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Identity involving symmetric sums of regularized multiple zeta-star values

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An identity involving symmetric sums of regularized multiple zeta-star values of harmonic type was proved by Hoffman. In this paper, we prove an identity of shuffle type. We use Bell polynomials appearing in the study of set partitions to prove the identity.

1. Introduction and statement of results

The multiple zeta value (MZV) and multiple zeta-star value (MZSV, or sometimes referred to as the nonstrict MZV) are real numbers defined by the nested series

$$\zeta(k_1, k_2, \ldots, k_r) = \sum_{0<m_1<m_2<\cdots<m_r} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_r^{k_r}}, \tag{1-1}$$

$$\zeta^\star(k_1, k_2, \ldots, k_r) = \sum_{0<m_1\le m_2\le\cdots\le m_r} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_r^{k_r}}, \tag{1-2}$$

respectively, where k_i $(1 \le i \le r)$ are arbitrary positive integers with $k_r > 1$. MZVs and MZSVs can also be given by integrals. These values have been actively studied for more than two decades, but Euler [1776] already mentioned them in a special case, $r = 2$. In this paper, we give an identity involving symmetric sums for a class of regularizations of (1-2).

The two expressions of series and integrals yield two different products $*$ and ш, called *harmonic* (or *stuffle*) and *shuffle*, respectively, for any real value in factored form written in terms of either MZVs or MZSVs. For example, the result of $*$ on MZSV for the value $\zeta^\star(2) \times \zeta^\star(2)$ is

$$\zeta^\star(2)\zeta^\star(2) = \zeta^\star((2)*(2)) = \zeta^\star((2,2)+(2,2)-(4)) = 2\zeta^\star(2,2) - \zeta^\star(4), \tag{1-3}$$

where, for notational simplicity, we think of the product $*$ as taking place among *indices*. (An index means a finite sequence $\boldsymbol{k} = (k_1, \ldots, k_r)$ of positive integers.) The result (1-3) follows from series expressions in (1-2) with

$$\Biggl(\sum_{0<m_1} \frac{1}{m_1^2}\Biggr)\Biggl(\sum_{0<m_2} \frac{1}{m_2^2}\Biggr) = \sum_{0<m_1\le m_2} \frac{1}{m_1^2 m_2^2} + \sum_{0<m_2\le m_1} \frac{1}{m_2^2 m_1^2} - \sum_{0<m} \frac{1}{m^4},$$

or with the division of the summation

$$\sum_{0<m_1,\, 0<m_2} = \sum_{0<m_1\le m_2} + \sum_{0<m_2\le m_1} - \sum_{0<m_1=m_2}. \tag{1-4}$$

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The other results for $\zeta(2)\zeta(2)=\zeta^\star(2)\zeta^\star(2)$ are similarly obtained such that $\zeta((2)*(2))=2\zeta(2,2)+\zeta(4)$, $\zeta((2)\mathbin{ш}(2))=2\zeta(2,2)+4\zeta(1,3)$, and $\zeta^\star((2)\mathbin{ш}(2))=2\zeta^\star(1,3)$. The case of $*$ on MZV follows from series expressions in (1-1) with division of the summation as in (1-4),

$$\sum_{0<m_1,\,0<m_2} = \sum_{0<m_1<m_2} + \sum_{0<m_2<m_1} + \sum_{0<m_1=m_2}.$$

The cases of ш on MZV and MZSV follow from integral expressions as

$$\zeta(2)=\int_{0<s<t<1}\frac{ds}{1-s}\frac{dt}{t},$$

with division of domains of integration

$$\int_{\substack{0<s_1<1\\0<s_2<1}} = \int_{0<s_1<s_2<1} + \int_{0<s_2<s_1<1},$$

where we require an extra technique [Kaneko and Yamamoto 2018] of the integral associated to 2-labeled posets for integral expressions of MZSVs. We omit details of $*$ and ш (for which see [Hoffman 1997; Ihara et al. 2006; Kaneko 2018; Kaneko and Yamamoto 2018; Reutenauer 1993; Zudilin 2003]),[1] since many notations are necessary for rigorous statements, though product rules are simply induced from dividing the summation and domain.

MZVs and MZSVs are divergent if $k_r=1$, but recently, the theory of regularization has been established. (For details, see [Ihara et al. 2006] and [Kaneko and Yamamoto 2018] for MZV and MZSV, respectively.) Four polynomials whose coefficients are $\mathbb{Q}$-linear combinations of MZVs and MZSVs, which we denote by

$$\zeta_*(\boldsymbol{k};T),\quad \zeta_{ш}(\boldsymbol{k};T),\quad \zeta_*^\star(\boldsymbol{k};T),\quad\text{and}\quad \zeta_{ш}^\star(\boldsymbol{k};T), \tag{1-5}$$

