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The mean square discrepancy in the circle problem

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We study the mean square of the error term in the Gauss circle problem. A heuristic argument based on the consideration of off-diagonal terms in the mean square of the relevant Voronoi-type summation formula leads to a precise conjecture for the mean square of this discrepancy.

1. Introduction

Let $r(n)$ denote the number of representations of the integer n as a sum of two squares of integers and let

$$P(x) = {\sum_{n \le x}}' r(n) - \pi x + 1, \tag{1-1}$$

where the prime superscript on the summation means that $r(x)$ is counted with weight $\frac{1}{2}$ if x is an integer. Finding the best estimate of the discrepancy $P(x)$ is known as Gauss' circle problem. It is trivial that $P(x) \ll x^{1/2}$, and it is conjectured that $P(x) \ll x^{1/4+\epsilon}$, where here and throughout ϵ denotes a small positive number that may be different at each occurrence. In the opposite direction, G. H. Hardy [1915; 1916b] proved that $P(x) = \Omega_+(x^{1/4})$ and $P(x) = \Omega_-((x \log x)^{1/4})$, and this has been improved slightly by a number of mathematicians; for example, see [Soundararajan 2003]. Here the notation $f(x) = \Omega_+(g(x))$ means there is a sequence of real numbers $x_n \to \infty$ and a positive constant c such that $f(x_n) \ge c|g(x_n)|$ for all n. Similarly, $f(x) = \Omega_-(g(x))$ means there is a sequence $x_n \to \infty$ and a positive constant c such that $f(x_n) \le -c|g(x_n)|$ for all n.

In spite of more than a century of effort, for example, by Sierpiński [1906], van der Corput [1923], Kolesnik [1985], Iwaniec and Mozzochi [1988], and Huxley [2003], Gauss's circle problem has resisted solution. In an attempt to understand it better, mathematicians have considered several variants of the problem that exploit the fact that the average of $P(x)$ is easier to analyze. For example, there has been considerable interest in the mean square of the discrepancy

$$\int_0^X P(x)^2 \, dx.$$

It is known that

$$\int_0^X P(x)^2 \, dx = C X^{3/2} + Q(X), \tag{1-2}$$

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where

$$C = \frac{1}{3\pi^2} \sum_{n=1}^{\infty} \frac{r^2(n)}{n^{3/2}} = \frac{16}{3\pi^2} \frac{\zeta_{\mathbb{Q}(i)}\left(\frac{3}{2}\right)^2}{\zeta(3)} (1+2^{-3/2})^{-1} = 1.69396\ldots \tag{1-3}$$

and $Q(X)$ is a function that is $o(X^{3/2})$. H. Cramér [1922] proved that $Q(X) \ll X^{5/4+\epsilon}$, E. Landau [1923] that $Q(X) \ll X^{1+\epsilon}$, A. Walfisz [1927] that $Q(X) \ll X \log^3 X$, and I. Kátai [1965] that $Q(X) \ll X \log^2 X$; see also the work of E. Preissmann [1988] for another proof. W. G. Nowak [2004] proved the estimate

$$Q(X) \ll X(\log X)^{3/2} \log\log X,$$

and Y.-K. Lau and K.-M. Tsang [2009] proved that

$$Q(X) \ll X \log X \log\log X, \tag{1-4}$$

the best estimate to date of which we are aware.

Our goal here is to conjecture a precise formula for $Q(X)$. We will then use this to determine how large and how small $Q(X)$ can be, and to uncover a previously unobserved phenomenon described below.

Conjecture. *There is a constant* $0 < \vartheta < 1$ *such that as* $X \to \infty$,

$$Q(X) = C(X)X - X + O(X^{\vartheta}), \tag{1-5}$$

where

$$C(X) = \lim_{N\to\infty} \frac{1}{2\pi^3} \sum_{1\le m,n\le N} \frac{r(n)r(m)\cos(2\pi(\sqrt{m}+\sqrt{n})\sqrt{X})}{(mn)^{3/4}(\sqrt{m}+\sqrt{n})}.$$

That is,

$$\int_0^X P(x)^2\,dx = CX^{3/2} + C(X)X - X + O(X^{\vartheta}), \tag{1-6}$$

where C *is given by* (1-3).

The phenomenon referred to above is the presence of the slowly oscillating function $C(X)$ in (1-6). This points to why it is so difficult to determine the exact size of $Q(X)$. A. Ivić [1996; 2001] has shown that the Laplace transform of $P(x)^2$ is

$$\int_0^{\infty} P(x)^2 e^{-x/T}\,dx = \frac{1}{4}\left(\frac{T}{\pi}\right)^{3/2} \sum_{n=1}^{\infty} r^2(n)n^{-3/2} - T + O(T^{2/3+\epsilon}).$$

Comparing this with (1-6), we see that the Laplace transform does not "see" the oscillating term $C(X)$.

It is not obvious that the limit defining $C(X)$ exists. We prove this in:

Proposition 1. *For* $X \ge 0$, $N \ge 1$ *let*

$$C_N(X) = \frac{1}{2\pi^3} \sum_{1\le m,n\le N} \frac{r(n)r(m)\cos(2\pi(\sqrt{m}+\sqrt{n})\sqrt{X})}{(mn)^{3/4}(\sqrt{m}+\sqrt{n})}. \tag{1-7}$$

Then for $X \ge 1$,

$$C(X) = \lim_{N\to\infty} C_N(X)$$

exists. Moreover, *for* $X \ge 1$ *we have*

$$|C(X) - C_N(X)| \ll \frac{X \log N}{N^{1/4}}. \tag{1-8}$$

In Section 9 we shall use Proposition 1 to prove:

Theorem 2. *As* $X \to \infty$,

$$|C(X)| \le \Big(\frac{16}{\pi} + o(1)\Big) \log X. \tag{1-9}$$

Hence, if the Conjecture is true,

$$|Q(X)| \le \Big(\frac{16}{\pi} + o(1)\Big) X \log X. \tag{1-10}$$

The upper bound (1-10) suggests that (1-4) is too large by a factor of at least $\log\log X$. However, we suspect that even (1-10) is larger than the true upper bound. It is possible that the lower bound for $Q(X)$ provided by the next theorem is closer to the actual upper bound.

Theorem 3. *We have*

$$\limsup_{X\to\infty} \frac{C(X)}{\log\log X} \ge \frac{1}{2\pi}.$$

Hence, if the Conjecture is true, then

$$Q(X) = \Omega_+(X \log\log X). \tag{1-11}$$

In view of the Conjecture one might ask whether one can model $C_N(X)$ by the sum

$$\frac{1}{2\pi^3} \sum_{m,n\le N} \frac{r(n)r(m)\cos(2\pi X_{m,n})}{(mn)^{3/4}(\sqrt{m}+\sqrt{n})} \qquad (X^4 \le N \le X^A),$$

where the $X_{m,n}$ are independent identically distributed random variables. This would be reasonable if the approximately $N^2/2$ numbers $\{\sqrt{m}+\sqrt{n}\}_{m\le n\le N}$ were linearly independent over the rationals. Using estimates of H. L. Montgomery and A. Odlyzko [1988] for large deviations of sums of random variables, one could then show that this sum is likely to be no larger than $O(\log\log X)$. Unfortunately, the numbers $\{\sqrt{m}+\sqrt{n}\}_{m\le n}$ are highly linearly dependent over the rationals in the sense that a relatively sparse subset of these numbers spans the set. For example, the $2N-1$ numbers $\sqrt{n}+\sqrt{2}$, $\sqrt{n}+\sqrt{3}$ $(n = 1, 2, \ldots, N)$ allow us to write an arbitrary one of the approximately N^2 elements $\sqrt{k}+\sqrt{l}$ as $(\sqrt{k}+\sqrt{2}) + (\sqrt{l}+\sqrt{3}) - (\sqrt{2}+\sqrt{3})$. Thus, such a model might not be very accurate.

Our method may also be applied to other well-known problems. For example, it may be used to conjecture a formula for the term $F(X)$ in

$$\int_0^X \Delta(x)^2\,dx = \frac{X^{3/2}}{6\pi^2} \sum_{n=1}^{\infty} \frac{d(n)^2}{n^{3/2}} + F(X), \tag{1-12}$$

the mean square of the error term in the Dirichlet divisor problem, where

$$\Delta(x) = {\sum_{n\le x}}' d(n) - x(\log x + 2\gamma - 1) - \tfrac{1}{4},$$

$d(n) = \sum_{d|n} 1$, and γ is Euler's constant. This will be the subject of a forthcoming paper by the first author and Dr. Fan Ge. The important feature shared by the circle and divisor problems that makes our method applicable is, of course, the existence of a Voronoi-type summation formula for their error terms.

2. Proof of Proposition 1

Before proving Proposition 1, we gather several formulae and a lemma.

Hardy [1916a] proved that for $x > 0$

$$P(x) = x^{1/2} \sum_{n=1}^{\infty} \frac{r(n)}{n^{1/2}} J_1(2\pi\sqrt{nx}), \tag{2-1}$$

where J_1 is a Bessel function of the first kind. Using the approximation

$$J_1(u) = \Big(\frac{2}{\pi u}\Big)^{1/2} \Big(\cos\Big(u - \frac{3\pi}{4}\Big) - \frac{3}{8u}\sin\Big(u - \frac{3\pi}{4}\Big)\Big) + O(u^{-5/2}), \tag{2-2}$$

valid for $u \geq 1$, we deduce that

$$P(x) = \frac{x^{1/4}}{\pi} \sum_{n=1}^{\infty} \frac{r(n)}{n^{3/4}} \cos\Big(2\pi\sqrt{nx} - \frac{3\pi}{4}\Big) - \frac{3x^{-1/4}}{16\pi^2} \sum_{n=1}^{\infty} \frac{r(n)}{n^{5/4}} \sin\Big(2\pi\sqrt{nx} - \frac{3\pi}{4}\Big) + O(x^{-3/4}) \tag{2-3}$$

for $x \geq 1$. From this we obtain

$$P_1(x) := \int_0^x P(u)\,du = \frac{x^{3/4}}{\pi^2} \sum_{n=1}^{\infty} \frac{r(n)}{n^{5/4}} \sin\Big(2\pi\sqrt{nx} - \frac{3\pi}{4}\Big) + O(x^{1/4}) \tag{2-4}$$

for $x \geq 1$. Note that

$$P_1(x) \ll x^{3/4} \tag{2-5}$$

for $x \geq 1$ follows immediately from this.

