cannot be omitted.

On the other hand, in Kalyagin's example (1-11), when $P(z) = z(z^2-1)$ and $m = 3$, (5-4) yields the following cubic equation for $w = u(z,t)$:

$$3z(z^2-1)w^3 + (3t-1)(3z^2-1)w^2 + 3t(3t-2)zw - 3t^2(1-t) = 0.$$

For $t \in (0,1)$, the end-points of the support of μ_t can be found from the branch points of w, which are solutions of the biquadratic equation

$$9z^4 + 3(9t^2-6t-2)z^2 - (1-t)(3t-1)^3 = 0;$$

see [Kalyagin 1979].

Finally, if P has complex roots, such as $P(z) = z^3 - 1$, (5-4) yields again a cubic equation for $w = u(z,t)$,

$$(z^3-1)w^3 - (1-3t)z^2w^2 - (2-3t)tzw - (1-t)t^2 = 0,$$

whose branch points are solutions of the equation

$$27t^3 - 54t^2 + 27t - 4z^3 = 0,$$

that is, the three cube roots

$$z = 3\sqrt[3]{\frac{t(1-t)^2}{4}}.$$

Appendix: Auxiliary results

Denote by $\mathcal{H}(R, M, c)$ the class of functions f satisfying the following conditions:

(i) f is analytic in the closed disc $\overline{\mathbb{D}}_R$;

(ii) $|f(z)| \leq M$ for $z \in \overline{\mathbb{D}}_R$;

(iii) $f'(0) = c > 0$.

Proposition A.1. *For $R, M, c > 0$, let*

$$r_1 := \min\left\{\frac{R}{2}, \frac{cR^2}{4M}\right\}. \tag{A-1}$$

Then, for $r \leq r_1$, any function $f \in \mathcal{H}(R, M, c)$ is univalent in $\overline{\mathbb{D}}_r$ and $f(\overline{\mathbb{D}}_r)$ covers the disc $\overline{\mathbb{D}}_\rho$ with $\rho = \frac{1}{2}cr$.

Consequently, f^{-1} is univalent in $\overline{\mathbb{D}}_\rho$ and $f^{-1}(\overline{\mathbb{D}}_\rho) \subset \mathbb{D}_r$.

Proof. Let

$$f(z) = cz + h(z) \in \mathcal{H}(R, M, c) \quad \text{and} \quad h(z) = \sum_{n=2}^{\infty} c_n z^n.$$

Since h is holomorphic in the closed disc $\overline{\mathbb{D}}_R$, the Cauchy inequalities imply $|c_n| \le M/R^n,\ n = 2, 3, \ldots,$ so that for any $r < R$

$$|h(z)| \le M \sum_{n=2}^{\infty} \left(\frac{r}{R}\right)^n = M \frac{(r/R)^2}{1-(r/R)}, \quad z \in \mathbb{D}_r.$$

If $r \le \frac{1}{2}R$ then $1 - r/R \ge \frac{1}{2}$, and

$$|h(z)| \le 2M \left(\frac{r}{R}\right)^2, \quad z \in \mathbb{D}_r.$$

Also,

$$r \le \frac{cR^2}{4M} \quad \Longleftrightarrow \quad cr \ge 4M \left(\frac{r}{R}\right)^2.$$

This means that if we assume $r \le r_1$, it implies, for $z \in C_r := \partial\mathbb{D}_r = \{z \in \mathbb{C} : |z| = r\}$, that $|cz| \ge 2|h(z)| > 0$. This means that the image of the circle C_r by f lies in the annulus $\frac{1}{2}cr = \rho \le |z| \le 3\rho$ and, by Rouché's theorem, its index with respect to the origin is one. The index of $f(C_r)$ with respect to any point $f(a) \notin f(C_r)$ is constant in any connected component of the complement to $f(C_r)$ and is equal to the multiplicity of $f(a)$ in $\mathbb{D}_r$. We conclude that this index is 1 for any $a \in \mathbb{D}_r$, concluding that f is univalent in $\mathbb{D}_r$. □

Proposition A.2. *Let $f \in \mathcal{H}(R, M, c)$ and*

$$r_0 := \min\left\{\frac{R}{2}, \frac{cR^2}{16M}\right\} \le r_1, \tag{A-2}$$

where r_1 was defined in (A-1).

Then for any $r \le r_0$ we have $|f'(z)| \ge \frac{1}{2}c$ for $z \in \mathbb{D}_r$.