are defined for any index $\boldsymbol{k}$ in the theory: $\zeta_*(\boldsymbol{k};T)$ and $\zeta_{ш}(\boldsymbol{k};T)$ are generalizations of $\zeta(\boldsymbol{k})$ involving products $*$ and ш, respectively; $\zeta_*^\star(\boldsymbol{k};T)$ and $\zeta_{ш}^\star(\boldsymbol{k};T)$ are those of $\zeta^\star(\boldsymbol{k})$. (Note that the polynomials in (1-5) are constant and equal to $\zeta(\boldsymbol{k})$ when $k_r>1$.) A key idea of the generalizations is roughly to regard the divergent value $\zeta(1)=\zeta^\star(1)=\frac{1}{1}+\frac{1}{2}+\cdots$ as the variable T when using product rule. For example, using the rule of $*$ on MZSV for $\zeta^\star(2)\zeta^\star(1)$ yields

$$\zeta^\star(2)\zeta^\star(1)=\zeta_*^\star((2)*(1))=\zeta_*^\star((2,1)+(1,2)-(3))=\zeta_*^\star(2,1;T)+\zeta^\star(1,2)-\zeta^\star(3),$$

which, together with $\zeta^\star(2)\zeta^\star(1)=\zeta^\star(2)T$, proves

$$\zeta_*^\star(2,1;T)=\zeta^\star(2)T-\zeta^\star(1,2)+\zeta^\star(3). \tag{1-6}$$

We can obtain

$$\begin{aligned}\zeta_*(2,1;T)&=\zeta(2)T-\zeta(1,2)-\zeta(3),\\ \zeta_{ш}(2,1;T)&=\zeta(2)T-2\zeta(1,2),\\ \zeta_{ш}^\star(2,1;T)&=\zeta^\star(2)T-\tfrac{1}{2}\zeta^\star(1,2)\end{aligned}$$

[1] We recommend [Kaneko 2018; Zudilin 2003] for nonspecialists.

in similar ways, where evaluating $\zeta^{\star}_{ш}(2, 1; T)$ requires extra computations because $\zeta^{\star}_{ш}(2, 1; T)$ is defined by means of the integral associated to 2-labeled posets. The regularized values $\zeta_*(\boldsymbol{k})$, $\zeta_{ш}(\boldsymbol{k})$, $\zeta^{\star}_*(\boldsymbol{k})$, and $\zeta^{\star}_{ш}(\boldsymbol{k})$ are defined by their constant terms; e.g., $\zeta_*(\boldsymbol{k}) = \zeta_*(\boldsymbol{k}; 0)$.

Fundamental theorems of regularization for MZVs and MZSVs were proved in [Ihara et al. 2006] and [Kaneko and Yamamoto 2018], respectively, which are stated as follows. For any index $\boldsymbol{k}$,

$$\rho(\zeta_*(\boldsymbol{k}; T)) = \zeta_{ш}(\boldsymbol{k}; T) \quad \text{and} \quad \rho^{\star}(\zeta^{\star}_*(\boldsymbol{k}; T)) = \zeta^{\star}_{ш}(\boldsymbol{k}; T), \tag{1-7}$$

where ρ and $\rho^{\star}$ are $\mathbb{R}$-linear endomorphisms on $\mathbb{R}[T]$ related to the gamma function $\Gamma(u)$. The detailed definition of $\rho^{\star}$ will be introduced in Section 2, and is necessary to prove our result, Theorem 1.1.

In order to state Theorem 1.1, we will recall Hoffman's identities involving symmetric sums of the polynomials $\zeta_*(\boldsymbol{k}; T)$ and $\zeta^{\star}_*(\boldsymbol{k}; T)$, which are shown in [Hoffman 1992; 2015]. Let $\mathcal{P}_r$ be the set of partitions of the set $\{1, \dots, r\}$. For any $\Pi = \{P_1, \dots, P_g\} \in \mathcal{P}_r$, we define integers $c(\Pi) = c_r(\Pi)$ and $c^{\star}(\Pi) = c^{\star}_r(\Pi)$ by

$$c_r(\Pi) = (-1)^{r-g} \prod_{i=1}^{g} (|P_i| - 1)! \quad \text{and} \quad c^{\star}_r(\Pi) = \prod_{i=1}^{g} (|P_i| - 1)!\,, \tag{1-8}$$

respectively, where $|P|$ is the number of the elements of a set P. We also define

$$H_*(\boldsymbol{k}, \Pi; T) := \prod_{i=1}^{g} \eta\biggl(\sum_{p \in P_i} k_p; T\biggr), \tag{1-9}$$

where[2]

$$\eta(k; T) = \begin{cases} \zeta(k), & k > 1, \\ T, & k = 1. \end{cases}$$

Let S_r denote the symmetric group of degree r. Hoffman's identities are then

$$\sum_{\sigma \in S_r} \zeta_*(k_{\sigma(1)}, \dots, k_{\sigma(r)}; T) = \sum_{\Pi \in \mathcal{P}_r} c(\Pi) H_*(\boldsymbol{k}, \Pi; T), \tag{1-10}$$

$$\sum_{\sigma \in S_r} \zeta^{\star}_*(k_{\sigma(1)}, \dots, k_{\sigma(r)}; T) = \sum_{\Pi \in \mathcal{P}_r} c^{\star}(\Pi) H_*(\boldsymbol{k}, \Pi; T). \tag{1-11}$$

Recently, a shuffle version of (1-10) was proved in [Machide 2017], which is obtained by replacing ζ_* and H_* with $\zeta_{ш}$ and $H_{ш}$, respectively:

$$\sum_{\sigma \in S_r} \zeta_{ш}(k_{\sigma(1)}, \dots, k_{\sigma(r)}; T) = \sum_{\Pi \in \mathcal{P}_r} c(\Pi) H_{ш}(\boldsymbol{k}, \Pi; T), \tag{1-12}$$

where $H_{ш}$ will be defined in (1-15).

