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On the distribution of values of Hardy's Z-functions in short intervals II: The q-aspect

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We continue our investigations regarding the distribution of positive and negative values of Hardy's Z-functions $Z(t,\chi)$ in the interval $[T, T+H]$ when the conductor q and T both tend to infinity. We show that for $q \leqslant T^\eta$, $H = T^\vartheta$, with $\vartheta > 0$, $\eta > 0$ satisfying $\frac{1}{2}+\frac{1}{2}\eta < \vartheta \leqslant 1$, the Lebesgue measure of the set of values of $t \in [T, T+H]$ for which $Z(t,\chi) > 0$ is $\gg (\varphi(q)^2/4^{\omega(q)}q^2)H$ as $T \to \infty$, where $\omega(q)$ denotes the number of distinct prime factors of the conductor q of the character χ, and φ is the usual Euler totient. This improves upon our earlier result. We also include a corrigendum for the first part of this article.

1. Introduction

Let χ be a primitive Dirichlet character of conductor $q > 1$. In this paper, we continue our investigations in [Mawia 2017] concerning the distribution of positive and negative values of Hardy's Z-function $Z(t,\chi)$ for t in the interval $[T, T+H]$. Let us recall some basic facts concerning the function $Z(t,\chi)$. First of all, Hardy's Z-function $Z(t,\chi)$ corresponding to the Dirichlet L-function $L(s,\chi)$ is defined by

$$Z(t,\chi) := \Psi\big(\tfrac{1}{2}+it,\chi\big)^{-1/2} L\big(\tfrac{1}{2}+it,\chi\big),$$

where

$$\Psi(s,\chi) = \mathfrak{w}(\chi)\left(\frac{\pi}{q}\right)^{s-1/2}\frac{\Gamma((1-s+\mathfrak{a})/2)}{\Gamma((s+\mathfrak{a})/2)}$$

is the factor from the functional equation $L(s,\chi) = \Psi(s,\chi)L(1-s,\bar{\chi})$. Here the quantity $\mathfrak{a} = (1-\chi(-1))/2$ measures the parity of the character χ and the number

$$\mathfrak{w}(\chi) = \frac{\tau(\chi)}{i^{\mathfrak{a}}\sqrt{q}}, \quad \text{with } \tau(\chi) = \sum_{a \pmod q} \chi(a) e\left(\frac{a}{q}\right),$$

is called the root number of χ. It is immediately seen from the definition that $Z(t,\chi)$ is a real-valued function of the real variable t and that $|Z(t,\chi)| = \big|L\big(\frac{1}{2}+it,\chi\big)\big|$, so that the real zeros of $Z(t,\chi)$ correspond to the zeros of $L(s,\chi)$ on the critical line. One of the main interests of Hardy's Z-functions comes from this fact. For a brief introduction to Hardy's Z-function for $\zeta(s)$, see [Karatsuba and Voronin 1992, III, §4]; a more comprehensive theory is developed in [Ivić 2013].

Although it is a well-known tool in the computations and study of the distributions of zeros of the Riemann zeta function (see for example [Karatsuba 1981]), research into its fine structures is a rather recent development, due mainly to results of Ivić [2004; 2010; 2017a; 2017b; 2017c], Korolev [2007;

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2008; 2017] and Jutila [2009; 2011]. Our interest here, following [Gonek and Ivić 2017] and later [Mawia 2017] is in the distribution of positive and negative values of $Z(t,\chi)$. In [Mawia 2017], we proved the following result:

Theorem. *Consider a fixed primitive Dirichlet character χ of conductor $q>1$. For $2\leqslant H\leqslant T$, let*

$$I_+(T,H;\chi)=\{T\leqslant t\leqslant T+H : Z(t,\chi)>0\},$$
$$I_-(T,H;\chi)=\{T\leqslant t\leqslant T+H : Z(t,\chi)<0\}.$$

Fix $0<\epsilon<\frac{1}{4}$ and let $T^{3/4+\epsilon}\leqslant H\leqslant T$. Then we have

$$\mu(I_+(T,H;\chi))\gg H \quad \text{and} \quad \mu(I_-(T,H;\chi))\gg H,$$

where μ is the Lebesgue measure on the line.

It should be remarked that the constants in the above result depend on the conductor q, as will be clear in the course of this paper. Our main concern here will be to take into account the variation of the conductor q and see to what extent results of the above type hold true when q is allowed to vary. For example, one may ask questions of the following type: does the above result remain valid when q is not held fixed but allowed to vary, say up to $q\leqslant T$, or for that matter, as $q\to\infty$ independently of T? The main result we prove in this article is the following.

Theorem 1. *Let $q\leqslant T^\eta$, $H=T^\vartheta$ with $\frac{1}{2}+\frac{1}{2}\eta<\vartheta\leqslant 1$. Then, as $T\to\infty$ we have*

$$\mu(I_\pm(T,H;\chi))\gg \frac{\varphi(q)^2}{4^{\omega(q)}q^2}H$$

uniformly in q, T and H, where the implied constants depend only on η, ϑ.

Thus, this result says that the main result of [Mawia 2017] holds for q almost as big as T and improves the interval as well by reducing $\frac{3}{4}$ to $\frac{1}{2}$. Note in particular that when q tends to infinity through integers with a bounded number of prime factors, $\varphi(q)^2/4^{\omega(q)}q^2\asymp 1$ and hence $\mu(I_\pm(T,H;\chi))\gg H$ as $T\to\infty$ and $q\to\infty$ along integers with a bounded number of prime factors, say $q\to\infty$ via primes, for example.

To prove this result, although the main argument for the result is the same as in [Mawia 2017], we have to explicitly show the dependence on q of the error terms in all our lemmas there, and that makes the argument more delicate. The improvement on the interval comes mainly from an application of the first derivative bound rather than the second derivative bound for exponential integrals in Lemmas 7 and 8. Following [Gonek and Ivić 2017], our main tool will be a study of various mollified integrals of $Z(t,\chi)$, using the mollifiers $B_X(s,\chi)$ analogous to the ones introduced for $\zeta(s)$ by Selberg [1942]. An obvious merit of this mollifier may be illustrated as follows. By following the steps of [Gonek and Ivić 2017] but without using the mollifiers they used, and applying the well-known bounds

$$\int_T^{2T} Z(t)\,\mathrm{d}t\ll T^{7/8},\qquad \int_T^{2T}|Z(t)|\,\mathrm{d}t\gg T,\qquad \int_T^{2T} Z(t)^2\,\mathrm{d}t=\int_T^{2T}\bigl|\zeta\bigl(\tfrac{1}{2}+it\bigr)\bigr|^2\,\mathrm{d}t\asymp T\log T,$$

one only obtains $\mu(I_+(T,2T))\gg T/\log T$, which is rather weak. In this light, it would be interesting to use other mollifiers and see what they yield. However, the mollifiers introduced by Selberg seem to be most effective in several circumstances. The function $F(t)=Z(t)\bigl|B_X\bigl(\frac{1}{2}+it\bigr)\bigr|^2$, which is a mollification of $Z(t)$, and is the main object in our lemmas for proving the above theorem, is also called the Hardy–Selberg function [Karatsuba and Voronin 1992, III, §5], and we shall follow this nomenclature introduced by Karatsuba [1984].

Once the lemmas on the mollified integrals are proved, the proof of the main result, Theorem 1 above, is the same as in [Gonek and Ivić 2017] (which is repeated in [Mawia 2017, §2]), so we content ourselves here with a precise formulation and proof of the requisite lemmas.

This paper is organised as follows. In Section 2, we restate and prove some lemmas on approximations of $L(s, \chi)$ by Dirichlet polynomials, and on the coefficients $\alpha_n(\chi)$ of our mollifiers. The three main lemmas essential for proving the above theorem are then proved in Section 3. Also, throughout this paper, we use, without comment, bounds on several sums which are either standard or which appeared already in [Mawia 2017], although they may not be obvious.