Lemma 4. *Let* $1 \leq A < B$ *and let* $0 < \epsilon < 1$. *Then uniformly for* $x \geq 1$ *and for* y *real*,

$$\sum_{A \leq k \leq B} \frac{r(k)}{k^{3/4}} \cos(x\sqrt{k} + y) \ll \frac{x^2}{A^{1/2}} + \frac{1}{A^{1/4}} + x^{\epsilon - 1/2} \min\Big(\frac{x}{A^{1/2}\|(x/2\pi)^2\|}, \log\frac{B}{A}\Big). \tag{2-6}$$

The implied constant depends at most on ϵ.

Proof. Denote the left-hand side of (2-6) by S. Then by (1-1) we may write

$$S = \pi \int_A^B u^{-3/4} \cos(x\sqrt{u} + y)\,du + \int_{A^-}^B u^{-3/4} \cos(x\sqrt{u} + y)\,dP(u) = S_1 + S_2.$$

Now

$$\begin{aligned} S_1 &= \frac{2\pi}{x} \int_A^B u^{-1/4}\, d(\sin(x\sqrt{u} + y)) \\ &= \frac{2\pi \sin(x\sqrt{u} + y)}{xu^{1/4}}\bigg|_A^B + \frac{\pi}{2x} \int_A^B \frac{\sin(x\sqrt{u} + y)}{u^{5/4}}\,du \ll A^{-1/4}x^{-1}. \end{aligned}$$

Using $P(u) \ll u^{1/2}$, we see that

$$\begin{aligned} S_2 &= \frac{P(u)\cos(x\sqrt{u} + y)}{u^{3/4}}\bigg|_{A^-}^B + \int_A^B \frac{P(u)}{u^{7/4}} \big(\tfrac{3}{4}\cos(x\sqrt{u} + y) + \tfrac{1}{2}x\sqrt{u}\sin(x\sqrt{u} + y)\big)\,du \\ &= \frac{x}{2} \int_{A^-}^B \frac{P(u)}{u^{5/4}} \sin(x\sqrt{u} + y)\,du + O(A^{-1/4}). \end{aligned}$$

Thus, for $x \geq 1$, we have

$$S = \frac{x}{2} \int_{A^-}^{B} \frac{P(u)}{u^{5/4}} \sin(x\sqrt{u} + y)\, du + O(A^{-1/4}).$$

Let $P_1(u)$ be as in (2-4). Then by (2-5)

$$\begin{aligned} S &= \frac{x}{2} \int_{A^-}^{B} \frac{\sin(x\sqrt{u}+y)}{u^{5/4}}\, dP_1(u) + O(A^{-1/4}) \\ &= \frac{x}{2}\left(P_1(u) \frac{\sin(x\sqrt{u}+y)}{u^{5/4}} \bigg|_{A^-}^{B} + \int_A^B P_1(u) \left(\frac{\frac{5}{4}\sin(x\sqrt{u}+y)}{u^{9/4}} - \frac{x}{2}\frac{\cos(x\sqrt{u}+y)}{u^{7/4}} \right) du \right) + O(A^{-1/4}) \\ &= -\left(\frac{x}{2}\right)^2 \int_A^B P_1(u) \frac{\cos(x\sqrt{u}+y)}{u^{7/4}}\, du + O(xA^{-1/2}) + O(A^{-1/4}). \end{aligned}$$

Next we insert the formula for $P_1(x)$ from (2-4) into the last integral. The $O(u^{1/4})$ term in (2-4) contributes $O(x^2/A^{1/2})$, so we obtain

$$S = -\frac{x^2}{4\pi^2} \sum_{n=1}^{\infty} \frac{r(n)}{n^{5/4}} \int_A^B \frac{\sin\left(2\pi\sqrt{nu} - \frac{3\pi}{4}\right)\cos(x\sqrt{u}+y)}{u}\, du + O(x^2 A^{-1/2}) + O(A^{-1/4}).$$

Writing the numerator in the integrand as a sum of two sines, we have

$$S = -\frac{x^2}{8\pi^2} \sum_{n=1}^{\infty} \frac{r(n)}{n^{5/4}} (I_1(n) + I_2(n)) + O(x^2 A^{-1/2}) + O(A^{-1/4}), \tag{2-7}$$

where

$$\begin{aligned} I_1(n) &= \int_A^B u^{-1} \sin\left((x + 2\pi\sqrt{n})\sqrt{u} + y - \frac{3\pi}{4}\right) du, \\ I_2(n) &= \int_A^B u^{-1} \sin\left((-x + 2\pi\sqrt{n})\sqrt{u} - y - \frac{3\pi}{4}\right) du. \end{aligned}$$

Integration by parts shows that

$$\begin{aligned} I_1(n) &\ll \frac{1}{A^{1/2}(x + 2\pi\sqrt{n})}, \\ I_2(n) &\ll \frac{1}{A^{1/2}|x - 2\pi\sqrt{n}|}. \end{aligned}$$

We also trivially have $I_2(n) \ll \log B/A$. It follows that

$$x^2 \sum_{n=1}^{\infty} \frac{r(n) I_1(n)}{n^{5/4}} \ll \frac{x}{A^{1/2}}$$

and

$$x^2 \sum_{n=1}^{\infty} \frac{r(n) I_2(n)}{n^{5/4}} \ll x^2 \sum_{n=1}^{\infty} \frac{r(n)}{n^{5/4}} \min\left(\frac{1}{A^{1/2}|x - 2\pi\sqrt{n}|}, \log\frac{B}{A} \right).$$

The terms in the last sum with $n \le \frac{1}{2}(x/2\pi)^2$ or $n > 2(x/2\pi)^2$ contribute

$$\ll \frac{x^2}{A^{1/2}}\Biggl(\sum_{n\le\frac{1}{2}(x/2\pi)^2} + \sum_{n>2(x/2\pi)^2}\Biggr)\frac{r(n)(x+2\pi\sqrt{n})}{n^{5/4}|x^2-4\pi^2 n|}$$

$$\ll \frac{x^2}{A^{1/2}}\Biggl(\frac{1}{x}\sum_{n\le\frac{1}{2}(x/2\pi)^2}\frac{r(n)}{n^{5/4}} + \sum_{n>2(x/2\pi)^2}\frac{r(n)}{n^{7/4}}\Biggr)$$

$$\ll \frac{x^2}{A^{1/2}}(x^{-1}+x^{-3/2}\log x) \ll \frac{x}{A^{1/2}}.$$

The remaining terms give

$$\ll \frac{x^2}{A^{1/2}}\sum_{\substack{\frac{1}{2}(x/2\pi)^2<n\le 2(x/2\pi)^2\\ n\ne[(x/2\pi)^2],\,[(x/2\pi)^2]+1}}\frac{r(n)}{n^{5/4}}\frac{x}{|(x/2\pi)^2-n|} \;+\; x^{\epsilon-1/2}\min\Biggl(\frac{x}{A^{1/2}\|(x/2\pi)^2\|},\log\frac{B}{A}\Biggr)$$

$$\ll \frac{x^{1/2+\epsilon}}{A^{1/2}} + x^{\epsilon-1/2}\min\Biggl(\frac{x}{A^{1/2}\|(x/2\pi)^2\|},\log\frac{B}{A}\Biggr).$$

Thus, from (2-7) we conclude that

$$S \ll \frac{x^2}{A^{1/2}} + \frac{1}{A^{1/4}} + x^{\epsilon-1/2}\min\Biggl(\frac{x}{A^{1/2}\|(x/2\pi)^2\|},\log\frac{B}{A}\Biggr).$$

This completes the proof of Lemma 4. □

We now prove Proposition 1. By definition

$$C_N(X) = \frac{1}{4\pi^3}\sum_{m\le N}\frac{r(m)^2\cos(4\pi\sqrt{X}\sqrt{m})}{m^2} + \frac{1}{\pi^3}\sum_{1\le m<n\le N}\frac{r(m)\,r(n)\;\cos(2\pi\sqrt{X}(\sqrt{m}+\sqrt{n}))}{(mn)^{3/4}\;(\sqrt{m}+\sqrt{n})}\;.$$

The second sum on the right equals

$$\sum_{1<n\le N} r(n)n^{-5/4}\sum_{1\le m\le n-1}\frac{r(m)\cos(2\pi\sqrt{X}(\sqrt{m}+\sqrt{n}))}{m^{3/4}(1+\sqrt{m/n})}\;.$$

Thus, for $M > N$ we have

$$|C_M(X)-C_N(X)| \le \sum_{N<m\le M}\frac{r(m)^2}{m^2} + \sum_{N<n\le M}\frac{r(n)}{n^{5/4}}\Biggl|\sum_{1\le m\le n-1}\frac{r(m)\cos(2\pi\sqrt{X}(\sqrt{m}+\sqrt{n}))}{m^{3/4}(1+\sqrt{m/n})}\Biggr|.$$

By partial summation, the sum over m within absolute values equals

$$\frac{S_{n-1}}{1+\sqrt{(n-1)/n}} + \sum_{1\le m\le n-2} S_m\Biggl(\frac{1}{1+\sqrt{m/n}} - \frac{1}{1+\sqrt{(m+1)/n}}\Biggr),$$

where

$$S_m = S_m(n) = \sum_{1\le k\le m}\frac{r(k)\cos(2\pi\sqrt{X}(\sqrt{k}+\sqrt{n}))}{k^{3/4}}\;.$$

Thus, for $X \geq 1$ we have

$$\sum_{1\leq m\leq n-1} \frac{r(m)\cos(2\pi\sqrt{X}(\sqrt{m}+\sqrt{n}))}{m^{3/4}(1+\sqrt{m/n})}$$
$$\ll \max_{1\leq m\leq n-1} |S_m(n)| \cdot \left(\frac{1}{1+\sqrt{(n-1)/n}} + \sum_{1\leq m\leq n-2} \left(\frac{1}{1+\sqrt{m/n}} - \frac{1}{1+\sqrt{(m+1)/n}} \right)\right)$$
$$\leq \max_{1\leq m\leq n-1} |S_m(n)|.$$

Now by Lemma 4 with $x = 2\pi\sqrt{X},\ y = 2\pi\sqrt{nX},\ A = 1$, and $B = m-1$, we see that

$$S_m(n) \ll_\epsilon X + X^{\epsilon-1/4}\log m$$

for any $0 < \epsilon < 1$. Therefore, since $r(n) \ll d(n)$, we find that for $M > N$,

$$|C_M(X) - C_N(X)| \ll \sum_{N<m\leq M} \frac{r(m)^2}{m^2} + \sum_{N<n\leq M} \frac{r(n)}{n^{5/4}} \left| \sum_{1\leq m\leq n-1} \frac{r(m)\cos(2\pi\sqrt{X}(\sqrt{m}+\sqrt{n}))}{m^{3/4}(1+\sqrt{m/n})} \right|$$
$$\ll_\epsilon \frac{(\log N)^3}{N} + \sum_{N<n\leq M} \frac{d(n)}{n^{5/4}} \left(X + \frac{\log n}{X^{1/4-\epsilon}} \right)$$
$$\ll_\epsilon \frac{(\log N)^3}{N} + \frac{X\log N}{N^{1/4}} + \frac{(\log N)^2}{N^{1/4}X^{1/4-\epsilon}} \ll \frac{X(\log N)}{N^{1/4}}.$$

In the last inequality, we have taken $\epsilon = \frac{1}{8}$, which allows us to make the implied constant absolute. It now follows from Cauchy's criterion that $\lim_{N\to\infty} C_N(X)$ exists as $N \to \infty$. The second assertion of Proposition 1 follows immediately from the last inequality.

