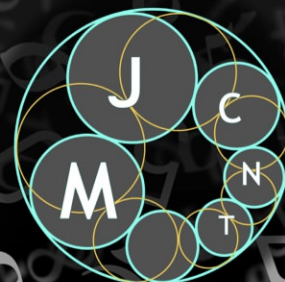


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Convex sequences may have thin additive bases

Imre Z. Ruzsa and Dmitrii Zhelezov





# Convex sequences may have thin additive bases

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For a fixed  $c > 0$  we construct an arbitrarily large set  $B$  of size  $n$  such that its sum set  $B + B$  contains a convex sequence of size  $cn^2$ , answering a question of Hegarty.

## Notation

The following notation is used throughout the paper. The expressions  $X \gg Y$ ,  $Y \ll X$ ,  $Y = O(X)$ ,  $X = \Omega(Y)$  all have the same meaning in that there is an absolute constant  $c$  such that  $|Y| \leq c|X|$ .

If  $X$  is a set then  $|X|$  denotes its cardinality.

For sets of numbers  $A$  and  $B$  the *sumset*  $A + B$  is the set of all pairwise sums

$$\{a + b : a \in A, b \in B\}.$$

## 1. Introduction

Let  $A = \{a_i\}$ ,  $i = 1, \dots, n$ , be a set of real numbers ordered in a way that  $a_1 \leq a_2 \leq \dots \leq a_n$ . (We also refer to  $A$  a sequence, if we wish to emphasize the ordering.) Recall that  $A$  is called *convex* if the gaps between consecutive elements of  $A$  are strictly increasing, that is

$$a_2 - a_1 < a_3 - a_2 < \dots < a_n - a_{n-1}.$$

Studies of convex sets were initiated by Erdős, who conjectured that any convex set must grow with respect to addition, so that the size of the set of sums  $A + A := \{a_1 + a_2 : a_1, a_2 \in A\}$  is significantly larger than the size of  $A$ .

The first nontrivial bound confirming the conjecture of Erdős was obtained by Hegyvári [1986]. The state of the art bound for the size of  $A + A$  of a convex sequence  $A$  is due to Shkredov [2015]:

$$|A + A| \gg |A|^{58/37} \log^{-20/37} |A|.$$

The best bound for the size of the difference set  $A - A$  is due to Schoen and Shkredov [2011], who proved that

$$|A - A| \gg |A|^{8/5} \log^{-2/5} |A|$$

if  $A$  is arbitrary convex sequence. It is conjectured that in fact

$$|A + A| \geq C(\epsilon) |A|^{2-\epsilon}$$

holds for any  $\epsilon > 0$  and some  $C > 0$  which depends only on  $\epsilon$ .

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In general, it is believed that convex sets cannot be additively structured. In particular, Hegarty [2012] asked whether there is a constant  $c > 0$  with the property that there is a set  $B$  of arbitrarily large size  $n$  such that  $B + B$  contains a convex set of size  $cn^2$ .

Recall that  $B$  is a *basis* (of order two) for a set  $A$  if  $A \subset B + B$ . In other words, Hegarty asked if a convex set of size  $n$  can have a thin additive basis (of order two) of size as small as  $O(n^{1/2})$ , which is clearly the smallest possible size up to a constant.

Perhaps contrary to the intuition that convex sets lack additive structure, we present a construction which answers Hegarty's question in the affirmative. Our main result is as follows.

**Theorem 1.** *There is  $c > 0$  such that for any  $m$  there is a set  $B$  of size  $n > m$  such that  $B + B$  contains a convex set of size  $cn^2$ .*

## 2. Construction

Assume  $n$  is fixed and large. We will construct a set  $B$  of size  $O(n)$  such that  $B + B$  contains a convex set of size  $\Omega(n^2)$ . Theorem 1 will clearly follow.

The following constants (we assume  $n$  is fixed) will be used throughout the proof:

$$\alpha := \frac{1}{n^2}, \quad \gamma := \frac{1}{1000n^3}, \quad \epsilon := 0.1.$$

Define

$$x_i = i + (\alpha + \gamma)i^2, \quad y_j = j - \alpha j^2.$$

Next, we define

$$B_k = \{x_i + y_j : i + j = k\},$$

where  $i$  and  $j$  are allowed to be negative.

