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Matiyasevich-type identities for hypergeometric Bernoulli polynomials and poly-Bernoulli polynomials

Ken Kamano

We give a Matiyasevich-type identity for hypergeometric Bernoulli polynomials and their generalizations. By using this identity, we also give an identity for poly-Bernoulli polynomials.

1. Introduction and main theorem

The Bernoulli polynomials $B_n(x)$ are defined by the generating function

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}\frac{B_n(x)}{n!}t^n. \tag{1}$$

When $x=0$, the numbers $B_n(0)=B_n$ are called Bernoulli numbers.

A well-known convolution identity for Bernoulli numbers is the following Euler's formula:

$$\sum_{i=0}^{n}\binom{n}{i}B_iB_{n-i}=-nB_{n-1}-(n-1)B_n \quad (n\geq 1).$$

There are many generalizations of this identity. For example, Dilcher [1996] gave an identity for sums of m products of Bernoulli polynomials ($m=2,3,\ldots$).

On the other hand, by a p-adic method, Miki [1978] proved the following interesting identity which relates two types of convolutions of Bernoulli numbers:

$$\sum_{i=2}^{n-2}\beta_i\beta_{n-i}-\sum_{i=2}^{n-2}\binom{n}{i}\beta_i\beta_{n-i}=2H_n\beta_n \quad (n\geq 4), \tag{2}$$

where $\beta_m:=B_m/m$ and $H_m:=\sum_{i=1}^{m}1/i$. Many alternative proofs and generalizations of this identity have been discovered by several authors; see, e.g., [Crabb 2005; Dilcher and Vignat 2016; Gessel 2005]. Matiyasevich [1997, Identity #0202] discovered the following identity, which also relates two types of convolutions of Bernoulli numbers:

$$(n+2)\sum_{i=2}^{n-2}B_iB_{n-i}-2\sum_{i=2}^{n-2}\binom{n+2}{i}B_iB_{n-i}=n(n+1)B_n \tag{3}$$

for any $n\geq 4$. We note that the identity (3) becomes trivial for odd $n>4$. It is known that Miki's and Matiyasevich's identities can be proved by using a difference operator [Pan and Sun 2006; Artamkin 2007]; see also [Sun and Pan 2006].

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Let N be a positive integer and $Q(t) \in t^N\mathbb{R}[\![t]\!]$. We introduce polynomials $f_{N,n}(x; Q) \in \mathbb{R}[x]$ ($n = 0, 1, 2, \dots$) by the generating function

$$\frac{Q(t)}{e^t - \sum_{i=0}^{N-1} t^i/i!} e^{xt} = \sum_{n=0}^{\infty} \frac{f_{N,n}(x; Q)}{n!} t^n.$$

When $Q(t) = t^N/N!$, the polynomials $f_{N,n}(x; Q)$ are nothing but the hypergeometric Bernoulli polynomials $B_{N,n}(x)$, which were first introduced by Howard [1967a; 1967b]. We note that $B_{1,n}(x)$ is the ordinary n-th Bernoulli polynomial $B_n(x)$ defined by (1). We denote $f_{N,n}(x; Q)$ by $f_n(x; Q)$ if there is no fear of confusion.

By definition, we have

$$f_n(x+y; Q) = \sum_{i=0}^{n} \binom{n}{i} f_i(y; Q) x^{n-i} \quad (n \geq 0), \tag{4}$$

$$\frac{d}{dx} f_n(x; Q) = n f_{n-1}(x; Q) \qquad (n \geq 1). \tag{5}$$

The purpose of this paper is to give a Matiyasevich-type identity for $f_{N,n}$ by using the difference operator. The following is the main theorem of this paper.

Theorem 1.1. *Let N, m and n be integers with $N, m \geq 1$ and $n \geq 0$. For $Q_u(t) \in t^N\mathbb{R}[\![t]\!]$ $(1 \leq u \leq m)$, we have*

$$\begin{aligned}
&\binom{n+N+m-1}{N} \sum_{\substack{i_1,\dots,i_m \geq 0 \\ i_1+\cdots+i_m = n}} \prod_{u=1}^{m} f_{N,i_u}(x+y_u; Q_u) \\
&= \sum_{p_1,\dots,p_m \geq 0} \binom{n+N+m-1}{P_m+m-1} B_{N,n-P_m+N}(x) \\
&\quad \times \left(\left(\prod_{u=1}^{m} f_{N,p_u}(y_u+1; Q_u) \right) - \sum_{\substack{j_1,\dots,j_m \geq 0 \\ 0 \leq j_1+\cdots+j_m \leq N-1}} \prod_{l=1}^{m} \binom{p_l}{j_l} f_{N,p_l-j_l}(y_l, Q_l) \right),
\end{aligned} \tag{6}$$

where P_m means $p_1 + \cdots + p_m$.

