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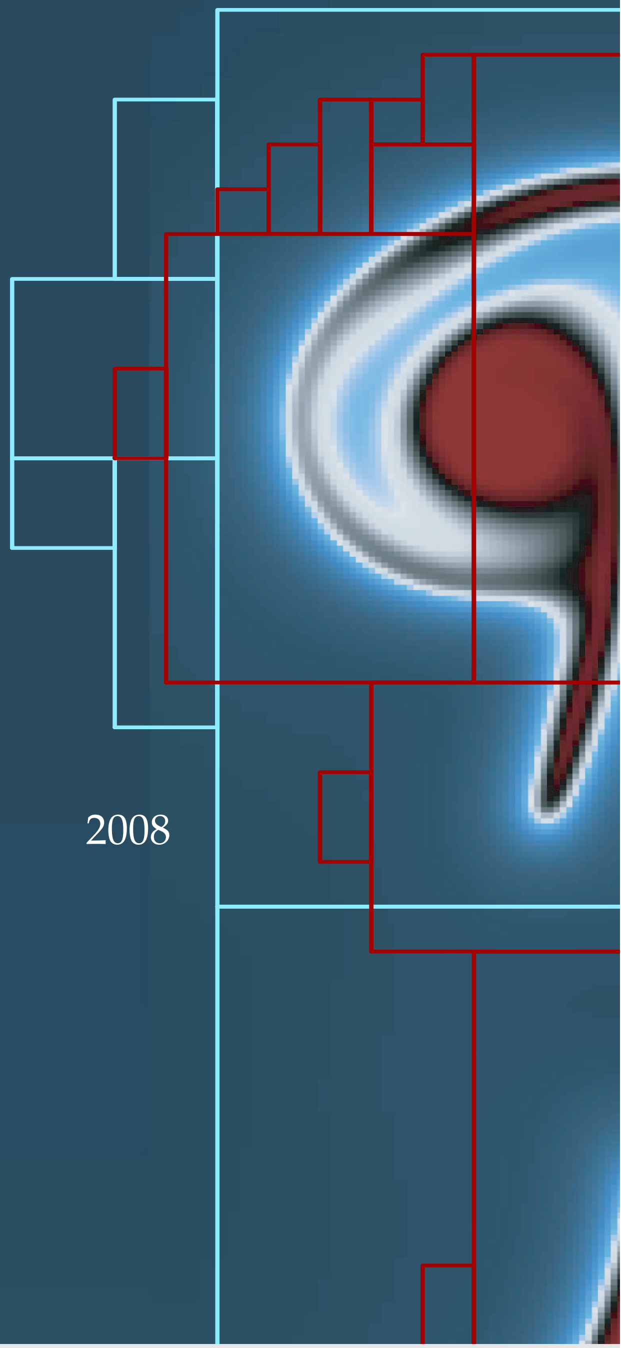

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REFLECTION OF VARIOUS TYPES OF WAVES BY LAYERED MEDIA

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The one-dimensional wave equation describing propagation and reflection of waves in a layered medium is transformed into an exact first-order system for the amplitudes of coupled counter-propagating waves. Any choice of such amplitudes, out of continuous multitude of them, allows one to get an accurate numerical solution of the reflection problem. We discuss relative advantages of particular choices of amplitude.

We also introduce the notion of reflection strength S of a plane wave by a nonabsorbing layer, which is related to the reflection intensity R by $R = \tanh^2 S$. We show that the total reflection strength by a sequence of elements is bounded above by the sum of the constituent strengths, and bounded below by their difference. Reflection strength is discussed for propagating acoustic waves and quantum mechanical waves. We show that the standard Fresnel reflection may be understood in terms of the variable S as a sum or difference of two contributions, one due to a discontinuity in impedance and the other due to a speed discontinuity.

1. Introduction

The reflection of waves by layered media is the subject of a large number of articles and monographs, such as [1; 2; 3; 4; 6; 9; 10; 12; 13; 14; 16; 17]. Here we undertake a theoretical study of the general properties of reflecting elements, supported by numerical computations. First we consider the reflection problem for the Schrödinger equation. We then discuss electromagnetic waves, allowing for variations in dielectric susceptibility $\varepsilon(z)$ and magnetic permeability $\mu(z)$. This is relevant in connection with new types of materials, including ones with $\varepsilon < 0$, $\mu < 0$; on this subject see [15]. Our study of electromagnetic waves also bears on the reflection of light by volume Bragg gratings, which have usually been studied in the slowly varying envelope approximation [7; 5; 18], but can be dealt with exactly with our treatment. We also discuss briefly the reflection of longitudinal acoustic waves.

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Keywords: reflection, electromagnetic waves, acoustic waves, continuous spectrum, Schrödinger equation, volume Bragg grating, reflectionless potential.

2. Reflection and transmission in the one-dimensional Schrödinger problem

We start with the one-dimensional stationary Schrödinger equation (see [4; 9] for instance):

$$\frac{d^2\psi}{dz^2} + \frac{2m}{\hbar^2}[E - U(z)]\psi(z) = 0. \tag{2.1}$$

Here $\hbar$ is Planck's constant, m is the particle's mass, E is the total energy, $U(z)$ is the potential energy, and $\psi(z)$ is the wavefunction. It is convenient to introduce the squared local wavenumber,

$$k^2(z) = \frac{2m}{\hbar^2}[E - U(z)] = k_0^2 - \frac{2m}{\hbar^2}U(z), \quad \text{where } k_0 = \frac{\sqrt{2mE}}{\hbar}. \tag{2.2}$$