The main result of this paper is the shuffle version of (1-11).

Theorem 1.1. *For any index* $\boldsymbol{k}$, *we have*

$$\sum_{\sigma \in S_r} \zeta^{\star}_{ш}(k_{\sigma(1)}, \dots, k_{\sigma(r)}; T) = \sum_{\Pi \in \mathcal{P}_r} c^{\star}(\Pi) H_{ш}(\boldsymbol{k}, \Pi; T), \tag{1-13}$$

[2] We note that $\eta(k; T) = \zeta_*(k; T) = \zeta_{ш}(k; T) = \zeta^{\star}_*(k; T) = \zeta^{\star}_{ш}(k; T)$.

where $H_{ш}(\boldsymbol{k}, \Pi; T)$ *is similar to* $H_{*}(\boldsymbol{k}, \Pi; T)$, *but the characteristic function*

$$\chi_{ш}(\boldsymbol{k}, P_i) := \begin{cases} 0 & \text{if } |P_i| > 1, \text{ and } k_p = 1 \text{ for all } p \in P_i, \\ 1 & \text{otherwise} \end{cases} \tag{1-14}$$

is added in each multiplicand; *that is*,

$$H_{ш}(\boldsymbol{k}, \Pi; T) := \prod_{i=1}^{g} \chi_{ш}(\boldsymbol{k}, P_i)\eta\Bigl(\sum_{p\in P_i} k_p; T\Bigr). \tag{1-15}$$

We give some examples of (1-13). The number of the terms on its right-hand side decreases as the number of k_i equal to 1 increases, because of (1-14).

Example 1.2. Let k and l be integers at least 2. Then

$$\zeta^{\star}(1, k) + \zeta^{\star}_{ш}(k, 1; T) = \zeta(k)T + \zeta(k+1),$$

$$\begin{aligned} &\zeta^{\star}(1, k, l) + \zeta^{\star}(1, l, k) + \zeta^{\star}(k, 1, l) + \zeta^{\star}(l, 1, k) + \zeta^{\star}_{ш}(k, l, 1; T) + \zeta^{\star}_{ш}(l, k, 1; T) \\ &\qquad = (\zeta(k)\zeta(l) + \zeta(k+l))T + \zeta(k)\zeta(l+1) + \zeta(l)\zeta(k+1) + 2\zeta(k+l+1), \end{aligned}$$

$$2(\zeta^{\star}(1, 1, k) + \zeta^{\star}_{ш}(1, k, 1; T) + \zeta^{\star}_{ш}(k, 1, 1; T)) = \zeta(k)T^2 + 2\zeta(k+1)T + 2\zeta(k+2).$$

In particular, we have a simple equation (1-16) when the number of k_i equal to 1 is $r-1$ (or equivalently, there is just one k_j that is greater than 1): the right-hand side is written in terms of only single zeta values.

Corollary 1.3. *For integers* $k \geq 2$ *and* $r \geq 1$, *we have*

$$\sum_{i=0}^{r-1} \zeta^{\star}_{ш}(\{1\}^i, k, \{1\}^{r-1-i}; T) = \sum_{j=0}^{r-1} \zeta(k+r-1-j)\frac{T^j}{j!}, \tag{1-16}$$

where $\{1\}^i$ *means* i *repetitions of* 1.

The method of the proof of Theorem 1.1 is an improvement to that used in [Machide 2017]. We will use complete exponential Bell polynomials to show Proposition 2.3, which are defined by

$$B_r(x_1, \dots, x_r) := r! \sum_{\substack{i_1, i_2, \dots, i_r \geq 0 \\ 1\cdot i_1 + 2\cdot i_2 + \cdots + r\cdot i_r = r}} \frac{1}{i_1!\, i_2! \cdots i_r!} \prod_{a=1}^{r} \Bigl(\frac{x_a}{a!}\Bigr)^{i_a}. \tag{1-17}$$

Bell polynomials [1927/28] first appear in the study of set partitions. Currently it is known that they have many relations to combinatorial numbers and applications to other areas; see, e.g., [Comtet 1974; Roman 1980]. We will mention an identity involving $\zeta^{\star}_{*}(1, 1, \dots, 1; T)$ in Remark 2.5, which is a variation of the identity $r! = B_r(0!, 1!, \dots, (r-1)!)$.

This paper is organized as follows. We prepare some propositions in Section 2, and prove Theorem 1.1 and Corollary 1.3 in Section 3.

2. Propositions

In this section, we introduce Propositions 2.1, 2.2, and 2.3, which will be used to prove Theorem 1.1. We will omit the proofs of Propositions 2.1 and 2.2 because these are almost the same as Lemmas 4.7 and 4.8 in [Machide 2017], respectively, where some notation and terminology are modified.