3. Beginning of the argument

To avoid technical difficulties, we shall estimate

$$\mathcal{I}(X) = \int_{X/2}^{X} P(x)^2\,dx$$

rather than

$$\int_0^X P(x)^2\,dx,$$

and then add the results for $(X/2, X]$, $(X/4, X/2]$, $(X/8, X/4]$ .... Hardy [1916a] proved that

$$P(x) = x^{1/2} \sum_{n=1}^{\infty} \frac{r(n)}{n^{1/2}} J_1(2\pi\sqrt{nx}) \quad (x > 0), \tag{3-1}$$

where

$$J_1(y) = \frac{1}{\pi}\int_0^\pi \cos(nx - y\sin x)\,dx$$

is a Bessel function of the first kind. To estimate $\mathcal{I}(X)$ we use a truncated version of (3-1) due to Ivić [1996], which we state as:

Lemma 5. *Let $X \ge 2$ and $X \le N \le X^A$ with $A > 1$. For $x > 0$ define $R(x, N)$ by*

$$P(x) = x^{1/2} \sum_{n \le N} \frac{r(n)}{n^{1/2}} J_1(2\pi\sqrt{nx}) + R(x, N). \tag{3-2}$$

Then for $X/2 \le x \le X$ and any $\epsilon > 0$, we have

$$R(x, N) \ll \begin{cases} x^{\epsilon} & \textit{always}, \\ \left(\frac{x}{N}\right)^{1/2} \frac{x^{\epsilon}}{\|x\|} + x^{3/4}\left(\frac{x}{N}\right)^{1/2} + \left(\frac{x}{N}\right)^{1/4} & \textit{if } x \notin \mathbb{Z}. \end{cases}$$

If we take $X/2 \le x \le X$ and impose the condition that $X^4 \le N \le X^A$ in Lemma 5, then

$$R(x, N) \ll \min\left\{x^{\epsilon},\ \left(\frac{x}{N}\right)^{1/2} \frac{x^{\epsilon}}{\|x\|} + \left(\frac{x}{N}\right)^{1/4}\right\}$$

and we easily see that

$$\int_{X/2}^{X} |R(x, N)|^2 \, dx \ll X^{3/2+\epsilon} N^{-1/2}.$$

From (1-2) and the known bounds for $Q(X)$, the mean square of the main term in (3-2) over $[X/2, X]$ is $O(X^{3/2})$. Thus, by the Cauchy–Schwarz inequality the contribution of the cross term to the mean square of (3-2) is $O(X^{3/2+\epsilon/2} N^{-1/4})$. Thus, if $N \ge X^4$,

$$\mathcal{I}(X) = \int_{X/2}^{X} \left(x^{1/2} \sum_{n \le N} \frac{r(n)}{n^{1/2}} J_1(2\pi\sqrt{nx})\right)^2 dx + O(X^{1/2+\epsilon}).$$

Of course, if we take N even larger, the error term will be smaller.

Next we use the approximation

$$J_1(u) = \left(\frac{2}{\pi u}\right)^{1/2} \left(\cos\left(u - \frac{3\pi}{4}\right) - \frac{3}{8u} \sin\left(u - \frac{3\pi}{4}\right)\right) + O(u^{-5/2}), \tag{3-3}$$

which is valid for $u \ge 1$, and find that

$$\begin{aligned} \mathcal{I}(X) &= \frac{1}{\pi^2} \int_{X/2}^{X} \left(x^{1/4} \sum_{n \le N} \frac{r(n)}{n^{3/4}} \cos\left(2\pi\sqrt{nx} - \frac{3\pi}{4}\right) - \frac{3}{16\pi x^{1/4}} \sum_{n \le N} \frac{r(n)}{n^{5/4}} \sin\left(2\pi\sqrt{nx} - \frac{3\pi}{4}\right)\right. \\ &\qquad\qquad \left. + O\left(\frac{1}{x^{3/4}} \sum_{n \le N} \frac{r(n)}{n^{7/4}}\right)\right)^2 dx + O(X^{1/2+\epsilon}) \\ &= \frac{1}{\pi^2} \int_{X/2}^{X} (A_1(x) - A_2(x) + A_3(x))^2 \, dx + O(X^{1/2+\epsilon}). \end{aligned} \tag{3-4}$$

The sums in $A_2(x)$ and $A_3(x)$ are $O(1)$ uniformly in N and x so

$$\int_{X/2}^{X} A_2(x)^2 \, dx \ll X^{1/2} \quad \text{and} \quad \int_{X/2}^{X} A_3(x)^2 \, dx \ll X^{-1/2}.$$

By (1-2) we therefore have

$$\int_{X/2}^{X} A_1(x)^2 \, dx \ll X^{3/2}.$$

From these estimates and the Cauchy–Schwarz inequality we now have

$$\mathcal{I}(X)=\frac{1}{\pi^2}\int_{X/2}^{X} A_1(x)^2-2A_1(x)A_2(x)\,dx+O(X^{1/2+\epsilon}). \tag{3-5}$$

Note that if we were to apply the Cauchy–Schwarz inequality to the integral of $A_1(x)A_2(x)$, we would obtain the estimate $O(X)$ for this term, which is the expected size of the lowest-order term in our main term. By isolating this term and treating it with a little more care, we shall show that it is in fact $O(X^{1/2}\log X)$.

The sums in the definitions of $A_1(x)$ and $A_2(x)$ contain trigonometric rather than Bessel functions. This makes it convenient to again work with integrals over $[0, X]$ rather than $[X/2, X]$, and then to take the difference of the results for $[0, X]$ and $[0, X/2]$ at the end of the argument. To proceed we use the identity

$$\cos a\ \cos b=\Re\tfrac{1}{2}[\exp(i(a-b))+\exp(i(a+b))]$$

and obtain

$$\begin{aligned}\frac{1}{\pi^2}\int_0^X A_1(x)^2\,dx&=\Re\frac{1}{2\pi^2}\sum_{m,n\le N}\frac{r(n)r(m)}{(mn)^{3/4}}\int_0^X x^{1/2}\exp(2\pi i\sqrt{x}(\sqrt{n}-\sqrt{m}))\,dx\\&\qquad+\Re\frac{i}{2\pi^2}\sum_{m,n\le N}\frac{r(n)r(m)}{(mn)^{3/4}}\int_0^X x^{1/2}\exp(2\pi i\sqrt{x}(\sqrt{n}+\sqrt{m}))\,dx\\&=\mathrm{I}(X)+\mathrm{J}(X).\end{aligned} \tag{3-6}$$

When $y\neq 0$, a substitution and two integrations by parts shows that

$$\int_0^X x^{1/2}\exp(2\pi iy\sqrt{x})\,dx=e^{2\pi iy\sqrt{X}}\left(\frac{X}{i\pi y}+\frac{\sqrt{X}}{\pi^2y^2}-\frac{1}{i2\pi^3y^3}\right)+\frac{1}{i2\pi^3y^3}.$$

When $y=0$ the integral equals $\frac{2}{3}X^{3/2}$ trivially. Thus

$$\begin{aligned}\mathrm{I}(X)&=\frac{X^{3/2}}{3\pi^2}\sum_{n\le N}\frac{r(n)^2}{n^{3/2}}+\frac{1}{2\pi^2}\sum_{m\neq n\le N}\frac{r(n)r(m)}{(mn)^{3/4}}i_X(\sqrt{m}-\sqrt{n}),\\ \mathrm{J}(X)&=\frac{1}{2\pi^2}\sum_{m,n\le N}\frac{r(n)r(m)}{(mn)^{3/4}}\mathrm{j}_X(\sqrt{m}+\sqrt{n}),\end{aligned}$$

where

$$\begin{aligned}\mathrm{i}_X(y)&=\frac{X\sin(2\pi y\sqrt{X})}{\pi y}+\frac{\sqrt{X}\cos(2\pi y\sqrt{X})}{\pi^2y^2}-\frac{\sin(2\pi y\sqrt{X})}{2\pi^3y^3},\\ \mathrm{j}_X(y)&=\frac{X\cos(2\pi y\sqrt{X})}{\pi y}-\frac{\sqrt{X}\sin(2\pi y\sqrt{X})}{\pi^2y^2}+\frac{1-\cos(2\pi y\sqrt{X})}{2\pi^3y^3}.\end{aligned} \tag{3-7}$$

Similarly, using

$$\cos a\ \sin b=-\Im\tfrac{1}{2}[\exp(i(a-b))-\exp(i(a+b))],$$

we obtain

$$\frac{1}{\pi^2}\int_0^X A_1(x)A_2(x)\,dx = -\Im\frac{3}{32\pi^3}\sum_{m,n\le N}\frac{r(n)r(m)}{n^{3/4}m^{5/4}}\int_0^X \exp(2\pi i\sqrt{x}(\sqrt{n}-\sqrt{m}))\,dx + \Im\frac{3i}{32\pi^3}\sum_{m,n\le N}\frac{r(n)r(m)}{n^{3/4}m^{5/4}}\int_0^X \exp(2\pi i\sqrt{x}(\sqrt{n}+\sqrt{m}))\,dx$$

$$= \mathrm{K}(X)+\mathrm{L}(X). \tag{3-8}$$

When $y\neq 0$,

$$\int_0^X \exp(2\pi i y\sqrt{x})\,dx = e^{2\pi i y\sqrt{X}}\left(\frac{\sqrt{X}}{i\pi y}+\frac{1}{2\pi^2y^2}\right)-\frac{1}{2\pi^2y^2},$$

whereas when $y=0$ the integral equals X. Thus

$$\mathrm{K}(X) = \frac{3}{32\pi^3}\sum_{m\neq n\le N}\frac{r(n)r(m)}{n^{3/4}m^{5/4}}\mathrm{k}_X(\sqrt{n}-\sqrt{m}),$$

$$\mathrm{L}(X) = \frac{3}{32\pi^3}\sum_{m,n\le N}\frac{r(n)r(m)}{n^{3/4}m^{5/4}}\mathrm{l}_X(\sqrt{n}+\sqrt{m}),$$

where

$$\mathrm{k}_X(y) = \frac{\sqrt{X}\cos(2\pi y\sqrt{X})}{\pi y}-\frac{\sin(2\pi y\sqrt{X})}{2\pi^2y^2},$$

$$\mathrm{l}_X = \frac{\sqrt{X}\sin(2\pi y\sqrt{X})}{\pi y}+\frac{\cos(2\pi y\sqrt{X})-1}{2\pi^2y^2}.$$

In Sections 4–7 we estimate $\mathrm{I}(X)$, $\mathrm{J}(X)$, $\mathrm{K}(X)$, and $\mathrm{L}(X)$. Our main terms come from $\mathrm{I}(X)$, which also requires the lengthiest treatment.

