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Cahen's constant is defined by the alternating sum of reciprocals of terms of Sylvester's sequence minus 1. Davison and Shallit proved the transcendence of the constant and Becker improved it. In this paper, we study rationality of functions satisfying certain functional equations and generalize the result of Becker by a variant of Mahler's method.

1. Introduction

Sylvester's sequence $\{S_n\}_{n\geq 0}$ is defined by the recurrence

$$S_0 = 2, \qquad S_{n+1} = S_n^2 - S_n + 1 \quad (n \geq 0).$$

It is well known that

$$\sum_{n=0}^{\infty} \frac{1}{S_n} = 1. \tag{1}$$

Cahen [1891] showed that the number

$$C = \sum_{n=0}^{\infty} \frac{(-1)^n}{S_n - 1}, \tag{2}$$

which is now called Cahen's constant, is irrational. Davison and Shallit [1991] established the transcendence of Cahen's constant. They constructed a class of alternating series each of which can be expanded in an explicit simple continued fraction having irrationality exponent greater than 2.5 and showed that the series (2) belongs to this class. Here, for an irrational number α, the irrationality exponent $\mu(\alpha)$ is defined by the least upper bound of the set of numbers μ for which the inequality

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{\mu}}$$

has infinitely many irreducible rational solutions p/q. Thus, the transcendence of Cahen's constant C follows from Roth's theorem. Becker [1992, Corollary 3] improved the result by a variant of Mahler's method. Indeed, he proved the following: Let $p(z)$ be a polynomial with algebraic coefficients and $\deg p(z) \geq 2$ and $q(z) = z - \gamma$ with an algebraic number γ. Let x be an algebraic number such that

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$\lim_{n\to\infty} p^n(x) = \infty$ and $q(p^n(x)) \neq 0$ for all $n \geq 0$, where $p^0(z) = z$, $p^n(z) = p(p^{n-1}(z))$ $(n \geq 1)$. Then, the number

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{q(p^n(x))}$$

is transcendental except when $q(p(z)) = \lambda^{-1}q(z)^2 + q(z) - \lambda$ for some constant $\lambda \neq 0$, in which case

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{q(p^n(z))} = \frac{1}{q(z)+\lambda}.$$

For example, if $p(z) = z^2 - z + 1$ and $\alpha = S_0$, then the number

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{S_n - \gamma}$$

is transcendental for any algebraic γ with $S_n \neq \gamma$ for all $n \geq 0$.

In this paper, we consider the function

$$f(z) = \sum_{n=0}^{\infty} \frac{a^n}{q(p^n(z))}, \tag{3}$$

where $a \neq 0$ is a complex number, $p(z) \in \mathbb{C}[z]$ with $\deg p(z) \geq 2$, and $q(z) \in \mathbb{C}[z]$ with $\deg q(z) \geq 1$. We note that the right-hand side of (3) is convergent at any $z \in \mathbb{C}$ for which $\lim_{n\to\infty} p^n(z) = \infty$ and $q(p^n(z)) \neq 0$ for all $n \geq 0$. Furthermore, there exists a constant $C_f > 1$ such that $f(z)$ is analytic in $\mathcal{D}_f = \{z \in \mathbb{C} \mid |z| > C_f\}$ and $f(\mathcal{D}_f) \subset \mathcal{D}_f$.

The function $f(z)$ satisfies the functional equation

$$af(p(z)) = f(z) - \frac{1}{q(z)}, \tag{4}$$

and more generally

$$f(p^n(z)) = \frac{1}{a^n}\left(f(z) - \sum_{j=0}^{n-1} \frac{a^j}{q(p^j(z))}\right) \quad (n \geq 1). \tag{5}$$

We now state our results.

Theorem 1.1. *Let $f(z)$ be the function defined by*

$$f(z) = \sum_{n=0}^{\infty} \frac{a^n}{q(p^n(z))},$$

where $a \in \mathbb{C}^\times$, $p(z) \in \mathbb{C}[z]$ with $\deg p(z) \geq 2$ and $q(z) \in \mathbb{C}[z]$ is monic with $\deg q(z) \geq 1$. Then, the function $f(z)$ is algebraic over the field $\mathbb{C}(z)$ of rational functions if and only if $\deg p(z) = 2$ and $p(z)$ and $q(z)$ satisfy the relation

$$b^l q(p(z)) - a = b^l q(z)(b^l q(z) - a), \tag{6}$$

where b is the leading coefficient of $p(z)$ and $l = \deg q(z)$, and if so

$$f(z) = \frac{b^l}{b^l q(z) - a}. \tag{7}$$

Theorem 1.2. *With the same notation as in Theorem 1.1, assume that a and the coefficients of $p(z)$ and $q(z)$ are algebraic. Then the number*

$$f(x) = \sum_{n=0}^{\infty} \frac{a^n}{q(p^n(x))}$$

is transcendental for any algebraic x with $\lim_{n\to\infty} p^n(x) = \infty$ and $q(p^n(x)) \neq 0$ for all $n \geq 0$, except when $d = 2$ and $p(z)$ and $q(z)$ satisfy the relation (6), in which case $f(z)$ is the rational function given by (7).

