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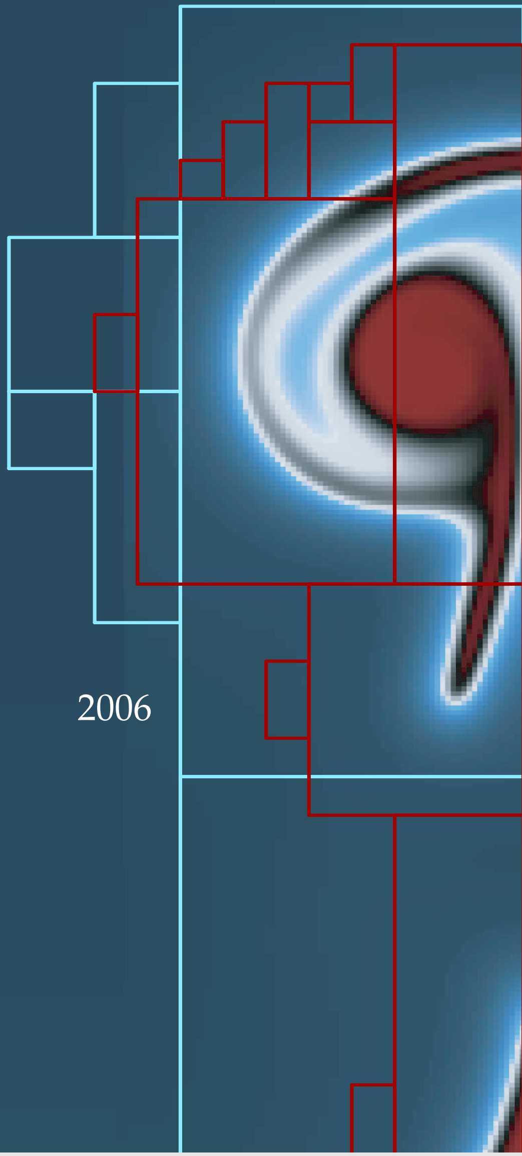

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BIFURCATED EQUILIBRIA AND MAGNETIC ISLANDS IN TOKAMAKS AND STELLARATORS

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The magnetohydrodynamic variational principle is employed to calculate equilibrium and stability of toroidal plasmas without two-dimensional symmetry. Differential equations are solved in a conservation form that describes force balance correctly across islands that are treated as discontinuities. The method is applied to both stellarators and tokamaks, and comparison with observations is favorable in both cases. Sometimes the solution of the equations turns out not to be unique, and there exist bifurcated equilibria that are nonlinearly stable when other theories predict linear instability. The calculations are consistent with recent measurements of high values of the pressure in stellarators. For tokamaks we compute three-dimensionally asymmetric solutions that are subject to axially symmetric boundary conditions.

1. Introduction

A community of industrialized nations is planning construction of the International Thermonuclear Experimental Reactor (ITER). A facility has been designed to test the concept of fusing deuterium and tritium ions so as to form helium and release energetic neutrons that can produce electric power at commercially viable cost [1]. This is to be achieved by confining a very hot plasma of ions and electrons in a strong magnetic field with toroidal geometry and a major radius of 6m. The magnetic fusion configuration preferred for ITER is a tokamak, which is axially symmetric and requires net toroidal current for confinement of the plasma. An alternate concept that seems to be more stable is the stellarator, which has fully three-dimensional geometry generating a poloidal field that eliminates the need for induced current.

Recent advances in high performance computing have led to significant progress in the theory of equilibrium, stability and transport for fusion plasmas in three dimensions. This has made it possible to design stellarators that are competitive

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with tokamaks as candidates for a fusion reactor. The work we shall describe in this direction is based on the NSTAB, VMEC and TRAN computer codes [2; 4; 9; 12; 17]. In particular, we consider simulations of anomalous thermal transport in tokamaks that result from calculations of bifurcated equilibria that do not have two-dimensional symmetry. For both tokamaks and stellarators difficult mathematical problems are encountered because accurate solutions of the relevant differential equations turn out to have discontinuities associated with islands and current sheets in the plasma (see Figure 1).