Corrigendum to [Mawia 2017]. There are a couple of minor errors in [Mawia 2017] which are not difficult to correct. They stem from an erroneous statement of the Riemann–Siegel formula for $Z(t, \chi)$, which is equation (5) in Lemma 3, p. 39. The error is that the variable n in the sum for $\Theta(t, \chi)$ should run up to $n \leqslant \sqrt{qt/(2\pi)}$, not up to $n \leqslant \sqrt{qT/(2\pi)}$. The correct statement is given in (2-1) of Lemma 2 below. A consequence of this error is that our treatment of the integrals in equation (10), p. 44 was erroneous. This problem is addressed in Lemma 9 below. Next, immediately before equation (11) on p. 45, "the first O-term in (3)" and "the first two integrals in (3)" should be replaced by "the first O-term in (10)" and "the first two integrals in (10)" respectively. Finally, from equation (12), p. 45 onwards the conditions $kn = lm$ and $kn \neq lm$ in the summation should be replaced by the conditions $km = ln$ and $km \neq ln$ respectively, and the fraction kn/lm should be replaced throughout by ln/km. As is clear, one does not need to change the subsequent argument.

2. Preliminary lemmas

As in [Mawia 2017], our main tool will be a form of the "Riemann–Siegel formula" for $Z(t, \chi)$, a version of which is stated in Lemma 3 in [Mawia 2017]; the proof of the "approximate functional equation" from which this can be derived is given in [Chandrasekharan and Narasimhan 1963; Suetuna 1932; Tchudakoff 1947]. The proof of the original Riemann–Siegel formula can be found in [Siegel 1932] (see also [Siegel 1966, pp. 275–310]). A more precise form for the terms of the asymptotic series for the remainder in the approximate functional equation for $L(s, \chi)$ is given in [Siegel 1943]. It should be remarked here that sharper forms of this formula are available, at least for the case $q = 1$ (see for example [Borwein et al. 2000; Gabcke 1979]); the approximate formula [Karatsuba and Voronin 1992, III, §4, Theorem 1] is also of interest, and similar formulas exist for the derivatives of $Z(t)$ as well [Karatsuba 1981]. We need a version which is uniform in both q- and t-aspects. Such an approximate functional equation for $L(s, \chi)$ is proved in [Lavrik 1968] (Corollary 1 to Theorem 1), from which it is easy to deduce the following corresponding result for $Z(t, \chi)$:

Lemma 2. *We have the following approximate equation for $Z(t, \chi)$:*

$$Z(t, \chi) = \Theta(t, \chi) + \overline{\Theta}(t, \chi) + O((q/t)^{1/4} \log(2t)), \tag{2-1}$$

where

$$\Theta(t, \chi) = \mathfrak{w}(\chi)^{-1/2} e^{(\pi i \mathfrak{a}/4)-(\pi i/8)} \left(\frac{qt}{2\pi e}\right)^{it/2} \sum_{n \leqslant \sqrt{qt/(2\pi)}} \frac{\chi(n)}{n^{1/2+it}}. \tag{2-2}$$

Here and in the following, we will restrict ourselves to $t > 0$, *since the values of the* Z*-function for* $L(s, \chi)$ *at* $t < 0$ *correspond to the values of the* Z*-function for* $L(s, \bar{\chi})$ *at* $t > 0$; *precisely, we have* $Z(-t, \chi) = Z(t, \bar{\chi})$.

As has already been noted in the corrigendum above, the expression (5) of Lemma 3 in [Mawia 2017] is erroneous and should be replaced by the above expression (2-1). As a result we have to modify our argument for proving [Mawia 2017, Lemma 8] accordingly; see Lemma 9 below.

Next, we shall need the following approximate functional equation for $L(s, |\chi|)$, whose proof is similar to the analogous formula for $\zeta(s)$.

Lemma 3. *Let* $x > 1$ *be a positive real number. We have*

$$L(s, |\chi|) = \sum_{n \leqslant x} \frac{|\chi(n)|}{n^s} - \frac{\varphi(q)}{q} \frac{x^{1-s}}{1-s} + O(2^{\omega(q)} |s| x^{-\sigma})$$

uniformly for $\sigma \geqslant \sigma_0 > 0$. *Also, as* $s \to 1$, *we have*

$$(s-1) L(s, |\chi|) = \frac{\varphi(q)}{q} + O(2^{\omega(q)} |s-1|).$$

Here, $\omega(q)$ *denotes the number of distinct prime factors of* q.

Proof. For $\sigma > 1$, we have

$$L(s, |\chi|) = \sum_{n \leqslant x} \frac{|\chi(n)|}{n^s} + \sum_{n > x} \frac{|\chi(n)|}{n^s}.$$

One writes the sum $\sum_{x < n \leqslant y} |\chi(n)| n^{-s}$ as an integral $\int_x^y u^{-s} \,\mathrm{d}A(u)$, where $A(u) = \sum_{n \leqslant u} |\chi(n)|$ counts the number of positive integers $\leqslant u$ which are coprime to q. Integrating by parts, we get that

$$\sum_{x < n \leqslant y} \frac{|\chi(n)|}{n^s} = \frac{A(y)}{y^s} - \frac{A(x)}{x^s} + \frac{\varphi(q)}{q} \frac{y^{1-s} - x^{1-s}}{1-s} s + s \int_x^y \frac{A(u) - \varphi(q) u/q}{u^{s+1}} \,\mathrm{d}u.$$

Letting $y \to \infty$, the resulting equation is valid for $\sigma \geqslant \sigma_0 > 0$. Noting that $A(u) = \varphi(q) u/q + O(2^{\omega(q)})$, the result follows. We observe here that in the formulas in this lemma, $2^{\omega(q)} \leqslant d(q) \ll q^{\epsilon}$. Note that $2^{\omega(q)} \ll 1$ if we let q tend to infinity along positive integers with a bounded number of distinct prime factors, which is the case for $q \to \infty$ via primes, for example. □

We remark here that we do not need any condition on the relative size of the imaginary part t of s and the length x of the approximating Dirichlet polynomial, unlike in the case of the slightly more refined result for $\zeta(s)$ found, for instance, in [Karatsuba and Voronin 1992, III, §2] or [Ivić 1985, §1.5], where the condition $|t| \leqslant \pi x$ is required.

We will again need a couple of lemmas regarding the Dirichlet coefficients of $L(s, \chi)^{-1/2}$, which we denote by $\alpha_n(\chi)$, following [Selberg 1942]. Recall that it may be explicitly expressed as

$$\alpha_n(\chi) = (-1)^{e_1 + \cdots + e_k} \binom{\frac{1}{2}}{e_1} \cdots \binom{\frac{1}{2}}{e_k} \chi(n), \tag{2-3}$$

where $n = p_1^{e_1} \cdots p_k^{e_k}$ is the factorisation of n. It is a multiplicative sequence satisfying $|\alpha_n(\chi)| \leqslant 1$ for all $n \geqslant 1$. Although not necessary for our applications, it is easily seen from the above explicit expression that its sign is essentially the same as $\mu(\mathrm{rad}(n))$, where $\mathrm{rad}(n)$ is the radical of n, the product of the distinct prime factors of n. Precisely, we have $\alpha_n(\chi)=\mu(\mathrm{rad}(n))|\alpha_n(\chi)|\chi(n)$. We recall other notation already used in [Mawia 2017]. For $1\leqslant n\leqslant X$ let

$$\beta_n\equiv\beta_n(\chi)=\alpha_n(\chi)\biggl(1-\frac{\log n}{\log X}\biggr)$$

and define the mollifying functions $B_X(s,\chi)$ by $B_X(s,\chi)=\sum_{n\leqslant X}\beta_n n^{-s}$, which correspond to the sums $\eta(t)$ used by Selberg [1942]. We write

$$B_X(s,\chi)^2=\sum_{n\leqslant X^2}b_n(\chi)n^{-s}, \tag{2-4}$$

where $b_n(\chi)=\sum_{d\,|\,n}\beta_d(\chi)\beta_{n/d}(\chi)$ with the sum running through those divisors d of n which satisfy both $d\leqslant X$ and $n/d\leqslant X$; note that $|b_n(\chi)|\leqslant d(n)$, where $d(n)$ is the usual divisor function.