4. Calculation of I(X)

We write

$$\mathrm{I}(X) = \frac{X^{3/2}}{3\pi^2}\sum_{n\le N}\frac{r(n)^2}{n^{3/2}}+\frac{1}{2\pi^2}\sum_{m\neq n\le N}\frac{r(n)r(m)}{(mn)^{3/4}}\mathrm{i}_X(\sqrt{m}-\sqrt{n}) = \mathrm{I_D}+\mathrm{I_O}. \tag{4-1}$$

Setting

$$C = \frac{1}{3\pi^2}\sum_{n=1}^{\infty}\frac{r(n)^2}{n^{3/2}},$$

we see that since $r(n)\ll_\epsilon n^\epsilon$ for any $\epsilon>0$ and $N\ge X^4$,

$$\mathrm{I_D} = CX^{3/2}+O_\epsilon(X^{3/2}N^{-1/2+\epsilon}) = CX^{3/2}+O(1). \tag{4-2}$$

To evaluate $\mathrm{I_O}(X)$ we write $m=n+h$ and use the symmetry in m and n to see that

$$\mathrm{I_O} = \frac{1}{\pi^2}\sum_{n<N}\sum_{1\le h\le N-n}\frac{r(n)r(n+h)}{(n(n+h))^{3/4}}\mathrm{i}_X(\sqrt{n+h}-\sqrt{n}).$$

We replace $\sqrt{n+h}-\sqrt{n}$ by the approximation $h/(2\sqrt{n})$, and replace $n+h$ by n in the denominator to obtain

$$\mathrm{I_O} \approx \frac{1}{\pi^2}\sum_{n<N}\sum_{1\le h\le N-n}\frac{r(n)r(n+h)}{n^{3/2}}\mathrm{i}_X(h/(2\sqrt{n})); \tag{4-3}$$

here and below $A \approx B$ means that $A = B + E$, with E an error term of order less than B. This is the first place in the argument where we have abandoned rigor. F. Chamizo [1999, Corollary 5.3] has shown that

$$\sum_{n\le x} r(n)r(n+h) = c(h)x + E(x,h),$$

where

$$c(h) = \frac{8(-1)^h}{h}\Bigl(\sum_{d|h}(-1)^d d\Bigr) \tag{4-4}$$

and

$$E(x,h) \ll_\epsilon x^{145/196+\epsilon}$$

uniformly for $h \le x$. This suggests that we may replace $r(n)r(n+h)$ in (4-3) by $c(h)$. We shall ignore the error terms; in a rigorous analysis, these would swamp our expected main terms. However, it is plausible to assume that the error terms for various h are independent and largely cancel one another. Supposing this to be the case and replacing the sum over n by an integral, we find that

$$\mathrm{I_O} \approx \frac{1}{\pi^2}\int_0^N\Bigl(\sum_{h=1}^{\infty} c(h)\mathrm{i}_X(h/(2\sqrt{u}))\Bigr)\frac{du}{u^{3/2}}.$$

From the definition of $c(h)$ in (4-4) we see that

$$\begin{aligned}\mathrm{I_O} &\approx \frac{8}{\pi^2}\int_0^N\Bigl(\sum_{h=1}^{\infty}\frac{(-1)^h}{h}\Bigl(\sum_{d|h}(-1)^d d\Bigr)\mathrm{i}_X(h/(2\sqrt{u}))\Bigr)\frac{du}{u^{3/2}}\\ &= \frac{8}{\pi^2}\int_0^N\sum_{k=1}^{\infty}\sum_{d=1}^{\infty}\frac{(-1)^{d(k+1)}}{k}\mathrm{i}_X(dk/(2\sqrt{u}))\frac{du}{u^{3/2}}.\end{aligned} \tag{4-5}$$

By (3-7) the sum over d is

$$\begin{aligned}&\sum_{d=1}^{\infty}(-1)^{d(k+1)}\mathrm{i}_X(dk/(2\sqrt{u}))\\ &\quad= \frac{2X\sqrt{u}}{k}\sum_{d=1}^{\infty}(-1)^{d(k+1)}\frac{\sin(\pi dk\sqrt{X/u})}{\pi d} + \frac{4u\sqrt{X}}{k^2}\sum_{d=1}^{\infty}(-1)^{d(k+1)}\frac{\cos(\pi dk\sqrt{X/u})}{\pi^2 d^2}\\ &\qquad - \frac{4u^{3/2}}{k^3}\sum_{d=1}^{\infty}(-1)^{d(k+1)}\frac{\sin(\pi dk\sqrt{X/u})}{\pi^3 d^3}.\end{aligned}$$

Using (A-7) of the Appendix to express these sums in terms of Bernoulli polynomials, we find that if k is odd, the right-hand side equals

$$-\frac{2X\sqrt{u}}{k}B_1(\{k\sqrt{X/4u}\}) + \frac{4u\sqrt{X}}{k^2}B_2(\{k\sqrt{X/4u}\}) - \frac{8u^{3/2}}{3k^3}B_3(\{k\sqrt{X/4u}\}),$$

and if k is even, it equals

$$-\frac{2X\sqrt{u}}{k}B_1\big(\{k\sqrt{X/4u}+\tfrac{1}{2}\}\big)+\frac{4u\sqrt{X}}{k^2}B_2\big(\{k\sqrt{X/4u}+\tfrac{1}{2}\}\big)-\frac{8u^{3/2}}{3k^3}B_3\big(\{k\sqrt{X/4u}+\tfrac{1}{2}\}\big).$$

Inserting these into (4-5), we obtain

$$\begin{aligned}\mathrm{I_O} &\approx \frac{8}{\pi^2}\int_0^N \sum_{\substack{k=1\\ k\text{ odd}}}^{\infty}\Big(-\frac{2X\sqrt{u}}{k^2}B_1(\{k\sqrt{X/4u}\})+\frac{4u\sqrt{X}}{k^3}B_2(\{k\sqrt{X/4u}\})-\frac{8u^{3/2}}{3k^4}B_3(\{k\sqrt{X/4u}\})\Big)\frac{du}{u^{3/2}}\\ &\quad+\frac{8}{\pi^2}\int_0^N \sum_{\substack{k=1\\ k\text{ even}}}^{\infty}\Big(-\frac{2X\sqrt{u}}{k^2}B_1(\{k\sqrt{X/4u}+\tfrac{1}{2}\})+\frac{4u\sqrt{X}}{k^3}B_2\big(\{k\sqrt{X/4u}+\tfrac{1}{2}\}\big)\\ &\qquad\qquad -\frac{8u^{3/2}}{3k^4}B_3\big(\{k\sqrt{X/4u}+\tfrac{1}{2}\}\big)\Big)\frac{du}{u^{3/2}}\\ &=\mathrm{I_{O,odd}}+\mathrm{I_{O,even}}.\end{aligned}\tag{4-6}$$

In both integrals we make the substitution $x=k\sqrt{X/4u}$ so that $2\sqrt{u}/k=\sqrt{X}/x$ and $du/u^{3/2}=-4\,dx/(k\sqrt{X})$. We then find that

$$\mathrm{I_{O,odd}}=\frac{32X}{\pi^2}\sum_{\substack{k=1\\ k\text{ odd}}}^{\infty}\frac{1}{k^2}\int_{k\sqrt{X/4N}}^{\infty}\Big(-\frac{B_1(\{x\})}{x}+\frac{B_2(\{x\})}{x^2}-\frac{B_3(\{x\})}{3x^3}\Big)\,dx,$$

$$I_{\mathrm{O,even}}=\frac{32X}{\pi^2}\sum_{\substack{k=1\\ k\text{ even}}}^{\infty}\frac{1}{k^2}\int_{k\sqrt{X/4N}}^{\infty}\Big(-\frac{B_1\big(\{x+\frac{1}{2}\}\big)}{x}+\frac{B_2\big(\{x+\frac{1}{2}\}\big)}{x^2}-\frac{B_3\big(\{x+\frac{1}{2}\}\big)}{3x^3}\Big)\,dx.$$

4.1. ***Calculation of $I_{\mathrm{O,odd}}$.*** We have

$$\mathrm{I_{O,odd}}=\frac{32X}{\pi^2}\sum_{\substack{k=1\\ k\text{ odd}}}^{\infty}\frac{1}{k^2}\int_{k\sqrt{X/4N}}^{\infty}\Big(-\frac{B_1(\{x\})}{x}+\frac{B_2(\{x\})}{x^2}-\frac{B_3(\{x\})}{3x^3}\Big)\,dx.$$

We split the sum over k into two sums according to whether $k\le 2\sqrt{N/X}$ or $k>2\sqrt{N/X}$. For terms in the second sum we have

$$\int_{k\sqrt{X/4N}}^{\infty}\frac{B_j(\{x\})}{x^j}\,dx \ll (k\sqrt{X/4N})^{-j}.$$

Hence, the contribution from the second sum is

$$\ll X\sum_{j=1}^{3}\sum_{k>2\sqrt{N/X}}\frac{1}{k^{j+2}}(N/X)^{j/2}\ll X^{3/2}N^{-1/2}.$$

Thus

$$\mathrm{I}_{O,\mathrm{odd}}=\frac{32X}{\pi^2}\sum_{\substack{k\le 2\sqrt{N/X}\\ k\text{ odd}}}\frac{1}{k^2}\int_{k\sqrt{X/4N}}^{\infty}\Big(-\frac{B_1(\{x\})}{x}+\frac{B_2(\{x\})}{x^2}-\frac{B_3(\{x\})}{3x^3}\Big)\,dx+O(X^{3/2}N^{-1/2}).$$