We begin with a study of weak solutions of the partial differential equations governing magnetohydrodynamic (MHD) equilibrium in three dimensions. Then we examine the role played by the magnetic spectrum in estimating the prompt loss of α particles in a reactor. Finally, we discuss candidates for a demonstration of the magnetic fusion concept after the ITER project is completed.

2. Computation of force balance

The KAM theory of dynamical systems predicts that smooth solutions of the partial differential equations describing MHD equilibrium of a toroidal plasma cannot be found in the absence of two-dimensional symmetry [2]. Let B be the magnetic field, $J = \nabla \times B$ be the current density, $p = p(s)$ be the scalar pressure, s be the toroidal flux, $\theta + \iota\phi$ and ϕ be invariant poloidal and toroidal angles, and ι be the rotational transform measuring how far a magnetic line circulates poloidally during one transit the long way around the torus. We call the Fourier coefficients B_{mn} in a representation

$$1/B^2 = \sum B_{mn}(s)\cos\bigl(m\theta - [n - \iota m]\phi\bigr)$$

of the magnetic field strength the magnetic spectrum. For stellarators an elementary manipulation of the MHD equations leads to a corresponding formula

$$\frac{J \cdot B}{B^2} = p' \sum \frac{m B_{mn}}{n - \iota m} \cos\bigl(m\theta - [n - \iota m]\phi\bigr)$$

for the parallel current in which the term with $m = n = 0$ is omitted. In this context the small denominators $n - \iota m$ explain why continuous solutions of the fully three-dimensional equilibrium problem do not exist under the hypothesis that the plasma is covered by nested toroidal flux surfaces s = const., which is important for good confinement.

To handle discontinuous solutions of the MHD equilibrium equations we write them in the conservation form

$$\nabla \cdot B = \nabla \cdot T = 0\,,$$

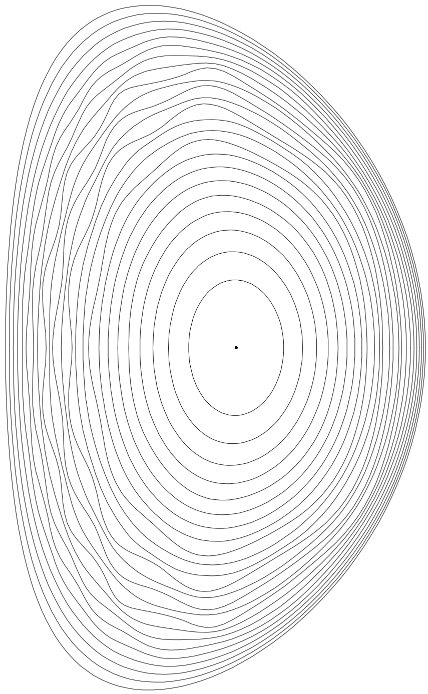

Figure 1. Poincaré section of the flux surfaces of a bifurcated ITER equilibrium at $\beta = 0.03$ with $p = p_0(1 - s^{1.1})^{1.1}$ and with net current bringing the rotational transform into the interval $0.8 > \iota > 0.2$. Ripples in the surfaces represent helical islands in this fully converged three-dimensional solution of an axially symmetric MHD problem.

where

$$T = BB - (B^2/2 + p)I$$

is the Maxwell stress tensor. Then force balance over any test volume of plasma reduces by the divergence theorem to the assertion that the surface integral

$$\iint T \cdot N \, dS = 0$$

vanishes over the boundary. Numerical methods that employ an analogous discrete conservation form of the equations provide an accurate approximation of force balance because when they are similarly summed over any collection of mesh points the result telescopes down to a corresponding statement at the boundary.