In Section 2, we give a proof of Theorem 1.1. In Section 3, we see that the identity (6) is a generalization of Matiyasevich's identity (3). Moreover, as an example of identity (6), we give an identity for poly-Bernoulli polynomials.

2. Proof of Theorem 1.1

For an integer $N \geq 1$, let us define a kind of difference operator Δ_N as

$$\Delta_N f(x) := f(x+1) - \sum_{i=0}^{N-1} \frac{f^{(i)}(x)}{i!} \quad (f(x) \in \mathbb{R}[\![x]\!]), \tag{7}$$

where $f^{(i)}$ is the i-th derivative of f. It is clear that Δ_1 is the ordinary difference operator.

Since

$$\Delta_N\left(\frac{e^{xt}}{e^t-\sum_{i=0}^{N-1}t^i/i!}\right)=e^{xt},$$

we have

$$\Delta_N B_{N,n+N}(x)=\binom{n+N}{N}x^n \quad (n\geq 0). \tag{8}$$

By definition, we have

$$\Delta_N x^m=\begin{cases}\sum_{i=N}^m\binom{m}{i}x^{m-i} & \text{for } m\geq N,\\ 0 & \text{for } 0\leq m\leq N-1.\end{cases}$$

Hence $\{\Delta_N x^N, \Delta_N x^{N+1}, \ldots\}$ provides a basis of the vector space $\mathbb{R}[x]$ over $\mathbb{R}$. Therefore $\Delta_N f(x)=0$ implies that $f(x)$ is a polynomial of degree $N-1$ and we obtain the following lemma.

Lemma 2.1. *Let $f(x), g(x)\in\mathbb{R}[x]$. If $\Delta_N f(x)=\Delta_N g(x)$, then $f(x)$ and $g(x)$ agree in their coefficients of x^j for $j\geq N$.*

By the identity

$$\sum_{i=0}^{\infty}\binom{i}{p}x^i=\frac{x^p}{(1-x)^{p+1}} \quad (p\geq 0),$$

we have

$$\sum_{i_1=0}^{\infty}\binom{i_1}{p_1}x^{i_1}\cdots\sum_{i_m=0}^{\infty}\binom{i_m}{p_m}x^{i_m}=\frac{x^{p_1+\cdots+p_m}}{(1-x)^{p_1+\cdots+p_m+m}}$$

for $m\geq 1$. By comparing the coefficients of both sides, we obtain the following lemma.

Lemma 2.2. *For integers $m\geq 1$, $n\geq 0$ and $p_1,\ldots,p_m\geq 0$, we have*

$$\sum_{\substack{i_1,\ldots,i_m\geq 0\\ i_1+\cdots+i_m=n}}\binom{i_1}{p_1}\cdots\binom{i_m}{p_m}=\binom{n+m-1}{p_1+\cdots+p_m+m-1}.$$

Now we prove our main theorem.

Proof of Theorem 1.1. For integers $i_1,\ldots,i_m\geq 0$, we have by (7)

$$\begin{aligned}
&\Delta_N\left(\prod_{u=1}^m f_{i_u}(x+y_u;Q_u)\right)\\
&=\left(\prod_{u=1}^m f_{i_u}(x+1+y_u;Q_u)\right)-\sum_{j=0}^{N-1}\frac{1}{j!}\frac{d^j}{dx^j}\prod_{u=1}^m f_{i_u}(x+y_u;Q_u)\\
&=\prod_{u=1}^m\left(\sum_{p_u=0}^{i_u}\binom{i_u}{p_u}f_{p_u}(y_u+1;Q_u)x^{i_u-p_u}\right)-\sum_{\substack{j_1,\ldots,j_m\geq 0\\ 0\leq j_1+\cdots+j_m\leq N-1}}\frac{f_{i_1}^{(j_1)}(x+y_1;Q_1)\cdots f_{i_m}^{(j_m)}(x+y_m;Q_m)}{j_1!\cdots j_m!},
\end{aligned} \tag{9}$$

where we have used the general Leibniz rule. For any $i, j\geq 0$ we have, by (4) and (5),

$$\frac{f_i^{(j)}(x+y;Q)}{j!}=\binom{i}{j}f_{i-j}(x+y;Q)=\binom{i}{j}\sum_{p=0}^{i-j}\binom{i-j}{p}f_p(y;Q)x^{i-j-p}=\sum_{p=j}^{i}\binom{i}{p}\binom{p}{j}f_{p-j}(y;Q)x^{i-p},$$