Let $[r]$ denote the set $\{1, \ldots, r\}$, and let A and B be its subsets. We denote by $\mathcal{P}(A)$ the set of partitions of A (i.e., $\mathcal{P}([r]) = \mathcal{P}_r$), and we define a subset $\mathcal{P}_B(A)$ in $\mathcal{P}(A)$ by

$$\mathcal{P}_B(A) := \{\Pi = \{P_1, \ldots, P_m\} \in \mathcal{P}(A) : P_i \not\subset B \text{ for all } i\}.$$

For example, if $(r, A, B) = (4, \{1, 2, 3\}, \{3, 4\})$, then

$$\mathcal{P}(A) = \{1|2|3,\ 12|3,\ 13|2,\ 23|1,\ 123\} \quad \text{and} \quad \mathcal{P}_B(A) = \{13|2,\ 23|1,\ 123\},$$

where $a_1 \cdots a_p | b_1 \cdots b_q | \cdots$ means a partition such as $12|3 = \{\{1, 2\}, \{3\}\}$.

Let $\Xi = \{P_1, \ldots, P_g\} \in \mathcal{P}(A)$, and let $s = |A|$. We will define a partition $\sigma_A(\Xi)$ in $\mathcal{P}_s$ as follows. Let $a_1 < \cdots < a_s$ be the increasing sequence of integers such that

$$A = \{a_1, \ldots, a_s\}$$

and let σ_A be the permutation of S_r that is uniquely determined by

$$\sigma_A^{-1}(i) = a_i \quad (i = 1, \ldots, s) \quad \text{and} \quad \sigma_A^{-1}(s+1) < \cdots < \sigma_A^{-1}(r);$$

by the definition,

$$\sigma_A(A) = \{\sigma_A(a_1), \ldots, \sigma_A(a_s)\} = [s].$$

We then define

$$\sigma_A(\Xi) := \{\sigma_A(P_1), \ldots, \sigma_A(P_g)\} \in \mathcal{P}_s.$$

For convenience, $\sigma_A(\Xi) = \phi$ if $A = \Xi = \phi$.

The propositions are as follows.

Proposition 2.1 [Machide 2017, Lemma 4.7]. *For any subset $B \subsetneqq [r]$, we have*

$$\bigsqcup_{A \subset B} \{\Xi \sqcup \Delta : (\Xi, \Delta) \in \mathcal{P}(A) \times \mathcal{P}_B([r] \setminus A)\} = \mathcal{P}_r, \tag{2-1}$$

where $\sqcup$ denotes the disjoint union, and $\bigsqcup_{A \subset B}$ ranges over all subsets in B which include ϕ.

Proposition 2.2 [Machide 2017, Lemma 4.8]. *Let A and B be subsets such that $A \subset B \subsetneqq [r]$, and let (Ξ, Δ) be in $\mathcal{P}(A) \times \mathcal{P}_B([r] \setminus A)$. Let the symbol $\bullet$ mean either $*$ or ш:*

(*i*) *We define $c^\star(\phi) = 1$. We have*

$$c^\star(\Xi \cup \Delta) = c^\star(\Xi) c^\star(\Delta). \tag{2-2}$$

(*ii*) *We define $H_\bullet(\phi, \phi; T) = 1$ and*

$$\mathbf{1}_s = (\underbrace{1, \ldots, 1}_{s}).$$

Suppose that $\boldsymbol{k} = (k_1, \ldots, k_r)$ is an index satisfying $B = \{a \in [r] : k_a = 1\}$ (and $\boldsymbol{k} \neq \mathbf{1}_r$). Then we have

$$H_\bullet(\boldsymbol{k}, \Xi \cup \Delta; T) = \left(\prod_{i=1}^{h} \zeta(k_{Q_i}) \right) H_\bullet(\mathbf{1}_{|A|}, \sigma_A(\Xi); T), \tag{2-3}$$

where $Q_1, \dots, Q_h$ *are the blocks of* Δ *(i.e.,* $\Delta = \{Q_1, \dots, Q_h\}$*), and*

$$k_{Q_i} = \sum_{q \in Q_i} k_q \quad (i = 1, \dots, h).$$

Note that $k_{Q_i} > 1$ *and* $\zeta(k_{Q_i})$ *is not infinity for any* i*, because* $\Delta \in \mathcal{P}_B([r] \setminus A)$ *and* $Q_i \not\subset B$.

Proposition 2.3 [Machide 2017, Lemma 4.9]. *For any positive integer* r*, we have*

$$\sum_{\Pi \in \mathcal{P}_r} c^\star(\Pi) H_*(\mathbf{1}_r, \Pi; T) = \rho^{\star -1}(T^r), \tag{2-4}$$

$$\sum_{\Pi \in \mathcal{P}_r} c^\star(\Pi) H_{\text{ш}}(\mathbf{1}_r, \Pi; T) = T^r. \tag{2-5}$$

The condition $B \neq [r]$ in the first two propositions is necessary for taking an element in $\mathcal{P}_B([r] \setminus A)$; see [Machide 2017, Remark 4.6] for details.