In the remaining sum $k\sqrt{X/4N} \le 1$ and we split the integral into two parts, one over $[k\sqrt{X/4N}, 1]$ and the other over $[1, \infty)$. By (A-6) of the Appendix the first integral is

$$\int_{k\sqrt{X/4N}}^{1} \left(\frac{\frac{1}{2} - x}{x} + \frac{x^2 - x + \frac{1}{6}}{x^2} - \frac{x^3 - \frac{3}{2}x^2 + \frac{1}{2}x}{3x^3} \right) dx = -\tfrac{1}{3} + O(k\sqrt{X/N}).$$

By Lemma 8

$$\int_{1}^{\infty} \left(-\frac{B_1(\{x\})}{x} + \frac{B_2(\{x\})}{x^2} - \frac{B_3(\{x\})}{3x^3} \right) dx = -\left(\tfrac{1}{2}\log 2\pi - 1\right) + \left(\log 2\pi - \tfrac{11}{6}\right) - \tfrac{1}{3}\left(\tfrac{3}{2}\log 2\pi - \tfrac{11}{4}\right) = \tfrac{1}{12}.$$

Hence

$$\mathrm{I}_{\mathrm{O,odd}} = \frac{32X}{\pi^2} \sum_{\substack{k \le 2\sqrt{N/X} \\ k \text{ odd}}} \frac{1}{k^2}\left(-\tfrac{1}{4} + O(k\sqrt{X/N})\right) + O(X^{3/2}N^{-1/2}).$$

The contribution of the O-term in the sum is $O(X^{3/2}N^{-1/2}\log(N/X))$. Hence

$$\begin{aligned} \mathrm{I}_{\mathrm{O,odd}} &= -\frac{8X}{\pi^2} \sum_{\substack{k \le 2\sqrt{N/X} \\ k \text{ odd}}} \frac{1}{k^2} + O(X^{3/2}N^{-1/2}\log(N/X)) \\ &= -\frac{8X}{\pi^2} \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{1}{k^2} + O(X^{3/2}N^{-1/2}\log(N/X)). \qquad (4\text{-}7) \end{aligned}$$

4.2. ***Calculation of*** $\mathbf{I}_{\mathbf{O,even}}$**.** The treatment of $\mathrm{I}_{\mathrm{O,even}}$ is similar to that of $\mathrm{I}_{\mathrm{O,odd}}$ so we will skip some of the details. We have

$$\mathrm{I}_{\mathrm{O,even}} = \frac{32X}{\pi^2} \sum_{\substack{k=1 \\ k \text{ even}}}^{\infty} \frac{1}{k^2} \int_{k\sqrt{X/4N}}^{\infty} \left(-\frac{B_1\left(\{x+\frac{1}{2}\}\right)}{x} + \frac{B_2\left(\{x+\frac{1}{2}\}\right)}{x^2} - \frac{B_3\left(\{x+\frac{1}{2}\}\right)}{3x^3} \right) dx.$$

Our first step is to split the sum over k according to whether $k \le \sqrt{N/X}$ or $k > \sqrt{N/X}$ (note that the division for odd k was at $2\sqrt{N/X}$). As before, the total contribution from the tail is $X^{3/2}N^{-1/2}$. Thus,

$$I_{\mathrm{O,even}} = \frac{32X}{\pi^2} \sum_{\substack{k \le \sqrt{N/X} \\ k \text{ even}}} \frac{1}{k^2} \int_{k\sqrt{X/4N}}^{\infty} \left(-\frac{B_1\left(\{x+\frac{1}{2}\}\right)}{x} + \frac{B_2\left(\{x+\frac{1}{2}\}\right)}{x^2} - \frac{B_3\left(\{x+\frac{1}{2}\}\right)}{3x^3} \right) dx + O(X^{3/2}N^{-1/2}).$$

Observe that for each k in the sum we have $k\sqrt{X/4N} \le \frac{1}{2}$. We may therefore split the integral over the intervals $\left[k\sqrt{X/4N}, \frac{1}{2}\right]$ and $\left[\frac{1}{2}, \infty\right)$. By (A-6) the first integral equals

$$\begin{aligned} &\int_{k\sqrt{X/4N}}^{1/2} \left(-\frac{x}{x} + \frac{\left(x+\frac{1}{2}\right)^2 - \left(x+\frac{1}{2}\right) + \frac{1}{6}}{x^2} - \frac{\left(x+\frac{1}{2}\right)^3 - \frac{3}{2}\left(x+\frac{1}{2}\right)^2 + \frac{1}{2}\left(x+\frac{1}{2}\right)}{3x^3} \right) dx \\ &\qquad = -\int_{k\sqrt{X/4N}}^{1/2} \tfrac{1}{3}\, dx = -\tfrac{1}{6} + O(k\sqrt{X/N}). \end{aligned}$$

By Lemma 8

$$\int_{1/2}^{\infty}\Big(-\frac{B_1(\{x+\frac12\})}{x}+\frac{B_2(\{x+\frac12\})}{x^2}-\frac{B_3(\{x+\frac12\})}{3x^3}\Big)\,dx$$
$$=-\big(-\tfrac12+\tfrac12\log 2\big)+\big(-\tfrac23+\log 2\big)-\tfrac13\big(-1+\tfrac32\log 2\big)=\tfrac16.$$

Hence,

$$\mathrm{I}_{\mathrm{O,even}}=\frac{32X}{\pi^2}\sum_{\substack{k\le\sqrt{N/X}\\ k\text{ even}}}\frac{1}{k^2}\big(-\tfrac16+\tfrac16+O(k\sqrt{X/N})\big)+O(X^{3/2}N^{-1/2})$$
$$\ll X^{3/2}N^{-1/2}\log(N/X). \tag{4-8}$$

4.3. ***Completion of the calculation of*** $\mathbf{I}(X)$**.** By (4-6), (4-7), and (4-8), we see that

$$\mathrm{I}_{\mathrm{O}}=-\frac{8X}{\pi^2}\sum_{\substack{k=1\\ k\text{ odd}}}^{\infty}\frac{1}{k^2}+O(X^{3/2}N^{-1/2}\log(N/X)).$$

Combining this with (4-1) and (4-2), we see that

$$\mathrm{I}(X)=CX^{3/2}-\frac{8X}{\pi^2}\sum_{\substack{k=1\\ k\text{ odd}}}^{\infty}\frac{1}{k^2}+O(X^{3/2}N^{-1/2}\log(N/X)),$$

where

$$C=\frac{1}{3\pi^2}\sum_{n=1}^{\infty}\frac{r(n)^2}{n^{3/2}}.$$

It is easy to see that

$$\sum_{\substack{k=1\\ k\text{ odd}}}^{\infty}\frac{1}{k^2}=\frac34\zeta(2)=\frac{\pi^2}{8}.$$

Using this and the assumption that $X^4\le N\le X^A$, we find that

$$\mathrm{I}(X)=CX^{3/2}-X+O(X^{1/2}\log X). \tag{4-9}$$

5. Calculation of $\mathbf{J}(X)$

Our treatment of $\mathrm{J}(X)$ is easier. We have

$$\mathrm{J}(X)=\frac{1}{2\pi^2}\sum_{m,n\le N}\frac{r(n)r(m)}{(mn)^{3/4}}\mathrm{j}_X(\sqrt{m}+\sqrt{n}),$$

where

$$\mathrm{j}_X(y)=\frac{X\cos(2\pi y\sqrt{X})}{\pi y}-\frac{\sqrt{X}\sin(2\pi y\sqrt{X})}{\pi^2y^2}+\frac{1-\cos(2\pi y\sqrt{X})}{2\pi^3y^3}.$$

The second and third terms of j_X contribute

$$\ll\sum_{m,n\le N}\frac{r(n)r(m)}{(mn)^{3/4}}\Big(\frac{\sqrt{X}}{(\sqrt{m}+\sqrt{n})^2}+\frac{1}{(\sqrt{m}+\sqrt{n})^3}\Big).$$

Since $(\sqrt{m}+\sqrt{n})^2 > 2\sqrt{mn}$, this is

$$\ll \sqrt{X} \sum_{m,n\le N} \frac{r(n)r(m)}{(mn)^{5/4}} + \sum_{m,n\le N} \frac{r(n)r(m)}{(mn)^{3/2}} \ll \sqrt{X}.$$

Hence, recalling (1-7),

$$\begin{aligned} \mathrm{J}(X) &= \frac{X}{2\pi^3} \sum_{m,n\le N} \frac{r(n)r(m)}{(mn)^{3/4}} \frac{\cos(2\pi(\sqrt{m}+\sqrt{n})\sqrt{X})}{(\sqrt{m}+\sqrt{n})} + O(X^{1/2}) \\ &= XC_N(X) + O(X^{1/2}). \end{aligned} \tag{5-1}$$

6. Estimation of K(X)

We have

$$\mathrm{K}(X) = \frac{3}{32\pi^3} \sum_{m\ne n\le N} \frac{r(n)r(m)}{n^{3/4}m^{5/4}} \mathrm{k}_X(\sqrt{n}-\sqrt{m}),$$

where

$$\mathrm{k}_X(y) = \frac{\sqrt{X}\cos(2\pi y\sqrt{X})}{\pi y} - \frac{\sin(2\pi y\sqrt{X})}{2\pi^2 y^2}.$$

Thus

$$\mathrm{K}(X) \ll \sum_{m\ne n\le N} \frac{r(n)r(m)}{n^{3/4}m^{5/4}} \left(\frac{\sqrt{X}}{|\sqrt{n}-\sqrt{m}|} + \frac{1}{(\sqrt{n}-\sqrt{m})^2} \right).$$

We split the sum over m and n according to whether $m < n$ or $m > n$ so that

$$\mathrm{K}(X) \ll \left(\sum_{n<N} \sum_{n<m\le N} + \sum_{m<N} \sum_{m<n\le N} \right) \cdots = K_1 + K_2.$$

In K_1 we write $m = n+h$ and further split the sum over h as

$$\begin{aligned} K_1 &= \sum_{n<N} \left(\sum_{1\le h\le n/2} + \sum_{n/2<h\le N-n} \right) \frac{r(n)r(n+h)}{n^{3/4}(n+h)^{5/4}} \left(\frac{\sqrt{X}}{|\sqrt{n+h}-\sqrt{n}|} + \frac{1}{(\sqrt{n+h}-\sqrt{n})^2} \right) \\ &= K_{11} + K_{12}. \end{aligned}$$