where an empty sum is taken to be zero. Hence

$$\sum_{\substack{j_1,\dots,j_m\ge 0\\ 0\le j_1+\cdots+j_m\le N-1}} \frac{f_{i_1}^{(j_1)}(x+y_1;\,Q_1)\cdots f_{i_m}^{(j_m)}(x+y_m;\,Q_m)}{j_1!\cdots j_m!} = \sum_{\substack{j_1,\dots,j_m\ge 0\\ 0\le j_1+\cdots+j_m\le N-1}} \prod_{u=1}^{m}\Bigg(\sum_{p_u=j_u}^{i_u}\binom{i_u}{p_u}\binom{p_u}{j_u} f_{p_u-j_u}(y_u;\,Q_u)x^{i_u-p_u}\Bigg).$$

Therefore, by Lemma 2.2, we have

$$\begin{aligned}
&\sum_{\substack{i_1,\dots,i_m\ge 0\\ i_1+\cdots+i_m=n}} \Delta_N\Bigg(\prod_{u=1}^{m} f_{i_u}(x+y_u;\,Q_u)\Bigg)\\
&= x^n \sum_{p_1,\dots,p_m\ge 0}\binom{n+m-1}{P_m+m-1}\Bigg(\prod_{u=1}^{m} f_{p_u}(y_u+1;\,Q_u)x^{-p_u}\Bigg)\\
&\qquad - x^n \sum_{0\le j_1+\cdots+j_m\le N-1}\ \sum_{p_1,\dots,p_m\ge 0}\binom{n+m-1}{P_m+m-1}\prod_{u=1}^{m}\binom{p_u}{j_u} f_{p_u-j_u}(y_u;\,Q_u)x^{-p_u}\\
&= \sum_{p_1,\dots,p_m\ge 0} x^{n-P_m}\binom{n+m-1}{P_m+m-1}\Bigg(\prod_{u=1}^{m} f_{p_u}(y_u+1;\,Q_u) - \sum_{0\le j_1+\cdots+j_m\le N-1}\ \prod_{u=1}^{m}\binom{p_u}{j_u} f_{p_u-j_u}(y_u;\,Q_u)\Bigg).
\end{aligned}$$

By the relation

$$x^{n-P_m} = \frac{\Delta_N B_{N,n-P_m+N}(x)}{\binom{n-P_m+N}{N}},$$

which comes from (8), we have, for $n\ge 0$,

$$\begin{aligned}
&\Delta_N \sum_{\substack{i_1,\dots,i_m\ge 0\\ i_1+\cdots+i_m=n}} \prod_{u=1}^{m} f_{i_u}(x+y_u;\,Q_u)\\
&= \Delta_N \sum_{p_1,\dots,p_m\ge 0} \frac{1}{\binom{n-P_m+N}{N}} B_{N,n-P_m+N}(x)\binom{n+m-1}{P_m+m-1}\\
&\qquad \times\Bigg(\Bigg(\prod_{u=1}^{m} f_{p_u}(y_u+1;\,Q_u)\Bigg) - \sum_{0\le j_1+\cdots+j_m\le N-1}\ \prod_{u=1}^{m}\binom{p_u}{j_u} f_{p_u-j_u}(y_u;\,Q_u)\Bigg).
\end{aligned}$$

Applying Lemma 2.1 to this last identity, with

$$\frac{1}{\binom{n-P_m+N}{N}}\binom{n+m-1}{P_m+m-1} = \frac{\binom{n+N+m-1}{P_m+m-1}}{\binom{n+N+m-1}{N}},$$

we see that (6) holds up to a polynomial in x of degree $N-1$. Finally, for any $n\ge 0$, by replacing n by $n+N$ in (6) and differentiating with respect to x both sides N times, we obtain (6) for n. □

3. Identities for poly-Bernoulli polynomials

In this section, we give some identities derived from Theorem 1.1. Firstly, we give identities for the ordinary Bernoulli polynomials.