We shall need the following analogue of Lemma 11 in [Selberg 1942].

Lemma 4. *Let ϱ be a positive integer and $0\leqslant\gamma\leqslant 1$, write*

$$f(s)=f(s;\gamma,\varrho,\chi)=\frac{1}{\sqrt{(s+i\gamma)L(1+s+i\gamma,|\chi|)}}\prod_{p\,|\,\varrho}(1-|\chi(p)|p^{-1-s-i\gamma})^{-1/2}$$

and let, for $r=2,3$,

$$f(s)=\sum_{j=0}^{r-1}\frac{f^{(j)}(0)}{j!}s^j+s^rR_r(s).$$

Then for $s=it$, $-2\leqslant t\leqslant 2$ and $j=0,1,2$, we have

$$f^{(j)}(0)\ll\frac{2^{\omega(q)}q^{1/2}}{\varphi(q)^{1/2}}\prod_{p\,|\,\varrho}(1+|\chi(p)|p^{-3/4}),$$

$$R_r(it)\ll\frac{2^{\omega(q)}q^{1/2}}{\varphi(q)^{1/2}}\prod_{p\,|\,\varrho}(1+|\chi(p)|p^{-3/4}).$$

Using this, one can prove the following, which is an analogue of [Selberg 1942, Lemma 12].

Lemma 5. *Let $1\leqslant d\leqslant X$, $0\leqslant\gamma\leqslant 2^{-\omega(q)}\varphi(q)/(q\sqrt{\log X})$ and ϱ be a positive integer. Then, for $r=1,2$,*

$$\sum_{\substack{n\leqslant X/d\\(n,\varrho)=1}}\frac{\alpha_n(|\chi|)}{n^{1+i\gamma}}\biggl(\log\frac{X}{dn}\biggr)^r=c_r\sqrt{\frac{q\gamma}{\varphi(q)}}\prod_{p\,|\,\varrho}(1-|\chi(p)|p^{-1-i\gamma})^{-1/2}\biggl(\log\frac{X}{d}\biggr)^r$$
$$+O\biggl(\frac{2^{\omega(q)}q^{1/2}}{\varphi(q)^{1/2}}(\log X)^{r-1/2}\prod_{p\,|\,\varrho}(1+|\chi(p)|p^{-3/4})\biggr), \tag{2-5}$$

where c_r is an absolute constant. Here and in the following, (m,n) denotes the gcd *of two integers m and n.*

We shall also need the following lemma, analogous to Lemma 13 in [Selberg 1942].

Lemma 6. *Let $X \geqslant 3$, $1 \leqslant d \leqslant X$ and $0 \leqslant \gamma \leqslant q^{\epsilon}/\log X$ and let ϱ be a positive integer. Then*

$$\sum_{\substack{n \leqslant X/d \\ (n,\varrho)=1}} \frac{\alpha_n(|\chi|)}{n} \left(\log \frac{X}{dn}\right) \frac{\sin(\gamma \log nd)}{\gamma} \ll \frac{2^{\omega(q)} q^{1/2}}{\varphi(q)^{1/2}} (\log X)^{3/2} \prod_{p \mid \varrho} (1 + |\chi(p)| p^{-3/4}),$$

where, for $\gamma = 0$, the left-hand side should be interpreted as its limit as $\gamma \to 0^+$.

Proof. For γ in this range, the O-term in Lemma 5 dominates the main term in absolute value, so we get, for $r = 1, 2$,

$$\sum_{\substack{n \leqslant X/d \\ (n,\varrho)=1}} \frac{\alpha_n(|\chi|)}{n^{1+i\gamma t}} \left(\log \frac{X}{dn}\right)^r = O\left(\frac{2^{\omega(q)} q^{1/2}}{\varphi(q)^{1/2}} (\log X)^{r-1/2} \prod_{p \mid \varrho} (1 + |\chi(p)| p^{-3/4})\right)$$

for $0 \leqslant t \leqslant 1$. Multiplying the formula for $r = 1$ by $\log X$ and subtracting the equation for $r = 2$, we obtain

$$\sum_{\substack{n \leqslant X/d \\ (n,\varrho)=1}} \frac{\alpha_n(|\chi|)}{n^{1+i\gamma t}} \left(\log \frac{X}{dn}\right) \log dn = O\left(\frac{2^{\omega(q)} q^{1/2}}{\varphi(q)^{1/2}} (\log X)^{3/2} \prod_{p \mid \varrho} (1 + |\chi(p)| p^{-3/4})\right).$$

Multiplying the formula by $d^{-i\gamma t}$ and integrating with respect to t between 0 and 1 and taking the real part, we get the lemma. □

3. Lemmas on the Hardy–Selberg function

In this section, we study the integrals of $Z(t, \chi)$ mollified using $B_X\left(\frac{1}{2} + it, \chi\right)$, namely the integrals of the function $Z(t, \chi)\left|B_X\left(\frac{1}{2} + it, \chi\right)\right|^2$, which, as we remarked earlier, is also called the Hardy–Selberg function, and was introduced and made use of in [Karatsuba 1984] to study the distribution of zeros of $\zeta(s)$ in short intervals of the critical line. Note that it depends on the parameter X as well. Our first lemma concerns the integral $\int_T^{T+H} Z(t, \chi)\left|B_X\left(\frac{1}{2} + it, \chi\right)\right|^2 \mathrm{d}t$, which should be $o(H)$, expecting some cancellation due to the sign changes of $Z(t, \chi)$, and in view of the fact that $B_X\left(\frac{1}{2} + it, \chi\right)$ looks like the partial sum of $L\left(\frac{1}{2} + it, \chi\right)^{-1/2}$ up to length X. The following lemma confirms this for a suitable range of H, and it supersedes Lemma 6 in [Mawia 2017].

Lemma 7. *Let $q \leqslant T^{\eta}$, $X = T^{\theta}$, $H = T^{\vartheta}$, with $0 < \eta, \theta, \vartheta \leqslant 1$. Then, as $T \to \infty$,*

$$\int_T^{T+H} Z(t, \chi)\left|B_X\left(\tfrac{1}{2} + it, \chi\right)\right|^2 dt = O(q^{1/4} T^{1/2} X \log^2 T + q^{1/4} T^{-1/4} H X \log X)$$

and hence the integral is $o(H)$ for $\frac{1}{2} + \frac{1}{4}\eta + \theta < \vartheta \leqslant 1$, $\eta + 4\theta < 1$.

Proof. Note first that this integral is $\ll d(q) q^{3/16} (\log q)^2 H \max(H^{1/4}, T^{1/4}) X$ for all values of q and H, using Heath-Brown's hybrid bound [1978, Theorem 1]. This is certainly interesting for small values of H, but is bigger than our bound for H in the above range. This is only natural, as any application of known bounds on $L\left(\frac{1}{2} + it, \chi\right)$ does not take into account the variations of $Z(t, \chi)$.