To prove Proposition 2.3, we need Lemma 2.4, which is the *star*-version of [Machide 2017, Lemma 4.10] in terms of Bell polynomials.

Lemma 2.4. *For any positive integer* r*, we have*

$$\sum_{\Pi \in \mathcal{P}_r} c^\star(\Pi) H_*(\mathbf{1}_r, \Pi; T) = B_r\big(0!\,\eta(1; T), 1!\,\eta(2; T), \dots, (r-1)!\,\eta(r; T)\big). \tag{2-6}$$

We will now prove Proposition 2.3, and then prove Lemma 2.4.

Proof of Proposition 2.3. We first recall the definition of $\rho^\star$, which is an $\mathbb{R}$-linear endomorphism on $\mathbb{R}[T]$ determined by the equality

$$\rho^\star(e^{Tt}) = A(-t)^{-1} e^{Tt} \tag{2-7}$$

in the formal power series algebra $\mathbb{R}[T][\![t]\!]$ on which $\rho^\star$ acts coefficientwise, see [Kaneko and Yamamoto 2018, Section 4], where

$$A(t) = \exp\bigg(\sum_{m=2}^{\infty} \frac{(-1)^m \zeta(m)}{m} t^m\bigg).$$

Note that $A(t) = e^{\gamma t}\Gamma(1+t)$, where γ is Euler's constant. We can see from (2-7) that the inverse endomorphism $\rho^{\star -1}$ exists and it satisfies

$$\rho^{\star -1}(e^{Tt}) = A(-t) e^{Tt} = \exp\bigg(Tt + \sum_{m=2}^{\infty} \frac{\zeta(m)}{m} t^m\bigg) = \exp\bigg(\sum_{m=1}^{\infty} \frac{\eta(m; T)}{m} t^m\bigg). \tag{2-8}$$

The exponential partial Bell polynomials can be defined by use of the generating function (see [Comtet 1974, Chapter 3]):

$$\exp\bigg(\sum_{m=1}^{\infty} x_m \frac{t^m}{m!}\bigg) = \sum_{r=0}^{\infty} B_r(x_1, \dots, x_r) \frac{t^r}{r!}. \tag{2-9}$$

Combining (2-8) and (2-9) with $x_m = (m-1)!\,\eta(m; T)$, we obtain

$$\rho^{\star -1}(e^{Tt}) = \sum_{r=0}^{\infty} B_r\big(0!\,\eta(1; T), 1!\,\eta(2; T), \dots, (r-1)!\,\eta(r; T)\big) \frac{t^r}{r!},$$

which, together with (2-6), gives

$$\rho^{\star -1}(e^{Tt}) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \sum_{\Pi \in \mathcal{P}_r} c^{\star}(\Pi) H_{*}(\mathbf{1}_r, \Pi; T).$$

Identity (2-4) follows from comparing the coefficients of t^r on both sides of this equation. We will give a proof of (2-5), which is a modification to that of (4-58) in [Machide 2017, Lemma 4.9].[3] Let $\Lambda = \Lambda_r$ be the partition in $\mathcal{P}_r$ defined by

$$\Lambda_r := 1|2|\cdots|r = \{\{1\}, \{2\}, \ldots, \{r\}\}.$$

We see from (1-14) and (1-15) that $H_{ш}(\mathbf{1}_r, \Pi; T) = 0$ for any $\Pi \in \mathcal{P}_r$ with $\Pi \neq \Lambda$, and so

$$\sum_{\Pi \in \mathcal{P}_r} c^{\star}(\Pi) H_{ш}(\mathbf{1}_r, \Pi; T) = c^{\star}(\Lambda) H_{ш}(\mathbf{1}_r, \Lambda; T).$$

Since

$$c^{\star}(\Lambda) = \prod_{i=1}^{r} 0! = 1 \quad \text{and} \quad H_{ш}(\mathbf{1}_r, \Lambda; T) = \prod_{i=1}^{r} \eta(1; T) = T^r,$$

we obtain (2-5). □

We will need the partial exponential Bell polynomials to prove Lemma 2.4, which we denote by $B_{r,k}(x_1, \ldots, x_{r-k+1})$ for integers r and k with $1 \le k \le r$. Complete and partial Bell polynomials have the relations

$$B_r(x_1, \ldots, x_r) = \sum_{k=1}^{r} B_{r,k}(x_1, \ldots, x_{r-k+1}). \tag{2-10}$$

Let $b_{r,k}(i_1, \ldots, i_{r-k+1})$ be the coefficients of $B_{r,k}(x_1, \ldots, x_{r-k+1})$ such that

$$B_{r,k}(x_1, \ldots, x_{r-k+1}) = \sum_{\substack{i_1, \ldots, i_{r-k+1} \ge 0 \\ i_1 + i_2 + \cdots + i_{r-k+1} = k \\ 1 \cdot i_1 + 2 \cdot i_2 + \cdots + (r-k+1) \cdot i_{r-k+1} = r}} b_{r,k}(i_1, \ldots, i_{r-k+1}) \prod_{a=1}^{r-k+1} x_a^{i_a}.$$