With this notation,

$$\frac{d^2\psi}{dz^2} + k^2(z)\psi(z) = 0.$$

The expression for the z-component of the flux J (with dimensions of $m^{-2}s^{-1}$) and the conservation law are

$$J(z) = \frac{i\hbar}{2m}\left(\psi\frac{d\psi^*}{dz} - \psi^*\frac{d\psi}{dz}\right), \qquad J(z) = \text{const}; \tag{2.3}$$

the conservation of flux is a consequence of the real-valuedness of $k^2(z)$. One must bear in mind, however, that in regions of classically forbidden motion one gets $k^2(z) < 0$, so a real-valued $k(z)$ can not be defined. We assume that $k^2(z)$ acquires positive limiting values at $z \to +\infty$ and $z \to -\infty$, so that

$$k_+ = \lim_{z\to+\infty} \sqrt{k^2}, \qquad k_- = \lim_{z\to-\infty} \sqrt{k^2}. \tag{2.4}$$

Our technical tool to solve the problem of reflection and transmission is to introduce amplitudes $A(z)$ and $B(z)$ through the definitions

$$\begin{aligned} A(z) &= \sqrt{\frac{\hbar}{4m}}e^{-i\varphi(z)}\left[Y(z)\psi(z) - \frac{i}{Y(z)}\frac{d\psi}{dz}\right], \\ B(z) &= \sqrt{\frac{\hbar}{4m}}e^{+i\varphi(z)}\left[Y(z)\psi(z) + \frac{i}{Y(z)}\frac{d\psi}{dz}\right]. \end{aligned} \tag{2.5}$$

Here $Y(z)$ is a (for now) arbitrary positive function of z, and $\phi(z)$ is a (for now) arbitrary real function of z. Neither $A(z)$ separately, nor $B(z)$ separately, constitute a solution of our system, either fundamental, or of any other type. But the amplitudes $A(z)$ and $B(z)$ do have the advantage that the flux (which is conserved) is expressed simply as

$$J = |A(z)|^2 - |B(z)|^2. \tag{2.6}$$

In this sense one may consider $A(z)$ and $B(z)$ as amplitudes of waves propagating in opposite directions.

Exact coupled first-order differential equations for $A(z)$ and $B(z)$ follow from the original Schrödinger equation (2.1); they are

$$\frac{d}{dz}\begin{pmatrix} A \\ B \end{pmatrix} = \hat{V}(z)\begin{pmatrix} A \\ B \end{pmatrix}, \tag{2.7}$$

and

$$\hat{V}(z) = \begin{pmatrix} i\left[\frac{1}{2}\left(Y^2 + \frac{k^2(z)}{Y^2}\right) - \frac{d\varphi}{dz}\right] & e^{-2i\varphi}\left[g - \frac{i}{2}\left(Y^2 - \frac{k^2(z)}{Y^2}\right)\right] \\ e^{2i\varphi}\left[g + \frac{i}{2}\left(Y^2 - \frac{k^2(z)}{Y^2}\right)\right] & -i\left[\frac{1}{2}\left(Y^2 + \frac{k^2(z)}{Y^2}\right) - \frac{d\varphi}{dz}\right] \end{pmatrix}, \tag{2.8}$$

$$g(z) = \frac{d \ln Y(z)}{dz}.$$

Later we will discuss the particular choices of $Y(z)$ and $\varphi(z)$ that provide for the existence of a definite limit for $\hat{M}(z)$ as $z \to \pm\infty$. Here we start by choosing some reasonable classes of functions $Y(z)$ and $\phi(z)$. It is natural to choose $Y(z)$ so that

$$Y(z \to +\infty) = \sqrt{k_+}, \qquad Y(z \to -\infty) = \sqrt{k_-}. \tag{2.9}$$

Then the coupling between the counter-propagating amplitudes vanishes at $z \to \pm\infty$. Besides that, it is advantageous to choose such $\phi(z)$, that

$$\frac{d\varphi}{dz}(z \to +\infty) = k_+, \qquad \frac{d\varphi}{dz}(z \to -\infty) = k_-. \tag{2.10}$$

Then, with the conditions (2.9) and (2.10) granted, the matrix $\hat{V}(z)$ of evolution has zero limits both at $z \to +\infty$ and $z \to -\infty$. In that distant regions the solution of the system (2.8) becomes constants $[A(-\infty), B(-\infty)]$ and $[A(+\infty), B(+\infty)]$. We can formulate our task as a Cauchy problem: given $A(-\infty)$, $B(-\infty)$, find $A(z)$ and $B(z)$. A numerical (or, with some luck, analytical) solution of that Cauchy problem may be represented as

$$\begin{pmatrix} A(z) \\ B(z) \end{pmatrix} = \begin{pmatrix} M_{AA}(z) & M_{AB}(z) \\ M_{BA}(z) & M_{BB}(z) \end{pmatrix}\begin{pmatrix} A(-\infty) \\ B(-\infty) \end{pmatrix}, \tag{2.11}$$

where the matrix $\hat{M}(z)$ satisfies the equation

$$\frac{d\hat{M}(z)}{dz} = \hat{V}(z)\hat{M}(z), \qquad \hat{M}(-\infty) = \hat{1}. \tag{2.12}$$

Using MathCAD and Maple, we integrated numerically the system of ordinary differential equations (2.12) for the matrix elements of $\hat{M}(z)$ in the Cauchy problem with initial condition $\hat{M}(z=-\infty)=\hat{1}$. Both software packages automatically adjust the step size, and they gave identical results to within about 10^{-10}.

In most cases one is interested in the problem of reflection and transmission. That requires the solution of a two-boundary problem, as opposed to a Cauchy problem. For example, for the wave incoming from $z\to-\infty$ one has the boundary conditions

$$A(z\to-\infty)=1, \qquad B(z\to+\infty)=0. \tag{2.13}$$

That allows one to find complex reflection and transmission coefficients

$$r(B\leftarrow A)=-M_{BA}/M_{BB}, \qquad t(A\leftarrow A)=1/M_{BB}. \tag{2.14}$$

Here $\hat{M}$ (without an argument) denotes the limit matrix at $z\to+\infty$:

$$\hat{M}=\lim_{z\to+\infty}\hat{M}(z).$$

It is worth noting that the matrix approach for one-dimensional Schrödinger reflection/transmission problems is mentioned in the textbook [4]. However, their matrix operates with free asymptotic waves only (the analog of our matrix $\hat{M}$), and no suggestions are made there on how to calculate such a matrix.