The proof of the lemma follows the same lines as in [Mawia 2017], where we use the definition of $Z(t, \chi)$ and move the contour to the segment going from $\frac{1}{2} + iT$ to $c + iT$ to $c + i(T+H)$ to $\frac{1}{2} + i(T+H)$, with $c = 1 + 1/\log T$. For the integral on the horizontal segments, we use the trivial convexity bound [Iwaniec and Kowalski 2004, (5.20)] for $L(s,\chi)$, which is

$$L(\sigma + it, \chi) \ll (qt)^{(1-\sigma)/2+\epsilon} \quad (0 \leqslant \sigma \leqslant 1,\ t \geqslant 1),$$

and the inequalities (easily seen from equations (8) and (9) in [Mawia 2017])

$$B_X(s,\chi)B_X(1-s,\bar{\chi}) \ll X \log X, \quad \Psi(s,\chi)^{-1/2} \ll (qT)^{(\sigma-1/2)/2}$$

for s in the horizontal segments, so that the overall contribution of the integral along the horizontal segments is

$$\ll \int_{1/2}^{c} (qT)^{(1-\sigma)/2+\epsilon}(X\log X)(qT)^{(\sigma-1/2)/2}\,\mathrm{d}\sigma \ll X(qT)^{1/4+\epsilon}.$$

Note that a bound sharper than the convexity bound (such as the Burgess bound or the hybrid bounds of Heath-Brown [1978; 1980]) will improve this lemma, but not the main theorem, as the main barrier comes from Lemma 9. The integral along the vertical segment is

$$\mathfrak{w}(\chi)^{-1/2}\int_T^{T+H}\biggl(\sum_{n\geqslant 1}\chi(n)n^{-c-it}\biggr)\biggl(\sum_{k\leqslant X}\beta_k(\chi)k^{-c-it}\biggr)\biggl(\sum_{\ell\leqslant X}\beta_\ell(\bar{\chi})\ell^{c+it-1}\biggr) \times \left(\frac{qt}{2\pi}\right)^{(c+it-1/2)/2} e^{-i(t+\pi(1-2\mathfrak{a})/4)/2}(1+O(1/t))\,\mathrm{d}t,$$

of which the O-term has a contribution

$$\ll \int_T^{T+H} (\log^2 T)X^{1-(1/\log T)}(qt)^{(c-1/2)/2}t^{-1}\,\mathrm{d}t \ll q^{1/4}XHT^{-3/4}\log^2 T.$$

Here, we have used the bounds

$$\biggl|\sum_{n\geqslant 1}\chi(n)n^{-c-it}\biggr| \leqslant L(c,|\chi|) = \zeta(c)\prod_{p\mid q}(1-p^{-c}) \ll \log T,$$
$$B_X(c+it,\chi) \ll \log T, \quad B_X(1-c-it,\bar{\chi}) \ll X.$$

Omitting the constant coefficients, the remaining expression can be rewritten as

$$\Sigma := \sum_{\substack{n\geqslant 1\\ k,\ell\leqslant X}} \frac{\chi(n)\beta_k(\chi)\beta_\ell(\bar{\chi})\ell^{c-1}}{(nk)^c}\int_T^{T+H}\left(\frac{qt}{2\pi}\right)^{(c-1/2)/2}\exp\biggl(\frac{it}{2}\log\biggl(\frac{qt\ell^2}{2\pi e n^2k^2}\biggr)\biggr)\,\mathrm{d}t.$$

To simplify matters, we will henceforward write $1/4$ in place of $(c-1/2)/2$ and 1 in place of c, since in any case $(c-1/2)/2 = 1/4 + 1/(2\log T)$ and the error incurred in our sums due to this simplification is clearly negligible. Let us further use the following notation:

$$F(t) = \left(\frac{qt}{2\pi}\right)^{1/4}, \quad f(t) = \frac{t}{2}\log\biggl(\frac{qt\ell^2}{2\pi e n^2 k^2}\biggr), \quad \tau = \sqrt{\frac{qT}{2\pi}},$$
$$\tau_0 = \sqrt{\frac{q(T-T^{3/4})}{2\pi}}, \quad \tau_1 = \sqrt{\frac{q(T+H)}{2\pi}}, \quad \tau_2 = \sqrt{\frac{q(T+H+T^{3/4})}{2\pi}}.$$

For each $k, \ell \leqslant X$ and $n < \tau_0\ell/k$ (resp. $n > \tau_2\ell/k$), the derivative $f'(t)$ of $f(t)$ has no zeros, is monotonic, and satisfies $f'(t) \geqslant \alpha T^{-1/4}$ (resp. $f'(t) \leqslant -\beta T^{-1/4}$) for some absolute constants $\alpha, \beta > 0$, so using the first derivative test for exponential integrals [Ivić 1985, Lemma 2.1], we immediately see that the above sum for n in either one of these ranges is $\ll q^{1/4}T^{1/2}X\log^2 T$. For $\tau_0\ell/k \leqslant n \leqslant \tau_2\ell/k$, the derivative $f'(t)$ vanishes at $t_0 = 2\pi n^2k^2/q\ell^2$, and t_0 may or may not lie in $[T, T+H]$. Let a and b be real numbers with $T - T^{3/4} \leqslant a \leqslant T \leqslant b \leqslant T + H + T^{3/4}$. Assume $a \leqslant t_0 \leqslant b$ (without loss). Following the proof of Lemma 2 from Chapter III, Section 1.3 of [Karatsuba and Voronin 1992] we get

$$\begin{aligned}\int_a^b F(t)\exp(if(t))\,\mathrm{d}t = \frac{2\sqrt{2}\pi}{q^{1/2}}\Big(\frac{nk}{\ell}\Big)^{3/2}\exp\Big(-\pi i\frac{n^2k^2}{q\ell^2}\Big) + O(qT)^{1/4}\\ + O\biggl(q^{-1/2}\Big(\frac{nk}{\ell}\Big)^{3/2}\min\Big\{1, \frac{1}{\sqrt{f(t_0)-f(a)}}\Big\}\biggr)\\ + O\biggl(q^{-1/2}\Big(\frac{nk}{\ell}\Big)^{3/2}\min\Big\{1, \frac{1}{\sqrt{f(b)-f(t_0)}}\Big\}\biggr).\end{aligned}$$

It follows that the integral is always $\ll q^{-1/2}(nk/\ell)^{3/2} + (qT)^{1/4}$. Therefore, the part of Σ contributed by n in the range $\tau\ell/k \leqslant n \leqslant \tau_1\ell/k$ (in which case $T \leqslant t_0 \leqslant T+H$) is

$$\begin{aligned}&\ll \sum_{k,\ell\leqslant X}\ \sum_{\tau\ell/k\leqslant n\leqslant\tau_1\ell/k}\Big\{\frac{(nk)^{1/2}}{\sqrt{q}\ell^{3/2}} + \frac{q^{1/4}T^{1/4}}{nk}\Big\}\\ &\ll q^{1/4}T^{-1/4}HX\log X + q^{1/4}T^{-3/4}HX^2 \ll q^{1/4}T^{-1/4}HX\log X.\end{aligned}$$

Finally, we have to consider the ranges $\tau_0\ell/k \leqslant n < \tau\ell/k$ and $\tau_1\ell/k < n \leqslant \tau_2\ell/k$ (in which two cases t_0 does not lie in $[T, T+H]$). Since the two cases are analogous, we only treat the first. We note that

$$\int_T^{T+H} F(t)\exp(if(t))\,\mathrm{d}t = \Big(\int_{t_0}^{T+H} - \int_{t_0}^{T}\Big)F(t)\exp(if(t))\,\mathrm{d}t \ll q^{-1/2}\Big(\frac{nk}{\ell}\Big)^{3/2} + (qT)^{1/4}.$$

Hence the contribution of this range to Σ is

$$\begin{aligned}&\ll \sum_{k,\ell\leqslant X}\ \sum_{\tau_0\ell/k\leqslant n<\tau\ell/k}\Big\{\frac{(nk)^{1/2}}{\sqrt{q}\ell^{3/2}} + \frac{(qT)^{1/4}}{nk}\Big\}\\ &\ll q^{1/4}T^{1/2}X\log X + q^{1/4}X\log X \ll q^{1/4}T^{1/2}X\log X.\end{aligned}$$ □

Remark. The referee has pointed out that one may directly apply Lemma 2 from Chapter III, Section 1.3 of [Karatsuba and Voronin 1992] in the treatment of the integral $\int_a^b F(t)\exp(if(t))$ in the last part of the above proof, since for n in the given range, we have $t_0 = 2\pi n^2k^2/q\ell^2 \asymp T$.