From combinatorial considerations, see, e.g., [Comtet 1974, Chapter 3], we know that $b_{r,k}(i_1, \ldots, i_{r-k+1})$ is the number of partitions with total k blocks in $\mathcal{P}_r$ which consist of i_a blocks of size a for $a \in [r-k+1]$. For instance,

$$b_{4,2}(1, 0, 1) = 4, \quad b_{4,2}(0, 2, 0) = 3, \quad \text{and} \quad B_{4,2}(x_1, x_2, x_3) = 4x_1x_3 + 3x_2^2,$$

since we have four partitions with blocks of size 1 and 3 and three partitions with 2 blocks of size 2 when a set with four elements is divided into two blocks. We note that $B_{r,k} = B_{r,k}(1, \ldots, 1)$ are Stirling numbers of the second kind; that is, they count the number of ways to partition a set of r elements into k nonempty subsets.

[3] There is a misprint in the proof of [Machide 2017, (4-58)]: $\underbrace{\{\{1\}, \ldots, \{1\}\}}_{n}$ should be $\{\{1\}, \ldots, \{n\}\}$.

Proof of Lemma 2.4. For any partition $\Pi = \{P_1, \dots, P_g\}$ in $\mathcal{P}_r$ and integer a in $[r]$, let $N_a(\Pi)$ be the number of the blocks whose cardinalities equal a; i.e.,

$$N_a(\Pi) := |\{j \in [g] : |P_j| = a\}|.$$

We see from the definition that

$$g = N_1(\Pi) + \cdots + N_r(\Pi) \quad \text{and} \quad r = 1 \cdot N_1(\Pi) + \cdots + r \cdot N_r(\Pi),$$

so that

$$c^\star(\Pi) H_*(\mathbf{1}_r, \Pi; T) = \prod_{i=1}^{g} (|P_i| - 1)!\, \eta(|P_i|; T) = \prod_{a=1}^{r} ((a-1)!\, \eta(a; T))^{N_a(\Pi)}.$$

It follows from the combinatorial meaning of $b_{r,k}(i_1, \dots, i_{r-k+1})$ that

$$\sum_{\substack{\Pi \in \mathcal{P}_r \\ N_a(\Pi) = i_a (\forall a)}} 1 = b_{r,k}(i_1, \dots, i_{r-k+1})$$

for nonnegative integers $i_1, \dots, i_{r-k+1}$ with $\sum_{a=1}^{r-k+1} i_a = k$ and $\sum_{a=1}^{r-k+1} a \cdot i_a = r$, and so

$$\begin{aligned}
&\sum_{\substack{i_1, i_2, \dots, i_r \geq 0 \\ 1 \cdot i_1 + 2 \cdot i_2 + \cdots + r \cdot i_r = r}} \sum_{\substack{\Pi \in \mathcal{P}_r \\ N_a(\Pi) = i_a (\forall a)}} c^\star(\Pi) H_*(\mathbf{1}_r, \Pi; T) \\
&\qquad = \sum_{\substack{i_1, i_2, \dots, i_r \geq 0 \\ 1 \cdot i_1 + 2 \cdot i_2 + \cdots + r \cdot i_r = r}} \left(\prod_{a=1}^{r} ((a-1)! \eta(a; T))^{i_a} \right) \sum_{\substack{\Pi \in \mathcal{P}_r \\ N_a(\Pi) = i_a (\forall a)}} 1 \\
&\qquad = \sum_{k=1}^{r} \sum_{\substack{i_1, i_2, \dots, i_{r-k+1} \geq 0 \\ i_1 + i_2 + \cdots + i_{r-k+1} = k \\ 1 \cdot i_1 + 2 \cdot i_2 + \cdots + (r-k+1) \cdot i_{r-k+1} = r}} \left(\prod_{a=1}^{r-k+1} ((a-1)! \eta(a; T))^{i_a} \right) b_{r,k}(i_1, \dots, i_{r-k+1}) \\
&\qquad = \sum_{k=1}^{r} B_{r,k}\big(0!\, \eta(1; T), 1!\, \eta(2; T), \dots, (r-k)!\, \eta(r-k+1; T)\big).
\end{aligned} \tag{2-11}$$

We thus obtain (2-6), since the first line of (2-11) is equal to the left-hand side of (2-6) because of [Machide 2017, (4-74)], or

$$\mathcal{P}_r = \bigsqcup_{\substack{i_1, i_2, \dots, i_r \geq 0 \\ 1 \cdot i_1 + 2 \cdot i_2 + \cdots + r \cdot i_r = r}} \{\Pi \in \mathcal{P}_r : N_a(\Pi) = i_a, \text{where } a \in [r]\},$$

and since the last line of (2-11) is equal to the right-hand side of (2-6) because of (2-10). □