We illustrate the way conservation form captures discontinuities in weak solutions of the MHD equations by considering a one-dimensional example of a reversed field pinch (RFP) in slab geometry [8]. In a rectangular coordinate system we conceive of x as a radius and y and z as toroidal and poloidal angles. Let $(0, \Psi_x, C)$, $(0, 0, \Psi_{xx})$ and $\eta(\Psi_{xxx}, 0, 0)$ represent the magnetic field, the current density and an artificial resistivity, respectively, where Ψ is a flux function depending only on x, and C and η are constants. The MHD equilibrium equations reduce to an ordinary differential equation that we write in the conservation form

$$(\Psi_x^2)_x = \eta\Psi_{xxx} ,$$

and we seek a solution on the interval $-1 \le x \le 1$ satisfying the boundary conditions

$$\Psi(-1) = \Psi(+1) = 0 , \quad \Psi_x(-1) = 1 .$$

The finite difference approximation

$$(\Psi_{n+1} - \Psi_n)^2 - (\Psi_n - \Psi_{n-1})^2 = \eta(\Psi_{n+2} - 3\Psi_{n+1} + 3\Psi_n - \Psi_{n-1})$$

of the RFP equation is in conservation form and defines iterations that converge in the limiting case $\eta = 0$ to the correct answer $\Psi = 1 - |x|$. This has a jump in its derivative at the origin, but satisfies force balance there because Ψ_x^2 remains continuous. It is easy to find less symmetrical difference schemes for the same boundary value problem that are not in conservation form and therefore give results that violate force balance significantly. The numerical example we have presented is also applied in computational fluid dynamics to show that conservation form is required to capture shock waves accurately [3].

The NSTAB computer code calculates toroidal equilibrium of stellarators and tokamaks by a numerical scheme that is in a conservation form associated with the MHD variational principle [4; 17]. Good convergence is achieved by applying the spectral method to describe dependence of the solution on the poloidal and toroidal angles and by using an exceptionally accurate finite difference approximation in the radial coordinate s. The high resolution of the radial scheme has been established by comparing numerical results with exact solutions [2]. The NSTAB code models magnetic fusion configurations effectively using a suitable Fourier series to represent the fixed boundary of the plasma.

Linear and nonlinear stability are tested by looking for bifurcated equilibria that do not have symmetries occurring in conventional models. This method has provided acceptable simulations of experiments for stellarators that exceed stability predictions of linear theory [6]. More specifically, our computations agree with recent observations [18] in the Large Helical Device (LHD) at the National Institute for Fusion Science (NIFS) in Japan of values of the performance parameter $\beta = 2p/B^2$ as high as 4%. The equilibria we examine for the LHD at such values of

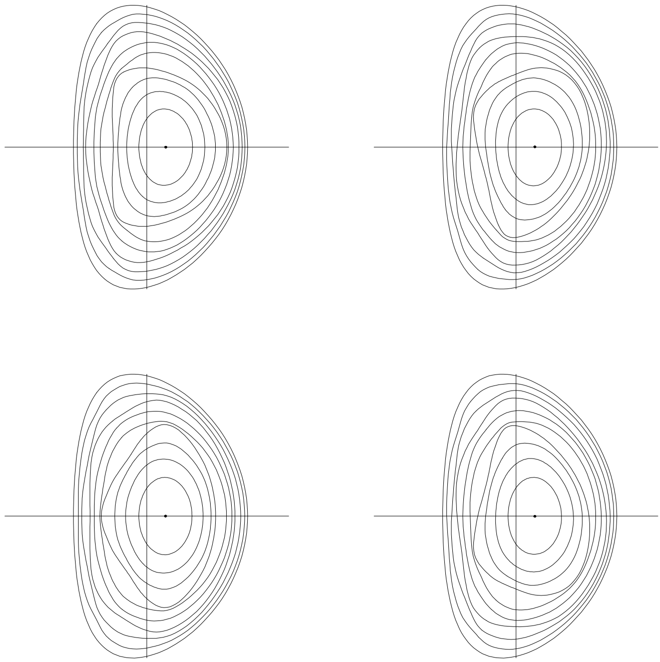

Figure 2. Four cross sections of the flux surfaces over half the torus of a bifurcated DIII-D equilibrium at $\beta = 0.02$ with $p = p_0(1-s^{1.1})^{1.1}$ and with net current bringing the rotational transform into the interval $0.9 > \iota > 0.3$. There is a large $m = 3,\ n = 2$ magnetic island at $\iota = 2/3$ in the solution that models an observed mode.