Theorem 1.3. *Let $f(z)$ be the function defined by*

$$f(z) = \sum_{n=0}^{\infty} \frac{a^n}{(p^n(z) - \gamma)^l}, \tag{8}$$

where $p(z) \in \mathbb{C}[z]$ with $\deg p(z) \geq 2$ and l is a positive integer. Assume that $a \neq 0$, γ, and the coefficients of $p(z)$ are algebraic numbers. Then the value $f(x)$ is transcendental for any algebraic x with $\lim_{n\to\infty} p^n(x) = \infty$ and $p^n(x) \neq \gamma$ for all $n \geq 0$, except in the following two cases:

(i) $l = 1,\ p(\gamma) - \gamma + b^{-1} p'(\gamma) = 0$ *and* $a = -p'(\gamma)$, *in which case*

$$f(x) = \frac{b}{b(x-\gamma) - a}. \tag{9}$$

(ii) $l = 2,\ p(\gamma) - \gamma = -2b^{-1},\ p'(\gamma) = 0$ *and* $a = 4$, *in which case*

$$f(x) = \frac{b^2}{b^2(x-\gamma)^2 - 4}. \tag{10}$$

Remark 1.4. The case (ii) can be obtained as a special case of (i). Indeed, if $a = 4$ in case (ii), we have by Taylor's formula $p(z) = b(x-\gamma)^2 - 4(x-\gamma) + \gamma + 4b^{-1}$, and therefore $p(z) - \gamma = b(x - \gamma - 2b^{-1})^2$. Hence

$$f(x) = \frac{1}{x-\gamma} + \sum_{n=1}^{\infty} \frac{4^n}{p(p^{n-1}(x)) - \gamma} = \frac{1}{x-\gamma} + 4b^{-1} \sum_{n=0}^{\infty} \frac{4^n}{(p^n(x) - \gamma - 2b^{-1})^2}.$$

Replacing $f(x)$ by using (9) and $\gamma + 2b^{-1}$ by γ yields

$$\frac{b}{b(x-\gamma) - 2} = \frac{b}{b(x-\gamma) + 2} + 4b^{-1} \sum_{n=0}^{\infty} \frac{4^n}{(p^n(x) - \gamma)^2},$$

which is exactly (10).

We give some examples of Theorem 1.3.

Example 1.5. Let $\{S_n\}_{n\geq 0}$ be Sylvester's sequence defined by

$$S_{n+1} = S_n^2 - S_n + 1 \quad (n \geq 0)$$

with arbitrary $S_0 \in \mathbb{Z} \setminus \{0, 1\}$. Here $p(z) = z^2 - z + 1$, $p'(z) = 2z - 1$ and $b = 1$. Let us study first case (i) in Theorem 1.3. The equation $p(\gamma) - \gamma + b^{-1}p'(\gamma) = 0$ is equivalent to $\gamma^2 = 0$. Therefore $\gamma = 0$ and $a = -p'(\gamma) = 1$. Case (ii) cannot occur. Hence for any algebraic numbers $a \neq 0$ and γ with $S_n \neq \gamma$ for all $n \geq 0$ and a positive integer l, the number

$$\sum_{n=0}^{\infty} \frac{a^n}{(S_n - \gamma)^l}$$

is transcendental except when $l = a = 1$ and $\gamma = 0$, and if so

$$\sum_{n=0}^{\infty} \frac{1}{S_n} = \frac{1}{S_0 - 1}.$$

Example 1.6. Let $\{T_n\}_{n\geq 0}$ be the recurrence

$$T_0 \in \mathbb{Z}, \quad |T_0| > 2, \qquad T_{n+1} = T_n^2 - 2 \quad (n \geq 0).$$