Next, the following lemma replaces Lemma 7 in [Mawia 2017] when $q \to \infty$ with T.

Lemma 8. *Let* $q \leqslant T^\eta,\ X = T^\theta,\ H = T^\vartheta$, *with* $0 < \eta, \theta, \vartheta \leqslant 1$, *such that* $\frac14 + \frac14\eta + \theta < \vartheta \leqslant 1$. *Then we have*

$$\int_T^{T+H} |Z(t,\chi)|\big|B_X\big(\tfrac12 + it, \chi\big)\big|^2\,dt \geqslant H + O(q^{1/4}T^{1/4}X) \quad (T \to \infty).$$

Proof. The proof is as in [Mawia 2017], but to take q into account, we use a different approximation for $L\left(\frac{1}{2}+it,\chi\right)$. Using Corollary 1 to Theorem 1 in [Lavrik 1968], we have the approximation

$$L\left(\tfrac{1}{2}+it,\chi\right)=\xi(t,\chi)+\Psi\left(\tfrac{1}{2}+it,\chi\right)\bar{\xi}(t,\chi)+O\left(\left(\frac{q}{T}\right)^{1/4}\log(2T)\right),$$

valid for $T\leqslant t\leqslant 2T$, where $\Psi\left(\frac{1}{2}+it,\chi\right)$ is the usual factor from the functional equation (mentioned in the first paragraph of the Introduction) and

$$\xi(t,\chi)=\sum_{n\leqslant\sqrt{qt/(2\pi)}}\frac{\chi(n)}{n^{1/2+it}}.$$

Using this, we get

$$\begin{aligned}\int_T^{T+H}|Z(t,\chi)|\left|B_X\left(\tfrac{1}{2}+it,\chi\right)\right|^2\mathrm{d}t&=\int_T^{T+H}\left|L\left(\tfrac{1}{2}+it,\chi\right)\right|\left|B_X\left(\tfrac{1}{2}+it,\chi\right)\right|^2\mathrm{d}t\\&\geqslant\left|\int_T^{T+H}L\left(\tfrac{1}{2}+it,\chi\right)B_X\left(\tfrac{1}{2}+it,\chi\right)^2\mathrm{d}t\right|\\&=\Bigg|\int_T^{T+H}\xi(t,\chi)B_X\left(\tfrac{1}{2}+it,\chi\right)^2\mathrm{d}t\\&\qquad\qquad+\int_T^{T+H}\Psi\left(\tfrac{1}{2}+it,\chi\right)\bar{\xi}(t,\chi)B_X\left(\tfrac{1}{2}+it,\chi\right)^2\mathrm{d}t\Bigg|\\&\quad+O\left(\left(\frac{q}{T}\right)^{1/4}\log(2T)\int_T^{T+H}\left|\sum_{n\leqslant X}\beta_n(\chi)n^{-1/2-it}\right|^2\mathrm{d}t\right).\end{aligned}$$

Let us first treat the O-term. Using the mean value theorem for Dirichlet polynomials (see Theorem 5.2 of [Ivić 1985]), the integral inside the O-term is easily seen to be

$$H\sum_{n\leqslant X}\frac{|\beta_n(\chi)|^2}{n}+O\left(\sum_{n\leqslant X}|\beta_n(\chi)|^2\right)=O(H\log X)+O(X)=O(H\log X).$$

Hence the O-term is $((q/T)^{1/4}\log(2T)H\log X)$. We now estimate the integral

$$\int_T^{T+H}\xi(t,\chi)B_X\left(\tfrac{1}{2}+it,\chi\right)^2\mathrm{d}t.$$

Using the expansion (2-4) for $B_X\left(\frac{1}{2}+it,\chi\right)^2$, we see that this integral is (we write $\tau_1=\sqrt{q(T+H)/2\pi}$ and $T(m)=\max(T,2\pi m^2/q)$)

$$\begin{aligned}&=\int_T^{T+H}\sum_{m\leqslant\sqrt{qt/(2\pi)}}\chi(m)m^{-1/2-it}\sum_{n\leqslant X^2}b_n(\chi)n^{-1/2-it}\,\mathrm{d}t\\&=H+\sum_{\substack{m\leqslant\tau_1;n\leqslant X^2\\mn>1}}\frac{\chi(m)b_n(\chi)}{\sqrt{mn}}\int_{T(m)}^{T+H}(mn)^{-it}\,\mathrm{d}t\\&=H+O\left(\sum_{\substack{m\leqslant\tau_1;n\leqslant X^2\\mn>1}}\frac{d(n)}{\sqrt{mn}\log(mn)}\right)=H+O(\sqrt{\tau_1}X).\end{aligned}$$

Next we look at the integral $\int_T^{T+H} \Psi\big(\frac{1}{2}+it,\chi\big)\bar{\xi}(t,\chi)B_X\big(\frac{1}{2}+it,\chi\big)^2\,\mathrm{d}t$. First, applying Stirling's approximation, we have

$$\frac{\Gamma\big(\frac{1}{4}-\frac{1}{2}it+\frac{1}{2}\mathfrak{a}\big)}{\Gamma\big(\frac{1}{4}+\frac{1}{2}it+\frac{1}{2}\mathfrak{a}\big)} = \left(\frac{2e}{t}\right)^{it} e^{\pi i(1/4-\mathfrak{a}/2)}\left(1+O\left(\frac{1}{T}\right)\right)$$

for $T \leqslant t \leqslant 2T$. Using this, we see that

$$\int_T^{T+H} \Psi\big(\tfrac{1}{2}+it,\chi\big)\bar{\xi}(t,\chi)B_X\big(\tfrac{1}{2}+it,\chi\big)^2\,\mathrm{d}t$$
$$= \mathfrak{w}(\chi)\sum_{m\leqslant\tau_1;n\leqslant X^2}\frac{\bar{\chi}(m)b_n(\chi)}{\sqrt{mn}}\int_{T(m)}^{T+H}\left(\frac{2\pi em}{qnt}\right)^{it}\mathrm{d}t + O\left(\frac{H}{T}\sqrt{\tau_1}X\log X\right).$$

We will apply a first derivative bound for exponential integrals [Ivić 1985, Lemma 2.1]. Write $f(t) = t\log(qnt/(2\pi em))$. Then $f'(t) = \log(qnt/(2\pi m))$, which is monotonic. Since

$$\frac{qnt}{2\pi m} \geqslant \frac{qT}{2\pi\tau_1} \gg (qT)^{1/2},$$

we see that $f'(t) \gg_\eta \log T$, as $q \leqslant T^\eta$. Therefore, the first derivative test gives $\int_{T(m)}^{T+H}\exp(if(t))\,\mathrm{d}t \ll 1/\log T$. Consequently,

$$\mathfrak{w}(\chi)\sum_{m\leqslant\tau_1;n\leqslant X^2}\frac{\bar{\chi}(m)b_n(\chi)}{\sqrt{mn}}\int_{T(m)}^{T+H}\left(\frac{2\pi em}{qnt}\right)^{it}\mathrm{d}t \ll \frac{1}{\log T}\sum_{m\leqslant\tau_1;n\leqslant X^2}\frac{d(n)}{\sqrt{mn}}$$
$$\ll \frac{\sqrt{\tau_1}}{\log T}X\log X \ll q^{1/4}T^{1/4}X.$$

The result follows. □

The following lemma is the most difficult part of the proof, and, as remarked earlier, the proof we gave in [Mawia 2017] had a minor error as the approximate functional equation we used was erroneous. But as will become clear, the proof remains substantially the same. To further reduce the length of the interval in the main theorem, it will be necessary to improve this lemma.