β tend to be linearly unstable, but nonlinearly stable, so that a better understanding of bifurcated solutions becomes desirable [11].

Our computational method has been applied to study neoclassical tearing modes (NTM) in the Doublet III-D (DIII-D) tokamak at General Atomics (GA) with the net current limited so that $\iota < 1$. Three-dimensional equilibria are calculated by at first imposing, but much later releasing, a suitable constraint in runs of the NSTAB code chosen to find bifurcated solutions that cannot be obtained without introducing discontinuous alterations in the topology of the magnetic surfaces. Figure 2 displays Poincaré sections of the flux surfaces of such a bifurcated equilibrium. Solutions like this are related to observations of NTM modes made in the experiment [5; 13]. On crude radial grids the computations are capable of capturing slender islands whose widths are comparable to the mesh size. The physical significance of finding many three-dimensional MHD equilibria in axially symmetric tokamaks needs

m	n	B_{mn}
0	0	0.997
1	0	0.525
2	0	0.144
3	0	0.058
4	0	0.025
0	1	0.015
3	2	0.010
1	1	0.009
4	2	0.008
1	−1	0.007

Table 1. Nontrivial coefficients in the spectrum of an optimized MHH2 configuration with a prompt loss of α particles below 10%.

further investigation. More specifically, one can ask how much their effect might contribute to the prompt loss of α particles or to disruptions.

3. Prompt loss of α particles

Neoclassical transport in tokamak and stellarator plasmas with three-dimensional geometry can be evaluated by tracking guiding center orbits of charged particles that are subjected to a random walk representing collisions. The TRAN computer code implements such a method that employs equilibria obtained from NSTAB calculations, which are needed to estimate the magnetic spectrum [9]. Substantial agreement has been found between computations of thermal transport from runs of the TRAN code and experimental observations in tokamaks and stellarators [7]. An algorithm determining the electric potential from quasineutrality in three-dimensional equilibria has been applied successfully. This theory has been used to demonstrate the advantage for stellarator transport of a magnetic spectrum with quasihelical symmetry (QHS), where only the diagonal coefficients B_{mm} are large, or with quasiaxial symmetry (QAS), where the first column of coefficients B_{m0} dominate [10; 16]. The computational approach facilitates designing new configurations that may bring the concept of magnetic fusion closer to construction of a commercially viable reactor.

The TRAN code has been modified to estimate the prompt loss of α particles in a fusion plasma at reactor conditions. This is defined to be the percentage of α particles that escape from the plasma during one slowing down time after they are born. Samples of between 128 and 1024 particles are adequate to give a meaningful answer, but that requires significant resources on a commodity work station. For stellarators the spectrum again plays a decisive role in the computations. Experience shows that only a significant improvement in the quasisymmetry required for satisfactory thermal transport can produce a loss of α particles as low as 10% that might be acceptable in the design of a fusion reactor. Table 1 lists averages with respect to s of the largest coefficients B_{mn} in a two field period configuration that has been optimized for such an application [8]. The three-dimensional asymmetry is seen to fall below half a percent if it is measured in units of the field strength B itself rather than $1/B^2$. To achieve this level of quasisymmetry presents a challenge not only to the accuracy of the codes that are used, but also to the precision of the hardware that must be fabricated.

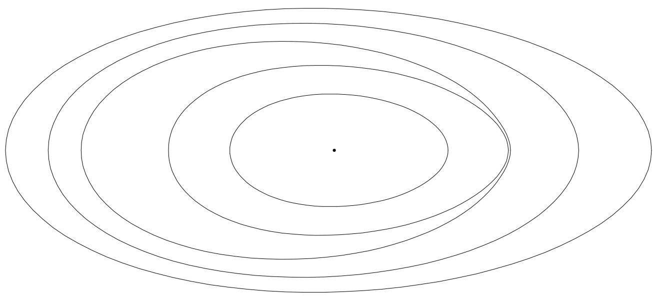

Figure 3. Zero β calculation of a Poincaré section of flux surfaces for a stellarator with reversed poloidal field. Two magnetic surfaces touch each other at an X-point where the rotational transform ι changes sign. They surround a magnetic island that would otherwise be obscured by the nested surface hypothesis implemented in the NSTAB code.