Here $p(z) = z^2 - 2$, $p'(z) = 2z$ and $b = 1$. By Theorem 1.3, we see that, for any algebraic numbers $a \neq 0$ and γ with $T_n \neq \gamma$ for all $n \geq 0$ and a positive integer l, the number

$$\sum_{n=0}^{\infty} \frac{a^n}{(T_n - \gamma)^l}$$

is transcendental except in the following three cases:

(i) $l = 1$, $\gamma = 1$, and $a = -2$, in which case

$$\sum_{n=0}^{\infty} \frac{(-2)^n}{T_n - 1} = \frac{1}{T_0 + 1}. \tag{11}$$

(ii) $l = 1$, $\gamma = -2$, and $a = 4$, in which case

$$\sum_{n=0}^{\infty} \frac{4^n}{T_n + 2} = \frac{1}{T_0 - 2}. \tag{12}$$

(iii) $l = 2$, $\gamma = 0$, and $a = 4$, in which case

$$\sum_{n=0}^{\infty} \frac{4^n}{T_n^2} = \frac{1}{(T_0 - 2)(T_0 + 2)}.$$

As mentioned in Remark 1.4, (iii) is intrinsically the same as (ii).

Example 1.7. Fermat numbers $F_n = 2^{2^n} + 1$ satisfy the recurrence relation

$$F_{n+1} = F_n^2 - 2F_n + 2 \quad (n \geq 0)$$

with $F_0 = 3$. By Theorem 1.3, for any algebraic numbers $a \neq 0$ and γ with $F_n \neq \gamma$ for all $n \geq 0$ and a positive integer l, the number

$$\sum_{n=0}^{\infty} \frac{a^n}{(F_n - \gamma)^l} \tag{13}$$

is transcendental except when $l = 1$, $a = 2$, and $\gamma = 0$, and if so

$$\sum_{n=0}^{\infty} \frac{2^n}{F_n} = \frac{1}{F_0 - 2} = 1. \tag{14}$$

Remark 1.8. Formulas (11), (12), and (14) are known; see formulas (2.22), (2.25), and (2.26) in [Duverney 2001]. In fact, let α and β with $|\alpha| > |\beta|$ be roots of the equations $x^2 - T_0 x - 1 = 0$. Then the Lucas-type sequence

$$T_n = \alpha^{2^n} + \beta^{2^n}$$

satisfies $T_{n+1} = T_n^2 - 2$ $(n \geq 1)$. Therefore the series (11) and (12), as well as (14), can also be seen as examples of exceptional cases related to the classical Mahler's method; see [Duverney et al. 2002, Theorem 1.3; Kanoko et al. 2009, Example 1].

2. Proof of Theorems 1.1 and 1.3

To prove the theorems, we study rational solutions of a functional equation which generalizes (4).

Lemma 2.1. *Let $a, c \in \mathbb{C}^\times$, $p(z) \in \mathbb{C}[z]$ with $d = \deg p(z) \geq 2$ and leading coefficient b, and $q(z) \in \mathbb{C}[z]$ be monic with $l = \deg q(z) \geq 1$. Assume that a rational function $g(z)$ satisfies the functional equation*

$$a g(p(z)) = g(z) - \frac{\delta}{q(z)}. \tag{15}$$

Then $d = 2$, and $p(z)$ and $q(z)$ satisfy the relation

$$b^l q(p(z)) - a = b^l q(z)(b^l q(z) - a), \tag{16}$$

in which case:

(i) *If $a \neq 1$, then (15) has one and only one rational solution, which is*

$$g(z) = \frac{\delta}{q(z) - ab^{-l}}. \tag{17}$$

(ii) *If $a = 1$, then (15) has infinitely many rational solutions given by*

$$g(z) = \alpha + \frac{\delta}{q(z) - b^{-1}} \quad (\alpha \in \mathbb{C}). \tag{18}$$

Proof. Let $R(z)$ and $S(z)$ be two coprime monic polynomials and $\alpha \in \mathbb{C}^\times$ be such that

$$g(z) = \alpha \frac{R(z)}{S(z)}.$$

As $g(z)$ satisfies (15), we have for $c=\delta\alpha^{-1}$

$$a\frac{R(p(z))}{S(p(z))}=\frac{R(z)}{S(z)}-\frac{c}{q(z)}. \tag{19}$$

Put for brevity $r=\deg R(z)$ and $s=\deg S(z)$. If $s=0$, then there is no solution satisfying (19) since $g(z)-ag(p(z))=c/(q(z))\notin\mathbb{C}[z]$. Hence, $s\geq 1$. The functional equation (19) can be written as

$$aR(p(z))S(z)q(z)=R(z)S(p(z))q(z)-cS(z)S(p(z)). \tag{20}$$

Since $(R(p(z)),S(p(z)))=1$, we have

$$S(p(z))\mid S(z)q(z). \tag{21}$$

Hence, $ds\leq s+l$. Therefore, we obtain

$$1\leq s\leq\frac{l}{d-1}. \tag{22}$$

Comparing the degrees of both sides of (20), we get $r\leq s$.