Remark 2.5. Bell polynomials are related to many combinatorial numbers. It may be worth noting the relation to the unsigned Stirling numbers of the first kind, which can be expressed as

$$c(r, k) = B_{r,k}(0!, \dots, (r-k)!).$$

The unsigned Stirling numbers are defined as coefficients of the rising factorial; that is,

$$x(x+1)\cdots(x+r-1)=\sum_{k=1}^{r}B_{r,k}(0!,\ldots,(r-k)!)x^{k}. \tag{2-12}$$

Substituting $x=1$ in this equation and combining it with (2-10), we thus obtain

$$r!=B_r(0!,1!,\ldots,(r-1)!). \tag{2-13}$$

We also have

$$r!\zeta_*^\star(\mathbf{1}_r;T)=B_r\big(0!\,\eta(1;T),1!\,\eta(2;T),\ldots,(r-1)!\,\eta(r;T)\big), \tag{2-14}$$

which follows from Hoffman's identity (1-11) with $\boldsymbol{k}=\mathbf{1}_r$ and (2-6). Equation (2-14) is a variation of (2-13) in the sense that we can obtain (2-14) from (2-13) by replacing $r!$ in the left-hand side with $r!\,\zeta_{ш}^\star(\mathbf{1}_r;T)$ and $j!$ in the right-hand side with $j!\,\eta(j+1;T)$. We note that (2-14) corresponds to an identity in terms of symmetric functions (see [Hoffman 1997, Theorem 5.1] and [Kaneko and Yamamoto 2018, Lemma 5.1]):

$$r!h_r=B_r(0!\,p_1,1!\,p_2,\ldots,(r-1)!\,p_r), \tag{2-15}$$

where h_i is the complete symmetric function of degree i and p_i is the i-th power sum symmetric function.

3. Proof

We will need (3-1) to prove (1-13), which is the star version of [Machide 2017, (4-51)].

Proposition 3.1. *For any index* $\boldsymbol{k}$,

$$\rho^\star\bigg(\sum_{\Pi\in\mathcal{P}_r}c^\star(\Pi)H_*(\boldsymbol{k},\Pi;T)\bigg)=\sum_{\Pi\in\mathcal{P}_r}c^\star(\Pi)H_{ш}(\boldsymbol{k},\Pi;T). \tag{3-1}$$

We can prove (3-1) in a quite similar way, as we see below.

Proof of Proposition 3.1. Let $B=\{j\in[r]:k_j=1\}\subset[r]$. We suppose that $B=[r]$. Then, $\boldsymbol{k}=\mathbf{1}_r$, and so we can see from Proposition 2.3 that

$$\rho^\star\bigg(\sum_{\Pi\in\mathcal{P}_r}c^\star(\Pi)H_*(\boldsymbol{k},\Pi;T)\bigg)\overset{(2\text{-}4)}{=}\rho^\star(\rho^{\star-1}(T^r))=T^r\overset{(2\text{-}5)}{=}\sum_{\Pi\in\mathcal{P}_r}c^\star(\Pi)H_{ш}(\boldsymbol{k},\Pi;T), \tag{3-2}$$

which proves (3-1) for this case.

We suppose that $B\neq[r]$. Let A be a subset in B. Then we have

$$\{\sigma_A(\Xi):\Xi\in\mathcal{P}(A)\}=\{\Xi':\Xi'\in\mathcal{P}_{|A|}\}, \tag{3-3}$$

because the restriction of σ_A to A is a bijection from A to $[|A|]$. From (1-8) we easily see that $c^\star(\Xi)=c^\star(\sigma_A(\Xi))$. Thus,

$$\begin{aligned}\sum_{\Pi\in\mathcal{P}_r}c^\star(\Pi)H_*(\boldsymbol{k},\Pi;T)&\overset{\text{Prop. 2.1}}{=}\sum_{A\subset B}\sum_{\substack{\Xi\in\mathcal{P}(A)\\ \Delta\in\mathcal{P}_B([r]\setminus A)}}c^\star(\Xi\cup\Delta)H_*(\boldsymbol{k},\Xi\cup\Delta;T)\\ &\overset{\text{Prop. 2.2}}{=}\sum_{A\subset B}\sum_{\Delta\in\mathcal{P}_B([r]\setminus A)}c^\star(\Delta)\bigg(\prod_{i=1}^{h}\zeta(k_{Q_i})\bigg)\sum_{\Xi\in\mathcal{P}(A)}c^\star(\Xi)H_*(\mathbf{1}_{|A|},\sigma_A(\Xi);T)\end{aligned}$$

$$\overset{(3\text{-}3)}{=}\sum_{A\subset B}\sum_{\Delta\in\mathcal{P}_B([r]\setminus A)}c^\star(\Delta)\biggl(\prod_{i=1}^{h}\zeta(k_{Q_i})\biggr)\sum_{\Xi'\in\mathcal{P}_{|A|}}c^\star(\Xi')H_*(\mathbf{1}_{|A|},\Xi';T)$$

$$\overset{(2\text{-}4)}{=}\sum_{A\subset B}\sum_{\Delta\in\mathcal{P}_B([r]\setminus A)}c^\star(\Delta)\biggl(\prod_{i=1}^{h}\zeta(k_{Q_i})\biggr)\rho^{\star-1}(T^{|A|}),$$

where $Q_1,\ldots,Q_h$ mean the blocks of Δ. Therefore,

$$\begin{aligned}\rho^\star\biggl(\sum_{\Pi\in\mathcal{P}_r}c^\star(\Pi)H_*(\boldsymbol{k},\Pi;T)\biggr)&=\sum_{A\subset B}\sum_{\Delta\in\mathcal{P}_B([r]\setminus A)}c^\star(\Delta)\biggl(\prod_{i=1}^{h}\zeta(k_{Q_i})\biggr)\rho^\star(\rho^{\star-1}(T^{|A|}))\\&=\sum_{A\subset B}\sum_{\Delta\in\mathcal{P}_B([r]\setminus A)}c^\star(\Delta)\biggl(\prod_{i=1}^{h}\zeta(k_{Q_i})\biggr)T^{|A|}.\end{aligned}\tag{3-4}$$