Any choices of $Y(z)$ and $\phi(z)$ satisfying the conditions (2.9), (2.10) yield the same results for the reflectivity $|r|^2$ and the transmissivity $|t|^2$. The current values of $|A(z)|^2$ and $|B(z)|^2$ in the reflection/transmission problem *do* depend on the choice of $Y(z)$ and $\phi(z)$, as will be shown below by numeric calculations. Nevertheless, one can find the explicit expression of the wavefunction of the relevant solution:

$$\psi(z)=\frac{1}{Y(z)}\sqrt{\frac{m}{\hbar}}\left[A(z)e^{+i\varphi(z)}+B(z)e^{-i\varphi(z)}\right], \tag{2.15}$$

and this solution, up to a constant factor, does *not* depend on the choice of $Y(z)$ and $\phi(z)$.

Let us discuss some variants of that choice, which yield a different structure for evolution matrix $\hat{V}$ and a different physical sense of coupled counter-propagating amplitudes $A(z)$ and $B(z)$.

1. Suppose that $k^2(z)>0$ for all z, so there is no classically forbidden region. Then the choice $Y(z)=\sqrt{k(z)}$ considerably simplifies the expression of the evolution matrix:

$$\hat{V}(z)=\begin{pmatrix} i\left[k(z)-\dfrac{d\varphi}{dz}\right] & e^{-2i\varphi}g \\ e^{2i\varphi}g & -i\left[k(z)-\dfrac{d\varphi}{dz}\right]\end{pmatrix},$$
$$g(z)=\frac{d\ln Y(z)}{dz}=-\frac{1}{4}\frac{1}{E-U(z)}\frac{dU(z)}{dz}. \tag{2.16}$$

We can say that the scattering processes $A \to B$ and $B \to A$ are generated (for the choice $Y(z) = \sqrt{k(z)}$) *by the potential gradient*.

1′. The choice $Y(z) = \sqrt{k(z)}$, $\varphi(z) = \int_{z_1}^{z} k(z')\,dz'$ allows us to eliminate completely the diagonal elements in the evolution matrix (2.16). This may or may not be advantageous in the numerical solution of the coupled wave equations (2.7), (2.12).

1″. The choice $Y(z) = \sqrt{k(z)}$, $\phi(z) = k_0 z$ leads to

$$\hat{V}(z) = \begin{pmatrix} i[k(z) - k_0] & g(z)e^{-2ik_0 z} \\ g(z)e^{-2ik_0 z} & -i[k(z) - k_0] \end{pmatrix}, \qquad g(z) = -\frac{1}{4}\,\frac{1}{E - U(z)}\,\frac{dU(z)}{dz}. \tag{2.17}$$

A choice analogous to this one will be used in Section 6 for the Maxwell equations.

2. Consider the special case when $U(+\infty) = U(-\infty) = 0$; then $k_+ = k_- = k_0$. Then the choice $Y(z) = \sqrt{k_0}$, still with arbitrary $\phi(z)$, yields for the evolution matrix with $g \equiv 0$ the expression

$$\hat{V}(z) = \begin{pmatrix} i\left(k_0\left[1 - \frac{U(z)}{2E}\right] - \frac{d\varphi}{dz}\right) & -ie^{-2i\varphi(z)}\frac{k_0}{2E}U(z) \\ ie^{2i\varphi(z)}\frac{k_0}{2E}U(z) & -i\left(k_0\left[1 - \frac{U(z)}{2E}\right] - \frac{d\varphi}{dz}\right) \end{pmatrix}. \tag{2.18}$$

Two features of this special choice ($Y(z) = \sqrt{k_0}$) should be mentioned. First, the passage of the particle under the potential barrier ($U(z) > E$) adds no extra computational difficulties here, since we do not have to take the square root of $E - U(z)$. Second, the scattering processes $A \to B$ and $B \to A$ for this special choice are governed *by the potential $U(z)$ itself*, and not by its gradients, as was the case in Equation (2.16).

2′. Choosing $\phi(z) = k_0 z$ for the same $Y(z) = \sqrt{k_0}$ affords an especially simple expression for the evolution matrix:

$$\hat{V}(z) = iU(z)\frac{k_0}{2E}\begin{pmatrix} -1 & -e^{-2ik_0 z} \\ e^{2ik_0 z} & 1 \end{pmatrix}. \tag{2.19}$$

This equation for the evolution matrix $\hat{V}(z)$ is especially convenient for the use of perturbation theory, since the whole matrix $\hat{V}(z)$ in (2.19) is of first order with respect to the potential $U(z)$. Taking $A^{(0)} = 1$ in the zeroth approximation and $B^{(1)}(+\infty) = 0$, one gets

$$B^{(1)}(z) = -i\frac{k_0}{2E}\int_z^{+\infty} U(z')e^{2ik_0 z'}\,dz', \tag{2.20}$$

and the first order reflection amplitude is given by (2.20) with the integral taken from $z = -\infty$.

This same integral taken from $-\infty$ to $+\infty$, which is to say, the Fourier component of the potential at k_0, may be transformed (by integration by parts) into

$$r^{(1)}(B \leftarrow A) = B^{(1)}(z = -\infty) = \frac{1}{4E} \int_{-\infty}^{+\infty} \frac{dU(z')}{dz'} e^{2ik_0 z'} dz'. \tag{2.21}$$

This last expression coincides with the perturbative use of (2.17) at $|U(z)| \ll E$.