If $r<s$, the degree of the first term of the right-hand side in (20) is greater than that of the left-hand side. Therefore, the degree of the first term of the right-hand side is equal to that of the second term of the right-hand side. Then, we have using (22)

$$0=r+ds+l-(s+ds)\geq r-s+(d-1)s=r+(d-2)s\geq 0.$$

Therefore, we deduce $d=2$ and $r=0$. This together with (20) leads to

$$aS(z)q(z)=S(p(z))q(z)-cS(z)S(p(z)). \tag{23}$$

The degree of the left-hand side is less than that of the first term of the right-hand side. Hence, the degrees of the two terms in the right-hand side are equal, and so $s=l$. This and (21) with $d=2$ imply

$$S(p(z))=b^{l}q(z)S(z). \tag{24}$$

Substituting (24) in (23), we get $a=b^{l}(q(z)-cS(z))$. Comparing the leading coefficients of both sides, we find $c=1$ and

$$S(z)=q(z)-ab^{-l}. \tag{25}$$

Substituting into (24) yields (16). In this case, as $R(z)$ is monic and $\deg R(z)=0$, we have $R(z)=1$ and

$$g(z)=\alpha\frac{R(z)}{S(z)}=c^{-1}\delta\frac{1}{q(z)-ab^{-l}},$$

which proves that (17) holds (also for $a=1$).

Now, let $r=s$. Then we get $a=1$ by comparing the leading coefficients of both sides in (20). Put $T(z)=R(z)-S(z)$. Then, by (20),

$$T(p(z))S(z)q(z)=T(z)S(p(z))q(z)-cS(z)S(p(z)). \tag{26}$$

Noting that $\deg T(z)<s$ and $(S(z),T(z))=1$, we apply the above discussion for $S(z)$ and $T(z)$, and thus we obtain $d=2$, $T(z)$ is a constant, and (24). Let $T(z)=k\neq 0$. Substituting (24) into (26), we get

$$1=b^{l}(q(z)-ck^{-1}S(z)).$$

Comparing the leading coefficients of both sides, we find $k = c$, and we see that (25) holds again. Therefore (16) holds. In this case $R(z) - S(z) = c$, whence

$$g(z) = \alpha \frac{R(z)}{S(z)} = \alpha + \frac{\alpha c}{q(z) - b^{-l}},$$

which proves (18). □

Now we prove Theorem 1.1 by using Lemma 2.1.

Proof of Theorem 1.1. Assume that the function (3) is algebraic over the field $\mathbb{C}(z)$ of rational functions. Then we have

$$(f(z))^{\delta} + g(z)(f(z))^{\delta-1} + \cdots = 0, \tag{27}$$

where the degree δ is chosen to be minimal and $g(z)$ is a rational function with complex coefficients. Replacing z by $p(z)$ in (27) yields

$$\left(\frac{1}{a}\left(f(z) - \frac{1}{q(z)}\right)\right)^{\delta} + g(p(z))\left(\frac{1}{a}\left(f(z) - \frac{1}{q(z)}\right)\right)^{\delta-1} + \cdots = 0$$

by using (4). This can be written as

$$f(z)^{\delta} + \left(ag(p(z)) - \frac{\delta}{q(z)}\right) f(z)^{\delta-1} + \cdots = 0. \tag{28}$$

As δ is minimal, comparison with (27) and (28) yields

$$ag(p(z)) = g(z) + \frac{\delta}{q(z)}.$$

Since $g(z)$ satisfies the functional equation (15), we can apply Lemma 2.1 and obtain (6). Replacing z by $p^n(z)$ in (6) yields

$$\frac{ab^l}{b^l q(p^{n+1}(z)) - a} = \frac{a}{q(p^n(z))(b^l q(p^n(z)) - a)} = \frac{b^l}{b^l q(p^n(z)) - a} - \frac{1}{q(p^n(z))}.$$

After multiplying by a^n, the function $f(z)$ appears as a telescoping series and we have

$$f(z) = b^l \sum_{n=0}^{\infty} \left(\frac{a^n}{b^l q(p^n(z)) - a} - \frac{a^{n+1}}{b^l q(p^{n+1}(z)) - a}\right) = \frac{b^l}{b^l q(z) - a}.$$ □