4. Magnetic islands

The NSTAB code captures islands successfully despite a nested surface hypothesis made in the coordinate system that is employed [11]. The resolution of the code can be checked by applying it to the vacuum field of stellarators where islands are known to exist in equilibria found by line tracing [14]. Figures 3 and 4 display calculations of an example of this phenomenon in which the rotational transform changes sign so that a sizeable island appears at $\iota = 0$. The same numerical construction produces helical islands in tokamaks like the DIII-D and ITER. When such three-dimensional solutions of the tokamak problem were used in computations of the energy confinement time, anomalous transport was not observed in the results [9].

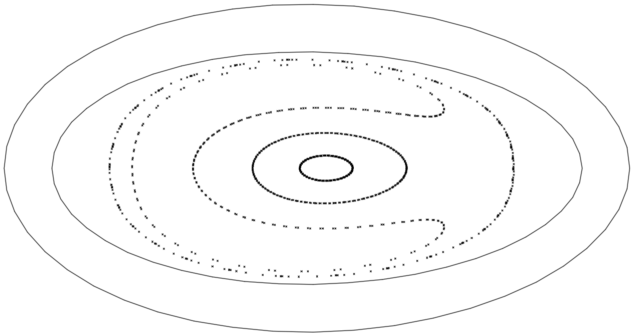

Figure 4. Tracing of magnetic lines through a Poincaré section of a stellarator with reversed poloidal field. A large magnetic island is seen where the rotational transform ι changes sign. The plasma surface is plotted together with a control surface used for Biot–Savart computation of the vacuum magnetic field.

The problem of ideal MHD equilibrium is singular in two dimensions and includes a continuous spectrum in the analysis of stability, and in three dimensions the KAM theory shows that no differentiable solutions exist [2]. So we introduce weak solutions of the kind constructed numerically by the NSTAB code, which have magnetic islands that appear as discontinuities like current sheets. Artificial resistivity implicit in the code captures the islands in a realistic fashion because of the conservation form of the MHD equations that is employed. That results in turn from a mixed Euler–Lagrange coordinate system featuring the toroidal flux as a radius. The method produces three-dimensional tokamak equilibria with small magnetic islands whose cumulative effect simulates experimental observations better than two-dimensional models do [9].

5. Configurations for a fusion reactor

A tokamak like ITER is the candidate of choice by the fusion community for a reactor. Disadvantages are that MHD instability tends to trigger disruptions, and it is hard to control the induced net current in a steady state. The calculations of NTM in the DIII-D at GA that we have described suggest that bifurcated equilibria with three-dimensional asymmetries may turn out to be important in attacking these problems [13]. Of special interest for reactors is that nuclear engineers may ultimately come to prefer a stellarator-tokamak hybrid with good quasisymmetry and small aspect ratio.

It is relatively easy to reduce the prompt loss of α particles in stellarators that have good QHS, such as the Helically Symmetric Experiment (HSX) at the University of Wisconsin with four field periods or the Wendelstein 7-X (W7-X) at Greifswald in Europe with five field periods [16]. We have studied a QHS version of the W7-X with rotational transform in the interval $1 < \iota < 5/4$ that has favorable properties of thermal transport and MHD stability. The prompt loss of α particles can be brought down to several percent by readjustment of the coefficients B_{mn} in the spectrum, but many twisted modular coils are required to maintain an equilibrium with low toroidal ripple of the magnetic field strength because the aspect ratio of the plasma is large.

Most of our theoretical work with the NSTAB and TRAN computer codes has been focused on QAS stellarators like the National Compact Stellarator Experiment (NCSX) at the Princeton Plasma Physics Laboratory (PPPL) with three field periods and the Modular Helias-like Heliac 2 (MHH2) with just two field periods [10]. The NCSX is a principal candidate for the ARIES_CS compact stellarator study of magnetic fusion reactors [15] funded by the United States Department of Energy (DOE). It is difficult to reduce the prompt loss of α particles in both the NCSX and the MHH2 because the necessary calculations are sensitive to small changes in the magnetic spectrum [8]. Net current that raises the rotational transform is helpful in these optimizations. For that one attractive configuration is a hybrid version of the MHH2 shown in Figure 5, which has major radius 8m and plasma radius 3m.