We may conclude that in the case $U(+\infty) = U(-\infty) = 0$ our transformation from $Y(z) = \sqrt{k_0}$ to $Y(z) = \sqrt{k(z)}$ is equivalent to integration by parts in perturbation theory. It should be emphasized, however, that we managed to perform such a transformation in exact equations, without the perturbative approach.

3. Particular examples of numerical modeling; low-reflecting potentials

We foresee various applications of the computational technique described above. One application, which we explored ourselves, is the problem of *reflectionless potentials*, those that do not reflect Schrödinger waves at any value of energy of incident particles. The best known example is the inverse square of the hyperbolic cosine potential well:

$$k^2(z) = k_0^2 + \frac{D}{\cosh^2 z}, \qquad D = s(1+s). \tag{3.1}$$

It is well known that a one-dimensional potential well with $\lim_{z\to+\infty} U(z) = \lim_{z\to-\infty} U(z) = 0$ always allows at least one state with negative energy eigenvalue (bound state). We call the lowest such state the zeroth bound state. There are certain values of the depth D of the well when the first bound state, then the second, then the third, and so on, first appear in the continuous spectrum. For the potential (3.1), these special values are (see [9]) $D_1 = 2$ (at $s = 1$), $D_2 = 6$ (at $s = 2$), $D_3 = 12$ (at $s = 3$), and so on for all positive integer values of s. Exactly for these values of the depth, the well is reflectionless: $R(k_0^2) \equiv 0$, at all k_0^2. This property plays an important role in the nonlinear theory of solitons; see [8], for example.

Using our technique for calculating the reflectivity $R(k_0^2)$, we tried to answer the question whether a similar property of "low reflection at all energies" might be valid for potential wells of other shapes. We studied, for example, the Gaussian potential well

$$k^2(z) = k_0^2 + D\exp(-z^2). \tag{3.2}$$

We found that for this well, the first few depth values D when new bound states appear are

$$\begin{aligned} &D_1 = 2.6840, \qquad D_2 = 8.6491, \qquad D_3 = 17.7957, \\ &D_4 = 30.1063, \qquad D_5 = 45.5735, \qquad D_6 = 64.1933. \end{aligned} \tag{3.3}$$

We then calculated the reflectivity $R(k_0^2)$ for a wide range of incident particle energies. While there is no value of D for which the reflectivity vanishes *identically*, we found remarkably low reflectivity in the whole energy range when the depth of the Gaussian well coincided with the values from (3.3).

As an example, Figure 1 shows the energy dependence of the reflection coefficient for Gaussian wells with depths $D = 2$, $D = 2.684005$, $D = 3$ and $D = 4$. At large k_0^2 reflection becomes exponentially small for any D. However, at small k_0^2 (less than or about $k_0^2 \sim 0.2$) reflectivity becomes close to 100%. A happy exception is the value $D_1 = 2.684005$, when reflectivity goes to zero at $k_0^2 \to 0$ as well. The maximum reflectivity for this D is a mere $R_{\max} = 1.1 \cdot 10^{-2}$, achieved for k_0^2 in the range 0.3–1.0. For the value $D_2 = 8.6491$ the maximum reflectivity is $R_{\max} = 1.4 \cdot 10^{-2}$.

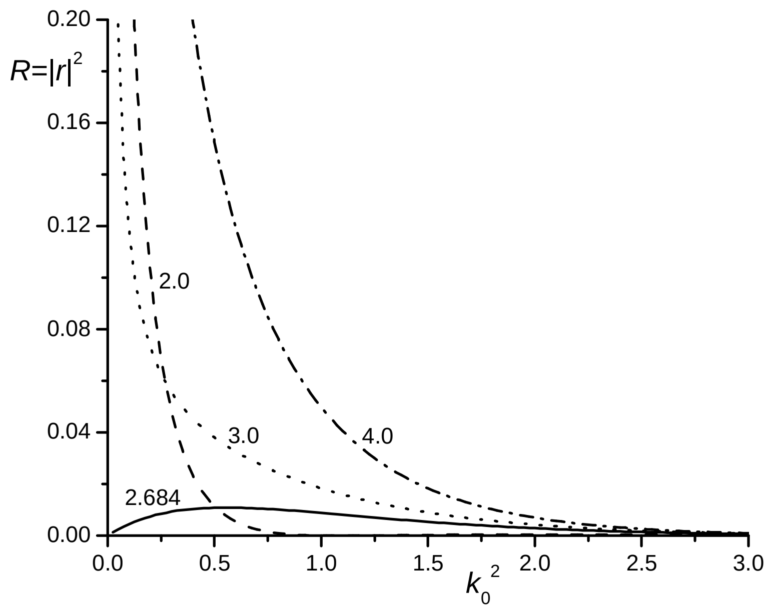

Figure 1. Reflection coefficient $R = |r|^2$ by a Gaussian potential well $\delta k^2(z) = D \cdot \exp(-z^2)$, with different values of the well's depth, $D = 2.0$, $D = D_1 = 2.684005$, $D = 3.0$ and $D = 4.0$, as functions of incident energy $\hbar^2 k_0^2/(2m)$. At $D > D_1$, z-odd bound state emerges in the well. For $D = D_1$ the maximum reflectivity is quite small, $R_{\max} = 1.1 \cdot 10^{-2}$.

It is worth nothing that the results for $R(k_0^2)$ were identical for both methods of computation: with $Y(z) = \sqrt{k(z)}$, $\varphi(z) = k_0 z$, and with $Y(z) = \sqrt{k_0}$, $\varphi(z) = k_0 z$. This confirms numerically the validity of different approaches. We have yet to explore the variant with $Y(z) = \sqrt{k(z)}$, $\varphi(z) = \int_{z_1}^{z} k(z')\,dz'$. Besides the solution of a system of ordinary differential equations, that variant would require calculating of the above integral with the continuous upper limit, and therefore we decided to postpone its study. We also postponed the study of the problems with $k_+ \neq k_-$, where the choice $\varphi(z) = k_0 z$ would be impossible.