In K_{11} we use $\sqrt{n+h}-\sqrt{n} \gg h/\sqrt{n}$ and $r(n) \ll n^\epsilon$ and find for ϵ small enough that

$$\begin{aligned} K_{11} &\ll \sum_{n<N} \sum_{1\le h\le n/2} \frac{r(n)r(n+h)}{n^{3/4}(n+h)^{5/4}} \left(\frac{\sqrt{nX}}{h} + \frac{n}{h^2} \right) \\ &\ll \sqrt{X} \sum_{n<N} \frac{1}{n^{3/2-2\epsilon}} \sum_{1\le h\le n/2} \frac{1}{h} + \sum_{n<N} \frac{1}{n^{1-2\epsilon}} \sum_{1\le h\le n/2} \frac{1}{h^2} \\ &\ll \sqrt{X} + N^{2\epsilon} \ll \sqrt{X}. \end{aligned}$$

In K_{12} we use the inequalities $\sqrt{n+h}-\sqrt{n} \gg \sqrt{h}$, $(n+h)^{5/4} \gg h^{5/4}$ and $r(n) \ll n^{\epsilon}$ and find that

$$
\begin{aligned}
K_{12} &\ll \sum_{n<N} \sum_{n/2<h\le N-n} \frac{1}{n^{3/4-\epsilon} h^{5/4-\epsilon}} \left(\sqrt{\frac{X}{h}} + \frac{1}{h} \right) \\
&\ll \sqrt{X} \sum_{n<N} \frac{1}{n^{3/4-\epsilon}} \sum_{n/2<h\le N-n} \frac{1}{h^{7/4-\epsilon}} + \sum_{n<N} \frac{1}{n^{3/4-\epsilon}} \sum_{n/2<h\le N-n} \frac{1}{h^{9/4-\epsilon}} \\
&\ll \sqrt{X} \sum_{n<N} \frac{1}{n^{3/2-2\epsilon}} + \sum_{n<N} \frac{1}{n^{2-2\epsilon}} \ll \sqrt{X}.
\end{aligned}
$$

Hence, $K_1 = K_{11} + K_{12} = O(\sqrt{X})$.

We treat K_2 in the same way and find that it is also $O(\sqrt{X})$. Thus

$$\mathrm{K}(X) \ll \sqrt{X}.$$

7. Estimation of L(X)

We treat $\mathrm{L}(X)$ as we did the last two terms in $\mathrm{J}(X)$. We have

$$\mathrm{L}(X) = \frac{3}{32\pi^3} \sum_{m,n\le N} \frac{r(n)r(m)}{n^{3/4}m^{5/4}} \, \mathrm{l}_X(\sqrt{n}+\sqrt{m}),$$

where

$$\mathrm{l}_X = \frac{\sqrt{X}\sin(2\pi y\sqrt{X})}{\pi y} + \frac{\cos(2\pi y\sqrt{X})-1}{2\pi^2 y^2}.$$

Using the inequality $(\sqrt{m}+\sqrt{n}) > 2\sqrt{mn}$, we find that

$$
\begin{aligned}
\mathrm{L}(X) &\ll \sum_{m,n\le N} \frac{r(n)r(m)}{n^{3/4}m^{5/4}} \left(\frac{\sqrt{X}}{(mn)^{1/4}} + \frac{1}{(mn)^{1/2}} \right) \\
&\ll \sqrt{X} \sum_{n\le N} \frac{r(n)}{n} \sum_{m\le N} \frac{r(m)}{m^{3/2}} + 1 \\
&\ll \sqrt{X} \log X.
\end{aligned}
$$

8. Completion of the argument for the Conjecture

By (3-6), (4-9), and (5-1)

$$\frac{1}{\pi^2} \int_0^X A_1(x)^2 \, dx = \mathrm{I}(X) + \mathrm{J}(X) = CX^{3/2} - X + C_N(X) + O(X^{1/2+\epsilon}).$$

By (3-8) and the estimates of the last two sections

$$\frac{1}{\pi^2} \int_0^X A_1(x)A_2(x) \, dx = K(X) + L(X) \ll \sqrt{X} \log X.$$

It now follows from (3-5) that

$$\mathcal{I}(X) = \int_{X/2}^{X} P(x)^2\,dx = C(X^{3/2} - (X/2)^{3/2}) - X/2 + (C_N(X) - C_N(X/2)) + O(X^{1/2+\epsilon}),$$

where

$$C = \frac{1}{3\pi^2}\sum_{n=1}^{\infty} r^2(n) n^{-3/2}.$$

We add this result (with the same value of $N \in [X^4, X^A]$) for the intervals $(X/2, X]$, $(X/4, X/2]$, ..., $(X/2^r, X/2^{r-1}]$, where $r = [3\log X/4\log 2]$. Then

$$X^{1/4} \le \frac{X}{2^r} < 2X^{1/4}.$$

Now $(X^{1/4})^{3/2} = X^{3/8} \ll X^{1/2+\epsilon}$, so

$$X/2^r C_N(X/2^r) \ll X/2^r \log N \ll X^{1/2+\epsilon}.$$

Hence

$$\int_{X/2^r}^{X} P(x)^2\,dx = CX^{3/2} - X + C_N(X) + O(X^{1/2+\epsilon}). \tag{8-1}$$

Finally,

$$\int_0^{X/2^r} P(x)^2 \ll (X/2^r)^{3/2} \ll X^{3/8},$$

so the integral on the left-hand side of (8-1) may be extended over the entire interval $[0, X]$. This completes the argument for the Conjecture.

9. Proof of Theorem 2

For $N \ge 1$ and $x \in \mathbb{R}$ we have

$$C_N(x^2) = \frac{1}{2\pi^3}\sum_{m,n\le N} \frac{r(n)r(m)\cos(2\pi(\sqrt{m}+\sqrt{n})x)}{(mn)^{3/4}(\sqrt{m}+\sqrt{n})}.$$

Clearly

$$\max_{x\in\mathbb{R}} |C_N(x^2)| = C_N(0) = \frac{1}{2\pi^3}\sum_{m,n\le N} \frac{r(n)r(m)}{(mn)^{3/4}(\sqrt{m}+\sqrt{n})}. \tag{9-1}$$

Lemma 6. *For $N \ge 2$ we have*

$$\frac{2}{\pi}\log N + O(1) \le C_N(0) \le \frac{4}{\pi}\log N + O(1).$$

Proof. Let

$$\sum_{n\le x} r(n) = \pi x + E(x),$$

where $E(x) \ll x^{1/3}$. By Riemann–Stieltjes integration, for $y < z$ we have

$$\sum_{y<n\le z} \frac{r(n)}{n^\sigma} = \pi\frac{z^{1-\sigma} - y^{1-\sigma}}{1-\sigma} + \frac{E(u)}{u^\sigma}\bigg|_y^z + \sigma\int_y^z \frac{E(u)}{u^{1+\sigma}}\,du.$$

Here the case $\sigma = 1$ is interpreted as a limit. In particular,

$$\sum_{n \le z} \frac{r(n)}{n} = \pi \log z + O(1) \tag{9-2}$$

and

$$\sum_{y<n\le z} \frac{r(n)}{n^\sigma} = \pi \frac{z^{1-\sigma} - y^{1-\sigma}}{1-\sigma} + O(\max(y^{1/3-\sigma}, z^{1/3-\sigma})). \tag{9-3}$$

Using this and the symmetry in m and n in the double sum defining $C_N(x^2)$, we find that

$$C_N(0) = \frac{1}{\pi^3} \sum_{1\le n<N} \frac{r(n)}{n^{3/4}} \Bigg(\sum_{n<m\le N} \frac{r(m)}{m^{3/4}(\sqrt{m}+\sqrt{n})} \Bigg) + O(1).$$

Now

$$\frac{1}{2} \sum_{n<m\le N} \frac{r(m)}{m^{5/4}} \le \sum_{n<m\le N} \frac{r(m)}{m^{3/4}(\sqrt{m}+\sqrt{n})} \le \sum_{n<m\le N} \frac{r(m)}{m^{5/4}},$$

so by (9-3) we have

$$2\pi n^{-1/4} + O(n^{-11/12}) \le \sum_{n<m\le N} \frac{r(m)}{m^{3/4}(\sqrt{m}+\sqrt{n})} \le 4\pi n^{-1/4} + O(n^{-11/12}).$$

Thus,

$$C_N(0) \le \frac{1}{\pi^3} \sum_{1\le n<N} \frac{r(n)}{n^{3/4}} (4\pi n^{-1/4} + O(n^{-11/12})) = \frac{4}{\pi} \log N + O(1).$$

Similarly,

$$C_N(0) \ge \frac{2}{\pi} \log N + O(1).$$

This completes the proof of the lemma. □

To prove Theorem 2, we note that by (1-8)

$$|C(X)| = |C_N(X)| + O\bigg(\frac{X \log N}{N^{1/4}}\bigg).$$

Thus, by (9-1) and Lemma 6

$$|C(X)| \le \frac{4}{\pi} \log N + O\bigg(\frac{X \log N}{N^{1/4}}\bigg) + O(1).$$

Taking $N = X^4 \log X$, say, we obtain

$$|C(X)| \le \Big(\frac{16}{\pi} + o(1)\Big) \log X.$$

This proves the first assertion of Theorem 2. The second follows from this and the Conjecture.

10. Proof of Theorem 3

We base the proof of Theorem 3 on a variant of a lemma of [Soundararajan 2003].