Lemma 2.2. *Make the same assumptions as in Lemma 2.1. Let $q(z) = (z-\gamma)^l$, where $l \geq 1$. Then $l = 1$ or 2:*

(i) *If $l = 1$, then $b(p(\gamma) - \gamma) + p'(\gamma) = 0$ and $a = -p'(\gamma)$.*

(ii) *If $l = 2$, then $p'(\gamma) = 0$, $p(\gamma) - \gamma = -2b^{-1}$, and $a = 4$.*

Proof. By Lemma 2.1, (16) holds and we get

$$(p(z) - \gamma)^l - ab^{-l} = b^l (z-\gamma)^{2l} - a(z-\gamma)^l. \tag{29}$$

Differentiating both sides of (29), we get

$$p'(z)(p(z)-\gamma)^{l-1} = 2b^l(z-\gamma)^{2l-1} - a(z-\gamma)^{l-1}. \tag{30}$$

If $l=1$, then taking $z=\gamma$ yields $p(z)-\gamma-ab^{-l}=0$ and $p'(\gamma)=-a$. Replacing a in the first equality gives $b(p(\gamma)-\gamma)+p'(\gamma)=0$, as claimed.

Let $l \geq 2$. By (29), we have

$$(p(\gamma)-\gamma)^l = ab^{-l} \neq 0. \tag{31}$$

Since $p(\gamma) \neq \gamma$ by (31), $(z-\gamma)^{l-1}$ divides $p'(z)$. Hence $l=2$, and so (30) is reduced to

$$p(z)-\gamma = b(z-\gamma)^2 - \tfrac{1}{2}ab^{-1}. \tag{32}$$

Substituting $z=\gamma$ in (32) and using (31), we find $a=4$ and $p(\gamma)-\gamma=-2b^{-1}$. Substituting $z=\gamma$ in (30) and using (31), we obtain $p'(\gamma)=0$. □

Finally, we prove Theorem 1.3 by using Lemma 2.2 and Theorem 1.2, which will be shown independently in the next section using Theorem 1.1.

Proof of Theorem 1.3. If the function $f(z)$ defined in Theorem 1.3 is not a rational function, then the value $f(x)$ is transcendental by Theorem 1.2. Assume to the contrary that $f(z)$ is a rational function. Then Lemma 2.2 with (7) yields the exceptional cases. □

3. Proof of Theorem 1.2

Becker's result mentioned in Section 1 is a special case of the main theorem in [Becker 1992], which establishes algebraic independence of the values of power series $f_1(z), \dots, f_m(z)$ satisfying the functional equations

$$f_i(z) = a_i(z) f_i(Tz) + b_i(z) \quad (i = 1, \dots, m),$$

where $a_i(z), b_i(z)$ are rational functions with algebraic coefficients and $Tz = p(z^{-1})^{-1}$ for a polynomial $p(z)$ with algebraic coefficients and $\deg p(z) \geq 2$. The proof of this theorem is based on a deep result due to Philippon [1986] on a criterion for algebraic independence of complex numbers and is rather involved. Although Theorem 1.2 can also be deduced from [Becker 1992, Theorem], we give here a self-contained proof for completeness.

We prove Theorem 1.2 by a variant of Mahler's method. In the proof we will have to estimate the denominators and houses of algebraic numbers. We will use the following lemmas.

Lemma 3.1. *Let $\mathbb{K}$ be any algebraic field of degree k, and let $h \in \mathbb{K}[z]$. Let $\delta = \deg h$. Then there exists $\mu = \mu(h) \geq 1$ such that, for every $\theta \in \mathbb{K}^\times$,*

(i) $\operatorname{den} h(\theta) \leq \mu (\operatorname{den}\theta)^\delta$,

(ii) $\overline{|h(\theta)|} \leq \mu(\max(1, \overline{|\theta|}))^\delta$.