Similar results of suppressed reflection were also obtained for the potential well with the double Lorentzian profile,

$$k^2(z) = k_0^2 + \frac{D}{(1+z^2/2)^2}. \tag{3.4}$$

Here the depth values D_s at which the s-th level (with z-parity $(-1)^s$) acquires zero energy are $D_s = s(1+s/2)$, so that $D_1 = 1.5$, $D_2 = 4$, $D_3 = 7.5$, $D_4 = 12$. We did observe the suppressed reflection in the whole energy range for these discrete values of well depth. For $D_1 = 1.5$, the maximum reflectivity equals $R_{\max} = 3.3\%$. It is possible that these results about suppressed reflectivity may have applications to soliton theory.

We also calculated the curves $|A(z)|^2$, $|B(z)|^2$ and $|\psi(z)|^2$ for the reflectionless potential well $k^2(z) = k_0^2 + 2/\cosh^2 z$, in the case $k_0^2 = 0.2$ (Figure 2). We see that

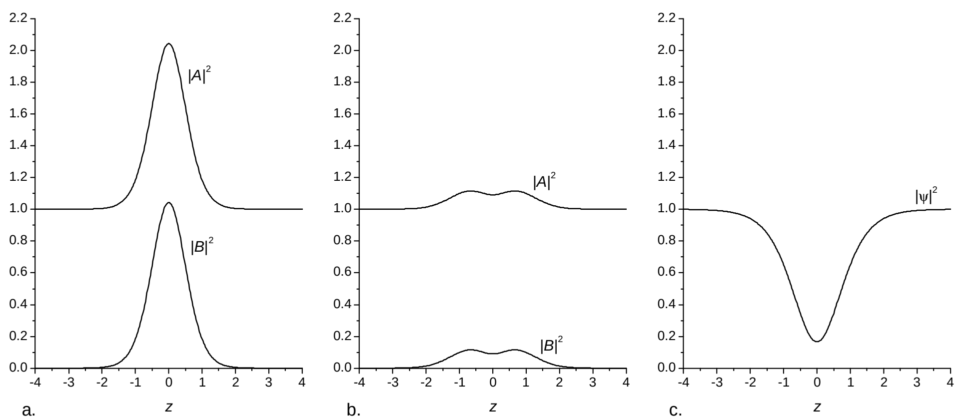

Figure 2. Transmission of Schrödinger wave over a reflectionless potential well, $k^2(z) = k_0^2 + 2/\cosh^2 z$, at $k_0^2 = 0.2$. Left: $|A(z)|^2$ and $|B(z)|^2$ for the choice (2′), when $Y(z) = \sqrt{k_0}$, $\varphi(z) = k_0 z$. Middle: $|A(z)|^2$ and $|B(z)|^2$ for the choice (1″), when $Y(z) = \sqrt{k(z)}$, $\varphi(z) = k_0 z$. Right: profile of $|\psi(z)|^2$ in this problem; it does not depend on the choice of $Y(z)$ and $\phi(z)$.

while the shapes of $|A(z)|^2$ and $|B(z)|^2$ depend on the choice of $Y(z)$, the shape of $|\psi(z)|^2$ does not depend on that choice, just as expected.

4. The matrix method and the notion of reflection strength

For a better perspective, we now consider transmission through a volume Bragg grating (VBG), which couples two plane electromagnetic waves, A and B, both having Poynting vector with positive z-component: $P_z = |A|^2 + |B|^2$. Here the z-axis is normal to the boundaries of VBG. The absence of absorption results in the conservation law $P_z = \text{const}$. Writing the matrix relationship for wave coupling in linear media, $A(z) = N_{AA} \cdot A(0) + N_{AB} \cdot B(0)$, $B(z) = N_{BA} \cdot A(0) + N_{BB} \cdot B(0)$, one comes to the conclusion that the matrix $\hat{N}(z)$ must be unitary: $\hat{N}(z) \in U(2)$.

Now consider a reflecting device where the waves A and B propagate in opposite directions with respect to z-axis, so that $P_z = |A|^2 - |B|^2$. The absence of absorption results in the conservation law $|A|^2 - |B|^2 = \text{const}$. Writing the matrix relationship for wave coupling in linear media (compare to Equation (2.11) above),

$$A(z) = M_{AA} \cdot A(0) + M_{AB} \cdot B(0), \quad B(z) = M_{BA} \cdot A(0) + M_{BB} \cdot B(0), \quad (4.1)$$

one can deduce from the assumption of energy conservation that the matrix $\hat{M}(z) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ satisfies

$$|\alpha|^2 - |\gamma|^2 = 1, \qquad |\delta|^2 - |\beta|^2 = 1, \qquad \alpha\beta^* = \gamma\delta^*. \quad (4.2)$$

The most general form of such a matrix $\hat{M}$ depends on four real parameters: a strength S, an inessential phase ψ and two phases ζ and η:

$$\hat{M} = e^{i\psi} \begin{pmatrix} e^{i\zeta} & 0 \\ 0 & e^{-i\zeta} \end{pmatrix} \begin{pmatrix} \cosh S & \sinh S \\ \sinh S & \cosh S \end{pmatrix} \begin{pmatrix} e^{-i\eta} & 0 \\ 0 & e^{i\eta} \end{pmatrix}. \quad (4.3)$$

The determinant of $\hat{M}$ equals $\exp(2i\psi)$, and so has modulus 1. The set of such matrices is closed under multiplication and inversion, and so forms a group, denoted $U(1, 1)$; it is the complex analog of Lorentz group SL(2) ($|A|^2$ being analogous to x^2 and $|B|^2$ analogous to $(ct)^2$).