Let  $k \in [0.999n, n]$  so that  $\alpha k^2 \in [0.99, 1]$ . For such an integer  $k$  writing  $j = k - i$  we have that the  $i$ -th element of  $B_k$  is given by

$$b_i^{(k)} = k + (\alpha + \gamma)i^2 - \alpha(k - i)^2 = (k - \alpha k^2) + \gamma i^2 + 2ik\alpha. \quad (1)$$

Now assume that  $i$  ranges in  $[-n, 2n]$ . The consecutive differences  $b_{i+1}^{(k)} - b_i^{(k)}$  are then given by

$$\Delta_i^{(k)} := \gamma(2i + 1) + 2k\alpha.$$

Observe that  $\Delta_i^{(k)}$  are positive and increasing, thus the block  $B_k := \{b_i^{(k)}\}_{-n}^{2n}$  is convex. Further, by (1) for sufficiently large  $n$  we have

$$b_{-n}^{(k)} = k - \alpha k^2 + \gamma n^2 - 2nk\alpha \in [k - 2.9, k - 3], \quad (2)$$

$$b_{2n}^{(k)} = k - \alpha k^2 + \gamma(2n)^2 + 4nk\alpha \in [k + 2.9, k + 3.1], \quad (3)$$

so  $B_k \subset [k - 3, k + 3] + [-\epsilon, \epsilon]$ .

Now we are going to build a large convex sequence out of blocks  $B_k$  with  $4 \mid k$ . Since each  $B_k$  is already convex, it remains to show how to glue together  $B_k$  and  $B_{k+4}$  so that the resulting set is again convex. We proceed with the following simple lemma.

**Lemma 2.** *Let  $X = \{x_i\}_{i=0}^N$  and  $Y = \{y_j\}_{j=0}^M$  be two convex sequences and there are indices  $u$  and  $v$  such that*

$$[x_u, x_{u+1}] \subset [y_v, y_{v+1}].$$

*Then*

$$Z := \{x_i\}_{i=0}^u \cup \{y_j\}_{j=v+1}^M$$

*is a convex sequence.*

*Proof.* Since  $[x_u, x_{u+1}] \subset [y_v, y_{v+1}]$  we have that

$$x_u - x_{u-1} < x_{u+1} - x_u < y_{v+1} - x_u.$$

On the other hand,

$$y_{v+1} - x_u < y_{v+1} - y_v < y_{v+2} - y_{v+1}. \quad \square$$

By Lemma 2, in order to merge  $B_k$  and  $B_{k+4}$  it suffices to find two consecutive elements  $b_i^{(k)}, b_{i+1}^{(k)} \in B_k$  in between two consecutive elements  $b_j^{(k+4)}, b_{j+1}^{(k+4)} \in B_{k+4}$ . Define

$$\delta := \max_{i \in [-n, 2n]} \Delta_i^{(k)}, \quad \Delta := \min_{i \in [-n, 2n]} \Delta_i^{(k+4)}.$$

We have

$$\delta < 4n\gamma + 2k\alpha < \frac{2.1}{n}, \quad (4)$$

$$\Delta - \delta > 8\alpha - 10n\gamma > \frac{6}{n^2}. \quad (5)$$

Let  $b_v^{(k)}$  be the least element in  $B_k$  greater than  $b_{-n}^{(k+4)}$  (such an element exists by (2) and (3)). We claim that with  $m := \lceil n/2 \rceil + 1$  holds  $b_{-n+m}^{(k+4)} > b_{v+m}^{(k)}$ , which in turn by the pigeonhole principle guarantees the arrangement of elements required by Lemma 2.

Indeed, by our choice of  $v$ ,

$$0 \leq d := b_v^{(k)} - b_{-n}^{(k+4)} \leq \delta. \quad (6)$$

But by (4) and (5),

$$b_{-n+m}^{(k+4)} - b_{v+m}^{(k)} > -d + m(\Delta - \delta) > \frac{3}{n} - \delta > 0, \quad (7)$$

so the claim follows.

It remains to note that by (3),

$$b_{v+m}^{(k)} < b_{-n}^{(k+4)} + m\Delta < (k+1+\epsilon) + \frac{2n^2\alpha}{2} + 4\gamma nm < k+2.2,$$

and thus  $v+m < 2n$  again by (3). This verifies that  $b_v^{(k)}, b_{v+m}^{(k)} \in B_k$ .

### 3. Putting everything together

Applying the procedure described in the previous section, we can glue together consecutive blocks  $B_{4l}$  with  $4l := k \in [0.999n, n]$ . Let  $A$  be the resulting convex sequence. First, observe there are  $\Omega(n)$  blocks being merged. Moreover, each interval  $[4l - 1 + \epsilon, 4l + 1 - \epsilon]$  is covered only by the block  $B_{4l}$  and by (2), (3), and (4) it contains  $\Omega(n)$  elements from  $B_{4l}$ , so  $|A| = \Omega(n^2)$ . On the other hand, by our construction,  $A$  is contained in the sumset  $B + B$  of  $B := \{x_i\}_{-2n}^{2n} \cup \{y_j\}_{-2n}^{2n}$  of size  $O(n)$ .

**Remark 3.** It follows from our construction that there are arbitrarily large convex sets  $A$  such that the equation

$$a_1 - a_2 = x : a_1, a_2 \in A$$

has  $\Omega(|A|^{1/2})$  solutions  $(a_1, a_2)$  for at least  $\Omega(|A|^{1/2})$  values of  $x$ .

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