Proof. Put $h(z) = \sum_{i=0}^{\delta} a_i z^i$, with $a_\delta \neq 0$. Then clearly

$$\operatorname{den} h(\theta) \leq D(\operatorname{den}\theta)^\delta,$$

where $D = \mathrm{LCM}(\operatorname{den} a_1, \operatorname{den} a_2, \ldots, \operatorname{den} a_\delta)$. Moreover, denote by $\sigma_1 = \mathrm{Id}, \sigma_2, \ldots, \sigma_k$ the monomorphisms of $\mathbb{K}$. Then for every $j = 1, 2, \ldots, k$, we have

$$\left|\sum_{i=0}^{\delta} \sigma_j(a_i)(\sigma_j(\theta))^i\right| \le \sum_{i=0}^{\delta} |\sigma_j(a_i)|(\overline{|\theta|})^i \le \left(\sum_{i=0}^{\delta} |\sigma_j(a_i)|\right)(\max(1, \overline{|\theta|}))^\delta, \qquad \square$$

Lemma 3.2. *Let $\mathbb{K}$ be any algebraic field of degree k, and let $h \in \mathbb{K}[z]$. Let $\delta = \deg h$. Then for every $\theta \in \mathbb{K}^\times$ such that $h(\theta) \neq 0$, there exist $\nu = \nu(h) \ge 1$ such that*

$$\max\left(\operatorname{den}\left(\frac{1}{h(\theta)}\right), \overline{\left|\frac{1}{h(\theta)}\right|}\right) \le \nu(\operatorname{den}\theta \times \max(1, \overline{|\theta|}))^{k\delta}.$$

Proof. First we have

$$\operatorname{den}\left(\frac{1}{h(\theta)}\right) = \operatorname{den}\left(\frac{\operatorname{den} h(\theta)}{\operatorname{den} h(\theta) \times h(\theta)}\right) = \operatorname{den}\left(\frac{\operatorname{den} h(\theta) \prod_{i \neq 1} \sigma_i(\operatorname{den} h(\theta) \times h(\theta))}{N(\operatorname{den} h(\theta) \times h(\theta))}\right),$$

where $N(\alpha)$ is the norm of $\alpha \in \mathbb{K}$ over $\mathbb{Q}$. The numerator of the fraction is an integer of $\mathbb{K}$, and therefore

$$\operatorname{den}\left(\frac{1}{h(\theta)}\right) \le |N(\operatorname{den} h(\theta) \times h(\theta))| \le (\operatorname{den} h(\theta))^k \times \overline{|h(\theta)|}^k,$$

which proves the first part of Lemma 3.2 by using Lemma 3.1(i).

For the second part, for every $i = 1, 2, \ldots, k$, we have

$$\left|\sigma_i\left(\frac{1}{h(\theta)}\right)\right| = \left|\frac{\operatorname{den} h(\theta)}{\operatorname{den} h(\theta) \times \sigma_i(h(\theta))}\right| = \left|\frac{(\operatorname{den} h(\theta))^k \times \prod_{j \neq i} \sigma_j(h(\theta))}{N(\operatorname{den} h(\theta) \times h(\theta))}\right|.$$

Now $|N(\operatorname{den} h(\theta) \times h(\theta))| \ge 1$ since $(\operatorname{den} h(\theta) \times h(\theta))$ is a nonzero integer of $\mathbb{K}$. Consequently

$$\left|\sigma_i\left(\frac{1}{h(\theta)}\right)\right| \le (\operatorname{den} h(\theta))^k \times \overline{|h(\theta)|}^{k-1} \quad (1 \le i \le k),$$

which proves Lemma 3.2 by using again Lemma 3.1. $\square$

Now we prove Theorem 1.2. For every $z \in \mathbb{C}$ satisfying $|z| > 1/C_f$ and every $n \ge 0$, put

$$q(p^n(z)) = \sum_{i=0}^{ld^n} \alpha_{n,i} z^i, \quad \alpha_{n,ld^n} = b^{(d^n-1)/(d-1)} \neq 0.$$

Then

$$\frac{a^n}{q(p^n(1/z))} = \frac{a^n z^{ld^n}}{\sum_{i=0}^{ld^n} \alpha_{n,i} z^{ld^n - i}}, \tag{33}$$

so that the function

$$F(z) = f\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{a^n}{q(p^n(1/z))} \tag{34}$$

is analytic in $\mathcal{E}_f = \{z \in \mathbb{C} \mid |z| < 1/C_f\}$.

If f is algebraic over $\mathbb{C}(z)$, we have the exceptional case by Theorem 1.1. From now on let f be not algebraic over $\mathbb{C}(z)$, and the coefficients of $p(z)$ and $q(z)$ be algebraic numbers, as well as x, a, and $f(x)$. We may assume without loss of generality that $x \in D_f$, since otherwise we can choose n_0 such that $p^n(x) \in D_f$ for all $n \geq n_0$ and consider the value $f(x')$ with $x' = p^{n_0}(x)$. To prove the theorem, we assume that the value $f(x)$ is algebraic and deduce a contradiction.