Lemma 9. *Let $q \leqslant T^\eta$, $X = T^\theta$, with $0 < \theta < \frac{1}{8}$ and $H = T^\vartheta$ with $\frac{1}{2}+\frac{1}{2}\eta+2\theta < \vartheta \leqslant 1$. Then*

$$\int_T^{T+H} Z(t,\chi)^2\big|B_X\big(\tfrac{1}{2}+it,\chi\big)\big|^4\,dt \ll \frac{4^{\omega(q)}q^2}{\varphi(q)^2}H \quad (T\to\infty).$$

Proof. We would like to point out first that this bound is much sharper in all aspects than we would get from a direct application of the Burgess bound or the hybrid bounds of [Heath-Brown 1978]. For, using Theorem 1 of [Heath-Brown 1978], we have

$$\int_T^{T+H} Z(t,\chi)^2\big|B_X\big(\tfrac{1}{2}+it,\chi\big)\big|^4\,\mathrm{d}t \ll d(q)^2q^{3/8}(\log q)^4H\max(H^{1/2},T^{1/2})X^2,$$

which is always bigger than our bound, for H in the stated range. As remarked earlier, this is due to the fact that the variations of $Z(t,\chi)$ are not taken into account when one applies bounds on $\big|L\big(\frac{1}{2}+it,\chi\big)\big|$. However, the merit of this bound is that it holds for *any* values of q and H without restriction.

We will now start the proof of the lemma. We will write, as before, $\tau = \sqrt{qT/(2\pi)}$. Using the approximate equation (2-1) of Lemma 2, we have

$$\begin{aligned}\int_T^{T+H} & Z(t,\chi)^2 \big|B_X\big(\tfrac12+it,\chi\big)\big|^4\,\mathrm{d}t \\ &= \int_T^{T+H} \Theta(t,\chi)^2 \big|B_X\big(\tfrac12+it,\chi\big)\big|^4\,\mathrm{d}t \\ &\quad + \int_T^{T+H} \overline{\Theta}(t,\chi)^2 \big|B_X\big(\tfrac12+it,\chi\big)\big|^4\,\mathrm{d}t + 2\int_T^{T+H} |\Theta(t,\chi)|^2 \big|B_X\big(\tfrac12+it,\chi\big)\big|^4\,\mathrm{d}t \\ &\quad + O\bigg(\Big(\frac{q}{T}\Big)^{1/4}\log(2T)\int_T^{T+H} |\Theta(t,\chi)|\,\big|B_X\big(\tfrac12+it,\chi\big)\big|^4\,\mathrm{d}t\bigg) \\ &\quad + O\bigg(\Big(\frac{q}{T}\Big)^{1/2}\log^2(2T)\int_T^{T+H} \big|B_X\big(\tfrac12+it,\chi\big)\big|^4\,\mathrm{d}t\bigg). \end{aligned} \tag{3-1}$$

Using the trivial bound $\big|B_X\big(\tfrac12+it,\chi\big)\big| \leqslant \sum_{n\leqslant X} 1/\sqrt{n} \ll \sqrt{X}$, we see that the two O-terms are respectively

$$O\bigg(\Big(\frac{q}{T}\Big)^{1/4}\log(2T)X^2\int_T^{T+H} |\Theta(t,\chi)|\,\mathrm{d}t\bigg) \quad\text{and}\quad O\bigg(\Big(\frac{q}{T}\Big)^{1/2} HX^2\log^2(2T)\bigg).$$

At this point, we apply the Cauchy–Schwarz inequality and get

$$\int_T^{T+H} |\Theta(t,\chi)|\,\mathrm{d}t \leqslant \sqrt{H}\bigg(\int_T^{T+H}\bigg|\sum_{n\leqslant\sqrt{qt/(2\pi)}} \frac{\chi(n)}{n^{1/2+it}}\bigg|^2\,\mathrm{d}t\bigg)^{1/2} \tag{3-2}$$

using the expression (2-2). The error in [Mawia 2017] lies in the subsequent application of the mean value theorem for Dirichlet polynomials at this point, which is not applicable in this form, since the summation is up to $n \leqslant \sqrt{qt/(2\pi)}$, and not up to $n \leqslant \sqrt{qT/(2\pi)}$ (t being the variable of integration, whereas T is the lower limit of the integral). However, this is easy to rectify. One method to correct this error is to use an analogue of Lemma 6 in [Selberg 1942], where the sum runs up to $n \leqslant \sqrt{T/(2\pi)}$ but the error is of size $T^{-3/20}$ instead of $T^{-1/4}$. We however follow another route. Let us write for brevity $T(m,n) = \max(T, 2\pi m^2/q, 2\pi n^2/q)$ and $\tau_1 = \sqrt{q(T+H)/(2\pi)}$. We then have

$$\begin{aligned}\int_T^{T+H}\bigg|\sum_{n\leqslant\sqrt{qt/(2\pi)}} \frac{\chi(n)}{n^{1/2+it}}\bigg|^2\,\mathrm{d}t &= \sum_{m,n\leqslant\tau_1} \frac{\chi(m)\bar\chi(n)}{\sqrt{mn}}\int_{T(m,n)}^{T+H}\Big(\frac{n}{m}\Big)^{it}\,\mathrm{d}t \\ &= H\sum_{n\leqslant\tau}\frac{|\chi(n)|}{n} + \sum_{\substack{m,n\leqslant\tau\\ m\neq n}} \frac{\chi(m)\bar\chi(n)}{\sqrt{mn}\log(n/m)}\bigg(\Big(\frac{n}{m}\Big)^{i(T+H)} - \Big(\frac{n}{m}\Big)^{iT}\bigg) \\ &\quad + \sum_{\tau<n\leqslant\tau_1}\frac{|\chi(n)|}{n}\Big(T+H-\frac{2\pi n^2}{q}\Big) \\ &\quad + \sum_{\substack{\tau<m,n\leqslant\tau_1\\ m\neq n}} \frac{\chi(m)\bar\chi(n)}{\sqrt{mn}\log(n/m)}\bigg(\Big(\frac{n}{m}\Big)^{i(T+H)} - \Big(\frac{n}{m}\Big)^{iT(m,n)}\bigg). \end{aligned}$$

Obviously, the first and third terms are $\ll H\log(qT)$. It is also easy to see that the second and last terms in the above expression are $\ll (qT)^{1/2}\log(qT)$ (see Lemma 1 in [Selberg 1942]). It follows from (3-2) that $\int_T^{T+H}|\Theta(t,\chi)|\,\mathrm{d}t \ll H\sqrt{\log(qT)}$ and the first O-term in (3-1) is $O((q/T)^{1/4}HX^2\sqrt{\log(qT)}\log(2T))$.

The main difficulty is in the treatment of the first three terms on the right side of (3-1). Since the first two integrals are bounded in absolute value by the third integral, it is enough to look at the third integral, namely

$$J=\int_T^{T+H}|\Theta(t,\chi)|^2\left|B_X\left(\tfrac{1}{2}+it,\chi\right)\right|^4\mathrm{d}t.$$

Using the expressions (2-4) and (2-1), we have

$$\begin{aligned}
J&=\sum_{\substack{k,\ell\leqslant\tau_1\\ m,n\leqslant X^2}}\frac{\chi(k)\bar{\chi}(\ell)b_m(\chi)b_n(\bar{\chi})}{\sqrt{k\ell mn}}\int_{T(k,\ell)}^{T+H}\left(\frac{\ell n}{km}\right)^{it}\mathrm{d}t\\
&=H\sum_{\substack{k,\ell\leqslant\tau;m,n\leqslant X^2\\ km=\ell n}}\frac{\chi(k)\bar{\chi}(\ell)b_m(\chi)b_n(\bar{\chi})}{\sqrt{k\ell mn}} &&(3\text{-}3)\\
&\quad+\left(\sum\nolimits_1+\sum\nolimits_2+\sum\nolimits_3\right)\frac{\chi(k)\bar{\chi}(\ell)b_m(\chi)b_n(\bar{\chi})}{\sqrt{k\ell mn}}(T+H-T(k,\ell)) &&(3\text{-}4)\\
&\quad+\sum_{\substack{k,\ell\leqslant\tau_1;m,n\leqslant X^2\\ km\neq\ell n}}\frac{\chi(k)\bar{\chi}(\ell)b_m(\chi)b_n(\bar{\chi})}{i\sqrt{k\ell mn}\log(\ell n/km)}\left(\left(\frac{\ell n}{km}\right)^{i(T+H)}-\left(\frac{\ell n}{km}\right)^{iT(k,\ell)}\right), &&(3\text{-}5)
\end{aligned}$$

where

$$\sum\nolimits_1=\sum_{\substack{\tau<k,\ell\leqslant\tau_1;m,n\leqslant X^2\\ km=\ell n}},\quad \sum\nolimits_2=\sum_{\substack{k\leqslant\tau<\ell\leqslant\tau_1;m,n\leqslant X^2\\ km=\ell n}},\quad \sum\nolimits_3=\sum_{\substack{\ell\leqslant\tau<k\leqslant\tau_1;m,n\leqslant X^2\\ km=\ell n}}.$$