The consecutive application of two elements of $U(1, 1)$, with parameters S_1, ψ_1, ζ_1, η_1 and S_2, ψ_2, ζ_2, η_2, yields the element described by the matrix $\hat{M}_3 = \hat{M}_2\hat{M}_1$, which is also of the same type (4.3). The expression for the resultant strength parameter S_3 is

$$S_3 = \operatorname{arcsinh} \sqrt{\sinh^2(S_1 + S_2)\cos^2\tau + \sinh^2(S_1 - S_2)\sin^2\tau}, \quad \tau = \zeta_1 - \eta_2, \quad (4.4)$$

which can vary due to the phase difference τ between the reflective elements.

Knowledge of the matrix $\hat{M}(z)$ allows one to find the reflection and transmission amplitudes. Thus, for the problem where the wave A is incident on the front layer, $z = 0$, and there is no wave B incident on the back, $z = L$, one uses the boundary conditions $A(0) = 1$, $B(L) = 0$, to get, just as for Schrödinger equation,

$$0 = M_{BA}(L) + M_{BB}(L) \cdot r,$$

which implies

$$r = r(B \leftarrow A) = -\frac{M_{BA}(L)}{M_{BB}(L)} = -e^{-2i\eta} \tanh S, \tag{4.5}$$

$$R = |r(B \leftarrow A)|^2 = \tanh^2 S. \tag{4.6}$$

The presence of the hyperbolic tangent function is very satisfying: as the strength S goes to infinity, the reflection coefficient goes to 1 asymptotically. The notion of reflection strength is applicable to the gratings in single-mode optical fibers as well. Kogelnik's theory of reflection by VBGs [7] predicts the following value for the resulting strength:

$$\begin{aligned} R_{\mathrm{VBG}} &= \tanh^2 S, \quad S = \arcsin h\left(S_0 \frac{\sinh\sqrt{S_0^2 - X^2}}{\sqrt{S_0^2 - X^2}}\right), \\ S_0 &= |\kappa| L, \qquad X = \left(\frac{\omega n}{c}\cos\theta_{\mathrm{inside}} - \frac{Q}{2}\right) L. \end{aligned} \tag{4.7}$$

Here S_0 is the strength of the VBG at perfect Bragg matching, i.e., when the detuning parameter X vanishes, the coupling parameter $|\kappa| = \frac{1}{2}(n_1\omega/c)\left|\cos(\boldsymbol{E}_A, \boldsymbol{E}_B)\right|$ corresponds to modulation of refractive index $\delta n(z) = n_1 \cos(Qz)$. The angle θ_{inside} is the propagation angle of the waves A and B inside the material of VBG. (Formula (4.7) is mathematically identical, for a lossless medium, to the result found in [7], but is written somewhat differently.)

If reflective VBG slab has certain residual reflection by the boundaries, $R_1 = |r_1|^2$ and $R_2 = |r_2|^2$, then the question arises about coherent interference between the main VBG refection from Equation (4.7) and these two extra contributions. Attentive consideration of the result (4.4) allows to predict, that at any particular wavelength and/or angle of the incident wave, the strength S_{tot} of the total element will be within the limits

$$S_{\mathrm{VBG}} - |S_1| - |S_2| \le S_{\mathrm{tot}} \le S_{\mathrm{VBG}} + |S_1| + |S_2|, \quad S_{1,2} = -\mathrm{arctanh}\, r_{1,2}. \tag{4.8}$$

This allows the easy estimation of the influence of Fresnel reflections.

5. Fresnel reflection and reflection of acoustic waves

Now consider a fundamental problem of electrodynamics: reflection of light by a sharp boundary between two media at an incidence angle θ_1, so that the refraction angle is θ_2. We denote by ε_1, μ_1, ε_2, μ_2 the dielectric permittivity and magnetic permeability in these two media, so the phase propagation speeds $v_{1,2}$ and impedances $Z_{1,2}$ are

$$v_j = \frac{c}{n_j}, \quad c = \frac{1}{\sqrt{\varepsilon_{\text{vac}}\mu_{\text{vac}}}}, \quad n_j = \sqrt{\frac{\varepsilon_j \mu_j}{\varepsilon_{\text{vac}}\mu_{\text{vac}}}}, \quad Z_j = \sqrt{\frac{\mu_j}{\varepsilon_j}}, \quad j = 1, 2. \tag{5.1}$$

The angles θ_1, θ_2 are related by Snell's law, which is governed by the ratio of propagation speeds, or, which is the same, the ratio of refractive indices n_1, n_2:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2. \tag{5.2}$$

The cases of total internal reflection (TIR) or an absorbing second medium require the definition

$$\cos\theta_2 = \sqrt{1-(n_1/n_2)^2 \sin^2\theta_2} = C_2' + iC_2'', \quad C_2'' > 0. \tag{5.3}$$

The condition $C_2'' > 0$ guarantees an exponential attenuation of the transmitted wave into the depth of second medium. The reflection amplitudes for TE (transverse electric) and TM (transverse magnetic) polarization can be found easily:

$$\begin{aligned} r_{\text{TE}} &\equiv r(E_y \leftarrow E_y) = \frac{\cos\theta_1/Z_1 - \cos\theta_2/Z_2}{\cos\theta_1/Z_1 + \cos\theta_2/Z_2}, \\ r_{\text{TM}} &\equiv r(E_x \leftarrow E_x) = -\frac{Z_1\cos\theta_1 - Z_2\cos\theta_2}{Z_1\cos\theta_1 + Z_2\cos\theta_2}. \end{aligned} \tag{5.4}$$

These expressions have two very instructive limiting cases. The first is when the two media have the same propagation speed $v_1 = v_2$ (and refractive indices), so Snell's law (5.2) yields $\theta_1 = \theta_2$. Surprisingly, the reflection coefficients in this case are equal and independent of the angle:

$$r_{\text{TE}} = r_{\text{TM}} \equiv r_{\Delta Z} = \frac{Z_2 - Z_1}{Z_2 + Z_1}. \tag{5.5}$$

The second limiting case is when the media have the same impedance, $Z_1 = Z_2$, but different propagation speeds, $n_1 \neq n_2$. Then the two reflection coefficients are equal up to a sign:

$$r_{\text{TE}} = -r_{\text{TM}} \equiv r_{\Delta v}(\theta_1) = \frac{\cos\theta_1 - \cos\theta_2}{\cos\theta_1 + \cos\theta_2}. \tag{5.6}$$

Thus there is no reflection at normal incidence for impedance-matched media (stealth technology).