For each $\boldsymbol{n} = (m, n) \in \mathbb{Z}^2$ let $a_{\boldsymbol{n}}$ and $\lambda_{\boldsymbol{n}}$ be nonnegative real numbers with the $\lambda_{\boldsymbol{n}}$ arranged in nondecreasing order. Assume that $\sum_{\boldsymbol{n}} a_{\boldsymbol{n}} < \infty$ and set

$$F(x) = \sum_{\boldsymbol{n}} a_{\boldsymbol{n}} \cos(2\pi \lambda_{\boldsymbol{n}} x).$$

Lemma 7. *Let $L \ge 2$ be an integer and let $\Lambda \ge 2$ be a real number. Let $\mathcal{M}$ be a subset of the double indices $\boldsymbol{n}$ for which $\lambda_{\boldsymbol{n}} \le \Lambda/2$, and let M be the cardinality of $\mathcal{M}$. Then for any real number $Y \ge 2$ there exists an x such that $Y/2 \le x \le (6L)^{M+1}Y$ and*

$$F(x) \ge \frac{1}{8} \sum_{\boldsymbol{n}\in\mathcal{M}} a_{\boldsymbol{n}} - \frac{1}{L-1} \sum_{\substack{\boldsymbol{n} \\ \lambda_{\boldsymbol{n}} \le \Lambda}} a_{\boldsymbol{n}} - \frac{4}{\pi^2 \Lambda Y} \sum_{\boldsymbol{n}} a_{\boldsymbol{n}}. \tag{10-1}$$

Proof. Let $K(u) = (\sin \pi u/\pi u)^2$ be Fejér's kernel and let $k(y) = \max(0, 1 - |y|)$ be its Fourier transform. Then the Fourier transform of $\Lambda K(\Lambda u)$ is $k(y/\Lambda) = \max(0, 1 - |y|/\Lambda)$. Consider the integral

$$\begin{aligned}\int_{-\infty}^{\infty} \Lambda K(\Lambda u) F(u+x)\, du &= \frac{1}{2} \sum_{\boldsymbol{n}} a_{\boldsymbol{n}} \int_{-\infty}^{\infty} \Lambda K(\Lambda u)(e^{2\pi i \lambda_{\boldsymbol{n}}(x+u)} + e^{-2\pi i \lambda_{\boldsymbol{n}}(x+u)})\, du \\ &= \frac{1}{2} \sum_{\boldsymbol{n}} a_{\boldsymbol{n}} e^{2\pi i \lambda_{\boldsymbol{n}} x} k(-\lambda_{\boldsymbol{n}}/\Lambda) + \frac{1}{2} \sum_{\boldsymbol{n}} a_{\boldsymbol{n}} e^{-2\pi i \lambda_{\boldsymbol{n}} x} k(\lambda_{\boldsymbol{n}}/\Lambda) \\ &= \sum_{\boldsymbol{n}} a_{\boldsymbol{n}} \cos(2\pi \lambda_{\boldsymbol{n}} x) k(\lambda_{\boldsymbol{n}}/\Lambda),\end{aligned}$$

where the last equality holds because $k(-y) = k(y)$. Defining

$$G(x) = \sum_{\boldsymbol{n}} a_{\boldsymbol{n}} \cos(2\pi \lambda_{\boldsymbol{n}} x) k(\lambda_{\boldsymbol{n}}/\Lambda),$$

we may write this as

$$\int_{-\infty}^{\infty} \Lambda K(\Lambda u) F(u+x)\, du = G(x).$$

Since F, G, and K are real-valued functions, and K and Λ are nonnegative, we find next that

$$\begin{aligned}G(x) &= \int_{-Y/2}^{Y/2} \Lambda K(\Lambda u) F(u+x)\, du + \int_{|u|>Y/2} \Lambda K(\Lambda u) F(u+x)\, du \\ &\le \max_{|u|\le Y/2} F(u+x) \int_{-\infty}^{\infty} \Lambda K(\Lambda u)\, du + \int_{|u|>Y/2} \frac{1}{\pi^2 \Lambda u^2} |F(u+x)|\, du \\ &\le \max_{|u|\le Y/2} F(u+x) + \frac{4}{\pi^2 \Lambda Y} \Bigl(\sum_{\boldsymbol{n}} a_{\boldsymbol{n}}\Bigr).\end{aligned} \tag{10-2}$$

Now let $\mathcal{M}$ be a subset of indices $\boldsymbol{n}$ with $\lambda_{\boldsymbol{n}} \le \Lambda/2$, and let M be its cardinality. By Dirichlet's theorem there is an x_0 with $Y \le x_0 \le (6L)^M Y$ such that $\|x_0 \lambda_{\boldsymbol{n}}\| \le 1/(6L)$ for each $\boldsymbol{n} \in \mathcal{M}$. Consider

$$\sum_{|l|\le L} G(x_0 l) k(l/L) = \frac{1}{2} \sum_{\boldsymbol{n}} a_{\boldsymbol{n}} k(\lambda_{\boldsymbol{n}}/\Lambda) \sum_{|l|\le L} k(l/L) \cos(2\pi \lambda_{\boldsymbol{n}} x_0 l).$$

The sum over l equals

$$\frac{1}{L}\left(\frac{\sin(\pi L\lambda_{\boldsymbol{n}}x_0)}{\sin(\pi\lambda_{\boldsymbol{n}}x_0)}\right)^2,$$

which is nonnegative. We may therefore drop any terms we wish to from the sum over $\boldsymbol{n}$ to obtain a lower bound. Moreover, for each $\boldsymbol{n}\in\mathcal{M}$, $\cos(2\pi\lambda_{\boldsymbol{n}}x_0 l)\geq\cos(2\pi l/(6L))\geq\cos(\pi/3)\geq\frac{1}{2}$, so

$$\sum_{|l|\leq L}k(l/L)\cos(2\pi\lambda_{\boldsymbol{n}}x_0 l)\geq\frac{1}{2}\sum_{|l|\leq L}k(l/L)=\frac{L}{2}.$$

Thus

$$\sum_{|l|\leq L}G(x_0 l)k(l/L)\geq\frac{L}{4}\sum_{\boldsymbol{n}\in\mathcal{M}}a_{\boldsymbol{n}}k(\lambda_{\boldsymbol{n}}/\Lambda).$$

Since $G(-x_0 l)=G(x_0 l)$, there is an $1\leq l_0\leq L$ such that

$$G(0)+2G(x_0 l_0)\sum_{1\leq l\leq L}k(l/L)\geq\frac{L}{4}\sum_{\boldsymbol{n}\in\mathcal{M}}a_{\boldsymbol{n}}k(\lambda_{\boldsymbol{n}}/\Lambda).$$

The sum over l equals $(L-1)/2$, so we see that for this l_0

$$G(x_0 l_0)\geq\frac{1}{4}\sum_{\boldsymbol{n}\in\mathcal{M}}a_{\boldsymbol{n}}k(\lambda_{\boldsymbol{n}}/\Lambda)-\frac{1}{L-1}\sum_{\substack{\boldsymbol{n}\\ \lambda_{\boldsymbol{n}}\leq\Lambda}}a_{\boldsymbol{n}}.$$

From this and (10-2) we obtain

$$\max_{|u|\leq Y/2}F(u+x_0 l_0)\geq\frac{1}{4}\sum_{\boldsymbol{n}\in\mathcal{M}}a_{\boldsymbol{n}}k(\lambda_{\boldsymbol{n}}/\Lambda)-\frac{1}{L-1}\sum_{\substack{\boldsymbol{n}\\ \lambda_{\boldsymbol{n}}\leq\Lambda}}a_{\boldsymbol{n}}-\frac{4}{\pi^2\Lambda Y}\sum_{\boldsymbol{n}}a_{\boldsymbol{n}}.$$

Now $\mathcal{M}\subset[0,\Lambda/2]$, so $k(\lambda_{\boldsymbol{n}}/\Lambda)\geq\frac{1}{2}$. Furthermore, for $|u|\leq Y/2$ we have

$$x_0 l_0+u\geq Y-Y/2=Y/2,$$
$$x_0 l_0+u\leq(6L)^M YL+X/2\leq(6L)^{M+1}Y.$$

Thus, there is an x with $Y/2\leq x\leq(6L)^{M+1}Y$ such that

$$F(x)\geq\frac{1}{8}\sum_{\boldsymbol{n}\in\mathscr{M}}a_{\boldsymbol{n}}-\frac{1}{L-1}\sum_{\substack{\boldsymbol{n}\\ \lambda_{\boldsymbol{n}}\leq\Lambda}}a_{\boldsymbol{n}}-\frac{4}{\pi^2\Lambda Y}\sum_{\boldsymbol{n}}a_{\boldsymbol{n}}.$$

This completes the proof of Lemma 7. □

We now prove Theorem 3. Let

$$C_N(X)=\frac{1}{2\pi^3}\sum_{m,n\leq N}\frac{r(n)r(m)\cos(2\pi(\sqrt{m}+\sqrt{n})\sqrt{X})}{(mn)^{3/4}(\sqrt{m}+\sqrt{n})}$$

with $X^4\leq N\leq X^A$. We replace X by x^2 and will first show that for all large Z, there exists an x with $Z^{1/2}\leq x\leq Z^{3/2}$ such that

$$C_N(x^2)\geq\left(\frac{8}{\pi}+o(1)\right)\log\log x. \tag{10-3}$$

Since $Z\leq X\leq Z^3$, we also need $N\geq(Z^3)^4=Z^{12}$. We take $N=[Z^{12}(\log Z)^4]$.

Write

$$C_N(x^2) = \frac{1}{2\pi^3} F(x) = \frac{1}{2\pi^3} \sum_{\boldsymbol{n}} a_{\boldsymbol{n}} \cos(2\pi \lambda_{\boldsymbol{n}} x),$$

with $\lambda_{\boldsymbol{n}} = \lambda_{m,n} = \sqrt{m} + \sqrt{n},\ m, n \le N$, and

$$a_{\boldsymbol{n}} = a_{m,n} = \frac{r(m)r(n)}{(mn)^{3/4}(\sqrt{m}+\sqrt{n})}.$$

We apply Lemma 7 to $F(x)$ with

$$\mathcal{M} = \{(m, n) : \lambda_{m,n} \le \Lambda/2\}$$

and Λ to be determined later. Observe that $M = |\mathcal{M}| \ll \Lambda^4$. If $L, Y \ge 2$ with L an integer, then by Lemma 7 there is an x with $Y/2 \le x \le (6L)^{M+1} Y$ such that

$$\begin{aligned}\sum_{m,n \le N} & \frac{r(m)r(n)}{(mn)^{3/4}(\sqrt{m}+\sqrt{n})} \cos(2\pi(\sqrt{m}+\sqrt{n})x) \\ & \ge \frac{1}{8} \sum_{\boldsymbol{n} \in \mathcal{M}} \frac{r(m)r(n)}{(mn)^{3/4}(\sqrt{m}+\sqrt{n})} - \frac{1}{L-1} \sum_{\substack{\boldsymbol{n} \\ \lambda_{m,n} \le \Lambda}} \frac{r(m)r(n)}{(mn)^{3/4}(\sqrt{m}+\sqrt{n})} \\ & \qquad - \frac{4}{\pi^2 \Lambda Y} \sum_{\boldsymbol{n}} \frac{r(m)r(n)}{(mn)^{3/4}(\sqrt{m}+\sqrt{n})}. \end{aligned} \tag{10-4}$$

By arguments similar to those in the proof of Lemma 6 we have

$$\begin{aligned}\sum_{\substack{\boldsymbol{n} \\ \lambda_{m,n} \le \Lambda}} \frac{r(m)r(n)}{(mn)^{3/4}(\sqrt{m}+\sqrt{n})} & \le 2 \sum_{\sqrt{n} \le \Lambda - 1} \frac{r(n)}{n^{3/4}} \sum_{\sqrt{n} < \sqrt{m} \le \Lambda - \sqrt{n}} \frac{r(m)}{m^{5/4}} + O(1) \\ & \le 2 \sum_{n \le (\Lambda-1)^2} \frac{r(n)}{n^{3/4}} \sum_{n < m \le (\Lambda - \sqrt{n})^2} \frac{r(m)}{m^{5/4}} + O(1) \\ & \le 8\pi \sum_{n \le (\Lambda-1)^2} \frac{r(n)}{n} + O(1) \\ & \le 16\pi^2 \log \Lambda + O(1). \end{aligned}$$