We shall treat the above sums one by one, starting from the easiest, namely (3-5), to the hardest, that is, (3-3). Using the fact that $|b_n(\chi)|\leqslant d(n)\ll T^{\epsilon}$ for $n\leqslant\tau_1X^2$, and applying [Selberg 1942, Lemma 1], we see that the sum in (3-5) above is $O(q^{1/2}T^{1/2+\epsilon}X^2)$ for any $\epsilon>0$.

Let us now look at the first sum in (3-4), namely the sum enclosed by $\sum_1$. We want to show that it is $\ll H$. This sum is essentially the same as the sum in (3-3), except that the range of k,ℓ is different. Although we can adapt the proof given in [Mawia 2017] that the sum in (3-3) is $\ll 1$, we give a simpler proof for this sum when $\vartheta<1$. Since $T+H-T(k,\ell)\leqslant H$ for k,ℓ in the given range, it is enough to show that

$$\sum_{\substack{\tau<k,\ell\leqslant\tau_1;m,n\leqslant X^2\\ km=\ell n}}\frac{d(m)d(n)}{\sqrt{k\ell mn}}\ll 1.$$

Note first that

$$\sum_{\substack{\tau<k,\ell\leqslant\tau_1;m,n\leqslant X^2\\ km=\ell n}}\frac{d(m)d(n)}{\sqrt{k\ell mn}}=\sum_{m,n\leqslant X^2}\frac{d(m)d(n)}{mn}g\sum_{\tau(m,n)<\nu\leqslant\tau_1(m,n)}\frac{1}{\nu}, \tag{3-6}$$

where $\tau(m,n)=g\tau/\min(m,n)$, $\tau_1(m,n)=g\tau_1/\max(m,n)$ and $g=(m,n)$ in each term. Since the sum on the right is symmetric in m and n, it is enough to look at the sum for $m\leqslant n\leqslant X^2$, which we do henceforward. Observe that the inner sum is zero unless $m/n > \tau/\tau_1$, in which case, the inner sum is of size $\ll \log(m\tau_1/(n\tau)) + (m+n)/(g\tau)$ (the term $\log(m\tau_1/(n\tau))$ comes from the main term of the harmonic sum, whereas the term $(m+n)/(g\tau)$ comes from the error term) and in the outer sum, for each $n \leqslant X^2$, the variable m ranges over $n\tau/\tau_1 < m \leqslant n$. Using the fact that $d(m)d(n) \ll T^\epsilon$, we see that the sum (3-6) is

$$\begin{aligned}
&\ll T^\epsilon \sum_{n\leqslant X^2} \frac{1}{n} \sum_{n\tau/\tau_1 < m \leqslant n} \frac{g}{m} \log \frac{m\tau_1}{n\tau} + \sum_{m,n\leqslant X^2} \frac{d(m)d(n)}{\tau}\left(\frac{1}{m}+\frac{1}{n}\right) \\
&\ll HT^{-1+\epsilon} \sum_{n\leqslant X^2} \sum_{n\tau/\tau_1 < m \leqslant n} \frac{g}{mn} + \frac{X^2 \log^3 X}{\tau} \\
&\ll H^2 T^{-2+\epsilon} (\log X)^3 + \frac{X^2 \log^3 X}{\tau} \ll 1
\end{aligned}$$

for small enough ϵ, as long as $\vartheta < 1$ and $\theta < \frac{1}{4} + \frac{1}{4}\eta$ (this latter condition is satisfied since $\theta < \frac{1}{8}$ by assumption). This completes the proof when $\vartheta < 1$. When $\vartheta = 1$, we can adapt the treatment of the sum (3-3), given in the ensuing paragraphs. It is to be noted that, although similar in appearance, the sum (3-4) needs a much less delicate treatment than the sum in (3-3) due to the restriction $\tau < k, \ell \leqslant \tau_1$.

Next we look at the sums enclosed by $\sum_2$ and $\sum_3$ in (3-4). Since the two sums are similar, it suffices to look at one of them, say $\sum_2$. Comparing it with $\sum_1$, we expect it to be roughly of the same size, as the range of ℓ is the same although k runs through a set of smaller integers, and we have exactly the same terms in the two sums. We observe that the sum $\sum_2$ is smaller in absolute value than

$$H \sum_{m,n\leqslant X^2} \frac{d(m)d(n)}{mn} g \sum_{g\tau/m < \nu \leqslant \min\{g\tau/n,\, g\tau_1/m\}} \frac{1}{\nu},$$

by following the same steps as in the treatment of $\sum_1$. This sum can now be estimated as (3-6).

It remains to treat the sum in (3-3). This sum was shown in [Mawia 2017] to be $\ll 1$ when q is fixed; the case when q is not fixed will be treated shortly. We first remark that in the treatment given of this sum in [Mawia 2017], the positions of k and l have to be interchanged throughout, so that the integral in the first line of equation (12) on p. 45 should be $\int_T^{T+H} (ln/km)^{it}\, dt$, and the conditions $kn = lm$ and $kn \neq lm$ should be replaced throughout by the conditions $km = ln$ and $km \neq ln$ respectively.

We now consider the same sum when q is not fixed. Note first that

$$\sum_{\substack{k,\ell\leqslant\tau;\, m,n\leqslant X^2 \\ km=\ell n}} \frac{\chi(k)\bar{\chi}(\ell) b_m(\chi) b_n(\bar{\chi})}{\sqrt{k\ell mn}} = \sum_{m,n\leqslant X^2} \frac{b_m(|\chi|) b_n(|\chi|)}{mn} g \sum_{\nu\leqslant\tau_0} \frac{|\chi(\nu)|}{\nu}, \tag{3-7}$$

where $\tau_0 = g\tau/\max(m,n)$ and $g = (m,n)$ in each term. As in [Mawia 2017], let us first define, for $\gamma \geqslant 0$,

$$S(\gamma) = \Re\left\{\tau^{i\gamma} \sum_{n_j\leqslant X} \frac{\beta_{n_1}(|\chi|)\beta_{n_2}(|\chi|)\beta_{n_3}(|\chi|)\beta_{n_4}(|\chi|)}{n_1 n_2 n_3 n_4} \frac{g^{1+i\gamma}}{(n_1 n_3)^{i\gamma}} \sum_{\nu\leqslant\tau_1} \frac{|\chi(\nu)|}{\nu^{1+i\gamma}}\right\},$$

where we have written $\tau_1 = g\tau/\max(n_1 n_2, n_3 n_4)$ with g now standing for $(n_1 n_2, n_3 n_4)$. Note that the right side of (3-7) is equal to $S(0)$. We will show that $S(\gamma) = O(4^{\omega(q)} q^2/\varphi(q)^2)$ for $0 < \gamma \leqslant 1/\log T$; it will follow by continuity that $S(0) = O(4^{\omega(q)} q^2/\varphi(q)^2)$ as well. As remarked in [Mawia 2017], the proof is essentially contained in the proof that $K(\gamma) = O(1)$ in [Selberg 1942, pp. 27–31]. Following the same reduction steps as in [Mawia 2017], we arrive at the expression

$$S(\gamma) = \Re\{\tau^{i\gamma} L(1+i\gamma, |\chi|) S_2(\gamma)\} + O(|S_2(\gamma)| \log T) + O(2^{\omega(q)} (qT)^{-1/2} X^2 \log^4 T), \tag{3-8}$$

with

$$S_2(\gamma) = \sum_{n_j \leqslant X} \frac{\beta_{n_1}(|\chi|)\beta_{n_2}(|\chi|)\beta_{n_3}(|\chi|)\beta_{n_4}(|\chi|)}{n_1 n_2 n_3 n_4} \frac{g^{1+i\gamma}}{(n_1 n_3)^{i\gamma}}. \tag{3-9}$$