The reflection strengths $S = -\operatorname{arctanh} r$ for these two limiting cases are

$$S_{\Delta Z} = \tfrac{1}{2}\ln\frac{Z_1}{Z_2}, \qquad S_{\Delta v}(\theta_1) = \tfrac{1}{2}\ln\frac{\cos\theta_2}{\cos\theta_1}. \tag{5.7}$$

Here is the relationship we found. *One can produce the reflection strengths* $S_{\text{TE}}(\theta_1)$ *and* $S_{\text{TM}}(\theta_1)$ *by simple addition (for* TE*) or subtraction (for* TM*) of the speed-governed and impedance-governed contributions from* (5.7):

$$S_{\text{TE}}(\theta_1) = S_{\Delta Z} + S_{\Delta v}(\theta_1), \qquad S_{\text{TM}}(\theta_1) = S_{\Delta Z} - S_{\Delta v}(\theta_1), \tag{5.8}$$

and according to Equation (4.5), $r = -\tanh S$. One can easily verify that expressions (5.7), (5.8) and (4.5) reproduce the standard formulae (5.4) identically.

It is interesting that the reflection in the TIR regime is also described by $S = -\text{arctanh}\, r$ and Equations (5.7), (5.8) are still valid. In this case we have

$$S_{\Delta Z} = \frac{1}{2}\ln\frac{Z_1}{Z_2}, \qquad S_{\Delta v}(\theta_1) = i\frac{\pi}{4} + \frac{1}{2}\ln\Big(\sqrt{(n_1/n_2)^2\sin^2\theta_1 - 1}/\cos\theta_1\Big). \tag{5.9}$$

As expected, $|r| = |\tanh(i\pi/4 + S')| = 1$ for the case of TIR.

It is instructive to consider the reflection of longitudinal acoustic waves from the boundary between two liquids having densities ρ_1 and ρ_2, propagation speeds c_1 and c_2 and therefore acoustic impedances $Z_1 = \rho_1 c_1$ and $Z_2 = \rho_2 c_2$. The well known expression for the reflection coefficient for the wave's pressure [3; 11] is

$$r_{\text{longitud}} \equiv r(p \leftarrow p) = \frac{\cos\theta_1/Z_1 - \cos\theta_2/Z_2}{\cos\theta_1/Z_1 + \cos\theta_2/Z_2}. \tag{5.10}$$

For this acoustic case we see that again, the reflection strength is given by the sum of two contributions,

$$r_{\text{longitud}} = -\tanh[S_p(\theta_1)], \qquad S_p(\theta_1) = S_{\Delta Z} + S_{\Delta c}(\theta_1), \tag{5.11}$$

similar to the case of TE polarization in electrodynamics.

6. Maxwell equations for coupled waves: exact approach

We have actually found the relationship (5.8) for ourselves not empirically; we have derived the result of additivity for reflection strength S directly from Maxwell equations. The idea is to formulate *exact Maxwell equations* for the layered medium in terms of two coupled amplitudes A and B propagating with $P_z > 0$ and $P_z < 0$ respectively. Let the incidence plane be the (x, z)-plane, for a monochromatic wave $\propto \exp(-i\omega t)$ incident upon a layered medium whose properties depend on z only. By θ_{air} we denote the incidence angle of the wave in air, so

$$\boldsymbol{k}_{\text{air}} = \hat{\boldsymbol{x}}k_x + \hat{\boldsymbol{z}}k_{\text{air},z}, \qquad k_x = \frac{\omega}{c}n_{\text{air}}\sin\theta_{\text{air}}, \qquad k_{\text{air},z} = \frac{\omega}{c}n_{\text{air}}\cos\theta_{\text{air}}. \tag{6.1}$$

Again here the waves separate naturally into TE and TM parts. We write the electric and magnetic vectors for each polarization by means of appropriately normalized components u_x, u_y, u_z and w_x, w_y, w_z:

$$\begin{aligned} \text{TE}: \boldsymbol{E}(\boldsymbol{r},t) &= -\hat{\boldsymbol{y}} u_y(z) e^{ik_x x - i\omega t}\sqrt{Z(\boldsymbol{r})}, \\ \boldsymbol{H}(\boldsymbol{r},t) &= [\hat{\boldsymbol{x}} w_x(z) + \hat{\boldsymbol{z}} w_z(z)] e^{ik_x x - i\omega t}/\sqrt{Z(\boldsymbol{r})}, \end{aligned} \tag{6.2}$$

$$\begin{aligned} \text{TM}: \boldsymbol{E}(\boldsymbol{r},t) &= [\hat{\boldsymbol{x}} u_x(z) + \hat{\boldsymbol{z}} u_z(z)] e^{ik_x x - i\omega t}\sqrt{Z(\boldsymbol{r})}, \\ \boldsymbol{H}(\boldsymbol{r},t) &= \hat{\boldsymbol{y}} w_y(z) e^{ik_x x - i\omega t}/\sqrt{Z(\boldsymbol{r})}. \end{aligned} \tag{6.3}$$