By using Propositions 2.1 and 2.2, and (3-3), and by using (2-5) instead of (2-4), we can similarly prove

$$\sum_{\Pi\in\mathcal{P}_r}c^\star(\Pi)H_{ш}(\boldsymbol{k},\Pi;T)=\sum_{A\subset B}\sum_{\Delta\in\mathcal{P}_B([r]\setminus A)}c^\star(\Delta)\biggl(\prod_{i=1}^{h}\zeta(k_{Q_i})\biggr)T^{|A|}.\tag{3-5}$$

Equating (3-4) and (3-5), we obtain (3-1) for $B\neq[r]$, and complete the proof. □

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Recall the *star-regularization theorem*, the second identity of (1-7), which we tag as (s-reg) here. We obtain

$$\begin{aligned}\sum_{\sigma\in S_r}\zeta^\star_{ш}(k_{\sigma(1)},\ldots,k_{\sigma(r)};T)&\overset{\text{(s-reg)}}{=}\rho^\star\biggl(\sum_{\sigma\in S_r}\zeta^\star_*(k_{\sigma(1)},\ldots,k_{\sigma(r)};T)\biggr)\\&\overset{(1\text{-}11)}{=}\rho^\star\biggl(\sum_{\Pi\in\mathcal{P}_r}c^\star(\Pi)H_*(\boldsymbol{k},\Pi;T)\biggr)\\&\overset{(3\text{-}1)}{=}\sum_{\Pi\in\mathcal{P}_r}c^\star(\Pi)H_{ш}(\boldsymbol{k},\Pi;T),\end{aligned}$$

which proves (1-13). □

Finally, we deduce Corollary 1.3 from Theorem 1.1.

Proof of Corollary 1.3. Let $\boldsymbol{k}=(k_1,\ldots,k_r)$ be the index $(k,1,\ldots,1)$, and let $\Pi=\{P_1,\ldots,P_m\}$ denote a partition in $\mathcal{P}_r$. Assume $1\in P_1$ through this proof, which does not lose the generality. We see from (1-14) that

$$H_{ш}(\boldsymbol{k},\Pi;T)=\begin{cases}0 & \text{if } |P_i|>1 \text{ for some } i\geq 2,\\ \zeta(k+|P_1|-1)T^{m-1} & \text{otherwise.}\end{cases}$$

Since

$$m-1=\sum_{i=2}^{m}|P_i|=r-|P_1|$$

if $|P_i| = 1$ for all $i \geq 2$, it follows from (1-13) that

$$(r-1)!\sum_{i=0}^{r-1}\zeta_{ш}^{\star}(\{1\}^i, k, \{1\}^{r-1-i}; T) = \sum_{j=1}^{r}\sum_{\Pi\in\mathcal{X}_j}(j-1)!\,\zeta(k+j-1)T^{r-j}$$
$$= \sum_{j=1}^{r}(j-1)!\,\zeta(k+j-1)T^{r-j}\sum_{\Pi\in\mathcal{X}_j}1, \tag{3-6}$$

where

$$\mathcal{X}_j = \{\{P_1, P_2, \dots, P_{r+1-j}\} \in \mathcal{P}_r : |P_1| = j, |P_2| = \cdots = |P_{r+1-j}| = 1\}.$$

Noting the assumption $1 \in P_1$, we have

$$\mathcal{X}_j = \big\{\{[r]\setminus\{a_2, \dots, a_{r+1-j}\}, \{a_2\}, \dots, \{a_{r+1-j}\}\} : 2 \leq a_2 < \cdots < a_{r+1-j} \leq r\big\},$$

where $[r]\setminus\{a_2, \dots, a_{r+1-j}\}$ corresponds to P_1 and $\{a_i\}$ $(2 \leq i \leq r)$ correspond to P_i. Thus $|\mathcal{X}_j|$ is the number of $(r-j)$-combinations of $\{2, \dots, r\}$, or

$$\sum_{\Pi\in\mathcal{X}_j}1 = \binom{r-j}{r-1}. \tag{3-7}$$

Combining (3-6) and (3-7), we obtain

$$\sum_{i=0}^{r-1}\zeta_{ш}^{\star}(\{1\}^i, k, \{1\}^{r-1-i}; T) = \sum_{j=1}^{r}\frac{1}{(r-j)!}\zeta(k+j-1)T^{r-j}.$$

Replacing j with $r-j$ in the right-hand side of this equation gives (1-16). □

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