Note that we have used Lemma 3 in the course of the reduction. Let us look at the first O-term in (3-8). We showed in [Mawia 2017] that $S_2(\gamma)$ is equal to

$$S_2(\gamma) = \sum_{d \leqslant X^2} \varphi_{i\gamma}(d) S_3(\gamma; d)^2, \tag{3-10}$$

where

$$S_3(\gamma; d) = \sum_{\substack{m,n \leqslant X \\ d \mid mn}} \frac{\beta_m(|\chi|)\beta_n(|\chi|)}{m n^{1+i\gamma}}$$

(see the second displayed equation on p. 48), and that $S_3(\gamma; d)$ is further equal to

$$S_3(\gamma; d) = \frac{1}{\log^2 X} \sum_{\substack{d_1, d_2 \leqslant X^2 \\ d \mid d_1 d_2 \mid d^\infty}} \frac{\alpha_{d_1}(|\chi|)\alpha_{d_2}(|\chi|)}{d_1 d_2^{1+i\gamma}} \Biggl\{ \sum_{\substack{n \leqslant X/d_1 \\ (n,d)=1}} \frac{\alpha_n(|\chi|)}{n} \log \frac{X}{d_1 n} \Biggr\} \Biggl\{ \sum_{\substack{n \leqslant X/d_2 \\ (n,d)=1}} \frac{\alpha_n(|\chi|)}{n^{1+i\gamma}} \log \frac{X}{d_2 n} \Biggr\}, \tag{3-11}$$

where, in the above sum, the only primes dividing d_1, d_2 are the prime factors of d, which is often symbolically written as $d_1 \mid d^\infty$, $d_2 \mid d^\infty$. We will now apply Lemma 5 with $r = 1$. Note that for $0 \leqslant \gamma \leqslant 1/\log X$, the O-term dominates the "main" term in (2-5). Using this fact in (3-11), we see that

$$\begin{aligned} S_3(\gamma; d) &\ll \frac{2^{\omega(q)} q}{\varphi(q) \log X} \sum_{\substack{d_1, d_2 \leqslant X^2 \\ d \mid d_1 d_2 \mid d^\infty}} \frac{1}{d_1 d_2} \prod_{p \mid d} (1 + |\chi(p)| p^{-3/4})^2 \\ &\ll \frac{2^{\omega(q)} q}{d \varphi(q) \log X} \prod_{p \mid d} (1 + p^{-3/4})^3. \end{aligned} \tag{3-12}$$

Using this in (3-10), we arrive at

$$\begin{aligned} S_2(\gamma) &\ll \frac{4^{\omega(q)} q^2}{\varphi(q)^2 \log^2 X} \sum_{d \leqslant X^2} \frac{|\varphi_{i\gamma}(d)|}{d^2} \prod_{p \mid d} (1 + p^{-3/4})^6 \\ &\ll \frac{4^{\omega(q)} q^2}{\varphi(q)^2 \log X}. \end{aligned}$$

Plugging this into (3-8), we have

$$S(\gamma) = \Re\{\tau^{i\gamma} L(1+i\gamma, |\chi|) S_2(\gamma)\} + O\biggl(\frac{4^{\omega(q)} q^2}{\varphi(q)^2} \biggr).$$

The treatment of the first term here is somewhat delicate. Using Lemma 3, we have $L(1+i\gamma,|\chi|)=\varphi(q)/qi\gamma+O(2^{\omega(q)})$, so that

$$S(\gamma)=\Im\left\{\tau^{i\gamma}\frac{\varphi(q)}{q\gamma}S_2(\gamma)\right\}+O\left(\frac{4^{\omega(q)}q^2}{\varphi(q)^2}\right). \tag{3-13}$$

Using the fact that the inequality $|\Im(ab^2)|=|y(u^2-v^2)+2yuv|\leqslant|\Im a||b|^2+2|a||b||\Im b|$ holds for $a=x+iy,\ b=u+iv$, we have, in view of (3-10) and (3-13),

$$\begin{aligned}|S(\gamma)|\leqslant\sum_{d\leqslant X^2}&\left|\Im\left(\frac{\tau^{i\gamma}\varphi_{i\gamma}(d)\varphi(q)}{q\gamma}\right)\right||S_3(\gamma;d)|^2\\&+2\sum_{d\leqslant X^2}|\varphi_{i\gamma}(d)||S_3(\gamma;d)|\left|\Im\left(\frac{S_3(\gamma;d)}{\gamma}\right)\right|+O\left(\frac{4^{\omega(q)}q^2}{\varphi(q)^2}\right).\end{aligned} \tag{3-14}$$

Let us look at the terms one by one. First of all, we observe that

$$\begin{aligned}\left|\Im\left(\frac{\tau^{i\gamma}\varphi_{i\gamma}(d)\varphi(q)}{q\gamma}\right)\right|&=\left|\Im\left(\frac{\tau^{i\gamma}\varphi(q)}{q\gamma}d^{1+i\gamma}\sum_{c\,|\,d}\frac{\mu(c)}{c^{1+i\gamma}}\right)\right|\\&=d\frac{\varphi(q)}{q}\left|\sum_{c\,|\,d}\frac{\mu(c)}{c}\frac{\sin(\gamma\log(\tau d/c))}{\gamma}\right|\\&\leqslant d\frac{\varphi(q)}{q}\log(\tau d)\prod_{p\,|\,d}(1+p^{-3/4}).\end{aligned}$$

Using the bound (3-12) for $S_3(\gamma;d)$, we see that the first term in (3-14) is

$$\ll\frac{2^{\omega(q)}q\log(\tau X^2)}{\varphi(q)\log^2 X}\sum_{d\leqslant X^2}\frac{1}{d}\prod_{p\,|\,d}(1+p^{-3/4})^7\ll\frac{2^{\omega(q)}q}{\varphi(q)}. \tag{3-15}$$

Similarly, using Lemmas 5 and 6 and the expression

$$\begin{aligned}&\Im\left(\frac{S_3(\gamma;d)}{\gamma}\right)\\&=\frac{1}{\log^2 X}\sum_{\substack{d_1,d_2\leqslant X\\ d\,|\,d_1d_2\,|\,d^\infty}}\frac{\alpha_{d_1}(|\chi|)\alpha_{d_2}(|\chi|)}{d_1d_2}\left\{\sum_{\substack{n\leqslant X/d_1\\(n,d)=1}}\frac{\alpha_n(|\chi|)}{n}\log\frac{X}{d_1n}\right\}\left\{\sum_{\substack{n\leqslant X/d_2\\(n,d)=1}}\frac{\alpha_n(|\chi|)}{n}\log\frac{X}{d_2n}\frac{\sin(\gamma\log(d_2n))}{\gamma}\right\},\end{aligned}$$

we get that the second term in (3-14) is

$$\ll\frac{2^{\omega(q)}q}{\varphi(q)}. \tag{3-16}$$

It follows from (3-14), (3-15) and (3-16) that $S(\gamma)\ll 4^{\omega(q)}q^2/\varphi(q)^2$. □

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