Here and below we use the quantities

$$k(z) = \frac{\omega n(z)}{c}, \qquad p(z) = \sqrt{k^2(z) - k_x^2} = k(z)\cos\theta(z), \tag{6.4}$$

$$g(z) = \frac{1}{2}\frac{d}{dz}\ln\frac{1}{Z(z)}, \qquad f(z) = \frac{1}{2}\frac{d}{dz}\ln\frac{p(z)}{k(z)} \equiv \frac{1}{2}\frac{d}{dz}\ln\cos\theta(z). \tag{6.5}$$

The Maxwell equations for the TE polarization amplitudes are

$$iku_y = \partial_z w_x - ik_x w_z + g w_x, \quad -ikw_x = -\partial_z u_y + g u_y, \quad -ikw_z = ik_x u_y; \tag{6.6}$$

they may be rewritten as

$$\partial_z u_y = g u_y + ikw_x, \qquad \partial_z w_x = ip^2/k u_y - g w_x. \tag{6.7}$$

It is convenient to introduce the amplitudes $A(z)$ and $B(z)$ for TE polarization by the definitions

$$\begin{aligned} A_{\text{TE}}(z) e^{ik_{\text{air},z} z} &= \frac{1}{\sqrt{8}}\left(\sqrt{\frac{p}{k}} u_y(z) + \sqrt{\frac{k}{p}} w_x(z)\right), \\ B_{\text{TE}}(z) e^{-ik_{\text{air},z} z} &= \frac{1}{\sqrt{8}}\left(\sqrt{\frac{p}{k}} u_y(z) - \sqrt{\frac{k}{p}} w_x(z)\right). \end{aligned} \tag{6.8}$$

This definition is analogous to the choice (1″) for the Schrödinger equation. The z-component of the Poynting vector for any incidence angle at any point z is $P_z = |A|^2 - |B|^2$. One may consider the transformation (6.8) as a transition to *slowly varying envelopes* $A(z)$ and $B(z)$. We emphasize, however, that no approximations have been made up to this point. Indeed, the exact Maxwell equations for TE polarization reduce to the very simple coupled pair

$$\begin{aligned} \frac{d}{dz}\begin{pmatrix} A_{\text{TE}}(z) \\ B_{\text{TE}}(z) \end{pmatrix} &= \hat{V}_{\text{TE}} \begin{pmatrix} A_{\text{TE}}(z) \\ B_{\text{TE}}(z) \end{pmatrix}, \\ \hat{V}_{\text{TE}} &= \begin{pmatrix} i(p(z) - k_{\text{air},z}) & (g(z) + f(z)) e^{-2ik_{\text{air},z} z} \\ (g(z) + f(z)) e^{2ik_{\text{air},z} z} & -i(p(z) - k_{\text{air},z}) \end{pmatrix}. \end{aligned} \tag{6.9}$$

A similar set of transformations may be done for TM polarization:

$$\begin{aligned} -iku_x &= -\partial_z w_y - g w_y, \\ -iku_z &= ik_x w_y, \\ -ikw_y &= ik_x u_z - \partial_z u_x + g u_x \iff \partial_z u_x = g u_x + ip^2/k, \\ \partial_z w_y &= iku_x - g w_y, \end{aligned} \tag{6.10}$$

with the same parameters $k(z)$, $g(z)$, $p(z)$. The coupled TM wave amplitudes are

$$\begin{aligned} A_{\mathrm{TM}}(z)e^{ik_{\mathrm{air},z}z} &= \frac{1}{\sqrt{8}}\left(\sqrt{\frac{k}{p}}u_x(z) + \sqrt{\frac{p}{k}}w_y(z)\right), \\ B_{\mathrm{TM}}(z)e^{-ik_{\mathrm{air},z}z} &= \frac{1}{\sqrt{8}}\left(\sqrt{\frac{k}{p}}u_x(z) - \sqrt{\frac{p}{k}}w_y(z)\right). \end{aligned} \tag{6.11}$$

Finally, the exact Maxwell equations for TM polarization are

$$\begin{aligned} \frac{d}{dz}\begin{pmatrix} A_{\mathrm{TM}}(z) \\ B_{\mathrm{TM}}(z) \end{pmatrix} &= \hat{V}_{\mathrm{TM}}\begin{pmatrix} A_{\mathrm{TM}}(z) \\ B_{\mathrm{TM}}(z) \end{pmatrix}, \\ \hat{V}_{\mathrm{TM}} &= \begin{pmatrix} i(p(z)-k_{\mathrm{air},z}) & (g(z)-f(z))e^{-2ik_{\mathrm{air},z}z} \\ (g(z)-f(z))e^{2ik_{\mathrm{air},z}z} & -i(p(z)-k_{\mathrm{air},z}) \end{pmatrix}, \end{aligned} \tag{6.12}$$

with the same parameters $f(z)$, $g(z)$ as in (6.5). The gradient functions $f(z)$ and $g(z)$ — related respectively to propagation speed and impedance — appear as sums (for TE polarization) or differences (for TM) in our "coupled" equations. The discontinuities in $n(z)$ and $Z(z)$ yield our resulting equations (5.8).

For brevity we omit the discussion of other possible choices of electromagnetic amplitudes $A(z)$ and $B(z)$ analogous to (1′), (2′) of the Schrödinger problem.

7. Conclusions

We have introduced the notion of reflection strength S, related to the reflection intensity R by $R = \tanh^2 S$. The total reflection strength S_{tot} from a sequence of two lossless elements equals at most the sum of the constituent strengths, and equals at least their difference. We have shown that the amplitudes of standard processes of Fresnel reflection may be understood in terms of S as a linear sum or difference of two independent contributions, one due to the discontinuity in impedance and the other due to the speed discontinuity. A similar result is obtained for the reflection of longitudinal acoustic waves. The one-dimensional Schrödinger equation is also treated with specially introduced amplitudes of coupled counter-propagating components.

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