Proof. Since $f'(z) = c + h'(z),\ h(z) = \sum_{n=2}^{\infty} c_n z^n$, the Cauchy inequalities imply again that, for $z \in \mathbb{D}_r$,

$$|h'(z)| \le \left|\sum_{n=2}^{\infty} n c_n z^{n-1}\right| \le M \sum_{n=2}^{\infty} n \frac{r^{n-1}}{R^n} = \frac{Mr}{R^2} \sum_{n=1}^{\infty} (n+1)\left(\frac{r}{R}\right)^{n-1}$$
$$\le \frac{2Mr}{R^2} \sum_{n=1}^{\infty} n\left(\frac{r}{R}\right)^{n-1} = \frac{2Mr}{R^2} \frac{1}{(1-r/R)^2} = \frac{2Mr}{(R-r)^2}.$$

If $r \le \frac{1}{2}R$, then $|h'(z)| \le 2Mr/r^2 = 2M/r$ for $z \in \mathbb{D}_r$. On the other hand, if additionally $r \le cR^2/(16M)$ then $c \ge 16Mr/R^2 \ge 4M/r$, which implies that, for $z \in \mathbb{D}_r$,

$$|f'(z)| \ge |c + h'(z)| \ge c - |h'(z)| \ge c - \frac{2M}{r} \ge c - \frac{8Mr}{R^2} \ge \frac{c}{2}.$$ □

Proposition A.3. *Let $f, g \in \mathcal{H}(R, M, c)$ and, for some $\epsilon > 0$, $|f(z) - g(z)| < \epsilon$ for $z \in \mathbb{D}_R$. Then there exist positive ρ and C, depending only on R, M, and c, such that the inverse functions f^{-1} and g^{-1} exist in $\mathbb{D}_\rho$ and*

$$m(\rho) := \max_{\zeta \in \overline{\mathbb{D}}_\rho} |f^{-1}(\zeta) - g^{-1}(\zeta)| \le C\epsilon. \tag{A-3}$$

Proof. Fix r_0 defined in (A-2) and consider $r \in (0, r_0)$, as long as the corresponding ρ equals $\frac{1}{2}cr$. By Proposition A.1, for any such r and ρ, the functions $f^{-1}(\zeta)$, $g^{-1}(\zeta)$ exist, are univalent in $\overline{\mathbb{D}}_\rho$, and map $\overline{\mathbb{D}}_\rho$ into $\mathbb{D}_r$. Thus, for any $\rho \le \frac{1}{2}c$ and $\zeta \in \mathbb{D}_\rho$,

$$\begin{aligned} |f^{-1} - g^{-1}|(\zeta) = \left| \int_0^\zeta \left[(f^{-1})'(w) - (g^{-1})'(w) \right] dw \right| &= \int_0^\zeta \left| \frac{1}{f'(f^{-1}(w))} - \frac{1}{g'(g^{-1}(w))} \right| |dw| \\ &\le \rho \max_{w \in \mathbb{D}_\rho} \left| \frac{g'(g^{-1}(w)) - f'(f^{-1}(w))}{g'(g^{-1}(w)) f'(f^{-1}(w))} \right| \\ &\le \frac{4\rho}{c^2} \max_{w \in \mathbb{D}_\rho} \left| g'(g^{-1}(w)) - f'(f^{-1}(w)) \right|, \quad \text{(A-4)} \end{aligned}$$

where we have used Proposition A.2. For $w \in \mathbb{D}_\rho$,

$$g'(g^{-1}(w)) - f'(f^{-1}(w)) = \big(g'(g^{-1}(w)) - f'(g^{-1}(w)) \big) + \big(f'(g^{-1}(w)) - f'(f^{-1}(w)) \big),$$

and thus

$$\left| g'(g^{-1}(w)) - f'(f^{-1}(w)) \right| \le \left| (g' - f')(g^{-1}(w)) \right| + \left| f'(g^{-1}(w)) - f'(f^{-1}(w)) \right|. \quad \text{(A-5)}$$

Observe that, by the Cauchy integral formula, for $f \in \mathcal{H}(R, M, c)$ and $z \in \mathbb{D}_r$, with any $r < R$, we have $|f'(z)| \le M/(R - r)$ and $|2f''(z)| \le M/(R - r)^2$. Thus, since $z = g^{-1}(w) \in \mathbb{D}_r$ for $w \in \mathbb{D}_\rho$, we have

$$\left| g'(g^{-1}(w)) - f'(g^{-1}(w)) \right| = |(g' - f')(z)| \le \frac{\epsilon}{R - r} \le 2\epsilon$$

and

$$\left| f'(g^{-1}(w)) - f'(f^{-1}(w)) \right| \le \max_{z \in \mathbb{D}_r} |f''(z)| \max_{w \in \mathbb{D}_\rho} |g^{-1}(w) - f^{-1}(w)| \le \frac{M}{2(R - r)^2} m(\rho) \le \frac{2M}{R^2} m(\rho).$$

Combining these bounds in (A-5) we obtain from (A-4)

$$m(\rho) \le \frac{4\rho}{c^2} \left(2\epsilon + \frac{2M}{R^2} m(\rho) \right).$$

Thus, selecting for instance $\rho = c^2 R^2/(16M)$ we obtain (A-3) with $C = M/R^2$. □

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