As in the last section, half of this, namely $8\pi^2 \log \Lambda + O(1)$, is a lower bound. Applying similar reasoning to all three sums in (10-4), we find that there is an x such that $Y/2 \le x \le (6L)^{M+1} Y$ for which

$$\sum_{m,n \le N} \frac{r(m)r(n)}{(mn)^{3/4}(\sqrt{m}+\sqrt{n})} \cos(2\pi(\sqrt{m}+\sqrt{n})x) \ge \pi^2 \log \Lambda - O\left(\frac{\log \Lambda}{L}\right) - O\left(\frac{\log Y}{\Lambda Y}\right).$$

We now choose $Y = 2Z^{1/2},\ L = [\log Z]$. Then

$$(6L)^{M+1} \le \exp(O(\Lambda^4 \log \log Z)).$$

We need this to be at most $Z/2$, and it will be if we take $\Lambda = (\log Z)^{1/4}/(\log\log Z)^2$ with Z sufficiently large. With these choices of the parameters, we see that there exists an x with $Z^{1/2} \le x \le Z^{3/2}$ such that

$$C_N(x^2) = \frac{1}{2\pi^3} \sum_{m,n \le N} \frac{r(m)r(n)}{(mn)^{3/4}(\sqrt{m}+\sqrt{n})} \cos(2\pi(\sqrt{m}+\sqrt{n})x) \ge (1+o(1))\frac{1}{2\pi} \log\log Z.$$

Recalling that $X = x^2$, we find that there is an $X \in [Z, Z^3]$ such that

$$C_N(X) \ge (1+o(1))\frac{1}{2\pi} \log\log X.$$

Since $N = [Z^{12}(\log Z)^4] \gg X^{12}$ and $Z \ge X^{1/3}$, we see from (1-8) that

$$C(X) = C_N(X) + O\left(\frac{X \log N}{N^{1/4}}\right) = C_N(X) + O(1). \tag{10-5}$$

Theorem 3 now follows.

Appendix: Facts about Bernoulli polynomials

We collect the formulas we need about Bernoulli polynomials $B_k(x)$ here. Appendix B of [Montgomery and Vaughan 2007] is a convenient reference.

The first three Bernoulli polynomials are

$$B_1(x) = x - \tfrac{1}{2}, \quad B_2(x) = x^2 - x + \tfrac{1}{6}, \quad \text{and} \quad B_3(x) = x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{2}x. \tag{A-6}$$

If we let $\{x\}$ denote the fractional part of the real number x and replace x by $\{x\}$ in B_j, the resulting functions are periodic with period 1. They therefore have Fourier series expansions, and these are given by

$$B_j(\{x\}) = -\frac{j!}{(2\pi i)^j} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^{2\pi i n x}}{n^j} \quad (j = 1, 2, 3).$$

These hold for all real x, except in the case of B_1 we need $x \notin \mathbb{Z}$. With this stipulation we immediately have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\sin(2\pi n x)}{\pi n} &= -B_1(\{x\}) = \tfrac{1}{2} - \{x\}, \\ \sum_{n=1}^{\infty} \frac{\cos(2\pi n x)}{\pi^2 n^2} &= B_2(\{x\}) = \{x\}^2 - \{x\} + \tfrac{1}{6}, \\ \sum_{n=1}^{\infty} \frac{\sin(2\pi n x)}{\pi^3 n^3} &= \tfrac{2}{3}B_3(\{x\}) = \tfrac{2}{3}\{x\}^3 - \{x\}^2 + \tfrac{1}{3}\{x\}. \end{aligned} \tag{A-7}$$

We collect the other formulas we need in:

Lemma 8. *We have*

$$\int_1^{\infty} \frac{B_1(\{x\})}{x}\,dx = \tfrac{1}{2}\log 2\pi - 1, \qquad \int_1^{\infty} \frac{B_2(\{x\})}{x^2}\,dx = \log 2\pi - \tfrac{11}{6},$$
$$\int_1^{\infty} \frac{B_3(\{x\})}{x^3}\,dx = \tfrac{3}{2}\log 2\pi - \tfrac{11}{4},$$

and

$$\int_{1/2}^{\infty} \frac{B_1\big(\{x+\frac{1}{2}\}\big)}{x}\,dx = -\tfrac{1}{2}+\tfrac{1}{2}\log 2, \quad \int_{1/2}^{\infty} \frac{B_2\big(\{x+\frac{1}{2}\}\big)}{x^2}\,dx = -\tfrac{2}{3}+\log 2,$$
$$\int_{1/2}^{\infty} \frac{B_3\big(\{x+\frac{1}{2}\}\big)}{x^3}\,dx = -1+\tfrac{3}{2}\log 2.$$

Proof. From [Montgomery and Vaughan 2007, p. 503 and exercise 23 on p. 508],[1] we find that

$$\int_{1}^{\infty} \frac{B_2(\{x\})}{x^2}\,dx = \log 2\pi - \tfrac{11}{6}. \tag{A-8}$$

We integrate the left-hand side by parts and find that

$$\int_{1}^{\infty} \frac{B_2(\{x\})}{x^2}\,dx = \frac{B_2(\{x\})}{-x}\bigg|_1^{\infty} + \int_1^{\infty} \frac{B_2'(\{x\})}{x}\,dx.$$

Now $B_k'(x) = kB_{k-1}(x)$ for $k \ge 1$, see [Montgomery and Vaughan 2007, p. 495], and $B_2(0) = \frac{1}{6}$ by (A-6). Hence, by (A-8),

$$\int_{1}^{\infty} \frac{B_1(\{x\})}{x}\,dx = \tfrac{1}{2}\log 2\pi - 1.$$

Similarly, we find that

$$\int_{1}^{\infty} \frac{B_3(\{x\})}{x^3}\,dx = \tfrac{3}{2}\int_1^{\infty} \frac{B_2(\{x\})}{x^2}\,dx = \tfrac{3}{2}\log 2\pi - \tfrac{11}{4}.$$

This establishes the first three formulas of the lemma.

The second set of formulas can be treated in the same way. However, we have not found a ready reference for the value of

$$\int_{1/2}^{\infty} \frac{B_1\big(\{x+\frac{1}{2}\}\big)}{x}\,dx = \int_1^{\infty} \frac{B_1(\{x\})}{x-\frac{1}{2}}\,dx,$$

so we derive it from scratch.

By Riemann–Stieltjes integration

$$\begin{aligned}
&\sum_{1\le n\le N} \log\big(n-\tfrac{1}{2}\big)\\
&= \int_1^N \log\big(x-\tfrac{1}{2}\big)\,dx + \int_{1^-}^N \log\big(x-\tfrac{1}{2}\big)\,d\big([x]-x+\tfrac{1}{2}\big)\\
&= \Big(\big(N-\tfrac{1}{2}\big)\log\big(N-\tfrac{1}{2}\big)-\big(N-\tfrac{1}{2}\big)\Big) - \big(\tfrac{1}{2}\log\tfrac{1}{2}-\tfrac{1}{2}\big) + \big([x]-x+\tfrac{1}{2}\big)\log\big(x-\tfrac{1}{2}\big)\Big|_{1^-}^N - \int_1^N \frac{[x]-x+\frac{1}{2}}{x-\frac{1}{2}}\,dx\\
&= N\log\big(N-\tfrac{1}{2}\big) - N + 1 + \int_1^N \frac{B_1(\{x\})}{x-\frac{1}{2}}\,dx.
\end{aligned} \tag{A-9}$$

[1] Note that $B_1(x)$ in exercise 23(a) should read $B_1(\{x\})$.

On the other hand,

$$\sum_{1\le n\le N} \log\big(n-\tfrac12\big) = \log\bigg(\prod_{1\le n\le N} \frac{2n-1}{2}\bigg) = \log\bigg(\frac{(2N)!}{2^{2N}N!}\bigg).$$

By Stirling's formula, $\log n! = n\log n - n + \frac12 \log 2\pi n + O(1/n)$, so

$$\begin{aligned}\sum_{1\le n\le N} \log\big(n-\tfrac12\big) &= \big(2N\log 2N - 2N + \tfrac12\log 4\pi N\big) - 2N\log 2 - \big(N\log N - N + \tfrac12 \log 2\pi N\big) + O(1/N)\\ &= N\log N - N + \tfrac12\log 2 + O(1/N).\end{aligned}$$

Combining this and (A-9) we obtain

$$\begin{aligned}\int_1^N \frac{B_1(\{x\})}{x-\frac12}\,dx &= \big(N\log N - N + \tfrac12\log 2 + O(1/N)\big) - \big(N\log\big(N-\tfrac12\big) - N + 1\big)\\ &= N\log\frac{N}{N-\frac12} + \tfrac12\log 2 - 1 + O(1/N).\end{aligned}$$

Letting $N\to\infty$, we deduce that

$$\int_{1/2}^\infty \frac{B_1\big(\big\{x+\frac12\big\}\big)}{x}\,dx = \int_1^\infty \frac{B_1(\{x\})}{x-\frac12}\,dx = -\tfrac12 + \tfrac12\log 2.$$

We now argue as above to find the value of the remaining two integrals. Integration by parts using $B_j'(x) = jB_{j-1}(x)$ reveals that

$$\int_{1/2}^\infty \frac{B_j\big(\big\{x+\frac12\big\}\big)}{x^j}\,dx = \frac{B_j(\{1\})2^{j-1}}{(j-1)} + \frac{j}{j-1}\int_{1/2}^\infty \frac{B_{j-1}\big(\big\{x+\frac12\big\}\big)}{x^{j-1}}\,dx.$$

Thus, using (A-6) we find that

$$\begin{aligned}\int_{1/2}^\infty \frac{B_3\big(\big\{x+\frac12\big\}\big)}{x^3}\,dx &= \tfrac32\int_{1/2}^\infty \frac{B_2\big(\big\{x+\frac12\big\}\big)}{x^2}\,dx,\\ \int_{1/2}^\infty \frac{B_2(\{x+\frac12\})}{x^2}\,dx &= \tfrac13 + 2\int_{1/2}^\infty \frac{B_1(\{x+\frac12\})}{x}\,dx\end{aligned}$$

Thus, the B_2 integral equals $-\frac23 + \log 2$ and the B_3 integral equals $-1 + \frac32\log 2$. This completes the proof of the lemma. □

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