Let $\mathbb{K} \subset \mathbb{C}$ be the number field generated by all these numbers, let $\mathbb{A}$ be the ring of integers of $\mathbb{K}$, and let $k = \deg \mathbb{K}$. It is clear from (33) and (34) that the power series expansions of $F(z)$ and all its powers, namely

$$(F(z))^j = \sum_{n=0}^{\infty} \gamma_{j,n} z^n, \tag{35}$$

satisfy $\gamma_{j,n} \in \mathbb{K}$ for all nonnegative integers j and n. Now let r be a fixed positive integer. We claim that there exist polynomials $P_0, P_1, \ldots, P_r \in \mathbb{A}[z]$ of degrees at most r, not all zero, such that

$$P_0(z) + P_1(z)F(z) + P_2(z)(F(z))^2 + \cdots + P_r(z)(F(z))^r = z^{r^2+\sigma} L_r(z), \tag{36}$$

where $\sigma = \sigma(r) \geq 0$, $L_r(z) \in \mathbb{K}[\![z]\!]$ with $L_r(0) \neq 0$. Indeed, the left-hand side is not identically 0 since F is not algebraic. To realize (36) we have to solve a system of r^2 homogeneous equations (the coefficients of the successive powers z^i of the left-hand side must be equal to 0 for i from 0 to $r^2 - 1$) with $(r+1)^2$ unknowns (the coefficients of the P_i's). Since $(F(z))^h \in \mathbb{K}[\![z]\!]$ for every nonnegative integer h, we know from an elementary result of linear algebra that the system has a nontrivial solution in $\mathbb{K}^{(r+1)^2}$, and hence in $\mathbb{A}^{(r+1)^2}$ if we multiply by a common denominator, which proves our claim.

Replacing z by $1/p^n(x)$ yields

$$\theta_{r,n} = \sum_{j=0}^{r} P_j\left(\frac{1}{p^n(x)}\right)(f(p^n(x)))^j = \left(\frac{1}{p^n(x)}\right)^{r^2+\sigma} L_r\left(\frac{1}{p^n(x)}\right). \tag{37}$$

Under our hypotheses, the left-hand side of (37), which we call $\theta_{r,n}$, belongs to $\mathbb{K}$. As usual, we will obtain a contradiction by letting n tend to infinity for a suitable value of r and applying the size inequality to $\theta_{r,n}$. In what follows, we denote by $C_1, C_2, \ldots$ real numbers greater than 1 which do not depend on n or r (they may depend on x, $p(x)$ or $f(x)$).

Lemma 3.3. *There exists C_1 such that*

$$\max\left(\operatorname{den}\left(\frac{1}{q(p^n(x))}\right), \overline{\left|\frac{1}{q(p^n(x))}\right|}\right) \leq C_1^{d^n}. \tag{38}$$

Proof. An easy induction using Lemma 3.1(i) shows that, for every $n \geq 1$,

$$\operatorname{den}(p^n(x)) \leq \mu(p)^{(d^n-1)/(d-1)}(\operatorname{den} x)^{d^n} \leq C_2^{d^n}. \tag{39}$$

Furthermore, we have by Lemma 3.1(ii)

$$\overline{|p^n(x)|} \leq \mu(p)^{(d^n-1)/(d-1)}(\max(1, \overline{|x|}))^{d^n} \leq C_3^{d^n}. \tag{40}$$

For $n \geq 2$, we see by Lemma 3.2 that

$$\max\left(\operatorname{den}\left(\frac{1}{p^n(x)}\right), \overline{\left|\frac{1}{p^n(x)}\right|}\right) \leq \nu(p)\big(\operatorname{den} p^{n-1}(x) \times \max(1, \overline{|p^{n-1}(x)|})\big)^{kd}.$$

Therefore by (39) and (40)

$$\max\left(\operatorname{den}\left(\frac{1}{p^n(x)}\right), \left|\overline{\frac{1}{p^n(x)}}\right|\right) \le C_4^{d^n}. \tag{41}$$

By Lemma 3.2, this implies (38). ☐

Lemma 3.4. *There exist C_5, C_6, and C_7 such that*

$$\operatorname{den}(\theta_{r,n}) \le C_5^{rd^n}, \tag{42}$$

$$\overline{|\theta_{r,n}|} \le (r+1)^2 \chi C_6^{rd^n}, \tag{43}$$

$$|\theta_{r,n}| \le 2L_r(0) C_7^{-r^2 d^n}, \tag{44}$$

where χ is the greatest house of all the coefficients of all the polynomials P_i, which depends on r.