It is hard to find modular coils that generate an external magnetic field compatible with a plasma equilibrium optimized to bring the loss of α particles below 10% at reactor conditions. One possibility is to determine the solution inside the plasma from an equilibrium calculation and then apply the Biot–Savart law to match that with a vacuum field defined by a distribution of current on a suitably chosen control surface where the coils are to be placed [14]. This method could be applied to smooth out unrealistic surface current on the separatrix of an alternate approximation found by solving a free boundary value problem. The analysis taxes the resolution of the best computer codes that are available because there is a high degree of magnetic quasisymmetry required in the answer. Moreover, the harmonics specifying the coils have to be filtered judiciously to eliminate erroneous excursions. The concept is elucidated by Runge's theorem, which asserts that an analytic function can be approximated by polynomials in any simply connected region of the complex plane.

The MHH2 configuration that has been optimized to reduce the prompt loss of α particles is a good candidate for a stellarator experiment to achieve ion temperatures competitive with those in tokamaks. Moreover, three-dimensional equilibria are found numerically in tokamaks, so two-dimensional models may be less realistic. Because truncation error in the computations is insignificant compared to sources hitting the plasma in an experiment, observations may exhibit effects associated with three-dimensional asymmetries in a bifurcated solution of the problem.

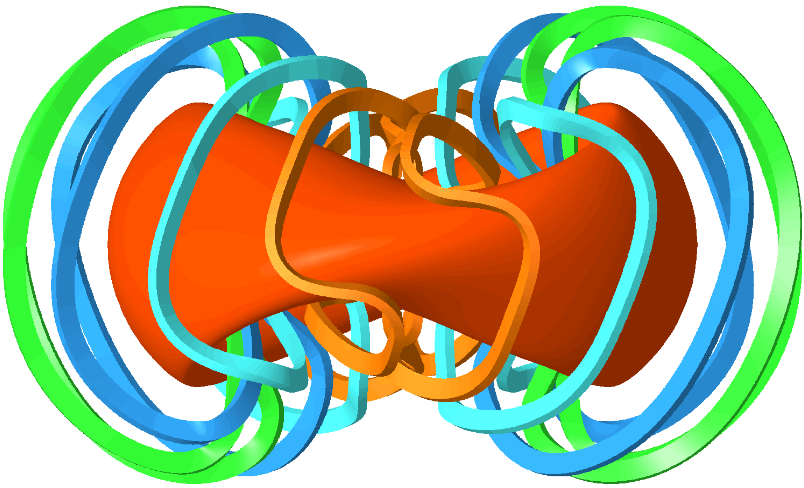

Figure 5. Diagram of a fusion reactor with low prompt loss of α particles in a magnetic field given by the Biot–Savart law. Sixteen moderately twisted modular coils produce robust flux surfaces that do not deteriorate when changes are made in the vertical and toroidal fields. This optimized configuration with two field periods has stellarator stability and tokamak transport. (Courtesy of Tak-Kuen Mau and Tsueren Wang.)

6. Conclusion

The NSTAB code has been applied to calculate a variety of bifurcated equilibria in tokamaks with axially symmetric boundary conditions. The KAM theory of dynamical systems displays small denominators at rational surfaces of 3D solutions, and analysis of the continuous spectrum shows that linear stability of tokamaks is singular. This is consistent with observations of sawtooth oscillations, NTM and ELMS, and disruptions. Desirable 3D solutions of the MHD equations for equilibrium may not exist, may not be unique, and may not depend continuously on the data. Yet success of the DIII-D and LHD experiments fosters a belief that it is possible to design a magnetic fusion reactor. Perhaps a QAS stellarator of very low aspect ratio is the answer, since it is helpful if some of the rotational transform comes from the external magnetic field.

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