Corollary 3.1. *The following identities hold*:

$$(n+2)\sum_{i_1+i_2=n} B_{i_1}(x)B_{i_2}(x) = \binom{n+2}{3}B_{n-1}(x)+2\sum_{p\ge 0}\binom{n+2}{p+2}B_pB_{n-p}(x) \qquad (n\ge 1), \quad (10)$$

$$(n+2)\sum_{i_1+i_2=n} B_{i_1}(y_1)B_{i_2}(y_2) = \sum_{p_1,p_2\ge 0}\binom{n+2}{p_1+p_2+1}B_{n-p_1-p_2+1} \times\bigl(B_{p_1}(y_1+1)B_{p_2}(y_2+1)-B_{p_1}(y_1)B_{p_2}(y_2)\bigr) \quad (n\ge 0). \quad (11)$$

Proof. We apply $N=1,\ m=2,\ Q_1(t)=Q_2(t)=t$ and $y_1=y_2=0$ in Theorem 1.1. Since $f_{1,n}(x;t)=B_n(x)$, we have

$$(n+2)\sum_{i_1+i_2=n} B_{i_1}(x)B_{i_2}(x) = \sum_{p_1,p_2\ge 0}\binom{n+2}{p_1+p_2+1}B_{n-p_1-p_2+1}(x)(B_{p_1}(1)B_{p_2}(1)-B_{p_1}B_{p_2}). \qquad (12)$$

It is well known that $B_p(1)=B_p+\delta_{1p}$ $(p\ge 0)$, where δ_{ij} is Kronecker's delta function. Therefore the right-hand side of (12) equals

$$\binom{n+2}{3}B_{n-1}(x)+2\sum_{p\ge 0}\binom{n+2}{p+2}B_{n-p}(x)B_p,$$

and this proves (10). Equation (11) can be also proved by applying $x=0$ in Theorem 1.1. □

Remark 3.2. (i) Matiyasevich's identity (3) can be obtained by setting $x=0$ in (10).

(ii) Agoh and Dilcher [2014, Theorem 1] gave an identity which includes (10). Pan and Sun [2006, Theorem 2.1] gave an identity for $\sum B_{i_1}(y_1)B_{i_2}(y_2)$ with $y_1\ne y_2$, but our identity (11) is different from theirs.

For any integer k, poly-Bernoulli polynomials $C_n^{(k)}(x)$ are defined by the generating function

$$\frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1}e^{xt} = \sum_{n=0}^{\infty}\frac{C_n^{(k)}(x)}{n!}t^n;$$

see, e.g., [Imatomi 2014, Chapter 6]. Here $\mathrm{Li}_k(z)$ is the k-th polylogarithm defined by $\mathrm{Li}_k(z)=\sum_{n=1}^{\infty} z^n/n^k$. The numbers $C_n^{(k)}(1)$ and $C_n^{(k)}(0)$ are poly-Bernoulli numbers $B_n^{(k)}$ and $C_n^{(k)}$ introduced by Kaneko [1997] and Arakawa and Kaneko [1999], respectively. When $k=1$, it can be checked that $C_n^{(1)}(x)=B_n(x)$ where $B_n(x)$ are the ordinary Bernoulli polynomials defined by (1). When $N=1$ and $Q(t)=\mathrm{Li}_k(1-e^{-t})$, we have $f_n(x;Q)=C_n^{(k)}(x)$. Hence the following corollary is obtained from Theorem 1.1.

Corollary 3.3. *For integers* k_1, k_2 *and* n *with* $n\ge 0$, *we have*

$$(n+2)\sum_{i_1+i_2=n} C_{i_1}^{(k_1)}(x)C_{i_2}^{(k_2)}(x) = \sum_{p_1,p_2\ge 0}\binom{n+2}{p_1+p_2+1}B_{n-p_1-p_2+1}(x)(B_{p_1}^{(k_1)}B_{p_2}^{(k_2)}-C_{p_1}^{(k_1)}C_{p_2}^{(k_2)}).$$

It is known that $B_n^{(k)} = C_n^{(k)} + C_{n-1}^{(k-1)}$ for $n \geq 0$. Here, when $n = 0$, we set $C_{-1}^{(k-1)} = 0$ for any k. Hence the identity above can be rewritten in the form using only $C_n^{(k)}$:

Corollary 3.4. *For integers k_1, k_2 and n with $n \geq 0$, we have*

$$(n+2) \sum_{i_1+i_2=n} C_{i_1}^{(k_1)}(x) C_{i_2}^{(k_2)}(x) = \sum_{p_1, p_2 \geq 0} \binom{n+2}{p_1+p_2+1} B_{n-p_1-p_2+1}(x) \big(C_{p_1}^{(k_1)} C_{p_2-1}^{(k_2-1)} + C_{p_2}^{(k_2)} C_{p_1-1}^{(k_1-1)} + C_{p_1-1}^{(k_1-1)} C_{p_2-1}^{(k_2-1)}\big).$$

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