Proof. First we prove the inequality (42). By using (5), we have

$$\begin{aligned}\operatorname{den}(f(p^n(x))) &= \operatorname{den}\left(\frac{1}{a^n}\left(f(x) - \sum_{j=0}^{n-1} \frac{a^j}{q(p^j(x))}\right)\right)\\ &\le \left(\operatorname{den}\left(\frac{1}{a}\right)\right)^n \times \operatorname{den}(f(x)) \times (\operatorname{den} a)^{n-1} \times \prod_{j=0}^{n-1} \operatorname{den}\left(\frac{1}{q(p^j(x))}\right).\end{aligned}$$

By using (41), we obtain

$$\operatorname{den}(f(p^n(x))) \le C_8^n \times \prod_{j=0}^{n-1} C_1^{d^j} \le C_9^{d^n}. \tag{45}$$

The polynomials P_i defined in (36) have integer coefficients and their degrees are at most r. Hence for every $i = 0, 1, \ldots, r$, we have by (41)

$$\operatorname{LCM}\left(\operatorname{den}\left(P_i\left(\frac{1}{p^n(x)}\right)\right)\right) \le \left(\operatorname{den}\left(\frac{1}{p^n(x)}\right)\right)^r \le C_4^{rd^n}. \tag{46}$$

Now we can give an upper bound for the denominator of $(\theta_{r,n})$:

$$\operatorname{den}(\theta_{r,n}) = \operatorname{den}\left(\sum_{j=0}^{r} P_j\left(\frac{1}{p^n(x)}\right)(f(p^n(x)))^j\right) \le C_4^{rd^n} \times C_9^{rd^n} \le C_5^{rd^n}.$$

Next, we prove the inequality (43). For every $i = 0, 1, \ldots, r$, we have by (41)

$$\left|\overline{P_i\left(\frac{1}{p^n(x)}\right)}\right| \le \chi \sum_{i=0}^{r} \left|\overline{\frac{1}{p^n(x)}}\right|^i \le (r+1)\chi C_4^{rd^n}. \tag{47}$$

For every $n \ge 0$, we have by (5) and (38) above

$$\overline{|f(p^n(x))|} \le \left|\overline{\frac{1}{a^n}}\right|\left(\overline{|f(x)|} + \left(\sum_{j=0}^{n-1} \overline{|a|}^j\right) C_1^{d^n}\right) \le C_{10}^{d^n}. \tag{48}$$

By using (47) and (48), we can give an upper bound for the house of $\theta_{r,n}$:

$$\begin{aligned}\overline{|\theta_{r,n}|} &\le \sum_{i=0}^{r} \overline{\left|P_i\left(\frac{1}{p^n(x)}\right)\right|} \times \overline{|[f(p^n(x))]^i|} \\ &\le (r+1)\chi \sum_{i=0}^{r} C_4^{rd^n} \times C_{10}^{rd^n} \le (r+1)^2 \chi C_6^{rd^n}.\end{aligned}$$

Finally, we show the inequality (44). By (37), we have

$$|\theta_{r,n}| = \left(\frac{1}{|p^n(x)|}\right)^{r^2+\sigma} \left|L_r\left(\frac{1}{p^n(x)}\right)\right|. \tag{49}$$

Since $|p^n(x)| \ge C_7^{d^n}$, we see that

$$\lim_{n\to\infty} \left|L_m\left(\frac{1}{p^n(x)}\right)\right| = |L_r(0)| \ne 0, \tag{50}$$

which proves that $\theta_{r,n} \ne 0$ for every large n. Moreover, by (49) we have (44). □

We come now to the conclusion. Define $\delta = \deg(\theta_{r,n})$. As $\theta_{r,n} \ne 0$ for every large n, it satisfies the size inequality:

$$|\theta_{r,n}| \ge (\mathrm{den}(\theta_{r,n}))^{-\delta} \times \overline{|\theta_{r,n}|}^{-\delta+1}. \tag{51}$$

Using (42), (43) and (44) yields

$$2(\gamma(r+1)^2)^{\delta-1} L_r(0) \ge \left(\frac{C_7^r}{C_5^\delta \times C_6^\delta}\right)^{rd^n}. \tag{52}$$

If we choose r such that $C_7^r > C_5^\delta \times C_6^\delta$ and fix it, we obtain a contradiction when n tends to infinity, which proves Theorem 1.2.

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